GEOPHYSICAL RESEARCH LETTERS, VOL. 19, NO. 10, PAGES 1065-1067, MAY 22, 1992

CONVENTIONAL SPHERICAL HARMONIC ANALYSIS FOR REGIONAL MODELLING OF THE GEOMAGNETIC FIELD

Angelo De Santis

Istituto Nazionale di Geofisica , Rome, Italy

Abstract. Spherical Harmonic Analysis (SHA) is normally used to model the three-dimensional global geomagnetic field. To address the same problem in regional modelling, Haines (1985) proposed Spherical Cap Harmonic Analysis (SCHA). This regional technique involves the computation of more complex Legendre functions with real (generally non-integer) harmonic degree. Here a new more practical technique is described; it is called Adjusted Spherical Harmonic Analysis (ASHA) because it is based on the expansion of conventional spherical harmonics after the colatitude interval is adjusted to that of a hemisphere. This kind of analysis can also be applied to modelling general two-dimensional functions.

Introduction

Haines (1985) introduced the elegant technique of Spherical Cap Harmonic Analysis (SCHA) to solve the problem of geomagnetic field regional modelling by means of spherical harmonic (SH) functions. When the usual system (R,Θ,Λ, i.e. radial distance, colatitude and longitude, respectively) is rotated to one (r,θ,λ) centred at the spherical cap which bounds the region of interest, and the new boundary conditions are applied to the basis functions which must be solutions of Laplace's equation, Haines found that the geomagnetic potential (and the associated magnetic components) can be expanded in SHs with integer order m but non-integer degree n_k:

$$V(r,\theta,\lambda) = a \sum_{k=0}^{Kmax} \sum_{m=0}^{k} (a/r)^{n_k+1} \cdot g_k^m \cos m\lambda + h_k^m \sin m\lambda) \cdot P_{n_k}^m(\cos\theta)$$

(a is a reference radius, usually the Earth's mean radius, i.e. 6371.2 km; n_k is placed after the second summation because it depends on the value of m).

Actually the central role in the new technique is played by the colatitude basis functions, i.e. the "fractional" Legendre functions. Here we will shed some further light on this technique, illustrate what it means in practice and show how it is possible to simplify it; all by means of an approximate expansion of conventional SH functions.

Some analogies with Fourier Analysis

The main idea that motivated our work was the application of some techniques similar to those normally used in Fourier analysis. For example, to decrease the boundary effects ("leakage" in the frequency domain) in fitting general functions, a common procedure is to apply a cosine tapering (e.g. Bloomfield, 1976) that in fact attenuates the weight of boundary values. It was with this purpose in mind that we introduced Translated Origin Spherical Cap harmonic Analysis (TOSCA; De Santis, 1991) as a step forward from SCHA: the vertical translation of the reference system gives a natural attenuation of edge effects, which are otherwise typical of SCHA when the data distribution is particularly sparse; often, for unfavourable data distributions, TOSCA can produce a model with a better fit than SCHA. Continuing with the analogy, when we want to represent data in a certain interval by means of a Fourier series we usually normalize the interval to 2π radians; this is obviously correct when the signal to be modelled is really periodic in the interval of interest. Unfortunately this kind of scaling is not applicable to Legendre functions because the colatitude is limited to the range 0 to π. This normalization is not possible for regional cap intervals, because otherwise it would mean physically assimilating all boundary values at one point ($\theta = \pi$). On the other hand it is also possible to scale the data interval to $[0,\pi/2]$; this would allow us to use conventional Legendre functions with integer degree and order. In this context they would represent two sets of appropriate basis functions that satisfy the same boundary conditions imposed by Haines (1985) in the regional cap modelling of the geomagnetic field. This fact has already been pointed out heuristically by De Santis (1991) and De Santis et al. (1991).

Scaling colatitudes to the hemisphere

The need to use basis functions which are simpler than Legendre functions of non-integer degree led us to consider the well-known Legendre polynomials with integer degree and order. They form a set of orthogonal functions for $0 \le \theta \le \pi$, but can still be used as 2 sets of orthogonal functions in $0 \le \theta \le \pi/2$ to fit any general functions defined in this interval. Thus we can consider the transformation from the spherical cap system (r,θ,λ) to a new hemispherical system (r',θ',λ') defined by:

$$r' = r \; ; \quad \lambda' = \lambda \; ; \quad \theta' = s \cdot \theta \qquad (1)$$

with $s = \pi/(2 \cdot \theta_0)$, θ_0= cap half-angle, r and r', the radial distances, λ and λ', the longitudes, in the old and new reference systems, respectively. The change of coordinates involves the corresponding transformations of the geomagnetic components:

$$X' = X/s \; ; \quad Y' = Y \sin\theta / \sin\theta' \; ; \quad Z' = Z \qquad (2)$$

(Note that $\theta' = \theta = 0$ is a singular point for the horizontal components X,X',Y and Y'; this results from their definition; see for example Figures 25,26,28,29 by Langel, 1987). Accordingly Laplace's equation changes only in the θ differential equation, i.e. Legendre's equation.

Paper number 92GL01068
0094-8534/92/92GL-01068$03.00

Consider it in the most common form:

$$\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})+[n(n+1)-\frac{m^2}{\sin^2\theta}]P=0 \quad (3)$$

where P is a function which is a solution of (3); as is well known it is the Legendre function $P_n^m(\theta)$. In equation (3) it is almost impossible to transform $\sin\theta$ in terms of $\sin\theta'$ (otherwise one should invoke de Moivre's theorem) but in the case of caps that are not too large, $\sin\theta \simeq \theta$ (this is a good approximation, to better than 99%, for $\theta_0 \leq 14°$, and to 98% for $\theta_0 = 20°$).

Legendre's equation (3) then becomes:

$$\frac{1}{\theta}\cdot\frac{d}{d\theta}(\theta\frac{dP}{d\theta})+[n(n+1)-\frac{m^2}{\theta^2}]P=0 \quad (4)$$

that is:

$$\frac{d^2P}{d\theta^2}+\frac{1}{\theta}\cdot\frac{dP}{d\theta}+[n(n+1)-\frac{m^2}{\theta^2}]P=0 \quad (5)$$

This equation represents an approximate version of Legendre's equation in the conventional colatitude θ, for small values of θ. It can now be easily transformed to the new colatitude θ', by means of the colatitude equation in the coordinate transformations (1). Let us recall that:

$$\frac{dP(\theta)}{d\theta}=s\cdot\frac{dP(\theta')}{d\theta'};\ \frac{d^2P(\theta)}{d\theta^2}=s^2\cdot\frac{d^2P(\theta')}{d\theta'^2};\ \frac{d\theta'}{d\theta}=s \quad (6)$$

Hence equation (5) becomes:

$$\frac{d^2P}{d\theta'^2}+\frac{1}{\theta'}\cdot\frac{dP}{d\theta'}+[n(n+1)/s^2-\frac{m^2}{\theta'^2}]P=0 \quad (7)$$

which is similar to (5) if one sets

$$k(k+1)=n(n+1)/s^2. \quad (8)$$

We know that the assumption of small angles is no longer valid for the angles θ' (they cover a range of 0 to $\pi/2$ radians), but it is reasonable to consider the Legendre functions with integer order m and degree k still as approximate solutions of equation (7). We define k as taking integer values 0,1,2 ...: this means that n takes real, non-integer values. The *old* degree n, hereafter indicated with n_k, because of its dependence on k, can be easily determined by solving equation (8) for n:

$$n_k=\sqrt{s^2k(k+1)+0.25}-0.5 \quad (9)$$

Table I gives, for a cap of $\theta_0 = 6°$ (s=15), some values of n_k as deduced from (9) for several values of k; we see that the values of n_k are very close to the roots introduced by Haines (1985) in conjunction with Legendre functions of non-integer degree when $m = 0$ (column 4 of Table I gives the percentage differences). This means that the variable k introduced here almost coincides with that of Haines. The differences can be partly explained by the approximation made in going from equation (3) to (4) and partly by the fact that the conventional Legendre functions used in the "expanded" hemispherical colatitude interval are not exactly solutions of Legendre's equation in θ. The differences could be taken into account by considering n_k dependent on k only, but independent of m, in contrast with Haines' functions. This fact could be simply explained by analogy with conventional global spherical harmonic analysis (SHA), where the degree n does not depend on m. The percentage differences depend on k and are almost independent of θ_0. To give a better agreement between Haines' roots and n_k as computed by (9) we can introduce a correction factor f_c (dividing by 100 the values listed in the percentage difference column of Table I). Thus n_k can be finally computed as follows:

Table I - Comparison between values of n_k calculated from eq. (9) and those of SCHA (Haines, 1985) for m=0, for a cap with a half-angle of 6°. Column 4 lists the percentage difference.

k	n_k	n_k(Haines)	Percentage Difference
0	0	0	0
1	20.7	22.5	7.8
2	36.2	36.1	0.4
3	51.5	52.2	1.4
4	66.6	66.5	0.0
5	81.7	82.1	0.6
6	96.7	96.7	0.0
7	111.8	112.1	0.0
8	126.8	126.7	0.0
9	141.8	142.1	0.0
10	156.8	156.8	0.0

$$n_k=\frac{\sqrt{s^2k(k+1)+0.25}-0.5}{(1-f_c)} \quad (10)$$

The use of the approximate formula given by Haines (1988):

$$n_k \simeq s(k+0.5)-0.5 \quad (11)$$

is also adequate.

Conclusions

From what we have found, SCHA corresponds in practice to an artificial enlargement of the cap to a hemisphere; the values of the spherical harmonic degree can be estimated using these new conditions, where conventional Legendre polynomials with integer degree k are the new basis functions. Owing to the approximations made, this comparison is only a crude approximation to what is found. Actually Haines preferred to apply the new boundary conditions directly to the cap; this involved the use of Legendre functions of non-integer degree. By induction we can say that the appropriate change of coordinates in equation (3) (rather than in equation (4)) would furnish the corresponding solutions in terms of conventional Legendre functions, with a more appropriate relation between n_k, k and m (e.g. Hobson, 1931; section 241), as Haines (1985, 1988) pointed out.

Hence, for caps that are not too large, the procedure of Haines (1985) can be alternatively applied in a simpler (approximate) way, which we can call "Adjusted Spherical Harmonic Analysis" (ASHA), owing to the fact that before modelling we "adjust" the colatitude interval to a hemisphere, in order to use conventional spherical harmonics. This practical procedure can be easily summarized in the following four steps:

- 1) Rotation of original polar coordinates to those of the spherical cap.
- 2) Change of colatitudes and magnetic components, according to (1) and (2), respectively, in order to "artificially" enlarge the cap to a hemisphere.
- 3) Application of conventional SH analysis as used in global modelling with the corresponding evaluation of Gauss coefficients.

The value of n_k to be used in the radial term of the potential expansion can be determined from equation (10) or (11) (the precision in determining n_k is not very important (Haines, 1985, page 2588) and in practice each n_k can also be approximated to the closer integer value (Haines, personal communication)). Then the potential can be expressed as follows:

$$V = a\sum_{k=0}^{Kmax}(a/r)^{(n_k+1)}\sum_{m=0}^{k}(g_k^m\cos m\lambda + h_k^m\sin m\lambda)\cdot P_k^m(\theta')$$

This last expression is very similar to that of SCHA; the only significant differences lie in the Legendre functions, here characterized by integer degree k, and in the fact that the n_k are no longer dependent on m, as in SHA.

- 4) Return to the original system (hemispherical to spherical cap and finally to original system) whenever we need to evaluate the magnetic field within the region of interest.

As in SCHA, the minimum wavelength portrayed by the model can be estimated as the ratio between the terrestrial circumference and n_k (Haines, 1988; De Santis et al., 1989), even though a more appropriate determination seems to be the ratio $4\theta_0 \cdot a/k$, with θ_0 in radians.
ASHA can be easily applied also to the case of modelling general two-dimensional fields $f(\theta,\lambda)$, expressed on the surface of a sphere, as SCHA has been applied to the ionospheric parameter f_0F_2 (De Santis et al., 1991). In this case, there is no need to determine the value of n_k, because it is only necessary for the radial term of the harmonic expansion. Hence any general function, once referred, by means of (1), to the hemispherical system (θ',λ), can be approximated by conventional spherical harmonics:

$$f(\theta',\lambda) = \sum_{k=0}^{Kmax}\sum_{m=0}^{k}(g_k^m\cos m\lambda + h_k^m\sin m\lambda)\cdot P_k^m(\theta')$$

Another significant advantage of the use of ASHA with respect to SCHA is that conventional Legendre functions can be more easily computed because they are formed from a finite number of terms, whereas fractional generalized Legendre functions are defined by a (convergent) series with an infinite number of terms, so that a truncation in the computation is needed. However, same care, as in SCHA, is needed when the technique is used to model fields with wavelengths much longer than the investigated interval. For example in the case of modelling secular variation fields, if the removal of a global field is not appropriate, it can be useful to use a cap much larger than the data interval, in order to take into account the proper wavelength content of the data set of interest (e.g. Torta et al., 1992).

Acknowledgements. The author thanks D.R. Barraclough and an unknown reviewer for their suggestions. Also the discussions with M. Chiappini, G.V. Haines and J.M. Torta have been of great help in improving some parts of the paper.

References

Bloomfield, P., Fourier Analysis of time series: an introduction, 258 pp., J. Wiley & Sons, New York, 1976.

De Santis, A., Translated Origin Spherical Cap Harmonic Analysis, Geophys. J. Int., 106, 253-263, 1991.

De Santis, A., D.J. Kerridge, D.R. Barraclough, A Spherical Cap Harmonic Model of the Crustal Magnetic Anomaly field in Europe observed by Magsat, in Geomagnetism and Palaeomagnetism, edited by F. J. Lowes, D.W. Collinson, J.H. Parry, S.K. Runcorn, D.C. Tozer, A. Soward, pp. 1-17, Kluwer, Dordrecht, 1989.

De Santis, A., G. De Franceschi, B. Zolesi, Lj. R. Cander, S. Pau, Regional mapping of the critical frequency of the F2 layer by spherical cap harmonic expansion, Annales Geophysicae, 9, 401-406, 1991.

Haines, G.V., Spherical cap harmonic analysis, J. Geoph. Res., 90, 2583-2591, 1985.

Haines, G.V., Computer programs for spherical cap harmonic analysis of potential and general fields, Computers & Geosciences, 14, 4, 413-447, 1985.

Hobson, E.W., The theory of spherical and ellipsoidal harmonics, 500 pp., Cambridge University Press, New York, 1931.

Langel, R.A., The Main Field, in Geomagnetism, vol.1, edited by J.A. Jacobs, pp. 249-512, Academic Press, London, 1987.

Torta, J.M., A. Garcia, J.J. Curto, A. De Santis, New representation of the geomagnetic secular variation over restricted regions by means of spherical cap harmonic analysis: application to the case of Spain. Phys. Earth Planet. Inter., in press, 1992.

A. De Santis, Istituto Nazionale di Geofisica, Via di Villa Ricotti 42, 00161, Rome, Italy.

(Received November 20, 1991;
revised January 29, 1992;
accepted March 31, 1992.)