



Alternative transformation from Cartesian to geodetic coordinates by least squares for GPS georeferencing applications

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ARTICLE INFO

Article history:

Received 25 August 2011

Received in revised form

27 October 2011

Accepted 28 October 2011

Available online 6 November 2011

Keywords:

Cartesian coordinates

Geodetic coordinates

Inverse transformation

Least squares

ABSTRACT

The inverse transformation of coordinates, from Cartesian to curvilinear geodetic, or symbolically $(x,y,z) \rightarrow (\lambda,\varphi,h)$ has been extensively researched in the geodetic literature. However, published formulations require that the application must be deterministically implemented point-by-point individually. Recently, and thanks to GPS technology, scientists have made available thousands of determinations of the coordinates (x,y,z) at a single point perhaps characterized by different observational circumstances such as date, length of occupation time, distance and geometric distribution of reference stations, etc. In this paper a least squares (LS) solution is introduced to determine a unique set of geodetic coordinates, with accompanying accuracy predictions all based on the given sets of individual (x,y,z) GPS-obtained values and their variance–covariance matrices. The (x,y,z) coordinates are used as pseudo-observations with their attached stochastic information in the LS process to simultaneously compute a unique set of (λ,φ,h) curvilinear geodetic coordinates from different observing scenarios.

Published by Elsevier Ltd.

1. Introduction

Many scientists have investigated the so-called, non-trivial, inverse transformation of coordinates from Cartesian (x,y,z) to curvilinear (orthogonal) geodetic coordinates (λ,φ,h) . Both sets of coordinates are defined with respect to any arbitrary geodetic Cartesian reference frame and, in the case of geodetic coordinates, a complementary rotational ellipsoid with center at the origin of the Cartesian frame, its semi-minor axis coincident with the z -axis, and its semi-major axis on the equatorial plane defined by the x and y axes. The selected rotational ellipsoid is typically the GRS80 ellipsoid as adopted by the International Association of Geodesy (Moritz, 1992). For an exhaustive study of the many inverse transformations available to the user, the readers may consult Featherstone and Claessens (2008) and Awange et al. (2010, p. 157) where they will find a partial list of approaches to solve this specific transformation problem. A recent article by Shu and Li (2010) cites newly developed algorithms to compute geodetic coordinates not mentioned in any of the above mentioned references. For completeness, it should also be mentioned that the International Earth Rotation and Reference System Service (IERS) recommends the use of Fukushima's (1999) iterative method. However, all these transformation equations and algorithms were analytically developed to compute coordinates in a one-by-one point basis, that is, given the

Cartesian coordinates of a point determine the equivalent curvilinear geodetic coordinates at the same point; therefore they are deterministic methods but not stochastic methods.

The alternative method presented herein takes advantage of the large number of (x,y,z) determinations that we sometimes have on hand these days at a single particular point when processing GPS data. The main idea that we are proposing is to obtain the most accurate curvilinear geodetic coordinates obtainable from the complete set of available GPS “detrended” time series (x,y,z) coordinates (e.g. Teza et al., 2010). Not only are the results going to be statistically meaningful, in a least squares (LS) sense, because as a byproduct the full variance–covariance (v – c) matrix for this uniquely derived triplet of coordinates can be determined. This statistical element is missing from the standard transformation formulas mentioned above. Other options, as for example, taking the weighted mean of all individually computed (x,y,z) values and finally transform them to (λ,φ,h) using any of the current methods will not be as complete, statistically speaking, as the procedure that will be outlined in this paper. If nothing else, because the v – c matrix of each individually processed GPS point (x,y,z) is known and taken into consideration in the LS solution. This information is not properly exploited when taking any other type of statistical sample averages.

As an immediate practical application of this procedure, one may think of the calculation of a single unique set of geodetic coordinates for a point, referred to a predefined datum ellipsoid, determined from a set of original GPS-processed (x,y,z) solutions. The intention here is to get the “best” (λ,φ,h) coordinates for each

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point that, subsequently, could be stored in the type of geodetic databases that are constantly revised and updated because of the introduction of new GPS positioning data.

Present-day GPS applications allow for the archiving of (x,y,z) coordinates of the same point perhaps determined at different times by many practitioners using a diverse spectrum of observing session durations (e.g. 1 h vs. 2 h, etc.). However, it is known that the accuracy of GPS processed solutions is highly dependent on the total time span of the observing session (Eckl et al., 2001; Soler et al., 2006); therefore, the input v-c matrix of the Cartesian coordinates implicitly contains the quality of the (x,y,z) pseudo-observations.

The recommended methodology can employ one and every one or all of these thousands of independent sets of (x,y,z) coordinates at a single point (perhaps obtained from different GPS campaigns) with their available stochastic model to compute, through a LS procedure, a unique value for the curvilinear geodetic coordinates at the same point making full use of the available statistics, which, as is well known, are highly dependent on many factors, such as total observation time span, atmospheric conditions, etc.

At the end, if required, a straight one-by-one direct transformation from geodetic to Cartesian (see Eq. (1) below) could be implemented by adding the corresponding statistics after transforming the final v-c matrix of the curvilinear coordinates just determined by LS to the Cartesian v-c matrix.

2. Mathematical formulation

The basic mathematical relationship between Cartesian and orthogonal curvilinear geodetic coordinates is attributed to Helmert (1880, p. 136) and can be written in matrix form as (see e.g. Soler, 1976):

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ (N(1-e^2)+h)\sin\varphi \end{Bmatrix} \quad (1)$$

where

$$N = \frac{a}{W} \text{ is the principal radius of curvature along the prime vertical} \quad (2)$$

$$M = \frac{a(1-e^2)}{W^3} \text{ is the principal radius of curvature along the meridian} \quad (3)$$

$$W^2 = 1 - e^2 \sin^2\varphi \quad (4)$$

$$e^2 = 2f - f^2 \quad (5)$$

In all the above equations, a and f are the semi-major axis and flattening of the selected ellipsoid, respectively.

2.1. Least squares methodology

The theory described herein follows the mathematical reasoning and matrix notation described in Leick (2004, p. 110).

The general functional expression (mathematical model) used in the adjustment is given by Eq. (1) and can be written symbolically as

$$\ell_a = \mathbf{f}(\mathbf{x}_a) \quad (6)$$

where ℓ_a denotes the vector of n adjusted observations and \mathbf{x}_a denotes u adjusted parameters (unknowns).

From Eq. (1) we can write explicitly the required set of variables as follows:

The column matrix containing the unknowns ($u=3$) is

$${}_{3 \times 1} \mathbf{x}_a = \begin{Bmatrix} \lambda \\ \varphi \\ h \end{Bmatrix} \quad (7)$$

while the column matrix of observables takes the form:

$${}_{3n \times 1} \ell_b = \begin{Bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_1 \\ \vdots \\ \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_n \end{Bmatrix} \quad (8)$$

Notice that the total number of repeated observations of the same point is n , and that the total number of equations in (6) is $r=3n$.

According to Leick (2004, p. 111), the least squares solution of the adjustment model can be written in matrix notation as

$${}_{3 \times 1} \hat{\mathbf{x}} = -\mathbf{N}^{-1} \mathbf{u} \quad (9)$$

where the normal equation matrices are given explicitly by

$${}_{3 \times 3} \mathbf{N} = \mathbf{A}^T \mathbf{P} \mathbf{A} \quad (10)$$

and

$${}_{3 \times 1} \mathbf{u} = \mathbf{A}^T \mathbf{P} \ell \quad (11)$$

The design matrix \mathbf{A} in Eqs. (10) and (11) is computed according to the following matrix expression:

$$\mathbf{A}_{3n \times 3} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\mathbf{x}_0, \ell_b} = \begin{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial h} \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial h} \\ \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial h} \end{bmatrix}_1 \\ \begin{bmatrix} \frac{\partial f_4}{\partial \lambda} & \frac{\partial f_4}{\partial \varphi} & \frac{\partial f_4}{\partial h} \\ \frac{\partial f_5}{\partial \lambda} & \frac{\partial f_5}{\partial \varphi} & \frac{\partial f_5}{\partial h} \\ \frac{\partial f_6}{\partial \lambda} & \frac{\partial f_6}{\partial \varphi} & \frac{\partial f_6}{\partial h} \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} \frac{\partial f_{r-2}}{\partial \lambda} & \frac{\partial f_{r-2}}{\partial \varphi} & \frac{\partial f_{r-2}}{\partial h} \\ \frac{\partial f_{r-1}}{\partial \lambda} & \frac{\partial f_{r-1}}{\partial \varphi} & \frac{\partial f_{r-1}}{\partial h} \\ \frac{\partial f_r}{\partial \lambda} & \frac{\partial f_r}{\partial \varphi} & \frac{\partial f_r}{\partial h} \end{bmatrix}_n \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_1 \\ \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_n \end{bmatrix} \quad (12)$$

The numerical value of the matrix \mathbf{A} is determined using any initial approximation of the parameters $\mathbf{x}_0 = \{\lambda_0 \ \varphi_0 \ h_0\}^T$. Because the model is not linear, the least squares process may need to be iterated if the initial approximation of the curvilinear coordinates is inaccurate. Fortunately, as our results corroborated, this is not the case when using GPS data.

The weight matrix is computed as usual:

$$\mathbf{P}_{3n \times 3n} = \sigma_0^2 \Sigma_{\ell_b}^{-1} = \Sigma_{\ell_b}^{-1} \quad (13)$$

where σ_0 is the a priori standard deviation of unit weight assumed equal to 1 (unitless) in this investigation, and the variance–covariance matrix of the observations is a symmetric block diagonal matrix of the form:

$$\Sigma_{\ell_b} = \begin{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \text{sym.} & & \sigma_z^2 \end{bmatrix}_1 & \dots & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \text{sym.} & & \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \text{sym.} & & \sigma_z^2 \end{bmatrix}_n \end{bmatrix} \quad (14)$$

$$= \begin{bmatrix} \Sigma_{x_1 y_1 z_1} & [0] & \dots & [0] \\ & \Sigma_{x_2 y_2 z_2} & \dots & [0] \\ & & \ddots & \vdots \\ \text{sym.} & & & \Sigma_{x_n y_n z_n} \end{bmatrix}$$

Notice that because the repeated observations for each point are practically independent of each other (the GPS data may have been collected with different instruments, on different days and/or campaigns, using different time spans, processed using different software, etc.) the non-diagonal block (cross-covariance) matrices are assumed zero.

The vector of residuals (given in the sense: computed minus observed values) is determined by the matrix equation:

$$\hat{\mathbf{v}}_{3n \times 1} = \mathbf{A}\hat{\mathbf{x}} + \ell \quad (15)$$

Then, the a posteriori variance of unit weight is determined by

$$\hat{\sigma}_0^2 = \frac{\hat{\mathbf{v}}^T \mathbf{P} \hat{\mathbf{v}}}{r-u} \quad (16)$$

and finally, the resulting variance–covariance matrix of the adjusted parameters is

$$\Sigma_{\hat{\mathbf{x}}} = \hat{\sigma}_0^2 \mathbf{N}^{-1} \quad (17)$$

Appendix B contains MATLAB routines implementing these equations.

3. An example based on actual GPS data

In the particular exercise presented in this paper, three GPS stations located on the stable region of the North American tectonic

plate were used to collect the observational data (RINEX files) for the investigation. These three stations belong to the GPS network Continuous Operating Reference Stations (CORS) managed by NOAA's National Geodetic Survey (NGS) (Snay and Soler, 2008). The position coordinates of the three stations are well known since they are based on several years of continuous GPS observations.

Due to the great accuracies achievable with today's GPS technology and methods, when a set of coordinates is given it is important to precisely specify to what coordinate frame they are referred to and to identify the epoch at which the coordinates were computed. The knowledge of the epoch and associated velocities will permit to correct the coordinates for the effects of crustal motion. Table 1 shows pertinent information about the sample of three stations GODE (at Goddard Space Flight Center, Maryland), MNLS (at Lesueur, Minnesota) and OKDN (at Duncan, Oklahoma) selected for this experiment. Also given are the coordinate frame and epoch of the NGS adopted curvilinear geodetic coordinates (longitude, latitude and ellipsoid height) with corresponding velocities. Notice that because the stations are located on the stable North American plate, two stations (MNLS and OKDN) have zero velocities relative to the plate, although at GODE minimal velocities, presumably due to minor local tectonic activity, are reported. It should be mentioned also that station GODE was purposely selected because its coordinates were independently calculated by the IERS, without direct input from NGS. The coordinates with respect to the NAD 83 (CORS96) frame reported in Table 1 were assumed as “true” coordinate values in this investigation.

To process the GPS data, the Web-based utility Online Positioning User Service, Rapid Static (OPUS-RS) mode was employed. This software was developed by NGS (<http://www.geodesy.noaa.gov/OPUS/>) and permits the automatic processing (using a relative point positioning solution) of GPS data observed almost anywhere in the United States (Schwarz et al., 2009). There are a few blackout areas where the baselines between the fixed CORS reference stations (A minimum of 3, a maximum of 9) and the “rover” are larger than 250 km. This is a maximum distance allowed by the program because it interpolates and/or under a less favorable scenario extrapolates local ionospheric and tropospheric delays measured at several CORS within the maximum distance to estimate the corresponding delays at the rover. However, the methodology described herein is also applicable to results from any GPS software including OPUS-S (where S denotes static), another program developed by NGS that is available internationally and that is essentially distance-independent, although it requires its users to submit at least 2 h of GPS data (Soler et al., 2011).

OPUS-RS solutions of GPS data for one full month (June, 2010) for the three selected CORS stations (GODE, MNLS, and OKDN) were processed for session lengths of 2 h, 1 h, 30 min and 15 min. The residuals from the adjustment in each case using the methodology described in Section 2, see Eq. (15), transformed to the local geodetic frame (E,N,U) are depicted in Fig. 1. Notice that the original residuals are given along the global (x,y,z) axes. To transform residuals from the global (geocentric) frame to a local geodetic frame, or vice versa, we can apply the following

Table 1
NGS published CORS geodetic coordinates and velocities for the three stations used. The geodetic coordinates refer to the NAD 83 (CORS96) frame, epoch 2002.0, and the GRS80 ellipsoid. The longitude is assumed positive toward the east. The components of the velocities are given along the local geodetic frame (E,N,U).

Station	λ	φ	h (m)	v_E (mm/yr)	v_N (mm/yr)	v_U (mm/yr)
GODE	283° 10' 23".42470	39° 01' 18".18995	15.868	1.9	−0.7	−2.2
MNLS	266° 05' 35".37989	44° 26' 28".13675	239.887	0.0	0.0	0.0
OKDN	262° 02' 00".43908	34° 28' 45".50157	315.462	0.0	0.0	0.0

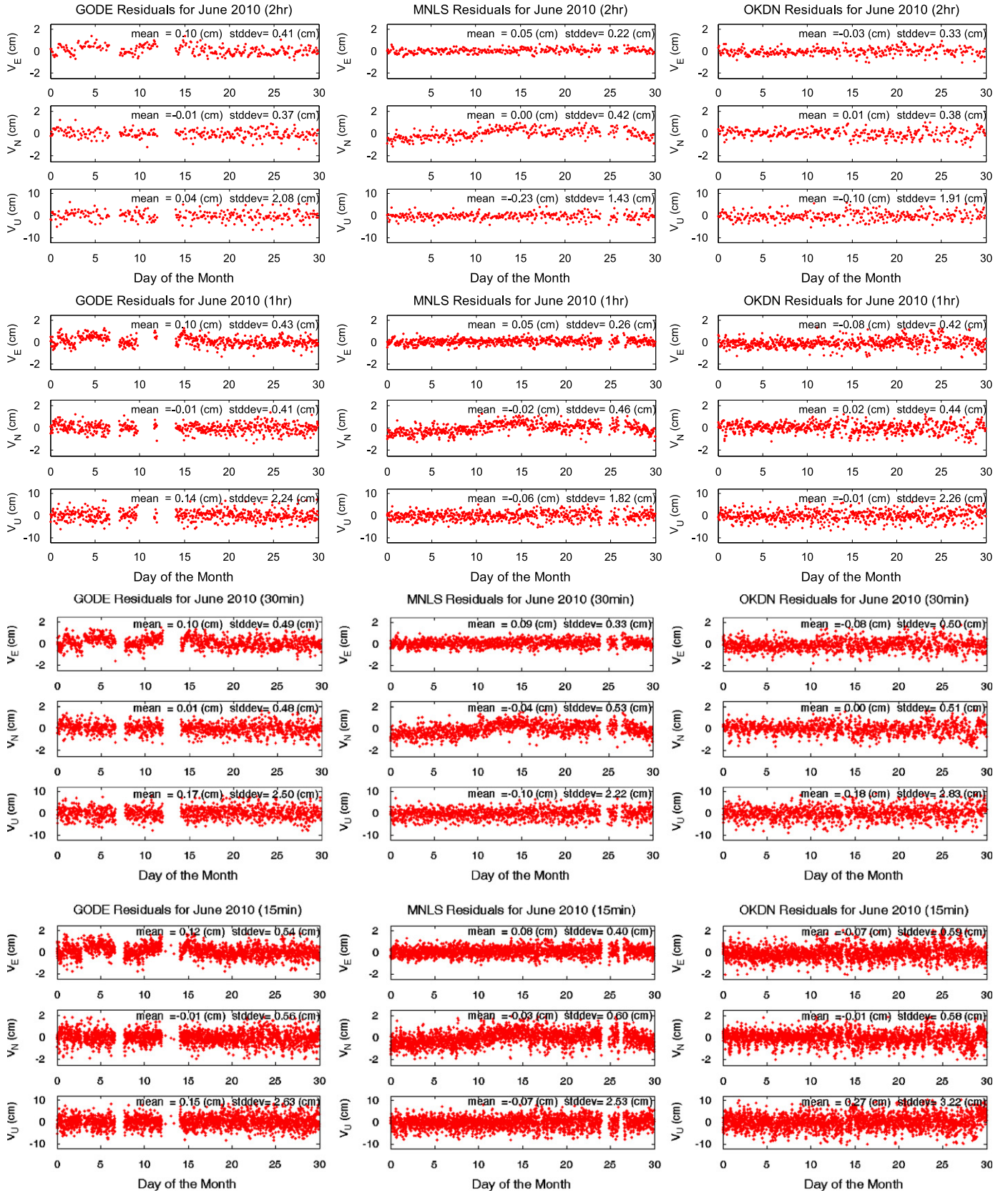


Fig. 1. Residuals from the LS solutions using data at different time spans. Each data set (2 h, 1 h, 30 min or 15 min) is used to compute a coordinate solution for the whole month.

equation:

$$\begin{bmatrix} \nu_E \\ \nu_N \\ \nu_U \end{bmatrix} = \mathbf{R} \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{bmatrix} \quad (18)$$

where the rotation matrix of the transformation between local geodetic (E,N,U) and global geocentric (x,y,z) frames at each station of coordinates (λ, φ, h) is

$$\mathbf{R} = \begin{bmatrix} -\sin \lambda & \cos \lambda & 0 \\ -\cos \lambda \sin \varphi & -\sin \lambda \sin \varphi & \cos \varphi \\ \cos \lambda \cos \varphi & \sin \lambda \cos \varphi & \sin \varphi \end{bmatrix} \quad (19)$$

Any approximate value for the coordinates (λ, φ) will suffice for the substitution in Eq. (19). In Fig. 1 the graphs for each station are stacked by columns, with the different time spans arranged by rows. In each plot, and to understand better the quality of the processed data, two important statistics of the scatter are also given: the mean and the standard deviation (std. dev.) of the plotted residuals. Initially, points with residuals larger than $3 \times (\text{std. dev.})$ were considered blunders and deleted; then a readjustment was performed and the observation residuals plotted. Table 2 shows the total number of observations and the number of points that were deleted from Fig. 1 for each specific station and time span. It should be mentioned here that the full variance–covariance matrix of the coordinates for each solution is also available from the OPUS-RS processing. However, the elements of these matrices are so-called “formal errors” implying that they are overly optimistic and do not realistically reflect the actual quality of the results. In the following section we introduce an approach to obtain reasonable statistics for the solved parameters based on the standard deviation of the residual scatter shown in Fig. 1 and the RMS errors attached to the coordinates on each OPUS-RS solution.

3.1. Scaling the optimistic variance–covariance matrix

As previously mentioned the resulting statistics of GPS-determined coordinates are consistently overly optimistic, that is, error estimates are frequently much smaller than they should be. For this reason, in the jargon of the GPS community, they are commonly referred to as “formal statistics” or “formal errors”. The main argument to simply explain this common situation is that the standard errors attached to the original GPS observables (phases and pseudoranges) do not consider various small systematic errors affecting GPS measurements. In order to get a more realistic set of statistics, the following scheme to scale the variance–covariance (v–c) matrix of the resulting coordinates was introduced in this investigation.

As previously described, all processing of GPS observations used in this experiment was done with OPUS-RS. This Web-based free service provides as output (in its “extended report”) the full v–c matrix of the (x,y,z) coordinates of the GPS solution. Granted, as it is generally the case, this v–c matrix is overly optimistic. However, the correlation between the Cartesian components holds valuable information that should not be lightly discarded. Consequently, we devised an empirical scaling scheme of the original v–c matrices based on the RMS errors (included on every OPUS-RS solution) that will not change the original correlations. The original v–c matrix from the OPUS-RS output was scaled by the following matrix equation that preserves correlations for each individual solution:

$$\Sigma_{xyz} = \mathbf{J}_s \bar{\Sigma}_{xyz} \mathbf{J}_s^T \quad (20)$$

In the above equation $\bar{\Sigma}_{xyz}$ is the original (optimistic; formal) v–c matrix from the output of OPUS-RS; Σ_{xyz} is the resulting scaled v–c matrix; and \mathbf{J}_s is a scale (Jacobian) propagation matrix, which is estimated by

$$\mathbf{J}_s = \begin{bmatrix} \frac{\sigma_x}{\bar{\sigma}_x} & 0 & 0 \\ 0 & \frac{\sigma_y}{\bar{\sigma}_y} & 0 \\ 0 & 0 & \frac{\sigma_z}{\bar{\sigma}_z} \end{bmatrix} \quad (21)$$

where σ_x , σ_y and σ_z are RMS errors along the x, y and z components provided in OPUS-RS short output. The values of $\bar{\sigma}_i$ ($i=x,y,z$) are the optimistic standard deviations extracted from the diagonal elements of matrix $\bar{\Sigma}_{xyz}$ from the OPUS-RS extended output. Since a block diagonal weight matrix is used and correlations between epochs are not considered here, the estimated LS solutions could be optimistic. To make the final standard errors realistically reflect the true dispersion of the estimated solution, the above procedure can be carried out again to scale the resulting v–c matrix based on the residual dispersions (standard deviation values) expressed in the local geodetic frame. The scaling equation is written as

$$\Sigma_{E N U} = \mathbf{J}_{s_e} \bar{\Sigma}_{E N U} \mathbf{J}_{s_e}^T \quad (22)$$

where

$$\bar{\Sigma}_{E N U} = \begin{bmatrix} (N+h)\cos \varphi & 0 & 0 \\ 0 & M+h & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\Sigma}_{\lambda, \varphi, h} \begin{bmatrix} (N+h)\cos \varphi & 0 & 0 \\ 0 & M+h & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \quad (23)$$

Table 2
Summary for observables used in the LS solutions.

Station	Time span	Number of original observations	Number of removed blunder observations	Percentage of removed blunder observations
CODE	2 h	298	8	2.68
	1 h	588	19	3.23
	30 min	1268	26	2.05
	15 min	2539	62	2.44
MNLS	2 h	331	10	3.02
	1 h	674	18	2.67
	30 min	1352	31	2.29
	15 min	2702	91	3.37
OKDN	2 h	356	13	3.87
	1 h	715	17	2.38
	30 min	1418	37	2.61
	15 min	2874	75	2.61

and

$$\mathbf{J}_{S_e} = \begin{bmatrix} \frac{\sigma_E}{\sigma_E} & 0 & 0 \\ 0 & \frac{\sigma_N}{\sigma_N} & 0 \\ 0 & 0 & \frac{\sigma_U}{\sigma_U} \end{bmatrix} \quad (24)$$

In (23), $\bar{\Sigma}_{\lambda,\varphi,h}$ is the v–c matrix for the estimated longitude, latitude and height from the LS solution. $\bar{\Sigma}_{E\ N\ U}$ is the corresponding v–c matrix expressed in linear units referred to the local geodetic frame. In (24), σ_i ($i = E, N, U$) are the standard deviation values of residual scatters as determined in Fig. 1. The values of $\bar{\sigma}_i$ ($i = E, N, U$) are the standard deviations extracted from the diagonal elements of matrix $\bar{\Sigma}_{E\ N\ U}$.

4. Results

Table 3 shows the determined coordinates and their internal consistency (precision) for each station obtained using the OPUS-RS software and the LS adjustment procedure introduced in this work. Tabulated are the curvilinear geodetic coordinates as determined at each GPS point for the four time spans tested in this experiment (2 h, 1 h, 30 min and 15 min). One immediate conclusion is that when large amounts of data over a period of 1 month are available, the differences between the coordinate solutions at different time spans are not significant. The LS adjustment arrives at a unique set of estimated parameters (the geodetic coordinates) and their full v–c matrix. At least in the three cases investigated here it can be said that when including all GPS-derived (x, y, z) data spanning 30 days, the final (λ, φ, h) results are not significantly different either when solving for 2 h or 15 min of (x, y, z) data. However, as it should be, the standard errors associated with each coordinate increase when the time span is shortened. It is appropriate at this point to remind the readers that, for example, the 15-min results were computed independently from the 2-h results. Of course, the same original archived GPS CORS data was used, but the program OPUS-RS was run independently twice, in one case automatically selecting 2 h time spans and in the other run only observation spans of 15 min. Although a single 15 min observation span may be greatly affected by atmospheric conditions, multipath and satellite geometry, when the quantity of 15 min data sets is significantly large, processed over different times and days, the LS simultaneous solution provides good estimates of the final parameters with a meaningful v–c matrix.

The other important observation that should be highlighted is that the alternative weighting scheme introduced in this investigation makes practical sense. The resulting *a posteriori* standard

deviation of unit weight confirms that the value of $\hat{\sigma}_0$ is, as it should be, close to one in all cases. Furthermore, although not shown for lack of space, the advantage of this proposed methodology is that, as explained before, the correlations of the original GPS-processed optimistic v–c matrix are preserved. That is not the case when weight matrices are simply multiplied by a scale factor in order to obtain *a posteriori* variances of unit weight as close to one as possible.

As Table 3 shows, the final sigmas attached to each coordinate are also realistic. Their values progressively increase as the observational time span decreases.

In order to better comprehend the kind of precisions expected from the different time spans selected for this investigation, the original GPS-derived data was also solved in a LS adjustment, not in a simultaneous adjustment for the whole month as obtained in Table 3, but at common local times. This may be considered another utilization of the procedure outline here amid the many thinkable applications of the suggested methodology. For example, for the 15 min case, a LS solution was determined using data for the first 15 min period (0:00–0:15) of local time for every day of the month of June; then the data for the second 15 min period (0:16–0:30) of local time for the month was used to generate another solution, etc. Subsequently the same logic was applied for the 2 h, 1 h and 30 min data sets. The results are depicted in Fig. 2. Notice that, as in Fig. 1, the results for each station are stacked by columns and the different time spans are arranged by rows. The plotted points in the graphs show the differences between the “reference values” obtained from the full month of 2 h data sets of Table 3 and the value of the resulting LS coordinates obtained when using the individual local time data set for each case.

One immediate conclusion that could be inferred from Fig. 2 is that the processed horizontal components are very precise (2–3 mm) for all selected time spans independently of the local time used. This is also reflected in the resulting standard deviations attached to each plot. There seems to be very little correlation between stations and local time of observation. As expected, the differences along the ellipsoid height component are noisier for shorter sessions. This is primarily due to unpredictable changes in meteorological conditions and the impossibility of accurately modeling random ionospheric and tropospheric biases below the 1 cm level. Nevertheless, only in a few instances do the actual errors in ellipsoid height exceed 2 cm, which may be considered a good result for height determinations when short spans of time, like the ones used in this case, are implemented.

Finally, it should be noted here that the precision of the coordinates determined by OPUS-RS is to some extent affected by the geometry and distance of the rover to the fixed CORS stations (see e.g. Soler et al., 2011). NGS developed an interactive

Table 3

Geodetic coordinates LS results using different time spans. The observed data for the whole month was used in each case.

Time span	λ	φ	h (m)	σ_E (cm)	σ_N (cm)	σ_U (cm)	$\hat{\sigma}_0$
CODE							
2 h	283° 10' 23".424495	39° 01' 18".190247	15.8643	0.41	0.37	2.08	1.15
1 h	283° 10' 23".424505	39° 01' 18".190252	15.8655	0.43	0.41	2.24	1.14
30 min	283° 10' 23".424501	39° 01' 18".190252	15.8625	0.49	0.48	2.50	1.27
15 min	283° 10' 23".424507	39° 01' 18".190247	15.8615	0.54	0.56	2.63	1.30
MNLS							
2 h	266° 05' 35".379638	44° 26' 28".136535	239.8863	0.22	0.42	1.43	0.96
1 h	266° 05' 35".379644	44° 26' 28".136535	239.8901	0.26	0.46	1.82	1.01
30 min	266° 05' 35".379659	44° 26' 28".136534	239.8919	0.33	0.53	2.22	1.05
15 min	266° 05' 35".379652	44° 26' 28".136531	239.8948	0.40	0.60	2.53	1.03
OKDN							
2 h	262° 02' 00".438848	34° 28' 45".501716	315.4748	0.33	0.38	1.91	1.12
1 h	262° 02' 00".438835	34° 28' 45".501726	315.4771	0.42	0.44	2.26	1.16
30 min	262° 02' 00".438832	34° 28' 45".501724	315.4799	0.50	0.51	2.83	1.22
15 min	262° 02' 00".438837	34° 28' 45".501722	315.4808	0.59	0.58	3.22	1.19

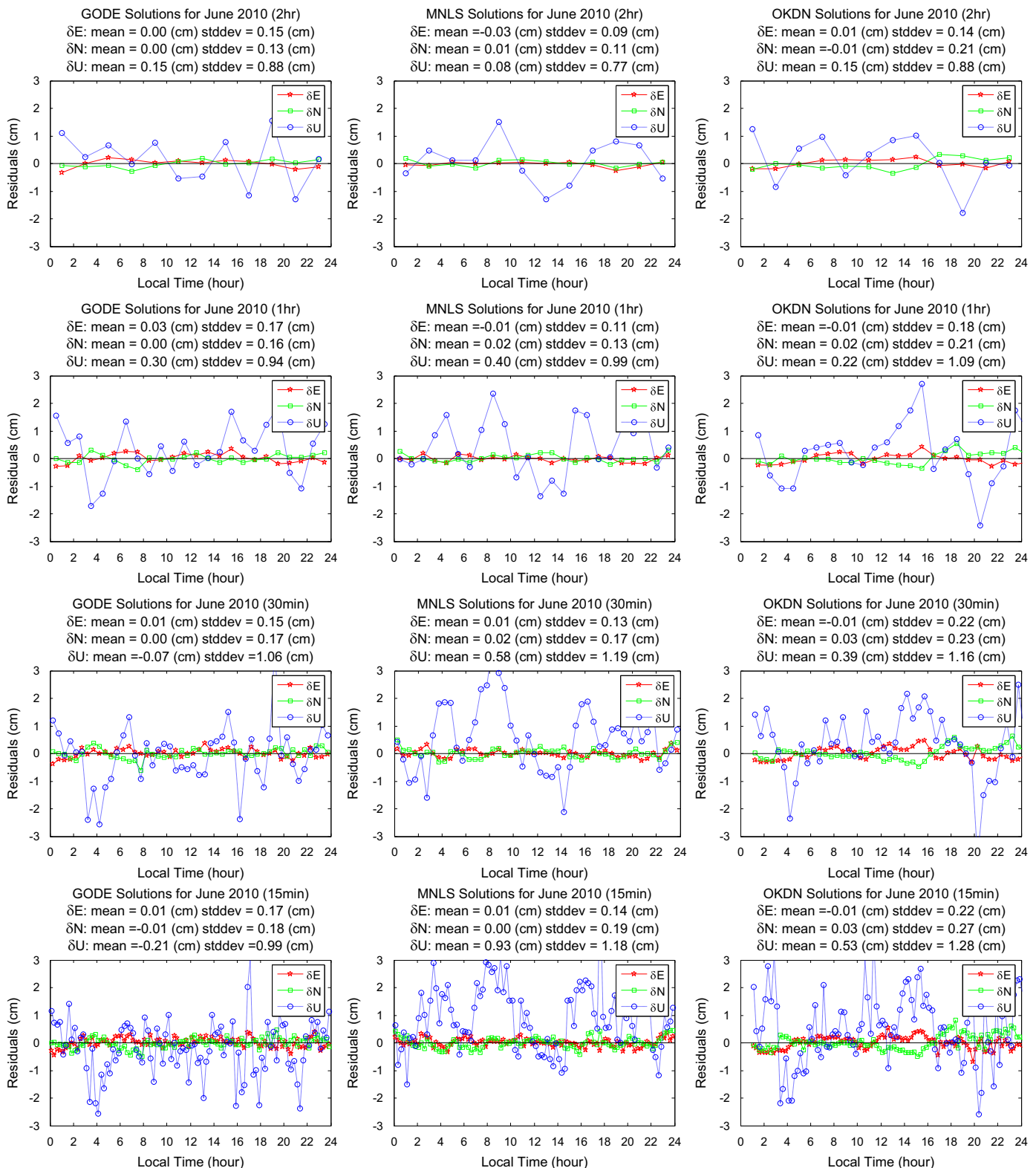


Fig. 2. Differences between the reference value (2 h solution for the whole month of June, Table 3) and each LS solution using different data spans (2 h, 1 h, 30 min and 15 min) simultaneously determined at common local times.

map that is available at (http://www.geodesy.noaa.gov/OPUS/Plots/Gmap/OPUSRS_sigmap.shtml) where the user can check the predicted accuracy of any OPUS-RS run anywhere in the U.S. even before doing the field observations. Using this Web-based page the values obtained for the three stations selected in this

investigation resulted in numbers very close to the 1σ values obtained in this experiment using real data. The map is limited only to the 1 h and 15 min cases. From this interactive map the following results were obtained for the ellipsoid height standard deviations for 1 h and 15 min, respectively: GODE, 1.6 and 2.8 cm;

Table 4

Geodetic coordinate differences between the NGS published values and the LS computed values from Table 3.

Time span	δE (cm)	δN (cm)	δU (cm)
CODE			
2 h	0.49	−0.91	0.37
1 h	0.47	−0.93	0.25
30 min	0.48	−0.93	0.55
15 min	0.46	−0.91	0.66
MNLS			
2 h	0.56	0.66	0.07
1 h	0.54	0.66	−0.31
30 min	0.51	0.67	−0.49
15 m	0.53	0.68	−0.78
OKDN			
2 h	0.59	−0.45	−1.28
1 h	0.62	−0.48	−1.51
30 min	0.63	−0.47	−1.79
15 min	0.62	−0.47	−1.88

MNLS, 1.9 and 3.0 cm; OKDN, 1.8 and 2.9 cm. These values compare very well with the standard deviations reported in Table 3 although the standard errors based on the real data processed in this investigation should be considered more realistic than the predicted values obtained from the map.

4.1. Coordinates accuracy estimates

Up to this point it is only known, from the results reported in Figs. 1 and 2, summarized in Table 3, that the internal consistency (the precision) of the LS results for the geodetic coordinates are very good for the horizontal components and, understandably, 4–5 times worse for the ellipsoid height component.

In order to estimate the actual accuracy of the LS results, the NGS published (adopted) “true” values given in Table 1 were differenced from the LS computed values reported in Table 3. These differences are given in linear units in Table 4, and are rigorously computed in a local geodetic frame with origin at each station, according to the equation (Soler, 1976):

$$\begin{Bmatrix} \delta E \\ \delta N \\ \delta U \end{Bmatrix} = \begin{bmatrix} (N+h)\cos\varphi & 0 & 0 \\ 0 & M+h & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \delta\lambda \\ \delta\varphi \\ \delta h \end{Bmatrix} \quad (25)$$

where the differences ($\delta\lambda, \delta\varphi$) are expressed in radians.

It may be surprising to see that for data spans as short as the ones investigated herein, the accuracy of the results, when compared to NGS published values, are so close. Thus, through the approach presented in this article, it is possible to obtain geodetic coordinates from a set of Cartesian coordinates at the millimeter level, a result that was unthinkable before the availability of GPS technology. The results of Table 4 show that the OPUS-RS solutions are consistent with these NGS adopted (published) coordinates at the 1–2 cm level. This is primarily due to the fact that CORS stations are used as reference (fiducial) points in the OPUS-RS software. Unexpected biases in the CORS Web-posted positions, if any, will be greatly reduced when new coordinates resulting from a 17 year reprocessing of all GPS CORS data replaces the coordinates currently published (Griffiths et al., 2010; see <http://geodesy.noaa.gov/CORS/coords.shtml>).

5. Conclusions

The methodology proposed in this investigation calculates, through a straightforward LS procedure, the best geodetic coordinates from a set of GPS-determined Cartesian (x, y, z) values. This

outcome represents the answer to a modern practical problem that involves the determination of a set of unique coordinates when repeated GPS solutions of the same point are available from a range of different circumstances: observers, receiver type, selection of time span and dates, post-processing software, etc. One advantage of the proposed method is the full use of the stochastic model available from the processed GPS observations.

For example, the alternative approach suggested here may be of interest to modern CORS or Real Time Network (RTN) managers who desire to monitor, from time to time, the value of the stations adopted coordinates. This procedure could be implemented automatically, at any desired periodical cycle, flagging the results when the differences with respect to the adopted coordinates surpass certain preselected tolerance. The advantage of this procedure is that the best geodetic coordinates of a specific station during a predetermined period of time could be calculated without recurring to a network adjustment involving observations from other nearby stations in the network. Currently, time series of station coordinates generally require a network adjustment involving GPS data from many stations. This approach is time consuming and requires more sophisticated adjustment software. The procedure advanced here, besides being much simpler, reaches similar final objectives. The best geodetic coordinates with realistic 1σ errors could be obtained independently at each station at any pre-specified GPS data time spans (30 min, 1 h, 2 h, 24 h, etc.) during a combined period of preselected time intervals (1 day, 2 day, 1 month, etc.).

Another possible application of the proposed concept could be that of obtaining a set of unique values from a collection of results gathered by different observers at different times. This situation is typical at some national geodetic organizations that have decided to archive in their data bases GPS results submitted by private GPS users. This appears to be the trend of the future considering the amount of time involved and the limited availability of money at the disposal of the geodetic organizations dedicated to gathering geodetic control. For example, the simplest procedure will be to assemble all submitted (x, y, z) coordinates and their v-c matrices referred to a common geodetic frame and epoch (e.g. NAD 83 (CORS96), epoch 2002.00) and implement the LS methodology outlined here to compute a unique value for the geodetic coordinates of the point in question that ultimately should be the representative value until superseded by the addition of future GPS observations.

Recapitulating, the authors envision the possibility of multiple applications of the presented methodology that, although conceptually simple, has been proven sufficiently accurate for calculating geodetic coordinates from a time series of GPS-processed (x, y, z) data.

Acknowledgments

The authors thank David Lehman for preliminary discussions about the topic and Richard Snay and Dru Smith for a thorough review of the first draft. The authors are also indebted to three anonymous reviewers for their constructive comments.

Appendix A. LS solution of the geocentric Cartesian coordinates (x, y, z)

Many readers may have asked themselves the question, why not to get the best estimates of the geodetic curvilinear coordinates using a linear model? Obviously, this is also possible. However, this will not be a direct approach; the geodetic coordinates and their v-c matrix will be obtained from the

resulting adjusted Cartesian coordinates and their statistics after a linear transformation, therefore the geodetic coordinates will not be the direct result of a LS adjustment, as we originally intended. This is important because a critical component such as the ellipsoid height greatly affected by atmospheric conditions, etc. will not be obtained directly from the original (x,y,z) pseudo-observations and their v-c matrix.

In any event, below is the approach determining directly the (x,y,z) adjusted values. The mathematical model of Eq. (1) will be replaced by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} x_\ell \\ y_\ell \\ z_\ell \end{Bmatrix} \quad (A1)$$

The matrix \mathbf{A}^T will take the simplified form of n blocks of 3×3 unit matrices, or explicitly:

$$\mathbf{A}^T_{3 \times 3n} = \begin{bmatrix} [I] & [I] & \cdots & [I] \end{bmatrix} \quad (A2)$$

Denoting now the weight matrix as

$$\mathbf{P}_{3n \times 3n} = \begin{bmatrix} [\mathbf{P}_1] & [0] & \cdots & [0] \\ & [\mathbf{P}_2] & \cdots & [0] \\ & & \ddots & \vdots \\ sym & & & [\mathbf{P}_n] \end{bmatrix} \quad (A3)$$

The following results are obtained:

$$\mathbf{N}_{3 \times 3} = \mathbf{A}^T \mathbf{P} \mathbf{A} = \sum_{i=1}^n [\mathbf{P}_i] \quad (A4)$$

$$\mathbf{u}_{3 \times 1} = \mathbf{A}^T \mathbf{P} \ell = \sum_{i=1}^n \left\{ [\mathbf{P}_i] \begin{Bmatrix} x_\ell \\ y_\ell \\ z_\ell \end{Bmatrix}_i \right\} \quad (A5)$$

The final v-c matrix along the local geodetic frame could be obtained using the following matrix transformation:

$$\begin{bmatrix} \sigma_E^2 & \sigma_{EN} & \sigma_{EU} \\ & \sigma_N^2 & \sigma_{NU} \\ sym. & & \sigma_U^2 \end{bmatrix} = \mathbf{R} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ & \sigma_y^2 & \sigma_{yz} \\ sym. & & \sigma_z^2 \end{bmatrix} \mathbf{R}^T \quad (A6)$$

where the matrix \mathbf{R} was given in Eq. (19).

A further simplification, if desired, could be made when $\forall i, [\mathbf{P}_i] = [I]$, then, as expected, we get the value of the mean for the values of the adjusted parameters:

$$\hat{x} = \frac{\sum x_i}{n}, \hat{y} = \frac{\sum y_i}{n}, \hat{z} = \frac{\sum z_i}{n} \quad (A7)$$

This Appendix was included here for completeness. However, it should be stressed that the authors believe that the mathematical model introduced in this paper determines the curvilinear coordinates (longitude, latitude, and ellipsoid height) directly and

Table B1

xyzfile='TEST.xyz';	% input x-y-z file
covfile='TEST.cov';	% input var-cov file
% initial setup	
a=6378137; f=1/298.257222101;	% GRS80 parameters
e=sqrt(2*f-f^2);	
% loading data	
xyz=load(xyzfile);	
n=length(xyz);	% number of points
x(:,1)=xyz(:,1);y(:,1)=xyz(:,2);z(:,1)=xyz(:,3);	% x-y-z coordinates
% modeling	
F1=sym('xi-(a/sqrt(1-(e*sin(p))^2)+h)*cos(p)*cos(l)');	
F2=sym('yi-(a/sqrt(1-(e*sin(p))^2)+h)*cos(p)*sin(l)');	
F3=sym('zi-(a/sqrt(1-(e*sin(p))^2*(1-e^2)+h)*sin(p)');	
A11=diff(F1,'l'); A12=diff(F1,'p'); A13=diff(F1,'h');	
A21=diff(F2,'l'); A22=diff(F2,'p'); A23=diff(F2,'h');	
A31=diff(F3,'l'); A32=diff(F3,'p'); A33=diff(F3,'h');	
% formig & solving equations	
it=0; del=[1 1 1];	
par=[atan(mean(y)/mean(x));atan(mean(z)/sqrt((mean(x))^2+(mean(y))^2));	% using mean spherical coordinates as initial values
sqrt((mean(x))^2+(mean(y))^2+(mean(z))^2)-a];	
cov=load(covfile);	% loading obs v-c matrix
for i=1:n	
Ell(3*i-2:3*i,3*i-2:3*i)=cov(3*i-2:3*i,1:3);	
End	
Qll=Ell;	% sig0=1
P=inv(Qll);	
while max(abs(del)) >= 1e-10 & it <= 10	% iteration (stops when corrections < 1e-10 or iteration # > 10)
l=par(1); p=par(2); h=par(3);	
for i=1:n	% filling A matrix
xi=x(i); yi=y(i); zi=z(i);	
A(3*i-2,1)=-eval(A11); A(3*i-2,2)=-eval(A12); A(3*i-2,3)=-eval(A13);	
A(3*i-1,1)=-eval(A21); A(3*i-1,2)=-eval(A22); A(3*i-1,3)=-eval(A23);	
A(3*i,1)=-eval(A31); A(3*i,2)=-eval(A32); A(3*i,3)=-eval(A33);	
f(3*i-2,1)=-eval(F1); f(3*i-1,1)=-eval(F2); f(3*i,1)=-eval(F3);	
end	
N=A'*P*A; t=A'*P*f;	
del=-inv(N)*t;	% delta x
par=par+del;	% adjusted parameters
it=it+1;	
end	
% quality analysis	
v=f+A*del;	% residual vector
sig0=sqrt(v'*P*v/(3*n-3));	% posterior stddev of unit weight
Qxx=inv(N); Exx=sig0^2*Qxx;	% v-c matrix for parameters

Table B2

%% xScaleVCM.m	% function name
function [covs]=xScaleVCM(inVCM,outVCM,sx,sy,sz)	% command syntax
% Required input:	
% inVCM: input var-cov filename	
% outVCM: output (scaled) var-cov filename	
% sx, sy, sz: reference stddev values for all stations	
cov=load(inVCM);	
covs=[];	
for i=1:length(cov)/3	
Js(1,1)=sx(i)/sqrt(cov(3*i-2,1));	
Js(2,2)=sy(i)/sqrt(cov(3*i-1,2));	
Js(3,3)=sz(i)/sqrt(cov(3*i,3));	
covp=Js*cov(3*i-2:3*i,1:3)*Js';	
covs=[covs;covp];	
end	
% output	
fid=fopen(outVCM,'w');	
fprintf(fid,'%15.10f %15.10f %15.10f\n',covs');	
fclose(fid)	

not through a deterministic transformation from the LS (x,y,z) results of the alternative linear case as described in the Appendix. In the proposed methodology, errors in the Cartesian components are absorbed directly into the parameters of the mathematical model (specially the ellipsoid height, the most affected by atmospheric conditions) and a final deterministic transformation (which one?) from Cartesian to curvilinear coordinates is not necessary. The same could be said about the final v - c matrix of the curvilinear parameters that are obtained directly from the adjustment of the suggested model and not through the approximated standard linear propagation of errors through a transformation matrix (see Eq. (A6)).

It is expected that a mathematical model that relates the (x,y,z) observables directly to the ellipsoid height will give better answers for this parameter than a series of deterministic transformations using the LS results of the (x,y,z) values obtained through the linear procedure described in this Appendix.

Appendix B. MATLAB codes for transforming Cartesian to geodetic coordinates by LS

To carry out the least squares estimation based on the proposed approach, a MATLAB program was created. This program contains a primary routine which reads input geocentric Cartesian coordinates and their associated variance-covariance matrix. Note that the input coordinate file (`***.xyz`) contains $n \times 3$ values where each row represents a set of (x,y,z) coordinates. The size of the variance-covariance file (`***.cov`) is $3n \times 3$. The i th 3×3 block represents the full variance-covariance matrix for the i th set of (x,y,z) coordinates. The code is presented in Table B1.

Additionally (See Table B2), a subroutine was also created to scale a variance-covariance matrix according to the proposed scale (Jacobian) propagation approach (Eqs. (20) and (22)).

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