Representation of the Earth Potential by Buried Masses

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Abstract. Some ways of representing the earth potential by point masses are explored. The 1969 standard earth model of the Smithsonian Astrophysical Observatory is used as a reference. Points regularly distributed on a sphere inside the earth have been considered, as well as nonuniform configurations. The values of the masses are derived by a least-squares fit to the SAO field. The values of the different representations obtained have been checked by computing the geoid heights, gravity anomalies, and spherical harmonics associated with the field of the masses. The best model obtained consists of 126 points, mostly at a depth of 1300 km, tied to the important gravity anomalies derived from the reference field. The mean of the residuals is 6 mgal.

The main results obtained so far in dynamical geodesy have been based on the expansion of the earth potential in spherical harmonics. The methods of analysis are complex, for they have to process numerous observations made on different satellites from many stations; in addition, the determination of the potential coefficients and station coordinates, computed at the same time, involves very large matrices that cannot be handled easily on present computers.

In the near future, the technical progress in satellite geodesy, as well as great improvements in type and accuracy of observational methods, will make it necessary to use new representations of the potential of the earth. The series of harmonic coefficients converges very slowly, and too many terms must be taken into account in order to represent small, or short-wave, variations of the field (as measured by an altimeter or determined from gravity measurements). It is desirable that new representations satisfy as many as possible of the following conditions:

1. Discrete formulation (one 'detail' for one function), which leads to simple programming. (Recursive formulas lead to important errors, which occur, for example, if one computes the Legendre functions up to (180, 180) to represent details of 1 degree on the geoid.)
2. Homogeneous representation on both the surface and exterior of the earth.
3. Possibility of increasing locally the number of parameters without perturbing distant regions.
4. The same formulation of the potential and its derivative on and outside the earth, so as to facilitate the combination of observations from different sources (like gravity anomalies and satellite-to-ocean altimetry).

The representation of the gravity field by buried masses, which was suggested by *Weightmann* [1967] several years ago, can satisfy these conditions under some restrictions on the configuration and depth of the points.

Theoretically, any distribution of masses inside the earth that keeps the geoid fixed gives the same field outside the geoid. Consequently, it is reasonable to distribute the mass points regularly. Nevertheless, the anomalies of the gravity field already known make us think of putting the masses at the places where particular features exist.

Taking the last standard earth model determined by *Gaposchkin and Lambeck* [1970], we have tried in our approach to attain an equivalence between this model and a configuration of mass points, in order to represent the gravimetric geoid as well as possible. The values of differently obtained representations outside the earth have been checked by integrating satellite orbits.

METHOD

One has the choice of taking a regular distribution on a sphere inside the earth, or putting the points at places corresponding to large gravity anomalies. If the anomalies of the field are not known (they are to be determined through observations), only the first point of view is valid, and a large number of mass points close to the earth's surface must be considered. One can also take surface density values [*Koch*, 1970], but the expression for the external field is not convenient to use for satellite motion.

Starting from an a priori configuration of points $(\mathbf{P}_i)_0$, $i = 1, 2, \cdots, N$, we determine parameters ϵ_i such that:

$$U_0 = \frac{GM}{r}\left[1 - \sum_{n=2}^{21}\left(\frac{a}{r}\right)^n J_n P_n(\sin\phi) + \sum_{n=2}^{22}\left(\frac{a}{r}\right)^n \sum_{k=1}^{n'} (C_{nk}\cos k\lambda + S_{nk}\sin k\lambda) P_{nk}(\sin\phi)\right] \quad (1)$$

$$n' = \min(n, 16)$$

$$U_M = E + T_M = E + \sum_{k=1}^{N} T_{Mk} = E + GM\sum_{k=1}^{N}\frac{\epsilon_k}{r_k}$$

$$\iiint_D |U_0 - U_M|^2 dv \quad \text{is a minimum} \quad (2)$$

where E is a reference potential included in U_m (because terms like

$$1/r \text{ or } (1/r)\ J_2(a/r)^2 P_2(\sin\phi)$$

cannot be easily represented otherwise, except by deep point masses, which are without physical meaning, and D is a region of space outside the earth (e.g., between 500- and 1500-km height).

D is represented by a large number $M(>N)$ of points S, and we thus have

$$\sum_{i=1}^{N}\left[\left(\frac{\partial T_M}{\partial \epsilon_i}\right)_S \Delta\epsilon_i + (\nabla T_{Mi})_S \cdot \mathbf{\Delta P}_i\right] = (U_0 - U_M)_S$$

with

$$\mathbf{\Delta P}_i = \mathbf{P}_i - (\mathbf{P}_i)_0$$

$$\Delta\epsilon_i = \epsilon_i - (\epsilon_i)_0$$

$$(\epsilon_i)_0 = 0$$

at the first iteration.

Different weights according to the heights of S can be chosen. In addition, we add the following constraints:

$$\sum_{i=1}^{N}\epsilon_i = 0$$

i.e., the total mass of the earth is unchanged, and

$$\sum_{i=1}^{N}\epsilon_i \boldsymbol{\varrho}_i = 0$$

i.e., the center of inertia is not modified.

In the above,

$$\boldsymbol{\varrho}_k = \mathbf{r} - \mathbf{r}_k$$

where $\mathbf{r}$ is the geocentric vector to S, the point at which the potential is being computed, and $\mathbf{r}_k$ is the vector from mass point P_k to S.

Finally, we solve

$$\begin{pmatrix} [\mathbf{A}^T\mathbf{\Pi} A] & \mathbf{J}^T \\ J & 0 \end{pmatrix}\begin{pmatrix} \mathbf{Y} \\ \Lambda \end{pmatrix} = \begin{pmatrix} [A^T\mathbf{\Pi}] & \mathbf{\Delta} \\ & 0 \end{pmatrix} \quad (3)$$

where

- $\mathbf{A}$ is the matrix of equations of condition.
- $\mathbf{J}$ is the jacobian of constraints.
- $\mathbf{y}$ is the vector of unknowns.
- $\mathbf{\Lambda}$ is the vector of Lagrange multipliers.
- $\mathbf{\Pi}$ is the matrix of weights.
- $\mathbf{\Delta}$ is the vector of the quantities $(U_0 - U_M)$.

Results obtained using (3) in conjunction with (1) and (2) did not yield a satisfactory representation of the field on the earth's surface. Hence, condition (2) was replaced by:

$$\iint_\Phi |\Delta g_0 - \Delta g_M|^2 ds \quad \text{is a minimum} \quad (4)$$

where Δg_0 is the field of the gravity anomalies derived from U_0, and

$$\Delta g_M = \frac{\partial T_M}{\partial n} - \frac{2T}{a}$$

n being the normal at the surface Φ of the reference ellipsoid whose potential here is exactly E and whose mean radius is a.

A natural weighting is obtained by writing the condition equations on the level curves of the gravity anomalies, which gives more importance to the large anomalies. Local studies have been made to determine optimal values ϵ_i, ρ_i for large anomalies.

The adequacy of a mass-point representation obtained via (2) or (4) can be checked in the following ways:

1. Computation of the residuals $U_0 - U_M$ in D.
2. Calculation of gravity anomalies on ϕ from the formula:

$$\Delta g_Q = GM \sum_{k=1}^{N} \frac{\epsilon_k}{r_k^3} (a - \rho_k \cos \psi_k)$$

where Q is a point on Φ, a is the geocentric radius to Q, and ψ_k is the geocentric angle between $\boldsymbol{\varrho}_i$ and $\mathbf{a}$; Δg_Q can then be compared with the actual anomaly at Q.

3. Determination of a harmonic coefficient set

$$\langle \gamma_{lm} \rangle = \sum_{k=1}^{N} \epsilon_k \left(\frac{\rho_k}{a}\right)^l \bar{P}_{lm}(\sin \phi_k) \cdot \left(\frac{(2 - \delta_0^{\ m})(l - m)!}{(2l + 1)(l + m)!}\right)^{1/2} \cos m \lambda_k$$

$$\langle \sigma_{lm} \rangle = \sum_{k=1}^{N} \epsilon_k \left(\frac{\rho_k}{a}\right)^l \bar{P}_{lm}(\sin \phi_k) \cdot \left(\frac{(2 - \delta_0^{\ m})(l - m)!)}{(2l + 1)(l + m)!}\right)^{1/2} \sin m \lambda_k$$

where the bar denotes normalization; these can be compared with the SAO coefficients [*Gaposchkin and Lambeck,* 1970].

4. Comparison of numerical integration of a given orbit of a satellite in the SAO field and in the field U_M.

RESULTS

Various tests using regular distributions of mass points and condition 2 proved unsatisfactory. A model that gave a good representation of the gravity anomalies on the ellipsoid was obtained in the following manner. The reference potential E in (1) was taken to be the earth ellipsoid harmonics. Ninety-two points were positioned at the extreme values of a map of gravity anomalies based on the SAO standard earth. Constraining the points to lie at the same depth, the best representation was obtained at a depth of 1300 km. This preliminary model was improved by adding additional points up to a total of 126 and individually adjusting the depths of the 20 largest anomalies. The results are given in Table 1. For this model,

TABLE 1. Model for 126 Mass Points

N	Lat. ϕ, deg	Long. λ, deg	Depth, km	Mass ϵ, 10^{-6} mass of earth
1	59	−152	−1100	1.753
2	3	−176	−1300	−0.511
3	−66	−177	−1300	−1.667
4	20	−160	−1300	0.810
5	−31	−159	−1300	1.176
6	−15	−158	−1300	−0.499
7	−2	−155	−1300	0.912
8	22	−143	−1300	−1.233
9	−27	−140	−1300	−1.734
10	32	−126	−1300	−1.603
11	3	−125	−1300	−1.159
12	−19	−121	−1300	0.762
13	−46	−121	−1500	2.122
14	47	−115	−1100	0.667
15	−12	−101	−1300	0.699
16	−49	−99	−1300	−1.632
17	38	−98	−1300	−1.334
18	−76	−95	−1300	−1.265
19	−35	−85	−1300	0.579
20	−15	−85	−1000	−1.507
21	60	−89	−1300	−2.065
22	39	−71	−1300	−0.645
23	0	−71	−1300	1.868
24	−19	−69	−1300	1.667
25	−54	−69	−1500	1.048
26	19	−64	−1300	−1.713
27	24	−28	−1300	−0.175
28	9	−49	−1300	−1.275
29	−14	−52	−1300	−1.015
30	−36	−42	−2000	−0.646
31	42	−32	−1100	1.876
32	−18	−24	−1200	−1.061
33	69	−20	−1300	0.977
34	60	−175	−1300	−0.933
35	9	−11	−1300	0.974
36	22	−4	−1300	−0.665
37	−64	−25	−1500	0.753
38	39	−3	−1000	1.218
39	−39	2	−1000	−0.616
40	−50	15	−1300	0.987
41	−66	21	−1300	−1.337
42	−42	24	−1300	−0.848
43	4	25	−1000	−1.482
44	18	29	−1300	1.940
45	30	25	−1300	−2.460
46	45	26	−1000	1.498

TABLE 1. (*continued*)

N	Lat. ϕ, deg	Long. λ, deg	Depth, km	Mass ϵ, 10^{-6} mass of earth
47	35	46	−1300	2.707
48	21	50	−1300	−2.213
49	−6	48	−1100	−1.360
50	−43	43	−1300	2.359
51	−44	61	−1300	−1.681
52	−67	73	−1300	1.682
53	−36	74	−1300	1.603
54	4	79	−1000	−2.143
55	41	68	−1300	−1.807
56	69	76	−1300	−1.792
57	35	90	−1300	1.308
58	−49	99	−1300	1.380
59	−25	98	−1300	−1.575
60	−1	112	−1300	1.590
61	47	100	−1300	−1.931
62	62	109	−1300	2.080
63	12	124	−1300	1.110
64	−21	121	−1300	0.808
65	−38	122	−1300	−2.029
66	−63	120	−1300	−1.442
67	−4	142	−1100	1.224
68	31	141	−1300	1.625
69	−42	152	−1500	−2.374
70	1	159	−1300	−0.434
71	39	160	−1300	−1.755
72	57	158	−1300	2.384
73	−47	135	−1300	−0.178
74	−25	174	−1300	0.766
75	22	101	−1300	−0.895
76	68	133	−1300	−1.734
77	−9	30	−1300	0.498
78	16	−94	−1300	0.707
79	10	50	−1300	1.573
80	40	−112	−1300	1.519
81	58	−40	−1300	−0.578
82	58	53	−1300	1.700
83	−18	−40	−1300	1.284
84	−46	167	−1300	1.113
85	85	100	−1100	1.052
86	−42	−35	−2000	−2.932
87	−42	−16	−1300	1.248
88	26	126	−1300	−1.006
89	40	131	−1300	0.942
90	−19	160	−1200	0.684
91	39	−163	−1300	−0.595
92	42	−17	−1100	−1.040
93	43	−135	−1300	−1.365
94	33	−144	−1300	0.933
95	17	−134	−1300	1.106
96	−51	−142	−1300	−0.318
97	−34	−108	−1300	−0.378
98	44	−89	−1300	1.252
99	77	−121	−1300	−0.334
100	82	−78	−1300	−0.857
101	85	−20	−1300	0.048
102	71	−52	−1300	1.519
103	37	−53	−1300	−0.470
104	27	−44	−1300	1.064
105	58	−5	−1300	0.667
106	−85	100	−1300	−0.010
107	4	−28	−1300	0.333
108	−6	−11	−1300	−0.127
109	−27	−16	−1300	−0.133
110	−73	−15	−1300	0.625
111	−61	−81	−1300	0.622
112	−33	−70	−1300	0.064
113	−23	−4	−1300	1.177
114	−10	4	−1300	−0.403
115	−19	30	−1300	0.091
116	28	65	−1300	0.961
117	48	43	−1300	−2.070
118	58	13	−1300	−1.057
119	16	145	−1300	−0.561
120	−47	137	−1300	0.899
121	−16	69	−1300	0.336
122	−15	112	−1300	−0.598
123	−29	145	−1300	0.153
124	23	165	−1300	−0.164
125	55	131	−1300	−0.872
126	15	14	−1300	0.990

the mean of the residuals is 6 mgal and the largest differences are of the order of 10 mgal, which is the estimated accuracy of the SAO standard earth.

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