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Distribution of the kurtosis statistic b_2 for normal samples

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SUMMARY

D'Agostino & Pearson (1973) gave percentage points of the distribution of b_2 for independent observations from a common univariate normal distribution. Their results can be adequately approximated, when the first three moments of the distribution of b_2 have been determined, by fitting a linear function of the reciprocal of a χ^2 variable and then using the Wilson–Hilferty transformation. Evidence is presented suggesting that the same method of approximation is satisfactory for the b_2 statistic calculated from any set of linear-least-squares residuals, on the hypothesis of normal homoscedastic errors.

Some key words: b_2 statistic; Kurtosis; Pearson type V distribution; Residuals.

1. INTRODUCTION

Over many years, effort has been devoted to determining the distribution of the kurtosis statistic b_2 or g_2 for a sample of independent observations from a common univariate normal distribution. The most detailed presentation has been by D'Agostino & Pearson (1973); they give charts showing 29 percentage points from 0·1% to 99·9% for sample sizes between 20 and 200.

The b_2 statistic is remarkable among well-known test statistics in that its distribution, though asymptotically normal, is very skew unless the sample size is very large. If only the mean and standard error of b_2 are calculated, and a test for normality of the population is made as if the distribution of b_2 were normal, the significance of high values will be exaggerated, and the significance of low values underrated. The matter has topical interest because of current enthusiasm for 'robust' estimation methods, suitable when error distributions have longer tails than the normal. It is desirable that evidence for above-normal kurtosis should not be much exaggerated, nor evidence for below-normal kurtosis be much underrated, so that balanced judgements can be formed concerning distribution shapes encountered in practice. It is desirable also that tests for shape of the error distribution, and in particular kurtosis, should be available for residuals from any least-squares fitting of a linear regression relation, and not only for the special case of fitting the mean of a homogeneous sample.

Effectiveness of the third-moment and fourth-moment statistics $\sqrt{b_1}$ and b_2, or g_1 and g_2, in testing for nonnormality has been indicated by Shapiro, Wilk & Chen (1968).

Two results are obtained in this paper. In § 2, an approximation to the distribution of b_2 for homogeneous normal samples is considered, namely, the distribution of a linear function of the reciprocal of a χ^2 variable, i.e. a Pearson type V distribution, chosen to have the same first three moments as b_2. The distribution of the χ^2 variable becomes in turn approximately normal with the Wilson–Hilferty cube-root transformation. Thus a readily computed test is available of ample accuracy for most practical purposes, at least between the lower 5% and upper 0·1% points. In § 3, some first steps are taken towards studying distribution shape for the b_2 statistic calculated from any set of linear-least-squares residuals. Formulae are given from which the first four moments of b_2 can be computed, on the hypothesis of normal homoscedastic errors. For the special case of residuals in the usual additive analysis of a two-way table, it is shown that the third and fourth moments of b_2 are related in almost the same way as for the homogeneous sample. This is ground for hope that the approximation of § 2 will be equally effective for b_2 determined from any linear-least-squares residuals. The matter has been directly verified, again for the special case of the two-way table, by a small-scale simulation, reported in § 4.

2. Homogeneous sample

Let $\{y_i\}$ $(i = 1, \ldots, n)$ be a 'sample of observations', independent random variables having a common normal Gauss–Laplace distribution. Let $\bar{y}$ be the sample mean, $n\bar{y} = \Sigma_i y_i$, and let the rth sample moment m_r be defined by $nm_r = \Sigma_i (y_i - \bar{y})^r$. Then Karl Pearson's b_2 statistic is the standardized, i.e. scale-free, fourth moment, $b_2 = m_4/m_2^2$. Fisher's kurtosis statistic g_2 is a linear function of b_2 suggested by an unbiasedness argument. For present purposes, whether b_2 or g_2 is considered is immaterial, and the former will be referred to. Asymptotically as $n \to \infty$, b_2 is normally distributed with mean 3 and variance $24/n$. Fisher (1930) showed that simple exact expressions could be found for low-order moments of standardized sample moments like b_2, because the standardized moments were independent of their denominators; and he gave results for g_2 from which may be immediately deduced the following, for $n \geqslant 4$:

$$E(b_2) = \frac{3(n-1)}{n+1}, \quad \operatorname{var}(b_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}, \tag{1}$$

and the standardized third moment,

$$\sqrt{\{\beta_1(b_2)\}} = \frac{E\{b_2 - E(b_2)\}^3}{\{\operatorname{var}(b_2)\}^{3/2}} = \frac{6(n^2-5n+2)}{(n+7)(n+9)} \sqrt{\left\{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}\right\}}. \tag{2}$$

The standardized fourth moment of b_2, $\beta_2(b_2)$, was determined by E. S. Pearson (1931) from a result by Wishart (1930). Pearson (1963) quotes a graph due to C. T. Hsu in which $(\beta_1(b_2), \beta_2(b_2))$ pairs are plotted for various n from 4 to 1000. For values of n above about 30, the points lie close to the curve representing Pearson type V distributions, that is distributions of a linear function of the reciprocal of a χ^2 variable.

Such a distribution is thus suggested as an approximation to the distribution of b_2; apparently the suggestion was made by C. T. Hsu in an unpublished thesis of 1939 (Pearson, 1963). The distribution can be fitted by the first three moments. If A denotes the number of degrees of freedom of the χ^2 variable, we choose A to equate the standardized third moments:

$$A = 6 + \frac{8}{\sqrt{\{\beta_1(b_2)\}}}\left[\frac{2}{\sqrt{\{\beta_1(b_2)\}}} + \sqrt{\left\{1 + \frac{4}{\beta_1(b_2)}\right\}}\right]. \tag{3}$$

As n ranges from 4 upwards, A is smallest when $n = 24$, and then it is just over 18. Having converted b_2 to a χ^2 variable with A degrees of freedom, we may now, since A is never small, convert the χ^2 to an equivalent normal deviate by the Wilson–Hilferty transformation; see, for example, Kendall & Stuart (1977, § 16·7). If x is the standardized b_2 statistic, that is

$$x = \frac{b_2 - E(b_2)}{\sqrt{\{\mathrm{var}\,(b_2)\}}}, \tag{4}$$

the equivalent normal deviate for b_2 is

$$\left(\left(1-\frac{2}{9A}\right)-\left[\frac{1-(2/A)}{1+x\sqrt{\{2/(A-4)\}}}\right]^{1/3}\right)\Big/\sqrt{\{2/(9A)\}}. \tag{5}$$

If, for example, the equivalent normal deviate for b_2 is 1·96, the approximation tells us that b_2 is at the upper $2\frac{1}{2}\%$ point of its distribution; if the equivalent normal deviate for b_2 is $-1{\cdot}96$, b_2 is at the lower $2\frac{1}{2}\%$ point. The five numbered equations specify how the equivalent normal deviate for b_2 is calculated; the procedure is easily programmed for computer.

The work of D'Agostino & Pearson permits the accuracy of this approximation to be tested rather thoroughly. If, for any n, one of their percentage point values for b_2 is inserted in the approximation, the corresponding equivalent normal deviate should emerge. Table 1 shows what happens. For seven values of n, all 21 percentage point

Table 1. *Errors* ($\times 100$) *in approximating the distribution of* b_2 *for homogeneous normal samples by a type V distribution and the Wilson–Hilferty transformation*

100 ×	Homogeneous samples of various sizes						
END	20	25	30	40	50	100	200
309	−5	−5	−4	−3	−2	1	11
281	0	−2	−2	−1	0	0	3
258	1	0	−1	−1	−1	−1	2
233	3	1	0	1	0	−1	0
196	3	2	1	0	−1	−2	−1
164	3	1	1	0	0	−1	−3
128	1	1	1	0	−1	−1	−1
84	−1	0	0	2	1	0	0
52	−3	−1	−1	1	1	1	0
25	−1	−2	−2	0	1	0	2
0	−3	−2	−1	1	3	2	2
−25	−2	−1	1	3	3	1	1
−52	−1	1	2	2	3	3	3
−84	1	1	1	2	1	0	3
−128	3	1	2	0	0	1	−1
−164	5	5	1	−4	−3	−4	−3
−196	4	2	1	−3	−7	−9	−10
−233	8	0	−3	−11	−17	−14	−15
−258	9	−1	−6	−15	−18	−25	−23
−281	14	3	−8	−20	−24	−27	−20
−309	19	6	−7	−21	−27	−34	−30

The first column shows correct equivalent normal deviates, END, ($\times 100$) corresponding to tail probabilities from upper 0·1%, 0·25%, 0·5%, ..., to lower 0·1%. The other columns show errors in approximate equivalent normal deviates derived from D'Agostino–Pearson tabulation.

values for b_2 were taken from a table in an unpublished Boston University research report from which D'Agostino & Pearson's published chart (1973) was drawn. The percentage points are, in each tail, 0·1%, 0·25%, 0·5%, 1%, 2·5%, 5%, 10%, then by steps of 10% to 50%. Corresponding correct equivalent normal deviates are shown in the column to the left of the table, multiplied by 100 and rounded to the nearest integer, going from 309, upper 0·1% point, to −309, lower 0·1% point. In the body of the table is shown the difference between the approximate and the correct equivalent normal deviates, calculated precisely and then multiplied by 100 and rounded to the nearest integer.

The negative entries appearing at the top left-hand corner of the table, extreme upper tail, lower n, are mostly due to inaccuracy in the Wilson–Hilferty approximation to the χ^2 distribution. Elsewhere that approximation contributes almost nothing to the errors shown, which are therefore due mainly either to the type V approximation or to simulation error in D'Agostino & Pearson's table. The extreme top right-hand entry of 11, upper 0·1% point for $n = 200$, is out of line. This could result from simulation error; the standard error estimated as at (10) below, which appears to be appropriate for D'Agostino & Pearson's determination of this percentage point, is about 7, and the entry of 11 can be judged not significantly different from zero. Ignoring this entry, we see that for all percentage points from the lower 5% point upwards all entries in the table are not greater than 5 in magnitude, and nearly all are not greater than 3. Thus the approximation gives an equivalent normal deviate correct to within 0·05, and usually correct to within 0·03, very adequate precision for most practical purposes. But in the lower tail below the 5% point, and for n above 40, the approximation exaggerates significance, giving equivalent normal deviates that may be 0·3 too low; −3·1 instead of −2·8. Note that what appears to be the lower 0·1% point should be the lower 0·25% point, etc. For some purposes this exaggerated significance would be worth remembering.

3. LEAST SQUARES RESIDUALS

Let the observations $\{y_i\}$ now be independent random variables having normal distributions with common variance σ^2 and means that are given linear functions of some regression coefficients. The homogeneous sample of §2 is a special case of this regression situation. Let $\{z_i\}$ be the residuals after the regression coefficients have been estimated by the method of least squares and the fitted values, i.e. estimated means, subtracted from $\{y_i\}$. Let $((q_{ij}))$ be the matrix transforming observations to residuals, $z_i = \Sigma_j q_{ij} y_j$ for all i. Suffixes i, j, k, l will be understood to take integer values from 1 to n. Let ν be the number of residual degrees of freedom; $\nu = \Sigma_i q_{ii}$. The residuals are jointly normally distributed in a singular normal distribution with zero means and variance matrix $((q_{ij}\sigma^2))$ of rank ν. Then a kurtosis statistic can be defined, by analogy with Pearson's b_2 for the homogeneous sample, as $b_2 = m_4/m_2^2$, where $nm_r = \Sigma_i z_i^r$; and one may also define an analogue of Fisher's g_2 (Anscombe, 1961). As before, b_2 is independent of its denominator m_2^2. The residual sum of squares, nm_2, is equal to σ^2 multiplied by a χ^2 variable with ν degrees of freedom, and expectations of its powers are well known. For positive integer r,

$$E(nm_2)^{2r} = \nu(\nu+2)(\nu+4)\dots(\nu+4r-2)\,\sigma^{4r}. \tag{6}$$

Since $E(b_2^r) = E(m_4^r)/E(m_2^{2r})$, moments of b_2 are available as soon as we have moments of m_4. The latter can be determined by expanding the moment generating function of the residuals. We find

$$E(b_2) = \frac{3n}{\nu(\nu+2)}\Sigma_i q_{ii}^2, \tag{7}$$

$$\operatorname{var}(b_2) = \frac{24n^2}{\nu(\nu+2)(\nu+4)(\nu+6)}\left\{3\,\Sigma_{ij} q_{ii} q_{ij}^2 q_{jj} + \Sigma_{ij} q_{ij}^4 - \frac{3(\nu+3)}{\nu(\nu+2)}(\Sigma_i q_{ii}^2)^2\right\}. \tag{8}$$

The standardized third moment of b_2, $\sqrt{\{\beta_1(b_2)\}}$, can be obtained easily from the above as soon as the third moment of m_4 has been found:

$$\begin{aligned}E(nm_4)^3 &= E(\Sigma_{ijk} z_i^4 z_j^4 z_k^4)\\ &= 27\sigma^{12}\{(\Sigma_i q_{ii}^2)^3 + 24(\Sigma_i q_{ii}^2)\Sigma_{jk} q_{jj} q_{jk}^2 q_{kk} + 8(\Sigma_i q_{ii}^2)\Sigma_{jk} q_{jk}^4\\ &\qquad + 96\,\Sigma_{ijk} q_{ii} q_{ij}^2 q_{jk}^2 q_{kk} + 64\,\Sigma_{ijk} q_{ii} q_{ij} q_{jj} q_{ik} q_{jk} q_{kk}\\ &\qquad + 128\,\Sigma_{ijk} q_{ii} q_{ij} q_{ik} q_{jk}^3 + 64\,\Sigma_{ijk} q_{ij}^2 q_{ik}^2 q_{jk}^2\}.\end{aligned} \tag{9}$$

These results suffice to give access to the approximation, of the equivalent normal deviate for b_2, of § 2 above. Calculation of $E(b_2)$ and var (b_2) requires storage space in the computer for not much more than the n values $\{q_{ii}\}$. The formula (9) for $E(m_4^3)$ can be expressed as operations on several $n \times n$ matrices and the vector $\{q_{ii}\}$; thus storage space of order n^2 is required, in general, for convenient evaluation.

Except for the small simulation described below, we have no direct information concerning the distribution of b_2 for general least squares residuals, like D'Agostino & Pearson's results for the homogeneous sample. A modest first step towards investigating the distribution shape would be to construct a chart of $(\beta_1(b_2), \beta_2(b_2))$ points like Hsu's, for some type of least squares residuals. That would require $E(m_4^4)$ to be given the lengthy formula of Table 2.

A computer has been used to verify that all the terms in Table 2 are distinct, no two differing only by a permutation of the subscripts, and to verify the numerical coefficients and that no term has been missed.

Table 2. *Explicit expression for* $E(nm_4)^4 = E(\Sigma_{ijkl} z_i^4 z_j^4 z_k^4 z_1^4)$

$$\begin{aligned}&27\sigma^{16}\{3(\Sigma_i q_{ii}^2)^4 + 144(\Sigma_i q_{ii}^2)^2 \Sigma_{kl} q_{kk} q_{kl}^2 q_{ll} + 48(\Sigma_i q_{ii}^2)^2 \Sigma_{kl} q_{kl}^4 + 1152(\Sigma_i q_{ii}^2)\Sigma_{jkl} q_{jj} q_{jk}^2 q_{kl}^2 q_{ll}\\ &\quad + 768(\Sigma_i q_{ii}^2)\Sigma_{jkl} q_{jj} q_{jk} q_{kk} q_{jl} q_{kl} q_{ll} + 1536(\Sigma_i q_{ii}^2)\Sigma_{jkl} q_{jj} q_{jk} q_{jl} q_{kl}^3 + 768(\Sigma_i q_{ii}^2)\Sigma_{jkl} q_{jk}^2 q_{jl}^2 q_{kl}^2\\ &\quad + 576(\Sigma_{ij} q_{ii} q_{ij}^2 q_{jj})^2 + 384(\Sigma_{ij} q_{ii} q_{ij}^2 q_{jj})\Sigma_{kl} q_{kl}^4 + 4608\,\Sigma_{ijkl} q_{ii} q_{ij}^2 q_{jk}^2 q_{kl}^2 q_{ll}\\ &\quad + 9216\,\Sigma_{ijkl} q_{ii} q_{ij}^2 q_{jk} q_{kk} q_{jl} q_{kl} q_{ll} + 6144\,\Sigma_{ijkl} q_{ii} q_{ij}^2 q_{jk} q_{jl} q_{kl}^3 + 2304\,\Sigma_{ijkl} q_{ii} q_{ij} q_{jj} q_{ik} q_{kk} q_{jl} q_{kl} q_{ll}\\ &\quad + 6144\,\Sigma_{ijkl} q_{ii} q_{ij} q_{jj} q_{ik} q_{jl} q_{kl}^3 + 9216\,\Sigma_{ijkl} q_{ii} q_{ij} q_{ik} q_{jk}^2 q_{jl} q_{kl} q_{ll} + 18{,}432\,\Sigma_{ijkl} q_{ii} q_{ij} q_{ik} q_{jk} q_{jl}^2 q_{kl}^2\\ &\quad + 64(\Sigma_{ij} q_{ij}^4)^2 + 2048\,\Sigma_{ijkl} q_{ij}^3 q_{ik} q_{jl} q_{kl}^3 + 2304\,\Sigma_{ijkl} q_{ij}^2 q_{ik}^2 q_{jl}^2 q_{kl}^2 + 9216\,\Sigma_{ijkl} q_{ij}^2 q_{ik} q_{jk} q_{il} q_{jl} q_{kl}^2\}.\end{aligned}$$

Computation of this expression seems to require storage space of order n^3 or n-fold generation of material occupying space of order n^2, in general, and will probably be deemed expensive if n is large enough to be interesting, i.e. 50 or more. But for designed experiments and some other sorts of data, the matrix $((q_{ij}))$ may have symmetry permitting much simplification of the general expression. This happens, in particular, when the data have the form of a two-way table with R rows and C columns, say ($n = RC$, $R \geqslant 2, C \geqslant 2$), and the residuals arise from the usual Fisherian additive analysis for such a table. Then $\nu = (R-1)(C-1)$. Each of the above expressions can be reduced to easily computed functions of R and C, not requiring the setting up of matrices. In particular, from (7) and (8),

$$E(b_2) = \frac{3\nu}{\nu+2},$$

$$\operatorname{var}(b_2) = \frac{24}{(\nu+2)(\nu+4)(\nu+6)}\left\{(R^2-3R+3)(C^2-3C+3) - \frac{3\nu^2}{\nu+2}\right\}.$$

Figure 1 shows $(\beta_1(b_2), \beta_2(b_2))$ points for two-way tables of various sizes, R and C taking values from 3 upwards. Also shown are points for homogeneous samples of various sizes, as previously shown by Pearson (1963); the point for a homogeneous sample of size n also refers to a $2 \times n$ or $n \times 2$ two-way table. A point for a homogeneous sample is marked in Fig. 1 as a circle with the value of n beside it, and a point for a two-way table is marked as a square with the values of R and C beside it. The upper curve refers to the type III, or χ^2, distribution, the lower curve to the type V, i.e. reciprocal of χ^2, distribution.

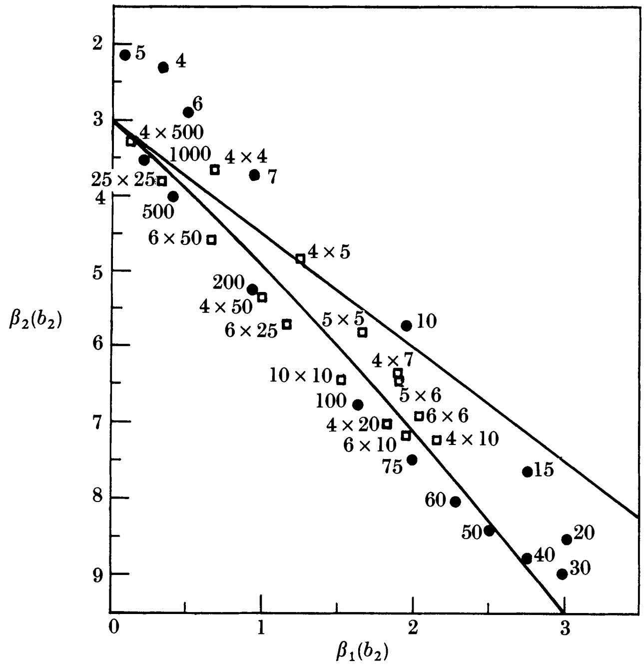

Fig. 1. (β_1, β_2) points for the distribution of b_2 in homogeneous samples and in two-way tables with homoscedastic normal errors.

It will be seen that when the number of residual degrees of freedom is above about 30, the points for a two-way table lie very close to the type V curve, just as for a homogeneous sample.

4. SIMULATION RESULTS FOR THE TWO-WAY TABLE

We obtained 960,000 random normal deviates using the program SUPERDUPER of G. Marsaglia and others, McGill University, 1973. They were used nine times over to form successively

(i) 20,000 homogeneous samples of size 48,
(ii) 20,000 two-way tables of size 6×8,
(iii) 20,000 two-way tables of size 3×16,
(iv) 10,000 homogeneous samples of size 96,
(v) 10,000 two-way tables of size 8×12,
(vi) 10,000 two-way tables of size 3×32,
(vii) 5000 homogeneous samples of size 192,
(viii) 5000 two-way tables of size 12×16,
(ix) 5000 two-way tables of size 3×64.

Sample sizes below 48 were not considered because they have little practical interest. The homogeneous samples were included for comparison with D'Agostino & Pearson's results for sample sizes 50, 100, 200.

The b_2 statistic was calculated for every homogeneous sample or two-way table. For each of the nine types of sample, estimated percentage points of the distribution of b_2 were determined by the simple procedure of ordering the whole set of b_2 values and finding the sample quantiles. The $(100p)$th percentage point was estimated from N random values of b_2 as halfway between the (Np)th and the $(Np+1)$th ordered values, Np being always an integer, i.e. no smoothing of the empirical distribution function was attempted. A standard error for such an estimated percentage point, $\hat{x}_p$, say, can be roughly assessed as the square root of

$$\operatorname{var}(\hat{x}_p) \sim \frac{p(1-p)}{N\{f(x_p)\}^2}, \tag{10}$$

where $f(x_p)$ is the density of the distribution at the percentage point. We have attempted to transform the distribution of b_2 to a standard normal distribution, and in that scale the standard error of the percentage point is assessed using the density function of the standard normal distribution in the above formula.

Table 3 shows differences between the approximate equivalent normal deviates obtained from these estimated percentage points and the corresponding correct normal deviates, in the same style as Table 1 for D'Agostino & Pearson's percentage points. Estimated standard errors from (10) are also shown after the $\pm$ sign. It should be borne in mind that adjacent percentage points are far from independent.

Table 3. *Errors* ($\times 100$) *in approximating the distribution of* b_2 *for homogeneous samples and for two-way tables by a type* V *distribution and the Wilson–Hilferty transformation*

	Homogeneous samples and two-way tables of various sizes											
100 × END	(i) 48	(ii) 6 × 8	(iii) 3 × 16	St. err.	(iv) 96	(v) 8 × 12	(vi) 3 × 32	St. err.	(vii) 192	(viii) 12 × 16	(ix) 3 × 64	St. err.
309	1	−10	−7	±7	2	13	−7	±9	6	−23	6	±13
281	−2	−6	−8	5	−3	1	−5	6	9	−10	−14	9
258	−3	−3	−9	3	−4	−7	−5	5	11	−1	−8	7
233	−3	−1	−3	3	−4	−2	−4	4	−4	−1	−8	5
196	−3	−2	−1	2	−4	0	−7	3	−6	−5	−12	4
164	−3	−1	−2	1	−6	−2	−6	2	−3	−6	−6	3
128	−2	−2	−2	1	−2	−3	−4	2	0	−4	−4	2
84	−1	−2	−1	1	−2	−1	0	1	−2	−5	−4	2
52	1	−1	−1	1	2	0	0	1	−2	0	−1	2
25	1	0	−1	1	2	−1	0	1	−1	−1	−2	2
0	0	1	0	1	1	1	1	1	0	0	−1	2
−25	1	0	0	1	0	0	2	1	1	0	1	2
−52	0	−1	1	1	1	0	3	1	0	0	1	2
−84	1	−1	2	1	1	−1	2	1	0	0	3	2
−128	−1	0	3	1	−4	−4	−1	2	−3	−4	2	2
−164	−5	−2	5	1	−6	−5	1	2	−5	−7	−3	3
−196	−7	−6	8	2	−7	−10	−2	3	−5	−13	−9	4
−233	−15	−12	7	3	−12	−16	−4	4	−12	−15	−17	5
−258	−23	−12	9	3	−19	−20	−7	5	−13	−15	−19	7
−281	−26	−16	8	5	−21	−28	−15	6	−23	−13	−34	9
−309	−35	−23	−6	±7	−41	−36	−24	±9	−58	−59	−38	±13

The first column shows correct equivalent normal deviates, END, ($\times 100$) corresponding to tail probabilities from upper 0·1%, 0·25%, 0·5%, ... to lower 0·1%. The other columns show errors in approximate equivalent normal deviates derived from a simulation, together with standard errors for the simulated values.

Results for the homogeneous samples are in quite good agreement with those in Table 1; just how good the agreement is, and whether any discrepancy should be judged 'significant', is debatable, because of the difficulty of assigning standard errors to Table 1. D'Agostino & Pearson smoothed some of their percentage points and obtained others by interpolation. Probably most entries in Table 1 have associated standard errors not very much lower than comparable entries in Table 3, in which case discrepancies between the tables can be dismissed as uninteresting.

As for the two-way tables, the three nearly square ones (6×8, 8×12, 12×16) have errors close to, or not significantly different from, those for homogeneous samples of the same size. It was remarked above that for a two-way table of size $2 \times n$ the b_2 statistic is distributed the same as for a homogeneous sample of size n; and we might therefore expect that for a two-way table of size $3 \times n$ the b_2 statistic would be distributed somewhat like that for a homogeneous sample of size not much greater than n, perhaps nearer to n than to $3n$. The entries in Table 3 for the 3×16 table resemble those in Table 1 for sample size 20 in having positive values in the lower tail; and the entries for the 3×32 and 3×64 tables seem a little less strongly negative in the lower tail than those for homogeneous samples of the same size.

As far as it goes, the simulation has confirmed the practical adequacy of our approximate distribution for b_2 for linear least squares residuals. Anscombe (1981, Appendices 2 and 3) has used the approximation freely in programs for various tests of residuals.

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