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Multiresolution approximation of the gravity field

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Abstract: In this paper, the idea of multiresolution approximation to the gravity field of the Earth is introduced, and a new approach is developed based on wavelet theory. In this approach, the modelling of the gravity field is done based on all available data with different resolutions, while a discrete wavelet transform is utilized as a bridge, effectively linking different resolution levels. The main advantage of this approach is that it allows us to consider not only optimal estimation of gravity field signals at multiple resolutions but also the fusion of measurements at multiple resolutions. Numerical results with simulated data show the applicability of the proposed approach in physical geodesy.

1. Introduction

Mapping of the earth's gravity field is still a primary goal in geodesy, geophysics and geodynamics, Colombo (1991). There are two possible approaches in the Earth's gravity field approximation, the model approach and the operational approach, Moritz (1980). The model approach is based on the solution of geodetic boundary value problems. A number of techniques for solving geodetic boundary problems have been developed, e.g. spherical harmonic expansion and integration techniques such as Stokes' formula and Molodensky's solution, Heiskanen and Moritz (1967). The operational approach is based on least - squares collocation, which is a technique for combining observational data of different types for an optimal estimation of the gravity field signal and other parameters. All these methods could be viewed as gravity field approximation at a specific resolution depending on the data spacing given. That is the case in gravity field modelling using airborne gravimetry and satellite altimetry data because they have a given specific resolution. It is also the case when mean data blocks are used. So it is natural to consider the Earth gravity field approximation from a multiresolution point of view. The conventional method to deal with multiresolution data is least-squares collocation, which combines all data with different resolution simultaneously and does not take any data structure into account. Therefore the objective of this paper is to develop a different approach based on wavelet theory which considers the data structure. Instead of combining all available measurements at the same time, this approach will estimate the gravity field signal at each resolution level by using the measurements from fine - to - coarse scale one by one. The wavelet transform is used as a tool to link different resolution levels.

This paper is organized as follows. The concepts of wavelet theory are introduced in Section 2 and the multiresolution approach to the local gravity field is developed in Section 3. Numerical examples are given in Section 4 to demonstrate the applicability of the proposed approach in physical geodesy.

2. Wavelet Theory and Its Application to Geodesy

In this section, the basic principles of wavelet theory will be briefly reviewed. Readers who are interested in wavelet theory are referred to Meyer (1992) and Daubechies (1992), for a more detailed description.

Wavelet theory is a relatively recent development in applied mathematics, e.g. Mallat (1989a), Meyer (1992) and Daubechies (1992). The concepts can be viewed as a synthesis of ideas originating during the last twenty or thirty years in pure mathematics (study of Calderon-Zygmund operator), physics (coherent states, renormalization group), and engineering (subband filtering). Wavelets and wavelet transforms were first proposed by Morlet as an alternative way to Fourier transforms for modelling seismic data, Grossman and Morlet (1984). Later, Meyer recognized this work to be part of the field of harmonic analysis, and came up with a family of wavelets, Meyer (1992). His work was further developed by other people, specifically Mallat (1989a, 1989b) and Daubechies (1988, 1992).

Wavelet theory provides a mathematical tool that decomposes data or functions or operators into different frequency components, and then studies each component with a resolution matched to its scale, Daubechies (1992). Roughly speaking the aim of wavelets is to obtain base functions (called wavelets) as localized as possible, both in time (or space) and frequency (spectral space). These functions are generated from a single "generating wavelet" or "mother wavelet" by translations and dilations. The wavelet transform has a form similar to that of a windowed Fourier transform. However, the basic function possesses windows of variable size, which enable adaptation to spatial phenomena at different scales, Daubechies (1992). Therefore wavelet transforms have advantages over windowed Fourier transforms, which do not have a good localization in the frequency domain. In this paper, only orthonormal wavelets and the corresponding discrete wavelet transforms are described. Orthonormal wavelets have been studied for only the last few years. It was difficult to construct an orthonormal wavelet base until a more systematic approach, i.e. multiresolution analysis, was found by Mallat (1989a, 1989b) and Meyer (1992). Daubechies constructed compactly supported orthonormal wavelets based on this approach. Multiresolution analysis can be interpreted as a successive approximation procedure, Vetterli, at al. (1992).

A multiresolution analysis is a sequence of subspaces of $L^2(R)$ which have the following properties:

1) $V_i \subset V_{i+1}\ ,\ i \in Z$;

2) $\bigcap_{i\in Z} V_j=\{0\},\ \bigcup_{i\in Z} V_j = L^2(R)$;

3) $f(x) \in V_j \longleftrightarrow f(2x) \in V_{j+1}$;

4) $f(x) \in V_0 \longleftrightarrow f(x-n) \in V_0\ ,\ n \in Z$;

5) $\exists f(x) \in V_0$ such that $\{f(x-n), n \in Z\}$ is an orthonormal basis of V_0.

Since $\{ f(x-n), n \in Z \}$ forms the basis of the space V_0, $\{2^{n/2}f(2x-n), n \in Z\}$ forms the basis of the space V_n. Therefore any function in $V_0 \subset V_1$ can be expressed in terms of the basis functions of V_1 . In particular,

$$\phi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} h(n)\phi(2x-n) \qquad (2.1)$$

with $h(n) = \int_{-\infty}^{\infty} \phi(x)\phi(2x-n)dx$ and $\sum_{n=-\infty}^{\infty} h^2(n) = 1$.

Equation (2.1) is often referred to the scaling function or the dilation function, which forms the basic function for generating wavelet functions.

The basic conclusion from multiresolution analysis is that whenever a collection of closed subspaces satisfies the above five properties, then there exists an orthonormal wavelet basis $\{ y_{n,k}\ ,\ n, k \in Z \}$, $y_{n,k}(x) = 2^{n/2}y(2^n x - k)$, of the orthogonal complement W_n of V_n in V_{n+1}, i.e.

$$V_{n+1} = V_n \oplus W_n .$$

This implies $L^2(R) = \bigoplus_{i\in Z} W_i$. Moreover, the wavelet function y(x) corresponding to the scale function f(x) can be constructed explicitly as follows:

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} g(n)\phi(2x-n) \quad (2.2)$$

where $g(n)=(-1)^{n-1}h(-n-1)$.

Figure 2.1 shows some examples of pairs of functions corresponding to different multiresolution analyses.

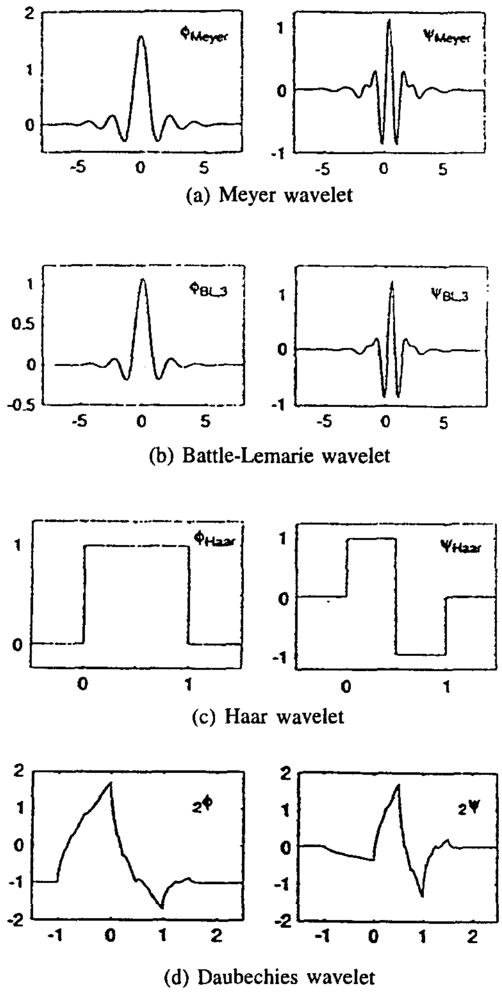


Fig. 2.1: Some examples of orthonormal wavelets (Daubechies, 1992)

It can be shown that the Meyer wavelet is C^∞, infinitely supported, symmetric, and decays faster than any inverse polynomial, while the Battle-Lemarie wavelets which are spline functions and can be chosen as C^k ($N \geq k+1$, N is the degree of the B-spline) have also infinite support, symmetry with exponential decay. The Daubechies wavelets, on the other hand, are compactly supported with width 2N-1 and asymmetric. Their smoothness increases with N, and they have N vanishing moments. Here N is the order of the Daubechies wavelets. The Haar wavelet can be viewed as the first order of the Daubechies wavelets.

More detailed discussion can be found in Daubechies (1988 and 1992).

Based on multiresolution analysis, the fast discrete wavelet transform was proposed by Mallat (1989b). It is a 'tree algorithm' or 'pyramid algorithm' that makes discrete wavelet transforms fast and simple. It does for the discrete wavelet transform what the FFT does for the discrete Fourier transform. The algorithm is fully recursive, Strange (1989). It was further improved by Beylkin et al (1991). Generally, a discrete wavelet transform algorithm corresponding to a multiresolution analysis can be described as follows:

For a given 1D sequence { $f_{i+1}(n)$, $n \in Z$ }of a signal f(t) at resolution level i+1, the lower resolution signal sequence { $f_i(n)$, $n \in Z$ } can be derived by lowpass filtering with a half band lowpass having impulse response h(n) (in this paper larger i corresponds to higher resolution or scale and smaller i corresponds to lower resolution or scale). At the same time the added detail $d_i(n)$, also called wavelet coefficients, can be computed by using a highpass filter with impulse g(n), i.e.

$$f_i(n) = \sum_k h(2n-k) f_{i+1}(k), \qquad (2.3a)$$
$$d_i(n) = \sum_k g(2n-k) f_{i+1}(k), \qquad (2.3b)$$

or

$$f_i = H f_{i+1}, \qquad (2.4a)$$
$$d_i = G f_{i+1}. \qquad (2.4b)$$

This process is referred to as the decomposition of the signal. The same decomposition procedure can be applied to a lower resolution signal until the lowest resolution of interest is reached.

Reversing this process, the synthesis form of wavelet transform is obtained in which finer and finer representation via a coarse - to - fine scale recursion is achieved, i.e.

$$f_{i+1}(n) = \sum_k h(2k-n) f_i(k) + \sum_k g(2k-n) d_i(k) \qquad (2.5a)$$

or

$$f_{i+1} = H^* f_i + G^* d_i \qquad (2.5b)$$

where * means conjugate operation.

This process is also referred to as the reconstruction of the signal. Figures 2.2 illustrate the block diagrams for decomposition and reconstruction.

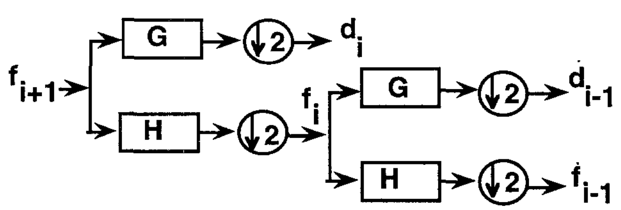


Fig. 2.2a: Decomposition of 1D signal

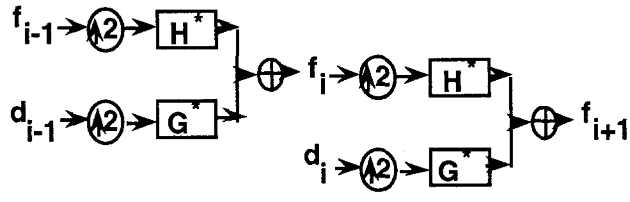


Fig. 2.2b: Reconstruction of 1D signal

The simplest example of a discrete wavelet transform is the Haar wavelet transform with

$$h(n) = \begin{cases} 1 & n = 0,1 \\ 0 & otherwise \end{cases}$$

and

$$g(n) = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & otherwise \end{cases} .$$

The 1D discrete wavelet transforms can be extended to 2D discrete wavelet transforms. In this case, the decomposition and reconstruction of a 2D signal takes the following form:

$$f_i(n,m) = \sum_k h(2n-k) h(2m-l) f_{i+1}(k,l), \qquad (2.6a)$$
$$d_{i,1}(n,m) = \sum_k g(2n-k) h(2m-l) f_{i+1}(k,l), \qquad (2.6b)$$
$$d_{i,2}(n,m) = \sum_k h(2n-k) g(2m-l) f_{i+1}(k,l), \qquad (2.6c)$$
$$f_{i,3}(n,m) = \sum_k g(2n-k) g(2m-l) f_{i+1}(k,l), \qquad (2.6d)$$

or

$$f_i = (H \otimes H) f_{i+1}, \qquad (2.7a)$$
$$d_{i,1} = (G \otimes H) f_{i+1}, \qquad (2.7b)$$
$$d_{i,2} = (H \otimes G) f_{i+1}, \qquad (2.7c)$$
$$d_{i,3} = (G \otimes G) f_{i+1}. \qquad (2.7d)$$

and

$$\begin{aligned} f_{i+1}(n,m) = & \sum_k h(2k-n) h(2l-m) f_i(k,l) \\ & + \sum_k g(2k-n) h(2l-m) d_{i,1}(k,l) \\ & + \sum_k h(2k-n) g(2l-m) d_{i,2}(k,l) \\ & + \sum_k g(2k-n) g(2l-m) d_{i,3}(k,l), \qquad (2.8a) \end{aligned}$$

or

$$\begin{aligned} f_{i+1} = & (H^* \otimes H^*) f_i + (G^* \otimes H^*) d_{i,1} \\ & + (H^* \otimes G^*)\, d_{i,2} + (G^* \otimes G^*) d_{i,3} \qquad (2.8b) \end{aligned}$$

where f_i, $d_{i,1}$, $d_{i,2}$ and $d_{i,3}$ represent vectors formed by stacking the rows of matrices from 2D signals, and $\otimes$ is Kronecker product.

One of the attractive features of wavelet transforms for the analysis of signals is that they cannot only be computed recursively in scale, from fine to coarse, but also be completely reconstructed from coarse to fine scale.

Therefore different scales can be related to each other. Such a feature is very useful for the development in this paper.

Wavelet theory and wavelet transforms have been successfully applied to many areas. In geodesy, the investigation of wavelet theory in geodetic applications has just started. Some possible applications of wavelets in geodesy were proposed by Blais (1993) and Benciolini (1994). The idea of the resolution of the geodetic boundary value problem by wavelet methods was developed by Freeden (1993), which can be considered as a first step towards the use of wavelets to boundary value problems on a smooth closed boundary surface in R^3 (Klees, 1994). Efficient numerical solutions to the integral equations using wavelet approximation were suggested by Klees (1995). Harmonic downward continuation using a Haar wavelet frame is developed by Keller (1995). Wavelet transforms were also used in GPS cycle slip detection, Collin, et al. (1995), polar motion analysis, Barthelmes, et al. (1995) and inverse problems, Blais (1995).

3. Multiresolution Approach to the Gravity Field

Multiresolution analysis decribed in Section 2 provides the basis for the development in this paper. There are three ways in which the multiresolution features can enter into the field of physical geodesy. First, multiresolution analysis allows for the approximation of the earth's gravity field in a natural way, which means that the gravity field is approximated successively from coarse to fine scales. Next, the data used in gravity field modelling have different resolutions. For example, the resolutions of data collected using airborne and satellite altimetry techniques are different. Finally, the multiresolution idea can be used to develop efficient algorithms for gravity field modelling, estimation and data fusion.
In the next subsection, the multiresolution approximation problem for gravity field modelling is formulated. Then a new approach for solving such a problem is described.

3.1 *Problem formulation*

The measurement equation for multiresolution measurements can be described by

$$z_i = l_i x_i + v_i = t_i + v_i, \quad i=0,\ 1,..,N . \qquad (3.1.1)$$

where z_i are measurements from N sensors or computed values at different resolution levels i (i=0, 1, 2, ...N), and x_i (i=0, 1,2,...,N) of the same gravity field quantity corresponding to different resolutions. The scale N corresponds to the highest resolution, while the scale 0 corresponding to the lowest scale. l_i is a linear operator, the v_i (i=1,2,...,N) are stochastic processes whose first and second moments are assumed to be known, i.e.

$$E(v_i)=0, \quad E(v_i v_i^T) = R_i, \quad i=0,\ 1,2,...N.$$

The problem of multiresolution approximation of gravity field modelling can be formulated as follows. Given measurements related to the gravity field at different resolutions, an optimal estimate of the gravity field signal at each scale is to be sought.

In the following subsection, we concentrate on developing a method to solve this problem.

3.2 *New approach*

To begin, some notations will be defined. $x_{i|i}$ denotes the estimate of x at scale i based on all measurements with resolution higher or equal to i. $x_{i|i+1}$ denotes the estimate of x at scale i based on all measurements with resolution higher or equal to i+1. A similar notation is used for other quantities. x_{si} denotes the optimal estimate of x at scale i based on all available measurements.

The new approach of estimating the gravity field quantity based on measurements of the form (3.1.1) is given in four steps:

Step 1 :Estimation of $x_{N|N}$ and the corresponding error covariance of $x_{N|N}$ at the finest scale N:

$x_{N|N}$ can be computed using least-squares collocation, i.e.

$$x_{N|N} = C_{x_N t_N}(C_{t_N t_N} + C_{v_N v_N})^{-1} z_N \qquad (3.2.1a)$$

with error covariance

$$C_{\varepsilon_{N|N}} = C_{x_N x_N} - C_{x_N t_N}(C_{t_N t_N} + C_{v_N v_N})^{-1} C_{t_N x_N}, \qquad (3.2.1b)$$

where $C_{t_N t_N}$ and $C_{v_N v_N}$ are the covariances of t_N and v_N, respectively, $C_{x_N t_N}$ is covariance between x_N and t_N, $C_{\varepsilon_{N|N}}$ is the error covariance of estimate $x_{N|N}$ and $C_{x_N x_N}$ the covariance of x_N. Readers who are not familiar with the theory of collocation are referred to Moritz (1980) for the details.

It can also be computed using a numerical integral formula if the type of measurements at the finest scale is the same, i.e.

$$x_{N|N} = \Delta x \Delta y \sum_{i,j} K(i,j) z_N(i,j) \qquad (3.2.2a)$$

with error covariance

$$C_{\varepsilon_{N|N}} = (\Delta x \Delta y)^2 \sum_{i,j} K^2(i,j) R_N(i,j), \qquad (3.2.2b)$$

where K is the kernel function, Δx and Δy are the grid increments along x and y.

It should be mentioned that the representation (3.2.2a) holds only in the case of planar approximation.

$\boxed{\text{Step 2}}$: Estimation of $x_{N-1|N}$ from $x_{N|N}$:

$x_{N-1|N}$ can be computed using the discrete wavelet transforms described in Section 2, i.e.

$$x_{N-1|N} = (H^* \otimes H^*)\, x_{N|N} \qquad (3.2.3a)$$

with error covariance

$$C_{\varepsilon_{N-1|N}} = (H^* \otimes H^*)^T\, C_{\varepsilon_{N|N}} (H^* \otimes H^*). \qquad (3.2.3b)$$

At the same time, the added details can also be computed, i.e.

$$d_{N-1,1} = (G \otimes H)\, x_{N|N}, \qquad (3.3.4a)$$

$$d_{N-1,2} = (H \otimes G)\, x_{N|N}, \qquad (3.3.4b)$$

$$d_{N-1,3} = (G \otimes G)\, x_{N|N}. \qquad (3.3.4c)$$

They will be used in Step 4.

$\boxed{\text{Step 3}}$: Measurement update:

$x_{N-1|N}$ can be updated using the measurements at scale N-1, i.e.

$$x_{N-1|N-1} = A_{N-1}x_{N-1|N} + K_{N-1}(z_{N-1} - B_{N-1}x_{N-1|N}),$$

$$C_{x_{N-1}} = C_{x_{N-1}x_{N-1}} + C_{\varepsilon_{N-1|N-1}},$$

$$A_{N-1} = C_{x_{N-1}x_{N-1}} C^{-1}_{x_{N-1}}$$

$$K_{N-1} = C_{N-1}\bar{C}^{-1}_{N-1}$$

$$C_{N-1} = C_{x_{N-1}t_{N-1}} + C^T_{x_{N-1}t_{N-1}} C^{-1}_{x_{N-1}} C_{x_{N-1}t_{N-1}}$$

$$\bar{C}_{N-1} = C_{t_{N-1}t_{N-1}} + C_{v_{N-1}v_{N-1}} + C^T_{x_{N-1}t_{N-1}} C^{-1}_{x_{N-1}} C_{x_{N-1}t_{N-1}},$$

$$B_{N-1} = C^T_{x_{N-1}t_{N-1}} C^{-1}_{x_{N-1}},$$

$$C_{\varepsilon_{N-1|N-1}} = C_{x_{N-1}x_{N-1}} - C_{x_{N-1}x_{N-1}} C^{-1}_{x_{N-1}} C_{x_{N-1}x_{N-1}} + \bar{C}_{x_{N-1}t_{N-1}} C^{-1}_{t_{N-1}} \bar{C}^T_{x_{N-1}t_{N-1}},$$

$$\bar{C}_{x_{N-1}t_{N-1}} = C_{x_{N-1}t_{N-1}} - C_{x_{N-1}x_{N-1}} C^{-1}_{x_{N-1}} C_{x_{N-1}t_{N-1}}. \qquad (3.3.5)$$

Formula (3.3.5) can be derived using a procedure similar to that of the stepwise collocation in Moritz (1980).

Step 2 and Step 3 are repeated until the lowest scale 0 is reached.

Step 3 provides the mechanism of combining data from two different resolution levels for the gravity field determination.

$\boxed{\text{Step 4}}$: Estimation of the optimal estimate based on all available data at scales between 0 and N:

Once the optimal estimate $x_{0|0}$ at scale 0 has been computed, the optimal estimates at other scales (i=1,2,...,N) can be derived using the reconstruction procedure of the wavelet transforms, i.e.

$$x_{s(i+1)} = (H^* \otimes H^*)\, x_{si} + (G^* \otimes H^*) d_{i,1} + (H^* \otimes G^*)\, d_{i,2} + (G^* \otimes G^*) d_{i,3}$$
$$(i=0,1,2,...,N-1). \qquad (3.3.6)$$

Note that this new approach provides not only the optimal estimation of gravity field signals at multiple resolutions but also the fusion of measurements at multiple resolutions.

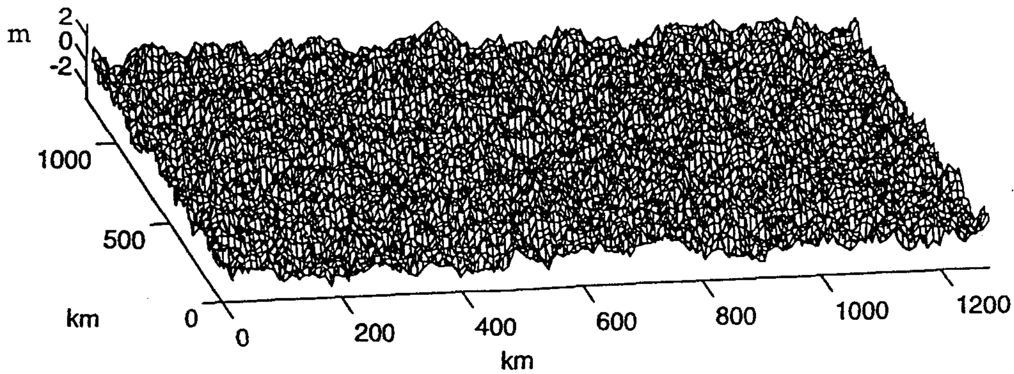
4. Numerical Examples

In this section, several examples of the application of the new approach described in this paper are provided. The purpose of the first example is to demonstrate that the new approach can be successfully applied to the determination of geoidal heights based on the measurements at two scales, i.e. fine-scale gravity anomalies and coarse-scale geoidal height. In the second example, an example of fusion of multiresolution measurements is presented using gravity anomalies at two different scales. Finally, in the third example, fusion of fine-scale gravity anomalies with limited coverage with full coverage coarse-scale gravity anomalies is considered.

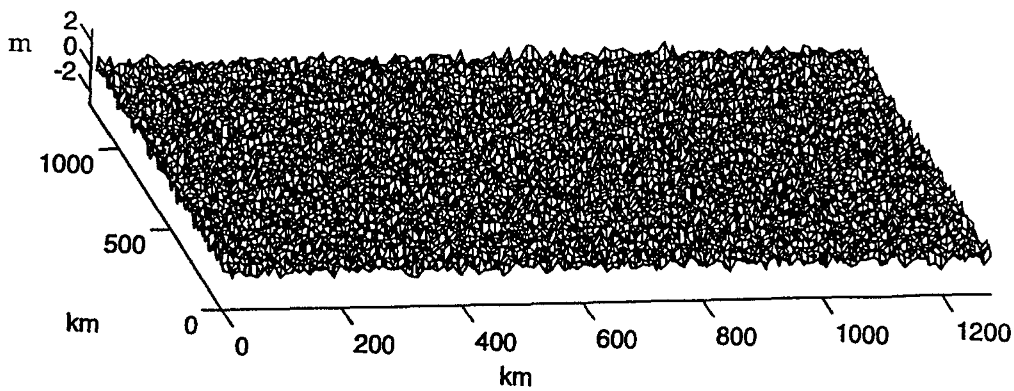
Since the purpose of the following three examples is to demonstrate the effectiveness of the proposed approach, only the Haar wavelet is used in the computations. The gravitational field generated by a point mass is chosen as the disturbing field.

4.1 *Geoidal height determination scale gravity anomalies and coarse-scale geoidal height data*

In the first example, measurements at fine scale are gravity anomalies with resolution 10 km x 10 km and coverage 1280 km x 1280 km. Measurement noise for the gravity anomalies is assumed to be white noise with covariance 25 $mGal^2$ and a bias of 2 mGal. At coarse scale, the measurements are geoidal heights with resolution 20 km x 20 km and the same coverage as gravity anomalies. The measurement noise for geoidal height is also assumed to be white noise with covariance 0.01 m^2. The optimal estimate of geoidal heights is computed using the approach developed in this paper. The initial estimate at fine scale is computed by the planar Stokes' formula. Figure 4.1 shows the errors of the initial estimate of geoidal height at fine scale using only the gravity anomalies, and the optimum estimate of geoidal height at fine scale using both the fine scale and coarse scale measurements. The statistics of this estimation is given in Table 4.1.



(a) Using fine scale measurements only



(b) Using both fine scale and coarse scale measurements

Fig. 4.1: The errors of the geoidal height estimates at fine scale

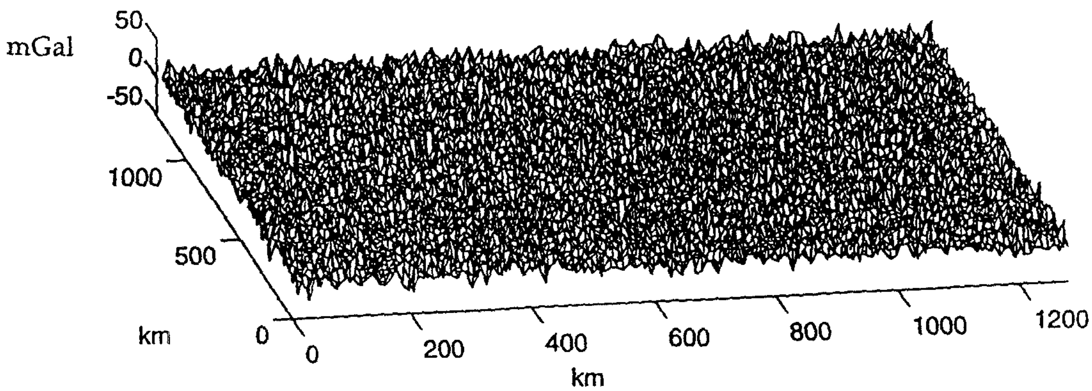
Geoidal height	Mean (m)	Std (m)	RMS (m)
estimate at fine scale (Δg only)	-0.54	0.83	0.99
measurement noise at coarse scale (ζ only)	0.00	0.10	0.10
fused estimate at fine scale	0.02	0.41	0.41
fused estimate at coarse scale	0.00	0.10	0.10

Table 4.1 Statistics of geoidal height estimation

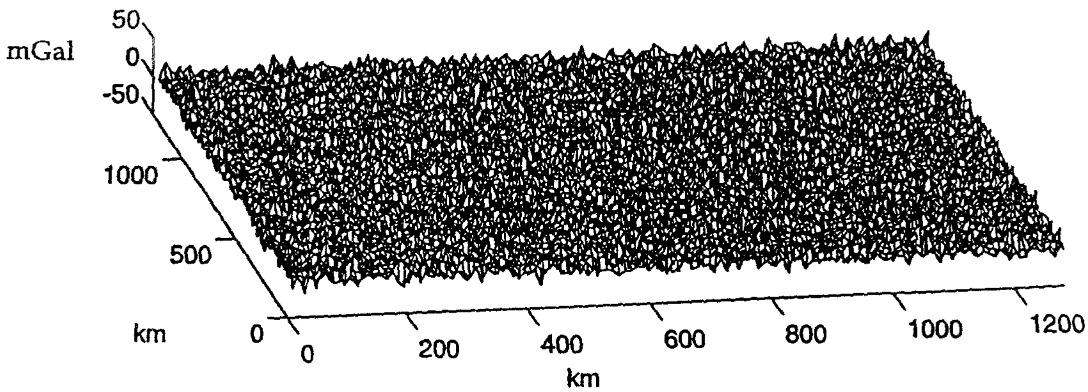
From Figure 4.1 and Table 4.1, it is easy to see that better results have been achieved after combining the two different measurement sets by the proposed method. The error of geoidal height at fine scale is reduced by 50% and the bias is removed as compared to the estimate computed from gravity anomalies only. The reason is that highly accurate coarse-scale measurements provide information which allows removal or reduction of the error in the low frequency part of the fine-scale estimate. The estimate at coarse scale is the same as given by the measurements because measurements at coarse scale are much better than the estimate at coarse scale obtained from the wavelet transform in this example.

4.2. *Fusion of gravity anomalies at two different scales*

In the second example, measurements in both fine scale and coarse scale are gravity anomalies with resolution 10 km x 10 km and 20 km x 20 km, respectively. The coverage for both of them is 1280 km x 1280 km. Measurement noise for the gravity anomalies is assumed to be white noise with covariance of 100 mGal^2 at the fine scale and 1 mGal^2 at coarse scale. There is also a bias of 2 mGal in the fine-scale measurements. The fusion of these two data sets is done using the proposed approach. Figure 4.2 shows the errors of the measurements at fine scale before and after data fusion. Table 4.2 summarizes the statistics of this estimation.



(a) Before data fusion



(b) After data fusion

Fig. 4.2: The errors of the measurements at fine scale before and after data fusion (Example 4.2)

gravity anomaly	Mean (mGal)	Std (mGal)	RMS (mGal)
measurement noise at fine scale	2.00	10.00	10.91
measurement noise at coarse scale	0.00	1.00	1.00
fused measurement at fine scale	0.11	7.74	7.75
fused measurement at coarse scale	0.00	0.10	0.10

Table 4.2 Statistics for the fusion of gravity anomalies

Similar conclusions to Section 4.1 can be drawn from Figure 4.2 and Table 4.2.

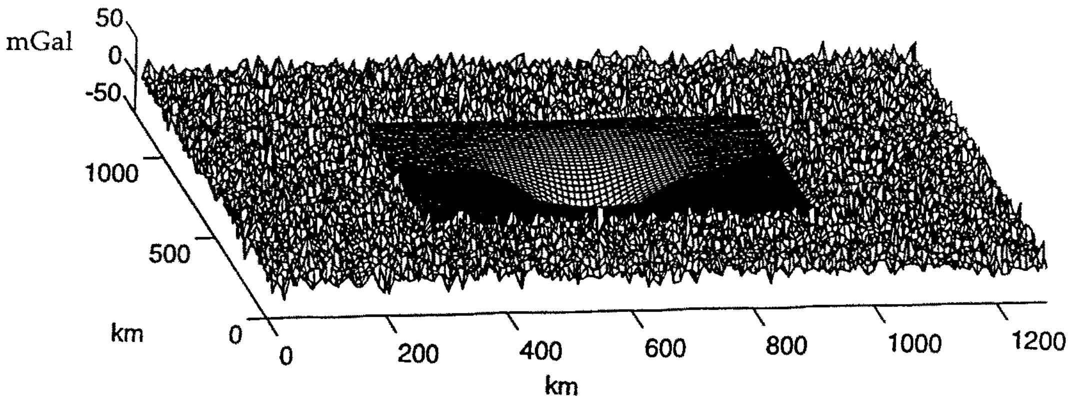
4.3. *Fusion of fine-scale gravity anomalies with limited coverage with full coverage coarse-scale gravity anomalies*

In the final example, the fusion of limited coverage gravity anomaly measurements of resolution 10 km x 10 km with

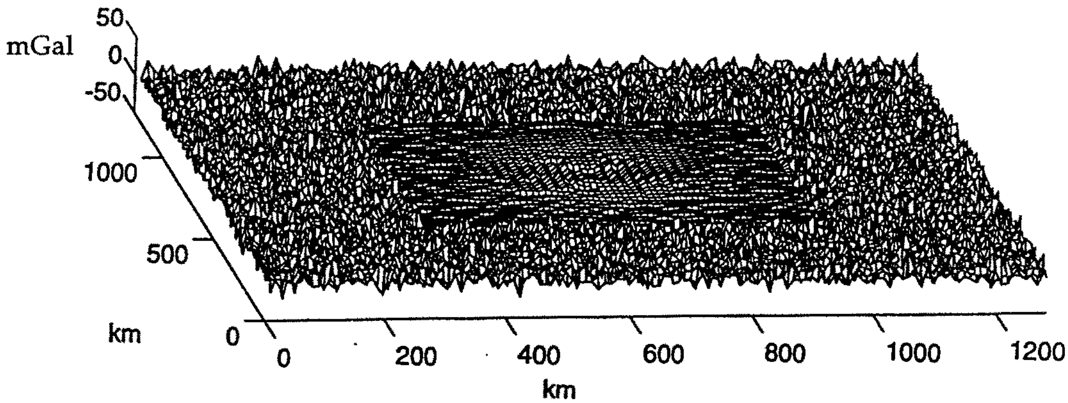
full coverage gravity anomalies of resolution 20 km x 20 km is considered. The coverage for coarse-scale gravity anomalies is 1280 km x 1280, while at fine scale the core area (32 km < x < 96 km, 32 km < y< 96 km) is blank. Measurement noise for the gravity anomalies is also assumed to be white noise with covariance of 100 mGal^2 and 1 mGal^2 at fine and coarse scales, respectively. There is a bias of 2 mGal in the fine-scale measurements. The fusion of these two measurements is done using the proposed approach, in which the gravity anomalies are set to zero with sufficient large error covariance in the core area. Figure 4.3 shows the errors of the original measurements at the fine scale and the fused measurements at the fine scale. Table 4.3 gives the statistics of this estimation.

gravity anomaly	Mean (mGal)	Std (mGal)	RMS (mGal)
fine-scale measurement noise (limited coverage)	1.83	21.41	21.59
coarse-scale measurement noise (full coverage)	0.00	1.00	1.00
fused estimate at fine scale	0.10	8.07	8.07
fused estimate at coarse scale	0.00	1.00	1.00

Table 4.3 Statistics for the fusion of gravity anomalies with different resolution and coverage



(a) Before data fusion



(b) After data fusion

Fig. 4.3: The errors of the measurements at fine scale before and after data fusion (Example 4.3)

Figure 4.3 and Table 4.3 show that good results for the fused measurements can be obtained. The reason is that the coarse-scale data provides information which supports the interpolation of the fine-scale data.
ion which supports the interpolation of the fine-scale data.

5. Conclusions

The following conclusions can be drawn from this paper:

• Wavelet theory can be used for multiresolution approximation of the gravity field. Its application in physical geodesy is a new area to be further explored, for example, for the combination of airborne gravity and satellite altimetry data.

• The new approach developed in this paper can be used for the optimal estimation of gravity field quantities based on multiresolution measurements and fusion of measurements with different resolution. It can also be used for the interpolation of fine-scale measurements in an area with limited fine-scale measurement coverage, provided full coverage coarse-scale measurements are available.

• Preliminary results show the good performance of the proposed method.

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