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A Robust Method for Relative Gravity Data Estimation with High Efficiency

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Abstract When gravimetric data observations have outliers, using standard least squares (LS) estimation will likely give poor accuracies and unreliable parameter estimates. One of the typical approaches to overcome this problem consists of using the robust estimation techniques. In this paper, we modified the robust estimator of Gervini and Yohai (2002) called REWLSE (Robust and Efficient Weighted Least Squares Estimator), which combines simultaneously high statistical efficiency and high breakdown point by replacing the weight function by a new weight function. This method allows reducing the outlier impacts and makes more use of the information provided by the data. In order to adapt this technique to the relative gravity data, weights are computed using the empirical distribution of the residuals obtained initially by the LTS (Least Trimmed Squares) estimator and by minimizing the mean distances relatively to the LS-estimator without outliers. The robustness of the initial estimator is maintained by adapted cut-off values as suggested by the REWLSE method which allows also a reasonable statistical efficiency. Hereafter we give the advantage and the pertinence of REWLSE procedure on real and semi-simulated gravity data by comparing it with conventional LS and other robust approaches like M- and MM-estimators.

Keywords Relative gravity · Weighted least squares · Outlier · Robust estimation · Efficiency · REWLSE

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1 Introduction

The typical approach of gravity data adjustment is the least squares method (LS), but LS estimation is sensitive to outliers, thus it will give unreliable results if the measurements are contaminated by outliers. Relative gravity measurements can be affected by three basic types of error: systematic (e.g. mismodelling errors, external effects, etc.), gross errors (often called outliers, e.g. incorrect gravimeter reading) and random errors. In this paper, we aim to reduce the influence of outliers under the normal-error model.

The classical solution is to apply the outlier detection and identification procedure. This approach is based on statistical tests like Baarda (1968) and Pope (1976) test which are usually used in geodesy and gravimetry, but when outliers appear in multiple observations, the task will not be easy because outliers are not correctly identified and the elimination of some observations may give rise to the problem of rank deficiency.

Robust estimation provides an alternative procedure. It allows best parameters estimation, without requiring any identification of specific observations as outliers or excluding them. Robust methods are basically developed by Huber's works (1964 and 1981) and Hampel (1973). Their use in geodesy is initiated by Carosio (1979), Mäkinen (1981) and Becker (1990) especially for gravimetric data. In statistic and geodesy literatures, different robust techniques are proposed. M-Estimators are largely used in geodesy and gravimetry (Wieser and Brunner 2001; Yang et al. 2002; Csapo et al. 2003; Aduol 2003; Berber et al. 2003). Some other robust methods with high breakdown point are rarely used in geodesy due to their complexity and low statistical efficiency such as LMS (Least Median Squares), LTS (Least Trimmed Squares) and S-Estimator (e.g., Rousseeuw 1984; Rousseeuw and Yohai 1984; Ruppert 1992; Hekimoglu 2005; Rousseeuw and Van Dressen 2006). Yohai (1987) combined the M-Estimator and S-Estimator for estimation with high efficiency and high breakdown point. However, tuning up these estimators for high efficiency leads to an increase in bias as an unpleasant side effect (Gervini and Yohai 2002). A new class of estimators that simultaneously reach the high breakdown point and high efficiency under normal errors called REWLSE is proposed by Gervini and Yohai (2002). We propose in this paper to modify this technique in order to adapt it to relative gravity measurements by replacing the weight function (with hard-rejection to outliers) by a new weight based on robust residuals obtained initially by LTS method and by minimizing the mean distances relatively to the LS-estimator without outliers, i.e., the solution obtained by REWLSE should be nearly that calculated by LS without outliers.

Given the availability of relative gravity data and by adding some simulated outliers, this paper aims at a comparative study between different methods of estimation (LS, M, MM and REWLSE) for parameter estimation with high breakdown point and efficiency. The paper starts with a presentation of model observation of relative gravity and LS estimation. After, summarizing briefly some robust methods and their principles, the REWLSE technique and its adaptation to relative gravity is discussed in detail. The paper ends with some results and discussion on robustness and efficiency performance of the considered methods and a special attention is given to REWLSE technique.

2 Least Squares Adjustment

In order to solve the problem of relative gravity network adjustment, we introduce the special linear Gauss–Markov model (Koch 1987).

2.1 Observational Model and Weighted Least Squares Solution

Least squares estimation of unknown parameters is carried out in the Gauss–Markov model. The relative gravity observation equation is written as (Hwang et al. 2002):

$$\Delta\mathbf{L}_{i,j} + \mathbf{V}_{i,j} = \left(\Delta\mathrm{F}(z_j) - \Delta\mathrm{F}(z_i)\right) + \left(\mathrm{D}(t_j) - \mathrm{D}(t_i)\right) + \Delta\mathbf{G} \tag{1}$$

where$\mathbf{V}_{i,j}$ residual of $\Delta\mathbf{L}_{i,j}$ relative gravity observation between ith and jth observations, t_i, t_j measurement times, $\Delta\mathrm{F}(z)$ calibration function to correct scale factor, z gravimeter reading in counter units (CU), $\mathrm{D}(t)$ unknown drift of gravimeter, $\Delta\mathbf{G}$ unknown relative gravity between jth and ith station.

We assume that there are m observations and n unknowns (gravity, drift and scale factor). If $\mathbf{W}(m \times m)$ is a weight matrix of $\Delta\mathbf{L}(m)$, the matrix representation of the observation equations become:

$$\Delta\mathbf{L} + \mathbf{V} = \mathbf{AX} \tag{2}$$

where $\mathbf{A}(m \times n)$ is the design matrix and $\mathbf{X}(n)$ is a vector of n unknowns. By inspecting the structure of $\mathbf{A}$, the solution by least squares method is not possible without constraint because $\mathbf{A}$ has rank defect of 1 (Hwang et al. 2002). This problem can be explained as follows:

Supposing that the number of gravimeter parameters is u, one can find a non-zero vector $\mathbf{y}$ fulfilling:

$$\mathbf{Ay} = 0 \tag{3}$$

From the structure of $\mathbf{A}$, one finds:

$$y^T = c\left[\underbrace{1 \quad 1 \quad \ldots \quad 1}_{n-u}\underbrace{0 \quad 0 \quad \ldots \quad 0}_{u}\right] \tag{4}$$

where c is an arbitrary non-zero constant (Hwang et al. 2002). Thus, $\mathbf{y}$ is only the element of the null space of the column vectors of $\mathbf{A}$ (Lancaster and Tismenetsky 1985). Therefore, the normal equation matrix $\mathbf{A}^{\mathrm{T}}\mathbf{WA}$ has a rank of $(\mathrm{n}-1)$ and is not positive definite.

In order to obtain a unique solution in the least squares sense, two methods can be used; gravimetric complementation and weighted constraints. The last one is selected to be used in this work and is given as follows:

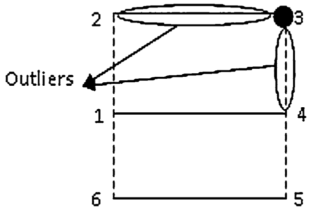
$$\hat{\mathbf{X}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{WA} + \mathbf{A}_{\mathrm{g}}^{\mathrm{T}}\mathbf{W}_{\mathrm{g}}\mathbf{A}_{\mathrm{g}}\right)^{-1}\left(\mathbf{A}^{\mathrm{T}}\mathbf{W}\Delta\mathbf{L} + \mathbf{A}_{\mathrm{g}}^{\mathrm{T}}\mathbf{W}_{\mathrm{g}}\mathbf{L}_{\mathrm{g}}\right) \tag{5}$$

where $\mathbf{A_g}$, $\mathbf{L_g}$ and $\mathbf{W_g}$ are, respectively, the design matrix, absolute gravity vector, and diagonal weight matrix of constraints and $\mathbf{A}_{\mathrm{g}}^{\mathrm{T}}\mathbf{W}_{\mathrm{g}}\mathbf{A}_{\mathrm{g}}$ is a diagonal matrix (Hwang et al. 2002).

2.2 Influence of Outliers on Relative Gravity Network Adjustment

The general problem in the adjustment of high precision gravity observations is the occurrence of outlying observations (Harnisch 1993; Becker 1990). This is done due to the fact that on the one hand the gravimeters are very delicate instruments leading a variety of error sources and on the other hand the observation scheme does not in many cases allow a perfect modeling of the instrumental behavior (Becker 1990).

Fig. 1 Outlier propagation in relative gravimetric network



In relative gravity data, one outlier in station reading may generate two outliers in adjacent bases; this can be explained by the structure of the gravimetric network as in Fig. 1; the bases of the network are subdivided into many polygons (2 polygons in this example with 4 stations each). The sequence of observation of two closures may be as: 1-2-3-4-1-6-5-4. Assuming that, in station 3, we have committed an error reading, consequently the gravity difference of bases (2–3 and 3–4) should be affected by outliers.

In geodesy and gravimetry, two statistical tests are generally used to handle suspect values; "Data Snooping technique" of Baarda (1968) and the τ-test of Pope (1976). For more than one outlier, we will encounter more trouble than in the case of only one outlier. Therefore, these tests based on the least squares residuals cannot always correctly detect and locate the outliers completely; also, we have no way to determine how many outliers are just correctly identified for a certain significance level. The difficulty is brought about in part by the least squares principle; i.e., the method cannot project the outlier completely onto the corresponding residual (Junhuan 2005; Hekimoglu 2005; Hekimoglu and Koch 2000).

3 Robust Estimates

Robust methods provide an alternative procedure, which do not require identifying specific observations as outliers or excluding them and make more use of the information provided by the data (Huber 1981; Hampel et al. 1986; Rousseeuw and Leroy 1987). They also should give the same result obtained by standard least squares method without outliers (Wicki 2001).

3.1 M-Estimator

Robust M-Estimation is the generalized form of the maximum likelihood estimation introduced by Huber (1964). Instead of minimizing the sum of the squares of the residuals, the M-Estimator $\hat{\mathbf{X}}_{\mathrm{M}}$ minimizes the sum of the score function $\rho(v)$ of the residuals.

$$\hat{\mathbf{X}}_{\mathrm{M}} = \underset{\mathbf{X}}{\operatorname{argmin}} \sum_{i=1}^{m} \rho(v(i)). \tag{6}$$

Now we can write the robust problem as an equivalent weighted least squares problem, where the weights are used to reduce the influence of bad data points as follows:

$$\mathrm{W}(i) = \frac{\Psi(\mathbf{v}(i)/\sigma)}{\mathbf{v}(i)/\sigma}, \tag{7}$$

where $\Psi(v) = \rho'(v)$ is the influence function (measure of robustness) and W(i) is the weight function. The many M-Estimators that have been proposed differ in the shape of their weights (Hampel et al. 1986). Three common weights functions are plotted in Fig. 2

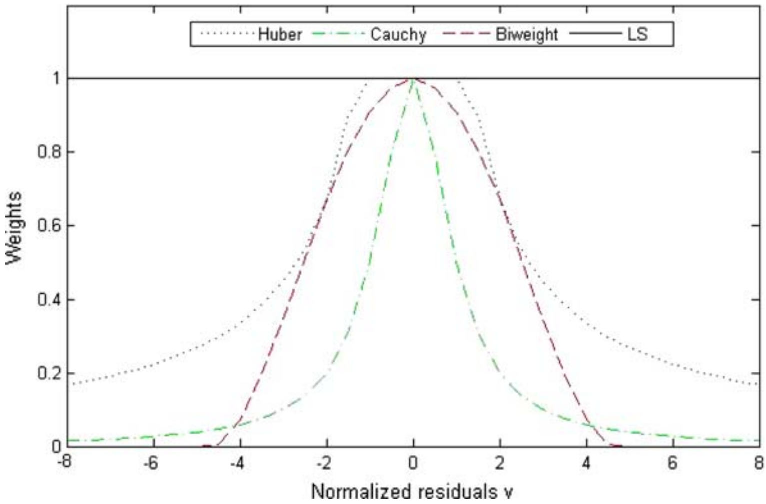


Fig. 2 Plot of three popular weight functions and standard least squares. The *x*-axis units are normalized residuals

with the standard least squares ($W(i) = 1$); Cauchy and Tukey's bi-weight functions allow the hardest rejection of outliers and Huber which is the most used in gravimetry and geodesy. The algorithms used to solve M-Estimators problem are known as Iteratively Reweighted Least Squares IRLS (Coleman et al. 1977; Koch 1987) and Newton's method (Chen and Pinar 1998).

3.2 LTS Estimation

Least Trimmed Squares (LTS) estimation is a high breakdown point method introduced by Rousseeuw (1984) and is defined as follows:

$$\hat{\mathbf{X}}_{\mathrm{LTS}} = \underset{\mathbf{X}}{\operatorname{argmin}} \sum_{i=1}^{h} v^2(i) \tag{8}$$

$v^2(1) \leq v^2(2) \leq \ldots \leq v^2(m)$ are the ordered squared residuals and the default h is defined in the range:

$$\frac{m+n+2}{2} \leq h \leq m \tag{9}$$

The performance of this method was improved by the FAST-LTS algorithm of Rousseeuw and Van Dressen (2006).

3.3 Estimation of the Scale Factor

Robust estimation of the scale factor is obtained by median method (Yang et al. 1999):

$$\hat{\sigma}_0 = 1.4826\mathrm{med}\{|v_i|\}, \tag{10}$$

where $\mathrm{med}\{|v_i|\}$ denotes the median of all $\{|v_i|\}$ and the value 1.4826 allows to have a null bias in the estimation of σ with Gaussian distribution (Hampel et al. 1986).

3.4 MM-Estimator

MM-Estimators introduced by Yohai (1987) are based on the combination of high breakdown techniques as LTS and efficient techniques as M-estimator. It has three steps (Chen 2002):

- Compute an initial solution by FAST-LTS techniques;
- Estimation of robust scale factor $\hat{\sigma}_0$;
- Compute the final solution $\hat{\mathbf{X}}_{\mathrm{MM}}$ by the M-Estimator.

4 Robust and Efficient Estimation

We present in this section the REWLSE method that simultaneously attains the maximum breakdown point and high efficiency under normal errors. First, the principle of REWLSE is given and then we show how it can be adapted to relative gravity data.

4.1 REWLSE Principle

Consider a pair of initial robust estimators of parameters and scale, $\hat{\mathbf{X}}_0$ and $\hat{\sigma}_0$ respectively. The standardized residuals are defined as:

$$\bar{v}_i = \frac{l_i - \mathbf{A}_i \hat{\mathbf{X}}_0^{\mathrm{T}}}{\hat{\sigma}_0} \tag{11}$$

A large value of $\bar{v}_i$ would suggest that the ith observation is an outlier. In order to maintain the breakdown point value of the initial estimator and to have a high efficiency, Gervini and Yohai (2002) proposed the use of an adapted cut-off value as follows:

$$t_m = \min\left\{t : \mathrm{F}_m^+(t) \geq 1 - d_m\right\} \tag{12}$$

where F_m^+ is the empirical cumulative distribution of the standardized absolute residuals, and d_{m} is the measure of the proportion of the outliers in the sample:

$$d_m = \sup_{t \geq t_0}\left\{\Phi^+(t) - \mathrm{F}_m^+(t)\right\}^+ \tag{13}$$

Φ^+ denotes the normal cumulative distribution of the random errors, $t_0 = 2.5$ is the initial cut-off value and $\{\}^+$ denotes the positive part between $\Phi^+(t)$ and $\mathrm{F}_m^+(t)$:

$$\mathrm{F}_m^+(t) = \frac{1}{m}\sum_{i=1}^{m} I(|\bar{v}_i| \leq t) \tag{14}$$

$$\Phi^+(t) = \frac{1}{\sqrt{2\pi}}\int_0^t e^{-\frac{x^2}{2}} dx \tag{15}$$

where $I(|\bar{v}_i| \leq t)$ is indicator function.

Note that if $|\bar{v}(1)| \leq |\bar{v}(2)| \leq \ldots \leq |\bar{v}(m)|$ are the order statistics of the standardized absolute residuals and $i_{\max} = \max\{i : |\bar{v}(i)| < t_0\}$, then

$$d_m = \max_{i > i_{\max}} \left\{ \mathrm{F}^{+}(\bar{v}_i) - \frac{(i-1)}{m} \right\} \tag{16}$$

More details about computing t_m are given in the Appendix.

Gervini and Yohai (2002) define the form of the weights **W** and the REWLSE estimator T_m as follows:

$$\mathrm{T}_m = \begin{cases} \left(\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{W}\mathbf{L} & \text{if } \hat{\sigma}_0 > 0 \\ \mathbf{X}_0 \text{ if } \hat{\sigma}_0 = 0 \end{cases} \tag{17}$$

The weight function **W** is chosen in order to have a hard-rejection to outliers

$$w_i = \begin{cases} 1 & \text{if } |\bar{v}_i| < t_m \\ 0 & \text{if } |\bar{v}_i| \geq t_m \end{cases} \tag{18}$$

$$\mathbf{W} = \mathrm{diag}\,(w_1, w_2, \ldots, w_m) \tag{19}$$

Note that with this weight, $\mathrm{T}_m$estimator maintains the same breakdown point value of the initial estimator T_0.

4.2 REWLSE Adaptation to Gravimetric Data

The adjustment model by weighted least squares as in Eq. 5 is modified in order to have a single design matrix (**M**) and single weight matrix (**P**):

$$\mathrm{A^TWA} + \mathrm{A_g^TW_gA_g} = \mathrm{M^TPM}, \tag{20}$$

where $M = \begin{bmatrix} A^T & A_g \end{bmatrix}$ and $P = \begin{bmatrix} W \\ W_g \end{bmatrix}$.

LS-estimator can be written then:

$$\mathrm{T} = \left(\mathrm{M^TPM}\right)^{-1}\mathrm{M^TPL} = \mathrm{QPL} \tag{21}$$

where $\mathbf{L} = \begin{bmatrix} \Delta\mathbf{L} \\ \mathbf{L_g} \end{bmatrix}$ and **Q** is a matrix of the same size as $\mathbf{M^T}$.

By applying REWLSE method to gravity data differences using previous weight function, the normal matrix became singular. Thus, it is necessary to investigate an adaptable weight function for gravimetric data in order to obtain not only a unique solution of the problem, but also to ensure the efficiency and the robustness of the estimated parameters.

If the observations are contaminated by outliers $\mathbf{L}_\delta = \mathbf{L} + \delta$, Zhu (1986) has deduced that all robust estimates used in surveying adjustment can be written as:

$$\mathbf{T}_\delta = \left(\mathbf{M^TP}_\delta\mathbf{M}\right)^{-1}\mathbf{M^TP}_\delta\mathbf{L} = \mathbf{QP}_\delta\mathbf{L}_\delta \tag{22}$$

The Eqs. 5 and 22 are both weighted least squares solutions but differ by their weights. In (22), we use a special weight instead the inverse of the variance–covariance matrix as in (5) in order to reduce the impact of the outliers.

Assuming that Q is approximately invariable, we obtain then:

$$\mathbf{T}_\delta = \mathbf{QP}_\delta\mathbf{L}_\delta \tag{23}$$

Taking difference between $\mathbf{T}_\delta$ and **T**:

$$\Delta = \mathrm{T} - \mathrm{T}_\delta = \mathrm{Q}(\mathrm{PL} - \mathrm{P}_\delta \mathrm{L}_\delta) \tag{24}$$

The mean distance between the two estimators is: $Dm = \mathrm{E}(\Delta^{\mathrm{T}}\Delta)$. In order to make $\mathbf{P}_\delta$ optimal in the robustness sense, Dm must be minimum, i.e.

$$\frac{\mathrm{dDm}}{\mathrm{d}p_{\boldsymbol{\delta}}(i)} = 0 \tag{25}$$

The corresponding weight function under stochastic model (Zhu 1996) will be:

$$p_{\boldsymbol{\delta}}(i) = \begin{cases} p_i & \text{if } |v_i| \le k \\ \frac{p_i \sigma_i^2}{v_i^2 / r_{ii}} & \text{if } |v_i| > k \end{cases} \tag{26}$$

σ_i^2: is the variance of the ith observation without outliers. Knowing that $p_i = 1/\sigma_i^2$ and opting to the adaptive cut-off value for k, we get:

$$p_\delta(i) = \begin{cases} p_i & \text{if } |\bar{v}_i| \le t_m \\ \frac{r_{ii}}{\hat{\sigma}_0 \bar{v}_i^2} & \text{if } |\bar{v}_i| > t_m \end{cases} \tag{27}$$

r_{ii} is the partial redundancy of the ith observation which can be determined from the so-called redundancy matrix or the reliability matrix (Ding and Coleman 1995) as follows:

$$\mathbf{R} = \mathbf{I} - \mathbf{M}\left(\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}\right)^{-\mathbf{1}}\mathbf{M}^{\mathrm{T}}\mathbf{P} \tag{28}$$

The a posteriori variance–covariance matrix of the estimated parameters can be obtained by covariance propagation as follows:

$$\hat{\Sigma}_{\hat{X}} = \hat{\sigma}_0^2 (\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M})^{-\mathbf{1}} \tag{29}$$

where $\hat{\sigma}_0$ is the a posteriori robust scale factor calculated using (10). The standard errors of the estimated parameters will be the square root of each diagonal element of $\hat{\Sigma}_{\hat{X}}$.

5 Results and Comparison

The sample of data contains 56 relative gravity observations with standard errors around 30 µgal and 6 absolutes points with standard errors between 2 and 3 µgal.

In order to investigate the robustness of different estimators proposed above (LS, M, MM and REWLSE), firstly, we apply these estimators to the original real data, and then we simulate some outliers and add them to the observations. We have used a normal QQ-plot tool (or quantile–quantile plot) for displaying clearly outliers. Normal QQ-plot consists of a plot of the ordered values of standardized residuals versus the corresponding quantiles of a *standard* normal distribution. The outliers are those points that fall outside 95% simulation envelopes for the normal QQ-plot, shown as dotted lines. Figure 3 displays a normal QQ-plot for both LS and robust residuals (MM-estimator). From this figure it follows the effectiveness of the robust methods in outlier detection (5, 6, 22 and 24). The LS method does not detect any outlier. The QQ-plot is fairly linear in both cases of LS and robust residuals; consequently the original sample of gravity data is reasonably Gaussian. It should be pointed out that the detected outliers are small and do not have significant impact on the estimated parameters; consequently, it is clear that the provided data is of good quality and can be set as a reference for comparison.

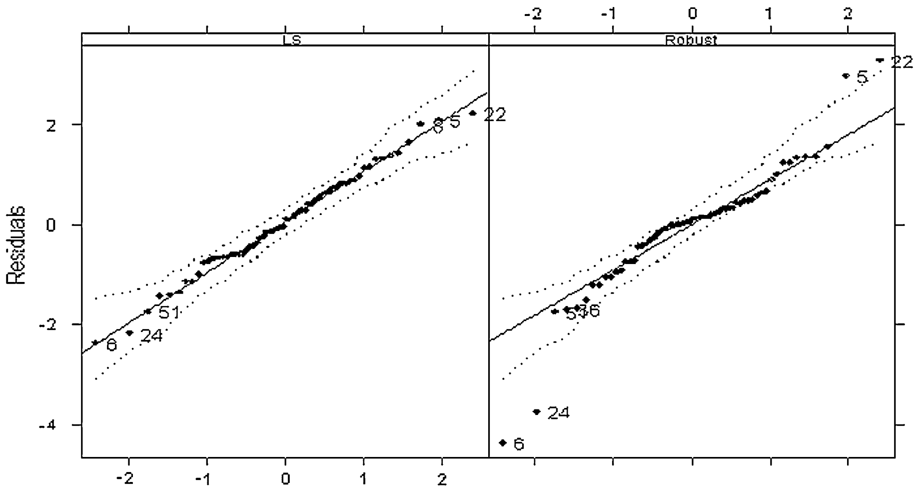


Fig. 3 Normal QQ-plot of residuals from original data, robust residuals obtained by MM-estimator. The x-axes units are quantiles of standard normal distribution

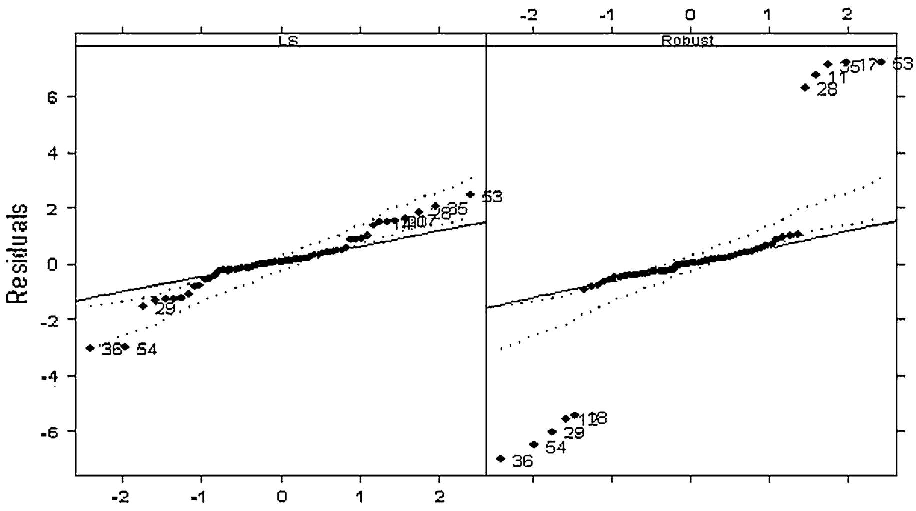


Fig. 4 Normal QQ-plot of residuals obtained by LS and robust method (MM) in case of relative gravity data contaminated by some simulated outliers

Using now a large number of outliers by simulation, the normal QQ-plot in Fig. 4 and the standard errors of the adjusted parameters in Fig. 5 can be achieved. In fact, only two small errors are detected (36, 54) using standard LS estimation and the standard deviations of the adjusted parameters are strongly affected by abnormal data which was well detected by most of robust techniques used in the present work. Furthermore, to show and compare the impact of outliers on the standard errors of the parameters obtained by different robust methods, from Fig. 5, it is clearly shown that the REWLSE method has much better results and it gave the same results as in conventional LS without outliers. The MM-Estimator is relatively more robust than the M-Estimator.

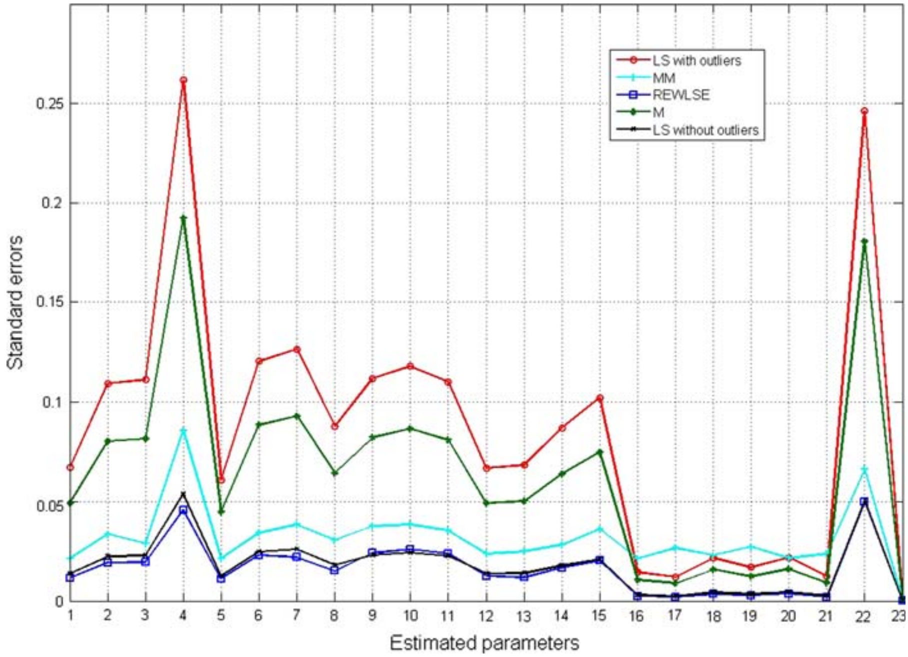


Fig. 5 Standard errors obtained from different estimators (LS with and without outliers, M, MM and REWLSE with outliers). Units are in mgal, mgal/day and mgal/CU for respectively gravity values (1–21), linear drift coefficient (22) and calibration factor error (23)

In Fig. 6, the root mean squares (RMS) values of differences $\left\{X^{\mathrm{LSE}}_{\text{without outlier}} - X^{\text{robust}}_{\text{with outlier}}\right\}$ of the parameters are plotted. This figure reveals that the gravimeter parameters (linear drift and calibration factor error) and the gravities of the six absolute stations (16–21) are efficiently estimated by REWLSE method. However, some relative gravity values of points (6, 10 and 11) are relatively deviated from the reference data (LSE without outlier). This is due to the gravimetric network structure as is explained in Sect. 2.2 and the failure of the initial estimator (LTS) to identify the correct outliers. From Table 1 we can deduce that some observations (19 and 22) are identified as outliers by FAST-LTS estimation instead of the 18th and 23rd observations in which we have added outliers. It is also important to notice that the ties (19 and 18) and (22 and 23) are acquired respectively in opposite ways.

In order to show the performance of REWLSE, 10 outliers are added to the observations and by increasing the magnitude of outliers using the following formula: $L_\delta = L + k \cdot (3\sigma)$, $k = 1, 2, \ldots$ with $\sigma = 0.03$ mgal. Next we compute for each case (k = 1, 2, and 3) the following statistics:

- three estimators T1, T2 and T3:

  T1: Weighted LS without outliers
  T2: Weighted LS with outliers
  T3: REWLSE with outliers

- two mean distances *Dm1* and *Dm2* according to $\mathbf{E}(\Delta^{\mathbf{T}}\Delta)$given in Eq. 24:

  Dm1: mean distance between T1 and T2
  Dm2: mean distance between T1 and T3

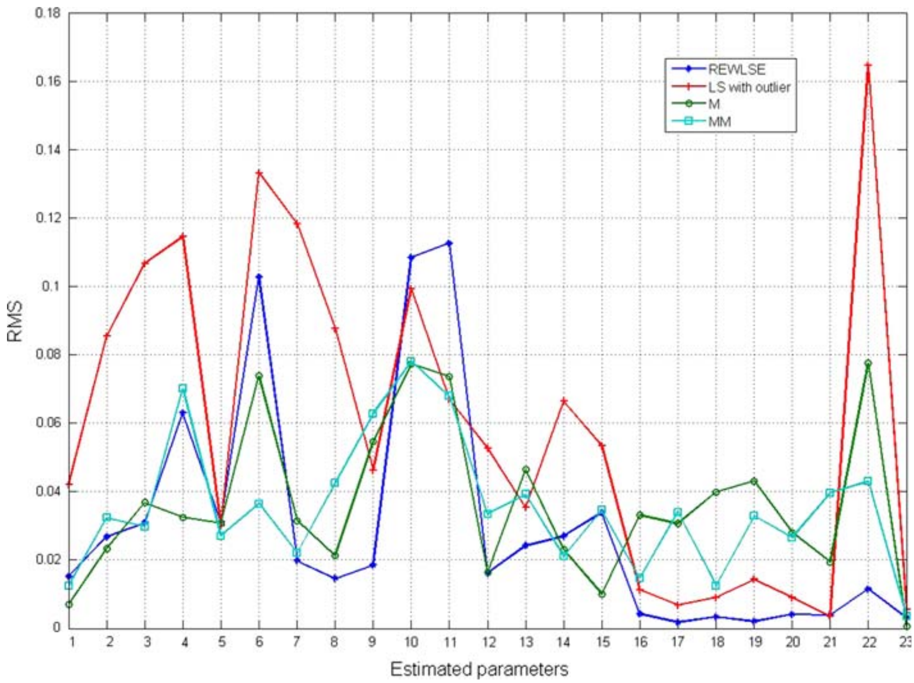


Fig. 6 RMS values obtained by different methods (LS, M, MM, and REWLSE with outliers)

- robust scale factor $\hat{\sigma}_0$ according to the Eq. 10 and standard error for each estimator.

The results are summarized in Tables 2, 3 and 4. The first values in bold *Dm1* and *Dm2* mean that the mean distances are computed as considering the completed vector of unknowns (the parameters of gravity vector and gravimeter) and the second values are referred to only parameters of gravity vector. Robust scale factor is computed only for T3 estimator.

By analyzing the values of the *Dm1* and *Dm2* in Tables 2, 3 and 4, it seems clearly the robustness of REWLSE notified by small values of *Dm2* which means that there is no significant difference between the estimated parameters by LS without outliers and REWLSE with outliers.

Other conclusions can be deduced from the tables, they concerns firstly the adapted cut-off values t_m which increase by increasing the magnitude of the outliers, and secondly, the percentage of outliers detected depends on the performance of the initial estimator (LTS in this case) which sometimes unable to detect small outliers Hekimoglu (2001) and (2005). Finally, we can conclude that the obtained results show how well the high efficiency and robustness of REWLSE is.

6 Conclusions

In this paper, we have realized a comparative study on robustness and efficiency between different robust estimators such as (M, MM and REWLSE) in the case of contaminated data. We have also shown from the normal QQ-plot that the conventional method LS is very sensitive to the outliers. The comparison between different robust estimators by adding simulating outliers allowed us to pick out some conclusions as follows: the outlier's

Table 1 Sample of relative gravity data and some results obtained by FAST-LTS estimation with 12 simulated outliers

No. observation	From–to	Gravity diff. (mgal)	Simulated outliers	Standardized robust residuals	Detected outliers
8	2–1	101.818		1.8073	
9	5–1	−79.760		1.0483	
10	1–5	79.755		−0.9734	
11	3–6	−18.975	+	11.6106	*
12	6–7	−91.643	+	−9.9558	*
13	7–6	91.356		−0.3210	
14	6–3	19.273		−0.3210	
15	7–8	47.110		0.4849	
16	8–7	−47.102		0.1639	
17	8–9	36.950	+	12.1563	*
18	9–10	29.118	+	0.3257	
19	10–9	−29.407		−10.6688	*
20	9–8	−36.639		−0.3257	
21	10–11	41.653		0.3257	
22	11–12	−17.164		−8.2446	*
23	12–11	16.925	+	−0.3257	
24	11–10	−42.023	+	−13.5650	*
25	12–13	−81.945		−0.7669	
26	13–4	−308.887		0.6818	
27	4–13	308.898		0.2126	
28	13–12	82.223	+	11.1740	*

+ observation with simulated outlier; * observation is detected as outlier by FAST-LTS

Table 2 Robustness and efficiency analysis of REWLSE with magnitude of outliers ($k = 1$)

Estimator	Standard error	$\hat{\sigma}$ (mgal)	*Dm1*	*Dm2*
T2	1.88			
T1	0.99	0.028	**0.008**/0.007	**0.008**/0.003
T3	1.02			

Initial cut-off value $t_0 = 2.5$, adapted cut-off value $t_m = 2.64$ and % of detected outliers = 8/10

Table 3 Robustness and efficiency analysis of REWLSE with magnitude of outliers ($k = 2$)

Estimator	Standard error	$\hat{\sigma}$ (mgal)	*Dm1*	*Dm2*
T2	3.08			
T1	0.99	0.030	**0.031**/0.023	**0.015**/0.014
T3	1.00			

Initial cut-off value $t_0 = 2.5$, adapted cut-off value $t_m = 5.48$ and % of detected outliers = 10/10

effect is remarkable on standard deviations computed by M-Estimator which do not tolerate a high percentage of outliers in the observations, which is not the case with the MM-Estimator that has a high breakdown point value, but this gain in robustness has induced

Table 4 Robustness and efficiency analysis of REWLSE with magnitude of outliers ($k = 3$)

Estimator	Standard error	$\hat{\sigma}$ (mgal)	*Dm1*	*Dm2*
T2	4.52			
T1	0.99	0.026	**0.078**/0.573	**0.003**/0.001
T3	0.97			

Initial cut-off value $t_0 = 2.5$, adapted cut-off value $t_m = 8.76$ and % of detected outliers = 10/10

undesired effects on efficiency noticed by an increase in standard deviations relatively to our proposed techniques (adapted REWLSE). In fact, the standard error values obtained by REWLSE method justifies the high efficiency of this estimator, and the use of an adapted cut-off values have allowed to keep the robustness of the initial estimator (LTS). In other part, the adapted weight function proposed in this paper does not only allow us to overcome the rank deficit problem but also to estimate the unknown of gravity vector and gravimeter parameters with high breakdown point. However the initial estimator (FAST-LTS) may fail because of the complexity of the gravimetric network structure as we have already seen above which consequently affect the values of the estimated gravity stations. Therefore, it is necessary to find a good strategy in the choice of the best initial estimator with high breakdown point. Finally, we can conclude that the performance of the adapted REWLSE to relative gravity measurements depends essentially on the robustness of the initial estimator and also on the design of gravimetric network.

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Appendix: algorithm for computing cutoff values t_m

begin
choose initial t_0
compute $\bar{v}_i$ using (11)
find **raord =** ascending order ($\bar{v}_i$)
compute $\Phi^+(\mathbf{raord})$
compute $\mathrm{F}_m^+ = \frac{1}{m}[0 \quad 1 \quad 2 \quad \ldots \quad m-1\,]$
compute **dif** $= 2\Phi^+(\mathbf{raord}) - 1 - \mathrm{F}_m^+$
  find **index** $= \{\mathbf{raord}(i) \geq t_0\}$
    if (**index** is empty)
      $d_m = 0$
    else
      $d_m = \max(\mathbf{dif}(index))$
    endif
    if ($d_m > 0$)
      $t_m = \mathbf{raord}(m\text{-integer}(md_m))$
    else
      $t_m = \infty$
    endif
  end

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