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DENSITY LAYER MODELS FOR THE GEOPOTENTIAL *

Abstract

Spherical harmonic representation of the geopotential suffers from convergence problems in both the mathematical and numerical senses ; it also is not suitable for use with the terrestrial gravity data, which is distributed very unevenly over the surface of the earth. Of the alternative mathematical models, the simple density layer offers the most mathematical and computational simplicity and avoids the problems associated with spherical harmonics. In practice low order spherical harmonic models have been combined with a density layer for a complete model of the geopotential. Results from satellite data alone and from combinations of satellite and gravity measurements have been published already ; improved models will be obtained as new data become available.

1. — Introduction

Even before the first satellites were launched, their usefulness in determining the earth's gravitational field was anticipated. [O'Keefe, (1957)]. Many surprises, however, greeted investigators at each step of the way. First, the oblateness (or flattening) of the earth was found to be much farther from Internationally accepted value of **1/297** than anticipated [Bucher (1958), O' Keefe, et al. (1958)]. Then the "pear–shape" term was discovered [O' Keefe et al. (1959a), (1959b)], and the ellipticity of the equator confirmed [Izsak (1961, 1964)]. Studies of hydrostatic flattening were resumed and the earth was found to be under more stress than previously assumed. [Henriksen (1960), O' Keefe (1960)]. A rotating body in hydrostatic equilibrium has centripetal force caused by rotation everywhere balanced by its self gravitation so that the surface is an equipotential. Obviously this is not locally true for the earth, but it was found to be not quite true on a global scale also.

The geopotential was represented by a truncated form of the series

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^n P_n^m \, (\cos\phi)\,(C_n^m \cos m\lambda + S_n^m \sin m\lambda)\right]$$

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which is a solution for Laplace's equation

$$\nabla^2 U = 0$$

valid outside a sphere of radius R centered at the origin ; the boundary conditions for this solution are (1.) $U(R, \phi, \lambda)$ is square–integrable ($\int U^2 \, d\sigma$ exists if the integration is taken over the sphere $r = R =$ constant) and (2.) $U(\infty) = 0$. Table 1 is a key to the terminology associated with this series. Very detailed discussions of this series and its applications may be found in Kaula (1966), sections 1.2, 1.3, 3.3. and Hotine (1969), chapter 21.

Table 1

Term	n	m
Oblateness	2	0
Pear–shapedness	3	0
Equatorial ellipticity	2	2
Zonal	>0	0
Tesseral	>0	$1 < m < n$ *or* $1 < n < m$
Sectorial	>0	n
"Forbidden" (= 0)	1	1, 0
"Forbidden" (= 0)	2	1

As data accumulated, higher and higher order zonal harmonics were determined ; more and more tesseral harmonics were found. [See, for example, Lundquist and Veis (1966); Gaposchkin and Lambeck (1970)].

In determining zonal harmonics, investigators found that each time a new, distinct orbit was used in a solution, a new harmonic coefficient should be added to the unknowns. Unless an additional coefficient were determined in the new solution, the fit of the orbits and the standard deviation of the parameters would deteriorate. The rate of decrease of the coefficients was very slow after order three. The phenomena, which Dr. John Vinti has called "the King–Hele Catastrophe", after one of the pioneering investigators, D. G. King Hele, has been found equally true for the tesseral harmonic coefficients. It has been quantified into a rule of thumb by Kaula (1963) :

$$\sigma_n = \sigma(\bar{C}_n^m, \bar{S}_n^m) = \frac{1}{\sqrt{2n+1}} \sum_{m=0}^{n} (\bar{C}_n^{m^2} + \bar{S}_n^{m^2})^{\frac{1}{2}} = 10^{-5} / n^2$$

$$\bar{C}_n^m, \bar{S}_n^m = \sqrt{\frac{(n+m)!}{(n-m)!\ (2n+1)(2-\delta m_0)}}\ (C_n^m, S_n^m)$$

Although geodesists had anticipated a very rapid decrease in the harmonic coefficients, there is no reason they should have since the coefficients of a series of spherical harmonics are known to have the property [Jeffries, (1959)] .

$$\bar{C}_n^m = 0\,(n^{-7/2}),\ \bar{S}_n^m = 0\,(n^{-7/2})$$

So

$$(\bar{C}_n^m)^2\ ,\ (\bar{S}_n^m)^2 = 0\,(n^{-7})$$

$$\sigma_n = \frac{0\,(\sqrt{n \times n^{-7}}}{\sqrt{n}} = 0\,(n^{-7/2})\,.$$

Indeed, the harmonic coefficients are not decreasing as rapidly as they should, but some geodesists must have expected, at least subconsciously, an exponential decay of σ_n with order

$$\sigma_n = 0\,(k^{-n}),\ \ k > 1$$

This would assure the convergence of the geopotential series in spherical harmonics everywhere outside the earth [see Morrison (1969)]– note that $(k^{-n})^{1/n} = k^{-1} = < 1 \ldots$; if one is familiar with the ***root test*** it is clear that the coefficients ***would*** have a strong effect favoring convergence). Moreover, the gravity anomalies computed from satellite orbits agree rather well with those obtained from astro–geodetic determinations, at least on a continent–wide scale. When the local datum is rotated and translated to conform to the coordinate system of the orbital–analysis, agreement between gravimetric and satellite results is satisfactory [Fischer, et al. (1970) ; Strange (1966)]. Astro–geodetic geoid maps –for continental areas– have been available for some time, so no one should have been surprised at the magnitude and very gradual decrease of the harmonic coefficients, especially since much more detailed information is indicated from the gravimetry than by satellite results. Someone could have foreseen that σ_n would decrease as a negative power of **n** , even though there was no way to anticipate the power is **–2** . But all this is easy to see in retrospect.

2. – The Alternatives

Several methods have been proposed for avoiding the difficulties inherent in spherical harmonics. One class of alternatives is the use of solutions of Laplace's equation in coordinate systems other than the spherical. General solutions for Laplace's equation have been found in spheroidal and ellipsoidal coordinate systems. For these coordinate systems the surface of equal radial coordinate corresponds to an ellipsoid of revolution (we should use the oblate spheroid development) and a tri–axial ellipsoid, respectively. Each solution is the natural one for problems with boundary values given over such a surface ; and each of these surfaces is a progressively better approximation of the earth.

Considerable detail is given on spheroidal coordinates by Hotine (1969), whose tensor calculus methods are very useful for coping with exotic coordinates systems. H. G. Walter (1970), among others, has been attempting to bring the ellipsoidal harmonics formulas to the same state of development enjoyed by spherical harmonic analysis. Spheroidal harmonics are very little more complex than spherical ones, but the ellipsoidal harmonics involve the non–elementary Lamé functions. While it is true that these more sophisticated harmonics improve the situation with respect to mathematical convergence, they are no help in the problem of numerical convergence. The higher coefficients will still decrease at the same slow rate. Since one can work only with truncated series, the use of spherical harmonics and the more exotic varieties will differ in detail but not in kind.

Discrete variable techniques also are available. No one has advocated the straightforward discrete variable method, which consists of covering the domain of definition (region where the partial differential equation holds) of the boundary value problem with a grid of points. Once the partial differential equations is replaced by a finite difference equation the problem reduces to a solution of a set of linear equations. Solving these, in turn, is equivalent to inversion of a matrix whose rank is equal to the number of grid points used [Smith (1965)]. Obviously this technique, so successful with many engineering problems, is unsuited for geodesy since the data is unevenly distributed and even if adequate data were available, the computations would be formidable due to their sheer size.

For Laplace's equation we have the solutions

a. Density Layer [Webster (1955)]

$$V(\bar{r}) = \int_S \rho \frac{d\sigma}{r},$$

where $\mathbf{S}$ is the boundary surface, $d\sigma$ an area element, ρ an integrable function defined on $\mathbf{S}$ and r the distance between the point of interest $\bar{r}$ and the area element.

b. Stokes' formula [Heiskanen and Moritz (1967)]

$$\Delta V(\bar{r}) = \frac{R}{4\pi} \int\int_\sigma \Delta g\, S(\psi)\, d\sigma,$$

$$S(\psi) = \frac{1}{\sin(\psi/2)} - 6 \sin\frac{\psi}{2} + 1 - 5\cos\psi$$

$$- 3\cos\psi \; \ell n \; (\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2})$$

Δg = gravity anomaly, ΔV = anomalous potential, ψ = angle between the fixed point $\bar{r}$ and the surface element $d\sigma$. In theory and practice these forms of the solution are a hybrid of the analytic and numerical forms. To find V at any point one must perform a ***quadrature***, which will have to utilize a discrete variable technique.

Solution **b** is appealing in that it uses gravity anomalies, which are the data obtained during a gravity survey. Solution **a**, in which ρ may be thought of as a density layer (mass per area), is somewhat less esthetically appealing since it's physical basis is a rather artificial one, but it is very much simpler to implement for computation, especially when satellite orbits are being used to determine the geopotential. Both formulas circumvent convergence problems, their inherent difficulty being discretization error from the quadrature formula used. With appropriate knowledge of the variation of the boundary values (in this case, gravity measurements) the errors may be estimated,

Each approach has had its advocates. John P. Vinci (1968) and Karl R. Koch (1968a, 1968b) independently have recommended the density layer model ; Vinti putting it on a sphere enclosing the earth and Koch, on the surface. Koch has pursued the application of the method while at the Geodetic Research and Development Laboratory of the National Ocean Survey [Koch and Morrison (1970); Koch and Witte, (1971)]. The investigation of the use of the Stokes' formula approach is being pursued by Richard Rapp and his students [Obenson (1970)] at the Ohio State University. Results from this method already have been published by Kurt Arnold (1968).

3. – The Surface Density Layer Model

Spherical harmonics are not completely without virtues, however. For one thing, they can be generated very handily by recurrence formulas [Gulick (1970)]. And for orbital analysis they are a very powerful representation. For one thing, the "damping factor", $(R/r)^{n+1}$, which is damning for surface representations of gravity, is favorable for convergence at satellite altitudes [Kaula (1969)]. Moreover, the potential is readily expanded in Keplerian elements, which is very useful in analyzing orbits qualitatively [Kaula (1966, 1969)] or integrating approximate orbits for long periods [Morrison (1970)]. This development is complicated but is based upon classical work on the theory of planetary motion and thereby is not obscure.

The advantages of spherical harmonics need not and should not be lost ; they may be used to represent the long–wave part of the geopotential. The dominant central force and oblateness terms are represented by zeroth and second harmonics, respectively. "Pear shapedness" is represented by the third zonal harmonic and the ellipticity of the equator by the harmonics of degree two and order two. Recent studies [Morrison (1971)] have shown that the effects of discretization errors are drastically reduced in orbits computed from geopotential representations if the density layer used has no zeroth, first, or second harmonic components. Higher frequencies are less readily identified with physical characteristics of the earth and for higher and higher orders become smaller, more random, and more suited for absorption into a density layer.

For the initial studies a spherical harmonic field of degree and order four was used as a reference and the higher frequency variations were represented by the surface density layer. To help define the surface for the density layer, the equipotential defined by

$$U_0 = \frac{\mu}{\sqrt{R^2 - b^2}} \arctan \frac{\sqrt{R^2 - b^2}}{b} + \frac{1}{3}\omega^2 R^2$$

was used. This is the potential on the surface of a level ellipsoid [Molodenskii, et al. (1962)]. A radius vector

$$r^* = \frac{\mu}{U_0}\left[1 - \sum_{n=1}^{N} \sum_{m=0}^{n} \left(\frac{R}{r^*}\right)^n P_n^m (\cos\phi) x \left(C_n^m \cos m\lambda + S_n^m \sin\lambda\right)\right]$$

$$\div \left[1 - \frac{1}{2U_0}\omega^2 r^{*2} \cos^2\phi\right]$$

for a particular ϕ and λ is computed by iteration and the topographic height is added to this to define the location of the surface layer. [Koch (1968) ; Koch and Morrison (1970)]. The surface was then divided into areas bounded by parallels and meridians within each division the density of the layer was assumed to be uniform. Each division was subdivided into four parts and the center of the parts was used to compute $\| \bar{r} - \bar{r}_x \|$, the distance from the satellite to the perturbing mass. The integral

$$\iint \frac{\rho \, d\sigma}{\| \bar{r} - \bar{r}_x \|}$$

is replaced by a sum

$$\sum_{i=1}^{4} \frac{m_i}{\| \bar{r} - \bar{r}_x \|}$$

where the masses $\mathbf{m}_i$ are weighted according to the area they represent. Physically, the layer is represented by four point masses lying on the surface. Varying the quadrature scheme according to the separation $||\bar{r} - \bar{r}_x||$ remains to be tried [Morrison (1971)].

For the initial determination of the values for the density layer [Koch and Morrison (1970)] the surface was divided up into 48 areas. Smithsonian Astrophysical Observatory optical data was used. After a model with only satellite data was obtained, the results were improved by the addition of ground gravity data. Subsequently, more and better gravity data were incorporated to form a new solution [Koch (1970)]. To obtain more satellite data, Doppler observations were tried next [Koch and Witte (1971)]. A density layer with 104 parameters was used.

Other improvements have been made in the determinations as well as the inclusion of more and better data and more parameters. A radiation pressure model and a drag model have been added to the luni–solar perturbations used in the first solution. The spherical harmonics reference potential is now of order 7 [Koch and Witte (1971)]. To increase efficiency, a density layer model is no longer used to determine the reference orbits. The best possible orbits attainable with the spherical harmonic model of the gravity field are determined and then the density parameters are computed from residuals and parameter–sensitivity equations computed along with the reference orbit.

For every determination an efficient technique for integrating the variational equations has been employed. Since the variational equations are *linear* differential equations, even though the trajectory equations are not, the corrector formula for integrating the variational equations (to obtain the state–transition matrix and parameter–sensitivity matrix) can be solved directly once the trajectory equations are integrated [Riley, et al. (1967)]. (Note : The state–transition matrix is the partials of position and velocity, $\bar{r}$ and $\dot{\bar{r}}$ with respect to initial conditions $\bar{r}_0$ and $\dot{\bar{r}}_0$; the parameter–sensitivity matrix is the partials of $\bar{r}$ and $\dot{\bar{r}}$ with respect to any physical parameters, such as $\mathbf{C}_n^m$, $\mathbf{S}_n^m$, μ , or density parameters).

There is little reason to suspect a gross error in these density layer models. In fact, Vinti (1970) has produced density layer models defined on an enclosing sphere using two different gravity models originally expressed in spherical harmonics and with a reference potential constructed only to obtain a good fit to a special functional form useful in orbit analysis. Agreement between these derived models and those obtained directly from analyzing the data is quite good, particularly in areas of steep density gradients.

Even better density layer models will be determined as data become available. Charles R. Schwarz (1970) has investigated the feasibility of doing a solution for density layer values using satellite–to–satellite tracking. When the anticipated order of magnitude improvements in geodetic data acquisition are available from satellite altimetry and other new data acquisition technology, such as satellite–to–satellite tracking, "drag–free" satellites, and laser tracking [Mit (1970)], an adequate model [Kaula (1969)] will be available.

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