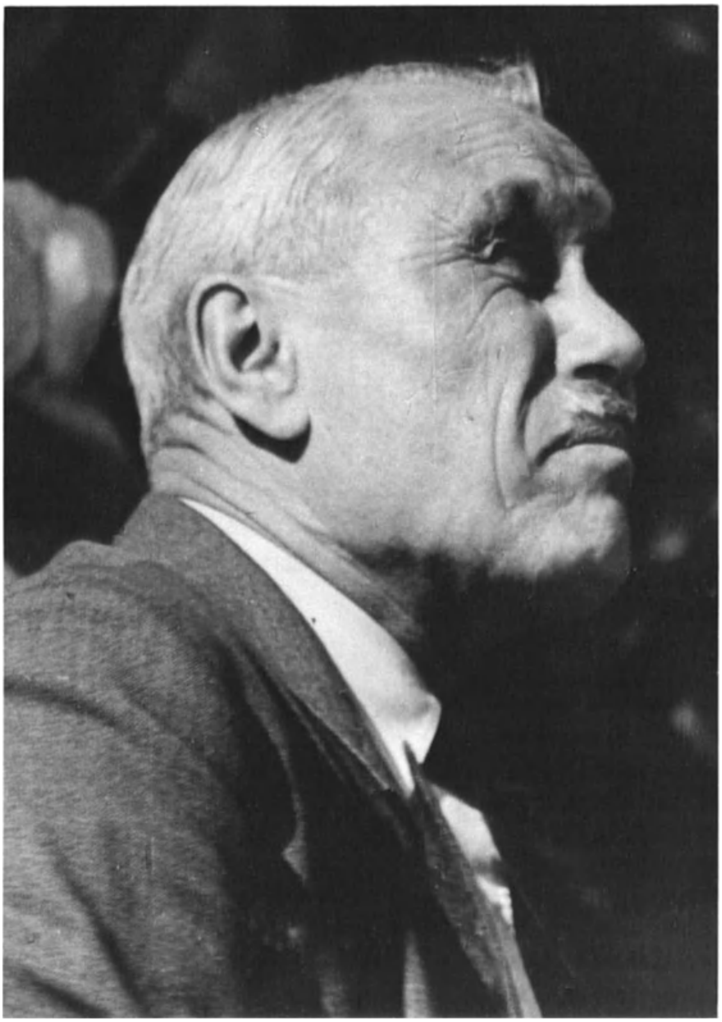

Martin Hotine in 1956

Martin Hotine

Differential Geodesy

Edited with Commentaries by Joseph Zund

Contributions by
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Foreword

After more than 20 years, the ideas expressed in Martin Hotine's treatise *Mathematical Geodesy* remain fresh and vibrant. Although geodesy has advanced remarkably over this span of time, this has been due mainly to increased precision of measurement and enlarged computing power. The bulk of the theory in *Mathematical Geodesy* still remains at the forefront of geodetic research. Examples of areas of current geodetic interest for which *Mathematical Geodesy* can provide a fruitful seedbed are precise coordinate systems linking various reference frames, and direct vector processing on modern computers.

We thus deem the appearance of this collection of Hotine's seminal papers which served as the basis for *Mathematical Geodesy* to be extremely providential. It is our hope that *Differential Geodesy* will motivate those interested in geodetic research to go forward, either by seeking out *Mathematical Geodesy*, or by continuing directly along the paths mapped out so elegantly by Hotine. We also consider it fortunate that the publication of this volume follows closely the issuance of *Intrinsic Geodesy* (Springer-Verlag 1985), which performed a similar service by making available the original research papers of Antonio Marussi, Hotine's friend and colleague. These two volumes are companions not only in time and subject matter, but also in spirit.

Martin Hotine inspired many persons during his lifetime, as we personally witnessed. But the true test of an outstanding scientist is his influence on those coming afterward. Joseph Zund, with a distinguished background in mathematical physics and differential geodesy, encountered *Mathematical Geodesy* by chance in 1981, and was immediately attracted by its creativity and richness of concepts. This happy interaction has yielded a series of publications advancing many of Hotine's ideas, and settling some long-standing problems posed by Hotine. We express our appreciation to Professor Zund for taking the time and effort to assemble, edit, and provide valuable comments to, the articles contained herein. We trust it was more a labour of love than a burden.

A debt of gratitude is also owed to John Nolton, who assembled the comprehensive bibliography from often quite remote sources. Finally, the willingness of Springer-Verlag to sponsor this undertaking deserves explicit recognition.

Charles A. Whitten   Bernard H. Chovitz

Editorial Introduction

Joseph Zund

The purpose of this monograph is to make available to the geodetic community a selection of the largely unpublished papers of Martin Hotine (1898–1968). It is intended to be a companion volume to the collection of Antonio Marussi's papers in his *Intrinsic Geodesy* (Springer 1985). Together, these two volumes contain the essential and original ideas underlying the Marussi-Hotine approach to mathematical geodesy.

The collection contains nine papers:

1. Trends in mathematical geodesy. Bolletino di Geodesia e Scienze Affini, anno XXIV (1965) 607–622
2. Adjustment of triangulation in space. (I.A.G. Symposium on European Triangulation, Munich, 1956)
3. Metrical properties of the Earth's gravitational field. (I.A.G. General Assembly, Toronto, 1957)
4. Geodetic coordinate systems. (I.A.G. General Assembly, Toronto, 1957)
5. A primer on non-classical geodesy. (First Symposium on Three-Dimensional Geodesy, Venice, 1959)
6. The third dimension in geodesy. (I.A.G. General Assembly, Helsinki, 1960)
7. Harmonic functions. (Second Symposium on Three-Dimensional Geodesy, Cortina d'Ampezzo, 1962)
8. Downward continuation of the gravitational potential. (I.A.G. General Assembly, Lucerne-Zürich, 1967)
9. Curvature corrections in electronic distance measurements. (I.A.G. General Assembly, Lucerne-Zürich, 1967).

Apart from papers 1 and 7, none of these has previously been published in the open literature, or has been generally available to non-participants of these meetings. Actually, paper 7 appeared only as an appendix in P.L. Baetsle's comprehensive report (Baetsle 1963) of a meeting. In toto they represent the primary source material for Hotine's bold and dramatic approach to mathematical geodesy. They clearly show the evolution of his ideas, and furnish valuable insight into the contents of his posthumous treatise *Mathematical Geodesy* (U.S. Department of Commerce, Washington, D.C. 1969). Henceforth, since we will frequently refer to this book, it will be convenient to denote it by MG, and likewise IG designates Marussi's *Intrinsic Geodesy.* For each of these nine papers we have added an editorial commentary which contains some introductory remarks, indicates the relation of the paper to the corresponding discussions in MG, and gives more recent results dealing with Hotine's work.

The above nine items represent virtually all of Hotine's work on differential geodesy intended for presentation/publication, except for three items, which are readily available in the Bulletin Géodésique (Hotine 1966a, 1966b; Hotine and Morrison 1969). The contents of Hotine (1966a, b) are briefly discussed in our editorial commentary on paper 1, and Hotine and Morrison (1969) is an addendum to Hotine's work on satellite geodesy in Chapter 28 of MG.

Paper 1 is a delightful expository article and is essentially a companion to Marussi's *From Classical Geodesy to Geodesy in Three Dimensions*, which appeared as the first paper in IG (see IG pp. 2–12). Unquestionably, the most definitive and significant of these reports is item 5. It is simply a masterpiece, and Marussi commented (1969):

> "It is written in a style which makes it difficult to know which to admire most: its conciseness, or its eloquence."

In effect, it is the pièce de résistance of Hotine's formulation of mathematical geodesy. It refines and synthesizes, but does not totally supersede the contents of items 2–4, and 6 is a natural continuation of it. Papers 7–9 stand apart, but are introductions to the material in Chapter 21 of MG.

The viewpoint and content of these papers differ somewhat from that contained in MG. In military parlance, which we think Hotine might appreciate, we could characterize the former as discovery and reconnaissance, and the latter as conquest and occupation. MG aimed to present a comprehensive and unified deduction of the theoretical aspects of geodesy from basic principles in the spirit of mathematical physics. While this is admirable and suitable for the cognoscente, for someone wishing to sample the flavour and scope of Hotine's approach, the size and conciseness of MG is rather overpowering. We believe that the present monograph may be useful as a preliminary introduction to MG. It is by no means a replacement for MG, but rather a complement and guide to it.

As indicated in Chovitz's preview paper (reprinted in this book on pp. 157–170), Hotine was uncertain about what to choose as a suitable title for his treatise. We faced a similar dilemma in selecting a title for the present monograph. The best, and obvious, title had already been used, and to employ it again would have led to confusion with MG. Our decision, which we hope that Hotine would have found acceptable, was to use the title *Differential Geodesy*. This choice was a compromise, but one which is not radical, and which Hotine had previously considered for MG.

In preparing the material for publication, we have made minor changes of an editorial character, but nothing of substance has been altered. We have added a few missing equation numbers (without disturbing the original numbering of his equations) and Hotine's references have been made more precise. The format of the papers has been adjusted to conform to that employed in MG.

I would like to express my profound gratitude to Bernard Chovitz and Charles Whitten for their cooperation and permission to include their valuable articles in the monograph. I am also grateful to John Nolton for his assistance in compiling the Bibliography of Hotine's papers. Special thanks are also due to Erik Grafarend for his advice and enthusiastic encouragement of this project. Finally,

I am indebted to the Hotine family for their help and for supplying the photograph which appears as the frontispiece.

During part of the time of the preparation of this material, I was under contract to the Geophysics Laboratory of the Air Force Systems Command and I would like to express my appreciation to that agency for allowing me to devote part of my research time to this project.

Unfortunately, both Hotine and Marussi were taken from us too soon, but they have left us, as a challenge and legacy, the foundations of a beautiful and imaginative approach to geodesy. It is my hope that the present monograph and the companion volume of Marussi will serve to make their ideas better known in the geodetic community, and in this spirit the monograph is dedicated to the memory of Martin Hotine and his friend Antonio Marussi.

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1 Trends in Mathematical Geodesy[1]

Rutherford, a physicist, is reputed to have said that there are only two sciences: physics and stamp-collecting. He meant, I suppose, that all other sciences do nothing but collect and classify, which is not quite true, because some of them speculate where they cannot accumulate, whereas physics is subject to the sterner rational discipline of mathematics. Indeed, apart from a certain amount of gadgetry, physics nowadays seems to be nothing but mathematics.

What sort of mathematics? Inglis, sometime Professor of Mechanical Sciences at Cambridge, England – it would be called Engineering in any other University – used to say that mathematics is a can-opener; but whereas the engineer should use it to open cans of beef, the mathematician uses it to open cans of can-openers. The gulf between the two is much wider nowadays. The mathematicians have retired into a shell of formal logic and existence theorems and are no longer concerned with opening any sort of can; whereas the engineer is interested only in binary arithmetic and tends to feed the entire can to a computer.

We geodesists get the worst of both worlds. As "snappers-up of unconsidered trifles" (Shakespeare 1611) in the shape of deflections and anomalies, we are reviled by anti-philatelic physicists; and since we mostly have a basic engineering education we are abused by all other scientists, who classify engineers as half-educated morons in boiler suits. Yet in my time academic engineering consisted entirely of physics, that is mathematics, classical mechanics, electricity and magnetism, hydraulics and thermodynamics. I hope the only change has been to bring all that up to date and perhaps to add some wave mechanics. We need also to start up an engine occasionally, but only for the purpose of measuring its inefficiency. And in the later years of an engineering course, when the pure physicists go "boundless inward to the atom", we geodesists should go outwards into bounded or unbounded space – it is still possible to choose which – until we meet again in a stationary or receding nebula.

Meanwhile, we have only ourselves to blame for departing from our highest traditions. Men like Newton, Lagrange, Laplace, Gauss (Todhunter 1873, passim) – to name only a few – were the geodesists of their day in the sense that they were primarily interested in measuring the whole Earth, its attributes and its surroundings. In the process they made considerable contributions to mathematics

[1] Manuscript dated 15 September 1964 (Washington D.C.), appendix dated 15 November 1964 (Washington D.C.). Published in Bolletino di Geodesia e Scienze Affini, anno XXIV, 1965, pp. 607–622.

and physics. Gauss, for example, worked out a complete differential geometry of surfaces and founded the theory of errors in conjunction with the reduction and projection of a survey of Hanover. Since then we have mostly basked in Gaussian glory. It is true that we have vastly improved the accuracy of measurement and from time to time have helped to confirm or destroy some theories of the constitution of the Earth. But until recently we have done little on the mathematical side beyond elaborating the old, old formulae. We have not even kept abreast of the development by Ricci, Riemann and others of Gaussian differential geometry to three or more dimensions, which Gauss would probably have originated himself had he moved from Hanover to the Himalayas. We have imported some of the jargon of modern statistics into Gaussian least squares, but have done little or nothing to examine modern probability theory and adapt it to our use.

If we are to do our job properly, and incidentally to be less lightly considered by the scientific community, we must get back to a stricter mathematical discipline as the rational basis of our work; and in the process it may well be that we can once more be of greater use to other sciences than by merely supplying them with data. We have spent too long on re-designing reference spheroids and have almost come to regard this activity as an end in itself. We shall not have done much more to advance the breadth and the depth of human knowledge by the time we have put a few reliable contours on the geoid.

To avoid the charge that all this is just talk; that there is nothing more to be done in the theory of measurement of the Earth; that this is an age of specialization and if there is any work to be done on mathematical geodesy it will have to be done by pure mathematicians – to avoid all this and to illustrate my point, I will give you a couple of examples. These are not epoch-making discoveries. They may even be wrong in the same way that anything can be wrong until it has been independently checked several times. But they are new basic ideas in fields which have been ploughed, sown, reaped and gleaned for centuries. And they come up at the first scratch with new tools, new, that is, in the sense that geodesists have not used them before. They will be justified if they do no more than suggest to you that, in geodesy as in much else, the Canadian poet is still right who said: "The highest peaks haven't been climbed yet; the best work hasn't been done".

The Gravitational Field

All geodetic measurements are necessarily made in a gravitational field which provides us with our ideas of "horizontal" and "vertical" and it is natural that we should turn our attention to that first. For some time I have had an uncomfortable feeling that there is something missing, either in the classical theory of gravitation itself, or in the mathematical handling of it. For instance, it is well known that the form of an equipotential surface settles the value of Newtonian gravity on the surface, apart from constants, which suggests that gravity is in some way intrinsic to the surface, except for the nature – flat or curved – of the surrounding space. It should be possible to express gravity in terms of the first- and second-order magnitudes of the surface, yet the actual relation has been formulated in a very few simple cases only, by using particular coordinate systems. Gravity on a rotating spheroid is, for example, found by using spheroidal coordinates as $g = AN + B/N$, where A and B are constants over the surface and N is the principal radius of curvature perpendicular to the meridian. We can similarly obtain a relation for a triaxial ellipsoid, but nothing more general. This could be due to limited knowledge of how to manipulate more general harmonic functions, but when we try a different approach, independent of any particular coordinate system, we find that the law of gravitation itself offers no help. We can use the law in its harmonic form to determine the vertical variation of gravity by means of Brun's formula, but not the horizontal variations.

There is a similar element of indetermination in Newtonian dynamics. The artificial satellite is providing us with a wonderful new tool. But unless we are prepared to restrict its use to a light in the sky, an elevated triangulation beacon visible over long distances, we are forced to consider motion in orbits closer to the Earth than we have ever dealt with before. The actual motion must, of course, be regulated by the nature of the gravitational field, yet the Newtonian equations of motion in a free orbit do not contain the law of gravitation at all. We derive particular solutions of the equations by assuming a harmonic potential, but the equations themselves do not require this and would presumably apply equally well to other laws of gravitational attraction.

On the other hand, the corresponding law in four-dimensional theories of gravitation is woven into the metric of space-time from which the equations of motion are derived; it is impossible to obtain any solutions independent of the law of gravitation. The main object of these four-dimensional systems is to allow for very large velocities approaching the speed of light at cosmic distances. They all reduce to the Newtonian system for small velocities or for weak central fields, and indeed are made to do so. There would accordingly be no point in facing the formidable complications of using four-dimensional methods on, say, the orbits of near-Earth satellites. But nowadays there may well be a requirement for a *three*-dimensional dynamical system more rigidly connected to the actual law of gravitation.

Newton's law of gravitation was originally formulated to describe planetary motion in the solar system, and there can be no doubt that in these circumstances, of great distances between nearly spherically symmetrical masses, it has proved remarkably accurate. A pious opinion before Newton – not a scientific hypothesis or even an article of faith – was that the Earth is continuously steered in towards the life-giving Sun by a cohort of angels. Newton himself would have been the last to deny such a poetic description of Divine Providence, and indeed for all we know now it could be literally true. All Newton wanted to do was to describe the *results* of these angelic labours in a form which could be programmed for computation, and in the process he introduced notions of gravitational force and potential which have no more tangible existence than the angels. His inverse-square law of supposed force is nevertheless far more than inspired guess. No other feasible law would result in closed orbits in a central field, and it has stood up to numerical tests in the solar system in a remarkable way for centuries. But we have no right to argue from the particular to the general and assert that, because the law is necessary and has been amply verified in a central symmetrical field, it is therefore true of the field close to a large unsymmetrical mass. We can certainly say that whatever the law is near such a mass, it must degenerate to an inverse square at great distances, but the converse is by no means necessarily true. The usual way of overcoming the difficulty is to assert that the unsymmetrical mass consists of particles, each of which sets up a central field, just as it would if it were alone. The more complicated unsymmetrical fields is accordingly obtained by superimposing an infinite number of central fields. Mathematically this may be true, but it still has to be verified experimentally that matter does, in fact, behave like a conglomeration of independent particles in its gravitational effects. This has never been done, at any rate to the degree of accuracy we now seek.

To provide experimental verification we need an alternative gravitational theory, which we have never had, based on slightly different but equally plausible assumptions. This should be used on the same observations as the Newtonian system and the results compared statistically. The difference is not likely to be large enough for laboratory comparisons and the effect, if any, would mostly disappear at astronomical distances, but the artificial satellite, which provides us with the means of exploring the entire near-Earth gravitational field to a high degree of accuracy, should soon provide us with enough observations for a practical test. We geodesists may in that way be able to make a more significant contribution to human knowledge, more in keeping with our traditional role, than by arguing about whose value of an eighth harmonic is the best and what it means. We should first provide some assurance that an eighth Newtonian harmonic has any meaning at all.

Generalized Gravitational Equations

The ideas of "horizontal" and "vertical", which we can materialize with bubbles and plumb lines, must find some place in any gravitational theory, at any rate in the near-Earth field. We accordingly assume that there exists in Nature a family of surfaces, which we define as "level"; and a family of lines everywhere orthogonal to these surfaces, which we define as "vertical". We assume further that the "level" surfaces can be expressed mathematically by means of a continuous differentiable scalar function of position V, which is constant over any particular surface; the value of the constant being, of course, different for different surfaces.

Two neighbouring level surfaces, V = const. and V+dV = const., are separated by a distance (dn) measured along a vertical line. We define the limit

$$g = -dV/dn \tag{1}$$

as the "distance function".

In index notation for vectors, we write υ_r for the unit vector in the direction of the vertical and V_r for the gradient of V. All the above definitions are then included in the vector equation

$$V_r = -g\upsilon_r \ . \tag{2}$$

All this is pure geometry. If V were the Newtonian potential, then g would be the gravitational acceleration or force per unit mass, but for the present we do not say this.

We shall follow most gravitational theories (including both Newton's and Einstein's) by making the assumption that free orbits or trajectories are subject to a certain minimum condition, which appears in other natural laws. Such a minimum condition is generally reckoned to be an expression of the basic economy of Nature, which we occasionally see in ourselves as a resistance to change. The Duke of Cambridge, still Commander-in-Chief of the British Army at the age of 80, said he had finally come to the conclusion that "all change is for the worse", and in this he was merely voicing a slightly military over-simplification of a fundamental law of Nature: that some change is inevitable and does occur, but should be as little as possible.

In classical three-dimensional mechanics the law takes the form of the Principle of Least Action (McConnell 1931). We define the Action as

$$A = \int v\,ds = \int v^2 dt \ , \tag{3}$$

that is, the integral of the velocity (v) between two points on the path of a freely moving particle of unit mass, or the time integral of twice the "kinetic energy". We then assert that the Action is less along the actual path than it would be along any neighbouring path between the two points considered.

Here I must digress and introduce the notion of curved space. We are all familiar with flat and curved spaces of two dimensions. The flat or Euclidean space is distinguished by the fact that we can find coordinates (x, y) such that the line element (ds) is given in the Pythagorean form

$$ds^2 = dx^2 + dy^2 \ .$$

The line element on a curved surface cannot be expressed as simply, whatever coordinates we choose. Now suppose we set up a one-to-one correspondence between points in a plane and the corresponding points on a curved surface. The same numbers (x, y) will now serve to define the position of points on the curved surface as well as on the plane, but the line element on the curved surface will be given by the more complicated expression

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \ ,$$

in which E, F, G are functions of position, that is of x and y. Those functions and their first and second derivatives are related to the intrinsic or specific curvature of the surface, which is a property of the surface itself without regard to the surrounding space, by means of the Gauss characteristic equation. Gauss himself called it his "egregious theorem".

We can go further with Gauss into Hanover and set up a one-to-one correspondence with points on a plane such that the line element on the surface is given in the simpler form

$$ds^2 = m^2 (dx^2 + dy^2) \ .$$

In other words, the ratio of the two line elements is a function of position m, which, of course, varies from point to point, but nevertheless implies that small corresponding figures around a point are similar. For this reason we call the transformation conformal, and we use it a great deal, as Gauss did, in the theory of map projections. In particular, it is evident that angles between corresponding directions around a point are preserved. We need not start with Cartesian coordinates x, y in the plane but can use any coordinates on either surface provided the two line elements are connected by the relation

$$d\bar{s}^2 = m^2 ds^2 \ .$$

But for such a transformation to be possible, the scale factor (m) must be related to the intrinsic curvature of the surface by means of the Gauss characteristic equation. Alternatively, we can use any continuous differentiable function of position we like for m and that will settle the curvature of the surface.

We now add a dimension to these ideas. We can have a flat space in which it is possible to find Cartesian coordinates x, y, z such that

$$ds^2 = dx^2 + dy^2 + dz^2$$

and a curved space whose line element is not expressible in this simple form. It will now contain six funtions instead of the three E, F, G. These functions and their derivatives provide, not one single function specifying the curvature as in two dimensions, but six distinct functions, known as the components of the curvature tensor. We can set up a conformal transformation from a flat space to a curved space, in which case there are six relations between derivatives of the scale factor and the components of the curvature tensor[2]. If we start with space of a defined curvature, then this drastically reduces the possible choice of scale factor. We can, however, start with any scale factor we like and this will settle all components of the curvature tensor.

The three-dimensional space in which we live is, so far as we know, flat. We cannot sense its flatness any more than a two-dimensional being, crawling along what it considers to be a straight line, could have any idea that this world surface is curved. Not so long ago (unfortunately, I have no reference), a room containing some simple furniture was actually constructed, using the geometry of a space of large constant curvature, and various observers were invited to look at it from the outside through two holes in the wall. None could see anything odd about it. However, if they had been allowed inside with a tape they would very soon have realized that what looked like a cube from the outside wasn't one. What a mess the photogrammetrists would make of things if they attempted to do without a floating mark!

For this reason we have started with flat space although we could almost as easily deal with two curved spaces. The curved space can be considered simply as a mathematical device without physical significance. In fact, we could obtain all the following results analytically by the calculus of variations, although not

[2] A first-order tensor in three dimensions is a vector having three components which transform in a particular way when we change coordinates. A second-order tensor is essentially a sum of products of vectors, and in general, has nine components. The curvature tensor is symmetrical and so has only six distinct components.

as simply. The approach via curved space moreover has other important applications as we shall see later.

We propose to set up a conformal transformation in which the scale factor is the velocity v, but before we can do this we must arrange for v to be defined not only along a particular trajectory but also throughout a region of space. We do this by taking a family of trajectories all having some feature in common; for example, all those trajectories starting from the same point in space with the same velocity, but in different directions. In a continuous field, one member of the family will then pass through every point in a region of space and we define velocity at the point as the velocity in that particular trajectory.

If we now set up the conformal transformation

$$d\bar{s} = v\,ds \ ,$$

integrate between two fixed points on a trajectory and require the Principle of Least Action [Eq. (3)] to hold, then we shall have

$$\bar{s} = A \ , \tag{4}$$

a minimum for the path, so that $\bar{s}$ or A is the length of the *geodesic* joining the two points in the curved space.

It can be shown (Levi-Civita 1926 p. 228, with some extensions) that the geodesics of the curved space correspond to curves in the original flat space which have the following properties:

(a) The arc rate of change of the logarithm of the scale factor in the direction of the principal normal to the curve is equal to the principal curvature of the curve, and
(b) There is no change in the scale factor in the direction of the binormal to the curve[3].

Now the gradient of the scale factor (v_r), like any other vector, is expressible in terms of three mutually perpendicular vectors which we shall take as l_r, m_r, n_r: respectively, the unit tangent to an trajectory of the family, its principal normal and binormal. Say

$$v_r = A l_r + B m_r + C n_r \ .$$

Multiplying this across in turn by l^r, m^r, n^r we find that $A = \partial v/\partial s$, if s is the arc length of the trajectory; B, from the condition (a) above is κv, where κ is the principal curvature of the trajectory; and C, from the condition (b) above is zero. So,

$$v_r = (\partial v/\partial s) l_r + (\kappa v) m_r \ , \tag{5}$$

[3] There is a similar proposition in two dimensions which is well known in the theory of conformal map projections. A geodesic on the curved surface corresponds to a curve on the plane surface whose curvature is similarly equal to the arc rate of change of the logarithm of the scale factor in the transverse direction. In two dimensions, a curve has no binormal.

which can also be written as

$$v_r = \frac{\delta(v l_r)}{\delta s} , \tag{6}$$

in which $\delta/\delta s$ implies intrinsic differentiation along the trajectory of the family which passes through the point under consideration. We may also multiply by $v = ds/dt$, t being the time, and write

$$\frac{\delta(v l_r)}{\delta t} = v v_r = \left(\frac{1}{2} v^2\right)_r . \tag{7}$$

Now the first member of this equation is a vector and it must also be the gradient of a scalar, say W, because the right-hand member is. So we can write further

$$\frac{\delta(v l_r)}{\delta t} = v v_r = \left(\frac{1}{2} v^2\right)_r = W_r . \tag{8}$$

This is the general differential vector equation of motion of all orbits or free trajectories satisfying the Principle of Least Action and depending on nothing but that principle. The vector $v l_r$ is evidently the velocity vector, in magnitude and direction, and is not to be confused with v_r the gradient of the scalar velocity. Equation (8) states that the time rate of change of the velocity vector, that is the acceleration vector, is equal to the gradient of $\frac{1}{2} v^2$.

There is one more property of a family of geodesics in curved space which we can use. It is well known (Eisenhart 1960 p. 57) that such a family cuts orthogonally a family of surfaces, known as geodesic parallels, and that these surfaces are parallel in the sense that any two of them cut off the same length measured along any geodesic of the family between them[4]. We can, accordingly, say that $\bar{s}$ or A measured from one of the surfaces is constant over any other of the surfaces. The geodesic parallels can accordingly be expressed as A = const., while the geodesics themselves are in the direction of the gradient $\bar{s}_r$ or A_r. Transforming the unit vector along the geodesics back to the flat space we can show without much difficulty that

$$v l_r = A_r . \tag{9}$$

This again is pure geometry resting solely on the Principle of Least Action. It is an integral of the equations of motion (8). It shows that the A = constant surfaces (corresponding to the geodesic parallels) contain the principal normal and binormal to l_r so that we have the following picture in the osculating plane of (l_r) (Fig. 1):

[4] This is a generalization of a theorem due to Gauss for parallel surfaces in flat space.

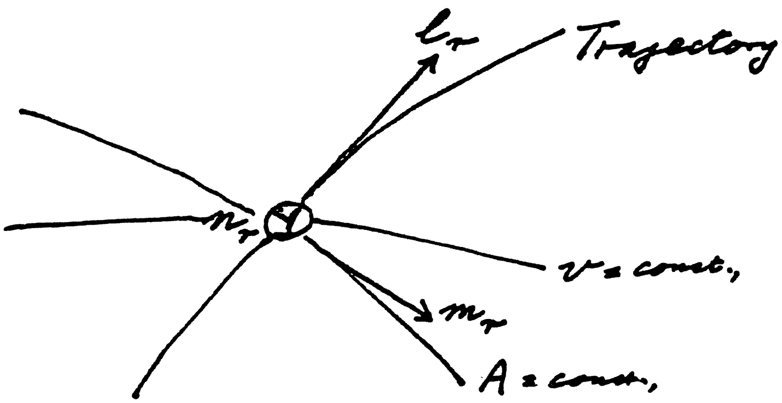

Fig. 1

From the fact that A, defined as above, is a scalar function of position, we can say that the curl of $v l_r$ is zero or that the curl of l_r is κn_r. This fact may sometimes be useful, although it conveys no more information than Eqs. (8) and (9).

Note that Eqs. (9) and (2) are entirely similar. We already have the differential geometry of the V-surface fully worked out without necessarily applying any restriction on V, such as requiring it to be a harmonic function (Hotine 1957a). All these properties can be applied to the A-surface by a simple change of notation.

We have said nothing so far about the gravitational field, except that bodies move in it in accordance with the Principle of Least Action. Nor have we provided any hook-up with the level surfaces and verticals. We can easily derive the Newtonian field from these Eq. (8). We then obtain a particular solution of these equations on the further assumption that V is a harmonic function.

There are, however, any number of other solutions. Suppose, for example, we make A equal to some function of V for a particular family of trajectories, which in that case would be the vertical lines. Then Eq. (9) tells us at cone that

$$v = g f(V) \ , \tag{10}$$

in which f(V) is some arbitrary function of V. This settles the velocity field for any family. By integrating the two last members of Eq. (8) (and this is a general result, corresponding to the Newtonian conservation of energy), we have

$$\tfrac{1}{2} v^2 - W = D \ , \tag{11}$$

in which D is an absolute constant for the family. We can further make the field Newtonian in the special case of spherical symmetry by putting

$$W = V = \mu/r \quad \text{and} \quad g = \mu/r^2 = V^2/\mu$$

so that the arbitrary function of V becomes

$$f(V) = \frac{\mu}{V^2}(2V+2D)^{1/2} \tag{12}$$

and

$$v = \frac{\mu g}{V^2}(2V+2D)^{1/2} \ . \tag{13}$$

In this equation g is the "distance function" for the V-surfaces and neither g nor V is Newtonian.

In solution for V of the equations of motion with this value of the velocity is certainly not as simple as the Newtonian case and would require the evaluation of more parameters from more observations, but nothing seems to daunt the binary arithmeticians these days. The resulting dynamical system is Newtonian in a central field. In a unsymmetrical field, it asserts that a body would fall along the physical vertical, whatever its velocity; whereas the Newtonians assert that the apple did not fall vertically but that the harmonic law of potential, derived from the central field, continues to hold. Which is nearer the facts and figures, we do not yet know.

This particular alternative to Newton is, however, merely an example. It is not my purpose to exhaust the entire field or even to pursue one solution to its numerical conclusion; I merely want to show how much remains to be done, not only in formulating alternative solutions but also in solving them. Equation (9), for instance, has not, so far as I know, entered into the calculations of geodesists and astronomers before. There is nevertheless a physical analogy in that, if the trajectories are considered as directions in which energy is propagated, then the surfaces A = const. are the wave-fronts. Perhaps we too should ride on the crest of a wave, with the help of all the work which has been done on wave propagation, instead of forever adding bits to a Kepler ellipse?

We have not yet dealt with the effect of diurnal rotation of the Earth, which, of course, affects the form of the V-surfaces as obtained from terrestrial observations. In any gravitational theory alternative to Newton's we must avoid such conceptions as centrifugal force and should treat the problem by means of rotating coordinate axes. The best way of doing so is probably to introduce a time dimension, although much more simply than Einstein does. The great advantage of the tensor methods outlined above is that they hold in any static coordinate system and are easily extended to more dimensions.

Static Measurements

We have so far considered measurements to a body in motion, relative to an observer on the Earth, with a view to using the orbital characteristics of the body; but important as that may be in the future for exploring the gravitational field, and even for position fixing, it is not yet an important method of geodetic measurement. Another geodetic use of artificial satellites, and perhaps at present the most important use, is to fix positions on the Earth by simultaneous observation of the satellite considered solely as a triangulation beacon, without requiring any accurate knowledge of its orbit. Satellite triangulation can bridge oceans and large desert areas on a single coordinate system, to greater accuracy and over longer distances than any previous system, and seems to be the best method in prospect for achieving a single world-wide datum and reference system. Present indications are that its accuracy is about the same as modern ground-to-ground measurements over land distances of about 1000 miles. Moreover, ground-to-ground methods will always be required – pace the advanced photogrammetrists – to provide a sufficient density of control for minor surveying over almost any land mass. Consequently, the sole use of satellite triangulation over most land masses is likely to be as over-all control on ground-to-ground networks and for this purpose it must be observed to the highest possible degree of accuracy.

It is hardly possible to use classical two-dimensional methods of reduction on the long sides and great altitudes of satellite triangulation, which must accordingly be reduced in three dimensions. There was, however, a move in this direction for ground-to-ground observations before the first sputnik went into orbit (Hotine 1956, 1957 a, b) and three-dimensional methods are now becoming standard.

Geodetic measurements of length, angles, astronomical latitude, longitude and azimuth are affected by the direction of gravity – the vertical – but not by the magnitude of gravity and were the first to go into three dimensions (Hotine 1958 et ante). Spirit-levels, which are but little affected by the magnitude of gravity have been added since (Hotine 1960).

Most ground observations must necessarily be made along the paths of light or radio waves affected by atmospheric refraction and this, more than anything else, restricts the accuracy of our results. Essentially, this is a physical problem: actually to measure the refraction, or the variations which cause it, throughout the line. There is a good change that this will be done during the next few years, and meanwhile we should re-examine our basic mathematical treatment of the subject. So far as a ray of light in a medium of refractive index (μ) is concerned, Fermat's principle tells us that

$$\int \mu \, ds$$

must be a minimum along the path compared with any adjoining path between fixed ends, and this principle has as much solid philosophical and experimental support as the Principle of Least Action which it is so closely resembles. Indeed, we have only to write μ for v and take the A-surfaces as the wave-fronts in the above derivations from the Principle of Least Action to have the whole story. We shall then discover that Snell's Law of Refraction holds only across a surface of discontinuity in μ and is usually misapplied in books on geodesy and astronomy.

There is a growing tendency to include all geodetic operations which are affected only by the direction of gravity in a branch of the subject known as *geometrical geodesy.* Operations which are concerned with, or affected by, the magnitude of gravity are included in *physical geodesy.* In my view, such an artificial subdivision is a great mistake and will retard both sides of the subject. The magnitude of gravity is just as much a geometrical entity as its direction, although both are subject to a physical law, and to split the two is as retrograde as refusing to use even elementary vector analysis in our basic mathematical discipline. "Physical" geodesy should follow its geometrical twin into the reality of three dimensions. Instead, there is the same old desire to get onto a surface as soon as possible, and a nice smooth surface at that. This in turn requires certain smooth assumptions as to underlying densities, sometimes based on the rigid application of theories which are no longer accepted as more than half-true by other Earth sciences, or else the operation of a mammoth bulldozer. A three-dimensional system has, in fact, been proposed by Bjerhammar (1963), who indeed calls it a five-dimensional system because he includes the potential and the time as well as three space coordinates, but much more work remains to be done before it can be said that this is the only possible system, or the best one.

Above all, the "geometrical" and the "physical" geodesists must work on the same law of gravity, which is set for the rise and fall of us all, and that goes for the "satellite" geodesist, too. If it should eventually appear that the number V, which defines the level surfaces, is not after all exactly a harmonic function modified by diurnal rotation of the frame of reference, then the bottom will drop out of "physical" geodesy. It could be a profitable exercise to consider now what to do then, and who knows what else might come out of the exercise?

Coordinate Systems

I have time for only one other example of the interaction between mathematics and geodesy and for this we will choose the question of three-dimensional coordinate systems.

Fifteen years ago Marussi (1949) proposed the use of a coordinate system consisting of the astronomical latitude and longitude and the Newtonian geopotential and worked out the basic differential geometry of the system. Unfortunately, this is fairly complicated and there is something to be said for trying to find a simpler system which may also be based on directly measurable quantities. If, for instance, we could find a triply orthogonal system which fits the gravitational field, the consequences could be considerable, both in theory and in practical computing.

Now we have already seen in Eq. (10) that a conformal transformation to a curved three-dimensional space with a scale factor g results in the lines corresponding to the verticals becoming geodesics. From our definition of g and V in Eq. (2), this means, moreover, that an element of length along the geodesics in the curved space is dV. The line element in the curved space can accordingly be written in what is known as the geodesic form of the metric (Eisenhart 1960, p. 47) as

$$ds^2 = \bar{E}\,dx^2 + 2\bar{F}\,dx\,dy + \bar{G}\,dy^2 + dV^2$$

and transforming this back to the flat space in the same coordinates

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 + (1/g^2)\,dV^2 \ , \tag{14}$$

in which E, F and G are components of the metric tensor of the level surfaces in the surface coordinates x, y. We do not yet know what these coordinates are, beyond the fact that they must be constant along the verticals; they are not the astronomical latitude and longitude. We have nevertheless shown that it is possible to find two coordinates, while retaining V as the third, which makes two of the six components of the metric tensor zero and reduces a third component to the directly measureable $1/g^2$. Neither this simplification nor any of the following argument depends on V being a harmonic function.

The question now arises whether we cannot also choose orthogonal surface coordinates, thereby making another component (F) of the metric tensor zero. In that case the coordinate surfaces (one family being the level surfaces V = const.) would cut one another everywhere at right angles. For this to be possible, the function which generates the level surfaces would have to satisfy a complicated third-order partial differential equation known as the Darboux equation.

It is well known that in a triply orthogonal system, each coordinate line would have to be a line of curvature of the two surfaces containing it; and since it would be normal to a coordinate surface, it would have to constitute what is known as a surface-normal field. Alternatively, we can say that one of the coordinates alone varies in the direction of each line of curvature and is constant in the two perpendicular directions, so that the lines of curvature must have the same directions as the gradient of some scalar, that is, of the appropriate coordinate.

If we express either of these conditions in the Marussi metric or in the simplified metric Eq. (14) we obtain the following simple formula:

$$\frac{\partial A}{\partial n} = \frac{\partial (\log g)}{\partial p} \tan \phi \ , \tag{15}$$

in which the first term is the vertical variation of the azimuth of the lines of curvature of the level surfaces: the second term is the variation of the logarithm of g in the direction of the astronomical parallel of latitude and ϕ is the astronomical latitude. It can be shown that this simple equation completely satisfies the complicated Darboux equation. It is a second-order differential equation in V and must therefore be an unsuspected integral of the third-order Darboux equation.

We have now to consider whether either Eq. (15) or the Darboux equation is a condition limiting the form of V, which would be the orthodox mathematical view, or whether it is true anyway in a space expressible in Marussi coordinates. For a start, Eq. (15) is automatically satisfied in an axially symmetrical field, which shows that any family of surfaces of revolution V = const. having a common axis of revolution is a possible member of a triply orthogonal system. This we already know.

We can, however, go further. *Each side* of Eq. (15) is zero in an axially symmetrical field, which means that the condition, if it is a condition, is over-filled in such a field and should therefore be satisfied in at any rate some unsymmetrical fields. The only way of resolving the matter beyond question is to find the other two coordinates, and this has not yet been done.

If Eq. (15) is generally true, it is a hitherto unsuspected relation between quantities in the field. It can be shown that it is equivalent to the following intrinsic level-surface tensor equation, true in any coordinates;

$$\left(\frac{1}{g}\right)_{\alpha\beta} n^\alpha t^\beta = 0 \ , \tag{16}$$

in which n^α, t^β are unit vectors in the direction of the lines of curvature. This may be, or may lead to, the missing relation between gravity and the form of the level surface, referred to earlier. It does not require the level surfaces to be Newtonian equipotentials.

If we are able to show by these means that the Darboux equation is automatically satisfied by any family of surfaces in Marussi space, then the mathematical consequences might well be very far-reaching. By Marussi space in this context I mean flat space containing a fixed axis to which the coordinates of latitude and longitude on the surfaces are referred; it does not have to be a gravitational field of any sort.

Equation (15) can be experimentally verified by torsion balance measurements over a range in altitude, say down a mineshaft. If it is satisfied over a sufficient range in the most gravitationally disturbed regions, we could accordingly use it in practice, and many of its consequences, whether it is rigorously true or not. This indicates a possible kickback to the data-production side of geodesy.

We must not underestimate the importance of data production, which is and must remain the bread and butter of geodesy. All I suggest is that we too use the data ourselves, primarily in furtherance of our own business of better data production, but also in order to see where it leads. There would, for instance, be far fewer advances in astronomy and in astrophysics if the observatories confined their activities to the production of ephemerides for the use of sailors and surveyors, important as that is. Never let us forget that Newton, Lagrange, Laplace, Gauss, among others who are now claimed as mathematicians, were geodesists who needed more and better mathematics to further their geodesy.

I hope therefore that I have said nothing to upset, much less to discourage stamp-collectors, or physicists, or those of us who need to be a little of both. The path to fame is beset by cut-throat bandits and it is no good being upset by them anyway.

I am grateful to Dr. John S. Rinehart of the Coast Survey, who gave me the basic idea for this paper quoting Rutherford, and much other encouragement to learn more physics; also to Mr. Bernard Chovitz of the Coast Survey who read the first draft and suggested several clarifications. He is not responsible for any errors and heresies which remain.

Appendix

Since this paper was written it has been found that the basic Eqs. (6) and (9) for free trajectories can be derived more simply.

The Principle of Least Action requires a particular family of trajectories in flat space to correspond to a family of geodesics in Riemannian space, a family of surfaces (geodesic parallels), orthogonal to the geodesics, therefore exists, on any one of which the action A is constant. It follows from the conformal properties of the transformation that corresponding constant A-surfaces exist in the flat space orthogonal to the trajectories. The gradient of A is accordingly in the direction of the trajectories, whose unit vector is l_r. We can therefore write

$$A_r = C l_r \ ,$$

in which C is a scalar function to be determined. Multiplying across by l_r, we have

$$A_r l^r = \partial A/\partial s = C \ ,$$

and therefore from Eq. (3), the definition of "action", $C = v$ and

$$v l_r = A_r \ , \tag{17}$$

which derives Eq. (9) without having to transform vectors between the two spaces.

Equation (6) can now be obtained by direct covariant differentiation of Eq. (17):

$$\begin{aligned} \delta(v l_r)/\delta s &= (v l_r)_s l^s \\ &= A_{rs} l^s \\ &= A_{sr} l^s \end{aligned}$$

(since A is a scalar function of position)

$$\begin{aligned} &= (v l_s)_r l^s \\ &= v_r l_s l^s + v l_{sr} l^s \\ &= v_r \end{aligned}$$

since l_s is a unit vector.

Equation (17), which as previously stated is an integral of Eq. (6), is accordingly a sufficient expression of the Principle of Least Action in this problem of free trajectories; it is unneccessary to use Eqs. (6) or (8) unless we wish to set up some relation between the acceleration vector and a scalar potential, as in the Newtonian theory.

The Cartesian form of Eq. (17) is

$$\frac{dx}{dt} = \frac{\partial A}{\partial x} \text{ etc.,}$$

and it might well be possible to solve these three equations numerically for A, expressed as a polynomial. The velocity is then given as the modulus of A_r. The function V which expresses the level surfaces, and which has to be compared with the Newtonian potential, would finally be obtained by inverse solution of Eq. (13).

Editorial Commentary

This paper represents Hotine's only expository work on mathematical geodesy. It is also unusual in that, unlike his other papers, it does not seem to have been prepared specifically in connection with a presentation at a geodetic conference or symposium. However, it was the basis of a talk that he gave an I. A. G. Symposium on Gravity Anomalies (Columbus 1964). Morever, it was written about a year after the I. A. G. General Assembly (Berkeley 1963), and it is also possible that it was an extended version of his presentation at that meeting. No text of his Berkeley report has been found, but this paper roughly seems to conform to the brief description of it given in Comptes rendus, Bull Geod (1963 p. 378). It offers an interesting companion to the expository paper *From Classical Geodesy to Geodesy in Three Dimensions* (1959) of Marussi, which is re-printed as the first paper in IG. Together these two papers admirably summarize the Marussi-Hotine approach to geodesy, and illustrate the subtle differences in their viewpoints.

Unlike Marussi's paper, which is largely historical, almost half of Hotine's paper is mathematical. In it, he suggested the use of conformal transformations, and the possibility of modelling the Earth's gravitational field on a triply orthogonal system of coordinates. Both of these were studied in more detail in his reports (Hotine 1966a, b) to the Third Symposium on Mathematical Geodesy (Turin 1965) which are not reprinted in this monograph.

Marussi regarded Hotine's ideas on the use of conformal transformations in geodesy as one of his most significant contributions. Indeed, in Parts V and VI of IG one can find five of Marussi's earlier papers on the same subject, and a valuable summary of his work is given in Bocchio (1978). The material in Hotine (1966a) was later presented in greater detail in MG: Chapter 10 was devoted to conformal transformations of space; Chapter 24 considered Fermat's principle in connection with atmospheric refraction. Nevertheless, the application of conformal techniques in three-dimensional geodesy is still in its nascent stage, and it is too early to know whether the great expectations of Hotine and Marussi will ultimately be fulfilled. A recent investigation dealing with the basic notions of conformal geometry is given in Zund (1987).

The final section of Hotine's paper was devoted to triply orthogonal coordinate systems. This was the subject of his publication (1966b) and Chapter 16 of MG. In particular, the former paper (1966b) contained his arguments for the existence of such coordinates, i.e the Hotine Conjecture, which was later shown to be invalid by Zund and Moore (1987).

Despite the informal and tentative nature of the material in this paper, it is rich in ideas – many of which have yet to be thought through to the end – and it vividly reveals the vibrant and exhuberant nature of Hotine's personality. As one of this anonymous obituary writers (see Memoir 1969) noted:

"Working with him was like dealing with the electric wiring of a house when the current has *not* been turned off at the main".

It is truly regrettable that we have so few examples of Hotine's engaging style and wit available to us. This paper is delightful, and deserves to be read and savoured. For the reader who has an interest in surveying, we recommend Hotine's five-part essay (Hotine 1952–53). An in-depth discussion of the mathematical background of the Marussi-Hotine approach to geodesy is given in Zund (1990).

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2 Adjustment of Triangulation in Space[1]

Introduction

The science of geodesy bears unmistakeable marks of its two-dimensional origin. It is true that geodesists no longer consider that the Earth is flat, in the sense of being expressible in two Euclidian dimensions – we have, in the course of 2000 or more years, progressed to the stage of two non-Euclidian dimensions – but whenever a third dimension obtrudes, as in Nature it must, it is to be got rid of immediately by means of "corrections", or simply ignored, so that all calculations may be done on a surface.

As an example, consider the current methods of calculating geodetic triangulation. "Horizontal" angles are measured at a point in 3-space in the tangent plane to the gravitational equipotential surface passing through the point. These are assumed to be the same as angles between curves of normal section on the surface of a spheroid, which by hallowed convention has come to be considered as representing the Earth. Here at the outset is a large assumption – that the plumb line coincides with the normal to the spheroid – which has never been satisfactorily justified, although we hope in time to have sufficient knowledge of the gravitational field to apply "corrections" to the observed angles to bring the procedure more into line with reality. The curves of normal section, of which there are two corresponding to each straight line in 3-space, or six to a triangle, are then replaced by spheroidal geodesics through correction of the angles, leading to a triangular excess which is more readily calculable. Next we remove the excess in such a way – usually by the simple one-third approximation – as to convert the geodesic triangle to a plane triangle having the same side lengths. The plane triangles are then solved and adjusted by plane trigonometry.

On the face of it, this seems a very odd way of solving and adjusting triangles which, apart from the effects of atmospheric refraction, were plane and straight-sided in the first place, although not coplanar. It should be noted here that atmospheric refraction is a physical fact, to be allowed for as best as we can, before treating the residue as an error of observation; we do not escape it by a particular method of calculation. In the method which follows, residual errors of refraction will to some extent vitiate the result; they do so just as much in the classical method described above, even though they do not enter explicitly into the classical method.

[1] Report dated 24 April 1956 (Tolworth) and presented to an I.A.G. Symposium on European Triangulation (Munich 1956).

Laxity of scientific discipline in this matter is still more evident in the case of the so-called Laplace adjustment for azimuth, which has been the subject of so much unresolved controversy. In its usual form, this adjustment is based on the statement, which hardly rises above the level of a rule-of-thumb, that the difference between the astronomical and spheroidal azimuths at a point (usually without any awareness of the fact that the latter needs special definition at a point in space) is equal to the corresponding difference in longitudes multiplied by the sine of the latitude. As we shall see, this is a simplified – perhaps over-simplified – approximation to an equation of coordinate transformation, which in general applies only between different systems of coordinates of the same point in space. It is a large assumption, which should certainly be justified before the method is used further, that it holds at all between terminal azimuths etc. computed on a spheroid by the usual processes, and direct astronomical measures at a point in space, indefinitely related at most to the terminal point on the spheroid.

Again, measured base-lines are supposed to be reduced to the length which is obtained between corresponding points on the reference spheroid. This presupposes some system of one-to-one correspondence between points in physical space and calculated points on the spheroid which is not clearly defined, and the resulting slack is adjusted into the triangulation.

The time has come for critical examination of the classical methods and conceptions against the background of a method which frankly recognizes that geodetic measures are carried out in flat space of three dimensions, and which clearly defines the coordinate systems used to express the space, even if one coordinate is small. The following method, which is given in outline only and will require modification as it is put into practice, indicates how this may be done without any knowledge or assumption relating to the gravitational field. It requires a much greater volume of stereotyped computation which nowadays need not involve much delay or more than a fraction of the total cost, and in any case is a small price to pay for a rigorous control on the classical methods. If the latter stand up to the test, preferably a searching test in mountainous country, there will be no need to abandon them – any more than Newton's law of gravity, now known to be scientifically incorrect, has been abandoned for ordinary purposes of computation.

Although all the formulae in this particular paper can be obtained from spherical figures, but not as simply, it has been decided to use vector methods, whose introduction even into surface geodesy is long overdue. Index notation, exactly as in McConnell (1931) or most other modern textbooks on the tensor calculus, is used because it has certain advantages (and no disadvantages) in non-orthogonal coordinate systems and is in any case required for other applications, notably to the gravitational field.

Coordinate Systems

We take a right-handed orthogonal triad of fixed unit vectors λ^r, μ^r, ν^r defining the directions of Cartesian axes, x, y, z. The physical axis of rotation of the Earth is parallel to ν^r; and λ^r is perpendicular to ν^r in a plane which defines the origin of longitudes in *all* coordinate systems involving latitude and longitude.

Latitude and longitude are taken as two of the three curvilinear *space* coordinates and are defined as follows. At any point in space, there will be a unit vector ξ^r normal to the third coordinate surface passing through the point, viz. the surface on which the third coordinate has a constant value equal to its value at the point considered. Latitude ϕ (positive North) and Longitude ω (positive East) are then defined by the following scalar products expressing the direction cosines of ξ^r:

$$\xi^r \upsilon_r = \sin\phi$$

$$\xi^r \lambda_r = \cos\phi \cos\omega$$

$$\xi^r \mu_r = \cos\phi \sin\omega$$

from which we obtain the following vector equation:

$$\xi^r = \lambda^r \cos\phi \cos\omega + \mu^r \cos\phi \sin\omega + \upsilon^r \sin\phi \ . \tag{1}$$

We have now to consider the third coordinate. If the first two are to be *astronomical* latitude and longitude, then ξ^r will be the direction of the physical plumb line; the third-coordinate surfaces will be the gravitationally level or equipotential surfaces, and the third-coordinate *must* be the geo-potential or some function of the geo-potential, including the kinetic potential of rotation around the axis. The properties of this coordinate system have been established by Marussi (1949), who also deals (Marussi 1950) with the special case of an axially symmetrical gravitational field to provide standard or *geodetic* coordinates in the space surrounding an equipotential surface of revolution, such as a spheroid.

There is, however, another class of coordinate systems which possess certain advantages in dealing with the space surrounding a given surface, or base surface. If we measure an equal distance h along all the straight normals to this given surface, then the terminal points will lie on a surface which is said to be geodesically parallel to the given surface (see e.g. Eisenhart 1960 p. 57). The two surfaces have common normals. We can take h as the third coordinate, and the geodesically parallel surfaces as third-coordinate surfaces, including the given surface as $h = 0$. Latitude and longitude will then refer to the common normals of the surfaces $h =$ const. and the point considered will have the same latitude and longitude as the foot of the perpendicular on the base surface.

As a special case, take as base surface a spheroid whose principal curvatures are ϱ (meridian) and υ. If the origin of Cartesian coordinates is at the centre of symmetry of this spheroid, whose minor axis (b) is in the direction υ^r, then it can be shown without difficulty that the transformation equations to Cartesian coordinates are

$$\begin{aligned} x &= (\upsilon + h) \cos\phi \cos\omega \\ y &= (\upsilon + h) \cos\phi \sin\omega \\ z &= (k'^2 \upsilon + h) \sin\phi \\ (k'^2 &= b^2/a^2 = 1 - e^2) \end{aligned} \tag{2}$$

giving rise to the following triply orthogonal metric:

$$ds^2 = (\varrho + h)^2 d\phi^2 + (\upsilon + h)^2 \cos^2\phi \, d\omega^2 + dh^2 \ . \tag{3}$$

The choice of coordinate system should make no difference whatever to the final adjustment of a triangulation and if a curvilinear system should be required during the process, or for any purpose not requiring a close overall approximation to the geoid, then there is much to be said for choosing a base sphere ($\varrho = \upsilon$; $k' = 1$).

It is important to realize that each coordinate system is a distinct entity, which must satisfy certain conditions in flat space; we can transform from one system to another, and certain quantities will be invariant under such a transformation, but we may not mix them up, e.g. by attempting to make up a coordinate system from astronomical latitude and longitude and orthometric height.

Azimuth and Zenith Distance

The meridian plane at a point in space is defined as parallel to ξ^r and v^r and a unit vector u^r in the meridian (towards North) and in the third coordinate surface will be given by

$$\begin{aligned} u^r &= -\xi^r \tan\phi + v^r \sec\phi \\ &= -\lambda^r \sin\phi\cos\omega - \mu^r \sin\phi\sin\omega + v^r\cos\phi \ . \end{aligned} \tag{4}$$

A unit vector (towards East) perpendicular to u^r in the third coordinate surface is given by

$$v^r = -\lambda^r \sin\omega + \mu^r \cos\omega \ . \tag{5}$$

A unit vector in azimuth α (measured from u^r towards v^r) and zenith distance β (measured from ξ^r) is accordingly given by

$$\begin{aligned} \sigma^r &= (u^r\cos\alpha + v^r\sin\alpha)\sin\beta + \xi^r\cos\beta \\ &= a\lambda^r + b\mu^r + c\upsilon^r \end{aligned} \tag{6}$$

by substituting Eqs. (1), (4) and (5) where

$$\begin{aligned} a &= -\sin\phi\cos\omega\cos\alpha\sin\beta - \sin\upsilon\sin\alpha\sin\beta + \cos\phi\cos\omega\cos\beta \\ b &= -\sin\phi\sin\omega\cos\alpha\sin\beta + \cos\omega\sin\alpha\sin\beta + \cos\phi\sin\omega\cos\beta \\ c &= \cos\phi\cos\alpha\sin\beta + \sin\phi\cos\beta \end{aligned} \tag{7}$$

or, by forming scalar products with ξ^r, u^r, v^r

$$\begin{aligned} \cos\beta &= a\cos\phi\cos\omega + b\cos\phi\sin\omega + c\sin\phi \\ \cos\alpha\sin\beta &= -a\sin\phi\cos\omega - b\sin\phi\sin\omega + c\cos\phi \\ \sin\alpha\sin\beta &= -a\sin\omega + b\cos\omega \ . \end{aligned} \tag{8}$$

Note that a, b, c are not independent since $a^2 + b^2 + c^2 = 1$. It will, however, be advisable to carry the computation of all three to provide a check.

Now a, b, c are the Cartesian components – with respect to the fixed Cartesian axes λ^r, μ^r, υ^r – of the unit vector σ^r. They are accordingly invariant both for displacements along the straight line σ^r and for any change in the curvilinear coordinates ϕ, ω at a point, provided α, β are measured in relation to the new third-coordinate surface. We may, for instance, substitute direct astronomical measures of ϕ, ω, α, β in Eq. (7); the result would be exactly the same it geodetic coordinates in the system given by Eq. (2) were substituted, provided that the origin of latitudes and longitudes is the same (i.e. υ^r, λ^r are fixed) and so long as the geodetic values of α, β are taken in relation to the surface $\hbar = \text{const.}$ through the point[2]. Again, we should obtain the same values for a, b, c at any other point on the straight line whose direction is σ^r; so that a, b, c have identical values at both ends of the side of a spatial triangle. Furthermore we can, at an initial point of a triangulation, substitute astronomical measures and use the resulting values of a, b, c to calculate differences of geodetic coordinates; for examnple, the changes in x, y, z along a straight line of length s are s a, s b, s c, which even though calculated from astronomical values can be used in such a geodetic system as Eq. (2).

To relate this result to the classical conception, differentiate Eq. (7) for small coordinate changes $\delta\phi$, $\delta\omega$ (whether changes in the coordinate system or due to displacements along the line) while retaining a, b, c fixed:

$$\delta\alpha = \sin\phi\,\delta\omega + \cos\beta(\sin\alpha\,\delta\phi - \cos\alpha\cos\phi\,\delta\omega)$$

$$\delta\beta = -\cos\alpha\,\delta\phi - \cos\phi\sin\alpha\,\delta\omega\ .$$

The first equation is the corrected form of the so-called Laplace azimuth equation for a non-horizontal azimuth line, both astronomical and geodetic azimuth at the same point in space being rigorously defined as above. The second equation has no counterpart in the classical methods of two-dimensional adjustment. We shall use both, but in the complete form (6) and (7).

Figural Adjustment

The best possible correction for atmospheric refraction is first made to observed zenith distances and thereafter it is assumed that, subject to observational error, we are dealing with straight lines between the observing stations.

Stations are numbered 1, 2, 3 etc., and we use the following notation:

α_{12} = Azimuth of 2 from 1.

β_{12} = Zenith distance of 2 from 1 corrected for refraction.

α_{123} = Measured "horizontal" angle at 1 between directions to 2 and 3.

[2] This azimuth is not the same as that of the projected direction on the base spheroid. If the latter azimuth is $\bar{\alpha}$, it can be shown that

$$\tan(\alpha - \bar{\alpha}) = \frac{\hbar(\varrho - \upsilon)\sin\alpha\cos\alpha}{\varrho\upsilon + \hbar(\varrho\cos^2\alpha + \upsilon\sin^2\alpha)} = \frac{\hbar(\varrho - \upsilon)\sin\bar{\alpha}\cos\bar{\alpha}}{\varrho\upsilon + \hbar(\varrho\sin^2\bar{\alpha} + \upsilon\cos^2\bar{\alpha})}\ .$$

We shall not, however, have occasion to apply this correction, since we shall not be working on the base spheroid.

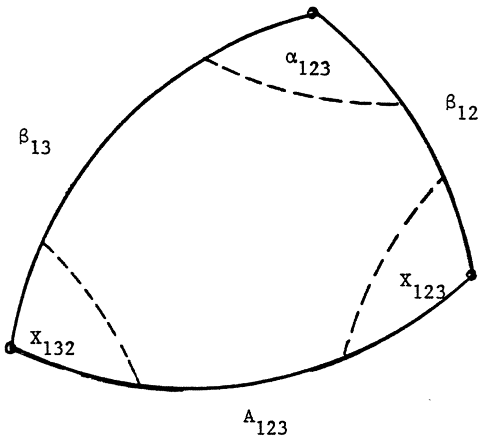

Fig. 1

A_{123} = "Facial" angle at 1 between directions to 2 and 3; that is the angle at 1 of the space triangle 123. This angle is shown in relation to the measured angles in the spherical Fig. 1, which also shows the base angles X_{123}, X_{132} of the spherical triangle.

A_{123}, X_{123}, X_{132} are first calculated between every pair of directions by a suitable formula for solving Fig. 1 from α_{123}, $\beta_{12}+\beta_{13}$. Note that a separate solution is required for every pair of directions. We cannot, for instance, add A_{123} to A_{134} to obtain A_{124} because these angles will not as a rule be in the same plane.

By differentiation of any such formula we relate errors or corrections to A_{123}, B_{12}, β_{13}, α_{123} as follows:

$$dA_{123} = \cos X_{123}\, d\beta_{12} + \cos X_{132}\, d\beta_{13} + \sin\beta_{12} \sin X_{123}\, d\alpha_{123} \ . \tag{9}$$

Angle and side equations are formed in the usual way, except that we incorporate closure corrections to the A_{123} angles instead of "horizontal" angles.

A typical angle equation would be:

$$dA_{123} + dA_{231} + dA_{312} = 180° - (A_{123} + A_{231} + A_{312}) \ ,$$

in which Eq. (9) should be substituted if it is desired to determine corrections to the actual observations; practical working alone can indicate whether this is necessary, or whether it would be sufficient to determine corrections to the facial angles. There are, of course, no corrections for spherical excess etc.

Side equations must be formed from one of the stations as pole – not the intersection of diagonals, which will usually be skew – and the sine of the A-angles of a particular spatial triangle must be substituted for the opposite sides of that same triangle before differentiation.

The X-angles will be nearly 90° and the coefficients of the $d\beta$'s in the condition equations will accordingly be small. This can be allowed for as usual by accurate computation of the X-angles and by altering the unit of the $d\beta$'s, which could also be weighted down to allow for the probably greater effect of residual refraction on these angles, if experience so indicates.

Adjustment for Scale

Measured base lengths are not reduced to sea level or to a spheroid but are computed as a traverse in a (vertical) plane to obtain the air-line distance between the terminals. Geodimeter measures are corrected solely for curvature of path and velocity; and radar measures are similarly reduced to provide the straight air-line distance between stations.

The base conditions are formed in the usual way by working through actual triangles between bases and expressing the resulting corrections to A-angles.

Adjustment of Unit Side Vectors

σ^r_{12} = Unit vector from 1 towards 2 = $-\sigma^r_{21}$ (Cartesian components a_{12}, b_{12}, c_{12})

τ^r_{123} = Vector product of σ^r_{12} and σ^r_{13} [Cartesian components $(b_{12}c_{13}-c_{12}b_{13})$, $(c_{12}a_{13}-a_{12}c_{13})$, $(a_{12}b_{13}-b_{12}a_{13})$].

A third side vector σ^r_{14} can be expressed as follows in terms of σ^r_{12}, σ^r_{13}, τ^r_{123} with only the facial angles in the coefficients. These will already have been computed and probably adjusted, and their use, in preference to formulae containing zenith distances, may serve to minimize the effect of residual refraction. It is assumed that σ^r_{12}, σ^r_{13}, σ^r_{14} are arranged in order of increasing azimuth, directions being reversed (and signs changed) as necessary to make this so.

$$\begin{aligned}
\sin^2 A_{123}\sigma^r_{14} &= (\cos A_{124}-\cos A_{123}\cos A_{134})\sigma^r_{12} \\
&\quad +(\cos A_{134}-\cos A_{123}\cos A_{124})\sigma^r_{13}-2n\tau^r_{123} \\
n^2 &= \sin s\sin(s-A_{123})\sin(s-A_{124})\sin(s-A_{134}) \\
2s &= A_{123}+A_{124}+A_{134}\ .
\end{aligned} \tag{10}$$

If the three side vectors are coplanar, in which case they could be parallel to the sides of a plane triangle, then $A_{124} = A_{123}+A_{134}$ and the formula reduces to:

$$\sin A_{123}\sigma^r_{14} = -\sin A_{134}\sigma^r_{12}+\sin A_{124}\sigma^r_{13}\ . \tag{11}$$

If σ^r_{14} is parallel to the third side σ^r_{23} of a plane triangle, then we may write $A_{134} = A_{312}$ and $A_{124} = 180°-A_{231}$ and the equation becomes

$$\sin A_{123}\sigma^r_{23} = -\sin A_{312}\sigma^r_{12}+\sin A_{231}\sigma^r_{13}\ . \tag{12}$$

If we replace the sines of the angles by the lengths of the opposite sides, this equation correctly expresses the fact that the change in Cartesian coordinates round the triangle is zero.

By means of Eq. (12), which enables us to deduce the closing vector of a triangle, and Eq. (10), which determines a side vector in the next triangle not necessarily coplanar with the first, we can express all side vectors in terms of the first two. The Cartesian components of the first two vectors are determined by substituting astronomical measures of ϕ, ω, α and β in Eq. (7), using the value of β corrected for refraction, and we are accordingly able to carry Cartesian components throughout the triangulation. A direct determination will be provided by Eq. (7) whenever we again have astronomical measures; and the difference between the direct and extended components is to be disturbed throughout the triangulation on any reasonable basis. It will probably be sufficient to carry the components along the shortest traverse between astronomical stations and to distribute the closing error evenly along this traverse, followed by correction of the figural adjustment, much as is usually done in the two-dimensional Laplace azimuth adjustment, which is of course included in theoretically correct form in the present vector adjustment. Alternatively, it may well be advantageous, by differentiation of Eqs. (10) and (12), to form closure conditions for solution simultaneously with the figural and base conditions. The best method will no doubt emerge in the course of practical operation. The adjustment should finally provide consistent values of the Cartesian components for use in the calculation of geodetic coordinates; it is not necessary to evaluate corrections to the observed angles unless this is found to be the best method of effecting the adjustment.

Geodetic Coordinates

An origin can be taken from previous work, provided this has similarly been properly adjusted to astronomical measures. Otherwise, we must start with precise astronomical measures and use them in Eq. (7) to determine the initial pair of side vectors. We can use any geodetic starting elements *provided that when these are substituted in Eq. (7) they give the same components for the initial pair of vectors as the astronomical measures.* A glance at Eq. (8) shows that any alteration in initial latitude and longitude implies a corresponding alteration to both azimuths and zenith distances, whereas in current practice it is seldom enough that even the effect on azimuth is considered.

The choice of third coordinate will then settle the geodetic system. An axially symmetric potential, approximating to the actual physical potential, would seem to be a logical choice, of the same kind as the astronomical system, but the fact is that the calculation of finite differences of coordinates in such a system is not simple, and the main object of adopting a standard geodetic system is to simplify the calculation of position. At the same time we need a system which does not depart too far from the physical plumb line. From this point of view, the system of Eq. (2) is as good as any, if a suitable choice of base spheroid is made, and it is just as easy to calculate standard gravity in this system.

The initial value of the third coordinate at the origin can be taken from previous work. If geodetic latitude and longitude are also taken from previous work, subject to the above proviso, and the shape and size of base spheroid in Eq. (2) are the same, then the entire coordinate system will be the same. Failing any previous work, we can use a spirit-levelled height, or for that matter an arbitrary datum, but it should be appreciated that this settles the coordinate system and any later alteration will require recomputation, or at any rate correction, of all three geodetic coordinates.

The Cartesian coordinates of the origin $\bar{x}$, $\bar{y}$, $\bar{z}$ can now be computed from Eq. (2). Differences of Cartesian coordinates along a side of length s and adjusted vector (a, b, c) are then (s a, s b, s c), so that we can calculate the Cartesian coordinates of all the triangulation stations, and substitute in Eq. (2) to obtain geodetic longitudes and latitudes. Longitude presents no difficulty, and latitude would probably best be obtained by iteration, starting with the close approximation

$$\tan\phi = \frac{3}{k'^2(x^2+y^2)^{1/2}} .$$

Geodetic azimuth is then obtained from Eq. (8).

Alternatively, we can derive formulae in terms of differences of Cartesian coordinates, but these are not likely to present any advantage if sufficiently large-capacity machines are available to take the full coordinates, which are of the same order as υ.

Conversely, if we are given the geodetic coordinates of two points, we can calculate their Cartesian coordinates and hence their distance apart and direction (a, b, c) in space. Azimuths follow from Eq. (8).

Deflection of the Vertical

If the third vector σ^r_{14} in Eq. (10) is the unit normal ξ^r (in the direction of the plumb line), then $A_{134} = \beta_{13}$ and $A_{124} = \beta_{12}$ and (see Fig. 1) the formula can be reduced to:

$$\sin\alpha_{123}\csc X_{123}\csc X_{132}\xi^r = \cos X_{132}\sigma^r_{12} + \cos X_{123}\sigma^r_{13} - \tau^r_{123} . \tag{13}$$

If ϕ, ω are now the astronomical latitude and longitude, the Cartesian components of ξ^r are ($\cos\phi\cos\omega$, $\cos\phi\sin\omega$, $\sin\phi$). Substitution of these in Eq. (13) together with the corresponding (a, b, c) components of σ^r_{13}, τ^r_{123}, accordingly gives three equations (two of which are independent) to determine ϕ, ω. Comparison with geodetic coordinates calculated as above gives a measure of deflection at each point.

Editorial Commentary

This was Hotine's first formal paper which employed tensorial methods. Its content was straightforward and elementary in that it made only modest mathematical demands on the reader. Its content was re-considered in his *A Primer on Non-Classical Geodesy* (1959), and *The Third Dimension in Geodesy* (1960), both of which are included in this monograph. A more complete discussion on the line of observation, internal and external adjustment of networks in space can be found in Chapters 25–27 of MG.

Due to his extensive experience in photogrammetry and his pioneering work in aerial surveying, the topic of triangulation was a natural area for him to test his newly acquired mathematical skills, i.e. tensor analysis. Hotine's plea for a three-dimensional formulation was eloquently and forcibly made in the introduction of the paper. For those of us who grew up in the age of Sputnik or afterward, this plea is a non-issue. However, this was certainly not the case with people of Hotine's generation (see Levallois 1963). Indeed, the present Hotine-Marussi Symposia on Mathematical Geodesy were initiated as a Symposium on Three-Dimensional Geodesy (Venice 1959) with the express purpose of developing and popularizing such techniques. By 1965, this issue had been settled and the third symposium became the Symposium on Mathematical Geodesy (Turin 1965).

Note that on p. 24 Hotine's use of symbols λ^r, μ^r, ν^r for the Cartesian basis vectors was unfortunate, since he subsequently denoted these by A^r, B^r, C^r in *all* his other work. He then employed λ^r, μ^r, ν^r as a more general set of vectors, see paper 3, p. 33.

The contemporary literature in this area is extensive and it has grown into elaborate theories of linear and non-linear adjustment. Many geodesists have contributed to it, and many make serious use of tensor-theoretic methods (see References).

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3 Metrical Properties of the Earth's Gravitational Field[1]

Introduction

Some, but not all, of the results in this paper have been obtained in an equivalent form by Professor Marussi (Marussi 1947, 1949) using a special coordinate system: the astronomical latitude and longitude, and the geopotential. The present treatment of the subject is independent of any coordinate system, and thereby achieves some simplification and completeness, although these three quantities, and others which can be directly measured, enter the analysis as scalar functions of position, defined either throughout space or along certain lines. All the properties of the field are shown to depend on five parameters, which are applied to such fundamental geodetic problems as variation of position and along refracted rays. Methods of measuring the parameters by torsion balance and by astro-geodetic observations are also considered. Finally, the results are used to derive briefly the properties of the Marussi metric as a special case of the general analysis and as an introduction to the similar derivation of geodetic coordinate systems in a further paper (Geodetic Coordinate Systems 1957).

The tensor notation used throughout is exactly as in McConnell (1947), and much the same, with obvious modification, as in Eisenhart (1949) and Levi-Civita (1926).

Base Vectors

1. We define as follows three mutually orthogonal unit vector fields A^r, B^r, C^r; right-handed in that order. C^r is parallel to the physical axis of rotation of the Earth; positive direction North. A^r is parallel to the plane determined by C^r and the zenith direction at the origin of longitudes, and is perpendicular to C^r; positive direction outwards from the Earth. B^r completes the right-handed orthogonal triad.

2. A second set of unit orthogonal vector fields λ^r, μ^r, ν^r, right-handed in that order, is defined as follows. ν^r is the direction of the zenith, or outward-drawn normal to the equipotential surface, or tangent to the line of force passing through the point in space under consideration. μ^r, to be known as the meridian

[1] Report dated 28 May 1957 (Tolworth) and presented to Study Group No. 1 at the I.A.G. Toronto Assembly 1957.

direction, lies in the equipotential surface and in the plane of υ^r and C^r; positive in a northerly direction. λ^r, also in the equipotential surface, and to be known as the parallel direction, completes the triad; positive easterly.

3. Astronomical longitude (ω) is the angle between the planes A^r, C^r and υ^r, μ^r, C^r; positive in the position rotation about C^r, that is from A^r to B^r, or East. Astronomical co-latitude ($\pi/2-\phi$) is the angle between C^r and υ^r; or latitude ϕ between υ^r and the plane A^r, B^r, positive North.

4. It should be noted that the direction of the meridian (parallel) as thus defined is not necessarily the same as the surface direction in which the longitude (latitude) is constant. We consider below the conditions for the two directions to coincide.

A curve on the equipotential surface at each point of which the tangent is in the meridian direction will be known as the meridian trace; and similarly for the parallel trace. Here again these curves are not necessarily the loci of points having the same longitude, or latitude.

5. Azimuth (α) is measured East from North, that is from μ^r towards λ^r. Accordingly a unit surface vector in azimuth α is given by $(\lambda^r \sin\alpha + \mu^r \cos\alpha)$. A unit space vector in azimuth α and zenith distance β will be given by $(\lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta)$.

6. The following covariant vector equations are easily verified for Cartesian axes parallel to A^r, B^r, C^r and are therefore true for the components in any coordinate system. The corresponding contravariant equations are obtained by simply raising the indices throughout.

$$\begin{aligned} \lambda_r &= -A_r \sin\omega + B_r \cos\omega \\ \mu_r &= -A_r \sin\phi\cos\omega - B_r \sin\phi\sin\omega + C_r\cos\phi \\ \upsilon_r &= A_r\cos\phi\cos\omega + B_r\cos\phi\sin\omega + C_r\sin\phi\ . \end{aligned} \tag{6.1}$$

If we consider the Cartesian coordinate x to be a scalar function of position, then the vector equation $x_r = A_r$ is true in Cartesian and therefore in any coordinates – and similarly for y and z – so that by solving the Eqs. (6.1) we have in any coordinate system:

$$\begin{aligned} x_r = A_r &= -\lambda_r\sin\omega - \mu_r\sin\phi\cos\omega + \upsilon_r\cos\phi\cos\omega \\ y_r = B_r &= \lambda_r\cos\omega - \mu_r\sin\phi\sin\omega + \upsilon_r\cos\phi\sin\omega \\ z_r = C_r &= \mu_r\cos\phi + \upsilon_r\sin\phi\ . \end{aligned} \tag{6.2}$$

7. If V is the geopotential (including the effect of uniform rotation, or of moving axes), then the resultant force (including centrifugal force) is the gradient of V and must act in the direction υ^r normal to the equipotential surface; its magnitude by definition is the gravity g. This gives rise to the vector equation

$$V_r = -g\upsilon_r\ , \tag{7.1}$$

in which the negative sign has been taken to make g positive when V decreases along the outward-drawn normal, or zenith.

Although mechanical definitions are given for V, g we are in fact dealing solely with the geometrical properties of a family of surfaces V = const. embedded in flat 3-space; V being any function of position and g the magnitude of its gradient, which is geometrically equivalent to what Weatherburn (1930, p. 32 et passim) called the "distance function." Later, V will be restricted to satisfy the modified Laplace or Poisson equation, but this also is essentially a geometrical consideration, analogous to Einstein's law of gravity obtained by contracting the curvature tensor of 4-space. The Laplacian of V, is indeed simply a contraction of the tensor V_{rs}.

First Derivatives of Base Vectors

8. Since A_r, B_r, C_r constitute parallel vector fields, defined throughout flat space, their intrinsic derivatives taken along *any* line are zero. Consequently, their covariant derivatives are zero.

Taking the covariant derivatives of Eqs. (6.1) we have, for example,

$$\lambda_{rs} = -(A_r \cos\omega + B_r \sin\omega)\,\omega_s = (\mu_r \sin\phi - \upsilon_r \cos\phi)\,\omega_s \tag{8.1}$$

on substituting for A_r, B_r from Eq. (6.2). In this tensor equation, true, of course, in any coordinate system, ω_s is the gradient of ω in space, that is $\partial\omega/\partial x^s$ in a space coordinate system x^s.

In the same way,

$$\mu_{rs} = -\sin\phi\,\lambda_r\omega_s - \upsilon_r\phi_s \tag{8,2}$$

$$\upsilon_{rs} = \cos\phi\,\lambda_r\omega_s + \mu_r\phi_s\ . \tag{8.3}$$

9. The covariant derivative of Eq. (7.1) gives

$$V_{rs} = -g_s\upsilon_r - g\upsilon_{rs} = -g\{\upsilon_{rs} + (\log g)_s\upsilon_r\} \tag{9.1}$$

leading to two important consequences. If we multiply by the metric tensor a^{rs} and contract, the left-hand side becomes the Laplacian Δ^2V of V.

If the elements of arc in the directions λ^r, μ^r, υ^r are respectively dp, dm, dn then we have for example $a^{rs}\lambda_r\omega_s = \lambda^s\omega_s = (\partial\omega/\partial p)$, and by substituting Eq. (8.3), the final contracted equation becomes

$$-\frac{1}{g}\Delta^2V = \cos\phi\left(\frac{\partial\omega}{\partial p}\right) + \frac{\partial\phi}{\partial m} + \frac{\partial\log g}{\partial n}\ . \tag{9.2}$$

10. The other consequence of (9.1) is that since V_r is the gradient of a scalar, V_{rs} is symmetrical, i.e. $V_{rs} = V_{sr}$. Using Eq. (8.3), we have the following tensor equation;

$$(\log g)_r\upsilon_s - (\log g)_s\upsilon_r = \cos\phi\,\lambda_r\omega_s + \mu_r\phi_s - \cos\phi\,\lambda_s\omega_r - \mu_s\phi_r\ , \tag{10.1}$$

which can conveniently be split into three vector equations by multiplying by λ^r, μ^r, υ^r in turn and contracting. Since the elements of arc in the directions λ^r, μ^r, υ^r are respectively dp, dm, dn, we have after some slight rearrangement the following equations for the gradient of ω, ϕ and g.

$$\omega_s \cos\phi = (\cos\phi\, \partial\omega/\partial p)\lambda_s + (\partial\phi/\partial p)\mu_s + (\partial \log g/\partial p)\upsilon_s \tag{10.2}$$

$$\phi_s = (\cos\phi\, \partial\omega/\partial m)\lambda_s + (\partial\phi/\partial m)\mu_s + (\partial \log g/\partial m)\upsilon_s \tag{10.3}$$

$$(\log g)_s = (\cos\phi\, \partial\omega/\partial n)\lambda_s + (\partial\phi/\partial n)\mu_s + (\partial \log g/\partial n)\upsilon_s\ . \tag{10.4}$$

Multiplying each equation in turn by λ^s, μ^s, υ^s we have, apart from identities, the following scalar equations:

$$\cos\phi\,(\partial\omega/\partial m) = (\partial\phi/\partial p) \tag{10.5}$$

$$\cos\phi\,(\partial\omega/\partial n) = (\partial \log g/\partial p) = \gamma_1 \quad \text{say} \tag{10.6}$$

$$(\partial\phi/\partial n) = (\partial \log g/\partial m) = \gamma_2 \quad \text{say}\ . \tag{10.7}$$

The last equation is often used for a spheroidal equipotential, for which case (10.5) and (10.6) are satisfied identically, but is now seen to be true in a wider context.

The Five Parameters of the Field

11. Consider a geodesic of the equipotential surface in the meridian direction μ^r. The geodesic torsion (τ) and normal curvature (χ_2) of the surface in this direction are by definition the torsion and principal curvature of this geodesic, whose principal normal is minus[2] υ^r and whose binormal is minus λ^r. The covariant form of the second Frenet formula for this geodesic is accordingly

$$-\upsilon_{rs}\mu^s = -\tau\lambda_r - \chi_2\mu_r\ ,$$

whence

$$\tau = \upsilon_{rs}\lambda^r\mu^s = \cos\phi\,(\partial\omega/\partial m) \quad \text{from (8.3)} \tag{11.1}$$

and

$$\chi_2 = \upsilon_{rs}\mu^r\mu^s = \partial\phi/\partial m \quad \text{from (8.3)}\ . \tag{11.2}$$

12. In the same way, the geodesic torsion (τ') and normal curvature (χ_1) in the parallel direction λ^r are given by the following equation, (the binormal to the geodesic in this case being plus μ^s):

$$-\upsilon_{rs}\lambda^s = +\tau'\mu_r - \chi_1\lambda_r\ ,$$

[2] By taking the principal normal of the geodesic in the opposite sense to the outward-drawn surface normal, we make the curvatures positive in the usual case of an equipotential convex to the outward-drawn normal.

whence

$$\chi_1 = \upsilon_{rs}\lambda^r\lambda^s = \cos\phi(\partial\phi/\partial p) \quad \text{from (8.3)} \tag{12.1}$$

and

$$\begin{aligned}\tau' &= -\upsilon_{rs}\mu^r\lambda^s \\ &= -(\partial\phi/\partial p) \quad \text{from (8.3)} \\ &= -\cos\phi(\partial\omega/\partial m) \quad \text{from (10.5)} \\ &= -\tau\ ;\end{aligned} \tag{12.2}$$

that is the negative of the geodesic torsion in the meridian direction; a well-known result connecting the geodesic torsions in any orthogonal directions, which enables us to use the same symbol for both.

13. These three parameters χ_1, χ_2, τ, together with the parallel and meridian components γ_1, γ_2 of the gradient of (log g) [Eqs. (10.6 and 10.7)], are fundamental in this treatment of the subject. Collecting results so far;

$$\begin{aligned}&\cos\phi(\partial\omega/\partial m) = \partial\phi/\partial p = \tau \quad \text{(geodetic torsion of meridian)} \\ &\cos\phi(\partial\omega/\partial n) = (\partial \log g)/\partial p = \gamma_1 \\ &\partial\phi/\partial n = (\partial \log g)/\partial m = \gamma_2 \\ &\cos\phi(\partial\omega/\partial p) = \chi_1 \quad \text{(normal curvature of parallel)} \\ &\partial\phi/\partial m = \chi_2\ .\end{aligned} \tag{13.1}$$

14. The divergence of λ^r is now given by

$$\begin{aligned}\lambda^r_r &= (\mu^r\omega_r)\sin\phi - (\upsilon^r\omega_r)\cos\phi \\ &= \sin\phi(\partial\omega/\partial m) - \cos\phi(\partial\omega/\partial n) \\ &= (\tau\tan\phi - \gamma_1)\ .\end{aligned} \tag{14.1}$$

Similarly,

$$\mu^r_r = -(\kappa_1\tan\phi + \gamma_2) \tag{14.2}$$

and

$$\upsilon^r_r = \chi_1 + \chi_2 = 2H\ , \tag{14.3}$$

if H is the mean curvature; a well-known result.

15. The *surface* divergence λ^α_α obtained by dropping the υ_r component of λ_{rs} is $\tau\tan\phi$; and similarly $\mu^\alpha_\alpha = -\chi_1\tan\phi$.

16. We are also able to rewrite Eqs. (10.2) and (10.3) as

$$\cos\phi(\omega_r) = \chi_1\lambda_r + \tau\mu_r + \gamma_1\upsilon_r \tag{16.1}$$

and

$$\phi_r = \tau\lambda_r + \chi_2\mu_r + \gamma_2\nu_r \ . \tag{16.2}$$

We can also rewrite Eqs. (9.2) and (10.4) as

$$(\partial \log g)/\partial n = -\chi_1 - \chi_2 - (\Delta^2 V)/g = -2H - (\Delta^2 V)/g \tag{16.3}$$

$$(\log g)_r = \gamma_1\lambda_r + \gamma_2\mu_r - \{\chi_1 + \chi_2 + (\Delta^2 V)/g\}\nu_r \ . \tag{16.4}$$

17. If $\tilde{\omega}$ is the angular rotation of the Earth (in radians per sidereal second), ϱ the density of matter at the point considered (zero for a point in free air), and k the gravitational constant, the modified Poisson equation enables us to write in conjunction with Eq. (16.3)

$$\Delta^2 V = 2\tilde{\omega}^2 - 4\pi k \varrho \ . \tag{17.1}$$

18. We can easily express the symmetrical tensor

$$V_{rs} = -g_s\nu_r - g\nu_{rs} = -g_s\nu_r - (g\cos\phi)\lambda_r\omega_s - g\mu_r\phi_s$$

in terms of the five parameters by forming such scalar products as $V_{rs}\lambda^r\lambda^s = -(g\cos\phi)(\partial\omega/\partial p)$, the final result being

$$\begin{aligned} -V_{rs}/g = {} & \chi_1\lambda_r\lambda_s + \tau\lambda_r\mu_s + \gamma_1\lambda_r\nu_s \\ & + \tau_1\mu_r\lambda_s + \chi_2\mu_r\mu_s + \gamma_2\mu_r\nu_s \\ & + \gamma_1\nu_r\lambda_s + \gamma_2\nu_r\mu_s - \{\chi_1 + \chi_2 + (\Delta^2 V)/g\}\nu_r\nu_s \ . \end{aligned} \tag{18.1}$$

The five parameters are accordingly components of the Marussi tensor W_{rs} (Marussi 1949 p. 433).

Properties of the Lines of Force

19. We now derive the usually required elements of the field in terms of the five parameters, starting with the properties of the lines of force. Their vector curvature is

$$\begin{aligned} \nu_{rs}\nu^s &= \cos\phi(\partial\omega/\partial n)\lambda_r + (\partial\phi/\partial n)\mu_r \quad \text{from (8.3)} \\ &= \gamma_1\lambda_r + \gamma_2\mu_r \quad \text{from (13.1)} \ , \end{aligned} \tag{19.1}$$

whence we conclude that the principal curvature is $(\gamma_1^2 + \gamma_2^2)^{1/2}$ and the principal normal is a surface vector in azimuth $\tan^{-1}(\gamma_1/\gamma_2)$ which from Eq. (16.4) is also the surface gradient of (log g). The binormal of the line of force is accordingly a surface vector along which (log g) is constant.

20. If $\alpha = \tan^{-1}(\gamma_1/\gamma_2)$, the principal normal is the unit vector $(\lambda_r \sin\alpha + \mu_r\cos\alpha)$ and the binormal is $(-\lambda_r\cos\alpha + \mu_r\sin\alpha)$. Writing τ_0 for the torsion of the line of force, we have from the third Frenet formula:

$$(-\lambda_r\cos\alpha + \mu_r\sin\alpha)_s\nu^s = -\tau_0(\lambda_r\sin\alpha + \mu_r\cos\alpha) \ ,$$

which, with the help of Eqs. (8.1) and (8.2) works out at

$$\begin{aligned}\tau_0 &= \sin\phi\,(\partial\omega/\partial n) - (\partial\alpha/\partial n)\\ &= \gamma_1 \tan\phi - \partial\alpha/\partial n \ .\end{aligned} \tag{20.1}$$

We investigate $\partial\alpha/\partial n$ later in § 38.

21. If the principal curvature is χ_0 and α is still the azimuth of the principal normal, we have also

$$\gamma_1 = \chi_0 \sin\alpha \ ; \quad \gamma_2 = \chi_0 \cos\alpha \ . \tag{21.1}$$

If the element of arc in the direction of the principal normal is du and in the direction of the binormal is dv, then

$$(\partial \log g)/\partial u = \gamma_1 \sin\alpha + \gamma_2 \cos\alpha = \chi_0 \tag{21.2}$$

$$(\partial \log g)/\partial n = -\gamma_1 \cos\alpha + \gamma_2 \sin\alpha = 0 \ , \tag{21.3}$$

agreeing with the remark at the end of § 19. These simple equations are often useful.

Properties of the Equipotential Surfaces

22. The normal curvature in any azimuth α, on the same lines as Eqs. (11.2) and (12.1, is

$$\begin{aligned}&\upsilon_{rs}(\lambda^r \sin\alpha + \mu^r \cos\alpha)(\lambda^s \sin\alpha + \mu^s \cos\alpha)\\ &\quad = \chi_1 \sin^2\alpha + 2\tau \sin\alpha \cos\alpha + \chi_2 \cos^2\alpha \ ,\end{aligned}$$

a generalized form of Euler's theorem.

23. The geodesic torsion in any azimuth α, on the same lines as Eq. (11.1) and remembering that the orthogonal surface vector is now in azimuth $(\pi/2+\alpha)$, is

$$\begin{aligned}&\upsilon_{rs}(\lambda^r \cos\alpha - \mu^r \sin\alpha)(\lambda^s \sin\alpha + \mu^s \cos\alpha)\\ &\quad = (\chi_1 - \chi_2) \sin\alpha \cos\alpha + \tau(\cos^2\alpha - \sin^2\alpha) \ .\end{aligned}$$

24. Since the geodesic torsion is zero in the principal directions, the azimuths of the latter are given by

$$\tan 2A = 2\tau/(\chi_2 - \chi_1) \ .$$

25. If κ_1, κ_2 are the principal curvatures of the surface, occurring respectively in azimuths $(\pi/2+A)$, A, we have from § 22

$$\kappa_1 = \chi_1 \cos^2 A - 2\tau \sin A \cos A + \chi_2 \sin^2 A$$

$$\kappa_2 = \chi_1 \sin^2 A + 2\tau \sin A \cos A + \chi_2 \cos^2 A$$

$$\kappa_1 + \kappa_2 = 2H = \chi_1 + \chi_2 \ , \quad \text{a well-known result} \ .$$

$$\begin{aligned}
\kappa_1 - \kappa_2 &= (\chi_1 - \chi_2)\cos 2\mathrm{A} - 2\tau \sin 2\mathrm{A} \\
&= (\chi_1 - \chi_2)\sec 2\mathrm{A} = -2\tau \csc 2\mathrm{A}\ , \quad \text{using § 24} \\
\kappa_1 \kappa_2 &= \text{Gauss curvature K} \\
&= \tfrac{1}{4}(\chi_1+\chi_2)^2 - \tfrac{1}{4}(\chi_1-\chi_2)^2 \sec^2 2\mathrm{A} = \chi_1\chi_2 - \tau^2\ .
\end{aligned}$$

The formulae may be inverted as

$$\begin{aligned}
\chi_1 &= \kappa_1 \cos^2\mathrm{A} + \kappa_2 \sin^2\mathrm{A} \\
\chi_2 &= \kappa_2 \sin^2\mathrm{A} + \kappa_2 \cos^2\mathrm{A} \\
\tau &= (\kappa_2 - \kappa_1)\cos\mathrm{A}\sin\mathrm{A}\ .
\end{aligned}$$

26. If $\varrho^s = \lambda^s \sin\alpha + \mu^s \cos\alpha$ is a unit surface vector in whose direction the longitude ω is constant, then multiplying Eq. (16.1) by ϱ^r and contracting, we have

$$0 = \chi_1 \sin\alpha + \tau\cos\alpha\ ,$$

so that ω is constant in azimuth $\tan^{-1}(-\tau/\chi_1)$; and similarly ϕ is constant in azimuth $\cos^{-1}(-\tau/\chi_2)$.

27. Similarly, the isozenithal direction in space, along which both ω and ϕ are constant, occurs in azimuth α, zenith distance β, if

$$0 = \chi_1 \sin\alpha \sin\beta + \tau\cos\alpha\sin\beta + \gamma_1\cos\beta\ ,$$

and

$$0 = \chi_2 \cos\alpha \sin\beta + \tau\sin\alpha\sin\beta + \gamma_2\cos\beta\ ,$$

whence

$$\tan\alpha = (\gamma_2\tau - \gamma_1\chi_2)/(\gamma_1\tau - \gamma_2\chi_1)\ ,$$

and

$$\begin{aligned}
\tan\beta &= -(\gamma_1\sin\alpha + \gamma_2\cos\alpha)/(\text{Normal curvature in azimuth } \alpha) \\
&= -(\gamma_1\cos\alpha - \gamma_2\sin\alpha)/(\text{Geodesic torsion in azimuth } \alpha)\ .
\end{aligned}$$

Alternatively, the isozenithal line must lie in a direction perpendicular to both ω_s and ϕ_s and is accordingly given by the vector equation

$$\begin{aligned}
(\cos\phi)\varepsilon^{rst}\omega_s\phi_t &= \varepsilon^{rst}(\chi_1\lambda_s + \tau\mu_s + \gamma_1 \nu_s)(\tau\lambda_t + \chi_2\mu_t + \gamma_2\nu_t) \\
&= (\gamma_2\tau - \gamma_1\chi_2)\lambda^r + (\gamma_1\tau - \gamma_2\chi_1)\mu^r + \mathrm{K}\nu^r\ ,
\end{aligned} \tag{27.1}$$

whence

$$\begin{aligned}
\sin\alpha\tan\beta &= (\gamma_2\tau - \gamma_1\chi_2)/\mathrm{K} \\
\cos\alpha\tan\beta &= (\gamma_1\tau - \gamma_2\chi_1)/\mathrm{K}\ .
\end{aligned}$$

28. The geodesic curvature (σ_1) of the parallel trace λ^r is by definition the component in the direction μ^r of its vector $\lambda_{rs}\lambda^s$, so that

$$\sigma_1 = \lambda_{rs}\lambda^s\mu^r = \sin\phi(\partial\omega/\partial p) \quad \text{from (8.1)}$$
$$= \chi_1 \tan\phi \quad \text{from (13.1)} \; .$$

Similarly the geodesic curvature (σ_2) of the meridian trace is given by

$$\sigma_2 = -\mu_{rs}\lambda^r\mu^s = \sin\phi(\partial\omega/\partial m) = \tau\tan\phi \; .$$

29. If $\varrho^r = (\lambda^r \cos\alpha - \mu^r \sin\alpha)$ is a unit surface vector in azimuth $(\pi/2+\alpha)$, the geodesic curvature in any azimuth α (arc length s) is given by

$$-(\lambda_r \sin\alpha + \mu_r \cos\alpha)_s \varrho^r(\lambda^s \sin\alpha + \mu^s \cos\alpha) \; ,$$

which an expansion and substitution of Eqs. (8.1) and (8.2) becomes

$$\sin\phi\sin\alpha(\partial\omega/\partial p) + \sin\phi\cos\alpha(\partial\omega/\partial m) - (\partial\alpha/\partial s)$$
$$= \sigma_1 \sin\alpha + \sigma_2 \cos\alpha - (\partial\alpha/\partial s)$$
$$= (\chi_1 \sin\alpha + \tau\cos\alpha)\tan\phi - (\partial\alpha/\partial s)$$

and since

$$(\partial\omega/\partial s) = \omega_r(\lambda^r \sin\alpha + \mu^r \cos\alpha)$$
$$= \sin\alpha(\partial\omega/\partial p) + \cos\alpha(\partial\omega/\partial m) \; ,$$

the geodesic curvature may also be written

$$\sin\phi(\partial\omega/\partial s) - (\partial\alpha/\partial s) \; .$$

30. The equation of the surface geodesic in azimuth α is accordingly

$$(\partial\alpha/\partial s) = (\chi_1 \sin\alpha + \tau\cos\alpha)\tan\phi = \sin\phi(\partial\omega/\partial s) \; .$$

The latter form of the equation is often used for spheroidal geodesics, but is here seen to be true for any surface V = const.

31. If τ is zero over a particular surface, then the meridian and parallel traces are lines of curvature (§ 24), and are also the surface directions in which ω and ϕ respectively are constant (§ 26); χ_1 and χ_2 are the principal curvatures (§ 25); the meridian traces are surface geodesics ($\sigma_2 = 0$) and are also plane curves, since their torsion as space curves is zero. Conversely, if any one of these statements is generally true, the others are true and $\tau = 0$. But these properties apply to an equipotential surface which is a surface revolution (such as a spheroid) symmetrical about the axis C^r. Accordingly, the parameter τ is a measure of a departure from this state.

Torsion Balance Measures

32. For the sake of completeness, the theory of the inclined torsion balance is reworked in the present notation, although this does not lead to any fresh results.

The line joining the two masses is of length 2l, azimuth α, zenith distance β, unit vector l^r. A unit surface vector in azimuth α is b^r and in $(3\pi/2+\alpha)$ is n^r.

The force acting on one mass m is $m V_r$; its resolved part perpendicular to the plane of v_r and l_r is $m V_r n^r$ and its turning moment about the suspension is $(m l \sin\beta) V_r n^r$. The force acting on the other mass is $m \bar{V}_r$ and the resultant moment of the two is

$$(m l \sin\beta)(V_r - \bar{V}_r) n^r \; .$$

But $(V_r - \bar{V}_r)$ is the intrinsic change in V_r over the short length 2l in the unit direction l^r so that

$$(V_r - \bar{V}_r) = (2l) V_{rs} l^s = -2 g l \{\upsilon_{rs} + (\log g)_s \upsilon_r\} l^s \quad \text{from (9.1)} \; ,$$

and remembering that $n^r \upsilon_r = 0$, since these two vectors are perpendicular, the resultant moment is

$$-2 m g l^2 \sin\beta\, \upsilon_{rs} n^r l^s = -2 m g l^2 \sin\beta\, \upsilon_{rs} n^r (v^s \cos\beta + b^s \sin\beta) \; . \tag{3.21}$$

Now from Eq. (11.1), minus $\upsilon_{rs} n^r b^s$ is the geodesic torsion in the direction b^s, i.e. in azimuth α (§ 23); $\upsilon_{rs} v^s$ is given by Eq. (19.1); and

$$n^r = -\lambda^r \cos\alpha + \mu^r \sin\alpha \; ,$$

so that the turning moment is finally

$$\begin{aligned} &+2 m g l^2 \sin^2\beta [(\chi_1 - \chi_2) \sin\alpha \cos\alpha + \tau(\cos^2\alpha - \sin^2\alpha) \\ &\quad + \gamma_1 \cos\alpha \cos\beta - \gamma_2 \sin\alpha \cos\beta] \; . \end{aligned} \tag{32.2}$$

33. Measurements in at least four azimuths will accordingly determine the parameters $(\chi_1 - \chi_2), \tau, \gamma_1, \gamma_2$ and with them many of the other local characteristics of the field, but this instrument will not separate χ_1 and χ_2. These two parameters could conceivably be obtained separately by measuring the turning moment about the *horizontal* axis n^r as suggested by Marussi (1947, p. 25). In that case, if m^r is a unit vector perpendicular to the balance arm in the vertical plane containing the balance arm [i.e. in azimuth α, zenith distance $(\pi/2+\beta)$], then the turning moment is

$$m l (\bar{V}_r - V_r) m^r = 2 m g l^2 \{\upsilon_{rs} + (\log g)_s \upsilon_r\} l^s m^r$$

which, with

$$m^r = l^r \sin\alpha \cos\beta + \mu^r \cos\alpha \cos\beta - \upsilon^r \sin\beta \; ,$$

works out at

$$\begin{aligned} &2 m g l^2 \sin\beta \cos\beta \{(\Delta^2 V)/g + \chi_1 (1 + \sin^2\alpha) + 2\tau \sin\alpha \cos\alpha \\ &\quad + \chi_2 (1 + \cos^2\alpha)\} + 2 m g l^2 \cos 2\beta (\gamma_1 \sin\alpha + \gamma_2 \cos\alpha) \end{aligned} \tag{33.1}$$

with, of course, $\Delta^2 V = 2\,\tilde{\omega}^2$ for a point in free air [Eq. (17.1)].

Derivatives of the Parameters

34. The conditions of integrability, corresponding to the Mainardi-Codazzi equations of coordinate surfaces, are easily obtained in this notation. If F is any scalar function of position,

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{\partial F}{\partial n}\right)-\frac{\partial}{\partial n}\left(\frac{\partial F}{\partial p}\right) &= (F_r \upsilon^r)_s \lambda^s - (F_r \lambda^r)_s \upsilon^s \\ &= F_{rs}\upsilon^r\lambda^s + F_r \upsilon^r_{,s}\lambda^s - F_{rs}\lambda^r\upsilon^s - F_r\lambda^r_{,s}\upsilon^s \\ &= F_r(\upsilon^r_{,s}\lambda^s - \lambda^r_{,s}\upsilon^s) \quad \text{(since } F_{rs} \text{ is symmetrical)} \\ &= \chi_1\frac{\partial F}{\partial p} + (\tau - \gamma_1 \tan\phi)\frac{\partial F}{\partial m} + \gamma_1\frac{\partial F}{\partial n},\end{aligned}$$

on substituting from Eqs. (8.3) and (8.1).

In the same way:

$$\frac{\partial}{\partial m}\left(\frac{\partial F}{\partial n}\right)-\frac{\partial}{\partial n}\left(\frac{\partial F}{\partial m}\right) = (\tau+\gamma_1\tan\phi)\frac{\partial F}{\partial p} + \chi_2\frac{\partial F}{\partial m} + \gamma_2\frac{\partial F}{\partial n}$$

$$\frac{\partial}{\partial m}\left(\frac{\partial F}{\partial p}\right)-\frac{\partial}{\partial p}\left(\frac{\partial F}{\partial m}\right) = \chi_1\tan\phi\frac{\partial F}{\partial p} + \tau\tan\phi\frac{\partial F}{\partial m}.$$

Writing $2H = (\chi_1+\chi_2)$ and putting $F = \phi$ we have

$$(\partial\gamma_2/\partial p)-(\partial\tau/\partial m) = 2H\tau - \gamma_1\chi_2\tan\phi + \gamma_1\gamma_2, \tag{34.1}$$

$$(\partial\gamma_2/\partial m)-(\partial\chi_2/\partial n) = \tau^2+\chi_2^2+\gamma_2^2+\gamma_1\tau\tan\phi, \tag{34.2}$$

$$(\partial\tau/\partial m)-(\partial\chi_2/\partial p) = 2H\tau\tan\phi \tag{34.3}$$

and the following three equations for $F = \omega$

$$(\partial\gamma_1/\partial p)-(\partial\chi_1/\partial n) = \chi_1^2+\gamma_1^2+\tau^2-2\gamma_1\tau\tan\phi+\chi_1\gamma_2\tan\phi \tag{34.4}$$

$$(\partial\gamma_1/\partial m)-(\partial\tau/\partial n) = 2H\tau+(\chi_1+\chi_2)\gamma_1\tan\phi+\gamma_1\gamma_2\tau\tan\phi \tag{34.5}$$

$$(\partial\chi_1/\partial m)-(\partial\tau/\partial p) = (\chi_1^2-\chi_1\chi_2+2\tau^2)\tan\phi. \tag{34.6}$$

These six equations are equivalent to the Codazzi equations in (ω, ϕ, V) coordinates, and we may similarly form corresponding equations for any other coordinates or for any other scalar. The following three equations, obtained from $F = (\log g)$ for displacements in free air for which $\Delta^2 V$ is constant, are, for instance, frequently useful.

$$(\partial\gamma_1/\partial n)+[\partial(2H)/\partial p] = 2\gamma_1\{(\Delta^2 V)/g\}+\gamma_1\chi_2-\gamma_2\tau+\gamma_1\gamma_2\tan\phi \tag{34.7}$$

$$(\partial\gamma_2/\partial n)+[\partial(2H)/\partial m] = 2\gamma_2\{(\Delta^2 V)/g\}+\gamma_2\chi_1-\gamma_1\tau-\gamma_1^2\tan\phi \tag{34.8}$$

$$(\partial\gamma_1/\partial m)-(\partial\gamma_2/\partial p) = (\chi_1\gamma_1+\gamma_2\tau)\tan\phi, \tag{34.9}$$

the last equation being equivalent to Eq. (34.5) minus Eq. (34.1).

35. By means of these equations we can readily find the Laplacians ($\Delta^2\phi$ etc.) of the main scalars. For example, covariant differentiation of Eq. (16.2) gives us

$$\phi_{rs} = (\tau)_s\lambda_r + (\chi_2)_s\mu_r + (\gamma_2)_s\upsilon_r + \tau\lambda_{rs} + \chi_2\mu_{rs} + \gamma_2\upsilon_{rs} \ ,$$

which becomes after multiplication by the metric tensor a^{rs} and substitution of Eq. (8.1) etc.

$$\begin{aligned} \Delta^2\phi &= \frac{\partial\tau}{\partial p} + \frac{\partial\chi_2}{\partial m} + \frac{\partial\gamma_2}{\partial n} + \chi_1\gamma_2 - \gamma_1\tau + \tau^2\tan\phi - \chi_1\chi_2\tan\phi \\ &= -(\chi_1^2 + \tau^2 + \gamma_1^2)\tan\phi + 2\gamma_2\{(\Delta^2 V)/g\} + 2\chi_1\gamma_2 - 2\gamma_1\tau \end{aligned}$$

on substituting Eqs. (34.6) and (34.8). And if we write

$$\nabla\omega = a^{rs}\omega_r\omega_s \quad \text{and} \quad \nabla(\log g, \phi) = a^{rs}(\log g)_r\phi_s \ ,$$

this can be expressed alternatively as

$$\Delta^2\phi = -\sin\phi\cos\phi\nabla\omega - 2\nabla(\log g, \phi) \ .$$

The equipotential *surface* Laplacian of ϕ is

$$\frac{\partial\tau}{\partial p} + \frac{\partial\chi_2}{\partial m} + \tau\lambda^{\alpha}_{,\alpha} + \chi_2\mu^{\alpha}_{,\alpha} = \frac{\partial(2H)}{\partial m} - (\chi_1^2 + \tau^2)\tan\phi$$

on substituting Eq. (34.6) and § 15.

36. In the same way, we have

$$\begin{aligned} (\cos\phi)\Delta^2\omega &= 2\gamma_1\{(\Delta^2 V)/g\} + 2\gamma_1\chi_2 - 2\gamma_2\tau + 2\tan\phi(\chi_1\tau + \chi_2\tau + \gamma_1\gamma_2) \\ &= 2\sin\phi\nabla(\phi, \omega) - 2\cos\phi\nabla(\log g, \omega) \end{aligned}$$

and the *surface* Laplacian of ω is

$$\sec\phi\frac{\partial(2H)}{\partial p} + 4H\tau\sec\phi\tan\phi \ .$$

37. We have also by the same means

$$\Delta^2(\log g) = -2K - 4H\{(\Delta^2 V)/g\} - \{(\Delta^2 V)/g\}^2 \ ,$$

which together with

$$\Delta^2(\log g) = a^{rs}(g_r/g)_s = (\Delta^2 g)/g - \nabla(\log g)$$

and

$$\nabla(\log g) = \gamma_1^2 + \gamma_2^2 + (\chi_1 + \chi_2)^2 + 2(\chi_1 + \chi_2)\{(\Delta^2 V)/g\} + \{(\Delta^2 V)/g\}^2$$

gives

$$(\Delta^2 g)/g = \gamma_1^2 + \gamma_2^2 + 4H^2 - 2K \ ,$$

which is equivalent to the sum of the squares of the principal curvatures of the equipotential surface and of the line of force – a rather remarkable result.

The *surface* Laplacian of (log g) is

$$\frac{\partial \gamma_1}{\partial \mathrm{p}}+\frac{\partial \gamma_2}{\partial \mathrm{m}}+\gamma_1 \tau \tan \phi-\gamma_2 \chi_1 \tan \phi \ ,$$

which with Eqs. (34.4), (34.2) and § 25 can be transformed to

$$\frac{\partial(2\mathrm{H})}{\partial \mathrm{n}}+(\gamma_1^2+\gamma_2^2)+(4\mathrm{H}^2-2\mathrm{K}) \ .$$

38. By means of the above equations, we can also find the derivatives of other scalars along the line of force. For instance, if α is the azimuth of the principal normal to the line of force, we have by differentiating Eq. (21.1) and using Eqs. (34.7), (34.8)

$$\begin{aligned}(\gamma_1^2+\gamma_2^2)(\partial\alpha/\partial \mathrm{n}) &= \gamma_2(\partial\gamma_1/\partial \mathrm{n})-\gamma_1(\partial\gamma_2/\partial \mathrm{n})\\ &= \gamma_1\gamma_2(\chi_2-\chi_1)+\tau(\gamma_1^2-\gamma_2^2)+\gamma_1 \tan\phi(\gamma_1^2+\gamma_2^2)\\ &\quad -\gamma_2\{\partial(2\mathrm{H})/\partial \mathrm{p}\}+\gamma_1\{\partial(2\mathrm{H})/\partial \mathrm{m}\} \ ,\end{aligned}$$

or, if $\chi_0=(\gamma_1^2+\gamma_2^2)^{1/2}$ is the curvature of the line of force,

$$\begin{aligned}(\partial\alpha/\partial \mathrm{n}) &= (\chi_2-\chi_1)\sin\alpha\cos\alpha+\tau(\sin^2\alpha-\cos^2\alpha)+\gamma_1\tan\phi\\ &= (2\mathrm{H})_{\mathrm{r}}(-\lambda^{\mathrm{r}}\cos\alpha+\mu^{\mathrm{r}}\sin\alpha)/\chi_0 \ .\end{aligned}$$

But $(-\lambda^{\mathrm{r}}\cos\alpha+\mu^{\mathrm{r}}\sin\alpha)$ is the binormal to the line of force (§ 20) and if dv is the element of arc in this direction the last term is $\{\partial(2\mathrm{H})/\partial \mathrm{v}\}\chi_0$. From Eq. (20.1) the torsion τ_0 of the line of force is

$$\tau_0=(\chi_1-\chi_2)\sin\alpha\cos\alpha+\tau(\cos^2\alpha-\sin^2\alpha)-\{\partial(2\mathrm{H})/\partial \mathrm{v}\}/\chi_0 \ ,$$

in which the first two terms are the geodesic torsion of the surface in azimuth α.

39. We can similarly derive an expression for $(\partial\chi_0/\partial \mathrm{n})$ from Eqs. (34.7), (34.8) together with $\gamma_1=\chi_0\sin\alpha$; $\gamma_2=\chi_0\cos\alpha$, as in Eq. (21.1)

$$\begin{aligned}\chi_0(\partial\chi_0/\partial \mathrm{n}) &= \gamma_1(\partial\gamma_1/\partial \mathrm{n})+\gamma_2(\partial\gamma_2/\partial \mathrm{n})\\ &= 4\chi_0^2\tilde{\omega}^2/\mathrm{g}+\chi_0^2(\chi_1\cos^2\alpha-2\tau\sin\alpha\cos\alpha+\chi_2\sin^2\alpha)\\ &\quad -\chi_0(2\mathrm{H})_{\mathrm{r}}(\lambda^{\mathrm{r}}\sin\alpha+\mu^{\mathrm{r}}\cos\alpha) \ ;\end{aligned}$$

and if du is the element of arc in the direction of the principal normal, this can be written

$$\begin{aligned}(\partial\log\chi_0/\partial \mathrm{n}) &= 4\tilde{\omega}^2/\mathrm{g}+(\chi_1\cos^2\alpha-2\tau\sin\alpha\cos\alpha+\chi_2\sin^2\alpha)\\ &\quad -\{\partial(2\mathrm{H})/\partial \mathrm{u}\}/\chi_0 \ .\end{aligned}$$

The second term is the normal curvature of the equipotential surface in azimuth $(\pi/2+\alpha)$ or $(3\pi/2+\alpha)$, i.e. in the direction of the binormal to the line of force.

Variation Along Lines

40. We now proceed to find the total change ΔF in a scalar function of position F over a finite length of a line whose unit tangent vector is l^r, principal normal m^r, binormal n^r, curvature $\bar{\chi}$, torsion $\bar{\tau}$ from the ordinary expansion

$$\Delta F = \left(\frac{\partial F}{\partial s}\right)_0 s + \frac{1}{2}\left(\frac{\partial^2 F}{\partial s^2}\right)_0 s^2 + \frac{1}{6}\left(\frac{\partial^3 F}{\partial s^3}\right)_0 s^3 + \ldots \text{ etc.} \tag{40.1}$$

for which we require to know the values of $(\partial F/\partial s)$ etc. at the initial point. By taking successive covariant derivatives, we have $(\partial F/\partial s) = F_r l^r$, F_r being as usual the space gradient of F,

$$(\partial^2 F/\partial s^2) = (F_r l^r)_s l^s = F_{rs} l^r l^s + \bar{\chi} F_r m^r \ ,$$

$$(\partial^3 F/\partial s^3) = F_{rst} l^r l^s l^t + 3\bar{\chi} F_{rs} m^r l^s + (\partial\bar{\xi}/\partial s) F_r m^r + \bar{\chi} F_r (\bar{\tau} n^r - \bar{\chi} l^r) \ ,$$

etc. using the Frenet formulae.

41. It may be noted in passing that changes in Cartesian coordinates (e.g. x) along a line are particularly easy to compute from these formulae, since all components of the tensors x_{rs}, x_{rst} etc. are zero in Cartesian coordinates and are therefore zero in any coordinates. The other, non-zero, invariants can, of course, be computed in any coordinates. For example, we take a Cartesian system with origin at one end of the line, x (respectively y) in the direction of λ^r (respectively μ^r) at the same end, z normal to the initial equipotential surface, and consider the change in these coordinates along the line of force. If α is the azimuth of the principal normal to the line of force, we have the following Cartesian components at the origin.

$$l^r = (0, 0, 1)$$

$$m^r = (\sin\alpha, \cos\alpha, 0)$$

$$n^r = (-\cos\alpha, \sin\alpha, 0)$$

and evaluating the above invariants at the origin in the Cartesian system,

$$(\partial x/\partial s)_0 = 0 \ ; \quad (\partial^2 x/\partial s^2)_0 = \chi_0 \sin\alpha = \gamma_1 \quad \text{as in (21.1)} \ ;$$

$$(\partial^3 x/\partial s^3)_0 = (\partial\chi_0/\partial n)\sin\alpha - \chi_0 \tau_0 \cos\alpha = (\partial\gamma_1/\partial n) - \gamma_1\gamma_2 \tan\phi \ ,$$

by differentiating $\gamma_1 = \chi_0 \sin\alpha$ along n and using Eq. (20.1). The values of γ_1, γ_2 are obtained from Eq. (34.7) in terms of the parameters and the form of the initial equipotential surface. We have finally

$$\Delta x = \tfrac{1}{2}\gamma_1 s^2 + \tfrac{1}{6}s^3(\partial\gamma_1/\partial n - \gamma_1\gamma_2 \tan\phi) + \ldots \quad \text{and similarly} \ ,$$

$$\Delta y = \tfrac{1}{2}\gamma_2 s^2 + \tfrac{1}{6}s^3(\partial\gamma_2/\partial n + \gamma_1^2 \tan\phi) + \ldots$$

$$\Delta z = s - \tfrac{1}{6}s^3(\gamma_1^2 + \gamma_2^2) + \ldots \ .$$

Variation Along Refracted Rays

42. Of particular importance in geodesy is the path of light refracted by the atmosphere, since all triangulation observations are made along such lines. We assume that the air density is a function of potential, since otherwise the model atmosphere would introduce lateral refraction. In setting up such a mathematical model, we do not, of course, assume a complete absence of lateral refraction, but merely that over a large number of lines it will occur as often to one side as the other.

43. Subject to this one assumption, it can be shown[3] that the principal normal m^r to the refracted ray (l^r, azimuth α, zenith distance β) lies in the plane of l^r and υ^r, the normal to the equipotential surface; so that m^r is in azimuth α, zenith distance $(\pi/2+\beta)$. The binormal n^r is a surface vector in azimuth $(3\pi/2+\alpha)$.

$$\begin{aligned} l^r &= \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta \ , \\ m^r &= \lambda^r \sin\alpha \cos\beta + \mu^r \cos\alpha \cos\beta - \upsilon^r \sin\beta \ , \\ n^r &= -\lambda^r \cos\alpha + \mu^r \sin\alpha \ . \end{aligned} \tag{43.1}$$

44. If μ is the refracting index, the principal curvature of the ray is given by

$$\bar{\chi} = (\log\mu)_r m^r = -\sin\beta\,\partial(\log\mu)/\partial n \ , \tag{44.1}$$

which is calculable for a standard atmosphere (Bomford 1952, p. 156).

45. To find the torsion $\bar{\tau}$, we multiply the third Frenet formula $n_{rs} l^s = -\bar{\tau} m_r$ by υ^r.

$$\begin{aligned} \bar{\tau} &= (\csc\beta)\, n_{rs} \upsilon^r l^s = -(\csc\beta)\, \upsilon_{rs} n^r l^s \\ &= (\chi_1 - \chi_2) \sin\alpha \cos\alpha + \tau(\cos^2\alpha - \sin^2\alpha) \\ &\qquad + \gamma_1 \cos\alpha \cos\beta - \gamma_2 \sin\alpha \cos\beta \quad \text{[cf. (32.1), (32.2)]} \ . \end{aligned} \tag{45.1}$$

Note that if the ray were straight, we could not put the latter expression equal to zero since the Frenet formula from which it is derived would not apply. Formulae derived for a curved ray in which the latter expression is substituted for $\bar{\tau}$ will nevertheless apply equally to straight rays.

46. Multiplying Eqs. (16.1), (16.2), (16.4) and (7.1) across by l^r we have the following first-order formulae for changes in ω etc. along the line

$$\cos\phi\,(\partial\omega/\partial s) = \chi_1 \sin\alpha \sin\beta + \tau \cos\alpha \sin\beta + \gamma_1 \cos\beta \tag{46.1}$$

[3] A rapid way of obtaining these results is to make a conformal transformation to a curved space $d\bar{s}^2 = \mu^2 ds^2$ (Levi-Civita 1926, p. 228) and to use Fermat's principle (Levi-Civita 1926, p. 334) by expressing the transformed rays as geodesics of the curved space.

$$\partial\phi/\partial s = \tau \sin\alpha \sin\beta + \chi_2 \cos\alpha \sin\beta + \gamma_2 \cos\beta \tag{46.2}$$

$$\partial(\log g)/\partial s = \gamma_1 \sin\alpha \sin\beta + \gamma_2 \cos\alpha \sin\beta - \{\chi_1 + \chi_2 + (\Delta^2 V)/g\} \cos\beta \tag{46.3}$$

$$\partial V/\partial s = -g \cos\beta \ . \tag{46.4}$$

Notice that, identically,

$$\bar{\tau} \sin\beta = \cos\alpha \cos\phi (\partial\omega/\partial s) - \sin\alpha (\partial\phi/\partial s) \ . \tag{46.5}$$

47. It will also be convenient to have equations for $(\partial\alpha/\partial s)$, $(\partial\beta/\partial s)$. Since n^r is defined along the line, we can differentiate it intrinsically as

$$n_{rs} l^s = (-\lambda_{rs} \cos\alpha + \mu_{rs} \sin\alpha) l^s + (\lambda_r \sin\alpha + \mu_r \cos\alpha)(\partial\alpha/\partial s) \ .$$

Now $b_r = (\lambda_r \sin\alpha + \mu_r \cos\alpha)$ is a surface vector in the plane of l_r and υ_r, so that it is also equal to $(l_r \csc\beta - \upsilon_r \cos\beta)$, and multiplying the last equation across by b^r, we have after substituting for λ_{rs} etc.

$$\begin{aligned} \partial\alpha/\partial s &= \sin\phi (\partial\omega/\partial s) + n_{rs}(l^r \csc\beta - \upsilon^r \cos\beta) l^s \\ &= \sin\phi (\partial\omega/\partial s) - \bar{\tau} \cos\beta \end{aligned} \tag{47.1}$$

[using Eq. (45.1) and noting that $n_{rs} l^r l^s = -\bar{\tau} m_r l^r = 0$] finally we have

$$\partial\alpha/\partial s = (\sin\phi - \cos\phi \cos\alpha \cos\beta)(\partial\omega/\partial s) + \sin\alpha \cos\beta (\partial\phi/\partial s) \tag{47.2}$$

from Eq. (46.5).

48. The simplest way to find $(\partial\beta/\partial s)$ is to differentiate $l^r \upsilon_r = \cos\beta$ intrinsically along l^s. We then have

$$\partial\beta/\partial s = -\upsilon_{rs} l^r l^s \csc\beta - (\bar{\chi} m^r) \upsilon_r \csc\beta \ ,$$

and writing

$$l^r = \upsilon^r \cos\beta + b^r \sin\beta$$

(b^r as above a unit surface vector in azimuth α), this is

$$\begin{aligned} \partial\beta/\partial s &= \bar{\chi} - \upsilon_{rs} \upsilon^s b^r \cos\beta - \upsilon_{rs} b^r b^s \sin\beta \\ &= \bar{\chi} - (\gamma_1 \cos\alpha + \gamma_2 \sin\alpha) \cos\beta \\ &\quad - (\chi_1 \sin^2\alpha + 2\tau \sin\alpha \cos\alpha + \chi_2 \cos^2\alpha) \sin\beta \end{aligned}$$

[using Eq. (19.1) and § 22] finally we have

$$\partial\alpha/\partial s = \bar{\chi} - \sin\alpha \cos\phi (\partial\omega/\partial s) - \cos\alpha (\partial\phi/\partial s) \tag{48.1}$$

from Eqs. (46.1) and (46.2).

49. The second differentials may be obtained from the invariant equations (§ 40) or, now that we have expressions for $(\partial\alpha/\partial s)$, $(\partial\beta/\partial s)$, by direct differentiation of Eq. (46.1) etc.

$$\begin{aligned}
\cos\phi(\partial^2\omega/\partial s^2) = {} & (\partial\chi_1/\partial s)\sin\alpha\sin\beta + (\partial\tau/\partial s)\cos\alpha\sin\beta + (\partial\gamma_1/\partial s)\cos\beta \\
& + (\partial\omega/\partial s)\{(\chi_1+\chi_2)\sin\phi\cos\alpha\sin\beta - \chi_1\cos\phi\cos\beta \\
& + \gamma_1\cos\phi\sin\alpha\sin\beta + \gamma_2\sin\phi\cos\beta\} \\
& + (\partial\phi/\partial s)(-\tau\cos\beta + \gamma_1\cos\alpha\sin\beta) \\
& + \bar{\chi}(\chi_1\sin\alpha\cos\beta + \tau\cos\alpha\cos\beta - \gamma_1\sin\beta)
\end{aligned}$$

$$\begin{aligned}
(\partial^2\phi/\partial s^2) = {} & (\partial\tau/\partial s)\sin\alpha\sin\beta + (\partial\chi_2/\partial s)\cos\alpha\sin\beta + (\partial\gamma_2/\partial s)\cos\beta \\
& + (\partial\omega/\partial s)\{\tau(\sin\phi\cos\alpha\sin\beta - \cos\phi\cos\beta) - \chi_2\sin\phi\sin\alpha\sin\beta \\
& + \gamma_2\cos\phi\sin\alpha\sin\beta\} + (\partial\phi/\partial s)\{-\chi_2\cos\beta + \gamma_2\cos\alpha\sin\beta\} \\
& + \bar{\chi}(\tau\sin\alpha\cos\beta + \chi_2\cos\alpha\cos\beta - \gamma_2\sin\beta)
\end{aligned}$$

$$\begin{aligned}
\partial^2(\log g)/\partial s^2 = {} & -\cos\beta\,\partial(\chi_1+\chi_2)/\partial s + (\partial\gamma_1/\partial s)\sin\alpha\sin\beta \\
& + (\partial\gamma_2/\partial s)\cos\alpha\sin\beta + 2\tilde{\omega}^2\cos\beta(\partial\log g/\partial s)/g \\
& + (\partial\omega/\partial s)\{\gamma_1(\sin\phi\cos\alpha\sin\beta - \cos\phi\cos\beta) \\
& - \gamma_2\sin\phi\sin\alpha\sin\beta - (\chi_1+\chi_2+2\tilde{\omega}^2/g)\cos\phi\sin\alpha\sin\beta\} \\
& + (\partial\phi/\partial s)\{-\gamma_2\cos\beta - (\chi_1+\chi_2+2\tilde{\omega}^2/g)\cos\alpha\sin\beta\} \\
& + \bar{\chi}\{\gamma_1\sin\alpha\cos\beta + \gamma_2\cos\alpha\cos\beta + (\chi_1+\chi_2+2\tilde{\omega}^2/g)\sin\beta\}
\end{aligned}$$

$$\partial^2 \upsilon/\partial s^2 = -g\cos\beta(\partial\log g/\partial s) + g\sin\beta(\partial\beta/\partial s)\ .$$

By using Eqs. (47.2) and (48.1), we can in the above expressions have terms in $(\partial\alpha/\partial s)$, $(\partial\beta/\partial s)$ instead of $(\partial\omega/\partial s)$, $(\partial\phi/\partial s)$. Many terms will in practice be negligible, but are given in full for the sake of completeness. If we know the five parameters at both ends of the line, say by torsion balance measures, we can calculate $(\partial\chi_1/\partial s)$ etc. on the assumption that the parameters vary uniformly over the line, and so the second differentials. Substitution of the values of the first and second differentials for the initial point in Eq. (40.1) will then give us to a second order the changes in ω, ϕ, (log g) and V over the line. The third-order terms will, however, contain $(\partial^2\chi_1/\partial s^2)$ etc., and there is no way of evaluating these over a single line. Nor is it likely that they could be determined with worthwhile accuracy from differences in a network necessarily measured at points on different equipotential surfaces. For these reasons it is unlikely that astronomical coordinates can ever be used to calculate geodetic triangulation to sufficient accuracy, even though the measured azimuths and zenith distances are in this gravitational field in which the measurements are necessarily made, fundamental to the whole subject, and in the course of another paper (Geodetic Coordinate Systems 1957) will be applied to geodetic calculations.

Astro-Geodetic Measures

50. It remains to consider conversely whether measures of astronomical latitude etc. over the lines of a triangulation can be used to determine the parameters as

a contribution to knowledge of the form of the field. The procedure would be to determine approximate values of the parameters from the first-order equations [(46.1) etc.] and use these to derive the second-order terms; then re-solving the first-order equations after applying the second-order terms as a correction. For the first-order solution we use observed differences for $(\partial\omega/\partial s)$ etc. and mean values of the coefficients computed from the values at both ends, azimuth, for instance, at the far end being the azimuth of the line produced. The results should give mean values of the parameters over the line, that is the values at some intermediate point on the line.

In the first place, it should be appreciated from Eqs. (47.2) and (48.1) that the azimuth and zenith distance equations are not independent of the latitude and longitude equations; two only of these equations should be used, probably latitude and azimuth, which are more easily measured precisely. These equations will not determine γ_1, γ_2 with any accuracy, although the terms containing γ_1, γ_2 are not necessarily negligible in any but flat country. We might, however, reckon on gravimeter observations at the stations to obtain (46.3) as a third equation. Accordingly, we need at least two lines in different azimuths to determine the five parameters. But the equations for each line will refer to the values at some intermediate point on that line, so that the problem in relation to one isolated point is soluble only on the implicit assumption that the parameters are sensibly constant over an area extending to something like half the length of the lines. If several lines radiating from a point are used in the solution, the residuals will indicate how far this assumption is justified.

51. Current methods of determining "geoidal sections" by similar means achieve a certain simplicity by assuming in addition that the stations are all on the same equipotential surface. They could hardly be expected to produce worthwhile results if the more rigorous methods discussed here do not. True, they do not use large differences of direct astronomical observations, but small "deflections", and that may lend an air of precision. At best, this procedure would amount to forming another first-order equation akin to e.g. Eq. (46.2) in a different set of coordinates and subtracting from (46.2), the precision of which is not thereby increased at all. The assumption that deflections measured and calculated at a point well above the geoid would remain the same at a point vertically below the geoid is probably very far from justifiable.

52. To test these matters, trial computations have been carried out at two points of a test triangulation which is free from observational error, one point, D, being a very disturbed area, which would be unlikely to occur in practice, and one, I, a much less severe case. In both cases the lines are long. Details of the two points and of the lines radiating from them are given in the annexed Table 1. A least-squares solution of the first-order latitude, longitude and gravity equations for all lines radiating from a point has been made and the residuals of each equation are given in Table 2, together with the magnitude of the second-order terms (in brackets). Residuals for each azimuth equation (not included in the solution) are also given. Even if allowance is made for such second-order terms as can be computed, it is apparent that the method does not give results commensurate with

Table 1.

Point	Approximate					Deflection		Gravity anomaly (milligals)
	Latitude (N)		Longitude (E)		Height (m)	Meridian	Prime vertical	
D	37°	00′	20°	40′	1300	−13.982″	−22.141″	−5876
B	36	35	20	25	0	−31.675	−49.379	−6115
C	36	48	19	41	1200	−24.753	+21.541	−6003
E	37	10	20	00	3900	−23.756	+1.051	−5913
G	37	40	20	11	4000	−13.367	−1.458	−5791
F	37	20	20	50	1100	−11.856	−8.861	−5801
I	38	00	20	40	3800	−4.617	−4.747	−5736
G	37	40	20	11	4000	−13.367	−1.458	−5791
F	37	20	20	50	1100	−11.856	−8.861	−5801
H	37	40	21	10	1000	−10.662	−3.430	−5768
J	38	00	21	50	1400	−12.368	−15.598	−5782
K	38	21	21	10	3500	+0.910	−8.611	−5727

the attainable accuracy of observation. The implicit assumption that the parameters are sensibly constant over the length of the lines is evidently too drastic.

53. Apart from uneconomic shortening of the lines, the only other possibility would seem to be simultaneous determination of the parameters over a network of triangulation on the assumption that they vary uniformly along each line. In that case the unknowns in the first-order observational equations [(46.1) etc.] might be written as the mean of the parameters at the two ends. Each additional point of the triangulation thus adds five unknowns, but since it must be connected to at least two points already included, it would provide at least six more equations, so that ultimately the system would be determinable or even redundant. Further consideration needs to be given, however, to the possibility of dependent equations on closing lines.

The Marussi Metric

54. If we take $(\omega, \phi, \mathrm{V})$ as coordinates $(1,2,3)$ the contravariant components of the base vectors, using Eq. (13.1), will be:

$$\begin{aligned}
\lambda^{\mathrm{r}} &= \{(\partial\omega/\partial \mathrm{p})\,,\quad (\partial\phi/\partial \mathrm{p})\,,\quad (\partial \mathrm{V}/\partial \mathrm{p})\} = (\chi_1 \sec\phi, \tau, 0)\\
\mu^{\mathrm{r}} &= \{(\partial\omega/\partial \mathrm{m})\,,\quad (\partial\phi/\partial \mathrm{m})\,,\quad (\partial \mathrm{V}/\partial \mathrm{m})\} = (\tau \sec\phi, \chi_2, 0)\\
\upsilon^{\mathrm{r}} &= \{(\partial\omega/\partial \mathrm{n})\,,\quad (\partial\phi/\partial \mathrm{n})\,,\quad (\partial \mathrm{V}/\partial \mathrm{n})\} = (\gamma_1 \sec\phi, \gamma_2, -\mathrm{g})\ .
\end{aligned}\tag{54.1}$$

55. By solving the nine equations

$$\lambda^{\mathrm{r}}\lambda_{\mathrm{s}} + \mu^{\mathrm{r}}\mu_{\mathrm{s}} + \upsilon^{\mathrm{r}}\upsilon_{\mathrm{s}} = \delta^{\mathrm{r}}_{\mathrm{s}}$$

Table 2.

Line	Longitude		Latitude		Gravity		Azimuth	
	Difference	Residual (Comp. − Obs.)	Difference	Residual (Comp. − Obs.)	Difference (Milligals)	Residual (Comp. − Obs.)	Difference	Residual (Comp. − Obs.)
DB	−913.776″	+23.01″ (+4.77″)	−1467.693″)	+6.45″ (−0.93″	+127	+88 (−61)	−546.914″	+13.42″ (+5.39″)
DC	−3575.378″	−33.86″ (+9.30″)	−710.771″	−19.44″ (−14.71″)	−113	−64 (−203)	−2146.662″	−20.31″ (+10.81″)
DE[a]	−2400.959″	−1.12″ (−5.01″)	+630.226″	−5.79″ (−6.78″)	−823	+3 (−95)	−1447.768″	+0.42″ (−5.92″)
DG	−1764.120″	+18.54″ (−14.99″)	+2400.616″	−6.26″ (−3.88″)	−689	+44 (−179)	−1070.511″	+10.97″ (−17.39″)
DF	+596.576″	−6.20″ (+2.77″)	+1242.126″	+5.34″ (−0.28″)	+167	+44 (−40)	+360.480″	−3.78″ (+3.08″)
IG	−1775.818″	−0.68″ (+8.22″)	−1238.750″	−3.19″ (−3.63″)	−147	−29 (−81)	−1089.234″	−0.41″ (+9.30″)
IF	+584.879″	−2.29″ (−5.13″)	−2397.239″	+2.87″ (−0.76″)	+709	+12 (−137)	+357.730″	−1.36″ (−5.87″)
IH[a]	+1781.691″	+7.91″ (−7.60″)	−1196.045″	−0.90″ (−3.93″)	+802	+16 (−79)	+1093.398″	+4.68″ (−8.77″)
IJ	+4156.230″	−0.30″ (+0.41″)	−27.751″	−8.15″ (−20.26″)	+693	−21 (−250)	+2559.257″	+0.06″ (+0.04″)
IK	+1805.046″	+0.01″ (+8.27″)	+1205.526	+0.07″ (−3.76″)	+131	−16 (−80)	+1115.578	+0.01″ (+9.31″)

[a] Not included in solution.

we can readily obtain the covariant components, beginning with the three equations $-g\upsilon_s = \delta^3_s$, which give υ_s at once. Writing $K = (\chi_1\chi_2 - \tau^2)$, the Gauss curvature of the equipotential surface, we have

$$
\begin{aligned}
K\lambda_r &= \{\chi_2 \cos\phi, -\tau, (\chi_2\gamma_1 - \tau\gamma_2)/g\} \\
K\mu_r &= \{-\tau\cos\phi, \chi_1, (\chi_1\gamma_2 - \tau\gamma_1)/g\} \qquad (55.1)\\
\upsilon_r &= \{0, 0, -1/g\} .
\end{aligned}
$$

56. From Eq. (16.4), we have

$$\partial(\log g)/\partial\omega = \gamma_1\lambda_1 + \gamma_2\mu_1 = (\chi_2\gamma_1 - \tau\gamma_2)\cos\phi/K \qquad (56.1)$$

and

$$\partial(\log g)/\partial\phi = \gamma_1\lambda_2 + \gamma_2\mu_2 = (\chi_1\gamma_2 - \tau\gamma_1)/K , \qquad (56.2)$$

so that we can write alternatively

$$\lambda_3 = -\sec\phi\{\partial(1/g)/\partial\omega\} \qquad (56.3)$$

and

$$\mu_3 = -\partial(1/g)/\partial\phi$$

and then

$$
\begin{aligned}
\partial g/\partial V &= g\gamma_1\lambda_3 + g\gamma_2\mu_3 - g\{\chi_1 + \chi_2 + (\Delta^2 V)/g\}\upsilon_3 \quad (16.4)\\
&= (\chi_1\gamma_2^2 + \chi_2\gamma_1^2 - 2\tau\gamma_1\gamma_2)/K + \{\chi_1 + \chi_2 + (\Delta^2 V)/g\} . \qquad (56.4)
\end{aligned}
$$

57. Components of the metric tensor a_{rs} or a^{rs} follow at once by substitution in the formulae:

$$a_{rs} = \lambda_r\lambda_s + \mu_r\mu_s + \upsilon_r\upsilon_s ,$$

or

$$a^{rs} = \lambda^r\lambda^s + \mu^r\mu^s + \upsilon^r\upsilon^s ,$$

true for any triad of orthogonal unit vectors.

58. If F is any scalar

$$
\begin{aligned}
(\partial F/\partial p) &= F_r\lambda^r = \chi_1\sec\phi(\partial F/\partial\omega) + \tau(\partial F/\partial\phi) \\
(\partial F/\partial m) &= F_r\mu^r = \tau\sec\phi(\partial F/\partial\omega) + \chi_2(\partial F/\partial\phi) \qquad (58.1)\\
(\partial F/\partial n) &= F_r\upsilon^r = \gamma_1\sec\phi(\partial F/\partial\omega) + \gamma_2(\partial F/\partial\phi) - g(\partial F/\partial V) .
\end{aligned}
$$

The Codazzi relations, which in the case of these coordinates reduce to five independent equations, can be obtained as derivatives of the five parameters χ_1 etc. with respect to ω, ϕ, V by means of the above equations and § 34, after any desired manipulation.

By solving the last set of equations or by evaluating the equation $F_r = (\partial F/\partial p)\lambda_r + (\partial F/\partial m)\mu_r + (\partial F/\partial n)\upsilon_r$ in (ω, ϕ, V) coordinates, we have also

$$
\begin{aligned}
K \sec\phi(\partial F/\partial\omega) &= \chi_2(\partial F/\partial p) - \tau(\partial F/\partial m)\\
K(\partial F/\partial\phi) &= -\tau(\partial F/\partial p) + \chi_1(\partial F/\partial m)\\
Kg(\partial F/\partial V) &= (\chi_2\gamma_1 - \gamma_2\tau)(\partial F/\partial p)\\
&\quad + (\chi_1\gamma_2 - \gamma_1\tau)(\partial F/\partial m) - K(\partial F/\partial n)\ .
\end{aligned} \tag{58.2}
$$

59. The remaining metrical properties of the space follow in the usual way from the metric tensor, but the foregoing general analysis will enable us to take several short-cuts. For instance, to find the Christoffel symbols, we have

$$V_{rs} = \frac{\partial^2 V}{\partial x^r \partial x^s} - \begin{Bmatrix} t \\ rs \end{Bmatrix} V_t = -\begin{Bmatrix} 3 \\ rs \end{Bmatrix}$$

so that, using Eqs. (9.1) and (8.3), the six symbols with superscript 3 are given by

$$
\begin{aligned}
\frac{1}{g}\begin{Bmatrix} 3 \\ rs \end{Bmatrix} &= \upsilon_{rs} + (\log g)_s \upsilon_r = \cos\phi\,\lambda_r\omega_s + \mu_r\phi_s + \upsilon_r(\log g)_s\\
&= \cos\phi\,\lambda_r\delta^1_s + \mu_r\delta^2_s - \delta^3_r(\log g)_s/g\ .
\end{aligned}
$$

For example,

$$\begin{Bmatrix} 3 \\ 12 \end{Bmatrix} = -\frac{g\tau\cos\phi}{K}\ ; \quad \begin{Bmatrix} 3 \\ 33 \end{Bmatrix} = -\frac{\partial \log g}{\partial V} \quad \text{etc.}$$

60. In the same way, we have, by differentiating Eq. (10.2),

$$
\begin{aligned}
-\cos\phi\begin{Bmatrix} 1 \\ rs \end{Bmatrix} &= \cos\phi\,\omega_{rs}\\
&= \phi_s\omega_r\sin\phi + (\chi_1)_s\lambda_r + (\tau)_s\mu_r + (\gamma_1)_s\upsilon_r + \chi_1\lambda_{rs} + \tau\mu_{rs} + \gamma_1\upsilon_{rs}\ ;
\end{aligned}
$$

and by substituting Eq. (8.1) etc. this is

$$
\begin{aligned}
-\cos\phi\begin{Bmatrix} 1 \\ rs \end{Bmatrix} &= \sin\phi\,\delta^1_r\delta^2_s + \lambda_r(\chi_1)_s + \mu_r(\tau)_s + \upsilon_r(\gamma_1)_s\\
&\quad + \chi_1(\mu_r\sin\phi - \upsilon_r\cos\phi)\delta^1_s - \tau(\sin\phi\,\lambda_r\delta^1_s + \upsilon_r\delta^2_s)\\
&\quad + \gamma_1(\cos\phi\,\lambda_r\delta^1_s + \mu_r\delta^2_s)
\end{aligned}
$$

so that, for example

$$\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{\tau}{K}\frac{\partial\tau}{\partial\phi} = \frac{\chi_2}{K}\frac{\partial\chi_1}{\partial\phi} + \frac{\gamma_1\tau}{K} - \tan\phi$$

or

$$\begin{Bmatrix} 1 \\ 21 \end{Bmatrix} = \frac{\tau\sec\phi}{K}\frac{\partial\chi_1}{\partial\omega} - \frac{\chi_1\sec\phi}{K}\frac{\partial\tau}{\partial\omega} - \frac{\chi_1^2\tan\phi}{K} - \frac{\tau^2\tan\phi}{K} + \frac{\gamma_1\tau}{K}\ ,$$

the two being shown to be identical by means of § 58 and § 34. Also,

$$\left\{ \begin{matrix} 2 \\ \mathrm{rs} \end{matrix} \right\} = -\phi_{\mathrm{rs}} = -(\tau)_{\mathrm{s}}\lambda_{\mathrm{r}} - (\chi_2)_{\mathrm{s}}\mu_{\mathrm{r}} - (\gamma_2)_{\mathrm{s}}\upsilon_{\mathrm{r}}$$
$$- \tau(\mu_{\mathrm{r}} \sin\phi - \upsilon_{\mathrm{r}} \cos\phi)\delta^1_{\mathrm{s}} + \chi_2(\sin\phi\,\lambda_{\mathrm{r}}\delta^1_{\mathrm{s}} + \upsilon_{\mathrm{r}}\delta^2_{\mathrm{s}})$$
$$- \gamma_2(\cos\phi\,\lambda_{\mathrm{r}}\delta^1_{\mathrm{s}} + \mu_{\mathrm{r}}\delta^2_{\mathrm{s}}) \ .$$

61. If we take ω, ϕ as coordinates (1,2) on the equipotential *surface*, the components of the base vectors, considered now as surface vectors in two dimensions, are

$$\lambda^\alpha = \{(\partial\omega/\partial \mathrm{p}), (\partial\phi/\partial \mathrm{p})\} = (\chi_1 \sec\phi, \tau)$$
$$\mu^\alpha = \{(\partial\omega/\partial \mathrm{m}), (\partial\phi/\partial \mathrm{m})\} = (\tau \sec\phi, \chi_2)$$

and by solving the tensor equation

$$\lambda^\alpha\lambda_\beta + \mu^\alpha\mu_\beta = \delta^\alpha_\beta$$

we have the covariant components

$$\mathrm{K}\lambda_\alpha = (\chi_2 \cos\phi, -\tau)$$
$$\mathrm{K}\mu_\alpha = (-\tau\cos\phi, \chi_1) \ ,$$

all of which are the same as the (1,2) space cmponents.

62. The surface metric tensor follows from

$$\bar{\mathrm{a}}^{\alpha\beta} = \lambda^\alpha\lambda^\beta + \mu^\alpha\mu^\beta \ ; \quad \bar{\mathrm{a}}_{\alpha\beta} = \lambda_\alpha\lambda_\beta + \mu_\alpha\mu_\beta \ .$$

63. In these coordinates the second-order magnitudes $\mathrm{b}_{\alpha\beta}$ of the surface are given by

$$\mathrm{b}_{\alpha\beta} = -\mathrm{a}_{\mathrm{mn}}\upsilon^{\mathrm{m}}_{,\alpha}(\partial \mathrm{x}^{\mathrm{n}}/\partial \mathrm{x}^\beta) = -\upsilon_{\beta\alpha}$$

(taken in relation to the space metric)

$$= -\cos\phi\,\lambda_\beta\lambda^1_\alpha - \mu_\beta\delta^2_\alpha \quad \text{from (8.3)}$$
$$= (-\chi_2\cos^2\phi/\mathrm{K}, \tau\cos\phi/\mathrm{K}, -\chi_1/\mathrm{K})$$

and the following tensor equation, true in any coordinates, is easily verified

$$\mathrm{b}_{\alpha\beta} = -\chi_1\lambda_\alpha\lambda_\beta - \tau(\lambda_\alpha\mu_\beta + \mu_\alpha\lambda_\beta) - \chi_2\mu_\alpha\mu_\beta \ .$$

64. The third-order magnitudes of the surface in these coordinates are

$$\mathrm{c}_{\alpha\beta} = \mathrm{a}_{\mathrm{mn}}\upsilon^{\mathrm{m}}_{,\alpha}\upsilon^{\mathrm{n}}_{,\beta} = \cos^2\phi\,\delta^1_\alpha\delta^1_\beta + \delta^2_\alpha\delta^2_\beta$$
$$= (\cos^2\phi, 0, 1)$$

with the tensor equation

$$\mathrm{c}_{\alpha\beta} = (\chi_1^2 + \tau^2)\lambda_\alpha\lambda_\beta + 2\mathrm{H}\tau(\lambda_\alpha\mu_\beta + \mu_\alpha\lambda_\beta) + (\chi_2^2 + \tau^2)\mu_\alpha\mu_\beta \ ,$$

showing that with the sign conventions used throughout this paper (to make χ_1, χ_2 positive),

$$c_{\alpha\beta}+2Hb_{\alpha\beta}+Ka_{\alpha\beta}=0 \ .$$

65. If barred quantities refer to the surface metric, and unbarred to the space metric, notice that

$$a_{\alpha\beta}=\bar{a}_{\alpha\beta} \quad \text{and} \quad a^{\alpha\beta}=\bar{a}^{\alpha\beta}+\upsilon^\alpha\upsilon^\beta(\alpha,\beta=1,2) \tag{65.1}$$

and since the (1,2) covariant components are the same, so are the Christoffel symbols of the first kind;

$$[\alpha\beta,\gamma]=\overline{[\alpha\beta,\gamma]} \ . \tag{65.2}$$

As regards the Christoffel symbols of the second kind, we have

$$\begin{aligned}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix} &= a^{\gamma m}[\alpha\beta,m]=a^{\gamma 3}[\alpha\beta,3]+a^{\gamma\delta}[\alpha\beta,\delta]\\ &= \bar{a}^{\gamma\delta}\overline{[\alpha\beta,\delta]}+\upsilon^\gamma\upsilon^3[\alpha\beta,3]+\upsilon^\gamma\upsilon^\delta[\alpha\beta,\delta] \ ,\end{aligned} \tag{65.3}$$

$$\begin{aligned}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}-\overline{\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}} &= \upsilon^\gamma\upsilon^m[\alpha\beta,m]=\upsilon^\gamma\upsilon_k a^{km}[\alpha\beta,m]=\upsilon^\gamma\upsilon_k\begin{Bmatrix}k\\ \alpha\beta\end{Bmatrix}\\ &= -\frac{1}{g}\upsilon^\gamma\begin{Bmatrix}3\\ \alpha\beta\end{Bmatrix}=-\upsilon^\gamma\upsilon_{\alpha\beta} \ , \quad \text{by § 59} \ ,\end{aligned} \tag{65.4}$$

a tensor equation which is true in any coordinates, provided only that the coordinates of the surface and of the space are the same. Subject to the same proviso, the (1,2) components of any surface vector ϱ^α, which must be a linear combination of λ^α and μ^α are the same, viz. $\varrho^\alpha=\bar{\varrho}^\alpha$ and $\varrho_\alpha=\bar{\varrho}_\alpha$. As regards the covariant derivative

$$\begin{aligned}\varrho_{\alpha\beta} &= \frac{\partial\varrho_\alpha}{\partial x^\beta}-\begin{Bmatrix}m\\ \alpha\beta\end{Bmatrix}\varrho_m=\frac{\partial\varrho_\alpha}{\partial x^\beta}-\overline{\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}}\bar{\varrho}_\gamma+\upsilon^\gamma\upsilon_{\alpha\beta}\varrho_\gamma-\begin{Bmatrix}3\\ \alpha\beta\end{Bmatrix}\varrho_3\\ &= \bar{\varrho}_{\alpha\beta}+\upsilon^\gamma\upsilon_{\alpha\beta}\varrho_\gamma+\upsilon^3\upsilon_{\alpha\beta}\varrho_3=\bar{\varrho}_{\alpha\beta}+(\upsilon^m\varrho_m)\upsilon_{\alpha\beta}\\ &= \bar{\varrho}_{\alpha\beta} \ , \quad \text{since} \quad \varrho_m \text{ is a surface vector and} \quad \upsilon^m\varrho_m=0 \ .\end{aligned}$$

As a special case, we have from Eqs. (8.1) and (8.2)

$$\begin{aligned}\bar{\lambda}_{\alpha\beta} &= \sin\phi\,\mu_\alpha\delta^1_\beta\\ \bar{\mu}_{\alpha\beta} &= -\sin\phi\,\lambda_\alpha\delta^1_\beta \ ,\end{aligned} \tag{65.6}$$

which are frequently useful in considering the differential geometry of the equipotential surface.

Variation of Position

66. We next investigate changes in the length (ds), azimuth ($d\alpha$), and zenith distance ($d\beta$) of a light ray in space due to changes in the coordinates dx^r and $d\bar{x}^r$ at both ends of the line. This leads to certain formulae which are required for the rigorous adjustment of triangulation, in whatever coordinates this may be carried out.

Quantities referring to the far end of the line are barred, e.g. $\bar{\lambda}_r$ is the base vector λ_r at the far end of the line. Azimuth and zenith distance at the far end refer to the production of the line, not to the back azimuth etc.

67. For the present purpose it will be sufficient to suppose that the refracted ray has been replaced by its chord, or in other words to suppose that the chord suffers the same changes as the ray. We denote the length of the chord by S and its unit vector by

$$l^r = \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \nu^r \cos\beta \ . \tag{67.1}$$

As before, we denote a unit surface vector in azimuth α by

$$b^r = \lambda^r \sin\alpha + \mu^r \cos\alpha \ ; \tag{67.2}$$

a unit surface vector in azimuth $3\pi/2 + \alpha$ perpendicular to l^r by

$$n^r = -\lambda^r \cos\alpha + \mu^r \sin\alpha \ ; \tag{67.3}$$

also a unit vector in azimuth α perpendicular to l^r by m^r

$$m^r = \lambda^r \sin\alpha \cos\beta + \mu^r \cos\alpha \cos\beta - \nu^r \sin\beta \ . \tag{67.4}$$

68. If the position vector [Cartesian components (x, y, z)] is ϱ^r, we have in any coordinates

$$s\,l^r = \bar{\varrho}^r - \varrho^r$$

and differentiating this for a change in coordinate dx^r (the barred end being for the moment held fixed), we have

$$l^r ds + s\,l^r_s dx^s = -\varrho^r_{,s} dx^s \ . \tag{68.1}$$

But $\varrho^r_{,s}$ in Cartesian coordinates is $\partial x^r/\partial s^s = \delta^r_s$ so that in any coordinates we have the tensor equation

$$\varrho^r_{,s} = \delta^r_s = \bar{\varrho}^r_s$$

and multiplying (68.1) by l_r, we find the variation in length of the line to be

$$ds = -\delta^r_s l_r dx^s = -b_s dx^s \ .$$

Substituting in (68.1) we obtain

$$s\,l^r_{,s} dx^s = (l^r l_s - \delta^r_s) dx^s \ . \tag{68.2}$$

Similarly for changes $d\bar{x}^r$ in coordinates at the barred end,

$$ds = \bar{l}_s \, d\bar{x}^s$$

and

$$s\bar{l}^r_{,s} d\bar{x}^s = (\delta^r_s - \bar{l}^r \bar{l}_s) \, d\bar{x}^s \qquad (68.3)$$

so that the total variation in length of the line is

$$ds = \bar{l}_s \, d\bar{x}^s - l_s \, dx^s \ . \qquad (68.4)$$

Note that $\bar{l}_s$, although the same vector as l_s, has not the same coordinates at the barred end, except in Cartesian coordinates.

69. Now consider changes in $\sin\alpha \sin\beta = l^r \lambda_r$ and in $\cos\alpha \sin\beta = l^r \mu_r$ due to $d\bar{x}^s$ and dx^s at the two ends of the line, and write e.g. (λ_r) for a unit vector parallel to λ_r at the barred end of the line:

$$\cos\alpha \sin\beta \, d\alpha + \sin\alpha \cos\beta \, d\beta = (l^r_s \lambda_r + l^r \lambda_{rs}) \, dx^s + \bar{l}^r_{,s} \, (\lambda_r) \, d\bar{x}^s \qquad (69.1)$$

$$-\sin\alpha \sin\beta \, d\alpha + \cos\alpha \cos\beta \, d\beta = (l^r_{,s} \mu_r + l^r \mu_{rs}) \, dx^s + \bar{l}^r_{,s} (\mu_r) \, d\bar{x}^s \ , \qquad (69.2)$$

in which (λ_r), (μ_r) have remained fixed in direction during the variation $d\bar{x}^s$.

Eliminating $d\beta$ and using Eqs. (67.3), (68.2), (68.3) and the usual expressions for λ_{rs} etc.

$$\begin{aligned} s \sin\beta \, d\alpha &= -s l^r_{,s} n_r \, dx^s - s \bar{l}^r_{,s} (n_r) \, d\bar{x}^s + s' l^r (\lambda_{rs} \cos\alpha - \mu_{rs} \sin\alpha) \, dx^s \\ &= + n_s \, dx^s - (n_s) \, d\bar{x}^s + s (\sin\phi \sin\beta - \cos\alpha \cos\beta \cos\phi) \, \omega_s \, dx^s \\ &\quad + s (\sin\alpha \cos\beta) \, \phi_s \, dx^s \ . \end{aligned} \qquad (69.3)$$

70. This equation is true in any coordinates, provided that we take the components n_s etc., in the same coordinates. In the (ω, ϕ, V) system, the components of $n_r = -\lambda_r \cos\alpha + \mu_r \sin\alpha$ using § 56 are

$$\begin{aligned} K n_1 &= -(\chi_2 \cos\alpha + \tau \sin\alpha) \cos\phi \\ K n_2 &= (\tau \cos\alpha + \chi_1 \sin\alpha) \\ g K n_3 &= -\gamma_1 (\chi_2 \cos\alpha + \tau \sin\alpha) + \gamma_2 (\tau \cos\alpha + \chi_1 \sin\alpha) \ . \end{aligned}$$

To evaluate (n_r) note that its Cartesian components are the same as those of n_r so that

$$(n_r) = (n_s A^s) \bar{A}_r + (n_s B^s) \bar{B}_r + (n_s C^s) \bar{C}_r$$

which on using Eq. (6.2) becomes

$$(n_r) = P \bar{\lambda}_r + Q \bar{\mu}_r + R \bar{\upsilon}_r$$

with

$$\begin{aligned} P &= -\cos\alpha \cos(\bar{\omega} - \omega) + \sin\phi \sin\alpha \sin(\bar{\omega} - \omega) \\ Q &= \sin\bar{\phi} \sin\phi \sin\alpha \cos(\bar{\omega} - \omega) + \sin\bar{\phi} \cos\alpha \sin(\bar{\omega} - \omega) + \cos\phi \cos\bar{\phi} \sin\alpha \\ R &= -\sin\phi \cos\bar{\phi} \sin\alpha \cos(\bar{\omega} - \omega) - \cos\bar{\phi} \cos\alpha \sin(\bar{\omega} - \omega) + \cos\phi \sin\bar{\phi} \sin\alpha \end{aligned}$$

and $P^2 + Q^2 + R^2 = 1$ as a check, since (n_r) is a unit vector.

The change in azimuth for changes in (ω, ϕ, V) coordinates is then given by

$$\begin{aligned} s \sin\beta \, d\alpha = {} & d\omega(n_1 + s\sin\phi\sin\beta - s\cos\alpha\cos\beta\cos\phi) \\ & + d\phi(n_2 + s\sin\alpha\cos\beta) + n_3 dN \\ & + d\bar{\omega}(-P\bar{\chi}_2\cos\bar{\phi}/\bar{K} + Q\bar{\tau}\cos\bar{\phi}/\bar{K}) \\ & + d\bar{\phi}(+P\bar{\tau}/\bar{K} - Q\bar{\chi}_1/\bar{K}) \\ & - d\bar{N}\{P(\bar{\chi}_2\bar{\gamma}_1 - \bar{\tau}\bar{\gamma}_2)/\bar{g}\bar{K} + Q(\bar{\chi}_1\bar{\gamma}_2 - \bar{\tau}\bar{\gamma}_1)/\bar{g}\bar{K} - R/\bar{g}\} \ . \end{aligned}$$

We can, of course, use changes in dynamic height $dh = -dN/g$ (and $d\bar{h} = -d\bar{N}/\bar{g}$) instead of dN (and $d\bar{N}$).

71. To obtain the change in zenith distance we eliminate $d\alpha$ between Eqs. (69.1) and (69.2) and use (67.2):

$$s\cos\beta\, d\beta = s l^r_{,s} b_r dx^s + s\bar{l}^r_s (b_r) d\bar{x}^s + s l^r(\lambda_{rs}\sin\alpha + \mu_{rs}\cos\alpha) dx^s \ ,$$

which, by means of (68.2) and (68.3), reduces to

$$s\, d\beta = -m_s dx^s + (m_s) d\bar{x}^s - s\cos\phi\sin\alpha\,\omega_s dn^s - s\cos\alpha\,\phi_s dx^s \ , \tag{71.1}$$

where (m_s) is a unit vector at the barred end parallel to m_s [Eq. (67.4)]. Again this equation is true in any coordinates.

72. In (ω, ϕ, V) the components of m_s are given by:

$$\begin{aligned} K m_1 &= \chi_2\cos\phi\sin\alpha\cos\beta - \tau\cos\phi\cos\alpha\cos\beta \\ K m_2 &= -\tau\sin\alpha\cos\beta + \chi_1\cos\alpha\cos\beta \\ gK m_3 &= (\chi_2\gamma_1 - \tau\gamma_2)\sin\alpha\cos\beta + (\chi_1\gamma_2 - \tau\gamma_1)\cos\alpha\cos\beta + K\sin\beta \end{aligned}$$

and

$$(m_s) = S\bar{\lambda}_s + T\bar{\mu}_s + U\bar{\nu}_s \ ,$$

where

$$\begin{aligned} S &= \sin\alpha\cos\beta\cos(\bar{\omega}-\omega) + \{\sin\phi\cos\alpha\cos\beta + \cos\phi\sin\beta\}\sin(\bar{\omega}-\omega) \\ &= \sin\bar{\alpha}\sin\bar{\beta}\cos\beta + \cos\phi\csc\beta\sin(\bar{\omega}-\omega) \\ T &= \{\sin\phi\cos\alpha\cos\beta + \cos\phi\sin\beta\}\sin\bar{\phi}\cos(\bar{\omega}-\omega) - \sin\phi\cos\bar{\phi}\sin\beta \\ &\quad - \sin\alpha\cos\beta\sin\bar{\phi}\sin(\bar{\omega}-\omega) + \cos\alpha\cos\beta\cos\phi\cos\bar{\phi} \\ &= \cos\bar{\alpha}\sin\bar{\beta}\cos\beta + \cos\phi\sin\bar{\phi}\csc\beta\cos(\bar{\omega}-\omega) - \sin\phi\cos\bar{\phi}\csc\beta \\ U &= -\{\sin\phi\cos\alpha\cos\beta + \cos\phi\sin\beta\}\cos\bar{\phi}\cos(\bar{\omega}-\omega) - \sin\phi\sin\bar{\phi}\sin\beta \\ &\quad + \sin\alpha\cos\beta\cos\bar{\phi}\sin(\bar{\omega}-\omega) + \cos\alpha\cos\beta\cos\phi\sin\bar{\phi} \\ &= \cos\bar{\beta}\cos\beta - \cos\phi\cos\bar{\phi}\csc\beta\cos(\bar{\omega}-\omega) - \sin\phi\sin\bar{\phi}\csc\beta \end{aligned}$$

the alternative expressions for S, T, U having been obtained from the fact that the Cartesian components of l^r are the same at both ends of the line, which leads to the following three useful formulae (two independent) for reverse azimuth and zenith distance.

$$\begin{aligned}
\sin\bar{\alpha}\sin\bar{\beta} &= \sin\alpha\sin\beta\cos(\bar{\omega}-\omega) \\
&\quad +\{\cos\alpha\sin\beta\sin\phi-\cos\beta\cos\phi\}\sin(\bar{\omega}-\omega) \\
\cos\bar{\alpha}\sin\bar{\beta} &= \{\cos\alpha\sin\beta\sin\phi-\cos\phi\cos\beta\}\sin\bar{\phi}\cos(\bar{\omega}-\omega) \\
&\quad -\sin\alpha\sin\beta\sin\bar{\phi}(\bar{\omega}-\omega)+(\cos\alpha\sin\beta\cos\phi+\sin\phi\cos\beta)\cos\bar{\phi} \\
\cos\bar{\beta} &= -\{\cos\alpha\sin\beta\sin\phi-\cos\phi\cos\beta\}\cos\bar{\phi}\cos(\bar{\omega}-\omega) \\
&\quad +\sin\alpha\sin\beta\cos\bar{\phi}\sin(\bar{\omega}-\omega) \\
&\quad +(\cos\alpha\sin\beta\cos\phi+\sin\phi\cos\beta)\sin\phi \ .
\end{aligned}$$

The following identities are easily obtained from various scalar and vector products and are frequently useful.

$$\begin{aligned}
&PS+QT+RU=0 \ , \\
&PT-QS=\cos\bar{\beta} \ , \\
&QU-RT=\sin\bar{\alpha}\sin\bar{\beta} \ , \\
&RS-PU=\cos\bar{\alpha}\sin\bar{\beta} \ , \\
&PQ+ST=-\sin\bar{\alpha}\cos\bar{\alpha}\sin^2\bar{\beta} \ , \\
&QR+TU=-\cos\bar{\alpha}\cos\bar{\beta}\sin\bar{\beta} \ , \\
&RP+US=-\sin\bar{\alpha}\cos\bar{\beta}\sin\bar{\beta} \ , \\
&P^2+Q^2+R^2=1 \\
&S^2+T^2+U^2=1 \ , \\
&P^2+S^2=\sin^2\bar{\alpha}\cos^2\bar{\beta}+\cos^2\bar{\alpha} \ , \\
&Q^2+T^2=\cos^2\bar{\alpha}\cos^2\bar{\beta}+\sin^2\bar{\alpha} \ , \\
&R^2+U^2=\sin^2\bar{\beta} \ , \\
&P\sin\bar{\alpha}\sin\bar{\beta}+Q\cos\bar{\alpha}\sin\bar{\beta}+R\cos\bar{\beta}=0 \ , \\
&S\sin\bar{\alpha}\sin\bar{\beta}+T\cos\bar{\alpha}\sin\bar{\beta}+U\cos\bar{\beta}=0 \ , \\
&S=R\cos\bar{\alpha}\sin\bar{\beta}-Q\cos\bar{\beta} \ , \\
&T=P\cos\bar{\beta}-R\sin\bar{\alpha}\sin\bar{\beta} \ , \\
&U=Q\sin\bar{\alpha}\sin\bar{\beta}-P\cos\bar{\alpha}\sin\bar{\beta} \ , \\
&P=T\cos\bar{\beta}-U\cos\bar{\alpha}\sin\bar{\beta} \ , \\
&Q=U\sin\bar{\alpha}\sin\bar{\beta}-S\cos\bar{\beta} \ , \\
&R=S\cos\bar{\alpha}\sin\bar{\beta}-T\sin\bar{\alpha}\sin\bar{\beta} \ , \\
&\bar{P}\sin\bar{\beta}=R\sin\alpha\cos\beta-U\cos\alpha \ ,
\end{aligned}$$

$$\bar{Q} \sin \bar{\beta} = R \cos \alpha \cos \beta + U \sin \alpha \ ,$$

$$\bar{R} \sin \bar{\beta} = -R \sin \beta \ ,$$

$$\bar{S} \sin \bar{\beta} = -R \cos \alpha - U \sin \alpha \cos \beta \ ,$$

$$\bar{T} \sin \bar{\beta} = R \sin \alpha - U \cos \alpha \cos \beta \ ,$$

$$\bar{U} \sin \bar{\beta} = U \sin \beta \ .$$

The change in zenith distance due to changes in (ω, ϕ, V) coordinates is finally given by

$$\begin{aligned} S\,d\beta &= d\omega(-m_1 - S\cos\phi\sin\alpha) + d\phi(-m_2 - S\cos\alpha) - m_3\,dV \\ &\quad + d\bar{\omega}(S\bar{\chi}_2\cos\bar{\phi}/\bar{K} - T\bar{\tau}\cos\bar{\phi}/\bar{K}) + d\bar{\phi}(-S\bar{\tau}/\bar{K} + T\bar{\chi}_1/\bar{K}) \\ &\quad + d\bar{V}\{S(\bar{\chi}_2\bar{\gamma}_1 - \bar{\tau}\bar{\gamma}_2)/\bar{g}\bar{K} + T(\bar{\chi}_1\bar{\gamma}_2 - \bar{\tau}\bar{\gamma}_1)/\bar{g}\bar{K} - U/\bar{g}\} \ . \end{aligned}$$

In all the above equations we can interchange the bars, that is bar the unbarred quantities and unbar the unbarred, provided only that we change the sign of S.

Index of Main Symbols

A_r, B_r, C_r	Cartesian vectors (§ 1)
λ_r	Unit vector in parallel direction (§ 2)
μ_r	Unit vector in meridian direction (§ 2)
ν_r	Unit vector in zenithal direction (§ 2)
ω	Astronomical longitude direction (§ 3)
ϕ	Astronomical latitude direction (§ 3)
α	Astronomical azimuth direction (§ 5)
β	Astronomical zenith distance (§ 5)
V	Potential (§ 7)
g	Gravity (§ 7)
dp, dm, dn	Elements of arc in direction λ_r, μ_r, ν_r (§ 9)
γ_1	Arc rate of change of (log g) along parallel (§ 10)
γ_2	Arc rate of change of (log g) along meridian (§ 10)
χ_1	Normal curvature of equipotential along parallel (§ 11)
χ_2	Normal curvature of equipotential along meridian (§ 11)
τ	Geodesic torsion in meridian direction (§ 11)
H	Mean curvature of equipotential surface (§ 14)
K	Gauss curvature of equipotential surface (§ 25)
$\tilde{\omega}$	Angular velocity of the Earth (§ 17)
χ_0	Curvature of line of force (§ 19, § 21)
τ_0	Torsion of line of force (§ 20)

Editorial Commentary

This was Hotine's first major paper on differential geodesy. In it he made full use of not only tensorial methods, but also classical differential geometry. Its goal, which he admirably attained, was to recast the original work of Marussi [see Hotine's references Marussi (1947, 1949), which were not included in IG] into tensor form. This was necessary, since many of Marussi's seminal ideas in these papers, as well as in Marussi (1951, 1988) were formulated in special coordinate systems and obtained by using the homographic calculus of Burali-Forti and Marcolongo. The valuable article of Reilly (Appendix IG) gives a brief discussion of this formalism and a guide of how homographic expressions may be translated into vector and tensor notation. Hotine never made any use of the homographic calculus, whereas it played a major role in Marussi's thinking.

Hotine's treatment of this material was a veritable tour de force and a striking example of his mathematical ingenuity and ability. It was quite remarkable, since it was the work of someone who some 9 years earlier had set out – at the age of 50 – to master the intricacies of tensor calculus and differential geometry on his own. As Whitten's memorial lecture (see the final paper in this monograph) relates, this task was done while he held responsible full-time administrative positions in the service of his country. Hotine's mastery of tensorial and geometrical methods, while not breaking new ground mathematically, transcended the standard techniques appearing in his references (McConnell 1947; Eisenhart 1949; Levi-Civita 1926; Weatherburn 1930). As discussed in some detail in Zund (1990), although his mathematical reading was selective, it was essentially adequate for his purposes. Indeed, he anticipated – but did not explicitly formalize – many of the ideas which form the basis of the contemporary theory.

Hotine was not content merely to transcribe Marussi's work into tensor notation, he also embarked on his own formulation of the theory and in doing so he obtained a number of new results. In particular, the contents of this report cut across much of the material in Chapters 7, 12, 20, 24, 25 and 26 of Parts II and III of MG. It contained a general covariant theory of the Marussi tensor, and demonstrated how the metrical properties of the Earth's gravitational field can be described in terms of five parameters (in Chap. 12 of MG these were called curvature parameters).

The notation employed in the report differs from that in MG. In particular the basic quantities:

$$\mathrm{N},\ \mathrm{n},\ \mathrm{k}_1,\ \mathrm{k}_2 \text{ and } \mathrm{t}_1$$

in MG correspond respectively to

$$-\mathrm{V},\ -\mathrm{g},\ \chi_1,\ \chi_2 \text{ and } \tau$$

in the report. Likewise, the arc length elements

$$\mathrm{d}\lambda,\ \mathrm{d}\mu \quad \text{and} \quad \mathrm{ds}$$

in MG, appear in the report as

$$\mathrm{dp},\ \mathrm{dm} \text{ and } \mathrm{dn}$$

respectively. The sign difference in N and V, and n and g results in numerous sign differences between equations in MG and the report. However, these pose no difficulties to anyone seeking to establish a correspondence between the discussions.

A few other minor discrepancies occur between the notations, but these are obvious and require no comment.

Generally speaking, the presentation in the report is clearcut and, being less comprehensive and exhaustive than that in MG, it is easier to assimilate on a first reading. Almost all of the equations and results of the report occur *somewhere* in MG; however, two noteworthy exceptions have been detected.

The first of these is the expression in Eq. (18.1) for the Marussi tensor in terms of products of the covariant components of the vector fields $\underset{\sim}{\lambda}$, $\underset{\sim}{\mu}$, $\underset{\sim}{\nu}$ (the Hotine triad, or 3-leg, of MG) and the five parameters (called the curvature parameters in MG). This is an important representation of the Marussi tensor which is not repeated in MG. There it appears only in contracted form, viz. [MG Eq. (12.162)], and although this is equivalent to Eq. (18.1), it is useful to have this representation in hand. In particular, by the definition of V_{rs} given in Eqs. (9.1), (18.1) also gives an expression for υ_{rs} which only occurs in contracted form like Eq. (19.1) in MG [i.e. MG Eq. (12.021)]. Reilly (1981, 1982, 1985) has given some elegant extensions of the notion of the Marussi tensor V_{rs} to include the case of a time-varying gravity field, and applied it to crustal deformations.

The second of these concerns the material in § 34 dealing with the Mainardi-Codazzi, or more simply the Codazzi, equations. This is a substantive topic, and Hotine considered it on no less than *three* occasions in MG: in Chapters 8 §§ 8–13; in Chapter 12 §§ 84–94, and again from an alternate viewpoint in §§ 95–97. None of these discussions precisely duplicates the contents of § 34 of the report. In particular, neither the general integrability conditions for the arbitrary function F, nor the specialized Eqs. (34.1)–(34.3) when $F = \phi$, or (34.4)–(34.6) when $F = \omega$, or (34.7)–(34.9) when $F = \log g$, appear in MG. The entire matter is one of some mathematical delicacy, since it concerns the existence and imbedding of the equipotential surfaces in Euclidean 3-space. A critical examination of this material was recently undertaken by the editor (Zund 1990).

A Test Triangulation (1957), cited in the report, was compiled by H. H. Brazier and L. M. Windsor and presented at the Toronto Assembly (Brazier and Windsor 1957).

A valuable report of the Toronto Assembly is given in Marussi (1958). Hotine's work was reported in Section I: *Triangulations*, under the presidency of C. A. Whitten, and with A. Marussi as secretary. Hotine himself was president of the special study group No. 1: *Problèmes théoretiques intéressant le calcul et la compensation des grandes triangulations, en prenant en considération la forme due géoïde*, which met on September 7, 1957. Its activities were discussed on pages 62–64 of Marussi (1958).

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4 Geodetic Coordinate Systems[1]

Introduction

1. For the present purpose, a coordinate system is defined as a set of three continuous, single-valued, differentiable functions of a Cartesian system (x, y, z) within a certain region of flat 3-space. Other means of defining position (e.g. Square X 56, or "follow the normal to the spheroid as far as the geoid and thence along the line of force") are better described as reference systems and are not considered here; they are not amenable to the ordinary processes of analysis.

Two coordinates in all the systems considered are the longitude and latitude, in relation to fixed Cartesian axes, of the normal to the third coordinate surface. Moreover, the directions of the Cartesian axes are arranged to be the same, in all four systems considered, by a suitable choice of origin. Azimuth (and zenith distance) are in all cases considered to be measured about (and from) the normal to the third coordinate surface. The origin of azimuth is the plane defined by the normal and the Cartesian axis C^r parallel to the axis of rotation of the Earth.

2. The following quantities, all scalar functions of position, are taken as third coordinate in each case:

i) The geopotential, in which case the normal is the astronomical zenith direction, and the latitude and longitude are directly measurable astronomically. The Cartesian axes (which will be common to all other systems) are parallel to unit vectors A^r, B^r, C^r, right-handed in that order.

This case has already been considered in a previous paper (Hotine 1957), to which frequent reference will be made and the notation of which, summarized in a list of symbols at the end of this paper, will be used for each coordinate system in turn. It was concluded (Hotine 1957, § 49) that this system is unsuitable for accurate geodetic calculations, but it does nevertheless provide the basis of all other systems, which must in any case be related to it, since most of the field measurements are necessarily made in this system.

ii) A standard potential, giving rise to a field symmetrical about the axis of rotation. The equipotentials of this field are surfaces of revolution and one of them may be a spheroid. It will be shown that this system is unsuitable for

[1] Report dated 29 June 1957 (Tolworth), and presented to Study Group No. 1 at the I. A. G. Toronto Assembly 1957. Hotine noted that it was to be read in conjunction with *Metrical Properties of the Earth's Gravitational Field*, which was also presented to the Study Group at the same time.

geodetic calculation of position, distances etc., and indeed has little to commend it for any purpose.

iii) The distance between third-coordinate surfaces, measured along their common normals, which are necessarily straight. The third-coordinate surfaces are geodesic parallels to a base surface which can be of any form. This system may be of use in problems involving reduction to a base surface.

iv) The same as (iii) with a spheroid whose minor axis is parallel to the axis of rotation, as base surface. This system leads to simple closed formulae which are very suitable for geodetic calculation, not differing violently from previous results obtained by classical methods. The rigorous three-dimensional adjustment of triangulation is considered in some detail in this system, together with the assimilation, without successive approximation and in a single process, of astronomical measurements, spirit-levels and measured bases. It is suggested that this provides a complete and rigorous answer to the fundamental problem posed to I. A. G. Study Group No. 1, although it does not require any explicit knowledge of the form of the geoid. The amount of computation is considerably greater than by classical methods but still costs only a small fraction of the cost of the field work, even without electronic aids.

Relations Between Coordinate Systems

3. We first consider a few general propositions relating any two of the above four coordinate systems, one of which is denoted by barred notation.

The fact that the Cartesian vectors A^r, B^r, C^r (Hotine 1957, § 1) are the same in all systems [we consider below (§ 7) how this can be ensured], enables us to relate the parallel, meridian and zenith vectors (Hotine 1957, § 2) at a point as follows [Hotine 1957, Eqs. (6.1, 6.2)]:

$$\begin{aligned}\bar{\lambda}_r &= -A_r \sin\bar{\omega} + B_r \cos\bar{\omega}\\ &= \lambda_r \cos(\bar{\omega}-\omega) + \mu_r \sin\phi \sin(\bar{\omega}-\omega) - \nu_r \cos\phi \sin(\bar{\omega}-\omega)\end{aligned} \tag{3.1}$$

and similarly,

$$\begin{aligned}\bar{\mu}_r = &-\lambda_r \sin\bar{\phi} \sin(\bar{\omega}-\omega) + \mu_r\,[\cos\phi \cos\bar{\phi} + \sin\phi \sin\bar{\phi} \cos(\bar{\omega}-\omega)]\\ &+ \nu_r\,[\sin\phi \cos\bar{\phi} - \cos\phi \sin\bar{\phi} \cos(\bar{\omega}-\omega)]\end{aligned} \tag{3.2}$$

$$\begin{aligned}\bar{\nu}_r = &\lambda_r \cos\bar{\phi} \sin(\bar{\omega}-\omega) + \mu_r\,[\cos\phi \sin\bar{\phi} - \sin\phi \cos\bar{\phi} \cos(\bar{\omega}-\omega)]\\ &+ \nu_r\,[\sin\phi \sin\bar{\phi} + \cos\phi \cos\bar{\phi} \cos(\bar{\omega}-\omega)]\ .\end{aligned} \tag{3.3}$$

4. Since the Cartesian components of a vector are the same as those of a parallel vector, these equations are also true *in Cartesian coordinates* if μ_r etc. is the meridian etc. vector at some other point in the same space whose position is given by the barred coordinates. Alternatively, if we define $\bar{\mu}_r$ etc. as parallel to the meridian etc. vector at the barred point and transported to the unbarred point,

then these vector equations are true at the unbarred point in Cartesian coordinates and therefore in any coordinates.

5. We next consider a unit vector l^r in azimuth α zenith distance β ($\bar{\alpha}, \bar{\beta}$ in the barred system) so that either of the following expressions represents the same vector at the same point in space:

$$\begin{aligned} l^r &= \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta \\ &= \bar{\lambda}^r \sin\bar{\alpha} \sin\bar{\beta} + \bar{\mu}^r \cos\bar{\alpha} \sin\bar{\beta} + \bar{\upsilon}^r \cos\bar{\beta} \ . \end{aligned}$$

Multiplying Eq. (3.1) etc. across by whichever expression is appropriate gives us the following equations (two independent) for azimuth and zenith distance resulting from a change of coordinates.

$$\begin{aligned} \sin\bar{\alpha}\sin\bar{\beta} = {} & \sin\alpha\sin\beta\cos(\bar{\omega}-\omega) + \cos\alpha\sin\beta\sin\phi\sin(\bar{\omega}-\omega) \\ & -\cos\beta\cos\phi\sin(\bar{\omega}-\omega) \ . \end{aligned} \tag{5.1}$$

$$\begin{aligned} \cos\bar{\alpha}\sin\bar{\beta} = {} & -\sin\alpha\sin\beta\sin\phi\sin(\bar{\omega}-\omega) \\ & +\cos\alpha\sin\beta\,[\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos[\bar{\omega}-\omega)] \\ & +\cos\beta\,[\sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)] \end{aligned} \tag{5.2}$$

$$\begin{aligned} \cos\bar{\beta} = {} & \sin\alpha\sin\beta\cos\bar{\phi}\sin(\bar{\omega}-\omega) \\ & +\cos\alpha\sin\beta\,[\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)] \\ & +\cos\beta\,[\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)] \ . \end{aligned} \tag{5.3}$$

It will be apparent from § 4 that these equations hold if the barred quantities refer to any other point on the straight line l^r in the same space. They are accordingly true as between two points on this line, whether there is also a change of coordinates or not.

6. If the changes are small and e.g. $\delta\phi = (\bar{\phi} - \phi)$, then the Eqs. (5.1) etc. reduce to the following first-order formulae:

$$\delta\alpha = \sin\phi\,\delta\omega + \cos\beta(\sin\alpha\,\delta\phi - \cos\alpha\cos\phi\,\delta\omega) \tag{6.1}$$

$$\delta\beta = -\cos\alpha\,\delta\phi - \cos\phi\sin\alpha\,\delta\omega \ . \tag{6.2}$$

These will usually be sufficient in practice for a change in coordinates, of the same order as "deflections of the vertical", but the full equations will be required to obtain reverse azimuth etc. at the other end of a long line. The equations contain the so-called Laplace azimuth equation, generalized for a line in 3-space and given a rigorous interpretation.

7. Conversely, if the barred and unbarred quantities, referring to a particular line at a particular point, satisfy two of the Eqs. (5.1)–(5.3), then the Cartesian vectors will be parallel in the two systems and the equations will hold at any other point. The simplest way of ensuring this is to make the barred and unbarred quan-

tities the same at a particular point, or origin. [It is not sufficient merely to make the barred and unbarred latitudes, longitudes and azimuths the same, since this might leave Eq. (6.2) unsatisfied.]

Relations Between Parameters

8. If we form barred and unbarred equations from [Hotine 1957, Eq. (16.2)], subtract, write $\delta\phi = \bar{\phi} - \phi$ and use Eqs. (3.1)–(3.3), we obtain the following vector equation:

$$\begin{aligned}(\delta\phi)_r = \lambda_r\{&\bar{\tau}\cos\delta\omega - \bar{\chi}_2\sin\bar{\phi}\sin\delta\omega + \bar{\gamma}_2\cos\bar{\phi}\sin\delta\omega - \tau\}\\ &+ \mu_r\{\bar{\tau}\sin\phi\sin\delta\omega + \bar{\alpha}_2(\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos\delta\omega)\\ &+ \bar{\gamma}_2(\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos\delta\omega) - \chi_2\}\\ &+ \upsilon_r\{-\bar{\tau}\cos\phi\sin\delta\omega + \bar{\chi}_2(\sin\phi\cos\bar{\phi} - \cos\phi\sin\phi\cos\delta\omega)\\ &+ \bar{\gamma}_2(\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos\delta\omega) - \gamma_2\}\end{aligned} \tag{8.1}$$

and forming scalar products with each of the base vectors,

$$\tau = \bar{\tau}\cos\delta\omega - \bar{\chi}_2\sin\bar{\phi}\sin\delta\omega + \bar{\gamma}_2\cos\bar{\phi}\sin\delta\omega - \partial(\delta\phi)/\partial p\ . \tag{8.2}$$

$$\begin{aligned}\chi_2 = {}&\bar{\tau}\sin\phi\sin\delta\omega + \bar{\chi}_2(\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos\delta\omega)\\ &+ \bar{\gamma}_2(\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos\delta\omega) - \partial(\delta\phi)/\partial m\ .\end{aligned} \tag{8.3}$$

$$\begin{aligned}\gamma_2 = {}&-\bar{\tau}\cos\phi\sin\delta\omega + \bar{\chi}_2(\sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos\delta\omega)\\ &+ \bar{\gamma}_2(\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos\delta\omega) - \partial(\delta\phi)/\partial n\ .\end{aligned} \tag{8.4}$$

These equations can be considerably simplified by a suitable choice of geodetic coordinates. For instance, if the barred system is that described in § 21 et seq., then both $\bar{\tau}$ and $\bar{\gamma}_2$ are zero and $\bar{\chi}_2 = 1/(\varrho + k)$. We then have some very simple equations for directly determining, say, the astronomical (unbarred) parameters from deflections, without requiring the deflections to be small. We should need to make the assumption that the deflections vary uniformly along finite lines, in order to evaluate e.g. $\partial(\delta\phi)/\partial m$, the change in latitude deflection per unit of meridian distance, or $\partial(\delta\phi)/\partial n$, the change in deflection per unit of height. The mean of longitude deflections and of geodetic parameters at the two ends of the line should be used in the coefficients, and we should expect the result to give a value for the parameter at some intermediate point along the line. The method is more practicable than the absolute method in (Hotine 1957, § 50) but suffers from much the same assumptions; at the moment of writing it has not been tried out. It should be noted, moreover, that the equations hold only for coordinate systems related as in § 7; any departure from this relation could have very serious effects.

9. In exactly the same way, from a pair of longitude equations [Hotine 1957, Eq. (16.1)], we have

$$\chi_1 \sec\phi = \bar{\chi}_1 \sec\bar{\phi} \cos\delta\omega - \bar{\tau}\tan\bar{\phi} \sin\delta\omega + \bar{\gamma}_1 \sin\delta\omega - \partial(\delta\omega)/\partial p \ , \tag{9.1}$$

$$\tau \sec\phi = \bar{\chi}_1 \sec\bar{\phi} \sin\phi \sin\delta\omega + \bar{\tau} \sec\phi\, (\cos\phi \cos\bar{\phi} + \sin\phi \sin\bar{\phi} \cos\delta\omega)$$
$$+ \bar{\gamma}_1 \sec\bar{\phi}\, (\cos\phi \sin\bar{\phi} - \sin\phi \cos\bar{\phi} \cos\delta\omega) - \partial(\delta\omega)/\partial m \ ,$$

$$\gamma_1 \sec\phi = -\bar{\chi}_1 \sec\bar{\phi} \cos\phi \sin\delta\omega + \bar{\tau} \sec\bar{\phi}\, (\sin\phi \cos\bar{\phi} - \cos\phi \sin\bar{\phi} \cos\delta\omega)$$
$$+ \bar{\gamma}_1 \sec\bar{\phi}\, (\sin\phi \sin\bar{\phi} + \cos\phi \cos\bar{\phi} \cos\delta\omega) - \partial(\delta\omega)/\partial n \ , \tag{9.3}$$

for the change in longitude deflections. For the geodetic coordinate system (§ 21 et seq.), $\bar{\tau}$ and $\bar{\gamma}_1$ are zero and $\bar{\chi}_1 = 1/(\upsilon + h)$. In the case of an oblique line, the two values of τ [from Eqs. (8.2) and (9.2)] should indicate how far the basic assumptions are justified. A line mainly in the meridian (or parallel) direction will not of course give a good determination of χ_1 (or χ_2) and γ_1, γ_2 will not be well-determined in view of the smallness of dn on most terrestrial lines. Better values of γ_1, γ_2 would be obtained by measuring gravity at the two ends of the line and evaluating the parameters directly from the equations [Hotine 1957, Eq. (16.4)]

$$\gamma_1 = \partial(\log g)/\partial p$$
$$\gamma_2 = \partial(\log g)/\partial m \ .$$

10. We can obtain vector relations involving the third coordinates from Eq. (3.3). For instance, if the barred system is astronomical, we can substitute $-\bar{V}_r/\bar{g}_r$ for $\bar{\upsilon}_r$ [Hotine 1957, Eq. (7.1)]. The change in potential along a line of length ds (geodetic azimuth α, geodetic zenith distance β) whose unit vector is $(\lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta)$ is then given by

$$-(1/\bar{g})(\partial\bar{V}/\partial s) = \cos\bar{\phi} \sin\delta\omega \sin\alpha \sin\beta$$
$$+ \cos\alpha \sin\beta\, [\cos\phi \sin\bar{\phi} - \sin\phi \cos\bar{\phi} \cos\delta\omega]$$
$$+ \cos\beta\, [\sin\phi \sin\bar{\phi} + \cos\phi \cos\bar{\phi} \cos\delta\omega] \ . \tag{10.1}$$

For small deflections this is

$$-(1/\bar{g})(\partial\bar{V}/\partial s) = \cos\phi \sin\alpha \sin\beta\, \delta\omega + \cos\alpha \sin\beta\, \delta\phi + \cos\beta \ . \tag{10.2}$$

Symmetrical Field Coordinates

11. If the gravitational field is symmetrical about the axis of rotation, all meridian sections will be plane and similar. Any line of force will lie in a meridian plane and its principal normal will be in the meridian direction μ_r. Consequently, (Hotine 1957, § 19) $\gamma_1 = 0$; and since (log g) will also be independent of longitude, $\tau = 0$ (Hotine 1957, § 56). All the properties in (Hotine 1957, § 31) follow; there will be no change in curvatures in the parallels direction; and the Codazzi etc., equations (Hotine 1957, § 34) reduce to four;

$$\frac{\partial\gamma_2}{\partial m} - \frac{\partial\chi_2}{\partial n} = \chi_2^2 + \gamma_2^2 \ , \tag{11.1}$$

$$-\frac{\partial \chi_1}{\partial n} = \chi_1^2 + \chi_1 \gamma_2 \tan\phi \ , \tag{11.2}$$

$$\frac{\partial \chi_1}{\partial m} = \chi_1 (\chi_1 - \chi_2) \tan\phi \ , \tag{11.3}$$

$$\frac{\partial(\chi_1+\chi_2)}{\partial m} + \frac{\partial \gamma_2}{\partial n} = \frac{4\tilde{\omega}^2}{g}\gamma_2 + \chi_2\gamma_2 \quad \text{in free air} \ . \tag{11.4}$$

12. These equations enable us to determine the three non-zero parameters χ_1, χ_2 and γ_2. Suppose we have a spheroid as base equipotential surface on which $\chi_1 = 1/\upsilon$; $\chi_2 = 1/\varrho$ (ϱ, υ being as usual the principal curvatures of the spheroid). The theoretical value of gravity on such a surface is known (§ 35 et seq.) in terms of latitude. Accordingly, we know

$$\gamma_2 = \partial(\log g_0)/(\varrho\,\partial\phi)$$

on the spheroid. Then at a distance n from the spheroid measured along the line of force, we have to a first order:

$$\chi_1 = \frac{1}{\upsilon} + n\left(\frac{\partial \chi_1}{\partial n}\right)_0 = \frac{1}{\upsilon} - n\left\{\frac{1}{\upsilon^2} + \frac{\tan\phi}{\varrho\upsilon}\frac{\partial \log g_0}{\partial\phi}\right\} \quad \text{from (11.2)} \tag{12.1}$$

$$\chi_2 = \frac{1}{\varrho} - n\left\{\frac{1}{\varrho^2} + \frac{1}{\varrho^2}\left(\frac{\partial \log g_0}{\partial\phi}\right)^2 - \frac{\partial}{\varrho\,\partial\phi}\left(\frac{\partial \log g_0}{\varrho\,\partial\phi}\right)\right\} \quad \text{from (11.1)} \tag{12.2}$$

$$\gamma_2 = \frac{\partial \log g_0}{\varrho\,\partial\phi} + n\left\{\frac{4\tilde{\omega}^2}{\varrho g_0}\frac{\partial \log g_0}{\partial\phi} + \frac{1}{\varrho\upsilon}\frac{\partial \log g_0}{\partial\phi} - \frac{\partial}{\varrho\,\partial\phi}\left(\frac{1}{\varrho} + \frac{1}{\upsilon}\right)\right\} \quad \text{from (11.4)} \tag{12.3}$$

Gravity at the same point is given by Hotine [1957, Eq. (16.3)] as

$$(\log g) = (\log g_0) - n\left\{\frac{1}{\varrho} + \frac{1}{\upsilon} + \frac{2\tilde{\omega}^2}{g_0}\right\} , \tag{12.4}$$

in which the last term corresponds to the "free air" reduction.

The latitude at a distance n along the line of force is greater than the latitude on the spheroid by n $(\partial\phi/\partial n)_0 = n(\gamma_2)_0 = (n/\varrho)(\partial \log g_0/\partial\phi)$ [Hotine 1957, Eq. (13.1)] and allowance for this should properly be made before looking up the spheroidal functions of latitude.

In all these formulae, to a comparable accuracy, n may be taken as the "orthometric" geodetic height.

The third Codazzi equation [Eq. (11.3)] becomes on the spheroid

$$\partial\upsilon/\partial\phi = (\upsilon - \varrho)\tan\phi \ , \tag{12.5}$$

a well-known result.

13. Any other required metrical properties of the field now follow by straight substitution in the corresponding formulae of Hotine (1957). For instance, the

components of the base vectors in this system of coordinates are (Hotine 1957, § 54);

$$\begin{aligned} \lambda^r &= (\chi_1 \sec\phi,\ 0,\ 0)\ , \\ \mu^r &= (0,\ \chi_2,\ 0)\ , \\ \nu^r &= (0,\ \gamma_2,\ -g)\ , \end{aligned} \tag{13.1}$$

$$\begin{aligned} \lambda_r &= (\cos\phi/\chi_1,\ 0,\ 0)\ , \\ \mu_r &= (0,\ 1/\chi_2,\ \gamma_2/(g\chi_2))\ , \\ \nu_r &= (0,\ 0,\ -1/g)\ , \end{aligned} \tag{13.2}$$

whence we can easily calculate the components of the metric tensor etc. in a form equivalent to that already given by Marussi (1950) for this system of coordinates.

14. As regards the expansion of coordinates along a line (Hotine 1957, § 42 et seq.), we need first to determine the variation of the three parameters along the line. We have, for example,

$$\begin{aligned} \frac{\partial\chi_1}{\partial s} &= \frac{\partial\chi_1}{\partial m}\cos\alpha\sin\beta + \frac{\partial\chi_1}{\partial n}\cos\beta \text{ (since } \chi_1 \text{ is independent of } \omega) \\ &= \frac{\partial\chi_1}{\partial\phi}\chi_2\cos\alpha\sin\beta + \frac{\partial\chi_1}{\partial n}\cos\beta\ , \end{aligned} \tag{14.1}$$

which can be evaluated from Eqs. (11.2) or (12.1), and similarly for $(\partial\chi_2/\partial s)$, $(\partial\gamma_2/\partial s)$. We can therefore calculate the second-order terms (Hotine 1957, § 49) and could theoretically obtain terms of still higher order. Accurate geodetic calculation is accordingly possible in this system, but by no means simple.

15. It is often supposed that this system leads to greater facility in handling gravity anomalies, but it is very doubtful if this is so. Exactly as in § 8, we can form a difference equation from Eq. (16.4) (Hotine 1957). If $\delta g = (\bar{g} - g)$ we have

$$\begin{aligned} \{\log(\bar{g}/g)\}_r = \{\log(1+\delta g/g)\}_r &= (\delta g/g)_r \quad \text{nearly} \\ &= \bar{\gamma}_1\bar{\lambda}_r + \bar{\gamma}_2\bar{\mu}_r - \{\bar{\chi}_1 + \bar{\chi}_2 + 2\tilde{\omega}^2/\bar{g}\}\bar{\nu}_r \\ &\quad - \gamma_1\lambda_r - \gamma_2\mu_r + \{\chi_1 + \chi_2 + 2\tilde{\omega}^2/g\}\nu_r \end{aligned}$$

so that, for example

$$\gamma_1 = \bar{\gamma}_1\cos\delta\omega - \bar{\gamma}_2\sin\bar{\phi}\sin\delta\omega - (\bar{\chi}_1 + \bar{\chi}_2 + 2\tilde{\omega}^2/\bar{g})\cos\bar{\phi}\sin\delta\omega - \frac{\partial}{\partial\varrho}\left(\frac{\delta g}{g}\right) .$$

If we substitute for the barred quantities the geodetic values in this coordinate system ($\bar{\gamma}_1 = 0$) from Eq. (12.1) etc. together with the change in gravity anomaly over a line, we should accordingly have equations to determine the astronomical parameters γ_1, γ_2 and $(\chi_1 + \chi_2)$. The difficulty of accurately computing geodetic positions in this system would, however, vitiate the accuracy of terms containing

$\delta\omega$ etc. and it is very doubtful if the method offers any advantage over § 8 and § 9, which are equally applicable in much simpler coordinate systems.

It is no easier, in fact it is more difficult, to calculate standard gravity accurately (§ 35 et seq.) at points whose positions are given in these coordinates.

Geodesic Parallel Coordinates

16. The next special case of the general gravitational metric is obtained by putting $g = -1$. In that case $\gamma_1 = \gamma_2 = 0$ and the normals are straight [Hotine 1957, Eq. (19.1)]. The contravariant unit normal (Hotine 1957, § 54) is (0, 0, 1), confirming that there is no change in ω or ϕ along the normal, and showing that the third coordinate can be interpreted as the distance h measured along the outward normal from some one of the third coordinate surfaces which we shall call the *base surface*. Since the third coordinate surfaces are equidistant along the straight normals, they are known as geodesic parallels (Eisenhart 1949, p. 57); they are no longer gravitational equipotentials, although the base surface can be chosen as such, e.g. we can take it to be the geoid.

17. The Codazzi equations (Hotine 1957, § 34) reduce to five [Eqs. (34.7)–(34.9), Hotine (1957) being no longer applicable]:

$$(\partial\chi_1/\partial h) = -\chi_1^2 - \tau^2 \;, \tag{17.1}$$

$$(\partial\chi_2/\partial h) = -\chi_2^2 - \tau^2 \;, \tag{17.2}$$

$$(\partial\tau/\partial h) = -2H\tau \;, \tag{17.3}$$

$$(\partial\tau/\partial m) - (\partial\chi_2/\partial p) = 2H\tau\tan\phi \;, \tag{17.4}$$

$$(\partial\chi_1/\partial m) - (\partial\tau/\partial p) = (\chi_1^2 + 2\tau^2 - \chi_1\chi_2)\tan\phi \;. \tag{17.5}$$

18. We next solve the first three Codazzi equations. By substracting the first two, we have

$$\frac{\partial \log(\chi_1 - \chi_2)}{\partial h} = -(\chi_1 + \chi_2) = -2H = \frac{\partial \log \tau}{\partial h} \quad \text{from (17.3)} \;.$$

Consequently $\tau/(\chi_1 - \chi_2)$ is constant along the normal, which implies that the azimuth A of the principal directions of the h-coordinate surfaces remains constant along the normal (Hotine 1957, § 24). Substituting the principal curvatures κ_1, κ_2 (Hotine 1957, § 25) in the first two Codazzi equations,

$$(\partial\kappa_1/\partial h)\cos^2 A + (\partial\kappa_2/\partial h)\sin^2 A = -\kappa_1^2\cos^2 A - \kappa_2^2\sin^2 A \;,$$

$$(\partial\kappa_1/\partial h)\sin^2 A + (\partial\kappa_2/\partial h)\cos^2 A = -\kappa_1^2\sin^2 A - \kappa_2^2\cos^2 A \;,$$

so that $(\partial\kappa_1/\partial h) = -\kappa_1^2$, and $(\partial\kappa_2/\partial h) = -\kappa_2^2$, or $\partial(1/\kappa_1)/\partial h = 1$, and $\partial(1/\kappa_2)/\partial h = 1$.

If the quantities on the base surface at the foot of the normal are barred, the integrated equations are

$$(1/\kappa_1) = (1/\bar{\kappa}_1) + h \quad \text{and} \quad (1/\kappa_2) = (1/\bar{\kappa}_2) + h \ .$$

Substituting back in the formulae at the end of (Hotine 1957, § 25), and writing K for the Gauss curvature of the h-coordinate surface, we have finally

$$(\chi_1/\mathrm{K}) = (\bar{\chi}_1/\bar{\mathrm{K}}) + h \ , \tag{18.1}$$

$$(\chi_2/\mathrm{K}) = (\bar{\chi}_2/\bar{\mathrm{K}}) + h \ , \tag{18.2}$$

$$(\tau/\mathrm{K}) = (\bar{\tau}/\bar{\mathrm{K}}) \ . \tag{18.3}$$

Also

$$(1/\mathrm{K}) = (1/\bar{\mathrm{K}} + (2\bar{\mathrm{H}}/\bar{\mathrm{K}})h + h^2 \tag{18.4}$$

and from (18.1) and 18.2)

$$(2\mathrm{H}/\mathrm{K}) = (2\bar{\mathrm{H}}/\bar{\mathrm{K}}) + 2h \ . \tag{18.5}$$

These formulae enable us to determine the parameters, and in consequence any metric property, at any point in space from values on the base surface for the same latitude and longitude, or vice versa. Notice that in any formula we can interchange the bars provided that we change the sign of h, since this procedure merely amounts to changing the base surface. For example, Eq. (18.4) becomes

$$(1/\bar{\mathrm{K}}) = (1/\mathrm{K}) - (2\mathrm{H}/\mathrm{K})h + h^2 \ ,$$

which can easily be verified from the other equations.

19. The base vectors (Hotine 1957, § 54) in these coordinates are

$$\begin{aligned} &\lambda^r = (\chi_1 \sec\phi,\ \tau,\ 0) \ , \quad && \mathrm{K}\lambda_r = (\chi_2 \cos\phi,\ -\tau,\ 0) \ , \\ &\mu^r = (\tau \sec\phi,\ \chi_2,\ 0) \ , \quad && \mathrm{K}\mu_r = (-\tau\cos\phi,\ \chi_1,\ 0) \ , \\ &\nu^r = (0,\ 0,\ 1) \ , && \nu_r \ = (0,\ 0,\ 1) \ , \end{aligned} \tag{19.1}$$

and if $\bar{\lambda}_r$ etc. are components of the base vectors at the foot of the normal, these can be combined with (18.1) etc. to give

$$\begin{aligned} \lambda_r &= \bar{\lambda}_r + h \cos\phi\, \delta_r^1 \\ \mu_r &= \bar{\mu}_r + h\ \delta_r^2 \\ \nu_r &= \bar{\nu}_r \ . \end{aligned} \tag{19.2}$$

The first two are not vector equations, true in any other coordinates, since λ_r, $\bar{\lambda}_r$ etc. are not defined at the same point in space. The space and surface metric tensors follow, together with all other results by straight substitution in the general formuale of Hotine (1957); except that, although we can write $\mathrm{V} = h$ for the third coordinate, $\Delta^2 h = 2\mathrm{H}$, which unlike $\Delta^2\mathrm{V}$ is not a constant; equations containing $\Delta^2\mathrm{V}$, or obtained by differentiating $\Delta^2\mathrm{V}$, should accordingly be re-worked. The following equations (from Hotine 1957, § 62, § 63, § 64) are of particular interest, giving the three fundamental forms of an h-coordinate surface in terms of those of the base surface at the foot of the normal

$$\begin{aligned} a_{\alpha\beta} &= \bar{a}_{\alpha\beta} - 2h\,\bar{b}_{\alpha\beta} + h^2\bar{c}_{\alpha\beta} \quad (\alpha,\beta = 1,2) \\ b_{\alpha\beta} &= \bar{b}_{\alpha\beta} - h\bar{c}_{\alpha\beta} \\ c_{\alpha\beta} &= \bar{c}_{\alpha\beta} = (\cos^2\phi, 0, 1)\ . \end{aligned} \tag{19.3}$$

The Laplacian of a scalar F in these coordinates can be found without difficulty as

$$\Delta^2 F = (\Delta^2 F) + 2H\frac{\partial F}{\partial h} + \frac{\partial^2 F}{\partial h^2}\ , \tag{19.4}$$

in which $(\Delta^2 F)$ is the *surface* Laplacian taken with respect to the metric of the h-coordinate surface passing through the point.

20. If the axes of Cartesian coordinates are as usual parallel to A^r, B^r, C^r (Hotine 1957, § 1), we have the Cartesian coordinates of a point distant h along the straight normal from $(\bar{x},\bar{y},\bar{z})$ on the base surface as

$$\begin{aligned} x &= \bar{x} + h\cos\phi\cos\omega \\ y &= \bar{y} + h\cos\phi\sin\omega \\ z &= \bar{z} + h\sin\phi\ , \end{aligned} \tag{20.1}$$

in which $\bar{x},\bar{y},\bar{z}$ are functions of (ω,ϕ) only. From Hotine 1957, Eq. (6.2) we have

$$\begin{aligned} \bar{x}_\alpha &= -\bar{\lambda}_\alpha \sin\omega - \bar{\mu}_\alpha \sin\phi\cos\phi\cos\omega \quad (\alpha = 1,2) \\ \bar{y}_\alpha &= \bar{\lambda}_\alpha \cos\omega - \bar{\mu}_\alpha \sin\phi\sin\omega \\ \bar{z}_\alpha &= \bar{\mu}_\alpha \cos\phi\ , \end{aligned}$$

where

$$\begin{aligned} \bar{\lambda}_\alpha &= -\bar{x}_\alpha \sin\omega + \bar{y}_\alpha \cos\omega \\ \bar{\mu}_\alpha &= -\bar{x}_\alpha \csc\phi\cos\omega - \bar{y}_\alpha \csc\phi\sin\omega = \bar{z}_\alpha \sec\phi\ , \end{aligned}$$

and substituting the components of § 19 of the vectors, these give

$$(\bar{x}_2\cos\phi)/\bar{K} = -\sin\omega(\partial\bar{x}/\partial\omega) + \cos\omega(\partial\bar{y}/\partial\omega) \tag{20.2}$$

$$\begin{aligned} \bar{\tau}/\bar{K} &= \sin\omega(\partial\bar{x}/\partial\phi) - \cos\omega(\partial\bar{y}/\partial\phi) \quad \text{or}\ , \\ &= \sec\phi\csc\phi\cos\omega(\partial\bar{x}/\partial\omega) + \sec\phi\csc\phi\sin\omega(\partial\bar{y}/\partial\omega) \quad \text{or} \\ &= -\sec^2\phi(\partial\bar{z}/\partial\omega)\ , \end{aligned} \tag{20.3}$$

$$\begin{aligned} \bar{\chi}_1/\bar{K} &= -\csc\phi\cos\omega(\partial\bar{x}/\partial\phi) - \csc\phi\sin\omega(\partial\bar{y}/\partial\phi)\ , \quad \text{or} \\ &= \sec\phi(\partial\bar{z}/\partial\phi)\ . \end{aligned} \tag{20.4}$$

Starting from Eq. (6.2) in Hotine (1957), the same equations obviously hold without the bars for any other third-coordinate surface and together with

Eq. (20.1) this enables us to verify (18.1) etc. These equations, together with $\bar{K} = \bar{\kappa}_1\bar{\kappa}_2 - \bar{\tau}^2$, will enable us to determine the parameters on the base surface [and therefore anywhere else from (18.1) etc.] in cases where the base surface is given as $\bar{x} = f(\omega, \phi)$ etc.

Geodesic Parallels to a Spheroid

21. If the base surface is a surface of revolution about an axis parallel to C^r (or z-axis), then the Cartesian coordinates of points on the base surface can take the form

$$\begin{aligned}\bar{x} &= f(\phi)\cos\omega\\ \bar{y} &= f(\phi)\sin\omega\\ \bar{z} &= g(\phi)\ ,\end{aligned}$$

which, on substitution in Eqs. (20.3), (20.2) and (20.4), gives

$$\bar{\tau} = 0\ ;\quad \bar{K} = \bar{\kappa}_1\bar{\kappa}_2\ ;\quad f(\phi) = \cos\phi/\bar{\kappa}_1\ ;\quad f'(\phi) = -\sin\phi/\bar{\kappa}_2\ ;$$
$$g'(\phi) = \cos\phi/\bar{\kappa}_2\ . \tag{21.1}$$

The fact that we have derived separate equations for $f(\phi)$ and $f'(\phi)$ does not introduce any limitation on $f(\phi)$, since these are reconciled by the Codazzi equation (17.5), which in this case takes the form (Hotine 1957, § 58)

$$\chi_2(\partial\chi_1/\partial\phi) = (\chi_1^2 - \chi_1\chi_2)\tan\phi\ . \tag{21.2}$$

The other Codazzi equation [17.4] is satisfied identically.

22. If the base surface is a spheroid of eccentricity e(with $\bar{e}^2 = 1 - e^2$), then $1/\bar{\chi}_1 = \upsilon$, $1/\bar{\chi}_2 = \varrho$, and the last equation is

$$\partial\upsilon/\partial\phi = (\upsilon - \varrho)\tan\phi \tag{22.1}$$

and since (21.1) $f(\phi) = \upsilon\cos\phi$; $f'(\phi) = -\varrho\sin\phi$; $g'(\phi) = \varrho\cos\phi$; $g(\phi) = \bar{e}^2\upsilon\sin\phi$ the coordinate equations (20.1) are

$$\begin{aligned}x &= (\upsilon + h)\cos\phi\cos\omega\\ y &= (\upsilon + h)\cos\phi\sin\omega\\ z &= (\bar{e}^2\upsilon + h)\sin\phi\ .\end{aligned} \tag{22.2}$$

The non-zero parameters at any point in space are (18.1) etc.:

$$\chi_1 = 1/(\upsilon + h)\ ;\quad \chi_2 = 1/(\varrho + h)\ . \tag{22.3}$$

Non-zero components of the base vectors are

$$\begin{aligned}\lambda^1 &= 1/\{(\upsilon + h)\cos\phi\}\\ \mu^2 &= 1/(\varrho + h)\\ \upsilon^3 &= 1\ ,\end{aligned} \tag{22.4}$$

$$\lambda_1 = (\nu + h)\cos\phi$$
$$\mu_2 = (\varrho + h) \tag{22.5}$$
$$\nu_3 = 1$$

and the metric is

$$ds^2 = (\nu + h)^2 \cos^2\phi\, d\omega^2 + (\varrho + h)^2 d\phi^2 + dh^2 \;. \tag{22.6}$$

The second-order magnitudes of the $h =$ const. surfaces (Hotine 1957, § 64) are

$$b_{\alpha\beta} = [(-\varrho\nu + h)\cos^2\phi, 0, -(\varrho - h)] \tag{22.7}$$

and the third-order magnitudes are

$$c_{\alpha\beta} = (\cos^2\phi, 0, 1) \;. \tag{22.8}$$

23. The equations for variation of coordinates along a line which is either straight or refracted in a medium whose density is now a function of h are obtained straight from (Hotine 1957, § 42) et seq. as

$$\cos\phi\,(\partial\omega/\partial s) = \sin\alpha \sin\beta/(\nu + h) \;,$$
$$(\partial\phi/\partial s) = \cos\alpha \sin\beta/(\varrho + h) \;,$$
$$(\partial h/\partial s) = \cos\beta \;, \tag{23.1}$$
$$(\partial\alpha/\partial s) = \tan\phi \sin\alpha \sin\beta/(\nu + h)$$
$$\qquad - \sin\alpha \cos\alpha \sin\beta \cos\beta(\varrho - \nu)/\{(\nu + h)(\varrho + h)\} \;,$$
$$(\partial\beta/\partial s) = \bar{\chi} - \sin\beta \sin^2\alpha/(\nu + h) - \sin\beta \cos^2\alpha/(\varrho + h) \;,$$

(in which $\bar{\chi}$ is the curvature of the refracted ray).

Since, in this case, we know the parameters $\chi_1 = 1/(\nu + h)$, $\chi_2 = 1/(\varrho + h)$, $\tau = 0$ as functions of the coordinates, we can compute $(\partial\chi_1/\partial s)$ etc. and therefore determine the second-order terms (Hotine 1957, § 49), but this operation is still by no means simple. Whenever the line can be considered straight, as it can be for azimuth calculations, and for others by removal of a standard refraction, it is better to use closed formulae obtained as follows.

24. It is evident from Eq. (22.2), that the position vector ϱ^r can be written as

$$\varrho^r = x A^r + y B^r + z C^r$$
$$= \{(a^2/\nu) + h\}\nu^r - (e^2 \nu \sin\phi \cos\phi)\mu^r \quad \text{[Eq. (6.2), Hotine 1957]} \tag{24.1}$$

a vector equation which is true in any coordinates. Now suppose that all quantities at the far end of a line of length s (unit vector $l^r = \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha$ $\sin\beta + \nu^r \cos\beta$) are barred, and that $\bar{\varrho}^r$ represents either the Cartesian components of the position vector at the barred end, or else a vector parallel to the latter at the unbarred end, then

$$\bar{\varrho}^r - \varrho^r = s l^r \tag{24.2}$$

with

$$\bar{\varrho}^{\mathrm{r}} = \{(\mathrm{a}^2/\bar{\upsilon}) + \bar{h}\}\bar{\upsilon}^{\mathrm{r}} - (\mathrm{e}^2 \bar{\upsilon} \sin \bar{\phi} \cos \bar{\phi})\bar{\mu}^{\mathrm{r}} \ ,$$

in which we can immediately substitute Eqs. (3.3) and (3.2) for $\bar{\upsilon}^{\mathrm{r}}$ and $\bar{\mu}^{\mathrm{r}}$.

Multiplying Eq. (24.2) across by λ^{r}

$$\begin{aligned} s \sin \alpha \sin \beta &= \{(\mathrm{a}^2/\bar{\upsilon}) + \bar{h}\} \cos \bar{\phi} \sin (\bar{\omega} - \omega) \\ &\quad + (\mathrm{e}^2 \bar{\upsilon} \sin^2 \bar{\phi} \cos \bar{\phi} \sin (\bar{\omega} - \omega) = (\bar{\upsilon} - \bar{h}) \cos \bar{\phi} \sin (\bar{\omega} - \omega) \end{aligned} \tag{24.3}$$

and similarly through multiplication by μ^{r} and υ^{r}

$$\begin{aligned} s \cos \alpha \sin \beta = &-(\bar{\upsilon} + \bar{h}) \sin \phi \cos \bar{\phi} \cos (\bar{\omega} - \omega) \\ &+ (\bar{\mathrm{e}}^2 \bar{\upsilon} + \bar{h}) \cos \phi \sin \bar{\phi} + \mathrm{e}^2 \upsilon \sin \phi \cos \phi \end{aligned} \tag{24.4}$$

or

$$\begin{aligned} &= (\bar{\upsilon} + \bar{h})\{\cos \phi \sin \bar{\phi} - \sin \phi \cos \bar{\phi} \cos (\bar{\omega} - \omega)\} \\ &\quad - \mathrm{e}^2 \cos \phi (\bar{\upsilon} \sin \bar{\phi} - \upsilon \sin \phi) \ . \end{aligned}$$

$$\begin{aligned} s \cos \beta &= (\bar{\upsilon} + \bar{h}) \cos \phi \cos \bar{\phi} \cos (\bar{\omega} - \omega) \\ &\quad + (\bar{\mathrm{e}}^2 \bar{\upsilon} + \bar{h}) \sin \phi \sin \bar{\phi} - \{(\mathrm{a}^2/\upsilon) + h\} \end{aligned} \tag{24.5}$$

or

$$\begin{aligned} &= (\bar{\upsilon} + \bar{h})\{\sin \phi \sin \bar{\phi} + \cos \phi \cos \bar{\phi} \cos (\bar{\omega} - \omega)\} \\ &\quad - (\upsilon + h) - \mathrm{e}^2 \sin \phi (\bar{\upsilon} \sin \bar{\phi} - \upsilon \sin \phi) \ . \end{aligned}$$

These three equations directly determine the length, azimuth and zenith distance of a line whose end geodetic coordinates are given. In these and all similar formulae we can interchange the bars provided only that we change the sign of s, assuming as usual that barred azimuth etc. refers to the line produced and not to the back azimuth. For example,

$$\mathrm{s} \sin \bar{\alpha} \sin \bar{\beta} = (\upsilon + h) \cos \phi \sin (\bar{\omega} - \omega) \tag{24.6}$$

and combining this with (24.3) we find that

$$(\upsilon + h) \cos \phi \sin \alpha \sin \beta \tag{24.7}$$

is a constant along the line, an analogous result in space to Clairaut's theorem for a geodesic on the spheroid.

25. For the inverse problem, given s, α, β and the coordinates of the unbarred end, we can reverse the above formulae as

$$\begin{aligned} (\bar{\upsilon} + \bar{h}) \cos \bar{\phi} \sin (\bar{\omega} - \omega) &= \mathrm{s} \sin \alpha \sin \beta \\ (\bar{\upsilon} + \bar{h}) \cos \bar{\phi} \cos (\bar{\omega} - \omega) &= -\mathrm{s} \cos \alpha \sin \beta \sin \phi + \mathrm{s} \cos \beta \cos \phi + (\upsilon + h) \cos \phi \\ (\bar{\mathrm{e}}^2 \bar{\upsilon} + \bar{h}) \sin \bar{\phi} &= \mathrm{s} \cos \alpha \sin \beta \cos \phi + \mathrm{s} \cos \beta \sin \phi + (\bar{\mathrm{e}}^2 \upsilon + h) \sin \phi \ , \end{aligned}$$

which could have been obtained from the difference of Cartesian components over the line, using a temporary origin for longitude at the unbarred end. The difference in longitude follows at once by division of the first two equations. We then have to solve by iteration

$$\begin{cases} (\bar{\nu}+\bar{h})\cos\bar{\phi} = A \\ (\bar{e}^2\bar{\nu}+\bar{h})\sin\bar{\phi} = B \end{cases}$$

for the latitude and height, starting with an approximate latitude given by $\bar{e}^2\tan\bar{\phi} = B/A$.

Adjustment of Triangulation in Space

26. As in Hotine (1957, § 66 et seq.), we consider the changes in azimuth α and zenith distance β of a line l^r due to changes in the positions of the ends. We use geodesic parallel coordinates (ω, ϕ, h) with a spheroidal base.

The components of the vector n_s in these coordinates (Hotine 1957, § 70) and § 22 are:

$$n_s = \{-(\nu+h)\cos\phi\cos\alpha, (\rho+h)\sin\alpha, 0\}$$

so that the change in azimuth is given by [Eq. (70.1), Hotine 1957]:

$$\begin{aligned} \sin\beta\, d\alpha = {} & d\omega\{-\cos\phi\cos\alpha(\nu+h)/s+\sin\phi\sin\beta-\cos\phi\cos\alpha\cos\beta\} \\ & +d\phi\{\sin\alpha(\rho+h)/s+\sin\alpha\cos\beta\} \\ & +d\bar{\omega}\{-P\cos\bar{\phi}(\bar{\nu}+\bar{h})/s\} \\ & +d\bar{\phi}\{-Q(\bar{\rho}+\bar{h})/s\} \\ & +d\bar{h}\{-R/s\} \end{aligned}$$

with

$$\begin{aligned} P &= -\cos\alpha\cos(\bar{\omega}-\omega)+\sin\phi\sin\alpha\sin(\bar{\omega}-\omega)\ , \\ Q &= +\sin\phi\sin\bar{\phi}\sin\alpha\cos(\bar{\omega}-\omega)+\sin\bar{\phi}\cos\alpha\sin(\bar{\omega}-\omega) \\ & \quad +\cos\phi\cos\bar{\phi}\sin\alpha \\ R &= -\sin\phi\cos\bar{\phi}\sin\alpha\cos(\bar{\omega}-\omega)-\cos\bar{\phi}\cos\alpha\sin(\bar{\omega}-\omega) \\ & \quad +\cos\phi\sin\bar{\phi}\sin\alpha \end{aligned}$$

and $P^2+Q^2+R^2 = 1$ as in check.

In these axially symmetrical coordinates (but not in general), the coefficients of $d\omega$ and $d\bar{\omega}$ in the above equation should obviously be equal and opposite in sign, so that there may be a single term in $(d\omega - d\bar{\omega})$, since it must be immaterial if the same quantity is added to both ω and $\bar{\omega}$. That this is so can easily be verified with the help of Eqs. (24.3), (24.6) and (5.1). This fact provides another check, or lessens the computation.

An alternative expression for R in these coordinates is

$$e^2(\bar{\nu}\sin\phi-\nu\sin\phi)\cos\phi\sin\alpha/(\bar{\nu}+\bar{\hbar})$$

obtainable from Eqs. (24.4) and (24.3).

If we change coordinates to the astronomical system by adding the (astronomic minus geodetic) deflections $\delta\omega$, $\delta\phi$ (at present unknown), there will be a further change in azimuth [Eq. (6.1)] of

$$\delta\alpha=(\sin\phi-\cos\phi\cos\alpha\cos\beta)\delta\omega+\sin\alpha\cos\beta\,\delta\phi\ ,$$

and if we then add a station correction $\Delta\alpha$ (at present unknown) to all rays at a station, the result should be the observed astronomical azimuth. The final observation equation, starting with a geodetic azimuth computed from provisional values of the geodetic coordinates, is accordingly:

$$\begin{aligned}
&\text{Observed Azimuth} - \text{Computed Azimuth}\\
&=\Delta\alpha+d\omega\{-(\nu+\hbar)\cos\phi\cos\alpha\csc\beta/s\} \qquad (26.1)\\
&\quad+d\phi\{(\varrho+\hbar)\sin\alpha\cos\beta/s\}\\
&\quad+d\bar{\omega}\{-P\cos\bar{\phi}\csc\beta(\bar{\nu}+\bar{\hbar})/s\}\\
&\quad+d\bar{\phi}\{-Q\csc\beta(\bar{\varrho}+\bar{\hbar})/s\}\\
&\quad+d\bar{\hbar}\{-R\csc\beta/s\}\\
&\quad+(d\omega+\delta\omega)(\sin\phi-\cos\phi\cos\alpha\cos\phi)\\
&\quad+(d\phi+\delta\phi)(\sin\alpha\cos\beta)\ .
\end{aligned}$$

If the deflections $\delta\omega$, $\delta\phi$ are unknown, it is permissible to take $(d\omega+\delta\omega)$, $(d\phi+\delta\phi)$ as composite unknowns and later to evaluate the deflections by subtracting the position corrections $d\omega$, $d\phi$. If, however, astronomical latitude and/or longitude has been measured, then $(d\phi+\delta\phi)$ and/or $(d\omega+\delta\omega)$ are the astronomical values minus the *preliminary* geodetic values and should be substituted in the equations before solution, thereby reducing the number of unknowns by two. Similarly, if astronomical azimuth has been measured at a station and it has accordingly not been necessary to assume an azimuth in order to orient the observed directions, then the $\Delta\alpha$ term should be dropped.

27. If the triangulation has been computed in Cartesian coordinates, some simplification is possible by determining the position corrections (dn, etc.) in Cartesian coordinates. In that case the vectors

$$n_r=-\lambda_r\cos\alpha+\mu_r\sin\alpha$$

and (n_s) [Eq. (69.3), Hotine 1957] both have the same Cartesian components (Hotine 1957, § 6):

$$\begin{aligned}
n_1&=\cos\alpha\sin\omega-\sin\phi\sin\alpha\cos\omega\\
n_2&=-\cos\alpha\cos\omega-\sin\phi\sin\alpha\sin\omega\\
n_3&=\cos\phi\sin\alpha
\end{aligned}$$

and [Eq. (69.3), Hotine 1957] the (Observed minus Computed) azimuth is

$$\begin{aligned}
&\Delta\alpha + (\mathrm{d}\bar{\mathrm{n}} - \mathrm{dn})(-\cos\alpha\sin\omega + \sin\phi\sin\alpha\cos\omega)\csc\beta/\mathrm{s} \\
&\quad + (\mathrm{d}\bar{\mathrm{y}} - \mathrm{dy})(\cos\alpha\cos\omega + \sin\phi\sin\alpha\sin\omega)\csc\beta/\mathrm{s} \\
&\quad - (\mathrm{d}\bar{\mathrm{z}} - \mathrm{dz})(\cos\phi\sin\alpha\csc\beta)/\mathrm{s} \\
&\quad + (\mathrm{d}\omega + \delta\omega)(\sin\phi - \cos\phi\cos\alpha\cos\beta) \\
&\quad + (\mathrm{d}\phi + \delta\phi)(\sin\alpha\cos\beta) \ .
\end{aligned} \tag{27.1}$$

28. In § 21 coordinates, the auxiliary vector m_r (Hotine 1957, § 71), required for the zenith distance equation, has the following components:

$$\begin{aligned}
m_1 &= (\upsilon + h)\cos\phi\sin\alpha\cos\beta \\
m_2 &= (\varrho + h)\cos\alpha\cos\beta \\
m_3 &= -\sin\beta
\end{aligned}$$

and the change in zenith distance due to change of coordinates to the astronomical system [Eq. (6.2)] is

$$\delta\beta = -\cos\phi\sin\alpha\,\delta\omega - \cos\alpha\,\delta\phi \ .$$

The (Observed minus Computed) zenith distance (Hotine 1957, § 71) is accordingly

$$\begin{aligned}
&-\mathrm{d}\omega\{(\upsilon + h)\cos\phi\sin\alpha\cos\beta/\mathrm{s}\} \\
&-\mathrm{d}\phi\{(\varrho + h)\cos\alpha\cos\beta/\mathrm{s}\} \\
&+\mathrm{d}h\,(\sin\beta/\mathrm{s}) \\
&+\mathrm{d}\bar{\omega}\{\mathrm{S}\cos\bar{\phi}\,(\bar{\upsilon} + \bar{h})/\mathrm{s}\} \\
&+\mathrm{d}\bar{\phi}\{\mathrm{T}(\bar{\varrho} + \bar{h})/\mathrm{s}\} \\
&+\mathrm{d}\bar{h}\,\{\mathrm{U}/\mathrm{s}\} \\
&-(\mathrm{d}\omega - \delta\omega)\cos\phi\sin\alpha \\
&-(\mathrm{d}\phi + \delta\phi)\cos\alpha
\end{aligned} \tag{28.1}$$

with S, T, U as in Hotine (1957, § 71).

Here again, in these symmetrical coordinates, the coefficients of $\mathrm{d}\omega$ and $\mathrm{d}\bar{\omega}$ must be equal and opposite in sign, so that

$$\mathrm{S}\cos\bar{\phi}\,(\bar{\upsilon} + \bar{h}) = (\upsilon + h)\cos\phi\sin\alpha\cos\beta + \mathrm{S}\cos\phi\sin\alpha \ ,$$

which is easily verified from Eqs. (24.3) and (24.6).

29. Again, if the triangulation has been computed in Cartesian, the Cartesian components of m_s or (m_s) are

$$\begin{aligned}
m_1 &= -\sin\alpha\cos\beta\sin\omega - \sin\phi\cos\alpha\cos\beta\cos\omega - \cos\phi\sin\beta\cos\omega \\
m_2 &= \sin\alpha\cos\beta\cos\omega - \sin\phi\cos\alpha\cos\beta\sin\omega - \cos\phi\sin\beta\sin\omega \\
m_3 &= \cos\phi\cos\alpha\cos\beta - \sin\phi\sin\beta
\end{aligned}$$

and the (Observed minus Computed) zenith distance takes the alternative form [Eq. (71.1) Hotine 1957]:

$$
\begin{aligned}
&+(\mathrm{d}\bar{n}-\mathrm{d}n)(m_1/s)\\
&+(\mathrm{d}\bar{y}-\mathrm{d}y)(m_2/s)\\
&+(\mathrm{d}\bar{z}-\mathrm{d}z)(m_3/s)\\
&-(\mathrm{d}\omega+\delta\omega)\cos\phi\sin\alpha\\
&-(\mathrm{d}\phi+\delta\phi)\cos\alpha\ .
\end{aligned}
\tag{29.1}
$$

30. The observed zenith distance should be freed from refraction as far as possible before forming all the above equations, whether the zenith distance occurs in the absolute term or in the coefficients of any of the observation equations. Even so, the observation equations for zenith distances should be weighted down, since zenith distances will certainly not be measured as accurately as the horizontal angles, but there seems to be no justification for omitting them altogether. If zenith distances are omitted, the adjustment would be weak (although still stable) and would tend not to alter the provisional values.

As shown in § 26, the adjustment can be stiffened by measuring astronomical azimuths and positions, and these are very easily assimilated wherever they have been measured. Current methods of adjustment in two dimensions, besides assuming that the third dimension can be treated quite independently, usually reduce the number of unknowns in effect by simply ignoring $\delta\phi$, $\delta\omega$, but this does nothing to increase the reliability of the result.

31. For adjustment to a known base length [Eq. (68.4), Hotine 1957], we require the components of the vector l_s in the § 21 system, which are in § 22:

$$
\begin{aligned}
l_1 &= (\nu+\hbar)\cos\phi\sin\alpha\sin\beta\\
l_2 &= (\varrho+\hbar)\cos\alpha\sin\beta\\
l_3 &= \cos\beta\\
\bar{l}_1 &= (\bar{\nu}+\bar{\hbar})\cos\bar{\phi}\sin\bar{\alpha}\sin\bar{\beta}\ \text{etc.}
\end{aligned}
$$

The (Observed minus Computed) length *of the air-line* is then [Eq. (68.4), Hotine 1957]

$$
-l_1\,\mathrm{d}\omega-l_2\,\mathrm{d}\phi-l_3\,\mathrm{d}\hbar+\bar{l}_1\,\mathrm{d}\bar{\omega}+\bar{l}_2\,\mathrm{d}\bar{\phi}+\bar{l}_3\,\mathrm{d}\bar{\hbar}\ . \tag{31.1}
$$

Notice once more that the symmetricality of the system requires $l_1=\bar{l}_1$, which is verified from Eq. (24.7).

A direct measure by invar, geodimeter or tellurometer (which in any current adjustment would be held fixed) would require this equation to be heavily weighted. (It needs, of course, to be divided by the length of the line to reduce it to the same terms as the azimuth and zenith distance equations).

Alternatively, it can be held fixed by treating Eq. (31.1) as a condition equation and using it to eliminate one of the unknowns from the other observation equations before solution, just as was proposed in § 26 for astronomical measures.

32. To obtain a length equation to solve with Eqs. (27.1) and (29.1), we need the Cartesian components of l_r, which are

$$l_1 = -\sin\alpha\sin\beta\sin\omega - \sin\phi\cos\alpha\sin\beta\cos\omega + \cos\phi\cos\beta\cos\omega$$

$$l_2 = \sin\alpha\sin\beta\cos\omega - \sin\phi\cos\alpha\sin\beta\sin\omega + \cos\phi\cos\beta\sin\omega$$

$$l_3 = \cos\phi\cos\alpha\sin\beta + \sin\phi\cos\beta$$

and the (Observed minus Computed) length becomes

$$+(d\bar{n} - dn)l_1 + (d\bar{y} - dy)l_2 + (d\bar{z} - dz)l_3 \ . \tag{32.1}$$

33. We have now obtained rigorous first-order observation equations to take care of all the quantities usually measured in triangulation, and we have done so without requiring any knowledge of the form of the geoid. In particular the method of § 31, § 32 completely eliminates any "reduction to sea level", for which the depth of the geoid in relation to the spheroid might be required. The only way in which a knowledge of the geoid could assist the adjustment would be to provide means of deducing $\delta\phi$ and $\delta\omega$ instead of making direct astronomical measures, and it will certainly be more accurate and economical to make these measures as required, rather than undertake the much greater programme of measurement necessary to determine the geoid. It should be noted, moreover, that what would be required is an accurate knowledge of the geoidal depth and slope at particular points and not a deliberately smoothed first-order result as currently proposed.

34. The only additional measurement, not as a rule undertaken at present, which would stiffen the adjustment seems to be spirit-levelling between stations. Combined with occasional measures of gravity, this in effect would give a direct measure of difference of potential V over the line. Spirit-levelled "heights" by themselves are not independent of the path; they are not point-functions like potential, or geodetic heights rigorously defined as in § 16, and we cannot properly speak of the "height" of a point obtained by spirit-levelling without also specifying the route by which it was measured. Much futile argument would be saved if this were more generally recognized.

For a rigorous treatment, we should accordingly have a relation between physical potential V and geodetic height h, analogous to the differences $\delta\phi$, $\delta\omega$ between the other coordinates in the two systems, and it would need to be more exact over a finite line than § 10. This might in time be provided by a knowledge at each station of N (the separation between geoid and spheroid or between geop and spherop) obtainable from the Stokes' theory, as recently modified by Levallois and de Graaff Hunter, but it is doubtful whether this is necessary to a first order, which is all the Stokes' treatment would give.

The change in potential due to a change in position dx^r is $V_r\,dx^r = -g(\nu_r)dx^r$, where (ν_r) is the astronomical zenith. But (ν_r) differs from ν_r, the normal to the $h = \text{const.}$ surface in the geodetic coordinates by a vector whose modulus is of the order $\delta\phi$ or $\delta\omega$, so that to a first order we can say the change of potential is $-g\nu_r dx^r = -g\,dh$ in § 21 coordinates. If we start with the spirit-levelled heights (perhaps after a circuit adjustment), as approximate geodetic coordinates (and use these for calculating geodetic azimuths and zenith distances), the observation equation is accordingly

$$0 = g\,dh - \bar{g}\,d\bar{h} \;, \tag{34.1}$$

which should be divided by mean gravity and by the side-length to reduce it to the same dimensions as the azimuth etc. equations, and then heavily weighted, as in the case of a directly measured base length. Alternatively, Eq. (34.1) can be used as a condition equation to eliminate one of the unknowns from the other observation equations before solution, as proposed for astronomical measures and bases. The final values will, as a result, be close to the spirit-levelling and cannot, of course, be considered as true coordinates in the geodetic system for any such purpose as determining the geoid. They may also affect derived values of $\delta\phi$, $\delta\omega$ for such purposes, but so far as the geometrical adjustment of the triangulation is concerned, these matters are of little consequence. At the time of writing, this method of absorbing the results of spirit-levelling has not yet been tried out in practice.

Standard Gravity

35. For certain purposes we use gravity "anomalies", defined as the difference between measured gravity and the value of gravity in some standard field at the same point. We now define the standard field rigorously as one in which the base spheroid of the geodetic coordinate system is an equipotential surface enclosing a mass M equal to that of the physical Earth; the field to rotate about the minor axis of the spheroid with the same angular velocity $\tilde{\omega}$ as the Earth does about its physical axis. The two axes will be parallel in the case of any coordinate system proposed in this paper, and will be made so in setting up the origin conditions of the coordinate system (§ 7); they will not necessarily coincide.

36. To determine the standard potential, we use (for this purpose only) ordinary spheroidal coordinates related to the usual Cartesian coordinates by the equations

$$x = c(1+u^2)^{1/2} \sin U \cos \omega$$

$$y = c(1+u^2)^{1/2} \sin U \sin \omega$$

$$z = c u \cos U \;.$$

The u = const. surfaces are the spheroids

$$\frac{x^2+y^2}{c^2(1+u^2)} + \frac{z^2}{c^2u^2} = 1$$

confocal with the base spheroid

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \;.$$

The constant F is half the inter-focal distance, or $c = ae = (a^2 - b^2)^{1/2}$; and the value of u for the base spheroid is

$$b/c = b/(ae) = \bar{e}/e \ .$$

The coordinate U is the *reduced* co-latitude on the u-spheroid passing through the point.

37. The normal solution, independent of longitude, of the Laplace equation in these coordinates (e.g. MacRobert 1947, p. 215) is easily found in terms of Legendre functions as

$$A_n Q_n(iu) P_n(\cos U) \ ,$$

so that the potential of the standard field is

$$V = \sum_{n=0}^{\infty} A_n Q_n(iu) P_n(\cos U) + \tfrac{1}{2}\tilde{\omega}^2 (x^2 + y^2)$$

$$= \sum_{n=0}^{\infty} A_n Q_n(iu) P_n(\cos U) + \tfrac{1}{3}\tilde{\omega}^2 c^2 (1 + u^2)\{1 - P_2(\cos U)\} \ .$$

On the base spheroid $u = b/c = \bar{e}/e$ and

$$V_0 = \sum_{n=0}^{\infty} A_n Q_n(i\bar{e}/e) P_n(\cos U) + \tfrac{1}{3}\tilde{\omega}^2 a^2 - \tfrac{1}{3}\tilde{\omega}^2 a^2 P_2(\cos U)$$

for all values of U, so that the coefficients of the P_n's may be equated to zero, giving

$$V_0 = A_0 Q_0(i\bar{e}/e) + \tfrac{1}{3}\tilde{\omega}^2 a^2 \tag{37.1}$$

and

$$A_2 Q_2(i\bar{e}/e) = \tfrac{1}{3}\tilde{\omega}^2 a^2 \tag{37.2}$$

and all other A's zero, whence finally

$$V = A_0 Q_0(iu) + A_2 Q_2(iu) P_2(\cos U) + \tfrac{1}{2}\tilde{\omega}^2 (x^2 + y^2) \ . \tag{37.3}$$

38. So far the treatment is equivalent to Somigliana's and others, but the coordinates (U, u), although valuable for this particular purpose, are of little use for other geodetic purposes and do not convert very easily to any of the systems proposed in this paper. Accordingly, we obtain the potential from Eq. (37.3) in ordinary spherical harmonics which are more readily computed from Cartesians and hence from (ω, ϕ, h) coordinates [Eq. (22.2)].

39. On the z-axis of symmetry, Eq. (37.3) gives the potential as

$$A_0 Q_0(iz/c) + A_2 Q_2(iz/c) \ ,$$

which can easily be expanded as

$$\sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n-1)(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{z^{2n-1}} . \tag{39.1}$$

To find the external attraction potential off the axis of symmetry, all we need do is to replace $(1/z^{2n-1})$ by $(1/r^{2n-1})P_{2n-2}(\cos\theta)$ (Ramsey 1952, p. 132), in which r is the radius vector and θ is the "geocentric" co-latitude. We must also restore the rotation term and obtain finally

$$V = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n-1)(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{r^{2n-1}} P_{2n-2}(\cos\theta)+\tfrac{1}{2}\tilde{\omega}^2(x^2+y^2) \tag{39.2}$$

with

$$r^2 = x^2+y^2+z^2 \;; \quad \cos\theta = 3/r .$$

40. For large values of z, Eq. (39.1) reduces to $-iA_0/(ae/z)$ and at the same large distance we can consider the mass M condensed at the origin to give rise to a potential kM/z so that

$$iA_0 = -kM/(ae) . \tag{40.1}$$

The other constant, iA_2, is obtained from Eq. (37.2). If $e = \sin\alpha$, this expands as

$$iA_2 = \tfrac{2}{3}\tilde{\omega}^2 a^2/\{(1+3\cos^2\alpha)\alpha-3\cos\alpha\} . \tag{40.2}$$

41. If λ is the geographical *co*-latitude of the line of force, we have, by differentiating Eq. (39.2) with respect to z, after some simplification

$$g\cos\lambda = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{r^{2n}} P_{2n-1}(\cos\theta) \tag{41.1}$$

and by differentiation with respect to x or y

$$g\sin\lambda = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n-1)(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{r^{2n}} P'_{2n-1}(\cos\theta)\sin\theta$$
$$-\tilde{\omega}^2 r\sin\theta . \tag{41.2}$$

These equations may be combined in the alternative forms:

$$g\cos(\lambda-\theta) = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{r^{2n}} P_{2n-2}(\cos\theta)$$
$$-\tilde{\omega}^2 r\sin^2\theta \tag{41.3}$$

$$g\sin(\lambda-\theta) = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n-1)(2n+1)} (-1)^n \frac{(ae)^{2n-1}}{r^{2n}} P'_{2n-2}(\cos\theta)$$
$$-\tilde{\omega}^2 r\sin\theta\cos\theta . \tag{41.4}$$

42. Gravity on the equator of the base spheroid ($\lambda = \theta = \pi/2$; $r = a$) is from Eq. (41.3)

$$g_e = \sum_{n=1}^{\infty} \frac{(2n+1)iA_0+(2n-2)iA_2}{(2n+1)} (-1)^n \frac{e^{2n-1}}{a} P_{2n-2}(0) ,$$

which relates g_e to M, or determines iA_0 in terms of g_e instead of M if required. An alternative formula, which can be shown to be equivalent, is obtained by differentiating Eq. (37.3) for $U = \pi/2$ with respect to

$$(x^2+y^2)^{1/2} = c(1+u^2)^{1/2} .$$

With $e = \sin\alpha$, the result after some manipulation is

$$a g_e = -iA_0 \tan\alpha + iA_2(\alpha - \tan\alpha) - \tfrac{2}{3}\tilde{\omega}^2 a^2 .$$

43. All the above formulae are rigorous and rapidly convergent and may often be required as a control on more approximate computation and interpolation. Gravity on the international spheroid tabulated in the Geodetic Tables (1956) agrees with results from these formulae, but the application of ordinary first-order height reductions can introduce errors up to 7 mg.

44. We can, however, expand gravity, or any other scalar, in terms of geodetic height h, by the method of Hotine (1957, § 40). Notice that since the geodetic normal $\bar{\upsilon}^r$ in § 22 coordinates is straight

$$\partial^2F/\partial h^2 = F_{rs}\bar{\upsilon}^r\bar{\upsilon}^s \quad \text{and} \quad \partial^3F/\partial h^3 = F_{rst}\bar{\upsilon}^r\bar{\upsilon}^s\bar{\upsilon}^t \text{ etc.}$$

For the line of force υ^r we have (from Hotine 1957, § 19) the following for substitution in the formulae of Hotine (1957, § 40)

$$\chi m^r = \gamma_1\lambda^r + \gamma_2\mu^r .$$

If we take an equipotential of the gravitational field as base surface ($h = 0$) of § 22 coordinates, than *at a point on this surface* we have from Hotine (1957, § 40)

$$\partial F/\partial h = \partial F/\partial n \tag{44.1}$$

$$\begin{aligned}\partial^2F/\partial h^2 &= \partial^2F/\partial n^2 - F_r(\gamma_1\lambda^r + \gamma_2\mu^r)\\ &= \partial^2F/\partial n^2 - \gamma_1(\partial F/\partial p) - \gamma_2(\partial F/\partial m)\end{aligned} \tag{44.2}$$

and the third differential follows from the substitution of results already given in Hotine (1957, § 38 and § 39). For example,

$$\frac{\partial \log g}{\partial n} = -2H - 2\tilde{\omega}^2/g \text{ [Eq. (16.3), Hotine 1957]}$$

$$\frac{\partial^2 \log g}{\partial n^2} = -\frac{\partial(2H)}{\partial n} + \frac{2\tilde{\omega}^2}{g}(-2H - 2\tilde{\omega}^2/g)$$

$$\frac{\partial^2 \log g}{\partial n^2} = -\frac{\partial(2H)}{\partial n} - 2H\left(\frac{2\tilde{\omega}^2}{g}\right) - \left(\frac{2\tilde{\omega}^2}{g}\right)^2 - \gamma_1^2 - \gamma_2^2$$

[from Eq. (44.2)]. In the last equation, which is true only for a point on the base surface, all quantities on the right have base surface values. Next, we have from Eqs. (34.2) and (34.4) from Hotine (1957)

$$\frac{\partial(2H)}{\partial n} = \frac{\partial \gamma_1}{\partial \varrho} + \frac{\partial \gamma_2}{\partial m} - \chi_1^2 - \chi_2^2 - 2\tau^2 - \gamma_1^2 - \gamma_2^2 + (\gamma_1 \tau - \chi_1 \gamma_2) \tan \phi$$

so that

$$\frac{\partial^2 \log g}{\partial n^2} = (\chi_1^2 + \chi_2^2 + 2\tau^2) - (\gamma_1 \tau - \chi_1 \chi_2) \tan \phi$$
$$-2H\left(\frac{2\tilde{\omega}^2}{g}\right) - \left(\frac{2\tilde{\omega}^2}{g}\right)^2 - \partial \gamma_1/\partial p - \partial \gamma_2/\partial m \ , \qquad (44.3)$$

which is calculable for any base surface if the form of the base surface, and gravity on it, are known. For a spheroidal base surface

$$\chi_1 = 1/\upsilon \ , \quad \chi_2 = 1/\varrho \ , \quad \tau = 0 \ , \quad \gamma_2 = \partial \log g/(\varrho \partial \phi)$$

etc. and the expression reduces to

$$\frac{\partial^2 \log g}{\partial h^2} = \left(\frac{1}{\upsilon^2} + \frac{1}{\varrho^2}\right) + \frac{\tan \phi}{\varrho \upsilon} \frac{\partial \log g}{\partial \phi} - \left(\frac{1}{\upsilon} + \frac{1}{\varrho}\right)\left(\frac{2\tilde{\omega}^2}{g}\right) - \left(\frac{2\tilde{\omega}^2}{g}\right)^2$$
$$- \frac{\partial}{\varrho \partial \phi}\left(\frac{\partial \log g}{\varrho \partial \phi}\right) .$$

Since

$$\frac{\partial^2 \log g}{\partial h^2} = -\left(\frac{\partial \log g}{\partial h}\right)^2 + \frac{1}{g}\frac{\partial^2 g}{\partial h^2} ,$$

we have finally the following expression as far as terms of the second-order for gravity at a geodetic height h:

$$g - h\left\{\left(\frac{1}{\upsilon} + \frac{1}{\varrho}\right) g + 2\tilde{\omega}^2\right\} + \frac{h^2}{2}\left[2g\left(\frac{1}{\upsilon^2} + \frac{1}{\varrho\upsilon} + \frac{1}{\varrho^2}\right)\right.$$
$$\left. + \left(\frac{1}{\upsilon} + \frac{1}{\varrho}\right)(2\tilde{\omega}^2) + \frac{\tan \phi}{\varrho \upsilon}\frac{\partial g}{\partial \phi} - \frac{g}{\varrho}\frac{\partial}{\partial \phi}\left(\frac{1}{\varrho g}\frac{\partial g}{\partial \varrho}\right)\right] , \qquad (44.4)$$

in which g and all spheroidal functions are to have their values on the base spheroid at the foot of the normal ($h = 0$) viz. at a point on the base spheroid having the same geodetic latitude.

We can similarly expand potential and latitude of the line of force ($\bar{\phi}$) as far as necessary, starting with

$$\frac{\partial V}{\partial n} = -g$$

and

$$\frac{\partial \bar{\phi}}{\partial n} = \gamma_2 \;.$$

These expansions will not have to be used often in practice, until the characteristics of the field are required at much greater altitudes than exist on the surface of the Earth, but they do provide yet another illustration of the power of the present theory, which can derive quite general expressions like Eq. (44.3).

Index of Main Symbols

A^r, B^r, C^r	Cartesian vectors, right-handed in that order (see Hotine 1957, § 1)
ω, ϕ	Longitude, latitude of ν^r in relation to A^r, B^r, C^r (two of the three coordinates in all systems)
ν^r	Outward-drawn unit normal to the third-coordinate surface
μ^r	Unit "meridian" vector, lying towards the North in the third-coordinate surface, coplanar with ν^r and C^r
λ^r	Unit "parallel" vector, perpendicular to ν^r and μ^r. λ^r, μ^r, ν^r is a right-handed set in that order
α	Azimuth, measured eastwards from μ^r
β	Zenith distance, measured from ν^r
V	Potential; the third coordinate in the astronomical system or in the standard field geodetic system
h	Geodetic height; the third coordinate in geodesic parallel systems
g	Gravity; physical in the astronomical system; standard or notional in other systems
dp, dm, dn	Elements of arc in directions λ^r, μ^r, ν^r
γ_1	Arc rate of change of (log g) in direction λ^r
γ_2	Arc rate of change of (log g) in direction μ^r
χ_1	Normal curvature of third-coordinate surface in direction λ^r
χ_2	Normal curvature of third-coordinate surface in direction μ^r
τ	Geodesic torsion of third-coordinate surface in direction μ^r
H	Mean curvature of third-coordinate surface $= (\chi_1 + \chi_2)/2$
K	Gauss curvature of third-coordinate surface $= \chi_1\chi_2 - \tau^2$
$\bar{\omega}$	Angular velocity; physical in the astronomical system; notional, same value but anout the axis of symmetry, in the case of the standard field (see § 11 and § 35)
ϱ, ν	Principal curvatures of base spheroid in systems § 12 and § 22
e	Eccentricity of spheroid. $\bar{e}^2 = (1 - e^2)$.

(*Note:* ω, ϕ, α, β etc. of the same point or line etc., will not of course be equal in different coordinate systems. Whenever two coordinate systems are considered together, one is denoted by barred notation, e.g. $\bar{\omega}$, $\bar{\phi}$, $\bar{\alpha}$, $\bar{\beta}$. In some cases, clear from the context, barred notation is also used for a different point in the same coordinate system.)

Editorial Commentary

This report is a direct continuation of the previous report to the Toronto Assembly. It addressed the complicated question of the construction of geodetic coordinate systems, and represented Hotine's first response to the challenge posed by Marussi's ideas on differential geodesy (see Marussi 1988; Zund 1989). As such is represents a preliminary account to the material in Part II of MG, in particular that in Chapters 12–15; and Part IIII of MG, viz. Chapters 22–27.

It furnishes a valuable, albeit admittedly brief account of this material, much of which is reconsidered in more depth and detail in the later two reports *A Primer of Non-Classical Geodesy* (Venice, 1959). and *The Third Dimension in Geodesy* (Helsinki, 1960). Nevertheless, one is pleased to have it, even if it is only of historical interest as indicative of the evolution of Hotine's ideas.

The wealth of material touched on in this report, i.e. not only the construction of geodetic coordinate systems (§§ 1–25), but the adjustment of triangulation in space (§§ 25–34), and standard gravity (§§ 35–44), is rather overpowering. It must have been – at least momentarily – overwhelming even to one of Hotine's ability, since it suggested the magnitude of the challenge awaiting him.

A particularly noteworthy derivation of the equations for the components of the potential and gravity is contained in §§ 37–40. This includes the higher order flattening expressions subsequently given by Cook (1959) and Lambert (1961). The corresponding discussion in MG is contained in Chapter 23.

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5 A Primer of Non-Classical Geodesy[1]

Introduction

1. This paper simplifies and extends some of the contents of two earlier papers[2] presented to the International Geodetic Association at Toronto in 1957. It is intended as a new approach to the basic mathematical theory of geodetic measurement, leading to rigorous methods of reduction and adjustment, unrestricted by the length of observed lines and suited to modern electronic computation. It is submitted as a complete answer to the problem posed to Study Group No. 1 of the International Association of Geodesy (Toronto, 1957), viz. the adjustment of large triangulation networks "taking into account the form of the geoid". The basic material theory contained in this paper shows that it is unnecessary for this purpose to know the form of the geoid at all.

2. The basis of the classical method is to project the observing stations orthogonally onto a spheroid. Observed directions, necessarily measured in relation to the plumb line, are (in some cases) corrected for "geoidal tilt" to derive directions which would be obtained if measurable in relation to the spheroidal normal. The actual line of observation – an optical or radio path – is considered to lie in a plane containing the spheroidal normal, and is projected to the spheroid as a curve of normal section. By applying a correction to the initial direction, this curve is then shifted to the normal section curve passing through the projection of the other station. A different curve in general results from observation at the other end of the line, and the two curves are replaced (through "correction" of the end directions) by the (unique) spheroidal geodesic joining the projected terminals. Precise calculation of the elements of this geodesic is often described as the fundamental problem of geodesy, despite the fact that it has no direct correspondence throughout with the actual line of observation. Geodesic triangles are solved by "correcting" their angles to those of a plane triangle having the same side-lengths. Two coordinates of the observing stations are finally computed either on the spheroid or on a plane-conformal transformation of the spheroid. For purposes of adjustment, astronomical measures, particularly of longitude and azimuth, made at points in physical space are compared with geodetic values derived at different points on the spheroid, without laying down a complete and unequivocal correspondence between the two systems.

[1] Presented at the first symposium on Three-Dimensional Geodesy (Venice 1959).
[2] See papers 3 and 4 in this monograph.

3. Determination of geodetic heights in the classical method receives scant theoretical attention. As a rule, not even corrections for "geoidal tilt" are applied to observed zenith distances, with the result that "trigonometric heights" are given in no defined system independent of the particular line of observation. It is usually assumed that heights should be provided by spirit-levelling. These are no doubt the best heights to show on maps of countries fortunate enough to have a network of precise levels, but they are related more to the astronomical system of coordinates and are not directly related to any possible analytical geodetic system.

4. Measured bases should be reduced to the length of the spheroidal geodesic between their projected terminals if they are to be in harmony with other classically reduced measures. This presupposes a knowledge of heights above the spheroid, which the classical method does not provide. If spirit-levels above mean sea level are used instead, the base length would be wrongly reduced to the general level of the geoid underlying it, not to the spheroid. This leads to an attempt to establish the relative depths of spheroid and geoid by integrating the effect of deflections along a section. The method rests on several not always clearly specified approximations, and assumptions relating to underground densities, whose validity may fairly be questioned. Determination of the form of the geoid in relation to a reference spheroid from gravity measures is another method which does not yet command universal acceptance in detail; it requires worldwide measures, which are not likely to be available for many years, to give a smoothed first-order result.

5. It may be said at once that the classical method does usually give as numerically satisfactory results as the rigorous methods described in this paper, at any rate over the normal side lengths of geodetic triangulation between terrestrial stations. So, at the time, did pre-Copernican methods of astronomical computation give satisfactory numerical results. The classical methods of azimuth and base adjustment have not yet been compared but they too may well be numerically adequate. There can be no doubt, however, that the theoretical basis of the classical method leads to quite unnecessary complication and confusion and should be replaced by more modern conceptions, which are vital to the further development of the subject, particularly the utilization of much longer lines and the incorporation of more frequent astronomical measures and measures of length.

6. The methods developed in this paper, and fully illustrated by application to most geodetic processes, are three-dimensional throughout. A base spheroid appears in the adopted geodetic 3-coordinate system, mainly to ensure results in fair sympathy with antecedent work[3], but it is not used for two-dimensional com-

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[3] For consideration of other possible systems, see paper 4 this Volume, where it is argued that the system adopted in the present paper is the most advantageous. The astronomical system is fully considered in paper 3. On these matters, the earlier papers are still current until they can be replaced.

putation on classical lines. No apology is offered for the inclusion of vector methods in index notation, which are much easier to acquire than to avoid[4].

7. The paper can be read and understood without prior knowledge of vector methods from the following very brief description. The notation l^r does not mean a number raised to the r-th power; it is shorthand for three numbers l^1, l^2, l^3 specifying a length in a certain direction, known as a *vector*, whose three *contravariant components* are l^1, l^2, l^3. If changes dx^r in the three coordinates x^1, x^2, x^3 occur in the direction of the vector, then these changes dx^1, dx^2, dx^3 are proportional to l^1, l^2, l^3 respectively. If the changes occur over a length ds (which may be infinitesimal) of the vector and if $l^r = dx^r/ds$, then l^r is known as a *unit* vector. A vector L^r of magnitude L in the direction l^r would have as its contravariant components Ll^1, Ll^2, Ll^3.

8. A vector l^r also has *covariant components*, denoted by l_r, which are the same as the contravariant components in Cartesian coordinates but are not as easy to define in more general curvilinear coordinates (such as latitude and longitude); wherever necessary their derivation will be illustrated in the text.

9. A product of two vectors L^rM_s is a second-order *tensor* having nine components (e.g. L^1M_1, L^1M_2, L^1M_3, L^2M_1, etc.). If the superscript and subscript are the same, the *summation convention* applies and the three remaining components of the tensor are summed to form a pure number or *scalar*, e.g. $L^rM_r = L^1M_1 + L^2M_2 + L^3M_3$. This is known as the *scalar product* of the two vectors and is equal to the product of their magnitudes and the cosine of the angle between them. If the two vectors are the same, or are parallel and of equal length, then their scalar product is equal to the square of their common magnitude. If they are perpendicular, their scalar product is zero. The scalar product is the same, or *invariant*, whatever the coordinate system associated with the vector components.

10. If a linear relation between vectors at a point in space is true in one coordinate system, then it is also true in any other coordinate system. A linear relation exists between any three *coplanar* vectors. Any vector can be expressed linearly in terms of three other non-coplanar vectors. If l^r, m^r, n^r are three mutually orthogonal (perpendicular) vectors of non-zero length and $Al^r + Bm^r + Cn^r = 0$, then by multiplying across by l_r, we find that $A = 0$, and similarly $B = 0$, $C = 0$.

11. Apologists for the classical methods have suggested that they were deliberately adopted in order to escape the effects of atmospheric refraction, and that this makes them superior to three-dimensional methods. Neither claim is valid. Any uncertainty in the form of the line of observation is bound to affect the same

[4] Two simple books on the subject are (1) McConnell's *Application of the Absolute Differential Calculus* and (2) Weatherburn's *Introduction to Riemannian Geometry and the Tensor Calculus.* The first 50 pages plus Chapter XI of the former are ample for the present application; or the first 40 pages of the latter.

results to the same extent, whatever method of computation is adopted. For instance, the classical method assumes implicitly that the line of observation lies in both normal section planes, which in general is impossible unless it is straight, so that the effect of refraction on the calculation of latitude and longitude is not overcome; it is simply ignored. True, the effect is small, but it is equally small in the three-dimensional method. The effect on heights is more serious but again does not favour one method over the other: equivalent assumptions have to be made in both methods as regards the form of the line, and the only difference between the two methods is that the three-dimensional method makes no other assumptions. Approximations equivalent to, or fewer than, those made or implied in the classical method can be made in the three-dimensional method, which then becomes just as easy to compute. In short, the classical method has no real comparative advantages. Historically, it has evolved from the flat Earth in two Euclidean dimensions to the round Earth in two non-Euclidean dimensions; it has made much use of Gaussian differential geometry of surfaces, but no use of the later extension of Gauss' methods by Ricci and others to three (or more) dimensions. Since Gauss, geodesy and geometry have gone their separate ways, to the disadvantage of geodesy.

Definitions

A diagram (Fig. 1) and list of symbols is provided at the end of the paper.

12. Three mutually orthogonal unit vector fields A^r, B^r, C^r – right-handed in that order – are defined as follows.

C^r at any point in space is parallel to the physical axis of rotation of the Earth; positive direction North.

A^r is parallel to the plane determined by C^r and the plumb line or astronomic zenith at the origin of the survey, and is perpendicular to C^r; positive direction outwards from the centre of the Earth.

B^r completes the triad; positive direction eastwards.

13. These three vector fields A^r, B^r, C^r are common to both the astronomic and geodetic systems of coordinates. In the geodetic system in § 31 C^r is parallel to the minor axis of the base spheroid. The plane A^r, C^r remains parallel to the astronomic meridian at the origin; it is not parallel to the geodetic meridian at the origin unless the astronomic and geodetic longitudes of the origin are made equal. Means of orienting the geodetic system to ensure parallelism of the three fundamental vectors are considered below (§ 24 et seq.).

The three vector fields A^r, B^r, C^r are also the base vectors of a Cartesian system of coordinates whose origin in the geodetic system is the geometrical centre of the base spheroid.

14. A second set of (variable) orthogonal unit vector fields λ^r, μ^r, ν^r – right handed in that order – is defined at any point in space as follows.

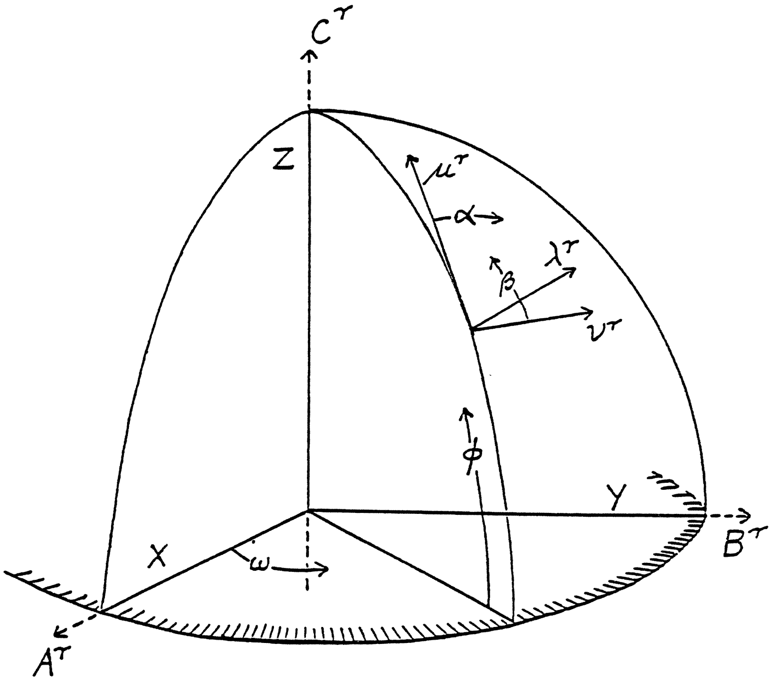

Fig. 1.

υ^r is the direction of the astronomic zenith, or outward-drawn normal to the equipotential surface (or tangent to the line of force).

μ^r in the direction of the astronomic meridian, lies in the equipotential surface and in the plane υ^r, C^r; positive northwards.

λ^r in the direction of the astronomic parallel, lies in the equipotential surface and completes the triad; positive eastwards.

A similar set of vectors, not in general parallel to the corresponding astronomic vectors, is defined in the geodetic system. In this case, υ^r is the straight outward-drawn normal to the base spheroid passing through the point in space under consideration; μ^r is in the plane υ^r, C^r; and λ^r completes the triad. Whenever the two sets are used together, the notation for one specified set will be barred, viz. $\bar{\lambda}^r$, $\bar{\mu}^r$, $\bar{\upsilon}^r$.

15. In both the astronomic and geodetic systems, longitude and latitude are defined as follows, using the appropriate vectors υ^r, μ^r.

Longitude (ω) is the angle between the planes A^r, C^r and υ^r, μ^r, C^r; positive in the positive rotation about C^r, that is from A^r to B^r, or *East.* This differs from the normal astronomic convention (and from some geodetic systems), but is necessary to preserve the ordinary analytical conventions. The sign of longitudes listed as positive to the West should be changed before entering any of the formulae in this paper and should also be abstracted by subtracting the astronomic longitude of the origin. The plane A^r, C^r thus becomes the zero of both astronomic and geodetic longitudes. The *geodetic* longitude of the origin will not necessarily be zero, unless it had been made so during orientation of the geodetic system (§ 24 et seq.).

Latitude (ϕ) is the angle between υ^r and the plane A^r, B^r; positive North. Co-latitude ($\pi/2-\phi$) is the angle between υ^r and C^r.

In the astronomic system, the direction of the meridian (or parallel) as defined in § 14 is not necessarily the same as the horizontal direction in which the longitude (or latitude) remains constant.

16. The third coordinate in the astronomic system will be the *geopotential*. In the geodetic system it will be the *geodetic height* (h), defined as the length of the straight outward-drawn normal to the base spheroid, intercepted between the base spheroid and the point in space under consideration.

Both systems can be shown to be analytical coordinate systems (as distinct from mere reference systems) in the sense of continuous, single-valued, differentiable functions of a Cartesian system in flat 3-space.

17. The third-coordinate surfaces in the astronomic system will be the equipotential or level surfaces, as determined by a spirit-level.

In the geodetic system, the constant-h surfaces are parallel to the base spheroid but are not themselves spheroids. The standard (or geodetic) gravitational field will be defined, apart from constants, as a field rotating about the minor axis of the base spheroid with the same angular velocity as the Earth; the base spheroid to be an equipotential surface of this field. The other equipotential surfaces are neither spheroids nor constant-h surfaces.

18. In both astronomic and geodetic systems, *azimuth* (α) is measured East from North, that is from μ^r towards λ^r, using the vectors appropriate.

A unit space vector in azimuth α and zenith distance β will accordingly be given by

$$l^r = \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta \ , \tag{18.1}$$

which can easily be verified by forming scalar products.

19. The *declination* (D) of a space vector l^r is the angle between l^r and the plane A^r, B^r; positive North. This follows the usual astronomic convention if we consider the vector l^r prolonged to meet the "celestial sphere" in a "star".

The *origin hour angle* (H) of a space vector l^r is the angle between the planes[5] A^r, C^r and l^r, C^r; positive in the positive rotation about C^r, that is from A^r towards B^r, or East. This reverses the sign of the usual astronomic convention. With the normal conventions for right ascension and sidereal time, H is the right ascension of the direction l^r *minus* the local sidereal time at the origin, both expressed in angular measure. The local sidereal time at the origin is the Greenwich sidereal time *plus* the astronomic longitude of the origin relative to Greenwich (measured as stated above positive eastwards).

[5] Although l^r, C^r do not intersect, lines parallel to them define the plane in question.

A unit space vector l^r in declination D and origin hour angle H is given by

$$l^r = A^r \cos H \cos D + B^r \sin H \cos D + C^r \sin D \ . \tag{19.1}$$

Relations Between Fundamental Vectors

20. We denote a general set of coordinates by x^r. If these are either the astronomic or the geodetic system, then they are to be taken in the order x^1 = longitude, x^2 = latitude, x^3 = geodetic height or the geopotential; if they are Cartesian coordinates, $x^1 = x$ measured in the direction A^r; $x^2 = y$ in the direction B^r; $x^3 = z$ in the direction C^r.

x, y, z can be considered as scalar functions of position, in which case such covariant vectors as x_r are, as usual, given by $\partial x/\partial x^r$. The vector equation $x_r = A_r$ is immediately verifiable in Cartesian coordinates and is accordingly true in any coordinate system. Similarly, $y_r = B_r$; $z_r = C_r$.

21. The following covariant vector equations, true in any coordinate system, are verified straight from the definitions by forming scalar produces (e.g. if we multiply the first equation across by B^r, we have $\lambda_r B^r = \cos\omega$ and can easily verify that this is so).

$$\begin{aligned}
\lambda_r &= -A_r \sin\omega + B_r \cos\omega \\
\mu_r &= -A_r \sin\phi \cos\omega - B_r \sin\phi \sin\omega + C_r \cos\phi \\
\upsilon_r &= A_r \cos\phi \cos\omega + B_r \cos\phi \sin\omega + C_r \sin\phi
\end{aligned} \tag{21.1}$$

and the reverse equations:

$$\begin{aligned}
A_r = x_r &= -\lambda_r \sin\omega - \mu_r \sin\phi \cos\omega + \upsilon_r \cos\phi \cos\omega \\
B_r = y_r &= \lambda_r \cos\omega - \mu_r \sin\phi \sin\omega + \upsilon_r \cos\phi \sin\omega \\
C_r = z_r &= \mu_r \cos\phi + \upsilon_r \sin\phi \ .
\end{aligned} \tag{21.2}$$

The corresponding contravariant equations, e.g.

$$\lambda^r = -A^r \sin\omega + B^r \cos\omega \ , \tag{21.3}$$

are obtained by simply raising the indices throughout.

22. In the above equations, λ_r, μ_r, υ_r, ϕ, ω naturally all refer to the same system, astronomic or geodetic. If the corresponding functions in the other system, *but at the same point in space*, are denoted by bars (e.g. $\bar{\lambda}_r$), then we should also have

$$\bar{\lambda}_r = -A_r \sin\bar{\omega} + B_r \cos\omega$$

etc. since the vectors A_r, B_r, C_r are common to both systems; and by eliminating A_r etc. with Eq. (21.2) we have the following relations applicable to a transformation between the two systems

$$\bar{\lambda}_r = -A_r \sin \bar{\omega} + B_r \cos \bar{\omega}$$
$$= \lambda_r \cos(\bar{\omega}-\omega) + \mu_r \sin\phi \sin(\bar{\omega}-\omega) - \nu_r \cos\phi \sin(\bar{\omega}-\omega) \tag{22.1}$$
$$\bar{\mu}_r = -\lambda_r \sin\bar{\phi}\sin(\bar{\omega}-\omega) + \mu_r[\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]$$
$$+\nu_r[\sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)] \tag{22.2}$$
$$\bar{\nu}_r = \lambda_r\cos\bar{\phi}\sin(\bar{\omega}-\omega) + \mu_r[\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]$$
$$+\nu_r[\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)] \; . \tag{22.3}$$

These last equations [(22.1), (22.2), (22.3)] should also be true if the barred quantities refer to the same coordinate system but to a different point in space, since we have merely used the constancy of the vectors A_r, B_r, C_r in order to derive them. Nevertheless, such vector equations apply in general only between the components of vectors defined at the same point in space. Accordingly, we define e.g. $\bar{\lambda}_r$ in the usual way at the barred point and then consider it translated parallel to itself to the unbarred point. Since the Cartesian components of equal and parallel vectors are the same, the vector Eqs. (22.1), (22.2), (22.3) will then hold in Cartesian coordinates at the unbarred point; and therefore, on this interpretation of $\bar{\lambda}_r$ etc. they will also hold at the unbarred point in any other coordinate system, even though the components of parallel vectors at different points are not the same in other coordinate systems. In the same way, we could consider λ_r etc. as defined in the usual way at the unbarred point and then translated parallel to themselves to the barred point; in which case the vector equations would hold in any coordinate system at the barred point.

23. If the Cartesian components of a unit vector l^r – in azimuth α, zenith distance β, declination D, origin hour angle H – are (a, b, c) then

$$l^r = aA^r + bB^r + cC^r \; , \tag{23.1}$$

and combining this with Eqs. (19.1) and (18.1), after substituting in the latter the contravariant form of (21.1), we have

$$\cos H \cos D = a = -\sin\phi\cos\omega\cos\alpha\sin\beta - \sin\omega\sin\alpha\sin\beta$$
$$+\cos\phi\cos\omega\cos\beta \; .$$
$$\sin H\cos D = b = -\sin\phi\sin\omega\cos\alpha\sin\beta + \cos\omega\sin\alpha\sin\beta \tag{23.2}$$
$$+\cos\phi\sin\omega\cos\beta \; .$$
$$\sin D = c = \cos\phi\cos\alpha\sin\beta + \sin\phi\cos\beta \; .$$

Conversely, by substituting Eqs. (21.2) in (23.1) and comparing the result with § 18 we have

$$\sin\alpha\sin\beta = -a\sin\omega + b\cos\omega$$
$$= \cos D\sin(H-\omega)$$
$$\cos\alpha\sin\beta = -a\sin\phi\cos\omega - b\sin\phi\sin\omega + c\cos\phi \tag{23.3}$$
$$= \sin D\cos\phi - \sin\phi\cos D\cos(H-\omega)$$

$$\cos\beta = a\cos\phi\cos\omega + b\cos\phi\sin\omega + c\sin\phi$$
$$= \sin D\sin\phi + \cos\phi\cos D\cos(H-\omega)\ .$$

These equations [(23.2) and (23.3)] clearly hold if α, β, ω, ϕ all refer either to the astronomic or to the geodetic system; since a, b, c, H, D are common to both systems. If we denote the other system by bars, we have, for example

$$\sin\bar{\alpha}\sin\bar{\beta} = -a\sin\bar{\omega} + b\cos\bar{\omega}$$

from Eq. (23.3) and substituting for a, b, c from (23.2) we have the following transformation equations:

$$\begin{aligned}
\sin\bar{\alpha}\sin\bar{\beta} &= \sin\alpha\sin\beta\cos(\bar{\omega}-\omega) + \cos\alpha\sin\beta\sin\phi\sin(\bar{\omega}-\omega)\\
&\quad -\cos\beta\cos\phi\sin(\bar{\omega}-\omega)\ ,\\
\cos\bar{\alpha}\sin\bar{\beta} &= -\sin\alpha\sin\beta\sin\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\quad +\cos\alpha\sin\beta[\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&\quad +\cos\beta[\sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\\
\cos\bar{\beta} &= \sin\alpha\sin\beta\cos\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\quad +\cos\alpha\sin\beta[\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&\quad +\cos\beta[\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\ .
\end{aligned} \tag{23.4}$$

Moreover, these equations [(23.4)] will also hold for a vector parallel to l^r at another point in the same space, denoted by bars, since a, b, c are the same for such a parallel vector. In particular, they hold for the straight line joining the unbarred and barred points and enable us to compute azimuth and zenith distance at the far end of such a line. But is should be noted that $\bar{\alpha}$, $\bar{\beta}$ refer to the same sense of the line as α, β; $\bar{\beta}$ should be subtracted from 180° and 180° should be added to $\bar{\alpha}$, if they are to refer to the back direction. The equations are true as between barred and unbarred points on the same straight line if $\bar{\alpha}$, $\bar{\beta}$, α, β, $\bar{\omega}$, $\bar{\phi}$, ω, ϕ all refer either to the astronomic or to the geodetic system, regardless of irregularities or anomalies affecting the plumb line.

Only two of the equations in each set [(23.2), (23.3) and (23.4)] are independent, since $a^2+b^2+c^2=1$; the third equation provides a check.

Origin Conditions

24. We now consider means of ensuring that the Cartesian vectors A_r, B_r, C_r are parallel in the astronomic and geodetic systems, since all the analysis so far depends on this.

It is evident that the Eqs. (23.4), considered as transformation equations from the astronomic system (unbarred) to the geodetic system (barred), must necessarily be satisfied at the origin, as at any other point in space. If we adopt measured astronomic values for α, β, ϕ, ω then we must choose initial geodetic values $\bar{\alpha}$, $\bar{\beta}$, $\bar{\phi}$, $\bar{\omega}$ to satisfy Eq. (23.4). The Cartesian components of the vector l^r (an actual physical direction in space between survey stations) will then be equal in the two systems. [This can easily be verified by forming the geodetic (barred) equations corresponding to (23.2) and using (23.4) to prove that $a = \bar{a}$ etc.] But this alone is not sufficient to ensure parallelism of the Cartesian vectors in the two systems, since it would still be possible to rotate the geodetic system complete with its Cartesian vectors about l^r. To ensure parallelism we must also equate the Cartesian components of a second direction.

25. This question is of such importance as to justify alternative and less intuitive consideration. Suppose that the Cartesian vectors of the geodetic system are *not* parallel to those of the astronomic system, but are denoted by $\bar{A}^r$ etc. yet nevertheless the Cartesian components of the unit vector l^r are the same in both systems. Then

$$l^r = aA^r + bB^r + cC^r = a\bar{A}^r + b\bar{B}^r + c\bar{C}^r$$

or

$$a(A^r - \bar{A}^r) + b(B^r - \bar{B}^r) + c(C^r - \bar{C}^r) = 0 \ ,$$

which merely requires the three vectors $(A^r - \bar{A}^r)$ etc. to be coplanar (and incidentally perpendicular to l^r, since $l^r(A_r - \bar{A}_r) = l^r A_r - l^r \bar{A}_r = a - a = 0$ etc.). But if, in addition, the Cartesian components (a^1, b^1, c^1) of a second, independent unit vector are the same in both systems, then we must also have

$$a^1(A^r - \bar{A}^r) + b^1(B^r - \bar{B}^r) + c_1(C^r - \bar{C}^r) = 0 \ ,$$

so that *either* $a/a^1 = b/b^1 = c/c^1$, in which case the second vector would coincide with the first *or* the three vectors $(A^r - \bar{A}^r)$ etc. are null and $A^r = \bar{A}^r$, $B^r = \bar{B}^r$, $C^r = \bar{C}^r$.

26. The difference between the astronomic and geodetic systems will usually be small. If we write $\bar{\phi} = (\phi + \delta\phi)$ etc. then to a first order the Eqs. (23.4) reduce to:

$$\delta\alpha = \sin\phi\,\delta\omega + \cos\beta\,(\sin\alpha\,\delta\phi - \cos\alpha\,\cos\phi\,\delta\omega) \tag{26.1}$$

$$\delta\beta = -\cos\phi\,\sin\alpha\,\delta\omega - \cos\alpha\,\delta\phi \ .$$

If, in addition, β is nearly 90°, then the first equation reduces to $\delta\alpha = \sin\phi\,\delta\omega$. This is the so-called Laplace azimuth equation, which alone is used in the classical method to orient the spheroid. It is probably sufficiently accurate in terrestrial triangulation to satisfy the first equation, but the classical method makes no attempt to satisfy the second equation, much less to do so for the necessary two distinct lines radiating from the origin.

The extra conditions in the present method do not arise from any special requirement in orienting the geodetic system. If astronomic and geodetic measures are to be used together in calculating and adjusting an extensive survey, then the relation between the two systems must be specified completely and the present specification is no more and no less exacting than is necessary for the purpose.

It may be objected that the second of the two equations (26.1) is vitiated by errors of atmospheric refraction in the measured (astronomic) zenith distances and ought not therefore to be used. Indeed, there is everything to be said for careful reciprocal observation of these angles over several lines to minimize the effect of refraction, but there is nothing to be said for omitting these observations altogether. We do not overcome the difficulty by simply ignoring the second equations of (26.1) without making any attempt to satisfy them.

27. The simplest way of satisfying the origin conditions is to accept, as the geodetic starting elements, astronomic measures of latitude, longitude and azimuth, plus two measured zenith distances corrected for refraction. This procedure will in effect have been adopted, without full consideration of the underlying theory, in choosing the origin conditions for some triangulations. In other cases, where the influence of vertical angles may have been ignored and where perhaps not even the Laplace azimuth equation has been satisfied at the origin, the matter should be investigated before the results are extended or further adjusted in conjunction with astronomic measures.

Means of correcting for refraction and of minimizing the effect of observational error are considered below in the section on triangulation adjustment.

28. The initial geodetic height may be assumed or be taken as a spirit-levelled height of the origin. It will settle the position of the base spheroid in relation to the actual point on the Earth's surface taken as origin. Whatever initial height is taken for the origin will not, however, affect the orientation of the spheroid and will therefore be without effect on any of the preceding formulae.

Differentiation of the Fundamental Vectors

29. Throughout sections § 29 and § 30, we assume that we are working in Cartesian coordinates, so that the components of the vectors A^r, B^r, C^r have the same values at all points in space. Accordingly, small changes in the components of the vector λ^r, during which the components of A^r and B^r remain constant, are from Eq. (21.3) given by

$$d(\lambda^r) = -(A^r \cos\omega + B^r \sin\omega)\, d\omega$$

and by substituting the contravariant forms of Eq. (21.2), this becomes

$$d(\lambda^r) = (\mu^r \sin\phi - \upsilon^r \cos\phi)\, d\omega \ . \tag{29.1}$$

Similarly,

$$d(\mu^r) = -\lambda^r \sin\phi\, d\omega - \upsilon^r d\phi \tag{29.2}$$

and

$$d(\upsilon^r) = \lambda^r \cos\phi\, d\omega + \mu^r d\phi \ . \tag{29.3}$$

If in the same way and using these last results we differentiate a general unit vector l^r in azimuth α and zenith distance β given by Eq. (18.1), that is

$$l^r = \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \upsilon^r \cos\beta \ , \tag{29.4}$$

we have

$$\begin{aligned} d(l^r) = {} & (\lambda^r \cos\alpha \sin\beta - \mu^r \sin\alpha \sin\beta)\,ds \\ & + (\lambda^r \sin\alpha \cos\beta + \mu^r \cos\alpha \cos\beta - \upsilon^r \sin\beta)\,d\beta \\ & + \sin\alpha \sin\beta(\mu^r \sin\phi - \upsilon^r \cos\phi)\,d\omega \\ & - \cos\alpha \sin\beta(\lambda^r \sin\phi\, d\omega + \upsilon^r d\phi) \\ & + \cos\beta(\lambda^r \cos\phi\, d\omega + \mu^r d\phi)\ . \end{aligned} \tag{29.5}$$

Denote a unit vector in azimuth α, zenith distance $(\pi/2+\beta)$ by m^r so that

$$m^r = \lambda^r \sin\alpha \cos\beta + \mu^r \cos\alpha \cos\beta - \upsilon^r \sin\beta\ ; \tag{29.6}$$

and a unit vector in azimuth $(3\pi/2+\alpha)$, zenith distance $\pi/2$ by

$$n^r = -\lambda^r \cos\alpha + m^r \sin\alpha\ . \tag{29.7}$$

The vectors l^r, m^r, n^r constitute a right-handed orthogonal set in that order. Substituting in (29.5) and simplifying, we have finally

$$\begin{aligned} d(l^r) = {} & m^r\{d\beta + \cos\phi \sin\alpha\, d\omega + \cos\alpha\, d\phi\} \\ & + n^r\{-\sin\beta\, d\alpha + (\sin\phi \sin\beta - \cos\phi \cos\alpha \cos\beta)\,d\omega + \sin\alpha \cos\beta\, d\phi\}\ . \end{aligned} \tag{29.8}$$

This equation shows that $d(l^r)$ is a vector in the plane of m^r, n^r and therefore at right-angles to l^r. The equation holds whether the changes in ω, ϕ, α, β are due to a change in position or a change from the astronomic to the geodetic system. Note that if there is no change in the vector l^r despite changes in ω, ϕ, α, β then the coefficients of m^r, n^r must both be zero and we have Eq. (26.1).

30. Next suppose that l^r is expressed in terms of declination D and origin hour angle H as Eq. (19.1), that is

$$l^r = A^r \cos H \cos D + B^r \sin H \cos D + C^r \sin D \tag{30.1}$$

and write

$$M^r = A^r \cos H \sin D + B^r \sin H \sin D - C^r \cos D \tag{30.2}$$

for a unit vector in origin hour angle H, declination $(D-\pi/2)$; also

$$N^r = -A^r \sin H + B^r \cos H \tag{30.3}$$

for a unit vector in origin hour angle $(\pi/2+H)$, declination zero. Here again l^r, M^r, N^r form a right-handed orthogonal set in that order and by differentiation of (30.1) we have at once, in view of the constancy of A^r, B^r, C^r,

$$d(l^r) = -M^r dD + N^r \cos D\, dH\ . \tag{30.4}$$

The Geodetic Coordinate System

31. The principal radii of curvature of the base spheroid are as usual denoted by ϱ, the meridian curvature; and υ, the length of the normal intercepted by the

minor axis. From the ordinary geometry of the elliptic meridian section of eccentricity e, we know that the normal strikes the minor axis at a distance $(\upsilon e^2 \sin\phi)$ below the centre of the spheroid, which we take as the origin of Cartesian coordinates (x, y, z). If the normal is extended a distance h, the geodetic height, above the base spheroid then we have at once:

$$\begin{aligned} x &= (\upsilon + h)\cos\phi\cos\omega \\ y &= (\upsilon + h)\cos\phi\sin\omega \\ z &= (\upsilon + h)\sin\phi - \upsilon e^2 \sin\phi \\ &= (\upsilon\bar{e}^2 + h)\sin\phi \ , \end{aligned} \tag{31.1}$$

writing $\bar{e}^2$ for $(1-e^2)$.

32. The *position vector* ϱ^r whose Cartesian components are (x, y, z) is evidently given by the following vector equation:

$$\begin{aligned} \varrho^r &= xA^r + yB^r + zC^r \\ &= (\upsilon + h)\upsilon^r - (\upsilon e^2 \sin\phi)C^r \\ &= (h + a^2/\upsilon)\upsilon^r - (\upsilon e^2 \sin\phi\cos\phi)\mu^r \ . \end{aligned} \tag{32.1}$$

33. Next, we find the components of the fundamental vectors λ^r, μ^r, υ^r in this system taken in the order $(\omega, \phi, h) = (1, 2, 3)$. The longitude only will vary in the direction λ^r along which an element of length is $(\upsilon + h)\cos\phi\, d\omega$, so that the only non-zero component of λ^r is $d\omega/ds = 1/\{(\upsilon + h)\cos\phi\}$. Proceeding in the same way, we find that the only non-zero contravariant components of the triad are

$$\begin{aligned} \lambda^1 &= 1/\{(\upsilon + h)\cos\phi\} \\ \mu^2 &= 1/(\varrho + h) \\ \upsilon^3 &= 1 \ . \end{aligned} \tag{33.1}$$

By evaluating such expressions as $\lambda^r\lambda_r = 1, \mu^r\lambda_r = 0, \upsilon^r\lambda_r = 0$, we find that the only non-zero covariant components are

$$\begin{aligned} \lambda_1 &= (\upsilon + h)\cos\phi \\ \mu_2 &= (\varrho + h) \\ \upsilon_3 &= 1 \ . \end{aligned} \tag{33.2}$$

The metric of the space follows as

$$ds^2 = (\upsilon + h)\cos^2\phi\, d\omega^2 + (\varrho + h)^2 d\phi^2 + dh^2 \ . \tag{33.3}$$

34. By substitution in Eq. (21.2), we find that the components of the Cartesian vectors are:

$$
\begin{aligned}
A^r &= \left(-\frac{\sin\omega}{(\nu+h)\cos\phi};\ -\frac{\sin\phi\cos\omega}{(\varrho+h)};\ \cos\phi\cos\omega\right)\\
B^r &= \left(\frac{\cos\omega}{(\nu+h)\cos\phi};\ -\frac{\sin\phi\sin\omega}{(\varrho+h)};\ \cos\phi\sin\omega\right)\\
C^r &= \left(0;\ \frac{\cos\phi}{(\varrho+h)};\ \sin\phi\right)
\end{aligned}
\tag{34.1}
$$

$$
\begin{aligned}
A_r &= (-(\nu+h)\cos\phi\sin\omega;\ -(\varrho+h)\sin\phi\cos\omega;\ \cos\phi\cos\omega)\\
B_r &= ((\nu+h)\cos\phi\cos\omega;\ -(\varrho+h)\sin\phi\sin\omega;\ \cos\phi\sin\omega)\\
C_r &= (0;\ (\varrho+h)\cos\phi;\ \sin\phi)\ .
\end{aligned}
\tag{34.2}
$$

35. From Eq. (18.1), we find that the components of a unit vector l^r in azimuth α, zenith distance β are:

$$
\begin{aligned}
l^r &= \left(\frac{\sin\alpha\sin\beta}{(\nu+h)\cos\phi};\ \frac{\cos\alpha\sin\beta}{(\varrho+h)};\ \cos\beta\right)\\
l_r &= [(\nu+h)\cos\phi\sin\alpha\sin\beta;\ (\varrho+h)\cos\alpha\sin\beta;\ \cos\beta]\ .
\end{aligned}
\tag{35.1}
$$

36. We shall need the covariant components of a parallel vector, which are the same in Cartesian coordinates but not in geodetic coordinates, at a different point in space. If L_r is a unit vector in azimuth A, zenith distance B at an unbarred point, then Eq. (35.1) combined with (23.4) shows that the components of the parallel vector at a barred point are as follows:

$$
\begin{aligned}
\bar{L}_1 = (\bar{\nu}+\bar{h})\cos\bar{\phi}\{&\sin A\sin B\cos(\bar{\omega}-\omega)\\
&+\cos A\sin B\sin\phi\sin(\bar{\omega}-\omega)\\
&-\cos B\cos\phi\sin(\bar{\omega}-\omega)\}
\end{aligned}
\tag{36.1}
$$

$$
\begin{aligned}
\bar{L}_2 = (\bar{\varrho}+\bar{h})\{&-\sin A\sin B\sin\bar{\phi}\sin(\bar{\omega}-\omega)\\
&+\cos A\sin B[\cos\phi\cos\bar{\phi}+\sin\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&+\cos B[\sin\phi\cos\bar{\phi}-\cos\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\}
\end{aligned}
\tag{36.2}
$$

$$
\begin{aligned}
\bar{L}_3 = {}&\sin A\sin B\cos\bar{\phi}\sin(\bar{\omega}-\omega)\\
&+\cos A\sin B[\cos\phi\sin\bar{\phi}-\sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&+\cos B[\sin\phi\sin\bar{\phi}+\cos\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\ .
\end{aligned}
\tag{36.3}
$$

The Straight Line in Space

37. If (a, b, c) are the Cartesian components of the unit vector l^r in the direction of the line from (x, y, z) to $(\bar{x}, \bar{y}, \bar{z})$ and s is the length of the line, then

$$(\bar{x}-x) = s\,a$$
$$(\bar{y}-y) = s\,b \tag{37.1}$$
$$(\bar{z}-z) = s\,c\ .$$

Substitute in (23.3) and use (31.1). Then we have after slight simplification,

$$\begin{aligned} s \sin\alpha \sin\beta &= -(\bar{x}-x)\sin\omega + (\bar{y}-y)\cos\omega \\ &= (\bar{\upsilon}+h)\cos\bar{\phi}\sin(\bar{\omega}-\omega) \end{aligned}$$

$$\begin{aligned} s\cos\alpha\sin\beta &= -(\bar{x}-x)\sin\phi\cos\omega-(\bar{y}-y)\sin\phi\sin\omega+(\bar{z}-z)\cos\phi \\ &= (\bar{\upsilon}+h)\{\sin\bar{\phi}\cos\phi-\cos\bar{\phi}\sin\phi\cos(\bar{\omega}-\omega)\} \\ &\quad -e^2\cos\phi(\bar{\upsilon}\sin\bar{\phi}-\upsilon\sin\phi) \end{aligned} \tag{37.3}$$

$$\begin{aligned} s\cos\beta &= (\bar{x}-x)\cos\phi\cos\omega+(\bar{y}-y)\cos\phi\sin\omega+(\bar{z}-z)\sin\phi \\ &= (\bar{\upsilon}+\bar{h})\{\sin\bar{\phi}\sin\phi+\cos\bar{\phi}\cos\phi\cos(\bar{\omega}-\omega)\} \\ &\quad -(\upsilon+h)-e^2\sin\phi(\bar{\upsilon}\sin\bar{\phi}-\upsilon\sin\phi)\ . \end{aligned} \tag{37.4}$$

These three equations directly determine the length, azimuth and zenith distance of a line whose end coordinates are given. The azimuth ($\bar{\alpha}$) and zenith distance ($\bar{\beta}$) at the other end of the line (produced through the barred point, not the back azimuth and back zenith distance) are obtained by interchanging the bars and changing the sign of s, since the length will now be measured in the direction opposite to l^r: e.g. Eq. (37.2) becomes

$$s\sin\bar{\alpha}\sin\bar{\beta} = (\upsilon+h)\cos\phi\sin(\bar{\omega}-\omega)$$

from which, by combining with Eq. (37.2) and (35.1), we infer that

$$(\upsilon+h)\cos\phi\sin\alpha\sin\beta = l_1 \tag{37.5}$$

is constant along the line.

38. The inverse problem, of finding the coordinates of the barred point given those of the unbarred together with the length and initial azimuth and zenith distance, is solved at once from Eq. (23.2). By taking a temporary origin for geodetic longitude at the unbarred point, we have

$$(\bar{\upsilon}+\bar{h})\cos\bar{\phi}\cos(\bar{\omega}-\omega) = (\upsilon+h)\cos\phi - s\sin\phi\cos\alpha\sin\beta + s\cos\phi\cos\beta \tag{38.1}$$

$$(\bar{\upsilon}+\bar{h})\cos\bar{\phi}\sin(\bar{\omega}-\omega) = s\sin\alpha\sin\beta \tag{38.2}$$

$$(\bar{e}^2\bar{\upsilon}+\bar{h})\sin\bar{\phi} = (\bar{e}^2\upsilon+h)\sin\phi + s\cos\phi\cos\alpha\sin\beta + s\sin\phi\cos\beta\ . \tag{38.3}$$

The difference of longitude follows at once by division of the first two equations. We then have to solve by iteration

$$(\upsilon+h)\cos\bar{\phi} = A$$

and

$$(\bar{e}^2\bar{\upsilon}+\bar{h})\sin\bar{\phi} = B$$

for the latitude and height, starting with an approximate latitude given by $\bar{e}^2 \tan \bar{\phi}$ B/A.

The Plane Triangle in Space

39. For the present we assume that deflections are zero at the vertices of the triangle, that is, the geodetic normal coincides with the plumb line at each vertex.

If l^r, $\bar{l}^r$ are the unit vectors in the directions of two adjacent sides of a plane triangle (astronomical azimuths and zenith distances α, β; $\bar{\alpha}, \bar{\beta}$), then the included angle L is given by Eq. (35.1) as

$$\begin{aligned} \cos L &= l^r \bar{l}_r \\ &= \sin\alpha \sin\beta \sin\bar{\alpha} \sin\bar{\beta} + \cos\alpha \sin\beta \cos\bar{\alpha} \sin\bar{\beta} + \cos\beta \cos\bar{\beta} \\ &= \cos\beta \cos\bar{\beta} + \sin\beta \sin\bar{\beta} \cos(\bar{\alpha} - \alpha) \ , \end{aligned} \tag{39.1}$$

in which $(\bar{\alpha} - \alpha)$ is simply the "horizontal" angle as measured between the two directions. The other angles of the triangle are found similarly. One side of the triangle will always be known by direct measurement or from antecedent work and the other sides can next be computed from the ordinary sine formula. In the case of a triangle containing the origin of a triangulation, the geodetic position of the origin together with a geodetic azimuth and two zenith distances will have been adopted to satisfy the conditions in § 24 et seq. and we can now compute the coordinates and azimuths at the other vertices of the triangle from the formulae in the last section. In all other cases we shall know the geodetic coordinates etc. of two vertices from antecedent work and can compute the coordinates etc. of the third point from either of these known points.

40. If angular measurements have been made at all three vertices, then before computing the sides, we should adjust the sum of the angles L to 180° by subtracting one-third of the excess from each. This, of course, has nothing to do with the "spherical excess" of the classical theory and the one-third rule arises simply from the fact that it is the most probable distribution of error among angles supposed to be measured with equal precision at the three vertices. In ordinary terrestrial triangulation, the β's are so nearly 90° that the resulting correction to the angles of the space triangle can be thrown straight into the difference of geodetic azimuths before calculating geodetic coordinates etc. round the triangle.

41. Observations are actually made, not along straight lines or the sides of plane triangles in space, but tangential to slightly curved optical paths refracted by the atmosphere. Before applying the formulae in this and the preceding section, we should accordingly add to observed zenith distances the angle or refraction as computed from whatever data may be available. Any error or defect of data in this operation will not affect the final calculation of geodetic coordinates any more than it does in the classical theory. The question is considered further in the section on triangulations adjustment below, where improved means of solving triangles are proposed in § 53 and the effect of refraction considered in more

detail in § 49. This more advanced method also enables full account to be taken of deflections at the vertices.

Variation of Geodetic Coordinates

42. We now proceed to find the changes in length, azimuth and zenith distance resulting from a change in the end coordinates of the straight line whose unit vector is l^r.

If for the moment we suppose we are working in Cartesian coordinates x^r, ($\bar{x}^r$ at the far end of the line), then

$$s\, l^r = \bar{x}^r - x^r \ ,$$

and differentiating this we have

$$s\, d(l^r) + l^r ds = d\bar{x}^r - dx^r \ . \tag{42.1}$$

Now we know from Eq. (29.8) that $d(l^r)$ is a vector perpendicular to l^r, so that by forming the scalar product of Eq. (42.1) with l_r we have

$$ds = (d\bar{x}^r - dx^r)\, l_r = \bar{l}_r d\bar{x}^r - l_r dx^r \ , \tag{42.2}$$

an invariant equation true in any coordinates provided we interpret $\bar{l}_r$ as a vector parallel to l_r at the barred point and take out its components in the same coordinate system. Using (35.1) and (37.5), the full equation in geodetic coordinates becomes

$$\begin{aligned} ds = {} & (\nu + h) \cos\phi \sin\alpha \sin\beta (d\bar{\omega} - \omega) \\ & + (\bar{\varrho} + \bar{h}) \cos\bar{\alpha} \sin\bar{\beta}\, d\bar{\phi} - (\varrho + h) \cos\alpha \sin\beta\, d\phi \\ & + \cos\bar{\beta}\, d\bar{h} - \cos\beta\, dh \ . \end{aligned}$$

In this equation, $\bar{\alpha}, \bar{\beta}$ refer, as usual, to the prolongation of the line through the barred point.

43. Eliminating ds between Eqs. (42.2) and (42.1), we have

$$\begin{aligned} s\, d(l^r) &= (d\bar{x}^r - dx^r) - (d\bar{x}^s - dx^s)\, l_s l^r \\ &= (d\bar{x}^s - dx^s)(\delta^r_s - l_s l^r) \\ &= (d\bar{x}^s - dx^s)(m_s m^r + n_s n^r) \ , \end{aligned} \tag{43.1}$$

where δ^r_s is the Kronecker delta, which is unity if $r = s$ and is otherwise zero. If l_r, m_r, n_r are *any* mutually orthogonal triad of unit vectors which can be taken as temporary Cartesian axes, then the tensor equation

$$\delta^r_s = l_s l^r + m_s m^r + n_s n^r$$

is easily verified in Cartesian coordinates and is therefore true in any coordinates. In this case we take m^r, n^r as the same vectors as (29.6) and (29.7), substitute (43.1) in (29.8), equate coefficients of m^r, n^r and find that

$$s\,d\beta = \bar{m}_s d\bar{x}^s - m_s dx^s - s\cos\phi\sin\alpha\,d\omega - s\cos\alpha\,d\phi \tag{43.2}$$

$$s\sin\beta\,d\alpha = -\bar{n}_s d\bar{x}^s + n_s dx^s + s(\sin\phi\sin\beta - \cos\beta\cos\phi\cos\alpha)d\omega + s\cos\beta\sin\alpha\,d\phi\ . \tag{43.3}$$

Here again, $\bar{m}_s, \bar{n}_s$ must be interpreted as parallel to m_s, n_s respectively but drawn through the barred point, and their components in the geodetic system taken out accordingly. Substituting the appropriate azimuths and zenith distances in § 36 (for $m_s, A = \alpha, B = \pi/2$; for $n_s, A = 3\pi/2 + \alpha, B = \pi/2$), we have

$$\begin{aligned}
\bar{m}_1 &= (\bar{\nu}+\bar{h})\cos\phi\{\sin\alpha\cos\beta\cos(\bar{\omega}-\omega)+\cos\alpha\cos\beta\sin\phi\sin(\bar{\omega}-\omega)\\
&\qquad +\sin\beta\cos\phi\sin(\bar{\omega}-\omega)\}\quad\text{or}\\
&= (\bar{\nu}+\bar{h})\cos\bar{\phi}\{\sin\bar{\alpha}\sin\bar{\beta}\cos\beta+\cos\phi\csc\beta\sin(\bar{\omega}-\omega)\}\quad\text{from (23.4)}\\
&= (\nu+h)\cos\phi\sin\alpha\cos\beta + s\cos\phi\sin\alpha\quad\text{from (37.5) and (37.2)}\\
&= m_1 + s\cos\phi\sin\alpha\quad\text{from § 35}
\end{aligned}$$

$$\begin{aligned}
\bar{m}_2 &= (\bar{\rho}+\bar{h})\{-\sin\alpha\cos\beta\sin\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\qquad +\cos\alpha\cos\beta[\cos\phi\cos\bar{\phi}+\sin\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&\qquad -\sin\beta[\sin\phi\cos\bar{\phi}-\cos\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]\}\quad\text{or}\\
&= (\bar{\rho}+\bar{h})\{\cos\bar{\alpha}\sin\bar{\beta}\cot\beta-\sin\phi\cos\bar{\phi}\csc\beta\\
&\qquad +\cos\phi\sin\bar{\phi}\csc\beta\cos(\bar{\omega}-\omega)\}\quad\text{from (23.4)}
\end{aligned}$$

$$\begin{aligned}
\bar{m}_3 &= \sin\alpha\cos\beta\cos\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\qquad +\cos\alpha\cos\beta[\cos\phi\sin\bar{\phi}-\sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\\
&\qquad -\sin\beta[\sin\phi\sin\bar{\phi}+\cos\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]\quad\text{or}\\
&= \cos\bar{\beta}\cot\beta-\cos\phi\cos\bar{\phi}\csc\beta\cos(\bar{\omega}-\omega)-\sin\phi\sin\bar{\phi}\csc\beta
\end{aligned}$$

$$m_1 = (\nu+h)\cos\phi\sin\alpha\cos\beta$$

$$m_2 = (\rho+h)\cos\alpha\cos\beta$$

$$m_3 = -\sin\beta$$

$$\begin{aligned}
\bar{n}_1 &= (\bar{\nu}+\bar{h})\cos\bar{\phi}\{-\cos\alpha\cos(\bar{\omega}-\omega)+\sin\alpha\sin\phi\sin(\bar{\omega}-\omega)\}\quad\text{or}\\
&= -(\nu+h)\cos\phi\cos\alpha + s\sin\phi\sin\beta - s\cos\beta\cos\phi\cos\alpha\\
&\qquad \text{using (38.1) and (38.2)}\\
&= n_1 + s\sin\phi\sin\beta - s\cos\beta\cos\phi\cos\alpha\ .
\end{aligned}$$

$$\begin{aligned}
\bar{n}_2 &= (\bar{\rho}+\bar{h})\{\cos\alpha\sin\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\qquad +\sin\alpha[\cos\phi\cos\bar{\phi}+\sin\phi\sin\bar{\phi}\cos(\bar{\omega}-\omega)]
\end{aligned}$$

$$\begin{aligned}
\bar{n}_3 &= -\cos\alpha\cos\bar{\phi}\sin(\bar{\omega}-\omega)\\
&\qquad +\sin\alpha[\cos\phi\sin\bar{\phi}-\sin\phi\cos\bar{\phi}\cos(\bar{\omega}-\omega)]
\end{aligned}$$

$$n_1 = -(\nu + h)\cos\phi\cos\alpha$$
$$n_2 = (\rho + h)\sin\alpha$$
$$n_3 = 0\ .$$

As a check, note that

$$\frac{(\bar{m}_1)^2}{(\bar{\nu}+\bar{h})^2\cos^2\bar{\phi}} + \frac{(\bar{m}_2)^2}{(\bar{\rho}+\bar{h})^2} + (\bar{m}_3)^2 = 1$$

and similarly for the components of the other vectors.

44. We could form similar equations giving the result of changes in the Cartesian coordinates (x, y, z) provided that the Cartesian components of the corresponding vectors are used. These components can be calculated from Eq. (23.2) using the appropriate azimuth and zenith distance of the vector; and any parallel vector at another point will have the same Cartesian components. Covariant and contravariant components will also be the same in Cartesian coordinates. Thus, for example, Eq. (42.2) expanded in Cartesian coordinates would be

$$ds = a(d\bar{x} - dx) + b(d\bar{y} - dy) + c(d\bar{z} - dz)\ .$$

The coordinates dx, $d\bar{x}$ etc. can still be varied quite independently at the two ends of the line, but will have the same coefficients; whereas in geodetic coordinates the coefficients of e.g. $d\phi$, $d\bar{\phi}$ are not the same (see § 48).

Adjustment of Triangulation and Traverse

45. We start with approximate geodetic positions (ω, ϕ, h) of the stations computed as in § 39 and § 53 and use these to compute accurately $s, \alpha, \beta, \bar{\alpha}, \bar{\beta}$ for each direction. Note once again that $\bar{\alpha}, \bar{\beta}$ refer to the same direction produced through the far station and not the back azimuth etc. We are then able to compute the coefficients of Eqs. (43.2) and (43.3), giving the changes in α and β resulting from corrections $ds^s, d\bar{x}^s$ (as yet unknown) which are to be applied to the initial approximate coordinates. In forming the equations for the reverse directions, we take $(180° + \bar{\alpha})$ and $(180° - \bar{\beta})$ as the initial unbarred α, β; and $(180° + \alpha)$, $(180° - \beta)$ as the far (barred) $\bar{\alpha}, \bar{\beta}$. The formulae then remain exactly the same, whereas confusion could result from writing down separate formulae applicable to the reverse direction[6].

46. Next we apply further changes to α, β given by Eq. (26.1) in order to transform to the astronomical system, so that we may have azimuths and zenith distances measured, as they are in practice, in relation to the plumb line. In these equations $\delta\omega, \delta\phi$ are the *deflection* at the station from which the direction is observed, with the sign (astronomic minus geodetic) longitude or latitude. Again note that longitude is positive eastwards. We may, or may not yet know, these deflections.

[6] For instance, new initial vectors m_r, n_r at what was the far end are not the same as $\bar{m}_r, \bar{n}_r$.

47. We may not have measured an astronomic azimuth at the station, in which case it will be necessary to assume a direction for the astronomic meridian. On this account we must add a *station correction* $\Delta\alpha$ to the assumed astronomic azimuths; or subtract it from the calculated azimuths. To reduce the observed zenith distance to the straight line joining the two end stations, we must also add the *angle of refraction* $\Delta\beta$; or subtract if from the calculated β.

48. We then have finally the following two observation equations for each observed direction; the coefficients $\bar{m}_1$ etc. being as given in § 43: (Observed minus Computed) Zenith Distance is

$$\begin{aligned}
&-\Delta\beta+\bar{m}_1\,d\bar{\omega}/s+\bar{m}_2\,d\bar{\phi}/s+\bar{m}_3\,d\bar{h}/s \\
&-m_1\,d\omega/s-m_2\,d\phi/s-m_3\,dh/s \\
&-(d\omega+\delta\omega)\cos\phi\sin\alpha \\
&-(d\phi+\delta\phi)\cos\alpha
\end{aligned} \tag{48.1}$$

(Observed minus Computed) Azimuth is

$$\begin{aligned}
&-\Delta\alpha-\bar{n}_1\,d\bar{\omega}\csc\beta/s-\bar{n}_2\,d\bar{\phi}\csc\beta/s-\bar{n}_3\,d\bar{h}\csc\beta/s \\
&+n_1\,d\omega\csc\beta/s+n_2\,d\phi\csc\beta/s \\
&+(d\omega+\delta\omega)(\sin\phi-\cos\alpha\cot\beta\cos\phi) \\
&+(d\phi+\delta\phi)\sin\alpha\cot\beta\ .
\end{aligned} \tag{48.2}$$

It is evident that $(d\omega+\delta\omega)$ is the astronomic minus the *initial* approximate geodetic longitude and is known if the astronomic longitude has been measured. If not, then it is permissible to treat $(d\omega+\delta\omega)$ as a single unknown and later to subtract $d\omega$, the correction to geodetic longitude, in order to determine the deflection $\delta\omega$. The same applies to $(d\phi-\delta\phi)$. For this reason it will usually be better to work in geodetic coordinates throughout, despite the fact that working in Cartesian coordinates as in § 44 for final conversion to geodetic coordinates otherwise offers some simplification.

49. We cannot treat $\Delta\beta$ as completely unknown at both ends of all lines if the β-equations are to contribute to the solution. There are two main possibilities at present, until more work has been done on atmospheric refraction, leading to an assurance that it can be accurately evaluated from physical data. We can assume that the refraction of all lines radiating from a particular station is the same per unit length, treat this as a single unknown for the station and express the $\Delta\beta$'s in terms of it. This procedure would be indicated where, as usual in practice, all vertical angles at a station are measured together, even though some complete measures may be taken at a different time. If, for instance, several complete sets were measured by day and several by night, then the refraction determined by this procedure would result from some intermediate atmospheric condition; it would have no physical significance, but the accuracy of the adjustment would be unaffected by thus including complete sets taken at such different times.

The other alternative would be to assume that $\Delta\beta$ is the same at the two ends of a line, in which case it would be eliminated by subtracting the β-equations in pairs before solving. This procedure would be indicated where reciprocal observations have been made.

50. The residual refraction must then be treated as a random error and for this reason the equations should properly be given less weight. There is, however, so little interaction between the two sets of equations *in normal terrestrial triangulation* that weighting has little effect, and indeed the α- and β-equations might well be solved separately. The coefficients $d\omega, d\phi, d\bar{\omega}, d\bar{\phi}$ in the β-equations are all small and these terms could be omitted in a first solution. The main function of the β-equations, assisted by frequent astronomic observations, is to determine $d\hbar, d\bar{\hbar}, (d\omega + \delta\omega)$ and $(d\phi + \delta\phi)$. The coefficients of these unknowns in the α-equations are, however, small, except the term $(d\omega + \delta\omega) \sin\phi$, which might be considered as incorporated in $\Delta\alpha$, so that fairly large errors in them would have little effect on the determination of $d\omega, d\phi, d\bar{\omega}, d\bar{\phi}$ from the α-equations.

51. If, by making assumptions equivalent to those normally made in classical methods, we were to solve the α-equations separately – ignore the effect of deflections, in view of the impossibility of measuring them at every station – and drop the small $d\bar{\hbar}$ term, then the α-equation boils down to

$$-\Delta\alpha + \bar{n}_1 \csc\beta (d\omega - d\bar{\omega})/s + n_2 \csc\beta\, d\phi/s - \bar{n}_2 \csc\beta\, d\bar{\phi}/s \ , \tag{51.1}$$

which is just as easy to compute as the usual classical equation for variation of position, whether on the spheroid or via the spheroid on a plane projection, without making as many other assumptions.

52. Incorporation of astronomic measures is simple. If azimuth is measured, the $\Delta\alpha$ term is dropped. If longitude is measured, we evaluate the $(d\omega + \delta\omega)$ terms and throw them into the absolute terms; and the same for latitude. This does not, of course, mean that astronomic measures are treated as error-free; errors in them will appear in the residuals of the observation equations after adjustment. It is not necessary to measure longitude and azimuth at the same station as for the classical Laplace adjustment. Indeed, except for the purpose of fixing an origin, it is not necessary to measure longitude at all; the full equations will amply serve to bridge gaps in the deflections between frequent latitude and azimuth measures, which are, of course, much easier to observe precisely.

53. Approximate geodetic coordinates to serve as initial data for the adjustment could be obtained by solving the azimuth (48.2) and zenith distance (48.1) equations for selected triangles taken one at a time. The two zenith distance equations for each line should be subtracted as proposed in § 49 to eliminate the refraction term. An electronic computer programmed to iterate the computation would itself require only very rough initial coordinates and would also deliver accurate coefficients of the azimuth and zenith-distance equations for the selected lines to use in the final adjustment.

To determine corrections to the three coordinates, the station correction and two deflections at an apex from two previously fixed points, we should have four azimuth and two (composite) zenith distance equations, so that the calculation is just determinate without astronomic observations at the apex.

54. In the case of rays radiating from the origin $d\omega, d\phi, d\not{h}$ are all zero and astronomic longitude, latitude and azimuth should all be measured so that the $\delta\omega, \delta\phi, \varDelta\alpha$ terms can be evaluated. (If astronomic values are accepted as the initial geodetic elements then $\delta\omega, \delta\phi, \varDelta\alpha$ are zero). The effect of this is to ensure that the conditions of § 24 et seq. are held satisfied (on at least two lines), apart from observational error. The remainder of these observational equations are left in the adjustment so as to minimize the effect of observational error by the inclusion of more lines and to assist the determination of $d\bar{\omega}, d\bar{\phi}, d\bar{\not{h}}$.

55. Measures of length are likely to be much more frequent in future through use of the tellurometer. Wherever a side has been measured, an observational equation is formed from Eq. (42.3), in which ds is the measured minus the computed length. The equation should be divided by a constant of the same order as the average side-lengths in the triangulation in order to reduce it to the same dimensions as the α- and β-equations; it is then taken straight into the general adjustment. Experience so far indicates that tellurometer measures (or for that matter invar measures, at any rate after base extension) are of much the same order of accuracy as the horizontal angular measures, both being about 1/300000. Consequently, there is no object in relative weighting. Sectionally measured bases are not reduced to the spheroid or to "sea level" but are "reduced" upwards to the air line between the terminals.

Flare Triangulation

56. The formulae derived in the last section can be used as they stand wherever angular observations of altitude and azimuth, or difference of azimuth are made in relation to the astronomic zenith. Such observations are made to parachute flares dropped from aircraft in order to provide geodetic connections between the non-intervisable shores of wide water gaps, as originally proposed by W. E. Browne.

57. A usual arrangement is to observe flares from three stations of known geodetic coordinates on one side of the gap, and from three stations whose geodetic coordinates are required in the same system on the far side of the gap. Observations at all six stations are synchronized by radio signals. In some cases the circle readings are photographed by cameras whose shutters are operated by the radio signals, so that all the observers need do is to keep the flare intersected. Flares, which can be dropped at different times, are required in at least two, preferably three, widely separated positions over the water, making reasonably well-conditioned figures with the ground stations.

58. Two observation equations [(48.1) and (48.2)] are formed for each line from approximate positions of the flares and of the unknown ground stations. Between six ground stations and three flares, there are then 36 equations (or 44 if connections between the three unknown ground stations are observed in both directions) to determine corrections to six sets of three coordinates, if full astronomical observations are taken at all ground stations.

In present practice, vertical angles would not be measured, and corrections to latitudes and longitudes only would be determined. Between six ground stations and three flares, 18 equations (22 if directions are observed both ways between the three unknown ground stations) would then be available to determine 12 corrections. Theoretically, there would be enough equations to determine heights but the equations would be too ill-conditioned for the purpose. The result would be an approximation whose validity should be demonstrated numerically from the full equations. It should be possible to measure vertical angles and to obtain reasonable refraction corrections from reciprocal observations between the ground stations, thereby obtaining a theoretically correct solution and carrying geodetic heights across the gap. If the flare is roughly mid-way between known and unknown ground stations, residual effects of refraction would tend to cancel as between the heights of ground stations, although perhaps seriously affecting the flare heights.

59. To minimize pointing errors, it is usual to make several observations to the same flare as it falls. We have no knowledge of the path of the flare and must accordingly treat its position as unknown at the time of every such additional observation. An extra set of observations accordingly introduces three more unknown corrections to the position of the flare and two extra equations for each ground station observing it. If the extra observations are made rapidly, the same coefficients would probably suffice. If vertical angles are not observed and heights are not to be determined, then an extra set of observations introduces two more unknowns in the flare position and one extra equation per ground station observing it.

Shoran or Hiran Measures

60. Radar measures of slant-ranges are made between each of two ground stations (S_1, S_2) and an aircraft (A) flying a straight and level course between them. The minimum sum of the radar ranges is used to determine the distance between the two ground stations, and hence geodetic positions by trilateration. In addition to the usual approximations inherent in classical geodesy, a particular assumption is made that the minimum position occurs when the plane S_1, A, S_2 is vertical at A. The limitations of this assumption can be seen at once by considering the aircraft course as tangential to a prolate spheroid whose foci are S_1, S_2. The sum $(S_1 P + P S_2)$ is the same for any point on this spheroid and less than for any point on the straight aircraft course external to the spheroid, so that the minimum position occurs at the point of contact of the course with the spheroid. In theory, the assumption is accordingly justified only if (a) the aircraft course is perpendicular

to S_1, S_2, which it usually is not, or (b) the aircraft crosses in the mid-way position. The problem can, however, be solved simply and rigorously in three dimensions without making any such assumptions.

61. Appropriate corrections, necessary for any method of reduction, should first be made to the radar readings to obtain the straight-line slant ranges.

Coordinates of S_1, A, S_2 respectively are denoted by $x^r, \bar{x}^r, \bar{\bar{x}}^r$. Unit vectors in the directions S_1A, AS_2 are p^r, q^r and the slant-ranges S_1A, AS_2 are u, v. The unit vectors for the aircraft course is $\bar{a}^r$. Parallel vectors at the three points are denoted by appropriate bars, e.g. a vector parallel to $\bar{a}^r$ at S_2 is $\bar{\bar{a}}^r$; and at S_1 is a^r.

Equations (42.2) for variation of the two slant-ranges are

$$du = \bar{p}_r \, d\bar{x}^r - p_r \, dx^r \tag{61.1}$$

$$dv = \bar{\bar{q}}_r \, d\bar{\bar{x}}^r - \bar{q}_r \, d\bar{x}^r \ . \tag{61.2}$$

62. To establish the minimum position, we assume first that S_1, S_2 are fixed and the aircraft only moves by $d\bar{x}^r$, while $dx^r = d\bar{\bar{x}}^r = 0$. At the minimum position, we have also $du + dv = 0$, and therefore from Eqs. (61.1) and (61.2)

$$\bar{p}_r \, d\bar{x}^r = \bar{q}_r \, d\bar{x}^r \ .$$

But $d\bar{x}^r$ is proportional to the (contravariant) course vector $\bar{a}^r$, so that

$$\bar{p}_r \bar{a}^r = \bar{q}_r \bar{a}^r \ . \tag{62.1}$$

This is the minimum condition and implies that the aircraft course makes equal angles with the directions S_1A, AS_2 when the minimum occurs, say $\cos P = \cos Q$.

63. Now suppose that the direction of the aircraft (a^r) is fixed but that the coordinates of S_1, A and S_2 are all varied by dx^r etc. from approximate initial positions to final positions, which final positions satisfy the minimum condition. The corresponding changes in $\cos P$ and $\cos Q$ are the given as follows, [see Eq. (42.1)].

$$\begin{aligned} u\text{x\{Final } (\cos P) \text{ minus Initial } (\cos P)\} &= u\,d(\cos P) = u\,d(a_r p^r) = u\,a_r\,d(p^r) \\ &= a_r(d\bar{x}^r - dx^r) - du \cos P \end{aligned}$$

and similarly

$$\begin{aligned} v\text{x\{Final } (\cos Q) \text{ minus Initial } (\cos Q)\} &= v\,d(\cos Q) \\ &= a_r(d\bar{\bar{x}}^r - d\bar{x}^r) - dv \cos Q \ . \end{aligned}$$

Substracting these two equations, and remembering that the final values of $\cos P$ and $\cos Q$ are to be equal, we have

$$\begin{aligned} \text{Initial } (\cos Q - \cos P) = {} & -\frac{1}{u} a_r \, dx^r + \left(\frac{1}{u} + \frac{1}{v}\right) \bar{a}_1 \, d\bar{x}^r - \frac{1}{v} \bar{\bar{a}}_r \, d\bar{\bar{x}}^r \\ & - \frac{du}{u} \cos P + \frac{dv}{v} \cos Q \ , \end{aligned} \tag{63.1}$$

in which P, Q are initial values computed from the initial assumed coordinates and du (or dv) is the final observed u (or v) minus the initial computed u (or v). This equation and (61.1) and (61.2) can be used either as condition equations, or as observation equations in conjunction with any other measurements which may have been made by triangulation or traverse etc., to fix S_1, S_2. As before in § 42, these equations are true in any coordinates provided that the components of $\bar{a}_r$ etc. and the parallel vectors $a_r, \bar{\bar{a}}_r$ are taken out in the same coordinates. P, Q, etc. must, of course, be accurately computed even though the aircraft course has been only roughly measured.

64. In the usual case, the azimuth of the aircraft course is $\bar{A}$ and the zenith distance for level flight is 90°. From § 35 and § 36 we then have for the coefficients in Eq. (63.1):

$$\bar{a}_1 = (\bar{\nu}+\bar{\hbar}) \cos \bar{\phi} \sin \bar{A}$$

$$\bar{a}_2 = (\bar{\varrho}+\bar{\hbar}) \cos \bar{A}$$

$$\bar{a}_3 = 0 \ .$$

$$a_1 = (\nu+\hbar) \cos \phi \{\sin \bar{A} \cos (\omega-\bar{\omega})+\cos \bar{A} \sin \bar{\phi} \sin (\omega-\bar{\omega})\}$$

$$a_2 = (\varrho+\hbar)\{-\sin \bar{A} \sin \phi \sin (\omega-\bar{\omega}) \\ +\cos \bar{A} [\cos \bar{\phi} \cos \phi+\sin \bar{\phi} \sin \phi \cos (\omega-\bar{\omega})]\}$$

$$a_3 = \sin \bar{A} \cos \phi \sin (\omega-\bar{\omega}) \\ +\cos \bar{A} [\cos \bar{\phi} \sin \phi-\sin \bar{\phi} \cos \phi \cos (\omega-\bar{\omega})]$$

$$\bar{\bar{a}}_1 = (\bar{\bar{\nu}}+\bar{\bar{\hbar}}) \cos \bar{\bar{\phi}}\{\sin \bar{A} \cos (\bar{\bar{\omega}}-\bar{\omega})+\cos \bar{A} \sin \bar{\phi} \sin (\bar{\bar{\omega}}-\bar{\omega})\}$$

$$\bar{\bar{a}}_2 = \bar{\bar{\varrho}}+\bar{\bar{\hbar}})\{-\sin \bar{A} \sin \bar{\bar{\phi}} \sin (\bar{\bar{\omega}}-\bar{\omega}) \\ +\cos \bar{A} [\cos \bar{\phi} \cos \bar{\bar{\phi}}+\sin \bar{\phi} \sin \bar{\bar{\phi}} \cos (\bar{\bar{\omega}}-\bar{\omega})]\}$$

$$\bar{\bar{a}}_3 = \sin \bar{A} \cos \bar{\bar{\phi}} \sin (\bar{\bar{\omega}}-\bar{\omega}) \\ +\cos \bar{A} [\cos \bar{\phi} \sin \bar{\bar{\phi}}-\sin \bar{\phi} \cos \bar{\bar{\phi}} \cos (\bar{\bar{\omega}}-\bar{\omega})]$$

and the term in e.g. dx_r in Eq. (63.1) expands as

$$-\frac{1}{u} a_1 d\omega-\frac{1}{u} a_2 d\phi-\frac{1}{u} a_3 d\hbar.$$

The coefficients need to be accurately computed even though $\bar{A}$ has been only roughly measured.

65. If α, β are the azimuth and zenith distance from S_1 to A as computed from the assumed initial coordinates of S_1 and A; and $\bar{\alpha}, \bar{\beta}$ refer as usual to the same direction at A and in the same sense, then the first Eq. (61.1) expands exactly as (42.3) (see also § 35 and § 64),

$$\cos \pi = \bar{p}^r \bar{a}_r = \sin \bar{A} \sin \bar{\alpha} \sin \bar{\beta}+\cos \bar{A} \cos \bar{\alpha} \sin \bar{\beta} = \sin \bar{\beta} \cos (\bar{A}-\bar{\alpha}) \ .$$

Similarly for the second Eq. (61.2) and $\cos Q = \bar{q}^r \bar{a}_r$.

66. If there are no other measures besides Shoran connecting the ground stations, then it will certainly be impossible to determine corrections to heights and the terms containing $dh, d\bar{h}, d\bar{\bar{h}}$ must be dropped. For instance, in a simple trilateration, where a third ground point is to be fixed from two previously known points, there would be only the above three equations for each side, and these could do no more than determine six unknowns, viz. corrections to latitude and longitude of the third point and of the two aircraft positions. Even though the latter may not be required, they must, of course, be left in the equations. The result is not very sensitive to height changes, but nevertheless the difficulty of height determination is a weakness of Shoran – not of the method of computation – since even if heights are not required the omission of the dh terms must to some extent affect the determination of latitude and longitude.

67. If the initial assumed positions are within 15 seconds of the truth, which can usually be arranged by rough spherical computation and by placing the aircraft along the line in simple proportion to the measured ranges, then a single solution provides results correct to about two feet. In a test case of a single trilateral, deliberately rough values were assumed which turned out to be 8 minutes and 5 minutes adrift in latitude and longitude respectively of the unknown station, and 3 minutes adrift to the final aircraft position. The first solution averaged about 14 seconds adrift, the largest difference being 47 seconds in longitude of the unknown ground station. A repeat computation gave results within 0.025 second of correct values. Movements of the aircraft are not very sensitive; Eq. (63.1) is soon satisfied, and when that occurs, the corrections to the aircraft position have the same coefficients in the remaining equations and can be eliminated. An electronic computer programmed to iterate the whole computation would soon satisfy all six equations in a trilateral, however rough the initial values, and at the same time produce coefficients of the observation equations to be used in a final net adjustment.

68. Results obtained by the classical method which have so far been checked work out very nearly the same, which indicates that the aircraft does in practice usually cross nearly mid-way. Nevertheless, it would be desirable in future to measure the aircraft course roughly and apply the rigorous method, at least as a final check.

Geodetic Positions from Rocket Flashes

69. This method – suggested in 1946 by Väisälä and more recently by R. d'E. Atkinson independently and in greater practical detail – of fixing geodetic positions over long distances, possibly across the oceans, is similar in principle to flare triangulation. Instead of observing a flare instrumentally, photographs of an "instantaneous" flash produced by a powerful rocket are taken in equatorially mounted long-focus cameras against a background of stars. Measurement of the plates provides the apparent right ascension (reducible to origin hour angle H by § 19), and declination D, of the flash from each of the ground observing stations. Accordingly, we need observation equations to give changes in H and D, rather

than changes in azimuth and zenith distance, arising from variation in position of the flash and of a ground station.

70. The required observation equations are obtained at once from Eqs. (30.4) and (43.1) as:

$$s\,dD = -\bar{M}_s d\bar{x}^s + M_s dx^s \;;$$

and

$$s \cos D\, dH = \bar{N}_s d\bar{x}^s - N_s dx^s \;, \tag{70.1}$$

in which the unit vectors M_s, N_s (and the parallel vectors $\bar{M}_s, \bar{N}_s$ at the barred point, which we shall assume to be the flash) are given by Eqs. (30.2) and (30.3). These equations are true in any coordinate system.

As usual, the displacements $d\bar{x}^s, dx^s$ are assumed to be made from initial (assumed) positions of the flash and the ground station, to final (observed) positions. We also compute initial values of H, D and s from Eqs. (23.2), (37.1) and (31.1) and use these in the coefficients of the observation equations. Thus:

$$\begin{aligned} s \cos H \cos D &= (\bar{x}-x) = (\bar{\nu}+\bar{h}) \cos\bar{\phi} \cos\bar{\omega} - (\nu+h) \cos\phi \cos\omega \\ s \cos D \cos H &= (\bar{y}-y) = (\bar{\nu}+\bar{h}) \cos\bar{\phi} \sin\bar{\omega} - (\nu+h) \cos\phi \sin\omega \\ s \sin D &= (\bar{z}-z) = (\bar{\nu}\bar{e}^2+\bar{h}) \sin\bar{\phi} - (\nu\bar{e}^2+h) \sin\phi \;. \end{aligned} \tag{70.2}$$

Finally we assume an approximate value of local sidereal time *at the origin* of the survey when the flash occurs; we use this to give us the computed right ascension from § 19, and we shall seek to correct it by determining an additive correction dt to the origin sidereal time of the flash, so that dt will be the same for all observed lines to the same flash. It will be clear from § 19 that we then have: (Observed minus Computed) Right ascension of the flash = dH + dt, all expressed in arc.

71. In geodetic coordinates the components of the vectors $M_2, \bar{M}_s$ etc., are obtained straight from Eqs. (30.2), (30.3) and (34.2) as:

$$\begin{aligned} \bar{M}_1 &= (\bar{\nu}+\bar{h}) \cos\bar{\phi} \sin D \sin(H-\bar{\omega}) \\ \bar{M}_2 &= -(\bar{\rho}+\bar{h})\{\cos\bar{\phi} \cos D + \sin\bar{\phi} \sin D \cos(H-\bar{\omega})\} \\ \bar{M}_3 &= -\sin\bar{\phi} \cos D + \cos\bar{\phi} \sin D \cos(H-\bar{\omega}) \\ M_1 &= (\nu+h) \cos\phi \sin D \sin(H-\omega) \\ M_2 &= -(\rho+h)\{\cos\phi \cos D + \sin\phi \sin D \cos(H-\omega)\} \\ M_3 &= -\sin\phi \cos D + \cos\phi \sin D \cos(H-\omega) \\ \bar{N}_1 &= (\bar{\nu}+\bar{h}) \cos\bar{\phi} \cos(H-\bar{\omega}) \\ \bar{N}_2 &= (\bar{\rho}+\bar{h}) \sin\bar{\phi} \sin(H-\bar{\omega}) \\ \bar{N}_3 &= -\cos\bar{\phi} \sin(H-\bar{\omega}) \end{aligned}$$

$$N_1 = (\nu + h) \cos\phi \cos(H - \omega)$$
$$N_2 = (\rho + h) \sin\phi \sin(H - \omega)$$
$$N_3 = -\cos\phi \sin(H - \omega)$$

leading to the expanded observation equations:
(Observed minus Computed) Declination is

$$\begin{aligned} &= -\bar{M}_1\, d\bar{\omega}/s - \bar{M}_2\, d\bar{\phi}/s - \bar{M}_3\, d\bar{h}/s \\ &\quad + M_1\, d\omega/s + M_2\, d\phi/s + M_3\, dh/s \end{aligned} \tag{71.1}$$

(Observed minus Computed) Right Ascension is

$$\begin{aligned} &= dt + \bar{M}_1 \sec D\, d\bar{\omega}/s + \bar{N}_2 \sec D\, d\bar{\phi}/s + \bar{N}_3 \sec D\, d\bar{h}/s \\ &\quad - N_1 \sec D\, d\omega/s - N_2 \sec D\, d\phi/s - N_3 \sec D\, dh/s \end{aligned} \tag{71.2}$$

72. To fix the position of a single unknown station, at least two known stations and two flashes (which need not be simultaneous) are required in suitably different positions to provide reasonably well-conditioned intersections. There are then 12 equations to determine 11 unknown viz. two dt's for the two flashes and three each of $d\omega$'s, $d\phi$'s and dh's for the two flashes and for the unknown station. Any number of other unknown stations can be fixed off the same two flashes, since each additional unknown station adds four equations and only three unknowns. Additional known stations and flashes will, of course, strengthen the network: an additional flash would add six equations and four unknowns, while an extra known station observing two flashes would add four equation and no unknowns.

It should be clear from the foregoing analysis that deflections can have no effect since the plumb line does not appear in the working, except at the origin of the survey.

73. If the geodetic coordinates of only one unknown station across a wide gap are fixed by this method, this single station can still be used to extend the survey in the same geodetic system by measuring astronomical latitude, longitude and azimuth at it, and by adopting geodetic azimuths and zenith distances in at least two directions radiating from it to satisfy Eqs. (23.4) or (26.1). The effect of refraction and of observational error is minimized by adopting the procedure given in § 54 during the adjustment of the extended survey.

74. The method is virtually free from geometrical errors of refraction. The flash will usually occur outside the refracting atmosphere and will therefore be subject to the same refraction as a star, so that its right ascension and declination will be determinable to the same degree of accuracy on this account as the apparent places of stars are determined by photographic methods. Most of the effect of refraction is taken out in determining the plate constants from the background star images, whose apparent places are known.

Changing refraction would affect the position of the "instantaneous" flash image relative to the star images, the effect of refraction on which is meaned over a longer exposure. Clear weather over such long distances would probably mean steady atmospheric conditions, but to minimize the effect of unsteady refraction, Atkinson suggests arranging the flash to appear at zenith distances not exceeding 70°. He also suggests the use of repeating flashes.

75. Instead of a rocket, it has been suggested that a flashing artificial satellite might be used as a beacon, observations from several ground stations being synchronized by means of the flashes. This probably offers the best chance of using an artificial satellite for fixing geodetic positions. It would move too fast to be used instead of a flare for accurate altazimuth measurements and its orbit is unlikely to be well enough known for use in the lunar methods now to be described.

Position Fixing by Lunar Methods

76. We consider next the method of position-fixation by photography of the moon against a background of stars, recently proposed and worked out in detail by W. Markowitz for wide use during the International Geophysical Year. The camera itself is mounted equatorially to hold the exposure of the stellar background. It also carries a parallel-plate filter (analogous to the parallel-plate micrometer of a precise level) which can be rotated to hold the photographic image of the moon fixed in relation to the stars. The time of the observation is considered to be when the rotating filter introduces no relative displacement between moon and stars. Measurement of the plate, which can be corrected for irregularities in the limb by referring all lunar profiles to a single datum, provides the apparent right ascension and declination of the moon's centre from the known positions of several stars. The observational data is accordingly the same as in Atkinson's method, with the moon in place of the flash, except that the observation must be accurately timed. Moreover, the position of the moon in space, unlike that of the flash, is reckoned to be known from the Lunar Ephermeris, so that it need not be fixed by photography from known ground stations. Photography of the moon in at least two different positions from the same unknown station will fix the position of that station in all three coordinates.

77. As in the case of Atkinson's method, we reduce right ascensions to origin hour angles H by § 19. The observation equations are then exactly the same as Eq. (70.1), with the moon at the barred point. For a reason which will presently appear, we retain the corrections $d\bar{x}^s$ to the position of the moon, even though this is supposed to be known.

78. The Ephermeris gives the position of the moon's centre by its right ascension (which we shall reduce to origin hour angle by § 19 and then call γ to distinguish it from H, the origin hour angle of the line joining the ground station to the moon); by its declination (which we shall call δ to distinguish it similarly from D, the declination of the line joining the ground station to the moon); and by its parallax (directly related to the radius vector r) – all of the line joining the centre

of the moon to the centre of mass of the Earth. Accordingly, we propose to use $(\gamma, \delta, r) = (1, 2, 3)$ as coordinates and to derive the position of the ground station in the same coordinates; δ will then be the geocentric latitude (the complement of the angle between the axis and the radius vector drawn from the physical centre of mass to the ground station); and γ will be the geocentric longitude (the angle between the axial plane containing the radius vector and the plane of the *astronomic* meridian of the origin). Note that these geocentric coordinates have nothing to do with the geodetic system and its base spheroid; the radius vector is drawn from the centre of mass of the Earth, not from the centre of the spheroid.

79. As usual, we assume approximate coordinates $(\bar{\gamma}, \bar{\delta}, \bar{r})$ for the moon and for the ground station (γ, δ, r) and compute the length S, declination D and origin hour angle H of the line from the following formulae, which are obtained by projection on the three Cartesian axes:

$$\begin{aligned} S \cos H \cos D &= \bar{r} \cos \bar{\gamma} \cos \bar{\delta} - r \cos \gamma \cos \delta \\ S \sin H \cos D &= \bar{r} \sin \bar{\gamma} \cos \bar{\delta} - r \sin \gamma \cos \delta \\ S \sin D &= \bar{r} \sin \bar{\delta} - r \sin \delta \ . \end{aligned} \tag{79.1}$$

80. Components of the fundamental Cartesian vectors in the geocentric coordinate system can be written down at once from (34.2) – even though the Cartesian origin is not the same – by substituting $\gamma = \omega$, $\delta = \phi$, $(\nu + h) = (\varrho + h) = r$, viz.

$$\begin{aligned} A_r &= (-r \cos \delta \sin \gamma \ ; \quad -r \sin \delta \cos \gamma \ ; \quad \cos \delta \cos \gamma) \\ B_r &= (r \cos \delta \cos \gamma \ ; \quad -r \sin \delta \sin \gamma \ ; \quad \cos \delta \sin \gamma) \\ C_r &= (0 \ ; \quad r \cos \delta \ ; \quad \sin \delta) \ . \end{aligned} \tag{80.1}$$

The components of the auxiliary vectors M_r, N_r follow by making the same substitutions in the formulae of § 71 or direct from Eqs. (30.2), (30.3) and (80.1), viz.

$$\begin{aligned} \bar{M}_1 &= \bar{r} \cos \bar{\delta} \sin D \sin (H - \bar{\gamma}) \\ \bar{M}_2 &= -\bar{r} \{\cos \bar{\delta} \cos D + \sin \bar{\delta} \sin D \cos (H - \bar{\gamma})\} \\ \bar{M}_3 &= -\sin \bar{\delta} \cos D + \cos \bar{\delta} \sin D \cos (H - \bar{\gamma}) \\ M_1 &= r \cos \delta \sin D \sin (H - \gamma) \\ M_2 &= -r \{\cos \delta \cos D + \sin \delta \sin D \cos (H - \gamma)\} \\ M_3 &= -\sin \delta \cos D + \cos \delta \sin D \cos (H - \gamma) \\ \bar{N}_1 &= \bar{r} \cos \bar{\delta} \cos (H - \bar{\gamma}) \\ \bar{N}_2 &= \bar{r} \sin \bar{\delta} \sin (H - \bar{\gamma}) \\ \bar{N}_3 &= -\cos \bar{\delta} \sin (H - \bar{\gamma}) \\ N_1 &= r \cos \delta \cos (H - \gamma) \end{aligned}$$

$$N_2 = r \sin \delta \sin (H-\gamma)$$
$$N_3 = -\cos \delta \sin (H-\gamma) \ ,$$

whence by substitution in (70.1), the expanded observational equations become: (Observed minus Computed) Declination of the line ground station to Moon's centre

$$= -M_1 \, d\bar{\gamma}/s - \bar{M}_2 \, d\bar{\delta}/s - \bar{M}_3 \, d\bar{r}/s$$
$$+ M_1 \, d\gamma/s + M_2 \, d\delta/s + M_3 \, dr/s \qquad (80.2)$$

(Observed minus Computed) Right Ascension (or Origin Hour Angle) of the same line

$$= \bar{N}_1 \sec D \, d\bar{\gamma}/s + \bar{N}_2 \sec D \, d\bar{\delta}/s + \bar{N}_3 \sec D \, d\bar{r}/s$$
$$- N_1 \sec D \, d\gamma/s - N_2 \sec D \, d\delta/s - N_3 \sec D \, dr/s \ . \qquad (80.3)$$

81. If the position of the moon really were known, we could put $d\bar{\gamma}$, $d\bar{\delta}$, $d\bar{r}$ equal to zero in these equations without further ado. Unfortunately, we cannot be sure that the Universal Time of the observation is the same as the Ephermeris Time used as argument in the Lunar Ephermeris; there is a difference between the two which is not tabulated and which varies slowly. The simplest way of overcoming the difficulty is to envisage a correction dt to the time of the observation, to find $(d\bar{\gamma}/dt)$ etc. from the tabular differences and to replace $d\bar{\gamma}$ etc. in the observational equation by $(d\bar{\gamma}/dt)\,dt$. This reduces the number of unknowns by two. The equations are still soluble from two separate positions of the moon, provided that dt is considered to be the same for both; but in practice many positions will be photographed from several stations, and it may also be possible to derive corrections to the elements of the orbit as well as by expressing $d\bar{\gamma}$ etc. in terms of these elements.

82. If π is the tabulated parallax, then $\bar{r} = \bar{S} \csc \pi$, in which the constant $\bar{S}$ is an assumed equatorial radius of the Earth. It may accordingly be necessary to write $\bar{r} = \csc \pi \ \{d\bar{r} = -\cos \pi \csc \pi \, d\pi\}$ and thereby in effect to determine $S/\bar{S}$ and $r/\bar{S}$ in a reduced scale model. In that case, $\bar{S}$ would be determined later from measured terrestrial distances between stations fixed by this method.

83. It has been suggested that a radar distance to the moon should also be measured, in which case there would be an additional observational equation obtainable from (42.2). If l_r is the unit vector in the direction of the moon from the ground station, then

$$l_r = (\cos H \cos D) A_r + (\sin H \cos D) B_r + (\sin D) C_r$$

and the components of this vector at the two ends of the line in $(\gamma, \bar{\delta}, r)$ coordinates are accordingly found from Eq. (80.1) to be:

$$\bar{l}_1 = \bar{r} \cos \bar{\delta} \cos D \sin (H-\bar{\gamma})$$
$$\bar{l}_2 = \bar{r} \{\cos \bar{\delta} \sin D - \sin \bar{\delta} \cos D \cos (H-\bar{\gamma})\}$$
$$\bar{l}_3 = \sin \bar{\delta} \sin D + \cos \bar{\delta} \cos D \cos (H-\bar{\gamma})$$

$$l_1 = r\cos\delta\cos D\sin(H-\gamma)$$
$$l_2 = r\{\cos\delta\sin D - \sin\delta\cos D\cos(H-\gamma)\}$$
$$l_3 = \sin\delta\sin D + \cos\delta\cos D\cos(H-\gamma)$$

and the required observational equation by substitution in Eq. (42.2) is: (Observed minus Computed) Distance

$$\begin{aligned} &= \bar{l}_1\,d\bar{\gamma} + \bar{l}_2\,d\bar{\delta} + \bar{l}_3\,d\bar{r} \\ &\quad - l_1\,d\gamma - l_2\,d\delta - l_3\,dr \ . \end{aligned} \tag{83.1}$$

With the help of (79.1) we find that $\bar{l}_1 = l_1$.

84. We have ensured that the Cartesian axes of the geocentric coordinates (γ, δ, r) are parallel to those of the geodetic system (ω, ϕ, h). If we know the position of a station in both systems, we can accordingly find the Cartesian coordinates (dx_0, dy_0, dz_0) of the centre of mass of the Earth relative to the centre of the spheroid from Eqs. (31.1) and (79.1);

$$\begin{aligned} dx_0 &= (\upsilon + h)\cos\phi\cos\omega - r\cos\delta\cos\gamma \\ dy_0 &= (\upsilon + h)\cos\phi\sin\omega - r\cos\delta\sin\gamma \\ dz_0 &= (\upsilon\bar{e}^2 + h)\sin\phi - r\sin\delta \ . \end{aligned} \tag{84.1}$$

The accuracy of the result will depend largely on the lunar theory on which the Ephermeris is based, and also it must be admitted on the method of computing the geodetic survey, but in time we may reasonably expect to obtain consistent values for dx_0, dy_0, dz_0 from a number of common stations. In that case, we can apply Eq. (84.1) in reverse to derive the geodetic coordinates of points whose geocentric coordinates have been obtained off the moon. We can then extend the geodetic survey on the same system (ultimately all round the Earth) as already proposed in § 73.

This is the only rigorously valid method of mixing lunar with other geodetic measurements.

85. Interest in methods involving stellar occultation has been revived in recent years by improvements in timing and photo-electric recording. All we get out of one timed occultation is an observation at a ground station for right ascension and declination of a particular point on the moon's limb where the occultation occurred. If these observations are reduced to right ascension and declination of the moon's centre, then we can form two observation equations [(80.2) and (80.3)] just as in the photographic method. Other equations can be added from different occultations observed at the same station, leading in exactly the same way to a solution for the geocentric coordinates of that station.

Unfortunately, the reduction to the moon's centre usually involves serious inaccuracies, owing to irregularities in the limb, which are largely avoided in the photographic method by locating the centre from several points on the limb. To overcome this difficulty, J. A. O'Keefe chooses two stations, one known and one unknown, where the same star is occulted by the same part of the limb. From the unknown coordinates of one station he is able in effect to provide data for the reduction to the moon's centre in this particular case, including any other uncertainty in lunar data which can be considered constant between the two observations, and to use this data in reducing the observations at the unknown station. Another such pair of occultations, including the same unknown station, provides two more observational equations and therefore the geocentric coordinates of the unknown station.

86. Solar eclipse observations could be reduced in much the same way, including the further reduction from the sun's centre to the point of contact with the moon. A single eclipse will not provide a complete fix, but the observational equations could nevertheless be of use in conjunction with other data.

Spirit-Levelling and the Geoid

87. Comparison of latitudes and longitudes in the astronomic and geodetic coordinate systems enters the argument at the outset, because all terrestrial angular measures are necessarily made in relation to the astronomic zenith, but we have not so far compared the third coordinates in the two systems.

The third coordinate in the astronomic system would naturally be the geopotential, or some function of the geopotential, since this is an independently variable function of position associated with the astronomic latitude and longitude, by reason of the fact that the astronomic zenith, whose direction defines latitude and longitude, is also the direction of the gradient of the geopotential. Moreover, the geopotential would be the best definition of natural "height" to use for practical purposes: water will not flow between two points of equal potential; starting from the same point, the same work would have to be done to climb two mountains whose summits are at the same potential; and the only physical notion of "horizontal" or "level" we can have is perpendicular to the astronomic zenith, in a direction in which the potential does not change.

88. If g is measured gravity and dl the difference in spirit-levelled heights between two near points, then the corresponding difference in potential is $g\,dl$, which can be integrated over a long line as $\int g\,dl$. The steps into which this integral is divided for numerical calculation can be quite large, and gravity measures correspondingly infrequent, unless the terrain is very hilly or geologically disturbed.

The potentials at junction-points in a level network should properly be obtained in this manner and used in the adjustment. Neat spirit-levels are not functions of position and are not independent of the path along which they are measured. Different values for the level of junction points can accordingly be obtained from different lines of levelling connecting them, even if the levelling is free of all error. The actual numerical difference depends on the extent to which gravity can be considered constant over the area of the network.

89. Unfortunately, the idea of potential is alien to most users of maps and survey data. We can, however, convey the same information in a form which looks more

like the conventional idea of height if we divide differences of potential by a constant having the dimensions of mean gravity. If we want to compare potentials over the whole Earth, then only one constant can be used for the whole Earth. Otherwise we can divide by mean gravity taken over a particular region and avoid comparison with "heights" obtained by the use of a different constant for another region. It is obvious that the more often the divisor is changed, the less the extent of the region over which comparisons can be made. To adopt a continuously variable divisor, as is sometimes suggested, rules out the possibility of making any valid "height" comparisons and stultifies the whole process. The important point to realize is that whatever system is used amounts to no more than a convention, which should be clearly recorded. Such expressions as "orthometric" and "dynamic" heights resulting from a particular convention and wrongly suggesting some measurable physical entity, are merely misleading and should be discontinued.

90. Suppose that we have determined the potential at a survey station by spirit-levelling in relation to the potential at mean sea level. If we sink a shaft, having depth but no other significant dimensions, under the station and make occasional gravity measurements down the shaft, we shall clearly be able to calculate the change in potential down the shaft, just as in surface levelling, and can thus find the depth at which the potential of mean sea level is recovered. By definition this will be the depth of the geoid below the station. The difference between this and the geodetic $\mathscr{h}$-coordinate of the station gives us at once the separation of the spheroid and geoid under the station, usually denoted $N = (d - \mathscr{h})$, subject to a cosine error of the deflection.

Since we cannot sink shafts under the station, some approximation is necessary. We could measure gravity at the surface and apply half the *free-air* correction (to simulate the conditions in a shaft) for an assumed depth to derive a mean value of gravity) half way down; then divide the difference in potential between the station and sea level by this mean value of gravity to obtain a better value of t. Except in very hilly country, it would probably be sufficient to ignore gravity altogether and to take d as the neat spirit-levelled height of the station above sea level.

91. The classical method of taking *geoidal sections* claims to obtain a difference of N between two stations as a product of the mean deflection in the direction of the line joining the two stations and the distance apart of the stations. Since neither d nor $\mathscr{h}$ enters the method at all, it is evident that some additional assumption is being made to determine their difference. Apart from the over-simplified plane geometry used in deriving the method, the main implicit assumption is that a deflection measured on the surface has the same value underground at the level of the spheroid (or can be shown that the change in deflection down a line of force is proportional to the *horizontal* gradient of gravity. This gradient is certainly subject to rapid local variation on the surface and there is no reason to suppose it would be any more predictable underground. There would also be discontinuity in the deflection on crossing dipping strata of different densities. The method may nevertheless be numerically adequate over short lines in flat country, despite its

theoretical defects. It should be tested in hilly country against the simpler and more rigorous method given above.

Change of Geodetic Coordinates

92. We next investigate the effect of changing the geodetic coordinate system, while retaining fixed the positions of points in space. Parallelism of the fundamental Cartesian vectors A^r, B^r, C^r is also retained, so that the basis of comparison between astronomic and geodetic measures may remain unimpaired. Accordingly, we seek first-order changes in the coordinates (ω, ϕ, h) resulting from changes (dx_0, dy_0, dz_0) in the Cartesian coordinates of the centre of the base spheroid; and changes da and dt ($= -d\bar{e}$) in the major semi-axis and the flattening $f = (1-\bar{e})$ of the base spheroid.

93. We first change a and f without altering the Cartesian origin, which means that the position vector p^r remains constant. If we write $p = h + a^2/\nu$ and $q = -\nu e^2 \sin\phi \cos\phi$, it follows from Eq. (32.1) that

$$p^r = p\nu^r + q\mu^r \ .$$

Differentiating this vector equation and substituting Eqs. (29.3) and (29.2), we have

$$0 = (p\cos\phi - q\sin\phi)\,d\omega\,\lambda^r + (dq + p\,d\phi)\mu^r + (dp - q\,d\phi)\nu^r \ ,$$

in which we can equate to zero the coefficients of the mutually orthogonal vectors λ^r, μ^r, ν^r. The coefficient of λ^r shows that $d\omega$ is zero since $(p\cos\phi - q\sin\phi) = (\nu + h)\cos\phi$ is not zero; in other words, as might have been anticipated, no change in geodetic longitude results from changes in the shape and size of the base spheroid. We are left with the two conditions

$$\begin{aligned} dq + p\,d\phi &= 0 \\ dp - q\,d\phi &= 0 \ . \end{aligned} \tag{93.1}$$

Considering p, q as functions of a, f, ϕ and h we find after some simplification that $\partial p/\partial\phi = q$ and $\partial q/\partial\phi = \varrho - a^2/\nu$, so that these two equations (93.1) reduce on expansion to

$$(\varrho + h)\,d\phi + (\partial q/\partial h)\,dh + (\partial q/\partial a)\,da + (\partial q/\partial f)\,df = 0$$

and

$$(\partial p/\partial h)\,dh + (\partial p/\partial a)\,da + (\partial p/\partial f)\,df = 0$$

which on differentiation of p + q, substitution and simplification become

$$dh = -(a/\nu)\,da + (\nu\bar{e}\sin^2\phi)\,df \tag{93.2}$$

$$(\varrho + h)\,d\phi = (\nu e^2 \sin\phi\cos\phi/a)\,da + (\nu\bar{e} + \varrho/\bar{e})\sin\phi\cos\phi\,df \ . \tag{93.3}$$

94. Next we introduce a change (dx_0, dy_0, dz_0) in the Cartesian origin, involving a corresponding translation of the spheroid. The effect will be the same if we keep the Cartesian origin and the spheroid fixed and alter the Cartesian coordinates of the point in space under consideration by ($-dx_0$, $-dy_0$, $-dz_0$). The corresponding changes in geodetic coordinates are given by

$$\begin{aligned} d\omega &= -\frac{\partial\omega}{\partial x}dx_0 - \frac{\partial\omega}{\partial y}dy_0 - \frac{\partial\omega}{\partial z}dz_0 \\ &= -A^1 dx_0 - B^1 dy_0 - C^1 dz_0 \ , \end{aligned}$$

which on substituting (34.1) becomes

$$(\upsilon + h)\cos\phi\, d\omega = \sin\omega\, dx_0 - \cos\omega\, dy_0 \ . \tag{94.1}$$

In the same way, we have

$$\begin{aligned} d\phi &= -A^2 dx_0 - B^2 dy_0 - C^2 dz_0 \\ dh &= -A^3 dx_0 - B^3 dy_0 - C^3 dz_0 \ , \end{aligned}$$

which on substituting (34.1) and including the terms (93.2) and (93.3), arising from change of spheroid, give us finally:

$$\begin{aligned} (\varrho + h)d\phi = {} & \sin\phi\cos\omega\, dx_0 + \sin\phi\sin\omega\, dy_0 - \cos\phi\, dz_0 \\ & + (\upsilon e^2 \sin\phi\cos\phi/a)da + (\upsilon\bar{e} + \varrho/\bar{e})\sin\phi\cos\phi\, df \end{aligned} \tag{94.2}$$

$$\begin{aligned} dh = {} & -\cos\phi\cos\omega\, dx_0 - \cos\phi\sin\omega\, dy_0 - \sin\phi\, dz_0 \\ & -(a/\beta)da + (\upsilon\bar{e}\sin^2\phi)df \ . \end{aligned} \tag{94.3}$$

Corresponding changes in azimuth and zenith distance are obtained by substituting (94.2) and (94.3) in (26.1), which apply just as much to a change of geodetic coordinates as to a change to astronomic coordinates. The resulting equations are not, of course, independent and may not be used as well as (94.2) and (94.3) when required as observational equations. Either can, however, be used instead of (94.2) and (94.3).

95. The three equations (94.1), (94.2) and (94.3) enable us to bring adjacent surveys, say P and Q, into sympathy, through the geodetic coordinates of their common points. If dx_0, dy_0, dz_0, da, df are "corrections" to the Q system of coordinates, then $d\omega$ is the longitude of a point in the P system minus the longitude of the same point in the Q system; and similarly for $d\phi$, dh. Three observational equations are thus obtained for each common point to determine the five unknowns[7], dx_0 etc. When dx_0 etc. have been found, the equations can be used to determine corrections $d\omega$, $d\phi$, dh to the old coordinates of all points in Q (including the geodetic origin of Q) to bring them into sympathy with P. It is, of course, assumed that both surveys have been computed and adjusted in

[7] Three, if the elements of both spheroids are known and the terms in da, df can be evaluated numerically.

accordance with the principles formulated in this paper, so that final discrepancies between common points can be considered local and random and not due to such systematic causes as faulty orientation of a base spheroid, arising from incorrect origin conditions and leading to a wrong use of astronomic observations in adjustment. Few surveys can be said to have avoided such pitfalls altogether and most will need a measure of re-computation before they can be joined with assurance.

No alteration is envisaged in the *orientation* of either geodetic system, although the satisfaction of the conditions (26.1) at the origin of either survey cannot have been error-free. If the procedure recommended in the section on triangulation adjustment has been followed, every astronomic observation, wherever made, will have contributed to setting up and maintaining correct orientation. The only way of further improving the result would be a complete re-adjustment together of the two surveys.

Figure of the Earth

96. The Eqs. (94.1), (94.2) and (94.3) also enable us to determine a geodetic system which departs as little as possible from the astronomic system – a problem which can be considered the three-dimensional extension of the old problem of finding a spheroid which best fits the geoid through minimizing the differences between astronomic and geodetic latitudes and longitudes. There are certain practical advantages in having such a geodetic system, provided it is not changed too often.

An extensive survey on a common geodetic coordinate system is necessary for the purpose, or several surveys which have been brought onto a common system by the method of the last section. Before long, surveys on a common geodetic system spanning the oceans may well have become available, running right round the the Earth and closing on themselves.

97. Wherever astronomical longitude has been measured, we write for $d\omega$ in (94.1) the astronomic minus the geodetic longitude; and similarly for $d\phi$ in (94.2). If astronomic azimuth has been measured, it may be used instead of, but not in addition to, one of these equations by substituting (94.1) and (94.2) in (26.1) and using the result as an azimuth equation in which dx is the astronomic minus the geodetic azimuth. In (94.3), $d\not{h}$ becomes the spirit-levelled minus the geodetic height, subject to the considerations discussed in § 90. Spirit-levels are most likely to be available in the flatter areas, where they can probably be used without gravity corrections. It may be argued that geodetic heights should not be included since these are vitiated by atmospheric refraction. If so, Eqs. (94.3) could be given a low weight or ignored. Consistent geodetic heights can nevertheless be obtained, despite atmospheric refraction, over large areas; and it is already becoming clear that the prejudice against "trigonometric heights" derives more from faulty methods of computing them than from any effect of refraction on properly conducted observations.

The resulting observational equations are solved for dx_0, etc., which are finally substituted in similar equations to derive the corrections $d\omega$ etc. to add to old values in order to derive new coordinates of all points on the new geodetic system.

98. The centre of the new spheroid will not necessarily coincide with the centre of mass of the Earth, although it is unlikely to be far away if the survey is sufficiently extensive. We could, however, obtain average values of dx_0, dy_0, dz_0 which would shift the centre of the spheroid to the centre of mass from lunar observations at a number of points (see § 84). These values of dx_0, dy_0, dz_0 would then be substituted in (94.1), (94.2) and (94.3) before solving to obtain values of the remaining two unknowns da and df. In this way we should obtain a new geodetic system which best fits the astronomic system, subject to the condition that the centre of the spheroid coincides with the centre of mass of the Earth, so far as this may be determinable from lunar observations. We should also obtain, by back substitution in (94.1), (94.2) and (94.3) and evaluation of residuals, the deflections and a value of N (see § 90) at every point in the survey in relation to the new geodetic system. These would be directly comparable with the same results obtained by Stokes' integration of gravity anomalies in relation to the same spheroid, apart form the smoothing and first-order approximation inherent in the Stokes's method.

Index of Main Symbols

(Figures in brackets are references to paragraphs where first used).

l^r	A general unit vector (§ 7)
l_r	Covariant components of l^r (§ 8)
l^r, m^r, n^r	A general set of mutually orthogonal unit vectors, right-handed in that order (§ 10)
A^r, B^r, C^r	A set of Cartesian unit vectors, right-handed in that order (§ 12)
λ^r, μ^r, ν^r	A right-handed set of unit vectors defined as follows: (§ 14)
λ^r	Unit "parallel" vector perpendicular to μ^r, ν^r (§ 14)
μ^r	Unit "meridian" vector lying towards the North in the third coordinate surface, coplanar with ν^r, C^r (§ 14)
ν^r	Outward drawn unit vector normal to the third coordinate surface (§ 14)
ω	Longitude of ν^r in relation to A^r, B^r, C^r (§ 15)
ϕ	Latitude of ν^r in relation to A^r, B^r, C^r (§ 15)
h	Geodetic height. This is the third coordinate in geodetic system of coordinates (§ 16)
α	Azimuth measured eastwards from μ^r (§ 18)
β	Zenith distance measured from ν^r (§ 18)
D	Declination of l^r. The angle between l^r and the plane A^r, B^r (§ 19)
H	Origin hour angle of l^r. The angle between the plane A^r, C^r and a plane parallel to C^r and l^r (§ 19)
x^r	A set of general coordinates in the following order; (§ 20)
x^1	Longitude
x^2	Latitude

x^3	Geodetic height or geopotential
x, y, z	Cartesian coordinates in the directions A^r, B^r, C^r (§ 20)
a, b, c	Cartesian components of the unit vector l^r (§ 23)
ϱ, υ	Principal curvatures of the base spheroid (§ 31)
a	Major semi-axis of the base spheroid (§ 31)
e	Eccentricity of the base spheroid (§ 31)
$\bar{e}$	$= \sqrt{1-e^2}$ (§ 31)
f	Flattening of the base spheroid. $e^2 = 2f - f^2 = 1 - \bar{e}^2$ (§ 92)
s	Length of a line (§ 37)
g	Measured gravity (§ 88)
N	Separation of spheroid and geoid (§ 90)
dx_0, dy_0, dz_0	Cartesian coordinates of the centre of mass of the Earth § 84)

(*Note:* ω, ϕ, α, β etc. of the same point or line etc. will not, of course, be equal in different coordinate systems. Whenever two coordinate systems are considered together one is denoted by barred notation, e.g. $\bar{\omega}$, $\bar{\phi}$, $\bar{\alpha}$, $\bar{\beta}$. In some cases, clear from the context, barred notation is also used for a different point in the same coordinate system).

Editorial Commentary

This is unquestionably the most elegant and polished of Hotine's unpublished reports, and it completely deserves the praise that Marussi lavished on it (see our Editorial Introduction and Marussi's tribute reprinted in Whitten's memorial lecture on pp. 171–183).

It encompasses much of the material contained in the two previous reports to the Toronto Assembly in 1957, but curiously *omits* any mention of the differential geometry of the Earth's gravitational field and the Marussi tensor. We believe this was not accidental, but indicative of the fact that first and foremost Hotine was a *practical geodesist*, not merely a desk-bound theorist. This fact is often missed by anyone who casually dips into MG and shrinks away, having been intimidated by its mathematical richness.

Indeed, the perfection of the *Primer* is its only imperfection. As in the writings of Gauss, here we see only the inexorable deduction of the results without any of the scaffolding which led the author to his approach and formulation. It is simply a masterpiece, and if asked which *one* paper of Hotine one should read to gauge the measure of the man and his thought, this would be the unequivocal choice. It is by no means an easy read, but one which repays careful study and consideration.

The content of the report is essentially reproduced with some abridgement in Chapters 25–27 of MG. The *Primer* has an advantage over MG in that it is almost self-contained – even to the extent that index notation of the tensor calculus is re-explained! Actually the use of the tensor calculus per se is very limited and really only index notation and the differentials of the coordinates are required. Roughly speaking, one could almost characterize the mathematical content of the report as three-dimensional analytic geometry as applied to practical geodesy and photogrammetry.

The Venice Symposium was Hotine's *finest hour*, and his report dominated the meeting. Although the symposium had only *19* participants, we are fortunate to have an excellent record of it in Dufour's report (1959a). Subsequent reactions and critiques of the *Primer* were given in Dufour (1959b), Wolf (1963), Näbauer (1965) and Dufour (1968). Chovitz (1974) assessed the relevance of Hotine's methods for the North American Datum.

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6 The Third Dimension in Geodesy[1]

Introduction

In a previous paper (Hotine 1957a, § 40 to § 49) the changes in various functions of position in the gravitational field along lines of finite length have been considered. Some simplification and added precision arise when certain measurements are made at both ends of the line and this case is now considered in particular relation to change in the potential, which is compared with the corresponding change in geodetic height. The results are used to demonstrate the approximations inherent in the method of geoidal sections. Practical tests of the formulae are proposed as a positive item of geodetic research.

A Lemma

1. A scalar function of position F can be expanded along a line of finite length s as follows:

$$(\bar{F}-F) = sF' + \frac{s^2}{2}F'' + \frac{s^3}{6}F''' + \frac{s^4}{24}F^{iv} + \dots$$

in which barred quantities (e.g. $\bar{F}$) refer to values at the far end of the line and unbarred quantities to the near end. Superscripts refer to successive differentiations with respect to s. If these differential coefficients are measured in the same sense at the far end, that is in the direction of the line produced and not in the back direction, then the corresponding expansion from the far end of the line is obtained by interchanging bars and changing the sign of s. Thus:

$$(\bar{F}-F) = s\bar{F}' - \frac{s^2}{6}\bar{F}'' + \frac{s^3}{6}\bar{F}''' - \frac{s^4}{24}\bar{F}^{iv} + \dots .$$

In the mean,

$$(\bar{F}-F) = \frac{s}{2}(F'+\bar{F}') + \frac{s^2}{4}(F''-\bar{F}'') + \frac{s^3}{12}(F'''+\bar{F}''') + \dots . \tag{1.1}$$

Now the differential coefficients can be considered as functions of position, defined at all points along the line, and similarly expanded as:

[1] Report dated 22 June 1960 (Tolworth) and presented to I.A.G. Helsinki Assembly 1960.

$$(\bar{F}'+F') = s\,F''+\frac{s^2}{2}\,F'''+\frac{s^3}{6}\,F^{iv}+\ldots$$

$$= s\,\bar{F}''-\frac{s^2}{2}\,\bar{F}'''+\frac{s^3}{6}\,\bar{F}^{iv}-\ldots$$

so that

$$0 = s\,(F''-\bar{F}'')+\frac{s^2}{2}\,(F'''+\bar{F}''')+\frac{s^3}{6}\,(F^{iv}-\bar{F}^{iv})+\ldots \tag{1.2}$$

and by direct expansion as in (1.1)

$$0 = (F''-\bar{F}'')+\frac{s}{2}\,(F'''+\bar{F}''')+\frac{s^2}{4}\,(F^{iv}-\bar{F}^{iv})+\ldots \tag{1.3}$$

with similar equations of higher order starting with the fourth differentials.

We can eliminate the terms containing either the second or the third differentials from (1.1), (1.2) and (1.3) *but not both*; and thereafter one of each succeedding pair of terms. Consequently, we may say that either of the following expansions

$$(\bar{F}-F) = \frac{s}{2}\,(F'+\bar{F}')+\frac{s^2}{12}\,(F''-\bar{F}'') \tag{1.4}$$

or

$$(\bar{F}-F) = \frac{s}{2}\,(F'+\bar{F}')-\frac{s^3}{24}\,(F'''+\bar{F}''') \tag{1.5}$$

is correct to a fifth order, subject to the usual conditions relating to continuity, differentiability and convergence. We can reasonably be assured from intuitive physical consideration that these conditions are met in the problems we are going to discuss. We may not, however, know how rapidly the original series converges and what will be the effect of neglecting terms of the fifth and higher orders. This will have to be determined experimentally.

When F is the potential, then the second-order terms (F'' etc.) can be measured at the two ends of the line, whereas the third-order terms cannot. Accordingly, we shall develop the terms in (1.4), but to the degree of accuracy at which we are working, the two Eqs. (1.4) and (1.5) are really equivalent and we can say that

$$(F''-\bar{F}'') = -\frac{s}{2}\,(F'''+\bar{F}''')\ .$$

The result is equally true for vector and tensor functions, provided the coefficients are obtained by successive covariant differentiations along the line.

Potential

2. If F in (1.4) is the potential V, then $\dfrac{\partial V}{\partial s}$ is the resolved part of the gravitational force in the direction of the line, so that

$$\frac{\partial V}{\partial s} = -g \cos \beta \ . \tag{2.1}$$

β is the zenith distance of the line measured from the astronomical zenith.

3. It has been shown in a previous paper (Hotine 1957a) how various quantities may be expressed in terms of the five parameters of the gravitational field, χ_1, χ_2, τ, γ_1, γ_2 (see list of symbols at end of this paper), and how these parameters may be measured (Hotine 1957a, §§ 32, 33). The variation of certain quantities along an atmospherically refracted ray have also been investigated (Hotine 1957a, § 42 to § 49) on the sole assumption that the gradient of atmospheric density is everywhere in the direction of the plumb line or astronomical zenith. This is the least mathematical assumption which can be made about atmospheric refraction and is equivalent to assuming that there is no lateral refraction in the model atmosphere. In particular, we have, after correcting a slight misprint in Hotine (1957a, § 48):

$$\begin{aligned}\frac{\partial \beta}{\partial s} &= \chi_0 - (\chi_1 \sin^2 \alpha + 2\tau \sin \alpha \cos \alpha + \chi_2 \cos^2 \alpha) \sin \beta \\ &\quad - (\gamma_1 \sin \alpha + \gamma_2 \cos \alpha) \cos \beta \\ &= \chi_0 - \chi_\alpha \sin \beta - (\gamma_1 \sin \alpha + \gamma_2 \cos \alpha) \cos \beta \ . \end{aligned} \tag{3.1}$$

In this expression α is the azimuth of the line; χ_0 is its curvature and χ_α is the normal curvature of the equipotential surface in azimuth α.

By differentiating (2.1) we have

$$\frac{\partial^2 V}{\partial s^2} = -g \cos \beta \left(\frac{\partial \log g}{\partial s} \right) + g \sin \beta \, \frac{\partial \beta}{\partial s} \ . \tag{3.2}$$

But from Eqs. (16.4) and § 43 (in Hotine 1957a)

$$\frac{\partial \log g}{\partial s} = \gamma_1 \sin \alpha \sin \beta + \gamma_2 \cos \alpha \sin \beta - \left(\chi_1 + \chi_2 + \frac{2\tilde{\omega}^2}{g} \right) \cos \beta \ , \tag{3.3}$$

in which $\tilde{\omega}$ is the angular velocity of the Earth's rotation and all points of the line are assumed to be free air.

Combining this last equation with (3.2), we have

$$\begin{aligned}\frac{\partial^2 V}{\partial s^2} &= (g \sin \beta) \chi_0 - (g \sin^2 \beta) \chi_\alpha - 2g \sin \beta \cos \beta (\gamma_1 \sin \alpha + \gamma_2 \cos \alpha) \\ &\quad + g \cos^2 \beta \left(\chi_1 + \chi_2 + \frac{2\tilde{\omega}^2}{g} \right) \ . \end{aligned} \tag{3.4}$$

Accordingly, Eq. (1.4) gives us the difference of potential along the line as

$$\begin{aligned}(\bar{V} - V) &= -\frac{s}{2} (g \cos \beta + \bar{g} \cos \bar{\beta}) \\ &\quad + \frac{s^2}{12} [\{(g \sin \beta) \chi_0 - (\bar{g} \sin \bar{\beta}) \bar{\chi}_0\} \end{aligned}$$

$$
\begin{aligned}
&-\{(g \sin^2 \beta)\chi_\alpha - (\bar{g} \sin^2 \bar{\beta})\bar{\chi}_\alpha\} \\
&-\{2g \sin\beta \cos\beta (\gamma_1 \sin\alpha + \gamma_2 \cos\alpha) \\
&-2\bar{g} \sin\bar{\beta} \cos\bar{\beta} (\bar{\gamma}_1 \sin\bar{\alpha} + \bar{\gamma}_2 \cos\bar{\alpha})\} \\
&+\{g \cos^2\beta (\chi_1 + \chi_2) - \bar{g} \cos^2\bar{\beta}(\bar{\chi}_1 + \bar{\chi}_2)\} \\
&+2\bar{\omega}^2 (\cos^2\beta - \cos^2\bar{\beta})] \ .
\end{aligned}
$$

Barred quantities refer as usual to their values at the far end of the line. Many of the second-order terms in this expression are likely to be negligible but are given in full here for the sake of completeness.

If we confuse g and $\bar{g}$, and divide (3.5) by either, then *to a first order* the difference spirit levels over the line as ordinarily computed would be $\frac{1}{2}$ s $(\cos\beta + \cos\bar{\beta})$, which is the same as the difference in "trigonometrical heights" as ordinarily computed, using zenith distances measured from the astronomical zenith or plumb line. Refraction does not enter the result computed to this order of accuracy. The second-order term on account of refraction would be very nearly $\frac{1}{12}\, s^2(\chi_0 - \bar{\chi}_0)$, which is zero if the curvature at the two ends is the same, as indeed is usually assumed. The second-order terms containing normal curvatures of the equipotential surfaces (χ_α) and variation in gravity (γ_1, γ_2) may well be more significant, particularly in gravitationally disturbed areas.

Spheroidal Heights

4. The full theory of three-dimensional geodetic coordinate systems has been developed in previous papers (Hotine 1957b, 1959). The recommended system consists of latitude ϕ and longitude ω measured on a base spheroid; and heights h measured along the straight normals to this spheroid, subject to stated conditions for the initial orientation of the spheroid (Hotine 1959, § 24 to § 28).

We can obtain the change in geodetic or spheroidal heights over the line by putting $V = h$; $g = -1$; and $\gamma_1 = \gamma_2 = 0$ in (3.1) and (3.2). (But the Laplacian of V is no longer $2\bar{\omega}^2$.) In the result we have

$$
\begin{aligned}
(\bar{h} - h) = \frac{s}{2}(\cos\beta + \cos\bar{\beta}) - \frac{s^2}{12}\ &[(\chi_0 \sin\beta - \bar{\chi}_0 \sin\bar{\beta}) \\
&-(\chi_\alpha \sin^2\beta - \bar{\chi}_\alpha \sin^2\bar{\beta})] \ . \qquad (4.1)
\end{aligned}
$$

In this expression, β, $\bar{\beta}$ are now measured from the geodetic zenith, that is the normal to the base spheroid; χ_0 remains the actual curvature of the ray; and χ_α is the normal curvature in azimuth α of the h = const. surface passing through the point:

$$
\chi_\alpha = \frac{\cos^2\alpha}{(\varrho + h)} + \frac{\sin^2\alpha}{(\nu + h)} \ ,
$$

in which ϱ, ν are, as usual, the principal curvatures of the base spheroid. This last expression may usually be approximated as the reciprocal of a mean radius of curvature.

5. Comparison with the exact formulae in previous papers indicates that Eq. (4.1) introduces a negligible computational error (over and above the effect of uncertain refraction curvature). For instance, over a line 50 miles in length in the worst azimuth, the error in height is no more than 3 mm in 2700 meters, that is about one part in a million.

Again note that the only effect of refraction is in the difference of end curvatures in the second-order term.

Potential and Spheroidal Heights

6. We *define* the vector deflection at a point in space as

$$\delta^r = (\nu^r) - \nu^r \ , \tag{6.1}$$

in which (ν^r) is a unit vector in the direction of the astronomical zenith and ν^r is a unit vector in the direction of the outward-drawn normal to the base spheroid.

7. This rigorous definition accords with the usual first-order theory of deflections. If we bracket astronomical quantities, then the (A minus G) latitude is $\delta\phi = (\phi) - \phi$ and the (A minus G) longitude is $(\omega) - \omega$. From Hotine (1959, § 22), we have

$$\begin{aligned}(\nu^r) = {} & \lambda^r \cos(\phi) \sin \delta\omega + \mu^r [\cos\phi \sin(\phi) - \sin\phi \cos(\phi) \cos\delta\omega] \\ & + \nu^r [\sin\phi \sin(\phi) + \cos\phi \cos(\phi) \cos\delta\omega] \ .\end{aligned} \tag{7.1}$$

The component of deflection in the direction of the geodetic parallel, λ^r, is accordingly

$$\delta^r \lambda_r = \cos(\phi) \sin\delta\omega = \cos\phi \, \delta\omega \quad \text{to a first order} \ .$$

The component of deflection in the direction of the geodetic meridian μ^r is

$$\begin{aligned}\delta^r \mu_r &= \cos\phi \sin(\phi) - \sin\phi \cos(\phi) \cos\delta\omega \\ &= \delta\phi \quad \text{to a first order} \ .\end{aligned}$$

And the component of deflection in the direction of the geodetic zenith ν^r is

$$\begin{aligned}\delta^r \nu_r &= \sin\phi \sin(\phi) + \cos\phi \cos(\phi) \cos\delta\omega - 1 \\ &= 0 \quad \text{to a first order.}\end{aligned}$$

8. Next we find the component of deflection in the direction of a unit vector l^r in astronomical azimuth (α) and zenith distance (β); equivalent to geodetic azimuth α and zenith distance β. The vector can be expressed (see Hotine 1959, § 18) in either of the following ways:

$$\begin{aligned}l^r &= (\lambda^r) \sin(\alpha) \sin(\beta) + (\mu^r) \cos(\alpha) \sin(\beta) + (\nu^r) \cos(\beta) \\ &= \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \nu^r \cos\beta \ .\end{aligned} \tag{8.1}$$

Accordingly, the component of deflection in the direction l^r is from (6.1):

$$\Delta = (\nu^r l_r - \nu^r l_r = \cos(\beta) - \cos\beta \ . \tag{8.2}$$

This result is rigorously true whatever the magnitude of the deflection, but it accords with the normal first-order theory, in which small deflections, equivalent to small rotations, are compounded like vectors. If we write $\delta\beta = (\beta) - \beta$ and use Hotine (1959) § 26, then (8.2) becomes to a first order

$$\begin{aligned} \Delta &= -\sin\beta\cdot\delta\beta = \sin\alpha\sin\beta(\cos\phi\cdot\delta\omega) + \cos\alpha\sin\beta\cdot\delta\phi \\ &= \sin\alpha(\cos\phi\cdot\delta\omega) + \cos\alpha\cdot\delta\phi \quad \text{if } \beta \text{ is near } 90^\circ \ , \end{aligned}$$

and this is the usual first-order expression.

9. We denote the component of deflection at the far end of a line of finite length s (not necessarily straight) by

$$\bar{\Delta} = (\bar{\nu}^r)\bar{l}_r - \bar{\nu}^r\bar{l}_r = \cos(\bar{\beta}) - \cos\bar{\beta} \ .$$

It is then evident from § 3 and (4.1), to a *first order* and subject to the limitations of § 3, that

$$\frac{s}{2}(\Delta + \bar{\Delta}) = \frac{s}{2}\{\cos(\beta) + \cos(\bar{\beta})\} - \frac{s}{2}\{\cos\beta + \cos\bar{\beta}\} \tag{9.1}$$

= Rise in spirit levels minus the rise in spheroidal heights in proceeding from the unbarred to the barred end of the line .

Geoidal Sections

10. In the normal classical theory $\frac{s}{2}(\Delta + \bar{\Delta})$ is reckoned to be the difference, over the length of the line, in the (underground) separation between geoid and spheroid. This clearly involves the further assumption that there is no change in deflection in proceeding from a point on the topographic surface to a point "vertically" below on the geoid (or spheroid). There is certainly no physical justification for this assumption. Geometrically, it would require the plumb line to have zero curvature underground, and the extent to which this is not so would introduce a direct first-order error in the first-order theory. It might be so in practice for points not much above the geoid (or spheroid) in flat country, but this does not seem to have been satisfactorily demonstrated.

Conclusion

11. Whether or not the less drastic assumptions in Eq. (9.1) ar justified depends on whether the first-order assumption, particularly as applied to Eq. (3.5), is justified. This can only be settled by experiments, which so far have not been done, by comparison between results from (3.5), with and without the second-order terms, and a direct measure of potential by means of spirit levels.

To run lines of spirit levels over mountains is a difficult and costly operation but it should be done to obtain essential topographic information. The use of (3.5) instead of spirit-levelling, if it leads to results of sufficient accuracy, might well have far-reaching practical and economic consequences.

To test Eq. (3.5), we need to measure the end-curvatures of the refracted ray, gravity, and the five parameters of the gravitational field at, say, every main triangulation station. Theoretically, all this is possible, but much research work would have to be done first on instrumentation. In addition to the economic results, such a programme of research would probably contribute far more to our knowledge of the gravitational field than the present collection of random data.

Index of Main Symbols

ϕ Latitude
ω Longitude
h Geodetic of spheroidal height
V Potential
g Gravity
$\tilde{\omega}$ Angular velocity of the Earth's rotation
χ Curvature
χ_0 Curvature of a refracted ray
χ_1 Normal curvature of the equipotential surface in the direction of the astronomical, parallel
χ_2 Normal curvature of the equipotential surface in the direction of the astronomical, meridian
τ Geodesic torsion in the direction of the astronomical, meridian
χ_α Normal curvature in azimuth α.

$$= \chi_1 \sin^2 \alpha + 2\tau \sin \alpha \cos \alpha + \chi_2 \cos^2 \alpha \text{ (see Hotine 1957a, § 22)}$$

γ_1 Arc rate of change of (log g) in the direction of the astronomical, parallel
γ_2 Arc rate of change of (log g) in the direction of the astronomical, meridian
α Azimuth
β Zenith distance
λ^r Unit vector in the parallel direction
μ^r Unit vector in the meridian direction
ν^r Unit vector in the zenith direction
l^r A general unit vector

Editorial Commentary

As Hotine said in his abstract, this report was a continuation and refinement of his two Toronto reports *Metrical Properties of the Earth's Gravitational Field, Geodetic Coordinate Systems*, and his *A Primer on Non-Classical Geodesy* (Venice, 1959). In particular, his concern was with the additional precision and simplification which occurs when certain measurements are made at both ends of a line of observation. Hence, in effect, this report dealt with the observational aspects of certain situations considered in these three reports. In particular, special attention is paid to the gravitational potential and spheroidal heights.

The material in this report is developed in somewhat greater detail in Chapter 25, §§ 16–25 of MG.

References to Paper 6

Hotine M (1957a) Metrical properties of the Earth's gravitational field. I.A.G. Toronto
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7 Harmonic Functions[1]

1. The only harmonic functions so far used in geodesy seem to be spherical and spheroidal harmonics, and very occasionally ellipsoidal harmonics. All these are special cases of a much more general class of harmonic functions, which may well find applications in modern geodesy, although I have not yet attempted to work out the applications in detail.

2. Suppose first that H is any scalar function of position, and that the $\mathbf{n}^{\text{th}}$-order tensor

$$H_{rst\ldots n} \tag{1}$$

is formed by $\mathbf{n}$ successive covariant differentiations of H.

3. The following tensor equation

$$H_{rts\ldots n} = H_{rst\ldots n} \; ,$$

in which any two indices have been interchanged, is clearly true in Cartesian coordinates, when the covariant derivatives become ordinary commutable derivatives, and is therefore true in any coordinates. The tensor (1) is accordingly symmetrical in any two indices and therefore has $\frac{1}{2}(\mathbf{n}+1)(\mathbf{n}+2)$ distinct components at most.

4. Now suppose that H is a harmonic function, and that g^{jk} is the metric tensor. Then the Laplacian of the tensor (1) is

$$g^{jk} H_{rst\ldots njk} = (g^{jk} H_{jk})_{rst\ldots n} = 0$$

so that all components of the tensor (1) are harmonic functions.

5. We may similarly write

$$g^{rs} H_{rst\ldots n} = 0 \; . \tag{2}$$

The tensor on the left-hand side is of order $(\mathbf{n}-2)$ with at most $\frac{1}{2}[(\mathbf{n}-2)+1][(\mathbf{n}-2] = \frac{1}{2}\mathbf{n}(\mathbf{n}-1)$ distinct components. If H is a harmonic function, there are then $\frac{1}{2}\mathbf{n}(\mathbf{n}-1)$ relations (2) between the components of the original tensor (1), which can accordingly have

[1] Undated report (Tolworth) presented at the Second Symposium on Three-Dimensional Geodesy (Cortina d'Ampezzo, 1962).

$$\tfrac{1}{2}(\mathbf{n}+1)(\mathbf{n}+2)-\tfrac{1}{2}\mathbf{n}(\mathbf{n}-1)=2\mathbf{n}+1$$

independent components at most.

6. Next, for the invariant

$$A^{rst\ldots n}H_{rst\ldots n} \tag{3}$$

in which the multiplying tensor $A^{rst\cdots n}$ is constant under covariant differentiation; that is, all its components are absolute constants in Cartesian coordinates, and are the transforms of Cartesian constants in other coordinates. Then each of the components to be summed in Eq. (3) is a harmonic function and there will be at most $(2\mathbf{n}+1)$ such independent functions. The constant $A^{rst\cdots n}$ should properly by symmetrical in any two indices and have its components further restricted to $(2\mathbf{n}+1)$ by some such relation as (2), but this is not essential. If these conditions are not fulfilled, then there will be more than $(2\mathbf{n}+1)$ terms in the summation (3), but some of them will not be independent.

7. Now consider the summation in series of all invariants such as (3):

$$K=\sum_{n=0}^{\infty} A^{rst\ldots n}H_{rst\ldots n}\ . \tag{4}$$

We assume, subject to test in individual cases, that the successive derivatives of H exist, and together with the constants $A^{rst\cdots n}$, form a convergent series having a finite sum K. It is evident by taking the Laplacian of both sides of Eq. (4) that K is a harmonic function; and considering there are $(2\mathbf{n}+1)$ degrees of freedom in each term of (4), and the function H is at choice, it is evident that K is a very general harmonic function.

8. It can be shown without difficulty that Eq. (3) represents all the *spherical* harmonics of $\mathbf{n}^{th}$-order if $H = 1/r$, the reciprocal of the radius vector, if the differentiations are performed in Cartesian coordinates, and if

$$g_{rs}A^{rst\ldots n}=0\ .$$

But it is well known that *any* harmonic function K can be expressed in a series of spherical harmonics, even though subject to these arbitrary conditions, in such a form as Eq. (4). It is reasonable to assume, therefore, that any harmonic function K can be expressed by Eq. (4), without the restrictions of spherical harmonics, provided the derivatives of H exist and provided Eq. (4) is a convergent series.

9. In geodetic problems, it might be more convenient to make $H=\tan^{-1}u$, where u is that parameter in spheroidal coordinates which is constant over any one of the spheroidal coordinate surfaces. But the main merit of the more generalized expansion (4) may be to make H the actual external attraction potential of the Earth, in which case K could be $1/r$ or any other desired material harmonic function. We can further introduce the geopotential W in a field rotating with angular velocity $\tilde{\omega}$ by making

$$H = W - \tfrac{1}{2}\,\tilde{\omega}^2(x^2+y^2)\ .$$

The second term on the right would vanish in the third order and subsequent terms of Eq. (4) in Cartesian, and therefore in any coordinates.

10. We could re-write the constant tensor in (3) as

$$A^{rst\ldots n} = A\lambda^r\mu^s\upsilon^t\ldots\tau^n\ ,$$

in which λ^r, μ^s, etc. are *fixed* arbitrary unit vectors and A is a Cartesian constant, so that $(2\mathbf{n}+1)$ independent components are still preserved. If all the fixed vectors are the same, say λ^r, and $A = s^n/n!$, then the expansion (4) becomes the tensor form of Taylor's series expressing the potential in the same field at another point distant s in the fixed direction λ^r. This may suggest that the expansion (4) is unnecessarily general in practice. Suppose, for example, that H and K are respectively the "normal" (or "geodetic") and the actual potentials at a point in space. It should always be possible to find another point in the "normal" field where the "normal" potential is K and so relate the two potentials by a Taylor series.

11. It should also be possible to find another point in the "normal" field where the "normal" potential is the actual potential at the original point. If the astronomical latitude and longitude at the original point are $(\phi,\ \omega)$, then the locus of points in the "normal" field having $(\phi,\ \omega)$ as the direction of the "normal" plumb line is known. At some point on this locus, the "normal" potential will be the actual potential at the original point. The whole procedure would be simplified by tables for the "normal" field.

12. The "normal" value of gravity at the second point would not, however, be the actual gravity at the first point, although the two might not differ very much in practice. If it were, then tables for the "normal" field would enable us to determine the actual potential at any point from a measure of gravity at that point without any process of integration, except what is implicit in the use of the "normal" field tables. The possibility, however, needs investigation, since this method might well give as accurate results as the first-order Stokes' theory.

13. The new theory of harmonic functions given in this paper does not stand or fall by this one possible application, which is given solely as an example to show the power of the new theory as a weapon of geodetic research. We are perhaps too much wedded to Green and Stokes, and to spherical harmonics, and may need to broaden our horizons.

Editorial Commentary

This brief paper was published as an appendix to Baetsle's comprehensive report (1963) of the Second Symposium on Three-Dimensional Geodesy (Cortina d'Ampezzo, 1962). Actually, Hotine's name does not appear on the paper, but his authorship was indicated in the text of Baetsle's report. See also the discussion of this paper in Baetsle (1963, pp. 43–44).

Essentially this paper was a preliminary version of the material in the first eight sections of Chapter 21 of MG. The treatment was somewhat less concise than that given in MG, and the content of § 9–§ 13 in it are not precisely duplicated in MG. Thus this paper furnishes a slight variant of the elaborate theory of spherical harmonics in Chapter 21 of MG. See also the comments of Chovitz in the second section of his preview article *Hotine's Mathematical Geodesy* re-printed in this monograph. As he noted, although the topic has been exhaustively discussed in the mathematical literature, see in particular Hobson (1911, 1955–65), much of Hotine's treatment was original and illustrated his personal touch. In MG (see § 7 p. 154), Hotine commented that his approach was based on the work of J.C. Maxwell, i.e. see Eq. (7.1) or MG, Chapter 21, Eq. (006). Maxwell's original exposition (1891) (MG Chap. IX, pp. 194–231), is difficult; however, a lucid elementary introduction to it can be found in MacRobert (1927) (MG Chap. XIII, pp. 231–240). In reading Hotine's discussion today, one feels its power, and senses that he has broken new ground, yet one does not quite know how to proceed further. Surely the last word has not been written on the subject.

References to Editorial Commentary

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8 Downward Continuation of the Gravitational Potential[1]

1. Clerk Maxwell's form of the attraction potential at an external point, distant r from an internal origin, can be written in the tensor form

$$-\frac{V}{G} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I^{stu\ldots(n)} \left(\frac{1}{r}\right)_{stu\ldots(n)} , \tag{1}$$

in which the lower indices imply successive covariant derivatives and the I's are constants, symmetrical in any two indices. The negative sign of the potential used in physics is adopted and G is the gravitational constant. This form of the potential is of great value in theoretical investigations; for example it shows at once the invariance of the potential under rotations of the coordinate axes about a fixed origin.

2. By working straight from the mass distribution, it can be shown that Eq. (1) is equivalent, not by individual terms but by partial sums of terms of the same order n, to the usual expansion in spherical harmonics:

$$-\frac{V}{G} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{P_n^m(\sin\phi)}{r^{n+1}} \{C_{nm}\cos m\omega + S_{nm}\sin m\omega\} , \tag{2}$$

in which (ϕ, ω) are spherical polar latitude and longitude from the same internal origin. At the same time, it can be shown that the I's in Eq. (1) – the constant inertia tensors – are given by

$$I^{stu\ldots(n)} = \sum \bar{m}\bar{x}^s\bar{x}^t\bar{x}^u\ldots\bar{x}^{(n)} , \tag{3}$$

in which the summation is taken over all particles of mass $\bar{m}$ located at the point whose coordinates are $\bar{x}^q$. An alternative expression for a continuous distribution of density ϱ is

$$I^{stu\ldots(n)} = \int \varrho\varrho^s\varrho^t\varrho^u\ldots\varrho^{(n)}dv , \tag{4}$$

in which ϱ^q is the position vector of the volume element dv and the integral is taken over the entire volume of the attracting body.

3. Again working straight from the mass distribution, it can be shown that a sufficient – but perhaps not necessary – condition for the convergence of Eqs. (1) or (2) is that the point where the potential is sought should lie outside a sphere,

[1] Undated report (Boulder) presented at the I.A.G. Lucerne Assembly 1967.

centered on the origin, which just contains all the attracting matter. The expression of the potental in spherical harmonics, obtained from satellites outside this sphere of convergence, is not necessarily valid at points on the Earth's surface inside this sphere. This question has been extensively and inconclusively argued.

4. We shall now consider expressions for the potential at external points when the origin is also external. In this case the potential in spherical harmonics is well known to be

$$-\frac{V}{G} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} r^n P_n^m (\sin\phi) \{[C_{nm}] \cos m\omega + [S_{nm}] \sin m\omega\} \tag{5}$$

and a sufficient condition for convergence is that the point where the potential is sought shall lie inside a sphere, centered on the origin, which just touches the attracting body. However, by working straight from the mass distribution and using Eq. (1), we find that the potential can also be expressed as

$$V = (V_0) + (V_s)_0 \varrho^s + \ldots + \frac{1}{n!} (V_{stu\ldots(n)})_0 \varrho^s \varrho^t \varrho^n \ldots \varrho^{(n)} + \ldots , \tag{6}$$

in which ϱ^q is the position vector from the external origin to the point where the potential is sought and the quantities within brackets are values of successive derivatives of the potential at the external origin. Moreover, Eqs. (5) and (6) are equivalent by partial sums of terms of the same order. But Eq. (6) is the same as the Taylor expansion of the potential from the external origin. We conclude that Eq. (6) is convergent within the sphere of convergence of Eq. (5).

5. Next, we propose to continue the potential analytically by means of a Taylor series from a Point P in the diagram (Fig. 1), where Eqs. (1) or (2) are certainly valid, to a Point T where Eqs. (5) or (6) are certainly valid. This operation gives the potential at T as

$$-\frac{V_T}{G} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!} \frac{(-1)^n}{n!} I^{stu\ldots(n)} \left(\frac{1}{r}\right)_{stu\ldots(n)pqr..(m)} \varrho^p \varrho^q \varrho^r \ldots \varrho^{(m)} . \tag{7}$$

The order of summation is important. To reflect the continuation process correctly we must first sum over n in order to derive the potential at P before differentiation and substitution in the Taylor series.

6. Equation (7) can be written as an infinite matrix in the form

$$\begin{matrix} M(1/r) , & -I^s(1/r)_s , & +\frac{1}{2} I^{st}(1/r)_{st} , \ldots \\ M(1/r)_p \varrho^p , & -I^s(1/r)_{sp} \varrho^p , & +\frac{1}{2} I^{st}(1/r)_{stp} \varrho^p , \ldots \\ \frac{1}{2} M(1/r)_{pq} \varrho^p \varrho^q , & \frac{1}{2} I^s(1/r)_{spq} \varrho^p \varrho^q , & +\frac{1}{4} I^{st}(1/r)_{stpq} \varrho^p \varrho^q , \ldots \end{matrix} \tag{8}$$

in which the inertia tensor of zero order is the total mass M. The first row summed represents the potential at P. The second row summed is the first derivative of the potential at P contracted with the fixed bounded vector ϱ^p, and so on.

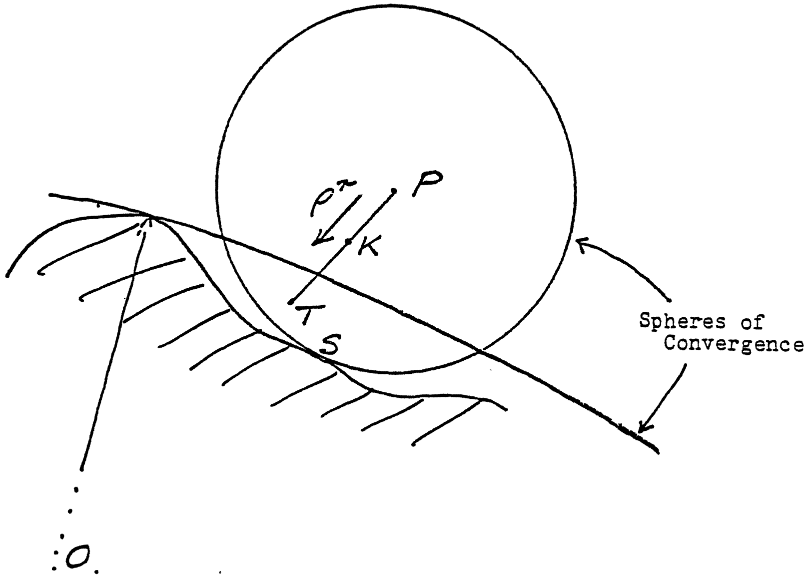

Fig. 1.

The fact that Eq. (7) is convergent implies that the matrix is convergent if the rows are summed first. If, on the other hand, we sum the columns first, and this process is not necessarily valid, then we find after some manipulation that the result would be

$$-\frac{V_T}{G} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I^{stu\ldots(n)} \left(\frac{1}{OT}\right)_{stu\ldots(n)}, \tag{9}$$

in which the derivatives are now evaluated at T. But this is the same as Eq. (1) evaluated at T, and if Eq. (9) correctly represents the potential at T, then Eq. (1) – or (2) – must be convergent at T, even though T lies inside the sphere of convergence of Eq. (1) – or (2) – and so does not satisfy the sufficient condition for convergence. The convergence of Eq. (1) – or (2) at T – accordingly depends on whether interchange of the order of summation in Eqs. (7) or (8) is valid. Sufficient conditions for this interchange have been worked out in most standard texts, and would not usually be satisfied, but necessary conditions do not seem to have been obtained.

7. Some light may be thrown on this question by considering a point K on PT which lies outside the sphere of convergence of Eq. (1) – or (2). In that case, Eq. (1) – or (2) – certainly represents the potential at K and the summation interchange in Eq. (7), leading to Eq. (9), is certainly valid at K. But the two continuation series, Eq. (7), for K and for T, both of which are convergent, must also have the same properties of absolute and uniform convergence because the coefficients of the vectors are the potential at P, and derivatives of the potential at P, and are

therefore the same for both series. The only difference between the two series is the magnitude, but not the direction, of the contracting vector and this does not affect the convergence of either series. Accordingly, if the necessary and sufficient conditions for the summation interchange depend solely on convergence properties, then these conditions would seem to be satisfied at T as well as at K. By suitable choice of the point P we could reach any point on the Earth's surface by this process.

8. Nevertheless, the proof of convergence of Eq. (7) depends on the absence of matter within the sphere PS, and could be invalidated if there were an alternative distribution of matter, nearer to P than the actual distribution, which gives the same potential, and derivatives of the potential, at P as the actual distribution. For example, if the attracting matter consists solely of a point mass at 0, then a homogeneous sphere of the same total mass and of radius less then OP would give rise to the same field at P. In this case, analytical continuation from P could be completely blocked. Whether or not there is such an alternative distribution in the case of a highly irregular body, such as the Earth, does not appear to have been investigated.

9. Suggestions have been made to overcome the difficulty by expressing the potential in spheroidal harmonics as

$$-\frac{V}{G} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_n^m(i \cot \alpha) P_n^m(\sin u)(A_{nm} \cos m\omega + B_{nm} \sin m\omega) \ , \tag{10}$$

in which the spheroidal coordinates (ω, u, α) are given by

$$\begin{aligned} x &= (ae) \csc \alpha \cos u \cos \omega \\ y &= (ae) \csc \alpha \cos u \sin \omega \\ z &= (ae) \cot \alpha \sin u \ , \end{aligned} \tag{11}$$

with (ae) an absolute constant for the (confocal) coordinate spheroids. This constant can be chosen to make one coordinate spheroid approximate closely to the actual surface of the Earth, while enclosing all the attracting matter, in which case Eq. (10) would converge outside this spheroid. However, direct determination of the A_{nm} and B_{nm} from current geodetic measurements is less simple than determination of the spherical harmonic coefficients C_{nm} and S_{nm}. Nevertheless, the two sets of coefficients must be related *for the same mass distribution* in a domain where both series are convergent. In fact, the relationship is linear and is given by

$$\begin{aligned} \begin{bmatrix} A_{nm} \\ B_{nm} \end{bmatrix} = \frac{1\cdot 3\cdot 5 \ldots (2n+1)}{(n+m)!} i^{m+n+1} \Bigg[& \frac{1}{(ae)^{n+1}} \begin{pmatrix} C_{nm} \\ S_{nm} \end{pmatrix} \\ & + \frac{(n-m)(n-m-1)}{2\cdot(2n-1)} \frac{1}{(ae)^{n-1}} \begin{pmatrix} C_{(n-2),m} \\ S_{(n-2),m} \end{pmatrix} \\ & + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4\cdot(2n-1)(2n-3)} \frac{1}{(ae)^{n-3}} \begin{pmatrix} C_{(n-1),m} \\ S_{(n-4),m} \end{pmatrix} + \ldots \Bigg] \end{aligned} \tag{12}$$

together with the inverse equations

$$
\begin{aligned}
i^{(m+n+1)}\begin{pmatrix} C_{nm} \\ S_{nm} \end{pmatrix} = \frac{(ae)^{n+1}(n-m)!}{1\cdot 3\cdot 5\ldots(2n+1)} \Bigg[& \frac{(n+m)!}{(n-m)!}\begin{pmatrix} A_{nm} \\ B_{nm} \end{pmatrix} \\
& + \frac{2n+1}{2}\,\frac{(n+m-2)!}{(n-m-2)!}\begin{pmatrix} A_{(n-2),m} \\ B_{(n-2),m} \end{pmatrix} \\
& + \frac{(2n+1)(2n-1)}{2\cdot 4}\,\frac{(n+m-4)!}{(n-m-4)!}\begin{pmatrix} A_{(n-4),m} \\ B_{(n-4),m} \end{pmatrix} \\
& + \frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6}\,\frac{(n+m-6)!}{(n-m-6)!}\begin{pmatrix} A_{(n-6),m} \\ B_{(n-6),m} \end{pmatrix} + \ldots \Bigg]\,.
\end{aligned} \tag{13}
$$

The same formulae give the zonal coefficients for $m = 0$. It is assumed that the spherical and spheroidal harmonics are both related to the same Cartesian system.

10. Equations (12) and (13) enable us easily to transform a potential, validly expressed by Eq. (2), to Eq. (10) and to continue the potential by means of Eq. (10) almost down to the Earth's surface. Even if Eq. (2) is finally agreed to be convergent down to the surface, it will not converge as rapidly as Eq. (10), which may accordingly be a better method of computation. The operation of transforming to spheroidal harmonics may also be a useful tool in further theoretical investigations.

Editorial Commentary

This report considered several of the standard mathematical questions in classical Newtonian potential theory, viz. analytical continuation and convergence. Both continue to be of contemporary interest, and relative to the latter, as Hotine noted at the end of § 3, "This question has been extensively and inconclusively argued". Over the last 25 years since he wrote those words, nothing but the number of memoirs seems to have significantly changed. An excellent survey of more recent work is given in Moritz (1980).

Following the spirit of his report *Harmonic Functions* (Cortina d'Ampezzo, 1962), Hotine attacked the issue using tensor-theoretic methods. This involved reformulating Maxwell's theory of spherical harmonics in tensor form (for references to the classical theory, see our Editorial Commentary on *Harmonic Functions* paper 7, this Vol.), and giving expressions for the inertia tensor of the attractive mass distributions. As no doubt he would have agreed, he solved neither question but merely displayed the difficulties in a new and tantalizing form. However, he would have been the first to maintain that nothing is lost – and something may well be gained – by looking at an old question in a new form. As Georg Christoph Lichtenberg (1742–1799) said

"Neue Blicke durch die alten Löcher".

A more exhaustive treatment of this material may be found in Chapter 21 of MG, beginning with § 12 and continuing through § 107. Note the footnotes on p. 165 of MG, which address different definitions of the inertia tensor. The definition of the inertia tensor employed by Hotine and McConnell (1931) (see § 1 of the latter's Chap. 18) still differ from those employed in standard physics texts on dynamics, e.g. Landau and Lifshitz (1960) (see § 32 of their Chap. 6), and does not seem as unusual as Hotine believed. It would appear that for potential-theoretic purposes (as Hotine observed) his approach is simply different – and more natural – than the definition employed in dynamical considerations. The important fact is to note this discrepancy.

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9 Curvature Corrections in Electronic Distance Measurements[1]

1. It can be shown (Hotine 1960) that if F is a continuous, differentiable scalar, the expansion

$$(\bar{F}-F) = \frac{1}{2}s(F'+\bar{F}') + \frac{1}{12}s^2(F''-\bar{F}'') \tag{1}$$

along a line of arc length s, is correct to a fourth order. The superscripts refer to successive derivatives of F with respect to s and over-bars indicate values at the far end of the line, while the absence of over-bars indicates values at the near end of the line. It is further assumed that the ordinary Taylor expansion of F along the line exists and is convergent, and that this condition is met, or justified by results, in practical cases.

2. The time (t) of propagation of light or other electromagnetic waves along a line in a medium of refractive index μ is given by the usual formula

$$ct = \int_{P}^{\bar{P}} \mu\, ds \;, \tag{2}$$

in which c is the (constant) velocity of propagation in vacuo and the integral is taken over the curved line from the initial point P to the far point $\bar{P}$. We take the basic scalar of Eq. (1) to be the *indefinite* integral

$$\int \mu\, ds$$

so that

$$ct = \int_{P}^{\bar{P}} \mu\, ds = \bar{F} - F \;. \tag{3}$$

We have also

$$\begin{aligned} F' &= \mu \\ \bar{F}' &= \bar{\mu} \end{aligned} \tag{4}$$

retaining the contention that the unbarred quantities refer to values at the initial point P and over-barred quantities refer to the far point $\bar{P}$.

[1] Undated report (Boulder) presented to I.A.G. Lucerne Assembly 1967.

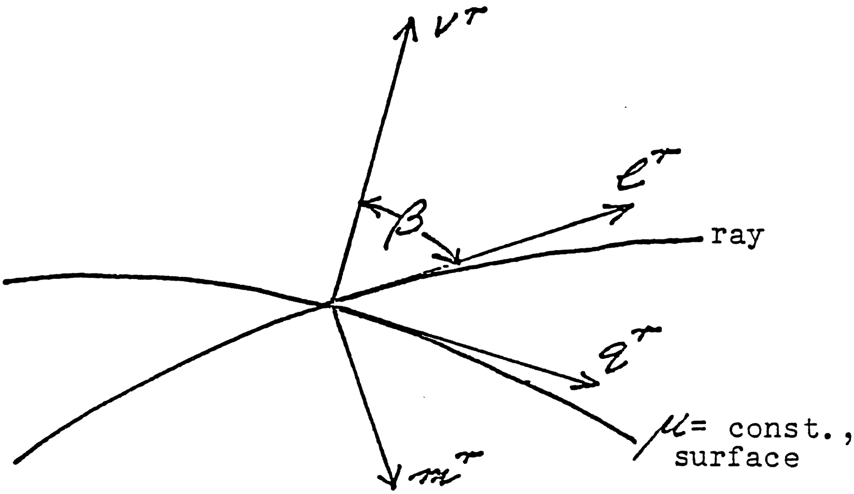

Fig. 1

3. To express the second derivatives we shall have to introduce the laws of refraction and the curvature of the line. In the diagram (Fig. 1) the unit tangent vector of the line in index notation is l^r and the unit normal of the line is m^r. The unit vector υ^r, usually assumed to be in the direction of the geodetic zenith, is normal to the surface of constant refractive index (μ) passing through the point under consideration, and therefore has the same direction as the gradient of μ. The laws of refraction, which can be directly derived from Fermat's variational principle, then require l^r, m^r, υ^r to be coplanar and

$$\chi = (\log n)_r m^r \ , \tag{5}$$

in which χ is the curvature of the line or ray. In this formula, (log n) is the natural logarithm of μ and $(\log n)_r$ in index notation is the gradient of (log n). From the geometry of the figure we have also

$$\begin{aligned} l^r &= \upsilon^r \cos\beta + q^r \sin\beta \\ m^r &= q^r \cos\beta - \upsilon^r \sin\beta \\ (\log n)_r q^r &= 0 \ , \end{aligned} \tag{6}$$

in which q^r is a unit vector tangential to the μ = constant surface in the plane of l^r, m^r, υ^r. Finally we have, for substitution in Eq. (1),

$$\begin{aligned} F'' = \frac{\partial\mu}{\partial s} &= \mu(\log n)_r l^r \\ &= \mu(\log n)_r \upsilon^r \cos\beta \\ &= -\mu(\log n)_r m^r \cot\beta \\ &= -\mu\chi \cot\beta \end{aligned}$$

using in succession Eqs. (4), (6) and (5). Equation (1) becomes finally

$$ct = \tfrac{1}{2}\,s(\bar{\mu}+\mu)+\tfrac{1}{12}\,s^2(\bar{\mu}\bar{\chi}\cot\bar{\beta}-\mu\chi\cot\beta)\ . \tag{7}$$

4. For measured t (half the time for the double journey along the line and return), Eq. (7) can easily be solved for s by iteration in the usual way. The terminal refractive indices are obtained from the standard Barrell and Sears (or Essen and Froome) formula, using measured temperature, pressure and humidity. Ignoring the second-order term, a preliminary value of s is obtained by dividing (ct) by the mean refractive index and this value of s is used to evaluate the second-order term. To complete the evaluation of the second-order term, β and $\bar{\beta}$ can be taken as the observed astronomical or geodetic zenith distance of the curved line, noting that $\bar{\beta}$ refers to the line produced through the far end and not to the reverse zenith distance. To determine the terminal curvatures, we can differentiate the Barrell and Sears (or Essen and Froome) formulae and use Eq. (5) or its equivalent

$$\begin{aligned}\chi &= (\log\mu)_r m^r = -(\log\mu)_r \upsilon^r \sin\beta\\ &= -\frac{1}{\mu}\,\frac{d\mu}{dh}\sin\beta\ ,\end{aligned} \tag{8}$$

in which h refers to geodetic height. In evaluating $d\mu/dh$ lapse rates would have to be assumed or obtained from local meteorological services.

5. The second-order term in Eq. (7) in various approximate forms is known in the literature as the "second velocity correction" or the "velocity component of the curvature correction" and it is of some interest to consider what assumptions are made in approximating this correction. If we differentiate the equation

$$\cos\beta = \upsilon_r l^r$$

covariantly along the line, we have

$$-\sin\beta(d\beta/ds) = \upsilon_{rs} l^r l^s + \upsilon_r l^r_s l^s = -k\sin^2\beta - \chi\sin\beta\ ,$$

in which k is the normal curvature of the μ = constant surface. To a first order we have also

$$\begin{aligned}(\cot\bar{\beta}-\cot\beta) &= d(\cot\beta) = -s\csc^2\beta(d\beta/ds)\\ &= -s(k\csc\beta+\chi\csc^2\beta)\ .\end{aligned}$$

If, in evaluating the small second-order correction, we consider that $\mu = \bar{\mu} = 1$ and χ is constant along the line, then the second-order term, considered as an additive correction to the preliminary value of s, is

$$\frac{1}{12}\,s^3\chi(k\csc\beta+\chi\csc^2\beta)\ .$$

If there is no considerable difference in height over the line, $\csc\beta$ is nearly unity; and if we confuse $-1/k$ with a mean radius R of the Earth and write $f = \chi R$ for the coefficient of refraction, the correction finally becomes

$$-\frac{1}{12}\frac{s^3}{R^2}f(1-f) \ , \tag{9}$$

which is the form given by Saastamoinen (1964) or Höpcke (1964). In micro-wave measurements, it is usual to assume $f = 0.25$. For precise geodimeter measurements, Saastamoinen recommends evaluating the coefficient of refraction from reciprocal zenith distances, or vertical angles, measured at the same time as the geodimeter observations. But if any such special measurements are to be made, the reciprocal zenith distances β, $180° - \bar{\beta}$ can just as easily enter the precise Eq. (7) together with a mean curvature derived also from the reciprocal zenith distance measurements, without making most of the assumptions implicit in Eq. (9).

6. It should be noted that Eq. (8) gives the actual length S of the curved path. A geometric arc-to-chord correction is necessary in addition to provide the length of the straight line joining the end points, and would also be necessary even if the effect of refraction on the velocity of propagation were removed altogether by a two-wavelength method. By expansion from one end of the line, the magnitude of this arc-to-chord correction is easily found to be

$$-\frac{1}{24}s^3\chi^2 \tag{10}$$

to a high degree of accuracy, whether the curve is plane or twisted. The curvature χ can be a mean curvature $\frac{1}{2}(\chi + \bar{\chi})$, or the correction can be evaluated at the two ends and meaned. Introducing the coefficient of refraction $f = \chi R$, the correction is

$$-\frac{1}{24}\frac{s^3}{R^2}f^2 \ ,$$

which can be combined with Eq. (9) in cases where the approximate formula (9) is sufficient.

7. For the adjustment of networks in three dimensions, the chord length is used and no other corrections are required.

Editorial Commentary

The title of this report is somewhat obscure as the "curvature" in question refers not to that of an equipotential surface, but of a refracted ray (a curve) in the Earth's atmosphere. In his characteristic manner of thinking in terms of geometry and basic physical principles, Hotine starts from Fermat's Principle in geometrical optics and applies the Frenet equations to the ray's trajectory. The result is expressed in Eq. (7) which is an elegant blend of geometry and optics. This result is approximate and the second-order term in it is known as the "second velocity correction" or more precisely as the "velocity component of the curvature correction." It is given in several forms and compared with previously known versions which had appeared in the literature.

In MG this material is developed in a more expansive and luxuriant manner in Chapter 24, and the contents of the report correspond to §§ 1–32 of that chapter. The remaining roughly two-thirds of Chapter 24 is concerned with measuring the refractive index and other basic physical properties, e.g. the pressure, and how they affect the problem.

Since this report was written the subject has continued to be of interest, especially due to the more recent use of satellite-to-ground applications like Doppler, GPS, satellite laser ranging, and VLBI. See for example the colloquium proceedings, (Brunner 1984), and the monograph by Iribane and Goodson (1981), which has some geodetic applications in Chapter 8.

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Editorial Commentary

The above Bibliography is an attempt to compile a list of the books and papers written by Martin Hotine. It makes no pretense at completeness, and is intended to highlight Hotine's activities and interests primarily *before* he became seriously involved in differential geodesy. We know of no previous effort to collect such a bibliography, and if Hotine made up a compilation for his own personal use or reference, it has not been found. Probably during his 40 years' service for the British government he contributed other papers and reports which appeared in various official documents, both unter his name and anonymously, especially during the war years. It would be virtually impossible now to locate such material, but we are confident that none of this material is likely to be relevant to his subsequent work in differential geodesy.

In addition to the listed items, Hotine also wrote about a dozen letters which appeared in *The Empire Survey Review* from 1937–1951. We have omitted these items from the Bibliography, since usually they referred to other letters and papers (not necessarily his own) which appeared in this journal, and, in retrospect, they are of limited interest.

The Bibliography was compiled by John Nolton based on a xerox copy of a card index of the Hotine items from the library of the Directory of Overseas Surveys furnished to Charles Whitten (in preparation for his memorial paper), and a list of Hotine papers in *The Empire Survey Review* provided by Herbert W. Stoughton. It is a pleasure for us to acknowledge the valuable assistance provided by these individuals. A special note of thanks is also due to Grace Sollers of the Technical Information Services of the National Geodetic Survey (Rockville, MD) for her help in obtaining copies of various Hotine papers. The collection of geodetic papers in her care is unique, and her skill in locating an obscure reference is matched only by her willingness to help a geodesist in need.

Finally, my thanks to John Nolton for compiling the final bibliography. His help has been invaluable.

Hotine's *Mathematical Geodesy*[1]

Bernard H. Chovitz

1. Martin Hotine's *Mathematical Geodesy* will be published by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. in October 1969.

This symposium has been dedicated to Martin Hotine. In turn, one might almost say that his book is dedicated to this series of symposia. Quoting from the Preface, "The author's main source of inspiration in the subject of this book has been Professor Antonio Marussi of the University of Trieste". But I am sure that Professor Marussi and all of you will agree as soon as you have the opportunity to become acquainted with this book, that it is the finest fruit of these Symposia. Each lends lustre to the other.

It is my intention to describe this work, and then to concentrate on specific instances in the volume which exemplify Hotine's genius. If Martin Hotine were with us now, I would have to extend compliments guardedly in deference to his personal sensibilities. But any words of praise that I put forth today are not meant to be lavish in the sense of speaking conventional phrases over the departed; they are expressions of my evaluation of Hotine's originality and his impact on the geodetic community.

Let me first make some general statements about the book, which I hope will serve to whet your appetite for it. It is comprehensive in that it covers everything substantive in theoretical geodesy today. However, it is completely unlike any other book on geodesy past or present, in English or any other language. Those of you who are familiar with Hotine's *A Primer of Non-Classical Geodesy* should understand what I mean. All of the basic principles of geodesy are completely derived directly from general mathematical considerations. In fact, one of Hotine's principal aims in writing this book was to put geodesy on a firm mathematical and physical foundation. Only in that way, he believed, could geodesy continue forward as a science in the spirit of previous great geodesists like Cassini, Gauss, Stokes and Helmert; otherwise it would deteriorate into technology, providing a means of sustenance for many able minds, but without the original spark that pure science provides. Hotine's book contains nothing concerning instrumentation or field procedure or, in the same sense, computational techniques or methods for the actual reduction of data. The title might well be "*Pure Geodesy*", although that would be much more subject to misinterpretation than its actual name. Indeed, I proposed the title "*Principia Geodesia*" to

[1] Reprinted with minor revisions from Proceedings of the IV Symposium on Mathematical Geodesy (Trieste, 1969), 11–24.

Hotine, half seriously because I thought that it was worthy of the name in the same sense that Newton's and Whitehead and Russell's masterpieces bore to physics and mathematics, and half in jest because I knew Hotine would never allow it.

Let me reiterate again the two points I have tried to make about the book. First, it is comprehensive in the broad sense, that is, in its scope of subject matter. A glance at the table of contents will verify this. Second, it is comprehensive in the deep sense, that is, these subjects are treated in depth; there are no equivocal derivations, and no concession is made anywhere, as is done so often (and in fact almost universally in other geodesy source books) to defer a proof to more advanced (and usually much more obscure) texts. Proofs are complete, direct, and without compromise.

If all this is so, how, you may ask, can so much be contained in a single volume of about 400 pages, even if the subject matter is restricted to what I have termed above "pure geodesy?" In the answer lies the essential strength of the book and the real inspiration of Martin Hotine. The book is developed from a basic single theme and in the most rigorous and economical fashion. The theme is that all geodetic measurements and concepts can be expressed as geometric properties of a three-dimensional manifold. The fashion is the employment from the beginning and throughout of the methods and notation of the tensor calculus.

Martin Hotine, whose sense of humour was well known to all, would anticipate now the immediate rejoinder of some wit "Well, of course, anything, including the collected works of Euler, could be written in 400 pages if converted into the tensor calculus". There is a measure of truth in this. However, this does not imply that if the book were translated into more elementary notation, the equivalent material would re-emerge in, say, 800 pages. The tensor calculus is not merely a notation. I learned this long ago myself from Hotine's old friend, Antonio Marussi. It may be possible to develop equivalent methods (like that of differential forms) which are as good or better; any such statements have yet to be proved for geodetic application. But it is not possible to obtain the same richness and elegance and completeness of results by methods which do not generalize coordinate systems and spatial curvature properties.

These, then, were the basic aims of Hotine: to liberate geodesy from its traditional bifurcation into horizontal and vertical, and to provide adequate tools for advancing geodesy as a respected science. Let us now proceed to particulars of the book itself, in order to see how these aims are fulfilled.

2. The word "tensor" has a forbidding connotation to many people, mainly because of unfamiliarity with its notational aspects. In this respect it is no better or worse than any language; facility comes with practice and application. Part I, the first 66 pages of the book, is devoted to an exposition of the basic principles of the tensor calculus and its application to the differential geometry of 2- and 3-space. On the one hand, Hotine does not pretend that this portion will suffice to make a novice self-sufficient in the tensor calculus; he cannot possibly contain the expository material of an entire textbook in 66 pages. On the other hand, he has managed ingeniously to insert original results, which belong in this section because of their general and abstract nature, as, for example, in his comprehensive discussion of conformal transformations of 3-space. The purpose, then, in

preparing this part was, first, to make the book self-contained, so that the person familiar with tensor calculus would have sufficient material with which to refresh his memory, and all the basic theorems needed for further reference; and, second, to present new or unconventional results not found in standard texts, which bear on the further geodetic development of the subject.

Understanding will not come easily. This book is not an elementary text. Those with no experience in the tensor calculus, who attempt to learn the discipline ab initio here, will find it rough, although not insuperable, going. In the Preface, Hotine himself recommends specific introductory texts for the beginner. If one responds "But I am now too old to learn tensor calculus," Hotine would answer unequivocally "then this book is not for you." He realized, and with full sympathy and understanding, that the older generation is usually well-set in its ways. The book was thus written with the next generation of geodesists in mind. If there was ever a book on geodesy that could be said to bubble with youthful vigour, it is this one. A not inapt parallel here would be Verdi's *Falstaff.*

The subject matter is intrinsically not as difficult as more general texts covering differential geometry, because Euclidean 3-space usually suffices as the arena of action. Since this space has zero Riemannian curvature, many of the general principles of Riemannian or non-Riemannian space are thereby simplified. (But one must remember that the 2-surfaces embedded in 3-space are usually not Euclidean, i.e. plane; otherwise geodesy would become too trivial an exercise). However, in Part I, Hotine makes occasional excursions into curved 3-space, notably in the chapter on conformal transformations. This, and the succeeding chapter on spherical representation, are probably the sections in Part I of most interest to the expert. Neither subject, to my knowledge, is documented to this extent in the literature, although, of course, the basic principles of conformal transformation can be found in Levi-Civita, and Marussi has done extensive research on the subject. Spherical representation – the mapping of a given surface onto a unit sphere – goes back to Gauss, and is discussed in many general texts. However, Hotine, in particular, sets up a train of theorems which prove valuable in applications later in the book. Indeed, the motivation for many apparently broad and abstract derivations is that they are subsequently put to good use in physical situations.

Part II covers about 70 pages. We have not arrived at geodesy in the accepted sense yet, but we are moving in that direction. Whereas Part I can be considered as a set of general instructions for building some useful instruments, in Part II such instruments are actually put together. (Then in Part III, which occupies the bulk of the book, they will be properly applied). The instruments I am referring to are completely delineated specific coordinate systems by which Euclidean 3-space can be described adequately for subsequent purpose. Hotine begins by constructing a specific, but very general, coordinate system, generated in the following fashion. Consider a scalar function of position, N, in 3-space, limited only by the usual continuity and differentiability considerations. The surface $N = \text{constant}$ can be described by two coordinates, say ω and ϕ, defined by the gradient of N, which are subject to no further restrictions than N. Families of $N = \text{constant}$, $\omega = \text{constant}$, and $\phi = \text{constant}$ surfaces thus provide a unique representation of any point in space as the intersection of three surfaces, one from each family. One might feel that such a specification is so general that little of use can be derived. It is true that just about any coordinate system one considers is subsumed under this description. For example, if we specify all the coordinate surfaces to be planes, we get Cartesian (although not necessarily rectangular) coordinates. But Hotine obtains a rich variety of results with definite geodetic application without further restriction of this (ω, ϕ, N) coordinate system. Such familiar concepts as azimuth, zenith distance, parallel, meridian, vertical, the various curvatures of arcs and surfaces, are precisely defined and specific formulae derived. Once such general results are available, particularization to specific coordinates like classical geodetic coordinates (geodetic latitude, longitude, and height) are straightforward, and in fact trivial.

Hotine's general (ω, ϕ, N) system should not be considered, however, as a mathematical playground in which he could display his skills at abstract games before settling down to more mundane tasks. If we consider the actual world in which we take our measurements, we must realize that to tie our mathematical models to reality, our coordinates should directly relate to our measured quantities. Astronomical latitude and longitude and spirit-levelling are examples. Such measurements can be taken as general (ω, ϕ, N) coordinates, but can hardly be specified further without intentional simplification. For example, there is no analytic general formula for the equation of a plumb line which expresses its true curvature in the entire region of interest. But Hotine's (ω, ϕ, N) system is broad enough to encompass this set, and all the results for the (ω, ϕ, N) system hold for the physically measured set. The basic ideas here were first put forth by Marussi, who derived analytic relations for the aforementioned physical set. What Hotine has done is to demonstrate how much can be derived with a bare minimum of particular restrictions.

In order to accomplish his aims, Hotine takes advantage of the fact that we are interested only in Euclidean 3-space. Thus a reference rectangular Cartesian system may be admitted to which the (ω, ϕ, N) system can be related. If this reference system is physically oriented, with, say, one axis parallel to the Earth's axis of rotation, then the subsequent definition of meridian, for example, becomes physically meaningful. But it is important to realize that the specification of a physically oriented Cartesian reference system at this stage is not a restriction; it simply provides a physical reference frame as a model for the results.

Continuing in Part II, Hotine now proceeds to provide more regular systems with which to work. Hotine terms a curve defined by the intersection of two surfaces $(\omega = C_1, \phi = C_2)$ an isozenithal line, that is, one on which only N varies. If all such lines are restricted to be normal to the $N =$ constant surfaces, then the resulting (ω, ϕ, N) system is called *normal*. The next step in simplification is to make the isozenithal lines straight. Following this, the ω, ϕ coordinates lines are set orthogonal to each other. In each of these three successive systems a series of formulae is generated which specify in turn more simply the geometric properties of the curves and surfaces relating to that particular coordinate system. If, in the last system mentioned, the $N = 0$ surface is taken to be an ellipsoid of revolution, we at last arrive at the familiar set of geodetic reference coordinates. Finally, in this part, Hotine takes up a topic which is essential for the geodetic applications:

means of transforming between different (ω, ϕ, N) systems. General conditions of orientation including a complete and rigorous discussion of the Laplace condition are covered here.

These are the basic essentials of Part II. However, Hotine has discussed other topics, which in a sense are digressions from the main theme, but are interesting in themselves. The application of generalized spherical representation to (ω, ϕ, N) coordinates occupies a chapter, as does the problem of variation of properties along the isozeniths. I call these disgressions because they emphasize projection onto two-dimensional surfaces, which runs counter to Hotine's general insistence on three-dimensional methods. Also, there is a chapter which explores the problem of constructing specific triply orthogonal systems. This is the final form of a paper presented by Hotine at the previous symposium at Turin in 1965.

Part III is the heart of the book – 200 pages which apply the machinery of Parts I and II to the field of geodesy. The subject matter of this part is more conventional than that of the preceding parts (although its treatment may not be!).

If the most general (ω, ϕ, N) system of Part II is restricted by making the N coordinate satisfy the Poisson equation of potential theory, then a unique representation of the Newtonian gravitational field is obtained. In Chapter 20, Hotine proves that this simple condition is necessary and sufficient for this purpose. We immediately begin to appreciate the value of the general resuls of Part II. They can be directly applied now with an insignificant amount of effort to obtain all the properties of the Newtonian field. The many-faceted aspects of Hotine's genius shine brightly here, as he discusses clearly and succinctly the relationship between the theoretical geometric properties of the field and means of actually measuring them by various instruments, especially the Eötvös torsion balance.

Chapter 21, on the presentation of the potential in spherical harmonics, is practically a treatise in itself. Although Hotine relies heavily on results from classical sources like Hobson, there is an extraordinary amount of material here which is original, or not readily available elsewhere. This chapter and the succeeding one on development into spheroidal harmonics comprise truly a source book on the subject. In addition to the usual exposition of the potential in terms of mass-functions and spherical harmonics, there is an equally complete development in terms of inertia tensors. This provides quick and elegant proofs of many results, such as MacCullagh's formula. Expansions for the internal as well as the external potential are developed. But since the latter is of most interest to geodesists, Hotine concentrates on this topic, considering origins both within and outside matter, and questions of convergence. Hobson's form of the external potential, using a modified cylindrical-type coordinate system, and Maxwell's theory of poles are covered. Gravity is represented in spherical harmonics by noting that although gravity is not a harmonic function, the derivatives of the potential with respect to Cartesian coordinates are; thus, for example, gravity times the sine of the astronomical latitude can be expanded in spherical harmonics. Hotine goes one step further and, by obtaining the second Cartesian derivatives of the potential, derives and exhibits rather involved formulae for the curvatures of the field.

Chapter 22 begins by developing, based on the formulae of Part II, the geometric properties of a spheroidal coordinate system: longitude, parametric latitude, and an eccentricity function (which is constant for a given ellipsoid). The standard solution for the potential in terms of Legendre functions of the first and second kinds is derived; the sufficient condition of convergence at the ellipsoidal surface depending on the ellipsoidal flattening is thoroughly discussed. Finally, the relationship between the spherical and spheroidal mass coefficients and the inertia tensor are explicitly exhibited, something I believe no other book on geodesy contains to this degree of completeness.

Chapter 23, the standard gravity field, handles the developments of Pizzetti and Somigliana by deriving expressions for the components of potential and gravity in terms of the geodetic coordinate system of Part II, the base ellipsoid of which is specified to be an equipotential surface. The development is, in my opinion, much more elegant and comprehensive than corresponding recent derivations to higher orders of the flattening by Lambert and Cook. Hotine's original derivation, incidentally, antedates these others; it can be found in his paper *Geodetic Coordinate Systems* presented at the I.A.G. assembly in Toronto in 1957. Also, the curvatures of the standard field are easily obtained from the general formulae of Chapter 21.

In the next four chapters the theme shifts from physical to geometric geodesy. Atmospheric refraction, contained in Chapter 24, was a topic of special interest to Hotine. His last oral presentation at the 1968 I.A.G. assembly in Lucerne was on a new formula for propagation time in a refracting medium. The treatment in this book is notable for its completeness and rigour, beginning from Fermat's principle. To explain his emphasis on this subject I quote: "Atmospheric refraction is particularly important in the three-dimensional methods used throughout this book, although no method of reducing the observations can overcome uncertainty in the refraction; three-dimensional methods are no better and no worse in this respect than any other. Accordingly, we shall treat the subject fully . . . ." After the theoretical development, involving analytic models for the ray and the refracting medium, all the explicit empirical formulae for index of refraction, curvature, and lapse rate are dealt with. Current formulae for astronomical refraction, which have become extremely important now for geodetic satellite work, are also discussed.

In the next chapter we enter the classic heart of geodesy – the adjustment of triangulation – and this is the area where Hotine's three-dimensional methods have been most publicized. It surprised and distressed Hotine that these methods remained misunderstood by many otherwise reputable geodesists. Because the measurement of low vertical angles is vitiated by atmospheric refraction, classical methods have been devised so as to bypass such data. In presenting general three-dimensional methods, Hotine did not compound this error. Although low vertical angles, if used, are very weak, it is no more accurate to ignore them, as is done in classical work. However, it is true that omitting them may be less burdensome. In these days of electronic computing, the extra computation involved in three-dimensional methods is slight compared to the benefit of generality and accuracy gained. And, of course, the question becomes academic in the case of satellite geodesy, where three-dimensional methods really come into their own. The point to be made is that the classical method of horizontal adjustment on a curved surface, treating the vertical dimension as a separate problem, is just a special case of the general three-dimensional method. The former may be simpler in certain cases, but it can never be more accurate, and for modern global geodesy it becomes inadequate.

Chapter 25 begins with the three-dimensional formulation and solution of the so-called direct and inverse problems of geodesy, that is, determining the relationship between two points, given either the coordinates of one and the distance and direction to the other, or the coordinates of both. Since the line joining the two points is a straight line in space, it is not hard to see that the solutions are simple and straightforward. It is a trivial application of the properties of the coordinate systems developed in Part II. A less trivial application arises next in discussing the difference in potential along a line, which is directly related to the determination of heights by levelling. The main problem is relating the general (ω, ϕ, N) system, which might be called the "astronomical" coordinate system, to the mathematically more regular geodetic coordinate system. Typically, Hotine devotes a large measure of this chapter to a little-used method for this purpose – that of employing the Eötvös torsion balance. A rigorous derivation is given and the practical advantages and limitations of the method are discussed.

I would like now to interpolate some remarks concerning Hotine's didactic aims, because they will apply to the remainder of the book. These chapters cover those sections of geodesy which are undergoing the most intense research and application today. It was Hotine's basic purpose to show that his methods provided a sound underpinning for these subject areas, and to open up vistas for further research. In this context, the actual solution of problems, involving standard mathematical techniques, would be out of place. Hotine believed that, in exposition, understanding precisely what we are trying to do and solve is more important than the solution itself. Thus his method is to begin with the most basic principles, either from elementary physical laws or the body of theorems developed in Parts I and II, and to proceed to develop carefully and in detail the exact formulation of the problem. The end result is usually a set of observation (or condition) equations. The formation and solution of normal equations are not dealt with; as Hotine remarks: "These matters are not peculiar to geodesy and are best studied in the standard literature".

In the internal adjustment of networks, the subject of Chapter 26, the basic method is that of variation of coordinates. This follows naturally from Hotine's development. The general coordinates of a spatial triangle, whose sides are straight lines, are specified. Variations in position or length are now easily formulated in whatever coordinate system desired. Specifically, using geodetic coordinates, the complete three-dimensional observation equations in terms of the observed angles (azimuth and zenith distance), observed lengths, and spirit levelling are derived. Observational problems which bear on the accuracy of the solution are not ignored. Corresponding equations in Cartesian coordinates are also presented. The equations thus formulated up to now apply principally to intervisible ground stations, measuring direction, distance and height directly between two points. The great trend in network adjustment over the past 20 years has been to expand geodetic horizons to long lines undreamed of before. It is only natural that Hotine, with his characteristic youthful outlook, should concentrate on such methods. The first step is flare triangulation; next, stellar triangulation which refers the flare to a special (i.e. the stellar) coordinate system; and then satellite triangulation in which the flare becomes a satellite. In discussing the case in which the observations are directions to a satellite, Hotine follows the photogrammetric procedure currently employed at the U.S. Coast & Geodetic Survey. I believe that this exposition is the clearest elementary one of this method available today. But the explanation is general enough to include an understanding of all variations of this method. Hotine next covers the case of distance being the measured quantity, and then the techniques employing lunar observations. In each case the specific observation equations for the method are carefully derived and precisely stated. Finally, equations for systems, such as hiran, which employ line-crossing techniques, are similarly set up.

It is no exaggeration to state that Chapters 25 and 26 provide a Bible – the ultimate sourcebook – for three-dimensional network adjustment techniques. It would be inconceivable for anyone who wishes to work in this field not to have this reference close at hand.

A short Chapter 27 on external adjustment of networks follows. By this is meant the placement of a local network within a world-wide system. Problems like change of shape, orientation, or position of the reference system, which have furiously occupied the energies of geodesists in the past, become almost trivial by Hotine's methods, and are dealt with completely in the space of about six pages. Hotine remarks near the end of this chapter: "In modern language, the old problem of determining a 'figure of the Earth' becomes the problem of finding a geodetic coordinate system which best fits the astronomic system". This statement succinctly exhibits the nub of his method.

The longest chapter of the book, and probably the one on which Hotine laboured hardest, is Chapter 28, on dynamic satellite geodesy. The youthful vigour to which I have already often referred is evident again here. Consider a person almost 60 years old when the first artificial satellite was launched, with no professional background in dynamical astronomy, and who, I personally know, did not become really interested in the subject until 1964. In a very short space of time he turned out a treatise which, first of all, carried out successfully the didactic aims I referred to previously, second, in no sense duplicates other texts as it consistently follows the theme and notational methods of the book as a whole, and third, actually, in my opinion, surpasses other books on the subject in its clear and straightforward treatment of some important but difficult topics. To be specific in this last point, let me mention the exposition of the integration of the canonical equation of motion employing the Hamilton-Jacobi method. One can compare this (paragraphs 103–114) with comparable discussions in Goldstein's *Classical Mechanics* or Plummer's *Dynamical Astronomy*, and then draw one's conclusion on elucidative superiority.

As in his practice, Hotine is very thorough in discussing the basic fundamentals of the subject. Such topics as the meaning of an inertial frame, the formulation of Newton's equations of motion in general coordinate systems and the relation between fixed and moving axes are handled with care. In developing the geometry and dynamics associated with the Kepler ellipse, he gives unusually complete treatment to the parameters α and β, quantities referred to the instantaneous position of the satellite which are analogous to azimuth and zenith distance, the latter being measured in the orbital plane, and the former in a plane perpendicular to the orbital plane. By this means, an astonishing amount of material from Part II can be applied. The discussion of perturbed orbits employs the standard method of variation of parameters, but it is refreshing to see it all translated into the general notation of the tensor calculus. Some might smile at this last remark. However, if one has noticed the difference between Danby's book on celestial mechanics, which uses vector notation consistently, to other standard works, then one can realize how much more is the distinction in passing to general tensor methods. I do not claim it is better, just refreshingly different. In essence, what Hotine has done is to derive generalized versions of the Gauss and Lagrange planetary equations, using elementary geometric methods. If one is concerned that tensor methods complicate matters, let me reassure him that Lagrange brackets never arise. Which would you rather undergo? The flavour of Hotine's treatment is aptly evident in a section which derives the curvature and torsion of the perturbed orbit.

Next the integration of the differential equations of motion is given similar treatment: complete, but novel. The first integral of energy is expounded in its general form, taking into account whether the coordinate system is inertial or rotating. Hotine does not flinch from the task of deriving and computing second-order perturbations. Other topics covered are the by-now classic technique of Kaula for integration of the Lagrange planetary equations to first order, problems of resonance, the aforementioned application of Hamilton-Jacobi theory to the canonical equation, the Vinti potential and von Zeipel's transformation.

There remain discussions of the various physical causes for orbital perturbations: drag, radiation pressure, etc. and a brief but explicit outline for setting up observation equations according to the different methods of measurement. Finally, the chapter is concluded in distinctive Hotine style by considering families of orbits under the conformal transformations properties discussed in Chapter 10. From basic geometric considerations, the familiar physical laws can be freshly derived and perhaps some insight gained from this different approach.

The last two chapters of the book – on the integration of gravity anomalies – continue in form and spirit the preceding portions. I especially stress this because these last chapters were written in 1968 under difficult circumstances. Incidentally, these chapters were circulated to a number of distinguished members of the geodetic community for comments and revisions. Hotine has been generous in his credits in the Preface, most of which follow from comments on these particular sections; but I think it is fair to say that the chapters remains essentially the same as his original version – nothing of substance has been changed.

These chapters overlap to a great extent material found in the excellent volume of Heiskanen and Moritz on *Physical Geodesy.* Hotine's treatment, of course, is more brief – the two chapters occupy less than 40 pages – but I doubt if there is a better and more complete derivation of the fundamental theorems of physical geodesy from basic principles. After one reads this, one really understands Stokes' formula. Beginning with elementary facts about spherical harmonics, and developing these on a purely mathematical basis, Pizzetti's extension of Stokes' function is derived, of which the latter is a special case.

The development of the conception and application of the term "gravity anomaly" is shown to arise from comparison of three coordinate systems; a general (ω, ϕ, N) system, the system corresponding to the standard gravitational field of Chapter 23 in which one coordinate surface is a level ellipsoid, and the geodetic coordinate system. Hotine precisely distinguishes between the "gravity disturbance" defined as the difference between gravity computed at the same point in the first two systems, and the "gravity anomaly" defined as the difference between gravity at a point according to the first system, and gravity in the second system computed at another point at which the potentials and direction of the geodetic normal of the two systems correspond. From these exact definitions, all of the well-known formulae are derived by stating exactly what approximations are involved. There is also a discussion, probably on the controversial side, on the relative value of the gravity disturbance versus the gravity anomaly.

Next, Poisson's integral, providing a means of upward continuation of potential from a reference surface, is derived in full generality, and its use illustrated by a number of applications. A similar treatment is provided by Stokes' integral along with Vening-Meinesz's adaptation to deflections. The limitations involved in applying these formulae are carefully explained. The names most closely associated with modern developments based on Stokes' and Poisson's integrals are Bjerhammer and Moritz. I am sure they will agree that Hotine does full justice to their methods in this chapter. Finally, an introduction to the theory of density layers is provided by considering a simple layer on a spherical surface. This topic will be developed further in the final chapter.

In the last chapter, developments associated with Molodensky and his school are analyzed. Whereas in the previous chapter, the reference surface was no more complicated than an ellipsoid, we now pass to surfaces which are more physically realizable. We begin with the concept of an S-surface, which can be considered to be the actual surface on the Earth, or some smoother version, like de Graaf Hunter's Model Earth. The machinery Hotine has developed in Part II proves invaluable here. All the geometric properties of these surfaces in terms of geodetic coordinates are developed in tensor form. An interesting digression is the discussion of the generation of a family of surfaces by progressive deformation of the S-surface, according to a specific, but very general, law. Green's theorem is applied to derive the basic integral formula for the relation of the S-surface to the potential. Hotine discusses this derivation in more generality and detail than even Heiskanen and Moritz. He is concerned, for example, with the continuity considerations involved if S is taken so that there is matter both internal and external to it. On the other hand, one must turn to Heiskanen and Moritz in order to obtain details on the solution of this integral equation, although Hotine indicates and discusses the work of Molodensky and others. Finally, Hotine tackles the problem of single and double layers on a surface, this time generalized to any S-surface.

That is a rough glance at *Mathematical Geodesy*. What I have tried to do is not so much to inform you on what the book contains, or to analyze it critically, but to try to build up your interest in it. What I fear is that, although you all undoubtedly have the greatest respect for Hotine and his work, you may be in such awe of the abstract-type approach that you may treat the book like a beautiful virgin – fascinating but untouchable. This would be a pity. The rewards to be gained are analogous. It is for this reason that I have interpolated deliberately controversial remarks. If you are curious enough to see if they have any substance, you must go to the book. Then so much the better.

3. What I have talked about up to now has been mostly generalities. I have given a narrative outline of the book and have heaped compliments on it here and there. For those of you who were familiar with Martin Hotine and his work, such praise, I am sure, did not surprise you. But actions speak louder than words, and I would now like to present two specific examples of Hotine's thought and techniques. It is not practical here to go into the development of a complicated or obscure argument. The examples I have chosen were selected primarily because of their simplicity (they can be laid out easily without recourse to tensor notation) and because of their familiarity; they should already be well-known to you. What I hope to convey is a measure of the flavour of his work, which bore unmistakeably his own personal stamp.

(1) The first example concerns the specification of the standard field, that is, the gravitational field at the surface of an ellipsoid of revolution which encloses all matter and is level.

Starting with Pizzetti's and Somigliana's treatment, a closed form for the gravitational potential V is obtained:

$$V = V_0 Q_0(iu) + A_2 Q_2(iu) P_2(\cos U) ,$$

where U, u are ellipsoidal coordinates. Transforming to Cartesian coordinates and computing V along the z-axis yields

$$V_2 = A_0 Q_0(iz/c) + A_2 Q_2(iz/c) ,$$

where c is an ellipsoidal parameter.

Expanding the Legendre polynomials above results in an expression in terms of $1/z^{2n-1}$; to find the external attraction off the axis of symmetry all that is needed is to replace $1/z^{2n-1}$ by $(1/r^{2n-1})P_{2n-2}(\sin\phi)$, and to substitute known expressions for A_0 and A_2. He thus obtains

$$V = \sum_{n=1}^{\infty} \left(\frac{-GM}{ae(2n-1)} + \frac{4(n-1)\omega^2 a^2}{3(2n-1)(2n+1)F(\alpha)} \right) (-1)^n \left(\frac{ae}{r} \right)^{2n-1} P_{2n-2}(\sin\phi) \tag{1}$$

for the gravitational potential V (*not* the gravity potential) in terms of spherical Earth-fixed coordinates r, ϕ, where e is the ellipsoidal eccentricity (not the flattening) and GM, a, and ω have their usual meaning. If we set $\alpha = \sin^{-1}(e)$, then $F(\alpha)$ is defined by

$$(1 + 3\cot^2\alpha)\alpha - 3\cot\alpha .$$

Equation (1) is compared directly with

$$V = \frac{GM}{r}\left[P_0 - \sum_{n=2}^{\infty} J_n \left(\frac{a}{r}\right)^n P_n(\sin\phi)\right] \tag{2}$$

in order to obtain the correspondence between the mass-functions J_n and the standard field given by Eq. (1).

For example n = 1 in Eq. (1) yields $\frac{GM}{r}P_0$, the same as the corresponding term in Eq. (2). For n = 2 in Eq. (1)

$$V_2 = \left(-\frac{GM}{3ae} + \frac{4\omega^2 a^2}{45F(\alpha)}\right)\left(\frac{ae}{r}\right)^3 P_2(\sin\phi) \ ,$$

which corresponds to

$$V_2 = -\frac{GM}{r} J_2 \left(\frac{a}{r}\right)^2 P_2(\sin\phi) \ .$$

Thus the exact relation between J_2 and the standard field is

$$J_2 = \frac{e^2}{3} - \frac{4\omega^2 a^3 e^3}{45GMF(\alpha)} \ .$$

It then becomes simply a matter of series expansion of $F(\alpha)$ to obtain a polynomial type formula for J_2 in powers of e. The same procedure holds for any mass-function (the odd ones are, of course, zero in the standard field).

Incidentally, matters of priority here are rather touchy. I previously mentioned analogous developments by Cook and Lambert which date from 1959 and 1960 respectively. Of course, the original solution in closed form in ellipsoid coordinates is due to Pizzetti. A general formula equivalent to Hotine's was derived by Caputo in 1963 and presented by him at the Turin symposium in 1965. The exhibited relation between J_2 and the parameters of the standard field can be found in the book of Molodensky, Eremeev and Yurkina. I only suggest that you look at Hotine's derivation, of which I have given just a brief outline, in the paper *Geodetic Coordinate Systems* or in *Mathematical Geodesy* in order to see that it has its own distinctive flavor.

(2) The next example is one in which Hotine takes the mystery out of the meaning of the Laplace condition. In the Preface he noted that his interest in the subject "was aroused some years ago by an argument in print between two leading European geodesists on the correct application of Laplace azimuth adjustment, between points not located on the reference surface, which showed that neither geodesists had clearly defined what he meant by a geodetic azimuth at points in space".

In 3-space, the Laplace conditions are simply a direct elementary consequence of the relation between the precisely defined quantities latitude, longitude, zenith distance, and azimuth as specified in the standard polar triangle. Consider two points P and $\bar{P}$ on a unit sphere with spherical coordinates (ϕ, ω) and $(\bar{\phi}, \bar{\omega})$ respectively. Let the azimuth and zenith distance to some zenith be (α, β) and $(\bar{\alpha}, \bar{\beta})$ respectively.

Then it is straightforward by the formulae of spherical trigonometry to derive the following exact relation:

$$\begin{pmatrix} \sin\bar{\alpha}\sin\bar{\beta} \\ \cos\bar{\alpha}\sin\bar{\beta} \\ \cos\bar{\beta} \end{pmatrix} = ||a_{ij}|| \begin{pmatrix} \sin\alpha\sin\beta \\ \cos\alpha\sin\beta \\ \cos\beta \end{pmatrix} ,$$

where

$$\Delta\omega = \bar{\omega} - \omega$$

$$\alpha_{11} = \cos\omega$$

$$\alpha_{21} = \sin\bar{\phi}\sin\Delta\omega$$

$$\alpha_{31} = \cos\bar{\phi}\sin\Delta\omega$$

$$\alpha_{12} = \sin\phi\sin\Delta\omega$$

$$\alpha_{22} = \cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos\Delta\omega$$

$$\alpha_{32} = \cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos\Delta\omega$$

$$\alpha_{13} = -\cos\phi\sin\delta\omega$$

$$\alpha_{23} = \sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos\Delta\omega$$

$$\alpha_{33} = \sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos\Delta\omega \ .$$

There are only two independent relations in the three-part vector equation above, because each term is equivalent to the component of a unit vector.

Now suppose that the change from (ϕ, ω) to $(\bar{\phi}, \bar{\omega})$ is small, so that we can write

$$\delta\phi = \bar{\phi} - \phi \ , \quad \delta\omega = \Delta\omega \ .$$

Then the above exact relations can be approximated by

$$\delta\alpha = \sin\phi\,\delta\omega + \cot\beta(\sin\alpha\,\delta\phi - \cos\alpha\cos\phi\,\delta\omega)$$

$$\delta\beta = -\cos\phi\sin\alpha\,\delta\omega - \cos\alpha\,\delta\omega \ .$$

If the zenith distance, β, is close to 90°, then the equation for $\delta\alpha$ can be further approximated by

$$\delta\alpha = \sin\phi\,\delta\omega \ ,$$

the usual Laplace condition of classical geodesy. But this does not affect the equation for $\delta\beta$ at all, so its neglect must be considered as a defect in the classical system. (It is covered by repeated application of the $\delta\alpha$ equation). The whole operation and the approximations involved become apparent in the three-dimensional approach. A clear account of this problem may also be found in Heiskanen and Moritz's *Physical Geodesy*, where Hotine is credited with first explicitly drawing up the above equations.

4. I should like to conclude with a few words on the impact of *Mathematical Geodesy.* I cannot foretell to what extent this book will influence trends in geodetic research and operations, but if this influence turns out to be insignificant, to me this will reflect more on the current state of geodesy and geodesists than on Hotine. He has provided new, sharper, more efficient tools to work with. In order to take advantage of these tools, as in any other technology, one must learn how to use them. The road to achievement is always uphill. A university provides the best place for systematic acquisition of knowledge in any discipline. It is my hope that this book will be considered seriously as the basis for university courses and seminars by those of you who are in a position to make such policy. My agency, the U.S. National Oceanic and Atmospheric Administration (NOAA), is proud that this book could be printed under its patronage.

Editorial Commentary

This valuable preview of MG appeared on the eve of its publication and was written by one of Hotine's closest associates. Indeed, although it is not mentioned in the paper, the very existence of MG is due in no small measure to the untiring efforts of Bernard Chovitz. He served, together with Ivan Mueller, as one of the official reviewers of the manuscript, and after Hotine's death he was responsible for seeing it through the press.

In addition to this preview, two other papers are especially noteworthy and deserve special notice for anyone interested in MG and Hotine's work. The first of these is the comprehensive review (Thomas 1975) which was written by another of Hotine's colleagues. The other is Chovitz (1982), which assesses the impact of MG during the dozen years following its publication.

MG was issued by the U.S. Government Printing Office in two printings in 1970 and 1971. The only difference between the first and second printing was the insertion of an Errata page in the latter. Currently the book is out of print.

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Martin Hotine: Friend and Pro[1]

Charles A. Whitten

"I hould every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit so ought they of duty to endeavour themselves by way of amends to be a help and ornament there unto". Bacon

Martin Hotine used this introductory paragraph from Francis Bacon's pen to preface a paper on *Professional Organization* which he presented in London at the 1947 Conference of the Commonwealth Survey Officers. I selected the same paragraph for this memorial lecture because of the insight it provides of Hotine's philosophy on professionalism – whether it be surveyor, photogrammetrist, geodesist, mathematician, administrator or statesman. His talents were great and he shared his tremendous enthusiasm with friends and colleagues throughout his entire life wherever he might be and whatever the responsibility he faced. All of us who were privileged to be associated with him in any way can recall many exciting incidents – some of action, some of discussion, some of debate – each in a very special way identifying this outstanding man as a true professional.

In his associations with friends and coworkers, he requested that titles of rank or distinction not be used; he much preferred the use of surnames for professional colleagues and reserved the first name or familiar nickname for his closer friends. Therefore, whenever I mention any of his colleagues, I will endeavour to follow this informal yet fully respectful pattern.

A playwright might describe Hotine's life as a continuous and logically planned series of episodes. Whether Hotine was preparing the script, setting the stage, leading the action, or greeting the guests after the performance, his keen sense of timing and his natural appreciation of the dramatic have made the story of his life unique for the international surveying and mapping fellowship.

Prologue: Early Life – First World War – Education – Family

We may consider the first third of his life as a fitting prologue to a full career of service to his government plus all the other governments associated with Great Britain. Born June 17, 1898, he received his early education at Southend High School in Essex and was graduated from the Royal Military Academy at Woolwich on June 6, 1917, being commissioned in the Corps of Royal Engineers, the "top cadet of his batch". In the First World War, following additional training at the School of Military Engineering at Chatham, he was sent to India for service in the Queen Victoria's Own Bombay Sappers and Miners. His company moved up

[1] A memorial lecture presented at the Annual Convention of the American Society of Photogrammetry, Washington, D.C., March 1973. Published in Photogrammetric Engineering and Remote Sensing 39 (1973):821–830; and reproduced with permission from the American Society for Photogrammetry and Remote Sensing.

to the Northwest Frontier, where there was incessant guerilla fighting with German-trained hill tribes. Later, his company moved on into Persia at the invitation of the Persian Government to suppress some unruly tribesman. Along with the skirmishing, his company was accomplishing some of the engineering requirements – building roads and railways, placing telegraph lines, and, of course, supplying the supporting surveying and mapping.

His mental challenge during those years was the mastery of unmasterable languages. He learned to read, write, and speak Urdu, the language of the Moslems in India, and also Persian. In 1920, his company was sent into Mesopotamia, now Iraq, where sporadic fighting with Arab guerillas was still going on along the banks of the Euphrates. Peace was slow in coming, but, finally, in 1922 he was ordered back to England for post-war courses at Magdalen College, Cambridge University, and advanced studies at Chatham.

The prologue must also include his marriage to Kate Amelia Pearson on August 9, 1924. I remember sitting under the stars in St. Mark's Square in Venice with Kate and Martin along with other friends listening to them recall their courtship days when they were participating in amateur theatricals. They told us that in 1916 Martin's school had decided to put on Shakespeare's *Merchant of Venice*. Kate had been chosen from a neighbouring girls' school to play Portia and, of course, Martin was playing Bassanio. At the dress rehearsal, before a mixed audience from both schools, Martin went beyond the action indicated in the lines and planted a kiss on Kate's cheek, leaving a tell-tale mark from his, at that time, grease paint mustache, much to her embarrassment, so she said, but, needless to say, to the general amusement of the others.

They could not have selected a better time and place for sharing their memories with friends. Those of you who have known these two, realize that they could have been tremendously successful as professionals in the theatre. However, the story unfolded in the more traditional way: military service, advanced education, home, family, profession. They had been engaged before his long tour of duty in India and Persia, but waited until he had completed his post-war training and education to be married.

Even though the family is basic to the whole drama of life, I should introduce them at this point in the prologue. The Hotines are blessed with three daughters: Margaret has devoted her life to theatrical work, Janet has given her life in service to the church, and Bridget has made a home for a family of her own. Religious faith was a tremendous force in their lives. They were devout Christians and active in their support of the Catholic Church.

Act I: Air Survey Committee – Research Officer – Photogrammetry – Methods – Texts

In 1925 Hotine was appointed to the Air Survey Committee of the British War Office serving as its Research Officer. Here he used the opportunity to devise practical methods of using aerial photographs for topographic mapping. His mathematical ability, his experimental aptitude, his tremendous energy, and his special gift of writing were all combined in producing four professional papers, each of book length, and finally, in 1931, a textbook, *Surveying from Air*

Photographs. The analytical procedures and graphic methods he devised became the basis for all mapping, civilian and military, and as we shall se later, for economic development. The concepts became identified as the *Arundel* Method. Generally, the proper noun identifier relates to a person, but in this case Arundel is the place where the method was first used. Arundel is a small village in Southern England just a few miles north of the Channel. The photogrammetric target would have been Arundel Castle, seat of the Duke of Norfolk, Earl Vice-Marshal of England.

Although this first episode or Act I of the Hotine Drama might be titled Photogrammetry, it was during this period that the prick of his pen began to be noted in the United States. Bowie and Hayford, working through the International Association of Geodesy, had been able to persuade the countries of the world to adopt a new ellipsoid, and, for greater persuasion and geodetic diplomacy, it was identified as the International Ellipsoid. But Hotine was quick to note that Bowie continued to use Clarke's 1866 Spheroid for North America. Also, Hotine expressed his concern about the use of Laplace azimuths in triangulation networks. In the Empire Survey Review, he wrote of the fact that the error of observation for azimuths was considerably larger than that of horizontal directions in triangulation. Therefore, he reasoned that the use of azimuth observations could distort the more accurate triangulation. Quoting in part: "Personally, I know very little about the subject, but I have an open mind and I can appreciate an argument. If the real geodesists can be induced to fight in the same ring, we may get at the truth. Therefore, I shall endeavour to prove that both sides are wrong". I do not know if any others entered the ring, but some months later, Hotine continued his discussion in the Empire Survey Review. He had relaxed somewhat, perhaps realizing that azimuths observed in the latitudes of the United States did not have as large errors as observed in the latitudes of Great Britain.

He also wondered why those responsible for the theoretical applications in the United States seemed to ignore the classical method of defining a datum, in particular, the origin. Bowie has chosen to use the same geographic coordinates for Meade's Ranch, the origin of the then new 1927 Datum, as had been calculated from the original New England Datum and later perpetuated into the U. S. Standard Datum. Quoting in part again from Hotine: "The fact that the constants of the Clarke 1866 Spheroid are just right for this purpose, and that the regional attraction varies with closely approximate uniformity from east to west, is probably the most amazing stroke of luck in the history of Geodesy. The Americans are to be congratulated on this; but their sagacity may be doubted when they suggest in effect that the same chance holds everywhere else". Again later, when writing about azimuths: "After some thought on the question, my own opinion is that the Americans may conceivably be right in using Laplase azimuths, but I am quite sure they are wrong in suggesting that everyone else should use this system of adjustment".

Act II: East Africa – 30th Meridian – Geodetic Engineering – Techniques – Rigours

Hotine's keen knowledge of theoretical geodesy was soon to be applied in the next episode of his career. Late in 1931 he was assigned the task of establishing an arc of triangulation in East Africa along the 30th Meridian from latitude 10° South to 4° South. Now we see his engineering capabilities: the utilization of native manpower, the development of precise base-measuring techniques, the adoption of rigorous observational procedures, and, in fact, the specification of a Laplace azimuth in almost every quadrilateral of the chain of triangulation. The design or strength of the survey was critical. He avoided the use of long lines for the mere purpose of getting from one place to another quickly. I particularly like this sentence of his written in 1933: "The god of least squares, with his unreasoning hatred of small angles and complete ignorance of field conditions, is not a just god and would most likely over-favour the long and possibly inaccurate line".

One can re-read accounts of this African project in the Empire Survey Review, but I must use one more quotation to show the complexity of geodetic problems in considering different continents: "for good or evil, the Clarke 1880 Spheroid has come to stay in Africa and will not be replaced any more than the Clarke 1866 figure has been displaced by Hayford in America".

Extending an arc of triangulation from Rhodesia northward along the east side of Lake Tanganyika through trackless brush and jungle, infested with tsetse flies, was a task that only humans could accomplish. Back-packing food, blankets (for some of the points were at high altitudes), building supplies, surveying equipment, and, of course, rifles, for protection from lions or for acquiring fresh game, required a small army of native porters, sometimes as many as 250. Hotine had the full responsibility. He had one junior officer to assist him with the observing and a few non-commissioned officers to mark stations, post lights, and assist in the training of natives to do some of the work. The task was completed in 2 years, working through two wet seasons and living under canvas the whole time – really a remarkable achievement!

Act III: Great Britain – Retriangulation – Organization – Adjustment – National Grid

The technical aspects of his assignment in East Africa proved to be the basis for the next major role in his service to his country. In the development of urban England after World War I, especially in Northern London, the need for resurvey (as opposed to revision) had become apparent. Any attempt to patch up the existing network to serve as control for the necessary breakdown surveys merely emphasized the inadequacy of the basic framework. In 1935 a decision was made to observe an entirely new primary net and subsequently to reestablish the lower-order networks. Hotine was asked to undertake this project at a time when the resources of the Ordnance Survey had been, to use their words, "pruned to the irreducible minimum". The Great Depression of the 1930's covered the globe, with England affected as much as any country, yet Hotine was called upon to accomplish within a few years a task which, when done the first time, had taken half a century.

The details of how this retriangulation was planned, personnel selected and trained, procedures for reconnaissance and observing established, a special station mark pedestal designed, and the actual work accomplished were well described by Hotine in several issues of the Empire Survey Review. World War II interrupted the work before it was completed. I refer those of you who are interested in reading a vivid account of the total programme to a 1966 publication of the Ordnance Survey, *History of the Retriangulation of Great Britain*, compiled primarily by John Kelsey.

The whole story has, however, never been published. The tales that are told by old-timers, when they sometimes meet, do not become part of an official government publication; but they do become part of geodetic legendry. Fortunately for this occasion, I was provided with a bit of the record which must be passed on to you.

Hotine and the observing party had been bogged down by the weather for weeks on the summit of Ben Mac Dui, a grim Scottish mountain and the second highest in Great Britain. Hotine had gone down to the base camp at Braemar, a distance of some 15 miles, and sent one of his men back to the summit with a supply of the finest Scotch whiskey obtainable. Within a short while the weather cleared, the observations were completed, and a triumphant party staggered down the Luibeg Glen to their trucks parked near a lovely spot, the Linn of Dee. The happy triangulators decided that a more extended celebration was in order, so they went to a nearby hotel. Too bad, though, that it happened to be the Sabbath and the Free Church of Scotland Community objected. A Black Maria provided transportation to the local "jug" for the merrymakers.

Next morning, Hotine, having heard about the incident, hastily made his way to the Court, arriving just at the time when the observer, who had accepted the responsibility for all, was pleading guilty. Hotine asked the Court if the observer was being provided legal counsel. When the Court replied that the defendant was entitled to such, Hotine proposed that inasmuch as he was the defendant's Commanding Officer he was qualified in his military capacity to speak for him. Thereupon, Hotine painted an excellent picture of the hardships of the field men, he emphasized the consequences of delay to the whole operation if the observer should be detained by imprisonment, he stressed the possible loss of rank and even ignominious discharge. Hotine's eloquence was partially effective. The Court ordered a nominal fine of £ 20 to pay his fine. After completing the transaction, Hotine ordered the man to another nearby and difficult station to do penance and work off his "nominal fine".

A few years before Hotine had been assigned this major task of retriangulation, he had expressed some concern, in fact some criticism, of the methods Hayford, Bowie, and others had used in the United States in devising the 1927 Datum for North America. When Hotine was confronted with a similar task in his own country, we must note that he also recognized that expediency sometimes must have primary consideration. Because of the desire not to disturb the graticule of the existing large-scale maps, many at 1/2500, the new triangulation was adjusted to fit the scale and orientation of the old. Thus, Laplace azimuths and precise bases were used for after-the-fact studies or investigations. In a practical sense, the distortions to the new observations were negligible, generally less than one part in 100000, and for the standards of that era, fully acceptable.

In addition to holding overall scale and orientation, the formerly used spheroid of reference, Airy's Figure of the Earth, had to be retained. Airy's parameters had been defined in feet, so there was the complex task of deriving the proper conversion ratios for expressing lengths in meters. There is some satisfaction in noting that we in the U.S. are not the only surveying and mapping group that has been subjected to this numbers game in the effort to achieve uniformity. A unique feature of the new British network is that all coordinates are expressed in metric units on a National Grid based on a single zone of a transverse Mercator projection. The published geographic coordinates were derived from the National Grid coordinates.

Another point of interest to us is that Hotine had acquired some Bilby towers from U.S. He institute a training programme for the erection and dismantling of these towers on the grounds of the Ordnance Survey at Southampton – somewhat different from the on-the-job training we have followed. Because of the area-type network and unusual requirement for tower heights in some of the Eastern counties, Hotine encountered the problem of needing more towers. Rather than face time delays and the cost factors involved when purchasing more towers from the U.S., he solved the problem with the assistance of the Geodetic Survey of Denmark, who generously loaned him two towers of similar design.

Act IV: Dunkirk – East Africa – Greece – Military Survey – Loper-Hotine-Agreement

I mentioned earlier that World War II interrupted the programme of retriangulation. At the outbreak of that war, Hotine was assigned to General Headquarters as the Deputy Director of Survey in the British Expeditionary Force. For those who wish to follow his record from Dunkirk to East Africa to Greece and returning to the War Office to serve the rest of the war as Director of Military Surveys, I suggest reading *Maps and Surveys*, published by the British War Office in 1952. From that historical report you will sense that Hotine and his colleagues could have used 3M as a trademark: not for Minnesota Mining and Manufacturing Company with its various products, but rather for a symbol of military strength – Men, Munitions and Maps – the third element being as essential as the other two.

Hotine was with the British forces in Belgium and France when they were fighting their way back to Dunkirk. He saw the acute problems which could develop if the maps that did exist were not in the hands of the troops needing them. Years later he referred to the episode as "when many of us were seabathing at Dunkirk" and, no doubt, using his keen sense of humour, could reminisce with his colleagues who were with him at the time about many of the little events contributing to this major military miracle.

Hotine was next assigned to survey operations in East Africa and later sent to Greece with a small survey team as a part of a British Expeditionary Force assisting the Greek Army. This mission failed, and many of the personnel were taken prisoner; Hotine was among those who escaped.

He was then ordered back to London to the War Office to be director of Military Surveys. In 1939 he must have been frustrated at times, for in one letter home he wrote: "I'd love to run my own show instead of being an eternally vibrating second string", but now he drew from his experiences of that earlier phase of the war. The requirements for surveys and maps and, in particular, the problems associated with printing and distributing maps were forcefully presented to the General Staff. Hotine came to Washington in May 1942 to discuss the mapping situation with H. B. Loper and members of the Intelligence Branch of the Chief of Engineers, U. S. Army. As a result of this conference, they drew up an agreement, known as the Loper-Hotine Agreement, dealing with the division of responsibility for map production, the exchange of mapping and other survey data, and the selection of military map grids.

Soon afterward, Herb Milwit was ordered to England, to serve as the Chief of the Engineer Intelligence Division, U. S. Army for the European Theatre of Operations. Hotine and Milwit worked closely and effectively throughout the war. Hotine's earlier experiences in aerial mapping, geodetic surveying, and mathematics of projections provided the technical base that, coupled with his tremendous vitality and dynamic and aggressive leadership, gave outstanding and truly professional direction to the Military Survey Office. Recognizing that in earlier years Hotine had been somewhat critical of survey practices in the United States, his ready acceptance of the U. S. competence in mapping with all of the supporting techniques adds to his professional stature. This spirit of cooperation and good will between the various surveying and mapping groups of Great Britain and the United States continues today. The Hotine-Loper Agreement had set the stage for broader international agreements in programme which would follow World War II.

Act V: Overseas Surveys – Statesman – Economist – Mathematician

In 1946, after he had retired from the British Army, Hotine became Survey Advisor to their Secretary of State for the Colonies. He was the first director of the newly formed Colonial Surveys, now known as the Directorate of Overseas Surveys. During Hotine's pre-war assignment in East Africa, he sensed the need for such an organization, and his close contact during the war years with all British territories as well as many others strengthened his position. In this new endeavour, he was to be statesman and economist, using his energy and ability to assist in the development of the Commonwealth. He organized a staff and provided the leadership in applying photogrammetric, geodetic and mathematical techniques to surveying and mapping. Before he retired in 1963, his organization had mapped nearly two million square miles, mostly at a scale of 1/50000.

Even though his administrative responsibilities were great, he continued his mathematical research. He published a series of papers in the Empire Survey Review, giving special treatment to various projections of the spheroid. This work has become a basic reference for all later writings in the field of mathematics dealing with projections. His development of the oblique Mercator projection of the spheroid is particularly unique. Sometimes this projection has been named skew or diagonal. In Hotine's original work, where it was applied to Malaya and Borneo, it is called rectified skew orthomorphic. The same technique was used for the State Plane Coordinate System in Southeast Alaska. Fortunately, the user of the plane coordinates in that part of Alaska has only to know that x and y are in Zone I. More recently, Ralph Berry has proposed the use of these skew projections for each of the Great Lakes, simplifying the cartographic operations where geodetic, photogrammetric and hydrographic surveys must be combined.

Off Stage: Conferences of Commonwealth Survey Officers – Symposia and Assemblies of International Association of Geodesy

During that same time, Hotine was also taking an active part in international organizations, giving unselfishly of his time and talent in planning and conducting conferences, even joining in hearty and friendly debate either in formal session or in the privacy of after-session places. This was the Martin we remember as friend and colleague.

As organizer and leader of several of the Commonwealth Survey Officers Conferences, he set standards which have been followed by many other groups. He enforced discipline on those who prepared papers, he insisted on open discussion, and he maintained a spirit of good will among all participants. At the 1963 Conference, while Sam Gamble was presiding, J.N.C. Rogers from Australia presented a motion "that this Conference of Commonwealth Survey Officers place on record its profound appreciation of the work of Martin Hotine for the distinguished service he has given as President of this Conference and of the previous Conferences of the Commonwealth Survey Officers in Cambridge in 1955 and 1959, and for the stimulation, encouragement, and leadership he has given to surveyors during a long and very distinguished career". The United States has been very fortunate to have been invited to participate in these conferences, even though we withdrew from the Commonwealth almost 200 years ago.

It was through his participation in and contributions to the programmes of the International Association of Geodesy that many of us in the United States learned to know him, grasp some inspiration from him, and join in hearty fellowship with him. I first met him at Oslo in 1948 at the General Assembly of I.A.G. It was at this first post-World War II Assembly that Antonio Marussi presented his classical paper on the differential geometry of the potential field of the earth. This modern treatment of the science of geodesy was appealing to the mathematically attuned Hotine. These two, Hotine and Marussi, initiated a series of symposia on mathematical geodesy – the two of them co-planning with Marussi hosting small groups for intimate discussion at such places as San Georgio in Venice, Cortina d'Ampezzo and Turin, to be followed by others at Trieste and Florence honoring Hotine. Marussi and Hotine seemed to challenge and inspire each other, and in turn do the same for those who had the good fortune to have been participants in these symposia. To show Hotine's personal appreciation for this association, I quote from his book, *Mathematical Geodesy*: "The author's main source of inspiration in the subject of this book has been Professor Antonio Marussi of the University of Trieste, not only for the range and originality of his ideas, but also for continual advice and encouragement". Marussi held the same respect for Hotine. You can grasp the full sincerity of this friendship by reading for yourself Antonio's tribute to Martin reprinted from The Survey Review No. 152, April 1969.

> "I feel inadequate to write the biography of Martin Hotine because I met him late in my life. I know little of his early professional activity, which I am sure was intensive; but in the past two decades through our connection with the International Association of Geodesy I have come to know him as a man, and

as a man of science. On first meeting, I understood many of his thoughts and feelings. Later I studied his work. Then he honoured me with a friendship which I reciprocated with admiration and affection.

"Our friendship was born on the advanced frontier of the discipline dear to us, Geodesy. I remember our first meeting at the General Assembly of the I.U.G.G. in 1948. It was there that the intrinsic and three-dimensional geodesy began to develop, anticipating the oncoming space age which was then knocking at the door. When a report on this subject was read, Martin said that he understood only very little about it, but that it broke with crystallized tradition and that it must therefore be important".

"Later I came to understand that in this thought was all of Martin. He was attracted to the new ideas because they challenged the scholastic framework of the time. Such ideas were perfectly congenial to his non-conformist and politely rebellious temperament.

"I did not meet him again for 4 years. When we did meet it was at the Assembly of the Association in Brussels. In the interim he had mastered tensor calculus, which he later used with rare insight and elegance, finding in it the instrument which permitted him to materialize his intuition of the fundamental problems of modern geodesy, giving life and substance to the creations of his vivid fantasy.

"His first work on the metric properties of the Earth's gravitational field, and on three-dimensional systems of geodetic coordinates appeared at the Toronto Assembly in 1957. Here, through the initiative of President Baeschlin, was born the idea of dedicating a special symposium to three-dimensional geodesy which had already anticipated the use of electromagnetic methods for the measurement of distances in space, and was now anticipating the launching of the first artificial Earth satellites.

"I considered Martin's proposal that this symposium should take place in my country, Italy, to be a token of his great friendship. Italy was very happy to host this symposium at Venice in 1959, and offered Martin its chairmanship. Later, two other symposia were held in Italy, at Cortina in 1963 and in Turin in 1965. The fourth symposium will also be held in Italy at Trieste in 1969, and will be dedicated to his memory.

"At Venice Martin introduced his *Primer of Non-Classical Geodesy*, a work whose title is decorously provocative. It is written in a style which makes it difficult to know which to admire most: its rigour, its conciseness, or its elegance.

"He thought that in the same way as in classical geodesy a clearer vision of many problems could be reached by representing the surface of the ellipsoid on the plane, if three instead of two dimensions were involved one could similarly represent with advantage the three-dimensional space on another three-dimensional space, not necessarily Euclidean, in order to make clearer and more immediate some aspects of the Earth's gravitational field. As in classical geodesy, conformal representations appeared to be the most promising. These ideas were gradually brought to perfection by Martin, and presented by him at Turin in 1965 at the third symposium held on the occasion of the celebration of the 100th anniversary of the Italian Geodetic Commission. In 1967 at Lucerne, Martin introduced new ideas about the downward continuation of the terrestrial potential field. That was the last time we had the pleasure of enjoying his incomparable personality.

"We would have expected that Martin, whose background was in the sphere of practical activity and whose career was anchored in the most deeply rooted traditional habits, would have contributed to geodesy by following the most classical and orthodox trends. However, the truth is exactly the contrary, since he was particularly anti-scholastic in facing the problems of speculative geodesy. This attitude originated in his deep dissatisfaction with compromises by which classical geodesy, burdened by prejudices accumulated during centuries, presented its own problems, managing a way between the rigor of the theoretical approach and the empiricism involved in its practical applications. At the same time, this attitude came from his innate horror for everything that was not simple and rational and, therefore, aesthetically satisfying.

"On this point Martin was intransigent. He detested any compromise whose purpose was to avoid conceptual difficulties if even to the least extent it impaired the rigour of the logical process. He hated approximations, reductions, corrections, or badly defined, unfounded, accommodating hypotheses. He was an aesthete and a purist.

"We must be grateful to Martin for the impetuous vigour with which he always defended his viewpoints, even when they clashed with the deepest-rooted convictions. We must be grateful to him for his courage, since even science sometimes needs courageous men in order to progress.

"Perhaps it is too early to assess how much his work has influenced modern thought concerning our discipline. It is probably too early because we have not had the privilege of reading his book on *Mathematical Geodesy*, which synthesizes the 20 years of thought and creative work with which Martin concluded his earthly spell. This book represents the last and most dignified satisfaction of a man whose work and family were his only reasons for life.

"If I am permitted to express my opinion, perhaps prematurely, I believe his influence will not be found in any particular results of immediate application, but rather in his effectiveness in changing convictions which, because of their venerableness, were removed from critical attack; in clearing away difficulties which hampered the free progress of ideas; and in showing how to get rid of the heavy superstructure heaped up in the long history of our discipline. Martin's spiritual heritage is for this reason embedded in our subconscious by the ways he demonstrated to us for researching the truth through the simplicity of the origin and the elegance of rational thinking".

Of course Hotine's interest extended to all aspects of geodetic work. He naturally had a keen interest in the coordination and improvement of the geodetic networks of Europe. The geographic location of the British Isles and the separation from the continent meant that the geometric impact on the continental networks might not be great, but his contributions to the development of the overall specifications were significant. When he was the leader, he was a strict disciplinarian, as we well know, but I must tell you of an incident I will always remember. Back in 1962 we were meeting in Munich, working on the plans for the readjustment of the European triangulation networks. At the middle of one morning session, a break was taken – not the customary coffee break – but a beer break, Munich's best along with open-faced sandwiches. We had relaxed for at least a half hour when I, who happened to be chairing that particular session, suggested we resume our work. Martin boomed out in his sonorous tone, "Whitten, I thought Abraham Lincoln had freed the slaves!". We did continue our work in good spirit, properly relaxed physically and mentally.

Thus it was that Hotine, through his spirit and vitality, supported with his engineering and scientific ability, enlivened the symposia and assemblies of the Geodetic Association, always supporting and encouraging the officers of that association in their various activities.

Epilogue: U. S. Coast and Geodetic Survey – Research – *Mathematical Geodesy* – Return to Weybridge – Honours

The year was the one that Hotine's own government had established as the point in his life when he should begin to enjoy the rewards of retirement. In that year he was fittingly honoured by his colleagues and countrymen for his long and distinguished service to his country, but he did not seek "the rocking chair". He accepted the invitation of Arnold Karo, the Director of the Coast and Geodetic Survey, to come to Washington and join the ranks of the Survey as a research scientist. I have referred to Hotine's numerous contributions to geodesy. These papers had been in the form of mimeographed reports – somewhat in sequence, all logically interrelated – yet all on the distinctive legal-size paper which would not conform to the standard U. S. notebook, file case or bookshelf. Karo persuaded Hotine to combine, refine and extend his work and produce a hardback monograph that would fit in our bookshelves.

For the next 5 years, Hotine worked toward this goal of producing a text which, in his words, would be "an attempt to free geodesy from its centuries-long bondage in two dimensions." Of course, those same 5 years were those in which geodesists were entering a new era of experimentation and development through the use of artificial satellites. Hotine's interest in and contributions to these developments were just as keen as all of his thinking had been throughout his whole life. Wherever activities of this type were going on, Hotine's advice was sought, so there were excursions from his main thrust. There were even administrative excursions involving agency reorganization. The Hotines accepted with pleasure the physical move to Boulder. He and Kate, with their love of nature and hiking, could explore the trails in the foothills of the Rockies. I was told that the photographic collection they made of Colorado wild flowers is probably equal to that of any ever compiled by the best of American naturalists.

I always noticed that Hotine never hesitated to use a familiar quotation if it helped to emphasize a particular point of interest. I do not think he would have objected to my use of another quote from Francis Bacon: "Reading maketh a full man, conference a ready man, and writing an exact man". All of these phrases apply to the man we honour and, in particular, to his preparation of the manuscript for his text, *Mathematical Geodesy*. Hotine's plan, which he followed explicitly, was to use the methods and notation of tensor calculus for the derivation of theorems and formulae which apply to all of mathematical geodesy. He included geometrical and physical, terrestrial and spheroidal, and, in his words, "internal adjustments" and "external adjustments" of geodetic networks. I urge you to read two outstanding reviews of this text. One is by Bernard Chovitz published in 1970 by the Italian Geodetic Commission in the Proceedings of the Fourth Symposium on Mathematical Geodesy, Trieste, Italy, May 28–30, 1969. The other was written by Paul Thomas and published in the International Hydrographic Review, January 1971. They were close friends and associates of Martin, had worked with him on many sections during the development stages, and Chovitz, in particular, during the final editing and printing.

Early in 1968, while the manuscript was nearing completion, Martin underwent a serious surgical operation. His friends in Boulder told me that he did some of his most effective writing after he returned to his office from the hospital. His friends around the world were concerned and Joe Edge has permitted me to quote from a personally penned letter Martin sent him in July of 1968: "I progress slowly after the extensive revisions, as the tailors might say and do say. It is an up-hill job, but according to the Medicos I have to work at it. If they would only let me off eating and let me concentrate on drinking, I should feel fine, but unfortunately they will do neither".

Martin and Kate Hotine had made their plans to return to England in August of 1968. The manuscript had been sent to the printer and now only the tedious task of checking the galley proof remained. Soon after the Hotines had returned to their apartment in Weybridge in Surrey, Martin wrote in his usual keen manner to Bernie Chovitz: "We have been struggling to get the place straight and in effect start a new life. It would be easier to do this in America; we have forgotten where the ropes are here, even what to do in the absence of yellow pages".

His interest in his work and his love and consideration for others never diminished, but his physical energy and strength, with which he had been blessed in abundance, left him. He died on November 12, 1968. Those who knew him mourned his passing, but gave thanks for the privilege of having known him and having been associated with him. The encouragement and inspiration he gave to others cannot be taken from them.

Governments, national engineering and scientific societies, and other groups recognize the honoured individuals of high distinction. Martin Hotine, modest and unassuming, yet forceful and energetic through his brilliant personality, was an example of true professionalism. Numerous awards were given him at various times throughout his career and significant memorials established after his death.

He was awarded the O.B.E., Commander of the Order of the British Empire in 1945, and the C.M.G., Companion of St. Michael and St. George in 1949.

Also, the Royal Geographical Society awarded him its Founders' Medal in 1947 and in 1955 he was the first recipient of the President's Medal of the British Photogrammetric Society. The Gold Medal of the Institution of Royal Engineers was awarded to him in 1964.

The United States Army made him an officer of the Legion of Merit in 1947 and in 1969, the U.S. Department of Commerce conferred, posthumously, on him its highest honour award, the Gold Medal.

During the 1971 Conference of Commonwealth Survey Officers, a tree, a Norway Maple, was planted on the grounds of the Ordnance Survey at Southampton as a living memorial. Kate Hotine assisted in the ceremony and her daughters and grandchildren were also present. Gifts, which had been received from friends in 31 countries, have been used to create in his honour a Scholarship Fund for graduate study at University College in London. I believe that this method of encouraging and helping young students would have pleased him the most.

Editorial Commentary

This memorial lecture really requires no comment except to say that it is unique and unquestionably the most detailed and authoritative account of Martin Hotine that we will ever have. Its author had the good fortune to be a close personal friend of both Hotine and Marussi, and an eyewitness to their development of mathematical geodesy. Indeed, together with Admiral Karo, Whitten was instrumental in persuading Hotine to come to the United States in 1963, and to write up his ideas in book form.

List of Symbols

Lists of main symbols were given by Hotine for paper 3, p. 61; paper 4, p. 88; paper 5, pp. 128–129; and paper 6, p. 137. His usage of these symbols was essentially the same in all of these papers.

However, some changes of notation, and different choices of signs, occur in MG. A brief indication of the most important of these is given in our Editorial Commentary on paper 3, see pp. 62–63.

In MG, an Index of Symbols was given on pp. 347–351, together with an exhaustive Summary of Formulae on pp. 353–397.

Note that in Chovitz's preview article reprinted in this monograph, see pp. 159–172, he employed the notation Hotine used in MG.

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