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Invited review

Integral formulas for transformation of potential field parameters in geosciences

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ABSTRACT

In this article boundary-value problems of potential theory, that are required for transforming and continuing selected parameters of a potential field, are discussed. By the potential field parameters we understand the potential and its gradients (potential gradients) up to the third order. In particular, integral equations defined for a spherical boundary transforming potential gradients of different orders and continuing potential gradients of the same order through the 3-D space are reviewed and classified. This mathematical apparatus can be used for any harmonic potential, such as electric, magnetic or gravitational, under the assumption of its conservativeness, i.e., neglecting possible temporal variations. Integral transforms are discussed in context of geoscience applications, namely in terms of the Earth's gravitational field; however, the article can serve as a general reference for integral transforms of potential field parameters in any scientific or engineering area where potential fields are used.

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1. Introduction

Potential theory is used for modelling physical fields that can be described by potential functions of 3-D position which satisfy Laplace's equation (MacMillan, 1930). In particular, potential theory is used for mathematical description of conservative potential fields, i.e., fields without temporal variability. Typical examples are static electric (electrostatics) and static gravitational (gravitostatics) fields. In a sense, potential theory relates to the theory of Laplace's differential equation as harmonic functions are the main subject of both.

Potential theory is also applied in studying and describing different potential fields in geosciences such as gravitational and magnetic fields of the Earth and other celestial bodies (Blakely, 1996). These potential fields are in fact varying in time as our planet and other celestial bodies move and change continuously; still potential theory can be used under a mild assumption of both fields being static, thus conservative, with scalar description in terms of a respective potential function. Typically, the potentials are directly unobservable (observing gravitational potential differences by atomic clocks is foreseen) but their gradients of different orders can often be measured with a very high relative and/or absolute accuracy. For example, vertical gradients of the gravitational potential can be measured with the relative accuracy of 1 ppb (e.g., Sasagawa et al., 1995). In recent years higher-order gradients of the gravitational potential have become observable. Currently, second-order gradients of the gravitational potential are observed not only at the Earth's surface but also by airborne and spaceborne sensors (e.g., Rummel, 1986; Dransfield and Christensen, 2013). Third-order gradients of the gravitational potential were already considered (Fantino and Casotto, 2009) and first physical experiments to measure them were already successfully performed (Rosi et al., 2015). Thus, both the potential and Laplace theories are important as ever and their extension to higher-order potential gradients, which would allow for their transformation and continuation, is needed.

The main purpose of the article is to review boundary-value problems of potential theory for boundary values in a form of the potential gradients up to the third order which are used as basis for deriving integral transforms between potential gradients of different orders, i.e., between lower- and higher-order potential gradients and vice versa. The integral transforms also include the so-called continuation of the potential field parameters for the case when the potential gradients of the same order are transformed between two different points of space. The respective boundary-value problems are classified, referenced wherever suitable references exist and selected examples of their applications in geosciences are given.

2. Fundamentals and nomenclature

This section reviews the background mathematical concepts, terminology and nomenclature used throughout the article. We start with an orthogonal coordinate system $\{\xi_i : i = 1,2,3\}$ defined in the 3-D (Euclidian) space with three scale factors $\{h_i : i = 1,2,3\}$. Two specific orthogonal coordinate systems are used in this article, see Fig. 1: the Cartesian one with three coordinates $\{x,y,z\}$ and unit scale factors, and the spherical (curvilinear) one with three coordinates $\{r,\varphi,\lambda\}$ being a radial distance, spherical latitude and longitude, and respective scale factors $h_1 = 1, h_2 = r$ and $h_3 = r\cos\varphi$.

In this article we consider the potential field parameters referring to the Cartesian system defined by a moving origin and a right-handed basis with three mutually orthogonal axes – x pointing to the North, y pointing to the West and z pointing radially outward, see Fig. 1. This coordinate system is realized by location and orientation of sensors collecting data; in geodesy this realization of the Cartesian coordinate system is referred to as a *local north-oriented reference frame*. As the Earth's gravitational field is used in the article as an example for application of integral transforms, a *geocentric reference frame* representing another specific realization of the right-handed Cartesian coordinate system is defined with the following properties: its origin is in the centre of the Earth's mass, its z-axis coincides with the mean axis of the Earth's rotation and x-axis is the intersection of the mean Greenwich meridian plane with the Earth's equator.

Integral formulas transforming the potential field parameters are used as the mathematical apparatus of our choice. In the geocentric reference frame, the geocentric position vector $\mathbf{x}$ designates the position of an evaluation (integration) point while the geocentric vector $\mathbf{x}'$ designates the position of an integrated infinitesimal surface element ("running point"). In all formulations the mean geocentric sphere S of radius R approximating the Earth plays an important role. Geocentric spherical coordinates $(r,\varphi,\lambda) = (r,\Omega)$ alternatively define the geocentric position of an evaluation point and the spherical coordinates (r',Ω') represent the position of an infinitesimal spherical surface element $\mathrm{d}S(\mathbf{x}') = R^2 \cos\varphi' \,\mathrm{d}\varphi' \,\mathrm{d}\lambda' = R^2 \mathrm{d}\Omega'$.

We also define and use the spherical polar coordinates $\psi = \psi(\mathbf{x},\mathbf{x}')$, $\alpha = \alpha(\mathbf{x},\mathbf{x}')$ and $\alpha' = \alpha'(\mathbf{x},\mathbf{x}')$, see Fig. 1. The symbol ψ stands for the spherical distance between the geocentric directions $\Omega = (\varphi,\lambda)$ and $\Omega' = (\varphi',\lambda')$

$$\cos\psi = \sin\varphi\,\sin\varphi' + \cos\varphi\,\cos\varphi'\,\cos\left(\lambda - \lambda'\right) = \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|\,|\mathbf{x}'|} = u(\mathbf{x},\mathbf{x}') = u\,. \tag{1}$$

The symbol α abbreviates the direct azimuth between the geocentric directions Ω and Ω' measured at the evaluation point clock-wise from the northern meridional segment. According to the rules of spherical geometry, see, e.g., Winch and Roberts (1995),

$$\cos\alpha = \frac{1}{\sqrt{1-u^2}}\,\left[\sin\varphi'\cos\varphi - \cos\varphi'\sin\varphi\cos\left(\lambda'-\lambda\right)\right], \tag{2}$$

$$\sin\alpha = \frac{1}{\sqrt{1-u^2}}\,\cos\varphi'\sin\left(\lambda'-\lambda\right)\;. \tag{3}$$

The symbol α' stands for the backward azimuth between the two geocentric directions Ω' and Ω as measured at the integration element clock-wise from the northern meridional segment. According to the rules of spherical geometry

$$\cos\alpha' = \frac{1}{\sqrt{1-u^2}}\,\left[\sin\varphi\cos\varphi' - \cos\varphi\sin\varphi'\cos(\lambda'-\lambda)\right], \tag{4}$$

$$\sin\alpha' = \frac{-1}{\sqrt{1-u^2}}\,\cos\varphi\sin(\lambda'-\lambda)\,. \tag{5}$$

Moreover, we define the following unitless substitution parameters: an attenuation factor

$$t = t(\mathbf{x},\mathbf{x}') = \frac{|\mathbf{x}|}{|\mathbf{x}'|} = \frac{R}{r}\,,\quad r \geq R\,, \tag{6}$$

and a normalized Euclidean distance between the points $\mathbf{x}$ and $\mathbf{x}'$

$$q = q\left(\mathbf{x},\mathbf{x}'\right) = \sqrt{1 + \left(\frac{R}{r}\right)^2 - \frac{2R}{r}\,\cos\psi} = \sqrt{1+t^2-2tu} = q(t,u)\,. \tag{7}$$

The substitution parameters u, t, q simplify presentation of forthcoming mathematical formulas. Spherical geometry corresponds well to problems that appear in geodesy while transforming various gravitational field parameters: some parameters are observed at the Earth's surface, others along aerial trajectories and/or satellite

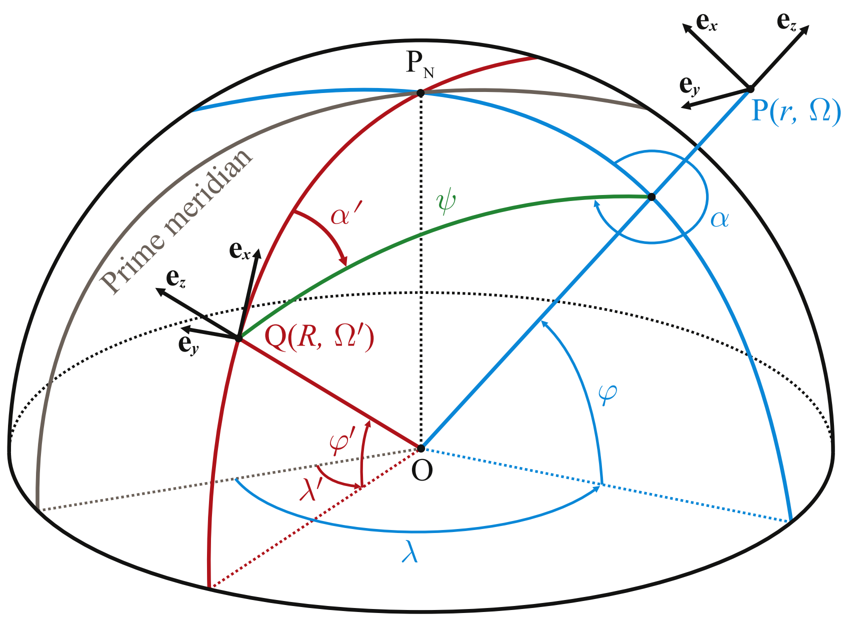

Fig. 1. Spherical coordinates φ and λ, distance ψ, forward azimuth α and backward azimuth α'.

orbits (geometry of sampling points as well as of the Earth' surface can be approximated by a sphere), from which the unobservable gravitational potential is sought (e.g., Torge and Müller, 2012).

Mathematical formulations presented in the article can be applied to any static potential field. We use the static gravitational field from now on as a typical representant of such a static field neglecting its temporal variations. As for the gravitational field, its potential V is used according to common practice in physics as a *continuous function* of the geocentric position $\mathbf{x}$ representing the amount of work done by the gravitational force to move a unit mass from its location in space to infinity. For the unit vector $\mathbf{e} = \{e_i : i = x, y, z\}$ in the Cartesian coordinate system, its gradient is generally defined as follows:

$$\text{grad}\ V(\mathbf{x}) = \sum_{i=x,y,z} \frac{e_i}{h_i} \frac{\partial V}{\partial x_i} = \sum_{i=x,y,z} V_i\ e_i\ . \tag{8}$$

Properties of the gravitational field in the Newtonian sense are governed by the so-called field equations with the fundamental one being the Laplace differential equation:

$$\text{div grad}\ V(\mathbf{x}) = -4\pi G \varrho(\mathbf{x})\,, \tag{9}$$

where G represents the universal gravitational constant and ϱ is the mass density distribution function. The gravitational potential satisfies the Laplace condition in mass-free space with $\varrho = 0$, i.e., it is a *harmonic function* there:

$$\text{div grad}\ V(\mathbf{x}) = 0\ . \tag{10}$$

Another fundamental property of the gravitational potential also includes *regularity* at infinity (direct consequence of the universal law of gravitation), i.e.:

$$\lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}) = 0\ . \tag{11}$$

The three properties of the gravitational potential – continuity, harmonicity and regularity – form fundamental conditions for all presented formulations. They also guarantee that respective solutions exist, are unique and stable (Hörmander, 1961).

3. Parametrization of the gravitational field

Let us consider – due to harmonicity of the gravitational potential – only the external gravitational field of the static Earth. This assumption allows for the application of a whole class of boundary-value problems with solutions defined in terms of surface integral equations. Any gravitational field is a vector field described at any point in the 3-D space by a vector called the gravitational acceleration. Following the gravitational theory of Newton (which is permissible due to limited velocities of all objects under consideration), the gravitational acceleration can be computed at any point given the 3-D mass density distribution of gravitating masses B is known. The static gravitational field can be represented at any point by the gravitational potential [$\mathrm{m^2\ s^{-2}}$]:

$$V(\mathbf{x}) = G \int_B \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, \mathrm{d}B(\mathbf{x}')\,, \tag{12}$$

with an infinitesimal volume $\mathrm{d}B(\mathbf{x}') = r'^2\ \mathrm{d}r'\ \mathrm{d}\Omega'$. This scalar parameter represents the simplest parametrization of the vector gravitational field; in fact, the gravitational potential is also a zero-order gravitational tensor. Considering this description as the simplest one, typical problems in geodesy solve for the gravitational potential (or unknown parameters in any of its possible representations such as numerical coefficients in harmonic series) from available discrete observations. In geodesy, the gravitational potential is particularly important as it defines the datum for physical heights (geoid). However, there also exists a whole family of other problems related to data validation and combination where transforms of various gravitational field parameters are needed (see, e.g., Vaníček and Krakiwsky, 1987, Sect. 21.1).

Values of the gravitational potential are not directly observable; its differences can be derived by combining spirit levelling with surface gravity observations (e.g., Vaníček and Krakiwsky, 1987, Sect. 16.4) or by applying theory of relativity when measuring gravitational time dilation in space (Chou et al., 2010). Observable parameters include the first-order gradients of the gravitational potential, i.e., components of the *first-order gravitational tensor* or the gravitational acceleration [m s^{-2}]:

$$\text{grad}\, V(\mathbf{x}) = \mathbf{g}(\mathbf{x})\,, \tag{13}$$

components of the *second-order gravitational tensor* or the gravitational gradient [s^{-2}]:

$$\text{grad grad}\, V(\mathbf{x}) = \Gamma(\mathbf{x})\,, \tag{14}$$

and components of the *third-order gravitational tensor* or the gravitational curvature [m^{-1} s^{-2}]:

$$\text{grad grad grad}\; V(\mathbf{x}) = \mathcal{G}(\mathbf{x}). \tag{15}$$

The components of the gravitational tensors defined through Eqs. (13)–(15) are further denoted as V_i, V_{ij} and V_{ijk} for $\{i, j, k = x, y, z\}$. Fig. 2 shows structures of the gravitational gradients of the second and third orders.

The question of "observability" of the gravitational field parameters defined above is very important for understanding the meaning or applicability of problems which are discussed later in the article. The most typical observational method to date is based on measuring components of the first-order gravitational tensor $\mathbf{g}$ (Torge, 1989) from which values of V, V_{ij} and eventually also V_{ijk} can be derived. These values are sampled in space and time at the surface of the Earth (terrestrial and marine gravimetry), by aircraft (aerial gravimetry) and in space (satellite gravimetry). In case of terrestrial gravimetry (with sensors rotating with the Earth), gravity rather than gravitation is observed with additional centrifugal acceleration affecting the observations. However, knowing the Earth's angular velocity and geocentric position of the sensor, the centrifugal component can precisely be computed and removed from the observations. Depending on a particular observation principle and sensor location, the relative accuracy of ground gravity observations can reach 1 ppb (absolutely 10^{-8} m s^{-2}).

Observation techniques providing values of the second-order gravitational tensor Γ (its components are the second-order gravitational gradients) can be traced back to the 19th century when the Hungarian physicist Loránd Eötvös developed the torsion balance and measured first terrestrial values of the second-order gravitational gradients, see Eötvös (1896). The renaissance of second-order gravitational gradient observations occurred at the beginning of the 21st century when sensors were designed to be borne by both aircraft and satellites (aerial and satellite gradiometry). The most important progress in the field of second-order gravitational gradient observations happened in 2009 when the Gravity field and steady-state Ocean Circulation Explorer (GOCE) was launched by the European Space Agency (Rummel, 2010). Until 2013 the mission had provided an abundant observation material covering almost the entire Earth's surface (except polar regions). The relative accuracy of these data was at the level of 1%; however, they were limited to a specific measurement bandwidth and only four second-order gravitational gradients were sampled with a relatively low noise level (Floberghagen et al., 2011).

In terms of the third-order gravitational tensor (its components are the third-order gravitational gradients), first attempts to measure its values have been performed recently: Rosi et al. (2015) proposed an observation method based on a principle of the atom interferometry. One can expect first operational observational devices to appear in course of time as further advances in technology will eventually result in successful observational techniques. Successful observations of the third-order gravitational gradients in space would counterbalance the attenuation of the gravitational field with an increasing distance from gravitating masses, which has a logarithmic effect in terms of individual frequencies of the gravitational potential, given a signal-to-noise ratio of these small values is maintained at a reasonable level.

This article will not make any further references to observations, data availability and/or characteristics such as spectral or stochastic. Its main goal is to provide comprehensively all possible integral transforms between the components of the gravitational tensors of the zero, first, second and third orders defined above.

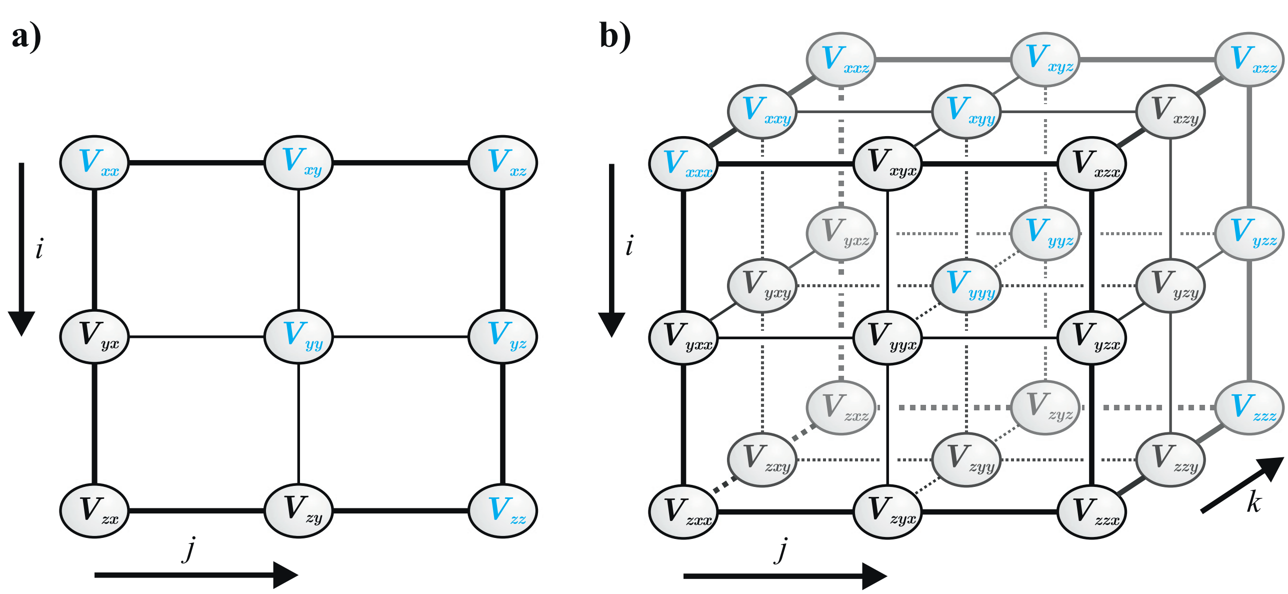

Fig. 2. Structures of the second-order (a) and third-order (b) gravitational gradients (defining components in blue).

4. Component decomposition of the gravitational tensors

In order to formulate mathematical models for transformation of the gravitational field parameters in the mass-free space, we proceed with discussing important properties of these parameters. As we use boundary-value problems of potential theory, components of the gravitational tensors will be decomposed into symmetric and anti-symmetric parts. In the most general case of the third-order gravitational tensor, the symmetric spherical triads $\mathbf{e}_{ijk} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \{i, j, k = x, y, z\}$ can be used to group the tensor components to be representable by orthonormal tensor-valued spherical harmonics (Zerilli, 1970; van Gelderen and Rummel, 2001; Šprlák and Novák, 2016a). Omitting the position parameter $\mathbf{x}$ the third-order gravitational tensor can be decomposed into four parts according to horizontal (x or y) and vertical (z) indices:

$$\begin{aligned}
\mathcal{G} = & V_{zzz}\ \mathbf{e}_{zzz} + V_{xzz}\ (\mathbf{e}_{xzz} + \mathbf{e}_{zxz} + \mathbf{e}_{zzx}) + V_{yzz}\ (\mathbf{e}_{yzz} + \mathbf{e}_{zyz} + \mathbf{e}_{zzy}) \\
& + \frac{1}{2}(V_{xxz} - V_{yyz})\ (\mathbf{e}_{xxz} + \mathbf{e}_{xzx} + \mathbf{e}_{zxx} - \mathbf{e}_{yyz} - \mathbf{e}_{yzy} - \mathbf{e}_{zyy}) \\
& + \frac{1}{2}(V_{xxz} + V_{yyz})\ (\mathbf{e}_{xxz} + \mathbf{e}_{xzx} + \mathbf{e}_{zxx} + \mathbf{e}_{yyz} + \mathbf{e}_{yzy} + \mathbf{e}_{zyy}) \\
& + V_{xyz}\ (\mathbf{e}_{xyz} + \mathbf{e}_{xzy} + \mathbf{e}_{yxz} + \mathbf{e}_{yzx} + \mathbf{e}_{zxy} + \mathbf{e}_{zyx}) \\
& + \frac{1}{4}\left[(V_{xxx} - 3V_{xyy})\ (\mathbf{e}_{xxx} - \mathbf{e}_{xyy} - \mathbf{e}_{yxy} - \mathbf{e}_{yyx})\right. \\
& \left. + (V_{yyy} - 3V_{xxy})(\mathbf{e}_{yyy} - \mathbf{e}_{xxy} - \mathbf{e}_{xyx} - \mathbf{e}_{yxx})\right] \\
& + \frac{1}{4}\left[(V_{xxx} + V_{xyy})\ (3\mathbf{e}_{xxx} + \mathbf{e}_{xyy} + \mathbf{e}_{yxy} + \mathbf{e}_{yyx})\right. \\
& \left. + (V_{yyy} + V_{xxy})\ (3\mathbf{e}_{yyy} + \mathbf{e}_{xxy} + \mathbf{e}_{xyx} + \mathbf{e}_{yxx})\right].
\end{aligned} \tag{16}$$

The first term on the right-hand side of Eq. (16) is the vertical-vertical-vertical (vvv) part. The second and third terms represent the horizontal-vertical-vertical (hvv) part. The horizontal-horizontal-vertical (hhv) part is composed of the fourth, fifth and sixth terms. The last two terms represent the horizontal-horizontal-horizontal (hhh) part of the third-order gravitational tensor. The first- and second-order gravitational tensors can be decomposed accordingly, see Table 1.

The first-order gravitational tensor $\mathbf{g}$ (gravitational acceleration), see Eq. (13), has three components in the 3-D space. In the Cartesian coordinate system these components can conveniently be grouped as follows:

- horizontal (h) if $\{i = x, y\}$,
- and vertical (v) if $i = z$.

The second-order gravitational tensor Γ (gravitational gradient), see Eq. (14), is more interesting being composed of 9 components. This tensor is symmetric with respect to the indices (ij), i.e., it holds $V_{ij} = V_{ji}$. Thus, only 6 components are different. Moreover, only 5 components are independent due to the Laplace condition of the diagonal components, i.e., $V_{xx} + V_{yy} + V_{zz} = 0$. In the Cartesian coordinate system, the 6 different components can be grouped as follows:

- horizontal-horizontal (hh) if $\{i, j = x, y\}$,
- horizontal-vertical (hv) if $\{i = x, y\}$ and $j = z$,
- and vertical-vertical (vv) if $\{i, j = z\}$.

Table 1
Component groups of the gravitational field parameters corresponding to orthonormal systems of spherical harmonics.

Order	Parameter	Group τ	Combination
1	Acceleration $\mathbf{g}$	v	V_z
		h	V_x, V_y
2	Gradient Γ	vv	V_{zz}
		hv	V_{xz}, V_{yz}
		hh	$V_{xx} - V_{yy}$, $2V_{xy}$
3	Curvature $\mathcal{G}$	vvv	V_{zzz}
		hvv	V_{xzz}, V_{yzz}
		hhv	$V_{xxz} - V_{yyz}$, $2V_{xyz}$
		hhh	$V_{xxx} - 3V_{xyy}$, $V_{yyy} - 3V_{xxy}$

Table 2
Properties of the gravitational field parameters.

Parameter	Notation	Order m	Components: Total	Defining	Independent	Groups	Laplacians
Potential	V	0	1	1	1	1	0
Acceleration	$\mathbf{g}$	1	3	3	3	2	0
Gradient	Γ	2	9	6	5	3	1
Curvature	$\mathcal{G}$	3	27	10	7	4	3

The third-order gravitational tensor $\mathcal{G}$ (gravitational curvature), see Eq. (15), contains already 27 components. The tensor is symmetric with respect to the pairs of indices (ij), (jk) and (ik), i.e., it holds $V_{ijk} = V_{jik} = V_{ikj} = V_{kij}$. By such symmetries the third-order gravitational tensor can be defined by only 10 components, see, e.g., (Tóth, 2005). Another simplification comes from the three Laplace conditions, i.e., $V_{xxk} + V_{yyk} + V_{zzk} = 0$; $\{k = x, y, z\}$, see Casotto and Fantino (2009). Thus, the third-order gravitational tensor is composed only of 7 independent components. The 10 different components may further be divided into the following groups, see Šprlák and Novák (2015):

- horizontal-horizontal-horizontal (hhh) if $\{i, j, k = x, y\}$,
- horizontal-horizontal-vertical (hhv) if $\{i, j = x, y\}$ and $k = z$,
- horizontal-vertical-vertical (hvv) if $\{i = x, y\}$ and $\{j, k = z\}$,
- and vertical-vertical-vertical (vvv) if $\{i, j, k = z\}$.

Properties of the gravitational tensors defined above are summarized in Table 2. There can be some general observations made regarding their components. It is obvious that the number of the components rapidly grows with the order of the gravitational tensor; denoting their order by m, the number of the components in the 3-D space is 3^m. The number of the defining components for the various tensors grows slowly due to their symmetries; the number is equal to the sum of the number of independent components and the number of Laplace conditions. The number of the independent components corresponds to a sequel of odd numbers ($2m + 1$). The number of the component groups raises only by one for each order of the tensor parameter, i.e., as ($m + 1$). The numbers in the last column reflect the Laplace conditions that apply to the diagonal components; of course, the number is zero for the tensors without any second-order gradients with one for the second-order tensor (one diagonal group) and three for the third-order tensor (three diagonal groups) with the general rule of 3^{m-2} for $m \geq 2$.

5. Transforms between the gravitational field parameters

In this section, we focus on formulating integral transforms between the gravitational field parameters defined in the previous section. The transforms are defined in the spatial domain as particular integral equations that solve boundary-value problems (BVPs) of potential theory. The apparatus of surface integral equations is a natural tool for solving BVPs that link gravitational field parameters defined as gradients of the harmonic gravitational potential. Considering sphericity of the Earth's gravitational field, the spherical boundary is used in all formulations. A biaxial ellipsoid in case of the oblate Earth would be more favourable; however, this approximation will be considered in future studies due to much higher complexity of the formulas. It is acknowledged that other solutions for transformation of the gravitational field parameters can be formulated and are used such as those based on variational calculus, e.g., Holota (2002).

In the following we shall distinguish between *transformation* of two gravitational field parameters of different orders, see Sections 5.1 and 5.2, and *continuation* of the same gravitational field parameter, see Section 5.3, in the mass-free space. One can also combine continuation and transformation of the gravitational field parameters in one integral (Novák, 2003); however, these combinations are not considered.

5.1. Forward transforms of the gravitational field parameters

Links in Table 3 indicated by arrows represent *forward transforms* which can be solved by respective BVPs of potential theory. The forward transforms generally reconstruct a harmonic gravitational potential in the 3-D space by integrating its gradients given on the known (in our case spherical) boundary. This is a natural property of integral equations which is used in solving differential equations. Each particular transform in Table 3 can be represented by one or more integral equations with tensor-valued integral kernels the order of which corresponds to that of the gravitational field parameter evaluated through the surface integration.

We start with the Dirichlet BVP (Kellogg, 1929, pp. 240–242) rigorously applicable to any harmonic function. To present its solution for the exterior of the mean geocentric sphere of radius R, we use the spherical coordinate system. For the harmonic gravitational potential the spherical Poisson integral equation reads (Heiskanen and Moritz, 1967, Eq. 1–89):

$$V(r,\Omega) = \frac{1}{4\pi}\int_{\Omega'} V(R,\Omega')\ \mathcal{K}(t,u)\ \mathrm{d}\Omega'. \tag{17}$$

The kernel function $\mathcal{K}$, also known as the Poisson kernel, can be found in Appendix B, see Eq. (123). In order to simplify the notation and keep the formulations as simple as possible, we shall rewrite Eq. (17) as follows:

$$V = \frac{1}{4\pi}\int_{\Omega'} V\ \mathcal{K}\ \mathrm{d}\Omega'. \tag{18}$$

This simplified notation is used hereafter in all integral equations presented in this article.

Among the transforms in Table 3 there are three fundamental ones, from which all the others can be derived. Basically, they represent transforms of the gravitational tensors $\mathbf{g}$, Γ and $\mathcal{G}$ onto the gravitational potential V. The first transform in the first line of Table 3 is probably the most frequently cited and used integral transform relating the gravitational potential V to its first-order gradients V_i. In this particular case, the *gravimetric BVP* can be formulated for the two component groups of the first-order gravitational tensor (Grafarend, 2001):

$$\text{grad}\ \ V(\mathbf{x}) = \mathbf{g}^{\tau}(\mathbf{x}),\ \ \text{for}\ \ |\mathbf{x}| = R,\ \ \tau = v, h\ . \tag{19}$$

Table 3
Forward transformation of the gravitational field parameters.

No.		1		2		3	
1	V	←	$\mathbf{g}$	←	Γ	←	$\mathcal{G}$
2	V		←		Γ		
3	V			←			$\mathcal{G}$
4			$\mathbf{g}$		←		$\mathcal{G}$

Other two equations required for the complete formulation of this particular BVP are Eqs. (10) and (11). The boundary conditions are:

$$\mathbf{g}^{v} = V_z\ \ \mathbf{e}_z\ , \tag{20}$$

$$\mathbf{g}^{h} = V_x\ \ \mathbf{e}_x + V_y\ \ \mathbf{e}_y\ . \tag{21}$$

Two integral equations for each of the two component groups $\mathbf{g}^v$ and $\mathbf{g}^h$ are defined separately (Grafarend, 2001; Hotine, 1969):

$$V = -\frac{R}{4\pi}\ \ \int_{\Omega'} V_z\ \ \mathcal{K}^{v}\ \ \mathrm{d}\Omega', \tag{22}$$

$$V = \frac{R}{4\pi}\ \int_{\Omega'} (V_x\ \ \cos\alpha' - V_y\ \ \sin\alpha')\ \ \mathcal{K}^{h}\ \ \mathrm{d}\Omega'. \tag{23}$$

The kernel functions $\mathcal{K}^v$ and $\mathcal{K}^h$ can be found in Appendix B, see Eqs. (124) and (125). The kernel function of Eq. (23) is anisotropic (depends on both the spherical distance ψ and the backward azimuth α' between the integration point and integration surface element) as it combines the isotropic function $\mathcal{K}^h$ with the goniometric functions of the backward azimuth, see Eqs. (4) and (5).

A slightly modified form of Eq. (22), also known as the *vertical gravimetric BVP*, was published by (Stokes, 1849). It transforms anomalous (not disturbing) gravity into the disturbing gravitational potential. After more than 160 years this solution still represents the fundamental integral transform between two gravitational field parameters in geodesy as it allows for evaluation of the geoid from gravity data measured on or outside the Earth's surface. While Stokes's BVP relies on gravity anomalies, Hotine's BVP (1969) uses gravity disturbances. The difference between the two gravitational field parameters is based on height information available for a particular observed value of gravity: if an orthometric height is available (height above the geoid/mean sea level usually obtained through levelling), then the gravity anomaly is used; if a geodetic height is available (height above the reference ellipsoid obtained through GNSS positioning), then the gravity disturbance is applied. The solution given by Eq. (23) represents the so-called *horizontal gravimetric BVP* which can be used for transforming measured values of deflections of the vertical (defined as horizontal derivatives of the disturbing gravitational potential scaled by inverse gravity) into the disturbing potential.

In case all three components of the gravitational vector are given, the transforms of Eqs. (22) and (23) represent an overdetermined system and some estimation method must be used for erroneous data. If only the horizontal components are known, then the unknown gravitational potential cannot completely be recovered: the zero-degree term in the harmonic series representation of the gravitational potential corresponding to a homogenous geocentric sphere of radius R must a priori be defined and respective deviations of the actual gravitational potential (so-called disturbing potential) are only estimated. For more details on completeness of the solution refer to Schreiner (1994).

The transform in the second line of Table 3, also known as the *gradiometric BVP*, transforms the second-order gravitational tensor Γ into the gravitational potential V with the help of the following boundary conditions (van Gelderen and Rummel, 2001):

$$\text{grad grad}\ V(\mathbf{x}) = \Gamma^{\tau}(\mathbf{x}),\ \ \text{for}\ \ |\mathbf{x}| = R,\ \ \tau = vv,\ hv,\ hh\ . \tag{24}$$

The boundary conditions are:

$$\Gamma^{vv} = V_{zz}\,\mathbf{e}_{zz}\ , \tag{25}$$

$$\Gamma^{hv} = 2\ \ V_{xz}\ \ \mathbf{e}_{xz} + 2\ \ V_{yz}\ \ \mathbf{e}_{yz}\ , \tag{26}$$

$$\Gamma^{hh} = \frac{1}{2}\,(V_{xx} - V_{yy})\,(\mathbf{e}_{xx} - \mathbf{e}_{yy}) + 2\,V_{xy}\,\mathbf{e}_{xy}\,. \tag{27}$$

The respective integral equations in the spherical coordinates can then be derived as follows (Martinec, 2003):

$$V = \frac{R^2}{4\pi}\int_{\Omega'} V_{zz}\ \mathcal{K}^{vv}\ \mathrm{d}\Omega', \tag{28}$$

$$V = -\frac{R^2}{4\pi}\int_{\Omega'} (V_{xz}\ \cos\alpha' - V_{yz}\ \sin\alpha')\ \mathcal{K}^{hv}\ \mathrm{d}\Omega', \tag{29}$$

$$V = \frac{R^2}{4\pi}\int_{\Omega'} [(V_{xx} - V_{yy})\ \cos 2\alpha' - 2\ V_{xy}\ \sin 2\alpha']\ \mathcal{K}^{hh}\ \mathrm{d}\Omega'. \tag{30}$$

Obviously, this transform consists of three integral equations defined for each of the three component groups Γ^{vv}, Γ^{vh} and Γ^{hh}. The particular BVPs are also known as the *vertical-vertical, horizontal-vertical and horizontal-horizontal gradiometric BVPs*. The respective kernel functions $\mathcal{K}^{vv}$, $\mathcal{K}^{hv}$ and $\mathcal{K}^{hh}$ can be found in Appendix B, see Eqs. (126)-(128). Goniometric functions of the double backward azimuth read (Abramowitz and Stegun, 1972, p. 72)

$$\cos 2\alpha' = 2\cos^2\alpha' - 1\,, \tag{31}$$

$$\sin 2\alpha' = 2\ \cos\alpha'\ \sin\alpha'\ . \tag{32}$$

The gradiometric BVP finds its applications in the gravitational field modelling (transforming measured values of the second-order gravitational gradients into the gravitational potential) or in validation of second-order gradients measured by the GOCE mission using GNSS/levelling and sea surface altimetry data. In case two or more of the three component groups are given, the transforms represent an over-determined problem. In case of the horizontal-vertical and horizontal-horizontal gradiometric BVPs, the gravitational potential can be recovered only partially with missing zero- and both zero- and first-order terms, respectively. The concept of the reference field and the respective disturbing gravitational potential must be applied. In this case, the reference gravitational field defined by the geocentric biaxial ellipsoid must be used in order to supply the missing terms in the harmonic expansion of the gravitational potential. This concept is applied in geodesy.

Finally, the transform in the third line of Table 3 transforms the third-order gravitational tensor $\mathcal{G}$ into the gravitational potential V. This problem is called the *gravitational curvature BVP* with the boundary conditions of the form (Šprlák and Novák, 2016a):

$$\text{grad grad grad}\ V(\mathbf{x}) = \mathcal{G}^{\tau}(\mathbf{x}),\ \text{ for } |\mathbf{x}| = R,\ \ \tau = vvv, hvv, hhv, hhh. \tag{33}$$

In particular, the boundary conditions are:

$$\mathcal{G}^{vvv} = V_{zzz}\ \mathbf{e}_{zzz}, \tag{34}$$

$$\mathcal{G}^{hvv} = V_{xzz}\ (\mathbf{e}_{xzz} + \mathbf{e}_{zxz} + \mathbf{e}_{zzx}) + V_{yzz}\ (\mathbf{e}_{yzz} + \mathbf{e}_{zyz} + \mathbf{e}_{zzy}), \tag{35}$$

$$\begin{aligned}\mathcal{G}^{hhv} = {} & [V_{xxz} - V_{yyz}]\ \ (\mathbf{e}_{xxz} + \mathbf{e}_{xzx} + \mathbf{e}_{zxx} - \mathbf{e}_{yyz} - \mathbf{e}_{yzy} - \mathbf{e}_{zyy}) \\ & + 2V_{xyz}\ \ (\mathbf{e}_{xyz} + \mathbf{e}_{xzy} + \mathbf{e}_{yxz} + \mathbf{e}_{yzx} + \mathbf{e}_{zxy} + \mathbf{e}_{zyx}),\end{aligned} \tag{36}$$

$$\begin{aligned}\mathcal{G}^{hhh} = {} & [V_{xxx} - 3V_{xyy}]\ \ (\mathbf{e}_{xxx} - \mathbf{e}_{xyy} - \mathbf{e}_{yxy} - \mathbf{e}_{xyy}) \\ & + [V_{yyy} - 3V_{xxy}]\ \ (\mathbf{e}_{yyy} - \mathbf{e}_{xxy} - \mathbf{e}_{xyx} - \mathbf{e}_{yxx}).\end{aligned} \tag{37}$$

The four surface integral equations can be formulated as follows (Šprlák and Novák, 2016a):

$$V = -\frac{R^3}{4\pi}\int_{\Omega'} V_{zzz}\ \mathcal{K}^{vvv}\ \mathrm{d}\Omega', \tag{38}$$

$$V = \frac{R^3}{4\pi}\int_{\Omega'} (V_{xzz}\ \cos\alpha' - V_{yzz}\ \sin\alpha')\ \mathcal{K}^{hvv}\ \mathrm{d}\Omega', \tag{39}$$

$$V = -\frac{R^3}{4\pi}\int_{\Omega'} [(V_{xxz} - V_{yyz})\ \cos 2\alpha' - 2\ V_{xyz}\sin 2\alpha']\ \mathcal{K}^{hhv}\ \mathrm{d}\Omega', \tag{40}$$

$$V = \frac{R^3}{4\pi}\int_{\Omega'} [(V_{xxx} - 3V_{xyy})\ \cos 3\alpha' + (V_{yyy} - 3V_{xxy})\ \sin 3\alpha']\ \mathcal{K}^{hhh}\ \mathrm{d}\Omega'. \tag{41}$$

This transform consists of the four surface integral equations called the *vertical-vertical-vertical, horizontal-vertical-vertical, horizontal-horizontal-vertical and horizontal-horizontal-horizontal curvature BVP*. It also represents an over-determined problem. In case of the horizontal-horizontal-horizontal curvature BVP the solution would additionally miss the third-order term in the harmonic expansion of the gravitational potential and a respective reference gravitational field would have to be used. The respective kernel functions $\mathcal{K}^{vvv}$, $\mathcal{K}^{hvv}$, $\mathcal{K}^{hhv}$ and $\mathcal{K}^{hhh}$ can be found in Appendix B, see Eqs. (129)–(132). Goniometric functions of the triple backward azimuth read (Abramowitz and Stegun, 1972, p. 72)

$$\cos 3\alpha' = \cos\alpha'\ \left(\cos^2\alpha' - 3\ \sin^2\alpha'\right)\ , \tag{42}$$

$$\sin 3\alpha' = -\sin\alpha'\ \left(\sin^2\alpha' - 3\ \cos^2\alpha'\right)\ . \tag{43}$$

Looking at the integral equations related to the three fundamental BVPs, see Eqs. (19), (24) and (33), it is obvious that increasing the order of the gravitational tensor results in the respective increase of the number and complexity of the integral equations. This applies mainly to horizontal gradients as the order of the radial gradient has much smaller impact on the complexity of the solution; the radial and mixed gradients can be derived rather easily. The radial derivative has also no impact on differentiating goniometric functions of the backward azimuth; thus, only the isotropic parts of the integral kernels must be differentiated.

The second transform in the first line of Table 3 transforming Γ into $\mathbf{g}$ can be obtained from Eqs. (28)–(30) by applying the differential operators of Eqs. (75)–(77) to their both sides, i.e., for $\{i = x, y, z\}$:

$$V_i = \frac{R}{4\pi}\int_{\Omega'} V_{zz}\ \mathcal{K}^{vv}_i\ \mathrm{d}\Omega', \tag{44}$$

$$V_i = -\frac{R}{4\pi}\int_{\Omega'} \left[V_{xz}\ \left(\cos\alpha'\,\mathcal{K}^{hv}\right)_i - V_{yz}\ \left(\sin\alpha'\,\mathcal{K}^{hv}\right)_i\right]\ \mathrm{d}\Omega', \tag{45}$$

$$V_i = \frac{R}{4\pi}\int_{\Omega'} \left[(V_{xx} - V_{yy})\ \left(\cos 2\alpha'\,\mathcal{K}^{hh}\right)_i - 2\ V_{xy}\ \left(\sin 2\alpha'\,\mathcal{K}^{hh}\right)_i\right]\ \mathrm{d}\Omega'. \tag{46}$$

These transforms consist of 9 integral equations which can be referred to as the *gravimetric-gradiometric BVP* relating first- and second-order gravitational gradients which can be used namely for data validation and combination. Sub-integral kernel functions must be re-ordered as differentiation concerns the entire kernel functions, i.e., both their isotropic and anisotropic parts, and one has to find derivatives of a composite function. The three kernel functions in

Table 4
Fundamental BVPs – spectral representations of the kernel functions.

BVP	Group τ	$(2n+1)/$	Legendre function
Gravimetric	v	$n+1$	$P_{n,0}$
	h	$n(n+1)$	$P_{n,1}$
Gradiometric	vv	$(n+1)(n+2)$	$P_{n,0}$
	hv	$n(n+1)(n+2)$	$P_{n,1}$
	hh	$(n-1)n(n+1)(n+2)$	$P_{n,2}$
Curvature	vvv	$(n+1)(n+2)(n+3)$	$P_{n,0}$
	hvv	$n(n+1)(n+2)(n+3)$	$P_{n,1}$
	hhv	$(n-1)n(n+1)(n+2)(n+3)$	$P_{n,2}$
	hhh	$(n-2)(n-1)n(n+1)(n+2)(n+3)$	$P_{n,3}$

Eq. (44) can be found in Appendix C, see Eqs. (134)–(136) for $\mu = vv$. The three kernel functions in Eq. (45) with cosine of the backward azimuth are defined by Eqs. (162)–(164) in Appendix D for $\mu = v$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). Finally, the three kernel functions in Eq. (46) with cosine of the double backward azimuth are defined by Eqs. (196)–(198) in Appendix E for $\mu = 0$ (kernel functions with sine of the double backward azimuth can again be derived analogously by substituting cosine by sine). References to some applications of integral equations transforming the second-order gradients Γ onto the first-order gradients $\mathbf{g}$ can be found in Table 5.

The transform of $\mathcal{G}$ onto $\mathbf{g}$ in the fourth line of Table 3 can be derived from Eqs. (38)–(41) by applying the differential operators of Eqs. (75)–(77):

$$V_i = -\frac{R^2}{4\pi}\int_{\Omega'} V_{zzz}\ \mathcal{K}_i^{vvv}\ \mathrm{d}\Omega', \tag{47}$$

$$V_i = \frac{R^2}{4\pi}\int_{\Omega'} \left[V_{xzz}\ \left(\cos\alpha'\ \mathcal{K}^{hvv}\right)_i - V_{yzz}\ \left(\sin\alpha'\ \mathcal{K}^{hvv}\right)_i\right]\ \mathrm{d}\Omega', \tag{48}$$

$$V_i = -\frac{R^2}{4\pi}\int_{\Omega'} \left[(V_{xxz} - V_{yyz})\ \left(\cos 2\alpha'\ \mathcal{K}^{hhv}\right)_i - 2\ V_{xyz}\ \left(\sin 2\alpha'\ \mathcal{K}^{hhv}\right)_i\right]\ \mathrm{d}\Omega', \tag{49}$$

$$V_i = \frac{R^2}{4\pi}\int_{\Omega'} \left[(V_{xxx} - 3V_{xyy})\ \left(\cos 3\alpha'\ \mathcal{K}^{hhh}\right)_i + (V_{yyy} - 3V_{xxy})\ \left(\sin 3\alpha'\ \mathcal{K}^{hhh}\right)_i\ \right]\ \mathrm{d}\Omega'. \tag{50}$$

These transforms consist of 12 integral equations which could be referred to as the *gravimetric-curvature BVP* relating first- and third-order gradient data. Their applications could cover validation of curvature measurements through gravimetric data or data combination. The three kernel functions in Eq. (47) can be found in Appendix C, see Eqs. (134)–(136) for $\mu = vvv$. The three kernel functions in Eq. (48) with cosine of the backward azimuth are defined by Eqs. (162)–(164) in Appendix D for $\mu = vv$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). The three kernel functions of Eq. (49) with cosine of the double backward azimuth are defined by Eqs. (196)–(198) in Appendix E for $\mu = v$ (kernel functions with sine of the double backward azimuth can again be derived analogously by substituting cosine by sine). The three kernel functions in Eq. (50) with cosine of the triple backward azimuth are defined by Eqs. (230)–(232) in Appendix F for $\mu = 0$ (kernel functions with sine of the triple backward azimuth can again be derived analogously by substituting cosine by sine).

Finally, the third transform in the first line of Table 3 transforming $\mathcal{G}$ into the gravitational gradient tensor Γ can be derived

Table 5
Forward, backward and continuation transformations of the gravitational field parameters.

BVP	From	To	Equations	References in geodetic literature
Gravimetric	$\mathbf{g}$	V	(22)–(23)	Stokes (1849), Pizzetti (1911), Kellogg (1929), Molodenskii et al. (1962), Heiskanen and Moritz (1967), Hotine (1969), Koch and Pope (1972), Pick et al. (1973), Bjerhammar and Svensson (1983), Grafarend et al. (1985), Vaníček and Krakiwsky (1987), Heck (1989), Moritz (1989), Rummel et al. (1989), Sansó (1995), Heck (1997), Schwarz and Li (1997), Hwang (1998), Grafarend (2001), van Gelderen and Rummel (2001), Novák and Heck (2002), Jekeli (2007), Freeden and Gerhards (2013)
Gradiometric	Γ	V	(28)–(30)	Heck (1979), Rummel (1975), Grafarend et al. (1985), Rummel et al. (1989, 1993), Brovelli and Sansó (1990), van Gelderen and Rummel (2001), Freeden et al. (2002), Tóth et al. (2002), Martinec (2003), Tóth (2003), Bölling and Grafarend (2005), Li (2005), Eshagh (2011a), Freeden and Gerhards (2013)
Curvature	$\mathcal{G}$	V	(38)–(41)	Moritz (1967), Šprlák and Novák (2016a)
	Γ	$\mathbf{g}$	(44)–(46)	Heck (1979), Petrovskaya and Zielinski (1997), van Gelderen and Rummel (2001), Tóth (2003), Li (2002, 2005), Šprlák and Novák (2014b)
	$\mathcal{G}$	$\mathbf{g}$	(47)–(50)	–
	$\mathcal{G}$	Γ	(51)–(54)	–
	V	$\mathbf{g}$	(55)	Lelgemann (1976), Rummel et al. (1978), Zhang (1993), Garcia (2002), Novák (2003, 2007), Ardalan and Grafarend (2004), Luying and Houze (2006), Novák et al. (2006), Jekeli (2007)
	V	Γ	(56)	Bölling and Grafarend (2005), Eshagh (2011b), Šprlák et al. (2015)
	V	$\mathcal{G}$	(57)	Šprlák and Novák (2015)
	$\mathbf{g}$	Γ	(58)–(59)	Reed (1973), Heck (1979), Thalhammer (1995), Denker (2003), Kern and Haagmans (2005), Wolf (2007), Wolf and Denker (2005), Janák et al. (2009), Šprlák and Novák (2014a),
	$\mathbf{g}$	$\mathcal{G}$	(60)–(61)	Šprlák and Novák (2015)
	Γ	$\mathcal{G}$	(62)–(64)	Šprlák and Novák (2016b)
	$\mathbf{g}$	$\mathbf{g}$	(65)–(66)	Vening-Meinesz (1928), Heiskanen and Moritz (1967), Hwang (1998), Sünkel (1981), Jekeli (2000), Grafarend (2001), Novák and Heck (2002), Luying and Houze (2006), Jekeli (2007), Šprlák and Novák (2014a)
	Γ	Γ	(68)–(70)	Bölling and Grafarend (2005), Tóth et al. (2005, 2006), Šprlák et al. (2014)
	$\mathcal{G}$	$\mathcal{G}$	(71)–(74)	–

from Eqs. (38)–(41) by applying the differential operators of Eqs. (81)–(86), i.e., for $\{ij = xx, xy, xz, yy, yz, zz\}$:

$$V_{ij} = -\frac{R}{4\pi}\int_{\Omega'} V_{zzz}\ \mathcal{K}_{ij}^{vvv}\ \mathrm{d}\Omega', \tag{51}$$

$$V_{ij} = \frac{R}{4\pi}\int_{\Omega'} \left[V_{xzz}\ \left(\cos\alpha'\ \mathcal{K}^{hvv}\right)_{ij} - V_{yzz}\ \left(\sin\alpha'\ \mathcal{K}^{hvv}\right)_{ij}\right]\ \mathrm{d}\Omega', \tag{52}$$

$$V_{ij} = -\frac{R}{4\pi}\int_{\Omega'} \left[(V_{xxz} - V_{yyz})\ \left(\cos 2\alpha'\ \mathcal{K}^{hhv}\right)_{ij} -2\ V_{xyz}\left(\sin 2\alpha'\ \mathcal{K}^{hhv}\right)_{ij}\right]\ \mathrm{d}\Omega', \tag{53}$$

$$V_{ij} = \frac{R}{4\pi}\int_{\Omega'} \left[(V_{xxx} - 3V_{xyy})\ \left(\cos 3\alpha'\ \mathcal{K}^{hhh}\right)_{ij} + (V_{yyy} - 3V_{xxy})\ \left(\sin 3\alpha'\ \mathcal{K}^{hhh}\right)_{ij}\right]\ \mathrm{d}\Omega'. \tag{54}$$

In this case, we get 24 integral equations which could be referred to as the *gradient-curvature BVP* relating the second- and third-order gradient data. The six kernel functions in Eq. (51) can be found in Appendix C, see Eqs. (137)–(142) for $\mu = vvv$. The six kernel functions in Eq. (52) with cosine of the backward azimuth are defined by Eqs. (165)–(170) in Appendix D for $\mu = vv$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). The six kernel functions in Eq. (53) with cosine of the double backward azimuth are defined by Eqs. (199)–(204) in Appendix E for $\mu = v$ (kernel functions with sine of the double backward azimuth can again be derived analogously by substituting cosine by sine). Finally, the six kernel functions in Eq. (54) with cosine of the triple backward azimuth are defined by Eqs. (233)–(238) in Appendix F for $\mu = 0$ (kernel functions with sine of the triple backward azimuth can again be derived analogously by substituting cosine by sine).

5.2. Backward transforms of the gravitational field parameters

The backward transforms of the gravitational field parameters can easily be deduced from Table 3 by changing the arrow sign from ← to →. In this case, the higher-order gradient tensor is computed from its lower-order counterparts. This operation seems to counteract the effect of integration as an inverse step of differentiation, i.e., the integral equation acts as a differentiator and the transforms have the character of integro-differential equations.

We start again with the three fundamental transforms that can be used for derivation of remaining combinations in Table 3; they transform the gravitational potential V onto the first-, second- and third-order gravitational tensors $\mathbf{g}$, Γ and $\mathcal{G}$. We start with the transform of V into $\mathbf{g}$ that can be derived by differentiating both sides of the original Poisson integral Eq. (17), i.e., we obtain three integral equations of the form:

$$V_i = \frac{1}{4\pi R}\int_{\Omega'} V\ \mathcal{K}_i\ \mathrm{d}\Omega'. \tag{55}$$

Similarly, by using the second-order differential operators, see Eqs. (81)–(86), the transform of V into Γ can be obtained:

$$V_{ij} = \frac{1}{4\pi R^2}\int_{\Omega'} V\ \mathcal{K}_{ij}\ \mathrm{d}\Omega', \tag{56}$$

and raising the order of differentiation one more time, see Eqs. (97)–(106), yields the transform of V into $\mathcal{G}$, i.e., for $\{ijk = xxx, xxy, xxz, xyy, xyz, xzz, yyy, yyz, yzz, zzz\}$:

$$V_{ijk} = \frac{1}{4\pi R^3}\int_{\Omega'} V\ \mathcal{K}_{ijk}\ \mathrm{d}\Omega'. \tag{57}$$

The kernel functions $\mathcal{K}_i$, $\mathcal{K}_{ij}$ and $\mathcal{K}_{ijk}$ can be found in Appendix C, see Eqs. (134)-(152) for $\mu = 0$.

Remaining inverse transforms in Table 3 can easily be formulated. We start with the second inverse transform in the first line of Table 3 which can be derived from Eqs. (22) and (23) by applying the differential operators of Eqs. (81)–(86):

$$V_{ij} = -\frac{1}{4\pi R}\int_{\Omega'} V_z\ \mathcal{K}_{ij}^{v}\ \mathrm{d}\Omega', \tag{58}$$

$$V_{ij} = \frac{1}{4\pi R}\int_{\Omega'} \left[V_x\ \left(\cos\alpha'\ \mathcal{K}^{h}\right)_{ij} - V_y\ \left(\sin\alpha'\ \mathcal{K}^{h}\right)_{ij}\right]\ \mathrm{d}\Omega', \tag{59}$$

where the integral kernels $\mathcal{K}^v$ and $\mathcal{K}^h$ transform $\mathbf{g}$ into the gravitational potential V. The six kernel functions in Eq. (58) can be found in Appendix C, see Eqs. (137)–(142) for $\mu = v$. The six kernel functions in Eq. (59) with cosine of the backward azimuth are defined by Eqs. (165)–(170) in Appendix D for $\mu = 0$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine).

Similarly, the inverse transform in the fourth line of Table 3 can also be formulated from Eqs. (22) and (23) by applying the differential operators of Eqs. (97)–(106):

$$V_{ijk} = -\frac{1}{4\pi R^2}\int_{\Omega'} V_z\ \mathcal{K}_{ijk}^{v}\ \mathrm{d}\Omega', \tag{60}$$

$$V_{ijk} = \frac{1}{4\pi R^2}\int_{\Omega'} \left[V_x\ \left(\cos\alpha'\ \mathcal{K}^{h}\right)_{ijk} - V_y\ \left(\sin\alpha'\ \mathcal{K}^{h}\right)_{ijk}\right]\ \mathrm{d}\Omega'. \tag{61}$$

The ten kernel functions in Eq. (60) can be found in Appendix C, see Eqs. (143)–(152) for $\mu = v$. The ten kernel functions in Eq. (61) with cosine of the backward azimuth are defined by Eqs. (171)–(180) in Appendix D for $\mu = 0$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine).

Finally, the third inverse transform in the first line of Table 3 can be defined by using the solution of the gradiometric BVP, i.e.:

$$V_{ijk} = \frac{1}{4\pi R}\int_{\Omega'} V_{zz}\ \mathcal{K}_{ijk}^{vv}\ \mathrm{d}\Omega', \tag{62}$$

$$V_{ijk} = -\frac{1}{4\pi R}\int_{\Omega'} \left[V_{xz}\ \left(\cos\alpha'\ \mathcal{K}^{hv}\right)_{ijk} - V_{yz}\ \left(\sin\alpha'\ \mathcal{K}^{hv}\right)_{ijk}\right]\ \mathrm{d}\Omega', \tag{63}$$

$$V_{ijk} = \frac{1}{4\pi R}\int_{\Omega'} \left[(V_{xx} - V_{yy})\ \left(\cos 2\alpha'\ \mathcal{K}^{hh}\right)_{ijk} -2\ V_{xy}\ \left(\sin 2\alpha'\ \mathcal{K}^{hh}\right)_{ijk}\right]\ \mathrm{d}\Omega'. \tag{64}$$

The ten kernel functions in Eq. (62) can be found in Appendix C, see Eqs. (143)–(152) for $\mu = vv$. The ten kernel functions in Eq. (63) with cosine of the backward azimuth are defined by Eqs. (171)–(180) in Appendix D for $\mu = v$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). The ten kernel functions in Eq. (64) with cosine of the double backward azimuth are defined by Eqs. (205)–(214) in Appendix E for

$\mu = 0$ (kernel functions with sine of the double backward azimuth can again be derived analogously by substituting cosine by sine).

5.3. Continuation of the gravitational field parameters

Transforms between two gravitational field parameters of the same order at two different levels (radii in case of spherical coordinates) represent the *continuation problem*: from the lower level (e.g., the geocentric sphere of radius R) upward the link is usually referred to as the upward continuation and as the downward continuation in case of its inverse. In general, the solution to the upward continuation problem is represented by the Dirichlet BVP that allows for continuing a harmonic function through the mass-free space given its values on a specific Lipschitz-type boundary surface are known (the sphere is such a boundary).

For the first-order gravitational tensor $\mathbf{g}$ we have in the Cartesian coordinate system 6 integral equations:

$$V_i = -\frac{1}{4\pi}\int_{\Omega'} V_z\ \mathcal{K}_i^{v}\ \mathrm{d}\Omega', \tag{65}$$

$$V_i = \frac{1}{4\pi}\int_{\Omega'} \left[V_x\ \left(\cos\alpha'\ \mathcal{K}^{h}\right)_i - V_y\ \left(\sin\alpha'\ \mathcal{K}^{h}\right)_i\right]\ \mathrm{d}\Omega'. \tag{66}$$

These equations can be derived by differentiating Eqs. (22) and (23). The three kernel functions in Eq. (65) can be found in Appendix C, see Eqs. (134)–(136) for $\mu = v$. The three kernel functions in Eq. (66) with cosine of the backward azimuth are defined by Eqs. (162)–(164) in Appendix D for $\mu = 0$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). In particular, the solution for continuation of the vertical gradient V_z is often used in geodesy (continuation of ground gravity and/or aerial gravitation), i.e.:

$$V_z = -\frac{R}{4\pi r}\int_{\Omega'} V_z\ \mathcal{K}\ \mathrm{d}\Omega', \tag{67}$$

see, e.g., Vaníček et al. (1996).

The solution for the continuation of the second-order gravitational tensor Γ represented by 18 integral equations was derived by Šprlák et al. (2014); for the six independent components ij and the three component groups τ it reads as follows:

$$V_{ij} = \frac{1}{4\pi}\int_{\Omega'} V_{zz}\ \mathcal{K}_{ij}^{vv}\,\mathrm{d}\Omega', \tag{68}$$

$$V_{ij} = -\frac{1}{4\pi}\int_{\Omega'} \left[V_{xz}\ \left(\cos\alpha'\ \mathcal{K}^{hv}\right)_{ij} - V_{yz}\ \left(\sin\alpha'\ \mathcal{K}^{hv}\right)_{ij}\right]\ \mathrm{d}\Omega', \tag{69}$$

$$V_{ij} = \frac{1}{4\pi}\int_{\Omega'} \left[(V_{xx}-V_{yy})\ \left(\cos 2\alpha'\ \mathcal{K}^{hh}\right)_{ij} - 2\ V_{xy}\ \left(\sin 2\alpha'\ \mathcal{K}^{hh}\right)_{ij}\right]\ \mathrm{d}\Omega'. \tag{70}$$

These equations are obtained by differentiating both sides of Eqs. (28)–(30) by the operators of Eqs. (81)–(86). The six kernel functions in Eq. (68) can be found in Appendix C, see Eqs. (137)–(142) for $\mu = vv$. The six kernel functions in Eq. (69) with cosine of the backward azimuth are defined by Eqs. (165)–(170) in Appendix D for $\mu = v$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). The six kernel functions in Eq. (70) with cosine of the double backward azimuth are defined by Eqs. (199)–(204) in Appendix E for $\mu = 0$ (kernel functions with sine of the double backward azimuth can be derived analogously by substituting cosine by sine).

The solution for continuation of the third-order gravitational tensor contains in total forty integral equations necessary for covering all ten independent components V_{ijk} and the four component groups:

$$V_{ijk} = -\frac{1}{4\pi}\int_{\Omega'} V_{zzz}\ \mathcal{K}_{ijk}^{vvv}\ \mathrm{d}\Omega', \tag{71}$$

$$V_{ijk} = \frac{1}{4\pi}\int_{\Omega'} \left[V_{xzz}\ \left(\cos\alpha'\ \mathcal{K}^{hvv}\right)_{ijk} - V_{yzz}\ \left(\sin\alpha'\ \mathcal{K}^{hvv}\right)_{ijk}\right]\ \mathrm{d}\Omega', \tag{72}$$

$$\begin{aligned} V_{ijk} = -\frac{1}{4\pi}\int_{\Omega'} \Big[&(V_{xxz}-V_{yyz})\ \left(\cos 2\alpha'\ \mathcal{K}^{hhv}\right)_{ijk} \\ &-2\ V_{xyz}\left(\sin 2\alpha'\ \mathcal{K}^{hhv}\right)_{ijk}\Big]\ \mathrm{d}\Omega', \end{aligned} \tag{73}$$

$$\begin{aligned} V_{ijk} = \frac{1}{4\pi}\int_{\Omega'} \Big[&(V_{xxx}-3V_{xyy})\ \left(\cos 3\alpha'\ \mathcal{K}^{hhh}\right)_{ijk} \\ &+(V_{yyy}-3V_{xxy})\ \left(\sin 3\alpha'\ \mathcal{K}^{hhh}\right)_{ijk}\Big]\ \mathrm{d}\Omega'. \end{aligned} \tag{74}$$

These equations are obtained by differentiating both sides of Eqs. (38)–(41) by the operators in Eqs. (97)–(106). The ten kernel functions in Eq. (71) can be found in Appendix C, see Eqs. (143)–(152) for $\mu = vvv$. The ten kernel functions in Eq. (72) with cosine of the backward azimuth are defined by Eqs. (171)–(180) in Appendix D for $\mu = vv$ (kernel functions with sine of the backward azimuth can be derived analogously by substituting cosine by sine). The ten kernel functions in Eq. (73) with cosine of the double backward azimuth are defined by Eqs. (205)–(214) in Appendix E for $\mu = v$ (kernel functions with sine of the double backward azimuth can be derived analogously by substituting cosine by sine). The ten kernel functions in Eq. (74) with cosine of the triple backward azimuth are defined by Eqs. (239)–(248) in Appendix E for $\mu = 0$ (kernel functions with sine of the triple backward azimuth can be derived analogously by substituting cosine by sine).

The above integrals solve the upward continuation problem. Their inverse after discretizing the boundary surface can be used for solving the respective downward continuation problems. In general, they represent Fredholm's integral equations of the first kind which are more or less ill conditioned. Their solutions are often numerically unstable and must be regularized.

6. Conclusions

This contribution summarizes the apparatus of integral equations transforming selected gravitational field parameters, see Table 5. The gravitational field parameters considered in the article include the gravitational potential and its gradients of the first, second and third orders. The integral transforms allow for their transformation and for their continuation in mass-free 3-D space. In all formulations the spherical approximation was used in terms of the boundary represented by the mean geocentric sphere.

The integral transforms are divided in the forward transforms, when the particular gravitational field parameter is integrated from known values of its gradients, and in the inverse transforms, when the parameter is differentiated through the application of the integro-differential equation. Integral equations, that allow for continuation of the potential field parameters, are also discussed in the article.

In practical applications of the presented formulas one has to use some caution. Integral transforms may increase an observation noise in input data (backward transforms) or smooth out the solution (forward transforms). The presented formulas were tested numerically using synthetic data; however, their practical applicability for processing of real data remains out of the scope of the article. Numerical studies can be found in selected references of the article.

All presented formulas were formulated using the spherical approximation. For oblate planets (such as the Earth) and some accurate applications a reference surface in the form of a biaxial ellipsoid could alternatively be used; however, the ellipsoidal approximation leads to more complicated mathematical expressions which will be formulated in our future studies.

The integral transforms contribute to the well-known Meissl scheme of physical geodesy (Meissl, 1971) by extending the links between the gravitational field parameters of up to third-order gradients of the gravitational potential. The integral transforms are discussed at the background of problems related to studying the Earth's gravitational field. However, they can be used in any scientific discipline or engineering problem where potential fields are used.

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Appendix A. Differential operators

The first-order gravitational tensor is obtained from the gravitational potential through application of the following differential operators:

$$\mathcal{D}^x = \frac{1}{r}\frac{\partial}{\partial\varphi}, \tag{75}$$

$$\mathcal{D}^y = -\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}, \tag{76}$$

$$\mathcal{D}^z = \frac{\partial}{\partial r}. \tag{77}$$

In terms of the substitution parameters t, u and the direct azimuth α the differential operators read:

$$\mathcal{D}^x = \frac{\cos\alpha}{R} t\sqrt{1-u^2} \quad \frac{\partial}{\partial u}, \tag{78}$$

$$\mathcal{D}^y = -\frac{\sin\alpha}{R} t\sqrt{1-u^2} \quad \frac{\partial}{\partial u}, \tag{79}$$

$$\mathcal{D}^z = -\frac{t^2}{R} \quad \frac{\partial}{\partial t}. \tag{80}$$

The second-order gravitational tensor is obtained from the gravitational potential through application of the following differential operators:

$$\mathcal{D}^{xx} = \frac{1}{r^2} \left(r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\varphi^2} \right), \tag{81}$$

$$\mathcal{D}^{xy} = -\frac{1}{r^2\cos\varphi} \left(\tan\varphi\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial\varphi\partial\lambda} \right), \tag{82}$$

$$\mathcal{D}^{xz} = -\frac{1}{r^2} \left(\frac{\partial}{\partial\varphi} - r\frac{\partial^2}{\partial r\partial\varphi} \right), \tag{83}$$

$$\mathcal{D}^{yy} = \frac{1}{r^2} \left(r\frac{\partial}{\partial r} - \tan\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} \right), \tag{84}$$

$$\mathcal{D}^{yz} = \frac{1}{r^2\cos\varphi} \left(\frac{\partial}{\partial\lambda} - r\frac{\partial^2}{\partial r\partial\lambda} \right), \tag{85}$$

$$\mathcal{D}^{zz} = \frac{\partial^2}{\partial r^2}. \tag{86}$$

In terms of the substitution parameters t and u, and the direct azimuth α the second-order differential operators read (Šprlák and Novák, 2014a,b):

$$\mathcal{D}^{xx} = \frac{1}{R^2} \left(\mathcal{D}_2^1 + \cos 2\alpha\, \mathcal{D}_2^2 \right), \tag{87}$$

$$\mathcal{D}^{xy} = -\frac{\sin 2\alpha}{R^2}\mathcal{D}_2^2, \tag{88}$$

$$\mathcal{D}^{xz} = \frac{\cos\alpha}{R^2}\mathcal{D}_2^3, \tag{89}$$

$$\mathcal{D}^{yy} = \frac{1}{R^2}\left(\mathcal{D}_2^1 - \cos 2\alpha\ \mathcal{D}_2^2\right), \tag{90}$$

$$\mathcal{D}^{yz} = -\frac{\sin\alpha}{R^2}\ \mathcal{D}_2^3, \tag{91}$$

$$\mathcal{D}^{zz} = \frac{1}{R^2}\ \mathcal{D}_2^4, \tag{92}$$

with the four isotropic differential operators defined as follows:

$$\mathcal{D}_2^1 = -t^3\frac{\partial}{\partial t} + \frac{t^2}{2}\left[\left(1-u^2\right)\frac{\partial^2}{\partial u^2} - 2u\frac{\partial}{\partial u}\right], \tag{93}$$

$$\mathcal{D}_2^2 = \frac{t^2\left(1-u^2\right)}{2}\frac{\partial^2}{\partial u^2}, \tag{94}$$

$$\mathcal{D}_2^3 = -t^2\ \sqrt{1-u^2}\ \left(\frac{\partial}{\partial u} + t\frac{\partial^2}{\partial t\partial u}\right), \tag{95}$$

$$\mathcal{D}_2^4 = t^3\ \left(2\frac{\partial}{\partial t} + t\frac{\partial^2}{\partial t^2}\right). \tag{96}$$

We note that both forms of the differential operators $\mathcal{D}^{xx}$, $\mathcal{D}^{xy}$, $\mathcal{D}^{xz}$, $\mathcal{D}^{yy}$, $\mathcal{D}^{yz}$ and $\mathcal{D}^{zz}$, i.e., in terms of $(r,\ \varphi,\ \lambda)$ or $(t,\ u,\ \alpha)$ are equivalent. Their application, however, depends on arguments of a function to which they are applied.

The third-order gravitational tensor is obtained by the successive application of the gradient operator to the gravitational potential. Therefore, the ten gravitational gradients are defined by differential operators which read in terms of the spherical geocentric coordinates as follows (Casotto and Fantino, 2009; Tóth, 2005):

$$\mathcal{D}^{xxx} = -\frac{1}{r^3}\left(2\ \frac{\partial}{\partial\varphi} - 3r\ \frac{\partial^2}{\partial r\partial\varphi} - \frac{\partial^3}{\partial\varphi^3}\right), \tag{97}$$

$$\mathcal{D}^{xxy} = -\frac{1}{r^3\cos\varphi}\left(2\tan^2\varphi\ \frac{\partial}{\partial\lambda} + r\ \frac{\partial^2}{\partial r\partial\lambda} + 2\tan\varphi\ \frac{\partial^2}{\partial\varphi\partial\lambda} + \frac{\partial^3}{\partial\varphi^2\partial\lambda}\right), \tag{98}$$

$$\mathcal{D}^{xxz} = -\frac{1}{r^3}\ \left(r\ \frac{\partial}{\partial r} - r^2\ \frac{\partial^2}{\partial r^2} + 2\ \frac{\partial^2}{\partial\varphi^2} - r\ \frac{\partial^3}{\partial r\partial\varphi^2}\right), \tag{99}$$

$$\mathcal{D}^{xyy} = -\frac{1}{r^3\cos\varphi}\ \left(\frac{1}{\cos\varphi}\ \frac{\partial}{\partial\varphi} - r\ \ \cos\varphi\ \frac{\partial^2}{\partial r\partial\varphi} + \sin\varphi\ \frac{\partial^2}{\partial\varphi^2}\ - \frac{2\sin\varphi}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{\cos\varphi}\frac{\partial^3}{\partial\varphi\partial\lambda^2}\right), \tag{100}$$

$$\mathcal{D}^{xyz} = \frac{1}{r^3\cos\varphi}\ \left(2\tan\varphi\ \frac{\partial}{\partial\lambda} - r\tan\varphi\ \frac{\partial^2}{\partial r\partial\lambda} + 2\ \frac{\partial^2}{\partial\varphi\partial\lambda} - r\ \frac{\partial^3}{\partial r\partial\varphi\partial\lambda}\right), \tag{101}$$

$$\mathcal{D}^{xzz} = \frac{1}{r^3}\ \left(2\ \frac{\partial}{\partial\varphi} - 2r\ \frac{\partial^2}{\partial r\partial\varphi} + r^2\ \frac{\partial^3}{\partial r^2\partial\varphi}\right), \tag{102}$$

$$\mathcal{D}^{yyy} = \frac{1}{r^3\cos\varphi}\left(\frac{2}{\cos^2\varphi}\frac{\partial}{\partial\lambda} - 3r\frac{\partial^2}{\partial r\partial\lambda}\ + 3\tan\varphi\frac{\partial^2}{\partial\varphi\partial\lambda} - \frac{1}{\cos^2\varphi}\frac{\partial^3}{\partial\lambda^3}\right), \tag{103}$$

$$\mathcal{D}^{yyz} = -\frac{1}{r^3}\left(r\frac{\partial}{\partial r} - 2\tan\varphi\frac{\partial}{\partial\varphi} - r^2\frac{\partial^2}{\partial r^2} + r\tan\varphi\frac{\partial^2}{\partial r\partial\varphi}\ + \frac{2}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{r}{\cos^2\varphi}\quad\frac{\partial^3}{\partial r\partial\lambda^2}\right), \tag{104}$$

$$\mathcal{D}^{yzz} = -\frac{1}{r^3\cos\varphi}\ \left(2\frac{\partial}{\partial\lambda} - 2r\ \frac{\partial^2}{\partial r\partial\lambda} + r^2\ \frac{\partial^3}{\partial r^2\partial\lambda}\right), \tag{105}$$

$$\mathcal{D}^{zzz} = \frac{\partial^3}{\partial r^3}. \tag{106}$$

The third-order differential operators defined in terms of the triad $(t,\ u,\ \alpha)$ are (Šprlák and Novák, 2015):

$$\mathcal{D}^{xxx} = \frac{1}{R^3}\left(\cos\alpha\ \mathcal{D}_3^1 + \cos 3\alpha\ \mathcal{D}_3^2\right), \tag{107}$$

$$\mathcal{D}^{xxy} = -\frac{1}{R^3}\left(\frac{1}{3}\sin\alpha\ \mathcal{D}_3^1 + \sin 3\alpha\ \mathcal{D}_3^2\right), \tag{108}$$

$$\mathcal{D}^{xxz} = \frac{1}{R^3}\left(\mathcal{D}_3^3 + \cos 2\alpha\ \mathcal{D}_3^4\right), \tag{109}$$

$$\mathcal{D}^{xyy} = \frac{1}{R^3}\left(\frac{1}{3}\cos\alpha\ \mathcal{D}_3^1 - \cos 3\alpha\ \mathcal{D}_3^2\right), \tag{110}$$

$$\mathcal{D}^{xyz} = -\frac{1}{R^3}\sin 2\alpha\ \mathcal{D}_3^4, \tag{111}$$

$$\mathcal{D}^{xzz} = \frac{1}{R^3}\cos\alpha\ \mathcal{D}_3^5, \tag{112}$$

$$\mathcal{D}^{yyy} = -\frac{1}{R^3}\left(\sin\alpha\ \mathcal{D}_3^1 - \sin 3\alpha\ \mathcal{D}_3^2\right), \tag{113}$$

$$\mathcal{D}^{yyz} = \frac{1}{R^3}\left(\mathcal{D}_3^3 - \cos 2\alpha\ \mathcal{D}_3^4\right), \tag{114}$$

$$\mathcal{D}^{yzz} = -\frac{1}{R^3}\sin\alpha\ \mathcal{D}_3^5, \tag{115}$$

$$\mathcal{D}^{zzz} = \frac{1}{R^3}\mathcal{D}_3^6, \tag{116}$$

with the isotropic differential operators:

$$\mathcal{D}_3^1 = -3t^3\sqrt{1-u^2}\left(\frac{\partial}{\partial u} + t\frac{\partial^2}{\partial t\partial u} + u\frac{\partial^2}{\partial u^2} - \frac{1-u^2}{4}\frac{\partial^3}{\partial u^3}\right), \tag{117}$$

$$\mathcal{D}_3^2 = \frac{t^3\sqrt{(1-u^2)^3}}{4}\frac{\partial^3}{\partial u^3}, \tag{118}$$

$$\mathcal{D}_3^3 = t^3\left[3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + tu\frac{\partial^2}{\partial t\partial u} + t^2\frac{\partial^2}{\partial t^2}\ -(1-u^2)\left(\frac{\partial^2}{\partial u^2} + \frac{t}{2}\frac{\partial^3}{\partial t\partial u^2}\right)\right], \tag{119}$$

$$\mathcal{D}_3^4 = -t^3(1-u^2)\left(\frac{\partial^2}{\partial u^2} + \frac{t}{2}\frac{\partial^3}{\partial t\partial u^2}\right), \tag{120}$$

$$\mathcal{D}_3^5 = t^3\sqrt{1-u^2}\left(2\frac{\partial}{\partial u} + 4t\frac{\partial^2}{\partial t\partial u} + t^2\frac{\partial^3}{\partial t^2\partial u}\right), \tag{121}$$

$$\mathcal{D}_3^6 = -t^4\left(6\frac{\partial}{\partial t} + 6t\frac{\partial^2}{\partial t^2} + t^2\frac{\partial^3}{\partial t^3}\right). \tag{122}$$

Appendix B. Kernel functions of the fundamental BVPs

The spherical Poisson kernel function reads (Kellogg, 1929):

$$\mathcal{K}(t,u) = \sum_{n=0}^{\infty} t^{n+1}\ (2n+1)\ P_{n,0}(u) = \frac{t\,(1-t^2)}{q^3}. \tag{123}$$

The kernel function in the vertical gravimetric BVP, see Eq. (22), is defined as follows (Hotine, 1969):

$$\mathcal{K}^{v}(t,u) = \sum_{n=0}^{\infty} t^{n+1}\ \frac{2n+1}{n+1}\ P_{n,0}(u) = \frac{2t}{q} - \ln\ \left(\frac{q+t-u}{1-u}\right), \tag{124}$$

and in the horizontal gravimetric BVP, see Eq. (23), as follows (Grafarend, 2001):

$$\mathcal{K}^{h}(t,u) = \sum_{n=1}^{\infty} t^{n+1}\ \frac{2n+1}{n(n+1)}\ P_{n,1}(u) = -\frac{1}{\sqrt{1-u^2}}\ \left(\frac{1-t^2}{q} - 1 - tu\right). \tag{125}$$

The kernel function in the vertical-vertical gradiometric BVP has the following form (Martinec, 2003):

$$\mathcal{K}^{vv}(t,u) = \sum_{n=0}^{\infty} t^{n+1}\ \frac{2n+1}{(n+1)(n+2)}\ P_{n,0}(u) = \frac{3}{t}\,(q-1) + \left(\frac{3u}{t} - 1\right)\ln\left(\frac{q+t-u}{1-u}\right), \tag{126}$$

in the horizontal-vertical gradiometric BVP:

$$\begin{aligned}\mathcal{K}^{hv}(t,u) = \sum_{n=1}^{\infty} t^{n+1}\ \frac{2n+1}{n(n+1)(n+2)}\ P_{n,1}(u) = \sqrt{1-u^2}\Bigg\{\frac{3}{2q} + \frac{t^2(q+1)}{2q\,(1+q-tu)} + \left(1 - \frac{3u}{2t}\right)\\ \times\left[\frac{1}{1-u} - \frac{q+t}{q\,(q+t-u)}\right] - \frac{3}{2t}\ln\left(\frac{q+t-u}{1-u}\right)\Bigg\},\end{aligned} \tag{127}$$

and in the horizontal-horizontal gradiometric BVP:

$$\mathcal{K}^{hh}(t,u) = \sum_{n=2}^{\infty} t^{n+1}\ \frac{2n+1}{(n-1)n(n+1)(n+2)}\ P_{n,2}(u) = -\frac{t}{2} + \frac{3}{2}ut^2 + qt + \frac{1}{t}\,(1-q) + \frac{u^2t^3}{1+q-tu} + \frac{u\,(u-t)}{t\,(1-u)} - \frac{u^2}{t\,(q+t-u)}. \tag{128}$$

The kernel function in the vertical-vertical-vertical gravitational curvature BVP has the following form (Šprlák and Novák, 2016a):

$$\mathcal{K}^{vvv}(t,u) = \sum_{n=0}^{\infty} t^{n+1}\ \frac{2n+1}{(n+1)(n+2)(n+3)}\ P_{n,0}(u) = \frac{1}{4t}\ \left\{7q - 12 - \frac{15u(q-1)}{t}\ - \frac{1}{t}\left[2t(t-6u) - 5(1-3u^2)\right]\ \ln\left(\frac{1+u}{q-t+u}\right)\right\}, \tag{129}$$

in the horizontal-vertical-vertical gravitational curvature BVP:

$$\begin{aligned}\mathcal{K}^{hvv}(t,u) = \sum_{n=1}^{\infty} t^{n+1}\ \frac{2n+1}{n(n+1)(n+2)(n+3)}\ P_{n,1}(u) = \frac{\sqrt{1-u^2}}{12t^2}\ \Bigg\{15(q-1) + \frac{t(13t-15u)}{q}\\ + \left[6t(t-3u) - 5(1-3u^2)\right]\ \left[\frac{1}{1+u} + \frac{t-q}{q(q-t+u)}\right] + \frac{2t^4(q+1)}{q(1+q-tu)} - 6(3t-5u)\ \ln\left(\frac{1+u}{q-t+u}\right)\Bigg\},\end{aligned} \tag{130}$$

in the horizontal-horizontal-vertical gravitational curvature BVP:

$$\begin{aligned}\mathcal{K}^{hhv}(t,u) = \sum_{n=2}^{\infty} t^{n+1}\ \frac{2n+1}{(n-1)n(n+1)(n+2)(n+3)}P_{n,2}(u) = \frac{1-u^2}{48t^2}\Bigg\{\frac{30t + t^2(11t-45u)}{q^3} + \frac{1}{1+u}\ \Bigg[12(2t-5u)\\ + \frac{12t(t+2) - 5(1-3u^2)}{1+u}\Bigg] - 30\ln\left(\frac{1+u}{q-t+u}\right) + \frac{12(5u-4t)(q-t)}{q(q-t+u)} - \frac{12t(t-2u) - 5(1-3u^2)}{q^2(q-t+u)}\\ \times\left[\frac{t^2}{q} + \frac{(q-t)^2}{q-t+u}\right] + \frac{6t^5}{1+q-tu}\left[1 - \frac{1}{3q^3} + \frac{(q+1)^2(3q-1)}{3q^2(1+q-tu)}\right]\Bigg\},\end{aligned} \tag{131}$$

and in the horizontal-horizontal-horizontal gravitational curvature BVP:

$$\begin{aligned}\mathcal{K}^{hhh}(t,u) = &\sum_{n=3}^{\infty} t^{n+1} \frac{2n+1}{(n-2)(n-1)n(n+1)(n+2)(n+3)} P_{n,3}(u) = \frac{(1-u^2)^{3/2}}{8}\left[\frac{1}{t^2(1+u)}\left[3+\frac{t-3u}{1+u}\right.\right.\\ &\left.+\frac{2t(3+2t)-(1-3u^2)}{3(1+u)^2}\right] - \frac{1}{2q^3}\left\{3(1-t^4)-\frac{t}{q^2}\left[5t-3u+t^3(1+3tu)\right]\right\}\\ &+\frac{3}{q(q-t+u)}\left\{\frac{t-q}{t^2}+\frac{u-t}{q^2}+\frac{(t-q)^2(u-t)}{t^2q(q-t+u)}+\frac{2t(2t-3u)-(1-3u^2)}{18q^2}\left[\frac{3t}{q^2}+\frac{3(t-q)}{q(q-t+u)}+\frac{2(t-q)^3}{t^2(q-t+u)^2}\right]\right\}\\ &+\frac{3t^4}{q(1+q-tu)}\left\{1+q+\frac{tu}{q^2}-\frac{1}{3q(1+q-tu)}\left[q^2-3tu(1+q)^2+\frac{1}{q}\right]-\frac{2+t^2(1-3u^2)}{18q^2}\right.\\ &\left.\left.\times\left[\frac{3}{q^2}+\frac{3(1+q)}{q(1+q-tu)}+\frac{2(1+q)^3}{(1+q-tu)^2}\right]-\frac{2(1+q)^3}{3q(1+q-tu)^2}\right\}\right]. \end{aligned} \tag{132}$$

Looking at the spectral representations of the kernel functions for the three fundamental BVPs, see Table 4, one easily generalize their form as follows:

$$\mathcal{K}^{\tau}(t,u) = \sum_{n=i}^{\infty} t^{n+1}(2n+1)\binom{n+m}{m+i} P_{n,i}(u), \quad \text{for} \quad i = 0,1,2,3, \tag{133}$$

where τ stands for the particular group, m for the order of the derivative in all coordinates and i for the order of the derivative in angular (horizontal) coordinates.

Appendix C. Kernel functions transforming the *v*, *vv* and *vvv* component groups

This appendix summarizes kernel functions required for transforming the gravitational potential as well as the vertical, vertical-vertical and vertical-vertical-vertical component groups onto the first-, second- and third-order gradients of the gravitational potential. In the formulas, the index μ is used. Its role is twofold: 1) on the left-hand sides of the equations, we substitute either null, v, vv or vvv to distinguish between the kernels $\mathcal{K}$, $\mathcal{K}^{v}$, $\mathcal{K}^{vv}$ or $\mathcal{K}^{vvv}$; 2) on the right-hand sides of the equations it is equal to zero (for null), one (for v), two (for vv) or three (for vvv).

Components of the first-order tensor of the kernel function $\mathcal{K}^{\mu}$ are:

$$\mathcal{K}^{\mu}_{x} = \cos\alpha \ \mathcal{K}^{\mu}_{o,2}, \tag{134}$$

$$\mathcal{K}^{\mu}_{y} = -\sin\alpha \ \mathcal{K}^{\mu}_{o,2}, \tag{135}$$

$$\mathcal{K}^{\mu}_{z} = \mathcal{K}^{\mu}_{o,1}. \tag{136}$$

Components of the second-order tensor of the integral kernel $\mathcal{K}^{\mu}$ are:

$$\mathcal{K}^{\mu}_{xx} = \mathcal{K}^{\mu}_{op,1} + \cos 2\alpha \ \mathcal{K}^{\mu}_{op,3}, \tag{137}$$

$$\mathcal{K}^{\mu}_{xy} = -\sin 2\alpha \ \mathcal{K}^{\mu}_{op,3}, \tag{138}$$

$$\mathcal{K}^{\mu}_{xz} = \cos\alpha \ \mathcal{K}^{\mu}_{op,2}, \tag{139}$$

$$\mathcal{K}^{\mu}_{yy} = \mathcal{K}^{\mu}_{op,1} - \cos 2\alpha \ \mathcal{K}^{\mu}_{op,3}, \tag{140}$$

$$\mathcal{K}^{\mu}_{yz} = -\sin\alpha \ \mathcal{K}^{\mu}_{op,2}, \tag{141}$$

$$\mathcal{K}^{\mu}_{zz} = -2 \ \mathcal{K}^{\mu}_{op,1}. \tag{142}$$

Components of the third-order tensor of the integral kernel $\mathcal{K}^{\mu}$ are:

$$\mathcal{K}^{\mu}_{xxx} = \cos\alpha \ \mathcal{K}^{\mu}_{opq,2} + \cos 3\alpha \ \mathcal{K}^{\mu}_{opq,4}, \tag{143}$$

$$\mathcal{K}^{\mu}_{xxy} = -\frac{1}{3}\sin\alpha\ \mathcal{K}^{\mu}_{opq,2} - \sin 3\alpha\ \mathcal{K}^{\mu}_{opq,4}, \tag{144}$$

$$\mathcal{K}^{\mu}_{xxz} = \mathcal{K}^{\mu}_{opq,1} + \cos 2\alpha\ \mathcal{K}^{\mu}_{opq,3}, \tag{145}$$

$$\mathcal{K}^{\mu}_{xyy} = \frac{1}{3}\cos\alpha\ \mathcal{K}^{\mu}_{opq,2} - \cos 3\alpha\ \mathcal{K}^{\mu}_{opq,4}, \tag{146}$$

$$\mathcal{K}^{\mu}_{xyz} = -\sin 2\alpha\ \mathcal{K}^{\mu}_{opq,3}, \tag{147}$$

$$\mathcal{K}^{\mu}_{xzz} = -\frac{4}{3}\cos\alpha\ \mathcal{K}^{\mu}_{opq,2}, \tag{148}$$

$$\mathcal{K}^{\mu}_{yyy} = -\sin\alpha\ \mathcal{K}^{\mu}_{opq,2} + \sin 3\alpha\ \mathcal{K}^{\mu}_{opq,4}, \tag{149}$$

$$\mathcal{K}^{\mu}_{yyz} = \mathcal{K}^{\mu}_{opq,1} - \cos 2\alpha\ \mathcal{K}^{\mu}_{opq,3}, \tag{150}$$

$$\mathcal{K}^{\mu}_{yzz} = \frac{4}{3}\sin\alpha\ \mathcal{K}^{\mu}_{opq,2}, \tag{151}$$

$$\mathcal{K}^{\mu}_{zzz} = -2\ \mathcal{K}^{\mu}_{opq,1}. \tag{152}$$

The isotropic parts of the kernel functions are defined as follows:

$$\mathcal{K}^{\mu}_{o,1} = -\sum_{n=0}^{\infty} t^{n+2}(2n+1)(n+1)\frac{n!}{(n+\mu)!}P_{n,0}(u), \tag{153}$$

$$\mathcal{K}^{\mu}_{o,2} = \sum_{n=1}^{\infty} t^{n+2}(2n+1)\frac{n!}{(n+\mu)!}P_{n,1}(u), \tag{154}$$

$$\mathcal{K}^{\mu}_{op,1} = -\frac{1}{2}\sum_{n=0}^{\infty} t^{n+3}(2n+1)(n+1)(n+2)\frac{n!}{(n+\mu)!}P_{n,0}(u), \tag{155}$$

$$\mathcal{K}^{\mu}_{op,2} = -\sum_{n=1}^{\infty} t^{n+3}(2n+1)(n+2)\frac{n!}{(n+\mu)!}P_{n,1}(u), \tag{156}$$

$$\mathcal{K}^{\mu}_{op,3} = \frac{1}{2}\sum_{n=2}^{\infty} t^{n+3}(2n+1)\frac{n!}{(n+\mu)!}P_{n,2}(u), \tag{157}$$

$$\mathcal{K}^{\mu}_{opq,1} = \frac{1}{2}\sum_{n=0}^{\infty} t^{n+4}(2n+1)(n+1)(n+2)(n+3)\frac{n!}{(n+\mu)!}P_{n,0}(u), \tag{158}$$

$$\mathcal{K}^{\mu}_{opq,2} = -\frac{3}{4}\sum_{n=0}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{n!}{(n+\mu)!}P_{n,1}(u), \tag{159}$$

$$\mathcal{K}^{\mu}_{opq,3} = -\frac{1}{2}\sum_{n=0}^{\infty} t^{n+4}(2n+1)(n+3)\frac{n!}{(n+\mu)!}P_{n,2}(u), \tag{160}$$

$$\mathcal{K}^{\mu}_{opq,4} = \frac{1}{4}\sum_{n=0}^{\infty} t^{n+4}(2n+1)\frac{n!}{(n+\mu)!}P_{n,3}(u). \tag{161}$$

Appendix D. Kernel functions transforming the *h*, *hv* and *hvv* component groups

This appendix summarizes kernel functions required for transforming the horizontal, horizontal-vertical and horizontal-vertical-vertical component groups onto the first-, second- and third-order gravitational gradients. The meaning of the index μ is the same as in Appendix C.

Components of the first-order tensor of the integral kernel $\left(\cos\alpha'\mathcal{K}^{h\mu}\right)$ are:

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_x = -\sin\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{o,2} + \cos\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{o,3}, \tag{162}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_y = -\sin\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{o,2} - \cos\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{o,3}, \tag{163}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_z = \cos\alpha'\mathcal{K}^{h\mu}_{o,1}. \tag{164}$$

Components of the second-order tensor of the integral kernel $\left(\cos\alpha'\mathcal{K}^{h\mu}\right)$ are:

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xx} = \cos\alpha'\left(\mathcal{K}^{h\mu}_{op,1} + \cos 2\alpha\ \mathcal{K}^{h\mu}_{op,2}\right) - \sin\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{op,3}, \tag{165}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xy} = -\cos\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{op,2} - \sin\alpha'\cos 2\alpha\ \mathcal{K}^{h\mu}_{op,3}, \tag{166}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xz} = \cos\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{op,4} + \sin\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{op,5}, \tag{167}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{yy} = \cos\alpha'\left(\mathcal{K}^{h\mu}_{op,1} - \cos 2\alpha\ \mathcal{K}^{h\mu}_{op,2}\right) + \sin\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{op,3}, \tag{168}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{yz} = -\cos\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{op,4} + \sin\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{op,5}, \tag{169}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{zz} = -2\cos\alpha'\mathcal{K}^{h\mu}_{op,1}. \tag{170}$$

Components of the third-order tensor of the integral kernel $\left(\cos\alpha'\mathcal{K}^{h\mu}\right)$ are:

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xxx} = -\cos\alpha'\left(\frac{3}{4}\cos\alpha\ \mathcal{K}^{h\mu}_{opq,2} - \cos 3\alpha\ \mathcal{K}^{h\mu}_{opq,6}\right) + \sin\alpha'\left(\frac{3}{4}\sin\alpha\ \mathcal{K}^{h\mu}_{opq,3} - \sin 3\alpha\ \mathcal{K}^{h\mu}_{opq,7}\right), \tag{171}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xxy} = \cos\alpha'\left(\frac{1}{4}\sin\alpha\ \mathcal{K}^{h\mu}_{opq,2} - \sin 3\alpha\ \mathcal{K}^{h\mu}_{opq,6}\right) + \sin\alpha'\left(\frac{1}{4}\cos\alpha\ \mathcal{K}^{h\mu}_{opq,3} - \cos 3\alpha\ \mathcal{K}^{h\mu}_{opq,7}\right), \tag{172}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xxz} = -\cos\alpha'\left(\frac{1}{2}\mathcal{K}^{h\mu}_{opq,1} + \cos 2\alpha\ \mathcal{K}^{h\mu}_{opq,4}\right) + \sin\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{opq,5}, \tag{173}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xyy} = -\cos\alpha\left(\frac{1}{4}\cos\alpha'\mathcal{K}^{h\mu}_{opq,2} + \cos 3\alpha\ \mathcal{K}^{h\mu}_{opq,6}\right) + \sin\alpha'\left(\frac{1}{4}\sin\alpha\ \mathcal{K}^{h\mu}_{opq,3} + \sin 3\alpha\ \mathcal{K}^{h\mu}_{opq,7}\right), \tag{174}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xyz} = \cos\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{opq,4} + \sin\alpha'\cos 2\alpha\ \mathcal{K}^{h\mu}_{opq,5}, \tag{175}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{xzz} = \cos\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{opq,2} - \sin\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{opq,3}, \tag{176}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{yyy} = \cos\alpha'\left(\frac{3}{4}\sin\alpha\ \mathcal{K}^{h\mu}_{opq,2} + \sin 3\alpha\ \mathcal{K}^{h\mu}_{opq,6}\right) + \sin\alpha'\left(\frac{3}{4}\cos\alpha\ \mathcal{K}^{h\mu}_{opq,3} + \cos 3\alpha\ \mathcal{K}^{h\mu}_{opq,7}\right), \tag{177}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{yyz} = -\cos\alpha'\left(\frac{1}{2}\mathcal{K}^{h\mu}_{opq,1} - \cos 2\alpha\ \mathcal{K}^{h\mu}_{opq,4}\right) - \sin\alpha'\sin 2\alpha\ \mathcal{K}^{h\mu}_{opq,5}, \tag{178}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{yzz} = -\cos\alpha'\sin\alpha\ \mathcal{K}^{h\mu}_{opq,2} - \sin\alpha'\cos\alpha\ \mathcal{K}^{h\mu}_{opq,3}, \tag{179}$$

$$\left(\cos\alpha'\mathcal{K}^{h\mu}\right)_{zzz} = \cos\alpha'\mathcal{K}^{h\mu}_{opq,1}. \tag{180}$$

The kernel functions $\left(\sin\alpha'\mathcal{K}^{h\mu}\right)_i$, $\left(\sin\alpha'\mathcal{K}^{h\mu}\right)_{ij}$ and $\left(\sin\alpha'\mathcal{K}^{h\mu}\right)_{ijk}$ can be obtained by substituting $\sin\alpha' \to -\cos\alpha'$ and $\cos\alpha' \to \sin\alpha'$ in Eqs. (162)–(180).

The isotropic parts of the above kernel functions are defined as follows:

$$\mathcal{K}^{h\mu}_{o,1} = -\sum_{n=1}^{\infty} t^{n+2}(2n+1)(n+1)\frac{(n-1)!}{(n+1+\mu)!}P_{n,1}(u), \tag{181}$$

$$\mathcal{K}^{h\mu}_{o,2} = \frac{1}{2}\sum_{n=1}^{\infty} t^{n+2}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n+1,10}(u) + P_{n+1,2}(u)\right], \tag{182}$$

$$\mathcal{K}^{h\mu}_{o,3} = -\frac{1}{2}\sum_{n=1}^{\infty} t^{n+2}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n,10}(u) - P_{n,2}(u)\right], \tag{183}$$

$$\mathcal{K}^{h\mu}_{op,1} = -\frac{1}{2}\sum_{n=1}^{\infty} t^{n+3}(2n+1)(n+1)(n+2)\frac{(n-1)!}{(n+1+\mu)!}P_{n,1}(u), \tag{184}$$

$$\mathcal{K}^{h\mu}_{op,2} = -\frac{1}{4}\sum_{n=1}^{\infty} t^{n+3}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-1)(n+2)P_{n,1}(u) - P_{n,3}(u)\right], \tag{185}$$

$$\mathcal{K}^{h\mu}_{op,3} = \frac{1}{4}\sum_{n=1}^{\infty} t^{n+3}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-1)nP_{n+1,1}(u) + P_{n+1,3}(u)\right], \tag{186}$$

$$\mathcal{K}^{h\mu}_{op,4} = \frac{1}{2}\sum_{n=1}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n,0}(u) - P_{n,2}(u)\right], \tag{187}$$

$$\mathcal{K}^{h\mu}_{op,5} = \frac{1}{2}\sum_{n=1}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n+1,0}(u) + P_{n+1,2}(u)\right], \tag{188}$$

$$\mathcal{K}^{h\mu}_{opq,1} = -\sum_{n=1}^{\infty} t^{n+4}(2n+1)(n+1)(n+2)(n+3)\frac{(n-1)!}{(n+1+\mu)!}P_{n,1}(u), \tag{189}$$

$$\mathcal{K}^{h\mu}_{opq,2} = -\frac{1}{2}\sum_{n=1}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n,0}(u) - P_{n,2}(u)\right], \tag{190}$$

$$\mathcal{K}^{h\mu}_{opq,3} = \frac{1}{2}\sum_{n=1}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-1)!}{(n+1+\mu)!}\left[n(n+1)P_{n+1,0}(u) + P_{n+1,2}(u)\right], \tag{191}$$

$$\mathcal{K}^{h\mu}_{opq,4} = -\frac{1}{4}\sum_{n=1}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-1)(n+2)P_{n,1}(u) - P_{n,3}(u)\right], \tag{192}$$

$$\mathcal{K}^{h\mu}_{opq,5} = \frac{1}{4}\sum_{n=1}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-1)nP_{n+1,1}(u) + P_{n+1,3}(u)\right], \tag{193}$$

$$\mathcal{K}^{h\mu}_{opq,6} = -\frac{1}{8}\sum_{n=1}^{\infty} t^{n+4}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-2)(n+3)P_{n,2}(u) - P_{n,4}(u)\right], \tag{194}$$

$$\mathcal{K}^{h\mu}_{opq,7} = \frac{1}{8}\sum_{n=1}^{\infty} t^{n+4}(2n+1)\frac{(n-1)!}{(n+1+\mu)!}\left[(n-1)(n+2)P_{n+1,2}(u) + P_{n+1,4}(u)\right]. \tag{195}$$

Appendix E. Kernel functions transforming the *hh* and *hhv* component groups

This appendix summarizes the integral kernels required for transforming the horizontal-horizontal or horizontal-horizontal-vertical component groups onto first-, second- and third-order gravitational gradients. The meaning of the index μ is the same as in Appendices C and D.

Components of the first-order tensor of the integral kernel $\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)$ are:

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_x = -2\sin 2\alpha' \sin\alpha\ \mathcal{K}^{hh\mu}_{o,2} + \cos 2\alpha' \cos\alpha\ \mathcal{K}^{hh\mu}_{o,3}, \tag{196}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_y = -2\sin 2\alpha' \cos\alpha\ \mathcal{K}^{hh\mu}_{o,2} - \cos 2\alpha' \sin\alpha\ \mathcal{K}^{hh\mu}_{o,3}. \tag{197}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_z = \cos 2\alpha' \mathcal{K}^{hh\mu}_{o,1}. \tag{198}$$

Components of the second-order tensor of the integral kernel $\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)$ are:

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xx} = \cos 2\alpha' \left(\mathcal{K}^{hh\mu}_{op,1} + \cos 2\alpha \ \mathcal{K}^{hh\mu}_{op,2}\right) - \sin 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{op,3}, \tag{199}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xy} = -\cos 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{op,2} - \sin 2\alpha' \cos 2\alpha \ \mathcal{K}^{hh\mu}_{op,3}, \tag{200}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xz} = \cos 2\alpha' \cos\alpha \ \mathcal{K}^{hh\mu}_{op,4} + \sin 2\alpha' \sin\alpha \ \mathcal{K}^{hh\mu}_{op,5}, \tag{201}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{yy} = \cos 2\alpha' \left(\mathcal{K}^{hh\mu}_{op,1} - \cos 2\alpha \ \mathcal{K}^{hh\mu}_{op,2}\right) + \sin 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{op,3}, \tag{202}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{yz} = -\cos 2\alpha' \sin\alpha \ \mathcal{K}^{hh\mu}_{op,4} + \sin 2\alpha' \cos\alpha \ \mathcal{K}^{hh\mu}_{op,5}, \tag{203}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{zz} = -2\cos 2\alpha' \mathcal{K}^{hh\mu}_{op,1}. \tag{204}$$

Components of the third-order tensor of the integral kernel $\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)$ are:

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xxx} = -\cos 2\alpha' \left(\frac{3}{4}\cos\alpha \ \mathcal{K}^{hh\mu}_{opq,2} - \cos 3\alpha \ \mathcal{K}^{hh\mu}_{opq,6}\right) + \sin 2\alpha' \left(\frac{3}{4}\sin\alpha \ \mathcal{K}^{hh\mu}_{opq,3} - \sin 3\alpha \ \mathcal{K}^{hh\mu}_{opq,7}\right), \tag{205}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xxy} = \cos 2\alpha' \left(\frac{1}{4}\sin\alpha \ \mathcal{K}^{hh\mu}_{opq,2} - \sin 3\alpha \ \mathcal{K}^{hh\mu}_{opq,6}\right) + \sin 2\alpha' \left(\frac{1}{4}\cos\alpha \ \mathcal{K}^{hh\mu}_{opq,3} - \cos 3\alpha \ \mathcal{K}^{hh\mu}_{opq,7}\right), \tag{206}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xxz} = -\cos 2\alpha' \left(\frac{1}{2}\mathcal{K}^{hh\mu}_{opq,1} + \cos 2\alpha \ \mathcal{K}^{hh\mu}_{opq,4}\right) + \sin 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{opq,5}, \tag{207}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xyy} = -\cos 2\alpha' \left(\frac{1}{4}\cos\alpha \ \mathcal{K}^{hh\mu}_{opq,2} + \cos 3\alpha \ \mathcal{K}^{hh\mu}_{opq,6}\right) + \sin 2\alpha' \left(\frac{1}{4}\sin\alpha \ \mathcal{K}^{hh\mu}_{opq,3} + \sin 3\alpha \ \mathcal{K}^{hh\mu}_{opq,7}\right), \tag{208}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xyz} = \cos 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{opq,4} + \sin 2\alpha' \cos 2\alpha \ \mathcal{K}^{hh\mu}_{opq,5}, \tag{209}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{xzz} = \cos 2\alpha' \cos\alpha \ \mathcal{K}^{hh\mu}_{opq,2} - \sin 2\alpha' \sin\alpha \ \mathcal{K}^{hh\mu}_{opq,3}, \tag{210}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{yyy} = \cos 2\alpha' \left(\frac{3}{4}\sin\alpha \ \mathcal{K}^{hh\mu}_{opq,2} + \sin 3\alpha \ \mathcal{K}^{hh\mu}_{opq,6}\right) + \sin 2\alpha' \left(\frac{3}{4}\cos\alpha \ \mathcal{K}^{hh\mu}_{opq,3} + \cos 3\alpha \ \mathcal{K}^{hh\mu}_{opq,7}\right), \tag{211}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{yyz} = -\cos 2\alpha' \left(\frac{1}{2}\mathcal{K}^{hh\mu}_{opq,1} - \cos 2\alpha \ \mathcal{K}^{hh\mu}_{opq,4}\right) - \sin 2\alpha' \sin 2\alpha \ \mathcal{K}^{hh\mu}_{opq,5}, \tag{212}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{yzz} = -\cos 2\alpha' \sin\alpha \ \mathcal{K}^{hh\mu}_{opq,2} - \sin 2\alpha' \cos\alpha \ \mathcal{K}^{hh\mu}_{opq,3}, \tag{213}$$

$$\left(\cos 2\alpha' \mathcal{K}^{hh\mu}\right)_{zzz} = \cos 2\alpha' \mathcal{K}^{hh\mu}_{opq,1}. \tag{214}$$

The isotropic parts of the above kernel functions are defined as follows:

$$\mathcal{K}^{hh\mu}_{o,1} = -\sum_{n=2}^{\infty} t^{n+2}(2n+1)(n+1)\frac{(n-2)!}{(n+2+\mu)!}P_{n,2}(u), \tag{215}$$

$$\mathcal{K}^{hh\mu}_{o,2} = \frac{1}{4}\sum_{n=2}^{\infty} t^{n+2}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)nP_{n+1,1}(u) + P_{n+1,3}(u)\right], \tag{216}$$

$$\mathcal{K}^{hh\mu}_{o,3} = -\frac{1}{2}\sum_{n=2}^{\infty} t^{n+2}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)(n+2)P_{n,1}(u) - P_{n,3}(u)\right], \tag{217}$$

$$\mathcal{K}^{hh\mu}_{op,1} = -\frac{1}{2}\sum_{n=2}^{\infty} t^{n+3}(2n+1)(n+1)(n+2)\frac{(n-2)!}{(n+2+\mu)!}P_{n,2}(u), \tag{218}$$

$$\mathcal{K}^{hh\mu}_{op,2} = \frac{1}{4}\sum_{n=2}^{\infty} t^{n+3}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)n(n+1)(n+2)P_{n,0}(u) + 8P_{n,2}(u) + P_{n,4}(u)\right] \tag{219}$$

$$\mathcal{K}^{hh\mu}_{op,3} = -\frac{1}{4}\sum_{n=2}^{\infty} t^{n+3}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)n(n+1)(n+2)P_{n+1,0}(u) + 4(n-1)P_{n+1,2}(u) - P_{n+1,4}(u)\right] \tag{220}$$

$$\mathcal{K}^{hh\mu}_{op,4} = \frac{1}{2}\sum_{n=2}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)(n+2)P_{n,1}(u) - P_{n,3}(u)\right], \tag{221}$$

$$\mathcal{K}^{hh\mu}_{op,5} = \frac{1}{2}\sum_{n=2}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)nP_{n+1,1}(u) + P_{n+1,3}(u)\right], \tag{222}$$

$$\mathcal{K}^{hh\mu}_{opq,1} = -\sum_{n=2}^{\infty} t^{n+4}(2n+1)(n+1)(n+2)(n+3)\frac{(n-2)!}{(n+2+\mu)!}P_{n,2}(u), \tag{223}$$

$$\mathcal{K}^{hh\mu}_{opq,2} = -\frac{1}{2}\sum_{n=2}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)(n+2)P_{n,1}(u) - P_{n,3}(u)\right] \tag{224}$$

$$\mathcal{K}^{hh\mu}_{opq,3} = \frac{1}{2}\sum_{n=2}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)nP_{n+1,1}(u) + P_{n+1,3}(u)\right], \tag{225}$$

$$\mathcal{K}^{hh\mu}_{opq,4} = \frac{1}{4}\sum_{n=2}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)n(n+1)(n+2)P_{n,0}(u) + 8P_{n,2}(u) + P_{n,4}(u)\right] \tag{226}$$

$$\mathcal{K}^{hh\mu}_{opq,5} = -\frac{1}{4}\sum_{n=2}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-1)n(n+1)(n+2)P_{n+1,0}(u) + 4(n-1)P_{n+1,2}(u) - P_{n+1,4}(u)\right] \tag{227}$$

$$\mathcal{K}^{hh\mu}_{opq,6} = \frac{1}{8}\sum_{n=2}^{\infty} t^{n+4}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-2)(n-1)(n+2)(n+3)P_{n,1}(u) + 18P_{n,3}(u) + P_{n,5}(u)\right] \tag{228}$$

$$\mathcal{K}^{hh\mu}_{opq,7} = -\frac{1}{8}\sum_{n=2}^{\infty} t^{n+4}(2n+1)\frac{(n-2)!}{(n+2+\mu)!}\left[(n-2)(n-1)n(n+3)P_{n+1,1}(u) + 6(n-2)P_{n+1,3}(u) - P_{n+1,5}(u)\right]. \tag{229}$$

The sub-integral kernels $\left(\sin 2\alpha' \mathcal{K}^{hh\mu}\right)_i$, $\left(\sin 2\alpha' \mathcal{K}^{hh\mu}\right)_{ij}$ and $\left(\sin 2\alpha' \mathcal{K}^{hh\mu}\right)_{ijk}$ can be obtained by substituting $\sin 2\alpha' \rightarrow -\cos 2\alpha'$ and $\cos 2\alpha' \rightarrow \sin 2\alpha'$ in Eqs. (196)–(214).

Appendix F. Kernel functions transforming the *hhh* component group

This appendix summarizes the kernel functions required for transforming the horizontal third-order tensor component group onto the first-, second- and third-order gravitational gradients.

Components of the first-order tensor of the integral kernel $\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)$ are:

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_x = -\sin 3\alpha' \sin\alpha \ \mathcal{K}^{hhh}_{o,2} + \cos 3\alpha' \cos\alpha \ \mathcal{K}^{hhh}_{o,3}, \tag{230}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_y = -\sin 3\alpha' \cos\alpha \ \mathcal{K}^{hhh}_{o,2} - \cos 3\alpha' \sin\alpha \ \mathcal{K}^{hhh}_{o,3}, \tag{231}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_z = \cos 3\alpha' \mathcal{K}^{hhh}_{o,1}. \tag{232}$$

Components of the second-order tensor of the integral kernel $\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)$ are:

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xx} = \cos 3\alpha' \left(\mathcal{K}^{hhh}_{op,1} + \cos 2\alpha \ \mathcal{K}^{hhh}_{op,2}\right) - \sin 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{op,3}, \tag{233}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xy} = -\cos 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{op,2} - \sin 3\alpha' \cos 2\alpha \ \mathcal{K}^{hhh}_{op,3}, \tag{234}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xz} = \cos 3\alpha' \cos \alpha \ \mathcal{K}^{hhh}_{op,4} + \sin 3\alpha' \sin \alpha \ \mathcal{K}^{hhh}_{op,5}, \tag{235}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{yy} = \cos 3\alpha' \left(\mathcal{K}^{hhh}_{op,1} - \cos 2\alpha \ \mathcal{K}^{hhh}_{op,2}\right) + \sin 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{op,3}, \tag{236}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{yz} = -\cos 3\alpha' \sin \alpha \ \mathcal{K}^{hhh}_{op,4} + \sin 3\alpha' \cos \alpha \ \mathcal{K}^{hhh}_{op,5}, \tag{237}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{zz} = -2\cos 3\alpha' \mathcal{K}^{hhh}_{op,1}. \tag{238}$$

Components of the third-order gravitational tensor of the integral kernel $\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)$ are:

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xxx} = -\cos 3\alpha' \left(\frac{3}{4}\cos \alpha \ \mathcal{K}^{hhh}_{opq,2} - \cos 3\alpha \ \mathcal{K}^{hhh}_{opq,6}\right) + \sin 3\alpha' \left(\frac{3}{4}\sin \alpha \ \mathcal{K}^{hhh}_{opq,3} - \sin 3\alpha \ \mathcal{K}^{hhh}_{opq,7}\right), \tag{239}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xxy} = \cos 3\alpha' \left(\frac{1}{4}\sin \alpha \ \mathcal{K}^{hhh}_{opq,2} - \sin 3\alpha \ \mathcal{K}^{hhh}_{opq,6}\right) + \sin 3\alpha' \left(\frac{1}{4}\cos \alpha \ \mathcal{K}^{hhh}_{opq,3} - \cos 3\alpha \ \mathcal{K}^{hhh}_{opq,7}\right), \tag{240}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xxz} = -\cos 3\alpha' \left(\frac{1}{2}\mathcal{K}^{hhh}_{opq,1} + \cos 2\alpha \ \mathcal{K}^{hhh}_{opq,4}\right) + \sin 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{opq,5}, \tag{241}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xyy} = -\cos 3\alpha' \left(\frac{1}{4}\cos \alpha \ \mathcal{K}^{hhh}_{opq,2} + \cos 3\alpha \ \mathcal{K}^{hhh}_{opq,6}\right) + \sin 3\alpha' \left(\frac{1}{4}\sin \alpha \ \mathcal{K}^{hhh}_{opq,3} + \sin 3\alpha \ \mathcal{K}^{hhh}_{opq,7}\right), \tag{242}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xyz} = \cos 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{opq,4} + \sin 3\alpha' \cos 2\alpha \ \mathcal{K}^{hhh}_{opq,5}, \tag{243}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{xzz} = \cos 3\alpha' \cos \alpha \ \mathcal{K}^{hhh}_{opq,2} - \sin 3\alpha' \sin \alpha \ \mathcal{K}^{hhh}_{opq,3}, \tag{244}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{yyy} = \cos 3\alpha' \left(\frac{3}{4}\sin \alpha \ \mathcal{K}^{hhh}_{opq,2} + \sin 3\alpha \ \mathcal{K}^{hhh}_{opq,6}\right) + \sin 3\alpha' \left(\frac{3}{4}\cos \alpha \ \mathcal{K}^{hhh}_{opq,3} + \cos 3\alpha \ \mathcal{K}^{hhh}_{opq,7}\right), \tag{245}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{yyz} = -\cos 3\alpha' \left(\frac{1}{2}\mathcal{K}^{hhh}_{opq,1} - \cos 2\alpha \ \mathcal{K}^{hhh}_{opq,4}\right) - \sin 3\alpha' \sin 2\alpha \ \mathcal{K}^{hhh}_{opq,5}, \tag{246}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{yzz} = -\cos 3\alpha' \sin \alpha \ \mathcal{K}^{hhh}_{opq,2} - \sin 3\alpha' \cos \alpha \ \mathcal{K}^{hhh}_{opq,3}, \tag{247}$$

$$\left(\cos 3\alpha' \mathcal{K}^{hhh}\right)_{zzz} = \cos 3\alpha' \mathcal{K}^{hhh}_{opq,1}. \tag{248}$$

The isotropic parts of the above kernel functions are defined as follows:

$$\mathcal{K}^{hhh}_{o,1} = -\sum_{n=3}^{\infty} t^{n+2}(2n+1)(n+1)\frac{(n-3)!}{(n+3)!}P_{n,3}(u), \tag{249}$$

$$\mathcal{K}^{hhh}_{o,2} = \frac{1}{2}\sum_{n=3}^{\infty} t^{n+2}(2n+1)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)P_{n+1,2}(u) + P_{n+1,4}(u)\right], \tag{250}$$

$$\mathcal{K}^{hhh}_{o,3} = -\frac{1}{2}\sum_{n=3}^{\infty} t^{n+2}(2n+1)\frac{(n-3)!}{(n+3)!}\left[(n+3)(n-2)P_{n,2}(u) - P_{n,4}(u)\right], \tag{251}$$

$$\mathcal{K}^{hhh}_{op,1} = -\frac{1}{2}\sum_{n=3}^{\infty} t^{n+3}(2n+1)(n+1)(n+2)\frac{(n-3)!}{(n+3)!}P_{n,3}(u), \tag{252}$$

$$\mathcal{K}^{hhh}_{op,2} = \frac{1}{4}\sum_{n=3}^{\infty} t^{n+3}(2n+1)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)(n+2)(n+3)P_{n,1}(u) + 18P_{n,3}(u) + P_{n,5}(u)\right], \tag{253}$$

$$\mathcal{K}^{hhh}_{op,3} = -\frac{1}{4}\sum_{n=3}^{\infty} t^{n+3}(2n+1)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)n(n+3)P_{n+1,1}(u) + 6(n-2)P_{n+1,3}(u) - P_{n+1,5}(u)\right], \tag{254}$$

$$\mathcal{K}^{hhh}_{op,4} = \frac{1}{2}\sum_{n=3}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n+3)P_{n,2}(u) - P_{n,4}(u)\right], \tag{255}$$

$$\mathcal{K}^{hhh}_{op,5} = \frac{1}{2}\sum_{n=3}^{\infty} t^{n+3}(2n+1)(n+2)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)P_{n+1,2}(u) + P_{n+1,4}(u)\right], \tag{256}$$

$$\mathcal{K}^{hhh}_{opq,1} = -\sum_{n=3}^{\infty} t^{n+4}(2n+1)(n+1)(n+2)(n+3)\frac{(n-3)!}{(n+3)!}P_{n,3}(u), \tag{257}$$

$$\mathcal{K}^{hhh}_{opq,2} = -\frac{1}{2}\sum_{n=3}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n+3)P_{n,2}(u) - P_{n,4}(u)\right], \tag{258}$$

$$\mathcal{K}^{hhh}_{opq,3} = \frac{1}{2}\sum_{n=3}^{\infty} t^{n+4}(2n+1)(n+2)(n+3)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)P_{n+1,2}(u) - P_{n+1,4}(u)\right], \tag{259}$$

$$\mathcal{K}^{hhh}_{opq,4} = \frac{1}{4}\sum_{n=3}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)(n+2)(n+3)P_{n,1}(u) + 18P_{n,3}(u) + P_{n,5}(u)\right], \tag{260}$$

$$\mathcal{K}^{hhh}_{opq,5} = -\frac{1}{4}\sum_{n=3}^{\infty} t^{n+4}(2n+1)(n+3)\frac{(n-3)!}{(n+3)!}\left[(n-2)(n-1)n(n+3)P_{n+1,1}(u) + 6(n-2)P_{n+1,3}(u) - P_{n+1,5}(u)\right], \tag{261}$$

$$\mathcal{K}^{hhh}_{opq,6} = -\frac{1}{8}\sum_{n=3}^{\infty} t^{n+4}(2n+1)\frac{(n-3)!}{(n+3)!}\Big[(n-2)(n-1)n(n+1)\,(n+2)(n+3)P_{n,0}(u) + 18(n-2)(n+3)P_{n,2}(u)$$
$$-36P_{n,4}(u) - P_{n,6}(u)\Big], \tag{262}$$

$$\mathcal{K}^{hhh}_{opq,7} = \frac{1}{8}\sum_{n=3}^{\infty} t^{n+4}(2n+1)\frac{(n-3)!}{(n+3)!}\Big[(n-2)(n-1)n(n+1)\,(n+2)(n+3)P_{n+1,0}(u) + 2(n-2)(n-1)(2n+11)$$
$$\times\, P_{n+1,2}(u) - 4(2n-7)P_{n+1,4}(u) + P_{n+1,6}(u)\Big]. \tag{263}$$

The sub-integral kernels $\left(\sin 3\alpha' \mathcal{K}^{hhh}\right)_i$, $\left(\sin 3\alpha' \mathcal{K}^{hhh}\right)_{ij}$ and $\left(\sin 3\alpha' \mathcal{K}^{hhh}\right)_{ijk}$ can be obtained by substituting $\sin 3\alpha' \rightarrow -\cos 3\alpha'$ and $\cos 3\alpha' \rightarrow \sin 3\alpha'$ in Eqs. (230)–(248).

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