Towards the Estimation of a Multi-Resolution Representation of the Gravity Field Based on Spherical Wavelets

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Abstract. Usually the gravity field of the Earth is modeled by means of a series expansion in terms of spherical harmonics. However, the computation of the series coefficients requires preferably homogeneous distributed global data sets. Since wavelet functions localize both in the spatial and in the frequency domain, regional and local structures may be modeled by means of a spherical wavelet expansion. Wavelet-based techniques allow the decomposition of a given data set into frequency-dependent detail signals. This paper deals with two methods to achieve a multi-resolution representation, namely a wavelet-only solution and a combined approach. The latter consists of a spherical harmonic expansion for the low-frequency part and a spherical wavelet expansion for the remaining medium- and high-frequency parts of the gravity field. Parameter estimation procedures are presented to determine the unknown series coefficients of both approaches.

Keywords. Spherical harmonics, spherical wavelets, multi-resolution analysis, parameter estimation methods.

1 Introduction

The determination and the representation of the gravity field of the Earth are some of the most important topics of physical geodesy. In the last years several approaches were pursued to generate a *multi-resolution representation* (MRR) of the geopotential; see e.g. Freeden (1999), Kusche (2002) and Beylkin and Cramer (2002).

The computation of the coefficients of a series expansion for the geopotential in terms of spherical harmonics requires preferably homogeneous distributed global data sets. Since a wavelet function is characterized by its ability to localize both in the spatial and in the frequency domain, regional or even local structures can be modeled by means of an appropriate wavelet expansion. Applying the wavelet transform, a given data set is decomposed into a certain number of frequency-dependent detail signals, i.e. an MRR is performed.

In this context there are generally two appropriate methods to model the geopotential, namely

1. the *wavelet-only approach*, i.e. the entire frequency spectrum is described by means of a series expansion in terms of wavelets and
2. the *combined approach*, i.e. the representation is split into an expansion in terms of spherical harmonics for the long-wavelength part and an expansion in terms of wavelets for the medium- and the high-frequency parts.

In order to consider the Laplacian differential equation our approaches are based on the spherical wavelet theory (Freeden 1999). In the next section some fundamental ideas of spherical signal representation are listed. The two methods mentioned before are explained in detail in the third section. We also outline the application of parameter estimation techniques in order to calculate the unknown model coefficients. Finally in the fourth section the MRRs of both an EGM 96 model data set and Champ in situ potential measurements are presented.

2 Foundations

First we introduce a real-valued signal $x(\mathbf{t})$, which is assumed to be harmonic in the exterior of the Earth, i.e. it fulfills the Laplacian differential equation. In general the geocentric position vector

$$\mathbf{t} = r\,[\,\cos\beta\cos\lambda,\ \cos\beta\sin\lambda,\ \sin\beta\,]' = r\,\mathbf{r} \tag{1}$$

of any arbitrary observation point $P = P(\mathbf{t})$ may be expressed by means of the spherical coordinates $\lambda =$ longitude, $\beta =$ latitude and $r =$ radial distance from the geocenter; $\mathbf{r}$ denotes the unit vector. The solution of *Dirichlet's problem* for the outer space Ω_R^{ext} of the sphere Ω_R with radius R can be expressed by the series expansion

$$x(\mathbf{t}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} X_{n,m}^R \, Y_{n,m}^R(\mathbf{t}) \tag{2}$$

in terms of the *solid spherical harmonics*

$$Y_{n,m}^R(\mathbf{t}) = = \frac{1}{\sqrt{4\pi}R}\left(\frac{R}{r}\right)^{n+1} \cdot \left\{ \begin{array}{l} P_{n,m}(\sin\beta)\ \cos m\lambda \\ \quad \text{for } m = 0,\ldots,n \\ P_{n,|m|}(\sin\beta)\ \sin|m|\lambda \\ \quad \text{for } m = -n,\ldots,-1 \end{array} \right\}; \tag{3}$$

see e.g. Heiskanen and Moritz (1967, p.34). $P_{n,m}(\cdot)$ are the associated Legendre functions of degree n and order m. Note, that $\overline{\Omega_R^{ext}} = \Omega_R^{ext} \cup \Omega_R$ means the union of the outer space Ω_R^{ext} and the sphere Ω_R. The *Stokes coefficients* $X_{n,m}^R$ are computed by the spherical *Fourier transform*

$$X_{n,m}^R = \int_{\Omega_R} x(\mathbf{t}_q)\, Y_{n,m}^R(\mathbf{t}_q)\, d\omega_R(\mathbf{t}_q)\,, \tag{4}$$

where $d\omega_R(\mathbf{t}) = R^2\cos\beta\, d\beta\, d\lambda$ denotes the surface element on Ω_R; $\mathbf{t}_q \in \Omega_R$. Now we assume, that the signal $x(\mathbf{t})$ is *band-limited*, i.e.

$$X_{n,m}^R = 0 \qquad \text{for} \qquad n > n_3\,. \tag{5}$$

With $(n_3+1)^2 =: \overline{n}$ we introduce the $\overline{n} \times 1$ vectors

$$\boldsymbol{\beta} = \left[\, X_{0,0}^R,\, X_{1,-1}^R,\, \ldots,\, X_{n_3,n_3}^R \,\right]'\,, \tag{6}$$

$$\mathbf{z}_\mathbf{t} = \left[\, Y_{0,0}^R(\mathbf{t}),\, Y_{1,-1}^R(\mathbf{t}),\, \ldots,\, Y_{n_3,n_3}^R(\mathbf{t}) \,\right]' \tag{7}$$

and obtain from Eq. (2) the representation

$$x(\mathbf{t}) = \mathbf{z}_\mathbf{t}'\,\boldsymbol{\beta}\,. \tag{8}$$

The *spherical convolution* of the band-limited function $x(\mathbf{t})$ with a kernel function

$$k^R(\mathbf{t},\mathbf{t}_k) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R^2}\left(\frac{R^2}{r r_k}\right)^{n+1} K(n)\, P_n(\mathbf{r}'\mathbf{r}_k) \tag{9}$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} K(n)\, Y_{n,m}^R(\mathbf{t})\, Y_{n,m}^R(\mathbf{t}_k) \tag{10}$$

is defined as

$$(k^R \star x)(\mathbf{t}_k) = \int_{\Omega_R} k^R(\mathbf{t},\mathbf{t}_k)\, x(\mathbf{t})\, d\omega_R(\mathbf{t}) \tag{11}$$

$$= \sum_{n=0}^{n_3} \sum_{m=-n}^{n} K(n)\, X_{n,m}^R\, Y_{n,m}^R(\mathbf{t}_k)\,. \tag{12}$$

The functions $P_n(\cdot) \equiv P_{n,0}(\cdot)$ are the Legendre polynomials of degree n. The Legendre coefficients $K(n)$ with $n \in \mathbb{N}_0$ reflect the frequency behavior of the kernel. If the kernel function is band-limited in the same manner as the function $x(\mathbf{t})$, i.e.

$$K(n) = 0 \qquad \text{for} \qquad n > n_3\,, \tag{13}$$

we rewrite the Eqs. (10) and (12) in matrix notation

$$k^R(\mathbf{t},\mathbf{t}_k) = \mathbf{z}_{\mathbf{t}_k}'\,\mathbf{K}\,\mathbf{z}_\mathbf{t}\,, \tag{14}$$

$$(k^R \star x)(\mathbf{t}_k) = \mathbf{z}_{\mathbf{t}_k}'\,\mathbf{K}\,\boldsymbol{\beta}\,. \tag{15}$$

If, in addition to (13), the Legendre coefficients $K(n)$ fulfill the condition

$$K(n) > 0 \qquad \text{for} \qquad 0 \le n \le n_3\,, \tag{16}$$

the $\overline{n} \times \overline{n}$ diagonal matrix $\mathbf{K} = \mathrm{diag}(K(0),\, K(1),\, K(1),\, \ldots,\, K(n_3))$ is positive definite. Next, we introduce the $N \times 1$ vector

$$\mathbf{k}_\mathbf{t} = \left[\, k^R(\mathbf{t},\mathbf{t}_1),\, k^R(\mathbf{t},\mathbf{t}_2),\, \ldots,\, k^R(\mathbf{t},\mathbf{t}_N) \,\right]' \tag{17}$$

and obtain with Eq. (14) the linear equation system

$$\mathbf{k}_\mathbf{t} = \mathbf{Z}\,\mathbf{K}\,\mathbf{z}_\mathbf{t}\,, \tag{18}$$

wherein

$$\mathbf{Z} = \left[\, \mathbf{z}_{\mathbf{t}_1},\, \mathbf{z}_{\mathbf{t}_2},\, \ldots,\, \mathbf{z}_{\mathbf{t}_N} \,\right]' \tag{19}$$

is an $N \times \overline{n}$ matrix. The $\overline{n}$ components of the vector $\mathbf{z}_\mathbf{t}$, Eq. (7), establish a complete orthonormal basis of an $\overline{n}$-dimensional space, denoted by $H_{0,\ldots,n_3}(\Omega_R)$. If for $N \ge \overline{n}$ the matrix $\mathbf{Z}$ is of full column rank, i.e. $\mathrm{rank}\,\mathbf{Z} = \mathrm{rank}(\mathbf{Z}\mathbf{K}) = \overline{n}$, the altogether N components $k^R(\mathbf{t},\mathbf{t}_k)$ with $k = 1,\ldots,N$ of the vector $\mathbf{k}_\mathbf{t}$, Eq. (17), establish another basis of the space $H_{0,\ldots,n_3}(\Omega_R)$. In this case the system

$$S_N(\Omega_R) = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_N\} \tag{20}$$

of points $P(\mathbf{t}_k)$ on the sphere Ω_R is called *admissible*. If even the equality $N = \overline{n}$ holds, the matrix $\mathbf{Z}$ is regular and $S_N(\Omega_R)$ is called a *fundamental* point system. In the following we always assume that the point system $S_N(\Omega_R)$ is at least admissible.

We assume now that the Legendre coefficients $K(n) =: K_1(n)$ of a kernel function $k_1^R(\mathbf{t},\mathbf{t}_k)$ fulfill the conditions (13) and (16). Since both functions $x(\mathbf{t})$ and $k_1^R(\mathbf{t},\mathbf{t}_k)$ are members of the space $H_{0,\dots,n_3}(\Omega_R)$, the convolution $(k_1^R \star x)(\mathbf{t})$ is a member of the same space and can be modeled as

$$(k_1^R \star x)(\mathbf{t}) = \mathbf{k}'_{1;\mathbf{t}}\,\mathbf{d}\,, \tag{21}$$

wherein $\mathbf{d} = (d_k)$ is an $N \times 1$ vector of initially unknown coefficients d_k with $k = 1,\dots,N$. Furthermore, substituting Eq. (18) with $\mathbf{K} =: \mathbf{K}_1$ for $\mathbf{k}_{1;\mathbf{t}}$ yields

$$(k_1^R \star x)(\mathbf{t}) = \mathbf{z}'_\mathbf{t}\,\mathbf{K}_1\,\mathbf{Z}'\,\mathbf{d}\,. \tag{22}$$

Comparing the right hand sides of the Eqs. (15) and (22) gives the expression

$$\boldsymbol{\beta} = \mathbf{Z}'\,\mathbf{d}\,. \tag{23}$$

Hence, according to (6), the Stokes coefficients $X_{n,m}$ with $n \le n_3$ and $-n \le m \le n$ are calculable by means of the coefficient vector $\mathbf{d}$. Next, we introduce a second kernel function $k_2^R(\mathbf{t},\mathbf{t}_k)$. Its Legendre coefficients $K_2(n)$ fulfill certainly the condition (13), but they are not restricted to the condition (16). To be more specific, we allow with

$$K_2(n) \ge 0 \qquad \text{for} \qquad 0 \le n \le n_3\,, \tag{24}$$

that the $\bar{n} \times \bar{n}$ matrix $\mathbf{K}_2$ is possibly just positive semidefinite. It follows from Eq. (15) under the consideration of the Eqs. (18) and (23)

$$(k_2^R \star x)(\mathbf{t}) = \mathbf{z}'_\mathbf{t}\,\mathbf{K}_2\,\boldsymbol{\beta} = \mathbf{z}'_\mathbf{t}\,\mathbf{K}_2\,\mathbf{Z}'\,\mathbf{d} = \mathbf{k}'_{2;\mathbf{t}}\,\mathbf{d}\,. \tag{25}$$

Thus, if the coefficient vector $\mathbf{d}$ is known once, it can be used to calculate both, the Stokes coefficients according to Eq. (23) and any convolution of the function $x(\mathbf{t})$ with kernel functions fulfilling the conditions (13) and (24). As will be explained in the next section, in spherical wavelet theory we identify the kernel functions $k_1^R(\mathbf{t},\mathbf{t}_k)$ and $k_2^R(\mathbf{t},\mathbf{t}_k)$ with the scaling function and the wavelet function, respectively.

If the function $x(\mathbf{t})$ is given in the points $P(\mathbf{t}_k)$ of the admissible system $S_N(\Omega_R)$, the components d_k of the coefficient vector $\mathbf{d} = (d_k)$ are computable from

$$d_k = w_k\,x(\mathbf{t}_k)\,, \tag{26}$$

wherein $w_k = w(\mathbf{t}_k)$ are the *integration weights*. A well-known example of an admissible points systems is the *standard longitude-latitude grid*, i.e. the points $P(\mathbf{t}_k) \equiv P(\lambda_i,\beta_j)$ are defined by $\lambda_i = i\frac{2\pi}{2L}$ for $i = 0,\dots,2L-1$ and $\beta_j = -\frac{\pi}{2} + j\frac{\pi}{2L}$ for $j = 0,\dots,2L$ with $N = (2L+1)\cdot 2L$. The corresponding integration weights $w_k =: w_j$ are merely latitude dependent (Driscoll and Healy 1994). For more details concerning appropriate point grids on the sphere, e.g. the *Reuter grid*, we refer to Freeden et al. (1998, p.171).

Eq. (26) requires that the function $x(\mathbf{t})$ is given in the points $P(\mathbf{t}_k)$ of the admissible system. In the next section we treat a more general approach, namely the computation of the coefficient vector $\mathbf{d}$ with the help of parameter estimation methods.

3 Multi-Resolution Representation

The basic idea of the MRR is to split a given input signal into a smoothed version and a certain number of band-pass signals by *successive low-pass filtering*. In the context of wavelet theory, this procedure consists of the *decomposition* of the signal into wavelet coefficients and the *(re)construction* of the (modified) signal by means of *detail signals*. The latter are the spectral components of the MRR because they are related to certain frequency bands.

3.1 Wavelet-Only Approach

As mentioned before we want to identify the kernel function $k^R(\mathbf{t},\mathbf{t}_k)$, which fulfills the conditions (13) and (16), with the spherical scaling function. To be more specific we introduce the convolution $(\theta_{I+1}^R \star x)(\mathbf{t})$, wherein $x(\mathbf{t})$ is a band-limited function according to Eq. (5) and

$$\theta_{I+1}^R(\mathbf{t},\mathbf{t}_k) = \sum_{n=0}^{2^{I+1}-1} \frac{2n+1}{4\pi R^2}\left(\frac{R^2}{r r_k}\right)^{n+1} \Theta_{I+1}(n)\,P_n(\mathbf{r}'\mathbf{r}_k) \tag{27}$$

the band-limited scaling function of highest resolution level $I+1$ with $I \in \mathbb{N}_0$. The Legendre coefficients $\Theta_{I+1}(n)$ fulfill the conditions (13) and (16), i.e.

$$\begin{aligned} \Theta_{I+1}(n) &> 0 \qquad \text{for} \quad n = 0,\dots,n_3 = 2^{I+1}-1\,, \\ \Theta_{I+1}(n) &= 0 \qquad \text{for} \quad n > n_3\,; \end{aligned} \tag{28}$$

see e.g. Freeden (1999, p.165). As an example we want to mention the *Shannon scaling function* defined by the Legendre coefficients

$$\Theta_i(n) = \left\{ \begin{array}{lll} 1 & \text{for} & n = 0,\dots,2^i-1 \\ 0 & \text{for} & n \ge 2^i \end{array} \right\} \tag{29}$$

of resolution level $i \in \mathbb{N}_0$. In order to perform an MRR we introduce the *spherical wavelet function*

$$\psi_i^R(\mathbf{t},\mathbf{t}_k) = \sum_{n=0}^{2^{i+1}-1} \frac{2n+1}{4\pi R^2} \left(\frac{R^2}{r r_k}\right)^{n+1} \Psi_i(n) P_n(\mathbf{r}'\mathbf{r}_k) \tag{30}$$

of resolution level $i \in \mathbb{N}_0$. Its Legendre coefficients $\Psi_i(n)$ are computed via the *two-scale relation*

$$\tilde{\Psi}_i(n)\Psi_i(n) = \Theta_{i+1}(n) - \Theta_i(n) \ , \tag{31}$$

wherein $\tilde{\Psi}_i(n)$ are the Legendre coefficients of the so-called *dual spherical wavelet function* $\tilde{\psi}_i^R(\mathbf{t},\mathbf{t}_k)$; see e.g. Freeden (1999, p.177). Note, that whereas the (primal) spherical wavelet function $\psi_i^R(\mathbf{t},\mathbf{t}_k)$ decomposes a function into its wavelet coefficients, the dual spherical wavelet function $\tilde{\psi}_i^R(\mathbf{t},\mathbf{t}_k)$ performs the (re)construction (Schmidt 2001, p.273).

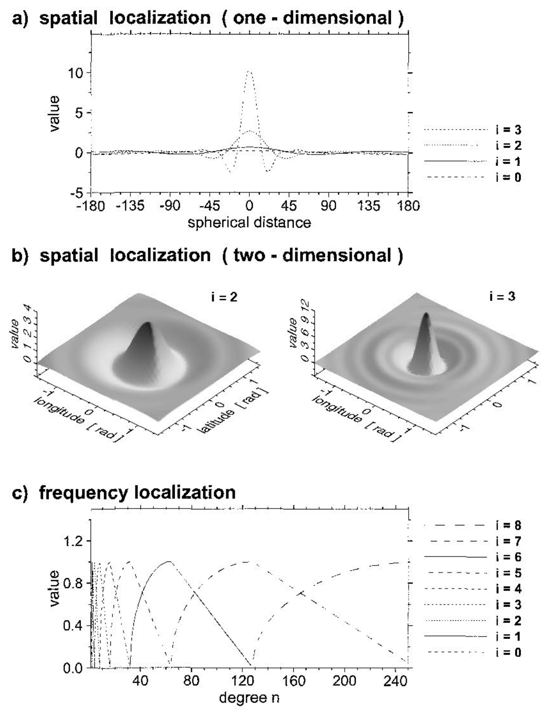

Fig. 1: Smoothed Shannon wavelet with shape parameter $\sigma = 0.5$; see Freeden (1999, p.165), Schmidt et al. (2002).

As an example for a spherical wavelet function with $\tilde{\psi}_i^R(\mathbf{t},\mathbf{t}_k) = \psi_i^R(\mathbf{t},\mathbf{t}_k)$ the *smoothed Shannon wavelet* is illustrated in Fig. 1. An important advantage of this function is the fact, that it is strictly band-limited, i.e. only a few Legendre coefficients are not equal to zero (Fig. 1c). Hence, according to the condition (24) the spherical wavelet function can be identified with the kernel function $k_2^R(\mathbf{t},\mathbf{t}_k)$.

The Shannon scaling function, Eq. (27) with Eq. (29) for $i = I+1$, is the *reproducing kernel* of the the space $H_{0,\ldots,n_3}(\Omega_R)$ with $n_3 = 2^{I+1}-1$. Hence, it follows with Eq. (21)

$$x(\mathbf{t}) = (\theta_{I+1}^R \star x)(\mathbf{t}) = \boldsymbol{\theta}_{I+1;\mathbf{t}}' \mathbf{d} \tag{32}$$

wherein the $N \times 1$ vector $\boldsymbol{\theta}_{I+1;\mathbf{t}}$ is defined analogously to (17). The highest resolution level $I+1$ depends on the maximum degree $n = n_2$ with $n_2 \le n_3$ of the function $x(\mathbf{t})$ in conformance with Eq. (5); we use the formula $I \ge \log_2(n_2+1)$. Furthermore, it follows from the remarks in the context of Eq. (19), that the number $N =: N_I$ of points $P(\mathbf{t}_{I,k})$ with $k = 1,\ldots,N_I$ of the level-I admissible system $S_{N_I}(\Omega_R)$ is restricted to $N_I \ge 2^{2I+2} =: \bar{n}_I$. In order to estimate the N_I unknown *scaling coefficients* $d_{I,k}$ with $k = 1,\ldots,N_I$ of the $N_I \times 1$ vector $\mathbf{d} =: \mathbf{d}_I$ from Eq. (32), we need the values $x(\mathbf{t}_p) =: x_p$ of the function $x(\mathbf{t})$ in altogether N discrete observation points $P(\mathbf{t}_p)$ with $p = 1,\ldots,N$ and $N \ge N_I$. Note, that henceforth N means the number of observations. However, *geodetic measurements* $y(\mathbf{t}_p) =: y_p$ are always erroneous, i.e. $x_p = y_p + e_p$, wherein $e_p := e(\mathbf{t}_p)$ denotes the measurement error. Under these assumptions Eq. (32) can be rewritten as the *observation equation*

$$y_p + e_p = \boldsymbol{\theta}_{I+1;p}' \mathbf{d}_I \tag{33}$$

for a single observation y_p. Note, that probably additional operators, like the Stokes operator, have to be considered in the vector $\boldsymbol{\theta}_{I+1;\mathbf{t}_p} =: \boldsymbol{\theta}_{I+1;p}$, if the observations are functionals of $x(\mathbf{t})$. Introducing the $N \times 1$ vectors $\mathbf{y} = (y_p)$ and $\mathbf{e} = (e_p)$ of the observations and the measurement errors, respectively, the $N \times N_I$ coefficient matrix

$$\mathbf{X} = \left[\boldsymbol{\theta}_{I+1;1},\ \boldsymbol{\theta}_{I+1;2},\ \ldots,\ \boldsymbol{\theta}_{I+1;N}\right]' \tag{34}$$

and the $N \times N$ covariance matrix $D(\mathbf{y})$ of the observations, the linear model

$$\mathbf{y} + \mathbf{e} = \mathbf{X}\,\mathbf{d}_I \quad \text{with} \quad D(\mathbf{y}) = \sigma^2 \boldsymbol{\Sigma}_y \tag{35}$$

is established. Herein σ^2 and $\boldsymbol{\Sigma}_y$ are denoted as the variance of unit weight and the matrix of cofactors, respectively. Analog to the matrix $\mathbf{Z}$, Eq. (19), the matrix $\mathbf{X}$ is of rank$\mathbf{X} = \bar{n}_I$, i.e. a rank deficiency of $N_I - \bar{n}_I$ exists. To overcome this rank deficiency problem, the normal equation system, resulting from the linear model (35), might be solved e.g. by means of *Bayesian Inference*, i.e. prior information about the coefficients is introduced; see e.g. Koch (1990). We assume in the following, that the general solution

$$\hat{\mathbf{d}}_I = \mathbf{B}_I \mathbf{y} \tag{36}$$

for the estimator $\widehat{\mathbf{d}}_I$ of the coefficient vector $\mathbf{d}_I$ exists; see e.g. Kusche (2003, p.40); a detailed treatment of regularization problems goes beyond the scope of this paper. Note, that the "hat" in Eq. (36) and in the following always marks an estimation of the corresponding parameter. The estimation of the coefficient vector $\mathbf{d}_I$ means the *initialization step* of the following pyramid algorithm. To be more specific the $(I-i)^{th}$ *pyramid step* consists of the two linear equations

$$\widehat{\mathbf{d}}_i = \mathbf{H}_i\,\widehat{\mathbf{d}}_{i+1} = \mathbf{H}_i\mathbf{H}_{i+1}\ldots\mathbf{H}_{I-1}\,\widehat{\mathbf{d}}_I\,, \tag{37}$$

$$\widehat{\mathbf{c}}_i = \mathbf{\Psi}_i\,\widehat{\mathbf{d}}_i\,, \tag{38}$$

where $\mathbf{H}_i, \mathbf{H}_{i+1}, \ldots, \mathbf{H}_{I-1}$ are *low-pass filter matrices*. In particular $\mathbf{H}_i$ is an $N_i \times N_{i+1}$ matrix, which transforms the $N_{i+1} \times 1$ scaling coefficient vector $\mathbf{d}_{i+1}$ of level $i+1$ into the $N_i \times 1$ scaling coefficient vector $\mathbf{d}_i$ of level i. In Eq. (38) $\mathbf{\Psi}_i$ denotes the $N_i \times N_i$ wavelet matrix of level i, which acts as a *band-pass filter*; $\widehat{\mathbf{c}}_i$ stands for the estimator of the $N_i \times 1$ vector of *level-i wavelet coefficients* $c_{i,k}$ with $k = 1,\ldots,N_i$.

In the *reconstruction process* we start from the wavelet coefficient vector $\mathbf{c}_i$ of level $i \in \{0,\ldots,I\}$ and compute the estimator

$$\widehat{\mathbf{g}}_i = \widetilde{\mathbf{\Psi}}_i\,\widehat{\mathbf{c}}_i \tag{39}$$

of the $M \times 1$ detail signal vector $\mathbf{g}_i$. Herein the $M \times N_i$ dual wavelet matrix $\widetilde{\mathbf{\Psi}}_i$ works as a *band-pass filter*. The elements $g_i(\mathbf{t}_q)$ of the vector $\mathbf{g}_i$ are related to the points $P(\mathbf{t}_q)$ with $\mathbf{t}_q \in \overline{\Omega_R^{ext}}$ and $q = 1,\ldots,M$, which do not necessarily coincide with the points $P(\mathbf{t}_{I,k})$ of the admissible system or the observation sites $P(\mathbf{t}_p)$. Finally the estimated MRR reads

$$\widehat{\mathbf{x}} = \widehat{\mathbf{x}}_0 + \sum_{i=0}^{I} \widehat{\mathbf{g}}_i\,. \tag{40}$$

Herein $\widehat{\mathbf{x}}$ means the $M \times 1$ vector of *filtered* or *predicted* signal values $\widehat{x}(\mathbf{t}_q)$. The components of the vector $\widehat{\mathbf{x}}_0 = (\widehat{x}_0(\mathbf{t}_q))$ are calculated from $\widehat{x}_0(\mathbf{t}_q) = \mathbf{\theta}'_{0;q}\,\widehat{\mathbf{d}}_0$. As an example for the signal $x(\mathbf{t})$ we want to mention the *disturbing potential*.

3.2 Combined Approach

In the following we assume that the unknown function $x(\mathbf{t})$, introduced in the previous subsection, can be expressed by means of a series expansion in terms of spherical harmonics (according to Eq. (2), but only until degree $n = n_1$) for the long-wavelength part and a spherical wavelet expansion for the remaining medium- and short-wavelength parts. Hence, the observation equation (33) can be replaced by

$$y_p + e_p = \mathbf{z}'_p\,\mathbf{\beta} + \mathbf{\theta}'_{I+1,n_1+1;p}\,\mathbf{d}_I\,, \tag{41}$$

wherein with $(n_1+1)^2 =: \overline{n}_1$ the $\overline{n}_1 \times 1$ vectors $\mathbf{z}_{\mathbf{t}_p} =: \mathbf{z}_p$ and $\mathbf{\beta}$ are defined by Eqs. (6) and (7) substituting n_1 for n_3. The $N_I \times 1$ vector $\mathbf{\theta}_{I+1,n_1+1;p}$ is computed analog to Eq. (32) with the spherical scaling function

$$\theta_{I+1,n_1+1}(\mathbf{t},\mathbf{t}_k) = \sum_{n=n_1+1}^{2^{I+1}-1} \frac{2n+1}{4\pi R^2}\left(\frac{R^2}{r r_k}\right)^{n+1} P_n(\mathbf{r}'\mathbf{r}_k) \tag{42}$$

according to the Eqs. (27) and (29). But in opposite to Eq. (27) the lower summation limit is now set to $n = n_1 + 1$. With the $N \times u_1$ coefficient matrix $\mathbf{X}_1 \equiv \mathbf{Z}$ according to (19) and the $N \times u_2$ coefficient matrix $\mathbf{X}_2$, analog to (34), we obtain the linear model

$$\mathbf{y} + \mathbf{e} = \mathbf{X}_1\,\mathbf{\beta}_1 + \mathbf{X}_2\,\mathbf{\beta}_2 \quad \text{with} \quad D(\mathbf{y}) = \sigma^2\mathbf{\Sigma}_y \tag{43}$$

of the combined approach, wherein $\mathbf{\beta}_1 = (X^R_{n,m})$ is the $u_1 \times 1$ vector of the $u_1 \equiv \overline{n}_1$ (unknown) Stokes coefficients $X^R_{n,m}$ and $\mathbf{\beta}_2 \equiv \mathbf{d}_I = (d_{I,k})$ the $u_2 \times 1$ vector of the $u_2 \equiv N_I$ unknown scaling coefficients $d_{I,k}$. Even if the matrix $\mathbf{X}_1$ is of full column rank, the rank of the $N \times u$ matrix $[\mathbf{X}_1, \mathbf{X}_2]$ with $u := u_1 + u_2$ is equal to the rank of the coefficient matrix $\mathbf{X}$ of the wavelet-only model (35). If we assume, that due to the definition (42) of the scaling function, the spherical wavelet model part is independent of the spherical harmonic model part, the normal equation system degenerates into two parts and allows a separate determination of the estimators $\widehat{\mathbf{\beta}}_1$ and $\widehat{\mathbf{\beta}}_2$ of the unknown parameter vectors $\mathbf{\beta}_1$ and $\mathbf{\beta}_2$. Note, that due to Eq. (23), it is also possible to compute both estimators together from the estimator $\widehat{\mathbf{d}}_I$, Eq. (36), of the wavelet-only model (35). The scaling function $\theta_{I+1,n_1+1}(\mathbf{t},\mathbf{t}_k)$, Eq. (42), fulfills the condition (24) just as the chosen wavelet functions of levels $i = i',\ldots,I$. According to the observation equation (41) the MRR (40) has to be replaced by

$$\widehat{\mathbf{x}} = \widehat{\mathbf{x}}_{0,\ldots,n_1} + \sum_{i=i'}^{I} \widehat{\mathbf{g}}_i\,, \tag{44}$$

wherein $\widehat{\mathbf{x}}_{0,\ldots,n_1}$ means the $M \times 1$ vector of the estimated spherical harmonic signal values computed in the points $P(\mathbf{t}_q)$.

4 Numerical Examples

In the first example the combined approach is applied to *gravity anomalies of EGM 96*. In order to demonstrate the method and to keep the numerical efforts

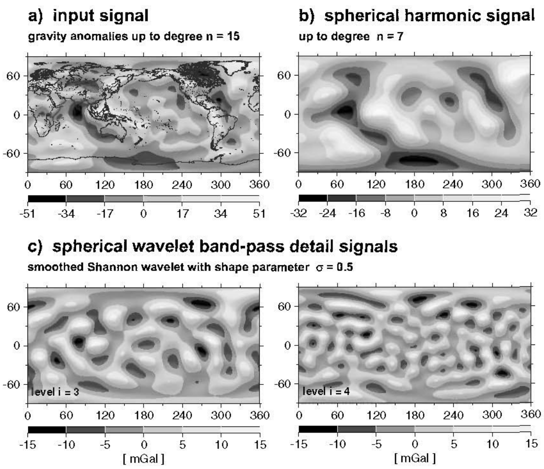

Fig. 2: Gravity anomalies of EGM 96 up to degree $n_3 = 15$ and results of the combined estimation.

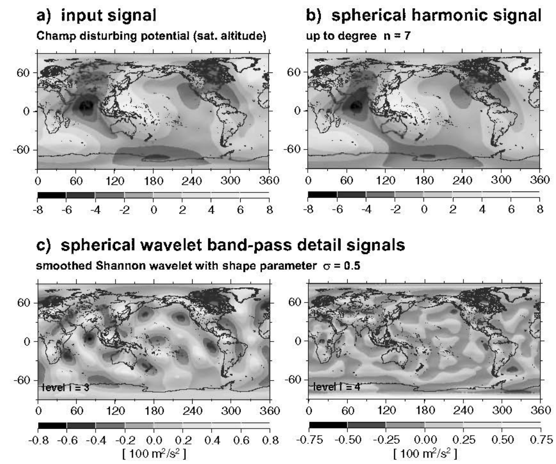

Fig. 3: Champ disturbing potential decomposition at satellite altitude (450 km).

low we choose a maximum degree $n_2 = 15$ for the input signal (Fig. 2a). With $n_1 = 7$ the series expansion in terms of spherical harmonics contains $u_1 = 64$ unknown Stokes coefficients. As spherical wavelet function we choose the smoothed Shannon wavelet shown in Fig. 1. Since, according to Fig. 1c the detail signals of the levels $i = 3 = i'$ and $i = 4 = I$ have to be considered, the MRR (44) reduces to

$$\widehat{\mathbf{x}} = \widehat{\mathbf{x}}_{0,\ldots,7} + \widehat{\mathbf{g}}_3 + \widehat{\mathbf{g}}_4 \,. \tag{45}$$

As admissible point system $S_{N_4}(\Omega_R)$ we use the Reuter grid, mentioned before. After computing the estimators $\widehat{\boldsymbol{\beta}}_1$ and $\widehat{\boldsymbol{\beta}}_2 \equiv \widehat{\mathbf{d}}_4$ from the linear model (43) in the initialization step, the following pyramid steps are performed according to the Eqs. (37) to (39). The Figs. 2b, c display the values of the three estimated vectors $\widehat{\mathbf{x}}_{0,\ldots,7}$, $\widehat{\mathbf{g}}_3$ and $\widehat{\mathbf{g}}_4$. According to Eq. (45) the sum of these vectors means an estimation of the MRR of the input gravity anomalies visualized in Fig. 2a. The deviations (not shown here) between the input signal and the MRR are in the range of the computational accuracy. Due to the ability of wavelet functions to localize, the block matrix $\mathbf{X}_2'\boldsymbol{\Sigma}_y^{-1}\mathbf{X}_2$ is, depending on the chosen wavelet function, possibly close to be of sparse form. The localizing property gives also reason to apply this procedure to regional data sets. Numerical investigations concerning this topic will be presented in an forthcoming paper of the authors.

In the second example the same procedure is applied to *Champ in situ disturbing potential values* in satellite altitude (Fig. 3a), computed from Champ data by means of the energy conservation principle. Again the sum of the estimated spherical harmonic signal $\widehat{\mathbf{x}}_{0,\ldots,7}$ (Fig. 3b) and the estimated detail signals $\widehat{\mathbf{g}}_3$ and $\widehat{\mathbf{g}}_4$ (Fig. 3c) yields the MRR of the input signal (Fig. 3a). But this time the deviations (not shown here) between the input signal and the MRR consist of both stochastic measurement errors and the detail signals of resolution levels higher than $I = 4$. The rms value of the deviations amounts less than 0.1 $\mathrm{m}^2/\mathrm{s}^2$.

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