HANDBOOK OF ELLIPTIC INTEGRALS FOR ENGINEERS AND PHYSICISTS

BY

PAUL F. BYRD

AND

MORRIS D. FRIEDMAN

SPRINGER-VERLAG
BERLIN HEIDELBERG GMBH

DIE GRUNDLEHREN DER
MATHEMATISCHEN
WISSENSCHAFTEN
IN EINZELDARSTELLUNGEN MIT BESONDERER
BERÜCKSICHTIGUNG DER ANWENDUNGSGEBIETE

BAND LXVII

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WITH 22 FIGURES

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Preface.

Engineers and physicists are more and more encountering integrations involving nonelementary integrals and higher transcendental functions. Such integrations frequently involve (not always in immediately recognizable form) elliptic functions and elliptic integrals.

The numerous books written on elliptic integrals, while of great value to the student or mathematician, are not especially suitable for the scientist whose primary objective is the ready evaluation of the integrals that occur in his practical problems. As a result, he may entirely avoid problems which lead to elliptic integrals, or is likely to resort to graphical methods or other means of approximation in dealing with all but the simplest of these integrals.

It became apparent in the course of my work in theoretical aerodynamics that there was a need for a handbook embodying in convenient form a comprehensive table of elliptic integrals together with auxiliary formulas and numerical tables of values. Feeling that such a book would save the engineer and physicist much valuable time, I prepared the present volume.

Although the book is not a text, an attempt has been made to write it in elementary terms so that no previous knowledge of elliptic integrals, theta functions or elliptic functions is needed. A collection of over 3000 integrals and formulas, designed to meet most practical needs, is presented using Legendre's and Jacobi's notations, rather than the less familiar Weierstrassian forms. Many of these formulas are substitutions and recurrence relations for evaluating additional integrals which are not explicitly written. Sufficient explanatory material and cross-references are given to permit the reader to obtain the answers he requires with a minimum of effort.

Short tables of numerical values are given for the elliptic integrals of the first and second kind, for Jacobi's "nome" q, for the function denoted by Heuman as Λ_0, and for K times the Jacobian Zeta function. Tables of the last three functions are useful in the numerical evaluation of elliptic integrals of the third kind.

Particular precautions, of course, have to be taken in a work of this kind to insure accuracy of the formulas. My co-author, Mr. Morris

D. FRIEDMAN, undertook the job of verifying each formula. Where ever possible, they were either derived independently in different ways or checked against more than one source. Criticisms of the material contained in the handbook and notice of any errors which may yet appear in it will be sincerely welcomed.

It is impossible to acknowledge properly all the sources to which debt is owed. The bibliography, however, lists many books in which the derivation of some of the formulas can be found or where related material may be obtained. For friendly advice and valuable suggestions, I am under obligation to Professors A. ERDÉLYI, W. MAGNUS and R. C. ARCHIBALD, and to colleagues in the Theoretical Aerodynamics Section, Ames Aeronautical Laboratory, NACA. To my colleague, DORIS COHEN, I am grateful for a critical reading of the manuscript and for many suggestions leading to improvement of exposition and organization. Hearty thanks are also extended to Mrs. ROSE CHIN BYRD and Mrs. MARY T. HUGGINS for assistance in the task of preparing the tables of numerical values and to Mr. DUANE W. DUGAN for help in reading the proofs.

On behalf of both authors I wish finally to express gratitude to Springer-Verlag and to Professor K. KLOTTER of Stanford University who kindly called their attention to our work. The appearance of this book is in no small measure due to their cooperative attitude, their encouragement, and their genuine interest in the promotion of technical publications.

Palo Alto, California. PAUL F. BYRD.

July, 1953.

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List of Symbols and Abbreviations.

The following table comprises a list of the principal symbols and abbreviations used in the handbook. Notations not listed are so well understood that explanation is unnecessary.

Symbol or Abbreviation	Meaning	Section
α^2	Parameter of elliptic integral of the third kind	**110**
$\mathrm{am}\,(u, k) \equiv \mathrm{am}\,u$	Amplitude u	**120**
$\mathrm{am}^{-1}\,(y, k)$	Inverse amplitude y	**130**
$\mathrm{cd}\,u$	$\equiv \dfrac{\mathrm{cn}\,u}{\mathrm{dn}\,u}$	**120**
$\mathrm{cd}^{-1}\,(y, k)$	—	**130**
$\mathrm{cn}\,(u, k) \equiv \mathrm{cn}\,u$	Cosine amplitude u; Jacobian elliptic function	**120**
$\mathrm{cn}^{-1}\,(y, k)$	—	**130**
$\cos^{-1}\varphi$	Inverse trigonometric function, often written *arc cos* φ	
$\mathrm{cs}\,u$	$\equiv \dfrac{\mathrm{cn}\,u}{\mathrm{sn}\,u}$	**120**
$\mathrm{cs}^{-1}\,(y, k)$	—	**130**
$\mathrm{dc}\,u$	$\equiv \dfrac{\mathrm{dn}\,u}{\mathrm{cn}\,u}$	**120**
$\mathrm{dc}^{-1}\,(y, k)$	—	**130**
$\mathrm{dn}\,u$	Delta amplitude u; Jacobian elliptic function	**120**
$\mathrm{dn}^{-1}\,(y, k)$	—	**130**
$\mathrm{ds}\,u$	$\equiv \dfrac{\mathrm{dn}\,u}{\mathrm{sn}\,u}$	**120**
$\mathrm{ds}^{-1}\,(y, k)$	—	**130**
e_1, e_2, e_3	Roots of polynomial written in Weierstrassian form	**1030**
$E\,(\varphi, k) \equiv E\,(u)$	Legendre's incomplete elliptic integral of the second kind; ($\varphi = \mathrm{am}\,u$)	**110**
$E'\,(\varphi, k) \equiv E\,(\varphi, k')$	Associated incomplete elliptic integral of the second kind	
$E\,(k) \equiv E \equiv E\,(\pi/2, k)$	Complete elliptic integral of the second kind	
$E' \equiv E\,(k')$	Associated complete elliptic integral of the second kind	
$F\,(\varphi, k) \equiv u$	Incomplete elliptic integral of the first kind	

Symbol or Abbreviation	Meaning	Section
$F(a, b; c; z)$	$\equiv \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m$, hypergeometric series	900
G	$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.91596559$, Catalan's constant	615
g_2, g_3	—	1030
$\Gamma(z)$	$\equiv \frac{1}{z} \prod_{m=1}^{\infty} \left[\left(1+\frac{1}{m}\right)^z \left(1+\frac{z}{m}\right)^{-1}\right]$, $z \neq 0, -1, -2, \ldots$, Gamma function	
H, H_1	Eta functions of Jacobi	1050
$I_\nu(z)$	$\equiv \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$ $\equiv e^{-\nu\pi i/2} J_\nu(iz)$, modified Bessel functions of first kind	560
Im	Imaginary part of a complex quantity	
$J_\nu(z)$	$\equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$, Bessel functions of first kind	560
k	Modulus of Jacobian elliptic functions and integrals	110
$k' = \sqrt{1-k^2}$	Complementary modulus	110
$K(k) \equiv K = F(\pi/2, k)$	Complete elliptic integral of the first kind	110
$K' \equiv K(k')$	Associated complete elliptic integral of the first kind	110
$\Lambda_0(\varphi, k)$	$\equiv \frac{2}{\pi}[EF(\varphi, k') + KE(\varphi, k') - KF(\varphi, k')]$, Heuman's Lambda function	150
$\ln z$	Natural logarithm of z	
m, n	Integers, unless otherwise stated	
$n!$	$= 1.2 \ldots n$; n factorial	
$\mathrm{nc}\, u$	$\equiv \frac{1}{\mathrm{cn}\, u}$	120
$\mathrm{nc}^{-1}(y, k)$	—	130
$\mathrm{nd}\, u$	$\equiv \frac{1}{\mathrm{dn}\, u}$	120
$\mathrm{nd}^{-1}(y, k)$	—	130
$\mathrm{ns}\, u$	$\equiv \frac{1}{\mathrm{sn}\, u}$	120
$\mathrm{ns}^{-1}(y, k)$	—	130
$\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$	—	430

Symbol or Abbreviation	Meaning	Section
$\wp(u)$	Weierstrassian elliptic function	**1030**
$\Pi(\varphi, \alpha^2, k) \equiv \Pi(u, \alpha^2)$	Legendre's incomplete elliptic integral of the third kind; ($\varphi = \operatorname{am} u$)	**110**
$\Pi(\alpha^2, k) \equiv \Pi(\pi/2, \alpha^2, k)$	Complete elliptic integral of the third kind	**110**
q	$\equiv e^{-\pi K'/K}$, referred to as the *nome*	**1050**
$Q_n(z)$	$\equiv \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n (z-t)^{-n-1}\, dt$, R.P. $(n+1) > 0$, Legendre functions (spherical harmonics)	**560**
R.P.	Real part of a complex quantity	
$\sigma(u)$	Weierstrassian Sigma function	**1030**
$\operatorname{sd} u$	$\equiv \frac{\operatorname{sn} u}{\operatorname{dn} u}$	**120**
$\operatorname{sd}^{-1}(y, k)$	—	**130**
$\sin^{-1}\varphi$	Inverse trigonometric function, often written *arc sin* φ	
$\operatorname{sn} u$	Sine amplitude u	**120**
$\operatorname{sn}^{-1}(y, k)$	—	**130**
$\operatorname{tn} u \equiv \operatorname{sc} u$	$\equiv \frac{\operatorname{sn} u}{\operatorname{cn} u}$	**120**
$\operatorname{tn}^{-1}(y, k)$	—	**130**
Θ, Θ_1	Jacobi's Theta functions	**1050**
$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$	Elliptic Theta functions	**1050**
y	Variable limit of integration in all integrals	
$\psi(z)$	$\equiv \xi - \frac{1}{z} + \sum_{m=1}^{\infty} \frac{z}{m(z+m)}$, digamma function; ξ is Euler's number ≈ 0.5772156665	**900**
$Z(u, k) \equiv Z(u) \equiv Z(\beta, k)$	$\equiv E(u) - \frac{E}{K} u$, Jacobian Zeta function; $(\operatorname{am} u = \beta)$	**140**
$\zeta(u)$	Weierstrassian Zeta function	**1030**
$(a)_n$	$= a(a+1)\ldots(a+n-1)$, for $n = 1, 2, \ldots$ $(a)_0 = 1$, Pochhammer's symbol	
$\binom{a}{n} = (-1)^n \frac{(-a)_n}{n!}$	$= \frac{a(a-1)\ldots(a-n+1)}{1.2.3\ldots n}$; $\binom{a}{0} = 1$.	

Introduction.

Integrals of the form $\int R[t, \sqrt{P(t)}]\, dt$, where $P(t)$ is a polynomial of the third or fourth degree and R is a rational function, have the simplest algebraic integrands that can lead to nonelementary[1] integrals. Equivalent integrals occur in trigonometric and other forms, in pure and applied mathematics. Such integrals are known as elliptic integrals because a special example of this type arose in the rectification of the arc of an ellipse. Although some early work on them was done by FRAGNANO, EULER, LAGRANGE and LANDEN, they were first treated systematically by LEGENDRE, who showed that any elliptic integral may be made to depend on three fundamental integrals which he denoted by $F(\varphi, k)$, $E(\varphi, k)$ and $\Pi(\varphi, n, k)$. These three integrals are called Legendre's canonical elliptic integrals of the first, second and third kind respectively. Legendre's normal forms are not the only standard forms possible, but they have retained their usefulness for over a century.

The elliptic functions of ABEL, JACOBI and WEIERSTRASS are obtained by the inversion of elliptic integrals of the first kind. As is shown in Vol. LV of this series[2], these inverse functions have numerous direct applications in problems of electrostatics, hydromechanics, aerodynamics etc. They are also highly useful for evaluating elliptic integrals, which is the primary concern of this handbook. The more modern theory of WEIERSTRASS is sometimes better suited to certain problems, but for most practical problems Jacobian elliptic functions appear in a more natural way, simplifying the formulas and facilitating numerical evaluation of the results. Thus, for obtaining a comprehensive table of elliptic integrals we find the older notation of JACOBI and LEGENDRE more convenient. (The Weierstrassian functions are treated briefly in the Appendix.)

The general plan of the handbook is as follows: The definitions and other basic information concerning the elliptic integrals and Jacobian

[1] The nonelementary character of elliptic integrals is briefly demonstrated in *Integration in Finite Terms* (Liouville's Theory of Elementary Methods) by J. F. Ritt, Columbia University Press, New York, 1948, pp. 35—37.

[2] F. OBERHETTINGER and W. MAGNUS, *Anwendung der elliptischen Funktionen in Physik und Technik*. (Grundlehren der mathematischen Wissenschaften, Band LV.) Springer-Verlag, 1949.

elliptic functions are given first [Items **100–199**]. Then elliptic integrals in the various algebraic or trigonometric forms in which they are encountered in practice are expressed in terms of integrals involving Jacobian elliptic functions [Items **200–299**]. These latter functions are integrated in Items **300–499**. Specific reference is made in each formula of section **200–299** to the applicable formula in the tables [**300–499**], in which the Jacobian forms are explicitly integrated. We have adopted this procedure because in this way it is possible to give in less space evaluations of a variety of elliptic integrals, particularly those leading to elliptic integrals of the third kind. The remainder of the handbook is devoted to auxiliary formulas and related integrals.

In our table of integrals involving algebraic integrands, one of the limits of integration is usually taken to be a zero of the polynomial under the radical sign, while the other limit may vary. The use of these tables, however, is not as restrictive as this may appear, because it is easy to evaluate integrals in which both limits are variable. Consider, for example, the integral

$$J = \int_{y_1}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}},$$

where $a > y > y_1 > b$. This integral may be reduced by using either **235.00** or **236.00**, since one can write

$$J = \int_{y_1}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} = \int_{b}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} - \int_{b}^{y_1} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}},$$

or

$$J = \int_{y_1}^{a} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} \quad \int_{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} .$$

As another example, take the integral

$$I = \int_{y_1}^{y} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}},$$

with $\infty > y > y_1 > a$. Here I may also be split into two integrals, e.g.,

$$\int_{y_1}^{y} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \int_{y_1}^{\infty} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} - \int_{y}^{\infty} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}.$$

Hence we may employ **215.00**.

The above also applies to the table of integrals involving trigonometric or hyperbolic integrands.

We give now the following five examples to illustrate in detail how the handbook may be used for rapid evaluation of elliptic integrals encountered in geometrical and physical problems.

Example I.

The length of arc of a hyperbola

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \tag{1}$$

measured from the vertex to any point (x, y) is determined by the integral

$$s = \sqrt{\frac{\alpha^2+\beta^2}{\beta^2}} \int_0^y \sqrt{\frac{b^2+t^2}{a^2+t^2}}\, dt, \tag{2}$$

where

$$a^2 = \beta^2, \qquad b^2 = \frac{\beta^4}{\alpha^2+\beta^2}, \qquad (b < a). \tag{3}$$

From **221.04** and **313.02**, it is seen that

$$\begin{cases} \displaystyle\int_0^y \sqrt{\frac{b^2+t^2}{a^2+t^2}}\, dt = \frac{b^2}{a}\int_0^{u_1} \mathrm{nc}^2 u\, du = \frac{b^2}{a\, k'^2}\Big[k'^2 u - E(u) + \mathrm{dn}\, u\, \mathrm{tn}\, u\Big]_0^{u_1} \\ \displaystyle\qquad = \frac{b^2}{a\, k'^2}\left[k'^2 F(\varphi, k) - E(\varphi, k) + \tan\varphi\sqrt{1-k^2\sin^2\varphi}\right], \end{cases} \tag{4}$$

where

$$\begin{cases} k = \sqrt{(a^2-b^2)/a^2} = \sqrt{\alpha^2/(\alpha^2+\beta^2)}, \\ \displaystyle\varphi = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\frac{y^2}{y^2+b^2}} = \sin^{-1}\sqrt{\frac{(\alpha^2+\beta^2)\, y^2}{\beta^4+(\alpha^2+\beta^2) y^2}}; \end{cases} \tag{5}$$

hence, finally

$$s = \sqrt{\alpha^2+\beta^2}\left[\frac{\beta^2}{\alpha^2+\beta^2} F(\varphi, k) - E(\varphi, k) + \frac{y}{\beta}\sqrt{\frac{(\alpha^2+\beta^2)(\beta^2+y^2)}{\beta^4+(\alpha^2+\beta^2)\, y^2}}\right]. \tag{6}$$

Example II.

An integral which arises in finding the decrease in lifting pressure near the tip of a sweptback wing flying at supersonic speed has the form

$$\begin{cases} \displaystyle I(X, Y) = \int_{t_0}^{t_1} \frac{dt}{\left(\frac{\beta Y}{X} - t\right)\sqrt{(t_1-t)(t_1+t)(1+t)(t-t_0)}}, \\ \displaystyle\qquad \left(-1 < -t_1 < \frac{\beta Y}{X} < t_0 < t_1\right). \end{cases} \tag{7}$$

From **257.39**, with $a=t_1$, $b=t_0$, $c=-t_1$, $d=-1$, $m=1$ and $p=\beta Y/X$, we obtain (noting that the integral is complete)

$$(8)\qquad I=\frac{2}{(p-a)\sqrt{(a-c)(b-d)}}\int_0^K \frac{1-\left(\frac{b-a}{b-d}\right)\mathrm{sn}^2 u}{1-\frac{(p-d)(a-b)}{(a-p)(b-d)}\mathrm{sn}^2 u}\,du,$$

where

$$(9)\qquad k^2=\frac{(a-b)(c-d)}{(a-c)(b-d)}=\frac{(t_1-t_0)(1-t_1)}{2t_1(t_0+1)}.$$

Since $1>\frac{(p-d)(a-b)}{(a-p)(b-d)}>k^2$, we use **413.06**. Equation (8) thus becomes

$$(10)\qquad \begin{cases} I=\dfrac{2K}{(p-d)\sqrt{(a-c)(b-d)}}-\dfrac{\pi\Lambda_0(\xi,k)}{\sqrt{(a-p)(p-d)(b-p)(p-c)}}\\[2ex] \quad=\dfrac{2XK}{(\beta Y+X)\sqrt{2(t_0+1)t_1}}-\dfrac{\pi X^2\Lambda_0(\xi,k)}{\sqrt{(X^2t_1^2-\beta^2Y^2)(X+\beta Y)(Xt_0-\beta Y)}},\end{cases}$$

where

$$(11)\qquad \xi=\sin^{-1}\sqrt{\frac{(p-c)(b-d)}{(p-d)(b-c)}}=\sin^{-1}\sqrt{\frac{(\beta Y+t_1X)(t_0+1)}{(\beta Y+X)(t_0+t_1)}}$$

and $\Lambda_0(\xi,k)$ is defined in **150**.

Example III.

To obtain the gravitational potential V of a homogeneous solid ellipsoid at the exterior point (X, Y, Z), one is led to the integral

$$(12)\qquad \begin{cases} V=\pi\varrho\alpha\beta\gamma\displaystyle\int_{y_1}^{\infty}\left(1-\frac{X^2}{\alpha^2+t}-\frac{Y^2}{\beta^2+t}-\frac{Z^2}{\gamma^2+t}\right)\times\\[2ex] \qquad\times\dfrac{dt}{\sqrt{(\alpha^2+t)(\beta^2+t)(\gamma^2+t)}}\end{cases}$$

where ϱ is the density and y_1 is the largest root of

$$(13)\qquad \frac{X^2}{\alpha^2+y_1}+\frac{Y^2}{\beta^2+y_1}+\frac{Z_2}{\gamma^2+y_1}=1,\quad (\alpha^2>\beta^2>\gamma^2).$$

Equation (12) thus involves the evaluation of four integrals of the form

$$(14)\qquad I=\int_{y_1}^{\infty}\frac{R(t)\,dt}{\sqrt{(t-a)(t-b)(t-c)}},$$

where $a=-\gamma^2$, $b=-\beta^2$, $c=-\alpha^2$, $(y_1>a>b>c)$ and $R(t)$ is a rational function of t. From **238.00** we have immediately

$$(15)\qquad \int_{y_1}^{\infty}\frac{dt}{\sqrt{(t-a)(t-b)(t-c)}}=\frac{2}{\sqrt{a-c}}\int_0^{u_1}du=\frac{2}{\sqrt{a-c}}u_1=\frac{2F(\varphi,k)}{\sqrt{a-c}}$$

where

$$(16)\quad \begin{cases} \varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\dfrac{a-c}{y_1-c}} = \sin^{-1}\sqrt{\dfrac{\alpha^2-\gamma^2}{y_1+\alpha^2}}\,, \\ k = \sqrt{\dfrac{b-c}{a-c}} = \sqrt{\dfrac{\alpha^2-\beta^2}{\alpha^2-\gamma^2}}\,. \end{cases}$$

With $m=1$, one obtains next, from **238.03** and **310.02**,

$$(17)\quad \begin{cases} \displaystyle\int_{y_1}^{\infty} \frac{dt}{(t-c)\sqrt{(t-a)(t-b)(t-c)}} = \frac{2}{(a-c)\sqrt{a-c}} \int_0^{u_1} \operatorname{sn}^2 u\, du \\ \displaystyle\qquad = \frac{2}{k^2(a-c)\sqrt{a-c}}\,[u - E(u)]_0^{u_1} \\ \displaystyle\qquad = \frac{2\,[u_1 - E(\operatorname{am} u_1, k)]}{k^2(a-c)\sqrt{a-c}} = \frac{2\,[F(\varphi,k) - E(\varphi,k)]}{k^2(a-c)\sqrt{a-c}}\,. \end{cases}$$

From **238.04** and **318.02** it is readily seen that

$$(18)\quad \begin{cases} \displaystyle\int_{y_1}^{\infty} \frac{dt}{(t-b)\sqrt{(t-a)(t-b)(t-c)}} = \frac{2}{(a-c)\sqrt{a-c}} \int_0^{u_1} \operatorname{sd}^2 u\, du \\ \displaystyle\qquad = \frac{2}{k^2 k'^2 (a-c)\sqrt{a-c}} \left[E(u_1) - k'^2 u_1 - k^2 \frac{\operatorname{sn} u_1 \operatorname{cn} u_1}{\operatorname{dn} u_1}\right] \\ \displaystyle\qquad = \frac{2}{k^2 k'^2 (a-c)\sqrt{a-c}} \left[E(\varphi,k) - k'^2 F(\varphi,k) - k^2 \frac{\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right], \end{cases}$$

and with **238.05** and **316.02** that

$$(19)\quad \begin{cases} \displaystyle\int_{y_1}^{\infty} \frac{dt}{(t-a)\sqrt{(t-a)(t-b)(t-c)}} = \frac{2}{(a-c)\sqrt{a-c}} \int_0^{u_1} \operatorname{tn}^2 u\, du \\ \displaystyle\qquad\qquad = \frac{2}{k'^2(a-c)^{\frac{3}{2}}} \left[\tan\varphi\sqrt{1-k^2\sin^2\varphi} - E(\varphi,k)\right], \end{cases}$$

where k and φ are given by (16). We finally have, making use of (15)–(19),

$$(20)\quad \begin{cases} \displaystyle V = \frac{2\pi\varrho\alpha\beta\gamma}{\sqrt{\alpha^2-\gamma^2}} \int_0^{u_1} [(\alpha^2-\gamma^2) - X^2\operatorname{sn}^2 u - Y^2 \operatorname{sd}^2 u - Z^2 \operatorname{tn}^2 u]\, du \\ \displaystyle\quad = \frac{2\pi\varrho\alpha\beta\gamma}{\sqrt{\alpha^2-\gamma^2}} \left\{\left[1 - \frac{X^2}{\alpha^2-\beta^2} + \frac{Y^2}{\alpha^2-\beta^2}\right] F(\varphi,k) + \right. \\ \displaystyle\quad + \left[\frac{X^2}{\alpha^2-\beta^2} + \frac{(\gamma^2-\alpha^2)\,Y^2}{(\alpha^2-\beta^2)(\beta^2-\gamma^2)} + \frac{Z^2}{\beta^2-\gamma^2}\right] E(\varphi,k) + \\ \displaystyle\quad \left. + \left[\frac{\gamma^2+y_1}{\beta^2-\gamma^2}\, Y^2 - \frac{\beta^2+y_1}{\beta^2-\gamma^2}\, Z^2\right] \sqrt{\frac{\alpha^2-\gamma^2}{(\alpha^2+y_1)(\beta^2+y_1)(\gamma^2+y_1)}}\right\}. \end{cases}$$

Example IV.

An integral which occurs in finding the spanwise distribution of circulation on a slender wing mounted on a cylindrical body is

$$(21)\qquad J=\int_{\alpha}^{y}\frac{(t^4-\beta^4)\,dt}{t^2\sqrt{(t^2-\alpha^2)\,(t^2-\beta^4/\alpha^2)}}\,,\qquad (y^2>\alpha^2>\beta^4/\alpha^2)\,,$$

or

$$(22)\qquad J=\int_{\alpha}\frac{t^2\,dt}{\sqrt{(t^2-\alpha^2)\,(t^2-\beta^4/\alpha^2)}}-\beta^4\int_{\alpha}^{y}\frac{dt}{t^2\sqrt{(t^2-\alpha^2)\,(t^2-\beta^4/\alpha^2)}}\,.$$

Using **216.06** and **216.09**, we may at once write (22) as

$$(23)\qquad J=\alpha\left[\int_0^{u_1}\mathrm{dc}^2\,u\,du-k^2\int_0^{u_1}\mathrm{cd}^2\,u\,du\right],$$

where

$$(24)\qquad k^2=\frac{\beta^4}{\alpha^4}\,,\qquad \varphi=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{(y^2-\alpha^2)\,\alpha^2}{\alpha^2\,y^2-\beta^4}}\,.$$

The two elliptic integrals in (23) are evaluated by employing **321.02** and **320.02**. We then have

$$(25)\qquad \begin{cases} J=\alpha\,[u_1-E(u_1)+\mathrm{dn}\,u_1\,\mathrm{tn}\,u_1-u_1+E(u_1)-k^2\,\mathrm{sn}\,u_1\,\mathrm{cd}\,u_1] \\ \quad=\alpha\,k'^2\,\mathrm{tn}\,u_1\,\mathrm{nd}\,u_1=\dfrac{\alpha\,k'^2\tan\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,, \end{cases}$$

or, applying (24),

$$(26)\qquad J=\sqrt{\frac{(y^2-\alpha^2)\,(\alpha^2\,y^2-\beta^4)}{\alpha^2\,y^2}}\,.$$

The final evaluation thus involves only elementary functions.

Integral (21) is an example of an elementary integral which has all the appearance of being elliptic. That this would lead to elementary functions, however, could have been shown in another way. For, if we substitute $t^2/\alpha^2=t_1$, it may be written

$$(27)\qquad J=\int_1^{y^2/\alpha^2}\frac{R(t_1)\,dt_1}{\sqrt{t_1\,(1-t_1)\,(1-k_1^2\,t_1)}}\,,$$

where

$$k_1^2=\alpha^4/\beta^4\quad\text{and}\quad R(t_1)=\frac{\alpha^4\,t_1^2-\beta^4}{2\alpha\,\beta^2\,t_1}\,.$$

Equation (27) now has the form of **1060.05** and satisfies the relation

$$R(t_1)+R(1/k_1^2\,t_1)=0\,;$$

hence it is *pseudo-elliptic*.

Example V.

In considering the motion of a simple pendulum whose angular displacement from the position of stable equilibrum is φ, one is led to an integral of the type

$$t = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{a + b\cos\vartheta}}, \tag{28}$$

where a and b are certain constants which depend on the acceleration of gravity, on the value of $d\varphi/dt$ at $\varphi = 0$, and on the length of the pendulum. There are three different types of motion according as $a > b$, $a < b$, or $a = b$:

If $a > b > 0$, one employs **289.00** and obtains immediately

$$t = \frac{2}{\sqrt{a+b}} F\left(\frac{\varphi}{2}, k\right) \tag{29}$$

where $k^2 = 2b/(a+b)$.

When $b > a > 0$, we have from **290.00**

$$t = \sqrt{\frac{2}{b}}\, F(\varepsilon, k) \tag{30}$$

where $k^2 = (a+b)/2b$, and $\varepsilon = \sin^{-1}\sqrt{\dfrac{b(1-\cos\varphi)}{a+b}}$.

Finally, if $a = b$, there results from either **289.00** or **290.00**

$$t = \frac{1}{\sqrt{2a}} F\left(\frac{\varphi}{2}, 1\right), \tag{31}$$

which, upon using **111.04**, reduces to

$$t = \frac{1}{\sqrt{2a}} \ln\left(\tan\frac{\varphi}{2} + \sec\frac{\varphi}{2}\right).$$

Definitions and Fundamental Relations.

Elliptic Integrals.

Definitions.

The integral[1]

110.01 $$I = \int R\left[t, \sqrt{a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4}\right] dt$$

is called an *elliptic integral* if the equation

$$a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4 = 0, \qquad (a_0 \text{ and } a_1 \text{ not both zero}),$$

has no multiple roots and if R is a rational function of t and of the square root $\sqrt{a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4}$.

The Three Canonical Forms of the Elliptic Integrals.

It is always possible to express **110.01** linearly in terms of elementary functions[2] and of the following three fundamental integrals:

The normal elliptic integral of the first kind:

110.02 $$\begin{cases} \displaystyle\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \\ \displaystyle\qquad = \int_0^{u_1} du = u_1 \equiv \operatorname{sn}^{-1}(y,k) \equiv F(\varphi,k) \\ \qquad\qquad [y = \sin\varphi;\ \varphi = \operatorname{am} u_1]. \end{cases}$$

The normal elliptic integral of the second kind:

110.03 $$\begin{cases} \displaystyle\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^\varphi \sqrt{1-k^2\sin^2\vartheta}\, d\vartheta = \int_0^{u_1} \operatorname{dn}^2 u\, du \\ \qquad = E(u_1) \equiv E(\operatorname{am} u_1, k) \equiv E(\varphi, k). \end{cases}$$

[1] There are special cases of integrals in this form that can be expressed in terms of a finite number of elementary functions. Such integrals are said to be pseudo-elliptic (see **1060**).

[2] Elementary functions are algebraic, trigonometric, inverse trigonometric, logarithmic and exponential functions.

The normal elliptic integral of the third kind[1]:

110.04
$$\left\{\begin{aligned}&\int_0^y \frac{dt}{(1-\alpha^2 t^2)\sqrt{(1-t^2)(1-k^2 t^2)}} = \int_0^\varphi \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}}\\ &= \int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \Pi(u_1,\alpha^2) \equiv \Pi(\operatorname{am} u_1,\alpha^2,k) \equiv \Pi(\varphi,\alpha^2,k)\\ &\qquad [-\infty<\alpha^2<\infty].\end{aligned}\right.$$

The second form of each of the above integrals is written in *Legendre's notation,* while the first and third forms are in *Jacobi's notation.*

In their algebraic forms, these three standard integrals possess the following important properties: The first is finite for all real or complex values of y, including infinity; the second has a simple pole of order 1 for $y=\infty$; and the third is logarithmically infinite for $y^2 = 1/\alpha^2$.

The Modulus.

The number k is called the *modulus.* This number may take any real or imaginary value in theoretical investigations. In applications to engineering and physics, however, transformations are generally employed to make $0<k<1$ (cf. **160** and **162**).

The Complementary Modulus.

The number k' is referred to as the *complementary modulus* and is related to k by

110.05
$$k' = \sqrt{1-k^2}\,.$$

The Argument.

The variable limit y or φ in **110.02—110.04** is the *argument* of the normal elliptic integrals. The argument may of course be either real or complex, but it is usually understood that $0<y\leqq 1$ or $0<\varphi\leqq\pi/2$.

Complete Elliptic Integrals.

When $y=1$, the integrals **110.02–110.04** are said to be *complete.* In that case, one writes:

110.06
$$\int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \int_0^K du = F(\pi/2, k) \equiv K(k) \equiv K,$$

[1] In the definition of $\Pi(\varphi,\alpha^2,k)$ some authors write $+n$ in the integrand, while we find it convenient to write $-(\alpha^2)$ with $-\infty<\alpha^2<+\infty$ (except in the special cases when $\alpha^2=k^2$, $\alpha^2=1$. or when the parameter α^2 is complex).

110.07 $$\int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}\,d\vartheta = \int_0^K \mathrm{dn}^2 u\,du = E(\pi/2,k) \equiv E(k) \equiv E,$$

110.08 $$\left\{\begin{aligned} &\int_0^{\pi/2} \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}} = \int_0^K \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ &\qquad = \Pi(\pi/2,\alpha^2,k) \equiv \Pi(\alpha^2,k), \qquad [\alpha^2 \neq 1]. \end{aligned}\right.$$

Associated Complete Elliptic Integrals.

110.09 $$\left\{\begin{aligned} K'(k) &\equiv K(k') \equiv K', \\ E'(k) &\equiv E(k') \equiv E'. \end{aligned}\right.$$

Legendre's Relation.

110.10 $$E K' + E' K - K K' = \pi/2, \qquad \text{[cf. 531.12]}.$$

Special Values.

111.00 $$\left\{\begin{aligned} E(0,k) &= 0, \\ F(0,k) &= 0, \\ \Pi(0,\alpha^2,k) &= 0. \end{aligned}\right.$$

111.01 $$\left\{\begin{aligned} &E(\varphi,0) = \varphi, \\ &F(\varphi,0) = \varphi, \\ &\Pi(\varphi,\alpha^2,0) = \varphi, \quad \text{if} \quad \alpha^2 = 0; \\ &\qquad = \frac{\tan^{-1}(\sqrt{1-\alpha^2}\tan\varphi)}{\sqrt{1-\alpha^2}}, \quad \text{if} \quad \alpha^2 < 1 \\ &\qquad = \frac{\tanh^{-1}(\sqrt{\alpha^2-1}\tan\varphi)}{\sqrt{\alpha^2-1}}, \quad \text{if} \quad \alpha^2 > 1. \end{aligned}\right.$$

111.02 $$\left\{\begin{aligned} K(0) &= K'(1) = \pi/2, \\ E(0) &= E'(1) = \pi/2. \end{aligned}\right.$$

111.03 $$\left\{\begin{aligned} &F\left(\sin^{-1}\frac{1}{\sqrt{1+k'}}, k\right) = \frac{K}{2}, \\ &E\left(\sin^{-1}\frac{1}{\sqrt{1+k'}}, k\right) = \frac{1}{2}\,[E + (1-k')]. \end{aligned}\right.$$

111.04 $$\left\{\begin{aligned} &E(\varphi,1) = \sin\varphi, \\ &F(\varphi,1) = \ln(\tan\varphi + \sec\varphi), \\ &\Pi(\varphi,\alpha^2,1) = \frac{1}{1-\alpha^2}\left[\ln(\tan\varphi+\sec\varphi) - \alpha\ln\sqrt{\frac{1+\alpha\sin\varphi}{1-\alpha\sin\varphi}}\right], \\ &\qquad\qquad [\alpha^2 \neq 1]. \end{aligned}\right.$$

111.05
$$\begin{cases} E(\pi/2, 1) = E(1) = E'(0) = 1, \\ F(\pi/2, 1) = K(1) = K'(0) = \infty, \\ \Pi(\pi/2, \alpha^2, 1) = \Pi(\alpha^2, 1) = \infty. \end{cases}$$

111.06
$$\begin{cases} \Pi(\varphi, 0, k) = F(\varphi, k), \\ \Pi(\varphi, 1, k) = \left[k'^2 F(\varphi, k) - E(\varphi, k) + \tan\varphi \sqrt{1 - k^2 \sin^2\varphi}\right] \big/ k'^2, \\ \Pi(\varphi, k^2, k) = \left[E(\varphi, k) - k^2 \sin\varphi \cos\varphi \big/ \sqrt{1 - k^2 \sin^2\varphi}\right] \big/ k'^2. \end{cases}$$

111.07
$$\begin{cases} F(i\varphi, 0) = iF(\vartheta, 1) = i \ln(\tan\vartheta + \sec\vartheta) = i\varphi, \quad [\sinh\varphi = \tan\vartheta] \\ E(i\varphi, 0) = i \ln(\tan\vartheta + \sec\vartheta) = i\varphi, \\ \Pi(i\varphi, \alpha^2, 0) = i\left[\ln(\tan\vartheta + \sec\vartheta) - \alpha^2 \Pi(\vartheta, 1 - \alpha^2, 1)\right] \big/ (1 - \alpha^2). \end{cases}$$

111.08
$$\begin{cases} F(i\varphi, 1) = i\vartheta = 2\tan^{-1} e^{\varphi} - \pi/2, \qquad [\sinh\varphi = \tan\vartheta] \\ E(i\varphi, 1) = i \tan\vartheta = i \sinh\varphi = \sin i\varphi, \\ \Pi(i\varphi, \alpha^2, 1) = i\left[\vartheta - \alpha^2 \Pi(\vartheta, 1 - \alpha^2, 0)\right] \big/ (1 - \alpha^2). \end{cases}$$

111.09
$$\begin{cases} F\left[\sin^{-1}(1/k), k\right] = F\left[\pi/2 + i\cosh^{-1}(1/k), k\right] = K + iK', \\ E\left[\sin^{-1}(1/k), k\right] = E + i(K' - E'). \end{cases}$$

111.10
$$\begin{cases} K' = K, & \text{when } k = \sqrt{2}/2, \\ K' = \sqrt{2}K & \text{,,} \quad k = \sqrt{2} - 1, \\ K' = 2K & \text{,,} \quad k = 3 - 2\sqrt{2}, \end{cases}$$

111.11
$$\left.\begin{cases} K' = \sqrt{3}K, \\ E = \dfrac{\pi\sqrt{3}}{12K} + \sqrt{\dfrac{2}{3}}\, k'K, \\ E' = \dfrac{\pi\sqrt{3}}{4K'} + \sqrt{\dfrac{2}{3}}\, kK'. \end{cases}\right\} \quad \text{when } k = (\sqrt{3} - 1)/2\sqrt{2}.$$

Limiting Values.

112.01
$$\lim_{k \to 1}\left(K - \ln\frac{4}{k'}\right) = 0.$$

112.02
$$\lim_{k \to 0}\frac{K - E}{k^2} = \lim_{k \to 0}\frac{E - k'^2 K}{k^2} = \pi/4.$$

112.03
$$\lim_{k \to 0}(E - K)K' = 0.$$

112.04 *
$$\lim_{k \to 0}\frac{e^{-(\pi K'/K)}}{k^2} = \lim_{k \to 1}\frac{e^{-(\pi K/K')}}{k'^2} = 1/16.$$

112.05
$$\lim_{\varphi \to 0}\frac{E(\varphi, k)}{\sin\varphi} = \lim_{\varphi \to 0}\frac{F(\varphi, k)}{\sin\varphi} = \lim_{\varphi \to 0}\frac{\Pi(\varphi, \alpha^2, k)}{\sin\varphi} = 1.$$

* The term $e^{-(\pi K'/K)}$, referred to as the *nome q*, appears in the evaluation of Theta functions (see Appendix). This function is tabulated in *Smithsonian Elliptic Functions Tables* (Smithson. Misc. Coll., v. 109), Washington 1947, by G. W. Spenceley and R. M. Spenceley. A shorter tabulation appears here in the Appendix.

Extension of the Range of φ and k.

The functions $E(\varphi, k)$ and $F(\varphi, k)$ are tabulated[1] for the range $0 \leqq \varphi \leqq \pi/2$ and $0 \leqq k \leqq 1$. In computing for other values of φ, use can be made of the following relationships:

113.01 $$\begin{cases} E(-\varphi, k) = -E(\varphi, k), \\ F(-\varphi, k) = -F(\varphi, k). \end{cases}$$

113.02 $$\begin{cases} E(m\pi \pm \varphi, k) = 2mE \pm E(\varphi, k), \\ F(m\pi \pm \varphi, k) = 2mK \pm F(\varphi, k). \end{cases}$$

Values of $E(\varphi, k)$ and $F(\varphi, k)$ may be obtained for $k > 1$ by employing formulas (see **162.01**) for the reciprocal modulus transformation:

114.01 $$\begin{cases} E(\varphi, k) = k_1 [k^2 E(\varphi_1, k_1) + k'^2 F(\varphi_1, k_1)], \\ F(\varphi, k) = k_1 F(\varphi_1, k_1), \end{cases}$$

where
$$k_1 = 1/k, \qquad \sin\varphi_1 = k \sin\varphi.$$

Imaginary Modulus.

See **160.02**.

Complex Argument.

115.01 $$\begin{cases} F(\vartheta \pm i\varphi, k) = F(\beta, k) \pm i F(A, k'), \\ E(\vartheta \pm i\varphi, k) = \left[E(\beta, k) + \dfrac{k^2 \sin\beta\cos\beta\sin^2 A\sqrt{1 - k^2\sin^2\beta}}{\cos^2 A + k^2\sin^2\beta\sin^2 A}\right] \pm \\ \qquad \pm i\left[F(A, k') - E(A, k') + \right. \\ \qquad \left. + \dfrac{(1 - k^2\sin^2\beta)\sin A\cos A\sqrt{1 - k'^2\sin^2 A}}{\cos^2 A + k^2\sin^2\beta\sin^2 A}\right], \end{cases}$$

where
$$\cosh\varphi\sin\vartheta = \frac{\sin\beta\sqrt{1 - k'^2\sin^2 A}}{\cos^2 A + k^2\sin^2\beta\sin^2 A},$$
$$\cos\vartheta\sinh\varphi = \frac{\cos\beta\cos A\sin A\sqrt{1 - k^2\sin^2\beta}}{\cos^2 A + k^2\sin^2\beta\sin^2 A}.$$

In the special case when $\sin\psi \equiv \sin(\vartheta + i\varphi) > 1$:

115.02 $$\begin{cases} F(\psi, k) = K + iF(A, k'), \\ E(\psi, k) = E + i\left[F(A, k') - E(A, k') + \dfrac{k'^2\sin A\cos A}{\sqrt{1 - k'^2\sin^2 A}}\right], \end{cases}$$

where
$$1 < \sin\psi \leqq 1/k, \quad A = \sin^{-1}\frac{\sqrt{\sin^2\psi - 1}}{k'\sin\psi}.$$

[1] For a 10-place tabulation, see K. Pearson's *Tables of the Complete and Incomplete Elliptic Integrals* (reissued from Tome II of Legendre's "Traité des Fonctions Elliptiques") London 1934; and for a short 6-place table, see Appendix here.

115.03
$$\begin{cases} F(\psi, k) = F(\beta, k) + i\,K', \\ E(\psi, k) = E(\beta, k) + \cot\beta\,\sqrt{1-k^2\sin^2\beta} + i\,(K'-E'), \end{cases}$$

where
$$1/k \leqq \sin\psi < \infty, \quad \beta = \sin^{-1}(1/k).$$

Addition Formulas.

116.01
$$\begin{cases} E(\vartheta, k) \pm E(\beta, k) = E(\varphi, k) \pm k^2 \sin\vartheta \sin\beta \sin\varphi, \\ F(\vartheta, k) \pm F(\beta, k) = F(\varphi, k), \end{cases}$$

where
$$\varphi = 2\tan^{-1}\left[\frac{\sin\vartheta\sqrt{1-k^2\sin^2\beta} \pm \sin\beta\sqrt{1-k^2\sin^2\vartheta}}{\cos\vartheta + \cos\beta}\right],$$

or
$$\varphi = \cos^{-1}\left[\frac{\cos\vartheta\cos\beta \mp \sin\vartheta\sin\beta\sqrt{(1-k^2\sin^2\vartheta)(1-k^2\sin^2\beta)}}{1-k^2\sin^2\vartheta\sin^2\beta}\right].$$

116.02
$$\begin{cases} \Pi(\vartheta, \alpha^2, k) \pm \Pi(\beta, \alpha^2, k) = \Pi(\varphi, \alpha^2, k) \pm \sqrt{\dfrac{\alpha^2}{(1-\alpha^2)(\alpha^2-k^2)}} \times \\ \qquad \times \tan^{-1}\left[\dfrac{\sin\vartheta\sin\beta\sin\varphi\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}{1-\alpha^2\sin^2\varphi + \alpha^2\sin\vartheta\sin\beta\cos\varphi\sqrt{1-k^2\sin^2\varphi}}\right], \end{cases}$$

if $0 < k^2 < \alpha^2 < 1$ or $0 < -\alpha^2 < \infty$, and φ is as given in **116.01**.

116.03
$$\begin{cases} \Pi(\vartheta, \alpha^2, k) \pm \Pi(\beta, \alpha^2, k) = \Pi(\varphi, \alpha^2, k) \pm \sqrt{\dfrac{\alpha^2}{(\alpha^2-1)(\alpha^2-k^2)}} \times \\ \qquad \times \tanh^{-1}\left[\dfrac{\sin\vartheta\sin\beta\sin\varphi\sqrt{\alpha^2(\alpha^2-1)(\alpha^2-k^2)}}{1-\alpha^2\sin^2\varphi + \alpha^2\sin\vartheta\sin\beta\cos\varphi\sqrt{1-k^2\sin^2\varphi}}\right], \end{cases}$$

if $0 < \alpha^2 < k^2 < 1$ or $\alpha^2 > 1$, and φ is as given in **116.01**.

Special Addition Formulas.

117.01
$$\begin{cases} F(\vartheta, k) + F(\beta, k) = K, \\ E(\vartheta, k) + E(\beta, k) = E + k^2\sin\vartheta\sin\beta. \end{cases} \quad \text{if } \cot\beta = k'\tan\vartheta$$

117.02
$$\begin{cases} \Pi(\alpha^2, k) + \Pi(k^2/\alpha^2, k) = K + \dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2}{(1-\alpha^2)(\alpha^2-k^2)}}, \\ \qquad\qquad \text{if } 0 < k^2 < \alpha^2 < 1 \text{ or } 0 < -\alpha^2 < \infty, \\ \Pi(\varphi, \alpha^2, k) + \Pi(\varphi, k^2/\alpha^2, k) \\ \quad = F(\varphi, k) + \sqrt{\dfrac{\alpha^2}{(1-\alpha^2)(\alpha^2-k^2)}}\tan^{-1}\sqrt{\dfrac{(1-\alpha^2)(\alpha^2-k^2)}{\alpha^2(1-k^2\sin^2\varphi)}}\tan\varphi. \end{cases}$$

117.03
$$\begin{cases} (1-\alpha^2)(k^2-\alpha^2)\Pi(\alpha^2,k)+\alpha^2 k'^2 \Pi[(\alpha^2-k^2)/(1-\alpha^2),k] \\ \qquad = k^2(1-\alpha^2)K, \\ (1-\alpha^2)(k^2-\alpha^2)\Pi(\varphi,\alpha^2,k)+\alpha^2 k'^2 \Pi[(\alpha^2-k^2)/(1-\alpha^2),k] \\ \qquad = k^2(1-\alpha^2)F(\varphi,k)+\sqrt{(1-\alpha^2)(\alpha^2-k^2)\alpha^2}\times \\ \qquad \times \tan^{-1}\left[\sqrt{\dfrac{\alpha^2-k^2}{(1-k^2\sin^2\varphi)(1-\alpha^2)\alpha^2}}\sin\varphi\cos\varphi\right], \\ \qquad [0<k^2<\alpha^2<1 \text{ or } 0<-\alpha^2<\infty]. \end{cases}$$

117.04
$$\begin{cases} (\alpha^2-1)K'\Pi(\alpha^2,k)+\alpha^2 K\Pi[k'^2/(\alpha^2-1),k']+ \\ \qquad +(1-\alpha^2)KK' = \dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2(\alpha^2-1)}{k^2-\alpha^2}}F(\beta,k'), \\ (1-\alpha^2)(k^2-\alpha^2)(K'-E')\Pi(\alpha^2,k)+ \\ \qquad +\alpha^2(\alpha^2-k^2)E\Pi[k'^2/(\alpha^2-1),k']+ \\ \qquad +(k^2-\alpha^2)(\alpha^2-1)EK' = \dfrac{\pi}{2}[F(\beta,k')- \\ \qquad -E(\beta,k')]\sqrt{\alpha^2(\alpha^2-1)(k^2-\alpha^2)}+\pi k^2(\alpha^2-1)/2, \\ \qquad \left[0<-\alpha^2<\infty,\ \beta=\sin^{-1}\sqrt{\alpha^2/(\alpha^2-k^2)}\right]. \end{cases}$$

117.05
$$\begin{cases} (\alpha^2-1)K'\Pi(\alpha^2,k)+\alpha^2 K\Pi[k'^2/(\alpha^2-1),k']+ \\ \qquad +(1-\alpha^2)KK' = \dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2(1-\alpha^2)}{k^2-\alpha^2}}F(\beta,k), \\ (\alpha^2-1)(K'-E')\Pi(\alpha^2,k)+\alpha^2 E\Pi[k'^2/(\alpha^2-1),k']+ \\ \qquad +(1-\alpha^2)EK' = \dfrac{\pi}{2}\left[1-\alpha^2+\sqrt{\alpha^2(1-\alpha^2)/(k^2-\alpha^2)}\,E(\beta,k)\right], \\ \qquad [0<\alpha^2<k^2<1,\ \beta=\sin^{-1}(\alpha/k)]. \end{cases}$$

117.06
$$\begin{cases} (k^2-\alpha^2)K'\Pi(\alpha^2,k)+k^2 K\Pi[(k^2-\alpha^2)/\alpha^2,k']-k^2KK' \\ \qquad = \dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2(k^2-\alpha^2)}{\alpha^2-1}}F(\beta,k'), \\ (k^2-\alpha^2)E'\Pi(\alpha^2,k)+k^2(K-E)\{\Pi[(k^2-\alpha^2)/\alpha^2,k']-K'\} \\ \qquad = \dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2(k^2-\alpha^2)}{\alpha^2-1}}E(\beta,k'), \\ \qquad 0<k^2<\alpha^2<1,\ \beta=\sin^{-1}\sqrt{\dfrac{\alpha^2-k^2}{\alpha^2 k'^2}}\Bigg]. \end{cases}$$

Differential Equations.

Elliptic integrals of the first kind and second kind satisfy the linear differential equations

118.01
$$\begin{cases} k\,k'^2 \dfrac{d^2}{dk^2} F(\varphi, k) + (1 - 3k^2) \dfrac{d}{dk} F(\varphi, k) - k F(\varphi, k) \\ \qquad = \dfrac{k \sin\varphi \cos\varphi}{(1 - k^2 \sin^2\varphi) \sqrt{1 - k^2 \sin^2\varphi}}, \\ k\,k'^2 \dfrac{d^2}{dk^2} E(\varphi, k) + k'^2 \dfrac{d}{dk} E(\varphi, k) + k\,E(\varphi, k) = \dfrac{k \sin\varphi \cos\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}. \end{cases}$$

If $\varphi = \pi/2$, the above equations reduce to

118.02
$$\begin{cases} k\,k'^2 \dfrac{d^2 K}{dk^2} + (1 - 3k^2) \dfrac{dK}{dk} - k\,K = 0, \\ k\,k'^2 \dfrac{d^2 E}{dk^2} + k'^2 \dfrac{dE}{dk} + k\,E = 0, \end{cases}$$

which are special cases of the *hypergeometric differential equation.* The complete elliptic integrals E, K, E', and K' can thus be expressed as *hypergeometric functions*:

$$K = \frac{\pi}{2} F(1/2, 1/2; 1; k^2), \qquad K' = \frac{\pi}{2} F(1/2, 1/2; 1; k'^2),$$

$$E = \frac{\pi}{2} F(-1/2, 1/2; 1; k^2), \qquad E' = \frac{\pi}{2} F(-1/2, 1/2; 1; k'^2).$$

Legendre's normal elliptic integral of the third kind $\Pi(\varphi, \alpha^2, k)$ satisfies the differential equation

118.03
$$\begin{cases} k^2 k'^2 (k^2 - \alpha^2) \dfrac{\partial^3 \Pi}{\partial k^3} + k(\alpha^2 + 4k^2 + 3\alpha^2 k^2 - 8k^4) \dfrac{\partial^2 \Pi}{\partial k^2} + \\ \qquad + (2k^2 - \alpha^2 - 13k^4) \dfrac{\partial \Pi}{\partial k} - 3k^3 \Pi = \dfrac{-3k^3 \sin\varphi \cos\varphi}{\sqrt{(1 - k^2 \sin^2\varphi)^5}}. \end{cases}$$

The right side of this equation vanishes when $\varphi = \pi/2$, thereby yielding a differential equation satisfied by the complete elliptic integral of the third kind $\Pi(\pi/2, \alpha^2, k)$.

Other Auxiliary Formulas.

Derivatives of Elliptic Integrals.

See **710**, **730**, and **733**, **734**.

Expansions in Series.

See **900** to **906**.

Transformation Formulas.

See **160** to **165**.

Integrals.

See **610** and **630**.

Sketches of $E(\varphi, k), F(\varphi, k), E(k)$ and $K(k)$.

The following graphs show the incomplete elliptic integrals of the first and second kind plotted as functions of the amplitude φ and the modular angle $\sin^{-1} k$, and the complete integrals $E(k)$ and $K(k)$ plotted against k^2. (These sketches should not be used to obtain values of the functions.)

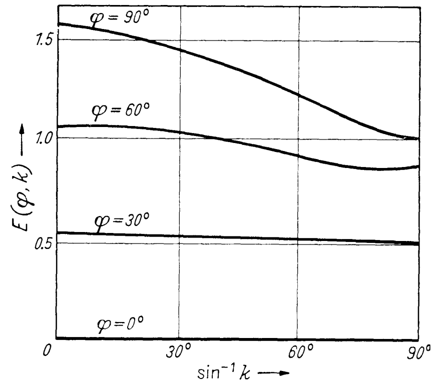

Fig. 1.

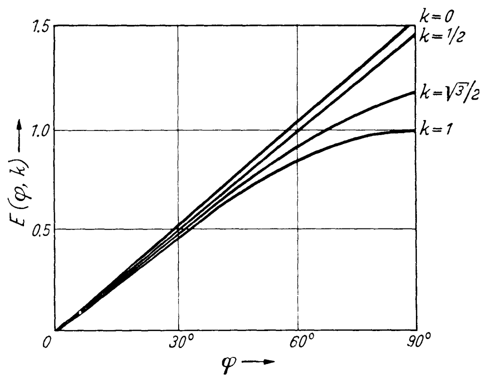

Fig. 2.

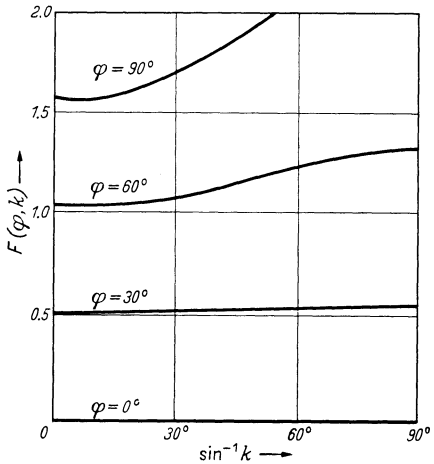

Fig. 3.

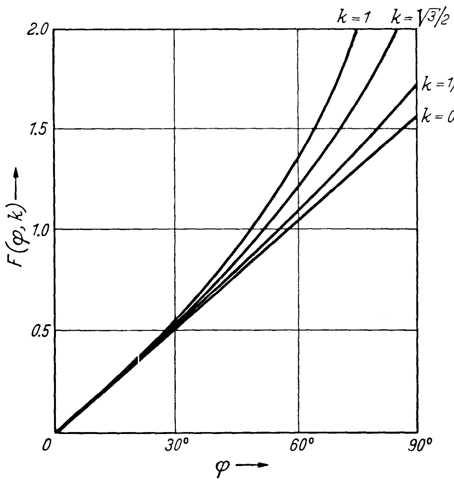

Fig. 4.

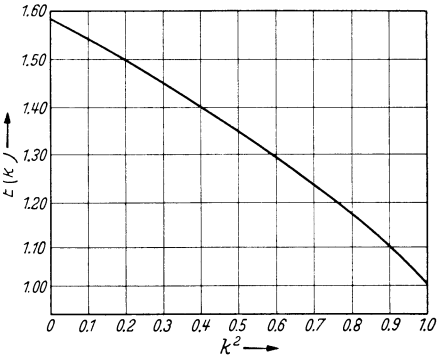

Fig. 5.

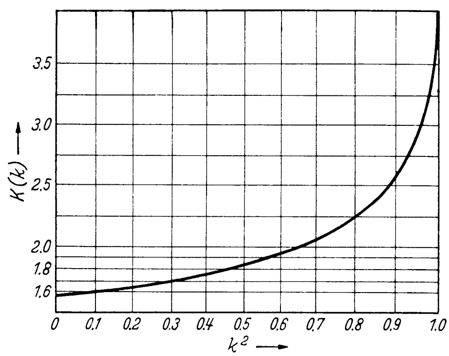

Fig. 6.

Conformal Mappings.

Some conformal mapping properties of the elliptic integrals are illustrated in the following sketches.

Mapping of the half-plane $Im(z) > 0$ onto a rectangle in the ζ-plane:

119.01
$$\zeta = \int_0^z \frac{dz_1}{\sqrt{(1-z_1^2)(1-k^2 z_1^2)}} = F(z, k); \qquad z = \operatorname{sn}\zeta.$$

ζ	$-K + iK'$	$-K$	0	K	$K + iK'$	iK'
z	$-1/k$	-1	0	1	$1/k$	∞

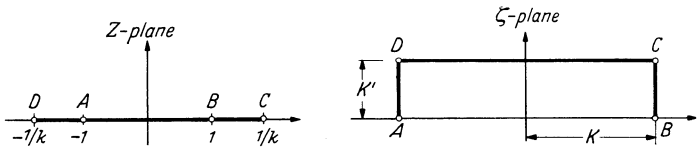

Fig. 7.

Mapping of the function

119.02
$$\zeta = \int_0^z \sqrt{\frac{1-k^2 z_1^2}{1-z_1^2}}\, dz_1 = E(z, k):$$

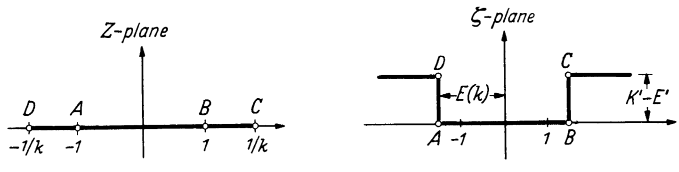

Fig. 8.

Mapping of two horizontal lines onto two vertical lines:

119.03 $$\zeta = \int_0^z \frac{(a^2 - z_1^2)\, dz_1}{\sqrt{(1 - z_1^2)(1 - k^2 z_1^2)}}, \quad 1 < a < 1/k;$$

$$t = \frac{2}{k}\left[E(k) - (1 - k^2 a^2) K(k)\right], \qquad b = \frac{2}{k}\left[E(k_1) - k^2 a^2 K(k_1)\right],$$

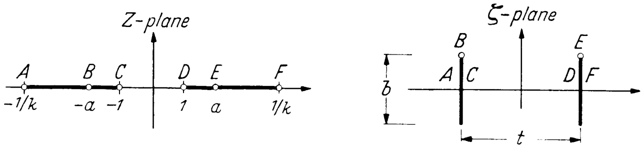

Fig. 9.

where

$$k_1 = \frac{k}{k'}\sqrt{a^2 - 1}$$

and a is determined for given k by the relation

$$a^2 = \frac{E(k')}{k^2 K(k')}, \quad k' = \sqrt{1 - k^2}.$$

Jacobian Elliptic Functions.

Definitions.

Instead of considering the elliptic integral

120.01 $$u(y_1, k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} = F(\varphi, k)$$

as the primary object of study, Abel and Jacobi reversed the problem and investigated the *inversion* of this integral. Inverse functions[1] may thus be defined by $y_1 = \sin\varphi = \operatorname{sn}(u, k)$ and $\varphi = \operatorname{am}(u, k)$, or briefly $y_1 = \operatorname{sn} u$, $\varphi = \operatorname{am} u$ when it is not necessary to emphasize the modulus; these may be read *sine amplitude u* and *amplitude u*.

The function $\operatorname{sn} u$ is an odd elliptic function of order two. It possesses a simple pole of residue $1/k$ at every point congruent to iK' (mod $4K$,

[1] This has an analogy with trigonometric functions. The singly-periodic function $y = \sin u$, for example, defines the inversion of the integral

$$u = \int_0^y \frac{dt}{\sqrt{1 - t^2}}, \quad [y \leq 1].$$

$2iK'$) and a simple pole of residue $-1/k$ at points congruent to $2K+iK'$ (mod $4K$, $2iK'$).

Two other functions can then be defined by $\operatorname{cn}(u,k)=\sqrt{1-y_1^2}=\cos\varphi$, $\operatorname{dn}(u,k)=\sqrt{1-k^2y_1^2}=\Delta\varphi=\sqrt{1-k^2\sin^2\varphi}$, requiring that $\operatorname{sn}(0,k)=0$, $\operatorname{cn}(0,k)=1$ and $\operatorname{dn}(0,k)=1$. The function $\operatorname{cn} u$ is an even function of order two. It has simple poles at every point congruent to iK' and $2K+iK'$ (mod $4K$, $2K+2iK'$) with residues $-i/k$, i/k respectively. We note also that $\operatorname{dn} u$ is an even elliptic function of order two, having singular points congruent to iK' or $3iK'$ (mod $2K$, $4iK'$). These points are simple poles of residue $-i$ and i respectively.

The functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ are called *Jacobian elliptic functions*[1] and are one-valued functions of the argument u. These functions, like the trigonometric functions, have a real period, and, like the hyperbolic functions, have an imaginary period. They are thus *doubly periodic*[2], having the periods $(4K, 2iK')$, $(4K, 2K+2iK')$ and $(2K, 4iK')$ respectively. The *modulus* is denoted by the number k and the *complementary modulus* by $k'=\sqrt{1-k^2}$.

The *quotients* and *reciprocals* of $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ are designated in *Glaisher's notation* by

120.02
$$\begin{cases} \operatorname{ns} u \equiv \dfrac{1}{\operatorname{sn} u}; & \operatorname{tn} u \equiv \operatorname{sc} u \equiv \dfrac{\operatorname{sn} u}{\operatorname{cn} u}; & \operatorname{sd} u \equiv \dfrac{\operatorname{sn} u}{\operatorname{dn} u}; \\ \operatorname{nc} u \equiv \dfrac{1}{\operatorname{cn} u}; & \dfrac{1}{\operatorname{tn} u} \equiv \operatorname{cs} u \equiv \dfrac{\operatorname{cn} u}{\operatorname{sn} u}; & \operatorname{cd} u \equiv \dfrac{\operatorname{cn} u}{\operatorname{dn} u}; \\ \operatorname{nd} u \equiv \dfrac{1}{\operatorname{dn} u}; & \operatorname{ds} u \equiv \dfrac{\operatorname{dn} u}{\operatorname{sn} u}; & \operatorname{dc} u \equiv \dfrac{\operatorname{dn} u}{\operatorname{cn} u}; \end{cases}$$

We thus have, in all, twelve Jacobian elliptic functions[3]. These functions have many direct applications in physical problems but are of interest to us principally for the purpose of evaluating elliptic integrals. The most important properties of the functions are summarized by the formulas given in the remainder of this section.

[1] The best tabulation of $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ appears in G. W. Spenceley and R. M. Spenceley's *Smithsonian Elliptic Functions Tables* (Smithson. Misc. Coll., v. 109) Washington, 1947.

[2] The Jacobian elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are of course only special cases of a more general class of functions which possess similar properties. If 2ω and $2\omega'$ are any two numbers whose ratio ω'/ω is not purely real, then any function $f(u)$, which is analytic except at poles and which has no singularities other than poles for any finite value of u, is called an *elliptic function* if the doubly-periodic relation

$$f(u+2\omega)=f(u+2\omega')=f(u)$$

is satisfied for all values of u for which $f(u)$ is defined.

[3] For an attractive and unique treatment of these functions, exhibiting them as functions constructed on a canonical lattice, see *Jacobian Elliptic Functions* by E. H. Neville, Clarendon Press, Oxford 1951.

Fundamental Relations.

121.00 $$\begin{cases} \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \\ k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \\ \operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u = k'^2, \\ k'^2 \operatorname{sn}^2 u + \operatorname{cn}^2 u = \operatorname{dn}^2 u. \end{cases}$$

121.01 $$\begin{cases} \operatorname{dn} u = \Delta\varphi = \sqrt{1 - k^2 \sin^2\varphi}, \\ u = \operatorname{am}^{-1}(\varphi, k) \\ \quad = \int\limits_0^{\varphi} \frac{d\vartheta}{\Delta\vartheta} = F(\varphi, k), \\ \operatorname{am} u \equiv \operatorname{am}(u, k) = \varphi. \end{cases}$$

121.02 $$\begin{cases} -1 \leqq \operatorname{sn} u \leqq 1, \\ -1 \leqq \operatorname{cn} u \leqq 1, \\ k' \leqq \operatorname{dn} u \leqq 1, \\ -\infty < \operatorname{tn} u < \infty. \end{cases}$$ [u real]

Special Values.

122.00 $$\begin{cases} \operatorname{am}(-u) = -\operatorname{am} u, \\ \operatorname{sn}(-u) = -\operatorname{sn} u, \\ \operatorname{cn}(-u) = \operatorname{cn} u, \\ \operatorname{dn}(-u) = \operatorname{dn} u, \\ \operatorname{tn}(-u) = -\operatorname{tn} u. \end{cases}$$

122.01 $$\begin{cases} \operatorname{am} 0 = 0, \\ \operatorname{sn} 0 = 0, \\ \operatorname{cn} 0 = 1, \\ \operatorname{dn} 0 = 1, \\ \operatorname{tn} 0 = 0. \end{cases}$$

122.02 $$\begin{cases} \operatorname{am} K = \pi/2, \\ \operatorname{sn} K = 1, \\ \operatorname{cn} K = 0, \\ \operatorname{dn} K = k', \\ \operatorname{tn} K = \infty. \end{cases}$$

122.03 $$\begin{cases} \operatorname{am}(u + K) \\ \quad = \tan^{-1}(k' \operatorname{tn} u) + \pi/2, \\ \operatorname{sn}(u + K) \\ \quad = \operatorname{cd} u = \operatorname{sn}(K - u), \\ \operatorname{cn}(u + K) = -k' \operatorname{sd} u, \\ \operatorname{dn}(u + K) = k' \operatorname{nd} u, \\ \operatorname{tn}(u + K) = -(\operatorname{cs} u)/k'. \end{cases}$$

122.04 $$\begin{cases} \operatorname{am}(u + 2K) = \pi + \operatorname{am} u, \\ \operatorname{sn}(u + 2K) = -\operatorname{sn} u, \\ \operatorname{cn}(u + 2K) = -\operatorname{cn} u, \\ \operatorname{dn}(u + 2K) = \operatorname{dn} u, \\ \operatorname{tn}(u + 2K) = \operatorname{tn} u. \end{cases}$$

122.05 $$\begin{cases} \operatorname{am}(u + 3K) \\ \quad = 3\pi/2 + \tan^{-1}(k' \operatorname{tn} u), \\ \operatorname{sn}(u + 3K) = -\operatorname{cd} u, \\ \operatorname{cn}(u + 3K) = k' \operatorname{sd} u, \\ \operatorname{dn}(u + 3K) = k' \operatorname{nd} u, \\ \operatorname{tn}(u + 3K) = -(\operatorname{cs} u)/k'. \end{cases}$$

122.06 $$\begin{cases} \operatorname{am}(u + 4K) = 2\pi + \operatorname{am} u, \\ \operatorname{sn}(u + 4K) = \operatorname{sn} u, \\ \operatorname{cn}(u + 4K) = \operatorname{cn} u, \\ \operatorname{dn}(u + 4K) = \operatorname{dn} u, \\ \operatorname{tn}(u + 4K) = \operatorname{tn} u. \end{cases}$$

122.07 $$\begin{cases} \operatorname{sn}(u + iK') = (\operatorname{ns} u)/k, \\ \operatorname{cn}(u + iK') = -i(\operatorname{ds} u)/k, \\ \operatorname{dn}(u + iK') = -i \operatorname{cs} u, \\ \operatorname{tn}(u + iK') = i \operatorname{nd} u. \end{cases}$$

122.08
$$\begin{cases} \operatorname{sn}(u, 0) = \sin u, \\ \operatorname{cn}(u, 0) = \cos u, \\ \operatorname{dn}(u, 0) = 1, \\ \operatorname{tn}(u, 0) = \tan u. \end{cases}$$

122.09
$$\begin{cases} \operatorname{sn}(u, 1) = \tanh u, \\ \operatorname{cn}(u, 1) = \operatorname{sech} u, \\ \operatorname{dn}(u, 1) = \operatorname{sech} u, \\ \operatorname{tn}(u, 1) = \sinh u. \end{cases}$$

122.10
$$\begin{cases} \operatorname{sn}(K/2) = 1/\sqrt{1+k'}, \\ \operatorname{cn}(K/2) = \sqrt{k'/(1+k')}, \\ \operatorname{dn}(K/2) = \sqrt{k'}, \\ \operatorname{tn}(K/2) = 1/\sqrt{k'}. \end{cases}$$

122.11
$$\begin{cases} \operatorname{sn}(i\,K'/2) = i/\sqrt{k}, \\ \operatorname{cn}(i\,K'/2) = \sqrt{(1+k)/k}, \\ \operatorname{dn}(i\,K'/2) = \sqrt{1+k}, \\ \operatorname{tn}(i\,K'/2) = i/\sqrt{1+k}. \end{cases}$$

122.12
$$\begin{cases} \operatorname{sn}(3K/2) = 1/\sqrt{1+k'}, \\ \operatorname{cn}(3K/2) = -\sqrt{k'/(1+k')}, \\ \operatorname{dn}(3K/2) = \sqrt{k'}, \\ \operatorname{tn}(3K/2) = -1/\sqrt{k'}. \end{cases}$$

122.13
$$\begin{cases} \operatorname{sn}(3\,iK'/2) = -i/\sqrt{k}, \\ \operatorname{cn}(3\,iK'/2) = -\sqrt{(1+k)/k}, \\ \operatorname{dn}(3\,iK'/2) = -\sqrt{1+k}, \\ \operatorname{tn}(3\,iK'/2) = i/\sqrt{1+k}. \end{cases}$$

122.14
$$\begin{cases} \operatorname{sn}(K/2 + i\,K'/2) = \left(\sqrt{1+k} + i\sqrt{1-k}\right)/\sqrt{2k}, \\ \operatorname{cn}(K/2 + i\,K'/2) = (1-i)\sqrt{k'/2k}, \\ \operatorname{dn}(K/2 + i\,K'/2) = \left(\sqrt{1+k'} - i\sqrt{1-k'}\right)\sqrt{k'/2}, \\ \operatorname{tn}(K/2 + i\,K'/2) = \left(\sqrt{1+k} + i\sqrt{1-k}\right)/(1-i)\sqrt{k'}. \end{cases}$$

122.15
$$\begin{cases} \operatorname{sn}(K/2 + i\,K') = 1/\sqrt{1-k'}, \\ \operatorname{cn}(K/2 + i\,K') = -i\sqrt{k'/(1-k')}, \\ \operatorname{dn}(K/2 + i\,K') = -i\sqrt{k'}, \\ \operatorname{tn}(K/2 + i\,K') = i/\sqrt{k'}. \end{cases}$$

122.16
$$\begin{cases} \operatorname{sn}(K/2 + 3\,i\,K'/2) = \sqrt{(k - i\,k')/k}, \\ \operatorname{cn}(K/2 + 3\,i\,K'/2) = -\sqrt{i\,k'/k}, \\ \operatorname{dn}(K/2 + 3\,i\,K'/2) = \sqrt{k'(k' + i\,k)}, \\ \operatorname{tn}(K/2 + 3\,i\,K'/2) = -\sqrt{(k - i\,k')/i\,k'}. \end{cases}$$

122.17
$$\begin{cases} \operatorname{sn}(3\,K/2 + 3\,i\,K'/2) = \sqrt{(k + i\,k')/k}, \\ \operatorname{cn}(3\,K/2 + 3\,i\,K'/2) = \sqrt{-i\,k'/k}, \\ \operatorname{dn}(3\,K/2 + 3\,i\,K'/2) = -\sqrt{k'(k' - i\,k)}, \\ \operatorname{tn}(3\,K/2 + 3\,i\,K'/2) = \sqrt{(k + i\,k')/(-i\,k')}. \end{cases}$$

$$\mathbf{122.18}\quad \begin{cases} \operatorname{sn}(u+K+iK') = (\operatorname{dc} u)/k, \\ \operatorname{cn}(u+K+iK') = -i(k'\operatorname{nc} u)/k, \\ \operatorname{dn}(u+K+iK') = i\,k'\operatorname{tn} u, \\ \operatorname{tn}(u+K+iK') = i\,(\operatorname{dn} u)/k'. \end{cases}$$

$$\mathbf{122.19}\quad \begin{cases} \operatorname{am}(u+2iK') = \pi + \operatorname{am} u, \\ \operatorname{sn}(u+2iK') = \operatorname{sn} u, \\ \operatorname{cn}(u+2iK') = -\operatorname{cn} u, \\ \operatorname{dn}(u+2iK') = -\operatorname{dn} u, \\ \operatorname{tn}(u+2iK') = -\operatorname{tn} u. \end{cases} \qquad \mathbf{122.20}\quad \begin{cases} \operatorname{am}(u+4iK') = \operatorname{am} u, \\ \operatorname{sn}(u+4iK') = \operatorname{sn} u, \\ \operatorname{cn}(u+4iK') = \operatorname{cn} u, \\ \operatorname{dn}(u+4iK') = \operatorname{dn} u, \\ \operatorname{tn}(u+4iK') = \operatorname{tn} u. \end{cases}$$

In the following m and n are integers including zero.

$$\mathbf{122.21}\quad \begin{cases} \operatorname{sn}(2mK+2inK') = 0, \\ \operatorname{cn}[(2m+1)K+2inK'] = 0, \\ \operatorname{dn}[(2m+1)K+(2n+1)iK'] = 0, \\ \operatorname{tn}(2mK+2inK') = 0. \end{cases}$$

$$\mathbf{122.22}\quad \begin{cases} \operatorname{sn}[2mK+(2n+1)iK'] = \infty, \\ \operatorname{cn}[2mK+(2n+1)iK'] = \infty, \\ \operatorname{dn}[2mK+(2n+1)iK'] = \infty, \\ \operatorname{tn}[(2m+1)K+2inK'] = \infty. \end{cases}$$

$$\mathbf{122.23}\quad \begin{cases} \operatorname{sn}[(2m-1)K+2inK'+u] = (-1)^{m+1}\operatorname{cd} u, \\ \operatorname{cn}[(2m-1)K+2inK'+u] = (-1)^{m+n}k'\operatorname{sd} u, \\ \operatorname{dn}[(2m-1)K+2inK'+u] = (-1)^{n}k'\operatorname{nd} u, \\ \operatorname{tn}[(2m-1)K+2inK'+u] = (-1)^{n+1}(\operatorname{cs} u)/k'. \end{cases}$$

$$\mathbf{122.24}\quad \begin{cases} \operatorname{sn}[(2m-1)K+(2n+1)iK'+u] = (-1)^{m+1}(\operatorname{dc} u)/k, \\ \operatorname{cn}[(2m-1)K+(2n+1)iK'+u] = (-1)^{m+n}(i\,k'\operatorname{nc} u)/k \\ \operatorname{dn}[(2m-1)K+(2n+1)iK'+u] = (-1)^{n}i\,k'\operatorname{tn} u, \\ \operatorname{tn}[(2m-1)K+(2n+1)iK'+u] = (-1)^{n}i\,(\operatorname{dn} u)/k'. \end{cases}$$

$$\mathbf{122.25}\quad \begin{cases} \operatorname{am}(u+2mK+2inK') = (m+n)\pi + (-1)^{m}\operatorname{am} u, \\ \operatorname{sn}(u+2mK+2inK') = (-1)^{m}\operatorname{sn} u, \\ \operatorname{cn}(u+2mK+2inK') = (-1)^{m+n}\operatorname{cn} u, \\ \operatorname{dn}(u+2mK+2inK') = (-1)^{n}\operatorname{dn} u, \\ \operatorname{tn}(u+2mK+2inK') = (-1)^{n}\operatorname{tn} u. \end{cases}$$

Addition Formulas.

123.01
$$\begin{cases} \operatorname{am}(u \pm v) = \tan^{-1}(\operatorname{tn} u \operatorname{dn} v) \pm \tan^{-1}(\operatorname{tn} v \operatorname{dn} u), \\ \operatorname{sn}(u \pm v) = [\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{cn}(u \pm v) = [\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{dn}(u \pm v) = [\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{tn}(u \pm v) = [\operatorname{tn} u \operatorname{dn} v \pm \operatorname{tn} v \operatorname{dn} u]/(1 \mp \operatorname{tn} u \operatorname{dn} v \operatorname{tn} v \operatorname{dn} u). \end{cases}$$

123.02
$$\begin{cases} \operatorname{sn}(u+v) + \operatorname{sn}(u-v) = 2 \operatorname{sn} u \operatorname{cn} v \operatorname{dn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{cn}(u+v) + \operatorname{cn}(u-v) = 2 \operatorname{cn} u \operatorname{cn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{dn}(u+v) + \operatorname{dn}(u-v) = 2 \operatorname{dn} u \operatorname{dn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{tn}(u+v) + \operatorname{tn}(u-v) = 2 \operatorname{tn} u \operatorname{nc} v \operatorname{dc} v/(1 - k'^2 \operatorname{tn}^2 u \operatorname{tn}^2 v). \end{cases}$$

123.03
$$\begin{cases} \operatorname{sn}(u+v) - \operatorname{sn}(u-v) = 2 \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{cn}(u+v) - \operatorname{cn}(u-v) = -2 \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{dn}(u+v) - \operatorname{dn}(u-v) = -2 k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{tn}(u+v) - \operatorname{tn}(u-v) = 2 \operatorname{tn} v \operatorname{nc} u \operatorname{dc} u/(1 - k'^2 \operatorname{tn}^2 u \operatorname{tn}^2 v). \end{cases}$$

123.04
$$\begin{cases} \operatorname{sn}(u+v) \cdot \operatorname{sn}(u-v) = (\operatorname{sn}^2 u - \operatorname{sn}^2 v)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{cn}(u+v) \cdot \operatorname{cn}(u-v) = (\operatorname{cn}^2 u - \operatorname{sn}^2 v \operatorname{dn}^2 u)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{dn}(u+v) \cdot \operatorname{dn}(u-v) = (\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u \operatorname{sn}^2 v)/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{tn}(u+v) \cdot \operatorname{tn}(u-v) = (\operatorname{sn}^2 u - \operatorname{sn}^2 v)/(\operatorname{cn}^2 u - \operatorname{sn}^2 v \operatorname{dn}^2 u). \end{cases}$$

123.05
$$\begin{cases} \operatorname{sn}(u \pm v) \cdot \operatorname{cn}(u \mp v) \\ \qquad = [\operatorname{sn} u \operatorname{cn} u \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} v \operatorname{dn} u]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{sn}(u \pm v) \cdot \operatorname{dn}(u \mp v) \\ \qquad = [\operatorname{sn} u \operatorname{dn} u \operatorname{cn} v \pm \operatorname{sn} v \operatorname{dn} v \operatorname{cn} u]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v), \\ \operatorname{cn}(u \pm v) \cdot \operatorname{dn}(u \mp v) \\ \qquad = [\operatorname{cn} u \operatorname{dn} u \operatorname{cn} v \operatorname{dn} v \mp k'^2 \operatorname{sn} u \operatorname{sn} v]/(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v). \end{cases}$$

123.06
$$\begin{cases} \operatorname{sn} u \pm \operatorname{sn} v = \dfrac{2 \operatorname{sn} \frac{1}{2}(u \pm v) \operatorname{cn} \frac{1}{2}(u \mp v) \operatorname{dn} \frac{1}{2}(u \mp v)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}, \\ \operatorname{cn} u + \operatorname{cn} v = \dfrac{2 \operatorname{cn} \frac{1}{2}(u+v) \operatorname{cn} \frac{1}{2}(u-v)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}, \\ \operatorname{dn} u + \operatorname{dn} v = \dfrac{2 \operatorname{dn} \frac{1}{2}(u+v) \operatorname{dn} \frac{1}{2}(u-v)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}, \\ \operatorname{tn} u \pm \operatorname{tn} v = \dfrac{2 \operatorname{nc} u \operatorname{nc} v \operatorname{sn} \frac{1}{2}(u \pm v) \operatorname{dn} \frac{1}{2}(u \mp v) \operatorname{cn} \frac{1}{2}(u \pm v)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}. \end{cases}$$

123.07
$$\begin{cases} \operatorname{cn} u - \operatorname{cn} v = \dfrac{2 \operatorname{sn} \frac{1}{2}(u+v) \operatorname{sn} \frac{1}{2}(u-v) \operatorname{dn} \frac{1}{2}(u+v) \operatorname{dn} \frac{1}{2}(u-v)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}, \\ \operatorname{dn} u - \operatorname{dn} v = \dfrac{2 k^2 \operatorname{sn} \frac{1}{2}(u+v) \operatorname{sn} \frac{1}{2}(v-u) \operatorname{dn} \frac{1}{2}(u+v) \operatorname{dn} \frac{1}{2}(v-u)}{1 - k^2 \operatorname{sn}^2 \frac{1}{2}(u+v) \operatorname{sn}^2 \frac{1}{2}(u-v)}. \end{cases}$$

Double and Half Arguments.

124.01
$$\begin{cases} \operatorname{am} 2u = 2\tan^{-1}(\operatorname{tn} u \operatorname{dn} u), \\ \operatorname{sn} 2u = 2\,[\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u]/(1 - k^2 \operatorname{sn}^4 u), \\ \operatorname{cn} 2u = [\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u]/(1 - k^2 \operatorname{sn}^4 u), \\ \operatorname{dn} 2u = [\operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u]/(1 - k^2 \operatorname{sn}^4 u), \\ \operatorname{tn} 2u = 2\,[\operatorname{tn} u \operatorname{dn} u]/(1 - \operatorname{tn}^2 u \operatorname{dn}^2 u). \end{cases}$$

124.02
$$\begin{cases} \operatorname{sn}^2(u/2) = (1 - \operatorname{cn} u)/(1 + \operatorname{dn} u), \\ \operatorname{cn}^2(u/2) = (\operatorname{dn} u + \operatorname{cn} u)/(1 + \operatorname{dn} u), \\ \operatorname{dn}^2(u/2) = (\operatorname{cn} u + \operatorname{dn} u)/(1 + \operatorname{cn} u), \\ \operatorname{tn}^2(u/2) = (1 - \operatorname{cn} u)/(\operatorname{dn} u + \operatorname{cn} u). \end{cases}$$

Complex and Imaginary Arguments.

125.01
$$\begin{cases} \operatorname{sn}(u \pm i v, k) = \dfrac{\operatorname{sn}(u,k)\operatorname{dn}(v,k') \pm i \operatorname{cn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{cn}(v,k')}{1 - \operatorname{sn}^2(v,k')\operatorname{dn}^2(u,k)}, \\[2ex] \operatorname{cn}(u \pm i v, k) = \dfrac{\operatorname{cn}(u,k)\operatorname{cn}(v,k') \mp i \operatorname{sn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{dn}(v,k')}{1 - \operatorname{sn}^2(v,k')\operatorname{dn}^2(u,k)}, \\[2ex] \operatorname{dn}(u \pm i v, k) = \dfrac{\operatorname{dn}(u,k)\operatorname{cn}(v,k')\operatorname{dn}(v,k') \mp i k^2 \operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{sn}(v,k')}{1 - \operatorname{sn}^2(v,k')\operatorname{dn}^2(u,k)}, \\[2ex] \operatorname{tn}(u \pm i v, k) = \dfrac{\operatorname{sn}(u,k)\operatorname{nc}(u,k)\operatorname{cn}(v,k')\operatorname{dn}(v,k') \pm i \operatorname{dn}(u,k)\operatorname{sn}(v,k')}{1 - \operatorname{sn}^2(u,k)\operatorname{dn}^2(v,k')}. \end{cases}$$

125.02
$$\begin{cases} \operatorname{sn}(i u, k) = i \operatorname{tn}(u, k'), \\ \operatorname{cn}(i u, k) = \operatorname{nc}(u, k'), \\ \operatorname{dn}(i u, k) = \operatorname{dc}(u, k'), \\ \operatorname{tn}(i u, k) = i \operatorname{sn}(u, k'). \end{cases}$$

Relation to Theta Functions.

126.01
$$\begin{cases} \operatorname{sn} u = \vartheta_1(v)/\sqrt{k}\,\vartheta_0(v), \\ \operatorname{cn} u = \sqrt{k'}\,\vartheta_2(v)/\sqrt{k}\,\vartheta_0(v), \\ \operatorname{dn} u = \sqrt{k'}\,\vartheta_3(v)/\vartheta_0(v), \\ \operatorname{tn} u = \vartheta_1(v)/\sqrt{k'}\,\vartheta_2(v), \end{cases}$$

where $v = \pi u/2K$ and the Theta functions are as defined in the Appendix.

Approximation Formulas.

127.01
$$\begin{cases} \operatorname{sn}(u,k) \approx \sin u - k^2 \cos u\,(u - \sin u \cos u)/4, & [k \ll 1] \\ \operatorname{cn}(u,k) \approx \cos u + k^2 \sin u\,(u - \sin u \cos u)/4, \\ \operatorname{dn}(u,k) \approx 1 - (k^2 \sin^2 u)/2. \end{cases}$$

127.02
$$\begin{cases} \operatorname{sn}(u,k) \approx \tanh u + & [k \text{ near one}] \\ \quad + k'^2 \operatorname{sech}^2 u (\sinh u \cosh u - u)/4, & \\ \operatorname{cn}(u,k) \approx \operatorname{sech} u - k'^2 \tanh u \operatorname{sech} u (\sinh u \cosh u - u)/4, & \\ \operatorname{dn}(u,k) \approx \operatorname{sech} u + k'^2 \tanh u \operatorname{sech} u (\sinh u \cosh u + u)/4. & \end{cases}$$

Differential Equations.

128.01
$$\begin{cases} \left[\dfrac{d}{du}(\operatorname{sn} u)\right]^2 = (1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u), \\ \left[\dfrac{d}{du}(\operatorname{cn} u)\right]^2 = (1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u), \\ \left[\dfrac{d}{du}(\operatorname{dn} u)\right]^2 = (1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2), \\ \left[\dfrac{d}{du}(\operatorname{tn} u)\right]^2 = (1 + \operatorname{tn}^2 u)(1 + k'^2 \operatorname{tn}^2 u). \end{cases}$$

Identities.

129.01
$$\begin{cases} \operatorname{cs}^2 u + 1 = \operatorname{ns}^2 u, \\ \operatorname{cs}^2 u + k'^2 = \operatorname{ds}^2 u, \\ k^2 \operatorname{cs}^2 u + k'^2 \operatorname{ns}^2 u = \operatorname{ds}^2 u, \\ \operatorname{ds}^2 u + k^2 = \operatorname{ns}^2 u. \end{cases}$$

129.02
$$\begin{cases} \operatorname{cd}^2 u + k'^2 \operatorname{sd}^2 u = 1, \\ k^2 \operatorname{sd}^2 u + 1 = \operatorname{nd}^2 u, \\ k^2 \operatorname{cd}^2 u + k'^2 \operatorname{nd}^2 u = 1, \\ \operatorname{sd}^2 u + \operatorname{cd}^2 u = \operatorname{nd}^2 u. \end{cases}$$

129.03
$$\begin{cases} \operatorname{tn}^2 u + 1 = \operatorname{nc}^2 u, \\ k'^2 \operatorname{tn}^2 u + 1 = \operatorname{dc}^2 u, \\ k'^2 \operatorname{nc}^2 u + k^2 = \operatorname{dc}^2 u, \\ k^2 \operatorname{tn}^2 u + \operatorname{dc}^2 u = \operatorname{nc}^2 u. \end{cases}$$

129.04
$$\begin{cases} (1 - \operatorname{cn} 2u)/(1 + \operatorname{cn} 2u) = \operatorname{tn}^2 u \operatorname{dn}^2 u, \\ (1 - \operatorname{dn} 2u)/(1 + \operatorname{dn} 2u) = k^2 \operatorname{sn}^2 u \operatorname{cd}^2 u. \end{cases}$$

Other Auxiliary Formulas.

Derivatives of Jacobian Elliptic Functions.

See **731.00–731.12** and **710.50–710.62.**

Expansions in Series.

See **907–908.**

Integrals.

See **310–365** and **400–440.**

Transformation Formulas.

See **160–165.**

Sketches of Jacobian Elliptic Functions.

The following sketches show only a cross section of each figure. For isometric drawings of modular surfaces, see *Tables of Functions with Formulas and Curves* by E. JAHNKE and F. EMDE.

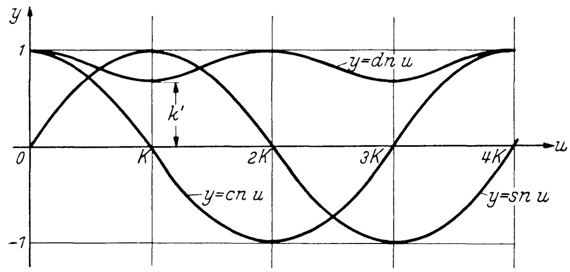

Fig. 10.

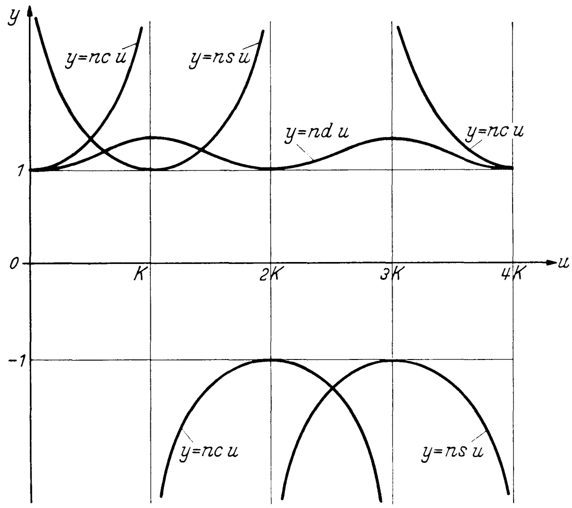

Fig. 11.

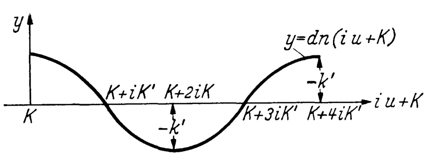

Fig. 12.

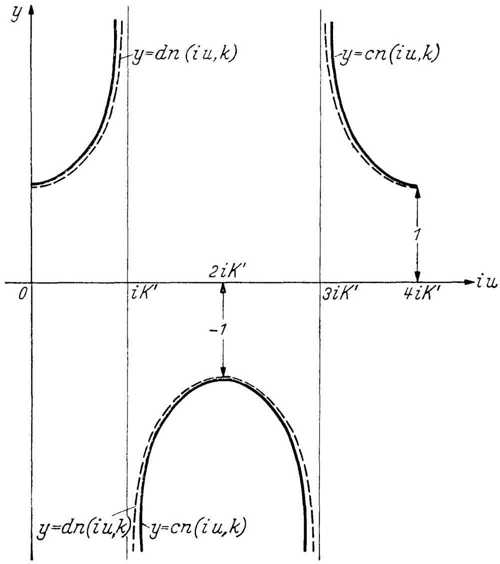

Fig. 13.

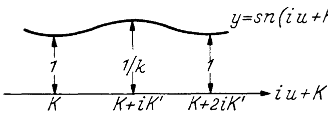

Fig. 14.

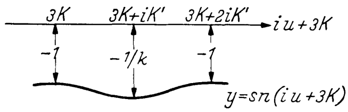

Fig. 15.

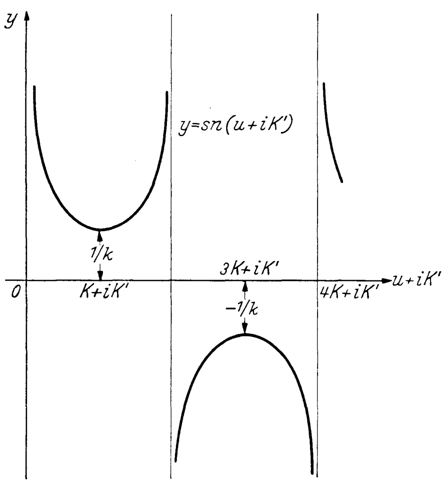

Fig. 16.

Conformal Mappings[1].

Mapping of the half-plane $Im(\zeta) > 0$ onto a rectangle in the z-plane:

129.50 $\zeta = \operatorname{sn} z.$ [cf. **119.01**]

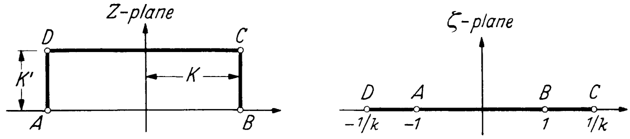

Fig. 17.

Mapping of a circular ring onto the unit circle with a symmetrical slit:

129.51 $$\zeta = \sqrt{k(q)}\,\operatorname{sn}\left[\left(\frac{2iK}{\pi}\ln\frac{z}{r} + K\right),\ k(q)\right],$$

where

$$k(q) = L^2 = 4\sqrt{q}\prod_{m=1}^{\infty}\left[\frac{1+q^{2m}}{1+q^{2m-1}}\right]^4; \quad r = e^{-\pi K'/4K}, \quad q = r^4.$$

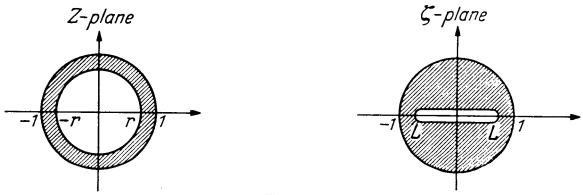

Fig. 18.

Other Mappings.

The function $\zeta = \operatorname{cn} z$ maps the rectangle $-K < \text{R.P.}(z) < K$, $0 < Im(z) < K'$ onto the right half-plane $\text{R.P.}(\zeta) > 0$ with a cut along the linear segment $0 < \zeta < 1$.

The function $\zeta = \operatorname{sn}^2 z$ maps the rectangle $0 < \text{R.P.}(z) < K$, $-K' < Im(z) < K'$ onto the entire ζ-plane with the slits $-\infty \leqq \zeta \leqq 0$, $1 \leqq \zeta \leqq \infty$.

The transformation $\zeta = (\operatorname{sn} z)/(1 + \operatorname{cn} z)$ maps the rectangle $-K < \text{R.P.}(z) < K$, $-K' < Im(z) < K'$ onto the unit circle $|\zeta| < 1$.

Applications.

Jacobian elliptic functions have numerous applications in both pure and applied mathematics. They arise, for example, in capillary phenomena, in rigid dynamics, in algebraic curves and surfaces (e.g., the

[1] For other useful mappings, see *Some Conformal Transformations Involving Elliptic Functions,* by CHARLES DARWIN, The Philosophical Magazine, Vol. 41 (Seventh Series) No. 312, 1950; and also *Dictionary of Conformal Representations* by H. KOBER, Dover Publications, Inc., New York, 1952.

wave surface in optics), and are particularly useful[1] in many problems in potential theory and conformal mapping. Our principal interest, however, is in applying them for the purpose of evaluating the variety of elliptic integrals which occur in almost every field of applied science.

By means of appropriate transformations, elliptic integrals having either algebraic or trigonometric integrands reduce readily to integrals involving rational functions of the Jacobian functions. Integrals with different integrands frequently lead to the same standard Jacobian form. Since recurrence formulas for the integral of these standard forms are easily obtained, it is possible to give in shorter space the evaluation of all sorts of elliptic integrals.

It is found, for instance, that the evaluation of the assorted complete elliptic integrals[2]

$$\int_0^{\pi} \frac{(a-b\cos\vartheta)^m}{\sqrt{a-b\cos\vartheta}}\,d\vartheta,\qquad \int_0^{\pi/4a} \frac{d\vartheta}{\cos^{2m} a\vartheta\sqrt{\cos 2a\vartheta}},\qquad \int_0^{a} \frac{(b^2+t^2)^m\,dt}{\sqrt{(a^2-t^2)(b^2+t^2)}},$$

$$\int_c^{b} \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(t-c)}},\qquad \int_b \frac{(t-d)^m\,dt}{(t-c)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}},$$

(m any integer)

may be given (except for certain constant factors and different moduli) by $\int_0^K \mathrm{nd}^{2m} u\,du$. It would be prohibitive to express the explicit evaluation of each of these integrals; however, by making specific reference in each case to the place where such a standard integral as $\int \mathrm{nd}^{2m} u\,du$ is fully written out, the above integrals, as well as hundreds of others having various Jacobian forms, can be included in a comprehensive table of elliptic integrals (see **200—299**).

Jacobi's Inverse Elliptic Functions.

Definitions.

Every Jacobian elliptic function is the inverse of an elliptic integral. Starting with the functions $\varphi = \mathrm{am}(u, k)$ etc as the direct functions, the *inverse Jacobian elliptic functions* may then be defined as follows:

130.01 $$\mathrm{am}^{-1}(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = F(\varphi, k), \qquad [0<\varphi \leqq \pi/2].$$

---

[1] See, for example, *Anwendung der elliptischen Funktionen in Physik und Technik*, by Oberhettinger and Magnus, Springer, 1949.

[2] In order to be most useful, our table of integrals will contain incomplete integrals for the most part.

130.02 $$\operatorname{sn}^{-1}(y,k)=\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=F(\sin^{-1}y,k),\ [0<y\leqq 1].$$

130.03 $$\operatorname{cn}^{-1}(y,k)=\int_y^1 \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}=F\left(\sin^{-1}\sqrt{1-y^2},k\right),\ [0\leqq y<1].$$

130.04 $$\operatorname{dn}^{-1}(y,k)=\int_y^1 \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}=F[\sin^{-1}\sqrt{(1-y^2)/k^2},k],\ [k'\leqq y<1].$$

130.05 $$\operatorname{tn}^{-1}(y,k)=\int_0^y \frac{dt}{\sqrt{(1+t^2)(1+k'^2t^2)}}=F[\sin^{-1}\sqrt{y^2/(1+y^2)},k],\ [0<y\leqq\infty].$$

130.06 $$\operatorname{nc}^{-1}(y,k)=\int_1^y \frac{dt}{\sqrt{(t^2-1)(k^2+k'^2t^2)}}=F[\sin^{-1}\sqrt{(y^2-1)/y^2},k],\ [\infty\geqq y>1].$$

130.07 $$\operatorname{dc}^{-1}(y,k)=\int_1^y \frac{dt}{\sqrt{(1-t^2)(k^2-t^2)}}=F[\sin^{-1}\sqrt{(1-y^2)/(k^2-y^2)},k],\ [\infty\geqq y>1].$$

130.08 $$\operatorname{sd}^{-1}(y,k)=\int_0^y \frac{dt}{\sqrt{(1-k'^2t^2)(1+k^2t^2)}}=F[\sin^{-1}\sqrt{y^2/(1+k^2y^2)},k],\ [1/k'\geqq y>0].$$

130.09 $$\operatorname{nd}^{-1}(y,k)=\int_1^y \frac{dt}{\sqrt{(t^2-1)(1-k'^2t^2)}}=F[\sin^{-1}\sqrt{(y^2-1)/k^2y^2},k],\ [1/k'\geqq y>1].$$

130.10 $$\operatorname{ns}^{-1}(y,k)=\int_y^\infty \frac{dt}{\sqrt{(t^2-1)(t^2-k^2)}}=F[\sin^{-1}(1/y),k],\ [y\geqq 1>k].$$

130.11 $$\operatorname{ds}^{-1}(y,k)=\int_y^\infty \frac{dt}{\sqrt{(t^2-k'^2)(t^2+k^2)}}=F[\sin^{-1}\sqrt{1/(y^2+k^2)},k],\ [y\geqq k'>0].$$

130.12 $$\operatorname{cs}^{-1}(y,k)=\int_y^{\infty}\frac{dt}{\sqrt{(1+t^2)(t^2+k'^2)}}=F[\sin^{-1}\sqrt{1/(1+y^2)},k],\qquad [\infty>y\geq 0].$$

130.13 $$\left\{\begin{aligned}\operatorname{cd}^{-1}(y,k)&=\int_y^{1}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\\&=F[\sin^{-1}\sqrt{(1-y^2)/(1-k^2y^2)},k],\qquad [0\leq y<1].\end{aligned}\right.$$

Identities.

131.01 $$\left\{\begin{aligned}\operatorname{am}^{-1}(\varphi,k)&=\operatorname{sn}^{-1}(\sin\varphi,k)\\&=\operatorname{cn}^{-1}(\cos\varphi,k)=\operatorname{dn}^{-1}\left(\sqrt{1-k^2\sin^2\varphi},k\right),\\\operatorname{sn}^{-1}(y,k)&=\operatorname{cn}^{-1}\left(\sqrt{1-y^2},k\right)\\&=\operatorname{dn}^{-1}\left(\sqrt{1-k^2y^2},k\right)=\operatorname{tn}^{-1}\left(y/\sqrt{1-y^2},k\right),\\\operatorname{cn}^{-1}(y,k)&=\operatorname{sn}^{-1}\left(\sqrt{1-y^2},k\right)\\&=\operatorname{dn}^{-1}\left(\sqrt{k'^2+k^2y^2},k\right)=\operatorname{tn}^{-1}\left(\sqrt{(1-y^2)/y^2},k\right),\\\operatorname{dn}^{-1}(y,k)&=\operatorname{sn}^{-1}\left(\sqrt{(1-y^2)/k^2},k\right)\\&=\operatorname{cn}^{-1}\left(\sqrt{(y^2-k'^2)/k^2},k\right)=\operatorname{tn}^{-1}\left[\sqrt{\frac{1-y^2}{y^2-k'^2}},k\right],\\\operatorname{tn}^{-1}(y,k)&=\operatorname{sn}^{-1}\left(\sqrt{y^2/(1+y^2)},k\right)\\&=\operatorname{cn}^{-1}\left(\sqrt{1/(1+y^2)},k\right)=\operatorname{dn}^{-1}\left[\sqrt{\frac{1+k'^2y^2}{1+y^2}},k\right].\end{aligned}\right.$$

Special Values.

132.01 $$\left\{\begin{aligned}\operatorname{am}^{-1}(\varphi,1)&=\ln(\sec\varphi+\tan\varphi)=\cosh^{-1}(\sec\varphi),\\\operatorname{sn}^{-1}(y,1)&=\tanh^{-1}y=\ln\left[\sqrt{(1+y)/(1-y)}\right],\\\operatorname{cn}^{-1}(y,1)&=\operatorname{sech}^{-1}y=\ln\left\{\left[1+\sqrt{1-y^2}\right]/y\right\},\\\operatorname{dn}^{-1}(y,1)&=\operatorname{sech}^{-1}y=\ln\left\{\left[1+\sqrt{1-y^2}\right]/y\right\},\\\operatorname{tn}^{-1}(y,1)&=\sinh^{-1}y=\ln\left[y+\sqrt{1+y^2}\right].\end{aligned}\right.$$

132.02 $$\left\{\begin{aligned}&\operatorname{am}^{-1}(\varphi,0)=\varphi, &&\operatorname{cn}^{-1}(y,0)=\cos^{-1}y,\\&\operatorname{sn}^{-1}(y,0)=\sin^{-1}y, &&\operatorname{tn}^{-1}(y,0)=\tan^{-1}y.\end{aligned}\right.$$

132.03 $$\left\{\begin{aligned}&\operatorname{sn}^{-1}(1/\sqrt{1+k'},k)=F[\sin^{-1}\sqrt{1/(1+k')},k]=\tfrac12\operatorname{sn}^{-1}(1,k)=K/2,\\&\operatorname{cn}^{-1}\left[\sqrt{k'/(1+k')},k\right]=\tfrac12\operatorname{cn}^{-1}(0,k)=K/2,\\&\operatorname{dn}^{-1}\left(\sqrt{k'},k\right)=\tfrac12\operatorname{dn}^{-1}(k',k)=K/2,\\&\operatorname{tn}^{-1}(1/\sqrt{k'},k)=\tfrac12\operatorname{tn}^{-1}(\infty,k)=K/2.\end{aligned}\right.$$

Addition Formulas.

133.01
$$\begin{cases} \operatorname{sn}^{-1}(y,k)+\operatorname{sn}^{-1}(x,k) \\ \quad =\operatorname{sn}^{-1}\left[\dfrac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2},k\right], \\ \operatorname{cn}^{-1}(y,k)+\operatorname{cn}^{-1}(x,k) \\ \quad =\operatorname{cn}^{-1}\left[\dfrac{xy-\sqrt{(1-x^2)(k'^2+k^2x^2)(1-y^2)(k'^2+k^2y^2)}}{1-k^2(1-x^2)(1-y^2)},k\right], \\ \operatorname{dn}^{-1}(y,k)+\operatorname{dn}^{-1}(x,k) \\ \quad =\operatorname{dn}^{-1}\left[\dfrac{k^2xy-\sqrt{(1-x^2)(x^2-k'^2)(1-y^2)(y^2-k'^2)}}{k^2-(1-y^2)(1-x^2)},k\right], \\ \operatorname{tn}^{-1}(y,k)+\operatorname{tn}^{-1}(x,k) \\ \quad =\operatorname{tn}^{-1}\left[\dfrac{y\sqrt{(1+y^2)(1+k'^2x^2)}+x\sqrt{(1+x^2)(1+k'^2y^2)}}{\sqrt{(1+y^2)(1+x^2)}-xy\sqrt{(1+k'^2y^2)(1+k'^2x^2)}},k\right], \\ \operatorname{sn}^{-1}(y,k)+\operatorname{cn}^{-1}(x,k) \\ \quad =\operatorname{sn}^{-1}\left[\dfrac{xy\sqrt{k'^2+k^2y^2}+\sqrt{(1-y^2)(1-x^2)(1-k^2x^2)}}{1-k^2x^2(1-y^2)},k\right], \\ \quad =\operatorname{cn}^{-1}\left[\dfrac{y\sqrt{1-x^2}-x\sqrt{(1-y^2)(k'^2+k^2y^2)(1-k^2x^2)}}{1-k^2x^2(1-y^2)},k\right]. \end{cases}$$

Special Addition Formulas.

134.01
$$\begin{cases} \operatorname{sn}^{-1}(y,k)+\operatorname{sn}^{-1}(x,k)=\operatorname{sn}^{-1}(1,k)=K, \\ \qquad \text{if } x=\sqrt{(1-y^2)/(1-k^2y^2)}, \\ \operatorname{cn}^{-1}(y,k)+\operatorname{cn}^{-1}(x,k)=\operatorname{cn}^{-1}(0,k)=K, \\ \qquad \text{if } x=\sqrt{k'^2(1-y^2)/(k'^2+k^2y^2)}, \\ \operatorname{dn}^{-1}(y,k)+\operatorname{dn}^{-1}(x,k)=\operatorname{dn}^{-1}(k',k)=K, \\ \qquad \text{if } x=k'/y, \\ \operatorname{tn}^{-1}(y,k)+\operatorname{tn}^{-1}(x,k)=\operatorname{tn}^{-1}(\infty,k)=K, \\ \qquad \text{if } x=1/k'y. \end{cases}$$

134.02
$$\begin{cases} 2\operatorname{sn}^{-1}(y,k)=\operatorname{sn}^{-1}[2y\sqrt{(1-y^2)(1-k^2y^2)}/(1-k^2y^4)], \\ 2\operatorname{cn}^{-1}(y,k)=\operatorname{cn}^{-1}\{(k^2y^4+2k'^2y^2-k'^2)/[1-k^2(1-y^2)^2]\}, \\ 2\operatorname{dn}^{-1}(y,k)=\operatorname{dn}^{-1}\{(y^4-2k'^2y^2+k'^2)/[k^2-(1-y^2)^2]\}, \\ 2\operatorname{tn}^{-1}(y,k)=\operatorname{tn}^{-1}[2y\sqrt{(1+y^2)(1+k'^2y^2)}/(1-k'^2y^4)]. \end{cases}$$

Other Auxiliary Formulas.

Derivatives.
See **732.00** to **732.12**.

Integrals.
See **390.01** to **392.04**.

Jacobian Zeta Function.

Definitions.

The Zeta function[1] of JACOBI is defined by

140.01
$$\begin{cases} Z(u_1) \equiv \int\limits_0^{u_1} \left[\operatorname{dn}^2 u - \frac{E}{K}\right] du = E(u_1) - \frac{E}{K} u_1 \\ \qquad = E(\beta, k) - \frac{E}{K} F(\beta, k), \quad [\beta = \operatorname{am} u_1], \end{cases}$$

or by the integrals

140.02
$$Z(\beta, k) \equiv -\frac{\cot\beta\sqrt{1 - k^2\sin^2\beta}}{K} \int\limits_0^K \frac{du}{1 - \csc^2\beta \operatorname{sn}^2 u},$$

140.03
$$Z(\beta, k) \equiv \frac{k^2 \sin\beta\cos\beta\sqrt{1 - k^2\sin^2\beta}}{K} \int\limits_0^K \frac{\operatorname{sn}^2 u\, du}{1 - k^2\sin^2\beta \operatorname{sn}^2 u}.$$

This function is an odd periodic function of u_1 with period $2K$ and appears in the evaluation for the hyperbolic cases of the elliptic integral of the third kind (see **400**).

Special Values

141.01
$$\begin{cases} Z(0, k) = 0, \\ Z(\pi/2, k) = 0, \\ Z(-u_1) = -Z(u_1), \\ Z(u_1) = 0, \quad \text{if} \quad k = 0, \\ Z(u_1) = \operatorname{sn}(u_1, 1) = \tanh u_1, \ \text{if} \ k = 1, \\ Z(u_1 + 2K) = Z(u_1), \\ Z(u_1 + iK') = Z(u_1) + \operatorname{cs} u_1 \operatorname{dn} u_1 - i\pi/2K, \\ Z(u_1 + 2iK') = Z(u_1) - i\pi/K, \\ Z(mK) = 0, \qquad m = 0, 1, 2, \ldots, \\ Z[\sin^{-1}\sqrt{1/(1 + k')}, k] = k^2/2(1 + k'), \\ 2Z(\beta, k) = Z(\varphi, k) + \dfrac{2k^2\sin^3\beta\cos\beta\sqrt{1 - k^2\sin^2\beta}}{1 - k^2\sin^4\beta}, \qquad \text{where} \\ \qquad \varphi = \cos^{-1}\left[\dfrac{1 - 2\sin^2\beta + k^2\sin^4\beta}{1 - k^2\sin^4\beta}\right]. \end{cases}$$

[1] The Zeta function is tabulated in *Jacobian Elliptic Function Tables* by MILNE-THOMSON, Dover, New York, 1950. A tabulation of $KZ(\beta, k)$ appears here in the Appendix.

Maximum Value.

141.25 $Z(\beta_1, k) = E(\beta_1, k) - \frac{E}{K} F(\beta_1, k), \quad \beta_1 = \sin^{-1}\left[\sqrt{(K-E)/k^2 K}\right].$

Limiting Value.

141.50 $$\lim_{\beta \to 0} \left[\frac{Z(\beta, k)}{\sin\beta}\right] = \frac{K-E}{K}.$$

Approximation Formula.

141.75 $$\begin{cases} Z(u) \approx \dfrac{4q \sin 2x}{1 + 2q(2 - \cos 2x) + 4q^2(1 - 2\cos 2x)}, \\ x = \pi u/2K \text{ (real)}, \quad q = e^{-(\pi K'/K)}. \end{cases}$$

Addition Formulas.

142.01 $$Z(\vartheta, k) \pm Z(\beta, k) = Z(\varphi, k) \pm k^2 \sin\vartheta \sin\beta \sin\varphi,$$

where
$$\varphi = 2\tan^{-1}\left[\frac{\sin\vartheta \sqrt{1 - k^2\sin^2\beta} \pm \sin\beta \sqrt{1 - k^2 \sin^2\vartheta}}{\cos\vartheta + \cos\beta}\right].$$

142.02* $$Z(u \pm v) = Z(u) \pm Z(v) \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u \pm v).$$

Special Addition Formula.

142.51 $$Z(\vartheta, k) + Z(\beta, k) = \frac{k^2 \sin\beta \cos\beta}{\sqrt{1 - k^2 \sin^2\beta}}, \quad \text{if } \cot\beta = k' \tan\vartheta.$$

Complex and Imaginary Arguments.

143.01 $$\begin{cases} Z(u \pm iv, k) = \left[Z(u,k) + \dfrac{k^2 \operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}^2(v,k')}{1 - \operatorname{sn}^2(v,k')\operatorname{dn}^2(u,k)}\right] \mp \\ \mp i\left[Z(v,k') + \dfrac{v\pi}{2KK'} - \dfrac{\operatorname{dn}^2(u,k)\operatorname{cn}(v,k')\operatorname{sn}(v,k')\operatorname{dn}(v,k')}{1 - \operatorname{sn}^2(v,k')\operatorname{dn}^2(u,k)}\right]. \end{cases}$$

143.02 $$Z(iu, k) = i\left[\operatorname{tn}(u,k')\operatorname{dn}(u,k') - Z(u,k') - \pi u/2KK'\right].$$

Relation to Theta Functions.

144.01 $$Z(u) = \frac{\Theta'(u)}{\Theta(u)}, \quad [\Theta(u) \text{ is defined in the Appendix}].$$

Other Auxiliary Formulas.

Derivatives.
See **710.04** and **730.03.**

Integrals.
See **630.**

* Frequently one writes $Z(u)$ for $Z(\vartheta, k)$ or $Z(\operatorname{am} u, k)$, and $Z(u+v)$ for $Z(\varphi, k)$ or $Z[\operatorname{am}(u+v), k]$. Similar notation is also used for the elliptic functions etc.

Sketch of the Zeta Function.

The following figures are sketches of $Z(\beta, k)$ plotted against the modulus k and the argument β:

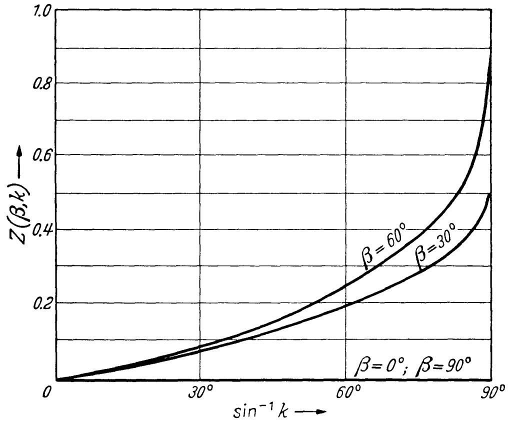

Fig. 19.

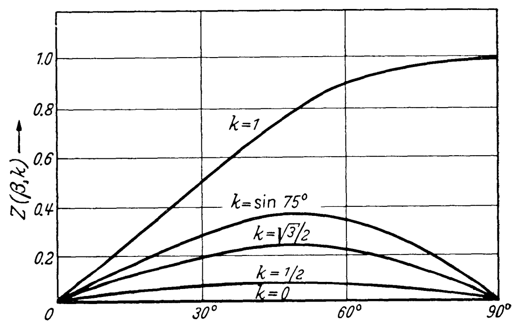

Fig. 20.

Heuman's Lambda Function.

Definitions.

The function $\Lambda_0(\beta, k)$ is defined by the integrals

150.01 $$\Lambda_0(\beta, k) \equiv \frac{2k'^2 \sin\beta\cos\beta}{\pi\sqrt{1-k'^2\sin^2\beta}} \int_0^K \frac{du}{1-\left(\dfrac{k^2}{1-k'^2\sin^2\beta}\right)\operatorname{sn}^2 u},$$

150.02 $$\Lambda_0(\beta, k) \equiv -\frac{2}{\pi}\sin\beta\sqrt{1+k^2\tan^2\beta}\int_0^K \frac{\operatorname{dn}^2 u\, du}{1+k^2\tan^2\beta\operatorname{sn}^2 u},$$

or simply by the formula

150.03 $\Lambda_0(\beta, k) \equiv \frac{2}{\pi}\,[E(k)\,F(\beta, k') + K(k)\,E(\beta, k') - K(k)\,F(\beta, k')]$.

This function appears in the evaluation for the circular cases of the elliptic integral of the third kind. It was first tabulated by CARL HEUMAN[1]. A shorter tabulation is found here in the Appendix.

Special Values.

151.01
$$\begin{cases} \Lambda_0(\beta, 0) = \sin\beta, \\ \Lambda_0(0, k) = 0, \\ \Lambda_0(\beta, 1) = 2\beta/\pi, \\ \Lambda_0(\pi/2, k) = 1, \\ \Lambda_0(m\pi/2, k) = m, \quad (m \text{ an integer including } 0), \\ \Lambda_0(-\beta, k) = -\Lambda_0(\beta, k), \\ \Lambda_0(m\pi \pm \beta, k) = 2m \pm \Lambda_0(\beta, k), \\ \Lambda_0\,[\sin^{-1}\sqrt{1/(1+k)}, k] = \tfrac{1}{2}\,[1 + 2(1-k)\,K/\pi]. \end{cases}$$

Limiting Value.

151.50
$$\lim_{\beta\to 0}\left[\frac{\Lambda_0(\beta, k)}{\sin\beta}\right] = 2E/\pi.$$

Addition Formula.

152.01 $\Lambda_0(\beta, k) \pm \Lambda_0(\vartheta, k) = \Lambda_0(\varphi, k) \pm \frac{2}{\pi}\,k'^2 K \sin\vartheta \sin\beta \sin\varphi$,

where
$$\cos\varphi = \cos\vartheta\cos\beta \mp \sin\vartheta\sin\beta\sqrt{1 - k'^2\sin^2\varphi}.$$

Special Addition Formulas.

153.01
$$\Lambda_0(\vartheta, k) + \Lambda_0(\beta, k) = 1 + \frac{2k'^2\sin\beta\cos\beta\,K}{\pi\sqrt{\cos^2\beta + k^2\sin^2\beta}},$$

if $\cot\vartheta = k\tan\beta$.

153.02
$$2\Lambda_0(\vartheta, k) = \Lambda_0(\varphi, k) + \frac{2}{\pi}\,k'^2 K\sin^2\vartheta\sin\varphi,$$

where
$$\varphi = \cos^{-1}\left[\frac{1 - 2\sin^2\vartheta + k'^2\sin^4\vartheta}{1 - k'^2\sin^4\vartheta}\right].$$

---

[1] *Tables of Complete Elliptic Integrals,* Journal of Mathematics and Physics, Vol. 20, 1941, pp. 127—206.

Relation to Theta Functions.

154.01
$$\Lambda_0(\beta, k) = \frac{\partial}{\partial w} \ln [\vartheta_2(i w)],$$

where
$$w = \frac{\pi F(\beta, k')}{2K},$$

and ϑ_2 is as defined in the Appendix.

Other Auxiliary Formulas.

Derivatives.
See **730.04** and **710.07.**

Integrals.
See **630.**

Sketch of the Function $\Lambda_0(\beta, k)$.

The following graphs show the Heuman function plotted as a function of the argument β and the modulus k.

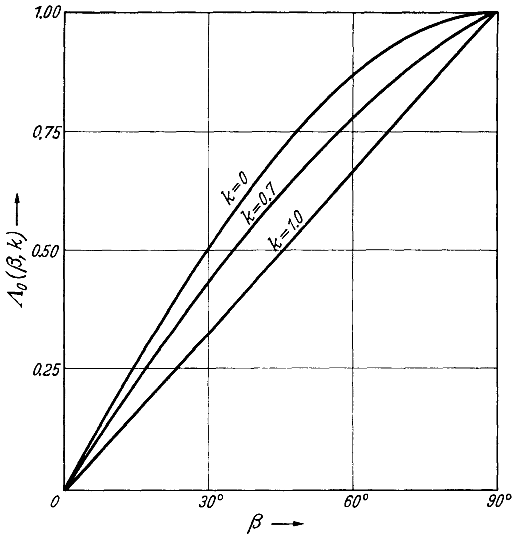

Fig. 21.

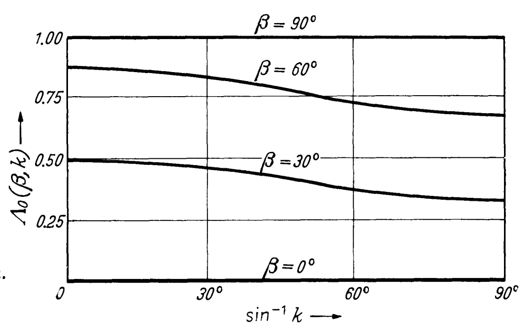

Fig. 22.

Transformation Formulas for Elliptic Functions and Elliptic Integrals.

A *transformation* of elliptic functions consists in the expression of elliptic functions with one modulus and argument in terms of those with a new modulus and argument. The following transformation formulas are found useful in problems arising in many branches of applied science:

Imaginary Modulus Transformation (cf. 282).

160.01
$$\begin{cases} \operatorname{sn}(u, i\,k) = k_1' \operatorname{sd}(u\sqrt{1+k^2}, k_1), & \left[k_1 = k/\sqrt{1+k^2}\right] \\ \operatorname{cn}(u, i\,k) = \operatorname{cd}(u\sqrt{1+k^2}, k_1), \\ \operatorname{dn}(u, i\,k) = \operatorname{nd}(u\sqrt{1+k^2}, k_1), \\ \operatorname{tn}(u, i\,k) = k_1' \operatorname{tn}(u\sqrt{1+k^2}, k_1). \end{cases}$$

160.02
$$\begin{cases} F(\varphi, i\,k) = k_1' F(\beta, k_1), \\ E(\varphi, i\,k) = \frac{1}{k_1'}\left[E(\beta, k_1) - k_1^2 \sin\beta\cos\beta/\sqrt{1-k_1^2\sin^2\beta}\right], \\ \Pi(\varphi, \alpha^2, i\,k) = k_1'\left[k_1^2 F(\beta, k_1) + k_1'^2\alpha^2\,\Pi(\beta, \alpha_1^2, k_1)\right]/(\alpha^2 k_1'^2 + k_1^2) \end{cases}$$

where

$$\alpha_1^2 = \alpha^2 k_1'^2 + k_1^2, \qquad k_1 = k/\sqrt{1+k^2}, \qquad \beta = \sin^{-1}\left[\frac{\sqrt{1+k^2}}{\sqrt{1+k^2\sin^2\varphi}}\sin\varphi\right].$$

Imaginary Argument Transformation.

161.01
$$\begin{cases} \operatorname{sn}(i\,u, k) = i\operatorname{tn}(u, k'), & \operatorname{sn}(u, k') = -i\operatorname{tn}(i\,u, k), \\ \operatorname{cn}(i\,u, k) = \operatorname{nc}(u, k'), & \operatorname{cn}(u, k') = \operatorname{nc}(i\,u, k), \\ \operatorname{dn}(i\,u, k) = \operatorname{dc}(u, k'), & \operatorname{dn}(u, k') = \operatorname{dc}(i\,u, k), \\ \operatorname{tn}(i\,u, k) = i\operatorname{sn}(u, k'), & \operatorname{tn}(u, k') = -i\operatorname{sn}(i\,u, k). \end{cases}$$

161.02
$$\begin{cases} F(i\varphi, k) = i\,F(\beta, k'), \\ E(i\varphi, k) = i\left[F(\beta, k') - E(\beta, k') + \tan\beta\sqrt{1-k'^2\sin^2\beta}\right], \\ \Pi(i\varphi, \alpha^2, k) = i\left[F(\beta, k') - \alpha^2\,\Pi(\beta, 1-\alpha^2, k')\right]/(1-\alpha^2), \end{cases}$$

where

$$\sinh\varphi = \tan\beta.$$

Reciprocal Modulus Transformation[1].

162.01
$$\begin{cases} \operatorname{sn}(k\,u, k_1) = k\operatorname{sn}(u, k), & [k_1 = 1/k], \\ \operatorname{cn}(k\,u, k_1) = \operatorname{dn}(u, k), \\ \operatorname{dn}(k\,u, k_1) = \operatorname{cn}(u, k), \\ \operatorname{tn}(k\,u, k_1) = k\operatorname{sd}(u, k). \end{cases}$$

---

[1] This transformation can be used when the modulus is greater than one; cf. **283**.

162.02
$$\begin{cases} F(\varphi, k_1) = k\,F(\beta, k), \qquad K(k_1) = k\,[K(k) + i\,K'(k)], \\ E(\varphi, k_1) = [E(\beta, k) - k'^2 F(\beta, k)]/k, \\ \Pi(\varphi, \alpha^2, k_1) = k\,\Pi(\beta, \alpha^2 k^2, k), \end{cases}$$

where

$$k_1 = 1/k, \qquad \beta = \sin^{-1}(k_1 \sin\varphi), \qquad k_1 \sin\varphi \leqq 1.$$

Landen's Transformation[1].

163.01
$$\begin{cases} \operatorname{sn}[(1+k')\,u, k_1] = (1+k')\operatorname{sn}(u,k)\operatorname{cd}(u,k), \\ \operatorname{cn}[(1+k')\,u, k_1] = \operatorname{nd}(u,k) - (1+k')\operatorname{sn}(u,k)\operatorname{sd}(u,k), \\ \operatorname{dn}[(1+k')\,u, k_1] = \operatorname{nd}(u,k) + (k'-1)\operatorname{sn}(u,k)\operatorname{sd}(u,k), \\ \operatorname{tn}[(1+k')\,u, k_1] = \dfrac{(1+k')\operatorname{sn}(u,k)\operatorname{cn}(u,k)}{1-(1+k')\operatorname{sn}^2(u,k)}. \end{cases}$$

163.02
$$\begin{cases} F(\varphi, k_1) = 2\,[F(\vartheta, k)]/(1+k_1) = (1+k')\,F(\vartheta, k), \\ E(\varphi, k_1) = [2E(\vartheta, k) + 2k'\,F(\vartheta, k') + (k'-1)\sin\varphi]/(1+k'), \\ \Pi(\varphi, \alpha_1^2, k_1) = (1+k')\,[(k^2-\alpha^2)\,\Pi(\vartheta, \alpha^2, k) + \\ \qquad + (\alpha_2^2 - k^2)\,\Pi(\vartheta, \alpha_2^2, k)]/(\alpha_2^2 - \alpha^2), \end{cases}$$

where

$$k_1 = (1-k')/(1+k'), \qquad \sin\varphi = \frac{(1+k')\sin\vartheta\cos\vartheta}{\sqrt{1-k^2\sin^2\vartheta}},$$

$$\alpha^2 = \frac{(1+k')^2\left[k_1+\alpha_1^2-\sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}\right]}{2}, \quad \alpha_2^2 = \frac{(1+k')^2\left[k_1+\alpha_1^2+\sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}\right]}{2},$$

Gauss' Transformation.

164.01
$$\begin{cases} \operatorname{sn}(u,k) = (1+k_1)\operatorname{sn}(u_1,k_1)/[1+k_1\operatorname{sn}^2(u_1,k_1)], \\ \operatorname{cn}(u,k) = \operatorname{cn}(u_1,k_1)\operatorname{dn}(u_1,k_1)/[1+k_1\operatorname{sn}^2(u_1,k_1)], \\ \operatorname{dn}(u,k) = [1-k_1\operatorname{sn}^2(u_1,k_1)]/[1+k_1\operatorname{sn}^2(u_1,k_1)], \\ \operatorname{tn}(u,k) = (1+k_1)\operatorname{tn}(u_1,k_1)\operatorname{nd}(u_1,k_1), \end{cases}$$

where

$$u_1 = u/(1+k_1), \qquad k_1 = (1-k')/(1+k').$$

164.02
$$\begin{cases} F(\varphi, k_1) = [F(\vartheta, k)]/(1+k_1) = (1+k')\,[F(\vartheta, k)]/2, \\ E(\varphi, k_1) = [E(\vartheta, k) + k'F(\vartheta, k) - \left(1-\sqrt{1-k^2\sin^2\vartheta}\right)\cot\vartheta]/(1+k'), \\ K(k_1) = (1+k')\,[K(k)]/2, \qquad K(k) = (1+k_1)\,K(k_1), \\ E(k_1) = [E(k) + k'\,K(k)]/(1+k'), \\ \Pi(\varphi, \alpha_1^2, k) \\ = \dfrac{[2(1-k')\,F(\vartheta, k) + (\alpha_1^2+k_1)\,(1+k')\,Y + (1+k')\,(\alpha_1^2-k_1)\,\Pi(\vartheta, \alpha^2, k)}{4\,(\alpha_1^2+k_1)} \end{cases}$$

[1] Successive application of this transformation permits the direct computation of the elliptic integrals, particularly when numerical tables are inadequate (cf. **810–811**).

where

$$k_1 = (1 - k')/(1 + k'), \qquad \sin\vartheta = \frac{(1 + k_1)\sin\varphi}{1 + k_1 \sin^2\varphi}, \qquad \alpha^2 = \frac{(1 + k')^2 (k_1 + \alpha_1^2)}{4\alpha_1^2},$$

and

$$Y = \frac{1}{\sqrt{1 - \alpha^2}} \tan^{-1}\left[\sqrt{1 - \alpha^2}\tan\vartheta\right], \quad \text{if} \quad \alpha^2 < 1;$$

$$= \frac{1}{\sqrt{\alpha^2 - 1}} \tanh^{-1}\left[\sqrt{\alpha^2 - 1}\tan\vartheta\right], \quad \text{if} \quad \alpha^2 > 1.$$

Other Transformations.

165.01
$$\begin{cases} \operatorname{sn}(u_1, k_1) = k' \operatorname{sd}(u, k), \\ \operatorname{cn}(u_1, k_1) = \operatorname{cd}(u, k), \\ \operatorname{dn}(u_1, k_1) = \operatorname{nd}(u, k), \\ \operatorname{tn}(u_1, k_1) = k' \operatorname{tn}(u, k), \end{cases}$$

where

$$u_1 = u\,k', \qquad k_1 = i\,k/k'.$$

165.02
$$\begin{cases} F(\varphi, k_1) = k' F(\vartheta, k), \qquad K(k_1) = k' K(k), \\ E(\varphi, k_1) = \left[E(\vartheta, k) - \dfrac{k^2 \sin\vartheta\cos\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}\right]\Big/k', \qquad E(k_1) = [E(k)]/k', \\ \Pi(\varphi, \alpha^2, k_1) = k'\left[k^2 F(\vartheta, k) + \alpha^2 k'^2\, \Pi(\vartheta, k^2 + \alpha^2 k'^2, k)\right]/(k^2 + \alpha^2 k'^2), \end{cases}$$

where

$$k_1 = i\,k/k', \qquad \sin\varphi = \frac{k'\sin\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}.$$

Following are five sets of transformation formulas for elliptic functions from which corresponding formulas for elliptic integrals may easily be developed:

165.03
$$\begin{cases} \operatorname{sn}(i\,k'\,u, 1/k') = i\,k' \operatorname{tn}(u, k), \\ \operatorname{cn}(i\,k'\,u, 1/k') = \operatorname{dc}(u, k), \\ \operatorname{dn}(i\,k'\,u, 1/k') = \operatorname{nc}(u, k), \\ \operatorname{tn}(i\,k'\,u, 1/k') = i\,k' \operatorname{sd}(u, k). \end{cases}$$

165.04
$$\begin{cases} \operatorname{sn}(u_1, k_1) = i\,(1 + k)\operatorname{tn}(u, k)\operatorname{nd}(u, k), \\ \operatorname{cn}(u_1, k_1) = [1 + k\operatorname{sn}^2(u, k)]\operatorname{nc}(u, k)\operatorname{nd}(u, k), \\ \operatorname{dn}(u_1, k_1) = [1 - k\operatorname{sn}^2(u, k)]\operatorname{nc}(u, k)\operatorname{nd}(u, k), \\ \operatorname{tn}(u_1, k_1) = [i\,(1 + k)\operatorname{sn}(u, k)]/[1 + k\operatorname{sn}^2(u, k)], \end{cases}$$

where

$$u_1 = i\,(1 + k)\,u, \qquad k_1 = (1 - k)/(1 + k).$$

165.05
$$\begin{cases} \operatorname{sn}(2\sqrt{k}\,u, k_1) = 2\sqrt{k}\operatorname{sn}(u,k)/[1+k\operatorname{sn}^2(u,k)], \\ \operatorname{cn}(2\sqrt{k}\,u, k_1) = [1-k\operatorname{sn}^2(u,k)]/[1+k\operatorname{sn}^2(u,k)], \\ \operatorname{dn}(2\sqrt{k}\,u, k_1) = \operatorname{cn}(u,k)\operatorname{dn}(u,k)/[1+k\operatorname{sn}^2(u,k)], \\ \operatorname{tn}(2\sqrt{k}\,u, k_1) = 2\sqrt{k}\operatorname{sn}(u,k)/[1-k\operatorname{sn}^2(u,k)], \end{cases}$$

where

$$k_1 = (1+k)/2\sqrt{k}.$$

165.06
$$\begin{cases} \operatorname{sn}[(1+k')\,i\,u, k_1] \\ \qquad = i\,(1+k')\operatorname{sn}(u,k)\operatorname{cn}(u,k)/[1-(1+k')\operatorname{sn}^2(u,k)], \\ \operatorname{cn}[(1+k')\,i\,u, k_1] \\ \qquad = \operatorname{dn}(u,k)/[1-(1+k')\operatorname{sn}^2(u,k)], \\ \operatorname{dn}[(1+k')\,i\,u, k_1] \\ \qquad = [1-(1-k')\operatorname{sn}^2(u,k)]/[1-(1+k')\operatorname{sn}^2(u,k)], \\ \operatorname{tn}[(1+k')\,i\,u, k_1] \\ \qquad = i\,(1+k')\operatorname{sd}(u,k)\operatorname{cn}(u,k), \end{cases}$$

where

$$k_1 = 2\sqrt{k'}/(1+k').$$

165.07
$$\begin{cases} \operatorname{sn}(u_1, k_1) = (k'+i\,k)\operatorname{sn}(u,k)\operatorname{dn}(u,k)/[1+k(k'i-k)\operatorname{sn}^2(u,k)], \\ \operatorname{cn}(u_1, k_1) = \operatorname{cn}(u,k)/[1+k(k'i-k)\operatorname{sn}^2(u,k)], \\ \operatorname{dn}(u_1, k_1) = [1-k(k+i\,k')\operatorname{sn}^2(u,k)]/[1+k(k'i-k)\operatorname{sn}^2(u,k)], \\ \operatorname{tn}(u_1, k_1) = (k'+i\,k)\operatorname{tn}(u,k)\operatorname{dn}(u,k), \end{cases}$$

where

$$u_1 = (k'+i\,k)\,u, \qquad k_1 = 2\sqrt{i\,k\,k'}/(k'+i\,k).$$

Reduction of Algebraic Integrands to Jacobian Elliptic Functions.

Introduction.

The most general elliptic integral encountered in practice may appear in the form

200.00 $$\Gamma = \int_{y_1}^{y} \frac{R_1 + R_2 \sqrt{P}}{R_3 + R_4 \sqrt{P}}\, dt,$$

where R_1, R_2, R_3 and R_4 are rational integral functions of t, and where P is a polynomial of the third or fourth degree with real coefficients and no repeated factors. It can easily be shown that **200.00** can be written

200.01 $$\Gamma = \int_{y_1}^{y} R_5\, dt + \int_{y_1}^{y} \frac{R_6}{\sqrt{P}}\, dt,$$

where

$$R_5 = \frac{R_1 R_3 - R_2 R_4 P}{R_3^2 - R_4^2 P}, \qquad R_6 = \frac{(R_2 R_3 - R_1 R_4) P}{R_3^2 - R_4^2 P},$$

R_5 and R_6 thus being rational functions of t. The first integral on the right in **200.01** is obviously an elementary integral. It is necessary therefore only to consider the reduction of integrals of the type

200.02 $$\xi = \int_{y_1}^{y} \frac{R(t)}{\sqrt{P}}\, dt,$$

where it is assumed that the polynomial P has already been expressed in factored[1] form as the product of sums and differences of squares, or as the product of three or four binomials.

The elliptic integral **200.02** may be reduced to *Jacobi's form* if, for real u, a transformation

200.03 $$\operatorname{sn} u = f(t)$$

allows us to write

200.04 $$\frac{dt}{\sqrt{P}} = g\, du,$$

where g is a constant.

[1] Factorization of a cubic or quartic polynomial may be found in most college algebra books.

The integral then becomes

200.05 $$\xi = g \int_{u_1}^{u_0} G_1(\operatorname{sn} u)\, du,$$

where $u_0 = \operatorname{sn}^{-1}[f(y)]$, $u_1 = \operatorname{sn}^{-1}[f(y_1)]$, and G_1 is a rational function of $\operatorname{sn} u$.

Elliptic integrals occurring in various algebraic forms are expressed in our table of integrals in terms of integrands involving Jacobian elliptic functions (**211–279**). Specific reference is then made in each formula to the applicable formula in the tables (**300–440**), in which the Jacobian forms are explicitly integrated. This method permits us to give in less space the evaluations of a variety of elliptic integrals, particularly of those leading to the third kind.

Integrands Involving the Square Roots of Sums and Differences of Squares, $\sqrt{a_0(t^2 \pm r_1^2)}$, $\sqrt{t^2 \pm r_2^2}$.

Introduction.

When the polynomial P has the special form[1]

$$P = a_0(t^2 \pm r_1^2)(t^2 \pm r_2^2),$$

it is advantageous to write **200.02** as

210.01 $$\int_{y_1}^{y} \frac{R(t)}{\sqrt{P}}\, dt = \int_{y_1}^{y} \frac{t\, R_0(t^2)}{\sqrt{P}}\, dt + \int_{y_1}^{y} \frac{R_7(t^2)}{\sqrt{P}}\, dt.$$

The first integral on the right side can be reduced to elementary form by the substitution $t^2 = \tau$; but the second integral is an elliptic integral and is the one we will now consider.

[1] If the two factors under the radical sign involve the fourth power of the variable of integration, and if a single power of this variable occurs in the numerator as a factor outside of the square root sign, the transformation $t^2 = \tau$, $Y^2 = y$, gives us at once an integral of the type in **210.01**. The integrals

$$\xi_1 = \int_{Y_1}^{Y} \frac{t\, dt}{\sqrt{a_0(t^4 \pm r_1^2)(t^4 \pm r_2^2)^3}} \quad \text{and} \quad \xi_2 = \int_{Y_1}^{Y} t \sqrt{\frac{t^4 \pm r_1^2}{a_0(t^4 \pm r_2^2)}}\, dt,$$

for instance, become simply

$$\xi_1 = \frac{1}{2} \int_{y_1}^{y} \frac{d\tau}{(\tau^2 \pm r_2^2)\sqrt{a_0(\tau^2 \pm r_1^2)(\tau^2 \pm r_2^2)}} \quad \text{and} \quad \xi_2 = \frac{1}{2} \int_{y_1}^{y} \sqrt{\frac{\tau^2 \pm r_1^2}{a_0(\tau^2 \pm r_2^2)}}\, d\tau,$$

which are now examples of **210.01** with $R_0(\tau^2) \equiv 0$. (See **575** for less obvious examples.)

By putting $R_7(t^2)$ into partial fractions, the last integral in **210.01** may be expressed linearly in terms of integrals of the form

$$\int_{y_1}^{y} \frac{t^{2m}\,dt}{\sqrt{P}}, \qquad \int_{y_1}^{y} \frac{dt}{(t^2-p^2)^n\sqrt{P}},$$

and these can all be evaluated in terms of the three integrals

$$\int_{y_1}^{y} \frac{dt}{\sqrt{P}}, \qquad \int_{y_1}^{y} \frac{t^2\,dt}{\sqrt{P}}, \qquad \int_{y_1}^{y} \frac{dt}{(t^2-p^2)\sqrt{P}}.$$

An integral of the type $\int dt/\sqrt{P}$, which is finite everywhere, leads in all cases to an integral of the first kind, while the integral $\int t^2\,dt/\sqrt{P}$, which is algebraically infinite of the first order for the value $t=\infty$, always gives rise to an integral of the second kind. The integral $\int dt/(t^2-p^2)\sqrt{P}$ is an integral of the third kind and becomes logarithmically infinite, for $t=p$, as

$$\pm\frac{1}{2\sqrt{P(p)}}\ln(t-p), \qquad [P(p)\neq 0]$$

and, for $t=-p$, as

$$\mp\frac{1}{2\sqrt{P(p)}}\ln(t+p).$$

(If $P(p)=0$, this latter integral is of the second kind and has an algebraic infinity of the one-half order at the point $t=\pm p$.)

Except for the factor $\sqrt{|a_0|}$, the radical $\sqrt{P}$ in **210.01** may be expressed in one of the following ways:

$$\sqrt{(a^2-t^2)(t^2-b^2)}, \qquad \sqrt{(a^2-t^2)(b^2-t^2)}, \qquad \sqrt{(a^2+t^2)(b^2-t^2)},$$
$$\sqrt{(a^2+t^2)(t^2-b^2)}, \qquad \sqrt{(a^2+t^2)(t^2+b^2)}, \qquad \sqrt{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)},$$

where a^2 and b^2 are real and ϱ^2, $\overline{\varrho^2}$ are conjugate complex numbers.

Substitutions for the reduction of these possible cases are given in the following table of integrals. The substitutions for the first five cases are of the form

210.02 $$t^2=\frac{A_1+A_2\,\mathrm{sn}^2 u}{A_3+A_4\,\mathrm{sn}^2 u}, \quad (0\le u\le K),$$

where A_1, A_2, A_3, A_4 are suitably chosen constants (see **211.00–222.14**). For the last case, use is made of the transformation

210.03 $$t^2=\frac{\alpha_1+\alpha_2\,\mathrm{cn}\,u}{\alpha_3+\alpha_4\,\mathrm{cn}\,u}, \quad (0\le u\le 2K),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are appropriate real constants (see **225**; also cf. **263–267**).

Table of Integrals.

One of the limits of integration in the following tables is usually taken to be one of the zeros of the polynomial under the radical sign, while the other limit is variable. As illustrated on page 2 of the Introduction, however, this procedure will not restrict the use of the tables for evaluating integrals in which both limits are variable. Although the table might appear to contain only definite integrals, it gives essentially a table of indefinite integrals.

Integrands involving $\sqrt{a^2+t^2}$ and $\sqrt{t^2-b^2}$, $(y>b>0)$

$$\boxed{\begin{array}{c} \operatorname{cn}^2 u = \dfrac{b^2}{t^2}, \qquad k^2 = \dfrac{a^2}{a^2+b^2}, \qquad g = \dfrac{1}{\sqrt{a^2+b^2}}, \\ \varphi = \operatorname{am} u_1 = \cos^{-1}(b/y), \qquad \operatorname{cn} u_1 = \cos\varphi. \end{array}}$$

211.00 $$\left\{\begin{aligned} \int_b^y \frac{dt}{\sqrt{(a^2+t^2)(t^2-b^2)}} &= g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\varphi, k) \\ &= g F(\varphi, k), \qquad \text{[cf. } \mathbf{212.00}]. \end{aligned}\right.$$

211.01 $$\int_b^y \frac{dt}{t^2}\sqrt{\frac{a^2+t^2}{t^2-b^2}} = \frac{g}{k'^2}\int_0^{u_1} \operatorname{dn}^2 u\, du = \frac{g}{k'^2} E(\varphi, k).$$

211.02 $$\left\{\begin{aligned} \int_b^y \frac{t^2\, dt}{(t^2-p)\sqrt{(a^2+t^2)(t^2-b^2)}} &= \frac{g b^2}{b^2-p}\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} \\ &= \frac{g b^2}{b^2-p}\Pi(\varphi, \alpha^2, k), \end{aligned}\right.$$

[See appropriate case in **400.**]

where $\alpha^2 = p/(p-b^2), \quad p \neq b^2.$

211.03 $$\int_b^y \sqrt{\frac{a^2+t^2}{t^2-b^2}}\, dt = \sqrt{a^2+b^2}\int_0^{u_1} \operatorname{dc}^2 u\, du. \qquad \text{[See } \mathbf{321.02.}]$$

211.04 $$\int_b^y \sqrt{\frac{t^2-b^2}{a^2+t^2}}\, dt = b^2 g \int_0^{u_1} \operatorname{tn}^2 u\, du. \qquad \text{[See } \mathbf{316.02.}]$$

211.05 $$\int_b^y \frac{dt}{(a^2+t^2)^m\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \operatorname{cd}^{2m} u\, du.$$

[See **320.05.**]

211.06 $$\int_b^y \frac{dt}{a^2+t^2}\sqrt{\frac{t^2-b^2}{a^2+t^2}} = k'^2 g \int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02.**]

211.07 $$\int_b^y \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{a^2+t^2}} = g \int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02.**]

211.08 $$\int_b^y \sqrt{(a^2+t^2)(t^2-b^2)}\, dt = b^2(a^2+b^2)\, g \int_0^{u_1} \mathrm{tn}^2 u\, \mathrm{dc}^2 u\, du.$$ [See **361.15.**]

211.09 $$\int_b^y \frac{t^{2m}\, dt}{\sqrt{(a^2+t^2)(t^2-b^2)}} = b^{2m} g \int_0^{u_1} \mathrm{nc}^{2m} u\, du.$$ [See **313.05.**]

211.10 $$\int_b^y (t^2-b^2)\sqrt{(a^2+t^2)(t^2-b^2)}\, dt = b^4(a^2+b^2)\, g \int_0^{u_1} \mathrm{tn}^4 u\, \mathrm{dc}^2 u\, du.$$ [See **361.23.**]

211.11 $$\int_b^y \frac{dt}{t^{2m}\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{b^{2m}} \int_0^{u_1} \mathrm{cn}^{2m} u\, du.$$ [See **312.05.**]

211.12 $$\int_b^y (a^2+t^2)\sqrt{(a^2+t^2)(t^2-b^2)}\, dt = b^2(a^2+b^2)^2 g \int_0^{u_1} \mathrm{tn}^2 u\, \mathrm{dc}^4 u\, du.$$ [See **361.26.**]

211.13 $$\int_b^y \frac{t^2\, dt}{(a^2+t^2)\sqrt{(a^2+t^2)(t^2-b^2)}} = k'^2 g \int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02.**]

211.14 $$\int_b^y \frac{dt}{(p-t^2)^m\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(p-b^2)^m} \int_0^{u_1} \frac{\mathrm{cn}^{2m} u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^m},$$ [See **338.04.**]

where $$\alpha^2 = p/(p-b^2),\quad p \neq b^2.$$

211.15 $$\int_b^y \frac{(p_1-t^2)\, dt}{(p-t^2)\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{p_1-b^2}{p-b^2}\, g \int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du,$$ [See **340.01.**]

where $$\alpha^2 = p/(p-b^2),\quad \alpha_1^2 = p_1/(p_1-b^2),\qquad p \neq b^2.$$

211.16 $$\int_b^y \frac{R(t^2)\, dt}{\sqrt{(a^2+t^2)(t^2-b^2)}} = g \int_0^{u_1} R(b^2 \mathrm{nc}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands Involving $\sqrt{a^2+t^2}$ and $\sqrt{t^2-b^2}$, $(\infty > y \geqq b > 0)$

$$\operatorname{sn}^2 u = \frac{a^2+b^2}{a^2+t^2}, \quad k^2 = \frac{a^2}{a^2+b^2}, \quad g = \frac{1}{\sqrt{a^2+b^2}},$$
$$\varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{a^2+b^2}{a^2+y^2}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

212.00 $$\int_y^\infty \frac{dt}{\sqrt{(a^2+t^2)(t^2-b^2)}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\varphi, k) = g\,F(\varphi, k), \quad \text{[cf. 211.00]}.$$

212.01 $$\int_y^\infty \frac{t^2\,dt}{(a^2+t^2)\sqrt{(a^2+t^2)(t^2-b^2)}} = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi, k).$$

212.02 $$\int_y^\infty \frac{dt}{t^2-p}\sqrt{\frac{a^2+t^2}{t^2-b^2}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\varphi, \alpha^2, k),$$
[See appropriate case in **400.**]

where $$\alpha^2 = (a^2+p)/(a^2+b^2).$$

212.03 $$\int_y^\infty \frac{dt}{(a^2+t^2)^m\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$ [See **310.05.**]

212.04 $$\int_y^\infty \frac{dt}{(t^2-b^2)^m\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \operatorname{tn}^{2m} u\,du, \quad (y \neq b).$$ [See **316.05.**]

212.05 $$\int_y^\infty \frac{dt}{(a^2+t^2)(t^2-b^2)\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^2}\int_0^{u_1} \operatorname{sn}^2 u \operatorname{tn}^2 u\,du, \quad (y \neq b).$$ [See **361.29.**]

212.06 $$\int_y^\infty \frac{dt}{t^{2m}\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \operatorname{sd}^{2m} u\,du.$$ [See **318.05.**]

212.07 $$\int_y^\infty \frac{dt}{t^2-b^2}\sqrt{\frac{a^2+t^2}{t^2-b^2}} = g\int_0^{u_1} \operatorname{nc}^2 u\,du, \quad (y \neq b).$$ [See **313.02.**]

212.08 $$\int_y^\infty \frac{dt}{a^2+t^2}\sqrt{\frac{t^2-b^2}{a^2+t^2}} = g\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02.**]

212.09 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{a^2+t^2}{t^2-b^2}} = g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02**.]

212.10 $$\int_y^\infty \frac{t^2\, dt}{(t^2-b^2)\sqrt{(t^2-b^2)(a^2+t^2)}} = g\int_0^{u_1} \mathrm{dc}^2 u\, du, \quad (y \neq b).$$ [See **321.02**.]

212.11 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{a^2+t^2}} = g\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02**.]

212.12 $$\int_y^\infty \frac{dt}{(t^2-p)^m\sqrt{(a^2+t^2)(t^2-b^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \frac{\mathrm{sn}^{2m} u\, du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}.$$ [See **337.04**.]

where $$\alpha^2 = (a^2+p)/(a^2+b^2)$$

212.13 $$\int_y^\infty \frac{(t^2-p_1)\, dt}{(t^2-p)\sqrt{(a^2+t^2)(t^2-b^2)}} = g\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\, du,$$ [See **340.01**.]

where $\alpha^2 = (a^2+p)/(a^2+b^2)$ and $\alpha_1^2 = (a^2+p_1)/(a^2+b^2)$.

212.14 $$\int_y^\infty \frac{R(t^2)\, dt}{\sqrt{(a^2+t^2)(t^2-b^2)}} = g\int_0^{u_1} R[(a^2+b^2)\,\mathrm{ds}^2 u]\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2+t^2}$ and $\sqrt{b^2-t^2}$, $(b > y \geq 0)$

$$\boxed{\begin{gathered} \mathrm{cn}^2 u = \frac{t^2}{b^2}, \quad k^2 = \frac{b^2}{a^2+b^2}, \quad g = \frac{1}{\sqrt{a^2+b^2}}, \\ \varphi = \mathrm{am}\, u_1 = \cos^{-1}(y/b), \quad \mathrm{cn}\, u_1 = \cos\varphi. \end{gathered}}$$

213.00 $$\int_y^b \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = g\int_0^{u_1} du = g\, u_1$$
$$= g\,\mathrm{cn}^{-1}(\cos\varphi, k) = g\, F(\varphi, k).$$

213.01 $$\left\{\begin{aligned} \int_y^b \sqrt{\frac{a^2+t^2}{b^2-t^2}}\, dt &= (a^2+b^2)\, g\int_0^{u_1} \mathrm{dn}^2 u\, du = g\,(a^2+b^2)\, E(u_1) \\ &= (a^2+b^2)\, g\, E(\varphi, k). \end{aligned}\right.$$

213.02 $$\left\{\begin{aligned} \int_y^b \frac{dt}{(p-t^2)\sqrt{(a^2+t^2)(b^2-t^2)}} &= \frac{g}{p-b^2}\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ &= \frac{g}{p-b^2}\, \Pi(\varphi, \alpha^2, k), \end{aligned}\right.$$ [See **400**.]

where $$\alpha^2 = b^2/(b^2-p), \quad p \neq b^2.$$

213.03 $$\int_y^b \sqrt{\frac{b^2-t^2}{a^2+t^2}}\,dt = b^2 g \int_0^{u_1} \mathrm{sn}^2 u\,du.$$ [See **310.02**.]

213.04 $$\int_y^b \frac{dt}{a^2+t^2}\sqrt{\frac{b^2-t^2}{a^2+t^2}} = k^2 g \int_0^{u_1} \mathrm{sd}^2 u\,du.$$ [See **318.02**.]

213.05 $$\int_y^b \frac{dt}{t^2}\sqrt{\frac{a^2+t^2}{b^2-t^2}} = \frac{g}{k^2}\int_0^{u_1} \mathrm{dc}^2 u\,du, \quad y \neq 0.$$ [See **321.02**.]

213.06 $$\int_y^b \frac{t^{2m}\,dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = b^{2m} g \int_0^{u_1} \mathrm{cn}^{2m} u\,du.$$ [See **312.05**.]

213.07 $$\int_y^b \frac{t^2\,dt}{(a^2+t^2)\sqrt{(a^2+t^2)(b^2-t^2)}} = k^2 g \int_0^{u_1} \mathrm{cd}^2 u\,du.$$ [See **320.02**.]

213.08 $$\int_y^b \frac{dt}{(a^2+t^2)^m\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{g}{(a^2+b^2)^m}\int_0^{u_1} \mathrm{nd}^{2m} u\,du.$$ [See **315.05**.]

213.09 $$\int_y^b \frac{dt}{t^{2m}\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{g}{b^{2m}}\int_0^{u_1} \mathrm{nc}^{2m} u\,du, \quad y \neq 0.$$ [See **313.05**.]

213.10 $$\int_y^b \frac{dt}{t^2}\sqrt{\frac{b^2-t^2}{a^2+t^2}} = g\int_0^{u_1} \mathrm{tn}^2 u\,du, \quad y \neq 0,$$ [See **316.02**.]

213.11 $$\int_y^b \frac{dt}{(p-t^2)^m\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{g}{(p-b^2)^m}\int_0^{u_1} \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **336.03**.]

where $$\alpha^2 = b^2/(b^2-p), \quad p \neq b^2$$

213.12 $$\int_y^b \frac{(p_1-t^2)\,dt}{(p-t^2)\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{p_1-b^2}{p-b^2}\,g\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$ [See **340.01**.]

where $$\alpha^2 = b^2/(b^2-p), \quad \alpha_1^2 = b^2/(b^2-p_1), \quad p \neq b^2.$$

213.13 $$\int_y^b \frac{R(t^2)\,dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = g\int_0^{u_1} R(b^2\,\mathrm{cn}^2 u)\,du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2-t^2}$ and $\sqrt{b^2+t^2}$, $(a \geqq y > 0)$

$$\operatorname{sn}^2 u = \frac{t^2(a^2+b^2)}{a^2(b^2+t^2)}, \quad k^2 = \frac{a^2}{a^2+b^2}, \quad g = \frac{1}{\sqrt{a^2+b^2}}$$

$$\psi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{y^2(a^2+b^2)}{a^2(y^2+b^2)}}, \quad \operatorname{sn} u_1 = \sin\psi$$

214.00
$$\left\{\begin{aligned}\int_0^y \frac{dt}{\sqrt{(a^2-t^2)(b^2+t^2)}} &= g\int_0^{u_1} du = g\,u_1\\ &= g\operatorname{sn}^{-1}(\sin\psi, k) = g\,F(\psi,k).\end{aligned}\right.$$

214.01
$$\left\{\begin{aligned}&\int_0^y \frac{dt}{(b^2+t^2)\sqrt{(a^2-t^2)(b^2+t^2)}}\\ &\qquad = \frac{g}{b^2}\int_0^{u_1}\operatorname{dn}^2 u\,du = \frac{g}{b^2}E(u_1) = \frac{g}{b^2}E(\psi,k).\end{aligned}\right.$$

214.02
$$\int_0^y \frac{dt}{p-t^2}\sqrt{\frac{b^2+t^2}{a^2-t^2}} = \frac{b^2 g}{p}\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{b^2 g}{p}\Pi(\psi,\alpha^2,k),$$
[See **400.**]

where $\alpha^2 = a^2(p+b^2)/p(a^2+b^2), \quad p \neq 0.$

214.03
$$\int_0^y \sqrt{\frac{a^2-t^2}{b^2+t^2}}\,dt = a^2 g\int_0^{u_1}\operatorname{cd}^2 u\,du.$$
[See **320.02.**]

214.04
$$\int_0^y \frac{t^2\,dt}{(t^2+b^2)\sqrt{(a^2-t^2)(b^2+t^2)}} = g\,k^2\int_0^{u_1}\operatorname{sn}^2 u\,du.$$
[See **310.02.**]

214.05
$$\int_0^y \frac{t^{2m}\,dt}{\sqrt{(a^2-t^2)(b^2+t^2)}} = g\,(k^2 b^2)^m\int_0^{u_1}\operatorname{sd}^{2m} u\,du.$$
[See **318.05.**]

214.06
$$\int_0^y \frac{dt}{t^{2m}\sqrt{(a^2-t^2)(b^2+t^2)}} = \frac{g}{(k^2 b^2)^m}\int_0^{u_1}\operatorname{ds}^{2m} u\,du.$$
[See **319.05.**]

214.07
$$\int_0^y \frac{t^2\,dt}{(a^2-t^2)\sqrt{(a^2-t^2)(b^2+t^2)}} = k'^2 g\int_0^{u_1}\operatorname{tn}^2 u\,du, \quad y \neq a.$$
[See **316.02.**]

214.08
$$\int_0^y \frac{dt}{b^2+t^2}\sqrt{\frac{a^2-t^2}{b^2+t^2}} = \frac{a^2 g}{b^2}\int_0^{u_1}\operatorname{cn}^2 u\,du.$$
[See **312.02.**]

214.09
$$\int_0^y \frac{dt}{a^2-t^2}\sqrt{\frac{b^2+t^2}{a^2-t^2}} = \frac{b^2 g}{a^2}\int_0^{u_1}\operatorname{nc}^2 u\,du, \quad y \neq a,$$
[See **313.02.**]

214.10 $$\int_0^y \frac{dt}{(a^2-t^2)^m\sqrt{(a^2-t^2)(b^2+t^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \mathrm{dc}^{2m} u\, du, \quad y \neq a.$$ [See **321.05.**]

214.11 $$\int_0^y \sqrt{\frac{b^2+t^2}{a^2-t^2}}\, dt = b^2 g \int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02.**]

214.12 $$\int_0^y \sqrt{(a^2-t^2)(b^2+t^2)}\, dt = a^2 b^2 g \int_0^{u_1} \mathrm{cd}^2 u\, \mathrm{nd}^2 u\, du.$$ [See **361.16.**]

214.13 $$\int_0^y \frac{dt}{(p-t^2)^m\sqrt{(a^2-t^2)(b^2+t^2)}} = \frac{g}{p^m}\int_0^{u_1} \frac{\mathrm{dn}^{2m} u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^m},$$ [See **339.04.**]

where $$\alpha^2 = (p+b^2)\, a^2 / p\,(a^2+b^2), \quad p \neq 0.$$

214.14 $$\int_0^y \frac{(p_1-t^2)\, dt}{(p-t^2)\sqrt{(a^2-t^2)(b^2+t^2)}} = \frac{p_1 g}{p}\int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du,$$ [See **340.01.**]

where

$$\alpha_1^2 = a^2(p_1+b^2)/p_1(a^2+b^2), \quad \alpha^2 = a^2(p+b^2)/p\,(a^2+b^2), \quad p \neq 0.$$

214.15 $$\int_0^y \frac{R(t^2)\, dt}{\sqrt{(a^2-t^2)(b^2+t^2)}} = g\int_0^{u_1} R(k^2 b^2 \mathrm{sd}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{t^2-a^2}$ and $\sqrt{t^2-b^2}$, $(\infty > y \geqq a > b > 0)$

$$\mathrm{sn}^2 u = \frac{a^2}{t^2}, \quad k^2 = \frac{b^2}{a^2}, \quad g = \frac{1}{a},$$
$$\varphi = \mathrm{am}\, u_1 = \sin^{-1}(a/y), \quad \mathrm{sn}\, u_1 = \sin\varphi.$$

215.00 $$\int_y^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} du = g\, u_1 = g\, \mathrm{sn}^{-1}(\sin\varphi, k) = g F(\varphi, k).$$

215.01 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{t^2-a^2}} = g\int_0^{u_1} \mathrm{dn}^2 u\, du = g\, E(u_1) = g\, E(\varphi, k).$$

215.02 $$\int_y^\infty \frac{t^2\, dt}{(t^2-p)\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} = g\, \Pi(\varphi, \alpha^2, k),$$ [See **400.**]

where $$\alpha^2 = p/a^2.$$

215.03 $$\int_y^\infty \frac{dt}{t^2-b^2}\sqrt{\frac{t^2-a^2}{t^2-b^2}} = g\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02.**]

215.04 $$\int_y^\infty \frac{dt}{(t^2-a^2)^m\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a^{2m+1}}\int_0^{u_1} \mathrm{tn}^{2m} u\, du, \quad y \neq a.$$ [See **316.05.**]

215.05 $$\int_y^\infty \frac{dt}{t^2-a^2}\sqrt{\frac{t^2-b^2}{t^2-a^2}} = g\int_0^{u_1} \mathrm{dc}^2 u\, du, \quad y \neq a.$$ [See **321.02.**]

215.06 $$\int_y^\infty \frac{dt}{(t^2-b^2)^m\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a^{2m+1}}\int_0^{u_1} \mathrm{sd}^{2m} u\, du.$$ [See **318.05.**]

215.07 $$\int_y^\infty \frac{dt}{t^{2m}\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a^{2m+1}}\int_0^{u_1} \mathrm{sn}^{2m} u\, du.$$ [See **310.05.**]

215.08 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{t^2-a^2}{t^2-b^2}} = g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02.**]

215.09 $$\int_y^\infty \frac{t^2\, dt}{(t^2-a^2)\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} \mathrm{nc}^2 u\, du, \quad y \neq a.$$ [See **313.02.**]

215.10 $$\left\{ \int_y^\infty \frac{dt}{(t^2-a^2)(t^2-b^2)\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a^5}\int_0^{u_1} \mathrm{tn}^2 u\, \mathrm{sd}^2 u\, du, \quad y \neq a. \right.$$ [See **361.24.**]

215.11 $$\int_y^\infty \frac{t^2\, dt}{(t^2-b^2)\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02.**]

215.12 $$\int_y^\infty \frac{dt}{(t^2-p)^m\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a^{2m+1}}\int_0^{u_1} \frac{\mathrm{sn}^{2m} u\, du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **337.04.**]

where $$\alpha^2 = p/a^2.$$

215.13 $$\int_y^\infty \frac{(t^2-p_1)\, dt}{(t^2-p)\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\, du,$$ [See **340.01.**]

where $$\alpha_1^2 = p_1/a^2; \quad \alpha^2 = p/a^2.$$

215.14 $$\int_y^\infty \frac{R(t^2)\, dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} R(a^2\,\mathrm{ns}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{t^2-a^2}$ and $\sqrt{t^2-b^2}$, $(y > a > b > 0)$

$$\operatorname{sn}^2 u = \frac{t^2-a^2}{t^2-b^2}, \qquad k^2 = \frac{b^2}{a^2}, \qquad g = 1/a,$$
$$\psi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{y^2-a^2}{y^2-b^2}}, \qquad \operatorname{sn} u_1 = \sin\psi.$$

216.00
$$\int_a^y \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} du = g\,u_1 = g\,\operatorname{sn}^{-1}(\sin\psi, k) = g\,F(\psi, k).$$

216.01
$$\int_a^y \frac{t^2\,dt}{(t^2-b^2)\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{g}{k'^2}\int_0^{u_1} \operatorname{dn}^2 u\,du = \frac{g}{k'^2}E(u_1) = \frac{g}{k'^2}E(\psi, k).$$

216.02
$$\int_a^y \frac{dt}{t^2-p}\sqrt{\frac{t^2-b^2}{t^2-a^2}} = \frac{a^2-b^2}{a^2-p}\,g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{a^2-b^2}{a^2-p}\,g\,\Pi(\psi, \alpha^2, k),$$ [See **400.**]

where $\alpha^2 = (p-b^2)/(p-a^2)$, $\quad p \neq a^2$.

216.03
$$\int_a^y \sqrt{\frac{t^2-b^2}{t^2-a^2}}\,dt = (a^2-b^2)\,g\int_0^{u_1}\operatorname{nc}^2 u\,du.$$ [See **313.02.**]

216.04
$$\int_a^y \sqrt{\frac{t^2-a^2}{t^2-b^2}}\,dt = (a^2-b^2)\,g\int_0^{u_1}\operatorname{tn}^2 u\,du.$$ [See **316.02.**]

216.05
$$\int_a^y \frac{dt}{t^2-b^2}\sqrt{\frac{t^2-a^2}{t^2-b^2}} = g\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

216.06
$$\int_a^y \frac{t^{2m}}{\sqrt{(t^2-a^2)(t^2-b^2)}}\,dt = a^{2m} g\int_0^{u_1}\operatorname{dc}^{2m} u\,du.$$ [See **321.05.**]

216.07
$$\int_a^y \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{t^2-a^2}} = k'^2 g\int_0^{u_1}\operatorname{nd}^2 u\,du.$$ [See **315.02.**]

216.08
$$\int_a^y \frac{dt}{t^2}\sqrt{\frac{t^2-a^2}{t^2-b^2}} = k'^2 g\int_0^{u_1}\operatorname{sd}^2 u\,du.$$ [See **318.02.**]

216.09 $$\int_a^y \frac{dt}{t^{2m}\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \mathrm{cd}^{2m} u\, du.$$ [See **320.05**.]

216.10 $$\int_a^y \sqrt{(t^2-a^2)(t^2-b^2)}\, dt = (a^2-b^2)^2 g \int_0^{u_1} \mathrm{tn}^2 u\, \mathrm{nc}^2 u\, du.$$ [See **361.07**.]

216.11 $$\int_a^y \frac{dt}{(t^2-b^2)^m\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{g}{(a^2-b^2)^m}\int_0^{u_1} \mathrm{cn}^{2m} u\, du.$$ [See **312.05**.]

216.12 $$\int_a^y \frac{dt}{(p-t^2)^m\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{g}{(p-a^2)^m}\int_0^{u_1} \frac{\mathrm{cn}^{2m} u\, du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **338.04**.]

where $$\alpha^2 = (p-b^2)/(p-a^2), \quad p \neq a^2.$$

216.13 $$\int_a^y \frac{(p_1-t^2)\, dt}{(p-t^2)\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{p_1-a^2}{p-a^2} g \int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\, du,$$ [See **340.01**.]

where $\alpha^2 = (p-b^2)/(p-a^2), \quad \alpha_1^2 = (p_1-b^2)/(p_1-a^2), \quad p \neq a^2.$

216.14 $$\int_a^y \frac{R(t^2)\, dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = g\int_0^{u_1} R(a^2\,\mathrm{dc}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2-t^2}$ and $\sqrt{t^2-b^2}$, $(a \geq y > b > 0)$

$$\mathrm{sn}^2 u = \frac{a^2(t^2-b^2)}{t^2(a^2-b^2)}, \qquad k^2 = \frac{a^2-b^2}{a^2}, \qquad g = 1/a,$$
$$\varphi = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\frac{a^2(y^2-b^2)}{y^2(a^2-b^2)}}, \qquad \mathrm{sn}\, u_1 = \sin\varphi.$$

217.00 $$\int_b^y \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = g\int_0^{u_1} du = g\, u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k) = g F(\varphi, k),$$ [cf. **218.00**].

217.01 $$\int_b^y \frac{dt}{t^2\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{g}{b^2}\int_0^{u_1} \mathrm{dn}^2 u\, du = \frac{g}{b^2} E(u_1) = \frac{g}{b^2} E(\varphi, k).$$

217.02
$$\begin{cases} \displaystyle\int_b^y \frac{t^2\,dt}{(t^2-p)\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{b^2 g}{b^2-p}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ \displaystyle\qquad = \frac{b^2 g}{b^2-p}\,\Pi(\varphi,\alpha^2,k), \quad \text{[See } \mathbf{400}.] \end{cases}$$

where $\alpha^2 = p(a^2-b^2)/a^2(p-b^2), \quad p \neq b^2.$

217.03 $$\int_b^y \sqrt{\frac{a^2-t^2}{t^2-b^2}}\,dt = a\,k^2\int_0^{u_1}\mathrm{cd}^2 u\,du.$$ [See **320.02**.]

217.04 $$\int_b^y \sqrt{\frac{t^2-b^2}{a^2-t^2}}\,dt = b^2 k^2 g\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02**.]

217.05 $$\int_b^y \frac{t^{2m}\,dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = b^{2m} g\int_0^{u_1}\mathrm{nd}^{2m} u\,du.$$ [See **315.05**.]

217.06 $$\int_b^y \frac{t^2\,dt}{(a^2-t^2)\sqrt{(a^2-t^2)(t^2-b^2)}} = g\,\frac{k'^2}{k^2}\int_0^{u_1}\mathrm{nc}^2 u\,du, \quad y \neq a.$$ [See **313.02**.]

217.07 $$\int_b^y \frac{dt}{a^2-t^2}\sqrt{\frac{t^2-b^2}{a^2-t^2}} = g\,k'^2\int_0^{u_1}\mathrm{tn}^2 u\,du, \quad y \neq a.$$ [See **316.02**.]

217.08 $$\int_b^y \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{a^2-t^2}} = g\,k^2\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02**.]

217.09 $$\int_b^y \sqrt{(a^2-t^2)(t^2-b^2)}\,dt = a^2 b^2 k^4 g\int_0^{u_1}\mathrm{sd}^2 u\,\mathrm{cd}^2 u\,du.$$ [See **361.27**.]

217.10
$$\begin{cases} \displaystyle\int_b^y \frac{dt}{(a^2-t^2)^m\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{g}{(a^2-b^2)^m}\int_0^{u_1}\mathrm{dc}^{2m} u\,du, \\ \qquad y \neq a. \end{cases}$$ [See **321.05**.]

217.11 $$\int_b^y \frac{dt}{t^2}\sqrt{\frac{a^2-t^2}{t^2-b^2}} = g\,\frac{k^2}{k'^2}\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

217.12 $$\int_b^y \frac{dt}{(p-t^2)^m\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{g}{(p-b^2)^m}\int_0^{u_1}\frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **339.04**.]

where $\alpha^2 = p(a^2-b^2)/a^2(p-b^2), \quad p \neq b^2.$

217.13 $$\int_b^y \frac{(p_1 - t^2)\,dt}{(p - t^2)\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \frac{(p_1 - b^2)\,g}{p - b^2}\int_0^{u_1} \frac{1 - \alpha_1^2 \operatorname{sn}^2 u}{1 - \alpha^2 \operatorname{sn}^2 u}\,du,$$ [See **340.01**.]

where $\alpha_1^2 = p_1(a^2 - b^2)/a^2(p_1 - b^2)$, $\alpha^2 = p(a^2 - b^2)/a^2(p - b^2)$, $p \neq b^2$.

217.14 $$\int_b^y \frac{R(t^2)\,dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = g\int_0^{u_1} R(b^2 \operatorname{nd}^2 u)\,du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2 - t^2}$ and $\sqrt{t^2 - b^2}$, $(a > y \geq b > 0)$

$$\boxed{\begin{array}{ccc} \operatorname{sn}^2 u = \dfrac{a^2 - t^2}{a^2 - b^2}, & k^2 = \dfrac{a^2 - b^2}{a^2}, & g = 1/a, \\ \psi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\dfrac{a^2 - y^2}{a^2 - b^2}}, & & \operatorname{sn} u_1 = \sin\psi \end{array}}$$

218.00 $$\left\{\begin{aligned} \int_y^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} &= g\int_0^{u_1} du = g\,u_1 = g \operatorname{sn}^{-1}(\sin\psi, k) \\ &= g F(\psi, k), \end{aligned}\right.$$ [cf. **217.00**.]

218.01 $$\int_y^a \frac{t^2\,dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = g a^2 \int_0^{u_1} \operatorname{dn}^2 u\,du = g a^2 E(u_1) = g a^2 E(\psi, k).$$

218.02 $$\left\{\begin{aligned} \int_y^a \frac{dt}{(t^2 - p)\sqrt{(a^2 - t^2)(t^2 - b^2)}} &= \frac{g}{a^2 - p}\int_0^{u_1} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u} \\ &= \frac{g}{a^2 - p}\Pi(\psi, \alpha^2, k), \end{aligned}\right.$$ [See **400**.]

where $\alpha^2 = (a^2 - b^2)/(a^2 - p^2)$, $p \neq a^2$.

218.03 $$\int_y^a \sqrt{\frac{t^2 - b^2}{a^2 - t^2}}\,dt = (a^2 - b^2)\,g\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02**.]

218.04 $$\int_y^a \frac{dt}{(t^2 - b^2)^m\sqrt{(t^2 - b^2)(a^2 - t^2)}} = \frac{g}{(a^2 - b^2)^m}\int_0^{u_1} \operatorname{nc}^{2m} u\,du,\quad y \neq b.$$ [See **313.05**.]

218.05 $$\int_y^a \frac{dt}{t^2 - b^2}\sqrt{\frac{a^2 - t^2}{t^2 - b^2}} = g\int_0^{u_1} \operatorname{tn}^2 u\,du,\quad y \neq b.$$ [See **316.02**.]

218.06 $$\int_y^a \frac{t^{2m}\,dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = g\,a^{2m}\int_0^{u_1} \mathrm{dn}^{2m} u\,du. \qquad \text{[See } \mathbf{314.05}.]$$

218.07 $$\int_y^a \frac{t^2\,dt}{(t^2-b^2)\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{g}{k^2}\int_0^{u_1} \mathrm{dc}^2 u\,du, \quad y \neq b.$$
[See **321.02**.]

218.08 $$\int_y^a \frac{dt}{t^2}\sqrt{\frac{t^2-b^2}{a^2-t^2}} = g\,k^2\int_0^{u_1} \mathrm{cd}^2 u\,du. \qquad \text{[See } \mathbf{320.02}.]$$

218.09 $$\int_y^a \sqrt{\frac{a^2-t^2}{t^2-b^2}}\,dt = (a^2-b^2)\,g\int_0^{u_1} \mathrm{sn}^2 u\,du. \qquad \text{[See } \mathbf{310.02}.]$$

218.10 $$\int_y^a \frac{dt}{t^2}\sqrt{\frac{a^2-t^2}{t^2-b^2}} = g\,k^2\int_0^{u_1} \mathrm{sd}^2 u\,du. \qquad \text{[See } \mathbf{318.02}.]$$

218.11 $$\int_y^a \sqrt{(a^2-t^2)(t^2-b^2)}\,dt = (a^2-b^2)\,g\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du.$$
[See **361.01**.]

218.12 $$\int_y^a \frac{dt}{t^{2m}\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a^{2m}}\int_0^{u_1} \mathrm{nd}^{2m} u\,du. \qquad \text{[See } \mathbf{315.05}.]$$

218.13 $$\int_y^a \frac{dt}{(t^2-p)^m\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{g}{(a^2-p)^m}\int_0^{u_1} \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$
[See **336.03**.]

where $\alpha^2 = (a^2-b^2)/(a^2-p)$, $p \neq a^2$.

218.14 $$\int_y^a \frac{(t^2-p_1)\,dt}{(t^2-p)\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{a^2-p_1}{a^2-p}\,g\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$
[See **340.01**.]

where $\alpha_1^2 = (a^2-b^2)/(a^2-p_1)$, $\alpha^2 = (a^2-b^2)/(a^2-p)$, $p \neq a^2$.

218.15 $$\int_y^a \frac{R(t^2)\,dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = g\int_0^{u_1} R\,(a^2\,\mathrm{dn}^2 u)\,du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2-t^2}$ and $\sqrt{b^2-t^2}$, $(a > b \geq y > 0)$

$$\boxed{\begin{array}{l} \operatorname{sn}^2 u = \dfrac{t^2}{b^2}, \quad k^2 = \dfrac{b^2}{a^2}, \quad g = 1/a, \\ \varphi = \operatorname{am} u_1 = \sin^{-1}(y/b), \quad \operatorname{sn} u_1 = \sin\varphi \end{array}}$$

219.00 $$\left\{ \begin{aligned} \int_0^y \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} &= g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\varphi, k) \\ &= g\,F(\varphi, k), \qquad \text{[cf. 220.00.]} \end{aligned} \right.$$

219.01 $$\int_0^y \sqrt{\frac{a^2-t^2}{b^2-t^2}}\,dt = a\int_0^{u_1} \operatorname{dn}^2 u\,du = a\,E(u_1) = a\,E(\varphi, k).$$

219.02 $$\int_0^y \frac{dt}{(p-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{g}{p}\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{g}{p}\Pi(\varphi, \alpha^2, k),$$
[See **400.**]

where $$\alpha^2 = b^2/p, \quad p \neq 0.$$

219.03 $$\int_0^y \sqrt{\frac{b^2-t^2}{a^2-t^2}}\,dt = b\,k\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02.**]

219.04 $$\int_0^y \frac{t^2\,dt}{(a^2-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} = k^2 g\int_0^{u_1} \operatorname{sd}^2 u\,du.$$ [See **318.02.**]

219.05 $$\int_0^y \frac{t^{2m}\,dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = b^{2m} g\int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$ [See **310.05.**]

219.06 $$\int_0^y \frac{dt}{(b^2-t^2)^m\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{g}{b^{2m}}\int_0^{u_1} \operatorname{nc}^{2m} u\,du, \quad y \neq b.$$
[See **313.05.**]

219.07 $$\int_0^y \frac{dt}{(a^2-t^2)^m\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \operatorname{nd}^{2m} u\,du.$$
[See **315.05.**]

219.08 $$\left\{ \begin{aligned} \int_0^y \frac{dt}{(a^2-t^2)(b^2-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} &= \frac{g}{a^2 b^2}\int_0^{u_1} \operatorname{nc}^2 u \operatorname{nd}^2 u\,du, \\ y &\neq b. \qquad \text{[See 361.12.]} \end{aligned} \right.$$

219.09 $$\int_0^y \frac{dt}{a^2-t^2}\sqrt{\frac{b^2-t^2}{a^2-t^2}} = k^2 g\int_0^{u_1} \operatorname{cd}^2 u\,du.$$ [See **320.02.**]

219.10 $$\int_0^y \frac{dt}{b^2-t^2}\sqrt{\frac{a^2-t^2}{b^2-t^2}} = \frac{g}{k^2}\int_0^{u_1} \mathrm{dc}^2 u\, du, \quad y \neq b. \qquad \text{[See } \mathbf{321.02}.]$$

219.11 $$\int_0^y \sqrt{(a^2-t^2)(b^2-t^2)}\, dt = a^2 b^2 g \int_0^{u_1} \mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du.$$
[See **361.03**.]

219.12 $$\int_0^y \frac{t^2\, dt}{(b^2-t^2)\sqrt{(a^2-t^2)\cdot(b^2-t^2)}} = g\int_0^{u_1} \mathrm{tn}^2 u\, du, \quad y \neq b.$$
[See **316.02**.]

219.13 $$\int_0^y \frac{dt}{(p-t^2)^m\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{g}{p^m}\int_0^{u_1} \frac{du}{(1-\alpha^2 \mathrm{sn}^2 u)^m},$$
[See **336.03**.]

where $$\alpha^2 = b^2/p, \quad p \neq 0.$$

219.14 $$\int_0^y \frac{(p_1-t^2)\, dt}{(p-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{p_1 g}{p}\int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du,$$
[See **340.01**.]

where $$\alpha_1^2 = b^2/p_1, \qquad \alpha^2 = b^2/p, \quad p \neq 0.$$

219.15 $$\int_0^y \frac{R(t^2)\, dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = g\int_0^{u_1} R(b^2 \mathrm{sn}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2-t^2}$ and $\sqrt{b^2-t^2}$, $(a > b > y \geqq 0)$

$$\boxed{\begin{gathered} \mathrm{sn}^2 u = \frac{a^2(b^2-t^2)}{b^2(a^2-t^2)}, \qquad k^2 = b^2/a^2, \qquad g = 1/a, \\ \psi = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\frac{a^2(b^2-y^2)}{b^2(a^2-y^2)}}, \qquad \mathrm{sn}\, u_1 = \sin\psi. \end{gathered}}$$

220.00 $$\int_y^b \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = g\int_0^{u_1} du = g\, u_1 = g\, \mathrm{sn}^{-1}(\sin\psi, k) = g F(\psi, k), \qquad \text{[cf. } \mathbf{219.00}.]$$

220.01 $$\left\{ \begin{aligned} \int_y^b \frac{dt}{(a^2-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} &= \frac{g}{a^2 k'^2}\int_0^{u_1} \mathrm{dn}^2 u\, du = \frac{g}{a^2 k'^2} E(u_1) \\ &= \frac{g}{a^2 k'^2} E(\psi, k). \end{aligned} \right.$$

220.02 $$\left\{\begin{aligned}\int_y^b \frac{dt}{t^2-p}\sqrt{\frac{a^2-t^2}{b^2-t^2}} &= \frac{(a^2-b^2)\,g}{b^2-p}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ &= \frac{k'^2\,a^2\,g}{b^2-p}\,\Pi(\psi,\alpha^2,k)\,,\end{aligned}\right.$$ [See **400.**]

where $\alpha^2 = b^2(p-a^2)/a^2(p-b^2),\quad p \neq b^2.$

220.03 $$\int_y^b \sqrt{\frac{a^2-t^2}{b^2-t^2}}\,dt = (a^2-b^2)\,g\int_0^{u_1}\mathrm{nd}^2 u\,du.$$ [See **315.02.**]

220.04 $$\int_y^b \sqrt{\frac{b^2-t^2}{a^2-t^2}}\,dt = k'^2\,b^2\,g\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02.**]

220.05 $$\int_y^b \sqrt{(a^2-t^2)\,(b^2-t^2)}\,dt = k^2\,a^4\,k'^4\,g\int_0^{u_1}\mathrm{sd}^2 u\,\mathrm{nd}^2 u\,du.$$ [See **361.19.**]

220.06 $$\int_y^b \frac{t^{2m}\,dt}{\sqrt{(a^2-t^2)\,(b^2-t^2)}} = b^{2m}\,g\int_0^{u_1}\mathrm{cd}^{2m} u\,du.$$ [See **320.05.**]

220.07 $$\int_y^b \frac{dt}{a^2-t^2}\sqrt{\frac{b^2-t^2}{a^2-t^2}} = k^2\,g\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02.**]

220.08 $$\int_y^b \frac{t^2\,dt}{(a^2-t^2)\sqrt{(a^2-t^2)\,(b^2-t^2)}} = \frac{k^2\,g}{k'^2}\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02.**]

220.09 $$\int_y^b \frac{dt}{t^{2m}\sqrt{(a^2-t^2)\,(b^2-t^2)}} = \frac{g}{b^{2m}}\int_0^{u_1}\mathrm{dc}^{2m} u\,du,\qquad y \neq 0.$$ [See **321.05.**]

220.10 $$\int_y^b \frac{dt}{(a^2-t^2)^m\sqrt{(a^2-t^2)\,(b^2-t^2)}} = \frac{g}{(a^2-b^2)^m}\int_0^{u_1}\mathrm{dn}^{2m} u\,du.$$ [See **314.05.**]

220.11 $$\int_y^b \frac{dt}{t^2}\sqrt{\frac{b^2-t^2}{a^2-t^2}} = k'^2\,g\int_0^{u_1}\mathrm{tn}^2 u\,du,\qquad y \neq 0.$$ [See **316.02.**]

220.12 $$\int_y^b \frac{dt}{t^2}\sqrt{\frac{a^2-t^2}{b^2-t^2}} = \frac{k'^2\,g}{k^2}\int_0^{u_1}\mathrm{nc}^2 u\,du,\qquad y \neq 0.$$ [See **313.02.**]

220.13 $$\int_y^b \frac{dt}{(p-t^2)^m\sqrt{(a^2-t^2)\,(b^2-t^2)}} = \frac{g}{(p-b^2)^m}\int_0^{u_1}\frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **339.03.**]

where $\alpha^2 = b^2(p-a^2)/a^2(p-b^2),\quad p \neq b^2.$

220.14 $$\int_y^b \frac{(p_1-t^2)\,dt}{(p-t^2)\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{(p_1-b^2)\,g}{p-b^2}\int_0^{u_1}\frac{1-\alpha_1^2\operatorname{sn}^2 u}{1-\alpha^2\operatorname{sn}^2 u}\,du,$$
[See **340.01**.]

where $\alpha_1^2 = b^2(p_1-a^2)/a^2(p_1-b^2)$, $\alpha^2 = b^2(p-a^2)/a^2(p-b^2)$, $p \neq b^2$.

220.15 $$\int_y^b \frac{R(t^2)\,dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = g\int_0^{u_1} R(b^2\operatorname{cd}^2 u)\,du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{t^2+a^2}$ and $\sqrt{t^2+b^2}$, $(y>0;\ a>b)$

$$\operatorname{tn}^2 u = t^2/b^2, \quad k^2 = \frac{a^2-b^2}{a^2}, \quad g = 1/a,$$
$$\varphi = \operatorname{am} u_1 = \operatorname{tn}^{-1}(y/b), \quad \operatorname{tn} u_1 = \tan\varphi.$$

221.00 $$\left\{\begin{aligned}\int_0^y \frac{dt}{\sqrt{(t^2+b^2)(t^2+a^2)}} &= g\int_0^{u_1} du = g\,u_1 = g\operatorname{tn}^{-1}(\tan\varphi, k)\\ &= g\,F(\varphi,k),\end{aligned}\right.$$
[cf. **222.00**].

221.01 $$\int_0^y \frac{dt}{t^2+b^2}\sqrt{\frac{t^2+a^2}{t^2+b^2}} = \frac{g}{k'^2}\int_0^{u_1}\operatorname{dn}^2 u\,du = \frac{g}{k'^2}E(u_1) = \frac{g}{k'^2}E(\varphi,k).$$

221.02 $$\int_0^y \frac{dt}{p-t^2}\sqrt{\frac{t^2+b^2}{t^2+a^2}} = \frac{b^2 g}{p}\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{b^2 g}{p}\Pi(\varphi,\alpha^2,k).$$
[See **400**.]

where $\alpha^2 = (p+b^2)/p$, $p \neq 0$.

221.03 $$\int_0^y \sqrt{\frac{t^2+a^2}{t^2+b^2}}\,dt = g\,a^2\int_0^{u_1}\operatorname{dc}^2 u\,du.$$
[See **321.02**.]

221.04 $$\int_0^y \sqrt{\frac{t^2+b^2}{t^2+a^2}}\,dt = g\,b^2\int_0^{u_1}\operatorname{nc}^2 u\,du.$$
[See **313.02**.]

221.05 $$\int_0^y \frac{dt}{(t^2+b^2)^m\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{g}{b^{2m}}\int_0^{u_1}\operatorname{cn}^{2m} u\,du.$$
[See **312.05**.]

221.06 $$\int_0^y \frac{dt}{(t^2+a^2)^m\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{g}{a^{2m}}\int_0^{u_1}\operatorname{cd}^{2m} u\,du.$$
[See **320.05**.]

221.07 $$\int_0^y \frac{dt}{(t^2+a^2)(t^2+b^2)\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{g}{a^2 b^2}\int_0^{u_1} \mathrm{cn}^2 u\, \mathrm{cd}^2 u\, du.$$ [See **361.28**.]

221.08 $$\int_0^y \sqrt{(t^2+a^2)(t^2+b^2)}\, dt = a^2 b^2 g \int_0^{u_1} \mathrm{dc}^2 u\, \mathrm{nc}^2 u\, du.$$ [See **361.13**.]

221.09 $$\int_0^y \frac{t^{2m}\, dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = g\, b^{2m}\int_0^{u_1} \mathrm{tn}^{2m} u\, du.$$ [See **316.05**.]

221.10 $$\int_0^y \frac{dt}{t^2+a^2}\sqrt{\frac{t^2+b^2}{t^2+a^2}} = g\, k'^2\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02**.]

221.11 $$\int_0^y \frac{t^2\, dt}{(t^2+a^2)\sqrt{(t^2+a^2)(t^2+b^2)}} = k'^2 g\int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02**.]

221.12 $$\int_0^y \frac{t^2\, dt}{(t^2+b^2)\sqrt{(t^2+a^2)(t^2+b^2)}} = g\int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02**.]

221.13 $$\int_0^y \frac{dt}{(p-t^2)^m\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{g}{p^m}\int_0^{u_1} \frac{\mathrm{cn}^{2m} u\, du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **338.04**.]

where $\alpha^2 = (p+b^2)/p, \quad p \neq 0.$

221.14 $$\int_0^y \frac{(p_1-t^2)\, dt}{(p-t^2)\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{p_1 g}{p}\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\, du,$$ [See **340.01**.]

where $\alpha_1^2 = (p_1+b^2)/p_1, \quad \alpha^2 = (p+b^2)/p, \quad p \neq 0.$

221.15 $$\int_0^y \frac{R(t^2)\, dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = g\int_0^{u_1} R(b^2\,\mathrm{tn}^2 u)\, du,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{a^2+t^2}$ and $\sqrt{b^2+t^2}$, $(\infty > y \geq 0;\ a > b)$

$$\mathrm{tn}^2 u = \frac{a^2}{t^2}, \quad k^2 = \frac{a^2-b^2}{a^2}, \quad g = \frac{1}{a}, \quad \mathrm{tn}\, u_1 = \tan\psi,$$
$$\psi = \mathrm{am}\, u_1 = \tan^{-1}(a/y).$$

222.00 $$\left\{ \begin{aligned} \int_y^\infty \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} &= g\int_0^{u_1} du = g u_1 = g\,\mathrm{tn}^{-1}(\tan\psi, k) \\ &= g F(\psi, k), \end{aligned} \right.$$ [cf. **221.00**].

222.01 $$\int_y^\infty \frac{dt}{t^2+a^2}\sqrt{\frac{b^2+t^2}{a^2+t^2}} = g\int_0^{u_1} \mathrm{dn}^2 u\, du = g\,E(u_1) = g\,E(\psi, k).$$

222.02 $$\int_y^\infty \frac{dt}{t^2-p}\sqrt{\frac{t^2+a^2}{t^2+b^2}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = g\,\Pi(\psi, \alpha^2, k);$$

[See **400.**]

where $$\alpha^2 = (a^2+p)/a^2.$$

222.03 $$\int_y^\infty \frac{dt}{(t^2+a^2)^m\sqrt{(a^2+t^2)(b^2+t^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \mathrm{sn}^{2m} u\, du.$$

[See **310.05.**]

222.04 $$\int_y^\infty \frac{dt}{t^{2m}\sqrt{(t^2+a^2)(b^2+t^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \mathrm{tn}^{2m} u\, du, \quad y \neq 0.$$

[See **316.05.**]

222.05 $$\int_y^\infty \frac{dt}{(b^2+t^2)^m\sqrt{(t^2+a^2)(b^2+t^2)}} = \frac{g}{a^{2m}}\int_0^{u_1} \mathrm{sd}^{2m} u\, du.$$

[See **318.05.**]

222.06 $$\int_y^\infty \frac{dt}{t^2+b^2}\sqrt{\frac{t^2+a^2}{t^2+b^2}} = g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02.**]

222.07 $$\int_y^\infty \frac{t^2\,dt}{(t^2+a^2)\sqrt{(t^2+a^2)(b^2+t^2)}} = g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02.**]

222.08 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{t^2+a^2}{b^2+t^2}} = g\int_0^{u_1} \mathrm{nc}^2 u\, du, \quad y \neq 0.$$ [See **313.02.**]

222.09 $$\int_y^\infty \frac{dt}{t^2}\sqrt{\frac{t^2+b^2}{a^2+t^2}} = g\int_0^{u_1} \mathrm{dc}^2 u\, du, \quad y \neq 0.$$ [See **321.02.**]

222.10 $$\int_y^\infty \frac{t^2\,dt}{(t^2+b^2)\sqrt{(t^2+b^2)(t^2+a^2)}} = g\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02.**]

222.11 $$\int_y^\infty \frac{dt}{(p-t^2)^m\sqrt{(t^2+b^2)(t^2+a^2)}} = \frac{g}{(-a^2)^m}\int_0^{u_1} \frac{\mathrm{sn}^{2m} u\, du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$

[See **337.04.**]

where $$\alpha^2 = (a^2+p)/a^2.$$

222.12 $$\int_y^\infty \frac{dt}{(t^2+b^2)(t^2+a^2)\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{g}{a^4}\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{sd}^2 u\, du.$$

[See **361.25.**]

222.13 $$\int_y^\infty \frac{(p_1 - t^2)\,dt}{(p - t^2)\sqrt{(t^2 + a^2)(b^2 + t^2)}} = g \int_0^{u_1} \frac{1 - \alpha_1^2 \operatorname{sn}^2 u}{1 - \alpha^2 \operatorname{sn}^2 u}\,du\,,$$ [See **340.01**.]

where $$\alpha_1^2 = (a^2 + p_1)/a^2, \qquad \alpha^2 = (a^2 + p)/a^2.$$

222.14 $$\int_y^\infty \frac{R(t^2)\,dt}{\sqrt{(t^2 + a^2)(t^2 + b^2)}} = g \int_0^{u_1} R(a^2 \operatorname{cs}^2 u)\,du\,,$$

where $R(t^2)$ is any rational function of t^2.

Integrands involving $\sqrt{t^2 + \varrho^2}$ and $\sqrt{t^2 + \bar{\varrho}^2}$; ϱ^2, $\bar{\varrho}^2$ conjugate complex numbers, $(0 \leqq y < \infty)$

$$\operatorname{cn} u = \frac{t^2 - \varrho\bar{\varrho}}{t^2 + \varrho\bar{\varrho}}, \qquad k^2 = \frac{-(\varrho - \bar{\varrho})^2}{4\varrho\bar{\varrho}}, \qquad g = \frac{1}{2\sqrt{\varrho\bar{\varrho}}},$$
$$\varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{y^2 - \varrho\bar{\varrho}}{y^2 + \varrho\bar{\varrho}}\right], \qquad \operatorname{cn} u_1 = \cos\varphi.$$

225.00 $$\left\{ \int_y^\infty \frac{dt}{\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}} = g \int_0^{u_1} du = g\,u_1 = g \operatorname{cn}^{-1}(\cos\varphi, k) \right. = g\,F(\cos\varphi, k)\,,$$ [cf. **263.00**].

225.01 $$\int_y^\infty \frac{\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}}{(t^2 + \varrho\bar{\varrho})^2}\,dt = g \int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi, k)\,.$$

225.02 $$\left\{ \int_y^\infty \frac{(t^2 + \varrho\bar{\varrho})^2\,dt}{[(t^2 + \varrho\bar{\varrho})^2 - 4\alpha^2\varrho\bar{\varrho}\,t^2]\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}} = g \int_0^{u_1} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u} \right. = g\,\Pi(\varphi, \alpha^2, k).$$ [See **400.**]

225.03 $$\int_y^\infty \frac{t^2\,dt}{(t^2 - \varrho\bar{\varrho})^2\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}} = \frac{g}{4\varrho\bar{\varrho}} \int_0^{u_1} \operatorname{tn}^2 u\,du\,.$$ [See **316.02.**]

225.04 $$\left\{ \int_y^\infty \frac{dt}{t^{2m}\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}} = \frac{g}{(\varrho\bar{\varrho})^m} \sum_{j=0}^{m} \frac{(-1)^{m+j}\,m!}{(m-j)!\,j!} \int_0^{u_1} \frac{du}{(1 + \operatorname{cn} u)^j}, \right. \quad y \neq 0.$$ [See **341.55.**]

225.05 $$\int_y^\infty \frac{t^2\,dt}{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)\sqrt{(t^2 + \varrho^2)(t^2 + \bar{\varrho}^2)}} = \frac{g}{4\varrho\bar{\varrho}} \int_0^{u_1} \operatorname{sd}^2 u\,du\,.$$ [See **318.02.**]

225.06 $$\int_y^\infty \frac{(t^2-\varrho\bar{\varrho})^2\,dt}{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)\sqrt{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)}} = g\int_0^{u_1} \operatorname{cd}^2 u\,du.$$ [See **320.02**.]

225.07 $$\int_y^\infty \frac{t^2\,dt}{(t^2+\varrho\bar{\varrho})^2\sqrt{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)}} = \frac{g}{4\varrho\bar{\varrho}}\int_0^{u_1} \operatorname{sn}^2 u\,du.$$ [See **310.02**.]

225.08 $$\int_y^\infty \frac{(t^2-\varrho\bar{\varrho})^2\,dt}{(t^2+\varrho\bar{\varrho})^2\sqrt{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)}} = g\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02**.]

225.09 $$\int_y^\infty \frac{R(t^2)\,dt}{\sqrt{(t^2+\varrho^2)(t^2+\bar{\varrho}^2)}} = g\int_0^{u_1} R\left[\varrho\bar{\varrho}\,\frac{1+\operatorname{cn} u}{1-\operatorname{cn} u}\right]du,$$

where $R(t^2)$ is a rational function of t^2.

Integrands Involving the Square Root of the Cubic, $\sqrt{a_0(t+r_1)(t+r_2)(t+r_3)}$.

Introduction.

We consider here the reduction of the integral[1]

230.00 $$\xi = \int_{y_1}^{y} \frac{R(t)}{\sqrt{P}}\,dt,$$

[1] On setting $t = t_1^2 - r_1$, (r_1 real), we may write

$$\int \frac{R(t)}{\sqrt{P}}\,dt = 2\int \frac{R_1(t_1^2)\,dt_1}{\sqrt{a_0(t_1^2+r_2-r_1)(t_1^2+r_3-r_1)}}.$$

The integrals in **230.00** can therefore be reduced to those of **211–225**.

One may occasionally encounter an integral in the form

$$\xi_1 = \int_{Y_1}^{Y} \frac{t\,R(t^2)\,dt}{\sqrt{a_0(t^2+r_1)(t^2+r_2)(t^2+r_3)}},$$

where the factors under the radical sign occur in even powers of t, with an odd power of t appearing outside either in the denominator or numerator. By applying the transformation $t^2 = \tau$, it is immediately seen that such integrals are equivalent to those given in **230.00**. In this case, we thus have

$$\xi_1 = \int_{y_1}^{y} \frac{R(\tau)\,d\tau}{2\sqrt{a_0(\tau+r_1)(\tau+r_2)(\tau+r_3)}}, \qquad (y = Y^2,\ y_1 = Y_1^2).$$

(cf. **575.10**).

where $P = a_0(t+r_1)(t+r_2)(t+r_3)$ is a polynomial whose three linear factors are all distinct.

By the method of partial fractions, the integral **230.00** may be expressed linearly in terms of the general integrals

$$\int_{y_1}^{y} \frac{t^m\,dt}{\sqrt{P}}, \qquad \int_{y_1}^{y} \frac{dt}{(t-p)^n\sqrt{P}}.$$

Now

230.01
$$\left\{\begin{aligned} \int_{y_1}^{y} \frac{t^m\,dt}{\sqrt{P}} &= \frac{1}{a_0(2m-1)}\Bigg\{2y^{m-2}\sqrt{a_0(y+r_1)(y+r_2)(y+r_3)} - \\ &\quad -2y_1^{m-2}\sqrt{a_0(y_1+r_1)(y_1+r_2)(y_1+r_3)} + \\ &\quad +2a_0(1-m)(r_1+r_2+r_3)\int_{y_1}^{y}\frac{t^{m-1}\,dt}{\sqrt{P}} + \\ &\quad +a_0(3-2m)(r_1r_2+r_1r_3+r_2r_3)\int_{y_1}^{y}\frac{t^{m-2}\,dt}{\sqrt{P}} + \\ &\quad +2a_0(2-m)\,r_1r_2r_3\int_{y_1}^{y}\frac{t^{m-3}\,dt}{\sqrt{P}}\Bigg\}; \end{aligned}\right.$$

and if p is not a zero of $P(t)$,

230.02
$$\left\{\begin{aligned} \int_{y_1}^{y} \frac{dt}{(t-p)^m\sqrt{P}} &= \frac{1}{2(m-1)\,P(p)}\Bigg\{\frac{2\sqrt{a_0(y_1+r_1)(y_1+r_2)(y_1+r_3)}}{(y_1-p)^{m-1}} - \\ &\quad -\frac{2\sqrt{a_0(y+r_1)(y+r_2)(y+r_3)}}{(y-p)^{m-1}} + (5-2m)\,a_0\int_{y_1}^{y}\frac{dt}{(t-p)^{m-3}\sqrt{P}} + \\ &\quad +2(2-m)(3p+r_1+r_2+r_3)\int_{y_1}^{y}\frac{dt}{(t-p)^{m-2}\sqrt{P}} + \\ &\quad +a_0(3-2m)\,[3p^2+2p(r_1+r_2+r_3)+r_1r_2+r_1r_3+r_2r_3]\times \\ &\quad \times\int_{y_1}^{y}\frac{dt}{(t-p)^{m-1}\sqrt{P}}\Bigg\}, \quad [m \neq 1]. \end{aligned}\right.$$

When $P(p) = 0$,

230.03
$$\left\{\begin{aligned}
&\int_{y_1}^{y}\frac{dt}{(t-p)^m\sqrt{P}}=\frac{1}{(2m-1)[3p^2+2p(r_1+r_2+r_3)+r_1r_2+r_1r_3+r_2r_3]}\times\\
&\times\Big\{2\sqrt{a_0(y_1+r_1)(y_1+r_2)(y_1+r_3)}-2\sqrt{a_0(y+r_1)(y+r_2)(y+r_3)}+\\
&+(3-2m)\,a_0\int_{y_1}^{y}\frac{dt}{(t-p)^{m-2}\sqrt{P}}+2a_0(1-m)(3p+r_1+r_2+r_3)\times\\
&\times\int_{y_1}^{y}\frac{dt}{(t-p)^{m-1}\sqrt{P}}\Big\}.
\end{aligned}\right.$$

Every elliptic integral of the form **230.00** thus depends on the three basic integrals $\int dt/\sqrt{P}$, $\int t\,dt/\sqrt{P}$ and $\int dt/(t-p)\sqrt{P}$. The integral $\int dt/\sqrt{P}$ is finite for all values of t and is always an integral of the first kind. An integral of the type $\int t\,dt/\sqrt{P}$ possesses an algebraic infinity at $t=\infty$ and leads to an integral of the second kind, while $\int dt/(t-p)\sqrt{P}$ is of the third kind, becoming logarithmically infinite at the point $t=p$ as $\pm[\ln(t-p)]/\sqrt{P(p)}$. [In case $P(p)=0$, this latter integral is of the second kind and is algebraically infinite at $t=p$.]

Considering a, b, c, a_1 and b_1 real, one may write the radicand $P(t)$ in **230.00** in one of the following ways:

$$|a_0|(t-a)(t-b)(t-c);\quad |a_0|(a-t)(t-b)(t-c);\quad |a_0|(a-t)(b-t)(t-c);$$
$$|a_0|(t-a)[(t-a_1)^2+b_1^2];\quad |a_0|(a-t)[(t-a_1)^2+b_1^2];\quad (a>b>c).$$

The roots of the equation $P=0$ are all real in the first three cases; but in the last two cases, two of the roots are complex.

Reduction to Jacobian normal form for the cases when the zeros of the radicand are all real is accomplished by means of substitutions of the type

230.04
$$t=\frac{A_1+A_2\operatorname{sn}^2u}{A_3+A_4\operatorname{sn}^2u},\quad (0\le u\le K),$$

where A_1, A_2, A_3 and A_4 are real constants chosen so that $dt/\sqrt{P}=g\,du$ and g is some real constant. If two roots of the equation $P=0$ are complex, the transformation

230.05
$$t=\frac{A_1+A_2\operatorname{cn}u}{A_3+A_4\operatorname{cn}u},\quad (0\le u\le 2K),$$

is employed.

Table of Integrals.

The following table of integrals gives appropriate substitutions and carries out the reduction for the various cases of integrands involving the square root of three linear factors. As in the previous section, one of the limits of integration will usually be taken as a zero of the polynomial under the radical sign, while the other limit is considered variable. The tables, however, may easily be used when neither limit is fixed. (See, for example, page 2 of the Introduction.)

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$ and $\sqrt{c-t}$, $(a > b > c \geq y)$

$$\boxed{\begin{array}{l} \operatorname{sn}^2 u = \dfrac{a-c}{a-t}, \quad k^2 = \dfrac{a-b}{a-c}, \quad g = \dfrac{2}{\sqrt{a-c}}, \\ \varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\dfrac{a-c}{a-y}}, \quad \operatorname{sn} u_1 = \sin\varphi. \end{array}}$$

231.00 $$\begin{cases} \displaystyle\int_{-\infty}^{y} \frac{dt}{\sqrt{(a-t)(b-t)(c-t)}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\varphi, k) \\ \qquad = g\,F(\varphi, k), \qquad \text{[cf. } \mathbf{232.00}.] \end{cases}$$

231.01 $$\int_{-\infty}^{y} \frac{dt}{a-t}\sqrt{\frac{b-t}{(c-t)(a-t)}} = g\int_0^{u_1} \operatorname{dn}^2 u\, du = g\,E(u_1) = g\,E(\varphi, k).$$

231.02 $$\int_{-\infty}^{y} \frac{dt}{p-t}\sqrt{\frac{a-t}{(b-t)(c-t)}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\varphi, \alpha^2, k), \qquad \text{[See } \mathbf{400}.]$$

where $$\alpha^2 = (a-p)/(a-c).$$

231.03 $$\int_{-\infty}^{y} \frac{dt}{b-t}\sqrt{\frac{a-t}{(c-t)(b-t)}} = g\int_0^{u_1} \operatorname{nd}^2 u\, du. \qquad \text{[See } \mathbf{315.02}.]$$

231.04 $$\int_{-\infty}^{y} \frac{dt}{c-t}\sqrt{\frac{b-t}{(c-t)(a-t)}} = g\int_0^{u_1} \operatorname{dc}^2 u\, du, \quad y \neq c. \qquad \text{[See } \mathbf{321.02}.]$$

231.05 $$\int_{-\infty}^{y} \frac{dt}{c-t}\sqrt{\frac{a-t}{(b-t)(c-t)}} = g\int_0^{u_1} \operatorname{nc}^2 u\, du. \qquad \text{[See } \mathbf{313.02}.]$$

231.06 $$\int_{-\infty}^{y} \frac{dt}{b-t}\sqrt{\frac{c-t}{(a-t)(b-t)}} = g\int_0^{u_1} \operatorname{cd}^2 u\, du. \qquad \text{[See } \mathbf{320.02}.]$$

231.07 $$\int_{-\infty}^{y}\frac{dt}{a-t}\sqrt{\frac{c-t}{(a-t)(b-t)}}=g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

231.08 $$\int_{-\infty}^{y}\frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^m}\int_0^{u_1}\mathrm{sn}^{2m} u\,du.$$ [See **310.05**.]

231.09 $$\int_{-\infty}^{y}\frac{dt}{(b-t)^m\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^m}\int_0^{u_1}\mathrm{sd}^{2m} u\,du.$$ [See **318.05**.]

231.10 $$\int_{-\infty}^{y}\frac{dt}{(c-t)^m\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^m}\int_0^{u_1}\mathrm{tn}^{2m} u\,du,\quad y\neq c.$$ [See **316.05**.]

231.11 $$\int_{-\infty}^{y}\frac{dt}{(a-t)(b-t)\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^2}\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{sd}^2 u\,du,$$ $y\neq c$. [See **361.25**.]

231.12 $$\int_{-\infty}^{y}\frac{dt}{(a-t)(c-t)\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^2}\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{tn}^2 u\,du,$$ $y\neq c$. [See **361.29**.]

231.13 $$\int_{-\infty}^{y}\frac{dt}{(b-t)(c-t)\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^2}\int_0^{u_1}\mathrm{sd}^2 u\,\mathrm{tn}^2 u\,du,$$ $y\neq c$. [See **361.24**.]

231.14 $$\int_{-\infty}^{y}\frac{dt}{(a-t)(b-t)(c-t)\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^3}\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{sd}^2 u\,\mathrm{tn}^2 u\,du,\quad y\neq c.$$ [See **361.30**.]

231.15 $$\int_{-\infty}^{y}\frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(c-t)}}=\frac{g}{(a-c)^m}\int_0^{u_1}\frac{\mathrm{sn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **337.04**.]

where $$\alpha^2=(a-p)/(a-c).$$

231.16 $$\int_{-\infty}^{y}\frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(c-t)}}=g\int_0^{u_1}\frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$ [See **340.01**.]

where $$\alpha_1^2=(a-p_1)/(a-c),\qquad \alpha^2=(a-p)/(a-c).$$

231.17 $$\int_{-\infty}^{y} \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(c-t)}} = g\int_0^{u_1} R\,[a-(a-c)\,\mathrm{ns}^2 u]\,du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$ and $\sqrt{c-t}$, $(a > b > c > y)$

$$\boxed{\begin{array}{lll} \mathrm{sn}^2 u = \dfrac{c-t}{b-t}, & k^2 = \dfrac{a-b}{a-c}, & g = \dfrac{2}{\sqrt{a-c}}, \\ \varphi = \mathrm{am}\,u_1 = \sin^{-1}\sqrt{\dfrac{c-y}{b-y}}, & & \mathrm{sn}\,u_1 = \sin\varphi. \end{array}}$$

232.00 $$\left\{\begin{aligned} \int_y^c \frac{dt}{\sqrt{(a-t)(b-t)(c-t)}} &= g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k) \\ &= g\,F(\varphi, k), \qquad \text{[cf. } \mathbf{231.00}\text{]}. \end{aligned}\right.$$

232.01 $$\left\{\begin{aligned} \int_y^c \frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(c-t)}} &= \frac{g}{k'^2}\int_0^{u_1} \mathrm{dn}^2 u\,du = \frac{g}{k'^2}E(u_1) \\ &= \frac{g}{k'^2}E(\varphi, k). \end{aligned}\right.$$

232.02 $$\left\{\begin{aligned} \int_y^c \frac{dt}{p-t}\sqrt{\frac{b-t}{(a-t)(c-t)}} &= \frac{b-c}{p-c}\,g\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ &= \frac{b-c}{p-c}\,g\,\Pi(\varphi, \alpha^2, k), \qquad \text{[See } \mathbf{400}.\text{]} \end{aligned}\right.$$

where $$\alpha^2 = (p-b)/(p-c), \qquad p \neq c.$$

232.03 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{c-t}{(a-t)(b-t)}} = \frac{b-c}{a-c}\,g\int_0^{u_1} \mathrm{sd}^2 u\,du. \qquad \text{[See } \mathbf{318.02}.\text{]}$$

232.04 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{c-t}{(a-t)(b-t)}} = g\int_0^{u_1} \mathrm{sn}^2 u\,du. \qquad \text{[See } \mathbf{310.02}.\text{]}$$

232.05 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(c-t)}} = k'^2 g\int_0^{u_1} \mathrm{nd}^2 u\,du. \qquad \text{[See } \mathbf{315.02}.\text{]}$$

232.06 $$\int_y^c \sqrt{\frac{a-t}{(b-t)(c-t)}}\,dt = (a-c)\,g\int_0^{u_1} \mathrm{dc}^2 u\,du. \qquad \text{[See } \mathbf{321.02}.\text{]}$$

232.07 $$\int_y^c \sqrt{\frac{b-t}{(a-t)(c-t)}}\,dt = (b-c)\,g\int_0^{u_1} \mathrm{nc}^2 u\,du.$$ [See **313.02**.]

232.08 $$\int_y^c \sqrt{\frac{c-t}{(a-t)(b-t)}}\,dt = (b-c)\,g\int_0^{u_1} \mathrm{tn}^2 u\,du.$$ [See **316.02**.]

232.09 $$\int_y^c \sqrt{\frac{(a-t)(b-t)}{c-t}}\,dt = (a-c)(b-c)\,g\int_0^{u_1} \mathrm{nc}^2 u\,\mathrm{dc}^2 u\,du.$$ [See **361.13**.]

232.10 $$\int_y^c \sqrt{\frac{(a-t)(c-t)}{b-t}}\,dt = (a-c)(b-c)\,g\int_0^{u_1} \mathrm{tn}^2 u\,\mathrm{dc}^2 u\,du.$$ [See **361.15**.]

232.11 $$\int_y^c \sqrt{\frac{(b-t)(c-t)}{a-t}}\,dt = (b-c)^2\,g\int_0^{u_1} \mathrm{tn}^2 u\,\mathrm{nc}^2 u\,du.$$ [See **361.07**.]

232.12 $$\left\{\begin{aligned} &\int_y^c \sqrt{(a-t)(b-t)(c-t)}\,dt \\ &\qquad = (a-c)(b-c)^2\,g\int_0^{u_1} \mathrm{tn}^2 u\,\mathrm{dc}^2 u\,\mathrm{nc}^2 u\,du \end{aligned}\right.$$ [See **361.17**.]

232.13 $$\int_y^c \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(c-t)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \mathrm{cd}^{2m} u\,du.$$ [See **320.05**.]

232.14 $$\int_y^c \frac{dt}{(b-t)^m\sqrt{(a-t)(b-t)(c-t)}} = \frac{g}{(b-c)^m}\int_0^{u_1} \mathrm{cn}^{2m} u\,du.$$ [See **312.05**.]

232.15 $$\left\{\begin{aligned} &\int_y^c \frac{dt}{(b-t)(a-t)\sqrt{(a-t)(b-t)(c-t)}} \\ &\qquad = \frac{g}{(a-c)(b-c)}\int_0^{u_1} \mathrm{cn}^2 u\,\mathrm{cd}^2 u\,du. \end{aligned}\right.$$ [See **361.28**.]

232.16 $$\int_y^c \frac{dt}{t^m\sqrt{(a-t)(b-t)(c-t)}} = \frac{g}{c^m}\int_0^{u_1} \frac{\mathrm{cn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **338.04**.]

where $$\alpha^2 = b/c.$$

232.17 $$\int_y^c \frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(c-t)}} = \frac{g}{(p-c)^m}\int_0^{u_1} \frac{\mathrm{cn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **338.04**.]

where $$\alpha^2 = (p-b)/(p-c), \qquad p \neq c.$$

232.18 $$\int_y^c \frac{(p_1 - t)\,dt}{(p - t)\sqrt{(a-t)(b-t)(c-t)}} = \frac{p_1 - c}{p - c}\, g \int_0^{u_1} \frac{1 - \alpha_1^2 \operatorname{sn}^2 u}{1 - \alpha^2 \operatorname{sn}^2 u}\, du,$$ [See **340.01**.]

where $\alpha_1^2 = (p_1 - b)/(p_1 - c)$; $\alpha^2 = (p - b)/(p - c)$, $p \neq c$.

232.19 $$\int_y^c \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(c-t)}} = g \int_0^{u_1} R\left[\frac{c - b \operatorname{sn}^2 u}{\operatorname{cn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$ and $\sqrt{t-c}$, $(a > b \geqq y > c)$

$$\operatorname{sn}^2 u = \frac{t-c}{b-c}, \quad k^2 = \frac{b-c}{a-c}, \quad g = \frac{2}{\sqrt{a-c}},$$
$$\varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{y-c}{b-c}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

233.00 $$\int_c^y \frac{dt}{\sqrt{(a-t)(b-t)(t-c)}} = g\int_0^{u_1} du = g\, u_1 = g \operatorname{sn}^{-1}(\sin\varphi, k) = g\, F(\varphi, k),$$ [cf. **234.00**].

233.01 $$\int_c^y \sqrt{\frac{a-t}{(b-t)(t-c)}}\, dt = (a-c)\, g \int_0^{u_1} \operatorname{dn}^2 u\, du = (a-c)\, g\, E(u_1) = (a-c)\, g\, E(\varphi, k).$$

233.02 $$\int_c^y \frac{dt}{(p-t)\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{p-c}\int_0^{u_1} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u} = \frac{g}{p-c}\, \Pi(\varphi, \alpha^2, k),$$ [See **400**.]

where $\alpha^2 = (b-c)/(p-c)$, $p \neq c$.

233.03 $$\int_c^y \sqrt{\frac{t-c}{(a-t)(b-t)}}\, dt = (b-c)\, g \int_0^{u_1} \operatorname{sn}^2 u\, du.$$ [See **310.02**.]

233.04 $$\int_c^y \sqrt{\frac{b-t}{(a-t)(t-c)}}\, dt = (b-c)\, g \int_0^{u_1} \operatorname{cn}^2 u\, du.$$ [See **312.02**.]

233.05 $$\int_c^y \sqrt{\frac{(a-t)(t-c)}{b-t}}\, dt = (a-c)(b-c)\, g \int_0^{u_1} \operatorname{sn}^2 u \operatorname{dn}^2 u\, du.$$ [See **361.02**.]

233.06 $$\int_c^y \sqrt{\frac{(t-c)(b-t)}{a-t}}\,dt = (b-c)^2 g\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du.$$ [See **361.01**.]

233.07 $$\int_c^y \sqrt{\frac{(a-t)(b-t)}{t-c}}\,dt = (b-c)(a-c)\,g\int_0^{u_1} \mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du.$$ [See **361.03**.]

233.08 $$\begin{cases}\displaystyle\int_c^y \sqrt{(a-t)(b-t)(t-c)}\,dt \\ \displaystyle\qquad = (a-c)(b-c)^2 g\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du.\end{cases}$$ [See **361.04**.]

233.09 $$\int_c^y \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \mathrm{nd}^{2m} u\,du.$$ [See **315.05**.]

233.10 $$\begin{cases}\displaystyle\int_c^y \frac{dt}{(b-t)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(b-c)^m}\int_0^{u_1} \mathrm{nc}^{2m} u\,du, \\ \qquad y\neq b.\end{cases}$$ [See **313.05**.]

233.11 $$\begin{cases}\displaystyle\int_c^y \frac{dt}{(a-t)(b-t)\sqrt{(a-t)(b-t)(t-c)}} \\ \displaystyle\qquad = \frac{g}{(a-c)(b-c)}\int_0^{u_1} \mathrm{nc}^2 u\,\mathrm{nd}^2 u\,du, \qquad y\neq b.\end{cases}$$ [See **361.12**.]

233.12 $$\int_c^y \frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(p-c)^m}\int_0^{u_1} \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **336.03**.]

where $$\alpha^2 = (b-c)/(p-c), \qquad p\neq c.$$

233.13 $$\int_c^y \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(t-c)}} = \frac{b-c}{a-c}\,g\int_0^{u_1} \mathrm{cd}^2 u\,du.$$ [See **320.02**.]

233.14 $$\int_c^y \frac{dt}{a-t}\sqrt{\frac{t-c}{(a-t)(b-t)}} = \frac{b-c}{a-c}\,g\int_0^{u_1} \mathrm{sd}^2 u\,du.$$ [See **318.02**.]

233.15 $$\int_c^y \frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(t-c)}} = \frac{a-c}{b-c}\,g\int_0^{u_1} \mathrm{dc}^2 u\,du, \qquad y\neq b.$$ [See **321.02**.]

233.16 $$\int_c^y \frac{dt}{b-t}\sqrt{\frac{t-c}{(a-t)(b-t)}} = g\int_0^{u_1} \mathrm{tn}^2 u\, du, \quad y \neq b. \quad \text{[See } \mathbf{316.02}.]$$

233.17 $$\int_c^y \frac{t^m\, dt}{\sqrt{(a-t)(b-t)(t-c)}} = c^m g \int_0^{u_1} \left[1 - \frac{c-b}{c}\,\mathrm{sn}^2 u\right]^m du.$$
[See **331.03**.]

233.18 $$\int_c^y \frac{dt}{t^m \sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{c^m}\int_0^{u_1} \frac{du}{\left[1 - \frac{c-b}{c}\,\mathrm{sn}^2 u\right]^m}.$$
[See **336.03**.]

233.19 $$\int_c^y \frac{(p_1-t)\, dt}{(p-t)\sqrt{(a-t)(b-t)(t-c)}} = \frac{p_1-c}{p-c}\, g \int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\, du,$$
[See **340.01**.]

where $\alpha_1^2 = (b-c)/(p_1-c), \qquad \alpha^2 = (b-c)/(p-c), \qquad p \neq c.$

233.20 $$\int_c^y \frac{R(t)\, dt}{\sqrt{(a-t)(b-t)(t-c)}} = g\int_0^{u_1} R\left[c - (c-b)\,\mathrm{sn}^2 u\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$ and $\sqrt{t-c}$, $(a > b > y \geqq c)$

$$\boxed{\begin{array}{l} \mathrm{sn}^2 u = \dfrac{(a-c)(b-t)}{(b-c)(a-t)}, \quad k^2 = \dfrac{b-c}{a-c}, \quad g = \dfrac{2}{\sqrt{a-c}}, \\ \varphi = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\dfrac{(a-c)(b-y)}{(b-c)(a-y)}}, \quad \mathrm{sn}\, u_1 = \sin\varphi. \end{array}}$$

234.00 $$\left\{ \begin{aligned} \int_y^b \frac{dt}{\sqrt{(a-t)(b-t)(t-c)}} &= g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k) \\ &= g\,F(\varphi, k), \qquad \text{[cf. } \mathbf{233.00}]. \end{aligned} \right.$$

234.01 $$\left\{ \begin{aligned} \int_y^b \frac{dt}{(a-t)\sqrt{(a-t)(b-t)(t-c)}} &= \frac{g}{a-b}\int_0^{u_1} \mathrm{dn}^2 u\, du \\ &= \frac{g}{a-b} E(u_1) = \frac{g}{a-b} E(\varphi, k). \end{aligned} \right.$$

234.02
$$\left\{\begin{aligned}\int_y^b \frac{dt}{p-t}\sqrt{\frac{a-t}{(b-t)(t-c)}} &= \frac{a-b}{p-b}\,g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u}\\ &= \frac{a-b}{p-b}\,g\,\Pi(\varphi,\alpha^2,k),\qquad \text{[See 400.]}\end{aligned}\right.$$

where $\alpha^2 = k^2(p-a)/(p-b(,\quad p\neq b.$

234.03
$$\left\{\begin{aligned}&\int_y^b \frac{dt}{(a-t)(t-c)\sqrt{(a-t)(b-t)(t-c)}}\\ &\quad=\frac{g}{(a-b)(b-c)}\int_0^{u_1}\operatorname{dn}^2 u\operatorname{dc}^2 u\,du,\quad y\neq c.\qquad \text{[See 361.31.]}\end{aligned}\right.$$

234.04
$$\int_y^b \frac{dt}{(t-c)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(b-c)^m}\int_0^{u_1}\operatorname{dc}^{2m} u\,du,\ y\neq c.\qquad \text{[See 321.05.]}$$

234.05
$$\int_y^b \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(a-b)^m}\int_0^{u_1}\operatorname{dn}^{2m} u\,du.\qquad \text{[See 314.05.]}$$

234.06
$$\int_y^b \sqrt{\frac{a-t}{(b-t)(t-c)}}\,dt = (a-b)\,g\int_0^{u_1}\operatorname{nd}^2 u\,du.\qquad \text{[See 315.02.]}$$

234.07
$$\int_y^b \sqrt{\frac{b-t}{(a-t)(t-c)}}\,dt = k'^2(b-c)\,g\int_0^{u_1}\operatorname{sd}^2 u\,du.\qquad \text{[See 318.02.]}$$

234.08
$$\int_y^b \sqrt{\frac{t-c}{(a-t)(b-t)}}\,dt = (b-c)\,g\int_0^{u_1}\operatorname{cd}^2 u\,du.\qquad \text{[See 320.02.]}$$

234.09
$$\int_y^b \sqrt{\frac{(a-t)(b-t)}{t-c}}\,dt = (b-c)(a-b)\,k'^2 g\int_0^{u_1}\operatorname{sd}^2 u\operatorname{nd}^2 u\,du.\qquad \text{[See 361.19.]}$$

234.10
$$\int_y^b \sqrt{\frac{(a-t)(t-c)}{b-t}}\,dt = (a-b)(b-c)\,g\int_0^{u_1}\operatorname{cd}^2 u\operatorname{nd}^2 u\,du.\qquad \text{[See 361.16.]}$$

234.11
$$\int_y^b \sqrt{\frac{(t-c)(b-t)}{a-t}}\,dt = (b-c)^2 k'^2 g\int_0^{u_1}\operatorname{sd}^2 u\operatorname{cd}^2 u\,du.\qquad \text{[See 361.27.]}$$

234.12
$$\begin{cases} \int\limits_y^b \sqrt{(a-t)(b-t)(c-t)}\,dt \\ = (a-b)(b-c)^2 k'^2 g \int\limits_0^{u_1} \mathrm{nd}^2 u\, \mathrm{sd}^2 u\, \mathrm{cd}^2 u\, du. \end{cases}$$
[See **361.18**.]

234.13
$$\int\limits_y^b \frac{dt}{t-c}\sqrt{\frac{a-t}{(b-t)(t-c)}}\,dt = \frac{k'^2 g}{k^2}\int\limits_0^{u_1} \mathrm{nc}^2 u\, du, \quad y \neq c.$$
[See **313.02**.]

234.14
$$\int\limits_y^b \frac{dt}{t-c}\sqrt{\frac{b-t}{(a-t)(t-c)}}\,dt = k'^2 g\int\limits_0^{u_1} \mathrm{tn}^2 u\, du, \quad y \neq c.$$
[See **316.02**.]

234.15
$$\int\limits_y^b \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(t-c)}}\,dt = k^2 g\int\limits_0^{u_1} \mathrm{sn}^2 u\, du.$$
[See **310.02**.]

234.16
$$\int\limits_y^b \frac{t^m\, dt}{\sqrt{(a-t)(b-t)(t-c)}}\,dt = g\, b^m \int\limits_0^{u_1} \frac{\left(1-\frac{a k^2}{b}\mathrm{sn}^2 u\right)^m}{(1-k^2 \mathrm{sn}^2 u)^m}\,du.$$
[See **340.04**.]

234.17
$$\int\limits_y^b \frac{dt}{t^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{b^m}\int\limits_0^{u_1} \frac{\mathrm{dn}^{2m} u\, du}{\left(1-\frac{a k^2}{b}\mathrm{sn}^2 u\right)^m}.$$
[See **339.04**.]

234.18
$$\int\limits_y^b \frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(t-c)}} = \frac{g}{(p-b)^m}\int\limits_0^{u_1} \frac{\mathrm{dn}^{2m} u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^m},$$
[See **339.04**.]

where $\alpha^2 = k^2(p-a)/(p-b), \quad p \neq b.$

234.19
$$\int\limits_y^b \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(t-c)}} = \frac{(p_1-b)}{p-b}\, g\int\limits_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\,du,$$
[See **340.01**.]

where

$$\alpha_1^2 = k^2(p_1-a)/(p_1-b), \qquad \alpha^2 = k^2(p-a)/(p-b), \quad p \neq b.$$

234.20
$$\int\limits_y^b \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(t-c)}} = g\int\limits_0^{u_1} R\left[\frac{b-a k^2 \mathrm{sn}^2 u}{\mathrm{dn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$ and $\sqrt{t-c}$, $(a \geqq y > b > c)$

$$\operatorname{sn}^2 u = \frac{(a-c)(t-b)}{(a-b)(t-c)}, \quad k^2 = \frac{a-b}{a-c}, \quad g = \frac{2}{\sqrt{a-c}}$$
$$\varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{(a-c)(y-b)}{(a-b)(y-c)}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

235.00 $$\int_b^y \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} = g\int_0^{u_1} du = g u_1 = g \operatorname{sn}^{-1}(\sin\varphi, k) = g F(\varphi, k). \quad \text{[cf. } \mathbf{236.00}\text{]}.$$

235.01 $$\int_b^y \frac{dt}{(t-c)\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{b-c}\int_0^{u_1} \operatorname{dn}^2 u\, du = \frac{g}{b-c} E(u_1) = \frac{g}{b-c} E(\varphi, k).$$

235.02 $$\int_b^y \frac{dt}{t-p}\sqrt{\frac{t-c}{(a-t)(t-b)}} = \frac{b-c}{b-p}\, g\int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{b-c}{b-p}\, g\, \Pi(\varphi, \alpha^2, k), \quad \text{[See } \mathbf{400}\text{.]}$$

where $\alpha^2 = (a-b)(c-p)/(a-c)(b-p), \quad p \neq b.$

235.03 $$\int_b^y \frac{dt}{t-c}\sqrt{\frac{t-b}{(t-c)(a-t)}} = g k^2 \int_0^{u_1} \operatorname{sn}^2 u\, du. \quad \text{[See } \mathbf{310.02}\text{.]}$$

235.04 $$\int_b^y \frac{dt}{(a-t)^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{(a-b)^m}\int_0^{u_1} \operatorname{dc}^{2m} u\, du, \quad y \neq a. \quad \text{[See } \mathbf{321.05}\text{.]}$$

235.05 $$\int_b^y \sqrt{\frac{t-c}{(a-t)(t-b)}}\, dt = (b-c)\, g\int_0^{u_1} \operatorname{nd}^2 u\, du. \quad \text{[See } \mathbf{315.02}\text{.]}$$

235.06 $$\int_b^y \sqrt{\frac{t-b}{(a-t)(t-c)}}\, dt = (b-c)\, g k^2 \int_0^{u_1} \operatorname{sd}^2 u\, du. \quad \text{[See } \mathbf{318.02}\text{.]}$$

235.07 $$\int_b^y \sqrt{\frac{a-t}{(t-b)(t-c)}}\, dt = (a-b)\, g\int_0^{u_1} \operatorname{cd}^2 u\, du. \quad \text{[See } \mathbf{320.02}\text{.]}$$

235.08 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-b}{(a-t)(t-c)}} = g k'^2 \int_0^{u_1} \operatorname{tn}^2 u\, du, \quad y \neq a, \quad \text{[See } \mathbf{316.02}\text{.]}$$

235.09 $$\int_b^y \sqrt{\frac{(a-t)(t-b)}{t-c}}\,dt = (a-b)(b-c)\,g\,k^2\int_0^{u_1} \mathrm{sd}^2 u\,\mathrm{cd}^2 u\,du.$$ [See **361.27**.]

235.10 $$\int_b^y \sqrt{\frac{(t-b)(t-c)}{a-t}}\,dt = (b-c)^2\,g\,k^2\int_0^{u_1} \mathrm{sd}^2 u\,\mathrm{nd}^2 u\,du.$$ [See **361.19**.]

235.11 $$\int_b^y \sqrt{\frac{(a-t)(t-c)}{t-b}}\,dt = (a-b)(b-c)\,g\int_0^{u_1} \mathrm{cd}^2 u\,\mathrm{nd}^2 u\,du.$$ [See **361.16**.]

235.12 $$\int_b^y \frac{dt}{t-c}\sqrt{\frac{a-t}{(t-b)(t-c)}} = \frac{a-b}{b-c}\,g\int_0^{u_1} \mathrm{cn}^2 u\,du.$$ [See **312.02**.]

235.13 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-c}{(t-b)(a-t)}} = \frac{b-c}{a-b}\,g\int_0^{u_1} \mathrm{nc}^2 u\,du, \quad y \neq a.$$ [See **313.02**.]

235.14 $$\int_b^y \sqrt{(a-t)(t-b)(t-c)}\,dt = g(a-b)(b-c)^2 k^2\int_0^{u_1} \mathrm{sd}^2 u\,\mathrm{cd}^2 u\,\mathrm{nd}^2 u\,du.$$ [See **361.18**.]

235.15 $$\left\{\begin{aligned} &\int_b^y \frac{dt}{(a-t)(t-c)\sqrt{(a-t)(t-b)(t-c)}} \\ &\qquad = \frac{g}{(a-b)(b-c)}\int_0^{u_1} \mathrm{dn}^2 u\,\mathrm{dc}^2 u\,du, \quad y \neq a. \end{aligned}\right.$$ [See **361.31**.]

235.16 $$\int_b^y \frac{t^m\,dt}{\sqrt{(a-t)(t-b)(t-c)}} = g\,b^m\int_0^{u_1} \frac{\left(1-\frac{c\,k^2}{b}\,\mathrm{sn}^2 u\right)^m}{(1-k^2\,\mathrm{sn}^2 u)^m}\,du.$$ [See **340.04**.]

235.17 $$\int_b^y \frac{dt}{(t-p)^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{(b-p)^m}\int_0^{u_1} \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$ [See **339.04**.]

where $\alpha^2 = (a-b)(c-p)/(a-c)(b-p), \quad p \neq b.$

235.18 $$\int_b^y \frac{(t-p_1)\,dt}{(t-p)\sqrt{(a-t)(t-b)(t-c)}} = \frac{(b-p_1)\,g}{b-p}\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$

where [See **340.01**.]

$$\alpha_1^2 = (a-b)(c-p_1)/(a-c)(b-p_1),$$
$$\alpha^2 = (a-b)(c-p)/(a-c)(b-p), \quad p \neq b.$$

235.19 $$\int_b^y \frac{dt}{t^m \sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{b^m} \int_0^{u_1} \frac{\mathrm{dn}^{2m} u\, du}{\left(1 - \frac{c k^2}{b} \mathrm{sn}^2 u\right)^m},$$

[See **339.04**.]

235.20 $$\int_b^y \frac{R(t)\, dt}{\sqrt{(a-t)(t-b)(t-c)}} = g \int_0^{u_1} R\left[\frac{b - c k^2 \mathrm{sn}^2 u}{\mathrm{dn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$ and $\sqrt{t-c}$, $(a > y \geq b > c)$

$$\mathrm{sn}^2 u = \frac{a-t}{a-b}, \quad k^2 = \frac{a-b}{a-c}, \quad g = \frac{2}{\sqrt{a-c}},$$
$$\psi = \mathrm{am}\, u_1 = \sin^{-1} \sqrt{\frac{a-y}{a-b}}, \quad \mathrm{sn}\, u_1 = \sin\psi.$$

236.00 $$\left\{\begin{aligned} \int_y^a \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} &= g \int_0^{u_1} du = g\, u_1 = g\, \mathrm{sn}^{-1}(\sin\psi, k) \\ &= g F(\psi, k). \end{aligned}\right.$$ [cf. **235.00**].

236.01 $$\left\{\begin{aligned} \int_y^a \sqrt{\frac{t-c}{(a-t)(t-b)}}\, dt &= (a-c)\, g \int_0^{u_1} \mathrm{dn}^2 u\, du = (a-c)\, g\, E(u_1) \\ &= (a-c)\, g\, E(\psi, k). \end{aligned}\right.$$

236.02 $$\left\{\begin{aligned} \int_y^a \frac{dt}{(t-p)\sqrt{(a-t)(t-b)(t-c)}} &= \frac{g}{a-p} \int_0^{u_1} \frac{du}{1 - \alpha^2 \mathrm{sn}^2 u} \\ &= \frac{g}{a-p} \Pi(\psi, \alpha^2, k), \end{aligned}\right.$$ [See **400**.]

where $$\alpha^2 = (a-b)/(a-p), \quad p \neq a.$$

236.03 $$\int_y^a \sqrt{\frac{t-b}{(a-t)(t-c)}}\, dt = (a-b)\, g \int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02**.]

236.04 $$\int_y^a \sqrt{\frac{a-t}{(t-b)(t-c)}}\, dt = (a-b)\, g \int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02**.]

236.05 $$\int_y^a \sqrt{\frac{(a-t)(t-b)}{t-c}}\, dt = (a-b)^2 g \int_0^{u_1} \mathrm{sn}^2 u\, \mathrm{cn}^2 u\, du.$$ [See **361.01**.]

236.06 $$\int_y^a \sqrt{\frac{(a-t)(t-c)}{t-b}}\,dt = (a-b)(a-c)\,g\int_0^{u_1} \mathrm{sn}^2 u\, \mathrm{dn}^2 u\, du.$$ [See **361.02**.]

236.07 $$\int_y^a \sqrt{\frac{(t-b)(t-c)}{a-t}}\,dt = (a-b)(a-c)\,g\int_0^{u_1} \mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du.$$ [See **361.03**.]

236.08 $$\int_y^a \sqrt{(a-t)(t-b)(t-c)}\,dt = (a-b)^2(a-c)\,g\int_0^{u_1} \mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du.$$ [See **361.04**.]

236.09 $$\int_y^a \frac{dt}{(t-b)^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{(a-b)^m}\int_0^{u_1} \mathrm{nc}^{2m} u\, du,\quad y \neq b.$$ [See **313.05**.]

236.10 $$\int_y^a \frac{dt}{(t-c)^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \mathrm{nd}^{2m} u\, du.$$ [See **315.05**.]

236.11 $$\int_y^a \frac{dt}{t-b}\sqrt{\frac{a-t}{(t-b)(t-c)}} = g\int_0^{u_1} \mathrm{tn}^2 u\, du,\quad y \neq b.$$ [See **316.02**.]

236.12 $$\int_y^a \frac{dt}{t-c}\sqrt{\frac{a-t}{(t-b)(t-c)}} = g k^2\int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02**.]

236.13 $$\int_y^a \frac{dt}{t-b}\sqrt{\frac{t-c}{(t-b)(a-t)}} = \frac{g}{k^2}\int_0^{u_1} \mathrm{dc}^2 u\, du,\quad y \neq b.$$ [See **321.02**.]

236.14 $$\int_y^a \frac{dt}{t-c}\sqrt{\frac{t-b}{(a-t)(t-c)}} = g k^2\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02**.]

236.15 $$\left\{\begin{aligned} &\int_y^a \frac{dt}{(t-b)(t-c)\sqrt{(a-t)(t-b)(t-c)}} \\ &\quad = \frac{g}{(a-b)(a-c)}\int_0^{u_1} \mathrm{nc}^2 u\, \mathrm{nd}^2 u\, du,\quad y \neq b. \end{aligned}\right.$$ [See **361.12**.]

236.16 $$\int_y^a \frac{t^m\,dt}{\sqrt{(a-t)(t-b)(t-c)}} = a^m g\int_0^{u_1}\left[1-\frac{a-b}{a}\,\mathrm{sn}^2 u\right]^m du.$$ [See **331.03**.]

236.17
$$\int_y^a \frac{dt}{(t-p)^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{(a-p)^m}\int_0^{u_1}\frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m},$$

[See **336.03**.]

where
$$\alpha^2 = (a-b)/(a-p), \quad p \neq a.$$

236.18
$$\int_y^a \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(t-b)(t-c)}} = \frac{p_1-a}{p-a}\,g\int_0^{u_1}\frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$

[See **340.01**.]

where
$$\alpha_1^2 = (a-b)/(a-p_1), \qquad \alpha^2 = (a-b)/(a-p), \qquad p \neq a.$$

236.19
$$\int_y^a \frac{dt}{t^m\sqrt{(a-t)(t-b)(t-c)}} = \frac{g}{a^m}\int_0^{u_1}\frac{du}{\left[1-\frac{a-b}{a}\,\mathrm{sn}^2 u\right]^m}.$$

[See **336.03**.]

236.20
$$\int_y^a \frac{R(t)\,dt}{\sqrt{(a-t)(t-b)(t-c)}} = g\int_0^{u_1} R\left[a-(a-b)\,\mathrm{sn}^2 u\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$ and $\sqrt{t-c}$, $(y>a>b>c)$

$$\mathrm{sn}^2 u = \frac{t-a}{t-b}, \quad k^2 = \frac{b-c}{a-c}, \quad g = \frac{2}{\sqrt{a-c}},$$
$$\varphi = \mathrm{am}\,u_1 = \sin^{-1}\sqrt{\frac{y-a}{y-b}}, \quad \mathrm{sn}\,u_1 = \sin\varphi.$$

237.00
$$\left\{\begin{aligned}\int_a^y \frac{dt}{\sqrt{(t-a)(t-b)(t-c)}} &= g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k)\\ &= g\,F(\varphi,k).\end{aligned}\right.$$

237.01
$$\left\{\begin{aligned}\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-c}{(t-a)(t-b)}} &= \frac{g}{k'^2}\int_0^{u_1}\mathrm{dn}^2 u\,du = \frac{g}{k'^2}E(u_1)\\ &= \frac{g}{k'^2}E(\varphi,k).\end{aligned}\right.$$

237.02
$$\left\{\begin{aligned} \int_a^y \frac{dt}{t-p}\sqrt{\frac{t-b}{(t-a)(t-c)}} &= \frac{a-b}{a-p}\, g \int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} \\ &= \frac{a-b}{a-p}\, g\, \Pi(\varphi, \alpha^2, k). \end{aligned}\right.$$ [See **400**.]

where $\alpha^2 = (b-p)/(a-p), \quad p \neq a.$

237.03 $$\int_a^y \sqrt{\frac{t-a}{(t-b)(t-c)}}\, dt = (a-b)\, g \int_0^{u_1} \operatorname{tn}^2 u\, du.$$ [See **316.02**.]

237.04 $$\int_a^y \sqrt{\frac{t-b}{(t-a)(t-c)}}\, dt = (a-b)\, g \int_0^{u_1} \operatorname{nc}^2 u\, du.$$ [See **313.02**.]

237.05 $$\int_a^y \sqrt{\frac{t-c}{(t-a)(t-b)}}\, dt = (a-c)\, g \int_0^{u_1} \operatorname{dc}^2 u\, du.$$ [See **321.02**.]

237.06 $$\int_a^y \sqrt{\frac{(t-a)(t-c)}{t-b}}\, dt = (a-b)(a-c)\, g \int_0^{u_1} \operatorname{tn}^2 u \operatorname{dc}^2 u\, du.$$ [See **361.15**.]

237.07 $$\int_a^y \sqrt{\frac{(t-a)(t-b)}{t-c}}\, dt = (a-b)^2\, g \int_0^{u_1} \operatorname{tn}^2 u \operatorname{nc}^2 u\, du.$$ [See **361.07**.]

237.08 $$\int_a^y \sqrt{\frac{(t-b)(t-c)}{t-a}}\, dt = (a-b)(a-c)\, g \int_0^{u_1} \operatorname{dc}^2 u \operatorname{nc}^2 u\, du.$$ [See **361.13**.]

237.09 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-a}{(t-b)(t-c)}} = g \int_0^{u_1} \operatorname{sn}^2 u\, du.$$ [See **310.02**.]

237.10 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{t-a}{(t-b)(t-c)}} = k'^2\, g \int_0^{u_1} \operatorname{sd}^2 u\, du.$$ [See **318.02**.]

237.11 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{t-b}{(t-a)(t-c)}} = k'^2\, g \int_0^{u_1} \operatorname{nd}^2 u\, du.$$ [See **315.02**.]

237.12 $$\int_a^y \frac{dt}{(t-b)^m \sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-b)^m} \int_0^{u_1} \operatorname{cn}^{2m} u\, du.$$ [See **312.05**.]

237.13 $$\int_a^y \frac{dt}{(t-c)^m \sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m} \int_0^{u_1} \mathrm{cd}^{2m} u\, du.$$ [See **320.05**.]

237.14 $$\left\{ \begin{aligned} &\int_a^y \frac{dt}{(t-b)(t-c)\sqrt{(t-a)(t-b)(t-c)}} \\ &\qquad = \frac{g}{(a-c)(a-b)} \int_0^{u_1} \mathrm{cn}^2 u\, \mathrm{cd}^2 u\, du. \end{aligned} \right.$$ [See **361.28**.]

237.15 $$\left\{ \begin{aligned} &\int_a^y \sqrt{(t-a)(t-b)(t-c)}\, dt \\ &\qquad = (a-b)^2 (a-c)\, g \int_0^{u_1} \mathrm{tn}^2 u\, \mathrm{nc}^2 u\, \mathrm{dc}^2 u\, du. \end{aligned} \right.$$ [See **361.17**.]

237.16 $$\int_a^y \frac{t^m\, dt}{\sqrt{(t-a)(t-b)(t-c)}} = a^m g \int_0^{u_1} \frac{\left(1 - \frac{b}{a} \mathrm{sn}^2 u\right)^m}{(1-\mathrm{sn}^2 u)^m} du.$$ [See **340.04**.]

237.17 $$\int_a^y \frac{dt}{(t-p)^m \sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-p)^m} \int_0^{u_1} \frac{\mathrm{cn}^{2m} u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^m},$$ [See **338.04**.]

where $$\alpha^2 = (b-p)/(a-p), \quad p \neq a.$$

237.18 $$\int_a^y \frac{(t-p_1)\, dt}{(t-p)\sqrt{(t-a)(t-b)(t-c)}} = \frac{a-p_1}{a-p} g \int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u} du,$$ [See **340.01**.]

where $\alpha_1^2 = (b-p_1)/(a-p_1), \qquad \alpha^2 = (b-p)/(a-p), \qquad p \neq a.$

237.19 $$\int_a^y \frac{dt}{t^m \sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{a^m} \int_0^{u_1} \frac{\mathrm{cn}^{2m} u\, du}{\left(1 - \frac{b}{a} \mathrm{sn}^2 u\right)^m}.$$ [See **338.04**.]

237.20 $$\int_a^y \frac{R(t)\, dt}{\sqrt{(t-a)(t-b)(t-c)}} = g \int_0^{u_1} R\left[\frac{a - b\,\mathrm{sn}^2 u}{\mathrm{cn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$ and $\sqrt{t-c}$, $(\infty > y \geqq a > b > c)$

$$\operatorname{sn}^2 u = \frac{a-c}{t-c}, \quad k^2 = \frac{b-c}{a-c}, \quad g = \frac{2}{\sqrt{a-c}}$$
$$\varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{a-c}{y-c}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

238.00 $$\begin{cases} \displaystyle\int_y^\infty \frac{dt}{\sqrt{(t-a)(t-b)(t-c)}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\varphi, k) \\ \qquad = g\,F(\varphi, k). \end{cases}$$

238.01 $$\int_y^\infty \frac{dt}{t-c}\sqrt{\frac{t-b}{(t-a)(t-c)}} = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi, k).$$

238.02 $$\int_y^\infty \frac{dt}{t-p}\sqrt{\frac{t-c}{(t-a)(t-b)}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\varphi, \alpha^2, k),$$

[See **400.**]

where $$\alpha = (p-c)/(a-c).$$

238.03 $$\int_y^\infty \frac{dt}{(t-c)^m\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$

[See **310.05.**]

238.04 $$\int_y^\infty \frac{dt}{(t-b)^m\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \operatorname{sd}^{2m} u\,du.$$

[See **318.05.**]

238.05 $$\int_y^\infty \frac{dt}{(t-a)^m\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \operatorname{tn}^{2m} u\,du, \quad y \neq a.$$

[See **316.05.**]

238.06 $$\int_y^\infty \frac{dt}{(t-p)^m\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \frac{\operatorname{sn}^{2m} u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^m},$$

[See **337.04.**]

where $$\alpha^2 = (p-c)/(a-c).$$

238.07 $$\int_y^\infty \frac{dt}{t-a}\sqrt{\frac{t-b}{(t-c)(t-a)}} = g\int_0^{u_1} \operatorname{dc}^2 u\,du, \quad y \neq a.$$ [See **321.02.**]

238.08 $$\int_y^\infty \frac{dt}{t-a}\sqrt{\frac{t-c}{(t-a)(t-b)}} = g\int_0^{u_1} \operatorname{nc}^2 u\,du, \quad y \neq a.$$ [See **313.02.**]

238.09 $$\int_y^\infty \frac{dt}{t-c}\sqrt{\frac{t-a}{(t-b)(t-c)}} = g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02**.]

238.10 $$\int_y^\infty \frac{dt}{t-b}\sqrt{\frac{t-a}{(t-b)(t-c)}} = g\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02**.]

238.11 $$\int_y^\infty \frac{dt}{t-b}\sqrt{\frac{t-c}{(t-a)(t-b)}} = g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02**.]

238.12 $$\int_y^\infty \frac{(t-p_1)\,dt}{(t-p)\sqrt{(t-a)(t-b)(t-c)}} = g\int_0^{u_1} \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du,$$ [See **340.01**.]

where $$\alpha_1^2 = (p_1 - c)/(a-c); \qquad \alpha^2 = (p-c)/(a-c).$$

238.13 $$\int_y^\infty \frac{dt}{(t-c)(t-b)\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^2}\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,du.$$ [See **361.25**.]

238.14 $$\left\{\int_y^\infty \frac{dt}{(t-c)(t-a)\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^2}\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{tn}^2 u\,du, \quad y \neq a.\right.$$ [See **361.29**.]

238.15 $$\left\{\int_y^\infty \frac{dt}{(t-a)(t-b)\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^2}\int_0^{u_1} \mathrm{tn}^2 u\,\mathrm{sd}^2 u\,du, \quad y \neq a.\right.$$ [See **361.24**.]

238.16 $$\left\{\begin{aligned}&\int_y^\infty \frac{dt}{(t-a)(t-b)(t-c)\sqrt{(t-a)(t-b)(t-c)}}\\ &\qquad = \frac{g}{(a-c)^3}\int_0^{u_1} \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,\mathrm{tn}^2 u\,du, \quad (y \neq a).\end{aligned}\right.$$ [See **361.30**.]

238.17 $$\int_y^\infty \frac{dt}{t^m\sqrt{(t-a)(t-b)(t-c)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \frac{\mathrm{sn}^{2m} u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}.$$ [See **337.04**.]

where $$\alpha^2 = c/(c-a).$$

238.18 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{(t-a)(t-b)(t-c)}} = g\int_0^{u_1} R\left[c + (a-c)\,\mathrm{ns}^2 u\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$ and $\sqrt{t-c}$; a real, b, c complex, $(y > a)$

$$\operatorname{cn} u = \frac{A+a-t}{A-a+t}, \quad k^2 = \frac{A+b_1-a}{2A}, \quad g = \frac{1}{\sqrt{A}},$$
$$A^2 = (b_1-a)^2 + a_1^2, \quad (t-b)(t-c) = (t-b)(t-\bar{b}) = (t-b_1)^2 + a_1^2;$$
$$a_1^2 = -\frac{(b-\bar{b})^2}{4}, \quad b_1 = \frac{b+c}{2} = \frac{b+\bar{b}}{2},$$
$$\varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{A+a-y}{A-a+y}\right], \quad \operatorname{cn} u_1 = \cos\varphi.$$

239.00 $$\int_a^y \frac{dt}{\sqrt{(t-a)[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\varphi, k) = g\,F(\varphi,k). \quad [\text{cf. } \mathbf{241.00}].$$

239.01 $$\int_a^y \frac{dt}{(A+t-a)^2}\sqrt{\frac{(t-b_1)^2+a_1^2}{t-a}} = g\int_0^{u_1} \operatorname{dn}^2 u\, du = g\,E(u_1) = g\,E(\varphi,k).$$

239.02 $$\int_a^y \frac{(A+t-a)^2\,dt}{[(A+t-a)^2 - 4A\,\alpha^2(t-a)]\sqrt{(t-a)[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\varphi,\alpha^2,k). \quad [\text{See } \mathbf{400}.]$$

239.03 $$\int_a^y \sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \sqrt{A}\int_0^{u_1}\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\,du. \quad [\text{See } \mathbf{361.53}.]$$

239.04 $$\int_a^y \frac{dt}{(A+t-a)^2}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\operatorname{sn}^2 u\,du. \quad [\text{See } \mathbf{310.02}.]$$

239.05 $$\int_a^y \frac{dt}{(A+a-t)^2}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\operatorname{tn}^2 u\,du. \quad [\text{See } \mathbf{316.02}.]$$

239.06 $$\int_a^y \frac{dt}{[(t-b_1)^2+a_1^2]}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\operatorname{sd}^2 u\,du. \quad [\text{See } \mathbf{318.02}.]$$

239.07
$$\left\{\begin{aligned}&\int_a^y \frac{t\,dt}{\sqrt{(t-a)\,[(t-b_1)^2+a_1^2]}}\\&\quad = g\,(a-A)\,F(\varphi,k) + 2gA\int_0^{u_1}\frac{du}{1+\operatorname{cn} u}.\end{aligned}\right.$$ [See **341.53**.]

239.08
$$\int_a^y \frac{R\,(t)\,dt}{\sqrt{(t-a)\,[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} R\left[\frac{a+A+(a-A)\operatorname{cn} u}{1+\operatorname{cn} u}\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t^3-1}$, $(y>1)$

Special case of **239**:	$\operatorname{cn} u = \dfrac{\sqrt{3}+1-t}{\sqrt{3}-1+t}$, $k^2 = \dfrac{2-\sqrt{3}}{4}$, $g = 1/\sqrt[4]{3}$, $\varphi = \operatorname{am} u_1 = \cos^{-1}\left[\dfrac{\sqrt{3}+1-y}{\sqrt{3}-1+y}\right]$, $\operatorname{cn} u_1 = \cos\varphi$.

240.00
$$\int_1^y \frac{dt}{\sqrt{t^3-1}} = g\int_0^{u_1} du = g\,u_1 = g\,F(\varphi,k).$$ [cf. **242.00**].

240.01
$$\int_1^y \frac{dt}{(\sqrt{3}+t-1)^2}\sqrt{\frac{t^2+t+1}{t-1}} = g\int_0^{u_1}\operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi,k).$$

240.02
$$\left\{\begin{aligned}&\int_1^y \frac{(\sqrt{3}+t-1)^2\,dt}{[(\sqrt{3}+t-1)^2-4\alpha^2\sqrt{3}\,(t-1)]\,\sqrt{t^3-1}} = g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u}\\&\qquad = g\,\Pi(\varphi,\alpha^2,k).\end{aligned}\right.$$ [See **400**.]

240.03
$$\int_1^y \sqrt{\frac{t-1}{t^2+t+1}}\,dt = \sqrt[4]{3}\int_0^{u_1}\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\,du.$$ [See **361.53**.]

240.04
$$\int_1^y \frac{dt}{(\sqrt{3}+t-1)^2}\sqrt{\frac{t-1}{t^2+t+1}} = \frac{g}{4\sqrt{3}}\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02**.]

240.05
$$\int_1^y \frac{t\,dt}{\sqrt{t^3-1}} = g\left(1-\sqrt{3}\right)F(\varphi,k) + 2g\sqrt{3}\int_0^{u_1}\frac{du}{1+\operatorname{cn} u}.$$ [See **341.53**.]

240.06 $$\int_1^y \frac{t^m\,dt}{\sqrt{t^3-1}} = g\left(1-\sqrt{3}\right)^m m! \sum_{j=0}^{m} \frac{\left(-3-\sqrt{3}\right)^j}{j!\,(m-j)!} \int_0^{u_1} \frac{du}{(1+\operatorname{cn} u)^j}$$
[See **341.55.**]

240.07 $$\left\{\begin{aligned} &\int_1^y \frac{dt}{(t-p)^m\sqrt{t^3-1}} = \frac{g}{\left(1-\sqrt{3}-p\right)^m} \sum_{j=0}^{m} \frac{m!\,(\alpha-1)^j}{j!\,(m-j)!} \int_0^{u_1} \frac{du}{(1+\alpha\operatorname{cn} u)^j}, \\ &\qquad p \neq 1. \end{aligned}\right.$$
[See **341.05.**]

where $$\alpha = \left(1-\sqrt{3}-p\right)/\left(1+\sqrt{3}-p\right).$$

240.08 $$\int_1^y \frac{dt}{\left(\sqrt{3}+1-t\right)^2}\sqrt{\frac{t-1}{t^2+t+1}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{tn}^2 u\,du.$$ [See **316.02.**]

240.09 $$\int_1^y \frac{dt}{(t^2+t+1)}\sqrt{\frac{t-1}{t^2+t+1}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{sd}^2 u\,du.$$ [See **318.02.**]

240.10 $$\int_1^y \frac{R(t)\,dt}{\sqrt{t^3-1}} = g\int_0^{u_1} R\left[\frac{1+\sqrt{3}+\left(1-\sqrt{3}\right)\operatorname{cn} u}{1+\operatorname{cn} u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$ and $\sqrt{t-c}$; a real, b, c complex ($\infty > y \geqq a$)

$$\operatorname{cn} u = \frac{t-a-A}{t-a+A}, \qquad k^2 = \frac{A+b_1-a}{2A}, \qquad A^2 = (b_1-a)^2 + a_1^2,$$
$$g = \frac{1}{\sqrt{A}}, \quad (t-b)(t-c) = (t-b)(t-\bar{b}) = (t-b_1)^2 + a_1^2, \quad b_1 = \frac{b+\bar{b}}{2},$$
$$a_1^2 = -\frac{(b-\bar{b})^2}{4}, \qquad \varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{y-a-A}{y-a+A}\right], \qquad \operatorname{cn} u_1 = \cos\varphi.$$

241.00 $$\left\{\begin{aligned} \int_y^\infty \frac{dt}{\sqrt{(t-a)\left[(t-b_1)^2+a_1^2\right]}} &= g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\varphi, k) \\ &= g\,F(\varphi, k), \end{aligned}\right.$$
[cf. **239.00**].

241.01 $$\int_y^\infty \frac{dt}{(t-a+A)^2}\sqrt{\frac{(t-b_1)^2+a_1^2}{t-a}} = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi, k).$$

241.02
$$\left\{\begin{aligned}&\int_y^\infty \frac{(t-a+A)^2\,dt}{[(t-a+A)^2-4A\,\alpha^2(t-a)]\sqrt{(t-a)[(t-b_1)^2+a_1^2]}}\\ &\qquad = g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = g\,\Pi(\varphi,\alpha^2,k). \qquad \text{[See } \mathbf{400}.]\end{aligned}\right.$$

241.03
$$\int_y^\infty \frac{dt}{(t-a)\sqrt{(t-a)[(t-b_1)^2+a_1^2]}} = \frac{g}{A}\int_0^u \frac{1-\mathrm{cn}\,u}{1+\mathrm{cn}\,u}\,du.$$
[See **361.53**.]

241.04
$$\int_y^\infty \frac{dt}{(t-p)\sqrt{(t-a)[(t-b_1)^2+a_1^2]}} = \frac{g}{(A+a-p)}\int_0^{u_1}\frac{1-\mathrm{cn}\,u}{1+\alpha\,\mathrm{cn}\,u}\,du.$$
[See **361.60**.]

$$\alpha = (A-a+p)/(A+a-p).$$

241.05
$$\int_y^\infty \frac{dt}{(t-a+A)^2}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\mathrm{sn}^2 u\,du. \qquad \text{[See } \mathbf{310.02}.]$$

241.06
$$\int_y^\infty \frac{dt}{(t-a-A)^2}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\mathrm{tn}^2 u\,du. \qquad \text{[See } \mathbf{316.02}.]$$

241.07
$$\int_y^\infty \frac{dt}{[(t-b_1)^2+a_1^2]}\sqrt{\frac{t-a}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1}\mathrm{sd}^2 u\,du.$$
[See **318.02**.]

241.08
$$\int_y^\infty \frac{R(t)\,dt}{\sqrt{(t-a)[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} R\left[\frac{a+A+(A-a)\,\mathrm{cn}\,u}{1-\mathrm{cn}\,u}\right]du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t^3-1}$, $(\infty > y \geqq 1)$

Special case of **241**:

$$\boxed{\begin{aligned}&\mathrm{cn}\,u = \frac{t-1-\sqrt{3}}{t-1+\sqrt{3}}, \quad k^2 = \frac{2-\sqrt{3}}{4}, \quad g = 1/\sqrt[4]{3},\\ &\varphi = \mathrm{am}\,u_1 = \cos^{-1}\left[\frac{y-1-\sqrt{3}}{y-1+\sqrt{3}}\right], \quad \mathrm{cn}\,u_1 = \cos\varphi.\end{aligned}}$$

242.00
$$\int_y^\infty \frac{dt}{\sqrt{t^3-1}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{cn}^{-1}(\cos\varphi,k) = g\,F(\varphi,k).$$
[cf. **240.00**].

242.01 $$\int_y^\infty \frac{(t^2+t+1)\,dt}{(t-1+\sqrt{3})^2\sqrt{t^3-1}} = g\int_0^{u_1} \mathrm{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi, k).$$

242.02 $$\left\{\begin{aligned} &\int_y^\infty \frac{(t-1+\sqrt{3})\,dt}{[(t-1+\sqrt{3})^2-4\sqrt{3}\alpha^2(t-1)]\sqrt{t^3-1}} = g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ &\qquad = g\,\Pi(\varphi, \alpha^2, k).\end{aligned}\right.$$ [See **400**.]

242.03 $$\int_y^\infty \frac{dt}{(t-1)\sqrt{t^3-1}} = \frac{g}{\sqrt{3}}\int_0^{u_1}\frac{1-\mathrm{cn}\,u}{1+\mathrm{cn}\,u}\,du, \quad y \neq 1.$$ [See **361.53**.]

242.04 $$\int_y^\infty \frac{dt}{(t-p)\sqrt{t^3-1}} = \frac{g}{(\sqrt{3}+1-p)}\int_0^{u_1}\frac{1-\mathrm{cn}\,u}{1+\alpha\,\mathrm{cn}\,u}\,du,$$ [See **361.60**.]

where $$\alpha = (\sqrt{3}-1+p)/(\sqrt{3}+1-p).$$

242.05 $$\int_y^\infty \frac{(t-1)\,dt}{(t-1+\sqrt{3})^2\sqrt{t^3-1}} = \frac{g}{4\sqrt{3}}\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02**.]

242.06 $$\left\{\begin{aligned} &\int_y^\infty \frac{dt}{t^m\sqrt{t^3-1}}\\ &\quad = \frac{g}{(\sqrt{3}-1)^m}\sum_{j=0}^{m}\frac{(-1-\alpha)^j\,m!}{j!\,(m-j)!}\int_0^{u_1}\frac{du}{(1+\alpha\,\mathrm{cn}\,u)^j},\end{aligned}\right.$$ [See **341.05**]

where $$\alpha = 2-\sqrt{3}.$$

242.07 $$\int_y^\infty \frac{(t-1)\,dt}{(t-1-\sqrt{3})^2\sqrt{t^3-1}} = \frac{g}{4\sqrt{3}}\int_0^{u_1}\mathrm{tn}^2 u\,du.$$ [See **316.02**]

242.08 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{t^3-1}} = g\int_0^{u_1} R\left[\frac{1+\sqrt{3}+(\sqrt{3}-1)\,\mathrm{cn}\,u}{1-\mathrm{cn}\,u}\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$ and $\sqrt{t-c}$; a real, b, c complex, $(y<a)$

$$\operatorname{cn} u=\frac{A-a+t}{A+a-t},\qquad k^2=\frac{A-b_1+a}{2A},\qquad g=\frac{1}{\sqrt{A}},\qquad A^2=(b_1-a)^2+a_1^2,$$
$$(t-b)(t-c)=(t-b)(t-\bar{b})=(t-b_1)^2+a_1^2;\quad a_1^2=-\frac{(b-\bar{b})^2}{4},$$
$$b_1=\frac{b+c}{2}=\frac{b+\bar{b}}{2},\quad \varphi=\operatorname{am} u_1=\cos^{-1}\left[\frac{A-a+y}{A+a-y}\right],\quad \operatorname{cn} u_1=\cos\varphi.$$

243.00 $$\begin{cases}\displaystyle\int_y^a\frac{dt}{\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}}=g\int_0^{u_1}du=g\,u_1=g\operatorname{cn}^{-1}(\cos\varphi,k)\\ \qquad\qquad\qquad\qquad=g\,F(\varphi,k).\qquad\text{[cf. \textbf{245}].}\end{cases}$$

243.01 $$\begin{cases}\displaystyle\int_y^a\frac{dt}{(t-a-A)^2}\sqrt{\frac{(t-b_1)^2+a_1^2}{a-t}}=g\int_0^{u_1}\operatorname{dn}^2u\,du=g\,E(u_1)\\ \qquad\qquad\qquad\qquad=g\,E(\varphi,k).\end{cases}$$

243.02 $$\begin{cases}\displaystyle\int_y^a\frac{(t-a+A)^2\,dt}{[(t-a-A)^2-4A^2\alpha^2(t-a)]\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}}\\ \displaystyle\qquad=g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2u}=g\,\Pi(\varphi,\alpha^2,k).\qquad\text{[See \textbf{400}.]}\end{cases}$$

243.03 $$\int_y^a\sqrt{\frac{a-t}{(t-b_1)^2+a_1^2}}\,dt=\sqrt{A}\int_0^{u_1}\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\,du.\qquad\text{[See \textbf{361.53}.]}$$

243.04 $$\int_y^a\frac{dt}{(t-a-A)^2}\sqrt{\frac{a-t}{(t-b_1)^2+a_1^2}}=\frac{g}{4A}\int_0^{u_1}\operatorname{sn}^2u\,du.\qquad\text{[See \textbf{310.02}.]}$$

243.05 $$\int_y^a\frac{dt}{(t-p)\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}}=\frac{g}{a-A-p}\int_0^{u_1}\frac{1+\operatorname{cn}u}{1+\alpha\operatorname{cn}u}\,du,$$
[See **361.59**.]

where $$\alpha=(a+A-p)/(a-A-p).$$

243.06 $$\int_y^a\frac{dt}{(t-a-A)^2}\sqrt{\frac{a-t}{(t-b_1)^2+a_1^2}}=\frac{g}{4A}\int_0^{u_1}\operatorname{tn}^2u\,du.\qquad\text{[See \textbf{316.02}.]}$$

243.07 $$\int_y^a\frac{t\,dt}{\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}}=g\,(a+A)\,F(\varphi,k)-2gA\int_0^{u_1}\frac{du}{1+\operatorname{cn}u}.$$
[See **341.53**.]

243.08 $$\int_y^a \frac{R(t)\,dt}{\sqrt{(a-t)[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} R\left[\frac{(a-A)+(a+A)\operatorname{cn} u}{1+\operatorname{cn} u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{1-t^3}$, $(y<1)$

Special case of **243**:

$$\operatorname{cn} u = \frac{\sqrt{3}-1+t}{\sqrt{3}+1-t}, \quad k^2 = \frac{2+\sqrt{3}}{4}, \quad g = 1/\sqrt[4]{3},$$
$$\varphi = \operatorname{am} u_1 = \operatorname{cn}^{-1}\left[\frac{\sqrt{3}-1+y}{\sqrt{3}+1-y}\right], \quad \operatorname{cn} u_1 = \cos\varphi.$$

244.00 $$\int_y^1 \frac{dt}{\sqrt{1-t^3}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\varphi, k) = g\,F(\varphi, k),$$ [cf. **246.00**].

244.01 $$\int_y^1 \sqrt{1-t^3}\,dt = \frac{3}{5} g\,F(\varphi,k) - \frac{2}{5} y\sqrt{1-y^3}.$$

244.02 $$\int_y^1 \frac{(t^2+t+1)\,dt}{(t-1+\sqrt{3})^2\sqrt{1-t^3}} = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\varphi,k).$$

244.03 $$\int_y^1 \frac{(1+\sqrt{3}-t)^2\,dt}{[(1+\sqrt{3}-t)^2 - 4\alpha^2\sqrt{3}(1-t)]\sqrt{1-t^3}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\varphi,\alpha^2,k).$$ [See **400.**]

244.04 $$\int_y^1 \frac{(1-t)\,dt}{(1+\sqrt{3}-t)^2\sqrt{1-t^3}} = \frac{g}{4\sqrt{3}}\int_0^{u_1} \operatorname{sn}^2 u\,du.$$ [See **310.02.**]

244.05 $$\int_y^1 \frac{t\,dt}{\sqrt{1-t^3}} = g\left(1+\sqrt{3}\right)F(\varphi,k) - 2g\sqrt{3}\int_0^{u_1} \frac{du}{1+\operatorname{cn} u}.$$ [See **361.51.**]

244.06 $$\int_y^1 \frac{t^2\,dt}{\sqrt{1-t^3}} = -\frac{2}{3}\sqrt{1-t^3}\,\Big]_y^1 = \frac{2}{3}\sqrt{1-y^3}.$$

244.07 $$\int_y^1 \frac{t^m\,dt}{\sqrt{1-t^3}} = \frac{2}{2m-1}\,y^{m-2}\sqrt{1-y^3} + \frac{2(m-2)}{2m-1}\int_y^1 \frac{t^{m-3}\,dt}{\sqrt{1-t^3}}.$$

244.08 $$\int_y^1 \frac{dt}{(t-p)\sqrt{1-t^3}} = \frac{g}{(1-\sqrt{3}-p)} \int_0^{u_1} \frac{1+\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\, du, \quad \text{[See } \mathbf{361.59}.]$$

where $$\alpha = (1+\sqrt{3}-p)/(1-\sqrt{3}-p) \qquad y \neq p.$$

244.09 $$\int_y^1 \frac{R(t)\, dt}{\sqrt{1-t^3}} = g \int_0^{u_1} R\left[\frac{1-\sqrt{3}+(1+\sqrt{3}) \operatorname{cn} u}{1+\operatorname{cn} u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$ and $\sqrt{t-c}$; a real, b, c complex, $(y \leqq a)$

$$\operatorname{cn} u = \frac{a-A-t}{a+A-t}, \quad k^2 = \frac{A-b_1+a}{2A}, \quad g = \frac{1}{\sqrt{A}}, \quad A^2 = (b_1-a)^2 + a_1^2,$$
$$(t-b)(t-c) = (t-b_1)^2 + a_1^2 = (t-b)(t-\bar{b}); \; a_1^2 = -\frac{(b-\bar{b})^2}{4},$$
$$b_1 = \frac{b+c}{2} = \frac{b+\bar{b}}{2}, \quad \varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{a-A-y}{a+A-y}\right], \quad \operatorname{cn} u_1 = \cos\varphi.$$

245.00 $$\int_{-\infty}^{y} \frac{dt}{\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} du = g\, u_1 = g \operatorname{cn}^{-1}(\cos\varphi, k) = g F(\varphi, k), \quad \text{[cf. } \mathbf{243.00}].$$

245.01 $$\int_{-\infty}^{y} \frac{dt}{(a+A-t)^2} \sqrt{\frac{(t-b_1)^2+a_1^2}{a-t}} = g\int_0^{u_1} \operatorname{dn}^2 u\, du = g E(u_1) = g E(\varphi, k).$$

245.02 $$\int_{-\infty}^{y} \frac{(a+A-t)^2\, dt}{[(a+A-t)^2 - 4A\alpha^2(a-t)]\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = g\int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = g\Pi(\varphi, \alpha^2, k). \quad \text{[See } \mathbf{400}.]$$

245.03 $$\int_{-\infty}^{y} \frac{dt}{(a+A-t)^2} \sqrt{\frac{a-t}{(t-b_1)^2+a_1^2}} = \frac{g}{4A}\int_0^{u_1} \operatorname{sn}^2 u\, du. \quad \text{[See } \mathbf{310.02}.]$$

245.04 $$\int_{-\infty}^{y} \frac{dt}{(a-t)\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = \frac{1}{A\sqrt{A}} \int_0^{u_1} \frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\, du.$$
$$y \neq a \qquad \text{[See } \mathbf{361.53}.]$$

245.05
$$\begin{cases} \int\limits_{-\infty}^{y} \frac{dt}{t\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = \frac{g}{a+A} F(\varphi, k) + \\ \qquad + \frac{2gA}{a^2-A^2} \int\limits_0^{u_1} \frac{du}{1+\alpha \operatorname{cn} u}, \qquad \text{[See 341.03.]} \end{cases}$$

where $\alpha = (A+a)/(A-a); \quad y \neq 0.$

245.06
$$\int\limits_{-\infty}^{y} \frac{dt}{(t-p)\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = \frac{g}{a-A-p} \int\limits_0^{u_1} \frac{1-\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\, du.$$
[See **361.60**.]

where $\alpha = (A+a-p)/(A-a+p); \quad y \neq p.$

245.07
$$\int\limits_{-\infty}^{y} \frac{R(t)\, dt}{\sqrt{(a-t)\,[(t-b_1)^2+a_1^2]}} = g \int\limits_0^{u_1} R\left[\frac{(a-A)-(a+A)\operatorname{cn} u}{1-\operatorname{cn} u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{1-t^3}$, $(y \leq 1)$

Special case of **245**:

$$\operatorname{cn} u = \frac{1-\sqrt{3}-t}{1+\sqrt{3}-t}, \quad k^2 = \frac{2+\sqrt{3}}{4}, \quad g = 1/\sqrt[4]{3},$$
$$\varphi = \cos^{-1}\left[\frac{1-\sqrt{3}-y}{1+\sqrt{3}-y}\right], \quad \operatorname{cn} u_1 = \cos\varphi.$$

246.00
$$\int\limits_{-\infty}^{y} \frac{dt}{\sqrt{1-t^3}} = g \int\limits_0^{u_1} du = g\, u_1 = g \operatorname{cn}^{-1}(\cos\varphi, k) = g\, F(\varphi, k),$$
[cf. **244.00**].

246.01
$$\int\limits_{-\infty}^{y} \frac{(t^2+t+1)\, dt}{(1+\sqrt{3}-t)^2\sqrt{1-t^3}} = g \int\limits_0^{u_1} \operatorname{dn}^2 u\, du = g\, E(u_1) = g\, E(\varphi, k).$$

246.02
$$\begin{cases} \int\limits_{-\infty}^{y} \frac{(1+\sqrt{3}-t)^2\, dt}{[(1+\sqrt{3}-t)^2 - 4\sqrt{3}\alpha^2(1-t)]\sqrt{1-t^3}} = g \int\limits_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} \\ \qquad = g\, \Pi(\varphi, \alpha^2, k). \qquad \text{[See 400.]} \end{cases}$$

246.03
$$\int\limits_{-\infty}^{y} \frac{dt}{t\sqrt{1-t^3}} = \frac{g}{1-\sqrt{3}} \int\limits_0^{u_1} \frac{1-\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\, du,$$
[See **361.61**.]

where $$\alpha = 2 + \sqrt{3}; \qquad y \neq 0.$$

246.04 $$\int_{-\infty}^{y} \frac{dt}{t^m \sqrt{1-t^3}} = \frac{g}{(1+\sqrt{3})^m} \sum_{j=0}^{m} \frac{(-1-\alpha)^j\, m!}{j!\,(m-j)!} \int_0^{u_1} \frac{du}{1+\alpha \operatorname{cn} u},$$ [See **341.03**.]

where $$\alpha = (1+\sqrt{3})/(\sqrt{3}-1); \qquad y \neq 0.$$

246.05 $$\int_{-\infty}^{y} \frac{dt}{(t-p)\sqrt{1-t^3}} = \frac{g}{(1-\sqrt{3}-p)} \int_0^{u_1} \frac{1-\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\, du,$$ [See **361.60**.]

where $$\alpha = (1+\sqrt{3}-p)/(\sqrt{3}-1+p); \qquad y \neq p.$$

246.06 $$\int_{-\infty}^{y} \frac{dt}{(1-t)\sqrt{1-t^3}} = \frac{g}{\sqrt{3}} \int_0^{u_1} \frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\, du, \quad y \neq 1.$$ [See **361.53**.]

246.07 $$\int_{-\infty}^{y} \frac{(1-t)\, dt}{(1+\sqrt{3}-t)^2 \sqrt{1-t^3}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{sn}^2 u\, du.$$ [See **310.02**.]

246.08 $$\int_{-\infty}^{y} \frac{(1-t)\, dt}{(1-\sqrt{3}-t)^2 \sqrt{1-t^3}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{tn}^2 u\, du.$$ [See **316.02**.]

246.09 $$\int_{-\infty}^{y} \frac{R(t)\, dt}{\sqrt{1-t^3}} = g \int_0^{u_1} R\left[\frac{(1-\sqrt{3})-(1+\sqrt{3})\operatorname{cn} u}{1-\operatorname{cn} u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands Involving the Square Root of the Quartic,

$\sqrt{a_0(t+r_1)(t+r_2)(t+r_3)(t+r_4)}$.

Introduction.

Consideration is given in this section to the elliptic integral

250.00 $$\xi = \int_{y_1}^{y} \frac{R(t)}{\sqrt{P}}\, dt,$$

where the polynomial P is of the form[1] $P(t) = a_0(t+r_1)(t+r_2)(t+r_3)\times$ $\times(t+r_4)$, having distinct factors.

If $R(t)$ is broken into partial fractions, **250.00** can be expressed in terms of the integrals

$$\int_{y_1}^{y} \frac{t^m\,dt}{\sqrt{P}}, \qquad \int_{y_1}^{y} \frac{dt}{(t-p)^n\sqrt{P}}.$$

A recurrence formula for the first integral is given by

250.01
$$\left\{\begin{aligned} \int_{y_1}^{y} \frac{t^m\,dt}{\sqrt{P}} &= \frac{1}{2(m-1)\,a_0}\Big\{2y^{m-3}\sqrt{P(y)} - 2y_1^{m-3}\sqrt{P(y_1)} + \\ &\quad + \sum_{j=1}^{4}(2+j-2m)\,\alpha_j \int_{y_1}^{y} \frac{t^{m-j}\,dt}{\sqrt{P}}\Big\}, \qquad [m \neq 1] \end{aligned}\right.$$

where α_j is obtained from

$$P = a_0(t+r_1)(t+r_2)(t+r_3)(t+r_4) = a_0 t^4 + \alpha_1 t^3 + \alpha_2 t^2 + \alpha_3 t + \alpha_4.$$

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[1] Integrals contained in **250.00** can be reduced to those of **211–225**, since it is possible to transform the polynomial P into a form which does not involve odd powers of t. This may be accomplished by means of the linear fractional transformation

$$t = \frac{f+g\tau}{1+\tau},$$

where f and g are obtained from

$$f+g = 2\,\frac{r_1 r_2 - r_3 r_4}{-r_1 - r_2 + r_3 + r_4}, \qquad f g = \frac{r_3 r_4 (r_1+r_2) - r_1 r_2 (r_3+r_4)}{-r_1 - r_2 + r_3 + r_4}$$

if $r_1 + r_2 \neq r_3 + r_4$. When $r_1 + r_2 = r_3 + r_4$, use is made of the transformation $t = \tau - (r_1+r_2)/2$.

Integrals which are equivalent to the type given in **250.00** frequently appear in some such form as the integral

$$\xi_1 = \int_{Y_1}^{Y} \frac{t\,dt}{\sqrt{a_0(t^2+r_1)(t^2+r_2)(t^2+r_3)(t^2+r_4)^3}},$$

where the factors under the radical sign occur in even powers, with an odd power of t outside of the radical sign. In such cases the transformation $t^2 = \tau$, $y = Y^2$, readily yields an integral whose radicand involves four binomials. For this example, we thus have

$$\xi_1 = \frac{1}{2}\int_{y_1}^{y} \frac{d\tau}{(\tau+r_4)\sqrt{a_0(\tau+r_1)(\tau+r_2)(\tau+r_3)(\tau+r_4)}}.$$

If p is not a root of $P(t)=0$, one has

250.02
$$\left\{\begin{aligned}&\int_{y_1}^{y}\frac{dt}{(t-p)^m\sqrt{P}}=\frac{1}{2(1-m)\,P(p)}\left\{\frac{2\sqrt{P(y)}}{(y-p)^{m-1}}-\frac{2\sqrt{P(y_1)}}{(y_1-p)^{m-1}}+\right.\\&\qquad\left.+\sum_{j=1}^{4}\frac{(2m-2-j)\,P^{(j)}(p)}{j!}\int_{y_1}^{y}\frac{dt}{(t-p)^{m-j}\sqrt{P}}\right\},\quad[m\neq 1]\end{aligned}\right.$$

where $P^{(j)}(t)=\frac{d^j}{dt^j}P(t)$; but when p is a zero of $P(t)$,

250.03
$$\left\{\begin{aligned}&\int_{y_1}^{y}\frac{dt}{(t-p)^m\sqrt{P}}=\frac{1}{4(1-2m)\,P'(p)}\left\{\frac{2\sqrt{P(y)}}{(y-p)^{m}}-\frac{2\sqrt{P(y_1)}}{(y_1-p)^{m}}+\right.\\&\qquad\left.+\sum_{j=1}^{3}\frac{(2m-j-1)}{(j+1)!}\,P^{(j+1)}(p)\int_{y_1}^{y}\frac{dt}{(t-p)^{m-j}\sqrt{P}}\right\}.\end{aligned}\right.$$

Thus every elliptic integral of the form **250.00** depends on the four fundamental integrals $\int dt/\sqrt{P}$, $\int t\,dt/\sqrt{P}$, $\int t^2\,dt/\sqrt{P}$ and $\int dt/(t-p)\sqrt{P}$. The first integral is in every case an elliptic integral of the first kind, while the last three are integrals of the third kind. [If $P(p)=0$, the integral $\int dt/(t-p)\sqrt{P}$ becomes an integral of the second kind.]

Taking $a, b, c, d, a_1, a_2, b_1$ and b_2 real, we may write $P(t)$ in **250.00** (except for the factor $|a_0|$) in one or other of the ways:

$$(t-a)(t-b)(t-c)(t-d); \qquad (a-t)(t-b)(t-c)(t-d);$$

$$(a-t)(b-t)(t-c)(t-d); \qquad (t-a)(t-b)\,[(t-b_1)^2+a_1^2];$$

$$(t-a)(b-t)\,[(t-b_1)^2+a_1^2];$$

$$[(t-b_1)^2+a_1^2]\,[(t-b_2)^2+a_2^2], \quad (a>b>c>d).$$

In the first three cases, the roots of the equation $P=0$ are all real, while in the last one all the roots are complex. For the other two cases there are two real roots and two conjugate complex roots.

In order to reduce the general integral **250.00** to Jacobi's normal form, we apply the transformation

250.04
$$t=\frac{A_1+A_2\operatorname{sn}^2 u}{A_3+A_4\operatorname{sn}^2 u}, \quad (0\leqq u\leqq K),$$

when all the zeros of $P(t)$ are real, and the substitution

250.05
$$t=\frac{A_1+A_2\operatorname{cn} u}{A_3+A_4\operatorname{cn} u}, \quad (0\leqq u\leqq 2K),$$

in the case when some of the zeros of the radicand are complex. If the real constants A_1, A_2, A_3, A_4 and the modulus k are chosen properly, one has

250.06 $$\frac{dt}{\sqrt{a_0(t+r_1)(t+r_2)(t+r_3)(t+r_4)}} = g\,du,$$

where g is also a real constant.

Table of Integrals.

Suitable substitutions (as well as the reductions) are given in the following table of integrals for integrands involving the square root of four linear factors. We again take one of the limits of integration as one of the zeros of the polynomial under the radical sign and refer the user of the tables to page 2 of the Introduction for an example illustrating how the tables may easily be employed to handle an integral when neither of the limits is a zero.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$, $\sqrt{c-t}$ and $\sqrt{d-t}$, $(a>b>c>d>y)$

$$\operatorname{sn}^2 u = \frac{(a-c)(d-t)}{(a-d)(c-t)}, \quad k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}, \quad g = \frac{2}{\sqrt{(a-c)(b-d)}},$$
$$\alpha^2 = \frac{a-d}{a-c} > 1, \quad \varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{(a-c)(d-y)}{(a-d)(c-y)}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

251.00 $$\int_y^d \frac{dt}{\sqrt{(a-t)(b-t)(c-t)(d-t)}} = g\int_0^{u_1} du = g u_1 = g\operatorname{sn}^{-1}(\sin\varphi, k) = g F(\varphi, k).$$

251.01 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{b-t}{(a-t)(c-t)(d-t)}} = \frac{b-d}{c-d}\, g\int_0^{u_1} \operatorname{dn}^2 u\, du = \frac{b-d}{c-d}\, g E(u_1) = \frac{b-d}{c-d}\, g E(\varphi, k).$$

251.02 $$\int_y^d \sqrt{\frac{c-t}{(a-t)(b-t)(d-t)}}\, dt = (c-d)\, g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = (c-d)\, g\,\Pi(\varphi, \alpha^2, k).$$ [See **400**.]

251.03 $$\int_y^d \frac{t^m\,dt}{\sqrt{(a-t)(b-t)(c-t)(d-t)}} = g\,d^m \int_0^{u_1} \frac{\left(1-\frac{c\,\alpha^2}{d}\operatorname{sn}^2 u\right)^m}{(1-\alpha^2\operatorname{sn}^2 u)^m}\,du.$$

[See **340.04.**]

251.04 $$\int_y^d \frac{dt}{t^m\sqrt{(a-t)(b-t)(c-t)(d-t)}} = \frac{g}{d^m}\int_0^{u_1} \frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{\left(1-\frac{c\,\alpha^2}{d}\operatorname{sn}^2 u\right)^m}\,du.$$

[See **340.04.**]

251.05 $$\int_y^d \sqrt{\frac{d-t}{(a-t)(b-t)(c-t)}}\,dt = (c-d)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **337.01.**]

251.06 $$\int_y^d \sqrt{\frac{a-t}{(b-t)(c-t)(d-t)}}\,dt = (a-d)\,g\int_0^{u_1}\frac{\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **338.01.**]

251.07 $$\int_y^d \sqrt{\frac{b-t}{(a-t)(c-t)(d-t)}}\,dt = (b-d)\,g\int_0^{u_1}\frac{\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **339.01.**]

251.08 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{a-t}{(b-t)(c-t)(d-t)}} = \frac{a-d}{c-d}\,g\int_0^{u_1}\operatorname{cn}^2 u\,du.$$

[See **312.02.**]

251.09 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{d-t}{(a-t)(b-t)(c-t)}} = \frac{c-d}{a-c}\,g\int_0^{u_1}\operatorname{tn}^2 u\,du.$$

[See **316.02.**]

251.10 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{d-t}{(a-t)(b-t)(c-t)}} = \alpha^2 g\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

251.11 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{d-t}{(a-t)(b-t)(c-t)}} = \frac{c-d}{b-d}\,\alpha^2 g\int_0^{u_1}\operatorname{sd}^2 u\,du.$$

[See **318.02.**]

251.12 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(c-t)(d-t)}} = \frac{a-d}{b-d}\,g\int_0^{u_1}\operatorname{cd}^2 u\,du.$$

[See **320.02.**]

251.13 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(c-t)(d-t)}} = \frac{b-d}{a-d}\,g\int_0^{u_1}\operatorname{dc}^2 u\,du.$$

[See **321.02.**]

251.14 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{c-t}{(a-t)(b-t)(d-t)}} = \frac{c-d}{b-d}\, g \int_0^{u_1} \mathrm{nd}^2 u\, du\,.$$
[See **315.02.**]

251.15 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{c-t}{(a-t)(b-t)(d-t)}} = \frac{c-d}{a-d}\, g \int_0^{u_1} \mathrm{nc}^2 u\, du\,.$$
[See **313.02.**]

251.16 $$\int_y^d \sqrt{\frac{(c-t)(d-t)}{(a-t)(b-t)}}\, dt = (c-d)^2 \alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\,.$$
[See **362.15.**]

251.17 $$\int_y^d \sqrt{\frac{(c-t)(a-t)}{(b-t)(d-t)}}\, dt = (a-c)(c-d)\alpha^2 g \int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\,.$$
[See **362.16.**]

251.18 $$\int_y^d \sqrt{\frac{(c-t)(b-t)}{(a-t)(d-t)}}\, dt = (c-d)(b-d)\, g \int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\,.$$
[See **362.17.**]

251.19 $$\int_y^d \sqrt{\frac{(a-t)(b-t)}{(c-t)(d-t)}}\, dt = (a-d)(b-d)\, g \int_0^{u_1} \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\, du\,.$$
[See **362.20.**]

251.20 $$\int_y^d \sqrt{\frac{(a-t)(d-t)}{(c-t)(b-t)}}\, dt = (a-d)(c-d)\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\,.$$
[See **362.18.**]

251.21 $$\int_y^d \sqrt{\frac{(b-t)(d-t)}{(c-t)(a-t)}}\, dt = (b-d)(c-d)\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\,.$$
[See **362.19.**]

251.22 $$\int_y^d \frac{dt}{(a-t)^m \sqrt{(a-t)(b-t)(c-t)(d-t)}} = \frac{g}{(a-d)^m} \int_0^{u_1} \frac{(1-\alpha^2 \mathrm{sn}^2 u)^m}{(1-\mathrm{sn}^2 u)^m}\, du\,.$$
[See **340.04.**]

251.23 $$\int_y^d \frac{dt}{(b-t)^m \sqrt{(a-t)(b-t)(c-t)(d-t)}} = \frac{g}{(b-d)^m} \int_0^{u_1} \frac{(1-\alpha^2 \mathrm{sn}^2 u)^m}{(1-k^2 \mathrm{sn}^2 u)^m}\, du\,.$$
[See **340.04.**]

251.24 $$\int_y^d \frac{dt}{(c-t)^m \sqrt{(a-t)(b-t)(c-t)(d-t)}} = \frac{g}{(c-d)^m} \int_0^{u_1} (1-\alpha^2 \mathrm{sn}^2 u)\, du\,.$$
[See **331.03.**]

251.25 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{(b-t)(d-t)}{(c-t)(a-t)}} = (b-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du.$$

[See **362.11**.]

251.26 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{(b-t)(d-t)}{(c-t)(a-t)}} = \frac{(b-d)(c-d)\,g}{a-c}\int_0^{u_1} \frac{\operatorname{dn}^2 u \operatorname{tn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.14**.]

251.27 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{(c-t)(d-t)}{(a-t)(b-t)}} = \frac{(c-d)^2\alpha^2 g}{a-d}\int_0^{u_1} \frac{\operatorname{tn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.06**.]

251.28 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{(c-t)(d-t)}{(a-t)(b-t)}} = \frac{(c-d)^2\alpha^2 g}{b-d}\int_0^{u_1} \frac{\operatorname{sd}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.07**.]

251.29 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{(c-t)(a-t)}{(d-t)(b-t)}} = \frac{(a-c)(c-d)\,\alpha^2 g}{b-d}\int_0^{u_1} \frac{\operatorname{cd}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.08**.]

251.30 $$\int_y^d \frac{dt}{a-t}\sqrt{\frac{(c-t)(b-t)}{(a-t)(d-t)}} = \frac{(c-d)(b-d)\,g}{a-d}\int_0^{u_1} \frac{\operatorname{dc}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.09**.]

251.31 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{(a-t)(b-t)}{(c-t)(d-t)}} = \frac{(a-d)(b-d)\,g}{c-d}\int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.12**.]

251.32 $$\int_y^d \frac{dt}{c-t}\sqrt{\frac{(a-t)(d-t)}{(c-t)(b-t)}} = (a-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.10**.]

251.33 $$\int_y^d \frac{dt}{b-t}\sqrt{\frac{(a-t)(d-t)}{(c-t)(b-t)}} = \frac{(a-d)(c-d)\,\alpha^2 g}{b-d}\int_0^{u_1} \frac{\operatorname{sd}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$

[See **362.13**.]

251.34 $$\left\{\begin{aligned} &\int_y^d \sqrt{\frac{(d-t)(b-t)(c-t)}{a-t}}\,dt \\ &\quad = (c-d)^2 (b-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}. \end{aligned}\right.$$

[See **362.21**.]

251.35
$$\begin{cases} \displaystyle\int_y^d \sqrt{\frac{(a-t)(c-t)(d-t)}{b-t}}\,dt \\ \displaystyle\qquad = (c-d)^2 (a-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}. \end{cases}$$
[See **362.23**.]

251.36
$$\begin{cases} \displaystyle\int_y^d \sqrt{\frac{(a-t)(b-t)(d-t)}{c-t}}\,dt \\ \displaystyle\qquad = (a-d)(b-d)(c-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u}{(1-\alpha^2 \operatorname{sn}^2 u)^3}\,du. \end{cases}$$
[See **362.24**.]

251.37
$$\begin{cases} \displaystyle\int_y^d \sqrt{\frac{(a-t)(b-t)(c-t)}{d-t}}\,dt \\ \displaystyle\qquad = (a-d)(b-d)(c-d)\,g \int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}. \end{cases}$$
[See **362.22**.]

251.38
$$\begin{cases} \displaystyle\int_y^d \sqrt{(a-t)(b-t)(c-t)(d-t)}\,dt \\ \displaystyle = (a-d)(b-d)(c-d)^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u}{(1-\alpha^2 \operatorname{sn}^2 u)^4}\,du. \end{cases}$$
[See **362.25**.]

251.39
$$\begin{cases} \displaystyle\int_y^d \frac{dt}{(p-t)^m \sqrt{(a-t)(b-t)(c-t)(d-t)}} \\ \displaystyle\qquad = \frac{g}{(p-d)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\alpha_3^2 \operatorname{sn}^2 u)^m}\,du, \end{cases}$$
[See **340.04**.]

where $\alpha_3^2 = (p-c)(a-d)/(p-d)(a-c), \quad p \neq d.$

251.40
$$\begin{cases} \displaystyle\int_y^d \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(c-t)(d-t)}} \\ \displaystyle\qquad = \frac{p_1-d}{p-d}\, g \int_0^{u_1} \frac{1-\alpha_1^2 \operatorname{sn}^2 u}{1-\alpha_3^2 \operatorname{sn}^2 u}\,du, \end{cases}$$
[See **340.01**.]

where $\alpha_1^2 = (p_1-c)(a-d)/(p_1-d)(a-c),$

$\alpha_3^2 = (p-c)(a-d)/(p-d)(a-c), \qquad p \neq d.$

251.41
$$\int_y^d \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(c-t)(d-t)}} = g \int_0^{u_1} R\left[\frac{d-c\,\alpha^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$, $\sqrt{c-t}$ and $\sqrt{t-d}$, $(a>b>c\geqq y>d)$

$$\operatorname{sn}^2 u=\frac{(a-c)(t-d)}{(c-d)(a-t)},\quad k^2=\frac{(a-b)(c-d)}{(a-c)(b-d)},\quad g=\frac{2}{\sqrt{(a-c)(b-d)}},$$
$$\varphi=\operatorname{am} u_1=\sin^{-1}\sqrt{\frac{(a-c)(y-d)}{(c-d)(a-y)}},\quad \alpha^2=\frac{d-c}{a-c}<0,\quad \operatorname{sn} u_1=\sin\varphi.$$

252.00 $$\begin{cases}\displaystyle\int_d^y\frac{dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}}=g\int_0^{u_1}du=g\,u_1=g\operatorname{sn}^{-1}(\sin\varphi,k)\\ \qquad=g\,F(\varphi,k), \qquad [\text{cf. } \mathbf{253.00}.]\end{cases}$$

252.01 $$\begin{cases}\displaystyle\int_d^y\frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}}=\frac{b-d}{a-d}\,g\int_0^{u_1}\operatorname{dn}^2 u\,du\\ \qquad=\dfrac{b-d}{a-d}\,g\,E(u)=\dfrac{b-d}{a-d}\,g\,E(\varphi,k).\end{cases}$$

252.02 $$\begin{cases}\displaystyle\int_d^y\sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}}\,dt=(a-d)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u}\\ \qquad=(a-d)\,g\,\Pi(\varphi,\alpha^2,k),\quad [\alpha^2 \text{ given above}]. \quad [\text{See } \mathbf{400}.]\end{cases}$$

252.03 $$\int_d^y\frac{dt}{c-t}\sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}}=\frac{b-d}{c-d}\,g\int_0^{u_1}\operatorname{dc}^2 u\,du,\quad y\neq c.$$ [See **321.02**.]

252.04 $$\int_d^y\frac{dt}{c-t}\sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}}=\frac{a-d}{c-d}\,g\int_0^{u_1}\operatorname{nc}^2 u\,du,\quad y\neq c.$$ [See **313.02**.]

252.05 $$\int_d^y\frac{dt}{a-t}\sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}}=\frac{c-d}{a-c}\,g\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02**.]

252.06 $$\int_d^y\frac{dt}{c-t}\sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}}=\frac{a-d}{a-c}\,g\int_0^{u_1}\operatorname{tn}^2 u\,du,\quad y\neq c.$$ [See **316.02**.]

252.07 $$\int_d^y\frac{dt}{b-t}\sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}}=\frac{a-d}{d-b}\,g\,\alpha^2\int_0^{u_1}\operatorname{sd}^2 u\,du.$$ [See **318.02**.]

252.08 $$\int_d^y \frac{dt}{a-t}\sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}} = \frac{c-d}{a-d}\, g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02**.]

252.09 $$\int_d^y \frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}} = \frac{a-d}{b-d}\, g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02**.]

252.10 $$\int_d^y \frac{dt}{b-t}\sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}} = \frac{c-d}{b-d}\, g\int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02**.]

252.11 $$\int_d^y \frac{t^m\, dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}} = d^m g\int_0^{u_1} \frac{(1-\alpha_1^2 \mathrm{sn}^2 u)^m}{(1-\alpha^2 \mathrm{sn}^2 u)^m}\, du,$$

[See **340.04**.]

where $$\alpha_1^2 = a(d-c)/(a-c)\, d.$$

252.12 $$\int_d^y \frac{dt}{t^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{d^m}\int_0^{u_1} \frac{(1-\alpha^2 \mathrm{sn}^2 u)^m}{\left(1-\frac{a\alpha^2}{d}\mathrm{sn}^2 u\right)^m}\, du.$$

[See **340.04**.]

252.13 $$\int_d^y \sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}}\, dt = (c-d)\, g\int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$

[See **338.01**.]

252.14 $$\int_d^y \sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}}\, dt = (d-a)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$

[See **337.01**.]

252.15 $$\int_d^y \sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}}\, dt = (b-d)\, g\int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$

[See **339.01**.]

252.16 $$\int_d^y \sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}}\, dt = (d-a)(c-d)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}.$$

[See **362.18**.]

252.17 $$\int_d^y \sqrt{\frac{(c-t)(a-t)}{(b-t)(t-d)}}\, dt = (a-d)(c-d)\, g\int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^2}.$$

[See **362.16**.]

252.18 $$\int_d^y \sqrt{\frac{(c-t)(b-t)}{(a-t)(t-d)}}\,dt = (b-d)(c-d)\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.20**.]

252.19 $$\int_d^y \sqrt{\frac{(a-t)(b-t)}{(c-t)(t-d)}}\,dt = (a-d)(b-d)\,g\int_0^{u_1} \frac{\operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.17**.]

252.20 $$\int_d^y \sqrt{\frac{(a-t)(t-d)}{(b-t)(c-t)}}\,dt = -(a-d)^2\alpha^2\,g\int_0^{u_1} \frac{\operatorname{sn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.15**.]

252.21 $$\int_d^y \sqrt{\frac{(b-t)(t-d)}{(a-t)(c-t)}}\,dt = (a-d)(d-b)\,\alpha^2\,g\int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.19**.]

252.22 $$\int_d^y \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(a-d)^m}\int_0^{u_1} (1-\alpha^2\operatorname{sn}^2 u)^m\,du.$$

[See **331.03**.]

252.23 $$\left\{\begin{aligned} &\int_d^y \frac{dt}{(b-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} \\ &\qquad = \frac{g}{(b-d)^m}\int_0^{u_1} \frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-k^2\operatorname{sn}^2 u)^m}\,du. \end{aligned}\right.$$

[See **340.04**.]

252.24 $$\left\{\begin{aligned} &\int_d^y \frac{dt}{(c-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} \\ &\qquad = \frac{g}{(c-d)^m}\int_0^{u_1} \frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m}\,du, \quad y \neq c. \end{aligned}\right.$$

[See **340.04**.]

252.25 $$\int_d^y \frac{dt}{a-t}\sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}} = (d-c)\,\alpha^2\,g\int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **362.10**.]

252.26 $$\int_d^y \frac{dt}{b-t}\sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}} = \frac{(d-a)(c-d)\,\alpha^2 g}{b-d}\int_0^{u_1} \frac{\operatorname{sd}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **362.13**.]

252.27 $$\int_d^y \frac{dt}{b-t}\sqrt{\frac{(c-t)(a-t)}{(b-t)(t-d)}} = \frac{(a-d)(c-d)}{b-d}\, g\int_0^{u_1} \frac{\mathrm{cd}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$
[See **362.08**.]

252.28 $$\int_d^y \frac{dt}{a-t}\sqrt{\frac{(c-t)(b-t)}{(a-t)(t-d)}} = \frac{(b-d)(c-d)}{a-d}\, g\int_0^{u_1} \frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$
[See **362.12**.]

252.29 $$\int_d^y \frac{dt}{c-t}\sqrt{\frac{(a-t)(b-t)}{(c-t)(t-d)}} = \frac{(a-d)(b-d)}{c-d}\, g\int_0^{u_1} \frac{\mathrm{dc}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}, \quad y \neq c.$$
[See **362.09**.]

252.30 $$\int_d^y \frac{dt}{b-t}\sqrt{\frac{(a-t)(t-d)}{(b-t)(c-t)}} = \frac{-(a-d)^2\alpha^2 g}{b-d}\int_0^{u_1} \frac{\mathrm{sd}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$
[See **362.07**.]

252.31 $$\int_d^y \frac{dt}{c-t}\sqrt{\frac{(a-t)(t-d)}{(b-t)(c-t)}} = \frac{(a-d)^2\alpha^2 g}{c-d}\int_0^{u_1} \frac{\mathrm{tn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}, \quad y \neq c.$$
[See **362.06**.]

252.32 $$\int_d^y \frac{dt}{a-t}\sqrt{\frac{(b-t)(t-d)}{(a-t)(c-t)}} = (d-b)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$
[See **362.11**.]

252.33 $$\int_d^y \frac{dt}{c-t}\sqrt{\frac{(b-t)(t-d)}{(a-t)(c-t)}} = \frac{(a-d)(d-b)}{c-d}\, g\int_0^{u_1} \frac{\mathrm{tn}^2 u\,\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}, \quad y \neq c.$$
[See **362.14**.]

252.34 $$\int_d^y \sqrt{\frac{(b-t)(c-t)(t-d)}{a-t}}\, dt = (a-d)(d-b)(c-d)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^3}\, du.$$
[See **362.24**.]

252.35 $$\int_d^y \sqrt{\frac{(a-t)(c-t)(t-d)}{b-t}}\, dt = (a-d)^2(d-c)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$
[See **362.23**.]

252.36 $$\int_d^y \sqrt{\frac{(a-t)(b-t)(t-d)}{c-t}}\, dt = (a-d)^2(d-b)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$
[See **362.21**.]

252.37
$$\begin{cases} \displaystyle\int_d^y \sqrt{(a-t)(b-t)(c-t)(t-d)}\, dt \\ \displaystyle\quad = (a-d)^2 (d-b)(c-d)\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u}{(1-\alpha^2 \operatorname{sn}^2 u)^4}\, du. \end{cases}$$ [See **362.25**.]

252.38
$$\int_d^y \sqrt{\frac{(a-t)(b-t)(c-t)}{t-d}}\, dt = (a-d)(b-d)(c-d)\, g \int_0^{u_1} \frac{\operatorname{dn}^2 u \operatorname{cn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$
[See **362.22**.]

252.39
$$\int_d^y \frac{dt}{(p-t)^m \sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(p-d)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\alpha_3^2 \operatorname{sn}^2 u)^m}\, du,$$
[See **340.04**.]

where α^2 is given above and $\alpha_3^2 = \alpha^2 (p-a)/(p-d)$, $p \neq d$.

252.40
$$\int_d^y \frac{(p_1 - t)\, dt}{(p-t)\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{p_1 - d}{p-d}\, g \int_0^{u_1} \frac{1-\alpha_2^2 \operatorname{sn}^2 u}{1-\alpha_3^2 \operatorname{sn}^2 u}\, du,$$
[See **340.01**.]

where $\alpha_2^2 = \alpha^2 (p_1 - a)/(p_1 - d)$, $\alpha_3^2 = \alpha^2 (p-a)/(p-d)$, $p \neq d$.

252.41
$$\int_d^y \frac{R(t)\, dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}} = g \int_0^{u_1} R\left[\frac{d - a\alpha^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$, $\sqrt{c-t}$ and $\sqrt{t-d}$, $(a > b > c > y \geq d)$

$$\operatorname{sn}^2 u = \frac{(b-d)(c-t)}{(c-d)(b-t)}, \quad k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad g = \frac{2}{\sqrt{(a-c)(b-d)}},$$
$$k^2 < \alpha^2 = \frac{c-d}{b-d} < 1, \; \varphi = \operatorname{am} u_1 = \sin^{-1} \sqrt{\frac{(b-d)(c-y)}{(c-d)(b-y)}}, \; \operatorname{sn} u_1 = \sin\varphi.$$

253.00
$$\begin{cases} \displaystyle\int_y^c \frac{dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}} = g \int_0^{u_1} du = g\, u_1 = g \operatorname{sn}^{-1}(\sin\varphi, k) \\ \qquad = g F(\varphi, k), \end{cases}$$ [cf. **252.00**].

253.01 $$\left\{\begin{aligned}\int_y^c \frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}} &= \frac{a-c}{b-c}\,g\int_0^{u_1}\mathrm{dn}^2 u\,du\\ &= \frac{a-c}{b-c}\,g\,E(u_1) = \frac{a-c}{b-c}\,g\,E(\varphi,k).\end{aligned}\right.$$

253.02 $$\left\{\begin{aligned}\int_y^c \sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}}\,dt &= (b-c)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ &= (b-c)\,g\,\Pi(\varphi,\alpha^2,k).\end{aligned}\right.$$ [See **400.**]

253.03 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}} = \frac{b-c}{c-d}\,g\int_0^{u_1}\mathrm{nc}^2 u\,du,\quad y \neq d.$$ [See **313.02.**]

253.04 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}} = \frac{a-c}{c-d}\,g\int_0^{u_1}\mathrm{dc}^2 u\,du,\quad y \neq d.$$ [See **321.02.**]

253.05 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}} = \frac{c-d}{a-c}\,g\int_0^{u_1}\mathrm{cd}^2 u\,du.$$ [See **320.02.**]

253.06 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}} = \frac{b-c}{b-d}\,g\int_0^{u_1}\mathrm{tn}^2 u\,du,\quad y \neq d.$$ [See **316.02.**]

253.07 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}} = \frac{c-d}{b-c}\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02.**]

253.08 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(c-t)(t-d)}} = \frac{b-c}{a-c}\,g\int_0^{u_1}\mathrm{nd}^2 u\,du.$$ [See **315.02.**]

253.09 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}} = g\,\alpha^2\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02.**]

253.10 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}} = \frac{b-c}{a-c}\,g\,\alpha^2\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02.**]

253.11 $$\int_y^c \frac{t^m\,dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}} = c^m g\int_0^{u_1}\frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^m}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}\,du,$$ [See **340.04.**]

where $$\alpha_1^2 = b(c-d)/c(b-d).$$

253.12 $$\int_y^c \frac{dt}{t^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{c^m}\int_0^{u_1}\frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-\alpha_1^2\operatorname{sn}^2 u)^m}\,du,$$

[See **340.04**.]

where $\alpha_1^2 = b\,\alpha^2/c$; α^2 given above.

253.13 $$\int_y^c \sqrt{\frac{c-t}{(a-t)(b-t)(t-d)}}\,dt = (b-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **337.01**.]

253.14 $$\int_y^c \sqrt{\frac{t-d}{(a-t)(b-t)(c-t)}}\,dt = (c-d)\,g\int_0^{u_1}\frac{\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **338.01**.]

253.15 $$\int_y^c \sqrt{\frac{a-t}{(b-t)(c-t)(t-d)}}\,dt = (a-c)\,g\int_0^{u_1}\frac{\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$

[See **339.01**.]

253.16 $$\int_y^c \sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}}\,dt = (b-c)\,(c-d)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.18**.]

253.17 $$\int_y^c \sqrt{\frac{(c-t)(a-t)}{(b-t)(t-d)}}\,dt = (a-c)\,(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{dn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.19**.]

253.18 $$\int_y^c \sqrt{\frac{(c-t)(b-t)}{(a-t)(t-d)}}\,dt = (b-c)^2\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.15**.]

253.19 $$\int_y^c \sqrt{\frac{(b-t)(a-t)}{(c-t)(t-d)}}\,dt = (a-c)\,(b-c)\,g\int_0^{u_1}\frac{\operatorname{dn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.17**.]

253.20 $$\int_y^c \sqrt{\frac{(a-t)(t-d)}{(b-t)(c-t)}}\,dt = (a-c)\,(c-d)\,g\int_0^{u_1}\frac{\operatorname{cn}^2 u\operatorname{dn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.20**.]

253.21 $$\int_y^c \sqrt{\frac{(b-t)(t-d)}{(a-t)(c-t)}}\,dt = (b-c)\,(c-d)\,g\int_0^{u_1}\frac{\operatorname{cn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^2}.$$

[See **362.16**.]

253.22 $$\int_y^c \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(a-c)^m}\int_0^{u_1}\frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-k^2\operatorname{sn}^2 u)^m}\,du.$$
[See **340.04**.]

253.23 $$\left\{\begin{aligned}&\int_y^c \frac{dt}{(t-d)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(c-d)^m}\int_0^{u_1}\frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m}\,du,\\ &y \neq d.\end{aligned}\right.$$
[See **340.04**.]

253.24 $$\int_y^c \frac{dt}{(b-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(b-c)^m}\int_0^{u_1}(1-\alpha^2\operatorname{sn}^2 u)^m\,du.$$
[See **331.03**.]

253.25 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}} = \frac{(b-c)(c-d)\alpha^2}{a-c}\,g\int_0^{u_1}\frac{\operatorname{sd}^2 u\operatorname{cn}^2 u}{1-\alpha^2\operatorname{sn}^2 u}\,du.$$
[See **362.13**.]

253.26 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{(c-t)(t-d)}{(a-t)(b-t)}} = (c-d)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$
[See **362.10**.]

253.27 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{(c-t)(a-t)}{(b-t)(t-d)}} = (a-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$
[See **362.11**.]

253.28 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{(c-t)(a-t)}{(b-t)(t-d)}} = \frac{(a-c)(b-c)}{b-d}\,g\int_0^{u_1}\frac{\operatorname{tn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y \neq d.$$
[See **362.14**.]

253.29 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{(c-t)(b-t)}{(a-t)(t-d)}} = \frac{(b-c)^2 g}{b-d}\int_0^{u_1}\frac{\operatorname{tn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y \neq d.$$
[See **362.06**.]

253.30 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{(c-t)(b-t)}{(a-t)(t-d)}} = \frac{(b-c)^2\alpha^2 g}{a-c}\int_0^{u_1}\frac{\operatorname{sd}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$
[See **362.07**.]

253.31 $$\int_y^c \frac{dt}{t-d}\sqrt{\frac{(a-t)(b-t)}{(c-t)(t-d)}} = \frac{(a-c)(b-c)\,g}{c-d}\int_0^{u_1}\frac{\operatorname{dc}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y \neq d.$$
[See **362.09**.]

253.32 $$\int_y^c \frac{dt}{b-t}\sqrt{\frac{(a-t)(t-d)}{(b-t)(c-t)}} = \frac{(a-c)(c-d)\,g}{b-c}\int_0^{u_1}\frac{\operatorname{cn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$
[See **362.12**.]

253.33 $$\int_y^c \frac{dt}{a-t}\sqrt{\frac{(b-t)(t-d)}{(a-t)(c-t)}} = \frac{(b-c)(c-d)g}{a-c}\int_0^{u_1}\frac{\mathrm{cd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$

[See **362.08.**]

253.34 $$\int_y^c \sqrt{\frac{(a-t)(c-t)(t-d)}{b-t}}\,dt = (a-c)(b-c)(c-d)\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}\,du.$$

[See **362.24.**]

253.35 $$\int_y^c \sqrt{\frac{(a-t)(b-t)(t-d)}{c-t}}\,dt = (a-c)(b-c)(c-d)g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$

[See **362.22.**]

253.36 $$\int_y^c \sqrt{\frac{(b-t)(c-t)(t-d)}{a-t}}\,dt = (b-c)^2(c-d)\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$

[See **362.23.**]

253.37 $$\int_y^c \sqrt{\frac{(a-t)(b-t)(c-t)}{t-d}}\,dt = (b-c)^2(a-c)\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$

[See **362.21.**]

253.38 $$\left\{\begin{aligned}&\int_y^c \sqrt{(a-t)(b-t)(c-t)(t-d)}\,dt\\ &= g(a-c)(c-d)(b-c)^2\alpha^2\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u}{(1-\alpha^2\,\mathrm{sn}^2 u)^4}\,du.\end{aligned}\right.$$ [See **362.25.**]

253.39 $$\int_y^c \frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(p-c)^m}\int_0^{u_1}\frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-\alpha_3^2\,\mathrm{sn}^2 u)^m}\,du,$$

[See **340.04.**]

where $\quad \alpha_3^2 = (c-d)(p-b)/(b-d)(p-c), \quad p \neq c.$

253.40 $$\int_y^c \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{p_1-c}{p-c}\cdot g\int_0^{u_1}\frac{1-\alpha_2^2\,\mathrm{sn}^2 u}{1-\alpha_3^2\,\mathrm{sn}^2 u}\,du,$$

[See **340.01.**]

where $\quad \alpha_2^2 = (c-d)(p_1-b)/(b-d)(p_1-c),$

$\alpha_3^2 = (c-d)(p-b)/(b-d)(p-c), \quad p \neq c.$

253.41 $$\int_y^c \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(c-t)(t-d)}} = g\int_0^{u_1} R\left[\frac{c-b\,\alpha^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$, $\sqrt{t-c}$ and $\sqrt{t-d}$, $(a>b\geqq y>c>d)$

$$\operatorname{sn}^2 u=\frac{(b-d)(t-c)}{(b-c)(t-d)},\quad k^2=\frac{(b-c)(a-d)}{(a-c)(b-d)},\quad g=\frac{2}{\sqrt{(a-c)(b-d)}},$$

$$0<\alpha^2=\frac{b-c}{b-d}<k^2,\quad \varphi=\operatorname{am} u_1=\sin^{-1}\sqrt{\frac{(b-d)(y-c)}{(b-c)(y-d)}},\quad \operatorname{sn} u_1=\sin\varphi.$$

254.00 $$\int_c^y\frac{dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}}=g\int_0^{u_1}du=g\,u_1=g\operatorname{sn}^{-1}(\sin\varphi,k)=gF(\varphi,k),\ \text{[cf. \textbf{255.00}.]}$$

254.01 $$\int_c^y\frac{dt}{t-d}\sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}}=\frac{a-c}{c-d}g\int_0^{u_1}\operatorname{dn}^2u\,du=\frac{a-c}{c-d}g\,E(u_1)=\frac{a-c}{c-d}g\,E(\varphi,k).$$

254.02 $$\int_c^y\sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}}\,dt=(c-d)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2u}=(c-d)\,g\,\Pi(\varphi,\alpha^2,k),\ (\alpha^2\text{ given above}).\ \text{[See \textbf{400}.]}$$

254.03 $$\int_c^y\frac{dt}{a-t}\sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}}=\frac{c-d}{a-c}g\int_0^{u_1}\operatorname{nd}^2u\,du.\ \text{[See \textbf{315.02}.]}$$

254.04 $$\int_c^y\frac{dt}{t-d}\sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}}=\alpha^2 g\int_0^{u_1}\operatorname{sn}^2u\,du.\ \text{[See \textbf{310.02}.]}$$

254.05 $$\int_c^y\frac{dt}{b-t}\sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}}=\frac{c-d}{b-c}g\int_0^{u_1}\operatorname{nc}^2u\,du,\quad y\neq b.\ \text{[See \textbf{313.02}.]}$$

254.06 $$\int_c^y\frac{dt}{b-t}\sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}}=\frac{a-c}{b-c}g\int_0^{u_1}\operatorname{dc}^2u\,du,\quad y\neq b.\ \text{[See \textbf{321.02}.]}$$

254.07 $$\int_c^y\frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}}=\frac{b-c}{a-c}g\int_0^{u_1}\operatorname{cd}^2u\,du.\ \text{[See \textbf{320.02}.]}$$

254.08 $$\int_c^y\frac{dt}{b-t}\sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}}=\frac{c-d}{b-d}g\int_0^{u_1}\operatorname{tn}^2u\,du,\quad y\neq b.\ \text{[See \textbf{316.02}.]}$$

254.09 $$\int_c^y \frac{dt}{a-t}\sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}} = \frac{c-d}{a-c}\alpha^2 g\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02**.]

254.10 $$\int_c^y \frac{t^m\,dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = c^m g\int_0^{u_1}\frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^m}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}\,du,$$ [See **340.04**.]

where $\alpha_1^2 = (b-c)\,d/c\,(b-d)$, and α^2 is given above.

254.11 $$\int_c^y \frac{dt}{t^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{c^m}\int_0^{u_1}\frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{\left(1-\frac{\alpha^2 d}{c}\,\mathrm{sn}^2 u\right)^m}\,du.$$ [See **340.04**.]

254.12 $$\int_c^y \sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}}\,dt = (c-d)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **337.01**.]

254.13 $$\int_c^y \sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}}\,dt = (a-c)\,g\int_0^{u_1}\frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **339.01**.]

254.14 $$\int_c^y \sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}}\,dt = (b-c)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **338.01**.]

254.15 $$\int_c^y \frac{dt}{t-d}\sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}} = \frac{b-c}{c-d}\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

254.16 $$\int_c^y \sqrt{\frac{(t-c)(t-d)}{(a-t)(b-t)}}\,dt = (c-d)^2\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.15**.]

254.17 $$\int_c^y \sqrt{\frac{(t-c)(a-t)}{(b-t)(t-d)}}\,dt = (a-c)(c-d)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.19**.]

254.18 $$\int_c^y \sqrt{\frac{(a-t)(b-t)}{(t-c)(t-d)}}\,dt = (a-c)(b-c)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.20**.]

254.19 $$\int_c^y \sqrt{\frac{(a-t)(t-d)}{(b-t)(t-c)}}\,dt = (c-d)(a-c)\,g\int_0^{u_1}\frac{\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.17**.]

254.20
$$\int^{y} \sqrt{\frac{(t-c)\,(b-t)}{(a-t)\,(t-d)}}\,dt = (b-c)\,(c-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$
[See **362.18**.]

254.21
$$\int_c^{y} \sqrt{\frac{(b-t)\,(t-d)}{(a-t)\,(t-c)}}\,dt = (b-c)\,(c-d)\,g \int_0^{u_1} \frac{\operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$
[See **362.16**.]

254.22
$$\int_c^{y} \frac{dt}{(a-t)^m \sqrt{(a-t)\,(b-t)\,(t-c)\,(t-d)}} = \frac{g}{(a-c)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-k^2 \operatorname{sn}^2 u)^m}\,du.$$
[See **340.04**.]

254.23
$$\int_c^{y} \frac{dt}{(b-t)^m \sqrt{(a-t)\,(b-t)\,(t-c)\,(t-d)}} = \frac{g}{(b-c)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m}\,du, \quad y \neq b.$$
[See **340.04**.]

254.24
$$\int_c^{y} \frac{dt}{(t-d)^m \sqrt{(a-t)\,(b-t)\,(t-c)\,(t-d)}} = \frac{g}{(c-d)^m} \int_0^{u_1} (1-\alpha^2 \operatorname{sn}^2 u)^m\,du.$$
[See **331.03**.]

254.25
$$\int_c^{y} \frac{dt}{b-t} \sqrt{\frac{(t-c)\,(a-t)}{(b-t)\,(t-d)}} = \frac{(a-c)\,(c-d)}{b-d}\, g \int_0^{u_1} \frac{\operatorname{tn}^2 u \operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}, \quad y \neq b.$$
[See **362.14**.]

254.26
$$\int_c^{y} \frac{dt}{t-d} \sqrt{\frac{(a-t)\,(t-c)}{(b-t)\,(t-d)}} = (a-c)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$
[See **362.11**.]

254.27
$$\int_c^{y} \frac{dt}{t-d} \sqrt{\frac{(a-t)\,(b-t)}{(t-c)\,(t-d)}} = \frac{(a-c)\,(b-c)\,g}{c-d} \int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$
[See **362.12**.]

254.28
$$\int_c^{y} \frac{dt}{a-t} \sqrt{\frac{(t-c)\,(b-t)}{(a-t)\,(t-d)}} = \frac{(b-c)\,(c-d)\,\alpha^2 g}{a-c} \int_0^{u_1} \frac{\operatorname{sd}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$
[See **362.13**.]

254.29
$$\int_c^{y} \frac{dt}{t-d} \sqrt{\frac{(t-c)\,(b-t)}{(a-t)\,(t-d)}} = (b-c)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$
[See **362.10**.]

254.30
$$\int_c^{y} \frac{dt}{a-t} \sqrt{\frac{(b-t)\,(t-d)}{(a-t)\,(t-c)}} = \frac{(b-c)\,(c-d)\,g}{a-c} \int_0^{u_1} \frac{\operatorname{cd}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$
[See **362.08**.]

254.31 $$\int_c^y \frac{dt}{b-t}\sqrt{\frac{(a-t)(t-d)}{(b-t)(t-c)}} = \frac{(c-d)(a-c)g}{b-c}\int_0^{u_1}\frac{\mathrm{dc}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}, \quad y\neq b.$$ [See **362.09**.]

254.32 $$\int_c^y \frac{dt}{a-t}\sqrt{\frac{(t-c)(t-d)}{(a-t)(b-t)}} = \frac{(c-d)^2\alpha^2 g}{a-c}\int_0^{u_1}\frac{\mathrm{sd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.07**.]

254.33 $$\int_c^y \frac{dt}{b-t}\sqrt{\frac{(t-c)(t-d)}{(a-t)(b-t)}} = \frac{(c-d)^2 g}{b-d}\int_0^{u_1}\frac{\mathrm{tn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}, \quad y\neq b.$$ [See **362.06**.]

254.34 $$\int_c^y \sqrt{\frac{(b-t)(t-c)(t-d)}{a-t}}\,dt = (c-d)^2(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$ [See **362.23**.]

254.35 $$\int_c^y \sqrt{\frac{(a-t)(t-c)(t-d)}{b-t}}\,dt = (c-d)^2(a-c)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$ [See **362.21**.]

254.36 $$\int_c^y \sqrt{\frac{(a-t)(b-t)(t-d)}{t-c}}\,dt = (a-c)(b-c)(c-d)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$ [See **362.22**.]

254.37 $$\int_c^y \sqrt{\frac{(a-t)(b-t)(t-c)}{t-d}}\,dt = (a-c)(b-c)(c-d)\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}\,du.$$ [See **362.24**.]

254.38 $$\left\{\begin{aligned}&\int_c^y \sqrt{(a-t)(b-t)(t-c)(t-d)}\,dt\\ &= (a-c)(b-c)(c-d)^2\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,\mathrm{dn}^2 u}{(1-\alpha^2\,\mathrm{sn}^2 u)^4}\,du.\end{aligned}\right.$$ [See **362.25**.]

254.39 $$\int_c^y \frac{dt}{(p-t)^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{(p-c)^m}\int_0^{u_1}\frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-\alpha_3^2\,\mathrm{sn}^2 u)^m}\,du,$$ [See **340.04**.]

where $\alpha_3^2 = (b-c)(p-d)/(b-d)(p-c), \quad p\neq c.$

254.40 $$\int_c^y \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{p_1-c}{p-c}\,g\int_0^{u_1}\frac{1-\alpha_2^2\,\mathrm{sn}^2 u}{1-\alpha_3^2\,\mathrm{sn}^2 u}\,du,$$ [See **340.01**.]

where $\quad \alpha_2^2 = (b-c)(p_1-d)/(b-d)(p_1-c),$

$\alpha_3^2 = (b-c)(p-d)/(b-d)(p-c), \qquad p \neq c.$

$$\textbf{254.41}\quad \int_c^y \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = g\int_0^{u_1} R\left[\frac{c-\alpha^2 d\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\right] du,$$

where α^2 is given above and $R(t)$ is a rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{b-t}$, $\sqrt{t-c}$ and $\sqrt{t-d}$, $(a > b > y \geqq c > d)$

$$\mathrm{sn}^2 u = \frac{(a-c)(b-t)}{(b-c)(a-t)}, \qquad k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}, \qquad g = \frac{2}{\sqrt{(a-c)(b-d)}},$$

$$0 < \alpha^2 = \frac{b-c}{a-c} < k^2, \quad \varphi = \mathrm{am}\,u_1 = \sin^{-1}\sqrt{\frac{(a-c)(b-y)}{(b-c)(a-y)}}, \quad \mathrm{sn}\,u_1 = \sin\varphi.$$

$$\textbf{255.00}\quad \left\{\begin{aligned} &\int_y^b \frac{dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k)\\ &\qquad = gF(\varphi, k), \ [\text{cf. } \textbf{254.00}].\end{aligned}\right.$$

$$\textbf{255.01}\quad \left\{\begin{aligned} &\int_y^b \frac{dt}{a-t}\sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}} = \frac{b-d}{a-b}\,g\int_0^{u_1}\mathrm{dn}^2 u\,du\\ &\qquad = \frac{b-d}{a-b}\,g\,E(u_1) = \frac{b-d}{a-b}\,g\,E(\varphi, k).\end{aligned}\right.$$

$$\textbf{255.02}\quad \left\{\begin{aligned} &\int_y^b \sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}}\,dt = (a-b)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ &\qquad = (a-b)\,g\,\Pi(\varphi, \alpha^2, k), \ (\alpha^2 \text{ given above}). \ [\text{See } \textbf{400}.]\end{aligned}\right.$$

$$\textbf{255.03}\quad \int_y^b \frac{dt}{t-c}\sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}} = \frac{a-b}{b-c}\,\alpha^2 g\int_0^{u_1}\mathrm{tn}^2 u\,du, \quad y \neq c.$$

[See **316.02**.]

$$\textbf{255.04}\quad \int_y^b \frac{dt}{t-c}\sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}} = \frac{a-b}{b-c}\,g\int_0^{u_1}\mathrm{nc}^2 u\,du, \quad y \neq c.$$

[See **313.02**.]

$$\textbf{255.05}\quad \int_y^b \frac{dt}{t-c}\sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}} = \frac{b-d}{b-c}\,g\int_0^{u_1}\mathrm{dc}^2 u\,du, \quad y \neq c.$$

[See **321.02**.]

255.06 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}} = \frac{a-b}{b-d}\alpha^2 g\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02**.]

255.07 $$\int_y^b \frac{dt}{a-t}\sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}} = \alpha^2 g\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02**.]

255.08 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}} = \frac{a-b}{b-d} g\int_0^{u_1}\mathrm{nd}^2 u\,du.$$ [See **315.02**.]

255.09 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}} = \frac{b-c}{b-d} g\int_0^{u_1}\mathrm{cd}^2 u\,du.$$ [See **320.02**.]

255.10 $$\int_y^b \frac{dt}{a-t}\sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}} = \frac{b-c}{a-b} g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

255.11 $$\int_y^b \sqrt{\frac{(t-c)(t-d)}{(a-t)(b-t)}}\,dt = (b-c)(b-d)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.20**.]

255.12 $$\int_y^b \sqrt{\frac{(t-c)(a-t)}{(b-t)(t-d)}}\,dt = (a-b)(b-c)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.16**.]

255.13 $$\int_y^b \sqrt{\frac{(t-c)(b-t)}{(a-t)(t-d)}}\,dt = (a-b)(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.18**.]

255.14 $$\int_y^b \sqrt{\frac{(a-t)(b-t)}{(t-c)(t-d)}}\,dt = (a-b)^2\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.15**.]

255.15 $$\int_y^b \sqrt{\frac{(a-t)(t-d)}{(b-t)(t-c)}}\,dt = (a-b)(b-d)\,g\int_0^{u_1}\frac{\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.17**.]

255.16 $$\int_y^b \sqrt{\frac{(b-t)(t-d)}{(a-t)(t-c)}}\,dt = (a-b)(b-d)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.19**.]

255.17 $$\int_y^b \frac{t^m\,dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = b^m g \int_0^{u_1} \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^m}{(1-\alpha^2 \operatorname{sn}^2 u)^m}\,du,$$ [See **340.04**].

where $$\alpha_1^2 = a(b-c)/b(a-c).$$

255.18 $$\int_y^b \frac{dt}{t^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{b^m}\int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{\left(1-\frac{a\alpha^2}{b}\operatorname{sn}^2 u\right)^m}\,du,$$ $y \neq c$. [See **340.04**.]

255.19 $$\int_y^b \sqrt{\frac{t-c}{(a-t)(b-t)(t-d)}}\,dt = (b-c)\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **338.01**.]

255.20 $$\int_y^b \sqrt{\frac{t-d}{(a-t)(b-t)(t-c)}}\,dt = (b-d)\,g\int_0^{u_1} \frac{\operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **339.01**.]

255.21 $$\int_y^b \sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}}\,dt = (a-b)\,\alpha^2 g\int_0^{u_1} \frac{\operatorname{sn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **337.01**.]

255.22 $$\int_y^b \frac{dt}{(t-c)^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{(b-c)^m}\int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m}\,du,$$ $y \neq c$. [See **340.04**.]

255.23 $$\int_y^b \frac{dt}{(t-d)^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{(b-d)^m}\int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-k^2 \operatorname{sn}^2 u)^m}\,du.$$ [See **340.04**.]

255.24 $$\int_y^b \frac{dt}{(a-t)^m\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{(a-b)^m}\int_0^{u_1} (1-\alpha^2 \operatorname{sn}^2 u)^m\,du.$$ [See **331.03**.]

255.25 $$\int_y^b \frac{dt}{a-t}\sqrt{\frac{(t-c)(t-d)}{(a-t)(b-t)}} = \frac{(b-c)(b-d)}{a-b}\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.12**.]

255.26 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{(t-c)(a-t)}{(b-t)(t-d)}} = \frac{(a-b)(b-c)}{b-d}\,g\int_0^{u_1} \frac{\operatorname{cd}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.08**.]

255.27 $$\int_y^b \frac{dt}{a-t}\sqrt{\frac{(t-c)(b-t)}{(a-t)(t-d)}} = (b-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.10**.]

255.28 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{(t-c)(b-t)}{(a-t)(t-d)}} = \frac{(a-b)(b-c)\,\alpha^2}{b-d}\,g\int_0^{u_1}\frac{\operatorname{sd}^2 u\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.13**.]

255.29 $$\int_y^b \frac{dt}{t-c}\sqrt{\frac{(a-t)(b-t)}{(t-c)(t-d)}} = \frac{(a-b)^2 g}{a-c}\int_0^{u_1}\frac{\operatorname{tn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y\neq c.$$ [See **362.06**.]

255.30 $$\int_y^b \frac{dt}{t-d}\sqrt{\frac{(a-t)(b-t)}{(t-c)(t-d)}} = \frac{(a-b)^2\alpha^2 g}{b-d}\int_0^{u_1}\frac{\operatorname{sd}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.07**.]

255.31 $$\int_y^b \frac{dt}{t-c}\sqrt{\frac{(a-t)(t-d)}{(b-t)(t-c)}} = \frac{(a-b)(b-d)\,g}{b-c}\int_0^{u_1}\frac{\operatorname{dc}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y\neq c.$$ [See **362.09**.]

255.32 $$\int_y^b \frac{dt}{a-t}\sqrt{\frac{(b-t)(t-d)}{(a-t)(t-c)}} = (b-d)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.11**.]

255.33 $$\int_y^b \frac{dt}{t-c}\sqrt{\frac{(b-t)(t-d)}{(a-t)(t-c)}} = \frac{(a-b)(b-d)\,g}{a-c}\int_0^{u_1}\frac{\operatorname{tn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y\neq c.$$ [See **362.14**.]

255.34 $$\int_y^b \sqrt{\frac{(a-t)(t-c)(t-d)}{b-t}}\,dt = (a-b)(b-c)(b-d)\,g\int_0^{u_1}\frac{\operatorname{cn}^2 u\operatorname{dn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^3}.$$ [See **362.22**.]

255.35 $$\int_y^b \sqrt{\frac{(b-t)(t-c)(t-d)}{a-t}}\,dt = g\alpha^2(a-b)(b-d)(b-c)\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn}^2 u}{(1-\alpha^2\operatorname{sn}^2 u)^3}\,du.$$ [See **362.24**.]

255.36 $$\int_y^b \sqrt{\frac{(a-t)(b-t)(t-c)}{t-d}}\,dt = (a-b)^2(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^3}.$$ [See **362.23**.]

255.37 $$\left\{\begin{aligned}&\int_y^b \sqrt{(a-t)(b-t)(t-c)(t-d)}\,dt\\ &\quad= (a-b)^2(b-d)(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn}^2 u}{(1-\alpha^2\operatorname{sn}^2 u)^4}\,du.\end{aligned}\right.$$ [See **362.25**.]

255.38 $$\int_y^b \frac{dt}{(p-t)^m \sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{g}{(p-b)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\alpha_3^2 \operatorname{sn}^2 u)^m} du.$$

[See **340.04**.]

where $\alpha_3^2 = (p-a)(b-c)/(p-b)(a-c), \quad p \neq b.$

255.39 $$\int_y^b \sqrt{\frac{(a-t)(b-t)(t-d)}{t-c}}\, dt = (a-b)^2 (b-d)\, \alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$

[See **362.21**.]

255.40 $$\int_y^b \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(b-t)(t-c)(t-d)}} = \frac{p_1-b}{p-b} g \int_0^{u_1} \frac{1-\alpha_2^2 \operatorname{sn}^2 u}{1-\alpha_3^2 \operatorname{sn}^2 u} du,$$

[See **340.01**.]

where $\alpha_2^2 = (p_1-a)(b-c)/(p_1-b)(a-c),$

$\alpha_3^2 = (p-a)(b-c)/(p-b)(a-c), \quad p \neq b.$

255.41 $$\int_y^b \frac{R(t)\,dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = g \int_0^{u_1} R\left[\frac{b-a\alpha^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$, $\sqrt{t-c}$ and $\sqrt{t-d}$, $(a \geq y > b > c > d)$

$$\operatorname{sn}^2 u = \frac{(a-c)(t-b)}{(a-b)(t-c)}, \quad k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad k^2 < \alpha^2 = \frac{a-b}{a-c} < 1,$$
$$g = \frac{2}{\sqrt{(a-c)(b-d)}}, \quad \varphi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{(a-c)(y-b)}{(a-b)(y-c)}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

256.00 $$\left\{\begin{aligned} &\int_b^y \frac{dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = g\int_0^{u_1} du = g\,u_1 = g \operatorname{sn}^{-1}(\sin\varphi, k) \\ &= g F(\varphi, k), \qquad \text{[cf. } \mathbf{257.00}\text{]}. \end{aligned}\right.$$

256.01 $$\left\{\begin{aligned} &\int_b^y \frac{dt}{t-c}\sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}} = \frac{b-d}{b-c} g \int_0^{u_1} \operatorname{dn}^2 u\, du \\ &= \frac{b-d}{b-c} g E(u_1) = \frac{b-d}{b-c} g E(\varphi, k). \end{aligned}\right.$$

256.02 $$\begin{cases} \int_b^y \sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}}\,dt = (b-c)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ \qquad = (b-c)\,g\,\Pi(\varphi,\alpha^2,k), \end{cases}$$ [See **400.**]

where α^2 is given above.

256.03 $$\int_b^y \frac{dt}{t-d}\sqrt{\frac{t-b}{(a-t)(t-d)(t-c)}} = \frac{b-c}{b-d}\,\alpha^2 g\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02.**]

256.04 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-b}{(a-t)(t-c)(t-d)}} = \frac{b-c}{a-c}\,g\int_0^{u_1}\mathrm{tn}^2 u\,du, \quad y \neq a.$$ [See **316.02.**]

256.05 $$\int_b^y \frac{dt}{t-d}\sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}} = \frac{a-b}{b-d}\,g\int_0^{u_1}\mathrm{cd}^2 u\,du.$$ [See **320.02.**]

256.06 $$\int_b^y \frac{dt}{t-d}\sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}} = \frac{b-c}{b-d}\,g\int_0^{u_1}\mathrm{nd}^2 u\,du.$$ [See **315.02.**]

256.07 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}} = \frac{b-c}{a-b}\,g\int_0^{u_1}\mathrm{nc}^2 u\,du, \quad y \neq a.$$ [See **313.02.**]

256.08 $$\int_b^y \frac{dt}{t-c}\sqrt{\frac{t-b}{(a-t)(t-d)(t-c)}} = \alpha^2 g\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02.**]

256.09 $$\int_b^y \frac{dt}{t-c}\sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}} = \frac{a-b}{b-c}\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02.**]

256.10 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}} = \frac{b-d}{a-b}\,g\int_0^{u_1}\mathrm{dc}^2 u\,du, \quad y \neq a.$$ [See **321.02.**]

256.11 $$\int_b^y \frac{t^m\,dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = b^m g\int_0^{u_1}\frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^m}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}\,du,$$ [See **340.04.**]

where α^2 is given above and $\alpha_1^2 = c(a-b)/b(a-c)$.

256.12 $$\int_b^y \frac{dt}{t^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{b^m}\int_0^{u_1} \frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{\left(1-\frac{c\,\alpha^2}{b}\,\mathrm{sn}^2 u\right)^m}\,du.$$

[See **340.04**.]

256.13 $$\int_b^y \sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}}\,dt = (b-d)\,g\int_0^{u_1}\frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **339.01**.]

256.14 $$\int_b^y \sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}}\,dt = (a-b)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **338.01**.]

256.15 $$\int_b^y \sqrt{\frac{t-b}{(a-t)(t-c)(t-d)}}\,dt = (b-c)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **337.01**.]

256.16 $$\int_b^y \sqrt{\frac{(t-c)(t-d)}{(a-t)(t-b)}}\,dt = (b-c)(b-d)\,g\int_0^{u_1}\frac{\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$

[See **362.17**.]

256.17 $$\int_b^y \sqrt{\frac{(t-c)(a-t)}{(t-b)(t-d)}}\,dt = (a-b)(b-c)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$

[See **362.16**.]

256.18 $$\int_b^y \sqrt{\frac{(t-c)(t-b)}{(a-t)(t-d)}}\,dt = (b-c)^2\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2},$$ [See **362.15**.]

256.19 $$\int_b^y \sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}}\,dt = (a-b)(b-c)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$

[See **362.18**.]

256.20 $$\int_b^y \sqrt{\frac{(a-t)(t-d)}{(t-b)(t-c)}}\,dt = (a-b)(b-d)\,g\int_0^{u_1}\frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$

[See **362.20**.]

256.21 $$\int_b^y \sqrt{\frac{(t-b)(t-d)}{(a-t)(t-c)}}\,dt = (b-c)(b-d)\,\alpha^2 g\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$

[See **362.19**.]

256.22 $$\int_b^y \frac{dt}{(t-c)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(b-c)^m}\int_0^{u_1}(1-\alpha^2\,\mathrm{sn}^2 u)^m\,du.$$

[See **331.03**.]

256.23 $$\int_b^y \frac{dt}{(t-d)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(b-d)^m}\int_0^{u_1}\frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-k^2\,\mathrm{sn}^2 u)^m}\,du.$$

[See **340.04**.]

256.24 $$\left\{ \int_b^y \frac{dt}{(a-t)^m \sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(a-b)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m} du, \quad y \neq a. \right.$$ [See **340.04**.]

256.25 $$\int_b^y \frac{dt}{a-t} \sqrt{\frac{(t-c)(t-d)}{(a-t)(t-b)}} = \frac{(b-c)(b-d)g}{a-b} \int_0^{u_1} \frac{\operatorname{dc}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}, \quad y \neq a.$$ [See **362.09**.]

256.26 $$\int_b^y \frac{dt}{t-d} \sqrt{\frac{(t-c)(a-t)}{(t-b)(t-d)}} = \frac{(b-c)(a-b)g}{b-d} \int_0^{u_1} \frac{\operatorname{cd}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u},$$ [See **362.08**.]

256.27 $$\int_b^y \frac{dt}{a-t} \sqrt{\frac{(t-c)(t-b)}{(a-t)(t-d)}} = \frac{(b-c)^2}{a-c} g \int_0^{u_1} \frac{\operatorname{tn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}, \quad y \neq a.$$ [See **362.06**.]

256.28 $$\int_b^y \frac{dt}{t-d} \sqrt{\frac{(t-c)(t-b)}{(a-t)(t-d)}} = \frac{(b-c)^2 \alpha^2 g}{b-d} \int_0^{u_1} \frac{\operatorname{sd}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.07**.]

256.29 $$\int_b^y \frac{dt}{t-c} \sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}} = (a-b)\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.10**.]

256.30 $$\int_b^y \frac{dt}{t-d} \sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}} = \frac{(a-b)(b-c)\alpha^2 g}{b-d} \int_0^{u_1} \frac{\operatorname{sd}^2 u \operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.13**.]

256.31 $$\int_b^y \frac{dt}{t-c} \sqrt{\frac{(a-t)(t-d)}{(t-b)(t-c)}} = \frac{(a-b)(b-d)g}{b-c} \int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u},$$ [See **362.12**.]

256.32 $$\int_b^y \frac{dt}{a-t} \sqrt{\frac{(t-b)(t-d)}{(a-t)(t-c)}} = \frac{(b-c)(b-d)g}{a-c} \int_0^{u_1} \frac{\operatorname{tn}^2 u \operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}, \quad y \neq a.$$ [See **362.14**.]

256.33 $$\int_b^y \frac{dt}{t-c} \sqrt{\frac{(t-b)(t-d)}{(a-t)(t-c)}} = (b-d)\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **362.11**.]

256.34 $$\int_b^y \sqrt{\frac{(t-b)(t-c)(t-d)}{a-t}}\, dt = (b-c)^2 (b-d) \alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$ [See **362.21**.]

256.35 $$\int_b^y \sqrt{\frac{(t-d)(a-t)(t-c)}{t-b}}\, dt = (b-d)(a-b)(b-c) g \int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$ [See **362.22**.]

256.36 $$\int_b^y \sqrt{\frac{(a-t)(t-b)(t-d)}{t-c}}\,dt = (a-b)(b-c)(b-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$
[See **362.24**.]

256.37 $$\int_b^y \sqrt{\frac{(a-t)(t-b)(t-c)}{t-d}}\,dt = (a-b)(b-c)^2 \alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^3}.$$
[See **362.23**.]

256.38 $$\left\{\begin{aligned} &\int_b^y \sqrt{(a-t)(t-b)(t-c)(t-d)}\,dt \\ &\quad = (b-c)^2 (a-b)(b-d)\,\alpha^2 g \int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u}{(1-\alpha^2 \operatorname{sn}^2 u)^4}\,du. \end{aligned}\right.$$
[See **362.25**.]

256.39 $$\int_b^y \frac{dt}{(t-p)^m \sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(b-p)^m} \int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m}{(1-\alpha_3^2 \operatorname{sn}^2 u)^m}\,du,$$
[See **340.04**.]

where α^2 is given above, $\alpha_3^2 = (p-c)(a-b)/(p-b)(a-c)$, $p \neq b$.

256.40 $$\int_b^y \frac{(t-p_1)\,dt}{(t-p)\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{(b-p_1)\,g}{b-p} \int_0^{u_1} \frac{1-\alpha_2^2 \operatorname{sn}^2 u}{1-\alpha_3^2 \operatorname{sn}^2 u}\,du,$$
[See **340.01**.]

where $\alpha_2^2 = (p_1-c)(a-b)/(p_1-b)(a-c)$,

$\alpha_3^2 = (p-c)(a-b)/(p-b)(a-c)$, $p \neq b$.

256.41 $$\int_b^y \frac{R(t)\,dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = g \int_0^{u_1} R\left[\frac{b-c\,\alpha^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$, $\sqrt{t-c}$ and $\sqrt{t-d}$, $(a > y \geq b > c > d)$

$$\operatorname{sn}^2 u = \frac{(b-d)(a-t)}{(a-b)(t-d)}, \quad k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad g = \frac{2}{\sqrt{(a-c)(b-d)}},$$
$$\alpha^2 = \frac{b-a}{b-d} < 0, \quad \varphi = \operatorname{am} u_1 = \sin^{-1} \sqrt{\frac{(b-d)(a-y)}{(a-b)(y-d)}}, \quad \operatorname{sn} u_1 = \sin\varphi.$$

257.00 $$\left\{\begin{aligned} &\int_y^a \frac{dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = g\int_0^{u_1} du = g\,u_1 = g \operatorname{sn}^{-1}(\sin\varphi, k) \\ &\qquad = g\,F(\varphi, k), \end{aligned}\right.$$
[cf. **256.00**.]

257.01
$$\begin{cases}\displaystyle\int_y^a \frac{dt}{t-d}\sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}} = \frac{a-c}{a-d}\,g\int_0^{u_1}\mathrm{dn}^2 u\,du \\ \displaystyle\qquad = \frac{a-c}{a-d}\,g\,E(u_1) = \frac{a-c}{a-d}\,g\,E(\varphi,k).\end{cases}$$

257.02
$$\begin{cases}\displaystyle\int_y^a \sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}}\,dt = (a-d)\,g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \\ \displaystyle\qquad = (a-d)\,g\,\Pi(\varphi,\alpha^2,k).\end{cases}$$
[See **400.**]

where α^2 is given above.

257.03
$$\int_y^a \frac{dt}{t-c}\sqrt{\frac{t-b}{(a-t)(t-c)(t-d)}} = \frac{a-b}{a-c}\,g\int_0^{u_1}\mathrm{cd}^2 u\,du.$$
[See **320.02.**]

257.04
$$\int_y^a \frac{dt}{t-c}\sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}} = \frac{d-a}{a-c}\,g\,\alpha^2\int_0^{u_1}\mathrm{sd}^2 u\,du.$$
[See **318.02.**]

257.05
$$\int_y^a \frac{dt}{t-d}\sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}} = -\,\alpha^2 g\int_0^{u_1}\mathrm{sn}^2 u\,du.$$
[See **310.02.**]

257.06
$$\int_y^a \frac{dt}{t-c}\sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}} = \frac{a-d}{a-c}\,g\int_0^{u_1}\mathrm{nd}^2 u\,du.$$
[See **315.02.**]

257.07
$$\int_y^a \frac{dt}{t-b}\sqrt{\frac{t-d}{(a-t)(t-b)(t-c)}} = \frac{a-d}{a-b}\,g\int_0^{u_1}\mathrm{nc}^2 u\,du, \quad y\neq b.$$
[See **313.02.**]

257.08
$$\int_y^a \frac{dt}{t-b}\sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}} = \frac{a-d}{b-d}\,g\int_0^{u_1}\mathrm{tn}^2 u\,du, \quad y\neq b.$$
[See **316.02.**]

257.09
$$\int_y^a \frac{dt}{t-d}\sqrt{\frac{t-b}{(a-t)(t-c)(t-d)}} = \frac{a-b}{a-d}\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$
[See **312.02.**]

257.10
$$\int_y^a \frac{dt}{t-b}\sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}} = \frac{a-c}{a-b}\,g\int_0^{u_1}\mathrm{dc}^2 u\,du, \quad y\neq b.$$
[See **321.02.**]

257.11
$$\int_y^a \frac{t^m\,dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = a^m g\int_0^{u_1}\frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^m}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}\,du,$$
[See **340.04.**]

where $\alpha_1^2 = (b-a)\,d/a\,(b-d)$ and α^2 is defined above.

257.12 $$\int_y^a \frac{dt}{t^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{a^m}\int_0^{u_1} \frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{\left(1-\frac{\alpha^2 d}{a}\,\mathrm{sn}^2 u\right)^m}\,du.$$

[See **340.04**.]

257.13 $$\int_y^a \sqrt{\frac{t-c}{(a-t)(t-b)(t-d)}}\,dt = (a-c)\,g\int_0^{u_1} \frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **339.01**.]

257.14 $$\int_y^a \sqrt{\frac{a-t}{(t-b)(t-c)(t-d)}}\,dt = (d-a)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **337.01**.]

257.15 $$\int_y^a \sqrt{\frac{t-b}{(a-t)(t-c)(t-d)}}\,dt = (a-b)\,g\int_0^{u_1} \frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **338.01**.]

257.16 $$\int_y^a \sqrt{\frac{(t-c)(t-d)}{(a-t)(t-b)}}\,dt = (a-c)(a-d)\,g\int_0^{u_1} \frac{\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.17**.]

257.17 $$\int_y^a \sqrt{\frac{(t-c)(a-t)}{(t-b)(t-d)}}\,dt = (d-a)(a-c)\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.19**.]

257.18 $$\int_y^a \sqrt{\frac{(t-c)(t-b)}{(a-t)(t-d)}}\,dt = (a-b)(a-c)\,g\int_0^{u_1} \frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.20**.]

257.19 $$\int_y^a \sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}}\,dt = (d-a)(a-b)\,\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.18**.]

257.20 $$\int_y^a \sqrt{\frac{(a-t)(t-d)}{(t-c)(t-b)}}\,dt = -(a-d)^2\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.15**.]

257.21 $$\int_y^a \sqrt{\frac{(t-b)(t-d)}{(a-t)(t-c)}}\,dt = (a-b)(a-d)\,g\int_0^{u_1} \frac{\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}.$$ [See **362.16**.]

257.22 $$\int_y^a \sqrt{\frac{(t-b)(t-c)(t-d)}{a-t}}\,dt = (a-b)(a-c)(a-d)\,g\int_0^{u_1} \frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$

[See **362.22**.]

257.23 $$\int_y^a \sqrt{\frac{(a-t)(t-c)(t-d)}{t-b}}\,dt = (c-a)(a-d)^2\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}.$$

[See **362.21**.]

257.24 $$\int_y^a \sqrt{\frac{(a-t)(t-b)(t-d)}{t-c}}\,dt = (b-a)(a-d)^2\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^3}.$$ [See **362.23**.]

257.25 $$\int_y^a \sqrt{\frac{(a-t)(t-b)(t-c)}{t-d}}\,dt = (d-a)(a-b)(a-c)\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn}^2 u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^3}.$$ [See **362.24**.]

257.26 $$\begin{cases}\displaystyle\int_y^a \sqrt{(a-t)(t-b)(t-c)(t-d)}\,dt \\ \displaystyle = (a-b)(c-a)(a-d)^2\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn}^2 u}{(1-\alpha^2\operatorname{sn}^2 u)^4}\,du.\end{cases}$$ [See **362.25**.]

257.27 $$\int_y^a \frac{dt}{(t-c)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(a-c)^m}\int_0^{u_1}\frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-k^2\operatorname{sn}^2 u)^m}\,du.$$ [See **340.04**.]

257.28 $$\int_y^a \frac{dt}{(t-d)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(a-b)^m}\int_0^{u_1}(1-\alpha^2\operatorname{sn}^2 u)^m\,du.$$ [See **331.03**.]

257.29 $$\begin{cases}\displaystyle\int_y^a \frac{dt}{(t-b)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(a-d)^m}\int_0^{u_1}\frac{(1-\alpha^2\operatorname{sn}^2 u)^m}{(1-\operatorname{sn}^2 u)^m}\,du, \\ y \neq b.\end{cases}$$ [See **340.04**.]

257.30 $$\int_y^a \frac{dt}{t-b}\sqrt{\frac{(t-c)(t-d)}{(a-t)(t-b)}} = \frac{(a-c)(a-d)}{a-b}\,g\int_0^{u_1}\frac{\operatorname{dc}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u},\quad y \neq b.$$ [See **362.09**.]

257.31 $$\int_y^a \frac{dt}{t-d}\sqrt{\frac{(t-c)(a-t)}{(t-b)(t-d)}} = (c-a)\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.11**.]

257.32 $$\int_y^a \frac{dt}{t-b}\sqrt{\frac{(t-c)(a-t)}{(t-b)(t-d)}} = \frac{(a-d)(a-c)g}{b-d}\int_0^{u_1}\frac{\operatorname{tn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.14**.]

257.33 $$\int_y^a \frac{dt}{t-d}\sqrt{\frac{(t-c)(t-b)}{(a-t)(t-d)}} = \frac{(a-b)(a-c)g}{a-d}\int_0^{u_1}\frac{\operatorname{cn}^2 u\operatorname{dn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.12**.]

257.34 $$\int_y^a \frac{dt}{t-c}\sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}} = \frac{(d-a)(a-b)\alpha^2 g}{a-c}\int_0^{u_1}\frac{\operatorname{sd}^2 u\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.13**.]

257.35 $$\int_y^a \frac{dt}{t-d}\sqrt{\frac{(a-t)(t-b)}{(t-c)(t-d)}} = (b-a)\alpha^2 g\int_0^{u_1}\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u}.$$ [See **362.10**.]

257.36 $\int_y^a \frac{dt}{t-c}\sqrt{\frac{(a-t)(t-d)}{(t-c)(t-b)}} = \frac{(a-d)^2\alpha^2 g}{c-a}\int_0^{u_1}\frac{\mathrm{sd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}$. [See **362.07.**]

257.37 $\int_y^a \frac{dt}{t-b}\sqrt{\frac{(a-t)(t-d)}{(t-b)(t-c)}} = \frac{(a-d)^2 g}{b-d}\int_0^{u_1}\frac{\mathrm{tn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}$, $y \neq b$. [See **362.06.**]

257.38 $\int_y^a \frac{dt}{t-c}\sqrt{\frac{(t-b)(t-d)}{(a-t)(t-c)}} = \frac{(a-b)(a-d)g}{a-c}\int_0^{u_1}\frac{\mathrm{cd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}$. [See **362.08.**]

257.39 $\int_y^a \frac{dt}{(p-t)^m\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{g}{(p-a)^m}\int_0^{u_1}\frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-\alpha_1^2\,\mathrm{sn}^2 u)^m}du$, [See **340.04.**]

where $\alpha_1^2 = (p-d)(a-b)/(a-p)(b-d)$, $p \neq a$.

257.40 $\int_y^a \frac{(p_1-t)\,dt}{(p-t)\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{p_1-a}{p-a}g\int_0^{u_1}\frac{1-\alpha_2^2\,\mathrm{sn}^2 u}{1-\alpha_1^2\,\mathrm{sn}^2 u}du$, [See **340.01.**]

where

$$\alpha_2^2 = (p_1-d)(b-a)/(p_1-a)(b-d),$$

$$\alpha_1^2 = (p-d)(b-a)/(p-a)(b-d), \qquad p \neq a.$$

257.41 $\int_y^a \frac{R(t)\,dt}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = g\int_0^{u_1} R\left[\frac{a-\alpha^2 d\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\right]du$,

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$, $\sqrt{t-c}$ and $\sqrt{t-d}$, $(y > a > b > c > d)$

$\mathrm{sn}^2 u = \frac{(b-d)(t-a)}{(a-d)(t-b)}$,	$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}$,	$g = \frac{2}{\sqrt{(a-c)(b-d)}}$,
$\alpha^2 = \frac{a-d}{b-d} > 1$,	$\varphi = \mathrm{am}\,u_1 = \sin^{-1}\sqrt{\frac{(b-d)(y-a)}{(a-d)(y-b)}}$,	$\mathrm{sn}\,u_1 = \sin\varphi$.

258.00 $\left\{ \int_a^y \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}} = g\int_0^{u_1} du = g u_1 = g\,\mathrm{sn}^{-1}(\sin\varphi, k) = g F(\varphi, k). \right.$

258.01 $$\left\{\begin{aligned}\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-c}{(t-a)(t-b)(t-d)}} &= \frac{a-c}{a-b}\, g \int_0^{u_1} \mathrm{dn}^2 u\, du = \frac{a-c}{a-b}\, g\, E(u_1)\\ &= \frac{a-c}{a-b}\, g\, E(\varphi, k).\end{aligned}\right.$$

258.02 $$\left\{\begin{aligned}\int_a^y \sqrt{\frac{t-b}{(t-a)(t-d)(t-c)}}\, dt &= (a-b)\, g \int_0^{u_1} \frac{du}{1-\alpha^2 \mathrm{sn}^2 u}\\ &= (a-b)\, g\, \Pi(\varphi, \alpha^2, k),\end{aligned}\right.$$ [See **400.**]

where α^2 is given above.

258.03 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{t-b}{(t-a)(t-c)(t-d)}} = \frac{a-b}{a-c}\, g \int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02.**]

258.04 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{t-a}{(t-b)(t-c)(t-d)}} = \frac{a-b}{a-c}\, \alpha^2 g \int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02.**]

258.05 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{t-a}{(t-b)(t-c)(t-d)}} = \frac{a-b}{b-d}\, g \int_0^{u_1} \mathrm{tn}^2 u\, du.$$ [See **316.02.**]

258.06 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{t-d}{(t-a)(t-b)(t-c)}} = \frac{a-d}{a-c}\, g \int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02.**]

258.07 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-d}{(t-a)(t-b)(t-c)}} = \frac{a-d}{a-b}\, g \int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02.**]

258.08 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-a}{(t-b)(t-c)(t-d)}} = \alpha^2 g \int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02.**]

258.09 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{t-b}{(t-a)(t-c)(t-d)}} = \frac{a-b}{a-d}\, g \int_0^{u_1} \mathrm{nc}^2 u\, du.$$ [See **313.02.**]

258.10 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{t-c}{(t-a)(t-b)(t-d)}} = \frac{a-c}{a-d}\, g \int_0^{u_1} \mathrm{dc}^2 u\, du.$$ [See **321.02.**]

258.11 $$\int_a^y \frac{t^m\,dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}} = a^m g \int_0^{u_1} \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^m}{(1-\alpha^2 \operatorname{sn}^2 u)^m}\,du.$$ [See **340.04.**]

where $$\alpha_1^2 = (a-d)\,b/a\,(b-d).$$

258.12 $$\int_a^y \frac{dt}{t^m\sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{g}{a^m}\int_0^{u_1} \frac{(1-\alpha^2 \operatorname{sn}^2 u)^m\,du}{\left(1-\frac{\alpha^2 b}{a}\operatorname{sn}^2 u\right)^m}.$$

[See **340.04.**]

258.13 $$\int_a^y \sqrt{\frac{t-c}{(t-a)(t-b)(t-d)}}\,dt = (a-c)\,g\int_0^{u_1} \frac{\operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **339.01.**]

258.14 $$\int_a^y \sqrt{\frac{t-d}{(t-a)(t-c)(t-b)}}\,dt = (a-d)\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **338.01.**]

258.15 $$\int_a^y \sqrt{\frac{t-a}{(t-b)(t-c)(t-d)}}\,dt = (a-b)\,\alpha^2 g\int_0^{u_1} \frac{\operatorname{sn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u}.$$ [See **337.01.**]

258.16 $$\int_a^y \sqrt{\frac{(t-c)(t-d)}{(t-a)(t-b)}}\,dt = (a-c)(a-d)\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$ [See **362.20.**]

258.17 $$\int_a^y \sqrt{\frac{(t-c)(t-a)}{(t-b)(t-d)}}\,dt = (a-b)(a-c)\,\alpha^2 g\int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.19.**]

258.18 $$\int_a^y \sqrt{\frac{(t-c)(t-b)}{(t-a)(t-d)}}\,dt = (a-b)(a-c)\,g\int_0^{u_1} \frac{\operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$ [See **362.17.**]

258.19 $$\int_a^y \sqrt{\frac{(t-a)(t-b)}{(t-c)(t-d)}}\,dt = (a-b)^2 g\,\alpha^2\int_0^{u_1} \frac{\operatorname{sn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$ [See **362.15.**]

258.20 $$\int_a^y \sqrt{\frac{(t-a)(t-d)}{(t-b)(t-c)}}\,dt = (a-d)(a-b)\,\alpha^2 g\int_0^{u_1} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$

[See **362.18.**]

258.21 $$\int_a^y \sqrt{\frac{(t-b)(t-d)}{(t-a)(t-c)}}\,dt = (a-b)(a-d)\,g\int_0^{u_1} \frac{\operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2}.$$ [See **362.16.**]

258.22 $$\int_a^y \frac{dt}{(t-c)^m\sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{g}{(a-c)^m}\int_0^{u_1} \frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-k^2\,\mathrm{sn}^2 u)^m}\,du.$$ [See **340.04**.]

258.23 $$\int_a^y \frac{dt}{(t-d)^m\sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{g}{(a-d)^m}\int_0^{u_1} \frac{(1-\alpha^2\,\mathrm{sn}^2 u)^m}{(1-\mathrm{sn}^2 u)^m}\,du.$$ [See **340.04**.]

258.24 $$\int_a^y \frac{dt}{(t-b)^m\sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{g}{(a-b)^m}\int_0^{u_1} (1-\alpha^2\,\mathrm{sn}^2 u)^m\,du.$$ [See **331.03**.]

258.25 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{(t-c)(t-d)}{(t-a)(t-b)}} = \frac{(a-c)(a-d)\,g}{a-b}\int_0^{u_1} \frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.12**.]

258.26 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{(t-c)(t-a)}{(t-b)(t-d)}} = \frac{(a-b)(a-c)\,\alpha^2 g}{a-b}\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.11**.]

258.27 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{(t-c)(t-b)}{(t-a)(t-d)}} = \frac{(a-b)(a-c)\,g}{a-d}\int_0^{u_1} \frac{\mathrm{dc}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.09**.]

258.28 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{(t-c)(t-a)}{(t-b)(t-d)}} = \frac{(a-b)(a-c)\,g}{b-d}\int_0^{u_1} \frac{\mathrm{tn}^2 u\,\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.14**.]

258.29 $$\int_a^v \frac{dt}{t-c}\sqrt{\frac{(t-b)(t-d)}{(t-a)(t-c)}} = \frac{(a-b)(a-d)\,g}{a-c}\int_0^{u_1} \frac{\mathrm{cd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.08**.]

258.30 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{(t-a)(t-d)}{(t-b)(t-c)}} = \frac{(a-d)(a-b)\,\alpha^2 g}{a-c}\int_0^{u_1} \frac{\mathrm{sd}^2 u\,\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.13**.]

258.31 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{(t-a)(t-d)}{(t-b)(t-c)}} = (a-d)^2\alpha^2 g\int_0^{u_1} \frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.10**.]

258.32 $$\int_a^y \frac{dt}{t-c}\sqrt{\frac{(t-a)(t-b)}{(t-c)(t-d)}} = \frac{(a-b)^2\alpha^2 g}{a-c}\int_0^{u_1} \frac{\mathrm{sd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}.$$ [See **362.07**.]

258.33 $$\int_a^y \frac{dt}{t-d}\sqrt{\frac{(t-a)(t-b)}{(t-c)(t-d)}} = \frac{(a-b)^2 g}{b-d}\int_0^{u_1} \frac{\mathrm{tn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u}.$$ [See **362.06**.]

258.34 $$\int_a^y \sqrt{\frac{(t-b)(t-c)(t-d)}{t-a}}\, dt = (a-b)(a-c)(a-d)\, g \int_0^{u_1} \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$ [See **362.22**.]

258.35 $$\int_a^y \sqrt{\frac{(t-a)(t-c)(t-d)}{t-b}}\, dt = (a-d)(a-b)(a-c)\,\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$ [See **362.24**.]

258.36 $$\int_a^y \sqrt{\frac{(t-a)(t-b)(t-d)}{t-c}}\, dt = (a-b)^2 (a-c)\,\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$ [See **362.23**.]

258.37 $$\int_a^y \sqrt{\frac{(t-a)(t-b)(t-c)}{t-d}}\, dt = (a-b)^2 (a-c)\,\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^3}.$$ [See **362.21**.]

258.38 $$\left\{\begin{aligned} &\int_a^y \sqrt{(t-a)(t-b)(t-c)(t-d)}\, dt \\ &\quad = (a-b)^2 (a-d)(a-c)\,\alpha^2 g \int_0^{u_1} \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^4}. \end{aligned}\right.$$ [See **362.25**.]

258.39 $$\int_a^y \frac{dt}{(p-t)^m \sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{g}{(p-a)^m}\int_0^{u_1} \frac{(1-\alpha^2 \mathrm{sn}^2 u)^m}{(1-\alpha_3^2 \mathrm{sn}^2 u)^m}\, du,$$ [See **340.04**.]

where $$\alpha_3^2 = (p-b)(a-d)/(p-a)(b-d), \quad p \neq a.$$

258.40 $$\int_a^y \frac{(p_1-t)\, dt}{(p-t)\sqrt{(t-a)(t-b)(t-c)(t-d)}} = \frac{p_1-a}{p-a}\, g \int_0^{u_1} \frac{1-\alpha_2^2 \mathrm{sn}^2 u}{1-\alpha_3^2 \mathrm{sn}^2 u}\, du,$$ [See **340.01**.]

where
$$\alpha_3^2 = (p-b)(a-d)/(p-a)(b-d), \qquad p \neq a$$
$$\alpha_2^2 = (p_1-b)(a-d)/(p_1-a)(b-d).$$

258.41 $$\int_a^y \frac{R(t)\, dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}} = g\int_0^{u_1} R\left[\frac{a-b\,\alpha^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{a-t}$, $\sqrt{t-b}$, $\sqrt{t-c}$ and $\sqrt{t-\bar{c}}$; $(a, b$ real, $a \geqq y > b$, $c, \bar{c}$ complex$)$

$$\sqrt{(a-t)(t-b)(t-c)(t-\bar{c})}=\sqrt{(a-t)(t-b)[(t-b_1)^2+a_1^2]}; \quad b_1=\frac{c+\bar{c}}{2},$$

$$a_1^2=-\frac{(c-\bar{c})^2}{4}, \quad \operatorname{cn} u=\frac{(a-t)B-(t-b)A}{(a-t)B+(t-b)A}, \quad A^2=(a-b_1)^2+a_1^2,$$

$$B^2=(b-b_1)^2+a_1^2, \quad g=\frac{1}{\sqrt{AB}}, \quad k^2=\frac{(a-b)^2-(A-B)^2}{4AB},$$

$$\operatorname{cn} u_1=\cos\varphi, \quad \varphi=\operatorname{am} u_1=\cos^{-1}\left[\frac{(a-y)B-(y-b)A}{(a-y)B+(y-b)A}\right].$$

259.00 $$\int_b^y \frac{dt}{\sqrt{(a-t)(t-b)(t-c)(t-\bar{c})}}=g\int_0^{u_1} du=g u_1=g \operatorname{cn}^{-1}(\cos\varphi, k) = g F(\varphi, k).$$

259.01 $$\int_b^y \frac{dt}{[(a-t)B+(t-b)A]^2}\sqrt{\frac{(t-c)(t-\bar{c})}{(a-t)(t-b)}}=\frac{g}{(a-b)^2}\int_0^{u_1}\operatorname{dn}^2 u\, du = \frac{g}{(a-b)^2}E(u_1)=\frac{g}{(a-b)^2}E(\varphi, k).$$

259.02 $$\int_b^y \frac{dt}{a-t}\sqrt{\frac{t-b}{(a-t)(t-c)(t-\bar{c})}}=\frac{Bg}{A}\int_0^{u_1}\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}du, \quad y \neq a.$$ [See **361.53**.]

259.03 $$\int_b^y \frac{t^m\, dt}{\sqrt{(a-t)(t-b)(t-c)(t-\bar{c})}}=\frac{g(aB+bA)^m}{(A-B)^m}\times \sum_{j=0}^{m}\frac{\alpha_2^{m-j}(\alpha-\alpha_2)^j\, m!}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\alpha \operatorname{cn} u)^j},$$ [See **341.05**.]

where $\alpha=(A-B)/(A+B), \quad \alpha_2=(bA-aB)/(aB+bA)$.

259.04 $$\int_b^y \frac{dt}{(t-p)^m\sqrt{(a-t)(t-b)(t-c)(t-\bar{c})}}=\frac{(A+B)^m}{[A(b-p)-B(a-p)]^m}\times \sum_{j=0}^{m}\frac{\alpha_1^{m-j}(\alpha-\alpha_1)^j\, m!}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\alpha \operatorname{cn} u)^j},$$ [See **341.05**.]

where $\alpha=(bA-aB+pB-pA)/(aB+bA-pA-pB)$,

$$\alpha_1=(A-B)/(A+B).$$

259.05 $$\int_b^y \frac{dt}{(t-c)(t-\bar{c})}\sqrt{\frac{(a-t)(t-b)}{(t-c)(t-\bar{c})}}=\frac{(a-b)^2 g}{4AB}\int_0^{u_1}\operatorname{sd}^2 u\, du.$$ [See **318.02**.]

259.06 $$\int_b^y \frac{dt}{[(a-t)B-(t-b)A]^2}\sqrt{\frac{(a-t)(t-b)}{(t-c)(t-\bar{c})}} = \frac{g}{4AB}\int_0^{u_1} \operatorname{tn}^2 u\, du.$$ [See **316.02.**]

259.07 $$\int_b^y \frac{R(t)\,dt}{\sqrt{(a-t)(t-b)(t-c)(t-\bar{c})}} = g\int_0^{u_1} R\left[\frac{aB+bA+(bA-aB)\operatorname{cn} u}{A+B+(A-B)\operatorname{cn} u}\right]du,$$

where $R(t)$ is a rational function of t.

In the special case when $a=-b$, and $c=-\bar{c}$, see **213**. An example of this appears in **259.75**.

Integrands involving $\sqrt{t-t^4}$, $(0<y\leqq 1)$

Special case of **259**:

$$\operatorname{cn} u = \frac{1-(1+\sqrt{3})\,t}{1+(\sqrt{3}-1)\,t}, \qquad k^2 = \frac{2-\sqrt{3}}{4}, \qquad g = \frac{1}{\sqrt[4]{3}},$$
$$\psi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{1-(1+\sqrt{3})\,y}{1+(\sqrt{3}-1)\,y}\right].$$

259.50 $$\int_0^y \frac{dt}{\sqrt{t(1-t^3)}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\psi, k) = \frac{1}{\sqrt[4]{3}}F(\psi,k),$$ [cf. **259.00**].

259.51 $$\int_0^y \frac{dt}{[1+(\sqrt{3}-1)\,t]^2}\sqrt{\frac{1+t+t^2}{t(1-t)}} = g\int_0^{u_1} \operatorname{dn}^2 u\, du = g\,E(u_1) = \frac{1}{\sqrt[4]{3}}E(\psi,k).$$

259.52 $$\int_0^y \frac{dt}{(1-t)}\sqrt{\frac{t}{(1-t)(1+t+t^2)}} = \frac{g}{\sqrt{3}}\int_0^{u_1} \frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\,du, \qquad y \neq 1.$$ [See **361.53.**]

259.53 $$\int_0^y \frac{t^m\,dt}{\sqrt{t(1-t^3)}} = \frac{g}{(\sqrt{3}-1)^m}\sum_{j=0}^{m} \frac{(-1)^{m+j}(3-\sqrt{3})^j\, m!}{(m-j)!\,j!}\int_0^{u_1} \frac{du}{[1+(\sqrt{3}-2)\operatorname{cn} u]^j}.$$ [See **341.05.**]

259.54 $$\begin{cases} \displaystyle\int_0^y \frac{dt}{(t-p)^m\sqrt{t(1-t^3)}} = \frac{(1+\sqrt{3})^m g}{(p-1-p\sqrt{3})^m} \times \\ \displaystyle\times \sum_{j=0}^{m} \frac{\alpha^{m-j}(\alpha_1-\alpha)^j\, m!}{(m-j)!\,j!}\int_0^{u_1} \frac{du}{(1+\alpha_1 \operatorname{cn} u)^j}, \end{cases}$$ [See **341.05.**]

$$\alpha = (\sqrt{3}-1)/(\sqrt{3}+1), \quad \alpha_1 = (p-1-p\sqrt{3})/(1-p-p\sqrt{3}).$$

259.55 $$\int_0^y \frac{dt}{(1+t+t^2)} \sqrt{\frac{t(1-t)}{1+t+t^2}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{sd}^2 u\, du. \qquad \text{[See } \mathbf{318.02}.]$$

259.56 $$\int_0^y \frac{R(t)\, dt}{\sqrt{t-t^4}} = \frac{1}{\sqrt[4]{3}} \int_0^{u_1} R\left[\frac{1-\operatorname{cn} u}{1+\sqrt{3}+(\sqrt{3}-1)\operatorname{cn} u}\right] du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{1-t^4}$, $(y<1)$

Another special case of **259**:

$$\boxed{\operatorname{cn} u = t, \quad k^2 = 1/2, \quad \varphi = \operatorname{am} u_2 = \cos^{-1}(y)}$$

259.75 $$\int_y^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} \int_0^{u_2} du = \frac{u_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \operatorname{cn}^{-1}(\cos\varphi, k) = \frac{1}{\sqrt{2}} F\left(\varphi, \sqrt{2}/2\right).$$
[cf. **213.00**].

259.76 $$\int_y^1 \frac{t^{2m}\, dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} \int_0^{u_2} \operatorname{cn}^{2m} u\, du. \qquad \text{[See } \mathbf{312.05}.]$$

259.77 $$\int_y^1 \frac{R(t)\, dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{2}} \int_0^{u_2} R(\operatorname{cn} u)\, du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$, $\sqrt{t-c}$ and $\sqrt{t-\bar{c}}$; $(b<a<y<\infty$, $c, \bar{c}$ complex$)$

$$\boxed{\begin{gathered}
\sqrt{(t-a)(t-b)(t-c)(t-\bar{c})} = \sqrt{(t-a)(t-b)\left[(t-b_1)^2+a_1^2\right]}, \quad b_1 = \frac{c+\bar{c}}{2}, \\
a_1^2 = -\frac{(c-\bar{c})^2}{4}, \quad g = 1/\sqrt{AB}, \quad k^2 = \frac{(A+B)^2-(a-b)^2}{4AB}, \\
A^2 = (a-b_1)^2 + a_1^2, \quad B^2 = (b-b_1)^2 + a_1^2, \quad \operatorname{cn} u = \frac{(t-b)A - B(t-a)}{(t-b)A + B(t-a)}, \\
\varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{(A-B)y + aB - bA}{(A+B)y - aB - bA}\right], \quad \operatorname{cn} u_1 = \cos\varphi.
\end{gathered}}$$

260.00 $$\left\{\begin{aligned} \int_a^y \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-\bar{c})}} &= g\int_0^{u_1} du = g\, u_1 = g \operatorname{cn}^{-1}(\cos\varphi, k) \\ &= g\, F(\varphi, k). \end{aligned}\right.$$

260.01
$$\begin{cases} \displaystyle\int_a^y \frac{dt}{[(t-b)A+B(t-a)]^2}\sqrt{\frac{(t-c)(t-\bar{c})}{(t-a)(t-b)}} = \frac{g}{(a-b)^2}\int_0^{u_1}\mathrm{dn}^2 u\,du \\ \displaystyle\qquad = \frac{g}{(a-b)^2}E(u_1) = \frac{g}{(a-b)^2}E(\varphi,k). \end{cases}$$

260.02 $$\int_a^y \frac{dt}{t-b}\sqrt{\frac{t-a}{(t-b)(t-c)(t-\bar{c})}} = \frac{A}{B}g\int_0^{u_1}\frac{1-\mathrm{cn}\,u}{1+\mathrm{cn}\,u}du.$$ [See **361.53.**]

260.03
$$\begin{cases} \displaystyle\int_a^y \frac{t^m\,dt}{\sqrt{(t-a)(t-b)(t-c)(t-\bar{c})}} = \frac{(Ba-bA)^m}{(B+A)^m}g\times \\ \displaystyle\qquad \times\sum_{j=0}^{m}\frac{\alpha_1^{m-j}(\alpha-\alpha_1)^j\,m!}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\alpha\,\mathrm{cn}\,u)^j}, \end{cases}$$ [See **341.05.**]

where $\alpha=(B+A)/(B-A), \qquad \alpha_1=(Ba+bA)/(Ba-bA).$

260.04
$$\begin{cases} \displaystyle\int_a^y \frac{dt}{(t-p)^m\sqrt{(t-a)(t-b)(t-c)(t-\bar{c})}} = \frac{(B-A)^m g}{(Ba+bA-pA-pB)^m}\times \\ \displaystyle\qquad \times\sum_{j=0}^{m}\frac{\alpha_2^{m-j}(\alpha-\alpha_2)^j\,m!}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\alpha\,\mathrm{cn}\,u)^j}, \end{cases}$$ [See **341.05.**]

where $\alpha=(Ba+bA-pA-pB)/(Ba-bA+pA-pB),$

$$\alpha_2=(B+A)/(B-A).$$

260.05 $$\int_a^y \frac{dt}{(t-c)(t-\bar{c})}\sqrt{\frac{(t-a)(t-b)}{(t-c)(t-\bar{c})}} = \frac{(a-b)^2 g}{4AB}\int_0^{u_1}\mathrm{sd}^2 u\,du.$$ [See **318.02.**]

260.06 $$\int_a^y \frac{dt}{[(t-b)A-B(t-a)]^2}\sqrt{\frac{(t-a)(t-b)}{(t-c)(t-\bar{c})}} = \frac{g}{4AB}\int_0^{u_1}\mathrm{tn}^2 u\,du.$$ [See **316.02.**]

260.07 $$\int_a^y \frac{R(t)\,dt}{\sqrt{(t-a)(t-b)(t-c)(t-\bar{c})}} = g\int_0^{u_1}R\left[\frac{Ba-bA+(Ba+bA)\,\mathrm{cn}\,u}{B-A+(A+B)\,\mathrm{cn}\,u}\right]du,$$

where $R(t)$ is a rational function of t.

In the special case when $c=-\bar{c}$ and $a=-b$,

260.08 $$\int_a^y \frac{R(t)\,dt}{\sqrt{(t^2-a^2)(t^2+a_1^2)}} = \frac{1}{\sqrt{a^2+a_1^2}} \int_0^{u_1} R(a \operatorname{nc} u)\,du.$$ [See **211**.]

Integrands involving $\sqrt{t(1+t^3)}$, $(y>0)$

Special case of **260**:

$$\boxed{\begin{array}{l} \operatorname{cn} u = \dfrac{1+(1-\sqrt{3})\,t}{1+(1+\sqrt{3})\,t}, \quad k^2 = \dfrac{2+\sqrt{3}}{4}, \quad g = 1/\sqrt[4]{3}, \\ \psi = \operatorname{am} u_1 = \cos^{-1}\left[\dfrac{1+(1-\sqrt{3})\,y}{1+(1+\sqrt{3})\,y}\right], \quad \operatorname{cn} u_1 = \cos\psi. \end{array}}$$

260.50 $$\int_0^y \frac{dt}{\sqrt{t(1+t^3)}} = g\int_0^{u_1} du = g\,u_1 = g \operatorname{cn}^{-1}(\cos\psi, k) = g\,F\left(\psi, \sqrt{\frac{2+\sqrt{3}}{4}}\right).$$

260.51 $$\int_0^y \frac{dt}{[1+(1+\sqrt{3})t]^2}\sqrt{\frac{t^2-t+1}{t(1+t)}} = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E\left(\psi, \sqrt{\frac{2+\sqrt{3}}{4}}\right).$$

260.52 $$\left\{\begin{array}{l} \displaystyle\int_0^y \frac{t^m\,dt}{\sqrt{t(1+t^3)}} = \frac{1}{(\sqrt{3}+1)^m \sqrt[4]{3}} \times \\ \displaystyle\quad \times \sum_{j=0}^{m} \frac{(-1)^{m+j}\,m!\,(3+\sqrt{3})^j}{(m-j)!\,j!} \int_0^{u_1} \frac{du}{[1+(2+\sqrt{3})\operatorname{cn} u]^j}. \end{array}\right.$$ [See **341.05**.]

260.53 $$\left\{\begin{array}{l} \displaystyle\int_0^y \frac{dt}{(t-p)^m \sqrt{t(1+t^3)}} = \frac{(\sqrt{3}-1)^m (-1)^m g}{(1+p+p\sqrt{3})^m} \times \\ \displaystyle\quad \times \sum_{j=0}^{m} \frac{(2+\sqrt{3})^{m-j}(\alpha-\alpha_2)^j}{(m-j)!\,j!} \int_0^{u_1} \frac{du}{(1+\alpha \operatorname{cn} u)^j}, \end{array}\right.$$ [See **341.05**.]

where $$\alpha = (p\sqrt{3}+1+p)/(p\sqrt{3}-1-p), \quad \alpha_2 = 2+\sqrt{3}.$$

260.54 $$\int_0^y \frac{dt}{(t^2-t+1)}\sqrt{\frac{t(1+t)}{t^2-t+1}} = \frac{g}{4\sqrt{3}} \int_0^{u_1} \operatorname{sd}^2 u\,du.$$ [See **318.02**.]

260.55 $$\int_0^y \frac{R(t)\,dt}{\sqrt{t(1+t^3)}} = \frac{1}{\sqrt[4]{3}} \int_0^{u_1} R\left[\frac{1-\operatorname{cn} u}{\sqrt{3}-1+(1+\sqrt{3})\operatorname{cn} u}\right] du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t^4-1}$, $(y>1)$

Another special case of **260**:

$$\boxed{\begin{gathered}\operatorname{cn} u = 1/t, \quad k^2 = 1/2, \quad \operatorname{cn} u_1 = \cos\varphi, \\ \varphi = \operatorname{am} u_1 = \cos^{-1}(1/y).\end{gathered}}$$

260.75 $$\int_1^y \frac{dt}{\sqrt{t^4-1}} = \frac{1}{\sqrt{2}} \int_0^{u_1} du = \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}} \operatorname{cn}^{-1}(\cos\varphi, k) = \frac{1}{\sqrt{2}} F(\varphi, \sqrt{2}/2),$$

[cf. **211.00**].

260.76 $$\int_1^y \frac{dt}{t^2} \sqrt{\frac{t^2+1}{t^2-1}} = \sqrt{2} \int_0^{u_1} \operatorname{dn}^2 u\, du = \sqrt{2} E(u_1) = \sqrt{2} E(\varphi, \sqrt{2}/2).$$

260.77 $$\int_1^y \frac{t^{2m}\,dt}{\sqrt{t^4-1}} = \frac{1}{\sqrt{2}} \int_0^{u_1} \operatorname{nc}^{2m} u\, du.$$ [See **313.05**.]

260.78 $$\int_1^y \frac{R(t)\,dt}{\sqrt{t^4-1}} = \frac{1}{\sqrt{2}} \int_0^{u_1} R(\operatorname{nc} u)\, du.$$

Integrands involving $\sqrt{t-a}$, $\sqrt{t-b}$ and $\sqrt{(2t-a-b)^2+4a_1^2}$, $(\infty > y \geqq a > b > 0)$

$$\boxed{\begin{gathered}\operatorname{sn}^2 u = \frac{4a_1^2+(a-b)^2}{4a_1^2+(2t-a-b)^2}, \quad k^2 = \frac{4a_1^2}{4a_1^2+(a-b)^2}, \quad g = \frac{1}{\sqrt{4a_1^2+(a-b)^2}}, \\ \varphi = \operatorname{am} u_1 = \sin^{-1}\left[\frac{2a_1}{k\sqrt{4a_1^2+(2y-a-b)^2}}\right], \quad \operatorname{sn} u_1 = \sin\varphi.\end{gathered}}$$

261.00 $$\left\{\begin{aligned}\int_y^\infty \frac{dt}{\sqrt{(t-a)(t-b)\left[(2t-a-b)^2+4a_1^2\right]}} &= g\int_0^{u_1} du = g\,u_1 \\ &= g\operatorname{sn}^{-1}(\sin\varphi, k) = g\,F(\varphi, k).\end{aligned}\right.$$

261.01 $$\int_y^\infty \frac{(2t-a-b)^2\,dt}{[4a_1^2+(2t-a-b)^2]\sqrt{(t-a)(t-b)[(2t-a-b)^2+4a_1^2]}} = g\int_0^{u_1} \mathrm{dn}^2 u\,du = g\,E(\varphi, k).$$

261.02 $$\int_y^\infty \frac{dt}{[4a_1^2+(2t-a-b)^2]}\sqrt{\frac{(t-a)(t-b)}{4a_1^2+(2t-a-b)^2}} = \frac{g}{4}\int_0^{u_1} \mathrm{cn}^2 u\,du.$$ [See **312.02**.]

261.03 $$\int_y^\infty \frac{dt}{[4a_1^2+(2t-a-b)^2]^m\sqrt{(t-a)(t-b)[4a_1^2+(2t-a-b)^2]}} = g^{2m+1}\int_0^{u_1} \mathrm{sn}^{2m} u\,du.$$ [See **310.05**.]

261.04 $$\int_y^\infty \frac{dt}{(t-a)^m(t-b)^m\sqrt{(t-a)(t-b)[4a_1^2+(2t-a-b)^2]}} = 4^m g^{2m+1}\int_0^{u_1} \mathrm{tn}^{2m} u\,du, \quad (y \neq a).$$ [See **316.05**.]

261.05 $$\int_y^\infty \frac{dt}{(2t-a-b)^{2m}\sqrt{(t-a)(t-b)[4a_1^2+(2t-a-b)^2]}} = g^{2m+1}\int_0^{u_1} \mathrm{sd}^{2m} u\,du.$$ [See **318.05**.]

261.06 $$\int_y^\infty \frac{(2t-a-b)^2\,dt}{(t-a)(t-b)\sqrt{(t-a)(t-b)[4a_1^2+(2t-a-b)^2]}} = 4g\int_0^{u_1} \mathrm{dc}^2 u\,du, \quad y \neq a.$$ [See **321.02**.]

261.07 $$\int_y^\infty \frac{dt}{(2t-a-b)^2}\sqrt{\frac{(t-a)(t-b)}{4a_1^2+(2t-a-b)^2}} = \frac{g}{4}\int_0^{u_1} \mathrm{cd}^2 u\,du.$$ [See **320.02**.]

261.08 $$\int_y^\infty \frac{dt}{(2t-a-b)^2}\sqrt{\frac{4a_1^2+(2t-a-b)^2}{(t-a)(t-b)}} = g\int_0^{u_1} \mathrm{nd}^2 u\,du.$$ [See **315.02**.]

261.09 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{(t-a)(t-b)[4a_1^2+(2t-a-b)^2]}} = g\int_0^{u_1} R\left[\frac{(a+b)g+\mathrm{ds}\,u}{2g}\right]du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t-1}$, $\sqrt{t}$ and $\sqrt{t^2-t+1}$, $(\infty > y \geqq 1)$

Special case of 261:

$$\operatorname{sn}^2 u = \frac{1}{t^2-t+1}, \quad k^2 = 3/4, \quad \operatorname{sn} u_1 = \sin\psi,$$
$$\psi = \operatorname{am} u_1 = \sin^{-1}\left[\frac{1}{\sqrt{y^2-y+1}}\right].$$

261.50 $$\int_y^\infty \frac{dt}{\sqrt{t(t-1)(t^2-t+1)}} = \int_0^{u_1} du = u_1 = \operatorname{sn}^{-1}(\sin\psi, k) = F\left(\psi, \sqrt{3}/2\right).$$

261.51 $$\int_y^\infty \frac{(2t-1)^2\,dt}{(t^2-t+1)\sqrt{t(t-1)(t^2-t+1)}} = 4\int_0^{u_1} \operatorname{dn}^2 u\,du = 4E(u_1) = 4E(\psi, k).$$

261.52 $$\int_y^\infty \frac{dt}{(t^2-t+1)^m\sqrt{t(t-1)(t^2-t+1)}} = \int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$ [See **310.05**.]

261.53 $$\int_y^\infty \frac{dt}{(t^2-t+1)}\sqrt{\frac{t(t-1)}{t^2-t+1}} = \int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02**.]

261.54 $$\int_y^\infty \frac{dt}{t^m(t-1)^m\sqrt{t(t-1)(t^2-t+1)}} = \int_0^{u_1} \operatorname{tn}^{2m} u\,du, \quad y \neq 1.$$ [See **316.05**.]

261.55 $$\int_y^\infty \frac{dt}{(2t-1)^{2m}\sqrt{t(t-1)(t^2-t+1)}} = \int_0^{u_1} \operatorname{sd}^{2m} u\,du.$$ [See **318.05**.]

261.56 $$\int_y^\infty \frac{(2t-1)^2\,dt}{t(t-1)\sqrt{t(t-1)(t^2-t+1)}} = 4\int_0^{u_1} \operatorname{dc}^2 u\,du, \quad y \neq 1.$$ [See **321.02**.]

261.57 $$\int_y^\infty \frac{dt}{(2t-1)^2}\sqrt{\frac{t(t-1)}{t^2-t+1}} = \frac{1}{4}\int_0^{u_1} \operatorname{cd}^2 u\,du.$$ [See **320.02**.]

261.58 $$\int_y^\infty \frac{dt}{(2t-1)^2}\sqrt{\frac{t^2-t+1}{t(t-1)}} = \frac{1}{4}\int_0^{u_1} \operatorname{nd}^2 u\,du.$$ [See **315.02**.]

261.59 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{t(t-1)(t^2-t+1)}} = \int_0^{u_1} R\left[\frac{1+\operatorname{ds} u}{2}\right] du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t^4+2b^2t^2+a^4}$, $(0<b<a;\ \infty>y\geqq 0)$

(Here all the zeros of $t^4+2b^2t^2+a^4$ are complex)

$$\boxed{\begin{aligned} &\operatorname{cn} u=\frac{t^2-a^2}{t^2+a^2}, \quad k^2=\frac{a^2-b^2}{2a^2}, \quad g=1/2a,\\ &\varphi=\operatorname{am} u_1=\cos^{-1}\left[\frac{y^2-a^2}{y^2+a^2}\right], \quad \operatorname{cn} u_1=\cos\varphi. \end{aligned}}$$

263.00 $$\int_y^\infty \frac{dt}{\sqrt{t^4+2b^2t^2+a^4}}=g\int_0^{u_1} du=g\,u_1=g\operatorname{cn}^{-1}(\cos\varphi,k)=g\,F(\varphi,k),$$ [cf. **225.00**].

263.01 $$\int_y^\infty \frac{\sqrt{t^4+2b^2t^2+a^4}}{(t^2+a^2)^2}\,dt=g\int_0^{u_1}\operatorname{dn}^2 u\,du=g\,E(u_1)=g\,E(\varphi,k).$$

263.02 $$\left\{\begin{aligned} &\int_y^\infty \frac{(t^2+a^2)^2\,dt}{[(t^2+a^2)^2-4a^2\alpha^2t^2]\sqrt{t^4+2b^2t^2+a^4}}=g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u}\\ &\qquad =g\,\Pi(\varphi,\alpha^2,k). \quad \text{[See \textbf{400.}]}\end{aligned}\right.$$

263.03 $$\int_y^\infty \frac{t^2\,dt}{(t^2+a^2)^2\sqrt{t^4+2b^2t^2+a^4}}=\frac{g}{4a^2}\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

263.04 $$\int_y^\infty \frac{(t^2-a^2)^2\,dt}{(t^2+a^2)^2\sqrt{t^4+2b^2t^2+a^4}}=g\int_0^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02.**]

263.05 $$\int_y^\infty \frac{t^2\,dt}{(t^2-a^2)^2\sqrt{t^4+2b^2t^2+a^4}}=\frac{g}{4a^2}\int_0^{u_1}\operatorname{tn}^2 u\,du, \quad y\neq a.$$ [See **316.02.**]

263.06 $$\left\{\begin{aligned} &\int_y^\infty \frac{dt}{t^{2m}\sqrt{t^4+2b^2t^2+a^4}}\\ &\quad =\frac{g}{a^{2m}}\sum_{j=0}^{m}\frac{(-1)^{m+j}\,m!\,2^j}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}, \quad y\neq 0.\end{aligned}\right.$$ [See **341.55.**]

263.07 $$\int_y^\infty \frac{\sqrt{t^4+2b^2t^2+a^4}}{(t^2-a^2)^2}\,dt=g\int_0^{u_1}\operatorname{dc}^2 u\,du, \quad y\neq a.$$ [See **321.02.**]

263.08 $$\int_y^\infty \frac{t^2\,dt}{(t^4+2b^2t^2+a^4)\sqrt{t^4+2b^2t^2+a^4}}=\frac{g}{4a^2}\int_0^{u_1}\operatorname{sd}^2 u\,du.$$ [See **318.02.**]

263.09 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{t^4+2b^2t^2+a^4}} = g\int_0^{u_1} R\left[\frac{a\,\mathrm{sn}\,u}{1-\mathrm{cn}\,u}\right]du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t^4+1}$, $(\infty > y \geqq 0)$

Special case of above:

$$\boxed{\begin{gathered}\mathrm{cn}\,u = \frac{t^2-1}{t^2+1}, \qquad k^2 = 1/2, \qquad \mathrm{cn}\,u_1 = \cos\psi, \\ \psi = \mathrm{am}\,u_1 = \cos^{-1}\left[\frac{y^2-1}{y^2+1}\right].\end{gathered}}$$

263.50 $$\int_y^\infty \frac{dt}{\sqrt{t^4+1}} = \frac{1}{2}\int_0^{u_1} du = \frac{1}{2}u_1 = \frac{1}{2}\,\mathrm{cn}^{-1}(\cos\psi, k) = \frac{1}{2}F\left(\psi, \sqrt{2}/2\right),$$ [cf. **264.50**].

263.51 $$\int_y^\infty \frac{\sqrt{t^4+1}}{(t^2+1)^2}\,dt = \frac{1}{2}\int_0^{u_1} \mathrm{dn}^2 u\,du = \frac{1}{2}E(u_1) = \frac{1}{2}E\left(\psi, \sqrt{2}/2\right).$$

263.52 $$\int_y^\infty \frac{(t^2+1)^2\,dt}{[(t^2+1)^2-4\alpha^2t^2]\sqrt{t^4+1}} = \frac{1}{2}\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{1}{2}\Pi\left(\psi, \alpha^2, \sqrt{2}/2\right).$$ [See **400**.]

263.53 $$\int_y^\infty \frac{t^2\,dt}{(t^2+1)^2\sqrt{t^4+1}} = \frac{1}{8}\int_0^{u_1} \mathrm{sn}^2 u\,du.$$ [See **310.02**.]

263.54 $$\int_y^\infty \frac{(t^2-1)^2\,dt}{(t^2+1)^2\sqrt{t^4+1}} = \frac{1}{2}\int_0^{u_1} \mathrm{cn}^2 u\,du.$$ [See **312.02**.]

263.55 $$\int_y^\infty \frac{t^2\,dt}{(t^2-1)^2\sqrt{t^4+1}} = \frac{1}{8}\int_0^{u_1} \mathrm{tn}^2 u\,du, \quad y \neq 1.$$ [See **316.02**.]

263.56 $$\int_y^\infty \frac{dt}{t\sqrt{t^4+1}} = \frac{1}{2}\int_0^{u_1} \frac{1-\mathrm{cn}\,u}{\mathrm{sn}\,u}\,du = \frac{1}{2}\ln\left[\frac{2(1+\mathrm{dn}\,u_1)}{\mathrm{cn}\,u_1+\mathrm{dn}\,u_1}\right] = \frac{1}{2}\ln\left[\frac{1+\sqrt{y^4+1}}{\sqrt{y^4+1}-1}\right].$$

263.57 $$\int_y^\infty \frac{dt}{t^{2m}\sqrt{t^4+1}} = \frac{1}{2}\sum_{j=0}^{m} \frac{(-1)^{m+j}\,m!\,2^j}{(m-j)!\,j!}\int_0^{u_1} \frac{du}{(1+\mathrm{cn}\,u)^j}.$$ [See **341.55**.]

263.58 $$\int_y^\infty \frac{\sqrt{t^4+1}}{(t^2-1)^2}\,dt = \frac{1}{2}\int_0^{u_1} \mathrm{dc}^2 u\,du, \quad y \neq 1.$$ [See **321.02**.]

263.59 $$\int_y^\infty \frac{t^2\,dt}{(t^4+1)\sqrt{t^4+1}} = \frac{1}{8}\int_0^{u_1} \mathrm{sd}^2 u\,du.$$ [See **318.02**.]

263.60 $$\int_y^\infty \frac{R(t)\,dt}{\sqrt{t^4+1}} = \frac{1}{2}\int_0^{u_1} R\left[\frac{\mathrm{sn}\,u}{1-\mathrm{cn}\,u}\right] du,$$

where $R(t)$ is any rational function of t.

Integrands involving $\sqrt{t^4+2b^2t^2+a^4}$, $(a>b;\ 0\leq y<a)$

$$\mathrm{tn}\,u = \frac{1}{\sqrt{k'}}\,\frac{a-t}{a+t}, \quad k^2 = \frac{4c}{(1+c)^2}, \quad c = \frac{1-k'}{1+k'} = \sqrt{\frac{a^2-b^2}{2a^2}},$$
$$g = \frac{1+k'}{2a}, \quad \varphi = \mathrm{am}\,u_1 = \tan^{-1}\left[\frac{1}{\sqrt{k'}}\,\frac{a-y}{a+y}\right], \quad \mathrm{tn}\,u_1 = \tan\varphi.$$

264.00 $$\int_y^a \frac{dt}{\sqrt{t^4+2b^2t^2+a^4}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{tn}^{-1}(\tan\varphi, k) = g\,F(\varphi, k).$$ [cf. **263.00**].

264.01 $$\int_y^a \frac{(t^2+2act+a^2)\,dt}{(t^2-2act+a^2)\sqrt{t^4+2b^2t^2+a^4}} = \frac{g}{k'}\int_0^{u_1} \mathrm{dn}^2 u\,du = \frac{g}{k'}E(u_1) = \frac{g}{k'}E(\varphi, k).$$

264.02 $$\int_y^a \frac{t^m\,dt}{\sqrt{t^4+2b^2t^2+a^4}} = g\,a^m \sum_{j=0}^{m} \frac{(-1)^{m+j}\,m!\,2^j}{(m-j)!\,j!}\int_0^{u_1} \frac{du}{(1+\sqrt{k'}\,\mathrm{tn}\,u)^j}.$$ [See **342.05**.]

264.03 $$\int_y^a \frac{(a-t)^m\,dt}{(a+t)^m\sqrt{t^4+2b^2t^2+a^4}} = g\,k'^m\int_0^{u_1} \mathrm{tn}^{2m} u\,du.$$ [See **316.05**.]

264.04 $$\int_y^a \frac{dt}{(t-p)^m\sqrt{t^4+2b^2t^2+a^4}} = \frac{(-1)^m g}{(a+p)^m(\sqrt{k'})^m} \times$$
$$\times \sum_{j=0}^{m} \frac{m!\,(\sqrt{k'})^{m+j}\,(2a)^j}{(m-j)!\,j!\,(p-a)^j}\int_0^{u_1} \frac{du}{(1+\alpha\,\mathrm{tn}\,u)^j},$$ [See **342.05**.]

where $\qquad \alpha = (a+p)\sqrt{k'}/(p-a), \quad p \neq a.$

264.05 $$\int_y^a \frac{(a-t)^2\,dt}{(t^2-2act+a^2)\sqrt{t^4+2b^2t^2+a^4}} = (1+k')\,g\int_0^{u_1} \mathrm{sn}^2 u\,du.$$ [See **310.02**.]

264.06 $$\int_y^a \frac{(a+t)^2\,dt}{(t^2-2act+a^2)\sqrt{t^4+2b^2t^2+a^4}} = \frac{(1+k')\,g}{k'}\int_0^{u_1} \mathrm{cn}^2 u\,du.$$ [See **312.02**.]

264.07 $$\int_y^a \frac{(a+t)^2\,dt}{(t^2+2act+a^2)\sqrt{t^4+2b^2t^2+a^4}} = g(1+k')\int_0^{u_1} \mathrm{cd}^2 u\,du.$$ [See **320.02**.]

264.08 $$\int_y^a \frac{R(t)\,dt}{\sqrt{t^4+2b^2t^2+a^4}} = g\int_0^{u_1} R\left[a\,\frac{1-\sqrt{k'}\,\mathrm{tn}\,u}{1+\sqrt{k'}\,\mathrm{tn}\,u}\right]du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t^4+1}$, $(0 \leq y < 1)$

Special case of above:

$$\mathrm{tn}\,u = \left(1+\sqrt{2}\right)\frac{1-t}{1+t}, \quad k^2 = 4\left(3\sqrt{2}-4\right), \quad k' = \sqrt{1-k^2},$$
$$g = 2-\sqrt{2}, \quad \psi = \mathrm{am}\,u_1 = \tan^{-1}\left[\left(1+\sqrt{2}\right)\frac{1-y}{1+y}\right],$$
$$\mathrm{tn}\,u_1 = \tan\psi.$$

264.50 $$\int_y^1 \frac{dt}{\sqrt{t^4+1}} = g\int_0^{u_1} du = g\,u_1 = \left(2-\sqrt{2}\right)\mathrm{tn}^{-1}(\tan\psi, k) = \left(2-\sqrt{2}\right)F(\psi, k),$$ [cf. **263.50**].

264.51 $$\int_y^1 \frac{\left(t^2+t\sqrt{2}+1\right)dt}{\left(t^2-t\sqrt{2}+1\right)\sqrt{t\ +1}} = \frac{g}{k'}\int_0^{u_1} \mathrm{dn}^2 u\,du = \frac{g}{k'}\,E(u_1) = \left(2+\sqrt{2}\right)E(\psi, k).$$

264.52 $$\int_y^1 \frac{t^m\,dt}{\sqrt{t\ +1}} = \left(2-\sqrt{2}\right)\sum_{j=0}^{m} \frac{(-1)^{m+j}\,m!\,2^j}{(m-j)!\,j!}\int_0^{u_1} \frac{du}{\left[1+\left(\sqrt{2}-1\right)\mathrm{tn}\,u\right]^j}.$$ [See **342.05**.]

264.53 $$\int_y^1 \frac{(1-t)^m\,dt}{(1+t)^m\sqrt{t^4+1}} = g\,k'^m\int_0^{u_1} \mathrm{tn}^{2m} u\,du.$$ [See **316.05**.]

264.54 $$\int_y^1 \frac{dt}{(t-p)^m\sqrt{t^4+1}} = \frac{(-1)^m g}{(1+p)^m(\sqrt{k'})^m}\sum_{j=0}^{m}\frac{m!(\sqrt{k'})^{m+j}2^j}{(m-j)!\,j!\,(p-1)^j}\int_0^{u_1}\frac{du}{(1+\alpha\,\mathrm{tn}\,u)^j}\,.$$ [See **342.05**.]

where $$\alpha=(\sqrt{2}-1)(1+p)/(1-p),\quad p\neq 1\,.$$

264.55 $$\int_y^1 \frac{(1-t)^2\,dt}{(t^2-t\sqrt{2}+1)\sqrt{t^4+1}} = 2(2-\sqrt{2})\,g\int_0^{u_1}\mathrm{sn}^2 u\,du.$$ [See **310.02**.]

264.56 $$\int_y^1 \frac{(1+t)^2\,dt}{(t^2-t\sqrt{2}+1)\sqrt{t^4+1}} = 2(2+\sqrt{2})\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

264.57 $$\int_y^1 \frac{(1+t)^2\,dt}{(t^2+t\sqrt{2}+1)\sqrt{t^4+1}} = (4-2\sqrt{2})\,g\int_0^{u_1}\mathrm{cd}^2 u\,du.$$ [See **320.02**.]

264.58 $$\int_y^1 \frac{R(t)\,dt}{\sqrt{t^4+1}} = g\int_0^{u_1} R\left[\frac{1-(\sqrt{2}-1)\,\mathrm{tn}\,u}{1+(\sqrt{2}-1)\,\mathrm{tn}\,u}\right]du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t^4+b\,t^2+a^4}$, $(b<2a^2,\ 0\leq y<\infty)$

$$\mathrm{cn}\,u=\frac{t^2-a^2}{t^2+a^2},\quad k^2=\frac{2a^2-b}{4a^2},\quad g=1/2a,$$
$$\xi=\mathrm{am}\,u_1=\cos^{-1}\left[\frac{y^2-a^2}{y^2+a^2}\right],\quad \mathrm{cn}\,u_1=\cos\xi.$$

266.00 $$\int_y^\infty \frac{dt}{\sqrt{t^4+b\,t^2+a^4}} = g\int_0^{u_1}du = g\,u_1 = g\,\mathrm{cn}^{-1}(\cos\xi,k) = g\,F(\xi,k),$$ [cf. **263.00**.]

266.01 $$\int_y^\infty \frac{\sqrt{t^4+b\,t^2+a^4}}{(t^2+a^2)^2}\,dt = g\int_0^{u_1}\mathrm{dn}^2 u\,du = g\,E(u_1) = g\,E(\xi,k).$$

266.02 $$\int_y^\infty \frac{t^{2m}\,dt}{(t^2-a^2)^{2m}\sqrt{t^4+b\,t^2+a^4}} = g^{2m+1}\int_0^{u_1}\mathrm{tn}^{2m} u\,du.$$ [See **316.05**.]

266.03 $$\int_y^\infty \frac{t^{2m}\,dt}{(t^4+b\,t^2+a^4)^m\sqrt{t^4+b\,t^2+a^4}} = g^{2m+1}\int_0^{u_1}\mathrm{sd}^{2m} u\,du.$$ [See **318.05**.]

$$\text{266.04}\quad \begin{cases} \displaystyle\int_y^\infty \frac{dt}{(t^2-p)^m\sqrt{t^4+b\,t^2+a^4}} = \frac{g}{(a^2+p)^m} \times \\ \displaystyle\qquad \times \sum_{j=0}^{m} \frac{(-1)^{m+j}(\alpha+1)^j\, m!}{(m-j)!\,j!} \int_0^{u_1} \frac{du}{(1+\alpha\,\mathrm{cn}\,u)^j}. \end{cases}$$

[See **341.05**.]

where $\alpha = (a^2+p)/(a^2-p), \quad p \neq a^2.$

$$\text{266.05}\quad \int_y^\infty \frac{t^{2m}\,dt}{(t^2+a^2)^{2m}\sqrt{t^4+b\,t^2+a^4}} = g^{2m+1}\int_0^{u_1} \mathrm{sn}^{2m} u\,du.$$ [See **310.05**.]

$$\text{266.06}\quad \int_y^\infty \frac{(t^2+a^2)^{2m}\,dt}{(t^4+b\,t^2+a^4)^m\sqrt{t^4+b\,t^2+a^4}} = g\int_0^{u_1} \mathrm{nd}^{2m} u\,du.$$ [See **315.05**.]

$$\text{266.07}\quad \int_y^\infty \frac{(t^2-a^2)^m\,dt}{(t^2+a^2)^m\sqrt{t^4+b\,t^2+a^4}} = g\int_0^{u_1} \mathrm{cn}^m u\,du.$$ [See **312.04** or **312.05**.]

$$\text{266.08}\quad \int_y^\infty \frac{(t^2-a^2)^{2m}\,dt}{(t^4+b\,t^2+a^4)^m\sqrt{t^4+b\,t^2+a^4}} = g\int_0^{u_1} \mathrm{cd}^{2m} u\,du.$$ [See **320.05**.]

$$\text{266.09}\quad \int_y^\infty \frac{R(t)\,dt}{\sqrt{t^4+b\,t^2+a^4}} = g\int_0^{u_1} R\left[\frac{a\,\mathrm{sn}\,u}{1-\mathrm{cn}\,u}\right]du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt{t-a}$, $\sqrt{t-\bar{a}}$, $\sqrt{t-c}$ and $\sqrt{t-\bar{c}}$; (a, $\bar{a}$, c, $\bar{c}$ all complex)

$$\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})} = \sqrt{[(t-b_1)^2+a_1^2]\,[(t-b_2)^2+a_2^2]},$$

$$b_1 = \frac{a+\bar{a}}{2}, \quad a_1^2 = -\frac{(a-\bar{a})^2}{4}, \quad b_2 = \frac{c+\bar{c}}{2}, \quad a_2^2 = -\frac{(c-\bar{c})^2}{4},$$

$$A^2 = (b_1-b_2)^2+(a_1+a_2)^2, \quad B^2 = (b_1-b_2)^2+(a_1-a_2)^2,$$

$$k^2 = \frac{4AB}{(A+B)^2}, \quad g = \frac{2}{A+B}, \quad g_1^2 = \frac{4a_1^2-(A-B)^2}{(A+B)^2-4a_1^2}, \quad y_1 = b_1 - a_1 g_1,$$

$$\varphi = \mathrm{am}\,u_1 = \tan^{-1}\left[\frac{y-b_1+a_1g_1}{a_1+g_1b_1-g_1y}\right]; \ \mathrm{tn}\,u = \frac{t-b_1+a_1g_1}{a_1+g_1b_1-g_1t}; \ \mathrm{tn}\,u_1 = \tan\varphi.$$

$$\text{267.00}\quad \int_{y_1}^{y} \frac{dt}{\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})}} = g\int_0^{u_1} du = g u_1 = g\,\mathrm{tn}^{-1}(\tan\varphi, k) = gF(\varphi, k).$$

267.01 $$\left\{\begin{aligned}&\int_{y_1}^{y} \frac{t^m\,dt}{\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})}}\\ &\qquad = \frac{g\,(b_1-g_1 a_1)^m}{g_1^m} \sum_{j=0}^{m} \frac{m!\,\alpha_1^{m-j}(g_1-\alpha_1)^j}{(m-j)!\,j!} \int_0^u \frac{du}{(1+g_1 \operatorname{tn} u)^j},\end{aligned}\right.$$

[See **342.05**.]

where $\alpha_1 = (a_1 + b_1 g_1)/(b_1 - a_1 g_1)$, and g_1 is given above.

267.02 $$\left\{\begin{aligned}&\int_{y_1}^{y} \frac{dt}{(t-p)^m \sqrt{(t-\bar{a})(t-a)(t-c)(t-\bar{c})}}\\ &\qquad = \frac{g}{(a_1+b_1 g_1-p g_1)^m} \sum_{j=0}^{m} \frac{m!\,g_1^{m-j}(\alpha_2-g_1)^j}{(m-j)!\,j!} \int_0^{u_1} \frac{du}{(1+\alpha_2 \operatorname{tn} u)^j},\end{aligned}\right.$$

[See **342.05**.]

where $\alpha_2 = (a_1 + b_1 g_1 - p\,g_1)/(b_1 - a_1 g_1 - p)$, and g_1 is given above.

267.03 $$\int_{y_1}^{y} \frac{(t-b_1)\,dt}{\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})}} = -a_1 g\,g_1 \int_0^{u_1} \frac{1-(1/g_1)\operatorname{tn} u}{1+g_1 \operatorname{tn} u}\,du.$$

[See **361.65**.]

267.04 $$\int_{y_1}^{y} \frac{(t-b_2)\,dt}{\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})}} = (b_1-b_2-a_1 g_1)\,g \int_0^{u_1} \frac{1+\alpha_3 \operatorname{tn} u}{1+g_1 \operatorname{tn} u}\,du.$$

[See **361.65**.]

where $\alpha_3 = (b_1 g_1 + a_1 - b_2 g_1)/(b_1 - b_2 - a_1 g_1)$, and g_1 is given above.

267.05 $$\int_{y_1}^{y} \frac{R(t)\,dt}{\sqrt{(t-a)(t-\bar{a})(t-c)(t-\bar{c})}} = g \int_0^{u_1} R\left[\frac{(b_1-a_1 g_1)(1+\alpha_1 \operatorname{tn} u)}{1+g_1 \operatorname{tn} u}\right] du,$$

where $R(t)$ is a rational function of t, and α_1 is as in **267.01**.

In special case when $b_1 = b_2$, let $t - b_1 = \tau$; then

267.50 $$\int_{y_1}^{y} \frac{R(t)\,dt}{\sqrt{[(t-b_1)^2+a_1^2][(t-b_1)^2+a_2^2]}} = \int_{y_1-b_1}^{y-b_1} \frac{R(\tau+b_1)\,d\tau}{\sqrt{(\tau^2+a_1^2)(\tau^2+a_2^2)}}.$$

[Hence see **221**.]

Integrands[1] Involving Miscellaneous Fractional Powers of Polynomials.

Integrands involving $\sqrt[4]{[(a-t)\,(t-b)]^3}$ and $\sqrt[4]{(a-t)\,(t-b)}$, $(a \geqq y > b)$

$$\operatorname{cn}^2 u = \frac{2\sqrt{(a-t)\,(t-b)}}{a-b}, \quad k^2 = 1/2, \quad g = \frac{2}{\sqrt{a-b}},$$
$$\varphi = \operatorname{am} u_1 = \cos^{-1}\sqrt[4]{\frac{4\,(a-y)\,(y-b)}{(a-b)^2}}, \quad \operatorname{cn} u_1 = \cos\varphi.$$

271.01 $$\int_b^y \frac{dt}{\sqrt[4]{[(a-t)\,(t-b)]^3}} = g\int_{-K}^{u_1} du = g\,(K+u_1) = g\,[K+\operatorname{cn}^{-1}(\cos\varphi, k)] = g\,[K+F(\varphi,k)].$$

271.02 $$\int_b^y \left[1+\frac{2}{a-b}\sqrt{(a-t)\,(t-b)}\right]\frac{dt}{\sqrt[4]{[(a-t)\,(t-b)]^3}} = 2g\int_{-K}^{u_1}\operatorname{dn}^2 u\,du = 2g\,[E+E(\varphi,k)].$$

271.03 $$\int_b^y \frac{dt}{[(a-b)\,(1-\alpha^2)+2\alpha^2\sqrt{(a-t)\,(t-b)}]\sqrt[4]{[(a-t)\,(t-b)]^3}} = \frac{g}{a-b}\int_{-K}^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{g}{a-b}\,\Pi(\alpha^2,k)+\frac{g}{a-b}\,\Pi(\varphi,\alpha^2,k).$$ [See **400**.]

271.04 $$\int_b^y \frac{(2t-a-b)^2\,dt}{\sqrt[4]{[(a-t)\,(t-b)]^3}} = 2\,(a-b)^2\,g\int_{-K}^{u_1}\operatorname{sn}^2 u\operatorname{dn}^2 u\,du.$$ [See **361.02**.]

271.05 $$\int_b^y \frac{dt}{\sqrt[4]{(a-t)\,(t-b)}} = \sqrt{a-b}\int_{-K}^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02**.]

271.06 $$\int_b^y \frac{(a-b)-2\sqrt{(a-t)\,(t-b)}}{\sqrt[4]{[(a-t)\,(t-b)]^3}}\,dt = (a-b)\,g\int_{-K}^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02**.]

271.07 $$\int_b^y \frac{R(t)\,dt}{\sqrt[4]{[(a-t)\,(t-b)]^3}} = g\int_{-K}^{u_1} R\left(\tfrac{1}{2}\left[a+b+\sqrt{2}\,(a-b)\operatorname{sn} u\operatorname{dn} u\right]\right)du,$$

where $R(t)$ is a rational function t.

[1] There are other cases in **575.14** and **575.15** which may also be reduced.

Integrands involving $\sqrt[4]{(1-t^2)^3}$ and $\sqrt[4]{1-t^2}$, $(1 \geqq y > 0)$

$$\operatorname{cn}^2 u = \sqrt{1-t^2}, \quad k^2 = 1/2, \quad g = \sqrt{2}$$
$$\psi = \operatorname{am} u_1 = \cos^{-1}\sqrt[4]{1-y^2}; \quad \operatorname{cn} u_1 = \cos\psi,$$

271.51 $$\int_0^y \frac{dt}{\sqrt[4]{(1-t^2)^3}} = g\int_0^{u_1} du = \sqrt{2}\,u_1 = \sqrt{2}\operatorname{cn}^{-1}(\cos\psi, k) = \sqrt{2}\,F(\psi, k),$$

271.52 $$\int_0^y \frac{1+\sqrt{1-t^2}}{\sqrt[4]{(1-t^2)^3}}\,dt = 2g\int_0^{u_1} \operatorname{dn}^2 u\,du = 2\sqrt{2}\,E(u_1) = 2\sqrt{2}\,E(\psi, k).$$

271.53 $$\int_0^y \frac{dt}{[(1-\alpha^2)+\alpha^2\sqrt{1-t^2}]\sqrt[4]{(1-t^2)^3}} = \sqrt{2}\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \sqrt{2}\,\Pi(\psi, \alpha^2, k).$$ [See **400.**]

271.54 $$\int_0^y \frac{t^2\,dt}{\sqrt[4]{(1-t^2)^3}} = 2\sqrt{2}\int_0^{u_1}\operatorname{sn}^2 u\operatorname{dn}^2 u\,du.$$ [See **361.02.**]

271.55 $$\int_0^y \frac{dt}{\sqrt[4]{1-t^2}} = \sqrt{2}\int_0^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02.**]

271.56 $$\int_0^y \frac{1-\sqrt{1-t^2}}{\sqrt[4]{(1-t^2)^3}}\,dt = \sqrt{2}\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

271.57 $$\int_0^y \frac{1-\sqrt{1-t^2}}{1+\sqrt{1-t^2}}\,\frac{dt}{\sqrt[4]{(1-t^2)^3}} = \frac{1}{\sqrt{2}}\int_0^{u_1}\operatorname{sd}^2 u\,du.$$ [See **318.02.**]

271.58 $$\int_0^y \frac{dt}{(1+\sqrt{1-t^2})\sqrt[4]{1-t^2}} = \frac{\sqrt{2}}{2}\int_0^{u_1}\operatorname{cd}^2 u\,du.$$ [See **320.02.**]

271.59 $$\int_0^y \frac{t^2\,dt}{\sqrt[4]{1-t^2}} = 2\sqrt{2}\int_0^{u_1}\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn}^2 u\,du.$$ [See **361.04.**]

271.60 $$\int_0^y \frac{R(t)\,dt}{\sqrt[4]{(1-t^2)^3}} = \sqrt{2}\int_0^{u_1} R\left(\sqrt{2}\operatorname{sn} u\operatorname{dn} u\right)du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt[4]{[(t-a)(t-b)]^3}$ and $\sqrt[4]{(t-a)(t-b)}$, $(\infty > y > a > b)$

$$\operatorname{cn} u = \frac{a-b-2\sqrt{(t-a)(t-b)}}{a-b+2\sqrt{(t-a)(t-b)}}, \quad k^2 = 1/2, \quad g = \sqrt{\frac{2}{a-b}},$$
$$\varphi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{a-b-2\sqrt{(y-a)(y-b)}}{a-b+2\sqrt{(y-a)(y-b)}}\right], \quad \operatorname{cn} u_1 = \cos\varphi.$$

272.00 $$\int_a^y \frac{dt}{\sqrt[4]{[(t-a)(t-b)]^3}} = g\int_0^{u_1} du = g\operatorname{cn}^{-1}(\cos\varphi, k) = gF(\varphi, k).$$

272.01 $$\int_a^y \frac{(2t-a-b)^2\,dt}{[(2t-a-b)^2+4(a-b)\sqrt{(t-a)(t-b)}]\sqrt[4]{[(t-a)(t-b)]^3}} = g\int_0^{u_1}\operatorname{dn}^2 u\,du = gE(u_1) = gE(\varphi,k).$$

272.02 $$\int_a^y \frac{[(2t-a-b)^2+4(a-b)\sqrt{(t-a)(t-b)}]\,dt}{[(2t-a-b)^2+4(a-b)(1-2\alpha^2)\sqrt{(t-a)(t-b)}]\sqrt[4]{[(t-a)(t-b)]^3}} = g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\Pi(\varphi, \alpha^2, k).$$ [See **400.**]

272.03 $$\int_a^y \frac{dt}{[(2t-a-b)^2+4(a-b)\sqrt{(t-a)(t-b)}]\sqrt[4]{(t-a)(t-b)}} = \frac{g}{8(a-b)}\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

272.04 $$\int_a^y \frac{[(2t-a-b)^2-4(a-b)\sqrt{(t-a)(t-b)}]\,dt}{[(2t-a-b)^2+4(a-b)\sqrt{(t-a)(t-b)}]\sqrt[4]{[(t-a)(t-b)]^3}} = g\int_0^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02.**]

272.05 $$\int_a^y \frac{dt}{\sqrt[4]{(t-a)(t-b)}} = \frac{1}{g}\int_0^{u_1}\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\,du.$$ [See **361.53.**]

272.06 $$\int_a^y \frac{dt}{(2t-a-b)^2 \sqrt[4]{(t-a)(t-b)}} = \frac{g}{8(a-b)} \int_0^{u_1} \operatorname{sd}^2 u \, du.$$ [See **318.02.**]

272.07 $$\int_a^y \frac{(2t-a-b)^2 - 4(a-b)\sqrt{(t-a)(t-b)}}{(2t-a-b)^2 \sqrt[4]{[(t-a)(t-b)]^3}} dt = g \int_0^{u_1} \operatorname{cd}^2 u \, du.$$ [See **320.02.**]

272.08 $$\begin{cases} \displaystyle\int_a^y \sqrt[4]{(t-a)(t-b)}\, dt \\ \displaystyle\quad = \frac{1}{g^3}\left[F(\varphi, k) - 4\int_0^{u_1} \frac{du}{1+\operatorname{cn} u} + 4\int_0^{u_1} \frac{du}{(1+\operatorname{cn} u)^2}\right]. \end{cases}$$ [See **341.53** and **341.54.**]

272.09 $$\int_a^y \frac{R(t)\, dt}{\sqrt[4]{[(t-a)(t-b)]^3}} = g \int_0^{u_1} R\left[\frac{(a+b)(1+\operatorname{cn} u) + 2(a-b)\operatorname{dn} u}{2(1+\operatorname{cn} u)}\right] du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt[4]{(t^2-1)^3}$ and $\sqrt[4]{t^2-1}$, $(\infty > y > 1)$

Special case of above:

$$\boxed{\begin{gathered} \operatorname{cn} u = \frac{1-\sqrt{t^2-1}}{1+\sqrt{t^2-1}}, \quad k^2 = 1/2, \quad \operatorname{cn} u_1 = \cos\psi, \\ \psi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{1-\sqrt{y^2-1}}{1+\sqrt{y^2-1}}\right]. \end{gathered}}$$

272.50 $$\int_1^y \frac{dt}{\sqrt[4]{(t^2-1)^3}} = \int_0^{u_1} du = u_1 = \operatorname{cn}^{-1}(\cos\psi, k) = F(\psi, k).$$

272.51 $$\int_1^y \frac{t^2\, dt}{\left[t^2 + 2\sqrt{t^2-1}\right] \sqrt[4]{(t^2-1)^3}} = \int_0^{u_1} \operatorname{dn}^2 u \, du = E(u_1) = E(\psi, k).$$

272.52 $$\int_1^y \frac{\left[t^2+2\sqrt{t^2-1}\right] dt}{\left[t^2+2(1-2\alpha^2)\sqrt{t^2-1}\right] \sqrt[4]{(t^2-1)^3}} = \int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = \Pi(\psi, \alpha^2, k).$$ [See **400.**]

272.53 $$\int_1^y \frac{dt}{\left[t^2+2\sqrt{t^2-1}\right] \sqrt[4]{t^2-1}} = \frac{1}{4} \int_0^{u_1} \operatorname{sn}^2 u \, du.$$ [See **310.02.**]

272.54 $$\int_1^y \frac{dt}{t^2\sqrt{t^2-1}} = \frac{1}{4}\int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02.**]

272.55 $$\int_1^y \frac{dt}{\sqrt[4]{t^2-1}} = \int_0^{u_1} \frac{1-\mathrm{cn}\, u}{1+\mathrm{cn}\, u}\, du.$$ [See **361.53.**]

272.56 $$\int_1^y \frac{t\,dt}{\sqrt[4]{(t^2-1)^3}} = 2\int_0^{u_1} \frac{\mathrm{dn}\, u\, du}{1+\mathrm{cn}\, u} = \frac{2\,\mathrm{sn}\, u_1}{1+\mathrm{cn}\, u_1} = 2\sqrt[4]{y^2-1}\,.$$

272.57 $$\int_1^y \sqrt[4]{t^2-1}\,dt = \left[F(\varphi, k) - 4\int_0^{u_1} \frac{du}{1+\mathrm{cn}\, u} + 4\int_0^{u_1} \frac{du}{(1+\mathrm{cn}\, u)^2}\right].$$
[See **341.53** and **341.54.**]

272.58 $$\int_1^y \frac{[t^2-2\sqrt{t^2-1}]\,dt}{[t^2+2\sqrt{t^2-1}]\,\sqrt{(t^2-1)^3}} = \int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02.**]

272.59 $$\int_1^y \frac{t^2-2\sqrt{t^2-1}}{t^2\sqrt[4]{(t^2-1)^3}}\,dt = \int_0^{u_1} \mathrm{cd}^2 u\, du.$$ [See **320.02.**]

272.60 $$\int_1^y \frac{R(t)\,dt}{\sqrt[4]{(t^2-1)^3}} = \int_0^{u_1} R\left[\frac{2\,\mathrm{dn}\, u}{1+\mathrm{cn}\, u}\right] du,$$
where $R(t)$ is a rational function of t.

Integrands involving $\sqrt[4]{[(t-b)^2+a^2]^3}$ and $\sqrt[4]{(t-b)^2+a^2}$, $(\infty > y > b)$

$$\mathrm{cn}^2 u = \sqrt{\frac{a^2}{(t-b)^2+a^2}}, \quad k^2 = 1/2, \quad g = \sqrt{\frac{2}{a}},$$
$$\varphi = \mathrm{am}\, u_1 = \cos^{-1}\left[\frac{\sqrt{a}}{\sqrt[4]{(y-b)^2+a^2}}\right], \quad \mathrm{cn}\, u_1 = \cos\varphi.$$

273.00 $$\int_b^y \frac{dt}{\sqrt[4]{[(t-b)^2+a^2]^3}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{cn}^{-1}(\cos\varphi, k) = g\,F(\varphi, k).$$

273.01 $$\int_b^y \frac{a+\sqrt{(t-b)^2+a^2}}{\sqrt[4]{[(t-b)^2+a^2]^5}}\,dt = 2g\int_0^{u_1} \mathrm{dn}^2 u\, du = 2g\,E(u_1) = 2g\,E(\varphi, k).$$

273.02 $$\left\{\begin{aligned}&\int_b^y \frac{dt}{[a\alpha^2 + (1-\alpha)\sqrt{(t-b)^2+a^2}]\sqrt[4]{(t-b)^2+a^2}} = g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ &\qquad = g\Pi(\varphi, \alpha^2, k).\qquad \text{[See } \mathbf{400.}]\end{aligned}\right.$$

273.03 $$\int_b^y \frac{(t-b)^2\,dt}{[(t-b)^2+a^2]\sqrt[4]{[(t-b)^2+a^2]^3}} = 2g\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du. \quad \text{[See } \mathbf{361.02.}]$$

273.04 $$\int_b^y \frac{dt}{[(t-b)^2+a^2]\sqrt[4]{(t-b)^2+a^2}} = \frac{g}{a}\int_0^{u_1}\mathrm{cn}^2 u\,du. \quad \text{[See } \mathbf{312.02.}]$$

273.05 $$\int_b^y \frac{dt}{\sqrt[4]{(t-b)^2+a^2}} = a\,g\int_0^{u_1}\mathrm{nc}^2 u\,du. \quad \text{[See } \mathbf{313.02.}]$$

273.06 $$\int_b^y \frac{(t-b)^2\,dt}{\sqrt[4]{[(t-b)^2+a^2]^5}} = 2a\,g\int_0^{u_1}\mathrm{tn}^2 u\,\mathrm{dn}^2 u\,du. \quad \text{[See } \mathbf{361.05.}]$$

273.07 $$\int_b^y \frac{a+\sqrt{(t-b)^2+a^2}}{\sqrt[4]{[(t-b)^2+a^2]^3}}\,dt = 2a\,g\int_0^{u_1}\mathrm{dc}^2 u\,du. \quad \text{[See } \mathbf{321.02.}]$$

273.08 $$\int_b^y \frac{-a+\sqrt{(t-b)^2+a^2}}{\sqrt[4]{[(t-b)^2+a^2]^3}}\,dt = a\,g\int_0^{u_1}\mathrm{tn}^2 u\,du. \quad \text{[See } \mathbf{316.02.}]$$

273.09 $$\int_b^y \frac{R(t)\,dt}{\sqrt[4]{[(t-b)^2+a^2]^3}} = g\int_0^{u_1} R\left(b + a\sqrt{2}\,\mathrm{tn}\,u\,\mathrm{dc}\,u\right)du,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt[4]{(t^2+1)^3}$ and $\sqrt[4]{t^2+1}$, $(\infty > y > 0)$

Special case of above:

$$\boxed{\begin{gathered}\mathrm{cn}^2 u = \frac{1}{\sqrt{t^2+1}},\quad k^2 = 1/2,\quad \mathrm{cn}\,u_1 = \cos\psi,\\ \psi = \mathrm{am}\,u_1 = \cos^{-1}\left[\frac{1}{\sqrt[4]{y^2+1}}\right].\end{gathered}}$$

273.50 $$\int_0^y \frac{dt}{\sqrt[4]{(t^2+1)^3}} = \sqrt{2}\int_0^{u_1} du = \sqrt{2}u_1 = \sqrt{2}\,\mathrm{cn}^{-1}(\cos\psi, k) = \sqrt{2}F(\psi, k).$$

273.51 $$\int_0^y \frac{1+\sqrt{t^2+1}}{(t^2+1)\sqrt[4]{t^2+1}}\,dt = 2\sqrt{2}\int_0^{u_1}\mathrm{dn}^2 u\,du = 2\sqrt{2}E(u_1) = 2\sqrt{2}E(\psi, k).$$

273.52 $$\int_0^y \frac{dt}{[\alpha^2+(1-\alpha)\sqrt{t^2+1}]\sqrt[4]{t^2+1}} = \sqrt{2}\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \sqrt{2}\,\Pi(\psi,\alpha^2,k).$$ [See **400.**]

273.53 $$\int_0^y \frac{t^2\,dt}{(t^2+1)\sqrt[4]{(t^2+1)^3}} = \frac{1}{\sqrt{2}}\int_0^{u_1} \operatorname{sn}^2 u \operatorname{dn}^2 u\,du.$$ [See **361.02.**]

273.54 $$\int_0^y \frac{dt}{(t^2+1)\sqrt[4]{t^2+1}} = \sqrt{2}\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02.**]

273.55 $$\int_0^y \frac{dt}{\sqrt[4]{t^2+1}} = \sqrt{2}\int_0^{u_1} \operatorname{nc}^2 u\,du.$$ [See **313.02.**]

273.56 $$\int_0^y \frac{t^2\,dt}{\sqrt[4]{(t^2+1)^5}} = 2\sqrt{2}\int_0^{u_1} \operatorname{tn}^2 u \operatorname{dn}^2 u\,du.$$ [See **361.05.**]

273.57 $$\int_0^y \frac{1+\sqrt{t^2+1}}{\sqrt[4]{(t^2+1)^3}}\,dt = 2\sqrt{2}\int_0^{u_1} \operatorname{dc}^2 u\,du.$$ [See **321.02.**]

273.58 $$\int_0^y \frac{-1+\sqrt{t^2+1}}{\sqrt[4]{(t^2+1)^3}}\,dt = \sqrt{2}\int_0^{u_1} \operatorname{tn}^2 u\,du.$$ [See **316.02.**]

273.59 $$\int_0^y \frac{R(t^2)\,dt}{\sqrt[4]{(t^2+1)^3}} = \sqrt{2}\int_0^{u_1} R(2\operatorname{tn}^2 u \operatorname{dc}^2 u)\,du,$$

where $R(t^2)$ is a rational function of t^2.

Integrands involving $\sqrt[3]{1+\tau^2}$, $(\infty > Y > 0)$

$$\tau = \sqrt{t^3-1},\quad t = \sqrt[3]{1+\tau^2},\quad d\tau = \frac{3}{2}\frac{t^2\,dt}{\sqrt{t^3-1}},\quad y = \sqrt[3]{Y^2+1},$$

$$\operatorname{cn} u = \frac{\sqrt{3}+1-t}{\sqrt{3}-1+t} = \frac{\sqrt{3}+1-\sqrt[3]{\tau^2+1}}{\sqrt{3}-1+\sqrt[3]{\tau^2+1}};\quad \tau = \frac{2\sqrt[4]{27}\operatorname{sn} u \operatorname{dn} u}{(1+\operatorname{cn} u)^2},\quad k^2 = \frac{2-\sqrt{3}}{4},$$

$$g = 1/\sqrt[4]{3},\quad \varphi = \cos^{-1}\left[\frac{\sqrt{3}+1-\sqrt[3]{Y^2+1}}{\sqrt{3}-1+\sqrt[3]{Y^2+1}}\right],\quad \operatorname{cn} u_1 = \cos\varphi.$$

274.00 $$\left\{\begin{aligned} \int_0^Y \frac{d\tau}{\sqrt[3]{(\tau^2+1)^2}} &= \frac{3}{2}\int_1^y \frac{dt}{\sqrt{t^3-1}} = \frac{3g}{2}\int_0^{u_1} du = \frac{3g}{2}u_1 = \frac{3g}{2}\operatorname{cn}^{-1}(\cos\varphi,k) \\ &= \frac{3}{2\sqrt[4]{3}}F(\varphi,k),\ [\text{cf. } \mathbf{240.00}]. \end{aligned}\right.$$

274.01 $$\int_0^Y \frac{d\tau}{\sqrt[3]{\tau^2+1}} = \frac{3}{2}\int_1^y \frac{t\,dt}{\sqrt{t^3-1}} = \frac{3}{2\sqrt[4]{3}}\left(1-\sqrt{3}\right)F(\varphi,k) + 3\sqrt[4]{3}\int_0^{u_1}\frac{du}{1+\operatorname{cn} u}.$$

[See **341.53**.]

274.02 $$\int_0^Y \sqrt[3]{\tau^2+1}\,d\tau = \frac{3}{2}\int_1^y \frac{t^3\,dt}{\sqrt{t^3-1}} = \frac{9(1-\sqrt{3})^3}{\sqrt[4]{3}}\sum_{j=0}^{3}\frac{(-3-\sqrt{3})^j}{j!\,(3-j)!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}.$$

[See **341.55**.]

274.03 $$\int_0^Y \frac{\tau\,d\tau}{\sqrt[3]{\tau^2+1}} = \frac{3}{2}\int_1^y t\,dt = \frac{3}{4}\left[\sqrt[3]{(Y^2+1)^2}-1\right].$$

274.04 $$\left\{\begin{aligned} &\int_0^Y \frac{\tau^2\,d\tau}{\sqrt[3]{\tau^2+1}} = \frac{3}{2}\int_1^y t\sqrt{t^3-1}\,dt = \frac{3g}{2}\Bigg\{\left(\sqrt{3}-1\right)F(\varphi,k) - 2\sqrt{3}\int_0^{u_1}\frac{du}{1+\operatorname{cn} u} + \\ &\quad + 24\left(1-\sqrt{3}\right)^4\sum_{j=0}^{4}\frac{(-3-\sqrt{3})^j}{j!\,(4-j)!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}\Bigg\}. \end{aligned}\right.$$ [See **341.53** and **341.55**.]

274.05 $$\left\{\begin{aligned} \int_0^Y \frac{\tau^{2m+1}\,d\tau}{\sqrt[3]{\tau^2+1}} &= \frac{3}{2}\int_1^y t(t^3-1)^m\,dt \\ &= \frac{3}{2}\sum_{j=0}^{m}\frac{(-1)^j\,m!\,\left[(Y^2+1)^{m-j}\sqrt[3]{(Y^2+1)^2}-1\right]}{j!\,(m-j)!\,(3m-3j+2)}. \end{aligned}\right.$$

274.06 $$\left\{\begin{aligned} &\int_0^Y \frac{\tau^{2m}\,d\tau}{\sqrt[3]{\tau^2+1}} = \frac{3}{2}\int_1^y \frac{t(t^3-1)^m}{\sqrt{t^3-1}}\,dt = \frac{3g}{2}\sum_{i=0}^{m}\frac{m!\,(-1)^i\,(1-\sqrt{3})^{3m-3i+1}}{i!\,(m-i)!}\times \\ &\quad \times\sum_{j=0}^{3m-3i+1}\frac{(-3-\sqrt{3})^j\,(3m-3i+1)!}{j!\,(3m-3i+1-j)!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}. \end{aligned}\right.$$ [See **341.55**.]

274.07 $$\left\{\begin{aligned} \int_0^Y \frac{R(\tau)\,d\tau}{\sqrt[3]{\tau^2+1}} &= \frac{3}{2}\int_1^y \frac{t\,R\left(\sqrt{t^3-1}\right)}{\sqrt{t^3-1}}\,dt \\ &= \frac{3g(1+\sqrt{3})}{2}\int_0^{u_1}\frac{1+(\sqrt{3}-2)\operatorname{cn} u}{1+\operatorname{cn} u}\,R\left[\frac{2\sqrt[4]{27}\operatorname{sn} u\operatorname{dn} u}{(1+\operatorname{cn} u)^2}\right]du, \end{aligned}\right.$$

where $R(\tau)$ is a rational function of τ.

Integrands involving $\sqrt[3]{\tau^2-1}$, $(\infty > Y > 0)$

$$\tau=\sqrt{1-t^3},\quad t=-\sqrt[3]{\tau^2-1},\quad d\tau=-\frac{3}{2}\frac{t^2\,dt}{\sqrt{1-t^3}},\quad y=-\sqrt[3]{Y^2-1},$$

$$\operatorname{cn} u=\frac{\sqrt{3}-1+t}{\sqrt{3}+1-t}=\frac{\sqrt{3}-1-\sqrt[3]{\tau^2-1}}{\sqrt{3}+1+\sqrt[3]{\tau^2-1}};\quad \tau=\frac{2\sqrt[4]{27}\operatorname{sn} u\operatorname{dn} u}{(1+\operatorname{cn} u)^2},\quad k^2=\frac{2+\sqrt{3}}{4},$$

$$g=\frac{1}{\sqrt[4]{3}},\quad \psi=\operatorname{am} u_1=\cos^{-1}\left[\frac{\sqrt{3}-1-\sqrt[3]{Y^2-1}}{\sqrt{3}+1+\sqrt[3]{Y^2-1}}\right],\quad \operatorname{cn} u_1=\cos\psi.$$

275.00
$$\begin{cases} -\displaystyle\int_0^Y\frac{d\tau}{\sqrt[3]{(\tau^2-1)^2}}=\frac{3}{2}\int_y^1\frac{dt}{\sqrt{1-t^3}}=\frac{3g}{2}\int_0^{u_1}du=\frac{3g}{2}\operatorname{cn}^{-1}(\cos\psi,k)\\ \qquad=\dfrac{3}{2\sqrt[4]{3}}F(\psi,k), \end{cases}$$
[cf. **244.00**].

275.01
$$-\int_0^Y\frac{d\tau}{\sqrt[3]{\tau^2-1}}=\frac{3}{2}\int_y^1\frac{t\,dt}{\sqrt{1-t^3}}=\frac{3}{2\sqrt[4]{3}}\left(1+\sqrt{3}\right)F(\psi,k)-3\sqrt[4]{3}\int_0^u\frac{du}{1+\operatorname{cn} u},$$
[See **341.53** and **361.51**.]

275.02
$$-\int_0^Y\frac{\tau\,d\tau}{\sqrt[3]{\tau^2-1}}=\frac{3}{2}\int_y^1 t\,dt=\frac{3}{4}\left[1-\sqrt[3]{(Y^2-1)^2}\right].$$

275.03
$$\begin{cases} -\displaystyle\int_0^Y\frac{\tau^2\,d\tau}{\sqrt[3]{\tau^2-1}}=\frac{3}{2}\int_y^1 t\sqrt{1-t^3}\,dt=\frac{3}{2}g\Bigg\{\left(1+\sqrt{3}\right)F(\psi,k)\\ \qquad -2\sqrt{3}\displaystyle\int_0^{u_1}\frac{du}{1+\operatorname{cn} u}+24\left(1+\sqrt{3}\right)^4\sum_{j=0}^{4}\frac{(\sqrt{3}-3)^j}{(m-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}\Bigg\}. \end{cases}$$
[See **341.53** and **341.55**.]

275.04
$$\begin{cases} -\displaystyle\int_0^Y\frac{\tau^{2m+1}\,d\tau}{\sqrt[3]{\tau^2-1}}=\frac{3}{2}\int_y^1 t(1-t^3)^m\,dt\\ \qquad=\dfrac{3}{2}\displaystyle\sum_{j=0}^{m}\frac{(-1)^j m!\left[1-(Y^2-1)^{m-j}\sqrt[3]{(Y^2-1)^2}\right]}{j!\,(m-j)!\,(3m-3j+2)}. \end{cases}$$

275.05
$$-\int_0^Y\sqrt[3]{\tau^2-1}\,d\tau=\frac{3}{2}\int_y^1\frac{t^3\,dt}{\sqrt{1-t^3}}=\frac{9\left(1+\sqrt{3}\right)^3}{\sqrt[4]{3}}\sum_{j=0}^{3}\frac{(\sqrt{3}-3)^j}{(3-j)!\,j!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j},$$
[See **341.55**.]

275.06
$$\left\{\begin{aligned} &-\int_0^Y \frac{\tau^{2m}\,d\tau}{\sqrt[3]{\tau^2-1}} = \frac{3}{2}\int_y^1 \frac{t\,(1-t^3)^m}{\sqrt{1-t^3}}\,dt = \frac{3g}{2}\sum_{i=0}^{m} \frac{m!\,(-1)^i(1+\sqrt{3})^{3m-3i+1}}{i!\,(m-i)!}\times \\ &\times \sum_{j=0}^{3m-3i+1} \frac{(\sqrt{3}-3)^j\,(3m-3i+1)!}{j!\,(3m-3i+1-j)!}\int_0^{u_1}\frac{du}{(1+\operatorname{cn} u)^j}, \quad \text{[See } \mathbf{341.55}.] \end{aligned}\right.$$

275.07
$$\left\{\begin{aligned} -\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt[3]{\tau^2-1}} &= \frac{3}{2}\int_y^1 \frac{t\,R\left(\sqrt{1-t^3}\right)dt}{\sqrt{1-t^3}} \\ &= \frac{3g\,(1-\sqrt{3})}{2}\int_0^{u_1} \frac{1-(2+\sqrt{3})\operatorname{cn} u}{1+\operatorname{cn} u}\,R\left[\frac{2\sqrt[4]{27}\operatorname{sn} u \operatorname{dn} u}{(1+\operatorname{cn} u)^2}\right]du, \end{aligned}\right.$$

where $R(\tau)$ is a rational function of τ.

Integrands involving $\sqrt[n]{(t-a)(t-b)}$, $(Z>a>b;\ n=3,6,8)$

$$\sqrt[n]{(t-a)(t-b)} = \sqrt[n]{(a-b)^2/4}\;\tau, \qquad t = \tfrac{1}{2}\left[a+b+(a-b)\sqrt{1+\tau^n}\right],$$
$$dt = \frac{n\,(a-b)}{4}\,\frac{\tau^{n-1}}{\sqrt{1+\tau^n}}\,d\tau, \qquad g_1 = n\left(\frac{a-b}{4}\right)\sqrt[n]{\frac{4}{(a-b)^2}},$$
$$Y = \sqrt[n]{4\,(Z-a)\,(Z-b)/(a-b)^2}, \qquad n = 3,6,8.$$

275.50
$$\int_a^Z \frac{dt}{\sqrt[n]{(t-a)(t-b)}} = g_1\int_0^Y \frac{\tau^{n-2}\,d\tau}{\sqrt{1+\tau^n}}.$$

275.51
$$\left\{\begin{aligned} &\int_a^Z \frac{dt}{\sqrt[3]{(t-a)(t-b)}} = \frac{3}{4}\sqrt[3]{4(a-b)}\int_0^Y \frac{\tau\,d\tau}{\sqrt{1+\tau^3}} = \frac{3}{4}\sqrt[3]{4(a-b)}\times \\ &\qquad\times\left[\int_0^1 \frac{t_1\,dt_1}{\sqrt{1-t_1^3}} - \int_{-Y}^1 \frac{t_1\,dt_1}{\sqrt{1-t_1^3}}\right] \qquad \text{[See } \mathbf{244.05}.] \end{aligned}\right.$$

275.52
$$\int_a^Z \frac{dt}{\sqrt[6]{(t-a)(t-b)}} = \frac{3}{2}\sqrt[6]{4(a-b)^4}\int_0^Y \frac{\tau^4\,d\tau}{\sqrt{1+\tau^6}} \qquad \text{[See } \mathbf{578.02}.]$$

275.53
$$\int_a^Z \frac{dt}{\sqrt[8]{(t-a)(t-b)}} = 2\sqrt[8]{4(a-b)^6}\int_0^Y \frac{\tau^6\,d\tau}{\sqrt{1+\tau^8}} \qquad \text{[See } \mathbf{584.04}.]$$

275.54 $$\int_a^Z \frac{R(t)\,dt}{\sqrt[n]{(t-a)\,(t-b)}} = g_1 \int_0^Y \frac{\tau^{n-2}}{\sqrt{1+\tau^n}}\, R\left(\frac{1}{2}\left[a+b+(a-b)\sqrt{1+\tau^n}\right]\right) d\tau\,,$$

where $R(t)$ is a rational function of t.

Integrands involving $\sqrt[n]{(1-\tau^n)^{n-1}}$, $(0 \leq Y < 1/\sqrt[n]{2},\ n = 3, 4, 6)$

$$\boxed{1 - t^n = (1-2\tau^n)^2, \quad t^n = 4\tau^n(1-\tau^n), \quad y = Y\sqrt[n]{4(1-Y^n)}.}$$

276.00 $$\int_Y^{1/\sqrt[n]{2}} \frac{d\tau}{\sqrt[n]{(1-\tau^n)^{n-1}}} = \frac{1}{\sqrt[n]{4}} \int_y^1 \frac{dt}{\sqrt{1-t^n}}\,.$$ [See **244.00, 213.00** and **576.00.**]

276.01 $$\begin{cases} \displaystyle\int_Y^{1/\sqrt[3]{2}} \frac{d\tau}{\sqrt[3]{(1-\tau^3)^2}} = \frac{1}{\sqrt[3]{4}} \int_y^1 \frac{dt}{\sqrt{1-t^3}} \\ \displaystyle\qquad = \frac{1}{\sqrt[3]{4}}\,\frac{1}{\sqrt[4]{3}}\,\mathrm{cn}^{-1}\left[\frac{\sqrt{3}-1+Y\sqrt[3]{4(1-Y^3)}}{\sqrt{3}+1-Y\sqrt[3]{4(1-Y^3)}},\ \sqrt{\frac{2+\sqrt{3}}{4}}\right]. \end{cases}$$

276.02 $$\begin{cases} \displaystyle\int_Y^{1/\sqrt[3]{2}} \frac{\tau^3\,d\tau}{\sqrt[3]{(1-\tau^3)^2}} = \frac{1}{2\sqrt[3]{4}} \int_y^1 \frac{\left[1-\sqrt{1-t^3}\right]}{\sqrt{1-t^3}}\,dt = \frac{1}{2\sqrt[3]{4}}\,\frac{1}{\sqrt[4]{3}} \times \\ \displaystyle\times\left\{\sqrt[4]{3}\left[Y\sqrt[3]{4(1-Y^3)}-1\right] + \mathrm{cn}^{-1}\left[\frac{\sqrt{3}-1+Y\sqrt[3]{4(1-Y^3)}}{\sqrt{3}+1-Y\sqrt[3]{4(1-Y^3)}},\ \sqrt{\frac{2+\sqrt{3}}{4}}\right]\right\}. \end{cases}$$

276.03 $$\int_Y^{1/\sqrt[4]{2}} \frac{d\tau}{\sqrt[4]{(1-\tau^4)^3}} = \frac{1}{\sqrt[4]{4}} \int_y^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt[4]{4}}\,\frac{1}{\sqrt{2}}\,\mathrm{cn}^{-1}\left[Y\sqrt[4]{4(1-Y^4)},\ \sqrt{2}/2\right].$$

276.04 $$\begin{cases} \displaystyle\int_Y^{1/\sqrt[4]{2}} \frac{\tau^4\,d\tau}{\sqrt[4]{(1-\tau^4)^3}} = \frac{1}{2\sqrt[4]{4}} \int_y^1 \frac{\left[1-\sqrt{1-t^4}\right]}{\sqrt{1-t^4}}\,dt = \frac{1}{2\sqrt[4]{4}}\,\frac{1}{\sqrt{2}} \times \\ \displaystyle\times\left\{\sqrt{2}\left[Y\sqrt[4]{4(1-Y^4)}-1\right] + \mathrm{cn}^{-1}\left[Y\sqrt[4]{4(1-Y^4)},\ \sqrt{2}/2\right]\right\}. \end{cases}$$

276.05 $$\begin{cases} \displaystyle\int_Y^{1/\sqrt[6]{2}} \frac{d\tau}{\sqrt[6]{(1-\tau^6)^5}} = \frac{1}{\sqrt[6]{4}} \int_y^1 \frac{dt}{\sqrt{1-t^6}} \\ \displaystyle\qquad = \frac{1}{2\sqrt[6]{4}}\,\frac{1}{\sqrt[4]{3}}\,\mathrm{cn}^{-1}\left[\frac{(\sqrt{3}+1)\,Y^2\sqrt[3]{4(1-Y^6)}-1}{(\sqrt{3}-1)\,Y^2\sqrt[3]{4(1-Y^6)}+1},\ \sqrt{\frac{2-\sqrt{3}}{4}}\right]. \end{cases}$$

276.06 $$\left\{\begin{aligned}&\int_Y^{1/\sqrt[6]{2}}\frac{\tau^6\,d\tau}{\sqrt[6]{(1-\tau^6)^5}}=\frac{1}{2\sqrt[6]{4}}\int_y^1\frac{[1-\sqrt{1-t^6}]}{\sqrt{1-t^6}}\,dt=\frac{1}{4\sqrt[6]{4}}\,\frac{1}{\sqrt[4]{3}}\times\\&\times\left\{\sqrt[4]{3}\left[Y\sqrt[6]{4(1-Y^6)}-1\right]+\mathrm{cn}^{-1}\left[\frac{(\sqrt{3}+1)Y^2\sqrt[3]{4(1-Y^6)}-1}{(\sqrt{3}-1)Y^2\sqrt[3]{4(1-Y^6)}+1},\sqrt{\frac{2-\sqrt{3}}{4}}\right]\right\}.\end{aligned}\right.$$

276.07 $$\left\{\begin{aligned}&\int_Y^{1/\sqrt[n]{2}}\frac{\tau^{2n}\,d\tau}{\sqrt[n]{(1-\tau^n)^{n-1}}}=\frac{1}{4\sqrt[n]{4}}\times\\&\qquad\times\left\{2\left[Y\sqrt[n]{4(1-Y^n)}-1\right]+2\int_y^1\frac{dt}{\sqrt{1-t^n}}-\int_y^1\frac{t^n\,dt}{\sqrt{1-t^n}}\right\}.\end{aligned}\right.$$

Integrands involving $\sqrt[3]{a_0(t_1-a)(t_1-b)(t_1-c)}$, $(a>b>c)$

$$t_1=\frac{a(b-c)\tau+ab+ac-2bc}{(b-c)\tau+2a-b-c},\quad dt_1=\frac{2(a-b)(a-c)(b-c)\,d\tau}{[(b-c)\tau+2a-b-c]^2},$$

$$G=\sqrt[3]{\frac{4}{a_0}(a-b)(a-c)(b-c)},\quad Y_1=\frac{ab+ac-2bc}{2a-b-c},$$

$$p=\frac{2a-b-c}{c-b},\quad y=\frac{(b+c-2a)Y+ab+ac-2bc}{(b-c)(Y-a)}.$$

277.00 $$\int_{Y_1}^{Y}\frac{dt_1}{\sqrt[3]{a_0(t_1-a)(t_1-b)(t_1-c)}}=\frac{G}{c-b}\int_0^y\frac{d\tau}{(\tau-p)\sqrt[3]{\tau^2-1}},$$

[For further reduction, see **275**.]

277.01 $$\left\{\begin{aligned}\int_{Y_1}^{Y}\frac{dt_1}{\sqrt[3]{(t_1-a)^2(t_1-b)^2(t_1-c)^2}}&=\frac{2}{\sqrt[3]{a_0}\,G}\int_0^y\frac{d\tau}{\sqrt[3]{(\tau^2-1)^2}}\\&=\frac{3}{\sqrt[3]{a_0}\,G\sqrt[4]{3}}\,\mathrm{cn}^{-1}(\cos\psi,k),\end{aligned}\right.$$

where $$\psi=\cos^{-1}\left[\frac{\sqrt{3}-1-\sqrt[3]{y^2-1}}{\sqrt{3}+1+\sqrt[3]{y^2-1}}\right],\quad k^2=\frac{2+\sqrt{3}}{4}.$$

277.02 $$\int_{Y_1}^{Y}\frac{R(t_1)\,dt_1}{\sqrt[3]{a_0(t_1-a)(t_1-b)(t_1-c)}}=\frac{G}{c-b}\int_0^y\frac{d\tau}{(\tau-p)\sqrt[3]{\tau^2-1}}\,R\left[a+\frac{2(a-b)(a-c)}{(c-b)(\tau-p)}\right],$$

[See **275**.]

where $R(t_1)$ is a rational function of t_1.

Integrands involving $\sqrt[3]{a_0(t_1-a)[(t_1-a_1)^2+b_1^2]}$, $(b_1>0)$

$$t_1=\frac{a b_1\tau+a a_1-a_1^2-b_1^2}{b_1\tau+a-a_1},\quad dt_1=\frac{b_1[(a-a_1)^2+b_1^2]}{(b_1\tau+a-a_1)^2}d\tau,$$
$$G=-\sqrt[3]{\frac{b_1}{a_0}[(a-a_1)^2+b_1^2]},\quad Y_1=\frac{a a_1-a_1^2-b_1^2}{a-a_1},\quad p=\frac{a_1-a}{b_1},$$
$$y=\frac{a a_1-a_1^2-b_1^2+(a-a_1)Y}{b_1(Y-a)}.$$

278.00 $$\int_{Y_1}^{Y}\frac{dt_1}{\sqrt[3]{a_0(t_1-a)[(t_1-a_1)^2+b_1^2]}}=\frac{G}{b_1}\int_0^y\frac{d\tau}{(\tau-p)\sqrt[3]{\tau^2+1}},$$

[For further reduction, see **274**.]

278.01 $$\int_{Y_1}^{Y}\frac{dt_1}{\sqrt[3]{(t_1-a)^2[(t_1-a_1)^2+b_1^2]^2}}=\frac{1}{\sqrt[3]{b_1[(a-a_1)^2+b_1^2]}}\int_0^y\frac{d\tau}{\sqrt[3]{(\tau^2+1)^2}}$$
$$=\frac{-3}{2\sqrt[4]{3}\sqrt[3]{a_0}\,G}\operatorname{cn}^{-1}(\cos\varphi,k),$$

where $$\varphi=\cos^{-1}\left[\frac{\sqrt{3}+1-\sqrt[3]{y^2+1}}{\sqrt{3}-1+\sqrt[3]{y^2+1}}\right],\quad k^2=\frac{2-\sqrt{3}}{4}.$$

278.02 $$\int_{Y_1}^{Y}\frac{R(t_1)\,dt_1}{\sqrt[3]{a_0(t_1-a)[(t_1-a_1)^2+b_1^2]}}=\frac{G}{b_1}\int_0^y\frac{d\tau}{(\tau-p)\sqrt[3]{\tau^2+1}}R\left[a-\frac{(a-a_1)^2+b_1^2}{b_1(\tau-p)}\right],$$

[See **274**.]

where $R(t_1)$ is a rational function of t_1.

Integrands involving $\sqrt[4]{a\tau^4+2b\tau^2+c}$, $(b^2-ac<0,\ \infty>Y>0)$

$$t=\frac{\sqrt[4]{a\tau^4+2b\tau^2+c}}{\tau},\quad \tau^2=\frac{b+\sqrt{ct^4+b^2-ac}}{t^4-a},\quad \tau t=t_1,$$
$$\frac{d\tau}{\tau}=-\frac{t^3[b+\sqrt{ct^4+b^2-ac}]}{(t^4-a)\sqrt{ct^4+b^2-ac}}dt,\quad y=\frac{\sqrt[4]{aY^4+2bY^2+c}}{Y}.$$

279.00 $$\int_0^Y\frac{d\tau}{\sqrt[4]{a\tau^4+2b\tau^2+c}}=\int_y^\infty\frac{t^2\,dt}{t^4-a}+b\int_y^\infty\frac{t^2\,dt}{(t^4-a)\sqrt{ct^4+b^2-ac}}=\frac{1}{2\sqrt[4]{a}}\times$$
$$\times\left\{-\tan^{-1}\left[\frac{\sqrt[4]{aY^4+2bY^2+c}}{Y\sqrt[4]{a}}\right]+\frac{\pi}{2}+\frac{1}{2}\ln\left[\frac{\sqrt[4]{aY^4+2bY^2+c}+\sqrt[4]{a}Y}{\sqrt[4]{aY^4+2bY^2+c}-\sqrt[4]{a}Y}\right]\right\}+$$
$$+b\int_y^\infty\frac{t^2\,dt}{(t^4-a)\sqrt{ct^4+b^2-ac}}.$$

[See **212** for further reduction of last integral.]

279.01
$$\left\{\begin{aligned}\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt[4]{a\tau^4+2b\tau^2+c}} &= \int_y^\infty \frac{t^2}{t^4-a} R_1\left[\frac{b+\sqrt{ct^4+b^2-ac}}{t^4-a}\right] dt + \\ &+ b\int_y^\infty \frac{t^2\,dt}{(t^4-a)\sqrt{ct^4+b^2-ac}} R_1\left[\frac{b+\sqrt{ct^4+b^2-ac}}{t^4-a}\right] + \\ &+ \int_{\sqrt[4]{c}}^{\sqrt[4]{aY^4+2bY^2+c}} \frac{t_1^2}{\sqrt{at_1^4+b^2-ac}} R_2\left[\frac{-b+\sqrt{at_1^4+b^2-ac}}{a}\right] dt_1.\end{aligned}\right.$$

279.02
$$\int_0^Y \sqrt{a\tau^4+2b\tau^2+c}\,d\tau = 2b\int_y^\infty \frac{t^4\,dt}{(t^4-a)^2} + \int_y^\infty \frac{t^4(ct^4+2b^2-ac)}{(t^4-a)^2\sqrt{ct^4+b^2-ac}}\,dt.$$

[The first integral on the right is elementary; see **212** for further reduction of the last integral.]

279.03
$$\left\{\begin{aligned}\int_0^Y R(\tau)\sqrt[4]{a\tau^4+2b\tau^2+c}\,d\tau &= 2b\int_y^\infty \frac{t^4}{(t^4-a)^2} R_1\left[\frac{b+\sqrt{ct^4+b^2-ac}}{t^4-a}\right] dt + \\ &+ \int_y^\infty \frac{t^4(ct^4+2b^2-ac)\,dt}{(t^4-a)^2\sqrt{ct^4+b^2-ac}} R_1\left[\frac{b+\sqrt{ct^4+b^2-ac}}{t^4-a}\right] + \\ &+ \int_{\sqrt[4]{c}}^{\sqrt[4]{aY^4+2bY^2+c}} \frac{t_1^4\,dt_1}{\sqrt{at_1^4+b^2-ac}} R_2\left[\frac{-b+\sqrt{at_1^4+b^2-ac}}{a}\right].\end{aligned}\right.$$

279.50
$$\left\{\begin{aligned}&\int_0^Y \sqrt[4]{1+\tau^4}\,d\tau = \int_y^\infty \frac{t^4\,dt}{(t^4-1)^2\sqrt{t^4-1}} \\ &= \frac{1}{2\sqrt{2}} F(\varphi, \sqrt{2}/2) + \frac{1}{2} Y\sqrt[4]{Y^4+1}; \quad \left(\varphi = \sin^{-1}\sqrt{\frac{2Y^2}{Y^2+\sqrt{Y^2+1}}}\right).\end{aligned}\right.$$

279.51
$$\left\{\begin{aligned}&\int_0^Y \frac{d\tau}{\sqrt[4]{1+\tau^4}} = \int_y^\infty \frac{t^2\,dt}{t^4-1} \\ &= \frac{1}{2}\left\{\frac{\pi}{2} - \tan^{-1}\left[\frac{\sqrt[4]{Y^4+1}}{Y}\right] + \frac{1}{2}\ln\left[\frac{\sqrt[4]{Y^4+1}+Y}{\sqrt[4]{Y^4+1}-Y}\right]\right\}.\end{aligned}\right.$$

Reduction of Trigonometric Integrands to Jacobian Elliptic Functions

Various elliptic integrals involving trigonometric integrands occur in many geometrical and physical problems. In order to evaluate a variety of these we again find it convenient to express them in terms of integrals involving Jacobian elliptic functions.

An elliptic integral of the form

$$L = \int_{\varphi_0}^{\varphi_1} \frac{R(\sin\vartheta, \cos\vartheta)\, d\vartheta}{\sqrt{a_0 + a_1\cos\vartheta + a_2\sin\vartheta + a_3\cos^2\vartheta + a_4\sin\vartheta\cos\vartheta + a_5\sin^2\vartheta}},$$

where R is a rational function, is reduced to *Jacobi's form* if it can be expressed as

$$L = \int_{u_0}^{u_1} R_1(\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u)\, du,$$

where R_1 is also a rational function. This is accomplished by an appropriate transformation of the type

$$\operatorname{sn} u = f(\vartheta), \qquad (u \text{ real})$$

yielding

$$u_1 = \operatorname{sn}^{-1}[f(\varphi_1)], \qquad u_0 = \operatorname{sn}^{-1}[f(\varphi_0)].$$

Reduction of Integrals in Legendre's Normal Form to Jacobi's Form

$\operatorname{sn}^2(u,k) \equiv \operatorname{sn}^2 u = t^2 = \sin^2\vartheta, \quad \varphi = \operatorname{am} u_1, \quad 0<\varphi<\pi/2,$
$Y = \operatorname{sn} u_2 = \sin\varphi_2,$
$y = \operatorname{sn} u_1 = \sin\varphi$; R is any rational function.

280.00 $$\left\{ \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \int_0^{y} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{u_1} du = u_1 = \operatorname{sn}^{-1}(\sin\varphi, k) = F(\varphi, k). \right.$$

280.01 $$\int_0^{\varphi}\sqrt{1-k^2\sin^2\vartheta}\,d\vartheta=\int_0^{y}\sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt=\int_0^{u_1}\mathrm{dn}^2u\,du=E(u_1)=E(\varphi,k).$$

280.02 $$\left\{\begin{aligned}&\int_0^{\varphi}\frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}}=\int_0^{y}\frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}\\&\qquad=\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2u}=\Pi(\varphi,\alpha^2,k).\end{aligned}\right.$$

281.01 $$\int_0^{\varphi}\frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_0^{y}\frac{R(t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{u_1}R(\mathrm{sn}^2u)\,du.$$

281.02 $$\int_0^{\varphi}\frac{R(\cos^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_0^{y}\frac{R(1-t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{u_1}R(\mathrm{cn}^2u)\,du.$$

281.03 $$\int_0^{\varphi}\frac{R(\tan^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_0^{y}\frac{R[t^2/(1-t^2)]\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{u_1}R(\mathrm{tn}^2u)\,du.$$

281.04 $$\int_0^{\varphi}\frac{R(1-k^2\sin^2\vartheta)}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta=\int_0^{y}\frac{R(1-k^2t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{u_1}R(\mathrm{dn}^2u)\,du.$$

281.05 $$\int_0^{\varphi}\frac{R[(\sin^2\vartheta)/(1-k^2\sin^2\vartheta)]}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta=\int_0^{y}\frac{R[t^2/(1-k^2t^2)]}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt=\int_0^{u_1}R(\mathrm{sd}^2u)\,du.$$

281.06 $$\int_{\varphi_2}^{\varphi}\frac{R(\cot^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_{Y}^{y}\frac{R[(1-t^2)/t^2]\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_{u_2}^{u_1}R(\mathrm{cs}^2u)\,du.$$

281.07 $$\int_0^{\varphi}\frac{R\left[\dfrac{\cos^2\vartheta}{1-k^2\sin^2\vartheta}\right]}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta=\int_0^{y}\frac{R[(1-t^2)/(1-k^2t^2)]}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt=\int_0^{u_1}R(\mathrm{cd}^2u)\,du.$$

281.08 $$\int_{\varphi_2}^{\varphi}\frac{R\left[\dfrac{1-k^2\sin^2\vartheta}{\sin^2\vartheta}\right]}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta=\int_{Y}^{y}\frac{R[(1-k^2t^2)/t^2]\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_{u_2}^{u_1}R(\mathrm{ds}^2u)\,du.$$

281.09 $$\int_{\varphi_2}^{\varphi}\frac{R(\csc^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_{Y}^{y}\frac{R(1/t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_{u_2}^{u_1}R(\mathrm{ns}^2u)\,du.$$

281.10 $$\int_0^{\varphi}\frac{R(\sec^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\int_0^{y}\frac{R[1/(1-t^2)]\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{u_1}R(\mathrm{nc}^2u)\,du.$$

281.11 $$\int_0^{\varphi} \frac{R[1/(1-k^2\sin^2\vartheta)]}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta = \int_0^{y} \frac{R[1/(1-k^2t^2)]}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt = \int_0^{u_1} R(\mathrm{nd}^2 u)\,du.$$

281.12 $$\int_0^{\varphi} \frac{R\left[\dfrac{1-k^2\sin^2\vartheta}{\cos^2\vartheta}\right]}{\sqrt{1-k^2\sin^2\vartheta}}\,d\vartheta = \int_0^{y} \frac{R[(1-k^2t^2)/(1-t^2)]}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt = \int_0^{u_1} R(\mathrm{dc}^2 u)\,du.$$

Table of Other Integrals.

As with the cases involving algebraic integrands, in the following[1] we fix one of the limits of integration and consider the other limit to vary. (By splitting an integral into two, however, one may easily evaluate integrals even when both limits are variable.)

Integrands involving $\sqrt{1+n^2\sin^2\vartheta}$, $(n^2 > 0)$

$$\mathrm{sn}^2 u = \frac{(1+n^2)\sin^2\vartheta}{1+n^2\sin^2\vartheta}, \quad k^2 = \frac{n^2}{1+n^2}, \quad \mathrm{sn}\,u_1 = \sin\psi, \quad 0 < \varphi \leqq \pi/2,$$
$$\psi = \mathrm{am}\,u_1 = \sin^{-1}\left[\frac{\sqrt{1+n^2}\sin\varphi}{\sqrt{1+n^2\sin^2\varphi}}\right].$$

282.00 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}} = k'\int_0^{u_1} du = k'u_1 = k'\,\mathrm{sn}^{-1}(\sin\psi, k) = k'F(\psi,k),$$ [cf. **160.02**.]

282.01 $$\int_0^{\varphi} \frac{d\vartheta}{(1+n^2\sin^2\vartheta)\sqrt{1+n^2\sin^2\vartheta}} = k'\int_0^{u_1} \mathrm{dn}^2 u\,du = k'E(u_1) = k'E(\psi,k).$$

282.02 $$\int_0^{\varphi} \frac{\sqrt{1+n^2\sin^2\vartheta}\,d\vartheta}{1+(n^2-\alpha^2 n^2-\alpha^2)\sin^2\vartheta} = k'\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = k'\,\Pi(\psi,\alpha^2,k).$$ [See **400**.]

282.03 $$\int_0^{\varphi} \sqrt{1+n^2\sin^2\vartheta}\,d\vartheta = k'\int_0^{u_1} \mathrm{nd}^2 u\,du.$$ [See **315.02**.]

282.04 $$\int_0^{\varphi} \frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}} = k'^{2m+1}\int_0^{u_1} \mathrm{sd}^{2m} u\,du.$$ [See **318.05**.]

282.05 $$\int_0^{\varphi} \frac{\cos^{2m}\vartheta\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}} = k'\int_0^{u_1} \mathrm{cd}^{2m} u\,du.$$ [See **320.05**.]

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[1] See **510** to **532** for additional integrals involving trigonometric integrands.

282.06 $$\int_0^{\varphi}\frac{\tan^{2m}\vartheta\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}}=k'^{2m+1}\int_0^{u_1}\mathrm{tn}^{2m}u\,du,\quad \varphi\neq\pi/2.$$ [See **316.05**.]

282.07 $$\int_0^{\varphi}\frac{d\vartheta}{(1-\alpha_1^2\sin^2\vartheta)\sqrt{1+n^2\sin^2\vartheta}}=k'\int_0^{u_1}\frac{\mathrm{dn}^2u\,du}{1-\alpha^2\,\mathrm{sn}^2u},$$

[See appropriate case in **410** and **430**.]

where $$\alpha^2=\alpha_1^2k'^2+k^2.$$

282.08 $$\int_0^{\varphi}\frac{(1+n^2\sin^2\vartheta)^{m-1}\sqrt{1+n^2\sin^2\vartheta}}{[1+(n^2-\alpha^2-\alpha^2n^2)\sin^2\vartheta]^m}\,d\vartheta=k'\int_0^{u_1}\frac{du}{(1-\alpha^2\,\mathrm{sn}^2u)^m}.$$

[See **336.03**.]

282.09 $$\int_0^{\varphi}\frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}}=k'\int_0^{u_1}R(k'^2\,\mathrm{sd}^2u)\,du,$$

where $R(\sin^2\vartheta)$ is any rational function of $\sin^2\vartheta$.

Integrands involving $\sqrt{1-n^2\sin^2\vartheta}$, $(n^2>1)$

$$\begin{array}{|c|}\hline \mathrm{sn}^2u=n^2\sin^2\vartheta,\quad k=1/n,\quad \mathrm{sn}\,u_1=\sin\beta,\\ \beta=\mathrm{am}\,u_1=\sin^{-1}[n\sin\varphi],\quad n\sin\varphi\leqq 1.\\ \hline\end{array}$$

283.00 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}du=k\,u_1=k\,\mathrm{sn}^{-1}(\sin\beta,k)=k\,F(\beta,k),$$

[cf. **162.02**].

283.01 $$\int_0^{\varphi}\frac{\cos^2\vartheta\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}\mathrm{dn}^2u\,du=k\,E(u_1)=k\,E(\beta,k).$$

283.02 $$\int_0^{\varphi}\frac{d\vartheta}{(1-\alpha_1^2\sin^2\vartheta)\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2u}=k\,\Pi(\beta,\alpha^2,k),$$

[See **410** and **430**.]

where $$\alpha^2=\alpha_1^2/n^2.$$

283.03 $$\int_0^{\varphi}\sqrt{1-n^2\sin^2\vartheta}\,d\vartheta=k\int_0^{u_1}\mathrm{cn}^2u\,du.$$ [See **312.02**.]

283.04 $$\int_0^{\varphi}\frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k^{2m+1}\int_0^{u_1}\mathrm{sn}^{2m}u\,du.$$ [See **310.05**.]

283.05 $$\int_0^{\varphi}\frac{\cos^{2m}\vartheta\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}\mathrm{dn}^{2m}u\,du.$$ [See **314.05.**]

283.06 $$\int_0^{\varphi}\frac{\tan^{2m}\vartheta\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k^{2m+1}\int_0^{u_1}\mathrm{sd}^{2m}u\,du.$$ [See **318.05.**]

283.07 $$\int_0^{\varphi}\frac{\sqrt{1-n^2\sin^2\vartheta}}{\cos^2\vartheta}\,d\vartheta=k\int_0^{u_1}\mathrm{cd}^2u\,du.$$ [See **320.02.**]

283.08 $$\int_0^{\varphi}\frac{d\vartheta}{(1-n^2\sin^2\vartheta)^m\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}\mathrm{nc}^{2m}u\,du.$$ [See **313.05.**]

283.09 $$\int_0^{\varphi}\frac{\sqrt{1-n^2\sin^2\vartheta}}{1-\alpha_1^2\sin^2\vartheta}\,d\vartheta=k\int_0^{u_1}\frac{\mathrm{cn}^2u\,du}{1-\alpha^2\mathrm{sn}^2u};\qquad \alpha^2=\alpha_1^2/n^2.$$ [See **410** and **430.**]

283.10 $$\int_0^{\varphi}\frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}=k\int_0^{u_1}R(k^2\,\mathrm{sn}^2u)\,du,$$

where $R(\sin^2\vartheta)$ is any rational function of $\sin^2\vartheta$.

Integrands involving $\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}$, $(1>n_1^2>n_2^2>0)$

$$\mathrm{sn}^2u=\frac{(1-n_2^2)\sin^2\vartheta}{1-n_2^2\sin^2\vartheta},\quad k^2=\frac{n_1^2-n_2^2}{1-n_2^2},\quad g=\frac{1}{\sqrt{1-n_2^2}},$$
$$\psi=\mathrm{am}\,u_1=\sin^{-1}\left[\frac{\sqrt{1-n_2^2}\sin\varphi}{\sqrt{1-n_2^2\sin^2\varphi}}\right],\quad \mathrm{sn}\,u_1=\sin\psi,\quad 0<\varphi\leqq\frac{\pi}{2}.$$

284.00 $$\begin{cases}\displaystyle\int_0^{\varphi}\frac{d\vartheta}{\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}}=g\int_0^{u_1}du=g\,u_1\\ \qquad =g\,\mathrm{sn}^{-1}(\sin\psi,k)=g\,F(\psi,k).\end{cases}$$

284.01 $$\int_0^{\varphi}\frac{d\vartheta}{1-n_2^2\sin^2\vartheta}\sqrt{\frac{1-n_1^2\sin^2\vartheta}{1-n_2^2\sin^2\vartheta}}=g\int_0^{u_1}\mathrm{dn}^2u\,du=g\,E(u_1)=g\,E(\psi,k).$$

284.02 $$\begin{cases}\displaystyle\int_0^{\varphi}\frac{d\vartheta}{[1+(n_2^2\alpha^2-n_2^2-\alpha^2)\sin^2\vartheta]}\sqrt{\frac{1-n_2^2\sin^2\vartheta}{1-n_1^2\sin^2\vartheta}}=g\int_0^{u_1}\frac{du}{1-\alpha^2\mathrm{sn}^2u}\\ \qquad =g\,\Pi(\psi,\alpha^2,k).\end{cases}$$ [See **400.**]

284.03 $$\int_0^{\varphi} \frac{\sin^{2m}\vartheta\, d\vartheta}{(1-n_2^2\sin^2\vartheta)^m \sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = \frac{g}{(1-n_2^2)^m}\int_0^{u_1} \mathrm{sn}^{2m} u\, du.$$

[See **310.05**.]

284.04 $$\int_0^{\varphi} \frac{d\vartheta}{1-n_1^2\sin^2\vartheta}\sqrt{\frac{1-n_2^2\sin^2\vartheta}{1-n_1^2\sin^2\vartheta}} = g\int_0^{u_1} \mathrm{nd}^2 u\, du.$$ [See **315.02**.]

284.05 $$\int_0^{\varphi} \frac{\cos^{2m}\vartheta\, d\vartheta}{(1-n_2^2\sin^2\vartheta)^m \sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = g\int_0^{u_1} \mathrm{cn}^{2m} u\, du.$$

[See **312.05**.]

284.06 $$\int_0^{\varphi} \frac{\sin^2\vartheta\, d\vartheta}{(1-n_1^2\sin^2\vartheta)\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = \frac{g}{1-n_2^2}\int_0^{u_1} \mathrm{sd}^2 u\, du.$$

[See **318.02**.]

284.07 $$\int_0^{\varphi} \frac{\tan^{2m}\vartheta\, d\vartheta}{\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = \frac{g}{(1-n_2^2)^m}\int_0^{u_1} \mathrm{tn}^{2m} u\, du.$$

[See **316.05**.]

284.08 $$\int_0^{\varphi} \frac{R(\cos^2\vartheta)\, d\vartheta}{\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = g\int_0^{u_1} R\left[\frac{(1-n_2^2)\,\mathrm{cn}^2 u}{1-n_2^2+n_2^2\,\mathrm{sn}^2 u}\right] du,$$

where $R(\cos^2\vartheta)$ is any rational function of $\cos^2\vartheta$.

284.09 $$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{(1-n_1^2\sin^2\vartheta)(1-n_2^2\sin^2\vartheta)}} = g\int_0^{u_1} R\left[\frac{\mathrm{sn}^2 u}{1-n_2^2+n_2^2\,\mathrm{sn}^2 u}\right] du,$$

where $R(\sin^2\vartheta)$ is any rational function of $\sin^2\vartheta$.

Integrands involving $\sqrt{a^2 \sin^2\vartheta - b^2}$, $[a > b,\ \pi/2 > \varphi \geqq \sin^{-1}(b/a)]$

$$\mathrm{sn}^2 u = \frac{a^2(1-\sin^2\vartheta)}{a^2-b^2}, \quad k^2 = \frac{a^2-b^2}{a^2}, \quad g = 1/a,$$
$$\psi = \mathrm{am}\, u_1 = \sin^{-1}[(\cos\varphi)/k], \quad \mathrm{sn}\, u_1 = \sin\psi.$$

285.00 $$\int_{\varphi}^{\pi/2} \frac{d\vartheta}{\sqrt{a^2\sin^2\vartheta - b^2}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\psi, k) = g\,F(\psi, k).$$

285.01 $$\int_{\varphi}^{\pi/2} \frac{\sin^2\vartheta\, d\vartheta}{\sqrt{a^2\sin^2\vartheta - b^2}} = g\int_0^{u_1} \mathrm{dn}^2 u\, du = g\,E(u_1) = g\,E(\psi, k).$$

285.02 $$\left\{\begin{aligned}\int_{\varphi}^{\pi/2}\frac{d\vartheta}{(1-\alpha_1^2\sin^2\vartheta)\sqrt{a^2\sin^2\vartheta-b^2}} &= \frac{g}{1-\alpha_1^2}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2\,u}\\ &= \frac{g}{1-\alpha_1^2}\,\Pi(\psi,\alpha^2,k)\,,\end{aligned}\right.$$ [See **400.**]

where $$\alpha^2=\alpha_1^2\,k^2\big/(\alpha_1^2-1)\,,\quad \alpha_1 \neq 1\,.$$

285.03 $$\int_{\varphi}^{\pi/2}\frac{\cos^{2m}\vartheta\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\,k^{2m}\int_0^{u_1}\mathrm{sn}^{2m}\,u\,du\,.$$ [See **310.05.**]

285.04 $$\int_{\varphi}^{\pi/2}\frac{\sin^2\vartheta\cos^2\vartheta\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\,k^2\int_0^{u_1}\mathrm{sn}^2\,u\,\mathrm{dn}^2\,u\,du\,.$$ [See **361.02.**]

285.05 $$\int_{\varphi}^{\pi/2}\frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\int_0^{u_1}\mathrm{dn}^{2m}\,u\,du\,.$$ [See **314.05.**]

285.06 $$\int_{\varphi}^{\pi/2}\sqrt{a^2\sin^2\vartheta-b^2}\,d\vartheta=(a^2-b^2)\,g\int_0^{u_1}\mathrm{cn}^2\,u\,du\,.$$ [See **312.02.**]

285.07 $$\left\{\begin{aligned}\int_{\varphi}^{\pi/2}\frac{d\vartheta}{(a^2\sin^2\vartheta-b^2)^m\sqrt{a^2\sin^2\vartheta-b^2}} &= \frac{g}{(a^2-b^2)^m}\int_0^{u_1}\mathrm{nc}^{2m}\,u\,du\\ &\varphi \neq \sin^{-1}(b/a)\,.\end{aligned}\right.$$ [See **313.05.**]

285.08 $$\int_{\varphi}^{\pi/2}\frac{\cot^{2m}\vartheta\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\,k^{2m}\int_0^{u_1}\mathrm{sd}^{2m}\,u\,du\,.$$ [See **318.05.**]

285.09 $$\int_{\varphi}^{\pi/2}\frac{(1-\alpha_1^2\sin^2\vartheta)^m\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\,(1-\alpha_1^2)^m\int_0^{u_1}(1-\alpha^2\,\mathrm{sn}^2\,u)^m\,du\,,$$ [See **331.03.**]

where $$\alpha^2=\alpha_1^2\,k^2\big/(\alpha_1^2-1)$$

285.10 $$\int_{\varphi}^{\pi/2}\frac{R\,(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2\sin^2\vartheta-b^2}}=g\int_0^{u_1}R(\mathrm{dn}^2\,u)\,du\,,$$

where $R(\sin^2\vartheta)$ is any rational function of $\sin^2\vartheta$.

Integrands involving $\sqrt{\cos 2a\vartheta}$, $(0 < a\varphi \leq \pi/4)$

$$\operatorname{sn}^2 u = 2\sin^2 a\vartheta, \quad k^2 = 1/2, \quad g = 1/a\sqrt{2},$$
$$\beta = \operatorname{am} u_1 = \sin^{-1}\left[\sqrt{2}\sin a\vartheta\right], \quad \operatorname{sn} u_1 = \sin\beta.$$

286.00 $$\int_0^\varphi \frac{d\vartheta}{\sqrt{\cos 2a\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\beta, k) = g\,F(\beta, k).$$

286.01 $$\int_0^\varphi \frac{\cos^2 a\vartheta}{\sqrt{\cos 2a\vartheta}}\,d\vartheta = g\int_0^{u_1} \operatorname{dn}^2 u\,du = g\,E(u_1) = g\,E(\beta, k).$$

286.02 $$\int_0^\varphi \frac{d\vartheta}{(1 - 2\alpha^2\sin^2 a\vartheta)\sqrt{\cos 2a\vartheta}} = \int_0^{u_1} \frac{du}{1 - \alpha^2\operatorname{sn}^2 u} = g\,\Pi(\beta, \alpha^2, k).$$ [See **400.**]

286.03 $$\int_0^\varphi \frac{\sin^{2m} a\vartheta\,d\vartheta}{\sqrt{\cos 2a\vartheta}} = \frac{g}{2^m}\int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$ [See **310.05.**]

286.04 $$\int_0^\varphi \sqrt{\cos 2a\vartheta}\,d\vartheta = g\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02.**]

286.05 $$\int_0^\varphi \frac{\sin^2 a\vartheta\,d\vartheta}{\cos 2a\vartheta\sqrt{\cos 2a\vartheta}} = \frac{g}{2}\int_0^{u_1} \operatorname{tn}^2 u\,du, \quad a\varphi \neq \pi/4.$$ [See **316.02.**]

286.06 $$\int_0^\varphi \frac{d\vartheta}{\cos^{2m} a\vartheta\sqrt{\cos 2a\vartheta}} = g\int_0^{u_1} \operatorname{nd}^{2m} u\,du.$$ [See **315.05.**]

286.07 $$\int_0^\varphi \frac{\tan^{2m} a\vartheta\,d\vartheta}{\sqrt{\cos 2a\vartheta}} = \frac{g}{2^m}\int_0^{u_1} \operatorname{sd}^{2m} u\,du.$$ [See **318.05.**]

286.08 $$\int_0^\varphi \frac{d\vartheta}{(\cos 2a\vartheta)^m\sqrt{\cos 2a\vartheta}} = g\int_0^{u_1} \operatorname{nc}^{2m} u\,du, \quad a\varphi \neq \pi/4.$$ [See **313.05.**]

286.09 $$\int_0^\varphi \frac{\sqrt{\cos 2a\vartheta}}{\cos^2 a\vartheta}\,d\vartheta = g\int_0^{u_1} \operatorname{cd}^2 u\,du.$$ [See **320.02.**]

286.10 $$\int_0^{\varphi} \frac{R(\cos^2 a\vartheta)}{\sqrt{\cos 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} R(\mathrm{dn}^2 u)\, du,$$

where $R(\cos^2 a\vartheta)$ is a rational function of $\cos^2 a\vartheta$.

286.11 $$\int_0^{\varphi} \frac{R(\sin^2 a\vartheta)}{\sqrt{\cos 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} R[(\mathrm{sn}^2 u)/2]\, du.$$

Integrands involving $\sqrt{-\cos 2a\vartheta}$, $(\pi/2 > a\varphi \geqq \pi/4)$

$$\boxed{\begin{array}{l} \mathrm{sn}^2 u = 2\cos^2 a\vartheta, \quad k^2 = 1/2, \quad g = 1/a\sqrt{2}, \\ \psi = \mathrm{am}\, u_1 = \sin^{-1}\left[\sqrt{2}\cos a\varphi\right], \quad \mathrm{sn}\, u_1 = \sin\psi. \end{array}}$$

287.00 $$\int_{\varphi}^{\pi/2a} \frac{d\vartheta}{\sqrt{-\cos 2a\vartheta}} = g\int_0^{u_1} du = g u_1 = g\,\mathrm{sn}^{-1}(\sin\psi, k) = g F(\psi, k).$$

287.01 $$\int_{\varphi}^{\pi/2a} \frac{\sin^2 a\vartheta}{\sqrt{-\cos 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} \mathrm{dn}^2 u\, du = g E(u_1) = g E(\psi, k).$$

287.02 $$\left\{\begin{aligned} \int_{\varphi}^{\pi/2a} \frac{d\vartheta}{(1 - 2\alpha^2\cos^2 a\vartheta)\sqrt{-\cos 2a\vartheta}} &= g\int_0^{u_1} \frac{du}{1 - \alpha^2\,\mathrm{sn}^2 u} \\ &= g\,\Pi(\psi, \alpha^2, k). \end{aligned}\right.$$ [See **400.**]

287.03 $$\int_{\varphi}^{\pi/2a} \frac{\cos^{2m} a\vartheta}{\sqrt{-\cos 2a\vartheta}}\, d\vartheta = \frac{g}{2^m}\int_0^{u_1} \mathrm{sn}^{2m} u\, du.$$ [See **310.05.**]

287.04 $$\int_{\varphi}^{\pi/2a} \sqrt{-\cos 2a\vartheta}\, d\vartheta = g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02.**]

287.05 $$\int_{\varphi}^{\pi/2a} \frac{\cos^2 a\vartheta\, d\vartheta}{\cos 2a\vartheta\sqrt{-\cos 2a\vartheta}} = -\frac{g}{2}\int_0^{u_1} \mathrm{tn}^2 u\, du, \quad a\varphi \neq \pi/4.$$ [See **316.02.**]

287.06 $$\int_{\varphi}^{\pi/2a} \frac{d\vartheta}{(\cos 2a\vartheta)^m\sqrt{-\cos 2a\vartheta}} = (-1)^m g\int_0^{u_1} \mathrm{nc}^{2m} u\, du, \quad a\varphi \neq \pi/4.$$ [See **313.05.**]

287.07 $$\int_{\varphi}^{\pi/2a} \frac{\cot^2 a\vartheta\, d\vartheta}{\sqrt{-\cos 2a\vartheta}} = \frac{g}{2}\int_0^{u_1} \mathrm{sd}^2 u\, du.$$ [See **318.02**.]

287.08 $$\int_{\varphi}^{\pi/2a} \frac{d\vartheta}{\sin^{2m} a\vartheta \sqrt{-\cos 2a\vartheta}} = g\int_0^{u_1} \mathrm{nd}^{2m} u\, du.$$ [See **315.05**.]

287.09 $$\int_{\varphi}^{\pi/2a} \frac{R(\cos^2 a\vartheta)}{\sqrt{-\cos 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} R[(\mathrm{sn}^2 u)/2]\, du,$$

where $R(\cos^2 a\vartheta)$ is any rational function of $\cos^2 a\vartheta$.

287.10 $$\int_{\varphi}^{\pi/2a} \frac{R(\sin^2 a\vartheta)}{\sqrt{-\cos 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} R(\mathrm{dn}^2 u)\, du.$$

Integrands involving $\sqrt{\sin 2a\vartheta}$, $(0 < a\varphi \leqq \pi/2)$

$$\mathrm{sn}^2 u = \frac{2\sin a\vartheta}{1+\cos a\vartheta + \sin a\vartheta}, \quad k^2 = 1/2, \quad g = \sqrt{2/a},$$

$$A = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\frac{2\sin a\varphi}{1+\cos a\varphi + \sin a\varphi}}, \quad \mathrm{sn}\, u_1 = \sin A.$$

287.50 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{\sin 2a\vartheta}} = g\int_0^{u_1} du = g\, u_1 = g\, \mathrm{sn}^{-1}(\sin A, k) = g F(A, k).$$

287.51 $$\int_0^{\varphi} \frac{(1+\cos a\vartheta)\, d\vartheta}{(1+\cos a\vartheta + \sin a\vartheta)\sqrt{\sin 2a\vartheta}} = g\int_0^{u_1} \mathrm{dn}^2 u\, du = g E(u_1) = g E(A, k).$$

287.52 $$\int_0^{\varphi} \frac{(1+\cos a\vartheta + \sin a\vartheta)\, d\vartheta}{[1+\cos a\vartheta + (1-2\alpha_1^2)\sin a\vartheta]\sqrt{\sin 2a\vartheta}} = g\int_0^{u_1} \frac{du}{1-\alpha_1^2\, \mathrm{sn}^2 u} = g\Pi(A, \alpha_1^2, k).$$ [See **400**.]

287.53 $$\int_0^{\varphi} \frac{(1+\cos a\vartheta - \sin a\vartheta)\, d\vartheta}{(1+\cos a\vartheta + \sin a\vartheta)\sqrt{\sin 2a\vartheta}} = g\int_0^{u_1} \mathrm{cn}^2 u\, du.$$ [See **312.02**.]

287.54 $$\int_0^{\varphi} \frac{\sin a\vartheta\, d\vartheta}{(1+\cos a\vartheta + \sin a\vartheta)\sqrt{\sin 2a\vartheta}} = \frac{g}{2}\int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02**.]

287.55 $$\int_0^{\varphi} \frac{\sin a\vartheta\, d\vartheta}{(1+\cos a\vartheta - \sin a\vartheta)\sqrt{\sin 2a\vartheta}} = \frac{g}{2}\int_0^{u_1} \operatorname{tn}^2 u\, du, \quad a\varphi \neq \pi/2.$$

[See **316.02.**]

287.56 $$\int_0^{\varphi} \frac{(1+\cos a\vartheta)\, d\vartheta}{(1+\cos a\vartheta - \sin a\vartheta)\sqrt{\sin 2a\vartheta}} = g\int_0^{u_1} \operatorname{dc}^2 u\, du, \quad a\varphi \neq \pi/2.$$

[See **321.02.**]

287.57 $$\begin{cases} \displaystyle\int_0^{\varphi} \frac{\sin a\vartheta\, d\vartheta}{\sqrt{\sin 2a\vartheta}} = 2g k^2 \int_0^{u_1} \operatorname{sn}^2 u\, du - 2k'^2 g F(A,k) + \\ \qquad + [C\Pi(A,\alpha^2,k) + \bar{C}\Pi(A,\overline{\alpha^2},k)]\, g, \end{cases}$$

[See **310.02** and **400.**]

where

$$\alpha^2 = (1+i)/2, \quad \overline{\alpha^2} = (1-i)/2, \quad C = (\alpha^2 - k^2)/\alpha^2, \quad \bar{C} = (\overline{\alpha^2} - k^2)/\overline{\alpha^2}.$$

287.58 $$\int_0^{\varphi} \frac{R(\sin a\vartheta)}{\sqrt{\sin 2a\vartheta}}\, d\vartheta = g\int_0^{u_1} R\left[\frac{4\operatorname{sn}^2 u \operatorname{dn}^2 u}{4\operatorname{dn}^4 u + \operatorname{sn}^4 u}\right] du = g\int_0^{u_1} R\left[\frac{2\operatorname{sn}^2 u \operatorname{dn}^2 u}{1+\operatorname{cn}^4 u}\right] du,$$

where $R(\sin a\vartheta)$ is a rational function of $\sin a\vartheta$.

Integrands involving $\sqrt{a+b\sin\vartheta}$, $(a>b>0)$

$$\boxed{\begin{gathered} \operatorname{sn}^2 u = \frac{1-\sin\vartheta}{2}, \quad k^2 = \frac{2b}{a+b}, \quad g = \frac{2}{\sqrt{a+b}}, \\ \xi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{1-\sin\varphi}{2}}, \quad \operatorname{sn} u_1 = \sin\xi, \\ \pi/2 > \varphi \geqq -\pi/2. \end{gathered}}$$

288.00 $$\int_{\varphi}^{\pi/2} \frac{d\vartheta}{\sqrt{a+b\sin\vartheta}} = g\int_0^{u_1} du = g u_1 = g \operatorname{sn}^{-1}(\sin\xi, k) = g F(\xi, k).$$

288.01 $$\begin{cases} \displaystyle\int_{\varphi}^{\pi/2} \sqrt{a+b\sin\vartheta}\, d\vartheta = (a+b)\, g\int_0^{u_1} \operatorname{dn}^2 u\, du = (a+b)\, g\, E(u_1) \\ \qquad\qquad = (a+b)\, g\, E(\xi, k). \end{cases}$$

288.02 $$\int_{\varphi}^{\pi/2} \frac{d\vartheta}{(2-\alpha^2+\alpha^2\sin\vartheta)\sqrt{a+b\sin\vartheta}} = \frac{g}{2}\int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{g}{2}\Pi(\xi, \alpha^2, k).$$

[See **400.**]

288.03 $$\int_{\varphi}^{\pi/2}\frac{\sin^m\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}}=g\int_0^{u_1}(1-2\,\mathrm{sn}^2 u)^m\,du.$$ [See **331.03**.]

288.04 $$\int_{\varphi}^{\pi/2}\frac{\cos^2\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}}=4g\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du.$$ [See **361.01**.]

288.05 $$\int_{\varphi}^{\pi/2}\frac{d\vartheta}{(a+b\sin\vartheta)^m\sqrt{a+b\sin\vartheta}}=\frac{g}{(a+b)^m}\int_0^{u_1}\mathrm{nd}^{2m}u\,du.$$ [See **315.05**.]

288.06 $$\int_{\varphi}^{\pi/2}\frac{1+\sin\vartheta}{\sqrt{a+b\sin\vartheta}}\,d\vartheta=2g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02**.]

288.07 $$\int_{\varphi}^{\pi/2}\frac{(1-\sin\vartheta)\,d\vartheta}{(1+\sin\vartheta)\sqrt{a+b\sin\vartheta}}=g\int_0^{u_1}\mathrm{tn}^2 u\,du,\quad \varphi\neq-\pi/2.$$ [See **316.02**.]

288.08 $$\int_{\varphi}^{\pi/2}\frac{\tan^2\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}}=\frac{g}{4}\int_0^{u_1}\mathrm{ns}^2 u\,\mathrm{nc}^2 u\,du-g\int_0^{u_1}\mathrm{nc}^2 u\,du+g\int_0^{u_1}\mathrm{tn}^2 u\,du,$$
$$\varphi\neq-\pi/2.$$ [See **361.10, 313.02, 316.02**.]

288.09 $$\int_{\varphi}^{\pi/2}\frac{R(\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta}}=g\int_0^{u_1}R(1-2\,\mathrm{sn}^2 u)\,du,$$
where $R(\sin\vartheta)$ is a rational function of $\sin\vartheta$.

Integrands involving $\sqrt{a+b\sin\vartheta}$, $(b>|a|>0)$

$$\mathrm{sn}^2 u=\frac{b(1-\sin\vartheta)}{a+b},\quad k^2=\frac{a+b}{2b},\quad g=\sqrt{\frac{2}{b}},\quad \mathrm{sn}\,u_1=\sin\psi,$$
$$\psi=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{b(1-\sin\varphi)}{a+b}};\quad \frac{\pi}{2}>\varphi\geqq-\sin^{-1}\left(\frac{a}{b}\right).$$

288.50 $$\int_{\varphi}^{\pi/2}\frac{d\vartheta}{\sqrt{a+b\sin\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\,\mathrm{sn}^{-1}(\sin\psi,k)=g\,F(\psi,k).$$

288.51 $$\int_{\varphi}^{\pi/2}\frac{(1+\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta}}=2g\int_0^{u_1}\mathrm{dn}^2 u\,du=2g\,E(u_1)=2g\,E(\psi,k).$$

288.52 $$\begin{cases} \displaystyle\int_{\varphi}^{\pi/2} \frac{d\vartheta}{(a+b-\alpha^2 b+\alpha^2 b\sin\vartheta)\sqrt{a+b\sin\vartheta}} = \frac{g}{a+b}\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} \\ \qquad = \dfrac{g}{a+b}\Pi(\psi,\alpha^2,k). \end{cases}$$ [See **400.**]

288.53 $$\int_{\varphi}^{\pi/2}\sqrt{a+b\sin\vartheta}\,d\vartheta = (a+b)\,g\int_0^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02.**]

288.54 $$\int_{\varphi}^{\pi/2}\frac{\sin^m\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}} = g\int_0^{u_1}(1-2k^2\operatorname{sn}^2 u)^m\,du.$$ [See **331.03.**]

288.55 $$\int_{\varphi}^{\pi/2}\frac{\cos^2\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}} = 4k^2 g\int_0^{u_1}\operatorname{sn}^2 u\operatorname{dn}^2 u\,du.$$ [See **361.02.**]

288.56 $$\int_{\varphi}^{\pi/2}\frac{d\vartheta}{(a+b\sin\vartheta)^m\sqrt{a+b\sin\vartheta}} = \frac{g}{(a+b)^m}\int_0^{u_1}\operatorname{nc}^{2m}u\,du,\quad \varphi \neq \sin^{-1}\left(\frac{a}{b}\right).$$ [See **313.05.**]

288.57 $$\int_{\varphi}^{\pi/2}\frac{(1-\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta}} = 2k^2 g\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

288.58 $$\int_{\varphi}^{\pi/2}\frac{\tan^2\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta}} = \frac{g}{4k^2}\int_0^{u_1}\operatorname{ns}^2 u\operatorname{nd}^2 u\,du - g\int_0^{u_1}\operatorname{nd}^2 u\,du + gk^2\int_0^{u_1}\operatorname{sd}^2 u\,du.$$ [See **361.11**, **315.02**, **318.02.**]

288.59 $$\int_{\varphi}^{\pi/2}\frac{R(\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta}} = g\int_0^{u_1}R(1-2k^2\operatorname{sn}^2 u)\,du,$$

where $R(\sin\vartheta)$ is any rational function of $\sin\vartheta$.

Integrands involving $\sqrt{a+b\cos\vartheta}$, $(a>b>0)$

$$\operatorname{sn}^2 u = \frac{1-\cos\vartheta}{2},\quad k^2 = \frac{2b}{a+b},\quad g = \frac{2}{\sqrt{a+b}},$$
$$\beta = \operatorname{am} u_1 = \varphi/2,\quad \operatorname{sn} u_1 = \sin\beta,\quad 0<\varphi\leq\pi.$$

289.00 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt{a+b\cos\vartheta}} = g\int_0^{u_1}du = g u_1 = g\operatorname{sn}^{-1}(\sin\beta,k) = gF(\beta,k).$$

289.01 $$\begin{cases}\displaystyle\int_0^{\varphi}\sqrt{a+b\cos\vartheta}\,d\vartheta=(a+b)\,g\int_0^{u_1}\mathrm{dn}^2 u\,du=(a+b)\,g\,E(u_1)\\ \qquad\qquad\qquad\qquad\qquad\qquad=(a+b)\,g\,E(\beta,k)\,.\end{cases}$$

289.02 $$\int_0^{\varphi}\frac{d\vartheta}{(2-\alpha^2+\alpha^2\cos\vartheta)\sqrt{a+b\cos\vartheta}}=\frac{g}{2}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}=\frac{g}{2}\Pi(\beta,\alpha^2,k)\,.$$ [See **400.**]

289.03 $$\int_0^{\varphi}\frac{\cos^m\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}(1-2\,\mathrm{sn}^2 u)^m\,du\,.$$ [See **331.03.**]

289.04 $$\int_0^{\varphi}\frac{\sin^2\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=4g\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du\,.$$ [See **361.01.**]

289.05 $$\int_0^{\varphi}\frac{d\vartheta}{(a+b\cos\vartheta)^m\sqrt{a+b\cos\vartheta}}=\frac{g}{(a+b)^m}\int_0^{u_1}\mathrm{nd}^{2m} u\,du\,.$$ [See **315.05.**]

289.06 $$\int_0^{\varphi}\frac{1+\cos\vartheta}{\sqrt{a+b\cos\vartheta}}\,d\vartheta=2g\int_0^{u_1}\mathrm{cn}^2 u\,du\,.$$ [See **312.02.**]

289.07 $$\int_0^{\varphi}\frac{(1-\cos\vartheta)\,d\vartheta}{(1+\cos\vartheta)\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}\mathrm{tn}^2 u\,du\,,\quad \varphi\neq\pi\,.$$ [See **316.02.**]

289.08 $$\begin{cases}\displaystyle\int_0^{\varphi}\frac{\cot^2\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}\\ \displaystyle\qquad=\frac{g}{4}\int_0^{u_1}\mathrm{ns}^2 u\,\mathrm{nc}^2 u\,du-g\int_0^{u_1}\mathrm{nc}^2 u\,du+g\int_0^{u_1}\mathrm{tn}^2 u\,du\,,\quad\varphi\neq\pi\,.\end{cases}$$ [See **361.10, 313.02, 316.02.**]

289.09 $$\int_0^{\varphi}\frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}R(1-2\,\mathrm{sn}^2 u)\,du\,,$$

where $R(\cos\vartheta)$ is any rational function of $\cos\vartheta$.

Integrands involving $\sqrt{a+b\cos\vartheta}$, $(b>|a|>0)$

$$\mathrm{sn}^2 u=\frac{b(1-\cos\vartheta)}{a+b}\,,\quad k^2=\frac{a+b}{2b}\,,\quad g=\sqrt{\frac{2}{b}}\,,\quad \mathrm{sn}\,u_1=\sin\xi\,,$$
$$\xi=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{b(1-\cos\varphi)}{a+b}}\,,\quad 0<\varphi\leqq\cos^{-1}(-a/b)\,.$$

290.00 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\,\mathrm{sn}^{-1}(\sin\xi,k)=g\,F(\xi,k)\,.$$

290.01 $\displaystyle\int_0^{\varphi}\frac{(1+\cos\vartheta)\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=2g\int_0^{u_1}\mathrm{dn}^2 u\,du=2g\,E(u_1)=2g\,E(\xi,k)\,.$

290.02 $$\begin{cases}\displaystyle\int_0^{\varphi}\frac{d\vartheta}{(a+b-\alpha^2 b+\alpha^2 b\cos\vartheta)\sqrt{a+b\cos\vartheta}}=\frac{g}{a+b}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}\\ \displaystyle\qquad=\frac{g}{a+b}\,\Pi(\xi,\alpha^2,k)\,.\end{cases}$$ [See **400.**]

290.03 $\displaystyle\int_0^{\varphi}\sqrt{a+b\cos\vartheta}\,d\vartheta=(a+b)\,g\int_0^{u_1}\mathrm{cn}^2 u\,du\,.$ [See **312.02.**]

290.04 $\displaystyle\int_0^{\varphi}\frac{\cos^m\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}(1-2k^2\,\mathrm{sn}^2 u)^m\,du\,.$ [See **331.03.**]

290.05 $\displaystyle\int_0^{\varphi}\frac{\sin^2\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=4k^2 g\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du\,.$ [See **361.02.**]

290.06 $\displaystyle\int_0^{\varphi}\frac{d\vartheta}{(a+b\cos\vartheta)^m\sqrt{a+b\cos\vartheta}}=\frac{g}{(a+b)^m}\int_0^{u_1}\mathrm{nc}^{2m}u\,du\,,\quad \varphi\neq\cos^{-1}(a/b)\,.$
[See **313.05.**]

290.07 $\displaystyle\int_0^{\varphi}\frac{(1-\cos\vartheta)\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=2k^2 g\int_0^{u_1}\mathrm{sn}^2 u\,du\,.$ [See **310.02.**]

290.08 $\displaystyle\int_0^{\varphi}\frac{\cot^2\vartheta\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=\frac{g}{4k^2}\int_0^{u_1}\mathrm{ns}^2 u\,\mathrm{nd}^2 u\,du-g\int_0^{u_1}\mathrm{nd}^2 u\,du+g\,k^2\int_0^{u_1}\mathrm{sd}^2 u\,du\,.$
[See **361.11, 315.02, 318.02.**]

290.09 $\displaystyle\int_0^{\varphi}\frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{a+b\cos\vartheta}}=g\int_0^{u_1}R(1-2k^2\,\mathrm{sn}^2 u)\,du\,,$

where $R(\cos\vartheta)$ is any rational integral function of $\sin\vartheta$.

Integrands involving $\sqrt{a-b\cos\vartheta}$, $(a>b>0)$

$$\mathrm{sn}^2 u=\frac{b(1-\cos\vartheta)}{k^2(a-b\cos\vartheta)}\,,\quad k^2=\frac{2b}{a+b}\,,\quad g=\frac{2}{\sqrt{a+b}}\,,$$
$$A=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{b(1-\cos\varphi)}{k^2(a-b\cos\varphi)}}\,,\quad \mathrm{sn}\,u_1=\sin A\,,\quad 0<\varphi\leqq\pi\,.$$

291.00 $\displaystyle\int_0^{\varphi}\frac{d\vartheta}{\sqrt{a-b\cos\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\,\mathrm{sn}^{-1}(\sin A,k)=g\,F(A,k)\,.$

291.01 $$\int_0^{\varphi}\frac{d\vartheta}{(a-b\cos\vartheta)\sqrt{a-b\cos\vartheta}}=\frac{g}{a-b}\int_0^{u_1}\mathrm{dn}^2 u\,du=\frac{g}{a-b}E(u_1)=\frac{g}{a-b}E(A,k).$$

291.02 $$\left\{\begin{aligned}&\int_0^{\varphi}\frac{\sqrt{a-b\cos\vartheta}}{1+p\cos\vartheta}\,d\vartheta=\frac{a-b}{1+p}\,g\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}=\frac{a-b}{1+p}\,g\,\Pi(A,\alpha^2,k),\\&\qquad[\alpha^2=a\,k^2 p/b(1+p),\ p\neq-1].\qquad\text{[See 400.]}\end{aligned}\right.$$

291.03 $$\int_0^{\varphi}\frac{\cos\vartheta\,d\vartheta}{\sqrt{a-b\cos\vartheta}}=g\int_0^{u_1}\frac{1-\frac{a\,k^2}{b}\,\mathrm{sn}^2 u}{1-k^2\,\mathrm{sn}^2 u}\,du.\qquad\text{[See 340.01.]}$$

291.04 $$\int_0^{\varphi}\frac{\sin^2\vartheta\,d\vartheta}{\sqrt{a-b\cos\vartheta}}=4g\,\frac{a-b}{a+b}\int_0^{u_1}\mathrm{sd}^2 u\,\mathrm{cd}^2 u\,du.\qquad\text{[See 361.27.]}$$

291.05 $$\int_0^{\varphi}\sqrt{a-b\cos\vartheta}\,d\vartheta=(a-b)\,g\int_0^{u_1}\mathrm{nd}^2 u\,du.\qquad\text{[See 315.02.]}$$

291.06 $$\int_0^{\varphi}\frac{(1-\cos\vartheta)\,d\vartheta}{(a-b\cos\vartheta)\sqrt{a-b\cos\vartheta}}=\frac{g\,k^2}{b}\int_0^{u_1}\mathrm{sn}^2 u\,du.\qquad\text{[See 310.02.]}$$

291.07 $$\left\{\begin{aligned}&\int_0^{\varphi}\frac{\tan^2\vartheta\,d\vartheta}{\sqrt{a-b\cos\vartheta}}=\frac{4(a-b)\,g}{(a+b)\,\alpha_1^4}\times\\&\times\left[-F(A,k)+(2-\alpha_1^2)\,\Pi(A,\alpha_1^2,k)-\int_0^{u_1}\frac{(1-\alpha_1^2)\,du}{(1-\alpha_1^2\,\mathrm{sn}^2 u)^2}\right],\qquad\text{[See 400.]}\end{aligned}\right.$$

where $\alpha_1^2=a\,k^2/b$.

291.08 $$\int_0^{\varphi}\frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{a-b\cos\vartheta}}=g\int_0^{u_1}R\left[\frac{1}{b}\,(b-a\,k^2\,\mathrm{sn}^2 u)\,\mathrm{nd}^2 u\right]du.$$

where $R(\cos\vartheta)$ is a rational function of $\cos\vartheta$.

Integrands involving $\sqrt{b\sin\vartheta + c\cos\vartheta}$

$$\operatorname{sn}^2 u = 1 - \frac{b}{p}\sin\vartheta - \frac{c}{p}\cos\vartheta, \quad g = \sqrt{\frac{2}{p}}, \quad k^2 = \frac{1}{2},$$

$$p = \sqrt{b^2 + c^2}, \quad \Gamma = \operatorname{am} u_1 = \sin^{-1}\sqrt{1 - \frac{b}{p}\sin\varphi - \frac{c}{p}\cos\varphi},$$

$$\varphi_1 = \sin^{-1}\frac{b}{p} = \cos^{-1}\frac{c}{p}; \quad \varphi_1 - \frac{\pi}{2} \leqq \varphi < \varphi_1.$$

292.00 $$\int_{\varphi}^{\varphi_1} \frac{d\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\Gamma, k) = g\,F(\Gamma, k).$$

292.01 $$\int_{\varphi}^{\varphi_1} \frac{(p + b\sin\vartheta + c\cos\vartheta)\,d\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}} = 2pg\int_0^{u_1} \operatorname{dn}^2 u\,du = 2pg\,E(u_1) = 2pg\,E(\Gamma, k).$$

292.02 $$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1} \frac{d\vartheta}{[p(1-\alpha^2) + \alpha^2(b\sin\vartheta + c\cos\vartheta)]\sqrt{b\sin\vartheta + c\cos\vartheta}} = \frac{g}{p}\int_0^{u_1} \frac{du}{1 - \alpha^2\operatorname{sn}^2 u} \\ \displaystyle\qquad = \frac{g}{p}\Pi(\Gamma, \alpha^2, k). \end{cases}$$ [See **400.**]

292.03 $$\int_{\varphi}^{\varphi_1} \frac{(p - b\sin\vartheta - c\cos\vartheta)\,d\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}} = g\,p\int_0^{u_1} \operatorname{sn}^2 u\,du.$$ [See **310.02.**]

292.04 $$\int_{\varphi}^{\varphi_1} \sqrt{b\sin\vartheta + c\cos\vartheta}\,d\vartheta = g\,p\int_0^{u_1} \operatorname{cn}^2 u\,du.$$ [See **312.02.**]

292.05 $$\int_{\varphi}^{\varphi_1} \frac{\sin\vartheta\,d\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}} = \frac{g\,b}{p}\int_0^{u_1} \operatorname{cn}^2 u\,du - \frac{\sqrt{2}\,c\,g}{p}\int_0^{u_1} \operatorname{sn} u\operatorname{dn} u\,du.$$ [See **312.02, 360.02.**]

292.06 $$\int_{\varphi}^{\varphi_1} \frac{\cos\vartheta\,d\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}} = \frac{g\,c}{p}\int_0^{u_1} \operatorname{cn}^2 u\,du + \frac{\sqrt{2}\,g\,b}{p}\int_0^{u_1} \operatorname{sn} u\operatorname{dn} u\,du.$$ [See **312.02, 360.02.**]

292.07 $$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1} \frac{c\sin\vartheta - b\cos\vartheta}{\sqrt{b\sin\vartheta + c\cos\vartheta}}\,d\vartheta = -g\,p\sqrt{2}\int_0^{u_1} \operatorname{sn} u\operatorname{dn} u\,du = g\,p\sqrt{2}\operatorname{cn} u\Big]_0^{u_1} \\ \displaystyle\qquad = -g\,p\sqrt{2} + g\sqrt{2p}\sqrt{b\sin\varphi + c\cos\varphi}. \end{cases}$$

292.08 $$\int_{\varphi}^{\varphi_1}\frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{b\sin\vartheta+c\cos\vartheta}}=g\int_0^{u_1}R\left[\frac{1}{p}\,(c\,\mathrm{cn}^2 u+b\sqrt{2}\,\mathrm{sn}\,u\,\mathrm{dn}\,u)\right]du,$$

where $R(\cos\vartheta)$ is a rational function of $\cos\vartheta$.

Integrands involving $\sqrt{a+b\sin\vartheta+c\cos\vartheta}$, $(0<|a|<\sqrt{b^2+c^2})$

$$\mathrm{sn}^2 u=\frac{1}{a+p}\,[p-b\sin\vartheta-c\cos\vartheta],\quad g=\sqrt{\frac{2}{p}},\quad k^2=\frac{a+p}{2p},$$
$$p=\sqrt{b^2+c^2},\quad \gamma=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{1}{a+p}\,(p-b\sin\varphi-c\cos\varphi)},$$
$$\varphi_1=\sin^{-1}(b/p)=\cos^{-1}(c/p);\quad \varphi_1-\cos^{-1}(-a/p)\leqq\varphi<\varphi_1.$$

293.00 $$\int_{\varphi}^{\varphi_1}\frac{d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\,\mathrm{sn}^{-1}(\sin\gamma,k)=g\,F(\gamma,k).$$

293.01 $$\int_{\varphi}^{\varphi_1}\frac{(p+b\sin\vartheta+c\cos\vartheta)}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}\,d\vartheta=2p\,g\int_0^{u_1}\mathrm{dn}^2 u\,du=2p\,g\,E(u_1)=2p\,g\,E(\gamma,k).$$

293.02 $$\begin{cases}\displaystyle\int_{\varphi}^{\varphi_1}\frac{d\vartheta}{[a+p-\alpha^2 p+\alpha^2\,(b\sin\vartheta+c\cos\vartheta)]\sqrt{a+b\sin\vartheta+c\cos\vartheta}}\\[2ex] \displaystyle\qquad=\frac{g}{a+p}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u}=\frac{g}{a+p}\,\Pi(\gamma,\alpha^2,k).\end{cases}$$ [See **400.**]

293.03 $$\int_{\varphi}^{\varphi_1}\sqrt{a+b\sin\vartheta+c\cos\vartheta}\,d\vartheta=(a+p)\,g\int_0^{u_1}\mathrm{cn}^2 u\,du.$$ [See **312.02.**]

293.04 $$\int_{\varphi}^{\varphi_1}\frac{(b\cos\vartheta-c\sin\vartheta)^2\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=2p\,(a+p)\,g\int_0^{u_1}\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du.$$ [See **361.02.**]

293.05 $$\begin{cases}\displaystyle\int_{\varphi}^{\varphi_1}\frac{\sin\vartheta\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=\frac{g}{p^2}\left[-a\,b\,F(\gamma,k)+b\,(a+p)\int_0^{u_1}\mathrm{cn}^2 u\,du\,-\right.\\[2ex] \displaystyle\qquad\left.-c\sqrt{2p\,(a+p)}\int_0^{u_1}\mathrm{dn}\,u\,\mathrm{sn}\,u\,du\right].\end{cases}$$ [See **312.02, 360.02.**]

293.06
$$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1} \frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}} \\ \displaystyle\qquad = g\int_0^{u_1} R\left[\frac{-ac+c(a+p)\,\mathrm{cn}^2 u + b\sqrt{2p(a+p)}\,\mathrm{dn}\,u\,\mathrm{sn}\,u}{p^2}\right] du, \end{cases}$$

where $R(\cos\vartheta)$ is a rational function of $\cos\vartheta$.

293.07
$$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1} \frac{R(\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}} \\ \displaystyle\qquad = g\int_0^{u_1} R\left[\frac{-ab+b(a+p)\,\mathrm{cn}^2 u - c\sqrt{2p(a+p)}\,\mathrm{dn}\,u\,\mathrm{sn}\,u}{p^2}\right] du, \end{cases}$$

where $R(\sin\vartheta)$ is a rational function of $\sin\vartheta$.

Integrands involving $\sqrt{a+b\sin\vartheta+c\cos\vartheta}$, $(a>\sqrt{b^2+c^2}>0)$

$$\boxed{\begin{gathered} \mathrm{sn}^2 u = \frac{1}{2p}\,[p-b\sin\vartheta-c\cos\vartheta], \quad k^2=\frac{2p}{a+p}, \quad g=\frac{2}{\sqrt{a+p}}, \\ p=\sqrt{b^2+c^2}, \quad \psi=\mathrm{am}\,u_1=\sin^{-1}\sqrt{\frac{1}{2p}\,(p-b\sin\varphi-c\cos\varphi)}, \\ \varphi_1=\sin^{-1}(b/p)=\cos^{-1}(c/p); \quad \varphi_1-\pi\leqq\varphi<\varphi_1. \end{gathered}}$$

294.00
$$\int_{\varphi}^{\varphi_1}\frac{d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\psi,k) = g\,F(\psi,k).$$

294.01
$$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1}\sqrt{a+b\sin\vartheta+c\cos\vartheta}\,d\vartheta = g(a+p)\int_0^{u_1}\mathrm{dn}^2 u \\ \displaystyle\qquad = g(a+p)\,E(u_1) = g(a+p)\,E(\psi,k). \end{cases}$$

294.02
$$\begin{cases} \displaystyle\int_{\varphi}^{\varphi_1}\frac{d\vartheta}{[2p-\alpha^2 p+\alpha^2(b\sin\vartheta+c\cos\vartheta)]\sqrt{a+b\sin\vartheta+c\cos\vartheta}} \\ \displaystyle\qquad = \frac{g}{2p}\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{g}{2p}\,\Pi(\psi,\alpha^2,k). \end{cases}$$
[See **400.**]

294.03 $$\int_{\varphi}^{\varphi_1}\frac{(b\cos\vartheta-c\sin\vartheta)^2}{\sqrt{a+b\sin\vartheta+c\sin\vartheta}}\,d\vartheta=4g\,p^2\int_0^{u_1}\operatorname{sn}^2 u\operatorname{cn}^2 u\,du.$$ [See **361.01**.]

294.04 $$\int_{\varphi}^{\varphi_1}\frac{(p+b\sin\vartheta+c\cos\vartheta)}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}\,d\vartheta=2g\,p\int_0^{u_1}\operatorname{cn}^2 u\,du.$$ [See **312.02**.]

294.05 $$\begin{cases}\displaystyle\int_{\varphi}^{\varphi_1}\frac{(b\cos\vartheta-c\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=-2p\,g\int_0^{u_1}\operatorname{sn} u\operatorname{cn} u\,du\\ \qquad\qquad=-2\left[\sqrt{a+p}-\sqrt{a+b\sin\varphi+c\cos\varphi}\right].\end{cases}$$

294.06 $$\int_{\varphi}^{\varphi_1}\frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=g\int_0^{u_1}R\left[\frac{c(a+p)\operatorname{dn}^2 u-2b\,p\operatorname{sn} u\operatorname{cn} u-ac}{p^2}\right]du,$$

where $R(\cos\vartheta)$ is a rational function of $\cos\vartheta$.

294.07 $$\int_{\varphi}^{\varphi_1}\frac{R(\sin\vartheta)\,d\vartheta}{\sqrt{a+b\sin\vartheta+c\cos\vartheta}}=g\int_0^{u_1}R\left[\frac{b(a+p)\operatorname{dn}^2 u+2c\,p\operatorname{sn} u\operatorname{cn} u-ab}{p^2}\right]du,$$

where $R(\sin\vartheta)$ is a rational function of $\sin\vartheta$.

Reduction of Hyperbolic Integrands to Jacobian Elliptic Functions.

In addition to the algebraic or trigonometric forms given in the foregoing sections, elliptic integrals encountered in practical problems may also involve hyperbolic integrands. The reduction of some important cases follows[1]:

295.00 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{1+k'^2 \sinh^2\vartheta}} = \int_0^{\operatorname{tn}^{-1}(\sinh\varphi, k)} du = \operatorname{tn}^{-1}(\sinh\varphi, k) = F[\sin^{-1}(\tanh\varphi), k]. \quad [0<\varphi<\infty.]$$

295.01 $$\int_0^{\varphi} \frac{R(\sinh\vartheta)\, d\vartheta}{\sqrt{1+k'^2 \sinh^2\vartheta}} = \int_0^{\operatorname{tn}^{-1}(\sinh\varphi, k)} R(\operatorname{tn} u)\, du. \quad [0<\varphi<\infty.]$$

295.10 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{k^2+k'^2 \cosh^2\vartheta}} = \int_0^{\operatorname{nc}^{-1}(\cosh\varphi, k)} du = \operatorname{nc}^{-1}(\cosh\varphi, k) = F[\sin^{-1}(\tanh\varphi), k]. \quad [0<\varphi<\infty.]$$

295.11 $$\int_0^{\varphi} \frac{R(\cosh\vartheta)\, d\vartheta}{\sqrt{k^2+k'^2 \cosh^2\vartheta}} = \int_0^{\operatorname{nc}^{-1}(\cosh\varphi, k)} R(\operatorname{nc} u)\, du. \quad [0<\varphi<\infty.]$$

295.20 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k'^2 \cosh^2\vartheta}} = \int_0^{\operatorname{nd}^{-1}(\cosh\varphi, k)} du = \operatorname{nd}^{-1}(\cosh\varphi, k) = F\left[\sin^{-1}\left(\frac{\tanh\varphi}{k}\right), k\right]. \quad [0<\varphi<\cosh^{-1}(1/k').]$$

295.21 $$\int_0^{\varphi} \frac{R(\cosh\vartheta)\, d\vartheta}{\sqrt{1-k'^2 \cosh^2\vartheta}} = \int_0^{\operatorname{nd}^{-1}(\cosh\varphi, k)} R(\operatorname{nd} u)\, du. \quad [0<\varphi<\cosh^{-1}(1/k').]$$

295.30 $$\int_{\varphi}^{\infty} \frac{d\vartheta}{\sqrt{\cosh^2\vartheta - k^2}} = \int_0^{\operatorname{ns}^{-1}(\cosh\varphi, k)} du = \operatorname{ns}^{-1}(\cosh\varphi, k) = F[\sin^{-1}(\operatorname{sech}\varphi), k]. \quad [\varphi>0.]$$

295.40 $$\int_{\varphi}^{\infty} \frac{d\vartheta}{\sqrt{\sinh^2\vartheta + k'^2}} = \int_0^{\operatorname{cs}^{-1}(\sinh\varphi, k)} du = \operatorname{cs}^{-1}(\sinh\varphi, k) = F[\sin^{-1}(\operatorname{sech}\varphi), k]. \quad [\varphi>0.]$$

[1] In these, $k'^2 = 1-k^2$; $0<k<1$. See **525** for other integrals.

Integrands involving $\sqrt{\cosh 2a\vartheta}$, $(0<a\varphi<\infty)$

$$\operatorname{sn}^2 u = \frac{\cosh 2a\vartheta - 1}{\cosh 2a\vartheta},\quad k^2 = 1/2,\quad g = 1/a\sqrt{2},$$
$$\xi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{\cosh 2a\varphi - 1}{\cosh 2a\varphi}},\quad \operatorname{sn} u_1 = \sin\xi.$$

296.00 $$\int_0^\varphi \frac{d\vartheta}{\sqrt{\cosh 2a\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\xi, k) = g\,F(\xi,k).$$

296.01 $$\int_0^\varphi \frac{(1+\cosh 2a\vartheta)\,d\vartheta}{\cosh 2a\vartheta\sqrt{\cosh 2a\vartheta}} = 2g\int_0^{u_1}\operatorname{dn}^2 u\,du = 2g\,E(u_1) = 2g\,E(\xi,k).$$

296.02 $$\int_0^\varphi \frac{\sqrt{\cosh 2a\vartheta}\,d\vartheta}{\alpha^2+(1-\alpha^2)\cosh 2a\vartheta} = g\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\xi,\alpha^2,k).$$ [See **400.**]

296.03 $$\int_0^\varphi \sqrt{\cosh 2a\vartheta}\,d\vartheta = g\int_0^{u_1}\operatorname{nc}^2 u\,du.$$ [See **313.02.**]

296.04 $$\int_0^\varphi \frac{d\vartheta}{\cosh^m 2a\vartheta\sqrt{\cosh 2a\vartheta}} = g\int_0^{u_1}\operatorname{cn}^{2m} u\,du.$$ [See **312.05.**]

296.05 $$\int_0^\varphi \frac{\tanh^2 2a\vartheta}{\sqrt{\cosh 2a\vartheta}}\,d\vartheta = 2g\int_0^{u_1}\operatorname{sn}^2 u\operatorname{dn}^2 u\,du.$$ [See **361.02.**]

296.06 $$\int_0^\varphi \frac{(\cosh 2a\vartheta - 1)\,d\vartheta}{\cosh 2a\vartheta\sqrt{\cosh 2a\vartheta}} = g\int_0^{u_1}\operatorname{sn}^2 u\,du.$$ [See **310.02.**]

296.07 $$\int_0^\varphi \frac{\sinh^2 2a\vartheta}{\sqrt{\cosh 2a\vartheta}}\,d\vartheta = 2g\int_0^{u_1}\operatorname{tn}^2 u\operatorname{dc}^2 u\,du.$$ [See **361.15.**]

296.08 $$\begin{cases}\displaystyle\int_0^\varphi \frac{\sinh 2a\vartheta}{\sqrt{\cosh 2a\vartheta}}\,d\vartheta = g\sqrt{2}\int_0^{u_1}\operatorname{tn} u\operatorname{dc} u\,du = \frac{1}{a}\left[\operatorname{nc} u_1 - 1\right] \\ \displaystyle\qquad = \frac{1}{a}\left[\sqrt{\cosh 2a\varphi} - 1\right].\end{cases}$$

296.09 $$\int_0^\varphi \frac{R(\cosh a\vartheta)\,d\vartheta}{\sqrt{\cosh 2a\vartheta}} = g\int_0^{u_1} R(\operatorname{dc} u)\,du,$$

where $R(\cosh a\vartheta)$ is a rational function of $\cosh a\vartheta$

Integrands involving $\sqrt{\sinh 2a\vartheta}$, $(0 < a\varphi < \infty)$

$$\operatorname{cn} u = \frac{1 - \sinh 2a\vartheta}{1 + \sinh 2a\vartheta}, \quad k^2 = 1/2, \quad g = 1/2a,$$
$$\psi = \operatorname{am} u_1 = \cos^{-1}\left[\frac{1 - \sinh 2a\varphi}{1 + \sinh 2a\varphi}\right], \quad \operatorname{cn} u_1 = \cos\psi$$

296.50 $$\int_0^\varphi \frac{d\vartheta}{\sqrt{\sinh 2a\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{cn}^{-1}(\cos\psi, k) = g\,F(\psi, k).$$

296.51 $$\int_0^\varphi \frac{\cosh^2 2a\vartheta\, d\vartheta}{(1 + \sinh 2a\vartheta)^2\sqrt{\sinh 2a\vartheta}} = g\int_0^{u_1} \operatorname{dn}^2 u\, du = g\,E(u_1) = g\,E(\psi, k).$$

296.52 $$\left\{\begin{aligned}&\int_0^\varphi \frac{(1 + \sinh 2a\vartheta)^2\, d\vartheta}{[(1 + \sinh 2a\vartheta)^2 - 4\alpha^2 \sinh 2a\vartheta]\sqrt{\sinh 2a\vartheta}} = g\int_0^{u_1} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u}\\ &\qquad = g\,\Pi(\psi, \alpha^2, k).\end{aligned}\right.$$ [See **400.**]

296.53 $$\int_0^\varphi \sqrt{\sinh 2a\vartheta}\, d\vartheta = g\int_0^{u_1} \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\, du.$$ [See **361.53.**]

296.54 $$\int_0^\varphi \frac{\sqrt{\sinh 2a\vartheta}\, d\vartheta}{(1 + \sinh 2a\vartheta)^2} = \frac{g}{4}\int_0^{u_1} \operatorname{sn}^2 u\, du.$$ [See **310.02.**]

296.55 $$\int_0^\varphi \frac{(1 - \sinh 2a\vartheta)^2\, d\vartheta}{(1 + \sinh 2a\vartheta)^2\sqrt{\sinh 2a\vartheta}} = g\int_0^{u_1} \operatorname{cn}^2 u\, du.$$ [See **312.02.**]

296.56 $$\int_0^\varphi \frac{\cosh 2a\vartheta}{\sqrt{\sinh 2a\vartheta}}\, d\vartheta = 2g\int_0^{u_1} \frac{\operatorname{dn} u\, du}{1 + \operatorname{cn} u} = 2g\,\frac{\operatorname{sn} u_1}{1 + \operatorname{cn} u_1} = \frac{1}{a}\sqrt{\sinh 2a\varphi}.$$

296.57 $$\int_0^\varphi \frac{R(\sinh 2a\vartheta)\, d\vartheta}{\sqrt{\sinh 2a\vartheta}} = g\int_0^{u_1} R\left[\frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}\right] du,$$

where $R(\sinh 2a\vartheta)$ is a rational function of $\sinh 2a\vartheta$.

Integrands involving $\sqrt{b\cosh\vartheta - a}$, $(b > a > 0; \infty > \varphi > 0)$

$$\operatorname{sn}^2 u = \frac{b(\cosh\vartheta - 1)}{b\cosh\vartheta - a}, \quad k^2 = \frac{a+b}{2b}, \quad g = \sqrt{\frac{2}{b}},$$
$$\xi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{b(\cosh\varphi - 1)}{b\cosh\varphi - a}}, \quad \operatorname{sn} u_1 = \sin\xi.$$

297.00 $$\int_0^{\varphi} \frac{d\vartheta}{\sqrt{b\cosh\vartheta - a}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}(\sin\xi, k) = g\,F(\xi, k).$$

297.01 $$\begin{cases} \displaystyle\int_0^{\varphi} \frac{(\cosh\vartheta + 1)\,d\vartheta}{(b\cosh\vartheta - a)\sqrt{b\cosh\vartheta - a}} = \frac{2g}{b-a}\int_0^{u_1} \operatorname{dn}^2 u\,du = \frac{2g}{b-a}E(u_1) \\ \displaystyle\qquad = \frac{2g}{b-a}E(\xi, k). \end{cases}$$

297.02 $$\int_0^{\varphi} \frac{\sqrt{b\cosh\vartheta - a}}{\alpha^2 b - a + b(1-\alpha^2)\cosh\vartheta}\,d\vartheta = g\int_0^{u_1} \frac{du}{1 - \alpha^2\operatorname{sn}^2 u} = g\,\Pi(\xi, \alpha^2, k).$$
[See **400.**]

297.03 $$\int_0^{\varphi} \frac{\cosh\vartheta\,d\vartheta}{\sqrt{b\cosh\vartheta - a}} = g\int_0^{u_1} \left[\operatorname{nc}^2 u - \left(\frac{a}{b}\right)\operatorname{tn}^2 u\right] du.$$
[See **313.02** and **316.02.**]

297.04 $$\begin{cases} \displaystyle\int_0^{\varphi} \frac{\sinh\vartheta\,d\vartheta}{\sqrt{b\cosh\vartheta - a}} = \frac{2\sqrt{b-a}}{b}\int_0^{u_1} \frac{\operatorname{sn} u\operatorname{dn} u}{\operatorname{cn}^2 u}\,du = \frac{2\sqrt{b-a}}{b}\left[\operatorname{nc} u_1 - 1\right] \\ \displaystyle\qquad = \frac{2}{b}\left[\sqrt{b\cosh\varphi - a} - \sqrt{b-a}\right]. \end{cases}$$

297.05 $$\int_0^{\varphi} \sqrt{b\cosh\vartheta - a}\,d\vartheta = (b-a)\,g\int_0^{u_1} \operatorname{nc}^2 u\,du.$$ [See **313.02.**]

297.06 $$\int_0^{\varphi} \frac{d\vartheta}{(b\cosh\vartheta - a)^m\sqrt{b\cosh\vartheta - a}} = \frac{g}{(b-a)^m}\int_0^{u_1} \operatorname{cn}^{2m} u\,du.$$ [See **312.05.**]

297.07 $$\int_0^{\varphi} \frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{b\cosh\vartheta - a}} = g\int_0^{u_1} R\left[\frac{b - a\operatorname{sn}^2 u}{b\operatorname{cn}^2 u}\right] du,$$

where $R(\cosh\vartheta)$ is a rational function of $\cosh\vartheta$.

Integrands involving $\sqrt{a + b\cosh\vartheta}$, $(0 < \Phi < \infty,\ a > b > 0)$

$$\operatorname{sn} u = \frac{\sinh\vartheta}{\cosh\vartheta + 1}, \quad k^2 = \frac{a-b}{a+b}, \quad g = \frac{2}{\sqrt{a+b}}, \quad \operatorname{sn} u_1 = \tanh\left(\frac{\Phi}{2}\right),$$

$$\psi = \operatorname{am} u_1 = \sin^{-1}\sqrt{\frac{\cosh\Phi - 1}{\cosh\Phi + 1}} = \sin^{-1}\left[\tanh\left(\frac{\Phi}{2}\right)\right].$$

297.25 $$\int_0^{\Phi} \frac{d\vartheta}{\sqrt{a + b\cosh\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\operatorname{sn}^{-1}\left[\tanh\left(\frac{\Phi}{2}\right), k\right] = g\,F(\psi, k).$$

297.26 $$\int_0^{\Phi} \frac{\sqrt{a + b\cosh\vartheta}}{\cosh\vartheta + 1}\,d\vartheta = \sqrt{a+b}\int_0^{u_1} \operatorname{dn}^2 u\,du = \sqrt{a+b}\,E(u_1) = \sqrt{a+b}\,E(\psi, k).$$

297.27 $$\int_0^{\Phi} \frac{(1 + \cosh\vartheta)\,d\vartheta}{[1+\alpha^2+(1-\alpha^2)\cosh\vartheta]\sqrt{a+b\cosh\vartheta}} = g\int_0^{u_1} \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = g\,\Pi(\psi, \alpha^2, k).$$

297.28 $$\int_0^{\Phi} \frac{\tanh^{2m}(\vartheta/2)\,d\vartheta}{\sqrt{a + b\cosh\vartheta}} = g\int_0^{u_1} \operatorname{sn}^{2m} u\,du.$$ [See **310.02.**]

297.29 $$\int_0^{\Phi} \sqrt{a + b\cosh\vartheta}\,d\vartheta = (a + b)\,g\int_0^{u_1} \operatorname{dc}^2 u\,du.$$ [See **321.02.**]

297.30 $$\int_0^{\Phi} \frac{d\vartheta}{(\cosh\vartheta + 1)^m\sqrt{a + b\cosh\vartheta}} = \frac{g}{2^m}\int_0^{u_1} \operatorname{cn}^{2m} u\,du.$$ [See **312.05.**]

297.31 $$\int_0^{\Phi} \frac{(\cosh\vartheta - 1)^m}{\sqrt{a + b\cosh\vartheta}}\,d\vartheta = 2^m g\int_0^{u_1} \operatorname{tn}^{2m} u\,du.$$ [See **316.05.**]

297.32 $$\left\{\begin{aligned} \int_0^{\Phi} \frac{\sinh\vartheta}{\sqrt{a + b\cosh\vartheta}}\,d\vartheta &= 2g\int_0^{u_1} \frac{\operatorname{sn} u}{\operatorname{cn}^2 u}\,du = \frac{2g}{k'^2}\,[\operatorname{dc} u_1 - 1] \\ &= \frac{2}{b}\left[\sqrt{a + b\cosh\Phi} - \sqrt{a+b}\right]. \end{aligned}\right.$$

297.33 $$\int_0^{\Phi} \frac{\cosh\vartheta\,d\vartheta}{\sqrt{a + b\cosh\vartheta}} = g\int_0^{u_1} [\operatorname{nc}^2 u + \operatorname{tn}^2 u]\,du.$$ [See **313.02** and **316.02**.]

297.34 $$\int_0^{\Phi} \frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{a + b\cosh\vartheta}} = g\int_0^{u_1} R\left[\frac{1 + \operatorname{sn}^2 u}{\operatorname{cn}^2 u}\right] du,$$

where $R(\cosh\vartheta)$ is a rational function of $\cosh\vartheta$.

Integrands involving $\sqrt{a-b\cosh\vartheta}$, $[a>b>0;\ 0<\Phi<\cosh^{-1}(a/b)]$

$$\operatorname{sn}^2 u=\frac{a-b\cosh\vartheta}{a-b},\quad k^2=\frac{a-b}{a+b},\quad g=\frac{2}{\sqrt{a+b}},\quad \Phi_1=\cosh^{-1}\left(\frac{a}{b}\right),$$
$$\psi=\operatorname{am}u_1=\sin^{-1}\sqrt{\frac{a-b\cosh\Phi}{a-b}},\quad \operatorname{sn}u_1=\sin\psi.$$

297.50 $$\int_\Phi^{\Phi_1}\frac{d\vartheta}{\sqrt{a-b\cosh\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\operatorname{sn}^{-1}(\sin\psi,k)=gF(\psi,k).$$

297.51 $$\int_\Phi^{\Phi_1}\frac{(1+\cosh\vartheta)}{\sqrt{a-b\cosh\vartheta}}d\vartheta=\frac{2\sqrt{a+b}}{b}\int_0^{u_1}\operatorname{dn}^2u\,du=\frac{2\sqrt{a+b}}{b}E(u_1)=\frac{2\sqrt{a+b}}{b}E(\psi,k).$$

297.52 $$\int_\Phi^{\Phi_1}\frac{d\vartheta}{(a-b-a\alpha^2+b\alpha^2\cosh\vartheta)\sqrt{a-b\cosh\vartheta}}=\frac{g}{a-b}\int_0^{u_1}\frac{du}{1-\alpha^2\operatorname{sn}^2u}=\frac{g}{a-b}\Pi(\psi,\alpha^2,k)$$ [See **400.**]

297.53 $$\int_\Phi^{\Phi_1}\frac{\sinh\vartheta\,d\vartheta}{\sqrt{a-b\cosh\vartheta}}=\frac{2\sqrt{a-b}}{b}\int_0^{u_1}\operatorname{cn}u\operatorname{dn}u\,du=\frac{2\sqrt{a-b}}{b}\operatorname{sn}u_1=\frac{2}{b}\sqrt{a-b\cosh\Phi}.$$

297.54 $$\int_\Phi^{\Phi_1}\sqrt{a-b\cosh\vartheta}\,d\vartheta=(a-b)\,g\int_0^{u_1}\operatorname{sn}^2u\,du.$$ [See **310.02.**]

297.55 $$\int_\Phi^{\Phi_1}\frac{(\cosh\vartheta-1)}{\sqrt{a-b\cosh\vartheta}}d\vartheta=\frac{a-b}{b}g\int_0^{u_1}\operatorname{cn}^2u\,du.$$ [See **312.02.**]

297.56 $$\int_\Phi^{\Phi_1}\frac{\cosh^m d\vartheta}{\sqrt{a-b\cosh\vartheta}}=\left(\frac{a}{b}\right)^m g\int_0^{u_1}(1-\alpha^2\operatorname{sn}^2u)^m du,$$ [See **331.03.**]

where $\alpha^2=(a-b)/a$.

297.57 $$\int_\Phi^{\Phi_1}\frac{d\vartheta}{\cosh^m\vartheta\sqrt{a-b\cosh\vartheta}}=\left(\frac{b}{a}\right)^m g\int_0^{u_1}\frac{du}{(1-\alpha^2\operatorname{sn}^2u)^m}.$$ [See **336.03.**]

297.58 $$\int_{\Phi}^{\Phi_1} \frac{d\vartheta}{(1+\cosh\vartheta)^m \sqrt{a-b\cosh\vartheta}} = \left(\frac{b}{a+b}\right)^m g \int_0^{u_1} \mathrm{nd}^{2m} u\, du.$$ [See **315.05**.]

297.59 $$\int_{\Phi}^{\Phi_1} \frac{R(\cosh\vartheta)\, d\vartheta}{\sqrt{a-b\cosh\vartheta}} = g \int_0^{u_1} R\left[\frac{a+(b-a)\,\mathrm{sn}^2 u}{b}\right] du,$$

where $R(\cosh\vartheta)$ is a rational function of $\cosh\vartheta$.

Integrands involving $\sqrt{b\cosh\vartheta - a}$, $[a > b > 0,$ $\cosh^{-1}(a/b) < \Phi < \infty]$

$$\mathrm{sn}^2 u = \frac{b\cosh\vartheta - a}{b(\cosh\vartheta - 1)}, \quad k^2 = \frac{2b}{a+b}, \quad g = \frac{2}{\sqrt{a+b}}, \quad \Phi_1 = \cosh^{-1}\left(\frac{a}{b}\right),$$
$$\varepsilon = \mathrm{am}\, u_1 = \sin^{-1}\sqrt{\frac{b\cosh\Phi - a}{b(\cosh\Phi - 1)}}, \quad \mathrm{sn}\, u_1 = \sin\varepsilon,$$

297.75 $$\int_{\Phi_1}^{\Phi} \frac{d\vartheta}{\sqrt{b\cosh\vartheta - a}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{sn}^{-1}(\sin\varepsilon, k) = g\,F(\varepsilon, k).$$

297.76 $$\int_{\Phi_1}^{\Phi} \frac{\coth^2(\vartheta/2)\, d\vartheta}{\sqrt{b\cosh\vartheta - a}} = \frac{g}{k'^2}\int_0^{u_1} \mathrm{dn}^2 u\, du = \frac{g}{k'^2} E(u_1) = \frac{g}{k'^2} E(\varepsilon, k).$$

297.77 $$\int_{\Phi_1}^{\Phi} \frac{\sqrt{b\cosh\vartheta - a}}{\cosh\vartheta - 1}\, d\vartheta = g\,b\int_0^{u_1} \mathrm{sn}^2 u\, du.$$ [See **310.02**.]

297.78 $$\int_{\Phi_1}^{\Phi} \frac{d\vartheta}{(\cosh\vartheta - 1)^m \sqrt{b\cosh\vartheta - a}} = \frac{b^m g}{(a-b)^m}\int_0^{u_1} \mathrm{cn}^{2m} u\, du.$$ [See **312.05**.]

297.79 $$\int_{\Phi_1}^{\Phi} \sqrt{b\cosh\vartheta - a}\, d\vartheta = (a-b)\,g\int_0^{u_1} \mathrm{tn}^2 u\, du.$$ [See **316.02**.]

297.80 $$\int_{\Phi_1}^{\Phi} \frac{d\vartheta}{(\cosh\vartheta + 1)^m \sqrt{b\cosh\vartheta - a}} = \frac{b^m g}{(a+b)^m}\int_0^{u_1} \mathrm{cd}^{2m} u\, du.$$ [See **320.05**.]

297.81 $$\begin{cases} \displaystyle\int_{\Phi_1}^{\Phi} \frac{\sinh\vartheta\, d\vartheta}{\sqrt{b\cosh\vartheta - a}} = \frac{\sqrt{a^2-b^2}}{b}\, g\int_0^{u_1} \frac{\mathrm{dn}\, u}{\mathrm{cn}^2 u}\, du = \frac{\sqrt{a^2-b^2}}{b}\, g\,\mathrm{tn}\, u_1 \\ \displaystyle\qquad = \frac{2}{b}\sqrt{b\cosh\vartheta - a}. \end{cases}$$

297.82 $$\int_{\Phi_1}^{\Phi}\frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{b\cosh\vartheta-a}}=g\int_0^{u_1}R\left[\frac{a+(a-b)\operatorname{tn}^2u}{b}\right]du,$$

where $R(\cosh\vartheta)$ is a rational function of $\cosh\vartheta$.

Integrands involving $\sqrt{a+b\sinh\vartheta}$, $[-\sinh^{-1}(a/b)<\Phi<\infty;\ a>0,\ b>0]$.

$$\operatorname{cn}u=\frac{\sqrt{a^2+b^2}-a-b\sinh\vartheta}{\sqrt{a^2+b^2}+a+b\sinh\vartheta},\qquad k^2=\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}},\qquad g=\frac{1}{\sqrt[4]{a^2+b^2}},$$
$$\varepsilon=\operatorname{am}u_1=\cos^{-1}\left[\frac{\sqrt{a^2+b^2}-a-b\sinh\Phi}{\sqrt{a^2+b^2}+a+b\sinh\Phi}\right];\quad \Phi_1=-\sinh^{-1}\left(\frac{a}{b}\right),$$
$$\operatorname{cn}u_1=\cos\varepsilon.$$

298.00 $$\int_{\Phi_1}^{\Phi}\frac{d\vartheta}{\sqrt{a+b\sinh\vartheta}}=g\int_0^{u_1}du=g\,u_1=g\operatorname{cn}^{-1}(\cos\varepsilon,k)=g\,F(\varepsilon,k).$$

298.01 $$\begin{cases}\displaystyle\int_{\Phi_1}^{\Phi}\frac{\cosh^2\vartheta\,d\vartheta}{\left[\sqrt{a^2+b^2}+a+b\sinh\vartheta\right]^2\sqrt{a+b\sinh\vartheta}}=\frac{g}{b^2}\int_0^{u_1}\operatorname{dn}^2u\,du\\ \displaystyle\qquad=\frac{g}{b^2}E(u_1)=\frac{g}{b^2}E(\varepsilon,k).\end{cases}$$

298.02 $$\int_{\Phi_1}^{\Phi}\sqrt{a+b\sinh\vartheta}\,d\vartheta=\sqrt[4]{a^2+b^2}\int_0^{u_1}\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\,du.$$ [See **361.53**.]

298.03 $$\int_{\Phi_1}^{\Phi}\frac{\sqrt{a+b\sinh\vartheta}}{\cosh^2\vartheta}\,d\vartheta=\frac{b^2g}{4\sqrt{a^2+b^2}}\int_0^{u_1}\operatorname{sd}^2u\,du.$$ [See **318.02**.]

298.04 $$\int_{\Phi_1}^{\Phi}\frac{\sqrt{a+b\sinh\vartheta}\,d\vartheta}{\left[\sqrt{a^2+b^2}-a-b\sinh\vartheta\right]^2}=\frac{g}{4\sqrt{a^2+b^2}}\int_0^{u_1}\operatorname{tn}^2u\,du.$$ [See **316.02**.]

298.05 $$\begin{cases}\displaystyle\int_{\Phi_1}^{\Phi}\frac{\cosh\vartheta\,d\vartheta}{\sqrt{a+b\sinh\vartheta}}=\frac{2\sqrt{a^2+b^2}}{b}g\int_0^{u_1}\frac{\operatorname{dn}u}{1+\operatorname{cn}u}\,du\\ \displaystyle\qquad=\frac{2\sqrt{a^2+b^2}}{b}g\frac{\operatorname{sn}u_1}{1+\operatorname{cn}u_1}=\frac{2}{b}\sqrt{a+b\sinh\Phi}.\end{cases}$$

298.06 $$\int_{\Phi_1}^{\Phi}\frac{R(\sinh\vartheta)\,d\vartheta}{\sqrt{a+b\sinh\vartheta}}=g\int_0^{u_1}R\left[\frac{\sqrt{a^2+b^2}-a-\left(\sqrt{a^2+b^2}+a\right)\operatorname{cn}u}{b(1+\operatorname{cn}u)}\right]du,$$

where $R(\sinh\vartheta)$ is a rational function of $\sinh\vartheta$.

Integrands involving $\sqrt{a\sinh\vartheta + b\cosh\vartheta}$, $(b > a > 0;\ \Phi_1 < \Phi < \infty)$

$$\mathrm{cn}^2 u = \frac{\sqrt{b^2-a^2}}{a\sinh\vartheta + b\cosh\vartheta}, \quad k^2 = \frac{1}{2}, \quad g = \sqrt[4]{\frac{4}{b^2-a^2}},$$

$$\Phi_1 = -\sinh^{-1}\left[\frac{a}{\sqrt{b^2-a^2}}\right], \quad \varepsilon = \mathrm{am}\, u_1 = \cos^{-1}\left[\frac{\sqrt[4]{b^2-a^2}}{\sqrt{a\sinh\Phi + b\cosh\Phi}}\right],$$

$$\mathrm{cn}\, u_1 = \cos\varepsilon.$$

299.00 $$\int_{\Phi_1}^{\Phi} \frac{d\vartheta}{\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = g\int_0^{u_1} du = g\,u_1 = g\,\mathrm{cn}^{-1}(\cos\varepsilon, k) = g\,F(\varepsilon, k).$$

299.01 $$\begin{cases} \displaystyle\int_{\Phi_1}^{\Phi} \frac{(\sqrt{b^2-a^2} + a\sinh\vartheta + b\cosh\vartheta)\,d\vartheta}{(a\sinh\vartheta + b\cosh\vartheta)\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = 2g\int_0^{u_1} \mathrm{dn}^2 u\,du \\ \qquad = 2g\,E(u_1) = 2g\,E(\varepsilon, k). \end{cases}$$

299.02 $$\int_{\Phi_1}^{\Phi} \sqrt{a\sinh\vartheta + b\cosh\vartheta}\,d\vartheta = g\sqrt{b^2-a^2}\int_0^{u_1} \mathrm{nc}^2 u\,du.$$ [See **313.02**.]

299.03 $$\int_{\Phi_1}^{\Phi} \frac{d\vartheta}{(a\sinh\vartheta + b\cosh\vartheta)^m\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = \frac{g}{\sqrt{(b^2-a^2)^m}}\int_0^{u_1} \mathrm{cn}^{2m} u\,du.$$

[See **312.05**.]

299.04 $$\int_{\Phi_1}^{\Phi} \frac{(a\sinh\vartheta + b\cosh\vartheta - \sqrt{b^2-a^2})\,d\vartheta}{\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = g\sqrt{b^2-a^2}\int_0^{u_1} \mathrm{tn}^2 u\,du.$$

[See **316.02**.]

299.05 $$\int_{\Phi_1}^{\Phi} \frac{(a\sinh\vartheta + b\cosh\vartheta + \sqrt{b^2-a^2})\,d\vartheta}{\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = 2g\sqrt{b^2-a^2}\int_0^{u_1} \mathrm{dc}^2 u\,du.$$

[See **321.02**.]

299.06 $$\int_{\Phi_1}^{\Phi} \frac{R(\sinh\vartheta)\,d\vartheta}{\sqrt{a\sinh\vartheta + b\cosh\vartheta}} = g\int_0^{u_1} R\left[\frac{b\sqrt{2}\,\mathrm{tn}\,u\,\mathrm{dc}\,u - a\,\mathrm{nc}^2 u}{\sqrt{b^2-a^2}}\right] du,$$

where $R(\sinh\vartheta)$ is a rational function of $\sinh\vartheta$.

299.07 $$\int_{\Phi_1}^{\Phi} \frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{a\sinh\vartheta + b\cosh\vartheta}}\,d\vartheta = g\int_0^{u_1} R\left[\frac{b\,\mathrm{nc}^2 u - a\sqrt{2}\,\mathrm{tn}\,u\,\mathrm{dc}\,u}{\sqrt{b^2-a^2}}\right] du,$$

where $R(\cosh\vartheta)$ is a rational function of $\cosh\vartheta$.

Table of Integrals of Jacobian Elliptic Functions.

In the foregoing sections, where elliptic integrals having diverse algebraic, trigonometric, and hyperbolic integrands were reduced to those involving elliptic functions, it is seen that certain standard integrals constantly recur. The following tables of integrals[1] give explicit evaluation to those Jacobian normal forms, to which specific reference was made in each formula of Item Nos. **200—299**.

Recurrence Formulas for the Integrals of the Twelve Jacobian Elliptic Functions.

Integrals of odd powers of the twelve Jacobian functions, it is to be noted, are expressible solely in terms of Jacobian elliptic functions and more elementary functions, while the evaluation of integrals of even powers requires in addition the two functions $E(u)$ and u.

$$A_m = \int \frac{t^m\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{\sin^m\varphi\,d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int \operatorname{sn}^m u\,du$$
$$[t = \sin\varphi = \operatorname{sn} u].$$

310.00 $$A_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

310.01 $$A_1 = \int \operatorname{sn} u\,du = \frac{1}{k}\ln(\operatorname{dn} u - k\operatorname{cn} u) = -\frac{1}{k}\cosh^{-1}[(\operatorname{dn} u)/k'].$$

310.02 $$A_2 = \int \operatorname{sn}^2 u\,du = \frac{1}{k^2}[u - E(u)], \qquad [E(u) = E(\varphi, k)].$$

310.03 $$A_3 = \int \operatorname{sn}^3 u\,du = \frac{1}{2k^3}[k\operatorname{cn} u\operatorname{dn} u + (1+k^2)\ln(\operatorname{dn} u - k\operatorname{cn} u)].$$

[1] With all the following indefinite integrals, the constant of integration is to be understood. Moreover, for brevity we write $E(u)$ for $E(\operatorname{am} u, k)$.

310.04 $$\begin{cases} A_4 = \int \operatorname{sn}^4 u\, du = \frac{1}{3k^4}\,[(2+k^2)\,u - 2(1+k^2)\,E(u) + \\ \qquad\qquad + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u]. \end{cases}$$

310.05 $$A_{2m+2} = \frac{\operatorname{sn}^{2m-1} u \operatorname{cn} u \operatorname{dn} u + 2m(1+k^2)\,A_{2m} + (1-2m)\,A_{2m-2}}{(2m+1)\,k^2}.$$

310.06 $$A_{2m+3} = \frac{\operatorname{sn}^{2m} u \operatorname{cn} u \operatorname{dn} u + (2m+1)\,(1+k^2)\,A_{2m+1} - 2m\,A_{2m-1}}{2(m+1)\,k^2}.$$

$$\boxed{\begin{gathered} B_m = -\int \frac{t^m\,dt}{\sqrt{(t^2-1)\,(t^2-k^2)}} = \int \frac{d\varphi}{\sin^m\varphi \sqrt{1-k^2\sin^2\varphi}} = \int \operatorname{ns}^m u\, du, \\ [t = 1/\sin\varphi = \operatorname{ns} u]. \end{gathered}}$$

311.00 $$B_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

311.01 $$B_1 = \int \operatorname{ns} u\, du = \int \frac{du}{\operatorname{sn} u} = \ln\left[\frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}\right].$$

311.02 $$B_2 = \int \operatorname{ns}^2 u\, du = u - E(u) - \operatorname{dn} u \operatorname{cs} u, \qquad [E(u) = E(\varphi, k)].$$

311.03 $$B_3 = \int \operatorname{ns}^3 u\, du = \frac{1}{2}\left[(1+k^2)\ln\left(\frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}\right) - \operatorname{cs} u \operatorname{dn} u \operatorname{ns} u\right].$$

311.04 $$\begin{cases} B_4 = \int \operatorname{ns}^4 u\, du \\ = \frac{1}{3}\,[(2+k^2)\,u - 2(1+k^2)\,E(u) - \operatorname{dn} u \operatorname{cs} u\,(\operatorname{ns}^2 u + 2 + 2k^2)]. \end{cases}$$

311.05 $$B_{2m+2} = \frac{2m(1+k^2)\,B_{2m} + (1-2m)\,k^2\,B_{2m-2} - \operatorname{cn} u \operatorname{dn} u \operatorname{ns}^{2m+1} u}{2m+1}.$$

311.06 $$B_{2m+3} = \frac{(2m+1)\,(1+k^2)\,B_{2m+1} - 2m\,k^2\,B_{2m-1} - \operatorname{cn} u \operatorname{dn} u \operatorname{ns}^{2m+2} u}{2(m+1)}.$$

$$\boxed{\begin{gathered} C_m = -\int \frac{t^m\,dt}{\sqrt{(1-t^2)\,(k'^2+k^2t^2)}} = \int \frac{\cos^m\varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int \operatorname{cn}^m u\, du, \\ [t = \cos\varphi = \operatorname{cn} u]. \end{gathered}}$$

312.00 $$C_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

312.01 $$C_1 = \int \operatorname{cn} u\, du = [\cos^{-1}(\operatorname{dn} u)]/k = [\sin^{-1}(k \operatorname{sn} u)]/k.$$

312.02 $C_2 = \int \mathrm{cn}^2 u\, du = \frac{1}{k^2}\left[E(u) - k'^2 u\right], \qquad [E(u) = E(\varphi, k)].$

312.03 $C_3 = \int \mathrm{cn}^3 u\, du = \frac{1}{2k^3}\left[(2k^2 - 1)\sin^{-1}(k\,\mathrm{sn}\,u) + k\,\mathrm{sn}\,u\,\mathrm{dn}\,u\right].$

312.04 $$\begin{cases} C_4 = \int \mathrm{cn}^4 u\, du \\ \quad = \frac{1}{3k^4}\left[(2 - 3k^2)k'^2 u + 2(2k^2 - 1)E(u) + k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\right]. \end{cases}$$

312.05 $C_{2m+2} = \frac{2m(2k^2 - 1)C_{2m} + (2m-1)k'^2 C_{2m-2} + \mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{cn}^{2m-1}u}{(2m+1)k^2}.$

312.06 $C_{2m+3} = \frac{(2m+1)(2k^2 - 1)C_{2m+1} + 2m\,k'^2 C_{2m-1} + \mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{cn}^{2m}u}{2(m+1)k^2}.$

$$D_m = \int \frac{t^m\, dt}{\sqrt{(t^2-1)(k^2 + k'^2 t^2)}} = \int \frac{d\varphi}{\cos^m\varphi\sqrt{1 - k^2\sin^2\varphi}} = \int \mathrm{nc}^m u\, du,$$
$$[t = 1/\cos\varphi = \mathrm{nc}\,u].$$

313.00 $D_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \mathrm{am}\,u].$

313.01 $D_1 = \int \mathrm{nc}\,u\, du = \int \frac{du}{\mathrm{cn}\,u} = \frac{1}{k'}\ln\left[\frac{k'\,\mathrm{sn}\,u + \mathrm{dn}\,u}{\mathrm{cn}\,u}\right].$

313.02 $D_2 = \int \mathrm{nc}^2 u\, du = \frac{1}{k'^2}\left[k'^2 u - E(u) + \mathrm{dn}\,u\,\mathrm{tn}\,u\right],$
$[E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\,u].$

313.03 $D_3 = \int \mathrm{nc}^3 u\, du = \frac{1}{2k'^3}\left[k'\,\mathrm{tn}\,u\,\mathrm{dc}\,u + (1 - 2k^2)\ln\left(\frac{k'\,\mathrm{sn}\,u + \mathrm{dn}\,u}{\mathrm{cn}\,u}\right)\right].$

313.04 $$\begin{cases} D_4 = \int \mathrm{nc}^4 u\, du = \frac{1}{3k'^4}\left[k'^2(2k'^2 - k^2)u + 2(2k^2 - 1)E(u) + \right. \\ \qquad \left. + (2 - 4k^2 + k'^2\,\mathrm{nc}^2 u)\,\mathrm{tn}\,u\,\mathrm{dn}\,u\right]. \end{cases}$$

313.05 $D_{2m+2} = \frac{(2m-1)k^2 D_{2m-2} + 2m(1 - 2k^2)D_{2m} + \mathrm{tn}\,u\,\mathrm{dn}\,u\,\mathrm{nc}^{2m}u}{(2m+1)k'^2}.$

313.06 $D_{2m+3} = \frac{2m\,k^2 D_{2m-1} + (2m+1)(1 - 2k^2)D_{2m+1} + \mathrm{tn}\,u\,\mathrm{dn}\,u\,\mathrm{nc}^{2m+1}u}{2(m+1)k'^2}.$

$$G_m = -\int \frac{t^m\,dt}{\sqrt{(1-t^2)(t^2-k'^2)}} = \int (1-k^2\sin^2\varphi)^{(m-1)/2}\,d\varphi = \int \mathrm{dn}^m u\,du,$$
$$\left[t = \sqrt{1-k^2\sin^2\varphi} = \mathrm{dn}\,u\right].$$

314.00 $G_0 = \int du = u = F(\varphi, k)$, $\qquad [\varphi = \mathrm{am}\,u]$.

314.01 $G_1 = \int \mathrm{dn}\,u\,du = \mathrm{am}\,u = \sin^{-1}(\mathrm{sn}\,u)$.

314.02 $G_2 = \int \mathrm{dn}^2 u\,du = E(u) = E(\varphi, k)$, $\quad [E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\,u]$.

314.03 $G_3 = \int \mathrm{dn}^3 u\,du = \frac{1}{2}\,[(1+k'^2)\,\mathrm{am}\,u + k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u]$.

314.04 $G_4 = \int \mathrm{dn}^4 u\,du = \frac{1}{3}\,[k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u - k'^2 u + 2(1+k'^2)\,E(u)]$.

314.05 $$G_{2m+2} = \frac{k^2\,\mathrm{dn}^{2m-1}u\,\mathrm{sn}\,u\,\mathrm{cn}\,u + (1-2m)\,k'^2 G_{2m-2} + 2m(2-k^2)\,G_{2m}}{2m+1}.$$

314.06 $$G_{2m+3} = \frac{k^2\,\mathrm{dn}^{2m}u\,\mathrm{sn}\,u\,\mathrm{cn}\,u - 2m\,k'^2 G_{2m-1} + (2m+1)(2-k^2)\,G_{2m+1}}{2(m+1)}.$$

$$I_m = \int \frac{t^m\,dt}{\sqrt{(t^2-1)(1-k'^2t^2)}} = \int \frac{d\varphi}{(1-k^2\sin^2\varphi)^{(m+1)/2}} = \int \mathrm{nd}^m u\,du,$$
$$\left[t = 1/\sqrt{1-k^2\sin^2\varphi} = \mathrm{nd}\,u\right].$$

315.00 $I_0 = \int du = u = F(\varphi, k)$, $\qquad [\varphi = \mathrm{am}\,u]$.

315.01 $$I_1 = \int \mathrm{nd}\,u\,du = \int \frac{du}{\mathrm{dn}\,u} = \frac{1}{k'}\tan^{-1}\left[\frac{k'\,\mathrm{sn}\,u - \mathrm{cn}\,u}{k'\,\mathrm{sn}\,u + \mathrm{cn}\,u}\right].$$

315.02 $I_2 = \int \mathrm{nd}^2 u\,du = \frac{1}{k'^2}\,[E(u) - k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u]$,
$\qquad [E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\,u]$.

315.03 $$\begin{cases} I_3 = \int \mathrm{nd}^3 u\,du = \frac{1}{2k'^3}\left[(2-k^2)\tan^{-1}\left(\frac{k'\,\mathrm{sn}\,u - \mathrm{cn}\,u}{k'\,\mathrm{sn}\,u + \mathrm{cn}\,u}\right) - \right. \\ \qquad\qquad \left. - k'\,k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{nd}^2 u\right]. \end{cases}$$

315.04 $\begin{cases} I_4 = \int \mathrm{nd}^4 u\, du = \frac{1}{3k'^4} [2(2-k^2)E(u) - \\ \qquad - k'^2 u - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u\, (k'^2 \mathrm{nd}^2 u + 4 - 2k^2)]. \end{cases}$

315.05 $I_{2m+2} = \frac{2m(2-k^2)I_{2m} + (1-2m)I_{2m-2} - k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{nd}^{2m+1} u}{(2m+1)k'^2}.$

315.06 $I_{2m+3} = \frac{(2m+1)(2-k^2)I_{2m+1} - 2m I_{2m-1} - k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{nd}^{2m+2} u}{2(m+1)k'^2}.$

$$J_m = \int \frac{t^m\, dt}{\sqrt{(1+t^2)(1+k'^2 t^2)}} = \int \frac{\tan^m \varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int \mathrm{tn}^m u\, du,$$
$$[t = \tan\varphi = \mathrm{tn}\, u].$$

316.00 $J_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \mathrm{am}\, u].$

316.01 $J_1 = \int \mathrm{tn}\, u\, du = \int \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u} du = \int \mathrm{sc}\, u\, du = \frac{1}{2k'} \ln\left[\frac{\mathrm{dn}\, u + k'}{\mathrm{dn}\, u - k'}\right].$

316.02 $J_2 = \int \mathrm{tn}^2 u\, du = \frac{1}{k'^2}[\mathrm{dn}\, u\, \mathrm{tn}\, u - E(u)],$
$\qquad [E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\, u].$

316.03 $J_3 = \int \mathrm{tn}^3 u\, du = \frac{1}{4k'^3}\left[2k'\, \mathrm{dc}\, u\, \mathrm{nc}\, u - (1+k'^2)\ln\left(\frac{\mathrm{dn}\, u + k'}{\mathrm{dn}\, u - k'}\right)\right].$

316.04 $\begin{cases} J_4 = \int \mathrm{tn}^4 u\, du = \frac{1}{3k'^4}[2(1+k'^2)E(u) - \\ \qquad - k'^2 u + \mathrm{tn}\, u\, \mathrm{dn}\, u\, (k'^2 \mathrm{nc}^2 u - 2 - 2k'^2)]. \end{cases}$

316.05 $J_{2m+2} = \frac{\mathrm{tn}^{2m-1} u\, \mathrm{dc}\, u\, \mathrm{nc}\, u + (1-2m)J_{2m-2} - 2m(1+k'^2)J_{2m}}{(2m+1)k'^2}.$

316.06 $J_{2m+3} = \frac{\mathrm{tn}^{2m} u\, \mathrm{dc}\, u\, \mathrm{nc}\, u - 2m J_{2m-1} - (2m+1)(1+k'^2)J_{2m+1}}{2(m+1)k'^2}.$

$$L_m = \int \frac{-t^m\, dt}{\sqrt{(1+t^2)(t^2+k'^2)}} = \int \frac{\cot^m \varphi\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int \mathrm{cs}^m u\, du,$$
$$[t = \cot\varphi = \mathrm{cs}\, u].$$

317.00 $L_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \mathrm{am}\, u].$

317.01 $L_1 = \int \mathrm{cs}\, u\, du = \int \frac{\mathrm{cn}\, u}{\mathrm{sn}\, u} du = \ln\left[\frac{1 - \mathrm{dn}\, u}{\mathrm{sn}\, u}\right].$

317.02 $L_2 = \int \mathrm{cs}^2 u\, du = -\,\mathrm{dn}\, u\, \mathrm{cs}\, u - E(u)$,
$[E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\, u]$.

317.03 $L_3 = \int \mathrm{cs}^3 u\, du = -\frac{1}{2}\left[(1 + k'^2) \ln\left(\frac{1 - \mathrm{dn}\, u}{\mathrm{sn}\, u}\right) + \mathrm{dn}\, u\, \mathrm{ns}^2 u\right]$.

317.04 $$\begin{cases} L_4 = \int \mathrm{cs}^4 u\, du \\ \quad = \frac{1}{3}[2(1 + k'^2)\, E(u) - k'^2 u + \mathrm{dn}\, u\, \mathrm{cs}\, u\, (2 + 2k'^2 - \mathrm{ns}^2 u)]. \end{cases}$$

317.05 $L_{2m+2} = \dfrac{(1 - 2m)\, k'^2 L_{2m-2} - 2m(1 + k'^2)\, L_{2m} - \mathrm{dc}\, u\, \mathrm{nc}\, u\, \mathrm{cs}^{2m+1} u}{2m + 1}$.

317.06 $L_{2m+3} = \dfrac{-2m k'^2 L_{2m-1} - (2m + 1)(1 + k'^2)\, L_{2m+1} - \mathrm{dc}\, u\, \mathrm{nc}\, u\, \mathrm{cs}^{2m+2} u}{2(m + 1)}$.

$$M_m = \int \frac{t^m\, dt}{\sqrt{(1 - k'^2 t^2)(1 + k^2 t^2)}} = \int \frac{\sin^m \varphi\, d\varphi}{(1 - k^2 \sin^2 \varphi)^{(m+1)/2}} = \int \mathrm{sd}^m u\, du,$$
$$[t = (\sin \varphi)/\sqrt{1 - k^2 \sin^2 \varphi} = \mathrm{sd}\, u].$$

318.00 $M_0 = \int du = u = F(\varphi, k)$ $[\varphi = \mathrm{am}\, u]$.

318.01 $M_1 = \int \mathrm{sd}\, u\, du = \int \frac{\mathrm{sn}\, u}{\mathrm{dn}\, u}\, du = \frac{1}{k' k} \tan^{-1}\left[\frac{k k' (1 + \mathrm{cn}\, u)}{k^2 \mathrm{cn}\, u - k'^2}\right]$.

318.02 $M_2 = \int \mathrm{sd}^2 u\, du = \frac{1}{k^2 k'^2}[E(u) - k'^2 u - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u]$,
$[E(u) = E(\varphi, k),\ \varphi = \mathrm{am}\, u]$.

318.03 $M_3 = \int \mathrm{sd}^3 u\, du = \frac{1}{2k^2 k'^2}\left[\tan^{-1}\left(\frac{k k' (1 + \mathrm{cn}\, u)}{k^2 \mathrm{cn}\, u - k'^2}\right) - k k'\, \mathrm{cd}\, u\, \mathrm{nd}\, u\right]$.

318.04 $$\begin{cases} M_4 = \int \mathrm{sd}^4 u\, du = \frac{1}{3k^4 k'^4}[2(2k^2 - 1)\, E(u) + k'^2 (2 - 3k^2)\, u - \\ \qquad - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u\, (k'^2 \mathrm{nd}^2 u + 4k^2 - 2)]. \end{cases}$$

318.05 $M_{2m+2} = \dfrac{2m(2k^2 - 1)\, M_{2m} + (2m - 1)\, M_{2m-2} - \mathrm{cd}\, u\, \mathrm{nd}\, u\, \mathrm{sd}^{2m+1} u}{(2m + 1)\, k^2 k'^2}$.

318.06 $M_{2m+3} = \dfrac{(2m + 1)(2k^2 - 1)\, M_{2m+1} + 2m M_{2m} - \mathrm{cd}\, u\, \mathrm{nd}\, u\, \mathrm{sd}^{2m+2} u}{2(m + 1)\, k^2 k'^2}$.

$$N_m = \int \frac{-t^m\,dt}{\sqrt{(t^2-k'^2)(t^2+k^2)}} = \int \frac{(1-k^2\sin^2\varphi)^{m/2}}{\sin^m\varphi\sqrt{1-k^2\sin^2\varphi}}\,d\varphi = \int \operatorname{ds}^m u\,du,$$
$$\left[t = \sqrt{1-k^2\sin^2\varphi}/\sin\varphi = \operatorname{ds} u\right].$$

319.00 $N_0 = \int du = u = F(\varphi, k)$, $[\varphi = \operatorname{am} u]$.

319.01 $N_1 = \int \operatorname{ds} u\,du = \int \frac{\operatorname{dn} u}{\operatorname{sn} u}\,du = \ln\left[\frac{\operatorname{sn} u}{1+\operatorname{cn} u}\right]$.

319.02 $N_2 = \int \operatorname{ds}^2 u\,du = k'^2 u - \operatorname{dn} u \operatorname{cs} u - E(u)$,
$[E(u) = (E\varphi, k),\ \varphi = \operatorname{am} u]$.

319.03 $N_3 = \int \operatorname{ds}^3 u\,du = \frac{1}{2}\left[(1-2k^2)\ln\left(\frac{\operatorname{sn} u}{1+\operatorname{cn} u}\right) - \operatorname{cn} u \operatorname{ns}^2 u\right]$.

319.04 $N_4 = \int \operatorname{ds}^4 u\,du = \frac{1}{3}\left[(2-3k^2)k'^2 u + 2(2k^2-1)E(u) -\right.$
$\left.- \operatorname{dn} u \operatorname{cs} u (\operatorname{ns}^2 u + 2 - 4k^2)\right]$.

319.05 $N_{2m+2} = \frac{k^2 k'^2 (2m-1) N_{2m-2} + 2m(1-2k^2) N_{2m} - \operatorname{cn} u \operatorname{nd}^2 u \operatorname{ds}^{2m+1} u}{2m+1}$.

319.06 $N_{2m+3} = \frac{2m k^2 k'^2 N_{2m-1} + (1-2k^2)(1+2m) N_{2m+1} - \operatorname{cn} u \operatorname{nd}^2 u \operatorname{ds}^{2m+2} u}{2(m+1)}$.

$$O_m = -\int \frac{t^m\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{\cos^m\varphi\,d\varphi}{(1-k^2\sin^2\varphi)^{(m+1)/2}} = \int \operatorname{cd}^m u\,du,$$
$$\left[t = (\cos\varphi)/\sqrt{1-k^2\sin^2\varphi} = \operatorname{cd} u\right].$$

320.00 $O_0 = \int du = u = F(\varphi, k)$, $[\varphi = \operatorname{am} u]$.

320.01 $O_1 = \int \operatorname{cd} u\,du = \int \frac{\operatorname{cn} u}{\operatorname{dn} u}\,du = \frac{1}{k}\ln\left[\frac{1+k\operatorname{sn} u}{\operatorname{dn} u}\right]$.

320.02 $O_2 = \int \operatorname{cd}^2 u\,du = \frac{1}{k^2}\left[u - E(u) + k^2 \operatorname{sn} u \operatorname{cd} u\right]$,
$[E(u) = E(\varphi, k), \varphi = \operatorname{am} u]$.

320.03 $O_3 = \int \operatorname{cd}^3 u\,du = \frac{1}{2k^3}\left[(1+k^2)\ln\left(\frac{1+k\operatorname{sn} u}{\operatorname{dn} u}\right) - k k'^2 \operatorname{sn} u \operatorname{nd}^2 u\right]$.

320.04 $$O_4 = \int \mathrm{cd}^4 u\, du = \frac{1}{3k^4}\left[(2+k^2)\,u - 2(1+k^2)\,E(u) + \right.$$
$$\left. + k^2 \operatorname{sn} u \operatorname{cd} u\,(2 + 2k^2 - k'^2 \operatorname{nd}^2 u)\right].$$

320.05 $$O_{2m+2} = \frac{2m(1+k^2)\,O_{2m} + (1-2m)\,O_{2m-2} - k'^2 \operatorname{sn} u \operatorname{nd}^2 u \operatorname{cd}^{2m-1} u}{(2m+1)\,k^2}.$$

320.06 $$O_{2m+3} = \frac{(1+2m)(1+k^2)\,O_{2m+1} - 2m\,O_{2m-1} - k'^2 \operatorname{sn} u \operatorname{nd}^2 u \operatorname{cd}^{2m} u}{2(m+1)\,k^2}.$$

$$P_m = \int \frac{t^m\,dt}{\sqrt{(1-t^2)(k^2-t^2)}} = \int \frac{(1-k^2\sin^2\varphi)^{m/2}\,d\varphi}{\cos^m\varphi\sqrt{1-k^2\sin^2\varphi}} = \int \mathrm{dc}^m u\,du,$$
$$\left[t = \sqrt{1-k^2\sin^2\varphi}\,/\cos\varphi = \operatorname{dc} u\right].$$

321.00 $$P_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

321.01 $$P_1 = \int \operatorname{dc} u\,du = \int \frac{\operatorname{dn} u}{\operatorname{cn} u}\,du = \ln\left[\frac{1+\operatorname{sn} u}{\operatorname{cn} u}\right].$$

321.02 $$P_2 = \int \mathrm{dc}^2 u\,du = u - E(u) + \operatorname{dn} u \operatorname{tn} u, \qquad [E(u) = E(\varphi, k)].$$

321.03 $$P_3 = \int \mathrm{dc}^3 u\,du = \frac{1}{2}\left[(1+k^2)\ln\left(\frac{1+\operatorname{sn} u}{\operatorname{cn} u}\right) + k'^2 \operatorname{sn} u \operatorname{nc}^2 u\right].$$

321.04 $$\left\{\begin{aligned} P_4 = \int \mathrm{dc}^4 u\,du = \frac{1}{3}\,&[(2+k^2)\,u - 2(1+k^2)\,E(u) + \\ &+ (2 + 2k^2 + k'^2 \operatorname{nc}^2 u) \operatorname{dn} u \operatorname{tn} u].\end{aligned}\right.$$

321.05 $$P_{2m+2} = \frac{2m(1+k^2)\,P_{2m} + (1-2m)\,k^2 P_{2m-2} + k'^2 \operatorname{sn} u \operatorname{nd}^2 u \operatorname{dc}^{2m+1} u}{2m+1}.$$

321.06 $$P_{2m+3} = \frac{(2m+1)(1+k^2)\,P_{2m+1} - 2m\,k^2\,P_{2m-1} + k'^2 \operatorname{sn} u \operatorname{nd}^2 u \operatorname{dc}^{2m+2} u}{2(m+1)}.$$

Additional Recurrence Formulas.

$$\sigma_m = \int \frac{(a_1+t)^m\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{(a_1+\sin\varphi)^m\,d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int (a_1 + \operatorname{sn} u)^m\,du,$$
$$[t = \sin\varphi = \operatorname{sn} u].$$

330.00 $$\sigma_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

330.01 $$\sigma_1 = \int (a_1 + \operatorname{sn} u)\,du = \frac{1}{k}\left[a_1 k u + \ln(\operatorname{dn} u - k \operatorname{cn} u)\right].$$

330.02 $$\begin{cases}\sigma_2 = \int (a_1 + \operatorname{sn} u)^2\,du = \frac{1}{k^2}\,[(1 + k^2 a_1^2)\,u - E(u) + \\ \qquad\qquad + 2a_1 k \ln(\operatorname{dn} u - k \operatorname{cn} u)].\end{cases}$$

330.03 $$\begin{cases}\sigma_3 = \int (a_1 + \operatorname{sn} u)^3\,du = \frac{1}{2k^3}\,[2a_1 k(3 + a_1^2 k^2)\,u - 6a_1 k E(u) + \\ + k \operatorname{cn} u \operatorname{dn} u + (6a_1^2 k^2 + k^2)\ln(\operatorname{dn} u - k \operatorname{cn} u)],\ [E(u) = (E\varphi, k)].\end{cases}$$

330.04 $$\begin{cases}\sigma_{m+3} = \frac{1}{(m+2)\,k^2}\Big[2(2m+3)\,a_1 k^2 \sigma_{m+2} + (m+1)\,(1 + k^2 - \\ \qquad - 6a_1^2 k^2)\,\sigma_{m+1} + (2m+1)\,(2a_1^2 k^2 - 1 - k^2)\,a_1 \sigma_m + \\ \qquad + m(a_1^2 - 1)\,(1 - k^2 a_1^2)\,\sigma_{m-1} + \frac{\operatorname{cn} u \operatorname{dn} u}{(a_1 + \operatorname{sn} u)^m}\Big].\end{cases}$$

330.50 $$\sigma_{-1} = \int \frac{du}{a_1 + \operatorname{sn} u} = \frac{1}{a_1}\,[\Pi(\varphi, 1/a_1^2, k) - f/a_1], \qquad [\varphi = \operatorname{am} u].$$

where f is defined in **361.58**.

330.51 $$\begin{cases}\sigma_{-2} = \int \frac{du}{(a_1 + \operatorname{sn} u)^2} = \frac{1}{(a_1^2 - 1)\,(a_1^2 k^2 - 1)}\Big[(2k^2 a_1^2 - 1 - k^2)\,\times \\ \qquad \times\,\Pi(\varphi, 1/a_1^2, k) - E(\varphi, k) + (1 - k^2 a_1^2)\,F(\varphi, k) + \\ \qquad + \frac{(1 + k^2 - 2a_1^2 k^2)\,\sigma_{-1}}{a_1} - \frac{\operatorname{cn} u \operatorname{dn} u}{a_1(a_1 + \operatorname{sn} u)}\Big].\end{cases}$$

$$\boxed{\begin{gathered}\gamma_m = \int \frac{(1 - \alpha^2 t^2)^m\,dt}{\sqrt{(1 - t^2)\,(1 - k^2 t^2)}} = \int \frac{(1 - \alpha^2 \sin^2\varphi)^m}{\sqrt{1 - k^2 \sin^2\varphi}}\,d\varphi = \int (1 - \alpha^2 \operatorname{sn}^2 u)^m\,du, \\ [t = \sin\varphi = \operatorname{sn} u].\end{gathered}}$$

331.00 $$\gamma_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

331.01 $$\begin{cases}\gamma_1 = \int (1 - \alpha^2 \operatorname{sn}^2 u)\,du = \frac{1}{k^2}\,[(k^2 - \alpha^2)\,u + \alpha^2 E(u)], \\ \qquad [E(u) = E(\varphi, k), \quad \varphi = \operatorname{am} u].\end{cases}$$

331.02 $$\begin{cases}\gamma_2 = \int (1 - \alpha^2 \operatorname{sn}^2 u)^2\,du = \frac{1}{3k^4}\,[(3k^4 - 6\alpha^2 k^2 + 2\alpha^4 + k^2 \alpha^4)\,u + \\ \qquad + 2(3\alpha^2 k^2 - \alpha^4 - k^2 \alpha^4)\,E(u) + \alpha^4 k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u].\end{cases}$$

331.03 $$\begin{cases}\gamma_{m+3} = \frac{1}{(2m+5)\,k^2}\,[2(m+2)\,(3k^2 - \alpha^2 - k^2 \alpha^2)\,\gamma_{m+2} + \\ \qquad + (2m+3)\,(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,\gamma_{m+1} + 2(m+1)\,\times \\ \qquad \times\,(\alpha^2 - 1)\,(k^2 - \alpha^2)\,\gamma_m + \alpha^4 (1 - \alpha^2 \operatorname{sn}^2 u)^{m+1} \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u].\end{cases}$$

$$T_m=\int\frac{[1-\alpha_1^2(1-t^2)]^m}{\sqrt{(1-t^2)(1-k^2t^2)}}dt=\int\frac{(1-\alpha_1^2\cos^2\varphi)^m}{\sqrt{1-k^2\sin^2\varphi}}d\varphi=\int(1-\alpha_1^2\operatorname{cn}^2u)^m du,$$
$$[t=\sin\varphi=\operatorname{sn}u].$$

332.01 $T_m=\int(1-\alpha_1^2\operatorname{cn}^2u)^m du=(1-\alpha_1^2)^m\int(1-\alpha^2\operatorname{sn}^2u)^m du=(1-\alpha_1^2)^m\gamma_m$, where $\alpha^2=\alpha_1^2/(\alpha_1^2-1)$, and γ_m is given above.

$$\tau_m=\int\frac{[1-\alpha_1^2(1-k^2t^2)]^m}{\sqrt{(1-t^2)(1-k^2t^2)}}dt=\int\frac{[1-\alpha_1^2(1-k^2\sin^2\varphi)]^m}{\sqrt{1-k^2\sin^2\varphi}}d\varphi$$
$$=\int(1-\alpha_1^2\operatorname{dn}^2u)^m du,\qquad [t=\sin\varphi=\operatorname{sn}u].$$

332.51 $\tau_m=\int(1-\alpha_1^2\operatorname{dn}^2u)^m du=(1-\alpha_1^2)\int(1-\alpha^2\operatorname{sn}^2u)^m du=(1-\alpha_1^2)^m\gamma_m$, where $\alpha^2=\alpha_1^2k^2/(\alpha_1^2-1)$, and γ_m is given above.

$$U_m=\int\frac{[1-(1+\alpha^2)t^2]^m dt}{(1-t^2)^m\sqrt{(1-t^2)(1-k^2t^2)}}=\int\frac{(1-\alpha^2\tan^2\varphi)^m}{\sqrt{1-k^2\sin^2\varphi}}d\varphi$$
$$=\int(1-\alpha^2\operatorname{tn}^2u)^m du,\qquad [t=\sin\varphi=\operatorname{sn}u].$$

334.00 $U_0=\int du=u=F(\varphi,k),\qquad [\varphi=\operatorname{am}u].$

334.01 $U_1=\int(1-\alpha^2\operatorname{tn}^2u)\,du=\frac{1}{k'^2}[k'^2u+\alpha^2E(u)-\alpha^2\operatorname{dn}u\operatorname{tn}u],$
$[E(u)=E(\varphi,k)].$

334.02
$$\left\{\begin{aligned}U_2=\int(1-\alpha^2\operatorname{tn}^2u)^2du&=\frac{1}{3k'^4}[k'^2(3k'^2-\alpha^4)u+\\&+2\alpha^2(3k'^2+\alpha^2+k'^2\alpha^2)E(u)-\\&-\alpha^2(6k'^2+2\alpha^2+2k'^2\alpha^2-k'^2\alpha^2\operatorname{nc}^2u)\operatorname{dn}u\operatorname{tn}u].\end{aligned}\right.$$

334.03
$$\left\{\begin{aligned}U_{m+3}=\frac{1}{(2m+5)k'^2}&[-2(m+2)(\alpha^2+k'^2\alpha^2+3k'^2)U_{m+2}+\\&+2(m+1)(1+\alpha^2)(\alpha^2+k'^2)U_m+\\&+(2m+3)(\alpha^4+2\alpha^2+2\alpha^2k'^2+3k'^2)U_{m+1}+\\&+\alpha^4\operatorname{tn}u\operatorname{dn}u\operatorname{nc}^2u(1-\alpha^2\operatorname{tn}^2u)^{m+1}].\end{aligned}\right.$$

$$V_m = \int \frac{dt}{(1-\alpha^2 t^2)^m \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{d\varphi}{(1-\alpha^2 \sin^2\varphi)^m \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^m} \qquad [t = \sin\varphi = \operatorname{sn} u].$$

336.00 $V_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$

336.01 $V_1 = \int \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = \Pi(\varphi, \alpha^2, k).$ [See **400**.]

336.02
$$\begin{cases} V_2 = \int \frac{du}{(1-\alpha^2\operatorname{sn}^2 u)^2} = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\Big[\alpha^2 E(u) + (k^2-\alpha^2)u + \\ \quad + (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,\Pi(\varphi,\alpha^2,k) - \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-\alpha^2\operatorname{sn}^2 u}\Big]. \end{cases}$$

336.03
$$\begin{cases} V_{m+3} = \frac{1}{2(m+2)(1-\alpha^2)(k^2-\alpha^2)}\Big[(2m+1)k^2 V_m + \\ + 2(m+1)(\alpha^2 k^2 + \alpha^2 - 3k^2) V_{m+1} + \\ + (2m+3)(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2) V_{m+2} + \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{(1-\alpha^2\operatorname{sn}^2 u)^{m+2}}\Big]. \end{cases}$$

$$W_m = \int \frac{t^{2m}\,dt}{(1-\alpha^2 t^2)^m \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{\sin^{2m}\varphi\, d\varphi}{(1-\alpha^2 \sin^2\varphi)^m \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{sn}^{2m} u\, du}{(1-\alpha^2 \operatorname{sn}^2 u)^m}, \qquad [t = \sin\varphi = \operatorname{sn} u].$$

337.00 $W_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$

337.01 $W_1 = \int \frac{\operatorname{sn}^2 u\, du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{1}{\alpha^2}[\Pi(\varphi,\alpha^2,k) - F(\varphi,k)],$ [See **400**].

337.02 $W_2 = \int \frac{\operatorname{sn}^4 u\, du}{(1-\alpha^2\operatorname{sn}^2 u)^2} = \frac{1}{\alpha^4}[u - 2V_1 + V_2].$

337.03 $W_3 = \int \frac{\operatorname{sn}^6 u\, du}{(1-\alpha^2\operatorname{sn}^2 u)^3} = \frac{1}{\alpha^6}[V_3 - 3V_2 + 3V_1 - u].$

337.04 $W_m = \int \frac{\operatorname{sn}^{2m} u\, du}{(1-\alpha^2\operatorname{sn}^2 u)^m} = \frac{1}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(-1)^{m+j}\, m!}{(m-j)!\, j!} V_j,$

where V_j is given by **336**.

$$\int \frac{t^{2m}\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{\sin^{2m}\varphi\,d\varphi}{(1-\alpha^2\sin^2\varphi)^n\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{sn}^{2m}u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^n}, \qquad [n>m;\ t=\sin\varphi=\operatorname{sn}u].$$

337.51 $$\int \frac{\operatorname{sn}^{2m}u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^n} = \frac{1}{\alpha^{2m}}\sum_{j=0}^{m}\frac{(-1)^{m+j}\,m!}{(m-j)!\,j!}V_{j+n-m},$$
[$n>m$; see **336**].

$$\int \frac{t^{2m}\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{\sin^{2m}\varphi\,d\varphi}{(1-\alpha^2\sin^2\varphi)^n\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{sn}^{2m}u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^n}, \qquad [n<m;\ t=\sin\varphi=\operatorname{sn}u].$$

337.75 $$\left\{\begin{aligned}\int \frac{\operatorname{sn}^{2m}u\,du}{(1-\alpha^2\operatorname{sn}^2 u)^n} &= \frac{1}{\alpha^{2m}}\sum_{j=0}^{m-n}\frac{(-1)^{m+j}\,m!}{(m-j)!\,j!}\gamma_{m-n-j} + \\ &+ \frac{1}{\alpha^{2m}}\sum_{j=m-n+1}^{m}\frac{(-1)^{m+j}\,m!}{(m-j)!\,j!}V_{n-m+j},\end{aligned}\right.$$
[$n<m$; see **331** and **336**].

$$X_m = \int \frac{(1-t^2)^m\,dt}{(1-\alpha^2t^2)^m\sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{\cos^{2m}\varphi\,d\varphi}{(1-\alpha^2\sin^2\varphi)^m\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{cn}^{2m}u\,du}{(1-\alpha^2\operatorname{sn}^2u)^m}, \qquad [t=\sin\varphi=\operatorname{sn}u].$$

338.00 $X_0 = \int du = u = F(\varphi, k)$, $\qquad [\varphi = \operatorname{am} u]$.

338.01 $X_1 = \int \frac{\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{1}{\alpha^2}\left[u + (\alpha^2-1)\,\Pi(\varphi,\alpha^2,k)\right].$ [See **400**.]

338.02 $X_2 = \int \frac{\operatorname{cn}^4 u\,du}{(1-\alpha^2\operatorname{sn}^2u)^2} = \frac{1}{\alpha^4}\left[u + 2(\alpha^2-1)V_1 + (\alpha^2-1)^2V_2\right].$

338.03 $$\left\{\begin{aligned}X_3 = \int \frac{\operatorname{cn}^6 u\,du}{(1-\alpha^2\operatorname{sn}^2u)^3} &= \frac{1}{\alpha^6}\left[(\alpha^2-1)^3V_3 + 3(\alpha^2-1)^2V_2 + \right.\\ &\left.+ 3(\alpha^2-1)V_1 + u\right].\end{aligned}\right.$$

338.04 $X_m = \int \frac{\operatorname{cn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^m} = \frac{1}{\alpha^{2m}} \sum_{j=0}^{m} (\alpha^2-1)^j \frac{m!}{j!\,(m-j)!} V_j,$

where V_j is given in **336**.

$$\int \frac{(1-t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{\cos^{2m}\varphi\,d\varphi}{(1-\alpha^2 \sin^2\varphi)^n \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{cn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^n}, \qquad [n > m; \quad t = \sin\varphi = \operatorname{sn} u].$$

338.51 $\int \frac{\operatorname{cn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^n} = \frac{1}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2-1)^j\,m!}{(m-j)!\,j!} V_{n-m+j}, \quad [n > m;$ see **336**.]

$$\int \frac{(1-t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{\cos^{2m}\varphi\,d\varphi}{(1-\alpha^2 \sin^2\varphi)^n \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{cn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^n}, \qquad [n < m; \quad t = \sin\varphi = \operatorname{sn} \mathrm{u}].$$

338.75
$$\begin{cases} \int \frac{\operatorname{cn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^n} = \frac{1}{\alpha^{2m}} \sum_{j=0}^{m-n} \frac{(\alpha^2-1)^j\,m!}{(m-j)!\,j!} \gamma_{m-n-j} + \\ \qquad + \frac{1}{\alpha^{2m}} \sum_{j=m-n+1}^{m} \frac{(\alpha^2-1)\ m!}{(m-j)!\,j!} V_{n-m+j}. \quad [n < m; \text{ see } \mathbf{336}.] \end{cases}$$

$$S_m = \int \frac{(1-k^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^m \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{(1-k^2\sin^2\varphi)^m\,d\varphi}{(1-\alpha^2 \sin^2\varphi)^m \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{dn}^{2m} u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^m}, \qquad [t = \sin\varphi = \operatorname{sn} u].$$

339.00 $S_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$

339.01 $S_1 = \int \frac{\operatorname{dn}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{\alpha^2}[k^2 u + (\alpha^2-k^2)\,\Pi(\varphi,\alpha^2,k)].$ [See **400**.]

339.02
$$\begin{cases} S_2 = \int \frac{\operatorname{dn}^4 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = \frac{1}{\alpha^4}[k^4 u + 2k^2(\alpha^2-k^2)\,\Pi(\varphi,\alpha^2,k) + \\ \qquad + (\alpha^2-k^2)^2 V_2]. \end{cases}$$

339.03 $$\begin{cases} S_3 = \int \frac{\mathrm{dn}^6 u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^3} = \frac{1}{\alpha^6}\,[k^6 u + 3k^4(\alpha^2-k^2)\,\Pi(\varphi,\alpha^2,k) + \\ \qquad + 3k^2(\alpha^2-k^2)^2 V_2 + (\alpha^2-k^2)^3 V_3]. \end{cases}$$

339.04 $$S_m = \int \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^m} = \frac{k^{2m}}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2-k^2)^j\, m!}{k^{2j}\, j!\,(m-j)!}\, V_j,$$

where V_j is given by **336**.

$$\int \frac{(1-k^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{(1-k^2\sin^2\varphi)^m\,d\varphi}{(1-\alpha^2\sin^2\varphi)^n\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^n}, \qquad [n > m;\ t = \sin\varphi = \mathrm{sn}\, u].$$

339.51 $$\int \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^n} = \frac{k^{2m}}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2-k^2)^j\, m!}{k^{2j}\,(m-j)!\,j!}\, V_{n-m+j},$$
[$n > m$; see **336**].

$$\int \frac{(1-k^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{(1-k^2\sin^2\varphi)^m\,d\varphi}{(1-\alpha^2\sin^2\varphi)^n\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^n}, \qquad [n < m;\ t = \sin\varphi = \mathrm{sn}\, u].$$

339.75 $$\begin{cases} \int \frac{\mathrm{dn}^{2m} u\,du}{(1-\alpha^2 \mathrm{sn}^2 u)^n} = \frac{k^{2m}}{\alpha^{2m}} \sum_{j=0}^{m-n} \frac{(\alpha^2-k^2)^j\, m!}{k^{2j}\,(m-j)!\,j!}\, \gamma_{m-n-j} + \\ \qquad + \frac{k^{2m}}{\alpha^{2m}} \sum_{j=m-n+1}^{m} \frac{(\alpha^2-k^2)^j\, m!}{k^{2m}\,(m-j)!\,j!}\, V_{n-m+j}, \end{cases}$$
[$n < m$; see **331** and **336**].

$$Z_m = \int \frac{(1-\alpha_1^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^m \sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{(1-\alpha_1^2\sin^2\varphi)^m\,d\varphi}{(1-\alpha^2\sin^2\varphi)^m\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{(1-\alpha_1^2 \mathrm{sn}^2 u)^m}{(1-\alpha^2 \mathrm{sn}^2 u)^m}\,du, \qquad [t = \sin\varphi = \mathrm{sn}\, u].$$

340.00 $$Z_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \mathrm{am}\, u].$$

340.01 $$Z_1 = \int \frac{1-\alpha_1^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{\alpha^2}\left[(\alpha^2-\alpha_1^2)\,\Pi(\varphi,\alpha^2,k)+\alpha_1^2 u\right].$$
[See **400**.]

340.02 $$Z_2 = \int \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^2}{(1-\alpha^2 \operatorname{sn}^2 u)^2}\,du = \frac{1}{\alpha^4}\left[\alpha_1^4 u + 2\alpha_1^2(\alpha^2-\alpha_1^2)V_1 + (\alpha^2-\alpha_1^2)^2 V_2\right].$$
[See **336**.]

340.03 $$\begin{cases} Z_3 = \displaystyle\int \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^3}{(1-\alpha^2 \operatorname{sn}^2 u)^3}\,du = \frac{1}{\alpha^6}\left[\alpha_1^6 u + 3\alpha_1^4(\alpha^2-\alpha_1^2)V_1 + \right. \\ \qquad\qquad \left. + 3\alpha_1^2(\alpha^2-\alpha_1^2)^2 V_2 + (\alpha^2-\alpha_1^2)^3 V_3\right]. \end{cases}$$

340.04 $$Z_m = \int \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^m}{(1-\alpha^2 \operatorname{sn}^2 u)^m}\,du = \frac{\alpha_1^{2m}}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2-\alpha_1^2)^j\, m!}{\alpha_1^{2j} j!\,(m-j)!} V_j,$$

where V_j is given in **336**. For the special case when $\alpha^2 = 1$, V_j becomes D_{2j} (see **313**); and when $\alpha^2 = k^2$, V_j reduces to I_{2j} (see **315**).

340.51 $$\begin{cases} \displaystyle\int \frac{(1-\alpha_1^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{(1-\alpha_1^2 \sin^2\varphi)^m\,d\varphi}{(1-\alpha^2 \sin^2\varphi)^n \sqrt{1-k^2 \sin^2\varphi}} \\ \qquad = \displaystyle\int \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^m}{(1-\alpha^2 \operatorname{sn}^2 u)^n}\,du, \qquad (n > m) \\ = \displaystyle\frac{\alpha_1^{2m}}{\alpha^{2m}} \sum_{j=0}^{m} \frac{(\alpha^2-\alpha_1^2)^j\, m!}{\alpha_1^{2j} j!\,(m-j)!} V_{n-m+j}, \qquad \text{[see } \mathbf{336}\text{]}. \end{cases}$$

340.75 $$\begin{cases} \displaystyle\int \frac{(1-\alpha_1^2 t^2)^m\,dt}{(1-\alpha^2 t^2)^n \sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{(1-\alpha_1^2 \sin^2\varphi)^m\,d\varphi}{(1-\alpha^2 \sin^2\varphi)^n \sqrt{1-k \sin^2\varphi}} \\ \qquad = \displaystyle\int \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^m}{(1-\alpha^2 \operatorname{sn}^2 u)^n}\,du, \qquad (n < m) \\ = \displaystyle\frac{\alpha_1^{2m}}{\alpha^{2m}} \sum_{j=0}^{m-n} \frac{(\alpha^2-\alpha_1^2)^j\, m!}{\alpha_1^{2j} j!\,(m-j)!} \gamma_{m-n-j} + \frac{\alpha_1^{2m}}{\alpha^{2m}} \sum_{j=m-n+1}^{m} \frac{(\alpha^2-\alpha_1^2)^j\, m!}{\alpha_1^{2j} j!\,(m-j)!} V_{n-m+j}. \\ \qquad\qquad \text{[See } \mathbf{331, 336}\text{.]} \end{cases}$$

$$\boxed{\begin{aligned} R_m &= \int \frac{-\,dt}{(1+\alpha t)^m \sqrt{(1-t^2)(k'^2+k^2 t^2)}} = \int \frac{d\varphi}{(1+\alpha\cos\varphi)^m \sqrt{1-k^2\sin^2\varphi}} \\ &= \int \frac{du}{(1+\alpha \operatorname{cn} u)^m}, \qquad [t = \cos\varphi = \operatorname{cn} u] \quad \alpha^2 \neq 1. \end{aligned}}$$

341.00 $$R_{-1} = \int (1+\alpha \operatorname{cn} u)\,du = u + \frac{\alpha}{k}\left[\cos^{-1}(\operatorname{dn} u)\right], \qquad [\varphi = \operatorname{am} u].$$

341.01 $$\begin{cases} R_{-2} = \int (1+\alpha\,\mathrm{cn}\,u)^2\,du = \frac{1}{k^2}\left[(k^2-\alpha^2 k'^2)\,u + \alpha^2 E(u) + \right. \\ \left. \qquad + 2\alpha k \cos^{-1}(\mathrm{dn}\,u)\right], \ [u = F(\varphi,k),\ E(u) = E(\varphi,k)]. \end{cases}$$

341.02 $$R_0 = \int du = u = F(\varphi,k).$$

341.03 $$R_1 = \int \frac{du}{1+\alpha\,\mathrm{cn}\,u} = \frac{1}{1-\alpha^2}\left[\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2-1}, k\right) - \alpha f_1\right], \quad \alpha^2 \neq 1.$$

[See **400**.]

where f_1 is defined in **361.54**.

341.04 $$\begin{cases} R_2 = \int \frac{du}{(1+\alpha\,\mathrm{cn}\,u)^2} = \frac{1}{(\alpha^2-1)(k^2+\alpha^2 k'^2)} \times \\ \qquad \times \left\{[\alpha^2(2k^2-1)-2k^2]R_1 + 2k^2 R_{-1} - k^2 R_{-2} + \frac{\alpha^2\,\mathrm{sn}\,u\,\mathrm{dn}\,u}{1+\alpha\,\mathrm{cn}\,u}\right\}. \end{cases}$$

341.05 $$\begin{cases} R_m = \frac{1}{(m-1)(\alpha^2-1)(k^2+\alpha^2 k'^2)}\left\{(3-2m)\,[\alpha^2(1-2k^2)+2k^2]\,R_{m-1} + \right. \\ \qquad + 2(5-2m)\,k^2 R_{m-3} + (m-2)\,(6k^2+\alpha^2-2k^2\alpha^2)\,R_{m-2} + \\ \qquad \left. + (m-3)\,k^2 R_{m-4} + \frac{\alpha^3\,\mathrm{sn}\,u\,\mathrm{dn}\,u}{(1+\alpha\,\mathrm{cn}\,u)^{m-1}}\right\}; \quad m \neq 1. \end{cases}$$

$$\varrho_m = \int \frac{-dt}{(1\pm t)^m\sqrt{(1-t^2)(k'^2+k^2t^2)}} = \int \frac{d\varphi}{(1\pm\cos\varphi)^m\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{du}{(1\pm\mathrm{cn}\,u)^m}, \qquad [t = \cos\varphi = \mathrm{cn}\,u].$$

341.51 $$\begin{cases} \varrho_{-1} = \int (1\pm\mathrm{cn}\,u)\,du = u \pm \frac{1}{k}\left[\cos^{-1}(\mathrm{dn}\,u)\right] \\ \qquad = F(\varphi,k) \pm \frac{1}{k}\left[\cos^{-1}\left(\sqrt{1-k^2\sin^2\varphi}\right)\right]. \end{cases}$$

341.52 $$\varrho_0 = \int du = u = F(\varphi,k).$$

341.53 $$\varrho_1 = \int \frac{du}{1\pm\mathrm{cn}\,u} = u - E(u) \pm \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{1\pm\mathrm{cn}\,u}.$$

341.54 $$\varrho_2 = \int \frac{du}{(1\pm\mathrm{cn}\,u)^2} = \frac{1}{3}\left[(4k^2+1)\,\varrho_1 - 2k^2 u \pm \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{(1\pm\mathrm{cn}\,u)^2}\right].$$

341.55 $$\begin{cases} \varrho_{m+1} = \frac{1}{2m+1}\left\{(4k^2+1)\,m\,\varrho_m - 2(2m-1)\,k^2\varrho_{m-1} + \right. \\ \qquad \left. + (m-1)\,k^2\varrho_{m-2} \pm \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{(1\pm\mathrm{cn}\,u)^{m+1}}\right\}. \end{cases}$$

$$\eta_m = \int \frac{dt}{(1+\alpha t)^m \sqrt{(1+t^2)(1+k'^2 t^2)}} = \int \frac{d\varphi}{(1+\alpha \tan\varphi)^m \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{du}{(1+\alpha \operatorname{tn} u)^m}, \qquad [t = \tan\varphi = \operatorname{tn} u].$$

342.00 $$\eta_{-1} = \int (1+\alpha \operatorname{tn} u)\, du = u + \frac{\alpha}{2k'} \ln\left[\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\right].$$

342.01 $$\left\{\begin{aligned} \eta_{-2} = \int (1+\alpha \operatorname{tn} u)^2 du = \frac{1}{k'^2}\Big[k'^2 u - \alpha^2 E(u) + \alpha^2 \operatorname{dn} u \operatorname{tn} u + \\ + \alpha k' \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\Big]. \end{aligned}\right.$$

342.02 $$\eta_0 = \int du = u = F(\varphi, k), \qquad [\varphi = \operatorname{am} u].$$

342.03 $$\eta_1 = \int \frac{du}{1+\alpha \operatorname{tn} u} = \frac{1}{1+\alpha^2}\left[u + \alpha^2 \Pi(\varphi, 1+\alpha^2, k) + \alpha(\alpha^2+1) f_2\right].$$

where f_2 is given in **361.64**. [See **400**.]

342.04 $$\left\{\begin{aligned} \eta_2 = \int \frac{du}{(1+\alpha \operatorname{tn} u)^2} = \frac{1}{(1+\alpha^2)(\alpha^2+k'^2)}\Big[(2k'^2 + 2\alpha^2 - \alpha^2 k^2)\eta_1 - \\ - 2k'^2 \eta_{-1} + k'^2 \eta_{-2} - \frac{\alpha^3 \operatorname{dc} u \operatorname{nc} u}{1+\alpha \operatorname{tn} u}\Big]. \end{aligned}\right.$$

342.05 $$\left\{\begin{aligned} \eta_m = \frac{1}{(m-1)(1+\alpha^2)(\alpha^2+k'^2)}\Big[(2m-3)(2k'^2 + 2\alpha^2 - \alpha^2 k^2)\eta_{m-1} + \\ + 2(2m-5)k'^2 \eta_{m-3} + (2-m)(6k'^2 - \alpha^2 k^2 + 2\alpha^2)\eta_{m-2} + \\ + (3-m)k'^2 \eta_{m-4} - \frac{\alpha^3 \operatorname{dc} u \operatorname{nc} u}{(1+\alpha \operatorname{tn} u)^{m-1}}\Big], \\ [m \neq 1,\ (1+\alpha^2)(\alpha^2+k'^2) \neq 0]. \end{aligned}\right.$$

342.50 $$\int \frac{du}{1 \pm i \operatorname{tn} u} = \frac{1}{k^2}\left[E(u) - k'^2 u \pm i \operatorname{dn} u\right].$$

342.75 $$\int \frac{du}{1 \pm i k' \operatorname{tn} u} = \frac{1}{k^2}\left[u - E(u) + k^2 \operatorname{sn} u \operatorname{cd} u \mp i k' \operatorname{nd} u\right].$$

$$\mu_m = \int \frac{t \sin^{-1} t\, dt}{(1-k^2 t^2)^m \sqrt{1-k^2 t^2}} = \int \frac{\varphi \sin\varphi \cos\varphi\, d\varphi}{(1-k^2\sin^2\varphi)^m \sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \frac{\operatorname{am} u \operatorname{sn} u \operatorname{cn} u\, du}{\operatorname{dn}^{2m} u}, \qquad [t = \sin\varphi = \operatorname{sn} u].$$

348.00 $$\mu_0 = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u\, du = \frac{1}{k^2}\left[E(u) - \operatorname{am} u \operatorname{dn} u\right],$$
$$[E(u) = E(\varphi, k),\ \varphi = \operatorname{am} u].$$

348.01 $\mu_1 = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u \operatorname{nd}^2 u\, du = \frac{1}{k^2}\left[\operatorname{am} u \operatorname{nd} u - u\right].$

348.02 $$\begin{cases} \mu_2 = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u \operatorname{nd}^4 u\, du \\ \quad = \frac{1}{3k^2 k'^2}\left[k'^2 \operatorname{nd}^3 u \operatorname{am} u - E(u) + k^2 \operatorname{sn} u \operatorname{cd} u\right]. \end{cases}$$

348.03 $$\begin{cases} \mu_m = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u \operatorname{nd}^{2m} u\, du \\ \quad = \frac{1}{(2m-1)k^2}\left[\operatorname{nd}^{2m-1} u \operatorname{am} u - I_{2m-2}\right], \end{cases}$$

where I_{2m-2} is given in **315**.

$$\nu_m = \int \frac{t(1-k^2t^2)^m \sin^{-1} t\, dt}{\sqrt{1-k^2t^2}} = \int \frac{\varphi \sin\varphi \cos\varphi (1-k^2\sin^2\varphi)^m\, d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \operatorname{am} u \operatorname{sn} u \operatorname{cn} u \operatorname{dn}^{2m} u\, du \qquad [t = \sin\varphi = \operatorname{sn} u].$$

348.50 $\nu_0 = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u\, du = \frac{1}{k^2}\left[E(u) - \operatorname{am} u \operatorname{dn} u\right],$
$[E(u) = E(\varphi, k), \varphi = \operatorname{am} u].$

348.51 $$\begin{cases} \nu_1 = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u \operatorname{dn}^2 u\, du \\ \quad = \frac{1}{9k^2}\left[2(1+k'^2)E(u) - k'^2 u + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u - 3 \operatorname{dn}^3 u \operatorname{am} u\right]. \end{cases}$$

348.52 $\nu_m = \int \operatorname{am} u \operatorname{cn} u \operatorname{sn} u \operatorname{dn}^{2m} u\, du = \frac{1}{(2m+1)k^2}\left[-\operatorname{dn}^{2m+1} u \operatorname{am} u + G_m\right],$

where G_m is given in **314**.

348.75 $$\begin{cases} \int \operatorname{am} u \operatorname{sn} u \operatorname{dn} u \operatorname{cn}^{2m} u\, du \\ = \frac{1}{2m+1}\left[-\operatorname{am} u \operatorname{cn}^{2m+1} u + \sum_{j=0}^{m} \frac{(-1)^j m!\, \operatorname{sn}^{2j+1} u}{(2j+1)(m-j)!\, j!}\right]. \end{cases}$$

348.85 $$\begin{cases} \int \operatorname{am} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^{2m} u\, du \\ = \frac{1}{2m+1}\left[\operatorname{am} u \operatorname{sn}^{2m+1} u + \operatorname{cn}^{2m+1} u \sum_{j=0}^{m} \frac{(-1)^m m!\, \operatorname{nc}^{2j} u}{(m-j)!\, j!\, (2m-2j+1)}\right]. \end{cases}$$

$$\lambda_m = \int \frac{t^{2m}\,dt}{(1-k^2 t^2)^{m+1}\sqrt{(1-t^2)(1-k^2 t^2)}} = \int \frac{\sin^{2m}\varphi\,d\varphi}{(1-k^2\sin^2\varphi)^{m+1}\sqrt{1-k^2\sin^2\varphi}}$$
$$= \int \mathrm{sd}^{2m} u\,\mathrm{nd}^2 u\,du, \qquad [t = \sin\varphi = \mathrm{sn}\,u].$$

350.00 $$\lambda_0 = \int \frac{du}{\mathrm{dn}^2 u} = \int \mathrm{nd}^2 u\,du = \frac{1}{k'^2}\left[E(u) - k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right],$$

[cf. **315.02**].

350.01 $$\begin{cases} \lambda_1 = \int \frac{\mathrm{sn}^2 u}{\mathrm{dn}^4 u}\,du = \int \mathrm{sd}^2 u\,\mathrm{nd}^2 u\,du = \frac{1}{3k^2 k'^4} \times \\ \times\left[(1+k^2)E(u) - k'^2 u - k^2(1+k^2+k'^2\mathrm{nd}^2 u)\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]. \end{cases}$$

350.02 $$\begin{cases} \lambda_2 = \int \mathrm{sd}^4 u\,\mathrm{nd}^2 u\,du = \frac{1}{15k^4 k'^6}\{(7k^2+3k^4-2)E(u) + \\ + 2k'^2(1-3k^2)u - k^2[3k^2k'^2 + 2(3k^2-1)\times \\ \times(1+k^2+k'^2\mathrm{nd}^2 u) + 3k'^4\mathrm{nd}^4 u]\,\mathrm{sn}\,u\,\mathrm{cd}\,u\}. \end{cases}$$

350.03 $$\begin{cases} \lambda_{m+2} = \frac{1}{(5+2m)k'^2k^2}\left[(2m+1)\lambda_m + \right. \\ \left. + 2(3k^2+2mk^2-m-1)\lambda_{m+1} - \mathrm{cn}\,u\,\mathrm{ns}^4 u\,\mathrm{sd}^{2m+5} u\right]. \end{cases}$$

351.01 $$\begin{cases} \int \frac{\mathrm{sn}^{2m} u\,\mathrm{cn}^{2m} u}{\mathrm{dn}^{2m} u}\,du = \int \mathrm{sn}^{2m} u\,\mathrm{cd}^{2m} u\,du = \int \mathrm{sd}^{2m} u\,\mathrm{cn}^{2m} u\,du \\ \quad = \int \mathrm{sn}^{2m} u\,\mathrm{cn}^{2m} u\,\mathrm{nd}^{2m} u\,du \\ \quad = \frac{1}{k^{4m}} \sum_{j=0}^{m} \sum_{\gamma=0}^{m} \frac{(-1)^{j+\gamma+m} k'^{2(m-\gamma)}\,m!\,m!}{(m-j)!\,j!\,(m-\gamma)!\,\gamma!} I_{2(m-j-\gamma)}, \end{cases}$$

where $I_{2(m-j-\gamma)}$ is **315**.

351.51 $$\begin{cases} \int \mathrm{sn}^{2m} u\,\mathrm{cn}^{2n} u\,\mathrm{nd}^{2p} u\,du \\ \quad = \frac{1}{k^{2(m+n)}} \sum_{j=0}^{m} \sum_{\gamma=0}^{m} \frac{(-1)^{j+\gamma+n} k'^{2(n-\gamma)}\,m!\,n!}{(m-j)!\,j!\,(n-\gamma)!\,\gamma!} I_{2(p-j-\gamma)}. \quad [\text{See } \mathbf{315}.] \end{cases}$$

352.01 $$\begin{cases} \int \frac{\mathrm{sn}\,u\,du}{\mathrm{dn}^{2m} u} = \int \mathrm{sn}\,u\,\mathrm{nd}^{2m} u\,du \\ \quad = \frac{1}{k'^{2m}} \sum_{j=0}^{m-1} \frac{k^{2j}(-1)^{j+1}(m-1)!}{(2j+1)(m-1-j)!\,j!}\,\mathrm{cd}^{2m+1} u, \qquad (m > 0). \end{cases}$$

352.51 $$\int \frac{\mathrm{cn}\,u\,du}{\mathrm{dn}^{2m} u} = \int \mathrm{cn}\,u\,\mathrm{nd}^{2m} u\,du = \sum_{j=0}^{m-1} \frac{k^{2j}(m-1)!}{(2j+1)(m-1-j)!\,j!}\,\mathrm{sd}^{2m+1} u,$$

$(m > 0)$.

353.01
$$\left\{\begin{aligned}
&\int \frac{\operatorname{sn} u\, du}{\operatorname{cn}^{2n} u \operatorname{dn}^{2m} u} = \int \operatorname{sn} u \operatorname{nc}^{2n} u \operatorname{nd}^{2m} u\, du = \int \operatorname{tn} u \operatorname{nc}^{2n-1} u \operatorname{nd}^{2m} u\, du \\
&= \frac{(-1)^{m+n} k^{2\; m+n-1)}}{k'^{2(m+n)}} \sum_{j=0}^{m+n-1} \frac{(-1)^j (m+n-1)!}{k^{2j}(m+n-1-j)!\, j!\, (2m-2j-1)} \times \\
&\times \operatorname{cd}^{2m-2j-1} u \qquad (m+n-1 \geqq 0).
\end{aligned}\right.$$

354.01
$$\left\{\begin{aligned}
&\int \frac{\operatorname{cn} u\, du}{\operatorname{sn}^{2n} u \operatorname{dn}^{2m} u} = \int \operatorname{cn} u \operatorname{ns}^{2n} u \operatorname{nd}^{2m} u\, du = \int \operatorname{cs} u \operatorname{ns}^{2n-1} u \operatorname{nd}^{2m} u\, du \\
&= \sum_{j=0}^{m+n-1} \frac{(m+n-1)!\, k^{2\; m+n-1-j)}}{(m+n-1-j)!\, j!\, (2m-2j-1)} \operatorname{sd}^{2m-2j-1} u, \\
&(m+n-1 \geqq 0).
\end{aligned}\right.$$

355.01
$$\left\{\begin{aligned}
&\int \frac{\operatorname{sn}^n u \operatorname{cn}^p u\, du}{\operatorname{dn}^{2m} u} = \int \operatorname{sn}^n u \operatorname{cn}^p u \operatorname{nd}^{2m} u\, du \\
&= \frac{1}{k^2(n+p-1-2m)} \Big\{\operatorname{sn}^{n-3} u \operatorname{cn}^{p+1} u \operatorname{nd}^{2m-1} u + \\
&+ [(n+p-2)+(n-2m-2)k^2] \int \operatorname{sn}^{n-2} u \operatorname{cn}^p u \operatorname{nd}^{2m} u\, du + \\
&+ (3-n) \int \operatorname{sn}^{n-4} u \operatorname{cn}^p u \operatorname{nd}^{2m} u\, du \Big\}, \qquad [n+p-1-2m \neq 0] \\
&= \frac{1}{k^2}\left[\int \operatorname{sn}^{n-2} u \operatorname{cn}^p u \operatorname{nd}^{2m} u\, du - \int \operatorname{sn}^{n-2} u \operatorname{cn}^p u \operatorname{nd}^{2m-1} u\, du\right] \\
&= \frac{1}{k^2}\left[\int \operatorname{sn}^n u \operatorname{cn}^{p-2} u \operatorname{nd}^{2m-1} u\, du - k'^2 \int \operatorname{sn}^n u \operatorname{cn}^{p-2} u \operatorname{nd}^{2m} u\, du\right].
\end{aligned}\right.$$

356.01
$$\left\{\begin{aligned}
&\int \frac{\operatorname{sn}^{2m} u\, du}{\operatorname{cn}^{2m} u \operatorname{dn}^{2m} u} = \int \frac{\operatorname{tn}^{2m} u\, du}{\operatorname{dn}^{2m} u} = \int \operatorname{tn}^{2m} u \operatorname{nd}^{2m} u\, du \\
&= \frac{1}{(2m-1)\, k'^2} \Big\{\operatorname{sn}^{2m+1} u \operatorname{nc}^{2m-1} u \operatorname{nd}^{2m-1} u + \\
&+ 2[2(1-m)k^2 - 1] \int \frac{\operatorname{sn}^{2m} u\, du}{\operatorname{cn}^{2\; m-1)} u \operatorname{dn}^{2m} u} + \\
&+ (2m-3)k^2 \int \frac{\operatorname{sn}^{2m} u\, du}{\operatorname{cn}^{2(m-2)} u \operatorname{dn}^{2m} u}\Big\}.
\end{aligned}\right.$$

357.01
$$\left\{\begin{aligned}
&\int \frac{\operatorname{sn}^n u\, du}{\operatorname{cn}^p u \operatorname{dn}^{2m} u} = \int \operatorname{sn}^n u \operatorname{nc}^p u \operatorname{nd}^{2m} u\, du \\
&= \frac{1}{(p-1)\, k'^2} \Big\{\operatorname{sn}^{n+1} u \operatorname{nc}^{p-1} u \operatorname{nd}^{2m-1} u + \\
&+ [p-n-2+(n-2p+4-2m)k^2] \int \operatorname{sn}^n u \operatorname{nc}^{p-2} u \operatorname{nd}^{2m} u\, du + \\
&+ k^2(p-n+2m-3) \int \operatorname{sn}^n u \operatorname{nc}^{p-4} u \operatorname{nd}^{2m} u\, du\Big\}, \qquad (p \neq 1).
\end{aligned}\right.$$

Integrals Involving Various Combinations of Jacobian Elliptic Functions.

360.01 $\int \operatorname{cn} u \operatorname{dn} u \, du = \operatorname{sn} u .$

360.02 $\int \operatorname{sn} u \operatorname{dn} u \, du = -\operatorname{cn} u .$

360.03 $\int \operatorname{sn} u \operatorname{cn} u \, du = -(\operatorname{dn} u)/k^2 .$

360.04 $\int \operatorname{sn} u \operatorname{cn}^2 u \, du = -\frac{1}{2k^3}\left[k'^2 \cosh^{-1}\left(\frac{\operatorname{dn} u}{k'}\right) + k \operatorname{cn} u \operatorname{dn} u\right].$

360.05 $\int \operatorname{sn}^2 u \operatorname{cn} u \, du = \frac{1}{2k^3}\left[\sin^{-1}(k \operatorname{sn} u) - k \operatorname{sn} u \operatorname{dn} u\right].$

360.06 $\int \operatorname{dn}^2 u \operatorname{sn} u \, du = \frac{1}{2k}\left[k'^2 \cosh^{-1}\left(\frac{\operatorname{dn} u}{k'}\right) - k \operatorname{cn} u \operatorname{dn} u\right].$

360.07 $\int \operatorname{sn}^2 u \operatorname{nc} u \, du = \frac{1}{k k'}\left[k \ln\left(\frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u}\right) - k' \sin^{-1}(k \operatorname{sn} u)\right].$

360.08 $\int \operatorname{dn}^2 u \operatorname{ns} u \, du = \ln\left[\frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}\right] - k \cosh^{-1}\left(\frac{\operatorname{dn} u}{k'}\right).$

360.09 $\int \operatorname{cn}^2 u \operatorname{ns} u \, du = \ln\left[\frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}\right] - \frac{1}{k} \cosh^{-1}\left(\frac{\operatorname{dn} u}{k'}\right).$

360.10 $\int \operatorname{dn}^2 u \operatorname{nc} u \, du = k \sin^{-1}(k \operatorname{sn} u) + k' \ln\left[\frac{\operatorname{dn} u + k' \operatorname{sn} u}{\operatorname{cn} u}\right].$

360.11 $\int \operatorname{sn} u \operatorname{cn} u \operatorname{dn}^m u \, du = \frac{-1}{(m+1)k^2} \operatorname{dn}^{m+1} u .$

360.12 $\int \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^m u \, du = \frac{\operatorname{sn}^{m+1} u}{m+1}.$

360.13 $\int \operatorname{sn} u \operatorname{dn} u \operatorname{cn}^m u \, du = -\frac{\operatorname{cn}^{m+1} u}{m+1}.$

360.14 $\int \operatorname{dn} u \operatorname{nc}^2 u \, du = \operatorname{tn} u .$

360.15 $\int \operatorname{cn} u \operatorname{dn} u \operatorname{ns} u \, du = \ln(\operatorname{sn} u) .$

360.16 $\int \operatorname{cn} u \operatorname{ns}^2 u \, du = -\operatorname{ds} u .$

360.17 $\int \operatorname{ns} u \operatorname{nc} u \, du = \ln\left[\frac{1 - \operatorname{dn} u}{\operatorname{sn} u}\right] + \frac{1}{k'} \ln\left[\frac{\operatorname{dn} u + k'}{\operatorname{cn} u}\right].$

360.18 $\int \operatorname{sn} u \operatorname{nc} u \operatorname{nd} u \, du = \frac{1}{k'^2} \ln(\operatorname{dc} u) .$

360.19 $\int \operatorname{cn} u \operatorname{nd}^2 u\, du = \operatorname{sd} u .$

360.20 $\int \operatorname{dn}^2 u \operatorname{tn} u\, du = \frac{k'}{2} \ln \left[\frac{k' + \operatorname{dn} u}{\operatorname{dn} u - k'}\right] - \operatorname{dn} u .$

360.21 $\int \operatorname{dn}^2 u \operatorname{cs} u\, du = \frac{1}{2} \ln \left[\frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}\right] + \operatorname{dn} u .$

360.22 $$\left\{\begin{aligned} &\int \operatorname{sn}^2 u \operatorname{dn}^2 u \operatorname{cn} u\, du \\ &\qquad = \frac{1}{8k^3} [\sin^{-1}(k \operatorname{sn} u) - k \operatorname{sn} u \operatorname{dn} u (1 - 2k^2 \operatorname{sn}^2 u)] . \end{aligned}\right.$$

360.23 $\int \operatorname{cs} u \operatorname{ds} u \operatorname{ns}^m u\, du = - \frac{\operatorname{ns}^{m+1} u}{m+1} .$

360.24 $\int \operatorname{tn} u \operatorname{dc} u \operatorname{nc}^m u\, du = \frac{\operatorname{nc}^{m+1} u}{m+1} .$

360.25 $\int \operatorname{sd} u \operatorname{cd} u \operatorname{nd}^m u\, du = \frac{\operatorname{nd}^{m+1} u}{(m+1) k^2} .$

360.26 $\int \operatorname{dn} u \operatorname{ns}^2 u \operatorname{cs}^m u\, du = - \frac{\operatorname{cs}^{m+1} u}{m+1} .$

360.27 $\int \operatorname{cn} u \operatorname{nd}^2 u \operatorname{sd}^m u\, du = \frac{\operatorname{sd}^{m+1} u}{m+1} .$

360.28 $\int \operatorname{cn} u \operatorname{ns}^2 u \operatorname{ds}^m u\, du = - \frac{\operatorname{ds}^{m+1} u}{m+1} .$

360.29 $\int \operatorname{nc} u \operatorname{dc} u \operatorname{tn}^m u\, du = \frac{\operatorname{tn}^{m+1} u}{m+1} .$

360.30 $\int \operatorname{sn} u \operatorname{nd}^2 u \operatorname{cd}^m u\, du = - \frac{\operatorname{cd}^{m+1} u}{(m+1) k'^2} .$

360.31 $\int \operatorname{sn} u \operatorname{nc}^2 u \operatorname{dc}^m u\, du = \frac{\operatorname{dc}^{m+1} u}{(m+1) k'^2} .$

361.01 $\int \operatorname{sn}^2 u \operatorname{cn}^2 u\, du = \frac{1}{3k^4} [(2 - k^2) E(u) - 2k'^2 u - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u] .$

361.02 $\int \operatorname{sn}^2 u \operatorname{dn}^2 u\, du = \frac{1}{3k^2} [(2k^2 - 1) E(u) + k'^2 u - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u] .$

361.03 $\int \operatorname{cn}^2 u \operatorname{dn}^2 u\, du = \frac{1}{3k^2} [(1 + k^2) E(u) - k'^2 u + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u] .$

361.04 $$\left\{\begin{aligned} &\int \operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u\, du = \frac{1}{15k^4} [k'^2 (k^2 - 2) u + \\ &\quad + 2(k^4 + k'^2) E(u) + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u (3k^2 \operatorname{sn}^2 u - 1 - k^2)] . \end{aligned}\right.$$

361.05 $\int \operatorname{tn}^2 u \operatorname{dn}^2 u \, du = u - 2E(u) + \operatorname{dn} u \operatorname{tn} u .$

361.06 $\int \operatorname{cs}^2 u \operatorname{dn}^2 u \, du = k'^2 u - 2E(u) - \operatorname{dn} u \operatorname{cs} u .$

361.07 $\begin{cases} \int \operatorname{tn}^2 u \operatorname{nc}^2 u \, du \\ = \frac{1}{3k'^4} [(1+k^2) E(u) - k'^2 u + \operatorname{dn} u \operatorname{tn} u (k'^2 \operatorname{nc}^2 u - 1 - k^2)] . \end{cases}$

361.08 $\int \operatorname{ds}^2 u \operatorname{dn}^2 u \, du = k'^2 u - (1+k^2) E(u) - \operatorname{dn} u \operatorname{cs} u .$

361.09 $\int \operatorname{cs}^2 u \operatorname{cn}^2 u \, du = \frac{1}{k^2} [k'^2 u - (1+k^2) E(u) - k^2 \operatorname{dn} u \operatorname{cs} u)] .$

361.10 $\int \operatorname{ns}^2 u \operatorname{nc}^2 u \, du = \frac{1}{k'^2} [2k'^2 u - (1+k'^2) E(u) + \operatorname{dn} u (\operatorname{tn} u - k'^2 \operatorname{cs} u)] .$

361.11 $\begin{cases} \int \operatorname{ns}^2 u \operatorname{nd}^2 u \, du \\ = \frac{1}{k'^2} [k'^2 u + (k^2 - k'^2) E(u) - \operatorname{cn} u (k'^2 \operatorname{ds} u + k^4 \operatorname{sd} u)] . \end{cases}$

361.12 $\int \operatorname{nc}^2 u \operatorname{nd}^2 u \, du = \frac{1}{k'^4} [k'^2 u - (1+k^2) E(u) + \operatorname{sn} u (\operatorname{dc} u + k^4 \operatorname{cd} u)] .$

361.13 $\begin{cases} \int \operatorname{nc}^2 u \operatorname{dc}^2 u \, du \\ = \frac{1}{3k'^2} [2k'^2 u + (k^2 - 2) E(u) + \operatorname{dn} u \operatorname{tn} u (2 - k^2 + k'^2 \operatorname{nc}^2 u)] . \end{cases}$

361.14 $\int \operatorname{ds}^2 u \operatorname{dc}^2 u \, du = 2k'^2 u - (1+k'^2) E(u) - \operatorname{dn} u (\operatorname{cs} u - k'^2 \operatorname{tn} u) .$

361.15 $\begin{cases} \int \operatorname{tn}^2 u \operatorname{dc}^2 u \, du = \frac{1}{3k'^2} [(1 - 2k^2) E(u) - k'^2 u + \\ \qquad + \operatorname{dc} u \operatorname{tn} u \operatorname{nc} u (k'^2 + \operatorname{cn}^2 u - 2k'^2 \operatorname{cn}^2 u)] . \end{cases}$

361.16 $\begin{cases} \int \operatorname{cd}^2 u \operatorname{nd}^2 u \, du = \frac{1}{3k^2 k'^2} [k'^2 u + (2k^2 - 1) E(u) + \\ \qquad + \operatorname{sn} u \operatorname{cd} u (k^2 - 2k^4 + k'^2 k^2 \operatorname{nd}^2 u)] . \end{cases}$

361.17 $\int \operatorname{tn}^2 u \operatorname{nc}^2 u \operatorname{dc}^2 u \, du = -k^2 D_2 + (k^2 - k'^2) D_4 + k'^2 D_6$

where D_2, D_4 and D_6 are given by **313**.

361.18 $\int \operatorname{sd}^2 u \operatorname{cd}^2 u \operatorname{nd}^2 u \, du = \frac{1}{k^4} [-I_2 + (1+k'^2) I_4 - k'^2 I_6] .$

where I_2, I_4 and I_6 are given by **315**.

361.19 $\begin{cases} \int \operatorname{sd}^2 u \operatorname{nd}^2 u \, du = \frac{1}{3k^2 k'^4} [(1+k^2) E(u) - k'^2 u + \\ \qquad + k^2 \operatorname{sn} u \operatorname{cd} u (k'^2 - k'^2 \operatorname{nd}^2 u - 2)] , \end{cases}$ [cf. **350.01**].

361.20 $\int \mathrm{cs}^2 u\,\mathrm{nd}^2 u\,du = u - 2E(u) + \mathrm{cn}\,u\,(k^2\,\mathrm{sd}\,u - \mathrm{ds}\,u)$.

361.21 $\int \mathrm{cs}^2 u\,\mathrm{ns}^2 u\,du = \frac{1}{3}\left[-k'^2 u + (1-2k^2)\,E(u) - \mathrm{dn}\,u\,\mathrm{cs}\,u\,(2k^2 + \mathrm{ns}^2 u - 1)\right]$.

361.22 $\int \mathrm{ds}^2 u\,\mathrm{ns}^2 u\,du = \frac{1}{3}\left[2k'^2 u + (k^2-2)\,E(u) - \mathrm{dn}\,u\,\mathrm{cs}\,u\,(\mathrm{ns}^2 u + 2 - k^2)\right]$.

361.23 $\int \mathrm{tn}^4 u\,\mathrm{dc}^2 u\,du = k^2 u + (k'^2 - 2k^2)\,D_2 + (k^2 - 2k'^2)\,D_4 + k'^2 D_6$,

where D_2, D_4 and D_6 are given by **313**.

361.24 $\int \mathrm{tn}^2 u\,\mathrm{sd}^2 u\,du = \frac{1}{k^2 k'^4}\left[k'^2 u - (1+k^2)\,E(u) + k^2\,\mathrm{sn}\,u\,(\mathrm{dc}\,u + \mathrm{cd}\,u)\right]$.

361.25 $\int \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,du = \frac{1}{k'^2 k^4}\left[(1+k'^2)\,E(u) - 2k'^2 u - k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]$.

361.26 $\int \mathrm{tn}^2 u\,\mathrm{dc}^4 u\,du = \frac{1}{3k'^6}\left[4k'^2 u - (k^2+7)\,E(u) + \mathrm{sn}\,u\,\mathrm{dc}\,u + {}\right.$
$\left.{} + (k^2 + 4 + k'^2\,\mathrm{nd}^2 u)\,k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]$.

361.27 $\int \mathrm{sd}^2 u\,\mathrm{cd}^2 u\,du = \frac{1}{3k^4 k'^2}\left[-2k'^2 + (1+k'^2)\,E(u) + k^2(k'^2\,\mathrm{nd}^2 u - 1 - k'^2)\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]$.

361.28 $\int \mathrm{cn}^2 u\,\mathrm{cd}^2 u\,du = \frac{1}{k^4}\left[-2k'^2 u + (1+k'^2)\,E(u) - k^2 k'^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]$.

361.29 $\int \mathrm{sn}^2 u\,\mathrm{tn}^2 u\,du = \frac{1}{k^2 k'^2}\left[(1-2k^2)\,E(u) + k^2\,\mathrm{dn}\,u\,\mathrm{tn}\,u - k'^2 u\right]$.

361.30 $\int \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,\mathrm{tn}^2 u\,du = \frac{1}{k'^4 k^4}\left[k'^2(2-k^2)\,u - 2(k'^2 + k^4)\,E(u) + k^2(\mathrm{cd}\,u + k^2\,\mathrm{dc}\,u)\,\mathrm{sn}\,u\right]$.

361.31 $\int \mathrm{dn}^2 u\,\mathrm{dc}^2 u\,du = k'^2 u + (1-2k'^2)\,E(u) + k'^2\,\mathrm{dn}\,u\,\mathrm{tn}\,u$.

361.32 $\int \mathrm{tn}^2 u\,\mathrm{nd}^2 u\,du = \frac{1}{k'^4}\left[k'^2 u - 2E(u) + (1 + k^2 - 2k^2\,\mathrm{sn}^2 u)\,\mathrm{tn}\,u\,\mathrm{nd}\,u\right]$.

361.33 $\int \mathrm{ns}^2 u\,\mathrm{nc}^2 u\,\mathrm{nd}^2 u\,du = \frac{1}{k'^4}\left[k'^2(1+k'^2)\,u - (1 + k^4 + k'^4)\,E(u) + {}\right.$
$\left.{} + (\mathrm{sn}^2 u - k'^4\,\mathrm{cn}^2 u + k^6\,\mathrm{sn}^2 u\,\mathrm{cd}^2 u)\,\mathrm{dc}\,u\,\mathrm{ns}\,u\right]$.

361.34 $\int \mathrm{sn}^2 u\,\mathrm{cd}^2 u\,du = \frac{1}{k^4}\left[(1+k'^2)\,u - 2E(u) + k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\right]$.

361.35 $$\begin{cases}\int \mathrm{sd}^4 u\,\mathrm{nd}^2 u\,du = \frac{1}{15k^4k'^6}\{(7k^2+3k^4-2)E(u)+2k'^2(1-3k^2)u- \\ -[3k^2k'^2+2(3k^2-1)(1+k^2+k'^2\mathrm{nd}^2 u)+3k'^4\mathrm{nd}^4 u]k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u\}.\end{cases}$$

361.36 $$\begin{cases}\int \mathrm{sd}^2 u\,\mathrm{tn}^2 u\,\mathrm{sn}^2 u\,du = \frac{1}{k^4k'^4}[k'^2(1+k'^2)u-2(k^2+k'^4)E(u)+ \\ \qquad + k^2(1+k^2-\mathrm{sn}^2 u - k^4\,\mathrm{sn}^2 u)\,\mathrm{tn}\,u\,\mathrm{nd}\,u].\end{cases}$$

361.37 $$\begin{cases}\int \mathrm{cd}^2 u\,\mathrm{cs}^2 u\,\mathrm{cn}^2 u\,du \\ = \frac{1}{k^4}[(2-k^2)k'^2u+2(k^2k'^2-1)E(u)+k^2(k'^4\mathrm{sd}\,u-k^2\,\mathrm{ds}\,u)\,\mathrm{cn}\,u].\end{cases}$$

361.38 $$\int \mathrm{ds}^2 u\,\mathrm{cs}^2 u\,\mathrm{ns}^2 u\,du = k^2B_2-(1+k^2)B_4+B_6,$$
[See **311** for B_2, B_4, B_6].

361.39 $$\begin{cases}\int \mathrm{dn}^2 u\,\mathrm{dc}^2 u\,\mathrm{ds}^2 u\,du \\ = k'^2(1+k'^2)u-2(k^4-k^2+1)E(u)+(k'^4\mathrm{tn}\,u-\mathrm{cs}\,u)\,\mathrm{dn}\,u.\end{cases}$$

361.50 $$\int \frac{du}{1\pm\mathrm{sn}\,u} = \frac{1}{k'^2}\left[k'^2u - E(u) \mp \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{1+\mathrm{sn}\,u}\right].$$

361.51 $$\int \frac{du}{1\pm\mathrm{cn}\,u} = u - E(u) \pm \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{1\pm\mathrm{cn}\,u},$$ [cf. **341**].

361.52 $$\int \frac{du}{1\pm k\,\mathrm{sn}\,u} = \frac{1}{k'^2}[E(u)+k(1\mp k\,\mathrm{sn}\,u)\,\mathrm{cd}\,u].$$

361.53 $$\int \frac{1-\mathrm{cn}\,u}{1+\mathrm{cn}\,u}du = u-2E(u)+\frac{2\,\mathrm{sn}\,u\,\mathrm{dn}\,u}{1+\mathrm{cn}\,u}.$$

361.54 $$\int \frac{du}{1+\alpha\,\mathrm{cn}\,u} = \frac{1}{1-\alpha^2}\left[\Pi\left(\varphi,\frac{\alpha^2}{\alpha^2-1},k\right)-\alpha f_1\right],$$
$[\varphi = \mathrm{am}\,u,\ \alpha^2 \neq 1]$ [See **400**; cf. **341**.]

where
$$f_1 = \sqrt{\frac{1-\alpha^2}{k^2+k'^2\alpha^2}}\tan^{-1}\left[\sqrt{\frac{k^2+k'^2\alpha^2}{1-\alpha^2}}\,\mathrm{sd}\,u\right],\quad \text{if } \alpha^2/(\alpha^2-1)<k^2;$$
$$= \mathrm{sd}\,u,\quad \text{if } \alpha^2/(\alpha^2-1)=k^2;$$
$$= \sqrt{\frac{\alpha^2-1}{k^2+k'^2\alpha^2}}\ln\left[\frac{\sqrt{k^2+k'^2\alpha^2}\,\mathrm{dn}\,u+\sqrt{\alpha^2-1}\,\mathrm{sn}\,u}{\sqrt{k^2+k'^2\alpha^2}\,\mathrm{dn}\,u-\sqrt{\alpha^2-1}\,\mathrm{sn}\,u}\right],$$
if $\alpha^2/(\alpha^2-1)>k^2$.

361.55 $$\int \frac{du}{1\pm\mathrm{dn}\,u} = \frac{1}{k^2}[u-E(u)-\mathrm{cs}\,u\,(\mathrm{dn}\,u\mp 1)].$$

361.56 $$\begin{cases}\int \frac{du}{1\pm\mathrm{tn}\,u} = \frac{1}{2}\Bigg[u+\Pi(\varphi,2,k)\pm \\ \qquad \pm\frac{1}{\sqrt{2(1+k'^2)}}\ln\frac{\sqrt{k'^2+1}-\sqrt{2}\,\mathrm{dn}\,u}{\sqrt{k'^2+1}+\sqrt{2}\,\mathrm{dn}\,u}\Bigg],\end{cases}$$ $[\varphi = \mathrm{am}\,u]$.

361.57 $\displaystyle\int \frac{du}{\operatorname{dn} u \pm \operatorname{cn} u} = -\frac{1}{k'^2}\,[\operatorname{cs} u \mp \operatorname{ds} u]\,.$

361.58 $\displaystyle\int \frac{du}{1+\alpha \operatorname{sn} u} = [\Pi(\varphi, \alpha^2, k) - \alpha f]\,, \quad (\alpha^2 \neq 1, k^2)\,,$

[See **400**; $\varphi = \operatorname{am} u$],

where

$$f = \frac{1}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}} \tan^{-1}\left[\frac{2(1-\alpha^2)(\alpha^2-k^2)+(1-\alpha^2 \operatorname{sn}^2 u)(2k^2-\alpha^2-\alpha^2 k^2)}{2\alpha^2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}\operatorname{cn} u \operatorname{dn} u}\right],$$

if $(1-\alpha^2)(\alpha^2-k^2) > 0$;

$$= \frac{1}{2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \ln\left|\frac{2(\alpha^2-1)(\alpha^2-k^2)+(1-\alpha^2\operatorname{sn}^2 u)(\alpha^2+\alpha^2k^2-2k^2)}{1-\alpha^2\operatorname{sn}^2 u} + \right.$$

$$\left. + \frac{2\alpha^2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}\operatorname{cn} u \operatorname{dn} u}{1-\alpha^2\operatorname{sn}^2 u}\right|, \qquad \text{if} \quad (1-\alpha^2)(\alpha^2-k^2) < 0.$$

361.59 $\displaystyle\int \frac{1+\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\,du = \frac{1}{\alpha}\left\{u - \frac{1}{1+\alpha}\left[\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2-1}, k\right) - \alpha f_1\right]\right\},$

$\alpha^2 \neq 1\,,\ \varphi = \operatorname{am} u\,,$

where f_1 is given in **361.54**.

361.60 $\displaystyle\int \frac{1-\operatorname{cn} u}{1+\alpha \operatorname{cn} u}\,du = \frac{1}{\alpha}\left\{-u + \frac{1}{1-\alpha}\left[\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2-1}, k\right) - \alpha f_1\right]\right\},$

$\alpha^2 \neq 1\,,\ \varphi = \operatorname{am} u$ [See **361.54** for f_1.]

361.61 $\displaystyle\int \frac{\operatorname{cn} u\,du}{1 \pm \operatorname{cn} u} = \pm E(u) - \frac{\operatorname{sn} u \operatorname{dn} u}{1 \pm \operatorname{cn} u}\,.$

361.62 $\displaystyle\int \frac{1+\alpha_1 \operatorname{cn} u}{1+\alpha \operatorname{cn} u}\,du = \frac{1}{\alpha}\left\{\alpha_1 u + \frac{\alpha-\alpha_1}{1-\alpha^2}\left[\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2-1}, k\right) - \alpha f_1\right]\right\},$

$\alpha^2 \neq 1\,,\ \varphi = \operatorname{am} u\,.$ [See **361.54** for f_1.]

361.63 $\displaystyle\int \frac{\operatorname{cn} u\,du}{1+\alpha \operatorname{cn} u} = \frac{1}{\alpha}\left\{u + \frac{1}{\alpha^2-1}\left[\Pi\left(\varphi, \frac{\alpha^2}{\alpha^2-1}, k\right) - \alpha f_1\right]\right\},$

$\alpha^2 \neq 1\,,\ \varphi = \operatorname{am} u\,.$ [See **361.54** for f_1.]

361.64 $\displaystyle\int \frac{du}{1+\alpha \operatorname{tn} u} = \frac{1}{1+\alpha^2}\,[u + \alpha^2\Pi(\varphi, 1+\alpha^2, k) + \alpha(\alpha^2+1) f_2]\,,\ \varphi = \operatorname{am} u\,,$

where

$$f_2 = \frac{1}{2\sqrt{(1+\alpha^2)(k'^2+\alpha^2)}} \ln\left[\frac{\sqrt{k'^2+\alpha^2} - \sqrt{1+\alpha^2}\operatorname{dn} u}{\sqrt{k'^2+\alpha^2} + \sqrt{1+\alpha^2}\operatorname{dn} u}\right],$$

$(1+\alpha^2)(k'^2+\alpha^2) > 0\,.$

361.65 $$\begin{cases} \displaystyle\int \frac{1+\alpha_1 \operatorname{tn} u}{1+\alpha \operatorname{tn} u}\,du = \frac{1}{1+\alpha^2} \times \\ \times\,[(1+\alpha\alpha_1)u + \alpha(\alpha-\alpha_1)\Pi(\varphi, 1+\alpha^2, k) + (\alpha-\alpha_1)(\alpha^2+1) f_2]\,, \\ \qquad \varphi = \operatorname{am} u\,. \qquad \text{[See } \mathbf{361.64} \text{ for } f_2.] \end{cases}$$

361.66 $\displaystyle\int \frac{\operatorname{tn} u\, du}{1+\alpha \operatorname{tn} u} = \frac{1}{1+\alpha^2}\left[\alpha u - \alpha \Pi(\varphi, 1+\alpha^2, k) - (\alpha^2+1) f_2\right].$

[See **361.64**; $\varphi = \operatorname{am} u$.]

361.67 $\displaystyle\int \frac{\operatorname{sn} u\, du}{1 \pm \operatorname{sn} u} = \frac{1}{k'^2}\left[\pm E(u) + \frac{\operatorname{cn} u \operatorname{dn} u}{1 \pm \operatorname{sn} u}\right].$

361.68 $\displaystyle\int \frac{\operatorname{sn} u\, du}{1+\alpha \operatorname{sn} u} = \frac{1}{\alpha}\left[u - \Pi(\varphi, \alpha^2, k) + \alpha f\right],$

[$\varphi = \operatorname{am} u$; see **361.58** for f].

361.69 $\displaystyle\int \frac{1+\alpha_1 \operatorname{sn} u}{1+\alpha \operatorname{sn} u}\, du = \frac{1}{\alpha}\{\alpha_1 u + (\alpha - \alpha_1)\left[\Pi(\varphi, \alpha^2, k) - \alpha f\right]\}.$

[See **361.58**; $\varphi = \operatorname{am} u$.]

361.70 $\displaystyle\int \frac{\operatorname{dn} u\, du}{1 \pm \operatorname{dn} u} = \frac{1}{k^2}\left[\pm E(u) \mp k'^2 u \pm (\operatorname{dn} u \mp 1) \operatorname{cs} u\right].$

361.71 $\displaystyle\int \frac{\operatorname{cn} u\, du}{1 - \operatorname{dn} u} = \frac{\operatorname{sn} u}{\operatorname{dn} u - 1}.$

361.72 $\displaystyle\int \frac{\operatorname{cn} u \operatorname{dn} u\, du}{1 - \operatorname{dn} u} = -\frac{1}{k^2 \operatorname{sn} u}\left[1 + \operatorname{dn} u + k \operatorname{sn} u \sin^{-1}(k \operatorname{sn} u)\right].$

361.73 $\displaystyle\int \frac{\operatorname{sn} u\, du}{1 - \operatorname{dn} u} = \frac{1}{k^2} \ln \frac{1 - \operatorname{cn} u}{\operatorname{cn} u + \operatorname{dn} u}.$

362.01 $\displaystyle\int \frac{\operatorname{ns}^2 u\, du}{1 - \alpha^2 \operatorname{sn}^2 u} = u - E(u) + \alpha^2 \Pi(u, \alpha^2) - \operatorname{dn} u \operatorname{cs} u;$

[$\varphi = \operatorname{am} u$, $\sin\varphi = \operatorname{sn} u$],

362.02 $\displaystyle\int \frac{\operatorname{nc}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{(1-\alpha^2) k'^2}\left[k'^2 u - E(u) - \alpha^2 k'^2 \Pi(u, \alpha^2) + \operatorname{dn} u \operatorname{tn} u\right].$

362.03 $\displaystyle\int \frac{\operatorname{nd}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{(k^2-\alpha^2) k'^2}\left[E(u) - \alpha^2 k'^2 \Pi(u, \alpha^2) - k^2 \operatorname{sn} u \operatorname{cd} u\right].$

362.04 $\displaystyle\int \frac{\operatorname{cs}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = u - E(u) + (\alpha^2 - 1) \Pi(u, \alpha^2) - \operatorname{dn} u \operatorname{cs} u.$

362.05 $\displaystyle\int \frac{\operatorname{ds}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = u - E(u) + (\alpha^2 - k^2) \Pi(u, \alpha^2) - \operatorname{dn} u \operatorname{cs} u.$

362.06 $\displaystyle\int \frac{\operatorname{tn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{(\alpha^2-1) k'^2}\left[E(u) - k'^2 u + k'^2 \Pi(u, \alpha^2) - \operatorname{dn} u \operatorname{tn} u\right].$

362.07 $\displaystyle\int \frac{\operatorname{sd}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{(\alpha^2-k^2) k'^2}\left[-E(u) + k'^2 \Pi(u, \alpha^2) + k^2 \operatorname{sn} u \operatorname{cd} u\right].$

362.08 $$\int \frac{\operatorname{cd}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{k^2-\alpha^2}\left[-E(u) + (1-\alpha^2)\,\Pi(u,\alpha^2) + k^2 \operatorname{sn} u \operatorname{cd} u\right].$$

362.09 $$\int \frac{\operatorname{dc}^2 u\,du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{\alpha^2-1}\left[E(u) - k'^2 u + (\alpha^2-k^2)\,\Pi(u,\alpha^2) - \operatorname{dn} u \operatorname{tn} u\right].$$

362.10 $$\int \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{k^2\alpha^4}\left[-\alpha^2 E(u) + (k^2+\alpha^2 k'^2)\,u + k^2(\alpha^2-1)\,\Pi(u,\alpha^2)\right].$$

362.11 $$\int \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{\alpha^4}\left[k^2 u - \alpha^2 E(u) + (\alpha^2-k^2)\,\Pi(u,\alpha^2)\right].$$

362.12 $$\int \frac{\operatorname{cn}^2 u \operatorname{dn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{\alpha^4}\left[\alpha^2 E(u) + (\alpha^2-1)\,k^2 u + (\alpha^2-1)(\alpha^2-k^2)\,\Pi(u,\alpha^2)\right].$$

362.13 $$\int \frac{\operatorname{sd}^2 u \operatorname{cn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{\alpha^2 k^2(\alpha^2-k^2)} \times \left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + k^2(\alpha^2-1)\,\Pi(u,\alpha^2) - \alpha^2 k^2 \operatorname{sn} u \operatorname{cd} u\right].$$

362.14 $$\int \frac{\operatorname{tn}^2 u \operatorname{dn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\,du = \frac{1}{\alpha^2(\alpha^2-1)} \times \left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + (\alpha^2-k^2)\,\Pi(u,\alpha^2) - \alpha^2 \operatorname{dn} u \operatorname{tn} u\right].$$

362.15 $$\int \frac{\operatorname{sn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = \frac{1}{2\alpha^2(\alpha^2-1)(k^2-\alpha^2)} \times \left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + (\alpha^4-k^2)\,\Pi(u,\alpha^2) - \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-\alpha^2 \operatorname{sn}^2 u}\right].$$

362.16 $$\int \frac{\operatorname{cn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = \frac{1}{2\alpha^2(k^2-\alpha^2)}\left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + (2k^2\alpha^2-\alpha^4-k^2)\,\Pi(u,\alpha^2) - \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-\alpha^2 \operatorname{sn}^2 u}\right].$$

362.17 $$\int \frac{\operatorname{dn}^2 u\,du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = -\frac{1}{2\alpha^2(\alpha^2-1)}\left[\alpha^2 E(u) + (k^2-\alpha^2)\,u + (2\alpha^2-\alpha^4-k^2)\,\Pi(u,\alpha^2) - \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-\alpha^2 \operatorname{sn}^2 u}\right].$$

362.18 $$\int \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{(1-\alpha^2 \operatorname{sn}^2 u)^2}\,du = \frac{1}{2\alpha^4(k^2-\alpha^2)}\left[\alpha^2 E(u) + (\alpha^2-k^2)\,u + (\alpha^4-2\alpha^2+k^2)\,\Pi(u,\alpha^2) - \frac{\alpha^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-\alpha^2 \operatorname{sn}^2 u}\right].$$

362.19 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\, du = \frac{1}{2\alpha^4(\alpha^2-1)}\Big[-\alpha^2 E(u) + (\alpha^2+k^2-2\alpha^2 k^2)\, u + (2\alpha^2 k^2 - \alpha^4 - k^2)\, \Pi(u,\alpha^2) + \frac{\alpha^4\, \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1-\alpha^2 \mathrm{sn}^2 u}\Big].$$

362.20 $$\int \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\, du = \frac{1}{2\alpha^4} \times \Big[-\alpha^2 E(u) + (\alpha^2+k^2)\, u + (\alpha^4-k^2)\, \Pi(u,\alpha^2) + \frac{\alpha^4\, \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1-\alpha^2 \mathrm{sn}^2 u}\Big].$$

362.21 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^3}\, du = \frac{1}{\alpha^4} \times [-k^2 \Pi(u,\alpha^2) + (2k^2-\alpha^2)\, V_2 + (\alpha^2-k^2)\, V_3].$$ [See **336.**]

362.22 $$\int \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^3}\, du = \frac{1}{\alpha^4} \times [k^2 \Pi(u,\alpha^2) + (k^2\alpha^2+\alpha^2-2k^2)\, V_2 + (\alpha^2-1)(\alpha^2-k^2)\, V_3.]$$ [See **336.**]

362.23 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^3}\, du = \frac{1}{\alpha^4}[-\Pi(u,\alpha^2) + (2-\alpha^2)\, V_2 + (\alpha^2-1)\, V_3].$$ [See **336.**]

362.24 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^3}\, du = \frac{1}{\alpha^6}[-k^2 u + (3k^2-\alpha^2k^2-\alpha^2)\, \Pi(u,\alpha^2) + (2\alpha^2k^2+2\alpha^2-3k^2-\alpha^4)\, V_2 + (\alpha^2-1)(\alpha^2-k^2)\, V_3].$$ [See **336.**]

362.25 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^4}\, du = \frac{1}{\alpha^6}[-k^2 \Pi(u,\alpha^2) + (3k^2-\alpha^2k^2-\alpha^2)\, V_2 + (2\alpha^2k^2+2\alpha^2-3k^2-\alpha^4)\, V_3 + (\alpha^2-1)(\alpha^2-k^2)\, V_4].$$ [See **336.**]

362.26 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^4} = \frac{1}{\alpha^4}[-V_2 + (2-\alpha^2)\, V_3 + (\alpha^2-1)\, V_4].$$ [See **336.**]

362.27 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^4} = \frac{1}{\alpha^4}[-k^2 V_2 + (2k^2-\alpha^2)\, V_3 + (\alpha^2-k^2)\, V_4].$$ [See **336.**]

362.28 $$\int \frac{\mathrm{cn}^2 u\, \mathrm{dn}^2 u\, du}{(1-\alpha^2 \mathrm{sn}^2 u)^4} = \frac{1}{\alpha^4} \times [k^2 V_2 + (\alpha^2k^2+\alpha^2-2k^2)\, V_3 + (\alpha^2-1)(\alpha^2-k^2)\, V_4].$$ [See **336.**]

362.29 $$\int \frac{\mathrm{sn}^2 u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u}{(1-\alpha^2 \mathrm{sn}^2 u)^5}\, du = \frac{1}{\alpha^6}[-k^2 V_2 + (3k^2-\alpha^2k^2-\alpha^2)\, V_3 + (2\alpha^2k^2+2\alpha^2-3k^2-\alpha^4)\, V_4 + (\alpha^2-1)(\alpha^2-k^2)\, V_5].$$ [See **336.**]

363.01 $$\int \frac{du}{1+k\,\mathrm{sn}^2 u} = \frac{u}{2} + \frac{1}{2(1+k)} \tan^{-1}[(1+k)\,\mathrm{tn}\,u\,\mathrm{nd}\,u].$$

363.02 $$\int \frac{du}{1-k\,\mathrm{sn}^2 u} = \frac{u}{2} + \frac{1}{2(1-k)} \tan^{-1}[(1-k)\,\mathrm{tn}\,u\,\mathrm{nd}\,u].$$

363.03 $$\left\{\begin{aligned} \int \frac{du}{(1+k\,\mathrm{sn}^2 u)^2} = \frac{1}{2(1+k)^2} \Big\{ & E(u) + k(1+k)\,u + \\ & + (1+k)\tan^{-1}[(1+k)\,\mathrm{tn}\,u\,\mathrm{nd}\,u] + \frac{k\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1+k\,\mathrm{sn}^2 u}\Big\}. \end{aligned}\right.$$

363.04 $$\left\{\begin{aligned} \int \frac{du}{(1-k\,\mathrm{sn}^2 u)^2} = \frac{1}{2(1-k)^2} \Big\{ & E(u) + k(k-1)\,u + \\ & + (1-k)\tan^{-1}[(1-k)\,\mathrm{tn}\,u\,\mathrm{nd}\,u] - \frac{k\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k\,\mathrm{sn}^2 u}\Big\}. \end{aligned}\right.$$

364.01 $$\left\{\begin{aligned} \int R(\mathrm{sn}\,u, \mathrm{cn}\,u, \mathrm{dn}\,u)\,du = & \int R_1(\mathrm{sn}\,u)\,du + \int R_2(\mathrm{sn}\,u)\,\mathrm{cn}\,u\,du + \\ & + \int R_3(\mathrm{sn}\,u)\,\mathrm{dn}\,u\,du + \int R_4(\mathrm{sn}\,u)\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,du, \end{aligned}\right.$$

where R, R_1, R_2, R_3, and R_4 are rational functions.

364.02 $$\int R_2(\mathrm{sn}\,u)\,\mathrm{cn}\,u\,du = 2\int R_2\left[\frac{2t}{1+k^2t^2}\right]\frac{dt}{1+k^2t^2},$$

where $$t = (\mathrm{sn}\,u)/(\mathrm{dn}\,u+1), \quad \mathrm{sn}\,u = 2t/(1+k^2t^2).$$

364.03 $$\int R_3(\mathrm{sn}\,u)\,\mathrm{dn}\,u\,du = 2\int R_3\left[\frac{2t}{1+t^2}\right]\frac{dt}{1+t^2},$$

where $$t = (\mathrm{sn}\,u)/(\mathrm{cn}\,u+1), \quad \mathrm{sn}\,u = 2t/(1+t^2).$$

364.04 $$\int R_4(\mathrm{sn}\,u)\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,du = \int R_4(t)\,dt,$$

where $$t = \mathrm{sn}\,u.$$

365.01 $$\int R(\mathrm{cn}\,u)\,\mathrm{sn}\,u\,\mathrm{dn}\,u\,du = -\int R(\tau)\,d\tau,$$

where $$\tau = \mathrm{cn}\,u.$$

365.02 $$\int R(\mathrm{dn}\,u)\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,du = -\frac{1}{k^2}\int R(x)\,dx,$$

where $$x = \mathrm{dn}\,u.$$

365.03 $$\int R(\mathrm{tn}\,u)\,\mathrm{nc}\,u\,\mathrm{dc}\,u\,du = \int R(z)\,dz,$$

where $$z = \mathrm{tn}\,u.$$

365.04 $$\int R(\mathrm{am}\,u)\,\mathrm{dn}\,u\,du = \int R(t_1)\,dt_1,$$

where $$t_1 = \mathrm{am}\,u.$$

Integrals of the Jacobian Inverse Elliptic Functions.

390.01 $$\begin{cases} \int \operatorname{sn}^{-1}(y,k)\,dy = \int F(\sin^{-1} y, k)\,dy \\ \qquad = y\operatorname{sn}^{-1}(y,k) - \frac{1}{k}\ln\left|k\sqrt{1-y^2} - \sqrt{1-k^2y^2}\right|. \end{cases}$$

390.02 $$\int \operatorname{cn}^{-1}(y,k)\,dy = y\operatorname{cn}^{-1}(y,k) - \frac{1}{k}\cos^{-1}\left[\sqrt{k'^2 + k^2y^2}\right].$$

390.03 $$\int \operatorname{dn}^{-1}(y,k)\,dy = y\operatorname{dn}^{-1}(y,k) - \sin^{-1}\left[\sqrt{(1-y^2)/k^2}\right].$$

390.04 $$\int \operatorname{tn}^{-1}(y,k)\,dy = y\operatorname{tn}^{-1}(y,k) - \frac{1}{k'}\ln\left[k'\sqrt{1+y^2} + \sqrt{1+k'^2y^2}\right].$$

390.05 $$\int \operatorname{nc}^{-1}(y,k)\,dy = y\operatorname{nc}^{-1}(y,k) - \frac{1}{k'}\ln\left[k'\sqrt{y^2-1} + \sqrt{k^2+k'^2y^2}\right].$$

390.06 $$\int \operatorname{nd}^{-1}(y,k)\,dy = y\operatorname{nd}^{-1}(y,k) - \frac{1}{k'}\tan^{-1}\left[\sqrt{k'^2(y^2-1)/(1-k'^2y^2)}\right].$$

390.07 $$\int \operatorname{dc}^{-1}(y,k)\,dy = y\operatorname{dc}^{-1}(y,k) - \ln\left[\sqrt{y^2-1} + \sqrt{y^2-k^2}\right].$$

390.08 $$\begin{cases} \int \operatorname{sd}^{-1}(y,k)\,dy \\ \qquad = y\operatorname{sd}^{-1}(y,k) - \frac{1}{k\,k'}\tan^{-1}\left[\sqrt{k^2(1-k'^2y^2)/k'^2(1+k^2y^2)}\right]. \end{cases}$$

390.09 $$\int \operatorname{ns}^{-1}(y,k)\,dy = y\operatorname{ns}^{-1}(y,k) + \ln\left[\sqrt{y^2-1} + \sqrt{y^2-k^2}\right].$$

390.10 $$\int \operatorname{ds}^{-1}(y,k)\,dy = y\operatorname{ds}^{-1}(y,k) + \ln\left[\sqrt{y^2-k'^2} + \sqrt{y^2+k^2}\right].$$

390.11 $$\int \operatorname{cs}^{-1}(y,k)\,dy = y\operatorname{cs}^{-1}(y,k) + \ln\left[\sqrt{1+y^2} + \sqrt{y^2+k'^2}\right].$$

390.12 $$\int \operatorname{cd}^{-1}(y,k)\,dy = y\operatorname{cd}^{-1}(y,k) + \frac{1}{k}\ln\left|k\sqrt{1-y^2} - \sqrt{1-k^2y^2}\right|.$$

391.01 $$\int y\operatorname{sn}^{-1}(y,k)\,dy = \frac{1}{2k^2}\left[(k^2y^2-1)\operatorname{sn}^{-1}(y,k) + E(\sin^{-1}y,k)\right].$$

391.02 $$\int y\operatorname{cn}^{-1}(y,k)\,dy = \frac{1}{2k^2}\left[(k^2y^2-k'^2)\operatorname{cn}^{-1}(y,k) + E(\cos^{-1}y,k)\right].$$

391.03 $$\int y\operatorname{dn}^{-1}(y,k)\,dy = \frac{1}{2}\left\{y^2\operatorname{dn}^{-1}(y,k) + E\left[\sin^{-1}\sqrt{(1-y^2)/k^2}, k\right]\right\}.$$

391.04 $$\begin{cases} \int y\operatorname{tn}^{-1}(y,k)\,dy = \frac{1}{2k'^2}\left\{k'^2y^2\operatorname{tn}^{-1}(y,k) + \right. \\ \qquad \left. + E\left[\sin^{-1}\sqrt{y^2/(1+y^2)}, k\right] - y\sqrt{(1+k'^2y^2)/(1+y^2)}\right\}. \end{cases}$$

392.01 $\int y^m \operatorname{sn}^{-1}(y, k)\, dy = \frac{1}{m+1}\left[y^{m+1} u - \int \operatorname{sn}^{m+1} u\, du\right];$

$[u = \operatorname{sn}^{-1}(y, k)].$

392.02 $\int y^m \operatorname{cn}^{-1}(y, k)\, dy = \frac{1}{m+1}\left[y^{m+1} u - \int \operatorname{cn}^{m+1} u\, du\right];$

$[u = \operatorname{cn}^{-1}(y, k)].$

392.03 $\int y^m \operatorname{dn}^{-1}(y, k)\, dy = \frac{1}{m+1}\left[y^{m+1} u - \int \operatorname{dn}^{m+1} u\, du\right];$

$[u = \operatorname{dn}^{-1}(y, k)].$

392.04 $\int y^m \operatorname{tn}^{-1}(y, k)\, dy = \frac{1}{m+1}\left[y^{m+1} u - \int \operatorname{tn}^{m+1} u\, du\right];$

$[u = \operatorname{tn}^{-1}(y, k)].$

Elliptic Integrals of the Third Kind.

Introduction.

Definition[1].

The *incomplete* elliptic integral of the third kind in *Legendre's canonical form* is defined by

400.01*
$$\begin{cases} \Pi(\varphi,\alpha^2,k) \equiv \displaystyle\int_0^y \frac{dt}{(1-\alpha^2 t^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}} \\ \qquad = \displaystyle\int_0^{u_1} \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} \equiv \Pi(u_1,\alpha^2), \\ \qquad [y = \sin\varphi = \mathrm{sn}\, u_1, \quad t = \sin\vartheta = \mathrm{sn}\, u; \quad \alpha^2 \neq 1 \text{ or } k^2]. \end{cases}$$

When $\varphi = \pi/2$, $y = 1$, $u_1 = K$, the integral is said to be *complete.*

In addition to the argument φ and the modulus k, this integral depends also on the *parameter* α^2. [We find it convenient to write $-\alpha^2$ in the integrand of $\Pi(\varphi, \alpha^2, k)$, while in much of the literature $-\alpha$ or $+n$ is written.] It is necessary to consider various cases according to the value of α^2.

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[1] Writing $\alpha^2 = k^2\,\mathrm{sn}^2 a$, JACOBI took

$$\Pi_1(u_1, a) = k^2\,\mathrm{sn}\, a\,\mathrm{cn}\, a\,\mathrm{dn}\, a \int_0^{u_1} \frac{\mathrm{sn}^2 u\, du}{1 - k^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u}$$

as his fundamental integral of the third kind, where the parameter a is imaginary in the circular cases and real in the hyperbolic cases. This integral is related to LEGENDRE's integral by the equation

$$\Pi(u_1, \alpha^2) = u_1 + \frac{\mathrm{sn}\, a}{\mathrm{cn}\, a\,\mathrm{dn}\, a}\,\Pi_1(u_1, a)\,.$$

Although addition formulas and many other relations may be simplified by use of JACOBI's $\Pi_1(u_1, a)$, we shall employ LEGENDRE's parameter except for sign.

* If the quantity $1/\alpha^2$ is not a root of $\sqrt{(1-y^2)(1-k^2y^2)}$, this normal integral becomes logarithmically infinite for $y^2 = 1/\alpha^2$; while if $1/\alpha^2$ is a root, the integral becomes algebraically infinite of the one-half order.

Possible cases.

$$\left.\begin{array}{ll}\text{Case I:} & 0 < -\alpha^2 < \infty \\ \text{Case II:} & k^2 < \alpha^2 < 1\end{array}\right\} \textit{Circular cases};$$

$$\left.\begin{array}{ll}\text{Case III:} & 0 < \alpha^2 < k^2 \\ \text{Case IV:} & 1 < \alpha^2 < \infty\end{array}\right\} \textit{Hyperbolic cases};$$

Case V: Complex parameter.

Evaluation.

Complete elliptic integrals of the third kind can be evaluated in terms of elementary functions and of the tabulated functions $\Lambda_0(\varphi, k)$ and $KZ(\varphi, k)$ (see **410—419**). For the *incomplete* integrals (**430—440**), however, there is an additional term involving Theta functions which requires evaluation by means of an infinite series. For computing this series, the circular cases are best broken down still further, as follows:

Case I: $0 < -\alpha^2 < k$ and $k < -\alpha^2 < \infty$.

Case II: $k < \alpha^2 < 1$ and $k^2 < \alpha^2 < k$.

Thus two sets of formulas are given for each of case I and II. These formulas are the source of the two alternate sets given for case I and II of the complete integrals.

In evaluating certain real elliptic integrals $\left[\text{e.g.,} \int_0^{u_1} \frac{du}{1-k \operatorname{sn} u \operatorname{cn} u}\right]$, one occasionally encounters either the sum $a\,\Pi(\varphi, \alpha^2, k) + \bar{a}\,\Pi(\varphi, \bar{\alpha}^2, k)$ or the difference $i\,[a\,\Pi(\varphi, \alpha^2, k) - \bar{a}\,\Pi(\varphi, \bar{\alpha}^2, k)]$, where $a, \bar{a}$ and the parameters $\alpha^2, \bar{\alpha}^2$ are two pairs of conjugate complex numbers. It is possible in these cases (**416—418** and **437—440**) to express the sum or difference in terms of elementary functions, of the function $F(\varphi, k)$ and of two integrals of the third kind having real parameters.

Table of Integrals.

The following short table gives for each of the above cases the evaluation[1] of $\Pi(\varphi, \alpha^2, k)$, as well as of certain integrals which occur frequently in practice. [Recurrence formulas are given in **336** to **342** for other integrals whose evaluations usually lead to a term involving $\Pi(\varphi, \alpha^2, k)$.]

[1] See **110** for addition formulas, special values and other basic information concerning the normal elliptic integral of the third kind, $\Pi(\varphi, \alpha^2, k)$.

Complete Integrals.

Case I: $0 < -\alpha^2 < \infty$.

$$\Lambda_0(\psi, k) = \frac{2}{\pi}\,[E\,F(\psi, k') + K\,E(\psi, k') - K\,F(\psi, k')],$$
$$\psi = \sin^{-1}\sqrt{\frac{\alpha^2}{\alpha^2 - k^2}}.$$

410.01* $$\Pi(\alpha^2, k) = \int_0^K \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{k^2 K}{k^2-\alpha^2} - \frac{\pi\,\alpha^2 \Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}, \quad (\alpha^2 < 0),$$

[cf. **411.01**].

Special case when $-\alpha^2 = k$:

$$\Pi(-k, k) = \int_0^K \frac{du}{1+k\,\mathrm{sn}^2 u} = \frac{1}{4(1+k)}\,[\pi + 2(1+k)\,K].$$

410.02 $$\int_0^K \frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{K}{k^2-\alpha^2} - \frac{\pi \Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

410.03 $$\int_0^K \frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{k'^2 K}{\alpha^2-k^2} + \frac{\pi(1-\alpha^2)\Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

410.04 $$\int_0^K \frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{\pi(k^2-\alpha^2)\Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

410.05 $$\int_0^K \frac{\mathrm{sd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{1}{(k^2-\alpha^2)\,k'^2}\left[E - \frac{k'^2 k^2 K}{k^2-\alpha^2} + \frac{\pi\,\alpha^2 k'^2 \Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\right].$$

410.06 $$\int_0^K \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du = \frac{k^2-\alpha_1^2}{k^2-\alpha^2}\,K + \frac{\pi(\alpha_1^2-\alpha^2)\Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

410.07 $$\left\{\begin{aligned} &\int_0^K \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2} = V_2 = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)} \times \\ &\times\left[\alpha^2 E + \frac{2k^4\alpha^2 - 2k^4 + \alpha^4 k'^2}{k^2-\alpha^2} K - \frac{\pi(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,\alpha^2 \Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\right].\end{aligned}\right.$$

* The function $\Lambda_0(\psi, k)$ is discussed in detail in **150**, and is tabulated in the Appendix.

410.08 $$\int_0^K \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^2}{(1-\alpha^2 \operatorname{sn}^2 u)^2}\, du = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2 (\alpha^2-\alpha_1^2)\, \Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2 V_2\right]$$

where $\Pi(\alpha^2, k)$ is given by **410.01** and V_2 is obtained from **410.07.**

Case I: $0 < -\alpha^2 < \infty$; alternate formulas:

$$\boxed{\begin{aligned} &\Lambda_0(\beta, k) = \frac{2}{\pi}\left[E\, F(\beta, k') + K\, E(\beta, k') - K\, F(\beta, k')\right], \\ &\qquad \beta = \sin^{-1}\frac{1}{\sqrt{1-\alpha^2}}. \end{aligned}}$$

411.01 $$\Pi(\alpha^2, k) = \int_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{K}{1-\alpha^2} + \frac{\pi \alpha^2 \left[\Lambda_0(\beta, k) - 1\right]}{2\sqrt{\alpha^2 (1-\alpha^2)(\alpha^2-k^2)}}, \quad (\alpha^2 < 0),$$
[cf. **410.01**].

Special case when $-\alpha^2 = k$:

$$\Pi(-k, k) = \int_0^K \frac{du}{1+k \operatorname{sn}^2 u} = \frac{1}{4(1+k)}\left[\pi + 2(1+k)K\right].$$

411.02 $$\int_0^K \frac{\operatorname{sn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{K}{1-\alpha^2} + \frac{\pi\left[\Lambda_0(\beta, k) - 1\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

411.03 $$\int_0^K \frac{\operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{\pi(1-\alpha^2)\left[1-\Lambda_0(\beta, k)\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

411.04 $$\int_0^K \frac{\operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{k'^2 K}{1-\alpha^2} + \frac{\pi(k^2-\alpha^2)\left[1-\Lambda_0(\beta, k)\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

411.05 $$\int_0^K \frac{\operatorname{sd}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{1}{k'^2(k^2-\alpha^2)}\left\{E - \frac{k'^2 K}{1-\alpha^2} - \frac{\pi k'^2 \alpha^2\left[\Lambda_0(\beta, k)-1\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\right\}.$$

411.06 $$\int_0^K \frac{1-\alpha_1^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\, du = \frac{1-\alpha_1^2}{1-\alpha^2} K + \frac{(\alpha_1^2-\alpha^2)\,\pi\left[1-\Lambda_0(\beta, k)\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

$$\textbf{411.07}\quad \begin{cases} \displaystyle\int_0^K \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2} = V_2 = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)} \times \Big\{\alpha^2 E - \\ \displaystyle\quad - \frac{2k^2-\alpha^2-\alpha^2 k^2}{1-\alpha^2} K + \frac{\alpha^2\pi\,(3k^2-2\alpha^2 k^2-2\alpha^2+\alpha^4)\,[\Lambda_0(\beta,k)-1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\Big\}. \end{cases}$$

$$\textbf{411.08}\quad \begin{cases} \displaystyle\int_0^K \frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^2}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}\,du \\ \displaystyle\qquad = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2(\alpha^2-\alpha_1^2)\,\Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2 V_2\right]. \end{cases}$$

where $\Pi(\alpha^2, k)$ is given by **411.01** and V_2 is obtained from **411.07**.

Case II: $(k^2 < \alpha^2 < 1)$.

$$\boxed{\begin{aligned} &\Lambda_0(\vartheta,k) = \frac{2}{\pi}\left[E\,F(\vartheta,k') + K\,E(\vartheta,k') - K\,F(\vartheta,k')\right], \\ &\qquad \vartheta = \sin^{-1}\sqrt{\frac{1-\alpha^2}{k'^2}}\,. \end{aligned}}$$

$$\textbf{412.01}\quad \Pi(\alpha^2,k) = \int_0^K \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = K + \frac{\pi\alpha\,[1-\Lambda_0(\vartheta,k)]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}, \quad [\text{cf. } \textbf{413.01}].$$

Special cases when $\alpha^2 = k^2$ and $\alpha^2 = k$:

$$\Pi(k^2,k) = \int_0^K \mathrm{nd}^2 u\,du = \frac{E}{k'^2};$$

$$\Pi(k,k) = \int_0^K \frac{du}{1-k\,\mathrm{sn}^2 u} = \frac{1}{4(1-k)}\left[\pi + 2(1-k)K\right].$$

$$\textbf{412.02}\quad \int_0^K \frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{\pi\,[1-\Lambda_0(\vartheta,k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

$$\textbf{412.03}\quad \int_0^K \frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = K - \frac{\pi(1-\alpha^2)\,[1-\Lambda_0(\vartheta,k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

$$\textbf{412.04}\quad \int_0^K \frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = K + \frac{\pi(\alpha^2-k^2)\,[1-\Lambda_0(\vartheta,k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

412.05 $$\int_0^K \frac{\operatorname{sd}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{1}{k'^2(k^2-\alpha^2)}\left\{E - k'^2 K - \frac{\pi\alpha k'^2[1-\Lambda_0(\vartheta,k)]}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}\right\}.$$

412.06 $$\int_0^K \frac{1-\alpha_1^2\operatorname{sn}^2 u}{1-\alpha^2\operatorname{sn}^2 u}\,du = K + \frac{\pi(\alpha^2-\alpha_1^2)[1-\Lambda_0(\vartheta,k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}.$$

412.07 $$\begin{cases}\displaystyle\int_0^K \frac{du}{(1-\alpha^2\operatorname{sn}^2 u)^2} = V_2 = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)}\Big\{\alpha^2 E + (2\alpha^2 k^2 - 2k^2 - \\ \displaystyle\quad -\alpha^4+\alpha^2)K - \frac{\pi\alpha(\alpha^4-2\alpha^2k^2-2\alpha^2+3k^2)[1-\Lambda_0(\vartheta,k)]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}\Big\}.\end{cases}$$

412.08 $$\int_0^K \frac{(1-\alpha_1^2\operatorname{sn}^2 u)^2}{(1-\alpha^2\operatorname{sn}^2 u)^2}\,du = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2(\alpha^2-\alpha_1^2)\,\Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2 V_2\right],$$

where $\Pi(\alpha^2, k)$ is given by **412.01** and V_2 by **412.07**.

Case II: $k^2 < \alpha^2 < 1$; alternate formulas:

$$\boxed{\begin{gathered}\Lambda_0(\xi,k) = \frac{2}{\pi}[E\,F(\xi,k') + K\,E(\xi,k') - K\,F(\xi,k')],\\ \xi = \sin^{-1}\sqrt{\frac{\alpha^2-k^2}{\alpha^2 k'^2}}.\end{gathered}}$$

413.01 $$\Pi(\alpha^2,k) = \int_0^K \frac{du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{\alpha\pi\Lambda_0(\xi,k)}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}, \qquad [\text{cf. } \mathbf{412.01}].$$

Special cases when $\alpha^2 = k^2$ and $\alpha^2 = k$:

$$\Pi(k^2,k) = \int_0^K \operatorname{nd}^2 u\,du = \frac{E}{k'^2};$$

$$\Pi(k,k) = \int_0^K \frac{du}{1-k\operatorname{sn}^2 u} = \frac{1}{4(1-k)}[\pi + 2(1-k)K].$$

413.02 $$\int_0^K \frac{\operatorname{sn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{\pi\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}} - \frac{K}{\alpha^2}.$$

413.03 $$\int_0^K \frac{\operatorname{cn}^2 u\,du}{1-\alpha^2\operatorname{sn}^2 u} = \frac{K}{\alpha^2} + \frac{\pi(\alpha^2-1)\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}}.$$

413.04 $$\int_0^K \frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{k^2 K}{\alpha^2} + \frac{\pi(\alpha^2-k^2)\,\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}}.$$

413.05 $$\int_0^K \frac{\mathrm{sd}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{1}{k'^2(k^2-\alpha^2)}\left[E - \frac{\alpha\pi k'^2\Lambda_0(\xi,k)}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}\right].$$

413.06 $$\int_0^K \frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du = \frac{\alpha_1^2 K}{\alpha^2} + \frac{\pi(\alpha^2-\alpha_1^2)\,\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}}.$$

413.07 $$\begin{cases} \displaystyle\int_0^K \frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2} = V_2 = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)} \\ \displaystyle\qquad\times\left[\alpha^2 E + (k^2-\alpha^2)K + \frac{\pi\alpha(2\alpha^2+2\alpha^2k^2-3k^2-\alpha^4)\,\Lambda_0(\xi,k)}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}\right]. \end{cases}$$

413.08 $$\int_0^K \frac{(1-\alpha_1^2\,\mathrm{sn}^2 u)^2}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}\,du = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2(\alpha^2-\alpha_1^2)\,\Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2 V_2\right].$$

[See **413.01** and **413.07**.]

Case III: $0<\alpha^2<k^2$.

$$\boxed{KZ(\beta,k) = KE(\beta,k) - EF(\beta,k); \quad \beta = \sin^{-1}\left(\frac{\alpha}{k}\right).}$$

414.01* $$\Pi(\alpha^2,k) = \int_0^K \frac{du}{1-\alpha^2\,\mathrm{sn}^2 u} = K + \frac{\alpha KZ(\beta,k)}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}}.$$

Special case when $\alpha^2=k^2$:

$$\Pi(k^2,k) = \int_0^K \mathrm{nd}^2 u\,du = \frac{E}{k'^2}.$$

414.02 $$\int_0^K \frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = \frac{KZ(\beta,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}.$$

414.03 $$\int_0^K \frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u} = K + \frac{(\alpha^2-1)\,KZ(\beta,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}.$$

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* A tabulation of $KZ(\beta,k)$ appears in the Appendix. See **140** for information about the function $Z(\beta,k)$.

414.04 $\displaystyle\int_0^K \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = K + \frac{(\alpha^2 - k^2)\, K Z(\beta, k)}{\sqrt{\alpha^2 (1-\alpha^2)(k^2-\alpha^2)}}.$

414.05 $\displaystyle\int_0^K \frac{\mathrm{sd}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{1}{k'^2 (k^2-\alpha^2)}\left[E - \frac{k'^2 \alpha K Z(\beta, k)}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}} - k'^2 K\right].$

414.06 $\displaystyle\int_0^K \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du = K + \frac{(\alpha^2 - \alpha_1^2)\, K Z(\beta, k)}{\sqrt{\alpha^2 (1-\alpha^2)(k^2-\alpha^2)}}.$

414.07 $$\left\{\begin{aligned}&\int_0^K \frac{du}{(1-\alpha^2 \mathrm{sn}^2 u)^2} = V_2 = \frac{1}{2(1-\alpha^2)(k^2-\alpha^2)}\Big[-\alpha^2 E + (\alpha^4 - 2\alpha^2 k^2 + \\ &\qquad + 2k^2 - \alpha^2) K + \frac{(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2)\, \alpha^2 K Z(\beta, k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}\Big].\end{aligned}\right.$$

414.08 $\displaystyle\int_0^K \frac{(1-\alpha_1^2 \mathrm{sn}^2 u)^2}{(1-\alpha^2 \mathrm{sn}^2 u)^2}\, du = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2(\alpha^2-\alpha_1^2)\Pi(\alpha^2, k) + (\alpha^2-\alpha_1^2)^2 V_2\right].$

[See **414.01** and **414.07**.]

Case IV: $\infty > \alpha^2 > 1$.

$$\boxed{K Z(A, k) = K E(A, k) - E F(A, k);\ A = \sin^{-1}(1/\alpha).}$$

415.01 $\displaystyle\Pi(\alpha^2, k) = \int_0^K \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} = -\frac{\alpha K Z(A, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}.$

415.02 $\displaystyle\int_0^K \frac{\mathrm{sn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = -\frac{K}{\alpha^2}\left[1 + \frac{\alpha Z(A, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].$

415.03 $\displaystyle\int_0^K \frac{\mathrm{cn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{K}{\alpha^2}\left[1 + \frac{\alpha(1-\alpha^2) Z(A, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].$

415.04 $\displaystyle\int_0^K \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{K}{\alpha^2}\left[k^2 + \frac{\alpha(k^2-\alpha^2) Z(A, k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].$

415.05 $\displaystyle\int_0^K \frac{\mathrm{sd}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{1}{k'^2 (k^2-\alpha^2)}\left[E + \frac{k'^2 \alpha Z(A, k) K}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].$

415.06 $$\int_0^K \frac{1-\alpha_1^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\, du = \frac{K}{\alpha^2}\left[\alpha_1^2 - \frac{(\alpha^2-\alpha_1^2)\,\alpha\, Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].$$

415.07 $$\left\{\begin{aligned} &\int_0^K \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = V_2 = \frac{1}{2(1-\alpha^2)(k^2-\alpha^2)} \times \\ &\quad\times\left[(\alpha^2-k^2)K - \alpha^2 E + \frac{(2\alpha^2+2\alpha^2 k^2 - 3k^2 - \alpha^4)\,\alpha K Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right].\end{aligned}\right.$$

415.08 $$\left\{\begin{aligned} &\int_0^K \frac{(1-\alpha_1^2 \operatorname{sn}^2 u)^2}{(1-\alpha^2 \operatorname{sn}^2 u)^2}\, du \\ &\quad = \frac{1}{\alpha^4}\left[\alpha_1^4 K + 2\alpha_1^2(\alpha^2-\alpha_1^2)\,\Pi(\alpha^2,k) + (\alpha^2-\alpha_1^2)^2 V_2\right],\end{aligned}\right.$$

where $\Pi(\alpha^2, k)$ is given by **415.01** and V_2 by **415.07.**

Case V: Complex parameter.

416.00 $$\left\{\begin{aligned} &a\,\Pi(\alpha^2,k) + \bar{a}\,\Pi(\bar{\alpha}^2,k) = a\int_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} + \bar{a}\int_0^K \frac{du}{1-\bar{\alpha}^2 \operatorname{sn}^2 u} \\ &\quad = \frac{2}{m_1 m_2 (s_1 t_2 - s_2 t_1)}\left\{[m_1(s_1 b_1 + t_1 a_1) - m_2(a_1 t_2 + b_1 s_2)]K + \right. \\ &\quad\quad \left. + m_1 m_2 [n_2(a_1 t_1 + s_1 b_1)\,\Pi(\alpha_2^2,k) - n_1(a_1 t_2 + b_1 s_2)\,\Pi(\alpha_1^2,k)]\right\}.\end{aligned}\right.$$

417.00 $$\left\{\begin{aligned} &a\,\Pi(\alpha^2,k) - \bar{a}\,\Pi(\bar{\alpha}^2,k) \\ &\quad = \frac{2i}{m_1 m_2 (s_1 t_2 - s_2 t_1)}\left\{[b_1(m_1 t_1 - m_2 t_2) - a_1(s_1 m_1 - s_2 m_2)]K + \right. \\ &\quad\quad \left. + m_1 m_2 [n_1(a_1 s_2 - b_1 t_2)\,\Pi(\alpha_1^2,k) + n_2(b_1 t_1 - a_1 s_1)\,\Pi(\alpha_2^2,k)]\right\},\end{aligned}\right.$$

where

$$a = a_1 + i\,b_1, \quad \bar{a} = a_1 - i\,b_1; \quad \alpha^2 = -\gamma_1 - i\gamma_2, \quad r^2 = \gamma_1^2 + \gamma_2^2,$$

$$\alpha_1^2 = \frac{k^2 m_1^2}{r^2}, \quad \alpha_2^2 = \frac{k^2 m_2^2}{r^2}, \quad s_1 = m_1 - m_1 n_1 - 1, \quad s_2 = m_2 - m_2 n_2 - 1,$$

$$n_1 = \frac{m_1[\alpha_1^4 - (2+m_1)\alpha_1^2 + (1-2\alpha_1^2)k^2] - r^2}{m_1(r^2 + 2\gamma_1\alpha_1^2 + \alpha_1^4)},$$

$$n_2 = \frac{m_2[\alpha_2^4 - (2+m_2)\alpha_2^2 + (1-2\alpha_2^2)k^2] - r^2}{m_2(r^2 + 2\gamma_1\alpha_2^2 + \alpha_2^4)},$$

$$t_1 = \frac{m_1^2 + (\gamma_1 + 2 - \alpha_1^2)\, m_1 + \gamma_1 + n_1 m_1 (\gamma_1 + \alpha_1^2)}{m_1 \gamma_2},$$

$$t_2 = \frac{m_2^2 + (\gamma_1 + 2 - \alpha_2^2)\, m_2 + \gamma_1 + n_2 m_2 (\gamma_1 + \alpha_2^2)}{m_2 \gamma_2},$$

m_1, m_2 being the two real roots of the quadratic equation

$$(r^2 + 2k^2\gamma_1 + k^2)\, m^2 + 2k'^2 r^2 m - r^2(r^2 + 2\gamma_1 + k^2) = 0.$$

Taking $m_2^2 > m_1^2$, we have $1 > \alpha_2^2 > k^2 > \alpha_1^2 > 0$. Hence case II applies to $\Pi(\alpha_2^2, k)$ and case III to $\Pi(\alpha_1^2, k)$.

Special case:

If $r^2 + 2\gamma_1 + k^2 = 0$, then $m_1 = 0$, $\alpha_1^2 = 0$; thus

418.00 $$\begin{cases} a\,\Pi(\alpha^2, k) + \bar a\,\Pi(\bar\alpha^2, k) = \dfrac{2}{m_2 r^2 (s_2 t_1 - t_2 s_1)} \{[a_1(k^2 t_2 m_2 - r^2 t_1) + \\ \qquad + b_1(k^2 s_2 m_2 - s_1 r^2)]\, K + n_2 m_2 r^2 (a_1 t_1 + b_1 s_1)\, \Pi(\alpha_2^2, k)\}. \end{cases}$$

419.00 $$\begin{cases} a\,\Pi(\alpha^2, k) - \bar a\,\Pi(\bar\alpha^2, k) = \dfrac{2i}{m_2 r^2 (s_2 t_1 - t_2 s_1)} \{[a_1(s_1 r^2 - k^2 s_2 m_2) + \\ \qquad + b_1(k^2 t_2 m_2 - r^2 t_1)]\, K + n_2 m_2 r^2 (a_1 s_1 - b_1 t_1)\, \Pi(\alpha_2^2, k)\}. \end{cases}$$

Incomplete Integrals.

Case I: $0 < -\alpha^2 < k$.

$$\Lambda_0(\psi, k) = \frac{2}{\pi}\,[E\,F(\psi, k') + K\,E(\psi, k') - K\,F(\psi, k')],\quad q = e^{-(\pi K'/K)},$$

$$\psi = \sin^{-1}\sqrt{\frac{\alpha^2}{\alpha^2 - k^2}},\qquad v = (\pi u_1)/2K,\qquad w = [\pi F(\psi, k')]/2K,$$

$$\Omega_1 = \frac{iK}{\pi} \ln \frac{H[u_1 - iF(\psi, k')]}{H[u_1 + iF(\psi, k')]} = \frac{iK}{\pi} \ln \frac{\vartheta(v - iw)}{\vartheta(v + iw)}$$

$$= \frac{2K}{\pi} \tan^{-1} \left\{ \frac{2 \sum_1^\infty (-1)^{m+1} q^{m^2} \sin(2mv) \sinh(2mw)}{1 + 2\sum_1^\infty (-1)^m q^{m^2} \cos(2mv) \cosh(2mw)} \right\}.$$

431.01 $$\Pi(u_1, \alpha^2) = \int_0^{u_1} \frac{du}{1 - \alpha^2 \operatorname{sn}^2 u} = \frac{u_1 k^2}{k^2 - a^2} - \frac{\pi \alpha^2 [u_1 \Lambda_0(\psi, k) + \Omega_1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}\, K},\quad (\alpha^2 < 0).$$

[See **363.01** for special case when $\alpha^2 = -k$.]

431.02 $$\int_0^{u_1} \frac{\operatorname{sn}^2 u\, du}{1 - \alpha^2 \operatorname{sn}^2 u} = \frac{u_1}{k^2 - \alpha^2} - \frac{\pi [u_1 \Lambda_0(\psi, k) + \Omega_1]}{2\sqrt{\alpha^2(1 - \alpha^2)(\alpha^2 - k^2)}\, K}.$$

431.03 $$\int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{k'^2 u_1}{\alpha^2 - k^2} + \frac{\pi(1-\alpha^2)\,[u_1 \Lambda_0(\psi,k) + \Omega_1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

431.04 $$\int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{\pi(k^2-\alpha^2)\,[u_1 \Lambda_0(\psi,k) + \Omega_1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

431.05 $$\int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du = \frac{k^2-\alpha_1^2}{k^2-\alpha^2} u_1 + \frac{\pi(\alpha_1^2-\alpha^2)\,[u_1 \Lambda_0(\psi,k) + \Omega_1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

431.06 $$\left\{\begin{aligned} &\int_0^{u_1} \frac{du}{(1-\alpha^2 \mathrm{sn}^2 u)^2} = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)} \left\{\alpha^2 E(u_1) + \frac{2k^4\alpha^2 - 2k^4 + \alpha^4 k'^2}{k^2-\alpha^2} u_1 - \right.\\ &\qquad \left. - \frac{(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,\pi\alpha^2\,[u_1\Lambda_0(\psi,k)+\Omega_1]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}K} - \frac{\alpha^4 \mathrm{sn}\, u_1 \mathrm{cn}\, u_1 \mathrm{dn}\, u_1}{1-\alpha^2 \mathrm{sn}^2 u_1}\right\}. \end{aligned}\right.$$

Case I: $k < -\alpha^2 < \infty$.

$$\Lambda_0(\beta,k) = \frac{2}{\pi}[EF(\beta,k') + KE(\beta,k') - KF(\beta,k')], \quad \beta = \sin^{-1}\frac{1}{\sqrt{1-\alpha^2}}.$$

$$v = (\pi u_1)/2K, \quad w = [\pi F(\beta,k')]/2K, \quad q = e^{-2p}, \quad p = \pi K'/2K,$$

$$\Omega_5 = \frac{iK}{\pi} \ln \frac{H[u_1 + iF(\beta,k')]}{H[iF(\beta,k') - u_1]} = \frac{iK}{\pi} \ln \frac{\vartheta_1(v+iw)}{\vartheta_1(iw - v)}$$

$$= u_1 + \tan^{-1}\left[\frac{2\sum_1^\infty (-1)^{m+1} q^{m^2} \sin(2mv) \sinh[2m(p-w)]}{1 + 2\sum_1^\infty (-1)^m q^{m^2} \cos(2mv) \cosh[2m(p-w)]}\right].$$

432.01 $$\Pi(u_1,\alpha^2) = \int_0^{u_1} \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{u_1}{1-\alpha^2} + \frac{\pi\alpha^2\,[u_1 \Lambda_0(\beta,k) - \Omega_5]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}, \quad (\alpha^2 < 0).$$

Special case when $\alpha^2 = -k$:

$$\Pi(u_1, -k) = \int_0^{u_1} \frac{du}{1 + k\, \mathrm{sn}^2 u} = \frac{u_1}{2} + \frac{1}{2(1+k)} \tan^{-1}[(1+k)\,\mathrm{tn}\, u_1 \mathrm{nd}\, u_1].$$

432.02 $$\int_0^{u_1} \frac{\mathrm{sn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{u_1}{1-\alpha^2} + \frac{\pi[u_1 \Lambda_0(\beta,k) - \Omega_5]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

432.03 $$\int_0^{u_1} \frac{\operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{\pi(1-\alpha^2)\,[\Omega_5 - u_1 \Lambda_0(\beta, k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

432.04 $$\int_0^{u_1} \frac{\operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{k'^2 u_1}{1-\alpha^2} + \frac{\pi(k^2-\alpha^2)\,[\Omega_5 - u_1 \Lambda_0(\beta, k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

432.05 $$\int_0^{u_1} \frac{1-\alpha_1^2 \operatorname{sn}^2 u}{1-\alpha^2 \operatorname{sn}^2 u}\, du = \frac{1-\alpha_1^2}{1-\alpha^2} u_1 + \frac{\pi(\alpha_1^2-\alpha^2)\,[\Omega_5 - u_1 \Lambda_0(\beta, k)]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}.$$

432.06 $$\begin{cases} \displaystyle\int_0^{u_1} \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)} \Big\{\alpha^2 E(u_1) + \frac{\alpha^2 k^2 + \alpha^2 - 2k^2}{1-\alpha^2} u_1 - \\ \displaystyle\quad - \frac{\alpha^4 \operatorname{sn} u_1 \operatorname{cn} u_1 \operatorname{dn} u_1}{1-\alpha^2 \operatorname{sn}^2 u_1} - \frac{\alpha^2 \pi (3k^2 - 2\alpha^2 k^2 - 2\alpha^2 + \alpha^4)[u_1 \Lambda_0(\beta, k) - \Omega_5]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\,K}\Big\}. \end{cases}$$

Case II: $0 < k < \alpha^2 < 1$.

$$\boxed{\begin{aligned} &\Lambda_0(\vartheta, k) = \frac{2}{\pi}\,[EF(\vartheta, k') + KE(\vartheta, k') - KF(\vartheta, k')], \quad \vartheta = \sin^{-1}\sqrt{\frac{1-\alpha^2}{k'^2}}, \\ &v = (\pi u_1)/2K, \quad w = [\pi F(\vartheta, k')]/2K, \quad p = \pi K'/2K, \quad q = e^{-2p}, \\ &\Omega_2 = \frac{iK}{\pi} \ln \frac{H_1[u_1 + iF(\vartheta, k')]}{H_1[u_1 - iF(\vartheta, k')]} = \frac{iK}{\pi} \ln \frac{\vartheta_2(v + iw)}{\vartheta_2(v - iw)} \\ &\quad = u_1 - \frac{2K}{\pi} \tan^{-1}\left[\frac{2\sum_1^\infty q^{m^2} \sin(2mv) \sinh[2m(p-w)]}{1 + 2\sum_1^\infty q^{m^2} \cos(2mv) \cosh[2m(p-w)]}\right]. \end{aligned}}$$

433.01 $$\Pi(u_1, \alpha^2) = \int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = u_1 + \frac{\pi\alpha\,[\Omega_2 - u_1 \Lambda_0(\vartheta, k)]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}\,K},$$

[cf. **434.01**].

Special case when $\alpha^2 = k$:

$$\Pi(u_1, k) = \int_0^{u_1} \frac{du}{1-k \operatorname{sn}^2 u} = \frac{u_1}{2} + \frac{1}{2(1-k)} \tan^{-1}[(1-k) \operatorname{tn} u_1 \operatorname{nd} u_1].$$

433.02 $$\int_0^{u_1} \frac{\operatorname{sn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{\pi\,[\Omega_2 - u_1 \Lambda_0(\vartheta, k)]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

433.03 $$\int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = u_1 + \frac{\pi(1-\alpha^2)\,[u_1 \Lambda_0(\vartheta, k) - \Omega_2]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

433.04 $$\int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = u_1 + \frac{\pi(\alpha^2-k^2)\,[\Omega_2 - u_1 \Lambda_0(\vartheta, k)]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

433.05 $$\int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du = u_1 + \frac{\pi(\alpha^2-\alpha_1^2)\,[\Omega_2 - u_1 \Lambda_0(\vartheta, k)]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

433.06 $$\begin{cases} \displaystyle\int_0^{u_1} \frac{du}{(1-\alpha^2 \mathrm{sn}^2 u)^2} = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)} \Big\{ \alpha^2 E(u_1) + (2\alpha^2 k^2 - 2k^2 - \alpha^4 + \alpha^2) u_1 + \\ \displaystyle + \frac{\pi\alpha(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\,[\Omega_2 - u_1 \Lambda_0(\vartheta, k)]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}\,K} - \frac{\alpha^4 \mathrm{sn}\, u_1 \mathrm{cn}\, u_1 \mathrm{dn}\, u_1}{1-\alpha^2 \mathrm{sn}^2 u_1} \Big\}. \end{cases}$$

Case II: $0 < k^2 < \alpha^2 < k$.

$$\boxed{\begin{aligned} &\Lambda_0(\xi, k) = \frac{2}{\pi}\,[E F(\xi, k') + K E(\xi, k') - K F(\xi, k')], \quad \xi = \sin^{-1}\sqrt{\frac{\alpha^2-k^2}{\alpha^2 k'^2}}, \\ &v = \pi u_1/2K, \quad w = [\pi F(\xi, k')]/2K, \quad q = e^{-2p}, \quad p = \pi K'/2K, \\ &\Omega_6 = \frac{K}{i\pi} \ln \frac{\Theta_1[u_1 - i F(\xi, k')]}{\Theta_1[u_1 + i F(\xi, k')]} = \frac{K}{i\pi} \ln \frac{\vartheta_3(v - i w)}{\vartheta_3(v + i w)} \\ &\quad = \frac{2K}{\pi} \tan^{-1}\left[\frac{2\sum_1^\infty q^m \sin(2mv)\sinh(2mw)}{1 + 2\sum_1^\infty q^{m^2}\cos(2mv)\cosh(2mw)}\right]. \end{aligned}}$$

434.01 $$\Pi(u_1, \alpha^2) = \int_0^{u_1} \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{\alpha\pi\,[u_1 \Lambda_0(\xi, k) - \Omega_6]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}\,K}, \qquad [\text{cf. } \mathbf{433.01}].$$

Special cases for $\alpha^2 = k$ and $\alpha^2 = k^2$:

$$\Pi(u_1, k^2) = \int_0^{u_1} \frac{du}{1-k^2 \mathrm{sn}^2 u} = \int_0^{u_1} \mathrm{nd}^2 u\, du = \frac{1}{k'^2}\,[E(u_1) - k^2 \mathrm{sn}\, u_1 \mathrm{cd}\, u_1].$$

$$\Pi(u_1, k) = \int_0^{u_1} \frac{du}{1-k\, \mathrm{sn}^2 u} = \frac{u_1}{2} + \frac{1}{2(1-k)} \tan^{-1}[(1-k)\,\mathrm{tn}\, u_1 \mathrm{nd}\, u_1].$$

434.02 $$\int_0^{u_1} \frac{\operatorname{sn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{\pi\,[u_1 \Lambda_0(\xi, k) - \Omega_6]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K} - \frac{u_1}{\alpha^2}.$$

434.03 $$\int_0^{u_1} \frac{\operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{u_1}{\alpha^2} + \frac{\pi(\alpha^2-1)\,[u_1 \Lambda_0(\xi, k) - \Omega_6]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

434.04 $$\int_0^{u_1} \frac{\operatorname{dn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{k^2 u_1}{\alpha^2} + \frac{\pi(\alpha^2-k^2)\,[u_1 \Lambda_0(\xi, k) - \Omega_6]}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}\,K}.$$

434.05 $$\left\{\begin{aligned} &\int_0^{u_1} \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)} \Big\{\alpha^2 E(u_1) + (k^2-\alpha^2)\,u_1 - \\ &- \frac{\alpha^4 \operatorname{sn} u_1 \operatorname{cn} u_1 \operatorname{dn} u_1}{1-\alpha^2 \operatorname{sn}^2 u_1} + \frac{\alpha\pi(2\alpha^2+2\alpha^2 k^2 - 3k^2 - \alpha^4)\,[u_1 \Lambda_0(\xi, k) - \Omega_6]}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}\,K}\Big\}. \end{aligned}\right.$$

Case III: $0 < \alpha^2 < k^2$.

$$Z(\beta, k) = E(\beta, k) - \frac{E}{K} F(\beta, k), \quad \beta = \sin^{-1}(\alpha/k), \quad v = (\pi u_1)/2K,$$
$$w = [\pi F(\beta, k)]/2K, \quad p = \pi K'/2K, \quad q = e^{-2p},$$
$$\Omega_3 = \frac{1}{2} \ln \frac{\Theta[F(\beta, k) - u_1]}{\Theta[F(\beta, k) + u_1]} = \frac{1}{2} \ln \frac{\vartheta_0(w+v)}{\vartheta_0(v-w)} = \sum_{m=1}^{\infty} \frac{\sin(2mw)\sin(2mv)}{m \sinh(2mp)}.$$

435.01 $$\Pi(u_1, \alpha^2) = \int_0^{u_1} \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} = u_1 + \frac{\alpha\,[u_1 Z(\beta, k) - \Omega_3]}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}}.$$

Special case when $\alpha^2 = k^2$:

$$\Pi(u_1, k^2) = \int_0^{u_1} \frac{du}{1-k^2 \operatorname{sn}^2 u} = \int_0^{u_1} \operatorname{nd}^2 u\, du = \frac{1}{k'^2}\,[E(u_1) - k^2 \operatorname{sn} u_1 \operatorname{cd} u_1].$$

435.02 $$\int_0^{u_1} \frac{\operatorname{sn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = \frac{u_1 Z(\beta, k) - \Omega_3}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}.$$

435.03 $$\int_0^{u_1} \frac{\operatorname{cn}^2 u\, du}{1-\alpha^2 \operatorname{sn}^2 u} = u_1 + \frac{(\alpha^2-1)\,[u_1 Z(\beta, k) - \Omega_3]}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}$$

435.04 $$\int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = u_1 + \frac{(\alpha^2 - k^2)\,[u_1 Z(\beta, k) - \Omega_3]}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}.$$

435.05 $$\int_0^{u_1} \frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u}\, du = u_1 + \frac{(\alpha^2 - \alpha_1^2)\,[u_1 Z(\beta, k) - \Omega_3]}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}.$$

435.06 $$\left\{\begin{aligned} &\int_0^{u_1} \frac{du}{(1-\alpha^2 \mathrm{sn}^2 u)^2} \\ &\quad = \frac{1}{2(1-\alpha^2)(k^2-\alpha^2)} \Big\{(\alpha^4 - 2\alpha^2 k^2 + 2k^2 - \alpha^2)\, u_1 - \alpha^2 E(u_1) + \\ &\quad + \frac{\alpha^2(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2)\,[u_1 Z(\beta, k) - \Omega_3]}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} + \frac{\alpha^4 \mathrm{sn}\, u_1 \mathrm{cn}\, u_1 \mathrm{dn}\, u_1}{1-\alpha^2 \mathrm{sn}^2 u_1}\Big\}. \end{aligned}\right.$$

Case IV: $\infty > \alpha^2 > 1$.

$$Z(A, k) = E(A, k) - \frac{E}{K} F(A, k), \quad A = \sin^{-1}(1/\alpha), \quad v = (\pi u_1)/2K,$$
$$w = [\pi F(A, k)]/2K, \quad p = \pi K'/2K, \quad q = e^{-2p},$$
$$\Omega_4 = \frac{1}{2} \ln \frac{\vartheta_1(w+v)}{\vartheta_1(w-v)} = \frac{1}{2} \ln \frac{H[F(A,k) + u_1]}{H[F(A,k) - u_1]}$$
$$= \frac{1}{2} \ln \frac{\sin(w+v)}{\sin(w-v)} + \sum_{m=1}^{\infty} \frac{q^m \sin(2mw)\sin(2mv)}{m \sinh(2mp)}.$$

436.01 $$\Pi(u_1, \alpha^2) = \int_0^{u_1} \frac{du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{-\alpha\,[u_1 Z(A, k) - \Omega_4]}{\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}}.$$

Special case when $\alpha^2 = 1$:

$$\Pi(u_1, 1) = \int_0^{u_1} \frac{du}{1-\mathrm{sn}^2 u} = \int_0^{u_1} \mathrm{nc}^2 u\, du = \frac{1}{k'^2}[k'^2 u_1 - E(u_1) + \mathrm{dn}\, u_1 \mathrm{tn}\, u_1].$$

436.02 $$\int_0^{u_1} \frac{\mathrm{sn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = -\frac{1}{\alpha^2}\left\{u_1 + \frac{[u_1 Z(A, k) - \Omega_4]\,\alpha}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right\}.$$

436.03 $$\int_0^{u_1} \frac{\mathrm{cn}^2 u\, du}{1-\alpha^2 \mathrm{sn}^2 u} = \frac{1}{\alpha^2}\left\{u_1 + \frac{\alpha(1-\alpha^2)\,[u_1 Z(A, k) - \Omega_4]}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right\}.$$

436.04 $$\int_0^{u_1} \frac{\mathrm{dn}^2 u\, du}{1-\alpha^2 \,\mathrm{sn}^2 u} = \frac{1}{\alpha^2}\left\{k^2 u_1 + \frac{\alpha(k^2-\alpha^2)\,[u_1 Z(A,k) - \Omega_4]}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right\}.$$

436.05 $$\int_0^{u_1} \frac{1-\alpha_1^2 \,\mathrm{sn}^2 u}{1-\alpha^2 \,\mathrm{sn}^2 u}\, du = \frac{1}{\alpha^2}\left\{\alpha_1^2 u_1 + \frac{(\alpha_1^2-\alpha^2)\,\alpha\,[u_1 Z(A,k) - \Omega_4]}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right\}.$$

436.06 $$\begin{cases} \displaystyle\int_0^{u_1} \frac{du}{(1-\alpha^2 \,\mathrm{sn}^2 u)^2} = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\Big\{\alpha^2 E(u_1) + (k^2-\alpha^2)\, u_1 - \\ \displaystyle - \frac{\alpha^4 \,\mathrm{sn}\, u_1 \,\mathrm{cn}\, u_1 \,\mathrm{dn}\, u_1}{1-\alpha^2 \,\mathrm{sn}^2 u_1} + \frac{(3k^2+\alpha^4-2\alpha^2-2\,\alpha^2 k^2)\,\alpha\,[u_1 Z(A,k)-\Omega_4]}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\Big\}. \end{cases}$$

Case V: Complex parameter.

437.00 $$\begin{cases} \displaystyle a\,\Pi(u_1,\alpha^2) + \bar a\,\Pi(u_1,\bar\alpha^2) = a\int_0^{u_1}\frac{du}{1-\alpha^2 \,\mathrm{sn}^2 u} + \bar a\int_0^{u_1}\frac{du}{1-\bar\alpha^2 \,\mathrm{sn}^2 u} \\ \displaystyle \quad = \frac{2}{m_1 m_2 (s_1 t_2 - s_2 t_1)}\big\{[m_1(s_1 b_1 + t_1 a_1) - m_2(a_1 t_2 + b_1 s_2)]\, u_1 + \\ \quad + m_1 m_2\,[n_2(a_1 t_1 + s_1 b_1)\,\Pi(u_1,\alpha_2^2) - n_1(a_1 t_2 + b_1 s_2)\,\Pi(u_1,\alpha_1^2)] + \\ \quad + m_1 m_2\,[(a_1 t_2 + b_1 s_2)\,\tau_1 - (s_1 b_1 + t_1 a_1)\,\tau_2]\big\}. \end{cases}$$

438.00 $$\begin{cases} a\,\Pi(u_1,\alpha^2) - \bar a\,\Pi(u_1,\bar\alpha^2) \\ \displaystyle \quad = \frac{2i}{m_1 m_2 (s_1 t_2 - t_1 s_2)}\big\{[b_1(m_1 t_1 - m_2 t_2) - a_1(s_1 m_1 - s_2 m_2)]\, u_1 + \\ \quad + m_1 m_2\,[n_1(a_1 s_2 - b_1 t_2)\,\Pi(u_1,\alpha_1^2) + n_2(b_1 t_1 - a_1 s_1)\,\Pi(u_1,\alpha_2^2)] + \\ \quad + m_1 m_2\,[(b_1 t_2 - a_1 s_2)\,\tau_1 + (a_1 s_1 - b_1 t_1)\,\tau_2]\big\}, \end{cases}$$

where $a_1, b_1, m_1, \alpha_1^2, \alpha_2^2, n_1, n_2, s_1, s_2, t_1, t_2$ are given in **417.00**; and where the elementary integrals τ_1 and τ_2 are

$$\tau_1 = \int_0^{p_1}\frac{dx}{1+h_1 x^2} = \frac{1}{\sqrt{-h_1}}\tanh^{-1}\!\left(p_1\sqrt{-h_1}\right), \quad h_1 = \frac{(\alpha_1^2-k^2)(k^4+2k^2\gamma_1+r^2)}{k^2 k'^2}$$

$$\tau_2 = \int_0^{p_2}\frac{dx}{1+h_2 x^2} = \frac{1}{\sqrt{h_2}}\tan^{-1}\!\left(p_2\sqrt{h_2}\right), \quad h_2 = \frac{(\alpha_2^2-k^2)(k^4+2k^2\gamma_1+r^2)}{k^2 k'^2}$$

with

$$p_1 = \frac{\mathrm{sn}\, u_1 \,\mathrm{cd}\, u_1}{1+m_1 \,\mathrm{sn}^2 u_1}, \qquad p_2 = \frac{\mathrm{sn}\, u_1 \,\mathrm{cd}\, u_1}{1+m_2 \,\mathrm{sn}^2 u_1},$$

Taking $m_1^2 < m_2^2$, then $h_1 < 0 < h_2$.

Special case:

When $r^2+2\gamma_1+k^2=0$, we have $m_1=0$, $\alpha_1^2=0$, $h_1=-r^2$, and hence the following:

439.00 $$\begin{cases} a\,\Pi(u_1,\alpha^2)+\bar{a}\,\Pi(u_1,\bar{\alpha}^2) \\ \quad = \dfrac{2}{m_2 r^2(s_2 t_1 - t_2 s_1)}\left\{[a_1(k^2 t_2 m_2 - r^2 t_1)+b_1(k^2 s_2 m_2 - s_1 r^2)]\,u_1 + \right. \\ \quad + n_2 m_2 r^2 (a_1 t_1 + b_1 s_1)\,\Pi(u_1,\alpha_2^2) + m_2 r^2(a_1 t_1 + b_1 s_1)\,\tau_2 + \\ \quad \left. + m_2 r(a_1 t_2 + b_1 s_2)\tanh^{-1}[r\,\mathrm{sn}\,u_1\,\mathrm{cd}\,u_1]\right\}. \end{cases}$$

440.00 $$\begin{cases} a\,\Pi(u_1,\alpha^2)-\bar{a}\,\Pi(u_1,\bar{\alpha}^2) \\ \quad = \dfrac{2i}{m_2 r^2(s_2 t_1 - t_2 s_1)}\left\{[b_1(k^2 t_2 m_2 - t_1 r^2)+a_1(r^2 s_1 - s_2 m_2 k^2)]\,u_1 + \right. \\ \quad + r^2 m_2 n_2 (a_1 s_1 - b_1 t_1)\,\Pi(u_1,\alpha_2^2) + m_2 r^2(b_1 t_1 - a_1 s_1)\,\tau_2 + \\ \quad \left. + m_2 r(a_1 s_2 - b_1 t_2)\tanh^{-1}[r\,\mathrm{sn}\,u_1\,\mathrm{cd}\,u_1]\right\}. \end{cases}$$

In all the above integrals, we take $m_1^2 < m_2^2$; hence $1 > \alpha_2^2 > k^2 > \alpha_1^2 > 0$, $h_1 < 0 < h_2$, and case II applies to $\Pi(u_1,\alpha_2^2)$ and case III to $\Pi(u_1,\alpha_1^2)$.

Miscellaneous Elliptic Integrals Involving Trigonometric and Hyperbolic Integrands.

Single Integrals[1].

510.01
$$\int_0^{\varphi} \frac{\cos^{2m-1}\vartheta\, d\vartheta}{\sqrt{a^2 + b^2\tan^2\vartheta}} = \frac{1}{a}\int_0^{\varphi} \frac{\cos^{2m}\vartheta\, d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}}, \quad [b^2 > a^2;\ \text{see } \mathbf{282.05} \text{ with } n^2 = (b^2-a^2)/a^2]$$
$$= \frac{1}{a}\int_0^{u_1} \mathrm{cn}^{2m} u\, du, \quad \left[a^2 > b^2;\ \text{see } \mathbf{312.05},\ k^2 = \frac{a^2-b^2}{a^2},\ \varphi = \mathrm{am}\, u_1\right].$$

510.02
$$\int_0^{\varphi} \frac{\cos^{2m-1}\vartheta\, d\vartheta}{\sqrt{a^2 - b^2\tan^2\vartheta}} = \frac{1}{a}\int_0^{\varphi} \frac{\cos^{2m}\vartheta\, d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}.$$
[See **283.05** with $n^2 = (a^2+b^2)/a^2 > 1$.]

511.01
$$\int_{\varphi}^{\pi/2} \frac{\cos^{2m-1}\vartheta\, d\vartheta}{\sqrt{\alpha^2\tan^2\vartheta - \beta^2}} = \int_{\varphi}^{\pi/2} \frac{\cos^{2m}\vartheta\, d\vartheta}{\sqrt{(\alpha^2+\beta^2)\sin^2\vartheta - \beta^2}}.$$
[See **285.03** with $a^2 = \alpha^2+\beta^2$, $b^2 = \beta^2$, $\pi/2 > \varphi > \sin^{-1}\left(\beta/\sqrt{\alpha^2+\beta^2}\right)$].

512.01
$$\int_{\varphi}^{\pi/2} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{\alpha^2\sin^2\vartheta - \beta^2\cos^2\vartheta}} = \int_{\varphi}^{\pi/2} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{(\alpha^2+\beta^2)\sin^2\vartheta - \beta^2}}.$$
$\left[\text{See } \mathbf{285.10} \text{ with } a^2 = \alpha^2+\beta^2,\ b^2 = \beta^2,\ \pi/2 > \varphi > \sin^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right).\right]$

512.02
$$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{a^2\sin^2\vartheta + b^2\cos^2\vartheta}} = \frac{1}{b}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}, \quad \left[b^2 > a^2;\ \text{see } \mathbf{281.01} \text{ with } k^2 = \frac{b^2-a^2}{b^2}\right],$$
$$= \frac{1}{b}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\, d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}}, \quad \left[a^2 > b^2;\ \text{see } \mathbf{282.09} \text{ with } n^2 = \frac{a^2+b^2}{a^2} > 1\right].$$

[1] See **280** to **299** for other such integrals.

512.03 $$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2\cos^2\vartheta - b^2\sin^2\vartheta}} = \frac{1}{a}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}}.$$

$\left[\text{See } \mathbf{283.10} \text{ with } n^2 = \frac{a^2+b^2}{a^2} > 1\right].$

513.01 $$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2 - b^2\cos^2\vartheta}} = \frac{1}{\sqrt{a^2-b^2}}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}},$$

$\left[a^2 > b^2;\ \text{see } \mathbf{282.09} \text{ with } n^2 = \frac{b^2}{a^2-b^2}\right].$

513.02 $$\int_{\varphi}^{\pi/2} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2-b^2\cos^2\vartheta}} = \frac{1}{b}\int_0^{u_1} R(\mathrm{dn}^2 u)\,du,$$ $\Big[b^2 > a^2;\ \text{see } \mathbf{314} \text{ with } k^2 = a^2/b^2,$

$\operatorname{sn} u = \frac{\cos\vartheta}{k},\quad \pi/2 > \varphi > \cos^{-1}(a/b),$

$\operatorname{am} u_1 = \sin^{-1}\left(\frac{\cos\varphi}{k}\right)\Big].$

513.03 $$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2+b^2\cos^2\vartheta}} = \frac{1}{\sqrt{a^2+b^2}}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}.$$

[See **281.01** with $k^2 = b^2/(a^2+b^2)$.]

513.04 $$\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{a^2\cos^2\vartheta - b^2}} = \frac{1}{\sqrt{a^2-b^2}}\int_0^{\varphi} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-n^2\sin^2\vartheta}},$$

$\left[a^2 > b^2;\ \text{see } \mathbf{283.10} \text{ with } n^2 = \frac{a^2}{a^2-b^2}\right].$

514.01 $$\int_0^{\varphi} \frac{\sin^{2m-1}\vartheta\,d\vartheta}{\sqrt{a^2+b^2\cot^2\vartheta}} = \frac{1}{b}\int_0^{\varphi} \frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{1}{b}\int_0^{u_1} \operatorname{sn}^{2m} u\,du,$$ $[b^2 > a^2;$

see **310.05** with $k^2 = (b^2-a^2)/b^2$, $\operatorname{am} u_1 = \varphi$, $\sin\vartheta = \operatorname{sn} u]$

$$= \frac{1}{b}\int_0^{\varphi} \frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{1+n^2\sin^2\vartheta}}$$ $[a^2 > b^2;$ see **282.04** with $n^2 = (a^2-b^2)/b^2]$.

514.02 $$\int_{\varphi}^{\pi/2} \frac{\sin^{2m-1}\vartheta\,d\vartheta}{\sqrt{\alpha^2-\beta^2\cot^2\vartheta}} = \int_{\varphi}^{\pi/2} \frac{\sin^{2m}\vartheta\,d\vartheta}{\sqrt{(\alpha^2+\beta^2)\sin^2\vartheta - \beta^2}}.$$

$\left[\text{See } \mathbf{285.05} \text{ with } a^2 = \alpha^2+\beta^2,\ b^2 = \beta^2,\ \pi/2 > \varphi > \sin^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right).\right]$

515.01 $$\int_0^{\varphi} \frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{1+2a_1\cos\vartheta+a_1^2}} = \frac{2}{1+a_1}\int_0^{u_1} R(1-2\,\mathrm{sn}^2 u)\,du,$$

$$\left[1+a_1^2 > 2a_1;\ \text{see } \mathbf{289.09} \text{ with } k^2=\frac{4a_1}{(1+a_1)^2},\right.$$

$$\left.\mathrm{sn}\,u_1=\sin(\varphi/2),\ 2\,\mathrm{sn}^2 u = 1-\cos\vartheta\right].$$

515.02 $$\int_0^{\varphi} \frac{R(\cos\vartheta)\,d\vartheta}{\sqrt{1-2a_1\cos\vartheta+a_1^2}} = \frac{2}{1+a_1}\int_0^{u_2} R\left[\frac{2a_1-(1+a_1^2)\,k^2\,\mathrm{sn}^2 u}{2a_1(1-k^2\,\mathrm{sn}^2 u)}\right]du,$$

$$\left[1+a_1^2 > 2a_1,\ \text{see } \mathbf{291.08} \text{ with } k^2=\frac{4a_1}{(1+a_1)^2},\right.$$

$$\left.\mathrm{sn}\,u_2=\sqrt{\frac{2a_1(1-\cos\varphi)}{k^2(1-2a_1\cos\varphi+a_1^2)}}\right].$$

516.01 $$\int_{\varphi_1}^{\varphi_2} \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{(a_1+b_1\sin^2\vartheta)(a_2+b_2\sin^2\vartheta)}} = \frac{1}{2}\int_{\sin^2\varphi_1}^{\sin^2\varphi_2} \frac{R(t)\,dt}{\sqrt{(a_1+b_1t)(a_2+b_2t)(1-t)t}},$$

$[\sin^2\vartheta = t.$ See **250.**]

517.01 $$\int_{\varphi_1}^{\varphi_2} \frac{R(\cos^2\vartheta)\,d\vartheta}{\sqrt{(a_1+b_1\cos^2\vartheta)(a_2+b_2\cos^2\vartheta)}} = \frac{1}{2}\int_{\cos^2\varphi_2}^{\cos^2\varphi_1} \frac{R(t)\,dt}{\sqrt{(a_1+b_1t)(a_2+b_2t)(1-t)t}},$$

$[\cos^2\vartheta = t.$ See **250**].

518.01 $$\int_{\varphi_1}^{\varphi_2} \frac{R(\tan^2\vartheta)\,d\vartheta}{\sqrt{(a_1+b_1\tan^2\vartheta)(a_2+b_2\tan^2\vartheta)}} = \frac{1}{2}\int_{\tan^2\varphi_1}^{\tan^2\varphi_2} \frac{R(t)\,dt}{(1+t)\sqrt{(a_1+b_1t)(a_2+b_2t)t}},$$

$[\tan^2\vartheta = t.$ See **250**].

519.01 $$\int_{\varphi_1}^{\varphi_2} \frac{R(\sin\vartheta,\cos\vartheta,\tan\vartheta)\,d\vartheta}{\sqrt{a_0+a_1\cos\vartheta+a_2\sin\vartheta+a_3\cos^2\vartheta+a_4\sin\vartheta\cos\vartheta+a_5\sin^2\vartheta}}$$

$$=2\int_{t_1}^{t_2} \frac{R\left[\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2},\ \frac{2t}{1-t^2}\right]dt}{\sqrt{b_0t^4+b_1t^3+b_2t^2+b_3t+b_4}},\quad \text{[See } \mathbf{250}]$$

where R is a rational function and

$$t=\tan(\vartheta/2),\ t_1=\tan(\varphi_1/2),\ t_2=\tan(\varphi_2/2),\ b_0=a_0-a_1+a_3,$$

$$b_1=2(a_2-a_4),\ b_2=2(a_0-a_3+2a_5),\ b_3=2(a_2+a_4),\ b_4=a_0+a_1+a_3.$$

520.01 $$\left\{\begin{aligned}&\int_0^{\varphi}\frac{\cos 2m\vartheta\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=F(\varphi,k)-\frac{4m^2}{2!}\int_0^{u_1}\operatorname{sn}^2u\,du+\frac{4m^2(4m^2-2^2)}{4!}\int_0^{u_1}\operatorname{sn}^4u\,du-\\&\quad-\frac{4m^2(4m^2-2^2)(4m^2-4^2)}{6!}\int_0^{u_1}\operatorname{sn}^6u\,du+\cdots\text{ to } m+1 \text{ terms},\\&\qquad[\varphi=\operatorname{am}u_1;\ \text{see } \mathbf{310}.]\end{aligned}\right.$$

521.01 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt[4]{1-k^2\sin^2\vartheta}}=\frac{1}{\sqrt{2(1+k')}}\operatorname{cn}^{-1}\left[\frac{\sqrt{1-k^2\sin^2\varphi}+\sqrt{k'}}{(1+\sqrt{k'})\sqrt[4]{1-k^2\sin^2\varphi}},\ \frac{1+\sqrt{k'}}{\sqrt{2(1+k')}}\right]+$$
$$+\frac{1}{\sqrt{2(1+k')}}\operatorname{cn}^{-1}\left[\frac{\sqrt{1-k^2\sin^2\varphi}-\sqrt{k'}}{(1-\sqrt{k'})\sqrt[4]{1-k^2\sin^2\varphi}},\ \frac{1-\sqrt{k'}}{\sqrt{2(1+k')}}\right].$$

521.02 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt[3]{\cos\vartheta}}=\frac{3}{2}\int_y^{1}\frac{dt}{\sqrt{1-t^3}},\quad \left[y=\sqrt[3]{\cos^2\varphi},\ t=\sqrt[3]{\cos^2\vartheta},\ 0\le y<1,\ 0<\varphi\le\pi/2.\quad \text{See } \mathbf{244.00}.\right]$$

521.03 $$\int_0^{\varphi}\frac{d\vartheta}{\sqrt[3]{\sin\vartheta}}=\frac{3}{2}\int_0^{y}\frac{dt}{\sqrt{1-t^3}}=\frac{3}{2}\int_0^{1}\frac{dt}{\sqrt{1-t^3}}-\frac{3}{2}\int_y^{1}\frac{dt}{\sqrt{1-t^3}},$$
$$\left[y=\sqrt[3]{\sin^2\varphi},\ t=\sqrt[3]{\sin^2\vartheta},\ 0<y\le 1,\ 0<\varphi\le\pi/2.\ \text{See } \mathbf{244.00}.\right]$$

525.01 $$\int_0^{\Phi}\frac{R(\sinh\vartheta)\,d\vartheta}{\sqrt{a^2+b^2\sinh^2\vartheta}}=\frac{1}{a}\int_0^{\Phi}\frac{R(\sinh\vartheta)\,d\vartheta}{\sqrt{1+k'^2\sinh^2\vartheta}}.$$
$$[a^2>b^2;\ \text{see } \mathbf{295.01} \text{ with } k'^2=b^2/a^2.]$$

525.02 $$\int_{\Phi}^{\infty}\frac{d\vartheta}{\sqrt{a^2+b^2\sinh^2\vartheta}}=\frac{1}{b}\int_0^{\infty}\frac{d\vartheta}{\sqrt{\sinh^2\vartheta+k'^2}}.$$
$$[b^2>a^2;\ \text{see } \mathbf{295.40} \text{ with } k'^2=a^2/b^2.]$$

525.03 $$\int_0^{\Phi}\frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{a^2+b^2\cosh^2\vartheta}}=\frac{1}{\sqrt{a^2+b^2}}\int_0^{\Phi}\frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{k^2+k'^2\cosh^2\vartheta}}.$$
$$\left[\text{See } \mathbf{295.11} \text{ with } k^2=\frac{a^2}{a^2+b^2},\quad k'^2=\frac{b^2}{a^2+b^2}.\right]$$

525.04 $$\int_0^{\Phi}\frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{a^2-b^2\cosh^2\vartheta}}=\frac{1}{a}\int_0^{\Phi}\frac{R(\cosh\vartheta)\,d\vartheta}{\sqrt{1-k'^2\cosh^2\vartheta}}.$$
$$[a^2>b^2;\ \text{see } \mathbf{295.21} \text{ with } k'^2=b^2/a^2.]$$

525.05 $$\int_{\Phi}^{\infty} \frac{d\vartheta}{\sqrt{b^2 \cosh^2\vartheta - a^2}} = \frac{1}{b} \int_{\Phi}^{\infty} \frac{d\vartheta}{\sqrt{\cosh^2\vartheta - k^2}}.$$

[$b^2 > a^2$; see **295.30** with $k^2 = a^2/b^2$.]

525.06 $$\int_0^{\Phi} \frac{\sinh^{2m-1}\vartheta \, d\vartheta}{\sqrt{\alpha^2 + \beta^2 \coth^2\vartheta}} = \frac{1}{\sqrt{\alpha^2+\beta^2}} \int_0^{\sinh\Phi} \frac{t^{2m}\, dt}{\sqrt{(t^2+1)\left(t^2 + \frac{\beta^2}{\alpha^2+\beta^2}\right)}}.$$

[See **221.09**.]

525.07 $$\int_0^{\Phi} \frac{\cosh^{2m-1}\vartheta \, d\vartheta}{\sqrt{\alpha^2 + \beta^2 \tanh^2\vartheta}} = \frac{1}{\sqrt{\alpha^2+\beta^2}} \int_1^{\cosh\Phi} \frac{t^{2m}\, dt}{\sqrt{(t^2-1)\left(t^2 - \frac{\beta^2}{\alpha^2+\beta^2}\right)}}.$$

[See **216.06**.]

525.08 $$\int_{\Phi_1}^{\Phi} \frac{R(\sinh^2\vartheta)\, d\vartheta}{\sqrt{(a_1+b_1\sinh^2\vartheta)(a_2+b_2\sinh^2\vartheta)}} = \frac{1}{2} \int_{\sinh^2\Phi_1}^{\sinh^2\Phi} \frac{R(t)\, dt}{\sqrt{(a_1+b_1 t)(a_2+b_2 t)(t+1)\, t}}.$$

[$\sinh^2\vartheta = t$. See **250**.]

525.09 $$\int_{\Phi_1}^{\Phi} \frac{R(\cosh^2\vartheta)\, d\vartheta}{\sqrt{(a_1+b_1\cosh^2\vartheta)(a_2+b_2\cosh^2\vartheta)}} = \frac{1}{2} \int_{\cosh^2\Phi_1}^{\cosh^2\Phi} \frac{R(t)\, dt}{\sqrt{(a_1+b_1 t)(a_2+b_2 t)(t-1)\, t}}.$$

[$\cosh^2\vartheta = t$. See **250**.]

525.10 $$\int_{\Phi_1}^{\Phi} \frac{R(\tanh^2\vartheta)\, d\vartheta}{\sqrt{(a_1+b_1\tanh^2\vartheta)(a_2+b_2\tanh^2\vartheta)}} = \frac{1}{2} \int_{\tanh^2\Phi_1}^{\tanh^2\Phi} \frac{R(t)\, dt}{(1-t)\sqrt{(a_1+b_1 t)(a_2+b_2 t)\, t}}.$$

[$\tanh^2\vartheta = t$. See **230**.]

525.11 $$\int_{\Phi_1}^{\Phi} \frac{R(\coth^2\vartheta)\, d\vartheta}{\sqrt{(a_1+b_1\coth^2\vartheta)(a_2+b_2\coth^2\vartheta)}} = \frac{1}{2} \int_{\coth^2\Phi_1}^{\coth^2\Phi} \frac{R(t)\, dt}{(t-1)\sqrt{(a_1+b_1 t)(a_2+b_2 t)\, t}}.$$

[$\coth^2\vartheta = t$. See **230**.]

526.01 $$\left\{\begin{aligned} &\int_{\Phi_1}^{\Phi} \frac{R(\sinh\vartheta, \cosh\vartheta, \tanh\vartheta)\, d\vartheta}{\sqrt{a_0 + a_1\cosh\vartheta + a_2\sinh\vartheta + a_3\cosh^2\vartheta + a_4\sinh\vartheta\cosh\vartheta + a_5\sinh^2\vartheta}} \\ &\qquad = 2\int_{y_1}^{y} \frac{R\left[\frac{t^2-1}{2t}, \frac{t^2+1}{2t}, \frac{t^2-1}{t^2+1}\right] dt}{\sqrt{b_0 t^4 + b_1 t^3 + b_2 t^2 + b_3 t + b_4}}, \quad \text{[See 250.]} \end{aligned}\right.$$

where R is a rational function and

$$t = e^{\vartheta}, \quad y_1 = e^{\Phi_1}, \quad y = e^{\Phi},$$

$$b_0 = a_3 + a_4 + a_5, \quad b_1 = 2(a_1 + a_2), \quad b_2 = 2(2a_0 + a_3 - a_5),$$

$$b_3 = 2(a_1 - a_2), \quad b_4 = a_3 - a_4 + a_5.$$

Multiple Integrals.

530.01 $$\int_0^k\int_0^{\pi/2}\frac{t\,dt\,d\vartheta}{\sqrt{1-t^2\sin^2\vartheta}}=E-k'^2K,$$ [cf. **610.01**].

530.02 $$\int_0^k\int_0^{\pi/2}t\sqrt{1-t^2\sin^2\vartheta}\,dt\,d\vartheta=\frac{1}{3}\left[(1+k^2)E-k'^2K\right],$$ [cf. **611.01**].

530.03 $$\int_0^k\int_0^{\pi/2}\frac{\sqrt{1-t^2\sin^2\vartheta}}{1-t^2}\,dt\,d\vartheta=kK,$$ [cf. **612.01**].

530.04 $$\int_0^k\int_0^{\varphi}\frac{t\,dt\,d\vartheta}{\sqrt{1-t^2\sin^2\vartheta}}=E(\varphi,k)-k'^2F(\varphi,k)+\left[\sqrt{1-k^2\sin^2\varphi}-1\right]\cot\varphi,$$ [cf. **613.01**].

530.05 $$\begin{cases}\displaystyle\int_0^k\int_0^{\varphi}t\sqrt{1-t^2\sin^2\vartheta}\,dt\,d\vartheta=\frac{1}{3}\Big[(1+k^2)E(\varphi,k)-\\ \qquad -k'^2F(\varphi,k)-\left(1-\sqrt{1-k^2\sin^2\varphi}\right)\cot\varphi\Big],\end{cases}$$ [cf. **613.02**].

530.06 $$\begin{cases}\displaystyle\int_0^k\int_0^{\varphi}\frac{t\,dt\,d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-t^2\sin^2\vartheta}}=(k^2-\alpha^2)\,\Pi(\varphi,\alpha^2,k)+E(\varphi,k)-\\ \qquad -F(\varphi,k)+\left[-1+\sqrt{1-k^2\sin^2\varphi}\right]\cot\varphi,\end{cases}$$ [cf. **613.03**].

530.07 $$\int_0^k\int_0^{\pi/2}\frac{t\,dt\,d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-t^2\sin^2\vartheta}}=(k^2-\alpha^2)\,\Pi(\alpha^2,k)-K+E,$$ [cf. **612.14**].

530.08 $$\int_0^{\varphi}\int_0^{\varphi_1}\frac{d\varphi_1\,d\vartheta}{\sqrt{1-k^2\sin^2\varphi_1}\sqrt{1-k^2\sin^2\vartheta}}=\frac{[F(\varphi,k)]^2}{2}$$ [cf. **630.01**].

530.09 $$\int_0^{\pi/2}\int_0^{\varphi_1}\frac{d\varphi_1\,d\vartheta}{\sqrt{1-k^2\sin^2\varphi_1}\sqrt{1-k^2\sin^2\vartheta}}=\frac{K^2}{2}.$$ [cf. **633.08**].

530.10 $$\int_0^{\varphi}\int_0^{\varphi_1}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi_1}}\,d\varphi_1\,d\vartheta=\ln\frac{\Theta[F(\varphi,k)]}{\Theta(0)}+\frac{[F(\varphi,k)]^2E}{2K},$$ [cf. **630.02**].

where $\Theta(0)$ and $\Theta(u)$ are defined in the appendix.

530.11 $$\int_0^{\pi/2}\int_0^{\varphi_1}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi_1}}\,d\varphi_1\,d\vartheta=\frac{E\cdot K}{2}-\ln\sqrt{k'},$$ [cf. **633.09**].

530.12 $$\int_0^{\varphi}\int_0^{\varphi_1}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi_1}}\,d\varphi_1\,d\vartheta-\frac{E}{K}\int_0^{\varphi}\int_0^{\varphi_1}\frac{d\varphi_1\,d\vartheta}{\sqrt{1-k^2\sin^2\varphi_1}\sqrt{1-k^2\sin^2\vartheta}}=\ln\frac{\Theta[F(\varphi,k)]}{\Theta(0)},$$ [cf. **630.03**].

530.13 $$\int_0^{\pi/2}\int_0^{\varphi_1}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi_1}}\,d\varphi_1\,d\vartheta-\frac{E}{K}\int_0^{\pi/2}\int_0^{\varphi_1}\frac{d\varphi_1\,d\vartheta}{\sqrt{1-k^2\sin^2\varphi_1}\sqrt{1-k^2\sin^2\vartheta}}=-\ln\sqrt{k'},$$ [cf. **633.10**].

530.14 $$\int_0^{\pi/2}\int_0^{\varphi}\frac{\cot\varphi\,d\varphi\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}=\pi K'/4+K\ln\sqrt{k}.$$

530.15 $$\int_0^{\pi/2}\int_0^{\varphi}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi}}\sin\varphi\cos\varphi\,d\varphi\,d\vartheta=\frac{1}{2k^2}\left[(2-k^2)K-\left(2+\ln\sqrt{k'}\right)E\right].$$

530.16 $$\int_0^{\pi/2}\int_0^{\varphi}\frac{\sin\varphi\cos\varphi\,d\varphi\,d\vartheta}{(1-k^2\sin^2\varphi)\sqrt{1-k^2\sin^2\vartheta}}=\frac{K}{2k^2}\ln\left(1/\sqrt{k'}\right).$$

530.17 $$\int_0^{\pi/2}\int_0^{\varphi}\sqrt{\frac{1-k^2\sin^2\vartheta}{1-k^2\sin^2\varphi}}\sin\varphi\,d\varphi\,d\vartheta=\pi/2k'.$$

531.01 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{d\vartheta\,d\varphi}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}}=K^2\left(\sqrt{2}/2\right),$$ [cf. **615.08**].

531.02 $$\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{1-\sin^2\vartheta\sin^2\varphi}\,d\vartheta\,d\varphi=\frac{1}{8}\left[4K^2\left(\sqrt{2}/2\right)+\frac{\pi^2}{K^2\left(\sqrt{2}/2\right)}\right],$$ [cf. **615.10**].

531.03 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{\cos^2\varphi\,d\vartheta\,d\varphi}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}}=\frac{1}{8}\left[4K^2\left(\sqrt{2}/2\right)-\frac{\pi^2}{K^2\left(\sqrt{2}/2\right)}\right].$$

531.04 $$\int_0^{\pi/2}\int_0^{\pi/2}\cos^2\varphi\sqrt{1-\sin^2\vartheta\sin^2\varphi}\,d\vartheta\,d\varphi=\frac{K^2\left(\sqrt{2}/2\right)}{3}.$$

531.05 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{\cos^2\vartheta\cos^2\varphi\,d\vartheta\,d\varphi}{\sqrt{1-\sin^2\vartheta\sin^2\varphi}}=\frac{1}{8}\left[4K^2\left(\sqrt{2}/2\right)-\frac{3\pi^2}{K^2\left(\sqrt{2}/2\right)}\right].$$

531.06 $$\int_0^{\pi/2}\int_0^{\pi/2}\cos^2\vartheta\cos^2\varphi\sqrt{1-\sin^2\vartheta\sin^2\varphi}\,d\vartheta\,d\varphi = \frac{1}{72}\left[20K^2(\sqrt{2}/2)-\frac{9\pi^2}{K^2(\sqrt{2}/2)}\right].$$

531.07 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{d\vartheta\,d\varphi}{\sqrt{1-k_1^2\sin^2\vartheta-k_2^2\sin^2\varphi}}=\frac{2}{1+k_2'}K(k_3)\cdot K(k_4),$$

where

$$k_1^2+k_2^2<1,\quad k_3=\frac{k_1'-\sqrt{1-k_1^2-k_2^2}}{1+k_2'},\qquad k_4=\frac{k_1'+\sqrt{1-k_1^2-k_2^2}}{1+k_2'}.$$

531.08 $$\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{d\varphi\,d\psi\,d\vartheta}{1-\cos\vartheta\cos\varphi\cos\psi}=4\pi K^2(\sqrt{2}/2).$$

531.09 $$\int_0^{\pi}\int_0^{\pi}\frac{d\varphi\,d\vartheta}{4-\alpha^2(1-\cos\vartheta)(1-\cos\varphi)}=\frac{\pi}{2\alpha}K(1/\alpha),\qquad [\alpha>1].$$

531.10 $$\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{d\vartheta\,d\varphi\,d\psi}{3-\cos\vartheta\cos\varphi-\cos\vartheta\cos\psi-\cos\varphi\cos\psi}=\sqrt{3}\pi K^2(k),\quad k^2=\frac{2-\sqrt{3}}{4}.$$

531.11 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{d\vartheta\,d\varphi}{1-k^2\sin\varphi\sin\vartheta}\sqrt{\frac{(1+\sin\vartheta)(1+\sin\varphi)}{\sin\varphi\sin\vartheta}}=2\pi K.$$

531.12 $$\int_0^{\pi/2}\int_0^{\pi/2}\frac{(k^2\cos^2\vartheta+k'^2\cos^2\varphi)\,d\vartheta\,d\varphi}{\sqrt{(1-k^2\sin^2\vartheta)(1-k'^2\sin^2\varphi)}}=EK'+KE'-KK'=\pi/2.$$

531.13 $$\int_0^{1}\int_0^{\pi/2}\frac{dt\,d\vartheta}{\sqrt{1-t^2\sin^2\vartheta}}=2G,\quad (G=0.91596559;\ \text{Catalan's constant}),$$ [cf. **615.01**].

531.14 $$\int_0^{1}\int_0^{\pi/2}\sqrt{1-t^2\sin^2\vartheta}\,dt\,d\vartheta=G+1/2,$$ [cf. **615.02**].

531.15 $$\int_0^{1}\int_0^{\pi/2}\frac{dt\,d\vartheta}{\sqrt{\cos^2\vartheta+t^2\sin^2\vartheta}}=\pi^2/4,$$ [cf. **615.03**].

531.16 $$\int_0^1\int_0^{\pi/2}\sqrt{\cos^2\vartheta+t^2\sin^2\vartheta}\,dt\,d\vartheta=\pi^2/8,$$ [cf. **615.04**].

531.17 $$\int_0^1\int_0^{\pi/2}\frac{dt\,d\vartheta}{t\sqrt{1-t^2\sin^2\vartheta}}-\frac{\pi}{2}\int_0^1\frac{dt}{t}=\pi\ln 2-2G,$$ [cf. **615.05**].

531.18 $$\int_0^1\int_0^{\pi/2}\frac{dt\,d\vartheta}{(1+t)\sqrt{1-t^2\sin^2\vartheta}}=\pi^2/8,$$ [cf. **615.09**].

531.19 $$\int_0^1\int_0^{\varphi}\frac{t\,dt\,d\vartheta}{\sqrt{1-t^2\sin^2\vartheta}}=\frac{1-\cos\varphi}{\sin\varphi},$$ [cf. **616.03**].

531.20 $$\int_0^1\int_0^{\varphi}t\sqrt{1-t^2\sin^2\vartheta}\,dt\,d\vartheta=\frac{1}{3}\left[\frac{\sin^2\varphi+1-\cos\varphi}{\sin\varphi}\right],$$ [cf. **616.04**].

531.21 $$\left\{\begin{aligned}&\int_0^1\int_0^{\varphi}\frac{t\,dt\,d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-t^2\sin^2\vartheta}}=(1-\alpha^2)\,\Pi(\varphi,\alpha^2,1)-\alpha^2\,\Pi(\varphi,\alpha^2,0)+\\&\qquad+(1-\cos\varphi)/\sin\varphi-\ln(\tan\varphi+\sec\varphi).\end{aligned}\right.$$
[See **112.02**, **111.04**, and cf. **616.05**.]

532.01 $$\int_0^k\int_0^{\infty}\frac{t\,dt\,d\vartheta}{\sqrt{\cosh^2\vartheta-t^2}}=E-k'^2K.$$

532.02 $$\int_0^k\int_0^{\infty}\frac{t^3\,dt\,d\vartheta}{\cosh^2\vartheta\sqrt{\cosh^2\vartheta-t^2}}=\frac{1}{3}\left[(2-k^2)E-2k'^2K\right].$$

532.03 $$\left\{\begin{aligned}&\int_0^{\ln(1+\sqrt{2})}\int_0^{\ln(1+\sqrt{2})}\frac{\cosh\vartheta\cosh\varphi\,d\vartheta\,d\varphi}{(1-k^2\sinh\vartheta\sinh\varphi)\sqrt{\sinh\vartheta\sinh\varphi\,(1-\sinh\vartheta)(1-\sinh\varphi)}}\\&\qquad=2\pi K.\end{aligned}\right.$$

532.04 $$\int_0^{\ln(1+\sqrt{2})}\int_0^{\ln(1+\sqrt{2})}\frac{\cosh\vartheta\cosh\varphi\,d\vartheta\,d\varphi}{\sqrt{(1-\sinh^2\varphi\sinh^2\vartheta)(1-\sinh^2\vartheta)(1-\sinh^2\varphi)}}=K^2\left(\sqrt{2}/2\right).$$

532.05 $$\left\{\begin{aligned}&\int_{\ln(-1+\sqrt{2})}^{\ln(1+\sqrt{2})}\int_{\ln(-1+\sqrt{2})}^{\ln(1+\sqrt{2})}\int_{\ln(-1+\sqrt{2})}^{\ln(1+\sqrt{2})}\frac{\cosh\vartheta\cosh\varphi\cosh\psi}{(1-\sinh\vartheta\sinh\varphi\sinh\psi)}\times\\&\qquad\times\frac{d\vartheta\,d\varphi\,d\psi}{\sqrt{(1-\sinh^2\vartheta)(1-\sinh^2\varphi)(1-\sinh^2\psi)}}=4\pi K^2\left(\sqrt{2}/2\right).\end{aligned}\right.$$

Elliptic Integrals Resulting from Laplace Transformations.

Finding the Laplace transform[1] of products of Bessel functions[2] often leads to the evaluation of elliptic integrals. We shall give here, however, only a short table of such integrals.

560.01 $$\int_0^\infty e^{-pt} J_0(rt)\, J_0(st)\, dt = \frac{1}{\pi\sqrt{rs}}\, Q_{-\frac{1}{2}}(Z) = \frac{k}{\pi\sqrt{rs}} \int_0^K du = \frac{kK}{\pi\sqrt{rs}},$$

where $Q_\gamma(Z)$ is Legendre's function[3] of the second kind of degree γ, and $k^2 = 2/(Z+1)$, $Z = (p^2 + r^2 + s^2)/2rs$.

560.02 $$\begin{cases} \displaystyle\int_0^\infty e^{-pt} J_1(rt)\, J_1(st)\, dt = \frac{1}{\pi\sqrt{rs}}\, Q_{\frac{1}{2}}(Z) = \frac{k^3}{\pi\sqrt{rs}} \int_0^K \operatorname{sn}^2 u \operatorname{cd}^2 u\, du \\ \displaystyle\qquad = \frac{1}{k\pi\sqrt{rs}} \left[(1 + k'^2) K - 2E\right]. \end{cases}$$

560.03 $$\begin{cases} \displaystyle\int_0^\infty e^{-pt} J_2(rt)\, J_2(st)\, dt = \frac{1}{\pi\sqrt{rs}}\, Q_{\frac{3}{2}}(Z) = \frac{k^5}{\pi\sqrt{rs}} \int_0^K \operatorname{sn}^4 u \operatorname{cd}^4 u\, du \\ \displaystyle\qquad = \frac{1}{3k^3\pi\sqrt{rs}} \left[(3 + 10k'^2 + 3k'^4) K + 8(1 + k'^2) E\right]. \end{cases}$$

---

[1] A function $f(p)$ is called the *Laplace Transform* of $g(t)$ if

$$f(p) = \int_0^\infty e^{-pt} g(t)\, dt.$$

[2] For the definition of Bessel functions, see, for example, N. W. McLachlan's *Bessel Functions for Engineers*, Oxford University Press, 1946.

[3] See *A Course of Modern Analysis* by E. T. Whittaker and G. N. Watson, Macmillan, New York, 1943.

560.04 $$\begin{cases}\displaystyle\int_0^\infty e^{-pt} J_\gamma(rt)\, J_\gamma(st)\, dt = \frac{1}{\pi\sqrt{rs}}\, Q_{\gamma-\frac{1}{2}}(Z) = \frac{k^{2\gamma+1}}{\pi\sqrt{rs}}\int_0^K \operatorname{sn}^{2\gamma} u \operatorname{cd}^{2\gamma} u\, du, \\ \qquad [\gamma \geqq 0;\quad \text{R.P. } (p \pm ir \pm is) > 0]\end{cases}$$

where k^2 is as in **560.01** and γ is an integer;

$$\int_0^K \operatorname{sn}^{2\gamma} u \operatorname{cd}^{2\gamma} u\, du = \frac{1}{k^{4\gamma}} \sum_{j=0}^{\gamma} \sum_{\lambda=0}^{\gamma} \binom{\gamma}{j}\binom{\gamma}{\lambda} (-1)^{j+\gamma+\lambda} (k'^2)^{\gamma-\lambda} \int_0^K \operatorname{nd}^{2\gamma-j-\lambda} u\, du$$

with the integral on the right given by **315.05**.

561.01 $$\int_0^\infty e^{-pt}[J_0(rt)]^2\, dt = \frac{2}{\pi\sqrt{p^2+4r^2}} \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{kK}{\pi r}, \qquad [p > 0;\ r \text{ real}],$$

where $k^2 = 4r^2/(p^2+4r^2)$.

561.02 $$\begin{cases}\displaystyle\int_0^\infty e^{-pt}[J_1(rt)]^2\, dt = -\frac{2}{\pi\sqrt{p^2+4r^2}} \int_0^{\pi/2} \frac{\cos 2\vartheta\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \\ \displaystyle\qquad = \frac{1}{\pi r K}\,[(1+k'^2)\, K - 2E].\end{cases}$$

561.03 $$\int_0^\infty e^{-pt}[J_n(rt)]^2\, dt = \frac{2(-1)^n}{\pi\sqrt{p^2+4r^2}} \int_0^{\pi/2} \frac{\cos 2n\vartheta\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}, \qquad \text{[See } \mathbf{806.01}.]$$

561.04 $$\int_0^\infty e^{-pt}\{[J_0(rt)]^2 - [J_1(rt)]^2\}\, dt = \frac{2k}{\pi r} \int_0^K \operatorname{cn}^2 u\, du = \frac{2}{\pi r k}\,[E - k'^2 K],$$

where $k^2 = 4r^2/(p^2+4r^2)$.

561.05 $$\int_0^\infty e^{-pt}\{[J_0(rt)]^2 + [J_1(rt)]^2\}\, dt = \frac{2k}{\pi r} \int_0^K \operatorname{sn}^2 u\, du = \frac{2}{\pi r k}\,[K - E].$$

561.06 $$\int_0^\infty t\, e^{-pt} J_0(rt)\, J_1(rt)\, dt = \frac{k^3}{2\pi r^2} \int_0^K \operatorname{sn}^2 u\, du = \frac{k}{2\pi r^2}\,[K - E].$$

561.07 $$\int_0^\infty e^{-pt}\{[J_0(rt)]^2 - 2rt\, J_0(rt)\, J_1(rt)\}\, dt = \frac{k}{\pi r} \int_0^K \operatorname{dn}^2 u\, du = \frac{k}{\pi r}\, E.$$

561.08 $$\int_0^\infty t^2 e^{-pt}[J_1(rt)]^2\,dt = \frac{k^5}{2r^3\pi}\int_0^K \mathrm{cd}^2 u\,du = \frac{k^3}{2r^3\pi}[K-E].$$

561.09 $$\int_0^\infty t^{2n} e^{-pt}[J_n(rt)]^2\,dt = \left(\frac{k^2}{r}\right)^{2n+1}\frac{\Gamma(2n+1/2)}{\pi\sqrt{\pi}}\int_0^K \mathrm{cd}^{2n} u\,du,$$
[See **320.05**.]

where Γ is the gamma function and k^2 is given in **561.04**.

562.01 $$\int_0^\infty t\,e^{-pt} I_0(rt)\,I_1(rt)\,dt = \frac{1}{\pi r p(p^2-4r^2)}[p^2 E + (4r^2-p^2)K]$$

where $k = 2r/p$, $p^2 \neq 4r^2$, and I_γ is the modified Bessel function of the first kind of order γ.

562.02 $$\int_0^\infty e^{-pt} I_0(t)\frac{dt}{\sqrt{t}} = \frac{2K}{\sqrt{\pi(p+1)}}, \qquad [k^2 = 2/(p+1);\ p > 1].$$

562.03 $$\left\{\begin{aligned}&\int_0^\infty e^{-pt}\{[I_0(rt)]^2 + 2rt\,I_0(rt)\,I_1(rt)\}\,dt \qquad [k = 2r/p.]\\ &\qquad = \frac{2}{\pi p(p^2-4r^2)}[2(p^2-2r^2)E + (4r^2-p^2)K].\end{aligned}\right.$$

562.04 $$\int_0^\infty e^{-pt}[I_0(rt)]^2\,dt = \frac{2E}{\pi p}, \quad [k = 2r/p;\ \text{R.P. }(p) > |\text{R.P. }(2r)|].$$

562.05 $$\int_0^\infty \frac{e^{-pt}}{t^2}[J_1(t)]^2\,dt = -\frac{p}{2} + \frac{1}{2\pi}\int_0^{2\pi}(1+\cos\vartheta)\sqrt{p^2+2-2\cos\vartheta}\,d\vartheta.$$
[See **290.58**.]

563.01 $$\left\{\begin{aligned}&\int_0^\infty e^{-pt} J_0(rt)\,J_1(st)\,dt = \frac{1}{s}[1-\Lambda_0(\beta,k)]\\ &\qquad = \frac{1}{s\pi}\{\pi - 2[E\,F(\beta,k') + K\,E(\beta,k') - K\,F(\beta,k')]\},\end{aligned}\right.$$

where

$$\beta = \sin^{-1}\sqrt{\frac{1}{2}\left[1 + \frac{p^2+r^2+s^2}{\sqrt{(p^2+r^2+s^2)^2+4p^2s^2}}\right]},$$

$$k^2 = \frac{\left[s^2-p^2-r^2+\sqrt{(p^2+r^2+s^2)^2+4p^2s^2}\right]\left[r^2-p^2-s^2+\sqrt{(p^2+r^2+s^2)^2+4p^2s^2}\right]}{\left[p^2+r^2-s^2+\sqrt{(p^2+r^2+s^2)^2+4p^2s^2}\right]\left[p^2-r^2+s^2+\sqrt{(p^2+r^2+s^2)^2+4p^2s^2}\right]}.$$

Hyperelliptic Integrals.

Introduction.

Definitions.

If

$$P(\tau) = a_0 \tau^n + a_1 \tau^{n-1} + a_2 \tau^{n-2} \cdots + a_n = a_0 (\tau - r_1)(\tau - r_2) \ldots (\tau - r_n)$$

and $R_1\left[\tau, \sqrt{P(\tau)}\right]$ is a rational function of τ and of $\sqrt{P(\tau)}$, the general integral[1]

575.00 $$H_1 = \int R_1\left[\tau, \sqrt{P(\tau)}\right] d\tau$$

is called a *hyperelliptic integral* when n is greater than four. If the degree of $P(\tau)$ is equal to $2p+2$, one can always obtain by means of a rational transformation (e.g., $\tau = r_1 + 1/t$) an equivalent integral in which the radicand is of degree $2p+1$. Since the resulting integral will lead to the same sort of transcendental functions as the original one, the number p may be used to designate the *class* of the hyperelliptic integral. (Integrals with $p=1$ are elliptic integrals, while those where $p=0$ are elementary.)

Special cases of hyperelliptic integrals may appear in some trigonometric or Jacobian form such as

575.01 $$H_2 = \int \frac{R_2(\sin^2\vartheta)\, d\vartheta}{\sqrt{(1-\alpha_1 \sin^2\vartheta)(1-\alpha_2\sin^2\vartheta)(1-\alpha_3\sin^2\vartheta)}},$$

or

575.02 $$H_3 = \int \frac{R_3(\operatorname{sn}^2 u)}{\sqrt{1+\operatorname{sn} u}}\, du.$$

Reduction of Hyperelliptic Integrals to three Types.

It is only necessary to consider the integral

575.03 $$H = \int \frac{R(\tau)\, d\tau}{\sqrt{P(\tau)}},$$

[1] This is a special case of an *Abelian* integral, that is, an integral of the form $\int R(\tau, z)\, d\tau$, where $R(\tau, z)$ is a rational function of τ and z, and z is an algebraic function of τ defined by $f(\tau, z) = 0$.

since the general integral **575.00** can always be written as the sum of an integral of a rational function and of an integral of the form given in **575.03**. The degree of the polynomial P is taken to be $2p+1$, i.e.,

$$P(\tau) = a_0(\tau - r_1)(\tau - r_2)\ldots(\tau - r_{2p+1}) = a_0\tau^{2p+1} + a_1\tau^{2p} + a_2\tau^{2p-1}\cdots + a_{2p+1}.$$

The function $R(\tau)$ may be broken into partial fractions so that

575.04
$$\begin{cases} \displaystyle\int\frac{R(\tau)\,d\tau}{\sqrt{P(\tau)}} = \int[A_0 + A_1\tau + A_2\tau^2 + \cdots]\frac{d\tau}{\sqrt{P(\tau)}} + \\ \displaystyle\qquad + B_1\int\frac{d\tau}{(\tau-\alpha)\sqrt{P(\tau)}} + B_2\int\frac{d\tau}{(\tau-\alpha)^2\sqrt{P(\tau)}} + \cdots + \\ \displaystyle\qquad + C_1\int\frac{d\tau}{(\tau-\alpha_1)\sqrt{P(\tau)}} + C_2\int\frac{d\tau}{(\tau-\alpha_1)^2\sqrt{P(\tau)}} + \cdots + \cdots, \end{cases}$$

where $A_0, A_1, A_2, \ldots, B_1, B_2, B_3, \ldots$ are constants and $\alpha, \alpha_1, \alpha_2, \ldots$ are poles of the rational function $R(\tau)$. Any hyperelliptic integral can thus be expressed linearly in terms of two general non-elementary integrals given by

575.05
$$\begin{cases} \displaystyle\Gamma_i = \int\frac{\tau^i\,d\tau}{\sqrt{P(\tau)}} \\ \displaystyle\lambda_m(\tau,\beta) = \int\frac{d\tau}{(\tau-\beta)^m\sqrt{P(\tau)}}, \end{cases} \qquad [m, i \text{ are integers, including zero}]$$

with

575.06
$$\begin{cases} \lambda_{-m}(\tau, 0) = \Gamma_m(\tau) \\ \displaystyle\lambda_{-m}(\tau,\beta) = \sum_{j=0}^{m}\frac{(-1)^{m-j}\,m!}{(m-j)!\,j!}\beta^{m-j}\Gamma_j(\tau). \end{cases}$$

When β is a root of $P(\tau) = 0$, one has

575.07
$$\begin{cases} \displaystyle\lambda_m(\tau,\beta) = \int\frac{d\tau}{(\tau-\beta)^m\sqrt{P(\tau)}} = \frac{2}{(1-2m)\,P'(\beta)}\times \\ \displaystyle\qquad\times\left[\frac{\sqrt{P(\tau)}}{(\tau-\beta)^m} + \sum_{j=1}^{2p}\frac{(2m-j-1)}{2(j+1)!}P^{(j+1)}(\beta)\int\frac{d\tau}{(\tau-\beta)^{m-j}\sqrt{P(\tau)}}\right], \end{cases}$$

where $P^{(j)}(\beta) = \dfrac{d^j}{d\tau^j}P(\tau)$. If, however, β is not a zero of $P(\tau)$,

575.08
$$\begin{cases} \displaystyle\lambda_m(\tau,\beta) = \frac{1}{2(1-m)\,P(\beta)}\left[\frac{2\sqrt{P(\tau)}}{(\tau-\beta)^{m-1}} + \right. \\ \displaystyle\qquad\left. + \sum_{j=1}^{2p+1}\frac{(2m-j-2)}{j!}P^{(j)}(\beta)\int\frac{d\tau}{(\tau-\beta)^{m-j}\sqrt{P(\tau)}}\right], \quad (m \neq 1). \end{cases}$$

For Γ_i, we have

575.09
$$\begin{cases} \Gamma_i = \int \frac{\tau^i \, d\tau}{\sqrt{P(\tau)}} = \frac{1}{(2i - 2p + 1)\, a_0} \times \\ \qquad \times \left[2\tau^{i-2p} \sqrt{P(\tau)} - \sum_{j=1}^{2p+1} (2i - 2p - j + 1)\, a_j \int \frac{\tau^{i-j}\, d\tau}{\sqrt{P(\tau)}} \right]. \end{cases}$$

The integral Γ_i may be divided into two types: p integrals of the form

$$\int \frac{d\tau}{\sqrt{P(\tau)}}, \quad \int \frac{\tau\, d\tau}{\sqrt{P(\tau)}}, \quad \int \frac{\tau^2\, d\tau}{\sqrt{P(\tau)}}, \ldots, \int \frac{\tau^{p-1}\, d\tau}{\sqrt{P(\tau)}},$$

and the p integrals

$$\int \frac{\tau^p\, d\tau}{\sqrt{P(\tau)}}, \quad \int \frac{\tau^{p+1}\, d\tau}{\sqrt{P(\tau)}}, \quad \int \frac{\tau^{p+2}\, d\tau}{\sqrt{P(\tau)}}, \ldots, \int \frac{\tau^{p-1}\, d\tau}{\sqrt{P(\tau)}}.$$

Integrals of the first set, which are everywhere finite on the two-sheeted Riemann surfaces, are called *integrals of the first kind*, while those of the latter set, which possess only algebraic discontinuities at the point $\tau = \infty$, are referred to as *integrals of the second kind*.

There exists another type of hyperelliptic integral called an *integral of the third kind*. Such integrals have a logarithmic infinity. If β is not a zero of $P(\tau)$, the integral

$$\int \frac{d\tau}{(\tau - \beta)\sqrt{P(\tau)}}$$

is of the third kind. [If the polynomial $P(\tau)$ is of degree $2p + 2$, the integral

$$\int \frac{\tau^p\, d\tau}{\sqrt{P(\tau)}}$$

is also an integral of the third kind.]

Hyperelliptic Integrals which Reduce to Elliptic Integrals.

For the evaluation of hyperelliptic integrals, one must usually resort to direct numerical integration or to the use of complicated series expansions. Many hyperelliptic integrals which occur in physical and geometric problems, however, may be reduced to elliptic integrals or the sum of elliptic integrals and can therefore be evaluated more simply. Some important examples follow.

An example of a hyperelliptic integral which readily reduces to elliptic integrals is

575.10
$$h_1 = \int \frac{R(\tau)\, d\tau}{\sqrt{a_0(\tau^2 + r_1)(\tau^2 + r_2)(\tau^2 + r_3)}}.$$

By writing $R(\tau) = 2R_1(\tau^2) + 2\tau R_2(\tau^2)$ and making use of the substitution $\tau^2 = t$, we immediately obtain

575.11 $$h_1 = \int \frac{R_1(t)\,dt}{\sqrt{a_0(t+r_1)(t+r_2)(t+r_3)\,t}} + \int \frac{R_2(t)\,dt}{\sqrt{a_0(t+r_1)(t+r_2)(t+r_3)}},$$

where the two integrals on the right are treated in **230** to **267**. Three special cases of **575.10** are the integrals

575.12 $$h_2 = \int \frac{R(\tau)\,d\tau}{\sqrt{1-\tau^6}}, \quad h_3 = \int \frac{R(\tau)\,d\tau}{\sqrt{\tau^6-1}}, \quad h_4 = \int \frac{R(\tau)\,d\tau}{\sqrt{1+\tau^6}}.$$

[See **576, 577** and **578.**]

The more general integral

575.13 $$h = \int R(\tau) \prod_{j=1}^{n} (\tau - r_j)^{\mu_j}\, d\tau,$$

where $\sum_{j=1}^{m} \mu_j$ is an integer but each μ_j is not an integer, furnishes numerous other examples of integrals which are reducible to simpler integrals[1]. For any arbitrary integers n and n', the following integrals reduce to those of class $p=1$, i.e., to elliptic integrals:

575.14
$$\left\{\begin{aligned}
h_5 &= \int R(\tau) \sqrt[m]{\frac{(\tau-r_1)^n(\tau-r_2)^{n'}}{(\tau-r_3)^{n+n'}}}\,d\tau, \qquad \text{for } m = 3, 4, 6, 8;\\
h_6 &= \int R(\tau) \sqrt[m]{\frac{(\tau-r_1)^n(\tau-r_2)^{n}}{(\tau-r_3)^{2n}}}\,d\tau, \qquad \text{for } m = 3, 4, 6, 8, 12;\\
h_7 &= \int R(\tau) \sqrt[2m]{(\tau-r_1)^{2n}(\tau-r_2)^{m-2n}(\tau-r_3)^{m}}\,d\tau,\\
&\qquad\qquad \text{for } m = 3, 4, 6, 8, 12;\\
h_8 &= \int R(\tau) \sqrt[12]{\frac{(\tau-r_1)^n(\tau-r_2)^{3n}}{(\tau-r_3)^{4n}}}\,d\tau;\\
h_9 &= \int R(\tau) \sqrt[2m]{\frac{(\tau-r_1)^{2n}(\tau-r_2)^{m+2n}}{(\tau-r_3)^{4n-m}}}\,d\tau, \quad \text{for } m = 3, 6, 8, 12, 16, 24;\\
h_{10} &= \int R(\tau) \sqrt[2m]{\frac{(\tau-r_1)^m(\tau-r_2)^{m}(\tau-r_3)^{2n}}{(\tau-r_4)^{2n}}}\,d\tau,\\
&\qquad\qquad \text{for } m = 2, 3, 4;\\
h_{11} &= \int R(\tau) \sqrt[m]{\frac{(\tau-r_1)^n(\tau-r_2)^{n}}{(\tau-r_3)^{n}(\tau-r_4)^{n}}}\,d\tau, \quad \text{for } m = 3, 4;\\
h_{12} &= \int R(\tau) \sqrt[2m]{(\tau-r_1)^{2n}(\tau-r_2)^{2n}(\tau-r_3)^{m-2n}(\tau-r_4)^{m-2n}}\,d\tau,\\
&\qquad\qquad \text{for } m = 3, 4, 6, 8, 12.
\end{aligned}\right.$$

[1] See H. LENZ: *Zurückführung einiger Integrale auf einfachere.* Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, No. 10, 1951, pp. 73—80.

The following integrals may also be reduced to elliptic integrals:

575.15
$$\begin{cases} h_{13} = \int R(\tau)\sqrt{1-\tau^{12}}\,d\tau. & \\ h_{14} = \int R\left[\tau, \sqrt[m]{a^m-\tau^m}\right]d\tau, & \text{for } m = 3, 4, 6, 8; \\ h_{15} = \int R(\tau)\sqrt{(a^m-\tau^m)(b^m-\tau^m)}\,d\tau, & \text{for } m = 3, 4; \\ h_{16} = \int R\left[\tau, \sqrt[m]{a^m-\tau^m}, \sqrt[m]{b^m-\tau^m}\right]d\tau, & \text{for } m = 3, 4; \\ h_{17} = \int R(\tau)\sqrt[2m]{\dfrac{(\tau^2-a^2)^m(\tau^2-b^2)^{2n}}{(\tau^2-c^2)^{2n}}}\,d\tau, & \text{for } m = 2, 3, 4. \end{cases}$$

The integrals **575.10** and **575.12** are special cases of the more general integral[1]

575.16 $$h_{18} = \int R(\tau)\sqrt{(\tau-r_1)(\tau-r_2)(\tau-r_3)(\tau-r_4)(\tau-r_5)(\tau-r_6)}\,d\tau,$$

which can be reduced to elliptic integrals if the six branch-points $r_1, r_2, r_3, \ldots, r_6$ form three pairs of points of an involution.

Table of Integrals.

The remainder of this section will give a short table of some special cases of the integrals mentioned in the above two pages. The integrands of the integrals in this table will not be reduced to Jacobian elliptic functions as was done in the previous sections, but will only be transformed to the simpler algebraic forms of those in **211** to **279**.

Integrands involving $\sqrt{1-\tau^6}$, $(0 < Y \leq 1)$

$$\boxed{\tau^2 = t, \quad d\tau = \frac{dt}{2\sqrt{t}}, \quad y = Y^2}$$

576.00
$$\begin{cases} \displaystyle\int_0^Y \frac{d\tau}{\sqrt{1-\tau^6}} = \frac{1}{2}\int_0^y \frac{dt}{\sqrt{t(1-t^3)}} = \frac{1}{2\sqrt{3}}F(\psi, k), \\ \qquad \left[k^2 = \dfrac{2-\sqrt{3}}{4}, \quad \psi = \cos^{-1}\left(\dfrac{1-(1+\sqrt{3})\,Y^2}{1+(\sqrt{3}-1)\,Y^2}\right)\right]. \end{cases}$$

[1] L. Koenigsberger: *Reduktion ultraellipt. Integr auf ellipt.*, J. f. Math. 67 (1867), p. 57.

576.01 $$\int_0^Y \frac{\tau^5\,d\tau}{\sqrt{1-\tau^6}} = \frac{1}{2}\int_0^y \frac{t^2\,dt}{\sqrt{1-t^3}} = \frac{1}{3}\left[1 - \sqrt{1-Y^6}\right].$$

576.02 $$\int_0^Y \frac{\tau^{2m}\,d\tau}{\sqrt{1-\tau^6}} = \frac{1}{2}\int_0^y \frac{t^m\,dt}{\sqrt{t(1-t^3)}}.$$ [See **259.53**.]

576.03 $$\int_0^Y \frac{\tau^{2m+1}\,d\tau}{\sqrt{1-\tau^6}} = \frac{1}{2}\int_0^y \frac{t^m\,dt}{\sqrt{1-t^3}} = \frac{1}{2}\left[\int_0^1 \frac{t^m\,dt}{\sqrt{1-t^3}} - \int_y^1 \frac{t^m\,dt}{\sqrt{1-t^3}}\right].$$ [See **244.07**.]

576.04 $$\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{1-\tau^6}} = \int_0^y \frac{R_1(t)\,dt}{\sqrt{t(1-t^3)}} + \int_0^y \frac{R_2(t)}{\sqrt{1-t^3}}\,dt - \int_y^1 \frac{R_2(t)}{\sqrt{1-t^3}}\,dt,$$ [See **259.56** and **244.09**.]

where $$R(\tau) = 2R_1(\tau^2) + 2\tau R_2(\tau^2).$$

Integrands involving $\sqrt{\tau^6 - 1}$, $(1 \leqq Y < \infty)$

$$\boxed{\tau^2 = \frac{1}{t}, \quad d\tau = -\frac{dt}{2t\sqrt{t}}, \quad y = \frac{1}{Y^2}}$$

577.00 $$\left\{\begin{aligned} \int_Y^\infty \frac{d\tau}{\sqrt{\tau^6-1}} &= \frac{1}{2}\int_0^y \frac{dt}{\sqrt{1-t^3}} = \frac{1}{2}\left[\int_0^1 \frac{dt}{\sqrt{1-t^3}} - \int_y^1 \frac{dt}{\sqrt{1-t^3}}\right] \\ &= \frac{1}{2\sqrt[4]{3}}\left[F(\psi_1, k) - F(\psi, k)\right], \end{aligned}\right.$$

where $k^2 = \dfrac{2+\sqrt{3}}{4}$, $\psi_1 = \cos^{-1}\left[2-\sqrt{3}\right]$, $\psi = \cos^{-1}\left[\dfrac{(\sqrt{3}-1)\,Y^2+1}{(\sqrt{3}+1)\,Y^2-1}\right]$.

577.01 $$\int_Y^\infty \frac{d\tau}{\tau^{2m}\sqrt{\tau^6-1}} = \frac{1}{2}\int_0^y \frac{t^m\,dt}{\sqrt{1-t^3}} = \frac{1}{2}\left[\int_0^1 \frac{t^m\,dt}{\sqrt{1-t^3}} - \int_y^1 \frac{t^m\,dt}{\sqrt{1-t^3}}\right].$$ [See **244.07**.]

577.02 $$\int_Y^\infty \frac{d\tau}{\tau^{2m+1}\sqrt{\tau^6-1}} = \frac{1}{2}\int_0^y \frac{t^{m+1}\,dt}{\sqrt{t(1-t^3)}}.$$ [See **259.53**.]

577.03 $$\int_Y^\infty \frac{d\tau}{\tau^4\sqrt{\tau^6-1}} = \frac{1}{2}\int_0^y \frac{t^2\,dt}{\sqrt{1-t^3}} = \frac{1}{3}\left[1 - \frac{\sqrt{Y^6-1}}{Y^3}\right].$$

577.04 $$\int_Y^\infty \frac{R(\tau)\,d\tau}{\sqrt{\tau^6-1}} = \int_0^y \frac{R_1(1/t)}{\sqrt{1-t^3}}\,dt + \int_0^y \frac{R_2(1/t)}{\sqrt{t(1-t^3)}}\,dt,$$ [See **259**.]

where $$R(\tau) = 2R_1(\tau^2) + 2\tau R_2(\tau^2).$$

Integrands involving $\sqrt{1+\tau^6}$, $(Y>0)$

$$\boxed{\tau^2 = t,\quad d\tau = \frac{dt}{2\sqrt{t}},\quad y = Y^2.}$$

578.00 $$\int_0^Y \frac{d\tau}{\sqrt{1+\tau^6}} = \frac{1}{2}\int_0^y \frac{dt}{\sqrt{t(1+t^3)}} = \frac{1}{2\sqrt{3}}F(\psi, k),$$ [cf. **260.50**].

where $$k^2 = \frac{2+\sqrt{3}}{4},\quad \psi = \cos^{-1}\left[\frac{1+(1-\sqrt{3})\,Y^2}{1+(1+\sqrt{3})\,Y^2}\right].$$

578.01 $$\int_0^Y \frac{\tau^5\,d\tau}{\sqrt{1+\tau^6}} = \frac{1}{2}\int_0^y \frac{t^2\,dt}{\sqrt{1+t^3}} = \frac{1}{3}\left[\sqrt{1+Y^6} - 1\right].$$

578.02 $$\int_0^Y \frac{\tau^{2m}\,d\tau}{\sqrt{1+\tau^6}} = \frac{1}{2}\int_0^y \frac{t^m\,dt}{\sqrt{(1+t^3)\,t}}.$$ [See **260.52**.]

578.03 $$\int_0^Y \frac{\tau^{2m+1}\,d\tau}{\sqrt{1+\tau^6}} = \frac{1}{2}\int_0^y \frac{t^m\,dt}{\sqrt{1+t^3}} = \frac{(-1)^m}{2}\left[\int_{-y}^1 \frac{t^m\,dt}{\sqrt{1-t^3}} - \int_0^1 \frac{t^m\,dt}{\sqrt{1-t^3}}\right].$$ [See **244.07**.]

578.04 $$\left\{\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{1+\tau^6}} = \frac{1}{2}\int_0^y \frac{R_1(t)\,dt}{\sqrt{t(1+t^3)}} + \frac{1}{2}\left[\int_{-y}^1 \frac{R_2(t)\,dt}{\sqrt{1-t^3}} - \int_0^1 \frac{R_2(t)\,dt}{\sqrt{1-t^3}}\right],\right.$$ [See **260** and **244**.]

where $$R(\tau) = R_1(\tau^2) + \tau R_2(\tau^2).$$

Integrands involving $\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}$, $(0<Y\leqq 1)$.

$$\frac{\tau^2+1}{\tau}=t,\quad \tau=\frac{1}{2}\left[t-\sqrt{t^2-4}\right],\quad y=\frac{1+Y^2}{Y^2},\quad -\frac{d\tau}{\tau\sqrt{\tau}}=\frac{1}{2}\left[\frac{1}{\sqrt{t+2}}+\frac{1}{\sqrt{t-2}}\right]dt.$$

579.00
$$\begin{cases}\displaystyle\int_0^Y\frac{d\tau}{\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}}\\ \displaystyle=\frac{1}{2}\int_y^\infty\frac{dt}{\sqrt{(t+2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+(a_3-2a_1)]}}+\\ \displaystyle+\frac{1}{2}\int_y^\infty\frac{dt}{\sqrt{(t-2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+(a_3-2a_1)]}},\quad\text{[See 250.]}\end{cases}$$

579.01
$$\begin{cases}\displaystyle\int_0^Y\frac{R(\tau)\,d\tau}{\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}}\\ \displaystyle=\frac{1}{2}\int_y^\infty\frac{[R_1(t)+(t+2)\,R_2(t)]\,dt}{\sqrt{(t+2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+(a_3-2a_1)]}}+\\ \displaystyle+\frac{1}{2}\int_y^\infty\frac{[R_1(t)+(t-2)\,R_2(t)]\,dt}{\sqrt{(t-2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+(a_3-2a_1)]}},\quad\text{[See 250.]}\end{cases}$$

where $R(\tau)$ is any rational integral function of τ. In this case, $R(\tau)=R_1(t)+R_2(t)\sqrt{t^2-4}$.

Integrands involving $\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}$, $(1<Y<\infty)$

$$\frac{\tau^2+1}{\tau}=t,\quad y=\frac{Y^2+1}{Y^2},\quad \frac{d\tau}{\tau\sqrt{\tau}}=-\frac{1}{2}\left[\frac{1}{\sqrt{t+2}}-\frac{1}{\sqrt{t-2}}\right]dt,$$
$$\tau=\frac{1}{2}\left[t+\sqrt{t^2-4}\right].$$

581.00
$$\begin{cases}\displaystyle\int_1^Y\frac{d\tau}{\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}}\\ \displaystyle=\frac{1}{2}\int_2^y\frac{dt}{\sqrt{(t-2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}-\\ \displaystyle-\frac{1}{2}\int_2^y\frac{dt}{\sqrt{(t+2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}.\quad\text{[See 250.]}\end{cases}$$

581.01
$$\left\{\begin{aligned}&\int_1^Y \frac{\tau\,d\tau}{\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}}\\&=\frac{1}{2}\int_2^y \frac{dt}{\sqrt{(t-2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}+\\&+\frac{1}{2}\int_2^y \frac{dt}{\sqrt{(t+2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}\,.\end{aligned}\right.$$
[See **250**.]

581.02
$$\left\{\begin{aligned}&\int_1^Y \frac{R(\tau)\,d\tau}{\sqrt{a_0\tau^6+a_1\tau^5+a_2\tau^4+a_3\tau^3+a_2\tau^2+a_1\tau+a_0}}\\&=\frac{1}{2}\int_2^y \frac{[R_1(t)+(t-2)\,R_2(t)]\,dt}{\sqrt{(t-2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}-\\&-\frac{1}{2}\int_2^y \frac{[R_1(t)+(t+2)\,R_2(t)]\,dt}{\sqrt{(t+2)\,[a_0t^3+a_1t^2+(a_2-3a_0)\,t+a_3-2a_1]}}\,,\end{aligned}\right.$$
[See **250**.]

where $R(\tau)$ is any rational function of τ, which in this case may be written $R(\tau)=R_1(t)-R_2(t)\sqrt{t^2-4}$.

Integrands involving $\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}$, $(0<Y\leqq 1)$

$$\boxed{\tau^2=t,\quad d\tau=\frac{dt}{2\sqrt{t}},\quad y=Y^2.}$$

582.00
$$\int_0^Y \frac{d\tau}{\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}}=\frac{1}{2}\int_0^y \frac{dt}{\sqrt{a_0t^5+a_2t^4+a_4t^3+a_2t^2+a_0t}}\,.$$
[See **579** with the a's redefined.]

582.01
$$\int_0^Y \frac{\tau\,d\tau}{\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}}=\frac{1}{2}\int_0^y \frac{dt}{\sqrt{a_0t^4+a_2t^3+a_4t^2+a_2t+a_0}}\,.$$
[See **250**.]

582.02
$$\left\{\begin{aligned}&\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}}\\&=\frac{1}{2}\int_0^y \frac{R_1(t)\,dt}{\sqrt{a_0t^5+a_2t^4+a_4t^3+a_2t^2+a_0t}}+\frac{1}{2}\int_0^y \frac{R_2(t)\,dt}{\sqrt{a_0t^4+a_2t^3+a_4t^2+a_2t+a_0}}\,,\end{aligned}\right.$$

[See **250** for further reduction of the last integral, and **579** for the second.]

where $$R(\tau)=R_1(\tau^2)+\tau\,R_2(\tau^2).$$

Integrands involving $\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}$, $(1<Y<\infty)$

$$\boxed{\tau^2=t,\quad d\tau=\frac{dt}{2\sqrt{t}},\quad y=Y^2.}$$

583.00 $$\int_1^Y\frac{d\tau}{\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}}=\frac{1}{2}\int_1^y\frac{dt}{\sqrt{a_0t^5+a_2t^4+a_4t^3+a_2t^2+a_0t}}.$$

[See **581** with the a's redefined.]

583.01 $$\int_1^Y\frac{\tau\,d\tau}{\sqrt{a_0\tau^8+a_2\tau^6+a_4\tau^4+a_2\tau^2+a_0}}=\frac{1}{2}\int_1^y\frac{dt}{\sqrt{a_0t^4+a_2t^3+a_4t^2+a_2t+a_0}}.$$

[See **250**.]

583.02 $$\left\{\begin{aligned}&\int_1^Y\frac{R(\tau)\,d\tau}{\sqrt{a_0\tau^8+a_2\tau^5+a_4\tau^4+a_2\tau^2+a_0}}\\&=\frac{1}{2}\int_1^y\frac{R_1(t)\,dt}{\sqrt{a_0t^5+a_2t^4+a_4t^3+a_2t^2+a_0t}}+\frac{1}{2}\int_1^y\frac{R_2(t)\,dt}{\sqrt{a_0t^4+a_2t^3+a_4t^2+a_2t+a_0}},\end{aligned}\right.$$

[See **250** for further reduction of the last integral, and **581** for the second.] $R(\tau)=R_1(\tau^2)+\tau R_2(\tau^2)$.

Integrands involving $\sqrt{1+\tau^8}$, $(0<Y\leqq 1)$

$$\boxed{\begin{aligned}&t=\frac{\tau^4+1}{\tau^2},\quad \tau=\frac{1}{2}\left[\sqrt{t+2}-\sqrt{t-2}\right],\quad y=\frac{Y^4+1}{Y^2},\\&d\tau=\frac{1}{4}\left[\frac{1}{\sqrt{t+2}}-\frac{1}{\sqrt{t-2}}\right]dt,\ \frac{1}{\sqrt{1+\tau^8}}=\frac{1}{2}\left[\frac{t+\sqrt{t^2-4}}{\sqrt{t^2-2}}\right].\end{aligned}}$$

584.00 $$\left\{\begin{aligned}&\int_0^Y\frac{d\tau}{\sqrt{1+\tau^8}}=\frac{1}{4}\int_y^\infty\frac{dt}{\sqrt{(t+2)(t^2-2)}}+\frac{1}{4}\int_y^\infty\frac{dt}{\sqrt{(t-2)(t^2-2)}}=\frac{\sqrt{4-2\sqrt{2}}}{8}\times\\&\times\left\{(1+\sqrt{2})K(\sqrt{2}-1)-\operatorname{sn}^{-1}\left[\frac{Y^8+1-2\sqrt{2}(Y^2+1)^2Y^2}{Y^8+1+2\sqrt{2}(Y^2+1)^2Y^2},\ (\sqrt{2}-1)\right]-\right.\\&\left.\qquad-\sqrt{2}\operatorname{sn}^{-1}\left[\frac{(Y^2-1)^2-Y^2\sqrt{2}}{(Y^2-1)^2+Y^2\sqrt{2}},\ (\sqrt{2}-1)\right]\right\}.\end{aligned}\right.$$

584.01 $$\int_0^Y\frac{\tau\,d\tau}{\sqrt{1+\tau^8}}=\frac{1}{2}\int_0^{Y^2}\frac{dt_1}{\sqrt{1+t_1^4}}=\frac{1}{4}\operatorname{cn}^{-1}\left[\frac{1-Y^4}{1+Y^4},\ \sqrt{2}/2\right].$$

584.02
$$\begin{cases} \displaystyle\int_0^Y \frac{\tau^2\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\int_y^\infty \frac{dt}{\sqrt{(t-2)(t^2-2)}} - \frac{1}{4}\int_y^\infty \frac{dt}{\sqrt{(t+2)(t^2-2)}} = \frac{\sqrt{4-2\sqrt{2}}}{8}\times \\ \displaystyle\times\left\{(\sqrt{2}-1)K(\sqrt{2}-1) - \sqrt{2}\,\mathrm{sn}^{-1}\left[\frac{(Y^2-1)^2-Y^2\sqrt{2}}{(Y^2-1)^2+Y^2\sqrt{2}},\ (\sqrt{2}-1)\right] + \right. \\ \displaystyle\left. + \mathrm{sn}^{-1}\left[\frac{Y^8+1-2\sqrt{2}(Y^2+1)^2Y^2}{Y^8+1+2\sqrt{2}(Y^2+1)^2Y^2},\ (\sqrt{2}-1)\right]\right\}. \end{cases}$$

[See **238.**]

584.03
$$\int_0^Y \frac{\tau^7\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\left[\sqrt{1+Y^8}-1\right].$$

584.04
$$\begin{cases} \displaystyle\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\left\{\int_y^\infty \frac{(t+2)R_2(t)+(t-2)R_3(t)}{\sqrt{(t^2-2)(t^2-4)}}\,dt + \right. \\ \displaystyle\left. + \int_y^\infty \frac{R_1(t)+(t-2)R_4(t)}{\sqrt{(t-2)(t^2-2)}}\,dt + \int_y^\infty \frac{R_1(t)+(t+2)R_4(t)}{\sqrt{(t+2)(t^2-2)}}\,dt + \int_y^\infty \frac{R_2(t)+R_3(t)}{\sqrt{t^2-2}}\,dt\right\}, \end{cases}$$

[See **238.**]

where

$$R(\tau) = R\left[\frac{1}{2}\left(\sqrt{t+2}-\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) + \sqrt{t-2}\,R_3(t) + \sqrt{t^2-4}\,R_4(t).$$

Integrands involving $\sqrt{1+\tau^8}$, $(1<Y<\infty)$

$$t = \frac{\tau^4+1}{\tau^2},\quad \tau = \frac{1}{2}\left[\sqrt{t+2}+\sqrt{t-2}\right],\quad y = \frac{Y^4+1}{Y^2}$$
$$d\tau = \frac{1}{4}\left[\frac{1}{\sqrt{t+2}} + \frac{1}{\sqrt{t-2}}\right]dt,\quad \frac{1}{\sqrt{1+\tau^8}} = \frac{1}{2}\left[\frac{t+\sqrt{t^2-4}}{\sqrt{t^2-2}}\right].$$

585.00
$$\begin{cases} \displaystyle\int_1^Y \frac{d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\int_2^y \frac{dt}{\sqrt{(t-2)(t^2-2)}} + \frac{1}{4}\int_2^y \frac{dt}{\sqrt{(t+2)(t^2-2)}} = \frac{\sqrt{4-2\sqrt{2}}}{8}\times \\ \displaystyle\times\left\{\sqrt{2}K(\sqrt{2}-1) - \mathrm{sn}^{-1}\left[\frac{Y^8+1-2\sqrt{2}Y^2(Y^2+1)^2}{Y^8+1+2\sqrt{2}Y^2(Y^2+1)^2},\ (\sqrt{2}-1)\right] + \right. \\ \displaystyle\left. + \mathrm{sn}^{-1}\left[\frac{1-4\sqrt{2}}{1+4\sqrt{2}},\ (\sqrt{2}-1)\right] + \sqrt{2}\,\mathrm{sn}^{-1}\left[\frac{(Y^2-1)^2-Y^2\sqrt{2}}{(Y^2-1)^2+Y^2\sqrt{2}},\ (\sqrt{2}-1)\right]\right\}. \end{cases}$$

585.01
$$\int_1^Y \frac{\tau\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{2}\int_1^{Y^2} \frac{dt_1}{\sqrt{1+t_1^4}} = \frac{1}{4}\left\{K(\sqrt{2}/2) - \mathrm{cn}^{-1}\left[\frac{1-Y^4}{1+Y^4},\ \sqrt{2}/2\right]\right\}.$$

585.02
$$\begin{cases}\displaystyle\int_1^Y \frac{\tau^2\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\int_2^y \frac{dt}{\sqrt{(t-2)(t^2-2)}} + \frac{1}{4}\int_2^y \frac{dt}{\sqrt{(t+2)(t^2-2)}} = \frac{\sqrt{4-2\sqrt{2}}}{8}\times \\ \times\left\{\sqrt{2}K(\sqrt{2}-1) + \operatorname{sn}^{-1}\left[\frac{Y^8+1-2\sqrt{2}\,Y^2(Y^2+1)^2}{Y^8+1+2\sqrt{2}\,Y^2(Y^2+1)^2},\ (\sqrt{2}-1)\right] - \right. \\ \left. -\operatorname{sn}^{-1}\left[\frac{1-4\sqrt{2}}{1+4\sqrt{2}},\ (\sqrt{2}-1)\right] + \sqrt{2}\operatorname{sn}^{-1}\left[\frac{(Y^2-1)^2-Y^2\sqrt{2}}{(Y^2-1)^2-Y^2\sqrt{2}},\ (\sqrt{2}-1)\right]\right\}.\end{cases}$$

585.03
$$\int_1^Y \frac{\tau^3\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\int_1^{Y^4} \frac{dt_2}{\sqrt{1+t_2^2}} = \frac{1}{4}\ln\left[(\sqrt{2}-1)\left(Y^8+\sqrt{Y^8+1}\right)\right].$$

585.04
$$\int_1^Y \frac{\tau^7\,d\tau}{\sqrt{1+\tau^8}} = \frac{1}{4}\left[\sqrt{Y^8+1}-\sqrt{2}\right],$$

585.05
$$\begin{cases}\displaystyle\int_1^Y \frac{R(\tau)\,d\tau}{\sqrt{1+\tau^8}} \\ \displaystyle = \frac{1}{4}\left\{\int_2^y \frac{(t+2)R_2(t)+(t-2)R_3(t)}{\sqrt{(t^2-4)(t^2-2)}}\,dt + \int_2^y \frac{R_1(t)+(t-2)R_4(t)}{\sqrt{(t-2)(t^2-2)}}\,dt - \right. \\ \displaystyle \left. -\int_2^y \frac{R_1(t)+(t+2)R_4(t)}{\sqrt{(t+2)(t^2-2)}}\,dt - \int_2^y \frac{R_2(t)+R_3(t)}{\sqrt{t^2-2}}\,dt\right\},\end{cases}$$

[See **237.**]

where

$$R(\tau) = R\left[\frac{1}{2}\left(\sqrt{t+2}+\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) - \sqrt{t-2}\,R_3(t) - \sqrt{t^2-4}\,R_4(t).$$

Integrands involving $\sqrt{(1-\tau^4)(1-n^4\tau^4)}$, $(0<Y\leqq 1<1/n)$

$$t = \frac{1+n^2\tau^4}{\tau^2 n}, \quad \tau = \frac{1}{2\sqrt{n}}\left[\sqrt{t+2}-\sqrt{t-2}\right], \quad y = \frac{1+n^2Y^4}{Y^2 n},$$
$$a = \frac{1+n^2}{n}.$$

586.00
$$\begin{cases}\displaystyle\int_0^Y \frac{d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t+2)(t^2-a^2)}} + \frac{1}{4\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t-2)(t^2-a^2)}}.\end{cases}$$

[See **238.**]

586.01 $$\int_0^Y \frac{\tau\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} = \frac{1}{2}\int_0^{Y^2} \frac{dt_1}{\sqrt{(1-t_1^2)(1-n^4t_1^2)}} = \frac{1}{2}F[\sin^{-1}(Y^2), n^2].$$

586.02 $$\begin{cases} \displaystyle\int_0^Y \frac{\tau^2\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4n\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t-2)(t^2-a^2)}} - \frac{1}{4n\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t+2)(t^2-a^2)}}. \quad \text{[See } \mathbf{238}.] \end{cases}$$

586.03 $$\int_0^Y \tau\sqrt{\frac{1-n^4\tau^4}{1-\tau^4}}\,d\tau = \frac{1}{2}\int_0^{Y^2}\sqrt{\frac{1-n^4t_1^2}{1-t_1^2}}\,dt_1 = \frac{1}{2}E[\sin^{-1}(Y^2), n^2].$$

586.04 $$\begin{cases} \displaystyle\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4\sqrt{n}}\left\{\int_y^\infty \frac{(t+2)R_2(t)+(t-2)R_3(t)}{\sqrt{(t^2-4)(t^2-a^2)}}\,dt + \int_y^\infty \frac{R_1(t)+(t-2)R_4(t)}{\sqrt{(t-2)(t^2-a^2)}}\,dt + \right. \\ \displaystyle \left. + \int_y^\infty \frac{R_1(t)+(t+2)R_4(t)}{\sqrt{(t+2)(t^2-a^2)}}\,dt + \int_y^\infty \frac{R_2(t)+R_3(t)}{\sqrt{t^2-a^2}}\,dt\right\}, \quad \text{[See } \mathbf{238}.] \end{cases}$$

where

$$R(\tau) = R\left[\frac{1}{2\sqrt{n}}\left(\sqrt{t+2}-\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) + \sqrt{t-2}\,R_3(t) + \sqrt{t^2-4}\,R_4(t).$$

Integrands involving $\sqrt{(1-\tau^4)(1-n^4\tau^4)}$, $(0 < 1/n < Y < \infty)$

$$t = \frac{1+n^2\tau^4}{\tau^2 n}, \quad \tau = \frac{1}{2\sqrt{n}}\left[\sqrt{t+2}+\sqrt{t-2}\right], \quad y = \frac{1+n^2Y^4}{Y^2 n},$$
$$a = \frac{1+n^2}{n}.$$

587.00 $$\begin{cases} \displaystyle\int_{1/n}^Y \frac{d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4\sqrt{n}}\int_a^y \frac{dt}{\sqrt{(t-2)(t^2-a^2)}} - \frac{1}{4\sqrt{n}}\int_a^y \frac{dt}{\sqrt{(t+2)(t^2-a^2)}}. \quad \text{[See } \mathbf{237}.] \end{cases}$$

587.01 $$\begin{cases} \displaystyle\int_{1/n}^{Y} \frac{\tau\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{2}\int_{1/n^2}^{Y^2} \frac{dt_1}{\sqrt{(1-t_1^2)(1-n^4 t_1^2)}} = \frac{1}{2} F\left[\sin^{-1}\sqrt{\frac{1-n^4 Y^4}{n^4(1-Y^4)}},\, n^2\right]. \end{cases}$$

587.02 $$\begin{cases} \displaystyle\int_{1/n}^{Y} \frac{\tau^2\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4n\sqrt{n}}\int_a^y \frac{dt}{\sqrt{(t-2)(t^2-a^2)}} + \frac{1}{4n\sqrt{n}}\int_a^y \frac{dt}{\sqrt{(t+2)(t^2-a^2)}}. \quad \text{[See } \mathbf{237}.\text{]} \end{cases}$$

587.03 $$\begin{cases} \displaystyle\int_{1/n}^{Y} \frac{R(\tau)\,d\tau}{\sqrt{(1-\tau^4)(1-n^4\tau^4)}} \\ \displaystyle = \frac{1}{4\sqrt{n}}\left\{\int_a^y \frac{[(t+2)R_2(t)+(t-2)R_3(t)]\,dt}{\sqrt{(t^2-4)(t^2-a^2)}} - \int_a^y \frac{[R_2(t)+R_3(t)]\,dt}{\sqrt{t^2-a^2}} + \right. \\ \displaystyle \left. + \int_a^y \frac{[R_1(t)+(t-2)R_4(t)]\,dt}{\sqrt{(t-2)(t^2-a^2)}} - \int_a^y \frac{[R_1(t)+(t+2)R_4(t)]\,dt}{\sqrt{(t+2)(t^2-a^2)}}\right\}, \\ \text{[See } \mathbf{237}.\text{]} \end{cases}$$

where

$$R(\tau) = R\left[\frac{1}{2\sqrt{n}}\left(\sqrt{t+2}+\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) - \sqrt{t-2}\,R_3(t) - \sqrt{t^2-4}\,R_4(t).$$

Integrands involving $\sqrt{(1+\tau^4)(1+n^4\tau^4)}$, $(0<Y\leq 1/\sqrt{n})$

$$t = \frac{1+n^2\tau^4}{n\tau^2}, \quad \tau = \frac{1}{2\sqrt{n}}\left[\sqrt{t+2}-\sqrt{t-2}\right], \quad y = \frac{1+n^2 Y^4}{n Y^2},$$
$$b = \frac{n^2-1}{n}.$$

588.00 $$\begin{cases} \displaystyle\int_0^{Y} \frac{d\tau}{\sqrt{(1+\tau^4)(1+n^4\tau^4)}} \\ \displaystyle = \frac{1}{4\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t+2)(t^2+b^2)}} + \frac{1}{4\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t-2)(t^2+b^2)}}. \quad \text{[See } \mathbf{241}.\text{]} \end{cases}$$

588.01
$$\int_0^Y \frac{\tau\,d\tau}{\sqrt{(1+\tau^4)(1+n^4\tau^4)}} = \frac{1}{2}\int_0^{Y^2} \frac{dt_1}{\sqrt{(1+t_1^2)(1+n^4 t_1^2)}}$$
$$= \frac{1}{2}\operatorname{tn}^{-1}\left(Y^2, \sqrt{1-n^4}\right) \text{ if } n<1$$
$$= \frac{1}{2n^2}\operatorname{tn}^{-1}\left[Y^2 n^2, \sqrt{\frac{n^4-1}{n^4}}\right] \text{ if } n>1.$$

588.02
$$\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{(1+\tau^4)(1+n^4\tau^4)}}$$
$$= \frac{1}{4\sqrt{n}}\left\{\int_y^\infty \frac{[(t+2)R_2(t)+(t-2)R_3(t)]\,dt}{\sqrt{(t^2-4)(t^2+b^2)}} + \int_y^\infty \frac{[R_2(t)+R_3(t)]\,dt}{\sqrt{t^2+b^2}} + \right.$$
$$\left. + \int_y^\infty \frac{[R_1(t)+(t-2)R_4(t)]\,dt}{\sqrt{(t-2)(t^2+b^2)}} + \int_y^\infty \frac{[R_1(t)+(t+2)R_4(t)]\,dt}{\sqrt{(t+2)(t^2+b^2)}}\right\},$$

[See **241.**]

where

$$R(\tau) = R\left[\frac{1}{2\sqrt{n}}\left(\sqrt{t+2}-\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) + \sqrt{t-2}\,R_3(t) + \sqrt{t^2-4}\,R_4(t).$$

Integrands involving $\sqrt{(1+\tau^4)(1+n^4\tau^4)}$, $(1/\sqrt{n}<Y<\infty)$

$$t = \frac{1+n^2\tau^4}{n\tau^2}, \quad \tau = \frac{1}{2\sqrt{n}}\left[\sqrt{t+2}+\sqrt{t-2}\right], \quad y = \frac{1+n^2Y^4}{nY^2},$$
$$b = \frac{n^2-1}{n}.$$

589.00
$$\int_{1/\sqrt{n}}^Y \frac{d\tau}{\sqrt{(1+\tau^4)(1+n^4\tau^4)}}$$
$$= \frac{1}{4\sqrt{n}}\int_2^y \frac{dt}{\sqrt{(t-2)(t^2+b^2)}} - \frac{1}{4\sqrt{n}}\int_2^y \frac{dt}{\sqrt{(t+2)(t^2+b^2)}}.$$

[See **239.**]

589.01
$$\int_{1/\sqrt{n}}^{Y}\frac{R(\tau)\,d\tau}{\sqrt{(1+\tau^4)(1+n^4\tau^4)}} = \frac{1}{4\sqrt{n}}\left\{\int_2^y \frac{[(t+2)R_2(t)+(t-2)R_3(t)]\,dt}{\sqrt{(t^2-4)(t^2+b^2)}} - \int_2^y\frac{[R_2(t)+R_3(t)]\,dt}{\sqrt{t^2+b^2}} + \right.$$
$$\left. + \int_2^y\frac{[R_1(t)+(t-2)R_4(t)]\,dt}{\sqrt{(t-2)(t^2+b^2)}} - \int_2^y\frac{[R_1(t)+(t+2)R_4(t)]\,dt}{\sqrt{(t+2)(t^2+b^2)}}\right\},$$

[See **239.**]

where

$$R(\tau) = R\left[\frac{1}{2\sqrt{n}}\left(\sqrt{t+2}+\sqrt{t-2}\right)\right]$$
$$= R_1(t) + \sqrt{t+2}\,R_2(t) - \sqrt{t-2}\,R_3(t) - \sqrt{t^2-4}\,R_4(t)$$

Integrands involving $\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}$, $(0<Y\leqq 1<1/n)$

$$t = \frac{1+n\tau^2}{\tau}, \quad \tau = \frac{1}{2n}\left[t-\sqrt{t^2-4n}\right], \quad y = \frac{1+nY^2}{Y},$$
$$\frac{d\tau}{\tau\sqrt{\tau}} = -\frac{1}{2}\left[\frac{1}{\sqrt{t+2\sqrt{n}}}+\frac{1}{\sqrt{t-2\sqrt{n}}}\right]dt.$$

595.00
$$\int_0^Y\frac{d\tau}{\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}} = -\frac{1}{2}\int_y^\infty\frac{dt}{\sqrt{(t+2\sqrt{n})[t^2-(1+n)^2]}} -$$
$$-\frac{1}{2}\int_y^\infty\frac{dt}{\sqrt{(t-2\sqrt{n})[t^2-(1+n)^2]}} = \frac{1}{\sqrt{2(1+n)}}\left\{K\left[\frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right]+\right.$$
$$+K\left[\frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right] - \mathrm{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1+Y\sqrt{n}},\ \frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right] -$$
$$\left.-\,\mathrm{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1-Y\sqrt{n}},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}.$$

595.01
$$\begin{cases} \displaystyle\int_0^Y \sqrt{\frac{\tau}{(1-\tau^2)(1-n^2\tau^2)}}\,d\tau = \frac{1}{2\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t-2\sqrt{n})\,[t^2-(1+n)^2]}} - \\ \displaystyle -\frac{1}{2\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t+2\sqrt{n})\,[t^2-(1+n)^2]}} = \frac{1}{\sqrt{2n(1+n)}}\left\{K\left[\frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right] - \right. \\ \displaystyle - K\left[\frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right] + \operatorname{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1+Y\sqrt{n}},\ \frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right] - \\ \displaystyle \left. - \operatorname{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1-Y\sqrt{n}},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}. \end{cases}$$

595.02
$$\begin{cases} \displaystyle\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}} \\ \displaystyle = \frac{1}{2}\int_y^\infty \frac{[R_1(t)+(t-2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t-2\sqrt{n})\,[t^2-(1+n)^2]}} + \frac{1}{2}\int_y^\infty \frac{[R_1(t)+(t+2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t+2\sqrt{n})\,[t^2-(1+n)^2]}}, \end{cases}$$

[See **238**.]

where $R(\tau) = R\left[\frac{1}{2n}\left(t-\sqrt{t^2-4n}\right)\right] = R_1(t) + \sqrt{t^2-4n}\,R_2(t)$.

Integrands involving $\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}$, $(1/n < Y < \infty)$

$$t = \frac{1+n\tau^2}{\tau}, \quad \tau = \frac{1}{2n}\left[t+\sqrt{t^2-4n}\right], \quad y = \frac{1+nY^2}{Y},$$
$$\frac{d\tau}{\tau\sqrt{\tau}} = \frac{1}{2}\left[\frac{1}{\sqrt{t-2\sqrt{n}}} - \frac{1}{\sqrt{t+2\sqrt{n}}}\right]dt.$$

596.00
$$\begin{cases} \displaystyle\int_{1/n}^Y \frac{d\tau}{\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}} \\ \displaystyle = \frac{1}{2}\int_{1+n}^y \frac{dt}{\sqrt{(t-2\sqrt{n})\,[t^2-(1+n)^2]}} - \frac{1}{2}\int_{1+n}^y \frac{dt}{\sqrt{(t+2\sqrt{n})\,[t^2-(1+n)^2]}} \\ \displaystyle = \frac{1}{\sqrt{2(1+n)}}\left\{\operatorname{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1+Y\sqrt{n}},\ \frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right] - \right. \\ \displaystyle \left. - \operatorname{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1-Y\sqrt{n}},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}. \end{cases}$$

596.01
$$\int_{1/n}^{Y}\sqrt{\frac{\tau}{(1-\tau^2)(1-n^2\tau^2)}}\,d\tau$$
$$=\frac{1}{2\sqrt{n}}\int_{1+n}^{y}\frac{dt}{\sqrt{(t-2\sqrt{n})[t^2-(1+n)^2]}}+\frac{1}{2\sqrt{n}}\int_{1+n}^{y}\frac{dt}{\sqrt{(t+2\sqrt{n})[t^2-(1+n)^2]}}$$
$$=\frac{1}{\sqrt{2(1+n)n}}\left\{\mathrm{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1+Y\sqrt{n}},\ \frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right]+\right.$$
$$\left.+\ \mathrm{sn}^{-1}\left[\frac{\sqrt{(1-Y)(1-nY)}}{1-Y\sqrt{n}},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}.$$

596.02
$$\int_{1/n}^{Y}\frac{R(\tau)\,d\tau}{\sqrt{\tau(1-\tau^2)(1-n^2\tau^2)}}$$
$$=\frac{1}{2}\int_{1+n}^{y}\frac{[R_1(t)+(t-2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t-2\sqrt{n})[t^2-(1+n)^2]}}-\frac{1}{2}\int_{1+n}^{y}\frac{[R_1(t)+(t+2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t+2\sqrt{n})[t^2-(1+n)^2]}}.$$

[See **239**.]

where $$R(\tau)=R\left[\frac{1}{2n}\left(t+\sqrt{t^2-4n}\right)\right]=R_1(t)-\sqrt{t^2-4n}\,R_2(t).$$

Integrands involving $\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}$, $(0<Y<1/\sqrt{n})$

$$t=\frac{1+n\tau^2}{\tau},\quad \tau=\frac{1}{2n}\left[t-\sqrt{t^2-4n}\right],\quad y=\frac{1+nY^2}{Y},$$
$$\frac{d\tau}{\tau\sqrt{\tau}}=-\frac{1}{2}\left[\frac{1}{\sqrt{t+2\sqrt{n}}}+\frac{1}{\sqrt{t-2\sqrt{n}}}\right]dt.$$

597.00
$$\int_{0}^{Y}\frac{d\tau}{\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}}$$
$$=\frac{1}{2}\int_{y}^{\infty}\frac{dt}{\sqrt{(t+2\sqrt{n})[t^2+(1-n)^2]}}+\frac{1}{2}\int_{y}^{\infty}\frac{dt}{\sqrt{(t-2\sqrt{n})[t^2+(1-n)^2]}}$$
$$=\frac{1}{2\sqrt{1+n}}\left\{\mathrm{sn}^{-1}\left[\frac{2(1+Y\sqrt{n})\sqrt{Y(1+n)}}{(1+Y\sqrt{n})^2+Y(1+n)},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]+\right.$$
$$\left.+\ \mathrm{sn}^{-1}\left[\frac{2(1-Y\sqrt{n})\sqrt{Y(1+n)}}{(1-Y\sqrt{n})^2+Y(1+n)},\ \frac{1-\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}.$$

597.01
$$\int_0^Y \sqrt{\frac{\tau}{(1+\tau^2)(1+n^2\tau^2)}}\,d\tau$$
$$= \frac{1}{2\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t-2\sqrt{n})\,[t^2+(1-n)^2]}} - \frac{1}{2\sqrt{n}}\int_y^\infty \frac{dt}{\sqrt{(t+2\sqrt{n})\,[t^2+(1-n)^2]}}$$
$$= \frac{1}{2\sqrt{n(1+n)}}\left\{\mathrm{sn}^{-1}\left[\frac{2(1-Y\sqrt{n})\sqrt{Y(1+n)}}{(1-Y\sqrt{n})^2+Y(1+n)},\ \frac{|1-\sqrt{n}|}{\sqrt{2(1+n)}}\right]-\right.$$
$$\left.-\,\mathrm{sn}^{-1}\left[\frac{2(1+Y\sqrt{n})\sqrt{Y(1+n)}}{(1+Y\sqrt{n})^2+Y(1+n)},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]\right\}.$$

597.02
$$\int_0^Y \frac{R(\tau)\,d\tau}{\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}}$$
$$=\frac{1}{2}\int_y^\infty \frac{[R_1(t)+(t-2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t-2\sqrt{n})\,[t^2+(1-n)^2]}}+\frac{1}{2}\int_y^\infty \frac{[R_1(t)+(t+2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t+2\sqrt{n})\,[t^2+(1-n)^2]}},$$

[See **241.**]

where $R(\tau) = R\left[\frac{1}{2n}\left(t-\sqrt{t^2-4n}\right)\right] = R_1(t) + \sqrt{t^2-4n}\,R_2(t)$.

Integrands involving $\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}$, $(1/\sqrt{n}<Y<\infty)$

$$t = \frac{1+n\tau^2}{\tau}, \quad \tau = \frac{1}{2n}\left[t+\sqrt{t^2-4n}\right], \quad y = \frac{1+nY^2}{Y},$$
$$\frac{d\tau}{\tau\sqrt{\tau}} = \frac{1}{2}\left[\frac{1}{\sqrt{t-2\sqrt{n}}}-\frac{1}{\sqrt{t+2\sqrt{n}}}\right]dt.$$

598.00
$$\int_{1/\sqrt{n}}^Y \frac{d\tau}{\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}}$$
$$= \frac{1}{2}\int_{2\sqrt{n}}^y \frac{dt}{\sqrt{(t-2\sqrt{n})\,[t^2+(1-n)^2]}} - \frac{1}{2}\int_{2\sqrt{n}}^y \frac{dt}{\sqrt{(t+2\sqrt{n})\,[t^2+(1-n)^2]}}$$
$$= \frac{1}{2\sqrt{1+n}}\left\{\mathrm{sn}^{-1}\left[\frac{2(1+Y\sqrt{n})\sqrt{Y(1+n)}}{(1+Y\sqrt{n})^2+Y(1+n)},\ \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right]-\right.$$
$$\left.-\,\mathrm{sn}^{-1}\left[\frac{2(1-Y\sqrt{n})\sqrt{Y(1+n)}}{(1-Y\sqrt{n})^2+Y(1+n)},\ \frac{|1-\sqrt{n}|}{\sqrt{2(1+n)}}\right]\right\}.$$

598.01
$$\begin{cases} \displaystyle\int_{1/\sqrt{n}}^{Y} \sqrt{\frac{\tau}{(1+\tau^2)(1+n^2\tau^2)}}\, d\tau \\ \displaystyle = \frac{1}{2\sqrt{n}} \int_{2\sqrt{n}}^{y} \frac{dt}{\sqrt{(t-2\sqrt{n})[t^2+(1-n)^2]}} + \frac{1}{2\sqrt{n}} \int_{2\sqrt{n}}^{y} \frac{dt}{\sqrt{(t+2\sqrt{n})[t^2+(1-n)^2]}} \\ \displaystyle = \frac{1}{2\sqrt{n(1+n)}} \left\{ \operatorname{sn}^{-1}\left[\frac{2(1+Y\sqrt{n})\sqrt{Y(1+n)}}{(1+Y\sqrt{n})^2+Y(1+n)}, \frac{1+\sqrt{n}}{\sqrt{2(1+n)}}\right] + \right. \\ \displaystyle \left. + \operatorname{sn}^{-1}\left[\frac{2(1-Y\sqrt{n})\sqrt{Y(1+n)}}{(1-Y\sqrt{n})^2+Y(1+n)}, \frac{|1-\sqrt{n}|}{\sqrt{2(1+n)}}\right]\right\}. \end{cases}$$

598.02
$$\begin{cases} \displaystyle\int_{1/\sqrt{n}}^{Y} \frac{R(\tau)\, d\tau}{\sqrt{\tau(1+\tau^2)(1+n^2\tau^2)}} \\ \displaystyle = \frac{1}{2} \int_{2\sqrt{n}}^{y} \frac{[R_1(t)+(t-2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t-2\sqrt{n})\,[t^2+(1-n)^2]}} - \frac{1}{2} \int_{2\sqrt{n}}^{y} \frac{[R_1(t)+(t+2\sqrt{n})R_2(t)]\,dt}{\sqrt{(t+2\sqrt{n})\,[t^2+(1-n)^2]}}, \end{cases}$$

[See **239**.]

where $R(\tau) = R\left[\frac{1}{2n}\left(t+\sqrt{t^2-4n}\right)\right] = R_1(t) - \sqrt{t^2-4n}\, R_2(t)$.

Integrals of the Elliptic Integrals.

With Respect to the Modulus.

Indefinite Integrals.

610.00 $$\int K\,dk = \frac{\pi k}{2}\left\{1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2 k^{2n+1}}{(2n+1)\, 2^{4n} (n!)^4}\right\}, \qquad [0 < k < 1].$$

610.01 $$\int k K\,dk = E - k'^2 K, \qquad [\text{cf. } \mathbf{530.01}].$$

610.02 $$\int k^3 K\,dk = \frac{1}{9}\left[(4+k^2)E - k'^2(4+3k^2)K\right].$$

610.03 $$\int k^5 K\,dk = \frac{1}{225}\left[(64+16k^2+9k^4)E - k'^2(64+48k^2+45k^4)K\right].$$

610.04 $$\begin{cases} \displaystyle\int k^{2m+3} K\,dk = \frac{1}{(2m+3)^2}\left[4(m+1)^2 \int k^{2m+1} K\,dk + \right. \\ \qquad\qquad \left. + k^{2m+2}(E - 2m k'^2 K - 3k'^2 K)\right]. \end{cases}$$

611.00 $$\int E\,dk = \frac{\pi k}{2}\left\{1 - \sum_{n=1}^{\infty} \frac{[(2n)!]^2 k^{2n+1}}{(4n^2-1)\, 2^{4n} (n!)^4}\right\}, \qquad [0 < k < 1].$$

611.01 $$\int k E\,dk = \frac{1}{3}\left[(1+k^2)E - k'^2 K\right], \qquad [\text{cf. } \mathbf{530.02}].$$

611.02 $$\int k^3 E\,dk = \frac{1}{45}\left[(4+k^2+9k^4)E - k'^2(4+3k^2)K\right].$$

611.03 $$\begin{cases} \displaystyle\int k^5 E\,dk \\ \displaystyle = \frac{1}{1575}\left[(64+16k^2+9k^4+225k^6)E - k'^2(64+48k^2+45k^4)K\right]. \end{cases}$$

611.04 $$\begin{cases} \displaystyle\int k^{2m+3} E\,dk = \frac{1}{4m^2+16m+15} \times \\ \displaystyle\times\left\{4(m+1)^2 \int k^{2m+1} E\,dk + k^{2m+2}\left[(3k^2 - 1 - 2m k'^2)E - k'^2 K\right]\right\}. \end{cases}$$

612.01 $\int \frac{E}{k'^2} dk = kK,$ [cf. **530.03**].

612.02 $\int \frac{E}{k^2} dk = \frac{1}{k}[k'^2 K - 2E].$

612.03 $\int \frac{E}{k^4} dk = \frac{1}{9k^3}[2(k^2-2)E + k'^2 K].$

612.04 $\int \frac{kE}{k'^2} dk = K - E.$

612.05 $\int \frac{K}{k^2} dk = -\frac{E}{k}.$

612.06 $\int \frac{K-E}{k} dk = -E.$

612.07 $\int \frac{K-E}{k^2} dk = \frac{E - k'^2 K}{k}.$

612.08 $\int (K-E)\, k\, dk = \frac{1}{3}[(2-k^2)E - 2k'^2 K].$

612.09 $\int \frac{E - k'^2 K}{k} dk = 2E - k'^2 K.$

612.10 $\int (E - k'^2 K)\, k\, dk = \frac{1}{9}[(10-2k^2)E - k'^2(10+3k^2)K].$

612.11 $\int \frac{k\,K\,dk}{(E-k'^2K)^2} = \frac{1}{k'^2 K - E}.$

612.12 $\int \frac{(1+k^2)K - E}{k} dk = -k'^2 K.$

612.13 $\int \frac{(1+k^2)E - k'^2 K}{k\,k'^4} dk = \frac{E}{k'^2}.$

612.14 $\int \Pi(\pi/2, \alpha^2, k)\, k\, dk = (k^2 - \alpha^2)\,\Pi(\pi/2, \alpha^2, k) - K + E,$ [cf. **530.07**].

612.15 $\int [K - \Pi(\pi/2, \alpha^2, k)]\, k\, dk = k^2 K - (k^2 - \alpha^2)\,\Pi(\pi/2, \alpha^2, k).$

612.16 $\int \left[\frac{E}{k'^2} + \Pi(\pi/2, \alpha^2, k)\right] k\, dk = (k^2 - \alpha^2)\,\Pi(\pi/2, \alpha^2, k).$

613.01 $\int F(\varphi, k)\, k\, dk = E(\varphi, k) - k'^2 F(\varphi, k) + \left[\sqrt{1 - k^2 \sin^2\varphi} - 1\right]\cot\varphi,$
[cf. **530.04**].

613.02 $$\left\{\begin{aligned} &\int E(\varphi, k)\, k\, dk \\ &\quad = \frac{1}{3}\left\{(1+k^2)\,E(\varphi, k) - k'^2 F(\varphi, k) - \left[1 - \sqrt{1 - k^2\sin^2\varphi}\right]\cot\varphi\right\}, \end{aligned}\right.$$
[cf. **530.05**].

613.03 $$\int \Pi(\varphi, \alpha^2, k)\, k\, dk = (k^2 - \alpha^2)\,\Pi(\varphi, \alpha^2, k) - F(\varphi, k) + E(\varphi, k) + \left[\sqrt{1 - k^2 \sin^2\varphi} - 1\right] \cot\varphi, \quad \text{[cf. } \mathbf{530.06}\text{]}.$$

613.04 $$\int [F(\varphi, k) - E(\varphi, k)]\, k\, dk = \frac{1}{3}\left\{(2 - k^2)\, E(\varphi, k) - 2k'^2 F(\varphi, k) + 2\left[\sqrt{1 - k^2 \sin^2\varphi} - 1\right] \cot\varphi\right\}.$$

613.05 $$\int [F(\varphi, k) - \Pi(\varphi, \alpha^2, k)]\, k\, dk = k^2 F(\varphi, k) - (k^2 - \alpha^2)\,\Pi(\varphi, \alpha^2, k).$$

613.06 $$\int [E(\varphi, k) - \Pi(\varphi, \alpha^2, k)]\, k\, dk = \frac{1}{3}\left\{(3 - k'^2)\, F(\varphi, k) + (k^2 - 2) \times E(\varphi, k) - 3\,(k^2 - \alpha^2)\,\Pi(\varphi, \alpha^2, k) + 2\left[1 - \sqrt{1 - k^2 \sin^2\varphi}\right] \cot\varphi\right\}.$$

Definite Integrals.

615.01 $$\int_0^1 K\, dk = 2G, \qquad \left[G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots + \cdots \approx \cdot 9159656\right].$$

615.02 $$\int_0^1 E\, dk = G + 1/2, \qquad \text{[cf. } \mathbf{531.14}\text{]}.$$

615.03 $$\int_0^1 K'\, dk = \pi^2/4, \qquad \text{[cf. } \mathbf{531.15}\text{]}.$$

615.04 $$\int_0^1 E'\, dk = \pi^2/8, \qquad \text{[cf. } \mathbf{531.16}\text{]}.$$

615.05 $$\int_0^1 (K - \pi/2)\, \frac{dk}{k} = \pi \ln 2 - 2G, \qquad \text{[cf. } \mathbf{531.17}\text{]}.$$

615.06 $$\int_0^1 (E - \pi/2)\, \frac{dk}{k} = \frac{1}{2}\,[2\pi \ln 2 - 4G + 2 - \pi].$$

615.07 $$\int_0^1 (E' - 1)\, \frac{dk}{k} = 2 \ln 2 - 1.$$

615.08 $$\int_0^1 K\, \frac{dk}{k'} = K^2\left(\sqrt{2}/2\right), \qquad \text{[cf. } \mathbf{531.01}\text{]}.$$

615.09 $$\int_0^1 \frac{K\, dk}{1 + k} = \pi^2/8, \qquad \text{[cf. } \mathbf{531.18}\text{]}.$$

615.10 $\int_0^1 E \frac{dk}{k'} = \frac{1}{8}\left[4K^2(\sqrt{2}/2) + \frac{\pi^2}{K^2(\sqrt{2}/2)}\right],$ [cf. **531.02**].

615.11 $m \int_0^1 k^m K'\, dk = (m-1)\int_0^1 k^{m-2} E'\, dk,$ $[m > 1]$.

615.12* $m^2(m+1)\int_0^1 k^m K\, dk = (m^2-1)(m-1)\int_0^1 k^{n-2} K\, dk + m + 1.$

615.13 $\int_0^1 [K' - \ln(4/k)]\frac{dk}{k} = \frac{1}{12}[24(\ln 2)^2 - \pi^2].$

615.14 $(m+2)\int_0^1 k^m E'\, dk = (m+1)\int_0^1 k^m K'\, dk,$ $[m > 1]$.

616.01 $\int_0^1 F(\varphi, k)\, dk = \sum_{m=0}^{\infty}\binom{-\frac{1}{2}}{m}\frac{(-1)^m}{2m+1} t_{2m}(\varphi).$ [See **902.00** for $t_{2m}(\varphi)$.]

616.02 $\int_0^1 E(\varphi, k)\, dk = \sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}\frac{(-1)^m}{2m+1} t_{2m}(\varphi).$ [See **902.00** for $t_{2m}(\varphi)$.]

616.03 $\int_0^1 F(\varphi, k)\, k\, dk = \frac{1-\cos\varphi}{\sin\varphi},$ [cf. **531.19**].

616.04 $\int_0^1 E(\varphi, k)\, k\, dk = \frac{1}{3}\left[\frac{\sin^2\varphi + 1 - \cos\varphi}{\sin\varphi}\right],$ [cf. **531.20**].

616.05 $\begin{cases} \int_0^1 \Pi(\varphi, \alpha^2, k)\, k\, dk = \frac{1-\cos\varphi}{\sin\varphi} - \\ -\alpha \ln\sqrt{\frac{1+\alpha\sin\varphi}{1-\alpha\sin\varphi}} - \alpha^2\Pi(\varphi, \alpha^2, 0), \end{cases}$ [cf. **531.21** and see **111.01**].

* A short 10-place table of $(m+1)\int_0^1 k^m K\, dk$ for $m = -.9(.1)2$ may be found in E. L. Kaplan: *Multiple Elliptic Integrals,* Journal of Mathematics and Physics, Vol. 29, 1950, pp. 69—75.

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With Respect to the Argument.

Indefinite Integrals.

630.01 $\displaystyle\int_0^{\varphi} \frac{F(\vartheta, k)\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{[F(\varphi, k)]^2}{2}, \quad 0 < \varphi \leq \pi/2,$ [cf. **530.08**].

630.02 $\displaystyle\int_0^{\varphi} \frac{E(\vartheta, k)\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{[F(\varphi, k)]^2 E}{2K} + \ln\frac{\Theta[F(\varphi, k)]}{\Theta(0)},$ [cf. **530.10**].

where $\Theta(0) = \sqrt{\dfrac{2k' K}{\pi}}$, and $\Theta(u)$ is defined in the Appendix.

630.03 $\displaystyle\int_0^{\varphi} \frac{Z(\vartheta, k)\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \ln\frac{\Theta[F(\varphi, k)]}{\Theta(0)},$ [cf. **530.12**].

630.04 $\displaystyle\int_0^{\varphi} \frac{\Lambda_0(\vartheta, k)\, d\vartheta}{\sqrt{1-k'^2\sin^2\vartheta}} = \frac{1}{2\pi K'}\left\{\pi[F(\varphi, k')]^2 + 4KK'\ln\frac{\Theta[F(\varphi, k')]}{\sqrt{2kK'/\pi}}\right\}.$

630.11 $\displaystyle\int_0^{\varphi} F(\vartheta, k)\sin\vartheta\, d\vartheta = -\cos\varphi\, F(\varphi, k) + \frac{1}{k}\sin^{-1}(k\sin\varphi).$

630.12 $$\left\{\begin{aligned}&\int_0^{\varphi} E(\vartheta, k)\sin\vartheta\, d\vartheta\\ &= -\cos\varphi\, E(\varphi, k) + \frac{1}{2k}\left[k\sin\varphi\sqrt{1-k^2\sin^2\varphi} + \sin^{-1}(k\sin\varphi)\right].\end{aligned}\right.$$

630.13 $$\left\{\begin{aligned}&\int_0^{\varphi} \Pi(\vartheta, \alpha^2, k)\sin\vartheta\, d\vartheta\\ &= -\cos\varphi\,\Pi(\varphi, \alpha^2, k) + \frac{1}{\sqrt{k^2-\alpha^2}}\tan^{-1}\left[\sqrt{\frac{k^2-\alpha^2}{1-k^2\sin^2\varphi}}\sin\varphi\right] \text{ if } \alpha^2 < k^2\\ &= -\cos\varphi\,\Pi(\varphi, \alpha^2, k) + \frac{1}{\sqrt{\alpha^2-k^2}}\tanh^{-1}\left[\sqrt{\frac{\alpha^2-k^2}{1-k^2\sin^2\varphi}}\sin\varphi\right] \text{ if } \alpha^2 > k^2.\end{aligned}\right.$$

630.14 $$\left\{\begin{aligned}&\int_0^{\varphi} Z(\vartheta, k)\sin\vartheta\, d\vartheta = -\cos\varphi\, Z(\varphi, k) +\\ &\qquad + \frac{1}{2k}\left[k\sin\varphi\sqrt{1-k^2\sin^2\varphi} + \left(1-\frac{2E}{K}\right)\sin^{-1}(k\sin\varphi)\right].\end{aligned}\right.$$

630.15 $$\left\{\begin{aligned}&\int_0^{\varphi} \Lambda_0(\vartheta, k)\sin\vartheta\, d\vartheta = -\cos\varphi\,\Lambda_0(\varphi, k) +\\ &\qquad + \frac{1}{\pi k'}\left[(2E-K)\sin^{-1}(k'\sin\varphi) + k'K\sin\varphi\sqrt{1-k'^2\sin^2\varphi}\right].\end{aligned}\right.$$

630.21 $$\begin{cases} \displaystyle\int_0^{\varphi} F(\vartheta, k)\cos\vartheta\, d\vartheta \\ \displaystyle\qquad = \sin\varphi\, F(\varphi, k) + \frac{1}{k}\cosh^{-1}\sqrt{\frac{1-k^2\sin^2\varphi}{k'^2}} - \frac{1}{k}\cosh^{-1}(1/k'). \end{cases}$$

630.22 $$\begin{cases} \displaystyle\int_0^{\varphi} E(\vartheta, k)\cos\vartheta\, d\vartheta = \sin\varphi\, E(\varphi, k) + \frac{1}{2k}\Big[k\cos\varphi\sqrt{1-k^2\sin^2\varphi} - \\ \displaystyle\qquad - k'^2\cosh^{-1}\sqrt{\frac{1-k^2\sin^2\varphi}{k'^2}} - k + k'^2\cosh^{-1}(1/k')\Big]. \end{cases}$$

630.23 $$\int_0^{\varphi} \Pi(\vartheta, \alpha^2, k)\cos\vartheta\, d\vartheta = \sin\varphi\, \Pi(\varphi, \alpha^2, k) - f + f_0,$$

where f is obtained from the expression in **361.58** on replacing $\operatorname{sn} u$ by $\sin\varphi$, $\operatorname{cn} u$ by $\cos\varphi$, and $\operatorname{dn} u$ by $\sqrt{1-k^2\sin^2\varphi}$; f_0 is the value of f at $\varphi = 0$.

630.24 $$\begin{cases} \displaystyle\int_0^{\varphi} Z(\vartheta, k)\cos\vartheta\, d\vartheta \\ \displaystyle = \sin\varphi\, Z(\varphi, k) + \frac{1}{2kK}\Big[kK\cos\varphi\sqrt{1-k^2\sin^2\varphi} - kK + \\ \displaystyle\quad + (2E+k'^2K)\cosh^{-1}(1/k') - (2E+k'^2K)\cosh^{-1}\sqrt{\frac{1-k^2\sin^2\varphi}{k'^2}}\Big]. \end{cases}$$

630.25 $$\begin{cases} \displaystyle\int_0^{\varphi} \Lambda_0(\vartheta, k)\cos\vartheta\, d\vartheta = \sin\varphi\, \Lambda_0(\varphi, k) + \frac{1}{\pi k'} \times \\ \displaystyle\times\Big\{k'K\cos\varphi\sqrt{1-k'^2\sin^2\varphi} - k'K - [2E-(2+k^2)K]\cosh^{-1}(1/k) + \\ \displaystyle\qquad + [2E-(2+k^2)K]\cosh^{-1}\sqrt{\frac{1-k'^2\sin^2\varphi}{k^2}}\Big\}. \end{cases}$$

630.31 $$\begin{cases} \displaystyle\int_0^{\varphi} F(\vartheta, k)\sqrt{1-k^2\sin^2\vartheta}\, d\vartheta \\ \displaystyle\qquad = F(\varphi, k)\, E(\varphi, k) - \frac{[F(\varphi, k)]^2 E}{2K} - \ln\frac{\Theta[F(\varphi, k)]}{\Theta(0)}. \end{cases}$$

630.32 $$\int_0^{\varphi} E(\vartheta, k)\sqrt{1-k^2\sin^2\vartheta}\, d\vartheta = \frac{[E(\varphi, k)]^2}{2}.$$

630.33 $$\int_0^{\varphi} Z(\vartheta, k)\sqrt{1-k^2\sin^2\vartheta}\, d\vartheta = \frac{1}{2}[Z(\varphi, k)]^2 + \frac{E}{K}\ln\frac{\Theta[F(\varphi, k)]}{\Theta(0)}.$$

630.34 $$\begin{cases} \displaystyle\int_0^{\varphi} \Lambda_0(\vartheta, k)\sqrt{1-k'^2\sin^2\vartheta}\, d\vartheta = \frac{E(\varphi, k')\,\Lambda_0(\varphi, k)}{2} + \\ \displaystyle\qquad + \frac{(E-K)\,F(\varphi, k')\,Z(\varphi, k')}{\pi} + \frac{2}{\pi}(K-E)\ln\frac{\Theta[F(\varphi, k')]}{\sqrt{2kK'/\pi}}. \end{cases}$$

630.41
$$\int_0^{\varphi} \frac{\sin^2\vartheta\, F(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2 K}\Big\{[F(\varphi, k)]^2 K - 2F(\varphi, k)\, E(\varphi, k)\, K + [F(\varphi, k)]^2 E + 2K \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.42
$$\int_0^{\varphi} \frac{\sin^2\vartheta\, E(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2 K}\Big\{[F(\varphi, k)]^2 E - [E(\varphi, k)]^2 K + 2K \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.43
$$\int_0^{\varphi} \frac{\sin^2\vartheta\, Z(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2}\Big\{-[Z(\varphi, k)]^2 + \frac{2(K-E)}{K} \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.44
$$\int_0^{\varphi} \frac{\sin^2\vartheta\, \Lambda_0(\vartheta, k)}{\sqrt{1-k'^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2\pi k'^2 K'}\Big\{\pi[F(\varphi, k')]^2 - \pi E(\varphi, k')\Lambda_0(\varphi, k) K' - 2(E-K)\, F(\varphi, k')\, K'\, Z(\varphi, k') + 4K'\, E \ln \frac{\Theta[F(\varphi, k')]}{\sqrt{2k K'/\pi}}\Big\}.$$

630.51
$$\int_0^{\varphi} \Pi(\vartheta, \alpha^2, k) \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}} = \frac{[\Pi(\varphi, \alpha^2, k)]^2}{2}.$$

630.61
$$\int_0^{\varphi} \frac{\cos^2\vartheta\, F(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2 K} \times \Big\{2E(\varphi, k) F(\varphi, k) K - (E + k'^2 K)\,[F(\varphi, k)]^2 - 2K \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.62
$$\int_0^{\varphi} \frac{\cos^2\vartheta\, E(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = -\frac{1}{2k^2 K} \times \Big\{k'^2[F(\varphi, k)]^2 E - [E(\varphi, k)]^2 K + 2k'^2 K \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.63
$$\int_0^{\varphi} \frac{\cos^2\vartheta\, Z(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2 K}\Big\{[Z(\varphi, k)]^2 K + 2(E - k'^2 K) \ln \frac{\Theta[F(\varphi, k)]}{\Theta(0)}\Big\}.$$

630.64
$$\int_0^{\varphi} \frac{\cos^2\vartheta\, \Lambda_0(\vartheta, k)}{\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{2\pi k'^2 K'}\Big\{\pi E(\varphi, k')\, \Lambda_0(\varphi, k)\, K' - \pi k^2 [F(\varphi, k')]^2 + 2(E-K)\, K'\, F(\varphi, k')\, Z(\varphi, k') + 4(K-E)\, K' \ln \frac{\Theta[F(\varphi, k')]}{\sqrt{2k K'/\pi}}\Big\}.$$

630.71 $$\int_0^{\varphi} \frac{F(\vartheta, k) \sin\vartheta \cos\vartheta}{\sqrt{1 - k^2 \sin^2\vartheta}}\, d\vartheta = \frac{1}{k^2}\left[\varphi - F(\varphi, k)\sqrt{1 - k^2 \sin^2\varphi}\right].$$

630.72 $$\left\{\begin{aligned} &\int_0^{\varphi} \frac{E(\vartheta, k) \sin\vartheta \cos\vartheta}{\sqrt{1 - k^2 \sin^2\vartheta}}\, d\vartheta \\ &\quad = \frac{1}{2k^2}\left[(1 + k'^2)\varphi + k^2 \sin\varphi \cos\varphi - 2E(\varphi, k)\sqrt{1 - k^2 \sin^2\varphi}\right]. \end{aligned}\right.$$

630.73 $$\left\{\begin{aligned} &\int_0^{\varphi} \frac{Z(\vartheta, k) \sin\vartheta \cos\vartheta}{\sqrt{1 - k^2 \sin^2\vartheta}}\, d\vartheta = \frac{1}{2k^2} \times \\ &\times\left\{-2Z(\varphi, k)\sqrt{1 - k^2 \sin^2\varphi} + \left[(1+k'^2) - \frac{2E}{K}\right]\varphi + k^2 \sin\varphi \cos\varphi\right\}. \end{aligned}\right.$$

630.74 $$\left\{\begin{aligned} &\int_0^{\varphi} \frac{\Lambda_0(\vartheta, k) \sin\vartheta \cos\vartheta}{\sqrt{1 - k'^2 \sin^2\vartheta}}\, d\vartheta = \frac{1}{\pi k'^2} \times \\ &\times\left[-\pi\Lambda_0(\varphi, k)\sqrt{1 - k'^2 \sin^2\varphi} + (2E - k'^2 K)\varphi + k'^2 K \sin\varphi \cos\varphi\right]. \end{aligned}\right.$$

Definite Integrals[1].

633.01 $$\int_0^K u \,\mathrm{sn}\, u \,\mathrm{dn}\, u \, du = \frac{1}{k} \sin^{-1} k, \qquad \text{[cf. } \mathbf{630.11}\text{]}.$$

633.02 $$\int_0^K E(u) \,\mathrm{sn}\, u \,\mathrm{dn}\, u \, du = \frac{1}{2k}\left[k k' + \sin^{-1} k\right], \qquad \text{[cf. } \mathbf{630.12}\text{]}.$$

633.03 $$\left\{\begin{aligned} &\int_0^K \Pi(u, \alpha^2) \,\mathrm{sn}\, u \,\mathrm{dn}\, u \, du = \frac{1}{\sqrt{k^2 - \alpha^2}} \tan^{-1}\sqrt{\frac{k^2 - \alpha^2}{k'^2}}, \quad \text{if } \alpha^2 < k^2; \\ &\qquad = \frac{1}{\sqrt{\alpha^2 - k^2}} \tanh^{-1}\sqrt{\frac{\alpha^2 - k^2}{k'^2}}, \quad \text{if } \alpha^2 > k^2, \qquad \text{[cf. } \mathbf{630.13}\text{]}. \end{aligned}\right.$$

633.04 $$\int_0^K Z(u) \,\mathrm{sn}\, u \,\mathrm{dn}\, u \, du = \frac{1}{2k}\left[k k' + \left(1 - \frac{2E}{K}\right)\sin^{-1} k\right], \qquad \text{[cf. } \mathbf{630.14}\text{]}.$$

633.05 $$\left\{\begin{aligned} &\int_0^{K'} \Lambda_0(\mathrm{am}\, u_1, k) \,\mathrm{sn}\, u_1 \,\mathrm{dn}\, u_1 \, du_1 = \frac{1}{\pi k'}\left[(2E - K)\sin^{-1} k' + k k' K\right], \\ &\qquad\qquad \text{[cf. } \mathbf{630.15}\text{]}. \\ &\qquad\qquad \left[d\vartheta = \sqrt{1 - k'^2 \sin^2\vartheta}\, du_1,\ \vartheta = \mathrm{am}(u_1, k')\right]. \end{aligned}\right.$$

[1] See **800** for other such integrals.

633.06 $\int_0^K u \operatorname{cn} u \operatorname{dn} u \, du = K - \frac{1}{k} \cosh^{-1}(1/k')$, [cf. **630.21**].

633.07 $\int_0^K E(u) \operatorname{cn} u \operatorname{dn} u \, du = E - \frac{1}{2} + \frac{k'^2}{2k} \cosh^{-1}(1/k')$, [cf. **630.22**].

633.08 $\int_0^K u \, du = \frac{K^2}{2}$, [cf. **630.01**].

633.09 $\int_0^K E(u) \, du = \frac{E \cdot K}{2} - \ln \sqrt{k'}$, [cf. **630.02**].

633.10 $\int_0^K Z(u) \, du = \ln\left(\sqrt{1/k'}\right)$, [cf. **630.03**].

633.11 $\int_0^{K'} \Lambda_0(\operatorname{am} u_1, k) \, du_1 = \frac{1}{2\pi} [\pi K' - 2K \ln k]$, [cf. **630.04**].
$\left[d\vartheta = \sqrt{1 - k'^2 \sin^2\vartheta} \, du_1, \ \vartheta = \operatorname{am}(u_1, k')\right]$.

633.12 $\int_0^K u \operatorname{dn}^2 u \, du = \frac{E \cdot K}{2} + \ln \sqrt{k'}$, [cf. **630.31**].

633.13 $\int_0^K E(u) \operatorname{dn}^2 u \, du = \frac{E^2}{2}$, [cf. **630.32**].

633.14 $\int_0^K Z(u) \operatorname{dn}^2 u \, du = \frac{E}{K} \ln\left(\sqrt{1/k'}\right)$, [cf. **630.33**].

633.15 $\int_0^{K'} \Lambda_0(\operatorname{am} u_1, k) \operatorname{dn}^2 u_1 \, du_1 = \frac{E'}{2} + \frac{2}{\pi}(K - E) \ln\left(\sqrt{1/k}\right)$,
[cf. **630.34**].
$\left[d\vartheta = \sqrt{1 - k'^2 \sin^2\vartheta} \, du_1, \ \vartheta = \operatorname{am}(u_1, k')\right]$.

633.16 $\int_0^K u \operatorname{sn}^2 u \, du = \frac{1}{2k^2} [(K - E) K - \ln(k')]$, [cf. **630.41**].

633.17 $\int_0^K E(u) \operatorname{sn}^2 u \, du = \frac{1}{2k^2} [(K - E) E - \ln(k')]$, [cf. **630.42**].

633.18 $\int_0^K Z(u) \operatorname{sn}^2 u \, du = \frac{(K - E)}{k^2 K} \ln\left(\sqrt{1/k'}\right)$, [cf. **630.43**].

633.19
$$\begin{cases} \displaystyle\int_0^{K'} \Lambda_0(\operatorname{am} u_1, k) \operatorname{sn}^2 u_1 \, du_1 = \frac{1}{2\pi k'^2}\left[\pi(K' - E') + 4E \ln\left(\sqrt{1/k}\right)\right], \\ \hfill [\text{cf. } \mathbf{630.44}]. \\ \hfill \left[d\vartheta = \sqrt{1 - k'^2 \sin^2\vartheta}\, du_1,\ \vartheta = \operatorname{am}(u_1, k')\right]. \end{cases}$$

633.20
$$\int_0^K \frac{\Pi(u, \alpha^2)}{1 - \alpha^2 \operatorname{sn}^2 u}\, du = \frac{[\Pi(\alpha^2, k)]^2}{2}, \qquad [\text{cf. } \mathbf{630.51}].$$

633.21
$$\int_0^K u \operatorname{cn}^2 u \, du = \frac{1}{2k^2}\left[(E - k'^2 K) K + \ln(k')\right], \qquad [\text{cf. } \mathbf{630.61}].$$

633.22
$$\int_0^K E(u) \operatorname{cn}^2 u \, du = \frac{1}{2k^2}\left[(E - k'^2 K) E + k'^2 \ln(k')\right], \qquad [\text{cf. } \mathbf{630.62}].$$

633.23
$$\int_0^K Z(u) \operatorname{cn}^2 u \, du = \frac{(E - k'^2 K)}{k^2 K} \ln\left(\sqrt{1/k'}\right), \qquad [\text{cf. } \mathbf{630.63}].$$

633.24
$$\begin{cases} \displaystyle\int_0^{K'} \Lambda_0(\operatorname{am} u_1, k) \operatorname{cn}^2 u_1 \, du_1 \\ \displaystyle\quad = \frac{1}{2\pi k'^2}\left[\pi(E' - k^2 K') + 4(K - E) \ln\left(\sqrt{1/k}\right)\right], \quad [\text{cf. } \mathbf{630.64}], \\ \hfill \left[d\vartheta = \sqrt{1 - k'^2 \sin^2\vartheta}\, du_1,\ \vartheta = \operatorname{am}(u_1, k')\right]. \end{cases}$$

633.25
$$\int_0^K u \operatorname{sn} u \operatorname{cn} u \, du = \frac{1}{2k^2}\left[\pi - 2k' K\right], \qquad [\text{cf. } \mathbf{630.71}].$$

633.26
$$\int_0^K E(u) \operatorname{sn} u \operatorname{cn} u \, du = \frac{1}{4k^2}\left[(1 + k'^2)\pi - 4k' E\right], \qquad [\text{cf. } \mathbf{630.72}].$$

633.27
$$\int_0^K Z(u) \operatorname{sn} u \operatorname{cn} u \, du = \frac{\pi}{4k^2}\left[(1 + k'^2) - \frac{2E}{K}\right], \qquad [\text{cf. } \mathbf{630.73}].$$

Derivatives.

With Respect to the Modulus.

Differentiation of the Elliptic Integrals.

710.00 $\frac{dK}{dk} = \frac{E - k'^2 K}{k k'^2}$; $\qquad \frac{d^2 K}{dk^2} = \frac{(3k^2 - 1)E + k'^2 (k'^2 - k^2) K}{k^2 k'^4}$.

710.01 $$\begin{cases} \dfrac{d^m K}{d(k^2)^m} = \dfrac{\pi [(2m!)]^2}{2(m!)^3 16^m} F\left(\dfrac{1}{2} + m,\ \dfrac{1}{2} + m,\ 1 + m;\ k^2\right) \\ \qquad = \dfrac{(2m)!}{4^m m!} \displaystyle\int_0^K \mathrm{sd}^{2m} u\, du, \end{cases}$$ [See **318.05**.]

where F is the hypergeometric function.

710.02 $\frac{dE}{dk} = \frac{E - K}{k}$; $\qquad \frac{d^2 E}{dk^2} = -\frac{1}{k}\frac{dK}{dk} = -\frac{E - k'^2 K}{k^2 k'^2}$.

710.03 $$\begin{cases} \dfrac{d^m E}{d(k^2)^m} = \dfrac{\pi (\frac{1}{2})_m (-\frac{1}{2})_m}{2(m!)} F\left(\dfrac{1}{2} + m,\ -\dfrac{1}{2} + m,\ 1 + m;\ k^2\right) \\ \qquad = -\dfrac{2[(2m-2)!]}{4^m (m-1)!} \displaystyle\int_0^K \dfrac{\mathrm{sn}^{2m} u}{\mathrm{dn}^{2m-2} u}\, du, \end{cases}$$ [See **355.01** with $p = 0$.]

where F is the hypergeometric function.

710.04 $\frac{d}{dk}(E - k'^2 K) = k K$.

710.05 $\frac{d}{dk}(E - K) = -\frac{kE}{k'^2}$.

710.06 $\frac{d}{dk} Z(\varphi, k) = -\frac{(E - k'^2 K) Z(\varphi, k)}{k k'^2 K} + \frac{k E \sin\varphi \cos\varphi}{k'^2 \sqrt{1 - k^2 \sin^2\varphi}\, K}$.

710.07 $\frac{d}{dk} F(\varphi, k) = \frac{E(\varphi, k) - k'^2 F(\varphi, k)}{k k'^2} - \frac{k \sin\varphi \cos\varphi}{k'^2 \sqrt{1 - k^2 \sin^2\varphi}}$.

710.08 $\frac{d^m}{d(k^2)^m} F(\varphi, k) = \frac{(2m)!}{4^m m!} \int_0^{u_1} \mathrm{sd}^{2m} u\, du$, $[\varphi = am\, u_1;$ see **318.05**.].

710.09 $$\frac{d}{dk} E(\varphi, k) = \frac{E(\varphi, k) - F(\varphi, k)}{k}.$$

710.10 $$\frac{d^m}{d(k^2)^m} E(\varphi, k) = -\frac{2[(2m-2)!]}{4^m (m-1)!} \int_0^{u_1} \frac{sn^{2m} u}{dn^{2m-2} u} du,$$

$$[\varphi = am\, u_1; \text{ see } \mathbf{355.01}.].$$

710.11 $$\frac{d}{dk} \Lambda_0(\varphi, k) = \frac{2(E-K)\sin\varphi\cos\varphi}{\pi k\sqrt{1-k'^2\sin^2\varphi}}.$$

710.12 $$\left\{\begin{aligned} &\frac{\partial}{\partial k} \Pi(\varphi, \alpha^2, k) \\ &\quad = \frac{k}{k'^2(k^2-\alpha^2)} \left[E(\varphi, k) - k'^2 \Pi(\varphi, \alpha^2, k) - \frac{k^2 \sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right]. \end{aligned}\right.$$

710.13 $$\left\{\begin{aligned} &\frac{\partial}{\partial k} \int_0^{\varphi} \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)^2\sqrt{1-k^2\sin^2\vartheta}} = \frac{dV_2}{dk} = \frac{k}{2(\alpha^2-1)(\alpha^2-k^2)^2 k'^2} \times \\ &\qquad \times \Big\{(k^2-\alpha^2) k'^2 F(\varphi, k) + (\alpha^2 k'^2 - 2\alpha^2 k^2 + 2k^2) E(\varphi, k) + \\ &\qquad\qquad + (2\alpha^2 k^2 + \alpha^4 - 3k^2) k'^2 \Pi(\varphi, \alpha^2, k) + \\ &\quad + \frac{[(2\alpha^2 k^4 - 2k^4 - \alpha^4 k'^2) + \alpha^2 k^2 (\alpha^2 k'^2 - 2\alpha^2 k^2 + 2k^2)\sin^2\varphi]\sin\varphi\cos\varphi}{(1-\alpha^2\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}\Big\}. \end{aligned}\right.$$

Differentiation of the Jacobian Elliptic Functions.

710.50 $$\frac{\partial}{\partial k}(\mathrm{am}\, u) = \frac{\mathrm{dn}\, u}{k k'^2}[-E(u) + k'^2 u + k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.51 $$\frac{\partial}{\partial k}(\mathrm{sn}\, u) = \frac{\mathrm{dn}\, u\, \mathrm{cn}\, u}{k k'^2}[-E(u) + k'^2 u + k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.52 $$\frac{\partial}{\partial k}(\mathrm{cn}\, u) = \frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{k k'^2}[-k'^2 u + E(u) - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.53 $$\frac{\partial}{\partial k}(\mathrm{dn}\, u) = \frac{k\, \mathrm{sn}\, u\, \mathrm{cn}\, u}{k'^2}[E(u) - k'^2 u - \mathrm{dn}\, u\, \mathrm{tn}\, u].$$

710.54 $$\frac{\partial}{\partial k}(\mathrm{tn}\, u) = \frac{\mathrm{nc}\, u\, \mathrm{dc}\, u}{k k'^2}[k'^2 u - E(u) + k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.55 $$\frac{\partial}{\partial k}(\mathrm{cs}\, u) = \frac{\mathrm{ns}\, u\, \mathrm{ds}\, u}{k k'^2}[E(u) - k'^2 u - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.56 $$\frac{\partial}{\partial k}(\mathrm{ns}\, u) = \frac{\mathrm{cs}\, u\, \mathrm{ds}\, u}{k k'^2}[E(u) - k'^2 u - k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.57 $$\frac{\partial}{\partial k}(\mathrm{nc}\, u) = \frac{\mathrm{tn}\, u\, \mathrm{dc}\, u}{k k'^2}[k'^2 u - E(u) + k^2 \mathrm{sn}\, u\, \mathrm{cd}\, u].$$

710.58 $\frac{\partial}{\partial k}(\operatorname{ds} u) = \frac{\operatorname{cs} u \operatorname{ns} u}{k\, k'^2}\,[E(u) - k'^2 u - k^2 \operatorname{dn} u \operatorname{tn} u]$.

710.59 $\frac{\partial}{\partial k}(\operatorname{sd} u) = \frac{\operatorname{cd} u \operatorname{nd} u}{k\, k'^2}\,[k'^2 u - E(u) + k^2 \operatorname{dn} u \operatorname{tn} u]$.

710.60 $\frac{\partial}{\partial k}(\operatorname{dc} u) = \frac{\operatorname{tn} u \operatorname{nc} u}{k}\,[k'^2 u - E(u)]$.

710.61 $\frac{\partial}{\partial k}(\operatorname{cd} u) = \frac{\operatorname{sd} u \operatorname{nd} u}{k}\,[E(u) - k'^2 u]$.

710.62 $\frac{\partial}{\partial k}(\operatorname{nd} u) = \frac{k \operatorname{sd} u \operatorname{cd} u}{k'^2}\,[k'^2 u - E(u) + \operatorname{dn} u \operatorname{tn} u]$.

The Jacobian elliptic functions are not only a function of k but of the argument u which also depends on the modulus. In order to obtain the total derivative with respect to the modulus, one may proceed, for example, as follows:

$$\frac{d}{dk}[\operatorname{dn}(u,k)] = \frac{\partial}{\partial k}[\operatorname{dn}(u,k)] + \frac{\partial}{\partial u}[\operatorname{dn}(u,k)]\frac{du}{dk} = -k \operatorname{sn}^2 u \operatorname{nd} u.$$

With Respect to the Argument.

Differentiation of Elliptic Integrals.

730.00 $\frac{d}{d\varphi} F(\varphi, k) = \frac{1}{\sqrt{1 - k^2 \sin^2\varphi}}$.

730.01 $\frac{d}{d\varphi} E(\varphi, k) = \sqrt{1 - k^2 \sin^2\varphi}$.

730.02 $\frac{d}{d\varphi} \Pi(\varphi, \alpha^2, k) = \frac{1}{(1 - \alpha^2 \sin^2\varphi)\sqrt{1 - k^2 \sin^2\varphi}}$.

730.03 $\frac{d}{d\varphi} Z(\varphi, k) = \frac{(1 - k^2 \sin^2\varphi) K - E}{\sqrt{1 - k^2 \sin^2\varphi}\, K}$.

730.04 $\frac{d}{d\varphi} \Lambda_0(\varphi, k) = \frac{2(E - k'^2 \sin^2\varphi\, K)}{\pi\sqrt{1 - k'^2 \sin^2\varphi}}$.

Differentiation of the Jacobian Elliptic Functions.

731.00 $\frac{\partial}{\partial u}(\operatorname{am} u) = \operatorname{dn} u$.

731.01 $\frac{\partial}{\partial u}(\operatorname{sn} u) = \operatorname{cn} u \operatorname{dn} u$.

731.02 $\frac{\partial}{\partial u}(\operatorname{cn} u) = -\operatorname{sn} u \operatorname{dn} u.$

731.03 $\frac{\partial}{\partial u}(\operatorname{dn} u) = -k^2 \operatorname{sn} u \operatorname{cn} u.$

731.04 $\frac{\partial}{\partial u}(\operatorname{ns} u) = -\operatorname{cs} u \operatorname{ds} u.$

731.05 $\frac{\partial}{\partial u}(\operatorname{nc} u) = \operatorname{tn} u \operatorname{dc} u.$

731.06 $\frac{\partial}{\partial u}(\operatorname{nd} u) = k^2 \operatorname{sd} u \operatorname{cd} u.$

731.07 $\frac{\partial}{\partial u}(\operatorname{cs} u) = -\operatorname{dn} u \operatorname{ns}^2 u.$

731.08 $\frac{\partial}{\partial u}(\operatorname{sd} u) = \operatorname{cn} u \operatorname{nd}^2 u.$

731.09 $\frac{\partial}{\partial u}(\operatorname{ds} u) = -\operatorname{cn} u \operatorname{ns}^2 u.$

731.10 $\frac{\partial}{\partial u}(\operatorname{tn} u) = \operatorname{nc} u \operatorname{dc} u.$

731.11 $\frac{\partial}{\partial u}(\operatorname{cd} u) = -k'^2 \operatorname{sn} u \operatorname{nd}^2 u.$

731.12 $\frac{\partial}{\partial u}(\operatorname{dc} u) = k'^2 \operatorname{sn} u \operatorname{nc}^2 u.$

Differentiation of the Jacobian Inverse Functions.

732.00 $\frac{d}{d\varphi} \operatorname{am}^{-1}(\varphi, k) = \frac{1}{\sqrt{1 - k^2 \sin^2\varphi}},$ $[0 < \varphi < \pi/2].$

732.01 $\frac{d}{dy} \operatorname{sn}^{-1}(y, k) = \frac{1}{\sqrt{(1-y^2)(1-k^2 y^2)}},$ $[0 < y < 1].$

732.02 $\frac{d}{dy} \operatorname{cn}^{-1}(y, k) = -\frac{1}{\sqrt{(1-y^2)(k'^2 + k^2 y^2)}},$ $[0 < y < 1].$

732.03 $\frac{d}{dy} \operatorname{dn}^{-1}(y, k) = -\frac{1}{\sqrt{(1-y^2)(y^2 - k'^2)}},$ $[k' < y < 1].$

732.04 $\frac{d}{dy} \operatorname{tn}^{-1}(y, k) = \frac{1}{\sqrt{(1+y^2)(1 + k'^2 y^2)}},$ $[0 < y < \infty].$

732.05 $\frac{d}{dy} \operatorname{nc}^{-1}(y, k) = \frac{1}{\sqrt{(y^2-1)(k^2 + k'^2 y^2)}},$ $[\infty > y > 1].$

732.06 $$\frac{d}{dy}\,\mathrm{nd}^{-1}(y,k) = \frac{1}{\sqrt{(y^2-1)(1-k'^2y^2)}}, \qquad \left[\frac{1}{k'} > y > 1\right],$$

732.07 $$\frac{d}{dy}\,\mathrm{dc}^{-1}(y,k) = \frac{1}{\sqrt{(y^2-1)(y^2-k^2)}}, \qquad [\infty > y > 0].$$

732.08 $$\frac{d}{dy}\,\mathrm{sd}^{-1}(y,k) = \frac{1}{\sqrt{(1-k'^2y^2)(1+k^2y^2)}}, \qquad \left[0 < y < \frac{1}{k'}\right],$$

732.09 $$\frac{d}{dy}\,\mathrm{ns}^{-1}(y,k) = -\frac{1}{\sqrt{(y^2-1)(y^2-k^2)}}, \qquad [y > 1 > k].$$

732.10 $$\frac{d}{dy}\,\mathrm{ds}^{-1}(y,k) = -\frac{1}{\sqrt{(y^2-k'^2)(y^2+k^2)}}, \qquad [y > k' > 0].$$

732.11 $$\frac{d}{dy}\,\mathrm{cs}^{-1}(y,k) = -\frac{1}{\sqrt{(1+y^2)(y^2+k'^2)}}, \qquad [\infty > y > 0].$$

732.12 $$\frac{d}{dy}\,\mathrm{cd}^{-1}(y,k) = -\frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}}, \qquad [0 < y < 1].$$

With Respect to the Parameter.

Differentiation of the Normal Elliptic Integral of the Third Kind.

733.00 $$\left\{\begin{aligned} &\frac{\partial}{\partial n}\Pi(u,\alpha^2,k) = \frac{\partial V_1}{\partial n} = \frac{V_2-V_1}{\alpha^2} = \frac{1}{2\alpha^2(1-\alpha^2)(\alpha^2-k^2)}\times\\ &\times\left[\alpha^2E(u)+(k^2-\alpha^2)u+(\alpha^4-k^2)\Pi(u,\alpha^2,k)-\frac{\alpha^4\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-\alpha^2\,\mathrm{sn}^2u}\right],\\ &\hfill[\alpha^2=n].\end{aligned}\right.$$

733.01 $$\frac{\partial^2}{\partial n^2}\Pi(u,\alpha^2,k) = 2\int\frac{\mathrm{sn}^4u\,du}{(1-\alpha^2\,\mathrm{sn}^2u)^3} = \frac{2}{\alpha^4}[V_1-2V_2+V_3].$$

733.02 $$\frac{\partial^3}{\partial n^3}\Pi(u,\alpha^2,k) = 6\int\frac{\mathrm{sn}^6u\,du}{(1-\alpha^2\,\mathrm{sn}^2u)^4} = \frac{6}{\alpha^6}[-V_1+3V_2-3V_3+V_4], \qquad [\alpha^2=n].$$

733.03 $$\left\{\begin{aligned} &\frac{\partial^j}{\partial n^j}\Pi(u,\alpha^2,k)\\ &\quad= (j)!\int\frac{\mathrm{sn}^{2j}u\,du}{(1-\alpha^2\,\mathrm{sn}^2u)^{j+1}} = \frac{(j)!}{(\alpha^2)^j}\sum_{\gamma=1}^{j}(-1)^{\gamma+j}V_{\gamma+1}\frac{(j)!}{(j-\gamma)!\,(\gamma)!},\end{aligned}\right.$$

where

$$V_{\gamma+1} = \int\frac{du}{(1-\alpha^2\,\mathrm{sn}^2u)^{\gamma+1}}$$

is given by **336**.

Differentiation of Other Elliptic Integrals.

734.01 $\frac{\partial}{\partial n}\int\frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}=\frac{2(V_3-V_2)}{\alpha^2}$, $[\alpha^2=n]$.

734.02 $\frac{\partial}{\partial n}\int\frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^3}=\frac{3(V_4-V_3)}{\alpha^2}$.

734.03 $\frac{\partial}{\partial n}\int\frac{du}{(1-\alpha^2\,\mathrm{sn}^2 u)^m}=\frac{m(V_{m+1}-V_m)}{\alpha^2}$. [See **336**.]

735.01 $\frac{\partial}{\partial n}\int\frac{\mathrm{sn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}=\int\frac{\mathrm{sn}^4 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}$. [See **337.02**.]

735.02 $\frac{\partial}{\partial n}\int\frac{\mathrm{cn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}=\int\frac{\mathrm{sn}^2 u\,\mathrm{cn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}$. [See **362.18**.]

735.03 $\frac{\partial}{\partial n}\int\frac{\mathrm{dn}^2 u\,du}{1-\alpha^2\,\mathrm{sn}^2 u}=\int\frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}$. [See **362.19**.]

735.04 $\frac{\partial}{\partial n}\int\frac{1-\alpha_1^2\,\mathrm{sn}^2 u}{1-\alpha^2\,\mathrm{sn}^2 u}\,du=\int\frac{\mathrm{sn}^2 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}-\alpha_1^2\int\frac{\mathrm{sn}^4 u\,du}{(1-\alpha^2\,\mathrm{sn}^2 u)^2}$.
[See **362.15** and **337.02**.]

Miscellaneous Integrals and Formulas.

While most of the integrals in the previous sections have at least one variable upper limit, both of the limits of integration of the integrals given here are fixed.

800.01
$$\left\{\begin{aligned}&\int_0^b \frac{\ln(t)\,dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\int_0^K \ln(b\,\mathrm{sn}\,u)\,du = \frac{1}{a}\left[K\ln\left(\sqrt{a\,b}\right)-\pi K'/4\right];\\ &\qquad k^2 = b^2/a^2,\ (a>b>0).\end{aligned}\right.$$

800.02
$$\left\{\begin{aligned}&\int_0^b \frac{\ln(t)\,dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{k}{b}\int_0^K \ln(b\,\mathrm{cn}\,u)\,du = \frac{k}{b}\left[K\ln\left(\sqrt{a\,b}\right)-\pi K'/4\right];\\ &\qquad k^2 = b^2\big/(a^2+b^2).\end{aligned}\right.$$

800.03
$$\left\{\begin{aligned}&\int_b^a \frac{\ln(t)\,dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\int_0^K \ln(a\,\mathrm{dn}\,u)\,du = \frac{1}{a}\int_0^K \ln(b\,\mathrm{nd}\,u)\,du\\ &\qquad = \frac{1}{a}\left[K\ln\left(\sqrt{a\,b}\right)\right];\ k^2=(a^2-b^2)\big/a^2,\ a>b>0.\end{aligned}\right.$$

800.04
$$\left\{\begin{aligned}&\int_0^\infty \frac{\ln(t)\,dt}{\sqrt{(a^2+t^2)(t^2+b^2)}} = \frac{1}{a}\int_0^K \ln(b\,\mathrm{tn}\,u)\,du\\ &\qquad = \frac{1}{a}\int_0^K \ln(a\,\mathrm{cs}\,u)\,du = \frac{1}{a}\,K\ln\left(\sqrt{a\,b}\right),\end{aligned}\right.$$

where $k^2 = (a^2-b^2)\big/a^2,\ a>b>0.$

800.05
$$\left\{\begin{aligned}&\int_a^\infty \frac{\ln(t)\,dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\int_0^K \ln(a\,\mathrm{ns}\,u)\,du = \frac{1}{a}\left[K\ln\left(\sqrt{a\,b}\right)+\pi K'/4\right];\\ &\qquad a>b>0,\ k^2 = b^2/a^2.\end{aligned}\right.$$

800.06
$$\left\{\begin{aligned}&\int_b^\infty \frac{\ln(t)\,dt}{\sqrt{(t^2+a^2)(t^2-b^2)}} = \frac{k}{a}\int_0^K \ln(b\,\mathrm{nc}\,u)\,du = \frac{k}{a}\int_0^K \ln\left(\sqrt{a^2+b^2}\,\mathrm{ds}\,u\right)du\\ &\qquad = \frac{k}{a}\left[K\ln\left(\sqrt{a\,b}\right)+\pi K'/4\right];\ b>0,\ k^2 = a^2\big/(a^2+b^2).\end{aligned}\right.$$

800.07 $$\int_0^K \frac{\ln(1-\alpha^2 t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^K \ln(1-\alpha^2 \operatorname{sn}^2 u)\,du = f,$$

where

$$f = -2K \ln \frac{\Theta[F(A,k)]}{\sqrt{2k' K/\pi}}, \quad \text{if } 0 < \alpha^2 < k^2;$$

$$= \pi F(\beta, k') + K \ln(k'/k) + \frac{\pi K'}{2} - \frac{\pi [F(\beta,k')]^2}{2K'} - $$
$$- 2K \ln \frac{\Theta[F(\beta,k')]}{\sqrt{2k K'/\pi}}, \quad \text{if } k^2 < \alpha^2 < 1.$$

$A = \sin^{-1}(\alpha/k)$, $\beta = \sin^{-1}\sqrt{(1-\alpha^2)/k'^2}$, and $\ln\Theta(u)$ is given in the Appendix.

801.01 $$\int_0^K \ln(\operatorname{sn} u \operatorname{cn} u)\,du = -K \ln\left(\sqrt{k^2/k'}\right) - \pi K'/2.$$

801.02 $$\int_0^K \ln(\operatorname{sn} u \operatorname{dn} u)\,du = K \ln\left(\sqrt{k'/k}\right) - \pi K'/4.$$

801.03 $$\int_0^K \ln(\operatorname{cn} u \operatorname{dn} u)\,du = K \ln\left(\sqrt{k'^2/k}\right) - \pi K'/4.$$

801.04 $$\int_0^K \ln(\operatorname{cn} u \operatorname{sd} u)\,du = K \ln(1/k) - \pi K'/2.$$

801.05 $$\int_0^K \ln(1+\operatorname{dn} u)\,du = K \ln\left(\sqrt{k}\right) + \pi K'/4.$$

801.06 $$\int_0^K \ln(1-\operatorname{dn} u)\,du = K \ln\left(\sqrt{k}\right) - 3\pi K'/4.$$

801.07 $$\int_0^K \ln\left[\frac{1+a\operatorname{dn} u}{1-a\operatorname{dn} u}\right] du = \pi F(\sin^{-1} a, k'), \quad a^2 < 1.$$

801.08 $$\int_0^K \operatorname{sn} u \operatorname{cn} u \ln(\operatorname{dn}^2 u)\,du = \frac{2}{k^2}\left[(1-\ln k')\,k' - 1\right].$$

801.09 $\int_0^K \operatorname{sn}^2 u \ln(\operatorname{dn}^2 u)\, du = \frac{1}{k^2}\left[(k^2 - 2 + \ln k')\, K + (2 - \ln k')\, E\right].$

801.10 $\int_0^K \operatorname{dn}^2 u \ln(\operatorname{dn}^2 u)\, du = (2 - k^2)\, K + (\ln k' - 2)\, E.$

801.11 $\int_0^K \ln(1 - k \operatorname{sn}^2 u)\, du = \frac{K}{2} \ln\left[2(1-k)/\sqrt{k}\right] - \pi K'/8.$

801.12 $\int_0^K \ln(1 + k \operatorname{sn}^2 u)\, du = \frac{K}{2} \ln\left[2(1+k)/\sqrt{k}\right] - \pi K'/8.$

802.01 $\int_0^K \cot^{-1}(\alpha \operatorname{dn} u)\, du = \frac{\pi}{2} F\left[\cot^{-1}(\alpha k'), k\right].$

802.02 $\int_0^K \tan^{-1}(\alpha \operatorname{dn} u)\, du = \frac{\pi}{2} F\left[\tan^{-1}(\alpha), k\right].$

802.03 $\int_0^{2K} u \operatorname{sn}^2 u\, du = \frac{2}{k^2} K[K - E].$

802.04 $\int_0^{2K} u \operatorname{cd}^4 u\, du = \frac{2K}{3k^4}\left[(2 + k^2)\, K - 2(1 + k^2)\, E\right].$

802.05 $\int_0^{2mK} \operatorname{am} u\, du = m^2 \pi K,$ [m an integer including zero].

802.06 $\int_0^{2mK} \operatorname{am} u \operatorname{sn} u\, du = \frac{(-1)^m m\pi}{k} \ln\left[(1 - k)/k'\right].$

802.07 $\int_0^K \operatorname{am} u \operatorname{sn} u \operatorname{cn} u\, du = \frac{2E - \pi k'}{2k^2},$ [cf. **348.00**].

802.08 $\int_0^K \operatorname{am} u \operatorname{sd} u \operatorname{cd} u\, du = \frac{\pi - 2k' K}{2k^2 k'},$ [cf. **348.01**].

802.09 $\int_0^{2K} u \operatorname{nd}^2 u\, du = \frac{2KE}{k'^2}.$

802.10 $\int_0^{2K} u \operatorname{nd}^3 u\, du = \frac{\pi}{2k'^3}\left[1 + k'^2\right] K.$

802.11 $\displaystyle\int_0^{2K} u \operatorname{nd}^4 u\, du = \frac{2K}{3k'^4}\,[2(1+k'^2)\,E - K].$

802.12 $\displaystyle\int_0^{K} u \operatorname{sn} u \operatorname{cn} u \operatorname{nd} u\, du = -\frac{1}{k^2}\,K \ln\sqrt{k'}.$

802.13 $\displaystyle\int_0^{K} u \operatorname{cn} u \operatorname{ds} u\, du = \frac{\pi}{4}\,K' + K\ln\sqrt{k}.$

804.01 $\displaystyle\int_{\sin^{-1}(b/a)}^{\pi/2} \ln\left[\frac{1+\sin\vartheta}{1-\sin\vartheta}\right]\frac{d\vartheta}{\sqrt{a^2\sin^2\vartheta - b^2}} = \frac{\pi}{a}\,K(b/a),\quad a > b > 0,$ [cf. **285**].

804.02 $\displaystyle\int_{\sin^{-1}(b/a)}^{\pi/2} \ln\left[\frac{1+\sin\vartheta}{1-\sin\vartheta}\right]\frac{\cos^2\vartheta\, d\vartheta}{\sqrt{a^2\sin^2\vartheta - b^2}} = \frac{\pi}{a}\,[E(b/a) - 1],\quad a > b > 0;$

[cf. **285**].

804.03 $\displaystyle\int_{\sin^{-1}(b/a)}^{\pi/2} \ln\left[\frac{1+\sin\vartheta}{1-\sin\vartheta}\right]\sqrt{a^2\sin^2\vartheta - b^2}\, d\vartheta = a\pi\left[1 + \frac{a^2-b^2}{a^2}K(b/a) - E(b/a)\right],$

$a > b > 0.$

804.04 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\, d\vartheta}{\vartheta\sqrt{1-k^2\sin^2\vartheta}} = K.$

804.05 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\, d\vartheta}{\vartheta\sqrt{1-k^2\cos^2\vartheta}} = K.$

804.06 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\cos\vartheta\, d\vartheta}{\vartheta\sqrt{1-k^2\sin^2\vartheta}} = \frac{1}{k^2}\,[E - k'^2 K].$

804.07 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\sqrt{1-k^2\sin^2\vartheta}}{\vartheta}\, d\vartheta = E.$

804.08 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\sqrt{1-k^2\cos^2\vartheta}}{\vartheta}\, d\vartheta = E.$

804.09 $\displaystyle\int_0^{\infty} \frac{\sin\vartheta\cos\vartheta\sqrt{1-k^2\sin^2\vartheta}}{\vartheta}\, d\vartheta = \frac{1}{3k^2}\,[(1+k^2)\,E - k'^2 K].$

804.10 $$\int_0^\infty \frac{\sin\vartheta\cos\vartheta\sqrt{1-k^2\cos^2\vartheta}}{\vartheta}\,d\vartheta = \frac{1}{3k^2}\left[(2k^2-1)E + k'^2K\right].$$

804.11 $$\alpha\int_0^\infty \frac{d\vartheta}{(\alpha^2+\vartheta^2)\sqrt{1-k^2\sin^2\vartheta}} = \coth\alpha\,\Pi(\pi/2, -\operatorname{csch}^2\alpha, k), \quad \alpha^2 \neq 0.$$

[See **400**.]

804.12 $$\begin{cases} \alpha\displaystyle\int_0^\infty \frac{\sqrt{1-k^2\sin^2\vartheta}}{\alpha^2+\vartheta^2}\,d\vartheta \\ = \coth\alpha\left[(1+k^2\sinh^2\alpha)\,\Pi(-\operatorname{csch}^2\alpha, k) - k^2K\sinh^2\alpha\right], \quad (\alpha^2 \neq 0). \end{cases}$$

806.01 $$\int_0^{\pi/2} \frac{\cos 2m\vartheta\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{(-1)^m\pi}{2}\sum_{j=m}^\infty \frac{[(2j-1)!]^2 k^{2j}}{4^{2j-1}(j-m)!\,(j+m)!\,[(j-1)!]^2}.$$

or

806.02 $$\begin{cases} \displaystyle\int_0^{\pi/2} \frac{\cos 2m\vartheta\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = K - \frac{4m^2}{2!}\int_0^K \operatorname{sn}^2 u\,du + \frac{4m^2(4m^2-2^2)}{4!}\int_0^K \operatorname{sn}^4 u\,du - \\ \displaystyle - \frac{4m^2(4m^2-2^2)(4m^2-4^2)}{6!}\int_0^K \operatorname{sn}^6 u\,du + \cdots \text{ to } m+1 \text{ terms.} \end{cases}$$

[See **310**.]

806.03 $$\begin{cases} \displaystyle\int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}\cos 2m\vartheta\,d\vartheta \\ \displaystyle\qquad = \frac{(-1)^{m+1}\pi}{2}\sum_{j=m}^\infty \frac{(2j-1)!\,(2j-3)!\,k^{2j}}{4^{2(j-1)}(j-m)!\,(j+m)!\,(j-1)!\,(j-3)!} \end{cases}$$

or

806.04 $$\begin{cases} \displaystyle\int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}\cos 2m\vartheta\,d\vartheta = K - \left[\frac{4m^2}{2!} + k^2\right]\int_0^K \operatorname{sn}^2 u\,du + \\ \displaystyle\quad + \left[\frac{4m^2(4m^2-2^2)}{4!} + \frac{4m^2k^2}{2!}\right]\int_0^K \operatorname{sn}^4 u\,du - \\ \displaystyle\quad - \left[\frac{4m^2(4m^2-2^2)(4m^2-4^2)}{6!} + \frac{4m^2(4m^2-2^2)k^2}{4!}\right]\int_0^K \operatorname{sn}^6 u\,du + \\ \quad + \cdots \text{ to } m+2 \text{ terms.} \end{cases}$$

[See **310**.]

806.05
$$\begin{cases} \displaystyle\int_0^{\pi/2} \frac{\sin(2m+1)\vartheta\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \\ \displaystyle = (2m+1)\int_0^K \operatorname{sn} u\, du - \frac{(2m+1)\,[(2m+1)^2-1^2]}{3!}\int_0^K \operatorname{sn}^3 u\, du + \\ \displaystyle \quad + \frac{(2m+1)\,[(2m+1)^2-1^2]\,[(2m+1)^2-3^2]}{5!}\int_0^K \operatorname{sn}^5 u\, du - \\ \quad - \cdots \text{to } m+1 \text{ terms.} \end{cases}$$

[See **310**.]

806.06
$$\begin{cases} \displaystyle\int_0^{\pi/2} \frac{\sin 2m\vartheta\, d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \\ \displaystyle = -\frac{2m}{k^2}\int_0^K d(\operatorname{dn} u) + \frac{2m(4m^2-2^2)}{3!\,k^4}\int_0^K (1-\operatorname{dn}^2 u)\, d(\operatorname{dn} u) - \\ \displaystyle - \frac{2m(4m^2-2^2)(4m^2-4^2)}{5!\,k^6}\int_0^K (1-\operatorname{dn}^2 u)^2 d(\operatorname{dn} u) + \cdots \text{to } m \text{ terms.} \end{cases}$$

807.01 $$\int_0^{\pi/2} \frac{d\vartheta}{\sqrt[4]{1-k^2\sin^2\vartheta}} = \sqrt{\frac{2}{(1+k')}}\, K(k_1), \text{ where } k_1 = \frac{1-\sqrt{k'}}{\sqrt{2(1+k')}}.$$

807.02 $$\int_0^{\pi/2} \frac{d\vartheta}{\sqrt{\sin\vartheta\,(1-k^2\sin^2\vartheta)}} = \frac{1}{\sqrt{2(1+k)}}\,[K(k_1)+K(k_2)], \qquad [0<k<1]$$

where

$$k_1 = \frac{1-\sqrt{k}}{\sqrt{2(1+k)}}, \quad k_2 = \frac{1+\sqrt{k}}{\sqrt{2(1+k)}}.$$

807.03 $$\int_0^{\pi/2} \sqrt{\frac{\sin\vartheta}{1-k^2\sin^2\vartheta}}\, d\vartheta = \frac{1}{\sqrt{2(1+k)\,k}}\,[K(k_2)-K(k_1)], \qquad [0<k<1]$$

where k_1 and k_2 are defined in **807.02**.

808.01 $$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{[(t-a)^2+b_1^2]\,[(t-b)^2+b_2^2]}} = \frac{2}{\sqrt{g\, b_1 b_2}}\, K(k), \qquad [b_1>b_2>0]$$

where

$$g = g_1 + \sqrt{g_1^2-1}, \quad g_1 = \frac{(a-b)^2+b_1^2+b_2^2}{2b_1 b_2}, \quad k^2 = \frac{g_1^2-1}{g^2}.$$

808.02
$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{t\,dt}{\sqrt{[(t-a)^2+b_1^2]\,[(t-b)^2+b_2^2]}} \\ \displaystyle = \frac{2}{\sqrt{g\,b_1\,b_2}}\left[a\,K(k) + \frac{b_1\,(b-a)\,\alpha^2}{b_2\,k^2\,g}\,\Pi(\alpha^2, k)\right], \quad [a \neq b,\ b_1 > b_2 > 0], \end{cases}$$

where k and g are given in **808.01** and

$$\alpha^2 = \frac{k^2\,b_2\,(b_2 - g\,b_1)}{(a-b)^2}.$$

808.03
$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{[(t-a)^2+b_1^2]\,[(t-a)^2+b_2^2]}} = \frac{2a}{b_1}\,K(k), \qquad [b_1 > b_2 > 0]$$

where k is given in **808.01**.

808.04
$$\begin{cases} \displaystyle\int_{-\infty}^{b} \frac{dt}{\sqrt{(t-a)\,(t-b)\,[(t-a_1)^2+b_1^2]}} + \int_{a}^{\infty} \frac{dt}{\sqrt{(t-a)\,(t-b)\,[(t-a_1)^2+b_1^2]}} \\ \displaystyle = 2\sqrt{\frac{2k\,k'}{b_1\,(a-b)}}\,K(k'), \quad [b_1 > 0,\ a > b] \end{cases}$$

where

$$k^2 = \frac{1}{2}\left[1 - \frac{b_1^2 + (a-a_1)\,(b-a_1)}{\sqrt{[b_1^2+(a-a_1)^2]\,[b_1^2+(b-a_1)^2]}}\right].$$

810.00
$$\begin{cases} \displaystyle P_{m-\frac{1}{2}}(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\vartheta}{[x+\sqrt{x^2-1}\cos\vartheta]^{\frac{1}{2}-m}} \\ \displaystyle = \frac{2}{\pi}\left(x+\sqrt{x^2-1}\right)^{m-\frac{1}{2}}\int_0^{K} \mathrm{dn}^{2m} u\,du, \quad [m \text{ is an integer; see } \mathbf{314.05}.] \end{cases}$$

where $k^2 = 2\left[x\sqrt{x^2-1} + 1 - x^2\right]$, $(x > 1)$; and $P_{m-\frac{1}{2}}(x)$ is Legendre's function.

811.01 $$K = \frac{1}{4i\pi}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}+t)\,\Gamma(\frac{1}{2}+t)\,\Gamma(-t)\,(-k^2)^t}{\Gamma(1+t)}\,dt,$$

where the path of integration is indented so as to separate the poles at $t = 0, 1, 2, \ldots$ from those at $t = -\frac{1}{2} - n$, $(n = 0, 1, 2, \ldots)$; and Γ is the gamma function.

811.02 $$E = \frac{1}{4i\pi}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}+t)\,\Gamma(-\frac{1}{2}+t)\,\Gamma(-t)\,(-k^2)^t}{\Gamma(1+t)}\,dt,$$

where the path of integration is chosen so that the poles at $t=-\frac{1}{2}-n$, $t=\frac{1}{2}-n$, $(n=0, 1, 2, 3, \ldots)$, lie on the left of the path and the poles at $t=0, 1, 2, \ldots$ lie on the right.

812.01 $$\iint \frac{dS}{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}} = \frac{4abc\pi}{\sqrt{a^2-c^2}} F(\varphi, k), \qquad (a>b>c)$$

where the integration is taken over the surface of the sphere $x^2+y^2+z^2=1$, and where $\varphi=\sin^{-1}\sqrt{(a^2-c^2)/a^2}$, $k^2=(a^2-b^2)/(a^2-c^2)$.

812.02 $$K=\frac{1}{2\pi}\int_0^1\int_0^1 \frac{dt\,d\tau}{(1-k^2 t\tau)\sqrt{t\tau(1-t)(1-\tau)}}.$$

813.01 $$F(\varphi, k) = k_n \sqrt{\frac{k_1\cdot k_2\cdot k_3\ldots k_{n-1}}{k}}\, F(\vartheta_n, k_n), \qquad \text{[cf. } \mathbf{163}]$$

where

$$k_1=\frac{2\sqrt{k}}{1+k}, \quad \sin(2\vartheta-\varphi)=k\sin\varphi,$$

$$k_m=\frac{2\sqrt{k_{m-1}}}{1+k_{m-1}}, \quad \sin(2\vartheta_m-\vartheta_{m-1})=k_{m-1}\sin\vartheta_{m-1}; \quad (m=2,3,\ldots,n).$$

813.02 $$\begin{cases} E(\varphi,k)=F(\varphi,k)\left\{1+k\left[1+\frac{2}{k_1}+\frac{2^2}{k_1k_2}+\frac{2^{n-1}}{k_1k_2\ldots k_{n-1}}-\right.\right. \\ \qquad \left.\left.-\frac{2^n}{k_1k_2\ldots k_{n-1}}\right]\right\}-k\left[\sin\varphi+\frac{2}{\sqrt{k}}\sin\vartheta_1+\frac{2^2}{\sqrt{k\,k_1}}\sin\vartheta_2+\cdots+\right. \\ \qquad \left.+\frac{2^{n-1}}{\sqrt{k\,k_1k_2\ldots k_{n-1}}}\sin\vartheta_{n-1}-\frac{2^n}{\sqrt{k\,k_1k_2\ldots k_n}}\sin\vartheta_n\right], \end{cases}$$

where k_1, k_m are defined in **813.01**.

Formulas **813.01** and **813.02** lend themselves readily to numerical computation when k is large. For small modulus, however, the following two formulas work more rapidly.

814.01 $$F(\varphi,k)=\frac{(1+k_1)(1+k_2)(1+k_3)\ldots(1+k_n)}{2^n}F(\psi_n,k_n), \qquad \text{[cf. } \mathbf{900.02}]$$

where

$$k_m=\frac{1-\sqrt{1-k_{m-1}^2}}{1+\sqrt{1-k_{m-1}^2}}, \quad \tan(\psi_m-\psi_{m-1})=\sqrt{1-k_{m-1}^2}\tan\psi_{m-1}, \qquad (m=2,\ldots,n)$$

$$k_1=\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}, \quad \tan(\psi_1-\varphi)=\sqrt{1-k^2}\tan\varphi.$$

$$
\mathbf{814.02}\quad \begin{cases} E(\varphi, k) = F(\varphi, k)\left[1 - \frac{k^2}{2}\left(1 + \frac{k_1}{2} + \frac{k_1 k_2}{2} + \frac{k_1 k_2 k_3}{2} + \cdots\right)\right] + \\ \qquad + k\left[\frac{\sqrt{k_1}}{2}\sin\psi_1 + \frac{\sqrt{k_1 k_2}}{2^2}\sin\psi_2 + \frac{\sqrt{k_1 k_2 k_3}}{2^3}\sin\psi_3 + \cdots\right], \end{cases}
$$

where k_1, k_m etc are given in **814.01**.

$$
\mathbf{815.01}\quad \begin{cases} 2F(\varphi, k) = F(\vartheta, k), \\ 2E(\varphi, k) = E(\vartheta, k) + \frac{2k^3 \sin^3\varphi \cos\varphi \sqrt{1 - k^2 \sin^2\varphi}}{1 - k^2 \sin^4\varphi}, \end{cases} \qquad \text{[cf. } \mathbf{116.01}\text{]}
$$

where $$\vartheta = \cos^{-1}\left[\frac{1 - 2\sin^2\varphi + k^2 \sin^4\varphi}{1 - k^2 \sin^4\varphi}\right].$$

$$
\mathbf{816.00}\qquad K'(k) = e^{iy/4} K\left(\sin\frac{y}{4}\right),
$$

where $$k = \cos\frac{y}{2} - i \sin\frac{y}{2} \text{ is complex; } 0 \leq y \leq \pi.$$

Expansions in Series.

Developments of the Elliptic Integrals.

Complete Elliptic Integrals of the First and Second Kind.

900.00
$$\begin{cases} K(k) \equiv K \\ \quad = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \cdots\right] \\ \quad = \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m}{m!\,m!}k^{2m} = \frac{\pi}{2}F(\tfrac{1}{2},\tfrac{1}{2};1;k^2), \qquad [k^2 < 1] \end{cases}$$

where F is the hypergeometric function (cf. **118.02**).

900.01 $$K(k) = \frac{\pi}{2}(1 + k_1)\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m}{m!\,m!}k_1^{2m}, \qquad [k_1^2 < k^2 < 1]$$

where $$k_1 = (1 - k')/(1 + k').$$

900.02 $$K(k) = \frac{\pi}{2}\prod_{m=1}^{\infty}(1 + k_m), \qquad [k^2 < 1]$$

where $k_1 = (1 - k')/(1 + k')$, $k_{m+1} = (1 - k'_m)/(1 + k'_m)$, $k'_m = \sqrt{1 - k_m^2}$.

900.03 $$\begin{cases} K = \frac{\pi^2}{4} + \frac{\pi^2}{4}\sum_{m=1}^{\infty}(-1)^m(4m+1)\left[\frac{(2m)!}{4^m\,m!\,m!}\right]^3 P_{2m}(\cos\vartheta), \\ \qquad (k' = \cos\vartheta), \quad [k^2 < 1] \end{cases}$$

where $P_{2m}(\cos\vartheta)$ is Legendre's function.

900.04 $$K = \frac{\pi}{2}\left[1 + 4\sum_{m=1}^{\infty}\frac{q^m}{1 + q^{2m}}\right].$$

900.05 $$K = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} [\ln(4/k') - b_m]\, k'^{2m}, \qquad [k^2 < 1]$$

where

$$b_0 = 0, \quad b_m = 2\sum_{j=1}^{2m} \frac{(-1)^{j-1}}{j} = b_{m-1} + \frac{2}{2m(2m-1)}.$$

900.06 $$K = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{2})_m}{m!\, m!} [\ln(1/k') - \psi(m+\tfrac{1}{2}) + \psi(m+1)]\, k'^{2m}, \quad [\text{cf. } \mathbf{900.05}] \qquad [k'^2 < 1]$$

where ψ is the di-gamma function.

900.07 $$\begin{cases} E(k) \equiv E \\ = \dfrac{\pi}{2}\left[1 - \dfrac{1}{4}k^2 - \dfrac{3}{64}k^4 - \dfrac{5}{256}k^6 - \dfrac{175}{16384}k^8 - \dfrac{441}{65536}k^{10} - \cdots\right] \\ = \dfrac{\pi}{2}\displaystyle\sum_{m=0}^{\infty} \frac{1}{1-2m}\binom{-\frac{1}{2}}{m}^2 k^{2m} = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2), \qquad [k^2 < 1] \end{cases}$$

where F is the hypergeometric function (cf. **118.02**).

900.08 $$E = \frac{\pi}{2(1+k_1)}\left[1 + \frac{1}{4}k_1^2 + \sum_{m=1}^{\infty}\left(\frac{(2m+2)!}{4^{m+1}(m+1)!\,(m+1)!}\right)^2 k_1^{2m+2}\right], \qquad [k_1^2 < k^2 < 1]$$

where $$k_1 = (1-k')/(1+k').$$

900.09 $$E = \frac{\pi^2}{8} + \frac{\pi^2}{8}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{4m+1}{(2m-1)(m+1)}\left[\frac{(2m)!}{4^m\, m!\, m!}\right]^3 P_{2m}(\cos\vartheta), \qquad [k^2 < 1]$$

where $\cos\vartheta = k' = \sqrt{1-k^2}$, and $P_{2m}(\cos\vartheta)$ is Legendre's function.

900.10 $$\begin{cases} E = 1 + \dfrac{1}{2}\left[\ln(4/k') - \dfrac{1}{2}\right]k'^2 + \dfrac{3}{16}\left[\ln(4/k') - \dfrac{13}{16}\right]k'^4 + \cdots \\ \quad = 1 + \dfrac{1}{4}\displaystyle\sum_{m=0}^{\infty}\frac{(1/2)_m (3/2)_m}{m!\,(m+1)!}\Big[2\ln(1/k') + \psi(m+2) - \\ \qquad - \psi\left(m+\dfrac{3}{2}\right) + \psi(m+1) - \psi\left(m+\dfrac{1}{2}\right)\Big] k'^{2m+2}, \qquad [k'^2 < 1] \end{cases}$$

where ψ is the di-gamma function.

Jacobi's Nome.

901.00 $$\begin{cases} q = e^{-\pi K'/K} \\ \quad = \dfrac{k^2}{16}\left[1 + 2\left(\dfrac{k}{4}\right)^2 + 15\left(\dfrac{k}{4}\right)^4 + 150\left(\dfrac{k}{4}\right)^6 + 1707\left(\dfrac{k}{4}\right)^8 + \cdots\right]^4, \quad [k^2 < 1] \end{cases}$$

901.01 $$q = \frac{1}{2} k_1\left[1 + 2\left(\frac{k_1}{2}\right)^4 + 15\left(\frac{k_1}{2}\right)^8 + 150\left(\frac{k_1}{2}\right)^{12} + 1707\left(\frac{k_1}{2}\right)^{16} + \cdots\right],$$

where $$k_1 = \left(1 - \sqrt{k'}\right)/\left(1 + \sqrt{k'}\right). \qquad [k_1^2 < k^2 < 1]$$

Incomplete Elliptic Integrals of the First and Second Kind.

902.00 $$F(\varphi, k) = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-k^2)^m \, t_{2m}(\varphi), \qquad [0 < \varphi < \pi/2, \; k^2 < 1]$$

where

$$t_0(\varphi) = \varphi, \quad t_2(\varphi) = \frac{1}{2}[\varphi - \sin\varphi\cos\varphi],$$

$$t_4(\varphi) = \frac{1}{8}[3\varphi - \sin\varphi\cos\varphi\,(3 + 2\sin^2\varphi)],$$

$$t_{2m}(\varphi) = \frac{2m-1}{2m} t_{2(m-1)}(\varphi) - \frac{1}{2m}\sin^{2m-1}\varphi\cos\varphi.$$

902.01 $$F(\varphi, k) = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} k'^{2m} \varrho_{2m}(\varphi), \qquad [0 < k'^2 \tan^2\varphi < 1, \; k^2 < 1]$$

where

$$\varrho_0(\varphi) = \ln\frac{1 + \sin\varphi}{\cos\varphi}, \quad \varrho_2(\varphi) = \frac{\sin\varphi\sec^2\varphi}{2} - \frac{1}{2}\ln\frac{1 + \sin\varphi}{\cos\varphi},$$

$$\varrho_4(\varphi) = \frac{1}{8}\left[2\sin^3\varphi\sec^4\varphi - 3\sin\varphi\sec^2\varphi + 3\ln\frac{1 + \sin\varphi}{\cos\varphi}\right],$$

$$\varrho_{2m}(\varphi) = \frac{1}{2m}[\sin^{2m-1}\varphi\sec^{2m}\varphi + (1 - 2m)\,\varrho_{2m-2}(\varphi)].$$

903.00 $$E(\varphi, k) = \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-k^2)^m \, t_{2m}(\varphi), \qquad [0 < \varphi < \pi/2, \; k^2 < 1]$$

where $t_{2m}(\varphi)$ is given in **902.00**.

903.01 $E(\varphi, k) = \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} k'^{2m} d_{2m}(\varphi)$, $\quad [0 < k'^2 \tan^2\varphi < 1;\ k^2 < 1]$

where

$$d_0(\varphi) = \sin\varphi, \quad d_2(\varphi) = -\sin\varphi + \ln\frac{1+\sin\varphi}{\cos\varphi},$$

$$d_4(\varphi) = \frac{1}{2}\left[\sin^3\varphi \sec^2\varphi + 3\sin\varphi - 3\ln\frac{1+\sin\varphi}{\cos\varphi}\right],$$

and

$$d_{2m}(\varphi) = \frac{1}{2(m-1)}\left[\sin^{2m-1}\varphi \sec^{2(m-1)}\varphi + (1-2m)\, d_{2(m-1)}(\varphi)\right], \quad m \neq 1.$$

Heuman's Lambda Function.

904.00 $\Lambda_0(\varphi, k) = \frac{2}{\pi}\left[a_0 t_0 - \sum_{m=1}^{\infty} a_{2m}(k)\, t_{2m}(\varphi)\right]$, $\quad [0 < \varphi < \pi/2,\ k^2 < 1]$

where $t_{2m}(\varphi)$ is given in **902.00**, and

$$a_0 = E, \quad a_2 = \frac{1}{2}(2K - E)\, k'^2,$$

$$a_4 = \frac{1 \cdot 1}{2 \cdot 4}(4K - 3E)\, k'^4, \quad a_6 \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}(6K - 5E)\, k'^6, \cdots.$$

Jacobian Zeta Function.

905.00
$$\begin{cases} Z(u, k) \equiv Z(u) \\ = \left(1 - \frac{E}{K}\right) u - 2k^2 \frac{u^3}{3!} + 8k^2(k^2+1)\frac{u^5}{5!} - 16k^2(2k^4 + 13k^2 + 2)\frac{u^7}{7!} + \\ \qquad + 128k^2(k^6 + 30k^4 + 30k^2 + 1)\frac{u^9}{9!} - \cdots, \quad [|u| < K']. \end{cases}$$

905.01 $Z(u, k) = \frac{\pi}{K}\sum_{m=1}^{\infty} \frac{\sin(m\pi u/K)}{\sinh(m\pi K'/K)} = \frac{2\pi}{K}\sum_{m=1}^{\infty} \frac{q^m}{1 - q^{2m}} \sin(m\pi u/K)$,

$[|\mathrm{Im}(u/K)| < \mathrm{Im}(i K'/K)]$.

The Elliptic Integral of the Third Kind.

906.00 $\Pi(\alpha^2, k) \equiv \Pi(\pi/2, \alpha^2, k) = \sum_{m=0}^{\infty} c_m k^{2m},$

$$[-\alpha^2 > 1,\ k^2 < 1;\quad |-\alpha^2| > 1,\ k^2 < |-\alpha^2|]$$

where

$$c_0 = \frac{\pi}{2\sqrt{1-\alpha^2}},\quad c_1 = \frac{\pi}{4\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}} - 1\right],\quad c_2 = \frac{3\pi}{32\alpha^4}\left[\frac{2}{\sqrt{1-\alpha^2}} - 2 - \alpha^2\right],$$

$$c_3 = \frac{5\pi}{256\alpha^6}\left[-2\alpha^2 - 3\alpha^4 - 8 + \frac{8}{\sqrt{1-\alpha^2}}\right],$$

$$2(m+1)\alpha^2 c_{m+1} = \frac{\pi}{2(2m-1)}\binom{-\frac{1}{2}}{m}^2 + (1-2m)c_{m-1} + (2m+1+2m\alpha^2)c_m.$$

906.01 $\Pi(\varphi, \alpha^2, k) = \sum_{m=0}^{\infty} b_m k^{2m},\quad [|-\alpha^2| > 1,\ k^2 < 1;\ |-\alpha^2| < 1,\ k^2 < |-\alpha^2|]$

where

$$b_0 = \frac{1}{\sqrt{1-\alpha^2}} \tan^{-1}\left[\sqrt{1-\alpha^2}\tan\varphi\right],\ \text{for } -\alpha^2 > -1;$$

$$= \frac{1}{\sqrt{\alpha^2 - 1}} \tanh^{-1}\left[\sqrt{\alpha^2-1}\tan\varphi\right],\ \text{for } -\alpha^2 < -1;$$

$$b_1 = \frac{b_0 - \varphi}{2\alpha^2},\quad b_2 = \frac{1}{16\alpha^4}\left[3\alpha^2 \sin\varphi\cos\varphi + 6b_0 - 3(2+\alpha^2)\varphi\right];$$

$$2(m+1)\alpha^2 b_{m+1} = (2m+1+2m\alpha^2)b_m + (1-2m)b_{m-1} -$$

$$-(-1)^m\binom{-\frac{1}{2}}{m-1}\sin^{2m-1}\varphi\cos\varphi - (-1)^m\binom{\frac{1}{2}}{m}t_{2m}(\varphi),$$

[$t_{2m}(\varphi)$ being given in **902.00**].

906.02
$$\begin{cases} \Pi(\varphi, \alpha^2, k) = \displaystyle\int_0^{\varphi} \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}} \\ \displaystyle = \sum_{m=0}^{\infty}\sum_{j=0}^{m} (\alpha^2)^m t_{2m}(\varphi)\binom{-\frac{1}{2}}{j}\left(\frac{k^2}{-\alpha^2}\right)^j, \\ \qquad [|-\alpha^2| < 1;\quad k^2 < 1] \end{cases}$$

where $t_{2m}(\varphi)$ is given in **902.00.**

906.03
$$\begin{cases} \Pi(\alpha^2, k) = \displaystyle\int_0^{\pi/2} \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{1-k^2\sin^2\vartheta}} = \frac{1}{1-\alpha^2}\left[\ln(4/k') + \right. \\ \left. + \sqrt{-\alpha^2}\tan^{-1}\left(\sqrt{-\alpha^2}\right)\right] + O(k'^2),\quad [-\alpha^2 > 0;\ k^2 < 1] \end{cases}$$

where the symbol $0(k'^2)$ means that the remaining terms are $< M\,k'^2$, M being a constant.

$$906.04\quad \Pi(\alpha^2, k) = K - \frac{E}{k'^2} + \frac{\pi(1 + k'^2 - k^2\alpha^2)}{4k'^3\sqrt{1-\alpha^2}} + 0(1-\alpha^2), \qquad (\alpha^2 \mp 1)$$

where $0(1-\alpha^2)$ means that the remaining terms are $<(1-\alpha^2)M$, M being a constant.

$$906.05\quad \Pi(\alpha^2, k) = \frac{\pi}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{(2m)!\,(2j)!}{4^m\,4^j\,(m!)^2\,(j!)^2}\,k^{2j}\,(\alpha^2)^{m-j}, \qquad [|-\alpha^2| < 1;\ k^2 < 1]$$

$$906.06\quad \begin{cases} \Pi(\alpha^2, k) = \dfrac{\pi}{2\sqrt{1-\alpha^2}}\displaystyle\sum_{m=0}^{\infty}\frac{(2m)!}{4^m\,(m!)^2}\left(\frac{k^2}{\alpha^2}\right)^m - \\ \qquad - \dfrac{\pi}{2}\displaystyle\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}\frac{(2m)!\,(2j)!}{4^m\,(m!)^2\,4^j\,(j!)^2}\,k^{2m}\,(\alpha^2)^{j-m}, \\ \qquad\qquad [-\alpha^2 > -1;\ k^2 < 1]. \end{cases}$$

Developments of Jacobian Elliptic Functions.

Maclaurin's Series.

$$907.00\quad \begin{cases} \operatorname{am} u = u - k^2\dfrac{u^3}{3!} + (4 + k^2)\,k^2\dfrac{u^5}{5!} - (16 + 44k^2 + k^4)\,k^2\dfrac{u^7}{7!} + \\ \qquad + (64 + 912k^2 + 408k^4 + k^6)\,k^2\dfrac{u^9}{9!} - \cdots, \qquad [|u| < K']. \end{cases}$$

$$907.01\quad \begin{cases} \operatorname{sn} u = u - (1 + k^2)\dfrac{u^3}{3!} + (1 + 14k^2 + k^4)\dfrac{u^5}{5!} - \\ \qquad - (1 + 135k^2 + 135k^4 + k^6)\dfrac{u^7}{7!} + \cdots. \end{cases}$$

$$907.02\quad \begin{cases} \operatorname{cn} u = 1 - \dfrac{u^2}{2!} + (1 + 4k^2)\dfrac{u^4}{4!} - (1 + 44k^2 + 16k^4)\dfrac{u^6}{6!} + \\ \qquad + (1 + 408k^2 + 912k^4 + 64k^6)\dfrac{u^8}{8!} - \cdots, \qquad [|u| < K']. \end{cases}$$

$$907.03\quad \begin{cases} \operatorname{dn} u = 1 - k^2\dfrac{u^2}{2!} + (4 + k^2)\,k^2\dfrac{u^4}{4!} - (16 + 44k^2 + k^4)\,k^2\dfrac{u^6}{6!} + \\ \qquad + (64 + 912k^2 + 408k^4 + k^6)\dfrac{u^8}{8!} - \cdots, \qquad [|u| < K']. \end{cases}$$

907.04 $$\begin{cases} \operatorname{ns} u = \dfrac{1}{u} + (1+k^2)\dfrac{u}{6} + (7 - 22k^2 + 7k^4)\dfrac{u^3}{360} + \\ \qquad + (31 - 15k^2 - 15k^4 + 31k^6)\dfrac{u^5}{15120} + \cdots . \\ \qquad\qquad [0 < |u| < \min(|2K'|, |2K|)] . \end{cases}$$

907.05 $$\begin{cases} \operatorname{cn} u \operatorname{dn} u = 1 - (1+k^2)\dfrac{u^2}{2!} + (1 + 14k^2 + k^4)\dfrac{u^4}{4!} - \\ \qquad - (1 + 135k^2 + 135k^4 + k^6)\dfrac{u^6}{6!} + \cdots, \quad [|u| < K'] . \end{cases}$$

907.06 $$\begin{cases} \operatorname{sn} u \operatorname{dn} u = u - (1+4k^2)\dfrac{u^3}{3!} + (1 + 44k^2 + 16k^4)\dfrac{u^5}{5!} - \\ \qquad - (1 + 408k^2 + 912k^4 + 64k^6)\dfrac{u^7}{7!} + \cdots, \quad [|u| < K'] . \end{cases}$$

907.07 $$\begin{cases} \operatorname{sn} u \operatorname{cn} u = u - (4+k^2)\dfrac{u^3}{3!} + (16 + 44k^2 + k^4)\dfrac{u^5}{5!} - \\ \qquad - (64 + 912k^2 + 498k^4 + k^6)\dfrac{u^7}{7!} + \cdots, \quad [|u| < K'] . \end{cases}$$

Fourier Series[1].

In the following, $q = e^{-\pi K'/K}$.

908.00 $$\begin{cases} \operatorname{am} u = \dfrac{\pi u}{2K} + 2\displaystyle\sum_{m=0}^{\infty} \dfrac{q^{m+1}}{(m+1)\,[1+q^{2(m+1)}]} \sin\left[(m+1)\dfrac{\pi u}{K}\right], \\ \qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(iK'/K)] . \end{cases}$$

908.01 $$\begin{cases} \operatorname{sn} u = \dfrac{2\pi}{kK}\displaystyle\sum_{m=0}^{\infty} \dfrac{q^{m+\frac{1}{2}}}{1-q^{2m+1}} \sin\left[(2m+1)\dfrac{\pi u}{2K}\right], \\ \qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(iK'/K)] . \end{cases}$$

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[1] In these series we may also write

$$\frac{q^m}{1-q^{2m}} = \frac{1}{2}\operatorname{csch}[m\pi K'/K], \qquad \frac{q^m}{1+q^{2m}} = \frac{1}{2}\operatorname{sech}[m\pi K'/K],$$

$$\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}} = \frac{1}{2}\operatorname{csch}[(2m+1)\pi K'/2K],$$

$$\frac{q^{m+\frac{1}{2}}}{1-q^{2m+1}} = \frac{1}{2}\operatorname{sech}[(2m+1)\pi K'/2K].$$

908.02
$$\left\{\begin{aligned}&\operatorname{cn} u = \frac{2\pi}{k K} \sum_{m=0}^{\infty} \frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}} \cos\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.03
$$\left\{\begin{aligned}&\operatorname{dn} u = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{m=0}^{\infty} \frac{q^{m+1}}{1+q^{2(m+1)}} \cos\left[(m+1)\frac{\pi u}{K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.04
$$\left\{\begin{aligned}&\operatorname{ns} u = \frac{\pi}{2K} \csc\frac{\pi u}{2K} + \frac{2\pi}{K} \sum_{m=0}^{\infty} \frac{q^{2m+1}}{1-q^{2m+1}} \sin\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/2K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.05
$$\left\{\begin{aligned}&\operatorname{nc} u = \frac{\pi}{2k' K} \sec\frac{\pi u}{2K} +\\ &\qquad + \frac{2\pi}{k' K} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{q^{2m+1}}{1+q^{2m+1}} \cos\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/2K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.06
$$\left\{\begin{aligned}&\operatorname{ds} u = \frac{\pi}{2K} \csc\frac{\pi u}{2K} - \frac{2\pi}{K} \sum_{m=0}^{\infty} \frac{q^{2m+1}}{1+q^{2m+1}} \sin\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/2K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.07
$$\left\{\begin{aligned}&\operatorname{dc} u = \frac{\pi}{2K} \sec\frac{\pi u}{2K} + \frac{2\pi}{K} \sum_{m=0}^{\infty} (-1)^{m} \frac{q^{2m+1}}{1-q^{2m+1}} \cos\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/2K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.08
$$\left\{\begin{aligned}&\operatorname{nd} u = \frac{\pi}{2k' K} + \frac{2\pi}{k' K} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{q^{m+1}}{1+q^{2(m+1)}} \cos\left[(m+1)\frac{\pi u}{K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.09
$$\left\{\begin{aligned}&\operatorname{cd} u = \frac{2\pi}{k K} \sum_{m=0}^{\infty} (-1)^{m} \frac{q^{m+\frac{1}{2}}}{1-q^{2m+1}} \cos\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.10
$$\left\{\begin{aligned}&\operatorname{sd} u = \frac{2\pi}{k k' K} \sum_{m=0}^{\infty} (-1)^{m} \frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}} \sin\left[(2m+1)\frac{\pi u}{2K}\right],\\ &\qquad\qquad\qquad [|\operatorname{Im}(u/K)| < \operatorname{Im}(i K'/K)].\end{aligned}\right.$$

908.11
$$\left\{\begin{aligned}&\operatorname{tn} u = \frac{\pi}{2k' K}\tan\frac{\pi u}{2K} + \frac{2\pi}{k' K}\sum_{m=1}^{\infty}(-1)^m \frac{q^{2m}}{1+q^{2m}}\sin\left[\frac{m\pi u}{K}\right],\\ &\qquad [|Im(u/2K)| < Im(i K'/K)].\end{aligned}\right.$$

908.12
$$\left\{\begin{aligned}&\operatorname{cs} u = \frac{\pi}{2K}\cot\frac{\pi u}{2K} - \frac{2\pi}{K}\sum_{m=1}^{\infty}\frac{q^{2m}}{1+q^{2m}}\sin\left[\frac{m\pi u}{K}\right],\\ &\qquad [|Im(u/2K)| < Im(i K'/K)].\end{aligned}\right.$$

908.50
$$\left\{\begin{aligned}&\operatorname{tn} u \operatorname{nd} u = \frac{\pi}{2k'^2 K}\tan\frac{\pi u}{2K} + \frac{2\pi}{k'^2 K}\sum_{m=1}^{\infty}(-1)^m \frac{q^{m}}{1+q^{m}}\sin\left[\frac{m\pi u}{K}\right],\\ &\qquad [|Im(u/2K)| < Im(i K'/K)].\end{aligned}\right.$$

908.51
$$\left\{\begin{aligned}&\operatorname{cs} u \operatorname{dn} u = \frac{\pi}{2K}\cot\frac{\pi u}{2K} - \frac{2\pi}{K}\sum_{m=1}^{\infty}\frac{q^{m}}{1+q^{m}}\sin\left[\frac{m\pi u}{K}\right],\\ &\qquad [|Im(u/2K)| < Im(iK'/K)].\end{aligned}\right.$$

908.52
$$\left\{\begin{aligned}&\operatorname{sd} u \operatorname{cn} u = \frac{4\pi}{k^2 K}\sum_{m=1}^{\infty}\frac{q^{2m-1}}{1-q^{2(2m-1)}}\sin\left[(2m-1)\frac{\pi u}{K}\right],\\ &\qquad [|Im(u/K)| < Im(i K'/K)].\end{aligned}\right.$$

Infinite Products.

909.00
$$\left\{\begin{aligned}&\operatorname{sn} u = \frac{2K}{\pi}\sin\frac{\pi u}{2K}\prod_{m=1}^{\infty}\left[\frac{(1-q^{2m-1})^2}{(1-q^{2m})^2}\times\right.\\ &\qquad \left.\times\frac{1-2q^{2m}\cos(\pi u/K)+q^{4m}}{1-2q^{2m-1}\cos(\pi u/K)+q^{4m-2}}\right],\\ &\qquad [\text{R.P.}\ (K'/K) > 0;\ |q| < 1].\end{aligned}\right.$$

909.01
$$\left\{\begin{aligned}&\operatorname{cn} u = \cos\frac{\pi u}{2K}\prod_{m=1}^{\infty}\left[\frac{(1-q^{2m-1})^2}{(1+q^{2m})^2}\cdot\frac{1+2q^{2m}\cos(\pi u/K)+q^{4m}}{1-2q^{2m-1}\cos(\pi u/K)+q^{4m-2}}\right],\\ &\qquad [\text{R.P.}\ (K'/K) > 0;\ |q| < 1].\end{aligned}\right.$$

909.02
$$\left\{\begin{aligned}&\operatorname{dn} u = \prod_{m=1}^{\infty}\left[\frac{(1-q^{2m-1})^2}{(1+q^{2m-1})^2}\cdot\frac{1+2q^{2m-1}\cos(\pi u/K)+q^{4m-2}}{1-2q^{2m-1}\cos(\pi u/K)+q^{4m-2}}\right],\\ &\qquad [\text{R.P.}\ (K'/K) > 0;\ |q| < 1].\end{aligned}\right.$$

909.03
$$\begin{cases} \operatorname{tn} u = \dfrac{2K}{\pi} \tan \dfrac{\pi u}{2K} \displaystyle\prod_{m=1}^{\infty} \left[\dfrac{(1+q^{2m})^2}{(1-q^{2m})^2} \cdot \dfrac{1-2q^{2m}\cos(\pi u/K)+q^{4m}}{1+2q^{2m}\cos(\pi u/K)+q^{4m}}\right], \\ \qquad [\text{R.P.}(K'/K) > 0;\ |q| < 1]. \end{cases}$$

Products for the other eight Jacobian elliptic functions may be obtained by taking the reciprocal and quotients of the above.

Other Developments.

910.01
$$\begin{cases} \operatorname{sn} u = \dfrac{\pi}{2kK} \displaystyle\sum_{m=-\infty}^{\infty} \csc \dfrac{\pi}{2K}[u-(2m-1)\,iK'], \\ \qquad [|Im(u/K)| < Im(iK'/K)]. \end{cases}$$

910.02
$$\begin{cases} \operatorname{cn} u = \dfrac{\pi i}{2kK} \displaystyle\sum_{m=-\infty}^{\infty} (-1)^m \csc \dfrac{\pi}{2K}[u-(2m-1)\,iK'], \\ \qquad [|Im(u/K)| < Im(iK'/K)]. \end{cases}$$

910.03
$$\begin{cases} \operatorname{dn} u = \dfrac{\pi i}{2K} \displaystyle\sum_{m=-\infty}^{\infty} (-1)^m \cot \dfrac{\pi}{2K}[u-(2m-1)\,iK'], \\ \qquad [|Im(u/K)| < Im(iK'/K)]. \end{cases}$$

911.01
$$\begin{cases} \operatorname{sn}^2 u = \dfrac{\pi}{4k^3K^3} \displaystyle\sum_{m=0}^{\infty} [4(1+k^2)K^2 - (2m+1)^2\pi^2] \times \\ \qquad \times \dfrac{q^{m+\frac{1}{2}}}{1-q^{2m+1}} \sin\left[(2m+1)\dfrac{\pi u}{2K}\right], \\ \qquad [|Im(u/2K)| < Im(iK'/K)]. \end{cases}$$

912.01
$$\begin{cases} \operatorname{ns}^2 u = \left[\dfrac{\pi}{2K}\csc\dfrac{\pi u}{2K}\right]^2 + \dfrac{K-E}{K} - \\ \qquad - \dfrac{\pi^2}{2K^2} \displaystyle\sum_{m=1}^{\infty} \dfrac{m\,q^{2m}}{1-q^{2m}} \cos\left[\dfrac{m\pi u}{2K}\right], \\ \qquad [|Im(u/2K)| < Im(iK'/K)]. \end{cases}$$

Appendix.

Weierstrassian Elliptic Functions and Elliptic Integrals.

It was found convenient in this handbook to evaluate elliptic integrals by first reducing them to Legendre-Jacobi normal form. This was accomplished by means of suitable homographic substitutions and the use of Jacobian elliptic functions. The more modern method of Weierstrass[1], however, employs a different notation which is sometimes advantageous when the polynomial under the radical sign is not resolved into factors. We will not discuss here the general modes of reducing the elliptic integral $\int dt/\sqrt{P_4(t)}$ to Weierstrassian form, but shall briefly give some of the more important formulas expressing the nature and properties of the Weierstrassian elliptic function $\wp(u)$ and the related functions $\zeta(u)$ and $\sigma(u)$.

Definition.

If

1030.00
$$\begin{cases} \wp^{-1}(y) \equiv u = \displaystyle\int_y^\infty \frac{dt_1}{\sqrt{4t_1^3 - g_2 t_1 - g_3}} = \int_y^\infty \frac{dt_1}{\sqrt{4(t_1-e_1)(t_1-e_2)(t_1-e_3)}}, \\ [e_1+e_2+e_3=0;\ g_2 = -4(e_2 e_3 + e_3 e_1 + e_1 e_2);\ g_3 = 4e_1 e_2 e_3] \end{cases}$$

then the inverse function $y = \wp(u, g_2, g_3) \equiv \wp(u)$ defines the *Weierstrassian elliptic function*[2]. If we set

$$k^2 = \frac{e_2-e_3}{e_1-e_3},\quad t_1 = e_1 + \gamma^2 \cot^2\vartheta,\quad \gamma^2 = e_1 - e_3,\quad t_1 = e_3 + \frac{\gamma^2}{t^2},\quad (e_1 > e_2 > e_3),$$

the integral **1030.00** becomes

1030.01
$$u = \frac{1}{\gamma}\int_0^\varphi \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{1}{\gamma}\int_0^Y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

which is now in Legendre's and Jacobi's form.

---

[1] An excellent treatment of the Weierstrassian functions is found in H. A. SCHWARZ, *Formeln und Lehrsätze zum Gebrauche der elliptischen Funktionen. Nach Vorlesungen und Aufzeichnungen des Herrn Prof.* K. WEIERSTRASS. (Berlin, 1893.)

[2] This function is tabulated in the *Proc. Roy. Artillery Inst.*, v. 17, pp. 181—216, 1889.

Relation to Jacobian Elliptic Functions.

From **1030.01**, it is thus seen that

$$Y = \operatorname{sn}(\gamma u), \quad \varphi = \operatorname{am}(\gamma u),$$

1031.01
$$\begin{cases} \operatorname{sn}(\gamma u) = \dfrac{\gamma}{\sqrt{\wp(u) - e_3}}, & \wp(u) = e_1 + \gamma^2 \operatorname{cs}^2(\gamma u), \quad [\gamma^2 = e_1 - e_3], \\ \operatorname{cn}(\gamma u) = \sqrt{\dfrac{\wp(u) - e_1}{\wp(u) - e_3}}, & \wp(u) = e_2 + \gamma^2 \operatorname{ds}^2(\gamma u), \\ \operatorname{dn}(\gamma u) = \sqrt{\dfrac{\wp(u) - e_2}{\wp(u) - e_3}}, & \wp(u) = e_3 + \gamma^2 \operatorname{ns}^2(\gamma u). \end{cases}$$

Unlike the Jacobian function $\operatorname{sn} u$, which has a simple pole, Weierstrass' elliptic function $\wp(u)$ has a pole of order 2 in any primitive period-parallelogram.

Fundamental Relations.

1032.01
$$\begin{cases} \wp(u + 2\omega) = \wp(u), \\ \wp(u + 2\omega') = \wp(u). \end{cases}$$

The function $\wp(u)$ is thus *doubly periodic*, having periods 2ω and $2\omega'$, where

1032.02
$$\left\{ \omega = \frac{K}{\sqrt{e_1 - e_3}} = \frac{K}{\gamma}, \qquad \omega' = \frac{i K'}{\sqrt{e_1 - e_3}} = \frac{i K'}{\gamma}; \right.$$

1032.03
$$\begin{cases} \omega = \displaystyle\int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_{e_3}^{e_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \\ \omega' = \displaystyle\int_{e_3}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_{e_2}^{e_1} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \\ \omega + \omega' = \displaystyle\int_{e_2}^{\infty} \frac{dt}{\sqrt{4t^2 - g_2 t - g_3}}. \end{cases}$$

Derivatives.

1032.04
$$\begin{cases} \wp'(u) = -\sqrt{4\wp^3(u) - g_2 \wp(u) - g_3}, \qquad \left[\wp'(u) = \dfrac{d}{du} \wp(u)\right], \\ \wp''(u) = 6\wp^2(u) - g_2/2, \\ \wp'''(u) = 12\wp(u)\wp'(u), \\ \wp^{\text{iv}}(u) = 12[\wp'^2(u) + \wp(u)\wp''(u)]. \end{cases}$$

Special Values.

1033.01 $$\begin{cases} \wp(-u) = \wp(u), \\ \wp(\omega) = e_1, \\ \wp(\omega') = e_3, \\ \wp(\omega + \omega') = e_2. \end{cases}$$

1033.02 $$\begin{cases} \wp'(\omega') = 0, \\ \wp'(\omega) = 0, \\ \wp'(\omega + \omega') = 0, \\ \wp^{(n)}(-u) = (-1)^n \wp^{(n)}(u). \end{cases}$$

1033.03 $$\begin{cases} \wp(\omega/2) = e_1 + \sqrt{(e_1 - e_3)(e_1 - e_2)}, \\ \wp(\omega'/2) = e_3 - \sqrt{(e_1 - e_3)(e_2 - e_3)}. \end{cases}$$

1033.04 $$\begin{cases} \wp'(\omega/2) = -2\left[(e_1 - e_3)\sqrt{e_1 - e_2} + (e_1 - e_2)\sqrt{e_1 - e_3}\right], \\ \wp'(\omega'/2) = -2i\left[(e_1 - e_3)\sqrt{e_2 - e_3} + (e_2 - e_3)\sqrt{e_1 - e_3}\right]. \end{cases}$$

1033.05 $$\begin{cases} \wp(u \pm \omega) = e_1 + \dfrac{(e_1 - e_2)(e_1 - e_3)}{\wp(u) - e_1}, \\ \wp(u \pm \omega') = e_3 + \dfrac{(e_3 - e_1)(e_3 - e_2)}{\wp(u) - e_3}, \\ \wp(u \pm \omega \pm \omega') = e_2 + \dfrac{(e_2 - e_1)(e_2 - e_3)}{\wp(u) - e_2}. \end{cases}$$

Addition Formulas.

1034.01 $$\wp(u \pm v) = -\wp(u) - \wp(v) + \frac{1}{4}\left[\frac{\wp'(u) \mp \wp'(v)}{\wp(u) - \wp(v)}\right]^2.$$

1034.02 $$\wp(u) - \wp(v) = \frac{\wp'\left(\frac{u+v}{2}\right)\wp'\left(\frac{u-v}{2}\right)}{\left[\wp\left(\frac{u+v}{2}\right) - \wp\left(\frac{u-v}{2}\right)\right]^2}.$$

1034.03 $$\wp(u + v) + \wp(u - v) = \frac{2\left[\wp(u)\wp(v) - g_2/4\right]\left[\wp(u) + \wp(v)\right] - g_3}{\left[\wp(u) - \wp(v)\right]^2}.$$

Relation to Theta Functions.

1035.01 $$\wp(u) = \frac{1}{12\omega^2}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - \frac{d^2}{du^2}\left[\ln \vartheta_1\left(\frac{u}{2\omega}\right)\right],$$

where $\vartheta_1(u)$ is defined in **1050.01**.

Weierstrassian Normal Elliptic Integrals.

A general elliptic integral of the form

1036.00 $$\int \frac{R_1(t_1)\,dt_1}{\sqrt{a_0 t_1^3 + b t_1^2 + c t_1 + d}},$$

where $R_1(t_1)$ denotes a rational function of t_1, may be written

1036.01 $$\int \frac{R(t)\,dt}{\sqrt{4t^3 - g_2 t - g_3}}$$

by means of the substitution

$$t_1 = \sqrt[3]{\frac{4}{a_0}}\,t - \frac{b}{3a_0},$$

with

$$g_2 = \left[\frac{b^2}{3a_0} - c\right]\sqrt[3]{\frac{4}{a_0}}, \quad g_3 = \frac{cb}{3a_0} - \frac{2b^3}{27a_0^2} - d.$$

The evaluation of this integral can then always be made to depend on the three integrals

$$\int_{\infty}^{\wp(u)} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \quad \int_{\wp(u_0)}^{\wp(u)} \frac{t\,dt}{\sqrt{4t^3 - g_2 t - g_3}}, \quad \int_{\infty}^{\wp(u)} \frac{dt}{(t-\alpha^2)\sqrt{4t^3 - g_2 t - g_3}},$$

which are Weierstrass' *normal elliptic integrals of the first*[1], *second* and *third kind* respectively. These may be evaluated as follows:

1036.02 $$\begin{cases} \displaystyle\int_{\infty}^{\wp(u)} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_0^u du = u, \\ \displaystyle\int_{\wp(u_0)}^{\wp(u)} \frac{t\,dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_{u_0}^{u} \wp(u)\,du = -\zeta(u) + \zeta(u_0), \\ \displaystyle\int_{\infty}^{\wp(u)} \frac{dt}{(t-\alpha^2)\sqrt{4t^3 - g_2 t - g_3}} = \int_0^u \frac{du}{\wp(u) - \wp(\alpha^2)} \\ \displaystyle\qquad = \frac{1}{\wp'(\alpha^2)} \ln\left[\frac{\sigma(\alpha^2 - u)}{\sigma(\alpha^2 + u)}\right] + 2u\,\frac{\zeta(\alpha^2)}{\wp'(\alpha^2)}, \end{cases}$$

where $\wp(u)$, and the *Weierstrassian Zeta* $\zeta(u)$ and *Sigma functions* $\sigma(u)$ are developed below.

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[1] For an alternate form of this, based on transforming the cubic under the radical sign to $t(t-m)(t-1)$, see A. R. Low's *Normal Elliptic Functions* (A Normalized Form of Weierstrass's Elliptic Functions), University of Toronto Press, Toronto, 1950.

Expansions of the functions $\wp(u)$, $\zeta(u)$ and $\sigma(u)$.

1036.50
$$\begin{cases} \wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{20} + \frac{g_3 u^4}{28} + \frac{g_2^2 u^6}{1200} + \frac{3 g_2 g_3 u^8}{6160} + \cdots, \\ \zeta(u) = -\int \wp(u)\, du = \frac{1}{u} - \frac{g_2 u^3}{60} - \frac{g_3 u^5}{140} - \frac{g_2^2 u^7}{8400} - \frac{3 g_2 g_3 u^9}{55440} - \cdots, \\ \sigma(u) = e^{\int \zeta(u) du} = u - \frac{g_2 u^5}{240} - \frac{g_3 u^7}{840} - \frac{g_2^2 u^9}{161280} - \cdots. \end{cases}$$

Other Integrals.

1037.01 $\int \wp^2(u)\, du = [\wp'(u)]/6 + g_2 u/12.$

1037.02 $\int \wp^3(u)\, du = [\wp'''(u)]/120 - 3 g_2 [\zeta(u)]/20 + g_3 u/10.$

1037.03
$$\begin{cases} \int \wp^m(u)\, du = \frac{1}{4(2m-1)} \left[(2m-3)\, g_2 \int \wp^{m-2}(u)\, du + \right. \\ \qquad \left. + 2(m-2)\, g_3 \int \wp^{m-3}(u)\, du + 2\wp^{m-2}(u)\, \wp'(u)\right]. \end{cases}$$

1037.04 $$\int \frac{\wp'(u)\, du}{[\wp(u) - \wp(a)]^m} = \frac{1}{1-m} \frac{1}{[\wp(u) - \wp(a)]^{m-1}}, \qquad (m \neq 1).$$

1037.05 $$\int \frac{\wp'(u)\, du}{\wp(u) - \wp(a)} = \ln [\wp(u) - \wp(a)].$$

1037.06
$$\begin{cases} \int \frac{du}{\wp(u) - \wp(a)} = \frac{1}{\wp'(a)} \left[\ln \frac{\sigma(u-a)}{\sigma(u+a)} + 2u\,\zeta(a)\right], \\ \qquad [\wp(a) \neq e_1, e_2 \text{ or } e_3]. \end{cases}$$

1037.07 $$\int \frac{du}{\wp(u) - e_1} = -\frac{1}{e_2 e_3 + 2e_1^2}\left[e_1 u + \zeta(u) + \frac{1}{2} \frac{\wp'(u)}{\wp(u) - e_1}\right].$$

1037.08 $$\int \frac{du}{\wp(u) - e_2} = -\frac{1}{e_3 e_1 + 2e_2^2}\left[e_2 u + \zeta(u) + \frac{1}{2} \frac{\wp'(u)}{\wp(u) - e_2}\right].$$

1037.09 $$\int \frac{du}{\wp(u) - e_3} = -\frac{1}{e_1 e_2 + 2e_3^2}\left[e_3 u + \zeta(u) + \frac{1}{2} \frac{\wp'(u)}{\wp(u) - e_3}\right].$$

1037.10 $$\int \frac{a\,\wp(u) + b}{c\,\wp(u) + d}\, du = \frac{a u}{c} + \frac{a d - b c}{c^2}\left[\frac{2\zeta(a)}{\wp'(a)} u + \frac{1}{\wp'(a)} \ln \frac{\sigma(u-a)}{\sigma(u+a)}\right],$$

1037.11
$$\begin{cases} \int \frac{du}{[\wp(u) - \wp(a)]^2} = \frac{\wp''(a)}{\wp'^3(a)} \ln \frac{\sigma(u+a)}{\sigma(u-a)} - \frac{1}{\wp'^2(a)} \zeta(u+a) - \\ \qquad - \frac{1}{\wp'^2(a)} \zeta(u-a) - \left[\frac{2\wp(a)}{\wp'^2(a)} + \frac{2\wp''(a)\zeta(a)}{\wp'^3(a)}\right] u, \\ \qquad [\wp(a) \neq e_1,\ e_2 \text{ or } e_3]. \end{cases}$$

1037.12
$$\left\{\begin{aligned}
&\int \frac{du}{[\wp(u)-\wp(a)]^m} = \frac{1}{2(1-m)\,[4\wp^3(a)-g_2\wp(a)-g_3]}\times\\
&\qquad\times\Big\{(2m-3)\,[12\wp^2(a)-g_2]\int\frac{du}{[\wp(u)-\wp(a)]^{m-1}}+\\
&\qquad+24(m-2)\,\wp(a)\int\frac{du}{[\wp(u)-\wp(a)]^{m-2}}+\\
&\qquad+4(2m-5)\int\frac{du}{[\wp(u)-\wp(a)]^{m-3}}+\frac{2\wp'(u)}{[\wp(u)-\wp(a)]^{m-}}\Big\},\\
&\qquad\qquad [m\neq 1;\ \wp(a)\neq e_1,\ e_2 \text{ or } e_3].
\end{aligned}\right.$$

1037.13
$$\left\{\begin{aligned}
&\int\frac{du}{[\wp(u)-e_1]^m}=\frac{1}{(1-2m)\,(e_2e_3+2e_1^2)}\Big\{6(m-1)\,e_1\int\frac{du}{[\wp(u)-e_1]^{m-1}}+\\
&\qquad+(2m-3)\int\frac{du}{[\wp(u)-e_1]^{m-2}}+\frac{1}{2}\,\frac{\wp'(u)}{[\wp(u)-e_1]^m}\Big\}.
\end{aligned}\right.$$

1037.14
$$\left\{\begin{aligned}
&\int\frac{du}{[\wp(u)-e_2]^m}=\frac{1}{(1-2m)\,(e_3e_1+2e_2^2)}\Big\{6(m-1)\,e_2\int\frac{du}{[\wp(u)-e_2]^{m-1}}+\\
&\qquad+(2m-3)\int\frac{du}{[\wp(u)-e_2]^{m-2}}+\frac{1}{2}\,\frac{\wp'(u)}{[\wp(u)-e_2]^m}\Big\}.
\end{aligned}\right.$$

1037.15
$$\left\{\begin{aligned}
&\int\frac{du}{[\wp(u)-e_3]^m}=\frac{1}{(1-2m)\,(e_1e_2+2e_3^2)}\Big\{6(m-1)\,e_3\int\frac{du}{[\wp(u)-e_3]^{m-1}}+\\
&\qquad+(2m-3)\int\frac{du}{[\wp(u)-e_3]^{m-2}}+\frac{1}{2}\,\frac{\wp'(u)}{[\wp(u)-e_3]^m}\Big\}.
\end{aligned}\right.$$

Illustrative Example.

We shall close this section with an example illustrating the Jacobian and Weierstrassian mode of procedure for the reduction of the elliptic integral

$$u=\int_3^y\frac{dt}{2\sqrt{t^3-5t^2+4t+6}},\qquad (y>3). \tag{A 1}$$

Reduction to Jacobian form: — The integral **(A 1)** may be written

$$\left.\begin{aligned}
u&=\int_3^y\frac{dt}{2\sqrt{(t-3)\,(t-1+\sqrt{3})\,(t-1-\sqrt{3})}}\\
&=\int_3^\infty\frac{dt}{2\sqrt{(t-3)\,(t-1+\sqrt{3})\,(t-1-\sqrt{3})}}-\\
&\quad-\int_y^\infty\frac{dt}{2\sqrt{(t-3)\,(t-1+\sqrt{3})\,(t-1-\sqrt{3})}}.
\end{aligned}\right\} \tag{A 2}$$

Using **238.00**, with $a=3$, $b=1+\sqrt{3}$, $c=1-\sqrt{3}$, the evaluation of the integral is found immediately to be

$$u=\frac{K}{\sqrt{2+\sqrt{3}}}-\frac{1}{\sqrt{2+\sqrt{3}}}\operatorname{sn}^{-1}(\sin\varphi,k)=\frac{1}{\sqrt{2+\sqrt{3}}}\,[K-F(\varphi,k)]\,, \qquad \textbf{(A 3)}$$

where

$$k^2=2\left(2\sqrt{3}-3\right),\quad \varphi=\sin^{-1}\sqrt{\frac{2+\sqrt{3}}{y-1+\sqrt{3}}}\,;$$

or

$$\operatorname{sn}^{-1}\sqrt{\frac{2+\sqrt{3}}{y-1+\sqrt{3}}}=K-u\sqrt{2+\sqrt{3}}\,. \qquad \textbf{(A 4)}$$

The inversion of this then yields

$$\sqrt{\frac{2+\sqrt{3}}{y-1+\sqrt{3}}}=\operatorname{sn}\left(K-u\sqrt{2+\sqrt{3}}\,\right)=\operatorname{cd}\left(u\sqrt{2+\sqrt{3}}\,\right), \qquad \textbf{(A 5)}$$

or

$$y=3+\left(2+\sqrt{3}\right)k'^2\operatorname{tn}^2\left(u\sqrt{2+\sqrt{3}}\,\right). \qquad \textbf{(A 6)}$$

Reduction to Weierstrassian form:—The integral

$$u=\int_3^y\frac{dt_1}{2\sqrt{t_1^3-5t_1^2+4t_1+6}}$$

becomes, upon making the substitution $t_1=t+5/3$, $Y=y-5/3$,

$$u=\int_{4/3}^{Y}\frac{dt}{\sqrt{4t^3-52t/3+368/27}}\,; \qquad \textbf{(A 7)}$$

so $g_2=52/3$, $g_3=-368/27$. Further, we may write equation **(A 7)** in the form

$$u=\int_{4/3}^{\infty}\frac{dt}{\sqrt{4t^3-52t/3+368/27}}-\int_{y-5/3}^{\infty}\frac{dt}{\sqrt{4t^3-52t/3+368/27}}\,, \qquad \textbf{(A 8)}$$

which, after using **1030.00** and **1032.03**, gives

$$u=\omega-\wp^{-1}[(y-5/3),\,52/3,\,-368/27] \qquad \textbf{(A 9)}$$

for the evaluation of the integral. Since

$$\left.\begin{aligned} &e_1 = 4/3, \quad e_2 = \sqrt{3} - 2/3, \quad e_3 = -\sqrt{3} - 2/3, \\ &k^2 = \frac{e_2 - e_3}{e_1 - e_3} = 2(2\sqrt{3} - 3), \quad \gamma = \sqrt{e_1 - e_3}, \end{aligned}\right\} \tag{A 10}$$

formula **1032.02** yields

$$\omega = \frac{K}{\gamma} = \frac{K}{\sqrt{2+\sqrt{3}}}. \tag{A 11}$$

Employing now **1033.05**, one has from **(A 9)**

$$y = 3 + \frac{1}{\wp(u) - 4/3}. \tag{A 12}$$

In order to compare this with the results obtained by the Jacobian process, formula **1032.01** is applied. We then have

$$y = 3 + (2 - \sqrt{3})\,\mathrm{tn}^2\left(u\sqrt{2+\sqrt{3}}\right) = 3 + (2+\sqrt{3})k'^2\,\mathrm{tn}^2\left(u\sqrt{2+\sqrt{3}}\right), \tag{A13}$$

which is the same as **(A 6)**.

Theta-Functions.

Definitions.

For obtaining definite numerical values in many problems involving elliptic functions, auxiliary functions called *Theta Functions* are found highly useful and at times indispensable. These functions[1] of Jacobi are defined by the FOURIER series

1050.01

$$\begin{cases} \Theta(u) = \vartheta_0(v) = 1 + 2\sum\limits_1^\infty (-1)^m q^{m^2} \cos(2mv), \\ H(u) = \vartheta_1(v) = 2\sum\limits_1^\infty (-1)^{m-1} q^{(m-1/2)^2} \sin(2m-1)v, \\ H_1(u) = \vartheta_2(v) = 2\sum\limits_1^\infty q^{(m-1/2)^2} \cos(2m-1)v, \\ \Theta_1(u) = \vartheta_3(v) = 1 + 2\sum\limits_1^\infty q^{m^2} \cos(2mv), \end{cases}$$

where

$$q = e^{-(\pi K'/K)}, \qquad v = \pi u/2K.$$

[1] A short tabulation which is related to these functions may be found in *Smithsonian Mathematical Formulas and Tables of Elliptic Functions*, by E. P. ADAMS and R. L. HIPPISLEY, Smithsonian Institution, Washington, D. C., 1922.

The four functions have quasi doubly-periodic properties[1]. They are single-valued analytic functions having no singularities except at infinity.

The logarithms of these functions are given by

1050.02
$$\begin{cases} \ln \Theta(u) = \ln \vartheta_0(v) = \ln \xi - 2\sum_{m=1}^{\infty} \frac{q^m}{m(1-q^{2m})} \cos(2m\pi v), \\ \qquad \left[|Im(v)| < \frac{1}{2} Im(iK'/K)\right]. \\ \ln H(u) = \ln \vartheta_1(v) \\ \quad = \ln\left[2\sqrt[4]{\xi}\sin(\pi v)\right] - 2\sum_{m=1}^{\infty} \frac{q^{2m}}{m(1-q^{2m})} \cos(2m\pi v), \\ \qquad [|Im(v)| < Im(iK'/K)]. \\ \ln H_1(u) = \ln \vartheta_2(v) \\ \quad = \ln\left[2\sqrt[4]{\xi}\sin(\pi v)\right] + 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1} q^{2m}}{m(1-q^{2m})} \cos(2m\pi v), \\ \qquad [|Im(v)| < Im(iK'/K)]. \\ \ln \Theta_1(u) = \ln \vartheta_3(v) = \ln \xi + 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1} q^m}{m(1-q^{2m})} \cos(2m\pi v), \\ \qquad \left[|Im\, v| < \frac{1}{2} Im(iK'/K)\right]. \end{cases}$$

where ξ is Euler's number ($\approx .57721566$), and $v = \pi u/2K$.

Special Values.

1051.01
$$\begin{cases} \Theta(0) = \Theta_1(K) = \vartheta_0(0) = \vartheta_3(\pi/2) = \sqrt{\frac{2k'K}{\pi}} \\ H(0) = -H_1(K) = \vartheta_1(0) = \vartheta_2(\pi/2) = 0 \\ H_1(0) = H(K) = \vartheta_2(0) = \vartheta_1(\pi/2) = \sqrt{\frac{2kK}{\pi}} \\ \Theta_1(0) = \Theta(K) = \vartheta_3(0) = \vartheta_0(\pi/2) = \sqrt{\frac{2K}{\pi}} \end{cases}$$

In the following, $v = \pi u/2\mathrm{K}$.

1051.02
$$\begin{cases} \Theta(-u) = \Theta(u), \\ H(-u) = H(u), \\ H_1(-u) = H_1(u), \\ \Theta_1(-u) = \Theta_1(u). \end{cases} \qquad \begin{cases} \vartheta_0(-v) = \vartheta_0(v), \\ \vartheta_1(-v) = \vartheta_1(v), \\ \vartheta_2(-v) = \vartheta_2(v), \\ \vartheta_3(-v) = \vartheta_3(v). \end{cases}$$

[1] For a full discussion of the properties of the Theta function see, for example, *Modern Analysis*, by Whittaker and Watson, Macmillan Co., New York, 1943.

1051.03
$$\begin{cases} \Theta(u+K) = \Theta_1(u), \\ H(u+K) = H_1(u), \\ H_1(u+K) = -H(u), \\ \Theta_1(u+K) = \Theta(u). \end{cases} \qquad \begin{cases} \vartheta_0(v+\pi/2) = \vartheta_3(v), \\ \vartheta_1(v+\pi/2) = \vartheta_2(v), \\ \vartheta_2(v+\pi/2) = -\vartheta_1(v), \\ \vartheta_3(v+\pi/2) = \vartheta_0(v). \end{cases}$$

1051.04
$$\begin{cases} \Theta(u+2K) = \Theta(u), \\ H(u+2K) = -H(u), \\ H_1(u+2K) = -H_1(u), \\ \Theta_1(u+2K) = \Theta_1(u). \end{cases} \qquad \begin{cases} \vartheta_0(v+\pi) = \vartheta_0(v), \\ \vartheta_1(v+\pi) = -\vartheta_1(v), \\ \vartheta_2(v+\pi) = -\vartheta_2(v), \\ \vartheta_3(v+\pi) = \vartheta_3(v). \end{cases}$$

1051.05
$$\begin{cases} \Theta(u+3K) = \Theta_1(u), \\ H(u+3K) = -H_1(u), \\ H_1(u+3K) = H(u), \\ \Theta_1(u+3K) = \Theta(u). \end{cases} \qquad \begin{cases} \vartheta_0(v+3\pi/2) = \vartheta_3(v), \\ \vartheta_1(v+3\pi/2) = -\vartheta_2(v), \\ \vartheta_2(v+3\pi/2) = \vartheta_1(v), \\ \vartheta_3(v+3\pi/2) = \vartheta_0(v). \end{cases}$$

1051.06
$$\begin{cases} \Theta(u+4K) = \Theta(u), \\ H(u+4K) = H(u), \\ H_1(u+4K) = H_1(u), \\ \Theta_1(u+4K) = \Theta_1(u). \end{cases} \qquad \begin{cases} \vartheta_0(v+2\pi) = \vartheta_0(v), \\ \vartheta_1(v+2\pi) = \vartheta_1(v), \\ \vartheta_2(v+2\pi) = \vartheta_2(v), \\ \vartheta_3(v+2\pi) = \vartheta_3(v). \end{cases}$$

1051.07
$$\begin{cases} \Theta[2mK + (2m+1)\,iK'] = 0, \\ H[2mK + 2n\,iK'] = 0, \\ H_1[(2m+1)K + 2n\,iK'] = 0, \\ \Theta_1[(2m+1)K + (2n+1)\,iK'] = 0, \\ \vartheta_0[m\pi + (2n+1)\tau] = 0, \\ \vartheta_1(m\pi + 2n\tau) = 0, \\ \vartheta_2[(m+\tfrac{1}{2})\pi + 2n\tau] = 0, \\ \vartheta_3[(m+\tfrac{1}{2})\pi + (2n+1)\tau] = 0, \end{cases} \qquad \left[\tau = i\pi K'/2K,\ m, n \text{ integers including } 0\right].$$

Quasi-Addition Formulas.

1051.25
$$\begin{cases} \vartheta_0(x+y)\,\vartheta_0(x-y)\,\vartheta_0^2(0) = \vartheta_3^2(x)\,\vartheta_3^2(y) - \vartheta_2^2(x)\,\vartheta_2^2(y), \\ \vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_0^2(0) = \vartheta_3^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_3^2(y), \\ \vartheta_2^2(x+y)\,\vartheta_2(x-y)\,\vartheta_0^2(0) = \vartheta_0^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_3^2(y), \\ \vartheta_3^2(x+y)\,\vartheta_3(x-y)\,\vartheta_0^2(0) = \vartheta_0^2(x)\,\vartheta_3^2(y) - \vartheta_1^2(x)\,\vartheta_2^2(y). \end{cases}$$

Differential Equation.

The Theta Functions are solutions of the partial differential equation of heat conduction in the form

1051.50
$$\frac{\partial^2 \vartheta}{\partial v^2} = \frac{4i}{\pi}\frac{\partial \vartheta}{\partial \varrho}, \qquad [q = e^{i\pi\varrho}].$$

Relation to Jacobian Elliptic Functions.

1052.01 $$\sqrt{k} = \frac{H_1(0)}{\Theta_1(0)} = \frac{\vartheta_2(0)}{\vartheta_3(0)}, \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)} = \frac{\vartheta_0(0)}{\vartheta_3(0)}.$$

1052.02 $$\begin{cases} \operatorname{sn} u = \dfrac{H(u)}{\sqrt{k}\,\Theta(u)} = \dfrac{1}{\sqrt{k}} \dfrac{\vartheta_1(v)}{\vartheta_0(v)}, \\ \operatorname{cn} u = \sqrt{\dfrac{k'}{k}} \dfrac{H_1(u)}{\Theta(u)} = \sqrt{\dfrac{k'}{k}} \dfrac{\vartheta_2(v)}{\vartheta_0(v)}, \\ \operatorname{dn} u = \sqrt{k'} \dfrac{\Theta_1(u)}{\Theta(u)} = \sqrt{k'} \dfrac{\vartheta_3(v)}{\vartheta_0(v)}, \\ \operatorname{tn} u = \dfrac{1}{\sqrt{k'}} \dfrac{H(u)}{H_1(u)} = \dfrac{1}{\sqrt{k'}} \dfrac{\vartheta_1(v)}{\vartheta_2(v)}. \end{cases}$$

Relation to Elliptic Integrals.

1053.01 $$\begin{cases} K = \dfrac{\pi}{2}[\Theta_1(0)]^2 = \dfrac{\pi}{2}[\vartheta_3(0)]^2 = \dfrac{\pi}{2}[\vartheta_0(\pi/2)]^2, \\ Z(u) = \dfrac{\Theta'(u)}{\Theta(u)} = \dfrac{\partial}{\partial u} \ln \vartheta_0(\pi u/2K), \qquad \left[\Theta'(u) = \dfrac{\partial}{\partial u}\Theta(u)\right] \\ E = \dfrac{\pi}{2}[\Theta_1(0)]^2 \left[1 - \dfrac{\Theta''(0)}{\Theta(0)}\right], \\ 1 - \dfrac{E}{K} = \dfrac{\Theta''(0)}{\Theta(0)}. \end{cases}$$

1053.02 $$\begin{cases} \dfrac{\pi\alpha}{2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \dfrac{\partial}{\partial w}\vartheta_0(w) = \dfrac{\alpha K Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}, \quad [\text{cf. } \mathbf{415.01}]. \\ \dfrac{\pi}{2\sqrt{\alpha^2(\alpha^2-1)(\alpha^2-k^2)}} \dfrac{\partial}{\partial w}\vartheta_1(w) = \dfrac{K}{\alpha^2}\left[1 + \dfrac{\alpha Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right], \\ \qquad [\text{cf. } \mathbf{415.02}], \\ -\dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2-1}{\alpha^2(\alpha^2-k^2)}} \dfrac{\partial}{\partial w}\vartheta_2(w) = \dfrac{K}{\alpha^2}\left[1 + \dfrac{\alpha(1-\alpha^2)Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right], \\ \qquad [\text{cf. } \mathbf{415.03}], \\ -\dfrac{\pi}{2}\sqrt{\dfrac{\alpha^2-k^2}{\alpha^2(\alpha^2-1)}} \dfrac{\partial}{\partial w}\vartheta_3(w) = \dfrac{K}{\alpha^2}\left[k^2 + \dfrac{\alpha(k^2-\alpha^2)Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\right], \\ \qquad [\text{cf. } \mathbf{415.04}], \end{cases}$$

where

$$\sin A = 1/\alpha, \quad w = \frac{\pi F(A,k)}{2K}, \quad \alpha^2 > 1;$$

$$Z(A,k) = E(A,k) - \frac{E}{K} F(A,k).$$

1053.03

$$\left\{\begin{aligned}
&\frac{\pi}{2\sqrt{(k^2-\alpha^2)(1-\alpha^2)\alpha^2}}\,\frac{\partial}{\partial w}\,\vartheta_0(w)=\frac{K\,Z(\beta,k)}{\sqrt{(k^2-\alpha^2)(1-\alpha^2)\alpha^2}}, && [\text{cf. } \mathbf{414.02}],\\
&\frac{\alpha\,\pi}{2\sqrt{(k^2-\alpha^2)(1-\alpha^2)}}\,\frac{\partial}{\partial w}\,\vartheta_1(w)=K+\frac{\alpha\,K\,Z(\beta,k)}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}}, && [\text{cf. } \mathbf{414.01}],\\
&\frac{\pi}{2\alpha}\sqrt{\frac{k^2-\alpha^2}{1-\alpha^2}}\,\frac{\partial}{\partial w}\,\vartheta_2(w)=K+\frac{(\alpha^2-k^2)\,K\,Z(\beta,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}, && [\text{cf. } \mathbf{414.04}],\\
&\frac{\pi}{2\alpha}\sqrt{\frac{1-\alpha^2}{k^2-\alpha^2}}\,\frac{\partial}{\partial w}\,\vartheta_3(w)=K+\frac{(\alpha^2-1)\,K\,Z(\beta,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}, && [\text{cf. } \mathbf{414.03}],
\end{aligned}\right.$$

where

$$\sin\beta=\alpha/k,\qquad w=\frac{\pi F(\beta,k)}{2K},\qquad 0<\alpha^2<k^2;$$

$$Z(\beta,k)=E(\beta,k)-\frac{E}{K}F(\beta,k).$$

1053.04

$$\left\{\begin{aligned}
&\frac{\pi}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\,\frac{\partial}{\partial w}\,\vartheta_0(i\,w)=\frac{K}{k^2-\alpha^2}-\frac{\pi\,\Lambda_0(\psi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}, && [\text{cf. } \mathbf{410.02}],\\
&\frac{\pi}{2}\,\frac{\alpha^2}{\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}\,\frac{\partial}{\partial w}\,\vartheta_1(i\,w)=\frac{k^2\,K}{k^2-\alpha^2}-\frac{\pi\,\alpha^2\,\Lambda_0(\psi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}, && [\text{cf. } \mathbf{410.01}],\\
&\frac{\pi}{2}\sqrt{\frac{\alpha^2-k^2}{\alpha^2(1-\alpha^2)}}\,\frac{\partial}{\partial w}\,\vartheta_2(i\,w)=\frac{\pi\,(k^2-\alpha^2)\,\Lambda_0(\psi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}, && [\text{cf. } \mathbf{410.04}],\\
&\frac{\pi}{2}\sqrt{\frac{1-\alpha^2}{\alpha^2(\alpha^2-k^2)}}\,\frac{\partial}{\partial w}\,\vartheta_3(i\,w)=\frac{k'^2\,K}{\alpha^2-k^2}+\frac{\pi\,(1-\alpha^2)\,\Lambda_0(\psi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}, && [\text{cf. } \mathbf{410.03}],
\end{aligned}\right.$$

where

$$\sin\psi=\sqrt{\frac{\alpha^2}{\alpha^2-k^2}},\qquad w=\frac{\pi F(\psi,k')}{2K},\qquad 0<-\alpha^2<\infty;$$

$$\Lambda_0(\psi,k)=\frac{2}{\pi}\,[E\,F(\psi,k')+K\,E(\psi,k')-K\,F(\psi,k')].$$

1053.05
$$\left\{\begin{aligned}
&-\frac{\pi}{2\alpha}\sqrt{\frac{1-\alpha^2}{\alpha^2-k^2}}\,\frac{\partial}{\partial w}\vartheta_0(i\,w)=\frac{K}{\alpha^2}+\frac{\pi(\alpha^2-1)\,\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}}\,, && \text{[cf. } \mathbf{413.03}]\\
&\frac{\pi}{2\alpha}\sqrt{\frac{\alpha^2-k^2}{1-\alpha^2}}\,\frac{\partial}{\partial w}\vartheta_1(i\,w)=\frac{k^2K}{\alpha^2}+\frac{\pi(\alpha^2-k^2)\,\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(\alpha^2-k^2)(1-\alpha^2)}}\,, && \text{[cf. } \mathbf{413.04}]\\
&\frac{\pi\alpha}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}\,\frac{\partial}{\partial w}\vartheta_2(i\,w)=\frac{\alpha\,\pi\,\Lambda_0(\xi,k)}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}\,, && \text{[cf. } \mathbf{413.01}]\\
&\frac{\pi}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\,\frac{\partial}{\partial w}\vartheta_3(i\,w)=\frac{\pi\,\Lambda_0(\xi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}-\frac{K}{\alpha^2}\,, && \text{[cf. } \mathbf{413.02}]
\end{aligned}\right.$$

where

$$\sin\xi=\sqrt{\frac{\alpha^2-k^2}{\alpha^2k'^2}}\,,\quad w=\frac{\pi F(\xi,k')}{2K}\,,\quad k^2<\alpha^2<1\,;$$

$$\Lambda_0(\xi,k)=\frac{2}{\pi}\,[E\,F(\xi,k')+K\,E(\xi,k')-K\,F(\xi,k')]\,.$$

Pseudo-elliptic Integrals.

Definition.

One frequently encounters integrals that have all the appearances of being elliptic but which can ultimately be expressed solely in terms of elementary functions. These special cases, of the form

$$P_s=\int R_1\left[t,\sqrt{a_0\,t^4+a_1\,t^3+a_2\,t^2+a_3\,t+a_4}\right]dt\,,$$

are called *pseudo-elliptic* integrals and can always be evaluated[1] by

1060.00
$$P_s=v_0(t)+\sum_{j=1}^{m}b_j\ln\,[v_j(t)]\,,$$

where each b is a constant and each $v(t)$ is an algebraic function.

---

[1] In *Oeuvres complète*, N. H. Abel showed that any algebraic function whose integral is elementary may be expressed as in **1060.00**. An equivalent theorem is attributed to Liouville.

One need not, however, be too concerned at the outset about whether or not his integral at hand is genuinely elliptic, because the substitutions given in this book for the reduction and evaluation of elliptic integrals will lead to the right results (without additional labor) even though the integral might turn out to be elementary. We give several important examples.

Examples.

The obviously pseudo-elliptic integral

$$\int_y^b \frac{t\,dt}{\sqrt{(a^2+t^2)(b^2-t^2)}},$$

for instance, is reduced by **213** to the integral

$$g\,b\int_0^{u_1} \operatorname{cn} u\,du = k\int_0^{u_1} \operatorname{cn} u\,du,$$

which, upon using **312.01**, yields the correct value

$$\cos^{-1}(\operatorname{dn} u_1) = \cos^{-1}\sqrt{1-k^2\sin^2\varphi} = \cos^{-1}\sqrt{(a^2+y^2)/(a^2+b^2)}\,.$$

This integral is a special case of the more general integral

1060.01 $$P_{s_1} = \int \frac{t\,R(t^2)\,dt}{\sqrt{a_0 t^4 + a_2 t^2 + a_4}},$$

which reduces to integrals involving odd powers of Jacobian elliptic functions and can thus always be ultimately evaluated in terms of elementary functions.

A general case of a pseudo-elliptic integral, which is less obvious than the foregoing one, is

1060.02 $$P_{s_2} = \int \frac{R(t^2)\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int \frac{R(\sin^2\vartheta)\,d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \int R(\operatorname{sn}^2 u)\,du,$$

when $R(t^2)$ is a rational function of t^2 such that for all values of t any one of the following relations holds:

1060.03 $$\begin{cases} R(t^2) + R(1/k^2t^2) = 0, \\ R(t^2) + R[(1-k^2t^2)/k^2(1-t^2)] = 0, \\ R(t^2) + R[(1-t^2)/(1-k^2t^2)] = 0. \end{cases}$$

A particular example of **1060.02** is the integral

$$\int_{y_1}^{y_2} \frac{(t_1^4 - b^2 a^2)\, dt_1}{t_1^2 \sqrt{(a^2 - t_1^2)(b^2 - t_1^2)}} = a \int_{y_1/b}^{y_2/b} \frac{(k^2 t^4 - 1)\, dt}{t^2 \sqrt{(1 - t^2)(1 - k^2 t^2)}}, \qquad [k^2 = b^2/a^2,\ a^2 > b^2 > y_2^2 > y_1^2]$$

$$= a\left[k^2 \int_{u_1}^{u_2} \operatorname{sn}^2 u\, du - \int_{u_1}^{u_2} \operatorname{ns}^2 u\, du\right]$$

$$= a\left[u - E(u) - u + E(u) + \operatorname{dn} u \operatorname{cs} u\right]_{u_1}^{u_2}$$

$$= a\left[\operatorname{dn} u_2 \operatorname{cs} u_2 - \operatorname{dn} u_1 \operatorname{cs} u_1\right]$$

$$= \sqrt{(b^2 - y_2^2)(a^2 - y_2^2)}/y_2 - \sqrt{(b^2 - y_1^2)(a^2 - y_1^2)}/y_1.$$

Written in the Jacobian forms to which they reduce, five other special cases are

1060.04
$$\begin{cases} k'^2 \int \operatorname{sc}^2 u\, du - \int \operatorname{cs}^2 u\, du = \operatorname{dn} u \operatorname{cs} u, \\ k^2 k'^2 \int \operatorname{sd}^2 u\, du + \int \operatorname{ds}^2 u\, du = -\operatorname{cd} u \operatorname{ns} u, \\ \int \operatorname{dc}^2 u\, du - k^2 \int \operatorname{cd}^2 u\, du = k'^2 \operatorname{tn} u \operatorname{nd} u, \\ \int \operatorname{dn}^2 u - k'^2 \int \operatorname{nd}^2 u\, du = k^2 \operatorname{sn} u \operatorname{cd} u, \\ k^2 \int \operatorname{cn}^2 u\, du + k'^2 \int \operatorname{nc}^2 u\, du = \operatorname{tn} u \operatorname{dn} u. \end{cases}$$

An alternate form of the pseudo-elliptic integral **1060.02** is given by

1060.05
$$P_{s_3} = \frac{1}{2} \int \frac{R(t)\, dt}{\sqrt{t(1-t)(1-k^2 t)}},$$

where $R(t)$ is a rational function of t and satisfies any one of the relations

$$R(t) + R(1/k^2 t) = 0; \quad R(t) + R[(1 - k^2 t)/k^2(1-t)] = 0;$$
$$R(t) + R[(1-t)/(1-k^2 t)] = 0.$$

A special instance of this occurred in the Introduction.

Tables of Numerical Values.

The following short, 6-place tables give values for the elliptic integrals of the first and second kind and for some functions which are useful in the numerical evaluation of elliptic integrals of the third kind. A comprehensive survey of other tables, as well as of all known errata, may be found in Alan Fletcher's *Guide to Tables of Elliptic Functions*, Mathematical Tables and Other Aids to Computation, Vol. III, No. 24, The National Research Council, Washington, D. C., 1948.

Values of the Complete Elliptic Integrals K and E, and of the Nome q.

$$K = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2\vartheta}}; \qquad E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\vartheta}\, d\vartheta; \qquad q = e^{-(\pi K'/K)}.$$

$\sin^{-1} k$	K	E	q	$\sin^{-1} k$	K	E	q
0°	1.570796	1.570796	.000000	45°	1.854075	1.350644	.043214
1°	1.570916	1.570677	.000019	46°	1.869148	1.341806	.045417
2°	1.571275	1.570318	.000076	47°	1.884809	1.332870	.047696
3°	1.571874	1.569720	.000171	48°	1.901083	1.323842	.050054
4°	1.572712	1.568884	.000305	49°	1.917998	1.314730	.052495
5°	1.573792	1.567809	.000477	50°	1.935581	1.305539	.055020
6°	1.575114	1.566497	.000687	51°	1.953865	1.296278	.057633
7°	1.576678	1.564948	.000935	52°	1.972882	1.286954	.060338
8°	1.578487	1.563162	.001222	53°	1.992670	1.277574	.063139
9°	1.580541	1.561142	.001549	54°	2.013267	1.268147	.066039
10°	1.582843	1.558887	.001914	55°	2.034715	1.258680	.069042
11°	1.585394	1.556400	.002318	56°	2.057062	1.249182	.072154
12°	1.588197	1.553681	.002762	57°	2.080358	1.239661	.075380
13°	1.591254	1.550732	.003245	58°	2.104658	1.230127	.078725
14°	1.594568	1.547555	.003769	59°	2.130021	1.220589	.082194
15°	1.598142	1.544150	.004333	60°	2.156516	1.211056	.085796
16°	1.601979	1.540522	.004938	61°	2.184213	1.201538	.089536
17°	1.606081	1.536670	.005585	62°	2.213195	1.192046	.093423
18°	1.610454	1.532597	.006272	63°	2.243549	1.182589	.097465
19°	1.615101	1.528306	.007002	64°	2.275376	1.173179	.101672
20°	1.620026	1.523799	.007775	65°	2.308787	1.163828	.106054
21°	1.625234	1.519079	.008590	66°	2.343905	1.154547	.110624
22°	1.630729	1.514147	.009449	67°	2.380870	1.145348	.115393
23°	1.636517	1.509007	.010353	68°	2.419842	1.136244	.120378
24°	1.642604	1.503662	.011301	69°	2.460999	1.127250	.125595
25°	1.648995	1.498115	.012295	70°	2.504550	1.118378	.131062
26°	1.655697	1.492369	.013335	71°	2.550731	1.109643	.136801
27°	1.662716	1.486427	.014421	72°	2.599820	1.101062	.142837
28°	1.670059	1.480293	.015556	73°	2.652138	1.092650	.149197
29°	1.677735	1.473970	.016740	74°	2.708068	1.084425	.155917
30°	1.685750	1.467462	.017972	75°	2.768063	1.076405	.163034
31°	1.694114	1.460774	.019256	76°	2.832673	1.068610	.170595
32°	1.702836	1.453908	.020591	77°	2.902565	1.061059	.178656
33°	1.711925	1.446869	.021978	78°	2.978569	1.053777	.187285
34°	1.721391	1.439662	.023419	79°	3.061729	1.046786	.196568
35°	1.731245	1.432291	.024915	80°	3.153385	1.040114	.206610
36°	1.741499	1.424760	.026467	81°	3.255303	1.033789	.217549
37°	1.752165	1.417075	.028077	82°	3.369868	1.027844	.229567
38°	1.763256	1.409240	.029745	83°	3.500422	1.022313	.242913
39°	1.774786	1.401260	.031474	84°	3.651856	1.017237	.257940
40°	1.786769	1.393140	.033265	85°	3.831742	1.012664	.275180
41°	1.799222	1.384887	.035120	86°	4.052758	1.008648	.295488
42°	1.812160	1.376504	.037040	87°	4.338654	1.005259	.320400
43°	1.825602	1.367999	.039028	88°	4.742717	1.002584	.353166
44°	1.839567	1.359377	.041085	89°	5.434910	1.000752	.403309
45°	1.854075	1.350644	.043214	90°	∞	1.000000	1.000000

Values of the Complete Elliptic Integrals K, K', E, E' and the Nomes q and q'.

$$K=\int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}},\quad K'=K(k');\quad E=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\vartheta}\,d\vartheta,\quad E'=E(k')$$

$$q=e^{-\pi K'/K};\quad q'=q(k').$$

k^2	K	K'	E	E'	q	q'	k'^2
0.00	1.570796	∞	1.570796	1.000000	.000000	1.000000	1.00
.01	1.574746	3.695637	1.566862	1.015994	.000628	.262196	.99
.02	1.578740	3.354141	1.562913	1.028595	.001263	.227935	.98
.03	1.582780	3.155875	1.558948	1.039947	.001904	.206880	.97
.04	1.586868	3.016112	1.554969	1.050502	.002551	.191496	.96
.05	1.591003	2.908337	1.550973	1.060474	.003206	.179316	.95
.06	1.595188	2.820752	1.546963	1.069986	.003867	.169208	.94
.07	1.599423	2.747073	1.542936	1.079121	.004536	.160554	.93
.08	1.603710	2.683551	1.538893	1.087938	.005211	.152981	.92
.09	1.608049	2.627773	1.534833	1.096478	.005894	.146244	.91
.10	1.612441	2.578092	1.530758	1.104775	.006585	.140173	.90
.11	1.616889	2.533335	1.526665	1.112856	.007283	.134646	.89
.12	1.621393	2.492635	1.522555	1.120742	.007989	.129571	.88
.13	1.625955	2.455338	1.518428	1.128451	.008703	.124880	.87
.14	1.630576	2.420933	1.514284	1.135998	.009425	.120517	.86
.15	1.635257	2.389016	1.510122	1.143396	.010156	.116439	.85
.16	1.640000	2.359264	1.505942	1.150656	.010895	.112610	.84
.17	1.644806	2.331409	1.501743	1.157787	.011643	.109002	.83
.18	1.649678	2.305232	1.497526	1.164798	.012401	.105589	.82
.19	1.654617	2.280549	1.493290	1.171697	.013167	.102352	.81
.20	1.659624	2.257205	1.489035	1.178490	.013943	.099274	.80
.21	1.664701	2.235068	1.484761	1.185183	.014728	.096338	.79
.22	1.669850	2.214022	1.480466	1.191781	.015524	.093533	.78
.23	1.675073	2.193971	1.476152	1.198290	.016329	.090848	.77
.24	1.680373	2.174827	1.471818	1.204714	.017146	.088271	.76
.25	1.685750	2.156516	1.467462	1.211056	.017972	.085796	.75
.26	1.691208	2.138970	1.463086	1.217321	.018810	.083413	.74
.27	1.696749	2.122132	1.458688	1.223512	.019659	.081117	.73
.28	1.702374	2.105948	1.454269	1.229632	.020520	.078902	.72
.29	1.708087	2.090373	1.449827	1.235684	.021393	.076761	.71
.30	1.713889	2.075363	1.445363	1.241671	.022277	.074690	.70
.31	1.719785	2.060882	1.440876	1.247595	.023175	.072685	.69
.32	1.725776	2.046894	1.436366	1.253458	.024085	.070741	.68
.33	1.731865	2.033369	1.431832	1.259263	.025009	.068854	.67
.34	1.738055	2.020279	1.427274	1.265013	.025946	.067023	.66
.35	1.744351	2.007598	1.422691	1.270707	.026898	.065242	.65
.36	1.750754	1.995303	1.418083	1.276350	.027864	.063510	.64
.37	1.757269	1.983371	1.413450	1.281942	.028845	.061825	.63
.38	1.763898	1.971783	1.408791	1.287484	.029842	.060182	.62
.39	1.770647	1.960521	1.404105	1.292979	.030854	.058582	.61
.40	1.777519	1.949568	1.399392	1.298428	.031883	.057020	.60
.41	1.784519	1.938908	1.394652	1.303832	.032929	.055496	.59
.42	1.791650	1.928526	1.389883	1.309192	.033993	.054008	.58
.43	1.798918	1.918410	1.385086	1.314511	.035075	.052554	.57
.44	1.806328	1.908547	1.380259	1.319788	.036175	.051133	.56
.45	1.813884	1.898925	1.375402	1.325024	.037296	.049742	.55
.46	1.821593	1.889533	1.370515	1.330223	.038436	.048382	.54
.47	1.829460	1.880361	1.365596	1.335382	.039597	.047050	.53
.48	1.837491	1.871400	1.360645	1.340505	.040780	.045745	.52
.49	1.845694	1.862641	1.355661	1.345592	.041985	.044467	.51
.50	1.854075	1.854075	1.350644	1.350644	.043214	.043214	.50
k'^2	K'	K	E'	E	q'	q	k^2

Values of the Incomplete Elliptic Integral of the First Kind, $F(\varphi, k)$.

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} .$$

$\sin^{-1} k$	φ 0°	5°	10°	15°	20°	25°	30°
0°	0.000000	0.087266	0.174533	0.261799	0.349066	0.436332	0.523599
1°	.000000	.087266	.174533	.261800	.349068	.436336	.523606
2°	.000000	.087267	.174534	.261803	.349074	.436349	.523626
3°	.000000	.087267	.174535	.261807	.349085	.436369	.523661
4°	.000000	.087267	.174537	.261814	.349100	.436397	.523709
5°	.000000	.087267	.174540	.261822	.349118	.436434	.523771
6°	.000000	.087268	.174543	.261832	.349141	.436478	.523847
7°	.000000	.087268	.174546	.261843	.349169	.436530	.523936
8°	.000000	.087269	.174550	.261857	.349200	.436591	.524038
9°	.000000	.087269	.174554	.261872	.349235	.436659	.524155
10°	.000000	.087270	.174559	.261888	.349275	.436735	.524284
11°	.000000	.087270	.174565	.261907	.349318	.436819	.524427
12°	.000000	.087271	.174571	.261927	.349366	.436910	.524583
13°	.000000	.087272	.174578	.261949	.349417	.437010	.524751
14°	.000000	.087273	.174585	.261972	.349472	.437116	.524933
15°	.000000	.087274	.174592	.261997	.349531	.437230	.525128
16°	.000000	.087275	.174600	.262024	.349594	.437351	.525334
17°	.000000	.087276	.174608	.262052	.349660	.437480	.525554
18°	.000000	.087277	.174617	.262082	.349730	.437615	.525785
19°	.000000	.087278	.174626	.262113	.349803	.437757	.526029
20°	.000000	.087279	.174636	.262146	.349880	.437906	.526284
21°	.000000	.087281	.174646	.262180	.349960	.438062	.526551
22°	.000000	.087282	.174657	.262215	.350044	.438224	.526829
23°	.000000	.087283	.174668	.262252	.350131	.438393	.527118
24°	.000000	.087285	.174679	.262290	.350220	.438567	.527418
25°	.000000	.087286	.174691	.262329	.350313	.438748	.527728
26°	.000000	.087288	.174703	.262370	.350409	.438934	.528049
27°	.000000	.087289	.174715	.262411	.350508	.439126	.528380
28°	.000000	.087291	.174728	.262454	.350609	.439324	.528720
29°	.000000	.087292	.174741	.262498	.350712	.439526	.529070
30°	.000000	.087294	.174754	.262542	.350819	.439734	.529429
31°	.000000	.087296	.174767	.262588	.350927	.439946	.529796
32°	.000000	.087298	.174781	.262635	.351038	.440163	.530172
33°	.000000	.087299	.174795	.262682	.351151	.440384	.530555
34°	.000000	.087301	.174809	.262731	.351266	.440609	.530946
35°	.000000	.087303	.174824	.262780	.351382	.440838	.531344
36°	.000000	.087305	.174839	.262829	.351501	.441071	.531749
37°	.000000	.087307	.174853	.262880	.351621	.441307	.532160
38°	.000000	.087308	.174868	.262931	.351742	.441546	.532577
39°	.000000	.087310	.174884	.262982	.351865	.441788	.533000
40°	.000000	.087312	.174899	.263034	.351989	.442032	.533427
41°	.000000	.087314	.174914	.263086	.352114	.442279	.533859
42°	.000000	.087316	.174930	.263138	.352239	.442528	.534295
43°	.000000	.087318	.174945	.263191	.352366	.442778	.534735
44°	.000000	.087320	.174961	.263244	.352493	.443030	.535178
45°	.000000	.087322	.174976	.263297	.352620	.443282	.535623
$\sin^{-1} k$	0°	5°	10°	15°	20°	25°	30° φ

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}.$$

$\sin^{-1} k$	φ = 0°	5°	10°	15°	20°	25°	30°
45°	0.000000	0.087322	0.174976	0.263297	0.352620	0.443282	0.535623
46°	.000000	.087324	.174992	.263350	.352747	.443536	.536070
47°	.000000	.087326	.175007	.263403	.352875	.443790	.536518
48°	.000000	.087328	.175023	.263456	.353003	.444044	.536968
49°	.000000	.087330	.175038	.263509	.353130	.444298	.537418
50°	.000000	.087331	.175054	.263562	.353257	.444552	.537868
51°	.000000	.087333	.175069	.263614	.353384	.444804	.538317
52°	.000000	.087335	.175084	.263666	.353510	.445056	.538764
53°	.000000	.087337	.175100	.263718	.353635	.445306	.539210
54°	.000000	.087339	.175115	.263769	.353758	.445555	.539654
55°	.000000	.087341	.175129	.263820	.353881	.445801	.540094
56°	.000000	.087343	.175144	.263870	.354003	.446045	.540531
57°	.000000	.087344	.175158	.263920	.354123	.446287	.540963
58°	.000000	.087346	.175173	.263968	.354241	.446525	.541391
59°	.000000	.087348	.175187	.264016	.354358	.446760	.541813
60°	.000000	.087350	.175200	.264064	.354472	.446991	.542229
61°	.000000	.087351	.175214	.264110	.354585	.447218	.542639
62°	.000000	.087353	.175227	.264155	.354695	.447442	.543041
63°	.000000	.087355	.175240	.264199	.354803	.447660	.543436
64°	.000000	.087356	.175252	.264243	.354908	.447873	.543822
65°	.000000	.087358	.175265	.264285	.355011	.448082	.544199
66°	.000000	.087359	.175276	.264326	.355111	.448285	.544567
67°	.000000	.087360	.175288	.264365	.355208	.448482	.544925
68°	.000000	.087362	.175299	.264404	.355302	.448673	.545272
69°	.000000	.087363	.175310	.264441	.355392	.448857	.545608
70°	.000000	.087364	.175320	.264476	.355480	.449035	.545932
71°	.000000	.087366	.175330	.264511	.355563	.449206	.546244
72°	.000000	.087367	.175339	.264543	.355644	.449370	.546543
73°	.000000	.087368	.175349	.264575	.355720	.449526	.546829
74°	.000000	.087369	.175357	.264604	.355793	.449675	.547102
75°	.000000	.087370	.175365	.264632	.355862	.449816	.547360
76°	.000000	.087371	.175373	.264659	.355927	.449949	.547604
77°	.000000	.087372	.175380	.264684	.355988	.450074	.547832
78°	.000000	.087373	.175387	.264707	.356045	.450190	.548046
79°	.000000	.087373	.175393	.264728	.356097	.450298	.548244
80°	.000000	.087374	.175399	.264748	.356146	.450397	.548425
81°	.000000	.087375	.175404	.264765	.356189	.450487	.548591
82°	.000000	.087375	.175408	.264781	.356229	.450568	.548739
83°	.000000	.087376	.175412	.264796	.356264	.450639	.548871
84°	.000000	.087376	.175416	.264808	.356294	.450702	.548986
85°	.000000	.087377	.175419	.264818	.356320	.450755	.549084
86°	.000000	.087377	.175421	.264827	.356341	.450798	.549163
87°	.000000	.087377	.175423	.264834	.356357	.450832	.549226
88°	.000000	.087377	.175425	.264838	.356369	.450856	.549270
89°	.000000	.087377	.175426	.264841	.356376	.450870	.549297
90°	.000000	.087377	.175426	.264842	.356379	.450875	.549306
$\sin^{-1} k$	φ = 0°	5°	10°	15°	20°	25°	30°

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}.$$

$\sin^{-1} k$	φ 30°	35°	40°	45°	50°	55°	60°
0°	0.523599	0.610865	0.698132	0.785398	0.872665	0.959931	1.047198
1°	.523606	.610876	.698147	.785420	.872694	.959968	1.047244
2°	.523626	.610908	.698194	.785485	.872780	.960080	1.047385
3°	.523661	.610962	.698273	.785594	.872925	.960267	1.047619
4°	.523709	.611037	.698382	.785746	.873128	.960528	1.047946
5°	.523771	.611133	.698523	.785941	.873388	.960864	1.048367
6°	.523847	.611251	.698695	.786180	.873706	.961275	1.048882
7°	.523936	.611390	.698898	.786462	.874082	.961760	1.049490
8°	.524038	.611550	.699132	.786786	.874516	.962319	1.050193
9°	.524155	.611731	.699396	.787154	.875007	.962953	1.050989
10°	.524284	.611933	.699692	.787565	.875555	.963662	1.051879
11°	.524427	.612156	.700017	.788018	.876161	.964445	1.052863
12°	.524583	.612399	.700374	.788514	.876824	.965302	1.053942
13°	.524751	.612663	.700760	.789052	.877544	.966234	1.055114
14°	.524933	.612947	.701176	.789632	.878321	.967240	1.056381
15°	.525128	.613251	.701622	.790254	.879154	.968320	1.057742
16°	.525334	.613575	.702097	.790918	.880044	.969474	1.059198
17°	.525554	.613919	.702602	.791622	.880990	.970703	1.060749
18°	.525785	.614281	.703135	.792368	.881992	.972005	1.062394
19°	.526029	.614663	.703697	.793155	.883049	.973380	1.064134
20°	.526284	.615064	.704287	.793981	.884162	.974830	1.065969
21°	.526551	.615483	.704905	.794848	.885330	.976352	1.067899
22°	.526829	.615921	.705550	.795754	.886553	.977948	1.069924
23°	.527118	.616376	.706223	.796699	.887829	.979617	1.072044
24°	.527418	.616849	.706922	.797683	.889160	.981358	1.074260
25°	.527728	.617339	.707647	.798705	.890544	.983171	1.076570
26°	.528049	.617845	.708398	.799765	.891981	.985057	1.078976
27°	.528380	.618368	.709174	.800861	.893470	.987014	1.081477
28°	.528720	.618907	.709975	.801994	.895011	.989042	1.084073
29°	.529070	.619461	.710799	.803162	.896603	.991141	1.086765
30°	.529429	.620030	.711647	.804366	.898245	.993311	1.089551
31°	.529796	.620614	.712518	.805604	.899937	.995550	1.092431
32°	.530172	.621211	.713411	.806876	.901679	.997857	1.095407
33°	.530555	.621822	.714326	.808180	.903468	1.000234	1.098476
34°	.530946	.622446	.715261	.809516	.905304	1.002677	1.101639
35°	.531344	.623082	.716216	.810883	.907187	1.005188	1.104895
36°	.531749	.623730	.717191	.812280	.909115	1.007764	1.108245
37°	.532160	.624389	.718183	.813706	.911087	1.010406	1.111686
38°	.532577	.625058	.719193	.815160	.913101	1.013110	1.115219
39°	.533000	.625737	.720220	.816641	.915158	1.015878	1.118843
40°	.533427	.626426	.721262	.818148	.917255	1.018706	1.122557
41°	.533859	.627122	.722319	.819678	.919391	1.021595	1.126359
42°	.534295	.627826	.723390	.821232	.921564	1.024541	1.130249
43°	.534735	.628537	.724473	.822808	.923773	1.027544	1.134225
44°	.535178	.629254	.725567	.824403	.926015	1.030602	1.138285
45°	.535623	.629977	.726672	.826018	.928290	1.033713	1.142429
$\sin^{-1} k$	30°	35°	40°	45°	50°	55°	60°
	φ						

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}.$$

$\sin^{-1} k$	φ 30°	35°	40°	45°	50°	55°	60°
45°	0.535623	0.629977	0.726672	0.826018	0.928290	1.033713	1.142429
46°	.536070	.630704	.727786	.827649	.930596	1.036874	1.146654
47°	.536518	.631434	.728908	.829296	.932929	1.040084	1.150957
48°	.536968	.632168	.730036	.830957	.935288	1.043339	1.155337
49°	.537418	.632903	.731170	.832630	.937671	1.046638	1.159791
50°	.537868	.633639	.732308	.834312	.940076	1.049977	1.164316
51°	.538317	.634376	.733448	.836003	.942499	1.053354	1.168909
52°	.538764	.635111	.734590	.837700	.944938	1.056764	1.173567
53°	.539210	.635845	.735731	.839401	.947390	1.060205	1.178284
54°	.539654	.636576	.736870	.841103	.949852	1.063672	1.183058
55°	.540094	.637304	.738006	.842805	.952321	1.067163	1.187884
56°	.540531	.638026	.739138	.844505	.954794	1.070671	1.192756
57°	.540963	.638743	.740262	.846199	.957267	1.074194	1.197670
58°	.541391	.639453	.741379	.847885	.959737	1.077725	1.202619
59°	.541813	.640156	.742485	.849561	.962200	1.081261	1.207597
60°	.542229	.640849	.743581	.851224	.964652	1.084794	1.212597
61°	.542639	.641533	.744663	.852871	.967089	1.088321	1.217611
62°	.543041	.642206	.745730	.854500	.969506	1.091834	1.222631
63°	.543436	.642867	.746781	.856108	.971901	1.095328	1.227650
64°	.543822	.643515	.747813	.857692	.974268	1.098796	1.232657
65°	.544199	.644149	.748825	.859249	.976602	1.102231	1.237642
66°	.544567	.644768	.749815	.860777	.978899	1.105625	1.242596
67°	.544925	.645371	.750781	.862271	.981155	1.108972	1.247506
68°	.545272	.645958	.751722	.863731	.983364	1.112264	1.252362
69°	.545608	.646526	.752636	.865151	.985522	1.115492	1.257151
70°	.545932	.647075	.753521	.866530	.987623	1.118649	1.261860
71°	.546244	.647604	.754375	.867864	.989662	1.121726	1.266475
72°	.546543	.648112	.755197	.869151	.991635	1.124715	1.270982
73°	.546829	.648598	.755985	.870388	.993537	1.127607	1.275367
74°	.547102	.649062	.756738	.871572	.995362	1.130394	1.279615
75°	.547360	.649502	.757454	.872699	.997105	1.133066	1.283710
76°	.547604	.649918	.758131	.873768	.998763	1.135616	1.287637
77°	.547832	.650309	.758768	.874776	1.000329	1.138034	1.291380
78°	.548046	.650674	.759364	.875720	1.001801	1.140313	1.294924
79°	.548244	.651013	.759917	.876599	1.003172	1.142444	1.298254
80°	.548425	.651324	.760426	.877408	1.004439	1.144419	1.301353
81°	.548591	.651607	.760891	.878148	1.005599	1.146231	1.304208
82°	.548739	.651863	.761309	.878815	1.006647	1.147873	1.306805
83°	.548871	.652089	.761681	.879408	1.007580	1.149338	1.309130
84°	.548986	.652286	.762005	.879925	1.008395	1.150620	1.311172
85°	.549084	.652454	.762280	.880365	1.009089	1.151715	1.312919
86°	.549163	.652591	.762506	.880727	1.009660	1.152617	1.314362
87°	.549226	.652698	.762682	.881009	1.010107	1.153322	1.315493
88°	.549270	.652775	.762808	.881211	1.010427	1.153828	1.316305
89°	.549297	.652821	.762884	.881333	1.010619	1.154133	1.316794
90°	.549306	.652837	.762910	.881374	1.010683	1.154235	1.316958
$\sin^{-1} k$	30°	35°	40°	45°	50°	55°	60° φ

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}.$$

$\sin^{-1} k$	φ						
	60°	65°	70°	75°	80°	85°	90°
0°	1.047198	1.134464	1.221730	1.308997	1.396263	1.483530	1.570796
1°	1.047244	1.134521	1.221799	1.309078	1.396357	1.483636	1.570916
2°	1.047385	1.134693	1.222005	1.309320	1.396637	1.483955	1.571275
3°	1.047619	1.134979	1.222348	1.309723	1.397104	1.484488	1.571874
4°	1.047946	1.135380	1.222828	1.310288	1.397758	1.485233	1.572712
5°	1.048367	1.135895	1.223446	1.311015	1.398599	1.486193	1.573792
6°	1.048882	1.136526	1.224202	1.311905	1.399629	1.487368	1.575114
7°	1.049490	1.137271	1.225096	1.312957	1.400848	1.488758	1.576678
8°	1.050193	1.138132	1.226128	1.314173	1.402256	1.490365	1.578487
9°	1.050989	1.139107	1.227299	1.315553	1.403855	1.492189	1.580541
10°	1.051879	1.140199	1.228610	1.317098	1.405645	1.494234	1.582843
11°	1.052863	1.141407	1.230061	1.318808	1.407628	1.496499	1.585394
12°	1.053942	1.142730	1.231652	1.320685	1.409806	1.498986	1.588197
13°	1.055114	1.144171	1.233384	1.322730	1.412179	1.501699	1.591254
14°	1.056381	1.145728	1.235258	1.324943	1.414749	1.504637	1.594568
15°	1.057742	1.147402	1.237275	1.327326	1.417518	1.507805	1.598142
16°	1.059198	1.149195	1.239435	1.329880	1.420487	1.511205	1.601979
17°	1.060749	1.151105	1.241739	1.332607	1.423660	1.514838	1.606081
18°	1.062394	1.153134	1.244188	1.335508	1.427037	1.518709	1.610454
19°	1.064134	1.155282	1.246784	1.338585	1.430622	1.522820	1.615101
20°	1.065969	1.157550	1.249526	1.341839	1.434416	1.527174	1.620026
21°	1.067899	1.159937	1.252417	1.345272	1.438422	1.531776	1.625234
22°	1.069924	1.162445	1.255457	1.348886	1.442644	1.536629	1.630729
23°	1.072044	1.165075	1.258647	1.352683	1.447084	1.541736	1.636517
24°	1.074260	1.167825	1.261990	1.356665	1.451745	1.547103	1.642604
25°	1.076570	1.170698	1.265485	1.360835	1.456630	1.552734	1.648995
26°	1.078976	1.173694	1.269134	1.365194	1.461744	1.558633	1.655697
27°	1.081477	1.176812	1.272939	1.369744	1.467089	1.568407	1.662716
28°	1.084073	1.180055	1.276900	1.374490	1.472670	1.571259	1.670059
29°	1.086765	1.183421	1.281021	1.379432	1.478490	1.577997	1.677735
30°	1.089551	1.186913	1.285301	1.384575	1.484555	1.585026	1.685750
31°	1.092431	1.190529	1.289742	1.389920	1.490868	1.592353	1.694114
32°	1.095407	1.194272	1.294346	1.395470	1.497434	1.599984	1.702836
33°	1.098476	1.198140	1.299115	1.401229	1.504258	1.607927	1.711925
34°	1.101639	1.202135	1.304049	1.407201	1.511346	1.616189	1.721391
35°	1.104895	1.206257	1.309151	1.413387	1.518703	1.624779	1.731245
36°	1.108245	1.210505	1.314422	1.419792	1.526335	1.633704	1.741499
37°	1.111686	1.214882	1.319864	1.426419	1.534248	1.642974	1.752165
38°	1.115219	1.219385	1.325478	1.433272	1.542447	1.652599	1.763256
39°	1.118843	1.224016	1.331265	1.440354	1.550941	1.662588	1.774786
40°	1.122557	1.228775	1.337228	1.447669	1.559734	1.672952	1.786769
41°	1.126359	1.233661	1.343368	1.455222	1.568836	1.683703	1.799222
42°	1.130249	1.238674	1.349685	1.463016	1.578253	1.694852	1.812160
43°	1.134225	1.243813	1.356182	1.471055	1.587993	1.706411	1.825602
44°	1.138285	1.249079	1.362860	1.479343	1.598065	1.718395	1.839567
45°	1.142429	1.254470	1.369719	1.487885	1.608477	1.730817	1.854075
$\sin^{-1} k$	60°	65°	70°	75°	80°	85°	90°
	φ						

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}.$$

$\sin^{-1} k$	φ = 60°	65°	70°	75°	80°	85°	90°
45°	1.142429	1.254470	1.369719	1.487885	1.608477	1.730817	1.854075
46°	1.146654	1.259985	1.376761	1.496684	1.619238	1.743693	1.869148
47°	1.150957	1.265623	1.383987	1.505746	1.630357	1.757037	1.884809
48°	1.155337	1.271382	1.391396	1.515074	1.641845	1.770868	1.901083
49°	1.159791	1.277261	1.398990	1.524673	1.653710	1.785203	1.917998
50°	1.164316	1.283258	1.406769	1.534546	1.665965	1.800062	1.935581
51°	1.168909	1.289370	1.414731	1.544698	1.678620	1.815464	1.953865
52°	1.173567	1.295594	1.422877	1.555134	1.691687	1.831431	1.972882
53°	1.178284	1.301927	1.431206	1.565855	1.705176	1.847986	1.992670
54°	1.183058	1.308366	1.439716	1.576867	1.719101	1.865154	2.013267
55°	1.187884	1.314906	1.448404	1.588172	1.733474	1.882961	2.034715
56°	1.192756	1.321541	1.457269	1.599774	1.748309	1.901436	2.057062
57°	1.197670	1.328268	1.466307	1.611674	1.763618	1.920608	2.080358
58°	1.202619	1.335079	1.475514	1.623874	1.779415	1.940509	2.104658
59°	1.207597	1.341968	1.484884	1.636376	1.795714	1.961174	2.130021
60°	1.212597	1.348926	1.494411	1.649179	1.812530	1.982640	2.156516
61°	1.217611	1.355946	1.504088	1.662282	1.829875	2.004946	2.184213
62°	1.222631	1.363018	1.513906	1.675684	1.847765	2.028136	2.213195
63°	1.227650	1.370131	1.523855	1.689379	1.866214	2.052255	2.243549
64°	1.232657	1.377273	1.533923	1.703364	1.885233	2.077352	2.275376
65°	1.237642	1.384432	1.544097	1.717629	1.904837	2.103482	2.308787
66°	1.242596	1.391594	1.554360	1.732165	1.925035	2.130701	2.343905
67°	1.247506	1.398743	1.564694	1.746958	1.945838	2.159071	2.380870
68°	1.252362	1.405862	1.575079	1.761991	1.967252	2.188658	2.419842
69°	1.257151	1.412934	1.585492	1.777243	1.989282	2.219536	2.460999
70°	1.261860	1.419938	1.595906	1.792687	2.011928	2.251780	2.504550
71°	1.266475	1.426854	1.606292	1.808293	2.035183	2.285473	2.550731
72°	1.270982	1.433659	1.616616	1.824023	2.059036	2.320704	2.599820
73°	1.275367	1.440330	1.626843	1.839830	2.083464	2.357566	2.652138
74°	1.279615	1.446840	1.636931	1.855662	2.108433	2.396156	2.708068
75°	1.283710	1.453164	1.646837	1.871454	2.133895	2.436576	2.768063
76°	1.287637	1.459273	1.656512	1.887133	2.159783	2.478927	2.832673
77°	1.291380	1.465138	1.665905	1.902614	2.186005	2.523307	2.902565
78°	1.294924	1.470732	1.674959	1.917798	2.212440	2.569803	2.978569
79°	1.298254	1.476022	1.683616	1.932576	2.238930	2.618477	3.061729
80°	1.301353	1.480980	1.691815	1.946822	2.265273	2.669350	3.153385
81°	1.304208	1.485576	1.699492	1.960402	2.291215	2.722372	3.255303
82°	1.306805	1.489780	1.706585	1.973167	2.316439	2.777367	3.369868
83°	1.309130	1.493565	1.713028	1.984960	2.340563	2.833963	3.500422
84°	1.311172	1.496904	1.718760	1.995621	2.363137	2.891467	3.651856
85°	1.312919	1.499774	1.723724	2.004988	2.383647	2.948689	3.831742
86°	1.314362	1.502153	1.727865	2.012905	2.401534	3.003709	4.052758
87°	1.315493	1.504024	1.731138	2.019229	2.416224	3.053629	4.338654
88°	1.316305	1.505370	1.733505	2.023841	2.427180	3.094489	4.742717
89°	1.316794	1.506183	1.734936	2.026647	2.433953	3.121698	5.434910
90°	1.316958	1.506454	1.735415	2.027589	2.436246	3.131301	∞
$\sin^{-1} k$	φ = 60°	65°	70°	75°	80°	85°	90°

Values of the Incomplete Elliptic Integral of the Second Kind, $E(\varphi,k)$.

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \vartheta}\, d\,\vartheta.$$

$\sin^{-1} k$ \ φ	0°	5°	10°	15°	20°	25°	30°
0°	0.000000	0.087266	0.174533	0.261799	0.349066	0.436332	0.523599
1°	.000000	.087266	.174533	.261798	.349064	.436328	.523592
2°	.000000	.087266	.174532	.261796	.349057	.436316	.523571
3°	.000000	.087266	.174531	.261791	.349047	.436296	.523537
4°	.000000	.087266	.174529	.261785	.349032	.436267	.523489
5°	.000000	.087266	.174526	.261777	.349013	.436231	.523427
6°	.000000	.087265	.174523	.261767	.348990	.436187	.523351
7°	.000000	.087265	.174520	.261756	.348963	.436134	.523262
8°	.000000	.087264	.174516	.261742	.348932	.436074	.523160
9°	.000000	.087264	.174511	.261727	.348896	.436006	.523044
10°	.000000	.087263	.174506	.261710	.348857	.435930	.522915
11°	.000000	.087262	.174501	.261692	.348814	.435847	.522773
12°	.000000	.087262	.174495	.261672	.348767	.435756	.522618
13°	.000000	.087261	.174488	.261650	.348715	.435657	.522451
14°	.000000	.087260	.174481	.261627	.348661	.435551	.522270
15°	.000000	.087259	.174474	.261602	.348602	.435438	.522078
16°	.000000	.087258	.174466	.261575	.348540	.435318	.521873
17°	.000000	.087257	.174458	.261547	.348474	.435190	.521657
18°	.000000	.087256	.174449	.261517	.348404	.435056	.521428
19°	.000000	.087255	.174440	.261486	.348331	.434916	.521189
20°	.000000	.087254	.174430	.261454	.348255	.434768	.520938
21°	.000000	.087252	.174420	.261420	.348175	.434615	.520676
22°	.000000	.087251	.174409	.261385	.348093	.434455	.520404
23°	.000000	.087250	.174398	.261348	.348007	.434289	.520121
24°	.000000	.087248	.174387	.261311	.347918	.434117	.519828
25°	.000000	.087247	.174375	.261272	.347826	.433940	.519526
26°	.000000	.087245	.174364	.261231	.347732	.433758	.519214
27°	.000000	.087244	.174351	.261190	.347635	.433570	.518894
28°	.000000	.087242	.174339	.261148	.347535	.433377	.518565
29°	.000000	.087240	.174326	.261104	.347433	.433180	.518227
30°	.000000	.087239	.174312	.261060	.347329	.432977	.517882
31°	.000000	.087237	.174299	.261015	.347222	.432771	.517529
32°	.000000	.087235	.174285	.260969	.347113	.432561	.517169
33°	.000000	.087234	.174271	.260922	.347003	.432347	.516803
34°	.000000	.087232	.174257	.260874	.346890	.432129	.516430
35°	.000000	.087230	.174243	.260826	.346776	.431908	.516052
36°	.000000	.087228	.174228	.260777	.346661	.431684	.515668
37°	.000000	.087226	.174213	.260727	.346544	.431457	.515280
38°	.000000	.087225	.174199	.260677	.346426	.431227	.514886
39°	.000000	.087223	.174183	.260626	.346307	.430996	.514489
40°	.000000	.087221	.174168	.260575	.346186	.430762	.514089
41°	.000000	.087219	.174153	.260524	.346065	.430527	.513685
42°	.000000	.087217	.174138	.260473	.345943	.430291	.513279
43°	.000000	.087215	.174122	.260421	.345821	.430053	.512870
44°	.000000	.087213	.174107	.260369	.345699	.429814	.512460
45°	.000000	.087211	.174092	.260317	.345576	.429575	.512049
$\sin^{-1} k$ \ φ	0°	5°	10°	15°	20°	25°	30°

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta.$$

$\sin^{-1} k$	φ = 0°	5°	10°	15°	20°	25°	30°
45°	0.000000	0.087211	0.174092	0.260317	0.345576	0.429575	0.512049
46°	.000000	.087209	.174076	.260265	.345453	.429336	.511638
47°	.000000	.087207	.174061	.260213	.345330	.429097	.511226
48°	.000000	.087205	.174045	.260161	.345207	.428858	.510815
49°	.000000	.087203	.174030	.260109	.345085	.428619	.510404
50°	.000000	.087202	.174015	.260058	.344963	.428382	.509995
51°	.000000	.087200	.174000	.260007	.344842	.428146	.509588
52°	.000000	.087198	.173984	.259956	.344722	.427911	.509183
53°	.000000	.087196	.173970	.259906	.344602	.427679	.508781
54°	.000000	.087194	.173955	.259856	.344484	.427448	.508383
55°	.000000	.087192	.173940	.259806	.344367	.427219	.507988
56°	.000000	.087190	.173926	.259758	.344252	.426994	.507598
57°	.000000	.087189	.173911	.259710	.344138	.426771	.507213
58°	.000000	.087187	.173897	.259662	.344025	.426551	.506833
59°	.000000	.087185	.173884	.259616	.343915	.426335	.506460
60°	.000000	.087183	.173870	.259570	.343806	.426123	.506092
61°	.000000	.087182	.173857	.259525	.343700	.425915	.505731
62°	.000000	.087180	.173844	.259481	.343596	.425711	.505378
63°	.000000	.087179	.173831	.259438	.343494	.425512	.505033
64°	.000000	.087177	.173819	.259397	.343395	.425317	.504695
65°	.000000	.087176	.173807	.259356	.343298	.425128	.504367
66°	.000000	.087174	.173795	.259316	.343204	.424944	.504047
67°	.000000	.087173	.173784	.259278	.343113	.424765	.503737
68°	.000000	.087171	.173773	.259241	.343025	.424592	.503437
69°	.000000	.087170	.173762	.259205	.342940	.424425	.503147
70°	.000000	.087169	.173752	.259171	.342858	.424265	.502868
71°	.000000	.087167	.173742	.259138	.342780	.424111	.502600
72°	.000000	.087166	.173733	.259106	.342704	.423963	.502344
73°	.000000	.087165	.173724	.259076	.342633	.423823	.502099
74°	.000000	.087164	.173716	.259048	.342565	.423689	.501866
75°	.000000	.087163	.173708	.259021	.342500	.423563	.501646
76°	.000000	.087162	.173700	.258995	.342440	.423444	.501439
77°	.000000	.087161	.173693	.258971	.342383	.423332	.501245
78°	.000000	.087161	.173687	.258949	.342330	.423228	.501064
79°	.000000	.087160	.173681	.258929	.342281	.423132	.500896
80°	.000000	.087159	.173675	.258910	.342236	.423044	.500742
81°	.000000	.087158	.173670	.258893	.342196	.422964	.500603
82°	.000000	.087158	.173665	.258877	.342159	.422892	.500477
83°	.000000	.087157	.173661	.258864	.342127	.422828	.500366
84°	.000000	.087157	.173658	.258852	.342099	.422773	.500269
85°	.000000	.087157	.173655	.258842	.342075	.422726	.500187
86°	.000000	.087156	.173653	.258834	.342055	.422687	.500120
87°	.000000	.087156	.173651	.258827	.342040	.422657	.500068
88°	.000000	.087156	.173649	.258823	.342029	.422635	.500030
89°	.000000	.087156	.173648	.258820	.342022	.422623	.500008
90°	.000000	.087156	.173648	.258819	.342020	.422618	.500000
$\sin^{-1} k$	φ = 0°	5°	10°	15°	20°	25°	30°

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta.$$

$\sin^{-1} k$	φ 30°	35°	40°	45°	50°	55°	60°
0°	0.523599	0.610865	0.698132	0.785398	0.872665	0.959931	1.047198
1°	.523592	.610854	.698116	.785376	.872636	.959894	1.047151
2°	.523571	.610822	.698069	.785311	.872549	.959782	1.047011
3°	.523537	.610769	.697991	.785203	.872404	.959595	1.046777
4°	.523489	.610694	.697881	.785051	.872202	.959335	1.046450
5°	.523427	.610597	.697741	.784856	.871942	.959000	1.046030
6°	.523351	.610480	.697569	.784618	.871625	.958591	1.045518
7°	.523262	.610341	.697367	.784337	.871251	.958108	1.044913
8°	.523160	.610182	.697134	.784014	.870820	.957553	1.044216
9°	.523044	.610001	.696871	.783649	.870333	.956925	1.043429
10°	.522915	.609801	.696578	.783242	.869790	.956225	1.042550
11°	.522773	.609579	.696255	.782793	.869192	.955453	1.041582
12°	.522618	.609338	.695902	.782303	.868539	.954610	1.040525
13°	.522451	.609077	.695521	.781773	.867831	.953697	1.039379
14°	.522270	.608796	.695110	.781203	.867070	.952715	1.038146
15°	.522078	.608496	.694672	.780593	.866256	.951664	1.036827
16°	.521873	.608176	.694205	.779945	.865390	.950545	1.035422
17°	.521657	.607838	.693711	.779258	.864473	.949360	1.033933
18°	.521428	.607482	.693190	.778533	.863505	.948109	1.032360
19°	.521189	.607108	.692643	.777771	.862487	.946793	1.030707
20°	.520938	.606717	.692070	.776974	.861421	.945414	1.028972
21°	.520676	.606308	.691471	.776141	.860306	.943972	1.027159
22°	.520404	.605882	.690848	.775273	.859145	.942470	1.025268
23°	.520121	.605441	.690201	.774372	.857939	.940908	1.023301
24°	.519828	.604983	.689531	.773437	.856688	.939287	1.021260
25°	.519526	.604511	.688838	.772471	.855393	.937610	1.019147
26°	.519214	.604023	.688123	.771474	.854057	.935877	1.016962
27°	.518894	.603521	.687387	.770447	.852680	.934090	1.014709
28°	.518565	.603006	.686631	.769391	.851263	.932252	1.012389
29°	.518227	.602478	.685855	.768307	.849808	.930363	1.010004
30°	.517882	.601937	.685060	.767196	.848317	.928425	1.007556
31°	.517529	.601384	.684248	.766060	.846790	.926440	1.005047
32°	.517169	.600820	.683418	.764899	.845230	.924411	1.002480
33°	.516803	.600245	.682573	.763715	.843637	.922338	.999857
34°	.516430	.599660	.681712	.762509	.842014	.920225	.997180
35°	.516052	.599066	.680837	.761283	.840362	.918072	.994452
36°	.515668	.598463	.679948	.760037	.838683	.915882	.991675
37°	.515280	.597852	.679048	.758773	.836979	.913658	.988852
38°	.514886	.597234	.678136	.757493	.835251	.911402	.985986
39°	.514489	.596610	.677214	.756197	.833501	.909115	.983079
40°	.514089	.595979	.676282	.754888	.831732	.906800	.980134
41°	.513685	.595343	.675343	.753566	.829944	.904460	.977155
42°	.513279	.594703	.674396	.752234	.828141	.902097	.974144
43°	.512870	.594059	.673444	.750892	.826323	.899714	.971104
44°	.512460	.593413	.672487	.749542	.824494	.897313	.968039
45°	.512049	.592764	.671525	.748187	.822654	.894897	.964951
$\sin^{-1} k$	30°	35°	40°	45°	50°	55°	60°
	φ						

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta .$$

$\sin^{-1} k$	φ						
	30°	35°	40°	45°	50°	55°	60°
45°	0.512049	0.592764	0.671525	0.748187	0.822654	0.894897	0.964951
46°	.511638	.592114	.670562	.746826	.820807	.892469	.961845
47°	.511226	.591464	.669597	.745463	.818954	.890030	.958723
48°	.510815	.590813	.668632	.744098	.817098	.887585	.955589
49°	.510404	.590164	.667667	.742733	.815240	.885136	.952446
50°	.509995	.589517	.666705	.741370	.813383	.882686	.949298
51°	.509588	.588872	.665746	.740011	.811530	.880237	.946149
52°	.509183	.588231	.664792	.738658	.809682	.877793	.943003
53°	.508781	.587594	.663843	.737311	.807842	.875357	.939863
54°	.508383	.586962	.662901	.735973	.806012	.872932	.936733
55°	.507988	.586336	.661968	.734645	.804195	.870521	.933617
56°	.507598	.585716	.661043	.733330	.802393	.868127	.930519
57°	.507213	.585104	.660129	.732028	.800608	.865753	.927444
58°	.506833	.584501	.659227	.730742	.798842	.863403	.924395
59°	.506460	.583906	.658338	.729474	.797099	.861079	.921377
60°	.506092	.583321	.657463	.728224	.795380	.858786	.918393
61°	.505731	.582747	.656603	.726995	.793688	.856525	.915449
62°	.505378	.582184	.655759	.725789	.792026	.854302	.912548
63°	.505033	.581633	.654933	.724607	.790395	.852118	.909695
64°	.504695	.581095	.654126	.723451	.788798	.849977	.906895
65°	.504367	.580571	.653338	.722322	.787238	.847883	.904151
66°	.504047	.580060	.652572	.721223	.785717	.845838	.901468
67°	.503737	.579565	.651828	.720154	.784237	.843846	.898850
68°	.503437	.579085	.651106	.719117	.782800	.841911	.896303
69°	.503147	.578622	.650409	.718115	.781409	.840035	.893830
70°	.502868	.578176	.649737	.717148	.780066	.838221	.891436
71°	.502600	.577747	.649091	.716217	.778773	.836473	.889126
72°	.502344	.577336	.648472	.715325	.777532	.834793	.886902
73°	.502099	.576945	.647880	.714473	.776345	.833186	.884771
74°	.501866	.576572	.647318	.713662	.775214	.831652	.882735
75°	.501646	.576219	.646785	.712893	.774142	.830196	.880800
76°	.501439	.575887	.646283	.712168	.773130	.828820	.878968
77°	.501245	.575575	.645812	.711487	.772179	.827527	.877244
78°	.501064	.575285	.645373	.710852	.771291	.826319	.875632
79°	.500896	.575016	.644967	.710264	.770469	.825198	.874134
80°	.500742	.574769	.644593	.709724	.769713	.824167	.872755
81°	.500603	.574545	.644254	.709232	.769025	.823228	.871497
82°	.500477	.574343	.643949	.708790	.768405	.822382	.870364
83°	.500366	.574164	.643678	.708398	.767856	.821631	.869357
84°	.500269	.574009	.643443	.708057	.767378	.820978	.868480
85°	.500187	.573877	.643244	.707768	.766972	.820422	.867734
86°	.500120	.573769	.643080	.707530	.766639	.819966	.867121
87°	.500068	.573685	.642952	.707345	.766379	.819611	.866642
88°	.500030	.573625	.642861	.707213	.766193	.819356	.866300
89°	.500008	.573589	.642806	.707133	.766082	.819203	.866094
90°	.500000	.573576	.642788	.707107	.766044	.819152	.866025
$\sin^{-1} k$	30°	35°	40°	45°	50°	55°	60°
	φ						

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta.$$

$\sin^{-1} k$	φ = 60°	65°	70°	75°	80°	85°	90°
0°	1.047198	1.134464	1.221730	1.308997	1.396263	1.483530	1.570796
1°	1.047151	1.134407	1.221662	1.308916	1.396170	1.483424	1.570677
2°	1.047011	1.134235	1.221456	1.308674	1.395890	1.483104	1.570318
3°	1.046777	1.133949	1.221114	1.308271	1.395424	1.482573	1.569720
4°	1.046450	1.133549	1.220634	1.307708	1.394772	1.481829	1.568884
5°	1.046030	1.133036	1.220019	1.306983	1.393934	1.480874	1.567809
6°	1.045518	1.132408	1.219267	1.306099	1.392910	1.479707	1.566497
7°	1.044913	1.131668	1.218380	1.305055	1.391702	1.478330	1.564948
8°	1.044216	1.130816	1.217358	1.303853	1.390311	1.476743	1.563162
9°	1.043429	1.129851	1.216202	1.302492	1.388736	1.474947	1.561142
10°	1.042550	1.128776	1.214913	1.300975	1.386979	1.472943	1.558887
11°	1.041582	1.127590	1.213491	1.299301	1.385041	1.470733	1.556400
12°	1.040525	1.126295	1.211937	1.297472	1.382923	1.468317	1.553681
13°	1.039379	1.124892	1.210254	1.295489	1.380627	1.465696	1.550732
14°	1.038146	1.123381	1.208441	1.293354	1.378153	1.462874	1.547555
15°	1.036827	1.121763	1.206500	1.291067	1.375504	1.459850	1.544150
16°	1.035422	1.120041	1.204432	1.288631	1.372680	1.456627	1.540522
17°	1.033933	1.118215	1.202239	1.286046	1.369684	1.453207	1.536670
18°	1.032360	1.116286	1.199923	1.283315	1.366518	1.449591	1.532597
19°	1.030707	1.114257	1.197484	1.280440	1.363183	1.445782	1.528306
20°	1.028972	1.112128	1.194925	1.277422	1.359682	1.441782	1.523799
21°	1.027159	1.109901	1.192248	1.274263	1.356017	1.437593	1.519079
22°	1.025268	1.107578	1.189455	1.270965	1.352190	1.433218	1.514147
23°	1.023301	1.105161	1.186547	1.267531	1.348203	1.428660	1.509007
24°	1.021260	1.102652	1.183526	1.263963	1.344059	1.423920	1.503662
25°	1.019147	1.100052	1.180396	1.260264	1.339761	1.419003	1.498115
26°	1.016962	1.097364	1.177157	1.256436	1.335311	1.413910	1.492369
27°	1.014709	1.094590	1.173814	1.252481	1.330713	1.408646	1.486427
28°	1.012389	1.091732	1.170367	1.248403	1.325970	1.403213	1.480293
29°	1.010004	1.088793	1.166821	1.244205	1.321084	1.397615	1.473970
30°	1.007556	1.085774	1.163177	1.239889	1.316058	1.391855	1.467462
31°	1.005047	1.082679	1.159438	1.235458	1.310897	1.385937	1.460774
32°	1.002480	1.079509	1.155608	1.230916	1.305604	1.379865	1.453908
33°	.999857	1.076269	1.151689	1.226267	1.300183	1.373642	1.446869
34°	.997180	1.072960	1.147685	1.221513	1.294636	1.367273	1.439662
35°	.994452	1.069585	1.143598	1.216659	1.288969	1.360762	1.432291
36°	.991675	1.066147	1.139433	1.211707	1.283185	1.354113	1.424760
37°	.988852	1.062650	1.135192	1.206662	1.277288	1.347331	1.417075
38°	.985986	1.059097	1.130879	1.201529	1.271283	1.340420	1.409240
39°	.983079	1.055490	1.126499	1.196310	1.265175	1.333385	1.401260
40°	.980134	1.051833	1.122054	1.191010	1.258967	1.326231	1.393140
41°	.977155	1.048130	1.117549	1.185634	1.252664	1.318963	1.384887
42°	.974144	1.044384	1.112988	1.180186	1.246272	1.311586	1.376504
43°	.971104	1.040599	1.108374	1.174671	1.239796	1.304106	1.367999
44°	.968039	1.036779	1.103713	1.169094	1.233240	1.296529	1.359377
45°	.964951	1.032927	1.099008	1.163458	1.226610	1.288859	1.350644
$\sin^{-1} k$	60°	65°	70°	75°	80°	85°	90° = φ

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta.$$

$\sin^{-1} k$	φ						
	60°	65°	70°	75°	80°	85°	90°
45°	0.964951	1.032927	1.099008	1.163458	1.226610	1.288859	1.350644
46°	.961845	1.029047	1.094265	1.157771	1.219912	1.281103	1.341806
47°	.958723	1.025143	1.089487	1.152036	1.213152	1.273268	1.332870
48°	.955589	1.021220	1.084680	1.146259	1.206334	1.265358	1.323842
49°	.952446	1.017282	1.079849	1.140446	1.199466	1.257382	1.314730
50°	.949298	1.013333	1.074998	1.134602	1.192553	1.249344	1.305539
51°	.946149	1.009378	1.070133	1.128733	1.185601	1.241253	1.296278
52°	.943003	1.005420	1.065259	1.122846	1.178619	1.233116	1.286954
53°	.939863	1.001465	1.060382	1.116946	1.171611	1.224939	1.277574
54°	.936733	.997518	1.055507	1.111040	1.164586	1.216730	1.268147
55°	.933617	.993584	1.050640	1.105134	1.157551	1.208497	1.258680
56°	.930519	.989666	1.045787	1.099236	1.150512	1.200247	1.249182
57°	.927444	.985771	1.040954	1.093352	1.143478	1.191990	1.239661
58°	.924395	.981904	1.036147	1.087489	1.136457	1.183733	1.230127
59°	.921377	.978070	1.031372	1.081655	1.129457	1.175486	1.220589
60°	.918393	.974274	1.026637	1.075857	1.122486	1.167257	1.211056
61°	.915449	.970521	1.021947	1.070103	1.115553	1.159057	1.201538
62°	.912548	.966818	1.017310	1.064401	1.108668	1.150894	1.192046
63°	.909695	.963169	1.012733	1.058760	1.101839	1.142778	1.182589
64°	.906895	.959582	1.008222	1.053188	1.095076	1.134721	1.173179
65°	.904151	.956060	1.003785	1.047694	1.088389	1.126734	1.163828
66°	.901468	.952611	.999430	1.042287	1.081790	1.118827	1.154547
67°	.898850	.949240	.995163	1.036975	1.075288	1.111012	1.145348
68°	.896303	.945953	.990994	1.031770	1.068895	1.103302	1.136244
69°	.893830	.942755	.986928	1.026680	1.062622	1.095709	1.127250
70°	.891436	.939654	.982976	1.021716	1.056482	1.088248	1.118378
71°	.889126	.936656	.979144	1.016889	1.050488	1.080931	1.109643
72°	.886902	.933765	.975441	1.012208	1.044651	1.073775	1.101062
73°	.884771	.930988	.971875	1.007685	1.038987	1.066794	1.092650
74°	.882735	.928331	.968454	1.003331	1.033510	1.060006	1.084425
75°	.880800	.925800	.965186	.999157	1.028233	1.053426	1.076405
76°	.878968	.923400	.962081	.995176	1.023173	1.047075	1.068610
77°	.877244	.921138	.959145	.991399	1.018346	1.040971	1.061059
78°	.875632	.919018	.956388	.987837	1.013769	1.035136	1.053777
79°	.874134	.917046	.953816	.984503	1.009459	1.029593	1.046786
80°	.872755	.915227	.951438	.981408	1.005433	1.024364	1.040114
81°	.871497	.913565	.949262	.978564	1.001711	1.019477	1.033789
82°	.870364	.912066	.947293	.975983	.998310	1.014959	1.027844
83°	.869357	.910733	.945539	.973676	.995250	1.010841	1.022313
84°	.868480	.909569	.944005	.971652	.992550	1.007157	1.017237
85°	.867734	.908579	.942698	.969922	.990228	1.003940	1.012664
86°	.867121	.907764	.941622	.968494	.988300	1.001230	1.008648
87°	.866642	.907129	.940780	.967375	.986783	.999065	1.005259
88°	.866300	.906673	.940177	.966571	.985689	.997484	1.002584
89°	.866094	.906399	.939814	.966087	.985029	.996519	1.000752
90°	.866025	.906308	.939693	.965926	.984808	.996195	1.000000
	60°	65°	70°	75°	80°	85°	90°
$\sin^{-1} k$	φ						

Values of the Function $KZ(\beta, k)$.

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1}k$ 1°	2°	3°	4°	5°	10°	15°
0°	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1°	.000004	.000017	.000038	.000067	.000104	.000418	.000942
2°	.000008	.000033	.000075	.000134	.000209	.000836	.001883
3°	.000013	.000050	.000113	.000200	.000313	.001252	.002822
4°	.000017	.000067	.000150	.000266	.000416	.001667	.003757
5°	.000021	.000083	.000187	.000332	.000519	.002080	.004688
6°	.000025	.000099	.000224	.000398	.000622	.002491	.005614
7°	.000029	.000115	.000261	.000463	.000724	.002898	.006533
8°	.000033	.000132	.000297	.000528	.000825	.003302	.007443
9°	.000037	.000148	.000333	.000592	.000924	.003702	.008346
10°	.000041	.000164	.000368	.000655	.001023	.004098	.009238
11°	.000045	.000179	.000403	.000717	.001121	.004488	.010119
12°	.000049	.000195	.000438	.000779	.001217	.004873	.010988
13°	.000052	.000210	.000472	.000839	.001312	.005253	.011844
14°	.000056	.000225	.000506	.000899	.001405	.005626	.012686
15°	.000060	.000239	.000538	.000957	.001496	.005992	.013513
16°	.000063	.000254	.000571	.001015	.001585	.006351	.014324
17°	.000066	.000268	.000602	.001071	.001673	.006702	.015118
18°	.000070	.000281	.000633	.001125	.001759	.007045	.015893
19°	.000074	.000295	.000663	.001179	.001842	.007380	.016650
20°	.000077	.000308	.000692	.001231	.001923	.007706	.017387
21°	.000080	.000320	.000721	.001281	.002002	.008023	.018103
22°	.000083	.000332	.000748	.001330	.002079	.008329	.018797
23°	.000086	.000344	.000775	.001377	.002153	.008626	.019469
24°	.000089	.000356	.000800	.001423	.002224	.008912	.020118
25°	.000092	.000367	.000825	.001467	.002292	.009188	.020743
26°	.000094	.000377	.000849	.001509	.002358	.009452	.021342
27°	.000097	.000387	.000871	.001549	.002421	.009705	.021917
28°	.000099	.000397	.000893	.001588	.002481	.009947	.022464
29°	.000101	.000406	.000913	.001624	.002538	.010176	.022985
30°	.000104	.000414	.000933	.001658	.002592	.010393	.023479
31°	.000106	.000422	.000951	.001691	.002643	.010597	.023944
32°	.000108	.000430	.000968	.001721	.002690	.010789	.024380
33°	.000109	.000437	.000984	.001749	.002734	.010967	.024787
34°	.000111	.000444	.000999	.001776	.002775	.011132	.025164
35°	.000112	.000450	.001012	.001800	.002813	.011284	.025510
36°	.000114	.000455	.001024	.001821	.002847	.011422	.025826
37°	.000115	.000460	.001035	.001841	.002878	.011546	.026111
38°	.000116	.000464	.001045	.001858	.002905	.011656	.026364
39°	.000117	.000468	.001053	.001873	.002928	.011751	.026585
40°	.000118	.000471	.001061	.001886	.002948	.011833	.026774
41°	.000118	.000474	.001067	.001897	.002965	.011900	.026930
42°	.000119	.000476	.001071	.001905	.002978	.011953	.027054
43°	.000119	.000477	.001074	.001911	.002987	.011991	.027145
44°	.000120	.000478	.001076	.001914	.002992	.012015	.027204
45°	.000120	.000479	.001077	.001916	.002994	.012023	.027228
β	1°	2°	3°	4°	5°	10°	15°
	$\sin^{-1}k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$						
	1°	2°	3°	4°	5°	10°	15°
45°	0.000120	0.000479	0.001077	0.001916	0.002994	0.012023	0.027228
46°	.000120	.000478	.001076	.001914	.002993	.012018	.027219
47°	.000119	.000477	.001075	.001911	.002987	.011997	.027178
48°	.000119	.000476	.001071	.001905	.002978	.011962	.027103
49°	.000118	.000474	.001067	.001897	.002966	.011913	.026995
50°	.000118	.000471	.001061	.001887	.002949	.011849	.026855
51°	.000117	.000468	.001054	.001874	.002930	.011770	.026681
52°	.000116	.000464	.001045	.001859	.002906	.011677	.026475
53°	.000115	.000460	.001035	.001842	.002879	.011570	.026236
54°	.000114	.000455	.001024	.001822	.002849	.011449	.025965
55°	.000113	.000450	.001012	.001800	.002815	.011313	.025662
56°	.000111	.000444	.000999	.001776	.002777	.011164	.025328
57°	.000109	.000437	.000984	.001750	.002737	.011001	.024962
58°	.000108	.000430	.000968	.001722	.002692	.010825	.024566
59°	.000106	.000423	.000951	.001692	.002645	.010635	.024139
60°	.000104	.000415	.000933	.001659	.002594	.010433	.023683
61°	.000101	.000406	.000914	.001625	.002541	.010217	.023197
62°	.000099	.000397	.000893	.001589	.002484	.009989	.022683
63°	.000097	.000387	.000872	.001550	.002424	.009749	.022141
64°	.000094	.000377	.000849	.001510	.002361	.009497	.021571
65°	.000092	.000367	.000825	.001468	.002295	.009233	.020975
66°	.000089	.000356	.000801	.001424	.002227	.008958	.020353
67°	.000086	.000344	.000775	.001378	.002155	.008672	.019705
68°	.000083	.000332	.000748	.001331	.002081	.008375	.019033
69°	.000080	.000320	.000721	.001282	.002005	.008068	.018338
70°	.000077	.000308	.000693	.001232	.001926	.007751	.017619
71°	.000074	.000295	.000663	.001180	.001845	.007425	.016879
72°	.000070	.000281	.000633	.001126	.001761	.007089	.016118
73°	.000066	.000268	.000602	.001072	.001676	.006745	.015337
74°	.000063	.000254	.000571	.001016	.001588	.006392	.014536
75°	.000060	.000239	.000539	.000958	.001498	.006032	.013718
76°	.000056	.000225	.000506	.000900	.001407	.005664	.012882
77°	.000052	.000210	.000472	.000840	.001314	.005289	.012030
78°	.000049	.000195	.000438	.000780	.001219	.004908	.011164
79°	.000045	.000179	.000404	.000718	.001123	.004520	.010283
80°	.000041	.000164	.000369	.000656	.001025	.004127	.009390
81°	.000037	.000148	.000333	.000592	.000926	.003729	.008484
82°	.000033	.000132	.000297	.000528	.000826	.003326	.007569
83°	.000029	.000115	.000261	.000464	.000725	.002920	.006643
84°	.000025	.000099	.000224	.000398	.000623	.002509	.005710
85°	.000021	.000083	.000187	.000333	.000520	.002096	.004769
86°	.000017	.000067	.000150	.000267	.000417	.001680	.003822
87°	.000013	.000050	.000113	.000200	.000313	.001262	.002871
88°	.000008	.000033	.000075	.000134	.000209	.000842	.001916
89°	.000004	.000017	.000038	.000067	.000105	.000421	.000959
90°	.000000	.000000	.000000	.000000	.000000	.000000	.000000
β	1°	2°	3°	4°	5°	10°	15°
	$\sin^{-1} k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$						
	15°	20°	25°	30°	35°	40°	45°
0°	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1°	.000942	.001679	.002633	.003809	.005217	.006869	.008785
2°	.001883	.003356	.005263	.007614	.010428	.013731	.017562
3°	.002822	.005030	.007887	.011411	.015628	.020579	.026321
4°	.003757	.006697	.010502	.015195	.020812	.027406	.035055
5°	.004688	.008357	.013105	.018962	.025973	.034205	.043755
6°	.005614	.010007	.015693	.022708	.031107	.040969	.052413
7°	.006533	.011645	.018263	.026430	.036208	.047692	.061019
8°	.007443	.013269	.020813	.030122	.041271	.054366	.069566
9°	.008346	.014879	.023338	.033781	.046289	.060984	.078045
10°	.009238	.016470	.025838	.037403	.051258	.067540	.086448
11°	.010119	.018043	.028308	.040983	.056172	.074027	.094766
12°	.010988	.019595	.030745	.044518	.061027	.080437	.102992
13°	.011844	.021123	.033148	.048003	.065816	.086765	.111115
14°	.012686	.022627	.035512	.051436	.070534	.093003	.119130
15°	.013513	.024105	.037836	.054811	.075176	.099145	.127026
16°	.014324	.025554	.040117	.058125	.079738	.105185	.134797
17°	.015118	.026973	.042352	.061374	.084213	.111115	.142434
18°	.015893	.028361	.044538	.064555	.088598	.116929	.149928
19°	.016650	.029715	.046673	.067664	.092886	.122622	.157273
20°	.017387	.031035	.048754	.070696	.097073	.128185	.164459
21°	.018103	.032318	.050779	.073649	.101155	.133614	.171480
22°	.018797	.033563	.052746	.076520	.105126	.138902	.178326
23°	.019469	.034768	.054652	.079304	.108982	.144053	.184992
24°	.020118	.035933	.056494	.081998	.112718	.149031	.191468
25°	.020743	.037055	.058271	.084599	.116329	.153860	.197748
26°	.021342	.038133	.059980	.087104	.119812	.158524	.203824
27°	.021917	.039165	.061619	.089510	.123162	.163017	.209688
28°	.022464	.040152	.063186	.091814	.126375	.167334	.215334
29°	.022985	.041091	.064680	.094013	.129447	.171469	.220755
30°	.023479	.041981	.066098	.096103	.132373	.175418	.225942
31°	.023944	.042822	.067438	.098083	.135150	.179175	.230890
32°	.024380	.043611	.068699	.099949	.137774	.182734	.235592
33°	.024787	.044348	.069879	.101700	.140242	.186091	.240041
34°	.025164	.045032	.070977	.103333	.142550	.189240	.244230
35°	.025510	.045662	.071991	.104844	.144695	.192178	.248154
36°	.025826	.046238	.072919	.106234	.146674	.194900	.251807
37°	.026111	.046758	.073761	.107498	.148483	.197401	.255181
38°	.026364	.047222	.074514	.108636	.150119	.199677	.258273
39°	.026585	.047629	.075179	.109646	.151581	.201725	.261075
40°	.026774	.047979	.075754	.110525	.152865	.203541	.263583
41°	.026930	.048270	.076238	.111273	.153969	.205120	.265792
42°	.027054	.048503	.076630	.111888	.154892	.206461	.267696
43°	.027145	.048678	.076929	.112369	.155630	.207559	.269291
44°	.027204	.048794	.077136	.112714	.156182	.208411	.270573
45°	.027228	.048850	.077249	.112924	.156547	.209016	.271538

β 15° 20° 25° 30° 35° 40° 45°
$\sin^{-1} k$

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$ 15°	20°	25°	30°	35°	40°	45°
45°	0.027228	0.048850	0.077249	0.112924	0.156547	0.209016	0.271538
46°	.027219	.048847	.077268	.112997	.156723	.209371	.272181
47°	.027178	.048784	.077192	.112932	.156709	.209473	.272499
48°	.027103	.048661	.077024	.112729	.156504	.209321	.272489
49°	.026995	.048479	.076760	.112388	.156107	.208913	.272148
50°	.026855	.048238	.076403	.111909	.155518	.208248	.271473
51°	.026681	.047938	.075951	.111292	.154736	.207325	.270462
52°	.026475	.047578	.075405	.110536	.153762	.206143	.269113
53°	.026236	.047160	.074766	.109643	.152595	.204702	.267425
54°	.025965	.046684	.074034	.108613	.151236	.203001	.265397
55°	.025662	.046150	.073210	.107447	.149686	.201042	.263028
56°	.025328	.045559	.072295	.106146	.147945	.198824	.260318
57°	.024962	.044912	.071290	.104710	.146015	.196336	.257266
58°	.024566	.044209	.070195	.103142	.143898	.193618	.253875
59°	.024139	.043451	.069012	.101442	.141595	.190632	.250145
60°	.023683	.042639	.067742	.099613	.139108	.187395	.246077
61°	.023197	.041774	.066386	.097656	.136439	.183908	.241675
62°	.022683	.040856	.064947	.095573	.133591	.180174	.236940
63°	.022141	.039888	.063425	.093367	.130567	.176197	.231877
64°	.021571	.038870	.061822	.091040	.127370	.171978	.226489
65°	.020975	.037803	.060141	.088594	.124003	.167527	.220781
66°	.020353	.036688	.058383	.086033	.120470	.162844	.214758
67°	.019705	.035527	.056550	.083360	.116776	.157934	.208426
68°	.019033	.034322	.054644	.080577	.112923	.152804	.201790
69°	.018338	.033073	.052669	.077688	.108918	.147459	.194859
70°	.017619	.031783	.050625	.074696	.104764	.141905	.187640
71°	.016879	.030453	.048517	.071605	.100466	.136149	.180141
72°	.016118	.029084	.046345	.068419	.096031	.130199	.172371
73°	.015337	.027678	.044114	.065142	.091463	.124059	.164340
74°	.014536	.026237	.041826	.061778	.086768	.117743	.156059
75°	.013718	.024763	.039483	.058332	.081953	.111254	.147536
76°	.012882	.023257	.037088	.054807	.077023	.104602	.138786
77°	.012030	.021722	.034646	.051208	.071986	.097797	.129819
78°	.011164	.020159	.032158	.047541	.066848	.090847	.120648
79°	.010283	.018571	.029629	.043809	.061615	.083763	.111286
80°	.009390	.016959	.027060	.040018	.056296	.076554	.101748
81°	.008484	.015325	.024456	.036173	.050897	.069231	.092048
82°	.007569	.013672	.021820	.032279	.045426	.061804	.082199
83°	.006643	.012002	.019156	.028341	.039891	.054284	.072219
84°	.005710	.010316	.016466	.024364	.034298	.046683	.062122
85°	.004769	.008617	.013755	.020354	.028657	.039011	.051923
86°	.003822	.006907	.011026	.016317	.022975	.031280	.041641
87°	.002871	.005188	.008282	.012257	.017260	.023502	.031290
88°	.001916	.003462	.005527	.008181	.011520	.015688	.020889
89°	.000959	.001732	.002766	.004093	.005764	.007850	.010453
90°	.000000	.000000	.000000	.000000	.000000	.000000	.000000
β	15°	20°	25°	30°	35°	40°	45°
	$\sin^{-1} k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$: 45°	50°	55°	60°	65°	70°	75°
0°	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1°	.008785	.010995	.013542	.016499	.019981	.024190	.029522
2°	.017562	.021979	.027073	.032985	.039946	.048364	.059025
3°	.026321	.032943	.040580	.049444	.059882	.072503	.088489
4°	.035055	.043877	.054052	.065862	.079771	.096592	.117897
5°	.043755	.054771	.067477	.082227	.099601	.120612	.147228
6°	.052413	.065614	.080843	.098525	.119354	.144547	.176463
7°	.061019	.076396	.094139	.114742	.139016	.168380	.205585
8°	.069566	.087108	.107353	.130865	.158572	.192094	.234572
9°	.078045	.097739	.120473	.146881	.178007	.215672	.263407
10°	.086448	.108280	.133487	.162776	.197305	.239097	.292070
11°	.094766	.118720	.146384	.178537	.216451	.262351	.320542
12°	.102992	.129049	.159152	.194150	.235431	.285418	.348803
13°	.111115	.139257	.171780	.209603	.254228	.308280	.376835
14°	.119130	.149335	.184255	.224881	.272828	.330920	.404619
15°	.127026	.159273	.196567	.239971	.291216	.353322	.432134
16°	.134797	.169060	.208704	.254860	.309375	.375468	.459362
17°	.142434	.178688	.220653	.269535	.327292	.397340	.486283
18°	.149928	.188145	.232404	.283981	.344950	.418922	.512878
19°	.157273	.197424	.243946	.298187	.362334	.440196	.539127
20°	.164459	.206513	.255266	.312138	.379430	.461145	.565367
21°	.171480	.215403	.266353	.325821	.396222	.481751	.590510
22°	.178326	.224086	.277197	.339223	.412694	.501997	.615605
23°	.184992	.232550	.287785	.352331	.428831	.521866	.640275
24°	.191468	.240788	.298106	.365131	.444618	.541341	.664501
25°	.197748	.248789	.308149	.377610	.460039	.560402	.688264
26°	.203824	.256545	.317903	.389754	.475079	.579033	.711542
27°	.209688	.264045	.327357	.401551	.489723	.597217	.734315
28°	.215334	.271283	.336499	.412988	.503955	.614934	.756565
29°	.220755	.278247	.345320	.424050	.517759	.632168	.778269
30°	.225942	.284929	.353807	.434726	.531121	.648900	.799407
31°	.230890	.291322	.361949	.445001	.544023	.665112	.819960
32°	.235592	.297415	.369737	.454862	.556452	.680787	.839905
33°	.240041	.303201	.377159	.464298	.568390	.695904	.859221
34°	.244230	.308670	.384205	.473293	.579823	.710447	.877888
35°	.248154	.313816	.390865	.481836	.590735	.724397	.895883
36°	.251807	.318630	.397127	.489914	.601109	.737734	.913185
37°	.255181	.323103	.402982	.497513	.610930	.750439	.929772
38°	.258273	.327228	.408420	.504621	.620182	.762495	.945622
39°	.261075	.330998	.413429	.511224	.628850	.773881	.960711
40°	.263583	.334405	.418002	.517310	.636916	.784577	.975016
41°	.265792	.337442	.422128	.522867	.644365	.794565	.988515
42°	.267696	.340102	.425797	.527882	.651182	.803824	1.001182
43°	.269291	.342379	.429000	.532341	.657349	.812334	1.012995
44°	.270573	.344265	.431728	.536234	.662850	.820074	1.023928
45°	.271538	.345755	.433972	.539547	.667669	.827024	1.033955
β	45°	50°	55°	60°	65°	70°	75°
	$\sin^{-1} k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1}k$						
	45°	50°	55°	60°	65°	70°	75°
45°	0.271538	0.345755	0.433972	0.539547	0.667669	0.827024	1.033955
46°	.272181	.346843	.435723	.542269	.671791	.833164	1.043051
47°	.272499	.347523	.436974	.544387	.675198	.838470	1.051190
48°	.272489	.347789	.437716	.545889	.677874	.842923	1.058344
49°	.272148	.347638	.437940	.546632	.679803	.846500	1.064485
50°	.271473	.347064	.437641	.547003	.680968	.849178	1.069585
51°	.270462	.346063	.436810	.546591	.681354	.850936	1.073613
52°	.269113	.344632	.435441	.545519	.680943	.851750	1.076541
53°	.267425	.342766	.433528	.543776	.679721	.851597	1.078336
54°	.265397	.340464	.431064	.541352	.677669	.850453	1.078966
55°	.263028	.337723	.428046	.538238	.674774	.848294	1.078397
56°	.260318	.334540	.424467	.534424	.671018	.845096	1.076595
57°	.257266	.330916	.420324	.529902	.666386	.840834	1.073525
58°	.253875	.326848	.415614	.524663	.660864	.835482	1.069147
59°	.250145	.322337	.410333	.518700	.654436	.829017	1.063425
60°	.246077	.317383	.404479	.512007	.647089	.821411	1.056317
61°	.241675	.311988	.398052	.504577	.638807	.812640	1.047783
62°	.236940	.306154	.391051	.496406	.629579	.802677	1.037777
63°	.231877	.299883	.383477	.487490	.619392	.791498	1.026257
64°	.226489	.293178	.375330	.477825	.608235	.779075	1.013174
65°	.220781	.286045	.366615	.467411	.596098	.765385	.998480
66°	.214758	.278486	.357334	.456246	.582972	.750402	.982124
67°	.208426	.270512	.347493	.444334	.568850	.734102	.964056
68°	.201790	.262127	.337097	.431676	.553726	.716464	.944221
69°	.194859	.253338	.326155	.418276	.537597	.697465	.922565
70°	.187640	.244154	.314676	.404143	.520463	.677086	.899033
71°	.180141	.234587	.302669	.389285	.502324	.655311	.873567
72°	.172371	.224646	.290147	.373712	.483186	.632127	.846114
73°	.164340	.214343	.277124	.357438	.463057	.607522	.816617
74°	.156059	.203691	.263614	.340479	.441949	.581493	.785025
75°	.147536	.192704	.249634	.322854	.419877	.554038	.751288
76°	.138786	.181397	.235203	.304584	.396862	.525166	.715364
77°	.129819	.169786	.220341	.285693	.372929	.494889	.677218
78°	.120648	.157887	.205070	.266209	.348109	.463232	.636825
79°	.111286	.145718	.189414	.246161	.322437	.430227	.594176
80°	.101748	.133299	.173397	.225584	.295957	.395917	.549278
81°	.092048	.120649	.157047	.204513	.268715	.360359	.502164
82°	.082199	.107788	.140392	.182987	.240765	.323622	.452890
83°	.072219	.094737	.123462	.161050	.212167	.285788	.401548
84°	.062122	.081519	.106287	.138746	.182988	.246954	.348265
85°	.051923	.068157	.088901	.116121	.153297	.207230	.293208
86°	.041641	.054672	.071337	.093227	.123172	.166741	.236585
87°	.031290	.041090	.053630	.070114	.092694	.125622	.178649
88°	.020889	.027435	.035814	.046835	.061947	.084020	.119687
89°	.010453	.013730	.017925	.023445	.031019	.042093	.060023
90°	.000000	.000000	.000000	.000000	.000000	.000000	.000000
β	45°	50°	55°	60°	65°	70°	75°
	$\sin^{-1}k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$						
	75°	80°	85°	86°	87°	88°	89°
0°	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1°	.029522	.036880	.049198	.053125	.058174	.065273	.077385
2°	.059025	.073738	.098370	.106224	.116320	.130515	.154736
3°	.088489	.110553	.147492	.159269	.174409	.195696	.232018
4°	.117897	.147303	.196536	.212233	.232413	.260785	.309198
5°	.147228	.183967	.245478	.265091	.290303	.325753	.386241
6°	.176463	.220523	.294293	.317814	.348052	.390568	.463113
7°	.205585	.256948	.342954	.370377	.405631	.455199	.539779
8°	.234572	.293222	.391436	.422752	.463012	.519618	.616197
9°	.263407	.329323	.439713	.474913	.520165	.583792	.692360
10°	.292070	.365230	.487761	.526833	.577064	.647691	.768206
11°	.320542	.400919	.535552	.578485	.633679	.711286	.843710
12°	.348803	.436370	.583061	.629842	.689982	.774544	.918838
13°	.376835	.471561	.630264	.680877	.745945	.837437	.993556
14°	.404619	.506470	.677133	.731563	.801539	.899932	1.067829
15°	.432134	.541075	.723644	.781873	.856736	.962000	1.141623
16°	.459362	.575354	.769769	.831781	.911507	1.023610	1.214904
17°	.486283	.609285	.815485	.881259	.965823	1.084731	1.287638
18°	.512878	.642846	.860764	.930280	1.019657	1.145333	1.359791
19°	.539127	.676016	.905581	.978818	1.072979	1.205385	1.431329
20°	.565367	.708771	.949910	1.026844	1.125761	1.264856	1.502217
21°	.590510	.741090	.993724	1.074332	1.177974	1.323716	1.572420
22°	.615605	.772951	1.036998	1.121254	1.229612	1.381933	1.641905
23°	.640275	.804330	1.079705	1.167583	1.280578	1.439476	1.710638
24°	.664501	.835206	1.121819	1.213292	1.330911	1.496316	1.778583
25°	.688264	.865556	1.163313	1.258352	1.380560	1.552420	1.845706
26°	.711542	.895357	1.204160	1.302736	1.429494	1.607757	1.911973
27°	.734315	.924585	1.244335	1.346416	1.477686	1.662296	1.977350
28°	.756565	.953219	1.283810	1.389365	1.525105	1.716006	2.041801
29°	.778269	.981235	1.322558	1.431553	1.571721	1.768855	2.105291
30°	.799407	1.008608	1.360551	1.472953	1.617506	1.820811	2.167786
31°	.819960	1.035317	1.397763	1.513535	1.662429	1.871843	2.229250
32°	.839905	1.061336	1.434166	1.553272	1.706459	1.921918	2.289648
33°	.859221	1.086642	1.469730	1.592133	1.749566	1.971004	2.348945
34°	.877888	1.111211	1.504429	1.630090	1.791720	2.019068	2.407104
35°	.895883	1.135017	1.538234	1.667113	1.832890	2.066078	2.464090
36°	.913185	1.158035	1.571115	1.703172	1.873043	2.111999	2.519866
37°	.929772	1.180241	1.603043	1.738235	1.912149	2.156800	2.574396
38°	.945622	1.201608	1.633989	1.772274	1.950174	2.200444	2.627643
39°	.960711	1.222110	1.663921	1.805255	1.987087	2.242898	2.679568
40°	.975016	1.241721	1.692810	1.837147	2.022854	2.284127	2.730134
41°	.988515	1.260413	1.720624	1.867917	2.057440	2.324095	2.779303
42°	1.001182	1.278159	1.747330	1.897533	2.090812	2.362766	2.827036
43°	1.012995	1.294929	1.772896	1.925960	2.122935	2.400103	2.873292
44°	1.023928	1.310695	1.797288	1.953163	2.154030	2.436068	2.918030
45°	1.033955	1.325428	1.820471	1.979107	2.183285	2.470622	2.961210
	75°	80°	85°	86°	87°	88°	89°
β	$\sin^{-1} k$						

$$KZ(\beta, k) = KE(\beta, k) - EF(\beta, k).$$

β	$\sin^{-1} k$						
	75°	80°	85°	86°	87°	88°	89°
45°	1.033955	1.325428	1.820471	1.979107	2.183285	2.470622	2.961210
46°	1.043051	1.339095	1.842410	2.003755	2.211437	2.503726	3.002790
47°	1.051190	1.351666	1.863068	2.027069	2.238190	2.535339	3.042724
48°	1.058344	1.363108	1.882408	2.049010	2.263502	2.565418	3.080970
49°	1.064485	1.373388	1.900390	2.069538	2.287332	2.593920	3.117479
50°	1.069585	1.382470	1.916972	2.088611	2.309636	2.620801	3.152206
51°	1.073613	1.390317	1.932114	2.106185	2.330370	2.646013	3.185101
52°	1.076541	1.396894	1.945770	2.122214	2.349485	2.669508	3.216112
53°	1.078336	1.402159	1.957894	2.136650	2.366934	2.691234	3.245185
54°	1.078966	1.406072	1.968437	2.149443	2.382664	2.711138	3.272266
55°	1.078397	1.408589	1.977347	2.160541	2.396620	2.729164	3.297295
56°	1.076595	1.409666	1.984571	2.169887	2.408745	2.745253	3.320209
57°	1.073525	1.409256	1.990051	2.177421	2.418979	2.759341	3.340945
58°	1.069147	1.407306	1.993724	2.183081	2.427255	2.771362	3.359432
59°	1.063425	1.403766	1.995526	2.186799	2.433504	2.781243	3.375597
60°	1.056317	1.398577	1.995386	2.188502	2.437651	2.788909	3.389359
61°	1.047783	1.391681	1.993229	2.188112	2.439616	2.794276	3.400634
62°	1.037777	1.383013	1.988972	2.185544	2.439312	2.797254	3.409329
63°	1.026257	1.372506	1.982530	2.180705	2.436643	2.797746	3.415344
64°	1.013174	1.360085	1.973795	2.173496	2.431505	2.795645	3.418568
65°	.998480	1.345674	1.962673	2.163806	2.423784	2.790834	3.418883
66°	.982124	1.329188	1.949042	2.151512	2.413355	2.783182	3.416154
67°	.964056	1.310536	1.932773	2.136481	2.400076	2.772546	3.410234
68°	.944221	1.289621	1.913724	2.118560	2.383792	2.758763	3.400957
69°	.922565	1.266337	1.891734	2.097582	2.364325	2.741653	3.388138
70°	.899033	1.240571	1.866624	2.073357	2.341478	2.721008	3.371563
71°	.873567	1.212199	1.838191	2.045669	2.315023	2.696594	3.351047
72°	.846114	1.181088	1.806205	2.014271	2.284700	2.668139	3.326147
73°	.816617	1.147095	1.770406	1.978881	2.250209	2.635400	3.296703
74°	.785025	1.110067	1.730491	1.939171	2.211199	2.597796	3.262280
75°	.751288	1.069839	1.686113	1.894760	2.167258	2.555104	3.222428
76°	.715364	1.026236	1.636870	1.845197	2.117896	2.506732	3.176603
77°	.677218	.979076	1.582290	1.789950	2.062527	2.452049	3.124149
78°	.636825	.928171	1.521818	1.728382	2.000437	2.390283	3.064251
79°	.594176	.873331	1.454802	1.659725	1.931185	2.320470	2.995890
80°	.549278	.814374	1.380465	1.583040	1.852380	2.241393	2.917759
81°	.502164	.751136	1.297888	1.497179	1.763945	2.151476	2.828150
82°	.452890	.683488	1.205985	1.400720	1.663679	2.048641	2.724765
83°	.401548	.611354	1.103485	1.291903	1.549280	1.930073	2.604411
84°	.348265	.534744	.988944	1.168555	1.417706	1.791846	2.462477
85°	.293208	.453784	.860811	1.028059	1.264912	1.628299	2.291948
86°	.236585	.368751	.717624	.867437	1.085558	1.430998	2.081462
87°	.178649	.280104	.558499	.683799	.872973	1.187061	1.810789
88°	.119687	.188503	.383568	.475564	.620358	.877290	1.440002
89°	.060023	.094799	.195601	.244833	.325432	.479563	.880203
90°	.000000	.000000	.000000	.000000	.000000	.000000	.000000
β	75°	80°	85°	86°	87°	88°	89°
	$\sin^{-1} k$						

Values of Heuman's Lambda Function $\Lambda_0\ (\beta,\ k)$

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\,[E\,F(\beta, k') + K\,E(\beta, k') - K\,F(\beta, k')].$$

$\sin^{-1} k$	β: 0°	5°	10°	15°	20°	25°	30°
0°	0.000000	0.087156	0.173648	0.258819	0.342020	0.422618	0.500000
1°	.000000	.087149	.173635	.258799	.341994	.422586	.499962
2°	.000000	.087129	.173595	.258740	.341916	.422490	.499848
3°	.000000	.087096	.173529	.258642	.341786	.422329	.499658
4°	.000000	.087050	.173437	.258504	.341604	.422104	.499391
5°	.000000	.086990	.173318	.258327	.341370	.421815	.499050
6°	.000000	.086917	.173173	.258111	.341084	.421462	.498633
7°	.000000	.086831	.173002	.257856	.340747	.421046	.498141
8°	.000000	.086732	.172804	.257562	.340359	.420566	.497574
9°	.000000	.086620	.172581	.257229	.339920	.420024	.496933
10°	.000000	.086495	.172332	.256858	.339430	.419419	.496219
11°	.000000	.086357	.172057	.256448	.338889	.418753	.495431
12°	.000000	.086206	.171757	.256001	.338299	.418024	.494572
13°	.000000	.086043	.171431	.255516	.337659	.417235	.493640
14°	.000000	.085866	.171080	.254994	.336969	.416385	.492638
15°	.000000	.085677	.170704	.254434	.336231	.415475	.491565
16°	.000000	.085476	.170303	.253838	.335445	.414506	.490424
17°	.000000	.085263	.169878	.253205	.334611	.413479	.489214
18°	.000000	.085037	.169429	.252536	.333729	.412394	.487937
19°	.000000	.084799	.168955	.251831	.332801	.411252	.486593
20°	.000000	.084549	.168458	.251092	.331827	.410054	.485184
21°	.000000	.084287	.167937	.250317	.330807	.408800	.483712
22°	.000000	.084013	.167393	.249509	.329743	.407492	.482176
23°	.000000	.083728	.166826	.248666	.328634	.406131	.480578
24°	.000000	.083432	.166236	.247790	.327483	.404717	.478920
25°	.000000	.083124	.165625	.246882	.326288	.403252	.477203
26°	.000000	.082806	.164991	.245941	.325052	.401736	.475428
27°	.000000	.082476	.164336	.244969	.323775	.400171	.473597
28°	.000000	.082136	.163661	.243966	.322458	.398558	.471710
29°	.000000	.081786	.162964	.242933	.321102	.396897	.469770
30°	.000000	.081425	.162247	.241870	.319707	.395191	.467777
31°	.000000	.081054	.161511	.240778	.318275	.393440	.465734
32°	.000000	.080674	.160755	.239657	.316806	.391645	.463642
33°	.000000	.080284	.159980	.238510	.315302	.389808	.461502
34°	.000000	.079884	.159187	.237335	.313764	.387930	.459316
35°	.000000	.079476	.158377	.236134	.312192	.386013	.457086
36°	.000000	.079058	.157548	.234908	.310587	.384057	.454813
37°	.000000	.078633	.156703	.233657	.308952	.382065	.452500
38°	.000000	.078198	.155842	.232383	.307286	.380037	.450147
39°	.000000	.077756	.154965	.231086	.305592	.377975	.447756
40°	.000000	.077307	.154073	.229767	.303869	.375880	.445330
41°	.000000	.076849	.153167	.228428	.302121	.373755	.442870
42°	.000000	.076385	.152246	.227068	.300346	.371600	.440378
43°	.000000	.075914	.151313	.225689	.298548	.369418	.437856
44°	.000000	.075436	.150367	.224292	.296727	.367209	.435306
45°	.000000	.074953	.149408	.222878	.294884	.364976	.432729

$\sin^{-1} k$ — β: 0° 5° 10° 15° 20° 25° 30°

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\left[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')\right].$$

$\sin^{-1} k$	β						
	0°	5°	10°	15°	20°	25°	30°
45°	0.000000	0.074953	0.149408	0.222878	0.294884	0.364976	0.432729
46°	.000000	.074463	.148439	.221447	.293022	.362720	.430127
47°	.000000	.073968	.147459	.220002	.291141	.360442	.427504
48°	.000000	.073469	.146470	.218543	.289242	.358145	.424860
49°	.000000	.072964	.145471	.217070	.287328	.355830	.422197
50°	.000000	.072455	.144464	.215587	.285399	.353500	.419519
51°	.000000	.071943	.143449	.214092	.283458	.351155	.416826
52°	.000000	.071426	.142428	.212589	.281505	.348799	.414121
53°	.000000	.070907	.141401	.211077	.279543	.346432	.411407
54°	.000000	.070385	.140370	.209558	.277573	.344057	.408685
55°	.000000	.069861	.139334	.208034	.275597	.341676	.405958
56°	.000000	.069336	.138295	.206506	.273616	.339290	.403228
57°	.000000	.068809	.137253	.204975	.271632	.336903	.400497
58°	.000000	.068281	.136211	.203443	.269648	.334516	.397769
59°	.000000	.067754	.135168	.201911	.267664	.332131	.395045
60°	.000000	.067226	.134126	.200380	.265684	.329751	.392328
61°	.000000	.066700	.133086	.198853	.263708	.327379	.389621
62°	.000000	.066175	.132049	.197331	.261739	.325015	.386926
63°	.000000	.065651	.131016	.195815	.259780	.322664	.384247
64°	.000000	.065131	.129989	.194307	.257832	.320328	.381586
65°	.000000	.064614	.128968	.192809	.255897	.318009	.378946
66°	.000000	.064100	.127955	.191324	.253979	.315710	.376331
67°	.000000	.063592	.126951	.189852	.252079	.313435	.373743
68°	.000000	.063088	.125958	.188396	.250200	.311185	.371186
69°	.000000	.062591	.124976	.186958	.248345	.308965	.368664
70°	.000000	.062100	.124009	.185540	.246517	.306778	.366180
71°	.000000	.061617	.123056	.184145	.244718	.304626	.363738
72°	.000000	.061143	.122121	.182774	.242952	.302515	.361342
73°	.000000	.060677	.121204	.181431	.241221	.300447	.358997
74°	.000000	.060223	.120307	.180119	.239531	.298427	.356706
75°	.000000	.059779	.119433	.178839	.237883	.296459	.354475
76°	.000000	.059348	.118583	.177596	.236282	.294547	.352309
77°	.000000	.058931	.117761	.176392	.234732	.292697	.350213
78°	.000000	.058528	.116967	.175231	.233238	.290914	.348194
79°	.000000	.058142	.116206	.174117	.231804	.289203	.346257
80°	.000000	.057773	.115479	.173054	.230436	.287571	.344410
81°	.000000	.057423	.114790	.172046	.229140	.286025	.342661
82°	.000000	.057095	.114143	.171099	.227922	.284573	.341017
83°	.000000	.056789	.113540	.170219	.226789	.283222	.339489
84°	.000000	.056508	.112988	.169410	.225750	.281983	.338088
85°	.000000	.056256	.112490	.168682	.224814	.280867	.336826
86°	.000000	.056034	.112053	.168043	.223992	.279887	.335718
87°	.000000	.055846	.111684	.167504	.223298	.279060	.334783
88°	.000000	.055698	.111392	.167078	.222751	.278408	.334046
89°	.000000	.055597	.111193	.166786	.222376	.277961	.333541
90°	.000000	.055556	.111111	.166667	.222222	.277778	.333333
	0°	5°	10°	15°	20°	25°	30°
$\sin^{-1} k$	β						

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\left[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')\right].$$

$\sin^{-1}k$	β						
	30°	35°	40°	45°	50°	55°	60°
0°	0.500000	0.573576	0.642788	0.707107	0.766044	0.819152	0.866025
1°	.499962	.573533	.642739	.707053	.765986	.819090	.865959
2°	.499848	.573402	.642592	.706891	.765811	.818903	.865762
3°	.499658	.573184	.642347	.706623	.765520	.818592	.865433
4°	.499391	.572878	.642006	.706247	.765113	.818157	.864975
5°	.499050	.572487	.641567	.705765	.764592	.817600	.864388
6°	.498633	.572009	.641032	.705177	.763956	.816922	.863674
7°	.498141	.571445	.640400	.704484	.763207	.816125	.862836
8°	.497574	.570795	.639674	.703687	.762347	.815210	.861876
9°	.496933	.570062	.638854	.702787	.761376	.814179	.860797
10°	.496219	.569244	.637940	.701786	.760298	.813034	.859602
11°	.495431	.568343	.636934	.700684	.759112	.811779	.858294
12°	.494572	.567360	.635836	.699484	.757822	.810416	.856877
13°	.493640	.566295	.634679	.698186	.756430	.808946	.855355
14°	.492638	.565150	.633373	.696794	.754937	.807375	.853731
15°	.491565	.563926	.632010	.695307	.753346	.805703	.852010
16°	.490424	.562623	.630561	.693729	.751660	.803935	.850194
17°	.489214	.561244	.629028	.692061	.749881	.802074	.848289
18°	.487937	.559789	.627412	.690306	.748011	.800123	.846297
19°	.486593	.558259	.625715	.688464	.746054	.798084	.844224
20°	.485184	.556657	.623939	.686540	.744012	.795963	.842073
21°	.483712	.554982	.622086	.684534	.741887	.793761	.839848
22°	.482176	.553238	.620157	.682450	.739683	.791483	.837553
23°	.480578	.551425	.618154	.680289	.737403	.789131	.835191
24°	.478920	.549546	.616080	.678054	.735049	.786709	.832766
25°	.477203	.547600	.613936	.675748	.732623	.784220	.830282
26°	.475428	.545591	.611725	.673372	.730130	.781667	.827743
27°	.473597	.543520	.609448	.670929	.727572	.779054	.825151
28°	.471710	.541389	.607107	.668422	.724951	.776384	.822510
29°	.469770	.539199	.604705	.665854	.722270	.773659	.819823
30°	.467777	.536953	.602244	.663225	.719533	.770883	.817093
31°	.465734	.534652	.599726	.660541	.716742	.768059	.814323
32°	.463642	.532297	.597153	.657801	.713900	.765190	.811517
33°	.461502	.529892	.594527	.655010	.711009	.762278	.808676
34°	.459316	.527437	.591851	.652170	.708073	.759326	.805804
35°	.457086	.524935	.589127	.649283	.705094	.756337	.802903
36°	.454813	.522388	.586356	.646351	.702074	.753314	.799976
37°	.452500	.519797	.583543	.643378	.699016	.750260	.797024
38°	.450147	.517165	.580687	.640365	.695923	.747177	.794052
39°	.447756	.514494	.577793	.637315	.692798	.744066	.791060
40°	.445330	.511786	.574862	.634231	.689642	.740932	.788051
41°	.442870	.509042	.571896	.631115	.686459	.737777	.785028
42°	.440378	.506266	.568898	.627970	.683251	.734602	.781992
43°	.437856	.503459	.565870	.624797	.680020	.731410	.778946
44°	.435306	.500622	.562815	.621600	.676769	.728203	.775891
45°	.432729	.497760	.559735	.618381	.673501	.724985	.772830
$\sin^{-1}k$	30°	35°	40°	45°	50°	55°	60°
	β						

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\,[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')]\,.$$

$\sin^{-1} k$	β 30°	35°	40°	45°	50°	55°	60°
45°	0.432729	0.497760	0.559735	0.618381	0.673501	0.724985	0.772830
46°	.430127	.494873	.556632	.615142	.670217	.721756	.769764
47°	.427504	.491963	.553508	.611886	.666920	.718519	.766696
48°	.424860	.489034	.550366	.608615	.663613	.715277	.763627
49°	.422197	.486087	.547209	.605332	.660297	.712032	.760560
50°	.419519	.483125	.544038	.602038	.656976	.708785	.757496
51°	.416826	.480150	.540857	.598738	.653651	.705540	.754437
52°	.414121	.477164	.537668	.595432	.650326	.702298	.751385
53°	.411407	.474170	.534473	.592124	.647002	.699061	.748342
54°	.408685	.471170	.531275	.588817	.643682	.695832	.745310
55°	.405958	.468167	.528076	.585512	.640369	.692612	.742291
56°	.403228	.465163	.524879	.582212	.637064	.689405	.739286
57°	.400497	.462161	.521687	.578921	.633771	.686213	.736298
58°	.397769	.459163	.518502	.575640	.630491	.683037	.733329
59°	.395045	.456173	.515328	.572373	.627229	.679880	.730380
60°	.392328	.453192	.512167	.569122	.623985	.676745	.727455
61°	.389621	.450225	.509021	.565890	.620763	.673633	.724554
62°	.386926	.447272	.505895	.562680	.617567	.670549	.721680
63°	.384247	.444339	.502791	.559495	.614397	.667493	.718836
64°	.381586	.441428	.499711	.556339	.611258	.664469	.716024
65°	.378946	.438541	.496661	.553214	.608153	.661480	.713246
66°	.376331	.435683	.493642	.550124	.605085	.658528	.710504
67°	.373743	.432856	.490659	.547072	.602057	.655617	.707802
68°	.371186	.430065	.487715	.544062	.599072	.652749	.705142
69°	.368664	.427313	.484813	.541097	.596134	.649929	.702527
70°	.366180	.424604	.481959	.538183	.593247	.647159	.699961
71°	.363738	.421943	.479156	.535321	.590415	.644443	.697446
72°	.361342	.419332	.476408	.532519	.587641	.641784	.694985
73°	.358997	.416778	.473720	.529778	.584932	.639188	.692584
74°	.356706	.414284	.471098	.527106	.582290	.636659	.690244
75°	.354475	.411857	.468546	.524506	.579721	.634200	.687972
76°	.352309	.409500	.466070	.521985	.577231	.631818	.685770
77°	.350213	.407222	.463676	.519548	.574826	.629517	.683645
78°	.348194	.405026	.461371	.517202	.572511	.627303	.681601
79°	.346257	.402922	.459161	.514955	.570294	.625184	.679645
80°	.344410	.400915	.457055	.512813	.568181	.623166	.677782
81°	.342661	.399014	.455061	.510786	.566183	.621256	.676021
82°	.341017	.397229	.453189	.508883	.564307	.619464	.674368
83°	.339489	.395570	.451449	.507114	.562564	.617800	.672833
84°	.338088	.394049	.449853	.505494	.560967	.616276	.671427
85°	.336826	.392679	.448417	.504034	.559529	.614903	.670162
86°	.335718	.391477	.447157	.502754	.558268	.613700	.669053
87°	.334783	.390462	.446093	.501674	.557205	.612685	.668117
88°	.334046	.389662	.445255	.500823	.556366	.611884	.667379
89°	.333541	.389114	.444680	.500239	.555791	.611336	.666874
90°	.333333	.388889	.444444	.500000	.555556	.611111	.666667
$\sin^{-1} k$	30°	35°	40°	45°	50°	55°	60° β

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\left[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')\right].$$

$\sin^{-1}k$	β 60°	65°	70°	75°	80°	85°	90°
0°	0.866025	0.906308	0.939693	0.965926	0.984808	0.996195	1.000000
1°	.865959	.906239	.939621	.965852	.984733	.996120	1.000000
2°	.865762	.906032	.939407	.965633	.984511	.995903	1.000000
3°	.865433	.905689	.939052	.965270	.984147	.995564	1.000000
4°	.864975	.905210	.938559	.964769	.983652	.995130	1.000000
5°	.864388	.904599	.937930	.964135	.983037	.994624	1.000000
6°	.863674	.903857	.937172	.963376	.982315	.994063	1.000000
7°	.862836	.902989	.936287	.962498	.981498	.993463	1.000000
8°	.861876	.901997	.935282	.961512	.980599	.992833	1.000000
9°	.860797	.900886	.934162	.960424	.979628	.992180	1.000000
10°	.859602	.899660	.932934	.959244	.978597	.991511	1.000000
11°	.858294	.898323	.931604	.957980	.977513	.990828	1.000000
12°	.856877	.896881	.930177	.956638	.976384	.990135	1.000000
13°	.855355	.895338	.928661	.955228	.975216	.989434	1.000000
14°	.853731	.893699	.927061	.953755	.974016	.988727	1.000000
15°	.852010	.891969	.925384	.952226	.972787	.988015	1.000000
16°	.850194	.890152	.923634	.950646	.971534	.987299	1.000000
17°	.848289	.888254	.921818	.949021	.970260	.986580	1.000000
18°	.846297	.886280	.919940	.947355	.968969	.985858	1.000000
19°	.844224	.884233	.918005	.945653	.967662	.985135	1.000000
20°	.842073	.882119	.916018	.943918	.966343	.984410	1.000000
21°	.839848	.879941	.913983	.942154	.965012	.983685	1.000000
22°	.837553	.877704	.911904	.940364	.963671	.982958	1.000000
23°	.835191	.875412	.909785	.938551	.962323	.982232	1.000000
24°	.832766	.873068	.907630	.936718	.960968	.981506	1.000000
25°	.830282	.870676	.905441	.934867	.959607	.980779	1.000000
26°	.827743	.868240	.903221	.933000	.958241	.980054	1.000000
27°	.825151	.865763	.900975	.931119	.956872	.979329	1.000000
28°	.822510	.863249	.898703	.929226	.955500	.978604	1.000000
29°	.819823	.860699	.896409	.927322	.954126	.977881	1.000000
30°	.817093	.858117	.894095	.925409	.952751	.977159	1.000000
31°	.814323	.855506	.891763	.923489	.951374	.976439	1.000000
32°	.811517	.852869	.889416	.921563	.949998	.975719	1.000000
33°	.808676	.850207	.887054	.919631	.948622	.975002	1.000000
34°	.805804	.847523	.884681	.917695	.947247	.974286	1.000000
35°	.802903	.844820	.882297	.915757	.945873	.973573	1.000000
36°	.799976	.842100	.879904	.913817	.944502	.972861	1.000000
37°	.797024	.839364	.877505	.911876	.943132	.972152	1.000000
38°	.794052	.836615	.875099	.909935	.941766	.971445	1.000000
39°	.791060	.833855	.872690	.907995	.940402	.970741	1.000000
40°	.788051	.831085	.870277	.906056	.939042	.970039	1.000000
41°	.785028	.828307	.867863	.904121	.937686	.969340	1.000000
42°	.781992	.825524	.865449	.902188	.936335	.968644	1.000000
43°	.778946	.822736	.863036	.900260	.934988	.967952	1.000000
44°	.775891	.819946	.860625	.898337	.933647	.967262	1.000000
45°	.772830	.817155	.858217	.896419	.932311	.966576	1.000000
$\sin^{-1}k$	60° β	65°	70°	75°	80°	85°	90°

$$\Lambda_0(\beta, k) = \frac{2}{\pi}\,[EF(\beta, k') + KE(\beta, k') - KF(\beta, k')]\,.$$

$\sin^{-1} k$	β						
	60°	65°	70°	75°	80°	85°	90°
45°	0.772830	0.817155	0.858217	0.896419	0.932311	0.966576	1.000000
46°	.769764	.814365	.855814	.894508	.930981	.965894	1.000000
47°	.766696	.811577	.853417	.892604	.929658	.965215	1.000000
48°	.763627	.808792	.851026	.890708	.928341	.964540	1.000000
49°	.760560	.806013	.848643	.888820	.927032	.963870	1.000000
50°	.757496	.803241	.846269	.886942	.925731	.963204	1.000000
51°	.754437	.800478	.843906	.885073	.924437	.962542	1.000000
52°	.751385	.797724	.841553	.883216	.923152	.961885	1.000000
53°	.748342	.794982	.839214	.881370	.921876	.961233	1.000000
54°	.745310	.792252	.836887	.879537	.920610	.960586	1.000000
55°	.742291	.789537	.834576	.877717	.919353	.959944	1.000000
56°	.739286	.786839	.832280	.875911	.918108	.959309	1.000000
57°	.736298	.784158	.830002	.874120	.916873	.958679	1.000000
58°	.733329	.781496	.827742	.872345	.915649	.958055	1.000000
59°	.730380	.778855	.825502	.870587	.914438	.957437	1.000000
60°	.727455	.776237	.823283	.868846	.913240	.956826	1.000000
61°	.724554	.773644	.821086	.867124	.912055	.956223	1.000000
62°	.721680	.771077	.818913	.865421	.910884	.955626	1.000000
63°	.718836	.768538	.816765	.863740	.909728	.955038	1.000000
64°	.716024	.766029	.814645	.862080	.908588	.954457	1.000000
65°	.713246	.763552	.812552	.860443	.907464	.953885	1.000000
66°	.710504	.761110	.810490	.858831	.906357	.953321	1.000000
67°	.707802	.758704	.808460	.857245	.905268	.952767	1.000000
68°	.705142	.756338	.806464	.855685	.904198	.952223	1.000000
69°	.702527	.754012	.804504	.854155	.903148	.951689	1.000000
70°	.699961	.751731	.802581	.852654	.902119	.951166	1.000000
71°	.697446	.749497	.800699	.851186	.901112	.950654	1.000000
72°	.694985	.747312	.798860	.849751	.900129	.950154	1.000000
73°	.692584	.745180	.797065	.848352	.899170	.949667	1.000000
74°	.690244	.743104	.795319	.846990	.898237	.949193	1.000000
75°	.687972	.741089	.793624	.845669	.897332	.948733	1.000000
76°	.685770	.739137	.791983	.844390	.896456	.948288	1.000000
77°	.683645	.737253	.790399	.843157	.895612	.947859	1.000000
78°	.681601	.735442	.788877	.841972	.894800	.947446	1.000000
79°	.679645	.733709	.787421	.840838	.894024	.947052	1.000000
80°	.677782	.732059	.786036	.839759	.893286	.946677	1.000000
81°	.676021	.730500	.784726	.838739	.892588	.946323	1.000000
82°	.674368	.729036	.783497	.837783	.891933	.945990	1.000000
83°	.672833	.727678	.782357	.836896	.891326	.945682	1.000000
84°	.671427	.726434	.781312	.836083	.890770	.945400	1.000000
85°	.670162	.725315	.780373	.835352	.890270	.945145	1.000000
86°	.669053	.724333	.779549	.834711	.889831	.944923	1.000000
87°	.668117	.723505	.778854	.834171	.889462	.944735	1.000000
88°	.667379	.722852	.778307	.833745	.889170	.944587	1.000000
89°	.666874	.722405	.777932	.833453	.888971	.944486	1.000000
90°	.666667	.722222	.777778	.833333	.888889	.944444	1.000000
	60°	65°	70°	75°	80°	85°	90°
$\sin^{-1} k$	β						

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Index.

(Numerals refer to pages. No symbols are included here. For these see List of Symbols and Abbreviations, p. X.)

Errata and Additions.

Page 13, line 4: Read $\sin^{-1}(1/k \sin \psi)$ for $\sin^{-1}(1/k)$

Page 17, line 4 from Figs. 5 and 6: Read Im for *Im*. In Fig. 7, the line AB of the rectangle should be drawn heavy.

Page 26, Fig. 12: Read $K + 2iK'$ for $K + 2iK$.

Page 28, line 2 from top, and lines 5, 7, and 10 from bottom: Read Im for *Im*. In Fig. 17, the line AB of the rectangle should be drawn heavy.

Page 118, formula 255.18: Read $y \neq 0$ for $y \neq c$.

Page 214, first term in bracket in formula 361.27: Read $-2k'^2 n$ for $-2k'^2$.

Page 224, footnote 1. The following should be added:

In this table of integrals, the integration is over real u. In case IV, however, when $\alpha > 1$, there are singularities on the real axis. We must thus make a remark on the correct interpretation of the results given.

The integrals 415.01—415.06 are interpreted as CAUCHY Principal Values. This is also the interpretation in 436.01—436.05 when $\alpha \operatorname{sn} u_1 > 1$. In 415.07—415.08, and also in 436.06 if $\alpha \operatorname{sn} u_1 > 1$, the integrals are interpreted as "generalized" principal values. Thus, for example, the principal part of the integral

$$\int_0^K \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} \quad \text{is} \quad \oint_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} + \frac{\alpha}{2}\frac{\partial}{\partial \alpha} \oint_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u},$$

where the symbol $\oint$ means that the CAUCHY Principal Value is to be taken.

Pages 245—248: In this section, an integral such as $\int_0^{\varphi}\int_0^{\varphi_1} F(\varphi_1, \vartheta)\, d\varphi_1\, d\vartheta$ means $\int_0^{\varphi}\left(\int_0^{\varphi_1} F(\varphi_1, \vartheta)\, d\vartheta\right) d\varphi_1$.

Page 276, line 2: Read "With Variable Upper Limit" for "Indefinite Integrals".

Pages 282 and 283, formulas 710.06—710.11: Read $\partial/\partial k$ for d/dk.

Page 284 formulas 730.00—730.04: Read $\partial/\partial \varphi$ for $d/d\varphi$.

Pages 285 and 286: In formula 732.00, read $\partial/\partial \varphi$ for $d/d\varphi$. In formulas 732.01—732.12, read $\partial/\partial y$ for d/dy.

Pages 305, 306 and 315: Read Im for *Im*.

Page 321: The numerical tables are accurate to six places. Calculations for $KZ(\beta, k)$ and $\Lambda_0(\beta, k)$ were made using corrected Legendre tables as source. The values given for $F(\varphi, k)$ and $E(\varphi, k)$ are abbreviations of Legendre's tables.

ERRATA

P. 2, Line 6 from bottom: For missing upper limit of 2nd integral, read a.

P. 6, formula (22): For missing upper limit of 1st integral, read y.

P. 14, formula 117.03: Line 1, read $\Pi[-(\alpha^2-k^2)/(1-\alpha^2),\ k]$ for $\Pi[(\alpha^2-k^2)/(1-\alpha^2),\ k]$; line 3, read $\Pi[\varphi,-(\alpha^2-k^2)/(1-\alpha^2),\ k]$ for $\Pi[(\alpha^2-k^2)/(1-\alpha^2),\ k]$; formulas 117.04 and 117.05, read $\Pi[-k'^2/(\alpha^2-1),\ k']$ for $\Pi[k'^2/(\alpha^2-1),\ k']$; formulas 117.06, read $\Pi[-(k^2-\alpha^2)/\alpha^2,\ k']$ for $\Pi[(k^2-\alpha^2)/\alpha^2,\ k']$.

P. 18, formulas 119.03: read $E(k_1, k')$ for $E(k_1)$, and $F(k_1, k')$ for $K(k_1)$, where, $k_1 = \sqrt{1-a^2k^2}/k'$.

P. 23, 1st formula in 123.07: Read -2 for 2; 2nd formula on the right, read cn for dn.

P. 24, last formula in 125.01: Read $\mathrm{cn}(u,k)$ for $\mathrm{nc}(u,k)$.

P. 26, Fig. 12: For distance from axis to upper edge of curve, read k' for $-k'$.

P. 28, formula 129.51: In the brackets, for modulus of sn, read q for $k(q)$.

P. 29, last integral in line 15: For missing upper limit, read a.

P. 32, last formula in 133.01: On the left, read $\mathrm{sn}^{-1}(x,k) + \mathrm{cn}^{-1}(y,k)$ for $\mathrm{sn}^{-1}(y,k) + \mathrm{cn}^{-1}(x,k)$.

P. 34, line 3 from bottom: Read 710.06 for 710.04.

P. 37, line 7: Read 710.11 for 710.07.

P. 40, 3rd formula in line 2: Read $(k_1+\alpha_1^2)^2$ for $(k_1+\alpha_1^2)$.

P. 57, formula 218.11: Read $(a^2-b^2)^2$ for (a^2-b^2).

P. 61, in 'box' above formula 221.00: Read $\tan^{-1}(y/b)$ for $\ln^{-1}(y/b)$.

P. 64, formula 225.00: Read $F(\varphi,k)$ for $F(\cos\varphi,k)$. Formula 225.04, under the summation sign, insert factor 2^j.

P. 67, line 2: First term, read $\dfrac{2}{(y_1-p)^m}$ for 2; second term, read $\dfrac{2}{(y-p)^m}$ for 2.

P. 75, line 3: Read $(p-b)$ for $(p-b($.

P. 84, formula 238.02: Read α^2 for α.

P. 85, formula 238.17: In denominator on the right, read t^m for ^{m}.

P. 86, formula 239.03: Under the integral sign, read dt.

P. 90, formula 242.06: On the right, multiply by the factor $(-1)^m$.

P. 91, formula 243.02, in denominator, read $4A$ for $4A^2$; formula 243.06, on the left, read $+A$ for $-A$.

P. 97, formula 250.03: Delete 4 in denominator.

P. 100, formula 251.24: Read $(1-\alpha^2\mathrm{sn}^2u)^m$ for $(1-\alpha^2\mathrm{sn}^2u)$.

P. 134, formula 259.53: Read $1-(\sqrt{3}-2)\,\mathrm{cn}\,u$ for $1+(\sqrt{3}-2)\,\mathrm{cn}\,u$.

P. 140, Formula 261.59, read $2\,\mathrm{ds}\,u$ for $\mathrm{ds}\,u$; formula 261.55 on the right, multiply by factor $1/4^m$.

P. 143, formula 264.04: Delete $(\sqrt{k'})^m$ and $(\sqrt{k'})^{m+j}$.

P. 145, formula 264.54: Delete $(\sqrt{k'})^m$ and $(\sqrt{k'})^{m+j}$; line 3, read $-\alpha$ for α.

P. 152, formula 272.54: Read $\sqrt[4]{t^2-1}$ for $\sqrt{t^2-1}$; formula 272.58, read $\sqrt[4]{(t^2-1)^3}$ for $\sqrt{(t^2-1)^3}$.

P. 154, formula 273.53: Read $4/\sqrt{2}$ for $1/\sqrt{2}$.

P. 156, formula 275.00, delete minus sign. Formula 275.03, read -24 for $+24$, and $(4-j)!$ for $(m-j)!$ Formula 275.04, multiply right side by $(-1)^m$, and read $-(1-Y^2)^{m-j}$ for $-(Y^2-1)^{m-j}$.

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P. 157, formula 275.06: Multiply right side by $(-1)^m$.
P. 160, formula 279.00, 1st line, 2nd integral on right: Read (t^4-a) for $(t-a)$.
P. 161: Delete formula 279.50.
P. 171, first line in 'box': Read $\sqrt{2}/a$ for $\sqrt{2/a}$.
P. 177, formula 291.02: Read $ak^2p\,(1+\frac{b}{ap})$ for ak^2p.
P. 180, formula 294.01: Under second integral sign, read du.
P. 187, formula 297.56: Read $\cosh^m\vartheta$ for $\cosh^m$.
P. 196, formula 318.05: Read $\mathrm{sd}^{2m-1}u$ for $\mathrm{sd}^{2m+1}u$; formula 318.06, read $\mathrm{sd}^{2m}u$ for $\mathrm{sd}^{2m+2}u$; formula 318.03, read $1/2k^3k'^3$ for $1/2k^2k'^2$, and multiply first term in brackets by $(2k^2-1)$.
P. 199, formula 330.04: Read $(a_1+\mathrm{sn}\,u)^{-m}$ for $(a_1+\mathrm{sn}\,u)^m$.
P. 200, formula 332.51: Read $(1-\alpha_1^2)^m$ for $(1-\alpha_1^2)$.
P. 204, formula 339.75: Under summation sign, 2nd term on right, read k^{2j} for k^{2m}.
P. 208, formula 348.52: Read G_{2m+2} for G_m; formula 348.85, read $(-1)^{m-j+1}$ for $(-1)^m$.
P. 209, formula 351.51: For upper index on 2nd summation sign, read n for m.
P. 210, formula 353.01: Read $k^{2(m+n-1)}$ for $k^{2\ m+n-1)}$; 5th and 6th line in formula 355.01, read $\mathrm{nd}^{2(m-1)}u$ for $\mathrm{nd}^{2m-1}u$; 3rd line in formula 356.01, read $\mathrm{cn}^{2(m-1)}u$ for $\mathrm{cn}^{2\ m-1)}u$.
P. 214, 1st term in brackets in formula 361.27: Read $-2k'^2u$ for $-2k'^2$.
P. 215, formula 361.50: On the right, read $1\pm\mathrm{sn}\,u$ for $1+\mathrm{sn}\,u$. Formula 361.54: After the 3rd equality sign in the expression for f_1, insert factor $1/2$.
P. 217, formula 362.03: Read $k^2E(u)$ for $E(u)$, and $k^4\mathrm{sn}\,u\,\mathrm{cd}\,u$ for $k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u$.
P. 232, line 3 in 'box': Read Θ for H, and ϑ_0 for ϑ.
P. 247, formula 531.07: Read $K'(k_4)$ for $K(k_4)$.
P. 254, last integral in line 7: Read τ^{2p-1} for τ^{p-1}.
P. 256, 1st line in formula 576.00: Read $\sqrt[4]{3}$ for $\sqrt{3}$.
P. 258, 1st line in formula 578.00: Read $\sqrt[4]{3}$ for $\sqrt{3}$.
P. 267, top line and also 1st line above 'box': Read $1<1/\sqrt{n}$ for $1<1/n$.
P. 268, line above 'box': Read $1/\sqrt{n}<Y<\infty$ for $1/n<Y<\infty$.
P. 272, formulas 610.00: and 611.00: Read k^{2n} for k^{2n+1}.
P. 275, formula 615.12: Read k^{m-2} for k^{n-2}.
P. 286, formula 732.07: Read $\infty>y>1$ for $\infty>y>0$.
P. 292, formula 806.03: Read $\frac{2}{(j-2)!}$ for $\frac{1}{(j-3)!}$.
P. 298, formula 900.05: Read $\binom{-\frac{1}{2}}{m}^2$ for $\binom{-\frac{1}{2}}{m}$; formula 900.08, under summation sign. read $k_1^{2m+2}/(2m+1)^2$ for k_1^{2m+2}.
P. 299, formula 902.01: Read $\binom{-\frac{1}{2}}{m}$ for $\left(\frac{-\frac{1}{2}}{m}\right)$.
P. 302, formula 907.03, last term: Read k^2u^8 for u^8.
P. 303, 3rd line in footnote: Read $1-q^{2m+1}$ for $1+q^{2m+1}$; in last line of footnote, read $1+q^{2m+1}$ for $1-q^{2m+1}$; formula 907.07, read 408 for 498.
P. 308, last line in formulas 1032.03: Read $4t^3$ for $4t^2$.
P. 311, formula 1037.10: In the brackets, replace a by α, with $\alpha=\wp^{-1}(-d)$.
P. 312, formula 1037.12: Read $m-1$ for m –.
P. 315, 2nd formula on right in 1051.02: Read $-H(u)$ for $H(u)$, and $-\vartheta_1(v)$ for $\vartheta_1(v)$.
Pages 317-319, formulas 1053.02-1053.05: The Theta functions in all these formulas should be replaced by the logarithm of the Theta functions.
P. 321, 1st formula in 1060.04: Read $\mathrm{dc}\,u\,\mathrm{ns}\,u$ for $\mathrm{dn}\,u\,\mathrm{cs}\,u$.