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The spherical horizontal and spherical vertical boundary value problem – vertical deflections and geoidal undulations – the completed Meissl diagram

E. W. Grafarend

Department of Geodesy and GeoInformatics, Stuttgart University, Geschwister-Scholl-Str. 24, D-70174 Stuttgart, Germany
e-mail: grafarend@gis.uni-stuttgart.de; Tel.: +49-711-1213390; Fax: +49-711-1213285

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Abstract. In a comparison of the solution of the spherical horizontal and vertical boundary value problems of physical geodesy it is aimed to construct downward continuation operators for vertical deflections (surface gradient of the incremental gravitational potential) and for gravity disturbances (vertical derivative of the incremental gravitational potential) from points on the Earth's topographic surface or of the three-dimensional (3-D) Euclidean space nearby down to the international reference sphere (IRS). First the horizontal and vertical components of the gravity vector, namely spherical vertical deflections and spherical gravity disturbances, are set up. Second, the horizontal and vertical boundary value problem in spherical gravity and geometry space is considered. The incremental gravity vector is represented in terms of vector spherical harmonics. The solution of horizontal spherical boundary problem in terms of the horizontal vector-valued Green function converts vertical deflections given on the IRS to the incremental gravitational potential external in the 3-D Euclidean space. The horizontal Green functions specialized to evaluation and source points on the IRS coincide with the Stokes kernel for vertical deflections. Third, the vertical spherical boundary value problem is solved in terms of the vertical scalar-valued Green function. Fourth, the operators for upward continuation of vertical deflections given on the IRS to vertical deflections in its external 3-D Euclidean space are constructed. Fifth, the operators for upward continuation of incremental gravity given on the IRS to incremental gravity to the external 3-D Euclidean space are generated. Finally, Meissl-type diagrams for upward continuation and regularized downward continuation of horizontal and vertical gravity data, namely vertical deflection and incremental gravity, are produced.

Keywords: Spherical Vertical Deflections – Spherical Geoidal Undulations – Boundary Value Problems – Upward Continuation – Downward Continuation – Meissl Diagram

Dedicated to the memory of G.G. Stokes whose fundamental geodetic contribution appeared 150 years ago.

1 Introduction

Over the last 50 years geodesists and mathematicians have tried to formulate and solve gravimetric boundary value problems, namely in terms of functionals of the gravitational potential on the Earth's topographic surface, of the Laplace equation.The target of those boundary value problems had been to determine the gravitational field outside the Earth's topographic surface, as well as in its interior, with a constraint to the Poisson equation (direct and inverse problem). Such a development had been initiated by Molodensky and other Russian scientists. A culmination point had been reached in the notion of the Telluroid which approximated in the pre-GPS era the Earth's topographic surface. For various reasons, at the time of a global positioning system (GPS: 'global problem solver') those gravimetric boundary value problems on the Earth's topographic surface failed to enter the platform of contemporary geodesy. For instance, when the gravitational field intensity (the modulus of the vector of the gravitational force) is given on a 'fairly smooth' Earth's topographic surface, its external boundary value problem is not unique, in general. Or the shape of the Earth's topographic surface is too irregular to produce a closed-form solution of the linearized gravimetric boundary value problem. Instead, geodesy took another direction. In terms of functionals of the gravitational disturbing potential (zero-order derivative: Dirichlet problem; first-order derivative: horizontal and vertical (Neumann) boundary value problems; second-order derivative: gradiometric boundary value problems) elementary boundary value problems of the Laplace equation on international reference figures had been formulated and solved in a closed form. The sphere and the ellipsoid of revolution had been chosen as international reference figures of the Earth. The problem that the gravitational potential is not a harmonic field outside the international reference sphere (IRS) or the international reference elliosoid (IRE) (WGS84) had been solved by the remove–restore technique: topographic masses had been reduced by means of the Newton integral which extends from the reference figure to the Earth's topographic surface (topography effect, terrain effect). While for those reductions review papers are available, there are no review papers of gravimetric boundary value problems on international reference figures which include downward-continuation operators of the horizontal and the vertical derivatives. Here we begin with a review of horizontal and vertical boundary value problems of the gravitational potential functionals of first order on the IRS. While the downward-continuation problem of spherical vertical derivatives (gravity disturbances) is well known, the downward-continuation problem of spherical horizontal derivatives (vertical deflections) is newly solved.

Stokes (1849) initiated the study of boundary value problems in physical geodesy when he succeeded in transforming gravity anomalies on the sphere $\mathbb{S}^2_R$ of radius R (nowadays called the IRS or 'Bjerhammar sphere') to gravitational potential disturbances (incremental potential). Pizzetti (1910) formulated the Stokes solution in terms of a boundary value problem: given gravity anomalies on the sphere $\mathbb{S}^2_R$, find the harmonic gravitational disturbing potential (incremental potential with respect to a reference potential of type gm/r) in the Euclidean three-dimensional (3-D) space *external* to $\mathbb{S}^2_R$ which approaches zero at infinity. When the incremental potential was converted by means of the Bruns (1878) formula to radial displacement, the Stokes–Pizzetti solution of the gravity anomaly boundary value problem in spherical approximation of gravity and geometry space produced undulations of the geoid (Gauss 1828, p. 49; Listing 1873, p. 45) as well as of other external equipotential surfaces.

The term IRS has to be defined more specifically. Such a sphere of reference is understood as that sphere which fits the geoid in a least-squares (LS) sense. Indeed, it is the particular sphere where the geoidal undulations vanish when integrated over the whole sphere. Later on we shall give a more analytical version of such a sphere which is partially inside the topographic masses.

The Stokes solution to the Stokes–Pizzetti boundary value problem has at least two deficiencies which were partially removed by Helmert (1880). First, the space outside the sphere $\mathbb{S}^2_R$ is *not* a domain of harmonicity. Thanks to topographic masses on the top of $\mathbb{S}^2_R$ the gravitational potential is *not* a harmonic function. Helmert (1880) proposed to condense those topographic masses on $\mathbb{S}^2_R$ or to remove them by means of terrain reduction, namely to generalize the Bouguer gravity anomaly (half-space approximation, Bouguer plate) to the Helmert gravity anomaly. Second, from a gravity survey on the topographic surface of the Earth we can only determine the gravity field intensity, which can be reduced by means of the centrifugal potential/centrifugal acceleration and the terrain effect to a gravity disturbance (incremental gravity) on the Earth's topographic surface. Accordingly the incremental gravity field intensity has to be downward continued, originally performed by generating the inverse free-air anomaly.

Less known is another geodetic boundary value problem which had been initiated by Stokes (1849) when he succeeded in transforming vertical deflections on the sphere $\mathbb{S}^2_R$ of radius R (IRS) to gravitational potential disturbances (incremental potential). The formulation and the solution of such a vertical boundary value problem with respect to the sphere $\mathbb{S}^2_R$ is our first target here: given vertical deflections on the sphere $\mathbb{S}^2_R$, find the harmonic gravitational disturbing potential (incremental potential with respect to a reference potential of type gm/r) in the Euclidean 3-D space external to $\mathbb{S}^2_R$ which approaches zero at infinity. Again, when the incremental potential is converted by means of the Bruns formula (Bruns 1878) to radial displacement, the solution of the horizontal boundary value problem in spherical approximation of gravity and geometry space produces undulations of the geoid as well as of other external equipotential surfaces. Conventionally the solution of the spherical horizontal boundary value problem may be called 'astrogeodetic geoid determination' since vertical deflections are used as input and geoid undulations as output. As a supplement we may add that vertical deflections can be uniquely determined by means of GPS-supported terrestrial direction observations, as was shown recently by Grafarend and Awange (2000). The conventional opinion is that astrogeodetic observations are needed to generate vertical deflections locally.

In potential theory, a boundary value problem which is similar to the vertical deflection boundary value problem (horizontal boundary value problem for the incremental potential) has been treated: given the surface gradient of the potential on the sphere $\mathbb{S}^2_R$, find the harmonic potential external to $\mathbb{S}^2_R$ which approaches zero at infinity. Since the horizontal boundary value problem, also called the vertical deflection boundary value problem, is formulated with respect to the incremental gravitational potential, its Green functions are remarkably different from those Green functions which are characteristic for the surface gradient boundary value problem. Due to the removal of a reference potential of type gm/r, the solution of the horizontal boundary value problem for the incremental gravitational potential is unique, while the solution of the surface gradient boundary value problem for the gravitational potential is not.

To illustrate such an argument, we outline in Sect. 2 the linearization procedure which leads us from a nonlinear observational functional, namely from astronomical longitude, astronomical latitude to vertical deflections, in particular a horizontal tangential linear operator acting on the incremental potential. The incremental potential is defined with respect to a spherical reference potential of type gm/r, thus eliminating the degree-zero term in the spherical harmonic expansion of the incremental gravitational potential.

The relative of the horizontal boundary value problem is the vertical boundary value problem: given gravity disturbances on the sphere $\mathbb{S}^2_R$, find the harmonic gravitational disturbing potential (incremental potential with respect to a reference potential of type gm/r) in the Euclidean 3-D space external to $\mathbb{S}^2_R$ which approaches zero at infinity. Again, when the incremental potential is converted by means of the Bruns formula (Bruns 1878) to radial displacement, the solution of the vertical boundary value problem, the second target of our contribution, is a generator of undulations of the geoid as well as of other external equipotential surfaces.

In potential theory, a boundary value problem which is similar to the vertical boundary value problem has been treated: given the radial derivative (vertical operator with respect to $\mathbb{S}^2_R$) of a scalar-valued potential on the sphere $\mathbb{S}^2_R$ (Neumann boundary value problem), find the harmonic potential external to $\mathbb{S}^2_R$ which approaches zero at infinity. Since the vertical boundary value problem is formulated with respect to the incremental gravitational potential, its Green functions are remarkably different from those Green functions which are characteristic for the Neumann boundary value problem. Due to the removal of a reference potential of type gm/r, the solution of the vertical boundary value problem is unique, while without any constraint the solution of the Neumann boundary value problem for the gravitational potential is not.

In order to prove our argument we refer to Bjerhammar and Svensson (1983), Grafarend and Keller (1995) and Grafarend et al. (1999), among others, where the linearization procedure is outlined, which leads us from a nonlinear observational functional, namely gravity field intensity (gravimeter observations) to spherical gravity disturbances, in particular a vertical/normal linear operator acting on the incremental potential. The special problem of removing degree terms of order zero and one appears.

The third target of our contribution is the upward and downward continuation of vertical deflections and gravity disturbances with respect to the IRS $\mathbb{S}^2_R$. If we have constructed the incremental potential functionals generated either by vertical deflections or by gravity disturbances on $\mathbb{S}^2_R$, an application of the horizontal/tangential or the vertical/normal linear operator leads to vertical deflections and gravity disturbances in the 3-D Euclidean space external to $\mathbb{S}^2_R$, and also at points on the Earth's topographic surface. In particular, we aim at a diagrammatic approach of upward and downward continuation (including regularization), extending the first results of the Meissl table (Meissl 1971, p. 46).

Our detailed analysis of the horizontal and vertical boundary value problem in the spherical approximation has to be generalized in two directions in the future. First, both boundary value problems have to be formulated and solved with respect to World Geodetic Datum 2000, namely the IRE, $\mathbb{E}^2_{a,a,b}$, and an expansion of the incremental potential and its linearized observational functionals in terms of scalar-valued and vector-valued ellipsoidal (spheroidal) harmonics. Indeed, we should introduce vertical deflections and gravity disturbances with respect to $\mathbb{E}^2_{a,a,b}$ (semi-major axis a, semi-minor axis b) *and* ellipsoidal (spheroidal) harmonics. Second, we introduce the linearized observational functions for vertical deflections and gravity disturbances, namely the horizontal/tangential and vertical/normal linear operators of first order. In a forthcoming paper we will present horizontal and vertical boundary value problems which are based upon gravity gradients, in particular horizontal, vertical as well as mixed horizontal–vertical linear operators of second order initiated by Rummel and van Gelderen (1999).

The remainder of this paper is origanized as follows. In Sect. 2 [Eqs. (1)–(63)] we set-up the horizontal and vertical components of the incremental gravity vector, namely for spherical vertical deflections and spherical gravity disturbance. Section 3 outlines the set-up and the solution of the horizontal and vertical boundary value problem in a spherical gravity/geometry space. Section 3.1 [Eqs. (64)–(99)] reviews the representation of the incremental gravity vector in terms of vector spherical harmonics. In contrast, Sect. 3.2 [Eqs. (100)–(176)] solves the spherical horizontal boundary value problem in terms of closed-form Green functions; a special case is the Stokes vertical deflection kernel function. Section 3.3 [Eqs. (177)–(217)] focuses on the solution of the spherical vertical boundary value problem in terms of closed-form modified (reduced) Neumann functions. The operators for upward and downward continuation of vertical deflections and gravity disturbances are constructed in Sects. 4 [Eqs. (218)–(256)] and 5 [Eqs. (257)–(277)]. A particular highlight is Sect. 5 [Eqs. (278)–(284)] on Meissl-type diagrams which systematically order upward and downward continuation inclusive of HAPS regularization. The already-oriented reader should just go through the Rummel–van Gelderen diagram (Fig. 15) and the Freeden diagram (Fig. 16) in the spectral domain as well as the upward continuation diagram (Fig. 17) and the regularized downward diagram (Fig. 18) in the space domain.

Key results for transforming (1) spherical coordinates (λ, ϕ) locally to meta-spherical/oblique coordinates (α^*, ψ^*), (2) (λ^*, ϕ^*) locally to (α, ψ) and (3) (α^*, ψ^*) locally to (α, ψ), as well as (4) the surface gradient from one spherical coordinate system to another meta-spherical/oblique spherical one, were originally given in four appendices. Due to restrictions on the length of this review paper postulated by the reviewers those appendices had to be partially removed. Instead the reader is advised to obtain this information from the Internet, namely from E. Grafarend (http://www.uni-stuttgart.de/gi/research/index. html#publications).

The review paper on the spherical horizontal and spherical vertical boundary value problem is based upon the contributions of Blaha et al. (1996), Ecker (1979), Freeden (1999), van Gelderen et al. (2000), Groten (1976, 1981), Hotine (1965), Koch (1971), Nashed (1971), Phillips (1962), Pizzetti (1911), Rummel et al. (1992), Schaffrin et al. (1977), Twomey (1963), Tykhonov (1963), Varshalovich (1988) and Xu et al. (1994a, b).

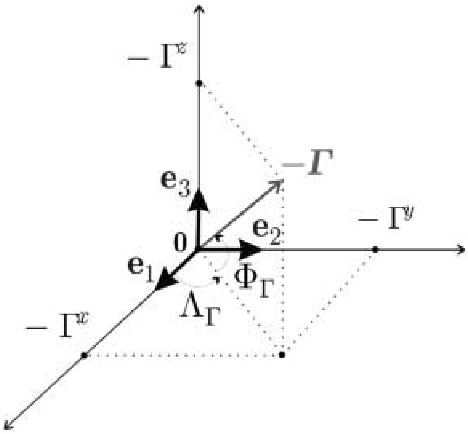

Fig. 1. Placement vector in gravity space, astronomical longitude Λ_Γ, astronomical latitude Φ_Γ, length $\|\mathbf{\Gamma}\|_2 =: \Gamma$ of the gravity vector $\mathbf{\Gamma}$ (ℓ_2 norm), and global frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$

2 Set-up of horizontal and vertical components of the gravity vector: spherical vertical deflections and spherical gravity disturbances

In order to set up the horizontal and vertical geodetic boundary value problem of the gravity vector we apply a two-step procedure. First, in order to linearize the gravity functionals, we decompose the vector-valued gravity field $\mathbf{\Gamma}(\mathbf{X}), \mathbf{\Gamma} \in \mathbb{R}^3, \mathbf{X} \in \mathbb{R}^3$, into a 'reference part' and a 'disturbing part'. Such a procedure is well known in perturbation theory and assumes a priori knowledge of the reference gravity field, which may relate to a band-limited spherical harmonic expansion of its gravitational potential field $U(\mathbf{x})$; the 'disturbing part' is left as an unknown. Second, as soon as we have chosen a proper operational coordinate system, both in gravity space as well as in geometry space, namely spherical coordinates $\{\lambda_\gamma, \phi_\gamma, \|\boldsymbol{\gamma}\|_2 = \gamma\}$ and $\{\lambda, \phi, \|\mathbf{x}\|_2 = r\}$, we are able to decompose the vector-valued disturbing gravity field $\boldsymbol{\delta\gamma}$ into a spherical horizontal/tangential part and a spherical vertical/normal part.

Let there be given the vector-valued gravity field $\mathbf{\Gamma}(\mathbf{x})$ at a point $\mathbf{P} \sim \mathbf{x}$ which, according to Fig. 1, we represent in an unimodular (orthonormal) global frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid 0\}$ attached to $\mathbf{0}$, the origin of a Cartesian coordinate system $[\Gamma^x, \Gamma^y, \Gamma^z]'$ which we identify with the mass centre of the Earth. The gravity vector $\mathbf{\Gamma}$ will be additively decomposed according to Eq. (1) into the reference gravity vector $\boldsymbol{\gamma}$, namely associated with a 'simple' gravity field to be specified soon, and into the disturbance gravity vector $\boldsymbol{\delta\gamma}$, also called the vector-valued Euler increment. As soon as we represent the reference gravity vector $\boldsymbol{\gamma}$ as well as the incremental gravity vector $\boldsymbol{\delta\gamma}$ in the chosen global frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid 0\}$, we are led to the 'left-basis representations' [Eqs. (2)–(7)] which we have collected in Table 1. Let us refer to $[\gamma^x, \gamma^y, \gamma^z]'$, $[\delta\gamma^x, \delta\gamma^y, \delta\gamma^z]'$ and $[\Gamma^x, \Gamma^y, \Gamma^z]'$ as column arrays of Cartesian coordinates of the vectors $\boldsymbol{\gamma}, \boldsymbol{\delta}_\gamma$ and $\mathbf{\Gamma}$, respectively. The transpose sign '′' is used.

In order to derive the modulus of the incremental gravity vector $\boldsymbol{\delta\gamma}$ as well as the vertical deflections $[\delta\lambda_\gamma, \cos\phi, \delta\phi_\gamma]$ we have to introduce spherical coordinates $[\lambda_\gamma, \phi_\gamma, \gamma]', [\delta\lambda_\gamma, \delta\phi_\gamma, \delta\gamma]'$ and $[\Lambda_\Gamma, \Phi_\Gamma, \Gamma]'$ of the gravity vectors $\boldsymbol{\gamma}, \boldsymbol{\delta\gamma}$ and $\mathbf{\Gamma}$. We depart from the inverse formula of Eq. (8) with respect to Eq. (1), namely the incremental gravity vector $-\boldsymbol{\delta\gamma}$.

As placement vectors in gravity space $\{-\mathbf{\Gamma}, -\boldsymbol{\gamma}\}$ are represented by means of Eqs. (9) and (10) in terms of spherical gravity coordinates: $[\Lambda_\Gamma, \Phi_\Gamma, \Gamma]'$ as a column array contains 'astronomical longitude Λ_Γ', 'astronomical latitude Φ_Γ' and 'modulus of the gravity vector Γ', also called ℓ_2-norm $\Gamma = \|\mathbf{\Gamma}\|_2$. In contrast, the column array $[\lambda_\gamma \phi_\gamma, \gamma]'$ contains 'geodetic longitude λ_γ', 'geodetic latitude ϕ_γ' and 'modulus of the reference gravity vector γ', also called ℓ_2-norm $\gamma = \|\boldsymbol{\gamma}\|_2$. Second, by means of Eqs. (11)–(13) we additively decompose the spherical

Table 1. Cartesian representation of the gravity vector $\boldsymbol{\delta\gamma}, \boldsymbol{\gamma}, \mathbf{\Gamma}$, Euler increments $(\delta\gamma^x, \delta\gamma^y, \delta\gamma^z)$

$$\mathbf{\Gamma} = \boldsymbol{\gamma} + \boldsymbol{\delta\gamma} \tag{1}$$

Cartesian representation of the gravity vector

$$\boldsymbol{\gamma} = \mathbf{e}_1\gamma^x + \mathbf{e}_2\gamma^y + \mathbf{e}_3\gamma^z = \sum_{i=1}^{3} \mathbf{e}_i\gamma^i \tag{2}$$

$$\boldsymbol{\delta\gamma} = \mathbf{e}_1\delta\gamma^x + \mathbf{e}_2\delta\gamma^y + \mathbf{e}_3\delta\gamma^z = \sum_{i=1}^{3} \mathbf{e}_i\delta\gamma^i \tag{3}$$

$$\mathbf{\Gamma} = \mathbf{e}_1\Gamma^x + \mathbf{e}_2\Gamma^y + \mathbf{e}_3\Gamma^z = \sum_{i=1}^{3} \mathbf{e}_i\Gamma^i \tag{4}$$

$$\boldsymbol{\gamma} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix}\gamma^x\\ \gamma^y\\ \gamma^z\end{bmatrix} \tag{5}$$

$$\boldsymbol{\delta\gamma} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix}\delta\gamma^x\\ \delta\gamma^y\\ \delta\gamma^z\end{bmatrix} \tag{6}$$

$$\mathbf{\Gamma} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix}\Gamma^x\\ \Gamma^y\\ \Gamma^z\end{bmatrix} \tag{7}$$

Table 2. Spherical representation of the incremental gravity vector, Euler increments $(\delta\lambda_\gamma, \delta\phi_\gamma, \delta\gamma)'$

$$\boldsymbol{\delta\gamma} = \mathbf{\Gamma} - \boldsymbol{\gamma} \tag{8}$$

$$-\mathbf{\Gamma} = \|\mathbf{\Gamma}\|_2\{\mathbf{e}_1\cos\Phi_\Gamma\cos\Lambda_\Gamma + \mathbf{e}_2\cos\Phi_\Gamma\sin\Lambda_\Gamma + \mathbf{e}_3\sin\Phi_\Gamma\} \tag{9}$$

$$-\boldsymbol{\gamma} = \|\boldsymbol{\gamma}\|_2\{\mathbf{e}_1\cos\phi_\gamma\cos\lambda_\gamma + \mathbf{e}_2\cos\phi_\gamma\sin\lambda_\gamma + \mathbf{e}_3\sin\phi_\gamma\} \tag{10}$$

$$\|\mathbf{\Gamma}\|_2 = \|\boldsymbol{\gamma} + \boldsymbol{\delta\gamma}\|_2 = \gamma + \delta\gamma \tag{11}$$

$$\Phi_\Gamma = \phi_\gamma + \delta\phi_\gamma \tag{12}$$

$$\Lambda_\Gamma = \lambda_\gamma + \delta\lambda_\gamma \tag{13}$$

$$\begin{aligned}-\boldsymbol{\delta\gamma} = {} & \mathbf{e}_1[(\gamma+\delta\gamma)\cos(\phi_\gamma+\delta\phi_\gamma)\cos(\lambda_\gamma+\delta\lambda_\gamma) - \gamma\cos\phi_\gamma\cos\lambda_\gamma]\\ & + \mathbf{e}_2[(\gamma+\delta\gamma)\cos(\phi_\gamma+\delta\phi_\gamma)\sin(\lambda_\gamma+\delta\lambda_\gamma) - \gamma\cos\phi_\gamma\sin\lambda_\gamma]\\ & + \mathbf{e}_3[(\gamma+\delta\gamma)\sin(\phi_\gamma+\delta\phi_\gamma) - \gamma\sin\phi_\gamma]\end{aligned} \tag{14}$$

$$\cos(\phi + \delta\phi) \doteq \cos\phi - \sin\phi\delta\phi \tag{15}$$

$$\cos(\lambda + \delta\lambda) \doteq \cos\lambda - \sin\lambda\delta\lambda \tag{16}$$

$$\sin(\phi + \delta\phi) = \sin\phi + \cos\phi\delta\phi \tag{17}$$

$$\sin(\lambda + \delta\lambda) = \sin\lambda + \cos\lambda\delta\lambda \tag{18}$$

$$\begin{aligned}-\boldsymbol{\delta\gamma} = {} & \mathbf{e}_1[\delta\gamma\cos\phi_\gamma\cos\lambda_\gamma - \gamma\sin\phi_\gamma\cos\lambda_\gamma\delta\phi_\gamma - \gamma\cos\phi_\gamma\sin\lambda_\gamma\delta\lambda_\gamma]\\ & + \mathbf{e}_2[\delta\gamma\cos\phi_\gamma\sin\lambda_\gamma - \gamma\sin\phi_\gamma\sin\lambda_\gamma\delta\phi_\gamma - \gamma\cos\phi_\gamma\cos\lambda_\gamma\delta\lambda_\gamma]\\ & + \mathbf{e}_3[\delta\gamma\sin\phi_\gamma - \gamma\cos\phi_\gamma\delta\phi_\gamma]\end{aligned} \tag{19}$$

$$\begin{aligned}-\boldsymbol{\delta\gamma} = {} & \delta\gamma[\mathbf{e}_1\cos\phi_\gamma\cos\lambda_\gamma + \mathbf{e}_2\cos\phi_\gamma\sin\lambda_\gamma + \mathbf{e}_3\sin\phi_\gamma]\\ & + \gamma\,\delta\phi_\gamma[-\mathbf{e}_1\sin\phi_\gamma\cos\lambda_\gamma - \mathbf{e}_2\sin\phi_\gamma\sin\lambda_\gamma + \mathbf{e}_3\cos\phi_\gamma]\\ & + \gamma\,\delta\lambda_\gamma\cos\phi_\gamma[-\mathbf{e}_1\sin\lambda_\gamma + \mathbf{e}_2\cos\lambda_\gamma]\end{aligned} \tag{20}$$

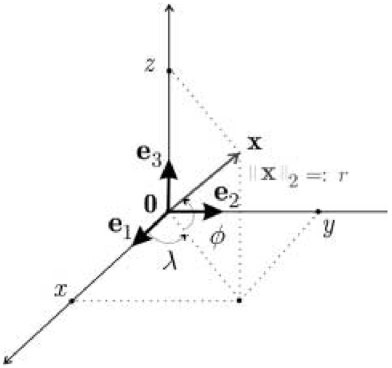

Fig. 2. Placement vector in geometry space, spherical longitude λ, spherical latitude ϕ, length $\|\mathbf{x}\|_2 =: r$ of the position vector $\mathbf{x}$ (ℓ_2 norm), and global frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \,|\, \mathbf{0}\}$

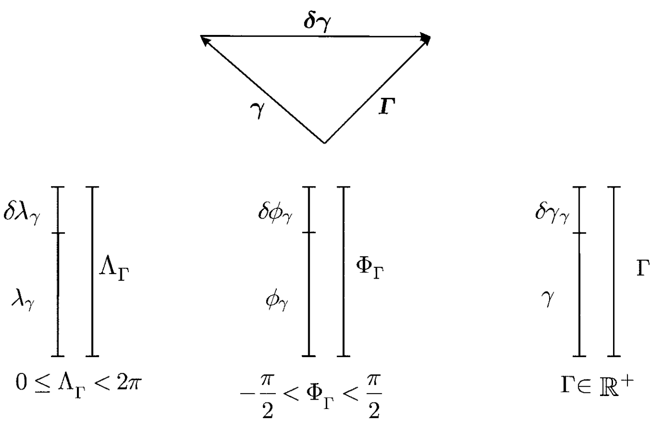

Fig. 3. Additive decomposition of the gravity vector $\boldsymbol{\Gamma}$ into the reference gravity vector $\boldsymbol{\gamma}$ and the disturbing gravity vector $\boldsymbol{\delta\gamma}$ (Euler δ-increment): $= \boldsymbol{\gamma} + \boldsymbol{\delta\gamma}$ decomposition of the spherical coordinates of the gravity vector $\Lambda_\Gamma = \lambda_\gamma + \delta\lambda_\gamma, \Phi_\Gamma = \phi_\gamma + \delta\phi_\gamma, \Gamma = \gamma + \delta\gamma, \|\|\|_2 =: \Gamma, \|\boldsymbol{\gamma}\|_2 =: \gamma$

coordinates $[\Lambda_\Gamma, \Phi_\Gamma, \Gamma]'$ of the gravity vector $\boldsymbol{\Gamma}$ into the spherical coordinates $[\lambda_\gamma, \phi_\gamma, \gamma]'$ of the reference gravity vector $\boldsymbol{\gamma}$ and the 'disturbed' spherical coordinates $[\delta\lambda_\gamma, \phi\lambda_\gamma, \delta\gamma]'$ – also called Euler increments within perturbation theory – of the disturbing gravity vector $\boldsymbol{\delta\gamma}$. Table 2 aims at a collection of formulae for the spherical representation of the incremental gravity vector $\boldsymbol{\delta\gamma}$ and Fig. 2 at a graphical illustration.

As soon as we implement the additively decomposed spherical coordinates $\Lambda_\Gamma = \lambda_\gamma + \delta\lambda_\gamma, \Phi_\gamma = \phi_\gamma + \delta\phi_\gamma$ and $\Gamma = \gamma + \delta\gamma$, of the gravity vector $\boldsymbol{\Gamma}$ into its spherical representation [Eq. (9)] we are led to Eq. (14), by means of the addition theorem of circular functions to Eqs. (15)–(18), and finally to Eqs. (19) and (20). The negative incremental gravity vector $-\boldsymbol{\delta\gamma}$ is represented additively by three terms, namely (1) $\delta\gamma$, (2) $\gamma\delta\phi_\gamma$ and (3) $\gamma\delta\lambda_\gamma \cos\phi_\gamma$, with respect to the global frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of equatorial type.

Table 3. Transformation of the global frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 | \mathbf{0}\}$ to the local frame of reference $\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r | P\}$

$$\mathbf{x} = \mathbf{e}_1 r\cos\phi\cos\lambda + \mathbf{e}_2 r\cos\phi\sin\lambda + \mathbf{e}_3 r\sin\phi \tag{21}$$

$$\mathbf{e}_\lambda := \mathrm{D}_\lambda\mathbf{x} \div \|\mathrm{D}_\lambda\mathbf{x}\|_2 \tag{22}$$

$$\mathbf{e}_\phi := \mathrm{D}_\phi\mathbf{x} \div \|\mathrm{D}_\phi\mathbf{x}\|_2 \tag{23}$$

$$\mathbf{e}_r := \mathrm{D}_r\mathbf{x} \div \|\mathrm{D}_r\mathbf{x}\|_2 \tag{24}$$

$$\mathbf{e}_\lambda = -\mathbf{e}_1\sin\lambda + \mathbf{e}_2\cos\lambda \tag{25}$$

$$\mathbf{e}_\phi = -\mathbf{e}_1\sin\phi\cos\lambda - \mathbf{e}_2\sin\phi\cos\lambda + \mathbf{e}_3\cos\phi \tag{26}$$

$$\mathbf{e}_r = \mathbf{e}_1\cos\phi\cos\lambda - \mathbf{e}_2\cos\phi\sin\lambda + \mathbf{e}_3\sin\phi \tag{27}$$

$$\begin{bmatrix} \mathbf{e}_\lambda \\ \mathbf{e}_\phi \\ \mathbf{e}_r \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\phi & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \mathbf{e}_\lambda \\ \mathbf{e}_\phi \\ \mathbf{e}_r \end{bmatrix} \tag{29}$$

Third, we make a choice of the coordinate system we operate in. Here we have chosen spherical coordinates $[\lambda, \phi, r]'$, namely spherical longitude λ, spherical latitude ϕ and length $\|\mathbf{x}\|_2 =: r$, called 'radius' or ℓ_2-norm $\|\mathbf{x}\|_2$ of the placement vector $\mathbf{x} = \chi(\lambda, \phi, r)$ in geometry space, illustrated in Fig. 3. Consult Eq. (21) in order to convince yourself that the partial derivatives $\mathrm{D}_\lambda\mathbf{x}, \mathrm{D}_\phi\mathbf{x}, \mathrm{D}_r\mathbf{x}$ as vectors being normalized generate the unimodular (orthonormal) local frame of reference [Eqs. (22)–(24)], namely $\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r \,|\, P\}$ at the point $P \sim \mathbf{x}$. Such a local frame of reference is also called 'spherical East, spherical North, spherical vertical'. $\mathbf{e}_\lambda$ is the unit vector directed to spherical East, $\mathbf{e}_\phi$ the unit vector directed to spherical North: $\{\mathbf{e}_\lambda, \mathbf{e}_\phi \,|\, P\}$ span the local tangent space $\mathbb{T}_P\mathbb{S}^2_r$ (tangent plane) at the point $P \sim \mathbf{x}$ of the sphere $\mathbb{S}^2_r$ of radius r. In contrast, $\mathbf{e}_r$ spans the local normal space $\mathbb{N}_P\mathbb{S}^2_r$ at the point $P \sim \mathbf{x}$ of the sphere $\mathbb{S}^2_r$. As soon as we differentiate the embedding function $\mathbf{x} = \chi(\lambda, \phi, r)$ *and* normalize by means of the ℓ_2-norms $\|\mathrm{D}_\lambda\mathbf{x}\|_2$, $\|\mathrm{D}_\phi\mathbf{x}\|_2$ and $\|\mathrm{D}_r\mathbf{x}\|_2$, we are led to the transformation $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]' \mapsto [\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r]'$ of the global frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \,|\, 0\}$ of equatorial type into the spherical local frame of reference of spherical horizontal type, namely Eqs. (25), (26), (27) and (28). Since such a transformation between orthonormal bases $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]', [\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r]'$, respectively, is generated by a rotation matrix $\mathbf{R} \in SO(3) := \{\mathbf{R} \in \mathbb{R}^{3\times 3} \,|\, \mathbf{R}\mathbf{R}' = \mathbf{I}_3, |\mathbf{R}| = +1\}$, its in-

Table 4. Representation of the incremental gravity vector in a local frame of reference $\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r | P\}$, horizontal versus vertical components

$$\begin{aligned} -\boldsymbol{\delta\gamma} = {} & \delta\gamma[\mathbf{e}_\lambda\cos\phi_\gamma(\cos\lambda\sin\lambda_\gamma - \cos\lambda_\gamma\sin\lambda) + \mathbf{e}_\phi(-\cos\phi_\gamma\sin\phi\cos\lambda_\gamma\cos\lambda \\ & - \cos\phi_\gamma\sin\phi\sin\lambda_\gamma\sin\lambda + \cos\phi\sin\phi_\gamma) + \mathbf{e}_r(\cos\phi_\gamma\cos\phi\cos\lambda_\gamma\cos\lambda \\ & + \cos\phi_\gamma\cos\phi\sin\lambda_\gamma\sin\lambda + \sin\phi_\gamma\sin\phi)] + \gamma\,\delta\phi_\gamma[\mathbf{e}_\lambda\sin\phi_\gamma(\cos\lambda_\gamma\sin\lambda \\ & - \cos\lambda\sin\lambda_\gamma) + \mathbf{e}_\phi(\sin\phi_\gamma\sin\phi\cos\lambda_\gamma\cos\lambda + \sin\phi_\gamma\sin\phi\sin\lambda_\gamma\sin\lambda \\ & + \cos\phi\cos\phi_\gamma) + \mathbf{e}_r(\cos\phi_\gamma\sin\phi - \cos\phi\sin\phi_\gamma\cos\lambda_\gamma\cos\lambda \\ & - \cos\phi\sin\phi_\gamma\sin\lambda_\gamma\sin\lambda)] + \gamma\,\delta\lambda_\gamma\cos\phi_\gamma[-\mathbf{e}_\lambda(\sin\lambda_\gamma\sin\lambda + \cos\lambda_\gamma\cos\lambda) \\ & + \mathbf{e}_\phi\sin\phi(\cos\lambda\sin\lambda_\gamma - \cos\lambda_\gamma\sin\lambda) + \mathbf{e}_r\cos\phi(\cos\lambda_\gamma\sin\lambda - \cos\lambda\sin\lambda_\gamma)] \end{aligned} \tag{30}$$

Table 5. Reference gravity field of type gm/r

$$\boldsymbol{\gamma} = \operatorname{grad} w \tag{31}$$

$$w = \frac{gm}{r}, \quad \mathrm{D}_i w = -\frac{gm}{r^3}x^i, \quad i \in \{1,2,3\} \tag{32}$$

$$\tan\lambda_\gamma = \frac{\gamma^y}{\gamma^x} = \frac{y}{x} = \tan\lambda \tag{33}$$

$$\boxed{\lambda_\gamma = \lambda \in [0, 2\pi]} \tag{34}$$

$$\tan\phi_\gamma = \frac{-\gamma^z}{\sqrt{(\gamma^x)^2 + (\gamma^y)^2}} = \frac{z}{\sqrt{x^2+y^2}} = \tan\phi \tag{35}$$

$$\boxed{\phi_\gamma = \phi \in]-\pi/2, +\pi/2[} \tag{36}$$

$$\gamma = \|\boldsymbol{\gamma}\|_2 = \sqrt{(\gamma^x)^2 + (\gamma^y)^2 + (\gamma^z)^2} \tag{37a}$$

$$\Rightarrow$$

$$\boxed{\gamma = \tfrac{gm}{r^2} \in \mathbb{R}^+} \tag{37b}$$

Table 6. Horizontal and vertical components of the disturbance gravity vector in a local spherical frame of reference and with respect to a reference potential of type gm/r (isotropic)

$$-\boldsymbol{\delta\gamma}(\lambda,\phi,\gamma) = \frac{gm}{r^2}\left\{\mathbf{e}_\lambda \delta\lambda_\gamma \cos\phi_\gamma + \mathbf{e}_\phi \delta\phi_\gamma\right\} + \mathbf{e}_r \delta\gamma \tag{38}$$

verse is just the transpose, $\mathbf{R}^{-1} = \mathbf{R}'$. Accordingly, Eq. (29) is the inverse transformation $[\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r]' \mapsto [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]'$, which we apply successfully later. Details of these transformations are collected in Table 3.

When we implement the local transformation $[\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r]' \mapsto [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]'$ from Eq. (29) to Eq. (20), we gain the general representation [Eq. (30)] of the negative disturbing gravity vector $-\boldsymbol{\delta\gamma}$ of Table 4. It is worth mentioning that all three coordinates $(\gamma\delta\lambda_\gamma\cos\phi_\gamma, \gamma\delta\phi_\gamma, \delta\gamma)$ have components in the tangent space $\mathbb{T}_P\mathbb{S}^2_r$ spanned by $\{\mathbf{e}_\lambda, \mathbf{e}_\phi \,|\, P\}$ *and* the normal space $\mathbb{N}_P\mathbb{S}^2_r$ spanned by $\{\mathbf{e}_r \,|\, P\}$ at the point $P \sim \mathbf{x}$. We shall specialize the general representation of $-\boldsymbol{\delta\gamma}$ [Eq. (30)] to two cases reviewed by Tables 5–8.

For case one, we have chosen a reference gravity field $\boldsymbol{\gamma}(\mathbf{x})$ which is derived from a gravitational potential field $w(\mathbf{x})$ of isotropic type, namely gm/r, where gm denotes the 'gravitational mass'. g is Newton's gravitational constant, also called the coupling constant of gravitational and inertial forces. m is the symbol for the total mass of the Earth. For such a gradient field of the type of Eqs. (31) and (32), geodetic longitude/latitude and spherical longitude/latitude coincide, namely $\lambda_\gamma = \lambda$, $\phi_\gamma = \phi$. As soon as we specialize Eq. (30) to such an isotropic gravity space we derive the standard representation of the negative disturbing gravity vector $-\boldsymbol{\delta\gamma}$ of the type of Eq. (38). Its horizontal components, illustrated by Fig. 4, are Easting $gm/r^2 * \delta\lambda_\gamma \cos\phi$ and Northing $gm/r^2 * \delta\phi_\gamma$, finally combined to the vertical deflection vector $\mathbf{e}_\lambda\delta\lambda_\gamma\cos\phi + \mathbf{e}_\phi\delta\phi_\gamma \in \mathbb{T}_P\mathbb{S}^2_r$. In contrast, its vertical component, illustrated by Fig. 5, is vertical $gm/r^2 * \delta\gamma$. All formulae are finally collected in Tables 5 and 6.

Finally, for case two we decided on $\boldsymbol{\gamma}(\mathbf{x})$ for a reference gravity field, which is derived from a gravity potential field $w(\mathbf{x})$ of anisotropic type, namely $gm/r + (\omega^2 r^2 \cos^2\phi)/2$. We have already explained the

Table 7. Reference gravity field of type $gm/r + \frac{1}{2}\omega^2 r^2 \cos^2\phi$

$$\boldsymbol{\gamma} = \operatorname{grad} w \tag{39}$$

$$w = \frac{gm}{\sqrt{x^2+y^2+z^2}} + \frac{1}{2}\omega^2(x^2+y^2) = \frac{gm}{r} + \frac{1}{2}\omega^2 r^2\cos^2\phi \tag{40}$$

$$\begin{bmatrix} \mathrm{D}_x w \\ \mathrm{D}_y w \\ \mathrm{D}_z w \end{bmatrix} = -\frac{gm}{(x^2+y^2+z^2)^{3/2}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \omega^2 \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{41}$$

$$\tan\lambda_\gamma = \frac{\gamma^y}{\gamma^x} = \frac{-\frac{gm}{r^3}y + \omega^2 y}{-\frac{gm}{r^3}x + \omega^2 x} = \frac{y}{x} = \tan\lambda \tag{42}$$

$$\Leftrightarrow$$

$$\boxed{\lambda_\gamma = \lambda} \tag{43}$$

$$\tan\phi_\gamma = \frac{-\gamma^z}{\sqrt{(\gamma^x)^2 + (\gamma^y)^2}} = \frac{z}{\sqrt{x^2+y^2}}\frac{1}{1 - \frac{\omega^2}{gm}(x^2+y^2+z^2)^{3/2}} \tag{44}$$

$$\tan\phi_\gamma = \frac{z}{\sqrt{x^2+y^2}}\left[1 + \frac{\omega^2}{gm}(x^2+y^2+z^2)^{3/2} + \mathcal{O}\left(\frac{\omega^4}{(gm)^2}\right)\right] \tag{45}$$

$$\tan\phi_\gamma = \tan\phi\left[1 + \frac{\omega^2}{gm}r^3 + \mathcal{O}\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{46}$$

$$\sin\phi_\gamma = \frac{\tan\phi_\gamma}{\sqrt{1+\tan^2\phi_\gamma}} \tag{47}$$

$$\cos\phi_\gamma = \frac{1}{\sqrt{1+\tan^2\phi_\gamma}} \tag{48}$$

$$1 + \tan^2\phi_\gamma = 1 + \tan^2\phi \frac{1}{(1 - \frac{\omega^2}{gm}r^3)^2} \tag{49}$$

$$1 + \tan^2\phi_\gamma = 1 + \tan^2\phi\left[1 + 2\frac{\omega^2}{gm}r^3 + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{50}$$

$$1 + \tan^2\phi_\gamma = 1 + \tan^2\phi + 2\frac{\omega^2}{gm}r^3\tan^2\phi + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right) \tag{51}$$

$$1 + \tan^2\phi_\gamma = (1 + \tan^2\phi)\left[1 + 2\frac{\omega^2}{gm}r^3\frac{\tan^2\phi}{1+\tan^2\phi} + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{52}$$

$$1 + \tan^2\phi_\gamma = (1 + \tan^2\phi)\left[1 + 2\frac{\omega^2}{gm}r^3\sin^2\phi + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{53}$$

$$\frac{1}{\sqrt{1+\tan^2\phi_\gamma}} = \frac{1}{\sqrt{1+\tan^2\phi}}\left[1 - \frac{\omega^2}{gm}r^3\sin^2\phi + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{54}$$

$$\frac{\tan\phi_\gamma}{\sqrt{1+\tan^2\phi_\gamma}} = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}}\left[1 + \frac{\omega^2}{gm}r^3 + \mathcal{O}_4\right] \times \left[1 - \frac{\omega^2}{gm}r^3\sin^2\phi + \mathcal{O}_4\right] \tag{55}$$

$$\frac{\tan\phi_\gamma}{\sqrt{1+\tan^2\phi_\gamma}} = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}}\left[1 + \frac{\omega^2}{gm}r^3\cos^2\phi + \mathcal{O}\left(\frac{\omega^2}{(gm)^2}\right)\right] \tag{56}$$

$$\boxed{\begin{aligned} \sin\phi_\gamma &= \left[1 + \frac{\omega^2}{gm}r^3\cos^2\phi + \mathcal{O}_4\left(\frac{\omega^2}{(gm)^2}\right)\right]\sin\phi & (57) \\ \cos\phi_\gamma &= \left[1 - \frac{\omega^2}{gm}r^3\sin^2\phi + \mathcal{O}\left(\frac{\omega^2}{(gm)^2}\right)\right]\cos\phi & (58) \end{aligned}}$$

Table 8. Horizontal and vertical components of the disturbance gravity vector in a local spherical frame of reference and with respect to a reference potential of type $gm/r + \omega^2 r^2 \cos^2\phi/2$ (anisotropic)

$$-\boldsymbol{\delta\gamma}(\lambda,\phi,r) = \delta\gamma\left[\mathbf{e}_\phi \frac{\omega^2}{gm} r^3 \cos\phi \sin\phi + \mathbf{e}_r\right] + \gamma\,\delta\phi_\gamma \times \left[\mathbf{e}_\phi - \mathbf{e}_r \frac{\omega^2}{gm} r^3 \cos\phi \sin\phi\right] + \gamma\,\delta\lambda_\gamma \cos\phi_\gamma\,\mathbf{e}_\lambda \tag{59}$$

$$\boxed{\begin{aligned} &\gamma = \sqrt{(\gamma^x)^2 + (\gamma^y)^2 + (\gamma^z)^2} && (60)\\ &\gamma = \frac{gm}{r^2}\sqrt{1 - 2\frac{\omega^2}{gm} r^3 \cos^2\phi + \frac{\omega^4}{(gm)^2} r^6 \cos^2\phi} && (61)\\ &\gamma = \frac{gm}{r^2}\left[1 - \frac{\omega^2}{gm} r^3 \cos^2\phi + \mathcal{O}\left(\frac{\omega^4}{(gm)^2}\right)\right] && (62)\end{aligned}}$$

$$\begin{aligned} -\delta\boldsymbol{\gamma}(\lambda,\phi,r) = {} & \delta\gamma[\mathbf{e}_\phi, \mathbf{e}_r]\begin{bmatrix} \frac{\omega^2}{gm} r^3 \cos\phi \sin\phi \\ 1 \end{bmatrix} \\ & + \frac{gm}{r^2}\left[1 - \frac{\omega^2}{gm} r^3 \cos^2\phi + \mathcal{O}\left(\frac{\omega^4}{(gm)^2}\right)\right] \\ & \times \delta\phi_\gamma [\mathbf{e}_\phi, \mathbf{e}_r]\begin{bmatrix} 1 \\ -\frac{\omega^2}{gm} r^3 \cos\phi \sin\phi \end{bmatrix} \\ & + \frac{gm}{r^2}\left[1 - \frac{\omega^2}{gm} r^3 \cos^2\phi + \mathcal{O}\left(\frac{\omega^4}{(gm)^2}\right)\right] \delta\lambda_\gamma \cos\phi_\gamma \mathbf{e}_\lambda \end{aligned} \tag{63}$$

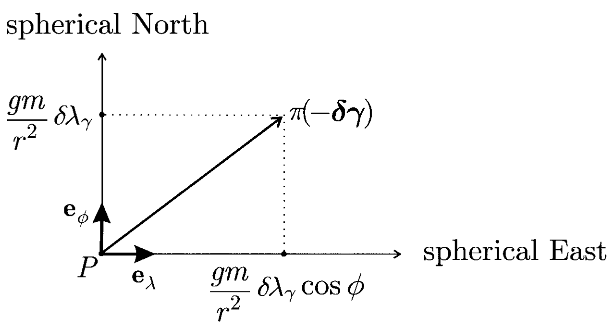

Fig. 4. Incremental gravity vector $-\boldsymbol{\delta\gamma}$ in the spherical tangent space $\mathbb{T}_P\mathbb{S}^2_r$ at point P, $\{\mathbf{e}_\lambda, \mathbf{e}_\phi \,|\, P\}$ local reference frame

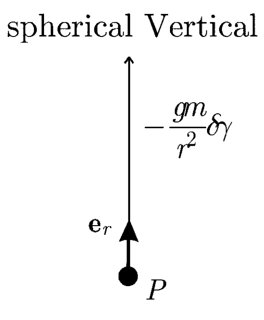

Fig. 5. Incremental gravity vector $-\boldsymbol{\delta\gamma}$ in the spherical normal space $\mathbb{N}_P\mathbb{S}^2_r$ at point P, $\{\mathbf{e}_r \,|\, P\}$ local reference frame

first term; the second term spherically represents the centrifugal potential. Such a centrifugal potential refers here to the three-axis as the Earth rotation axis with ω its modulus of rotational speed. By differential calculus, we derive from Eqs. (39) and (40) the coordinates $[\gamma^x, \gamma^y, \gamma^z]'$ of the type of Eq. (41) of the reference gravity vector. If the Cartesian coordinates $[\gamma^x, \gamma^y, \gamma^z]'$ of such a reference gravity space are converted into spherical coordinates $[\lambda_\gamma, \phi_\gamma, \gamma]'$ via Eqs. (42)–(58) of Table 7 and Eqs. (60)–(62) of Table 8, $\lambda_\gamma = \lambda, \sin\phi_\gamma, \cos\phi_\gamma$ [Eq. (58)] and $\gamma(\phi, r)$ [Eqs. (61) and (62)] are found. Accordingly, case two demonstrates nicely the anisotropy of the reference gravity space $\boldsymbol{\gamma}$! In consequence we have found the related specialized representation of the negative disturbing gravity vector $-\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ of the type of Eq. (63) of Table 8: incremental gravity $\delta\gamma$ is related to the 'small' spherical North component $\mathbf{e}_\phi \cos\phi \sin\phi r^3 w^2/(gm)$ and the spherical Vertical component $\mathbf{e}_r$. The incremental gravity latitude $\delta\phi_\gamma$ has the spherical North component $\mathbf{e}_\phi$ and the 'small' spherical Vertical component $-\mathbf{e}_r \cos\phi \sin\phi r^3 \omega^2/(gm)$, both multiplied by the modulus of reference gravity $[1 - r^3\cos^2\phi * \omega^2/(gm)^2 + \mathcal{O}(\omega^4/(gm)^2)]gm/r^2$. In contrast, due to the rotational symmetry of the reference gravity field of choice, the incremental gravity longitude $\delta\lambda_\gamma \cos\phi_\gamma$ relates only to the spherical East component $\mathbf{e}_\lambda[1 - r^3 \cos^2\phi * \omega^2/(gm)^2 + \mathcal{O}(\omega^4/(gm)^2)]$, where $[\cdot]$ represents up to order four of $\omega/\sqrt{gm}$, the modulus of reference gravity. For more details of such an anisotropic gravity field we refer to Bode and Grafarend (1981, 1982). For a more advanced reference gravity field of band-limited scalar spherical harmonic expansion type (say of degree/order 360/360), the general representation of Eq. (30) of the negative disturbing gravity vector is achieved.

A brief summary of the set-up of the horizontal and vertical components of the disturbing gravity vector follows.

Lemma 1 (incremental gravity vector). *If the gravity space and the geometry space are locally represented by means of spherical coordinates* $[\lambda_\gamma, \phi_\gamma, \gamma]'$ *and* $[\lambda, \phi, r]'$, *respectively, the general representation of horizontal and vertical components in terms of a spherical local reference frame* $\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r \,|\, P\}$ *at the point* $P \sim \mathbf{x} \sim [\lambda, \phi, r]'$ *of the incremental gravity field* $-\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ *is given by Eq.* (30) *of Table* 4.

Case one ($w = gm/r$). *If the reference gravity potential field is chosen isotropically, namely of type gm/r, the spherical horizontal as well as spherical vertical components of the increment gravity vector* $-\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ *amount to Easting* $\delta\lambda_\gamma \cos\phi * gm/r^2$, *Northing* $\delta\phi_\gamma * gm/r^2$ *and Vertical* $\delta\gamma$ *given by Eq.* (38) *of Table* 6, *illustrated by Figs.* 4 *and* 5.

Case two ($w = gm/r + (\omega^2 r^2 \cos^2\phi)/2$). *If the reference gravity potential field is chosen an isotropically* (*although three-axis rotational symmetric*), *namely of type* $(gm/r) + (\omega^2 r^2 \cos^2\phi)/2$, *the spherical horizontal as well as spherical vertical components of the incremental gravity vector* $-\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ *amount to Eq.* (63) *of Table* 8. $\delta\gamma$ *has a Northern component in addition to radial component*; $\delta\phi_\gamma$, *has a radial component in addition to a Northern one*; *and* $\delta\lambda_\gamma \cos\phi_\gamma$ *has only an Eastern component* (*due to rotational symmetry of the reference gravity field*).

3 Set-up and solution of the horizontal and vertical boundary value problem in a spherical gravity/geometry space

As a tool we present first the representation of the incremental gravity vector in terms of vector spherical harmonics. Second, we are going to formulate and solve the spherical horizontal boundary value problem: by means of a particular Green operator we map vertical deflections given on a sphere of reference to the disturbing potential in the space external to the sphere of reference. Third, we aim at the formulation and solution of the special vertical boundary value problem: again we succeed to construct a particular Green operator which maps gravity disturbances given on sphere of reference $\mathbb{S}^2_R$ to the disturbing potential in the space external to the sphere of reference $\mathbb{S}^2_R$. Both Green operators are analysed or modified with respect to a fixed spherical harmonics coefficient of very low degree and order. Special attention is paid to the problem of permuting summation, integration and differentiation of infinite series.

3.1 Representation of the incremental gravity vector in terms of vector spherical harmonics

Due to the harmonicity of the incremental gravity potential outside the sphere $\mathbb{S}^2_R$ of radius R, we are able to represent the incremental gravitational potential in terms of external spherical harmonics of the type of Eq. (64) with $(R/r)^{\ell+1}$ and $\mathbf{e}^{\ell m}(\lambda,\phi)$ of the type of Eq. (65) as orthonormal base functions. Indeed, $P_{\ell m}(\sin\phi)$ refer to Ferrer's functions associated with the Legendre functions of degree ℓ and order m. (Sansone 1959, p. 246). The gradient of the incremental gravitational potential as represented in spherical coordinates $\{\lambda,\phi,r\}$ by means of Eq. (66) is decomposed into two parts. In the first or surface normal part the radial/normal derivative is collected in terms of the vector-valued radial base function $\mathbf{R}^{\ell m}$ of the type of Eq. (68). In contrast, the spherical surface gradient produces the second or surface tangential part with respect to the vector-valued spheroidal base function $\mathbf{S}^{\ell m}$ of the type of Eq. (69). The superposition of the radial and spheroidal constituents leads to the representation of the incremental gravitational vector field of Eq. (67). As vector spherical harmonic, the vector-valued base functions $\{\mathbf{R}^{\ell m},\mathbf{S}^{\ell m}\}$ are orthonormal in the sense of Eqs. (70)–(73). The longitudinal as well as lateral derivatives of the base functions $\mathbf{e}^{\ell m}$ follow the rules of Eqs. (74), (75), (76) and (77). In toto, our results on the representation of the incremental gravity vector field $\boldsymbol{\delta\gamma}(\mathbf{x})$ in terms of vector spherical harmonics are collected in Table 9.

In contrast, in Table 10 we have collected detailed representations of the incremental gravity vector decomposed into vertical deflections $\{\eta,\xi\}$ and incremental gravity $\delta\gamma$ and developed into series of vector spherical harmonics. By means of the definitions $\{\zeta,\eta,\xi\}$ as radial gravity increment [Eq. (78)], and vertical deflections [Eqs. (79) and (80)] with respect to a spherical reference gravity field, the spherical incremental gravity field $\boldsymbol{\delta\gamma}$ enjoys the additive decompositions of Eq. (81) subject to the horizontal and vertical operators of Eqs. (82)–(84). As soon as we implement the incremental potential field $\delta w(\lambda,\phi,r)$ in the series expansion of external spherical harmonic type, we are led to the series expansions of $\{\zeta(\lambda,\phi,r),\eta(\lambda,\phi,r),\xi(\lambda,\phi,r)\}$ of the type of Eqs. (85)–(87). They constitute the tangential/horizontal incremental gravity field of Eq. (88) as well as the normal/vertical incremental gravity field of Eq. (89)

Table 9. Representation of the incremental gravity vector field $\boldsymbol{\delta\gamma}(\mathbf{x})$ in terms of vector spherical harmonics (spherical reference gravity space, spherical reference geometry space)

$$\delta w(\lambda,\phi,r)=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)\delta w_{\ell m} \tag{64}$$

$$\begin{aligned}\mathbf{e}^{\ell m}(\lambda,\phi)&=\begin{bmatrix}P^*_{\ell m}(\sin\phi)\cos m\lambda & \text{for } m>0\\ P^*_{\ell 0}(\sin\phi) & \text{for } m=0\\ P^*_{\ell m}(\sin\phi)\sin|m|\lambda & \text{for } m<0\end{bmatrix}\\ &=\begin{bmatrix}\sqrt{2(2\ell+1)}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell m}(\sin\phi)\cos m\lambda & \text{for } m>0\\ \sqrt{2\ell+1}P_{\ell 0}(\sin\phi) & \text{for } m=0\\ \sqrt{2(2\ell+1)}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_{\ell|m|}(\sin\phi)\sin|m|\lambda & \text{for } m<0\end{bmatrix}\end{aligned} \tag{65}$$

$$\boxed{\boldsymbol{\delta\gamma}=\operatorname{grad}\delta w=\mathbf{e}_\lambda\frac{1}{r\cos\phi}\mathrm{D}_\lambda\delta w+\mathbf{e}_\phi\frac{1}{r}\mathrm{D}_\phi\delta w+\mathbf{e}_r\mathrm{D}_r\delta w} \tag{66}$$

$$\begin{aligned}\boldsymbol{\delta\gamma}=\operatorname{grad}\delta w=\frac{1}{R}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+2}\\ *\left\{-(\ell+1)\mathbf{R}^{\ell m}+\sqrt{\ell(\ell+1)}\mathbf{S}^{\ell m}\right\}\delta w_{\ell m}\end{aligned} \tag{67}$$

radial vector spherical harmonics

$$\mathbf{R}^{\ell m}(\lambda,\phi):=\mathbf{e}_r\mathbf{e}^{\ell m}(\lambda,\phi) \tag{68}$$

spheroidal vector spherical harmonics

$$\begin{aligned}\mathbf{S}^{\ell m}(\lambda,\phi)&:=\left(\mathbf{e}_\lambda\frac{1}{\cos\phi}\mathrm{D}_\lambda+\mathbf{e}_\phi\mathrm{D}_\phi\right)\frac{\mathbf{e}^{\ell m}(\lambda,\phi)}{\sqrt{\ell(\ell+1)}}\\ &=\frac{r}{\sqrt{\ell(\ell+1)}}\operatorname{grad}e^{\ell m}(\lambda,\phi)\end{aligned} \tag{69}$$

orthonormality

$$\begin{aligned}\left\langle\mathbf{e}^{\ell_1 m_1}|\mathbf{e}^{\ell_2 m_2}\right\rangle&:=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\lambda\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi\cos\phi\, e^{\ell_1 m_1}(\lambda,\phi)e^{\ell_2 m_2}(\lambda,\phi)\\ &=\delta^{\ell_1\ell_2}\delta^{m_1 m_2}\end{aligned} \tag{70}$$

$$\left\langle\mathbf{R}^{\ell_1 m_1}|\mathbf{R}^{\ell_2 m_2}\right\rangle=\delta^{\ell_1\ell_2}\delta^{m_1 m_2} \tag{71}$$

$$\left\langle\mathbf{R}^{\ell_1 m_1}|\mathbf{S}^{\ell_2 m_2}\right\rangle=0 \tag{72}$$

$$\left\langle\mathbf{S}^{\ell_1 m_1}|\mathbf{S}^{\ell_2 m_2}\right\rangle=\delta^{\ell_1\ell_2}\delta^{m_1 m_2} \tag{73}$$

$$\mathrm{D}_\lambda\mathbf{e}^{\ell m}(\lambda,\phi)=\begin{bmatrix}-mP^*_{\ell m}(\sin\phi)\sin m\lambda & \text{for } m>0\\ 0 & \text{for } m=0\\ |m|P^*_{\ell m}(\sin\phi)\cos|m|\lambda & \text{for } m<0\end{bmatrix} \tag{74}$$

$$\mathrm{D}_\phi\mathbf{e}^{\ell m}(\lambda,\phi)=\begin{bmatrix}\frac{\partial}{\partial\phi}P^*_{\ell m}(\sin\phi)\cos m\lambda & \text{for } m>0\\ \frac{\partial}{\partial\phi}P^*_{\ell m}(\sin\phi) & \text{for } m=0\\ \frac{\partial}{\partial\phi}P^*_{\ell|m|}(\sin\phi)\sin|m|\lambda & \text{for } m<0\end{bmatrix} \tag{75}$$

$$\mathrm{D}_\phi P^*_{\ell m}=\mathrm{D}_\phi P_{\ell|m|}\begin{bmatrix}\sqrt{2(2\ell+1)}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}} & \text{for } m>0\\ \sqrt{2\ell+1} & \text{for } m=0\\ \sqrt{2(2\ell+1)}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}} & \text{for } m<0\end{bmatrix} \tag{76}$$

$$\mathrm{D}_\phi P_{\ell m}=P_{\ell,m+1}-m\tan\phi\, P_{\ell m} \tag{77}$$

Table 10. Representation of the incremental gravity vector field $\boldsymbol{\delta\gamma}(\mathbf{x})$ in horizontal and vertical components (spherical reference gravity space, spherical reference geometry space)

$$\boldsymbol{\delta\gamma}(\lambda,\phi,r) = \mathbf{e}_\lambda \frac{gm}{r^2}\delta\lambda_\gamma \cos\phi_\gamma - \mathbf{e}_\phi \frac{gm}{r^2}\delta\phi_\gamma - \mathbf{e}_r \delta\gamma \tag{38}$$

$$\zeta(\lambda,\phi,r) := \frac{1}{gm/r^2}\delta\gamma(\lambda,\phi,r) \tag{78}$$

$$\eta(\lambda,\phi,r) := \delta\lambda_\gamma \cos\phi_\gamma(\lambda,\phi,r) \tag{79}$$

$$\xi(\lambda,\phi,r) := \delta\phi_\gamma(\lambda,\phi,r) \tag{80}$$

$$\boxed{\boldsymbol{\delta\gamma}(\lambda,\phi,r) = -\tfrac{gm}{r^2} * \{\mathbf{e}_\lambda \eta + \mathbf{e}_\phi \xi + \mathbf{e}_r \zeta\}} \tag{81}$$

$$\zeta(\lambda,\phi,r) = -\frac{1}{gm/r^2}\mathrm{D}_r \delta w \tag{82}$$

$$\eta(\lambda,\phi,r) = -\frac{1}{gm/r^2}\frac{1}{r\cos\phi}\mathrm{D}_\lambda \delta w \tag{83}$$

$$\xi(\lambda,\phi,r) = -\frac{1}{gm/r^2}\frac{1}{r}\mathrm{D}_\phi \delta w \tag{84}$$

$$\zeta(\lambda,\phi,r) = -\frac{1}{gm/r^2}\frac{1}{R}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}(\ell+1)\left(\frac{R}{r}\right)^{\ell+2} * \mathbf{e}^{\ell m}(\lambda,\phi)\delta w_{\ell m} \tag{85}$$

$$\eta = -\frac{1}{gm/r^2}\frac{1}{r\cos\phi}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1} * \begin{bmatrix} -mP^*_{\ell m}(\sin\phi)\sin m\lambda\,\delta w_{\ell m} & \text{for } m>0 \\ 0 & \text{for } m=0 \\ |m|P^*_{\ell m}(\sin\phi)\cos|m|\lambda\,\delta w_{\ell m} & \text{for } m<0 \end{bmatrix} \tag{86}$$

$$\xi = -\frac{1}{gm/r^2}\frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1} * \mathrm{D}_\phi P^*_{\ell m}(\sin\phi)\begin{bmatrix} \cos m\lambda\;\delta w_{\ell m} & \text{for } m>0 \\ \delta w_{\ell m} & \text{for } m=0 \\ \sin|m|\lambda\;\delta w_{\ell m} & \text{for } m<0 \end{bmatrix} \tag{87}$$

$$\mathbf{t}(\lambda,\phi,r) \in \mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_r$$

$$\mathbf{t}(\lambda,\phi,r) := \mathbf{e}_\lambda \eta(\lambda,\phi,r) + \mathbf{e}_\phi \xi(\lambda,\phi,r) \tag{88}$$

$$\mathbf{n}(\lambda,\phi,r) \in \mathbb{N}_{(\lambda,\phi)}\mathbb{S}^2_r$$

$$\mathbf{n}(\lambda,\phi,r) := \mathbf{e}_r \zeta(\lambda,\phi,r) \tag{89}$$

$$\boldsymbol{\delta\gamma}(\lambda,\phi,r) = -\frac{gm}{r^2} * \{\mathbf{t}(\lambda,\phi,r) + \mathbf{n}(\lambda,\phi,r)\} \tag{90}$$

$$\mathbf{t}(\lambda,\phi,r) = -\frac{1}{\frac{gm}{r^2}}\frac{1}{R}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\sqrt{\ell(\ell+1)}\mathbf{S}^{\ell m}(\lambda,\phi)\delta w_{\ell m} \tag{91}$$

$$\mathbf{n}(\lambda,\phi,r) = +\frac{1}{\frac{gm}{r^2}}\frac{1}{R}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+2}(\ell+1)\mathbf{R}^{\ell m}(\lambda,\phi)\delta w_{\ell m} \tag{92}$$

$$\delta\gamma(\lambda,\phi,r) = \frac{1}{R}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+2} * (\ell+1)\mathbf{R}^{\ell m}(\lambda,\phi)\delta w_{\ell m} \tag{93}$$

$$\boxed{\boldsymbol{\delta\gamma}(\lambda,\phi,r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\{\mathbf{R}^{\ell m}(\lambda,\phi)r_{\ell m}(r) + \mathbf{S}^{\ell m}(\lambda,\phi)s_{\ell m}(r)\}} \tag{94}$$

subject to

$$r_{\ell m}(r) = -(\ell+1)\frac{1}{\frac{gm}{r^2}}\frac{1}{R}\left(\frac{R}{r}\right)^{\ell+2} \tag{95}$$

$$s_{\ell m}(r) = \sqrt{\ell(\ell+1)}\frac{1}{\frac{gm}{r^2}}\frac{1}{R}\left(\frac{R}{r}\right)^{\ell+2} \tag{96}$$

$$r_{\ell m}(R) = -(\ell+1)\frac{1}{\frac{gm}{R^2}}\frac{1}{R} \tag{97}$$

$$s_{\ell m}(R) = \sqrt{\ell(\ell+1)}\frac{1}{\frac{gm}{R^2}}\frac{1}{R} \tag{98}$$

$$\boldsymbol{\delta\gamma}(\lambda,\phi,r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\mathbf{R}^{\ell m}r_{\ell m}(R)\left(\frac{R}{r}\right)^{\ell+2} + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\mathbf{S}^{\ell m}s_{\ell m}(R)\left(\frac{R}{r}\right)^{\ell+2} \tag{99}$$

superposed to $\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ of the type of Eq. (90). As soon as we substitute the first-order derivatives into the tangential–normal field components, we are led to the famous representations of Eqs. (91) and (92) of the $\{\mathbf{t}(\lambda,\phi,r), \mathbf{n}(\lambda,\phi,r)\}$ field components in external vector spherical harmonics. Let us emphasize the eigenvalue structure $\sqrt{\ell(\ell+1)}$ and $(\ell+1)$, respectively, of $\{\delta\gamma,\eta,\xi\}$ within Eqs. (91)–(93). If we think of a series expansion of the incremental gravity vector field in terms of external vector spherical harmonics of the type of Eq. (94) we produce the radial coefficients $r_{\ell m}$ of the type of Eq. (95) and the spheroidal coefficients $s_{\ell m}$ of the type of Eq. (96). Indeed, the radial coefficients depend on the spectrum $(\ell+1)$ and the radial decrease $(R/r)^{\ell+2}$, normalized by the spherical gravitational factor gm/r^2. However, the spherical coefficients turn out to depend on the spectrum $\sqrt{\ell(\ell+1)}$ and the radial decrease $(R/r)^{\ell+2}$, normalized by the spherical gravitational factor gm/r^2. If we express the incremental gravitational field $\boldsymbol{\delta\gamma}(\lambda,\phi,r)$ in terms of the coefficients $r_{\ell m}(R)$ and $s_{\ell m}(R)$ gauged to the sphere $\mathbb{S}^2_R$, namely Eqs. (97) and (98), we are led to the $\mathbb{S}^2_R$-boundary representation [Eq. (99)] of $\boldsymbol{\delta\gamma}(\lambda,\phi,r)$.

3.2 The spherical horizontal boundary value problem

Next we are going to formulate the spherical horizontal boundary value problem by means of Table 11 and Definition 1. First, as a boundary operator we assume that the horizontal/tangential field $L_t\delta w(\lambda,\phi,r)$ of Eqs. (100) and (101) is given on the sphere $\mathbb{S}^2_R$ of radius R. Second, the incremental gravitational potential $\delta w(\lambda,\phi,r)$ is assumed to fulfil as a region operator the Laplace–Beltrami equation [Eq. (102)] with $2\omega^2$ as the constant inhomogeneity which can be reduced to zero. Since all field quantities are formulated in a quasi-Earth fixed frame (ITRF) we have built in the centrifugal potential δw_{inh} of the type of Eq. (103) which fulfils the inhomogeneous Laplace–Beltrami equation. In addition, we have restricted the harmonic domain to coincide with the region outside the sphere $\mathbb{S}^2_R$ of R. Accordingly we have to assume that all topographic masses have been removed from the gravitational potential, for instance by means of the terrain effect. Third, by means of a 'radiation condition' [Eq. (105)] we postulate that the incremental 'homogeneous' gravitational potential approaches zero at infinity, $r \to \infty$.

In geodetic terms, the horizontal boundary operator of Eq. (106) expresses the vertical deflection vector to be given in the tangent space $\mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_R$ at (λ,ϕ) of the sphere $\mathbb{S}^2_R$ of radius R. In addition, we have to define the notion of the incremental gravitational potential δw with respect to a reference gravitational potential w of type gm/r. Finally, the solution of Eq. (108) of the spherical horizontal boundary value problem (given the surface gradient of the incremental gravitational potential) is represented as a linear functional with respect to the Green functions $G_\lambda(P,P^*)$ and $G_\phi(P,P^*)$, where P is the symbol for the evaluation point and P^* the symbol for moving point, also called data point or source point.

Table 11. Spherical horizontal boundary value problem

(1) Boundary operator (spherical surface gradient)

$$L_t \delta w(\lambda, \phi, r) = \mathbf{t}(\lambda, \phi, r) \quad \text{for } r = R, (\lambda, \phi) \in \mathbb{S}^2_R \tag{100}$$

$$L_t := \mathbf{e}_\lambda \frac{1}{r \cos\phi} \mathrm{D}_\lambda + \mathbf{e}_\phi \frac{1}{r} \mathrm{D}_\phi \tag{101}$$

(2) Region operator (spherical Laplace operator)

$$L \delta w(\lambda, \phi, r) = 2\omega^2 \quad \text{for } (\lambda, \phi, r) \in G_e = \mathbb{R}^3 \backslash G_i \tag{102}$$

(i) Special solution of the inhomogeneous differential equation

$$\delta w_{inh} = \frac{1}{2} \omega^2 r^2 \cos^2\phi \tag{103}$$

(ii) General solution of the homogeneous differential equation

$$L \delta w_{\mathrm{hom}}(\lambda, \phi, r) = 0 \tag{104}$$

(3) Radiation condition

$$\lim_{r \to \infty} \delta w_{\mathrm{hom}}(\lambda, \phi, r) = 0 \tag{105}$$

Definition 1 (*horizontal boundary value problem*)

Given the vertical deflection vector (the spherical surface gradient)

$$\begin{aligned} \mathbf{t}(\lambda, \phi, r = R) &:= \mathbf{e}_\lambda \delta\lambda_\gamma \cos\phi + \mathbf{e}_\phi \delta\phi_\gamma \\ &= \mathbf{e}_\lambda \eta + \mathbf{e}_\phi \xi = \mathbf{e}_\lambda \frac{1}{R \cos\phi} \mathrm{D}_\lambda \delta w + \mathbf{e}_\phi \frac{1}{R} \mathrm{D}_\phi \delta w \\ &= \mathbf{e}_\lambda L_\lambda \delta w + \mathbf{e}_\phi L_\phi \delta w = L_t \delta w \end{aligned} \tag{106}$$

on the sphere $\mathbb{S}^2_R$ of radius R with respect to the disturbing potential δw (incremental gravitational potential), namely $W = w + \delta w$, subject to $w = gm/r$, $\mathrm{D}_r w = -gm/r^2$, which is a harmonic function up to the rotational term, namely

$$\mathrm{Lap}\ \delta w(\lambda, \phi, r) = 2\omega^2 \tag{107}$$

in the space outside the sphere. The orthonormal base vectors $\{\mathbf{e}_\lambda, \mathbf{e}_\phi | P\}$ span the tangent space $\mathbb{T}_{\lambda,\phi}\mathbb{S}^2_R$, the spherical local horizontal plane at the point $P \sim (\lambda, \phi, R)$. Find the disturbing potential, namely the Green functions $G_\lambda(P, P^*)$ and $G_\phi(P, P^*)$, represented by

$$\begin{aligned} \delta w(\lambda, \phi, r) &= \int_{\mathbb{S}^2_R} \mathrm{d}\omega^*_R G_\lambda(\lambda, \phi, r | \lambda^*, \phi^*, R)\ \eta(\lambda^*, \phi^*, R) \\ &+ \int_{\mathbb{S}^2_R} \mathrm{d}\omega^*_R G_\phi(\lambda, \phi, r | \lambda^*, \phi^*, R)\ \xi(\lambda^*, \phi^*, R) \\ &+ \frac{1}{2} \omega^2 r^2 \cos^2\phi \end{aligned} \tag{108}$$

The integration of the linear functional $\delta w[\eta, \xi]$ is extended over the surface ω_R of $\mathbb{S}^2_R$.

As outlined by Table 12, we now aim to solve the spherical horizontal boundary value problem in the spectral domain. The 'ansatz' of the type of Eq. (109) introduces the spheroidal vector harmonics as those functions for the horizontal field. Equation (110) defines the spheroidal coefficients $s_{\ell m}$ as generated by means of scalar product $\langle \mathbf{t} \,|\, \mathbf{S}_{\ell m} \rangle$, integrated over the unit sphere $\mathbf{S}^2$ with respect to the spheroidal base $\mathbf{S}_{\ell m}$. Applying the surface gradient to the incremental potential δw represented in the spectral domain by means of spherical harmonics, we are led to the tangential/horizontal field/vertical deflection vector field of the type of Eq. (112). In particular, by means of Eqs. (113)–(115) the spheroidal coefficients $s_{\ell m}$ are related to the potential coefficients $\delta w_{\ell m}$ with the elements $\sqrt{\ell(\ell+1)}$ of the eigenspace. A commutative diagram illustrates the transformation $s_{\ell m} \mapsto \delta w_{\ell m}$ in the spectral domain as well as $\delta w_{lm} \mapsto \delta w$ from the spectral domain to incremental potential space. Finally, Eq. (117) relates the incremental potential $\delta w(\lambda, \phi, r)$ to the spectral coefficients $s_{\ell m}$ of the vertical deflection vector divided by the spectrum $\sqrt{\ell(\ell+1)}$ and a decay rate $(R/r)^{\ell+1}$. Figure 6 illustrates the related spectrum $\lambda(\ell) \sim 1/\sqrt{\ell(\ell+1)}$, for all $\ell \in \{0, 1, \ldots, \infty\}$.

Is it possible to represent the solution of Eq. (112) in terms of a linear functional as an integral over the data points attached with the vertical deflection vector and a Green function in a closed form? Such a representation is achieved as outlined in Table 13!

The procedure for solving the spherical horizontal boundary value problem in a closed form by means of a

Table 12. Solution of the spherical horizontal boundary value problem in the spectral domain

'Ansatz'

$$\mathbf{t}(\lambda, \phi, R) = \mathbf{e}_\lambda \eta(\lambda, \phi, R) + \mathbf{e}_\phi \xi(\lambda, \phi, R) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \mathbf{S}^{\ell m}(\lambda, \phi) s_{\ell m} \tag{109}$$

$$s_{\ell m} := \frac{1}{4\pi} \int_{\mathbb{S}} \mathrm{d}\omega \langle \mathbf{t} | \mathbf{S}_{\ell m} \rangle \tag{110}$$

$$\mathrm{d}\omega := \mathrm{d}\lambda\ \mathrm{d}\phi \cos\phi \quad 0 \le \lambda \le 2\pi,\ -\pi/2 < \phi < +\pi/2 \tag{111}$$

$\boxed{\text{comparison of } \mathbf{t}(\lambda, \phi, r) \text{ (Table 10) and } \mathbf{t}(\lambda, \phi, r) \text{ (Table 12)}}$

$$\begin{aligned} \mathbf{t}(\lambda, \phi, R) &= \frac{1}{\frac{gm}{R^2}} * \frac{1}{R} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sqrt{\ell(\ell+1)} \mathbf{S}^{\ell m}(\lambda, \phi) \delta w_{\ell m} \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \mathbf{S}^{\ell m}(\lambda, \phi) s_{\ell m} \end{aligned} \tag{112}$$

$$\Leftrightarrow$$

$$\boxed{s_{\ell m} = \frac{1}{\frac{gm}{R^2}} * \frac{1}{R} \sqrt{\ell(\ell+1)} \delta w_{\ell m}} \tag{113}$$

$$\Leftrightarrow$$

$$\boxed{\delta w_{\ell m} = -\frac{gm}{R^2} \frac{R}{\sqrt{\ell(\ell+1)}} s_{\ell m}} \tag{114}$$

Eigenspace

$$\delta w_{\ell m} = \lambda_{\ell m} s_{\ell m} \tag{115}$$

subject to

$$\boxed{\lambda_{\ell m} = -\frac{gm}{R^2} \frac{R}{\sqrt{\ell(\ell+1)}}} \tag{116}$$

$$\begin{tikzcd} s_{\ell m} \arrow[r] \arrow[dr] & \delta w_{\ell m} \arrow[d] \\ & \delta w \end{tikzcd}$$

$$\boxed{\begin{aligned} \delta w(\lambda, \phi, r) &= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{+\ell} \left(\frac{R}{r}\right)^{\ell+1} \mathbf{e}^{\ell m}(\lambda, \phi) \delta w_{\ell m} \\ &= -\frac{gm}{R^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{R}{\sqrt{\ell(\ell+1)}} \left(\frac{R}{r}\right)^{\ell+1} \mathbf{e}^{\ell m} s_{\ell m} \end{aligned}} \tag{117}$$

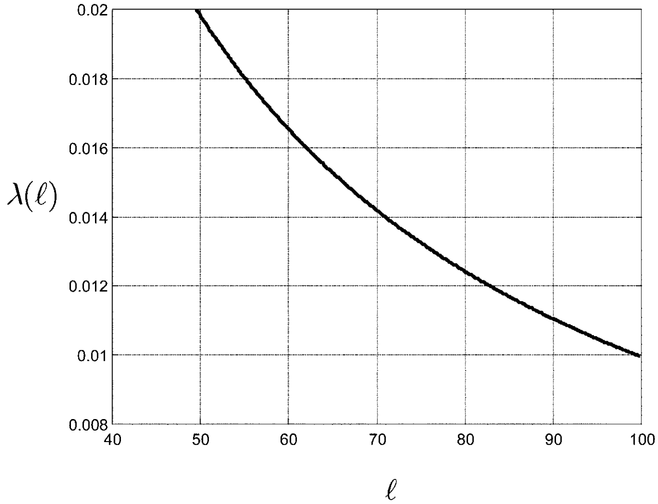

Fig. 6. $\lambda(\ell)$, spherical horizontal boundary value problem $\frac{GM}{R}+\frac{1}{3}\Omega^2 R^2 = W_0$; $GM = 3.986004415 \times 10^{14}\ \mathrm{km}^3\,\mathrm{s}^{-2}, \Omega = 7.292\,115\ \mathrm{rad\,s}^{-1}$, $W_0 = 60\,636\,855.8\ \mathrm{m}^2\,\mathrm{s}^{-2}, R = 6\,370\,991.248\ \mathrm{m}$, eigenspace $\lambda(\ell) \sim 1/\sqrt{\ell(\ell+1)},\ \ell \in \{50,\ldots,100\}$

vector-valued Green function is lengthy. It is for this reason that we have split the derivation into various steps reviewed in Figure 7. Step one by means of Eq. (118) expresses the spherical harmonic coefficients $\delta w_{\ell m}$, the coordinates of the Hilbert space of spherical harmonics, by their reproducing property (scalar product) in terms of the spherical surface integral over the incremental gravitational potential and the spherical base functions $\mathbf{e}^{\ell m}(\lambda,\phi)$. In turn, in step two we relate the incremental gravitational potential $\delta w(\lambda,\phi,r)$ in Eq. (119) in the spectral domain to the spherical harmonic coefficients $\delta w_{\ell m}$ and, thanks to Eq. (114), to the spheroidal harmonic coefficients $s_{\ell m}$ [Eq. (112)] of the vertical deflection vector. Step three uses the substitution of the scalar product of Eq. (110) into the functional relation $\delta w(s_{\ell m})$ in terms of Eq. (120). In step four we replace the tangential/horizontal field $\mathbf{t}(\lambda^*,\phi^*)$, the vertical deflection field, at the moving point $P^* \in \mathbb{S}^2_R$ in Eq. (120) in order to generate the lengthy integral relation of Eq. (121). Next, for Eq. (122), we apply in step five the permutation rule for infinite summation and integration: if the series expansion is uniformly convergent we are permitted to interchange summation and integration. Indeed, in the admissible space outside $\mathbb{S}^2_R$, the series of Eq. (121) are uniformly convergent. Step six involves an application of the additive theorem of spherical harmonics, namely Eq. (123), where $\cos\psi$ as the cosine of oblique colatitude of the type of Eq. (124) is represented through Eqs. (124)–(129) in terms of the spherical coordinates of the points P and P^*, respectively. Step seven aims at a longitudinal and lateral derivative [Eqs. (130), (131)] of the addition theorem of spherical harmonics. By chain-rule differentiation we gain Eqs. (132), (133) and (134)–(137). Such a procedure leads finally to the longitudinal kernel [Eqs. (138), (139)] as well as to the lateral kernel [Eqs. (140), (141)] in step eight. Step nine originates from Eqs. (122) where all longitudinal/lateral kernels are used to produce the

Table 13. Solution of the spherical horizontal boundary value problem

spectral representation of the Green function

$\boxed{\text{reproducing property}}$

$$\delta w_{\ell m} := \frac{1}{4\pi R^2}\int\limits_{\mathbb{S}^2_R} \mathrm{d}\omega_R \mathbf{e}^{\ell m}(\lambda,\phi)\ \delta w(\lambda,\phi,R) \tag{118}$$

$$\begin{aligned}\delta w(\lambda,\phi,r) &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)\delta w_{\ell m}\\ &= -\frac{gm}{R^2}\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R^{\ell+2}}{r^{\ell+1}}\frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{e}^{\ell m}(\lambda,\phi)s_{\ell m}\end{aligned} \tag{119}$$

$$\begin{aligned}&\delta w(\lambda,\phi,r)\\ &\quad= -\frac{gm}{R^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R^{\ell+2}}{r^{\ell+1}}\frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{e}^{\ell m}(\lambda,\phi)\frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\mathrm{d}\omega^*\langle \mathbf{S}_{\ell m}|\mathbf{t}^*\rangle\end{aligned} \tag{120}$$

$$\begin{aligned}&\delta w(\lambda,\phi,r)\\ &\quad= -\frac{gm}{R^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R}{4\pi}\int\limits_{\mathbb{S}^2}\mathrm{d}\omega^*\left(\frac{R}{r}\right)^{\ell+1}\frac{1}{\sqrt{\ell(\ell+1)}}\mathbf{e}^{\ell m}(\lambda,\phi)\frac{1}{\sqrt{\ell(\ell+1)}}\\ &\quad\times\Big\langle \mathbf{e}_\lambda\frac{1}{\cos\phi^*}\mathrm{D}_{\lambda^*}e^{\ell m}(\lambda^*,\phi^*) + \mathbf{e}_{\phi^*}\mathrm{D}_{\phi^*}e^{\ell m}(\lambda,\phi)\,|\,\mathbf{e}_{\lambda^*}\eta(\lambda^*,\phi^*,R)\\ &\qquad + \mathbf{e}_\phi\xi(\lambda^*,\phi^*,R)\Big\rangle\end{aligned} \tag{121}$$

$$`\sum\int = \int\sum'$$

$$\begin{aligned}\delta w(\lambda,\phi,r) &= -\frac{gm}{R^2}\frac{R}{4\pi}\int\limits_{\mathbb{S}^2}\mathrm{d}\omega^*\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+m}\left(\frac{R}{r}\right)^{\ell+1}\frac{1}{\ell(\ell+1)}\\ &\quad * \{\mathbf{e}^{\ell m}(\lambda,\phi)\frac{1}{\cos\phi^*}\mathrm{D}_{\lambda^*}\mathbf{e}^{\ell m}(\lambda^*,\phi^*)\eta(\lambda^*,\phi^*,R)\\ &\quad + \mathbf{e}^{\ell m}(\lambda,\phi)\mathrm{D}_{\phi^*}\mathbf{e}^{\ell m}(\lambda^*,\phi^*)\xi(\lambda^*,\phi^*,R)\}\end{aligned} \tag{122}$$

addition theorem of spherical harmonics

$$(2\ell+1)P_\ell(\cos\psi) = \sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) \tag{123}$$

$$\cos\psi := \frac{\langle \mathbf{x}\,|\,\mathbf{x}^*\rangle}{\|\mathbf{x}\|_2\|\mathbf{x}^*\|_2} \tag{124}$$

$$\mathbf{x} = \mathbf{e}_1 r\cos\phi\cos\lambda + \mathbf{e}_2 r\cos\phi\sin\lambda + \mathbf{e}_3 r\sin\phi \tag{125}$$

$$\mathbf{x}^* = \mathbf{e}_1 r^*\cos\phi^*\cos\lambda^* + \mathbf{e}_2 r^*\cos\phi^*\sin\lambda^* + \mathbf{e}_3 r^*\sin\phi^* \tag{126}$$

$$\|\mathbf{x}\|_2 = r,\quad r^* = \|\mathbf{x}^*\|_2$$

$$\begin{aligned}\cos\psi &= \cos\phi\cos\phi^*\cos\lambda\cos\lambda^* + \cos\phi\cos\phi^*\sin\lambda\sin\lambda^*\\ &\quad + \sin\phi\sin\phi^*\end{aligned} \tag{127}$$

$$\cos\psi = \cos\phi\cos\phi^*(\cos\lambda\cos\lambda^* + \sin\lambda\sin\lambda^*) + \sin\phi\sin\phi^* \tag{128}$$

$$\cos\psi = \cos\phi\cos\phi^*\cos(\lambda-\lambda^*) + \sin\phi\sin\phi^* \tag{129}$$

$$\mathrm{D}_{\lambda^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) = (2\ell+1)\mathrm{D}_{\lambda^*}P_\ell(\cos\psi) \tag{130}$$

$$\mathrm{D}_{\phi^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) = (2\ell+1)\mathrm{D}_{\phi^*}P_\ell(\cos\psi) \tag{131}$$

$$\mathrm{D}_{\lambda^*}P_\ell(\cos\psi) = \mathrm{D}_\psi P_\ell(\cos\psi)\mathrm{D}_{\lambda^*}\psi \tag{132}$$

$$\mathrm{D}_{\phi^*}P_\ell(\cos\psi) = \mathrm{D}_\psi P_\ell(\cos\psi)\mathrm{D}_{\phi^*}\psi \tag{133}$$

$$\mathrm{D}_{\lambda^*}P_\ell(\cos\psi) = \mathrm{D}_\psi P_\ell(\cos\psi)\left(-\frac{1}{\sin\psi}\right)\mathrm{D}_{\lambda^*}\cos\psi \tag{134}$$

Table 13. Contd.

$$\mathrm{D}_{\phi^*}P_\ell(\cos\psi) = \mathrm{D}_\psi P_\ell(\cos\psi)\left(-\frac{1}{\sin\psi}\right)\mathrm{D}_{\phi^*}\cos\psi \tag{135}$$

$$\begin{aligned}\mathrm{D}_{\lambda^*}\cos\psi &= +\cos\phi\cos\phi^*(-\cos\lambda\sin\lambda^* + \sin\lambda\cos\lambda^*)\\ &= \cos\phi\cos\phi^*\sin(\lambda-\lambda^*)\end{aligned} \tag{136}$$

$$\mathrm{D}_{\phi^*}\cos\psi = -\cos\phi\sin\phi^*\cos(\lambda-\lambda^*) + \sin\phi\cos\phi^* \tag{137}$$

$$\boxed{\text{longitudinal kernel}}$$

$$\begin{aligned}&\mathrm{D}_{\lambda^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*)\\ &= (2\ell+1)\mathrm{D}_{\lambda^*}P_\ell(\cos\psi) = (2\ell+1)D_\psi P_\ell(\cos\psi)\mathrm{D}_{\lambda^*}\psi\\ &= -(2\ell+1)\frac{1}{\sin\psi}\mathrm{D}_\psi P_\ell(\cos\psi)\mathrm{D}_{\lambda^*}\cos\psi\\ &= -(2\ell+1)\frac{1}{\sin\psi}\;\mathrm{D}_\psi P_\ell(\cos\psi)\cos\phi\cos\phi^*\sin(\lambda-\lambda^*)\end{aligned} \tag{138}$$

$$\boxed{\frac{1}{\cos\phi^*}\mathrm{D}_{\lambda^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) = +(2\ell+1)\sin\alpha^*\mathrm{D}_\psi P_\ell(\cos\psi)} \tag{139}$$

$$\boxed{\text{lateral kernel}}$$

$$\begin{aligned}&\mathrm{D}_{\phi^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*)\\ &= (2\ell+1)\mathrm{D}_{\phi^*}P_\ell(\cos\psi) = (2\ell+1)\mathrm{D}_\psi P_\ell(\cos\psi)\mathrm{D}_{\phi^*}\psi\\ &= -(2\ell+1)\frac{1}{\sin\psi}\mathrm{D}_\psi P_\ell(\cos\psi)\mathrm{D}_{\phi^*}\cos\psi\\ &= -(2\ell+1)\frac{1}{\sin\psi}\mathrm{D}_\psi P_\ell(\cos\psi)\\ &\quad * [-\cos\phi\sin\phi^*\cos(\lambda-\lambda^*) + \sin\phi\cos\phi^*]\end{aligned} \tag{140}$$

$$\mathrm{D}_{\phi^*}\sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) = +(2\ell+1)\cos\alpha^*\mathrm{D}_\psi P_\ell(\cos\psi) \tag{141}$$

$$\boxed{\text{Green function}}$$

$$\begin{aligned}\delta w(\lambda,\phi,r) = &+\frac{gm}{R^2}\frac{R}{4\pi}\int_{\mathbb{S}^2}\mathrm{d}\omega^*\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{2\ell+1}{\ell(\ell+1)} * \mathrm{D}_\psi P_\ell(\cos\psi)\\ &* \{\sin\alpha^*\eta(\lambda^*,\phi^*,R) + \cos\alpha^*\xi(\lambda^*,\phi^*,R)\}\end{aligned} \tag{142}$$

$$\begin{aligned}\delta w(\lambda,\phi,r) = &+\frac{gm}{R^2}\frac{R}{4\pi}\int_{\mathbb{S}^2}\mathrm{d}\omega^*\mathrm{D}_\psi\left\{\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{2\ell+1}{\ell(\ell+1)}P_\ell(\cos\psi)\right\}\\ &* \{\sin\alpha^*\eta(\lambda^*,\phi^*,R) + \cos\alpha^*\xi(\lambda^*,\phi^*,R)\}\end{aligned} \tag{143}$$

$$\mathrm{G}_\lambda(\lambda,\phi,r|\lambda^*,\phi^*,R) := +\frac{gm}{R^2}\frac{R}{4\pi}\sin\alpha^*\mathrm{D}_\psi\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{2\ell+1}{\ell(\ell+1)}P_\ell(\cos\psi) \tag{144}$$

$$\mathrm{G}_\phi(\lambda,\phi,r|\lambda^*,\phi^*,R) := +\frac{gm}{R^2}\frac{R}{4\pi}\cos\alpha^*\mathrm{D}_\psi\sum_{\ell=2}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{2\ell+1}{\ell(\ell+1)}P_\ell(\cos\psi) \tag{145}$$

Green functions of Eqs. (142), (143), (144) and (145). An essential feature in the derivation is the substitution of $\sin\alpha^*$ and $\cos\alpha^*$ into Eq. (138) as well as Eq. (140). The corresponding relations are derived in Appendix A, where the transformation of spherical longitude/spherical latitude (λ,ϕ) into meta-longitude (oblique longitude, North (left-handed) azimuth α^*)/meta-colatitude ψ^* (negative spherical distance, $\psi^* = -\psi$) is given. Here

Table 14. Partial fractional decomposition

'Ansatz'

$$\frac{2\ell+1}{\ell(\ell+1)} = \frac{a}{\ell} + \frac{b}{\ell+1} \tag{146}$$

$$\frac{a}{\ell} + \frac{b}{\ell+1} = \frac{a(\ell+1)+b\ell}{\ell(\ell+1)} = \frac{\ell(a+b)+a}{\ell(\ell+1)} \tag{147}$$

$$\begin{gathered}\ell(a+b)+a = 2\ell+1\\ \Leftrightarrow a=1,\ b=\ell\\ \boxed{\frac{2\ell+1}{\ell(\ell+1)} = \frac{1}{\ell} + \frac{1}{\ell+1}}\end{gathered} \tag{148}$$

we took advantage of the spherical sine law [Eq. (A25)], the spherical sine–cosine law [Eq. (A24)] and the spherical side cosine law [Eq. (A25)], derived in a way to apply it also for ellipsoidal symmetry.

The next procedure to generate the Green function in a closed form is based on 'partial fractional decomposition' of $(2\ell+1)/\ell(\ell+1)$, outlined in Table 14 [Eqs. (146)–(148)]. In addition, intermediate results, namely product–sum formulae, are collected in Table 15, Eqs. (149) and (150) following Pick et al. (1973, p. 478) and Eqs. (151) and (152) following Pick et al. (1973, p. 477), finally to derive Eq. (153) with the $\ell = 2$ reduction of Eqs. (154) and (155).

Table 16 aims at a closed-form representation of the ψ-derivative of the kernel $(2\ell+1)/\ell(\ell+1)\ (R/r)^{\ell+1}$ $P_\ell(\cos\psi)$ through Eqs. (157)–(163). Indeed, the first term of the Green function is found by Eqs. (164) and (165), and the second term by Eq. (166). $G_1(r,R,\psi)$ is the first highlight Green function of Eqs. (167) and (168).

Table 17 aims at a closed-form representation of the ψ-derivative of the kernel $(2\ell+1)/\ell(\ell+1)(R/r)^{\ell+1}$ $P_\ell(\cos\psi)$ where the degree-one term is removed. $G_2(r,R,\psi)$ is the second highlight Green function of Eqs. (169) and (170).

In deriving the closed-form Green functions G_1 and G_2, we took advantage of the permutation rule of termwise differentiation and summation of an infinite series of the type of Eqs. (157) and (169). Such a procedure is permitted if (1) the series are uniformly converging and (2) the differentiated series of the type of Eqs. (149)–(155) are uniformly converging. Since these conditions apply we could check Eq. (168) as well as Eq. (170) by differentiating Eq. (153) as well as Eq. (155) to prove the correctness of our approach.

At the end of our solution of the spherical horizontal boundary value problem in terms of closed-form Green functions to be multiplied by $(\sin\alpha^*,\cos\alpha^*)$, respectively, we want to discuss the special case $r = R$ where we transform vertical deflections given on $\mathbb{S}^2_R$ to the incremental potential on $\mathbb{S}^2_R$: as outlined by Table 18, the Green functions of Eqs. (171) and (172) specialized to $G_1(\psi) = -\cot\psi/2$, $G_2(\psi) = -\cot\psi/2 + 1.5\sin\psi$. Indeed, after a careful study of Stokes' (1849) epochal contribution, we found $G_1(\psi)\sin\alpha^*$, $G_1\cos\alpha^*$, on p. 163, formula (40), and p. 164, formula (141), fine proof of the correctness of our more general results. Finally, we have plotted the Green functions $G_1(\psi)$, which may be called a vertical deflection Stokes function, and $G_2(\psi)$ in Fig. 8.

Table 15. Solution of the spherical horizontal boundary value problem

Green function (1st part)

$$\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) = \sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)+\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) \tag{149}$$

$$\sum_{\ell=0}^{\infty}\frac{1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) = \ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)} \tag{150}$$

[Pick et al. 1973, p. 478, Eq. (1567)]

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) &= \sum_{\ell=0}^{\infty}\frac{1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)-\frac{R}{r}\\ &= \ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}-\frac{R}{r}\end{aligned} \tag{151}$$

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) &= \frac{R}{r}\sum_{\ell=0}^{\infty}\frac{1}{\ell}\left(\frac{R}{r}\right)^{\ell}P_\ell(\cos\psi)\\ &= -\frac{R}{r}\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+r-R\cos\psi}{2r}\end{aligned} \tag{152}$$

[Pick et al. 1973, p. 477, Eq. (1567)]

$$\boxed{\begin{aligned}\sum_{\ell\neq0\,\ell=1}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) &= \ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}-\frac{R}{r}\\ &\quad -\frac{R}{r}\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+r-R-R\cos\psi}{2r}\\ &= \ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{\left[\sqrt{r^2+R^2-rR\cos\psi}+r-R\cos\psi\right]^{R/r}}\\ &\quad +\ln\frac{(2r)^{R/r}}{r(1-\cos\psi)}-\frac{R}{r}\end{aligned}} \tag{153}$$

$\boxed{\ell = 2 \text{ reduction}}$

$$\sum_{\ell=2}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) = \sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)-\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi \tag{154}$$

$$\boxed{\begin{aligned}\sum_{\ell=2}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) &= \ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\\ &\quad -\frac{R}{r}\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+r-R-R\cos\psi}{2r}\\ &\quad -\frac{R}{r}-\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi\end{aligned}} \tag{155}$$

$$\|\mathbf{x}-\mathbf{x}^*\|_2 := \sqrt{r^2+R^2-2rR\cos\psi} \tag{156}$$

(to be replaced in the two kernels in order to produce the convolution terms)

In order to show the correctness of the permutation rule of termwise differentiation and summation of an infinite series we complete the proof by the example of the vertical deflection Stokes function $G_1(\psi)$, namely by means of Table 18, eventually represented in the closed form of Eq. (174). As soon as we apply the ψ-derivative to the closed-form expression of Eq. (174), we are led to $G_1(\psi)$ in Eqs. (175) and (176), a result we have already obtained by the alternative approach of Table 16 and 17.

Table 16. Solution of the spherical horizontal boundary value problem

Green function (2nd part)

$$\begin{aligned}G_1(r,R,\psi) &= \mathrm{D}_\psi\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)\\ &= \sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}\mathrm{D}_\psi P_\ell(\cos\psi)\end{aligned} \tag{157}$$

$$x=\cos\psi \tag{158}$$

$$\mathrm{D}_\psi P_\ell(\cos\psi)=\frac{\mathrm{d}}{\mathrm{d}\psi}P_\ell(\cos\psi)=\frac{\mathrm{d}x}{\mathrm{d}\psi}\frac{\mathrm{d}}{\mathrm{d}x}P_\ell(x)=-\sin\psi P'_\ell(x) \tag{159}$$

$$(x^2-1)P'_\ell(x)=\frac{\ell(\ell+1)}{2\ell+1}[P_{\ell+1}(x)-P_{\ell-1}(x)] \tag{160}$$

[Freeden et al. 1998, p. 34; Sansone 1959, p. 178, Eq. (13) and p. 179, Eq. (17)]

$$\mathrm{D}_\psi P_\ell(\cos\psi)=\frac{\ell(\ell+1)}{2\ell+1}\frac{1}{\sin\psi}[P_{\ell+1}(\cos\psi)-P_{\ell-1}(\cos\psi)] \tag{161}$$

$$\mathrm{D}_\psi\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell(\ell+1)}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi) = \frac{1}{\sin\psi}\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}[P_{\ell+1}(\cos\psi)-P_{\ell-1}(\cos\psi)] \tag{162}$$

$$\sum_{\ell=0}^{\infty}\left(\frac{R}{r}\right)^{\ell}P_\ell(\cos\psi)=\frac{r}{\sqrt{r^2+R^2-2rR\cos\psi}} \tag{163}$$

[Pick et al. 1973, p. 478, Eq. (1576)]

1st term

$$\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}P_{\ell+1}(\cos\psi) = \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n}P_n(\cos\psi)=\sum_{\ell=0}^{\infty}\left(\frac{R}{r}\right)^{\ell}P_\ell(\cos\psi)-1-\frac{R}{r}\cos\psi \tag{164}$$

$$\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}P_{\ell+1}(\cos\psi)=\frac{r}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{r+R\cos\psi}{r} \tag{165}$$

2nd term

$$\begin{aligned}-\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}P_{\ell-1}(\cos\psi) &= -\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+2}P_n(\cos\psi)\\ &= -\left(\frac{R}{r}\right)^2\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n}P_n(\cos\psi)\\ &= -\left(\frac{R}{r}\right)^2\frac{r}{\sqrt{r^2+R^2-2rR\cos\psi}}\end{aligned} \tag{166}$$

$$\begin{aligned}G_1(r,R,\psi) &= \frac{1}{\sin\psi}\frac{r}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{r+R\cos\psi}{r\sin\psi}\\ &\quad -\frac{1}{\sin\psi}\left(\frac{R}{r}\right)^2\frac{r}{\sqrt{r^2+R^2-2rR\cos\psi}}\end{aligned} \tag{167}$$

$$\boxed{G_1(r,R,\psi)=\frac{1}{r\sin\psi}\frac{r^2-R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{r+R\cos\psi}{\sin\psi}} \tag{168}$$

3.3 The spherical vertical boundary value problem

While we have previously formulated and solved the horizontal geodetic boundary problem for vertical deflections, here we are going to formulate and solve the vertical geodetic boundary value problem for incremental gravity. Following Table 19 and Definition 2, we assume at first as a boundary operator the vertical/normal field $L_n \delta w(\lambda, \phi, r)$ of Eqs. (177) and (178) to be given on the sphere $\mathbb{S}^2_R$ of radius R. Second the incremental gravitational potential $\delta w(\lambda, \phi, r)$ is assumed to fulfil as a region operator the Laplace–Beltrami equation [Eq. (177)] with $2\omega^2$ as the constant inhomogeneity which can be reduced to zero. Since all field quantities are formulated in a quasi-Earth fixed frame (ITRF), we have built in the centrifugal potential δw_{inh} of the type of Eq. (180) which fulfils the inhomogeneous Laplace–Beltrami equation. In addition, we have restricted the harmonic domain to coincide with the region outside the sphere $\mathbb{S}^2_R$ of R. Accordingly we have to assume that all topographic masses have been removed from the gravitational potential, for instance by means of the terrain effect. Third, by means of a 'radiation condition' [Eq. (180)] we postulate that the incremental 'homogeneous' gravitational potential approaches zero at infinity, $r \to \infty$.

In geodetic terms, the vertical boundary operator of Eq. (181) expresses the radial derivative of the disturbing potential (incremental gravitational potential) to be given in the tangent space $\mathbb{N}_{(\lambda,\phi)}\mathbb{S}^2_R$ at (λ, ϕ) of the sphere $\mathbb{S}^2_R$ of radius R. In addition, we have to define the notion of the incremental gravitational potential δw with respect to a reference gravitational potential w of type gm/r, $\mathrm{D}_r w = -gm/r^2$. Finally the solution of Eq. (183) of the spherical vertical boundary value problem [given the surface radial derivative of the incremental gravitational potential called $\delta\gamma(\lambda^*, \phi^*, R)$] is represented as a linear functional with respect to the Green function $G(P, P^*)$, where P is the symbol for the evaluation point, P^* for the moving point, also called the data point or source point. The integration of the linear functional $\delta w[\delta\gamma]$ is extended over the surface ω_R of $\mathbb{S}^2_R$.

As outlined in Table 20, we aim now to solve the spherical vertical boundary value problem in the spectral domain. The 'ansatz' of the type of Eq. (184) introduces the radial vector harmonics as test functions for the vertical field. Equation (185) defines the radial coeffi-

Table 17. Solution of the spherical horizontal boundary value problem

Green function (3rd part)

$$G_2(r,R,\psi) = \mathrm{D}_\psi \sum_{\ell=2}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\psi)$$
$$= \mathrm{D}_\psi \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\psi) - \frac{3}{2}\left(\frac{R}{r}\right)^2 \mathrm{D}_\psi P_\ell(\cos\psi)$$
$$= \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell+1} \left(\frac{R}{r}\right)^{\ell+1} \mathrm{D}_\psi P_\ell(\cos\psi) + \frac{3}{2}\left(\frac{R}{r}\right)^2 \sin\psi \quad (169)$$

$$\boxed{G_2(r,R,\psi) = \frac{1}{r\sin\psi}\frac{r^2-R^2}{\sqrt{r^2+R^2-2rR\cos\psi}} - \frac{r+R\cos\psi}{r\sin\psi} + \frac{3}{2}\left(\frac{R}{r}\right)^2 \sin\psi} \quad (170)$$

Table 18. Solution of the spherical horizontal boundary value problem, Green function, special case: $r = R$ [Stokes 1849, p. 163, formula (40) and p. 164, formula (41)]

$r = R$

$$\sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} P_\ell(\cos\psi) = -\ln\frac{\sqrt{2}\sqrt{1-\cos\psi}+(1-\cos\psi)}{2} + \ln\frac{\sqrt{2}\sqrt{1-\cos\psi}+(1-\cos\psi)}{1-\cos\psi} - 1 \quad (171)$$

$$\boxed{\sum_{\ell=1}^{\infty} \tfrac{2\ell+1}{\ell+1} P_\ell(\cos\psi) = -\ln(1-\cos\psi) + \ln 2 - 1} \quad (172)$$

$$G_1(\psi) = \mathrm{D}_\psi \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell+1} P_\ell(\cos\psi) = \mathrm{D}_\psi \ln(1-\cos\psi) \quad (173)$$

$$\boxed{G_1(\psi) = \mathrm{D}_\psi \sum_{\ell=1}^{\infty} \tfrac{2\ell+1}{\ell+1} P_\ell(\cos\psi) = -\tfrac{\sin\psi}{1-\cos\psi} = -\cot\tfrac{\psi}{2}} \quad (174)$$

Table 19. Spherical vertical boundary value problem

(1) Boundary operator (spherical surface normal operator)

$$L_n \delta w(\lambda,\phi,r) = \mathbf{n}(\lambda,\phi,r) \quad \text{for } r = R, (\lambda,\phi) \in \mathbb{S}^2_R \quad (175)$$
$$L_n := \mathbf{e}_r \mathrm{D}_r \quad (176)$$

(2) Region operator (spherical Laplace operator)

$$L\delta w(\lambda,\phi,r) = 2\omega^2 \quad \text{for } (\lambda,\phi,r) \in G_e = \mathbb{R}^3/G_i \quad (177)$$

(i) Special solution of the inhomogeneous differential equation

$$\delta w_{inh} = \tfrac{1}{2}\omega^2 r^2 \cos^2\phi \quad (178)$$

(ii) General solution of the homogeneous differential equation

$$L\delta w_{\mathrm{hom}}(\lambda,\phi,r) = 0 \quad (179)$$

(3) 'Radiation condition'

$$\lim_{r\to\infty} \delta w_{\mathrm{hom}}(\lambda,\phi,r) = 0 \quad (180)$$

Definition 2 (*vertical boundary value problem*)

Given the modulus of incremental field intensity (the spherical normal derivative, the radical derivative)

$$\mathbf{n}(\lambda,\phi,r=R) := \mathbf{e}_r \zeta = L_n \delta w \quad (181)$$

on the sphere $\mathbb{S}^2_R$ of radius R with respect to the disturbing potential δw (incremental gravitational potential), namely $W = w + \delta w$, subject to $w = gm/r$, $\mathrm{D}_r w = -gm/r^2$, which is a harmonic function up to the rotational term, namely

$$\mathrm{Lap}\ \delta w(\lambda,\phi,r) = 2\omega^2 \quad (182)$$

in the space outside the sphere. The normalized base vectors $\{\mathbf{e}_r|P\}$ spans the normal space $\mathbb{N}_{\lambda,\phi}\mathbb{S}^2_R$, the spherical normal vector at the point $P \sim (\lambda,\phi,R)$

Find the disturbing potential, namely the Green functions $G(\mathbf{x},\mathbf{x}^*)$, represented by

$$\delta w(\lambda,\phi,r) = \int_{\mathbb{S}^2_R} \mathrm{d}\omega^* G(\lambda,\phi,r\,|\,\lambda^*,\phi^*,r^*)\delta\gamma(\lambda^*,\phi^*,R) + \frac{1}{2}\omega^2 r^2\cos^2\phi \quad (183)$$

Table 20. Solution of the spherical vertical boundary value problem in the spectral domain

'Ansatz'

$$\mathbf{n}(\lambda,\phi,R)=\mathbf{e}_r\zeta(\lambda,\phi,R)=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\mathbf{R}^{\ell m}(\lambda,\phi)r_{\ell m} \tag{184}$$

$$r_{\ell m}:=\frac{1}{4\pi}\int_{\mathbb{S}^2}\mathrm{d}\omega\,\langle\mathbf{n}\,|\,\mathbf{R}_{\ell m}\rangle \tag{185}$$

$$\mathrm{d}\omega:=\mathrm{d}\lambda\,\mathrm{d}\phi\cos\phi \qquad 0\le\lambda\le 2\pi,\quad -\pi/2<\phi<+\pi/2 \tag{186}$$

$\boxed{\text{comparison of } \mathbf{n}(\lambda,\phi,R) \text{ (Table 10) and } \mathbf{t}(\lambda,\phi,R) \text{ (Table 10)}}$

$$\begin{aligned}\mathbf{n}(\lambda,\phi,R)&=\frac{1}{\frac{gm}{R^2}}*\frac{1}{R}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}(\ell+1)\mathbf{R}^{\ell m}(\lambda,\phi)\delta w_{\ell m}\\&=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\mathbf{R}^{\ell m}(\lambda,\phi)r_{\ell m}\end{aligned} \tag{187}$$

$$\Leftrightarrow$$

$$r_{\ell m}=\frac{1}{\frac{gm}{R^2}}*\frac{1}{R}(\ell+1)\delta w_{\ell m} \tag{188}$$

$$\Leftrightarrow$$

$$\delta w_{\ell m}=-\frac{gm}{R^2}\frac{R}{(\ell+1)}r_{\ell m} \tag{189}$$

Eigenspace

$$\delta w_{\ell m}=\lambda_{\ell m}r_{\ell m} \tag{190}$$

subject to

$$\boxed{\lambda_{\ell m}=-\frac{gm}{R^2}\frac{R}{\ell+1}} \tag{191}$$

$$\begin{tikzcd} r_{\ell m} \arrow[r] \arrow[dr] & \delta w_{\ell m} \arrow[d] \\ & \delta w \end{tikzcd}$$

$$\boxed{\begin{aligned}\delta w(\lambda,\phi,r)&=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)\delta w_{\ell m}\\&=-\frac{gm}{R^2}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}r_{\ell m}\end{aligned}} \tag{192}$$

cients $r_{\ell m}$ as generated by means of scalar product $\langle\mathbf{n}\,|\,\mathbf{R}_{\ell m}\rangle$, integrated over the unit sphere $\mathbb{S}^2$ with respect to the radial base $\mathbf{R}_{\ell m}$. Applying the radial derivative to the incremental potential δw represented in the spectral domain by means of spherical harmonics, we are led to the normal/vertical field/incremental gravity field of the type of Eq. (187). In particular, by means of Eqs. (188)–(189) the radial coefficients $r_{\ell m}$ are related to the potential coefficients $\delta w_{\ell m}$ with the elements $(\ell+1)$ of the eigenspace. A commutative diagram illustrates the transformation $r_{\ell m}\mapsto\delta w_{\ell m}$ in the spectral domain as well as $\delta w_{\ell m}\mapsto\delta w$ from the spectral domain to incremental potential space. Finally, Eq. (192) relates the incremental potential $\delta w(\lambda,\phi,r)$ to the spectral coefficients $r_{\ell m}$ of the incremental gravity divided by the spectrum

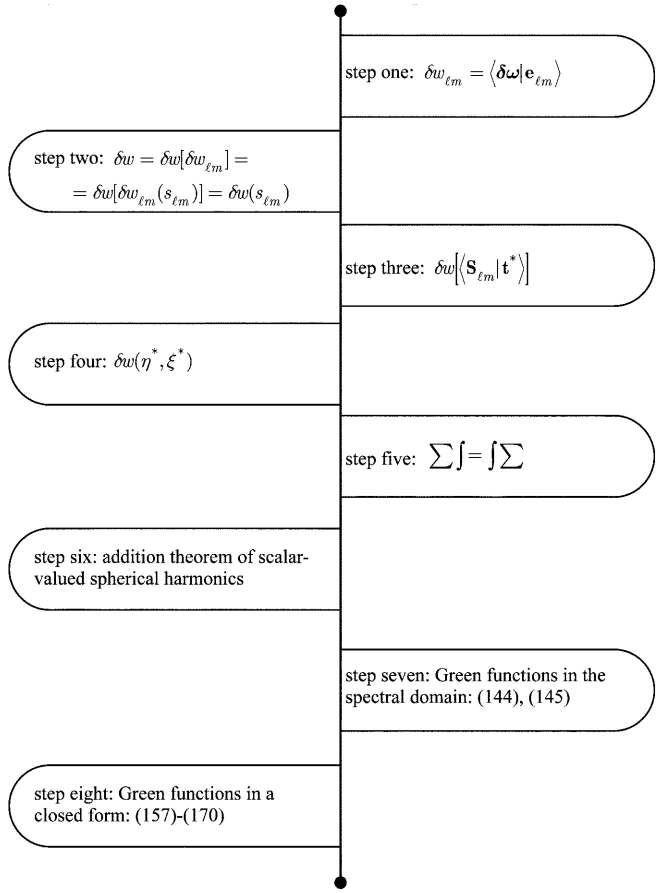

Fig. 7. The eight steps which generate a closed-form solution of the spherical horizontal boundary value problem

$(\ell+1)$ and a decay rate $(R/r)^{\ell+1}$. Figure 10 illustrates the related spectrum $\lambda(\ell)\sim 1/(\ell+1)$, for all $\ell\in\{0, 1, \ldots, \infty\}$.

Is it possible to represent the solution of Eq. (192) in terms of a linear functional as an integral over the data points attached with the gravity values and a Green function in a closed form? Such a representation is achieved as outlined in Table 21!

The procedure for solving the spherical vertical boundary value problem in a closed form by means of a vector-valued Green function is lengthy. It is for this reason that we have split the derivation into various steps (see Fig. 9). Step one, by means of Eq. (193), expresses the spherical harmonic coefficients $\delta w_{\ell m}$, the coordinates of the Hilbert space of spherical harmonics, by their reproducing property (scalar product) in terms of the spherical surface integral over the incremental gravitational potential and the spherical base functions $\mathbf{e}^{\ell m}(\lambda,\phi)$. In step two we relate the incremental gravitational potential $\delta w(\lambda,\phi,r)$ in Eq. (194) in the spectral domain to the spherical harmonic coefficients $\delta w_{\ell m}$ and, thanks to Eq. (189), to the radial harmonic coefficients $r_{\ell m}$ [Eq. (185)] of the incremental gravity. Step three uses the substitution of the scalar product of Eq. (193) into the functional relation $\delta w(s_{\ell m})$ in terms of Eq. (195). In step four we replace the normal/vertical field $\mathbf{n}(\lambda^*,\phi^*)$, the incremental gravity field, at the moving point

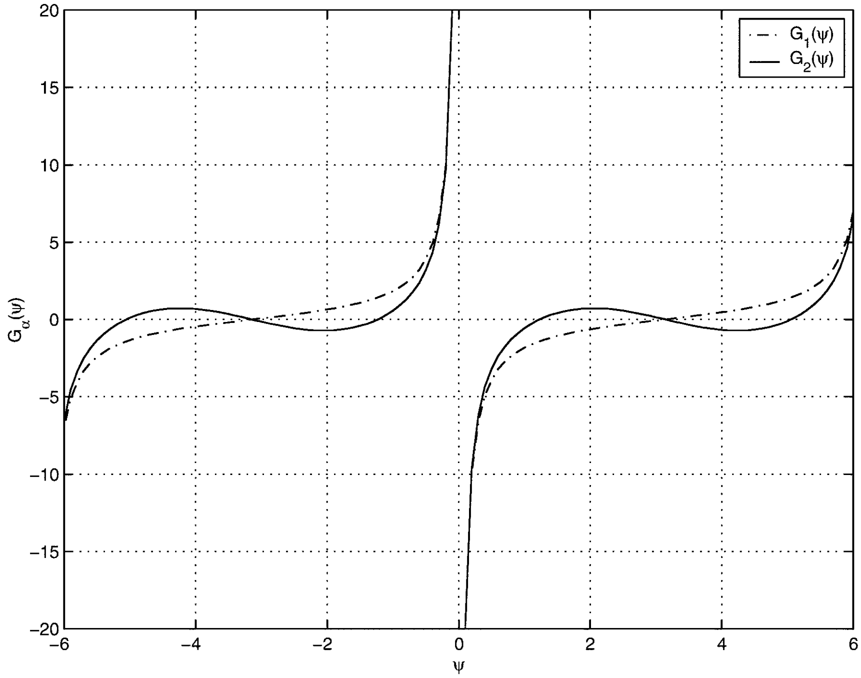

Fig. 8. Spherical horizontal boundary value problem, Green function $G_\alpha(\psi)$, special case: $r = R$, $\alpha \in \{1, 2\} : G_1(\psi) = -\cot(\psi/2)$, $G_2(\psi) = -\cot(\psi/2) + 1.5 \sin \psi$, $\psi \in [-\pi/2/ + \pi/2]$

$P^* \in \mathbb{S}^2_R$ in Eq. (195) in order to generate the lengthy integral relation of Eq. (196). Next, for Eq. (197), we apply in step five the permutation rule for infinite summation and integration. If the series expansion is uniformly convergent we are permitted to interchange summation and integration. Indeed, in the admissible space outside $\mathbb{S}^2_R$, the series of Eq. (196) are uniformly convergent. Step six involves an application of the additive theorem of spherical harmonics, namely Eq. (198), to Eq. (197), which finally generates Eqs. (199) and (200), namely the vertical Green function $G(\mathbf{x}, \mathbf{x}^*)$ in the spectral domain, the target of step seven. In contrast, in step eight we aim at a formulation of the vertical Green function in closed form. Table 22 shows a systematic treatment of the produce-sum formulae involved. From Pick et al. (1973, p. 478) we deduce the conversion (201) of an infinite sum to a closed form. Special attention has to be paid to the degree term ℓ, $\ell \in \{0, \ldots, \infty\}$. There are two reductions of interest. If we exclude the zero-degree term by means of the first reduction of the type of Eq. (202) we fix the solution of the spherical vertical boundary problem to the chosen (gm) value. In contrast, as soon as we exclude the zero- and first-degree term by means of Eq. (203) we relate the solution of the spherical vertical boundary value problem to the fixed value of gravitational mass as well as to the reference mass centre. The mass centres of the reference gravity field and of the incremental gravity field coincide. Martinec (1998, p. 31–32) has studied a related effect. Accordingly, we have come up with three different vertical Green functions:

(1) Neumann kernel N_0: external Neumann function, Hotine function [Eq. (204)]
(2) 1st modified Neumann kernel N_1 [Eq. (205)]
(3) 2nd modified Neumann kernel N_2 [Eqs. (206), (207)]

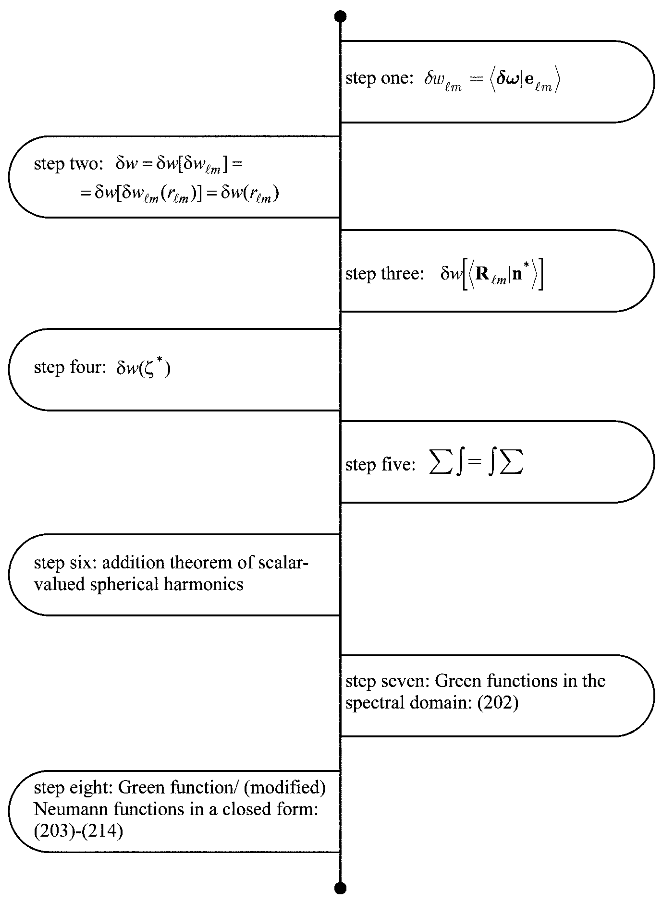

Fig. 9. The eight steps which generate a closed-form solution of the spherical vertical boundary value problem

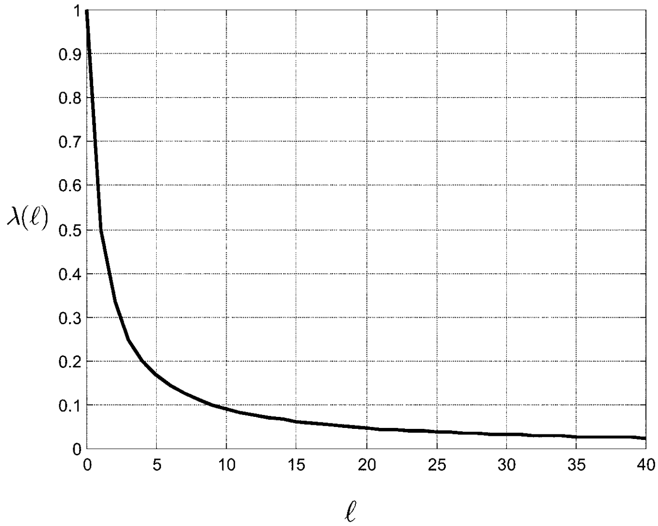

Fig. 10. $\lambda(\ell)$, Spherical vertical boundary value problem $\frac{GM}{R}+\frac{1}{3}\Omega^2 R^2 = W_0$, $GM = 3.986004415 \times 10^{14}\,\mathrm{km^3\,s^{-2}}$, $\Omega = 7.292\,115\,\mathrm{rad\,s^{-1}}$, $W_0 = 60\,636\,855.8\,\mathrm{m^2\,s^{-2}}$; $R = 6,370,991.248\,\mathrm{m}$, eigenspace $\lambda(\ell) \sim 1/(\ell+1), \ell \in \{0,1,\ldots,40\}$

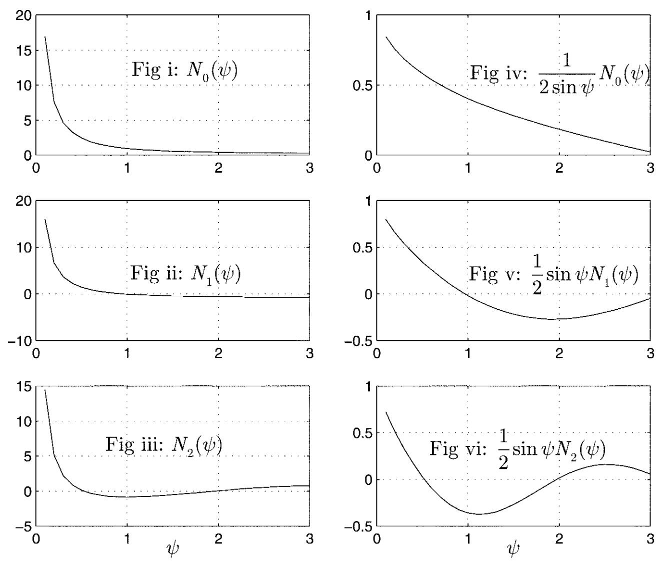

Fig. 11. Spherical vertical boundary value problem, Green functions. Special case: $r = R$, $N_\alpha(\psi)$, $\frac{1}{2}\sin\psi N_\alpha(\psi)\ \forall\,\alpha \in \{0,1,2\}, \psi \in \{0,\pi\}$

Such a procedure leads us finally to step nine, where by means of Eqs. (208)–(212) we present the solution of the spherical vertical boundary value problem in terms of (modified) Neumann functions N_α $(\alpha = 0,1,2)$ and of the incremental gravity $\delta\gamma(\psi, R)$ with specific 1st and 2nd terms. The particular form of Eq. (210) of the (modified) Neumann functional is achieved if we specify the surface element $d\omega^* = d\alpha^*\, d\psi \sin\psi$ in terms of oblique spherical coordinates $(\alpha, \psi^* = -\psi)$ and integrate over the azimuth. The (modified) Neumann kernels are isotropic; they do not depend on the North azimuth α^*.

At the end of our solution of the special vertical boundary value problem in terms of closed-form Green functions/(modified) Neumann functions we want to discuss the special case $r = R$. Here we transform, by means of Table 23, Eqs. (212)–(215), incremental gravity values given on $\mathbb{S}^2_R$ to the incremental potential on $\mathbb{S}^2_R$. In detail we refer to $N_0(\psi), N_1(\psi)$, and $N_2(\psi)$ as illustrated by Fig. 11.

Table 21. Solution of the spherical vertical boundary value problem. Spectral representation of the Green function

$$\boxed{\text{reproducing property}}$$

$$\delta w_{\ell m} := \frac{1}{4\pi R^2}\int_{\mathbb{S}^2_R} d\omega_R \mathbf{e}^{\ell m}(\lambda,\phi)\delta w(\lambda,\phi,R) \tag{193}$$

$$\delta w(\lambda,\phi,r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)\delta w_{\ell m}$$

$$= -\frac{gm}{R^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)r_{\ell m} \tag{194}$$

$$\delta w(\lambda,\phi,r) = -\frac{gm}{R^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R^{\ell+2}}{r^{\ell+1}}\frac{1}{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)\frac{1}{4\pi}\int_{\mathbb{S}^2} d\omega^*\langle \mathbf{R}_{\ell m}|\mathbf{n}^*\rangle \tag{195}$$

$$\delta w(\lambda,\phi,r) = -\frac{gm}{R^2}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{R}{4\pi}\int_{\mathbb{S}^2} d\omega^*\left(\frac{R}{r}\right)^{\ell+1}\frac{1}{\ell+1}\mathbf{e}^{\ell m}(\lambda,\phi)$$

$$*\left\langle \mathbf{e}_r\mathbf{e}^{\ell m}(\lambda,\phi)|\mathbf{e}_r\zeta(\lambda^*,\phi^*,R)\right\rangle \tag{196}$$

$$\sum\int = \int\sum$$

$$\delta w(\lambda,\phi,r) = -\frac{gm}{R^2}\frac{R}{4\pi}\int_{\mathbb{S}^2} d\omega^*\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+m}\left(\frac{R}{r}\right)^{\ell+1}\frac{1}{\ell+1}$$

$$*\{\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda,\phi)\zeta(\lambda^*,\phi^*,R)\} \tag{197}$$

addition theorem of spherical harmonics

$$(2\ell+1)P_\ell(\cos\psi) = \sum_{m=-\ell}^{+\ell}\mathbf{e}^{\ell m}(\lambda,\phi)\mathbf{e}^{\ell m}(\lambda^*,\phi^*) \tag{198}$$

$$\delta w(\lambda,\phi,R) = -\frac{R}{4\pi}\int_{\mathbb{S}^2} d\omega^*\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)\delta\gamma(\lambda,\phi,R) \tag{199}$$

$$\boxed{\mathrm{G}(\mathbf{x},\mathbf{x}^*) := \mathrm{G}(\lambda,\phi,r|\lambda^*,\phi^*,R) = -\frac{R}{4\pi}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)} \tag{200}$$

4 Upward and downward continuation of vertical deflections: spherical horizontal component of incremental gravity vector

The solution of the spherical horizontal boundary value problem $\delta w = G_\lambda\eta + G_\phi\xi$ which is written here in operator form can be favourably used to generate upward as well as downward continuation operators. Such a genesis is the target of this chapter.

According to Table 24 we depart by means of Eqs. (216) and (217) from the representation of the coordinates of the vertical operator. As soon as we apply such a surface gradient to the solution of Eq. (218) of the spherical horizontal boundary value problem the Green functions $G_\alpha(r,R,\psi)$ are not a function of the spherical coordinates (λ,ϕ). However, by chain-rule differentiation [Eqs. (219) and (220)] subject to the spherical cosine law [Eqs. (221)–(223)] we are led via Eqs. (224) and (225)

Table 22. Solution of the spherical horizontal boundary value problem Green function (1st part)

$$\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=2\frac{R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{201}$$

[M. Pick et al. 1973, p. 478 (1572)]

$\boxed{\text{1st reduction}}$

$$\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=2\frac{R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{R}{r}-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{202}$$

$\boxed{\text{2nd reduction}}$

$$\sum_{\ell=2}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=2\frac{R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{R}{r}-\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{203}$$

Neumann kernel (external Neumann function, Hotine function)

$$N_0(r,R,\psi):=\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=\frac{2R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{204}$$

1st modified Neumann kernel

$$N_1(r,R,\psi):=\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=\frac{2R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{R}{r}-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{205}$$

2nd modified Neumann kernel

$$N_2(r,R,\psi):=\sum_{\ell=2}^{\infty}\frac{2\ell+1}{\ell+1}\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\psi)=\frac{2R}{\sqrt{r^2+R^2-2rR\cos\psi}}-\frac{R}{r}-\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi-\ln\frac{\sqrt{r^2+R^2-2rR\cos\psi}+R-r\cos\psi}{r(1-\cos\psi)}\tag{206}$$

$$\sqrt{r^2+R^2-2rR\cos\psi}=:\|\mathbf{x}-\mathbf{x}^*\|_2\tag{207}$$

(to be replaced in the kernels in order to produce the convolution term)

$\forall\,\alpha\in\{0,1,2\}$

$$\boxed{\delta w(\lambda,\phi,r)=-\frac{R}{4\pi}\int_{\mathbb{S}^2}\mathrm{d}\omega^* N_\alpha(r,R,\psi)\,\delta\gamma(\lambda^*,\phi^*,R)}\tag{208}$$

$$G_\alpha(r,R,\psi):=\frac{1}{2}\sin\psi N_\alpha(r,R,\psi)\tag{209}$$

$$\boxed{\delta w(\lambda,\phi,r)=-R\int_0^\pi \mathrm{d}\omega^* G_\alpha(r,R,\psi)\,\delta\gamma(\psi,R)}\tag{210}$$

$\boxed{\text{1st term}}$

$$\delta w(\lambda,\phi,r)=\frac{R}{r}\int_{\mathbb{S}^2}\mathrm{d}\omega^*\,\delta\gamma(\lambda^*,\phi^*,R)\tag{211}$$

Table 22. Contd.

$\boxed{\text{2nd term}}$

$$\frac{3}{2}\left(\frac{R}{r}\right)^2\int_{\mathbb{S}^2}\mathrm{d}\omega^* P_1(\cos\psi)\delta\gamma(\lambda^*,\phi^*,R)=\frac{3}{2}\left(\frac{R}{r}\right)^2\int_0^{2\pi}\mathrm{d}\lambda\int_{-\pi/2}^{+\pi/2}\mathrm{d}\phi\cos\phi[\cos\phi\cos\phi^*\cos(\lambda-\lambda^*)+\sin\phi\sin\phi^*]\mathrm{d}\delta(\lambda^*,\phi^*,R)\tag{212}$$

Table 23. Solution of the spherical vertical boundary value problem, Green function. Special case: $r=R$

$$N_0(\psi)=\frac{1}{\sin\psi}-\ln\frac{1+\sin\frac{\psi}{2}}{\sin\frac{\psi}{2}}\tag{213}$$

$$N_1(\psi)=N_0(\psi)-1,\quad N_2(\psi)=N_0(\psi)-1-\frac{3}{2}\cos\psi\tag{214}$$

$\forall\,\alpha\in\{0,1,2\}$

$$\boxed{\delta w(\lambda,\phi,R)=-R\int_0^\pi \mathrm{d}\psi\,\tfrac{1}{2}\sin\psi N_\alpha(\psi)\delta\gamma(\psi,R)}\tag{215}$$

Table 24. Upward continuation of vertical deflections, transformation of horizontal operators

$$-\frac{gm}{r^2}\delta\lambda_\gamma\cos\phi_\gamma=-\frac{gm}{r^2}\eta=\frac{1}{r\cos\phi}\mathrm{D}_\lambda\delta w\tag{216}$$

$$-\frac{gm}{r^2}\delta\phi_\gamma=-\frac{gm}{r^2}\xi=\frac{1}{r}\mathrm{D}_\phi\delta w\tag{217}$$

$$\delta w(\lambda,\phi,r)=\frac{gm}{R^2}*\frac{R}{4\pi}*\int_{\mathbb{S}^2}\mathrm{d}\omega^* G_\alpha(r,R,\psi)\{\sin\alpha^*\eta^*+\cos\alpha^*\xi^*\}\tag{218}$$

transformation of horizontal operators from spherical coordinates to meta-spherical coordinates (oblique spherical coordinates)

$$\mathrm{D}_\lambda=\mathrm{D}_\lambda\psi\mathrm{D}_\psi\tag{219}$$

$$\mathrm{D}_\phi=\mathrm{D}_\phi\psi\mathrm{D}_\psi\tag{220}$$

$$\cos\psi=\cos\phi\cos\phi^*\cos(\lambda-\lambda^*)+\sin\phi\sin\phi^*\tag{221}$$

$$\mathrm{d}\cos\psi=-\sin\psi\,\mathrm{d}\psi\tag{222}$$

$$\Leftrightarrow \mathrm{d}\psi=-\frac{1}{\sin\psi}\mathrm{d}\cos\psi\tag{223}$$

$$\mathrm{D}_\lambda\psi=-\frac{1}{\sin\psi}\mathrm{D}_\lambda\cos\psi\tag{224}$$

$$\mathrm{D}_\phi\psi=-\frac{1}{\sin\psi}\mathrm{D}_\phi\cos\psi\tag{225}$$

$$\mathrm{D}_\lambda\psi=+\frac{1}{\sin\psi}[\cos\phi\cos\phi^*\sin(\lambda-\lambda^*)]\tag{226}$$

$$\mathrm{D}_\phi\psi=+\frac{1}{\sin\psi}[\sin\phi\cos\phi^*\cos(\lambda-\lambda^*)-\cos\phi\sin\phi^*]\tag{227}$$

to the longitudinal derivative of Eq. (226) and to the lateral derivative of Eq. (227). Note that in differentiating the Green operators we have interchanged differentiation and integration. Such a procedure is permitted as long as the Green kernels are continuous.

Table 25 collects the derivational formulae of Eqs. (228)–(231) for $\mathrm{D}_\lambda\psi$ and $\mathrm{D}_\phi\psi$ where we have substituted $(\cos\alpha\sin\alpha,\cos\alpha)$ by means of Appendix B, Eqs. (B24) and (B25) (see http://www.uni-stuttgart.de/gi/research/index.html#publications). Finally we derive the Green function derivatives $\mathrm{D}_\psi G_1$ and $\mathrm{D}_\psi G_2$ by means of Table 26, Eqs. (234)–(237).

The singular integrals which relate $\{\eta(\lambda,\phi,r),\ \xi(\lambda,\phi,r)\}$ in the external space of $\mathbb{S}^2_R$ to

Table 25. Upward continuation of vertical deflections, horizontal operators in meta-spherical coordinates

$$\mathrm{D}_\lambda \psi = \frac{1}{\sin\psi}[\cos\phi\cos\phi^* \sin(\lambda-\lambda^*)] = -\cos\phi\sin\alpha \tag{228}$$

$$\mathrm{D}_\phi \psi = \frac{1}{\sin\psi}[\sin\phi\cos\phi^*\cos(\lambda-\lambda^*) - \cos\phi\sin\phi^*] = -\cos\phi \tag{229}$$

$$\frac{1}{\cos\phi}\mathrm{D}_\lambda = -\sin\alpha \mathrm{D}_\psi \tag{230}$$

$$\mathrm{D}_\phi = -\cos\alpha \mathrm{D}_\psi \tag{231}$$

Table 26. Upward continuation of spherical vertical deflections, derivative of Green functions

$$\frac{1}{\cos\phi}\mathrm{D}_\lambda G_\alpha(r,R,\psi) = -\sin\alpha \mathrm{D}_\psi G_\alpha(r,R,\psi) \tag{232}$$

$$\mathrm{D}_\phi G_\alpha(r,R,\psi) = -\cos\alpha G_\alpha(r,R,\psi) \tag{233}$$

$$\boxed{\begin{aligned}\mathrm{D}_\psi G_1(r,R,\psi) = &-\frac{R(r^2-R^2)}{(r^2+R^2-2rR\cos\psi)^{3/2}} \\ &-\frac{1}{r\sin^2\psi}\frac{(r^2-R^2)\cos\psi}{\sqrt{r^2+R^2-2rR\cos\psi}} \\ &+\frac{R+r\cos\psi}{r\sin^2\psi} =: H_1(r,R,\psi)\end{aligned}} \tag{234}$$

$$\boxed{\begin{aligned}\mathrm{D}_\psi G_2(r,R,\psi) = &-\frac{R(r^2-R^2)}{(r^2+R^2-2rR\cos\psi)^{3/2}} \\ &-\frac{1}{r\sin^2\psi}\frac{(r^2-R^2)\cos\psi}{\sqrt{r^2+R^2-2rR\cos\psi}} \\ &+\frac{R+r\cos\psi}{r\sin^2\psi}+\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\psi =: H_2(r,R,\psi)\end{aligned}} \tag{235}$$

$\{\eta(\lambda,\phi,R),\xi(\lambda,\phi,R)\}$ on $\mathbb{S}^2_R$ are finally presented in Table 27. The East component $\eta(\lambda,\phi,r)$ is given by Eq. (236), the North component $\xi(\lambda,\phi,r)$ by Eq. (237). We have to implement the height Green functions $H_\alpha(r,R,\psi)$ as well as the four components $\sin\alpha\sin\alpha^*\eta^*$, $\sin\alpha\cos\alpha^*\xi^*$, $\cos\alpha\sin\alpha^*\eta^*$ and $\cos\alpha\cos\alpha^*\xi^*$.

The vector-valued upward continuation operator is symbolically written by Eqs. (238) and (239). In order to introduce spherical heights $h, r = R+h$, into the height Green functions, we take advantage of the representations/expansions of Eqs. (240)–(252) to derive the kernels $H_1(h/R,\psi)$ and $H_2(h/R,\psi)$ in Eqs. (253) and (254). These height Green functions are illustrated in Figs. 12 and 13. The downward operators $(\eta,\xi)\mapsto(\eta^*,\xi^*)$ are formulated in Sect. 6.

5 Upward and downward continuation of incremental gravity: spherical vertical component of incremental gravity vector

The solution of the spherical vertical boundary value problem $\delta w = N\delta_\gamma$, which is written here in operator form, can be efficiently used to generate upward as well as downward continuation operators. Such a genesis is the target of this section.

In Table 28 we depart by means of Eq. (255) from the representation of the incremental value as a vertical/normal operator. As soon as we apply such an operator to the solution [Eq. (258)] of the spherical vertical boundary value problem we have the compute the radial derivative of the Neumann functions $N_\alpha(r,R,\psi)$ as outlined by Eqs. (258)–(260). Such a procedure leads to the external Abel–Poisson kernel functions of Eq. (263) of the type $P_0(r,R,\psi)$, Eq. (264) of the type $P_1(r,R,\psi)$ and Eq. (265) of the type $P_2(r,R,\psi)$ defined by Eqs. (266)–(270). In toto, we find the functional of Eq. (271), $\delta\gamma(\delta\gamma^*)$, and in particular the Abel–Poisson functional of Eq. (272), $(r\delta\gamma)(R\delta\gamma^*)$, subject to the kernels of

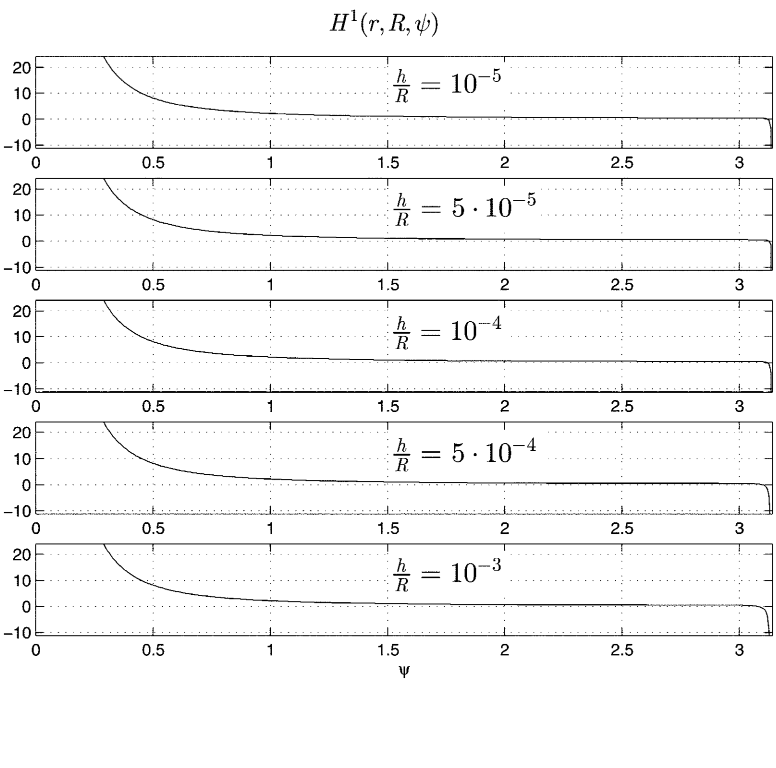

Fig. 12. Upward continuation of spherical vertical deflections, kernel function $H_1(r,R,\psi)$, at fixed $r = R+h = R(1+\frac{h}{R})$ values based on $\frac{h}{R}\in\{10^{-5}, 5\cdot10^{-5}, 10^{-4}, 5\cdot10^{-4}, 10^{-3}\}$, and for polar distance $\psi\in[0,\pi]$

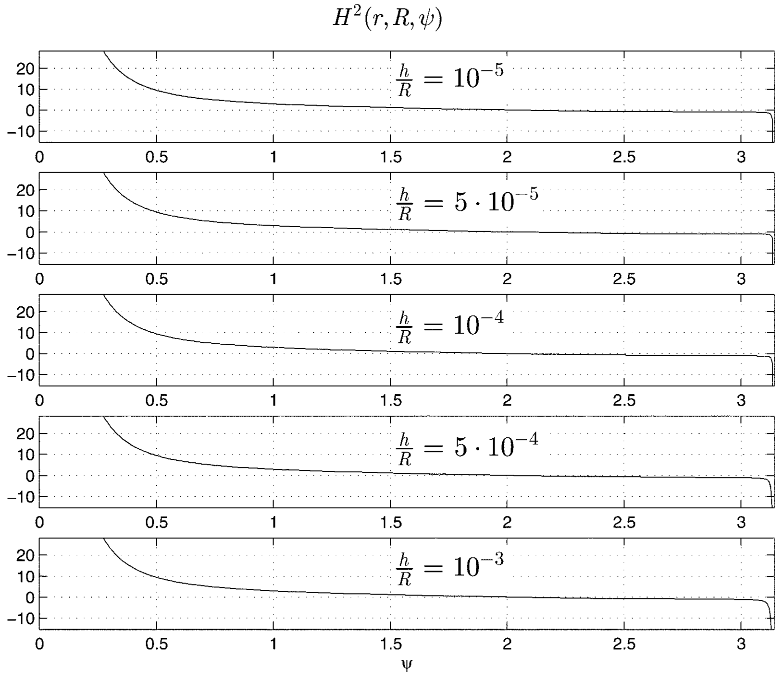

Fig. 13. Upward continuation of spherical vertical deflections, kernel function $H_2(r, R, \psi)$ for various values of polar distance $\psi \in \{10°, 20°, \ldots, 90°\}$, $r = R + h = R\left(1 + \frac{h}{R}\right)$, $\frac{h}{R} \in \{10^{-5}, 5 \cdot 10^{-5}, 10^{-4}, 5 \cdot 10^{-4}, 10^{-3}\}$

Eqs. (273)–(275), k_0, k_1, k_2, and Eqs. (263)–(265), P_0, P_1, P_2. These Abel–Poisson kernels are plotted in Fig. 14. Finally, it is necessary to mention that in differentiating the Neumann operators we have interchanged differentiation and integration. Such a procedure is permitted as long as the Neumann kernels are continuous.

6 Meissl-type diagrams

The reader may have experienced the large number of formulae for upward as well as downward continuation of vertical deflections as well as incremental gravity. Meissl (1971, p. 46) had the idea to present the spectral representation of the incremental potential and its radial derivatives, directly and inversely, in form of a table. Such a table has been extended, namely for upward continuation, by Rummel and van Gelderen (1995, p. 383) and Freeden and Windheuser (1997, p. 35) to a diagram form, beside the radial/vertical derivatives also including the lateral/horizontal derivatives (surface gradient) of the incremental potential. Figures 15 (Rummel–van Gelderen diagram) and 16 (Freeden diagram) give an illustrative survey of the behaviour of horizontal and vertical operators in the discrete spectral domain. Corresponding diagrams for regularised downward continuation of horizontal and vertical first-order derivative operators in the spectral domain have been presented by Xu and Rummel (1994a, b) as well as Keller and Hirsch (1994) and for the so-called flat approximation by Keller (1999, pp. 244–245).

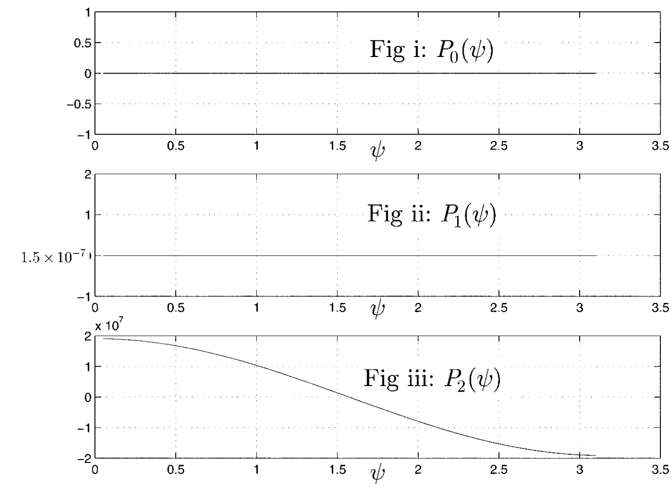

Fig. 14. Upward continuation of incremental gravity. **i** Abel–Poisson kernel P_0; **ii** modified Abel–Poisson kernel P_1; **iii** modified Abel–Poisson kernel P_2; special case: $r = R$

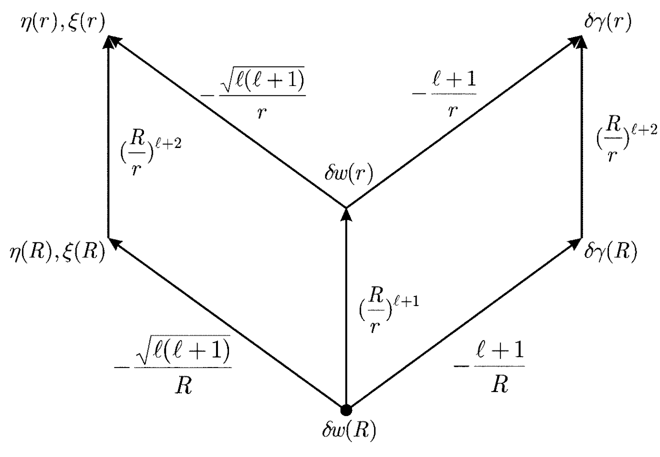

Fig. 15. Rummel–van Gelderen diagram, horizontal and vertical derivative (spherical, spectral domain)

Table 27. Upward continuation of spherical vertical deflections, singular integrals

East component

$$\eta(\lambda,\phi,r) = -\frac{1}{\frac{gm}{r^2}}\frac{1}{r\cos\phi}\mathrm{D}_\lambda\delta w$$
$$= \frac{1}{4\pi}\frac{r}{R} * \left\{ \int\limits_{\mathbb{S}^2} \mathrm{d}\omega^* H_\alpha(r_0,R,\psi)[\sin\alpha\sin\alpha^*\eta(\lambda^*,\phi^*,R) + \sin\alpha\cos\alpha^*\xi(\lambda^*,\phi^*,R)] \right\} \tag{236}$$

North component

$$\xi(\lambda,\phi,r) = -\frac{1}{\frac{gm}{r^2}}\frac{1}{r}\mathrm{D}_\phi\delta w$$
$$= \frac{1}{4\pi}\frac{r}{R} * \left\{ \int\limits_{\mathbb{S}^2} \mathrm{d}\omega^* H_\alpha(r,R,\psi)[\cos\alpha\sin\alpha^*\eta(\lambda^*,\phi^*,R) + \cos\alpha\cos\alpha^*\xi(\lambda^*,\phi^*,R)] \right\} \tag{237}$$

Vector-valued upward continuation operator

$$\begin{bmatrix}\eta\\ \xi\end{bmatrix} = \mathbf{Z}\begin{bmatrix}\eta^*\\ \xi^*\end{bmatrix} \tag{238}$$

Vertical deflection upward continuation operator

$$\mathbf{Z} := \frac{1}{4\pi}\frac{r}{R}\int\limits_{\mathbb{S}^2_R}\mathrm{d}\omega^* H_\alpha\begin{bmatrix}\sin\alpha\sin\alpha^* & \sin\alpha\cos\alpha^*\\ \cos\alpha\sin\alpha^* & \cos\alpha\cos\alpha^*\end{bmatrix} \tag{239}$$

$$r = R + h = R\left(1+\frac{h}{R}\right) \tag{240}$$

$$\frac{1}{r} = \frac{1}{R}\frac{1}{1+\frac{h}{R}} \tag{241}$$

$$\frac{1}{r} = \frac{1}{R}\left[1-\frac{h}{R}+\mathcal{O}(2)\right] \tag{242}$$

$$(r^2+R^2-2rR\cos\psi)^{-3/2}$$
$$= \frac{1}{4}\sqrt{2}\frac{1}{R^3}\left[1-\cos\psi+\frac{h}{R}(1-\cos\psi)+\frac{1}{2}\left(\frac{h}{R}\right)^2\right]^{-3/2} \tag{243}$$

$$(r^2+R^2-2rR\cos\psi)^{-3/2}$$
$$= \frac{1}{4}\sqrt{2}\frac{1}{R^3}(1-\cos\psi)^{-3/2} + \left[1-\frac{3}{2}\frac{h}{R}+\mathcal{O}(2)\right] \tag{244}$$

$$(r^2+R^2-2rR\cos\psi)^{-1/2}$$
$$= \frac{1}{2}\sqrt{2}\frac{1}{R}\left[1-\cos\psi+\frac{h}{R}(1-\cos\psi)+\frac{1}{2}\left(\frac{h}{R}\right)^2\right]^{-1/2} \tag{245}$$

$$(r^2+R^2-2rR\cos\psi)^{-1/2}$$
$$= \frac{1}{2}\sqrt{2}\frac{1}{R}\left[1-\frac{1}{2}\frac{h}{R}+\mathcal{O}(2)\right](1-\cos\psi)^{-1/2} \tag{246}$$

$$r^2 - R^2 = 2Rh + h^2 \tag{247}$$

$$r^2 - R^2 = 2R^2\frac{h}{R} + \mathcal{O}(2) \tag{248}$$

$$R + r\cos\psi = R(1+\cos\psi) + h\cos\psi \tag{249}$$

$$R + r\cos\psi = R\left(1+\cos\psi+\frac{h}{R}\cos\psi\right) \tag{250}$$

$$\left(\frac{R}{r}\right)^2 = \frac{1}{(1+\frac{h}{R})^2} \tag{251}$$

$$\left(\frac{R}{r}\right)^2 = 1 - 2\frac{h}{R} + \mathcal{O}(2) \tag{252}$$

if $\frac{h}{R} \ll 1$, then

Table 27. Contd.

$$\boxed{\begin{aligned} H_1\left(\frac{h}{R},\psi\right) &= -\frac{1}{2}\sqrt{2}\frac{h}{R}(1-\cos\psi)^{-3/2}\left[1-\frac{3}{2}\frac{h}{R}+\mathcal{O}(2)\right] \\ &\quad -\frac{\sqrt{2}}{\sin^2\psi}\frac{h}{R}(1-\cos\psi)^{-1/2}\left[1-\frac{3}{2}\frac{h}{R}+\mathcal{O}(2)\right] \\ &\quad +\frac{1}{\sin^2\psi}\left[1+\cos\psi-\frac{h}{R}+\mathcal{O}(2)\right]\end{aligned}} \tag{253}$$

$$\begin{aligned} H_2\left(\frac{h}{R},\psi\right) &= -\frac{1}{2}\sqrt{2}\frac{h}{R}(1-\cos\psi)^{-3/2}\left[1-\frac{3}{2}\frac{h}{R}+\mathcal{O}(2)\right] \\ &\quad -\frac{\sqrt{2}}{\sin^2\psi}\frac{h}{R}(1-\cos\psi)^{-1/2}\left[1-\frac{3}{2}\frac{h}{R}+\mathcal{O}(2)\right] \\ &\quad +\frac{1}{\sin^2\psi}\left[1+\cos\psi-\frac{h}{R}+\mathcal{O}(2)\right] \\ &\quad +\frac{3}{2}\cos\psi\left[1-2\frac{h}{R}+\mathcal{O}(2)\right]\end{aligned} \tag{254}$$

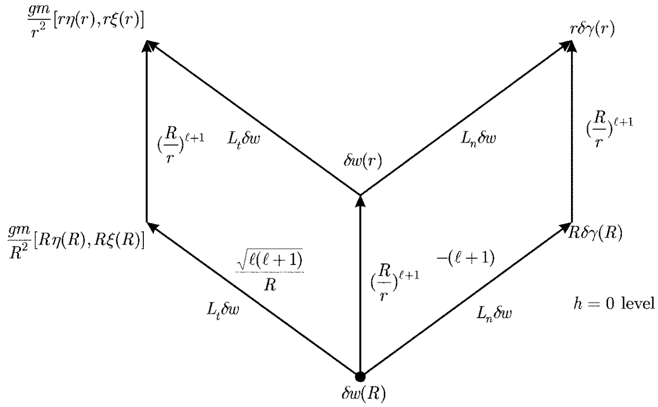

Fig. 16. Freeden diagram, horizontal and vertical derivative (spherical, spectral domain)

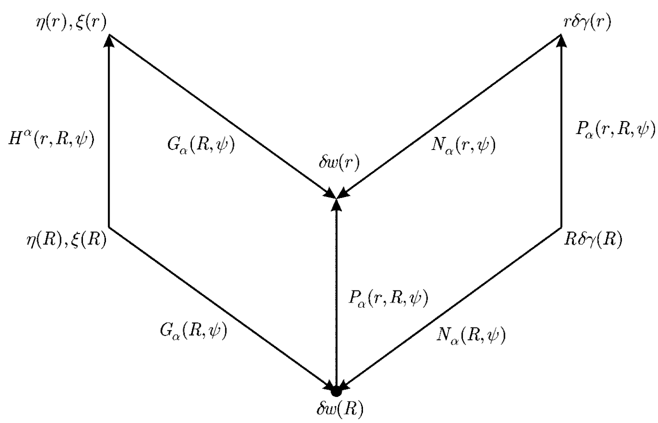

Fig. 17. Upward continuation of the horizontal and vertical derivative (spherical, spatial domain)

Here we extend these diagrams into the space domain, namely by taking advantage of the various Green functions which we have already derived from the horizontal and vertical boundary value problem. Figure 17 introduces the diagram for upward continuation of the horizontal and vertical derivative of the incremental

Table 28. Upward continuation of incremental gravity, vertical derivative of Green function

$$\begin{bmatrix} -\delta\gamma = \mathrm{D}_r \delta w \\ \delta\gamma = \mathrm{D}_r(-\delta w) \end{bmatrix} \tag{255}$$

$\forall\, \alpha \in \{0, 1, 2\}$

$$-\delta w(\lambda, \phi, r) = \frac{R}{4\pi} \int_{\mathbb{S}^2} \mathrm{d}\omega^* N_\alpha(r, R, \psi)\delta\gamma(\lambda^*, \phi^*, R)$$

$$\times R \int_0^\pi \mathrm{d}\omega \frac{1}{2} \sin\psi N_\alpha(r, R, \psi)\delta\gamma(\lambda^*, \phi^*, R) \tag{256}$$

$$\boxed{\delta\gamma(\lambda, \phi, r) = \frac{R}{4\pi} \int_{\mathbb{S}^2} \mathrm{d}\omega^* \mathrm{D}_r N_\alpha(r, R, \psi)\delta\gamma(\lambda^*, \phi^*, R)} \tag{257}$$

vertical derivative of the Neumann kernel

$$N_0(r, R, \psi) = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{\ell + 1} \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\psi) \tag{258}$$

$$\mathrm{D}_r N_0(r, R, \psi) = -\sum_{\ell=0}^{\infty} (2\ell + 1) \left(\frac{R}{r}\right)^{\ell+2} \frac{1}{R} P_\ell(\cos\psi) \tag{259}$$

$$\mathrm{D}_r N_0(r, R, \psi) = -\frac{1}{r} \sum_{\ell=0}^{\infty} (2\ell + 1) \left(\frac{R}{r}\right)^{\ell+1} \frac{1}{R} P_\ell(\cos\psi) \tag{260}$$

$$\sum_{\ell=0}^{\infty} 2\ell + 1 \left(\frac{R}{r}\right)^{\ell+1} \frac{1}{R} P_\ell(\cos\psi) = R \frac{r^2 - R^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} \tag{261}$$

[Pick et al. 1973, p. 478, Eq. (1570)]

$$\mathrm{D}_r N_0 = \frac{R}{r} \frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} \tag{262}$$

Abel–Poisson kernel (external Abel–Poisson kernel function)

$$P_0(r, R, \psi) := \frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} \tag{263}$$

1st modified Poisson kernel

$$P_1(r, R, \psi) = \frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} + \frac{1}{r} \tag{264}$$

2nd modified Poisson kernel

$$P_2(r, R, \psi) = \frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} + \frac{1}{r} + 3\left(\frac{R}{r}\right)^3 r\cos\psi \tag{265}$$

$$\mathrm{D}_r N_0 = \frac{R}{r} P_0(r, R, \psi) \tag{266}$$

$$\mathrm{D}_r N_1 = \mathrm{D}_r N_0 + \frac{R}{r}\frac{1}{r} \tag{267}$$

$$\mathrm{D}_r N_1 = \frac{R}{r} P_1 \tag{268}$$

$$\mathrm{D}_r N_2 = \frac{R}{r} P_0 + \frac{R}{r}\frac{1}{r} + 3\frac{R}{r}\left(\frac{R}{r}\right)^3 r\cos\psi \tag{269}$$

$$\mathrm{D}_r N_2 = \frac{R}{r} P_2 \tag{270}$$

$\forall\, \alpha \in \{0, 1, 2\}$

$$\delta\gamma(\mathbf{x}) = \int_{\mathbb{S}^2} \mathrm{d}\omega^* K_\alpha(\mathbf{x}, \mathbf{x}^*)\delta\gamma(\mathbf{x}^*) \tag{271}$$

$$r\delta\gamma(\mathbf{x}) = \frac{R}{4\pi} \int_{\mathbb{S}^2} \mathrm{d}\omega^* P_\alpha(\mathbf{x}, \mathbf{x}^*) R\delta\gamma(\mathbf{x}^*) \tag{272}$$

$$K_0 := K_0(r, R, \psi) = \frac{R}{4\pi}\frac{R}{r} \frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\psi)^{3/2}} \tag{273}$$

$$K_1 = K_0 + \frac{R}{4\pi}\frac{R}{r}\frac{1}{r} \tag{274}$$

$$K_2 = K_0 + \frac{R}{4\pi}\frac{R}{r}\left[\frac{1}{r} + 3\left(\frac{R}{r}\right)^3 r\cos\psi\right] \tag{275}$$

Table 29. Downward continuation of spherical vertical deflections, λ-HAPS inverse

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = \mathbf{Z} \begin{bmatrix} \eta^* \\ \xi^* \end{bmatrix} \tag{276}$$

$$\begin{bmatrix} \eta^{**} \\ \xi^{**} \end{bmatrix} = (\lambda\mathbf{I} + \mathbf{Z}^*\mathbf{Z})^{-1}\mathbf{Z}^* \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \mathbf{Z}^\lambda \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{277}$$

$\mathbf{Z}^\lambda$ is the λ-HAPS inverse of $\mathbf{Z}$

$$(\eta^{**}, \xi^{**}) = \arg \inf_{\eta^* \xi^*} \left\{ \left\| \begin{bmatrix} \eta \\ \xi \end{bmatrix} - \mathbf{Z} \begin{bmatrix} \eta^* \\ \xi^* \end{bmatrix} \right\|^2 + \lambda \|(\eta^*, \xi^*)'\|^2 \right\}$$

$$\mathbf{Z}^\lambda := (\lambda\mathbf{I} + \mathbf{Z}^*\mathbf{Z})^{-1}\mathbf{Z}^* \tag{278}$$

hybrid approximation solution; λ-HAPS $\mathbf{Z}^*$ adjoint operator of $\mathbf{Z}$

Table 30. Downward continuation of incremental gravity, λ-HAPS inverse

$$\delta_\gamma = K\delta_\gamma^* \tag{279}$$

$$r\delta_\gamma = NR\delta_\gamma^* \tag{280}$$

$$\delta_\gamma^{**} = (\lambda\mathbf{I} + \mathbf{K}^*\mathbf{K})^{-1}\mathbf{K}^*\delta_\gamma \tag{281}$$

$$(R\delta_\gamma^{**}) = (\lambda\mathbf{I} + \mathbf{N}^*\mathbf{N})^{-1}\mathbf{N}^*(r\delta_\gamma) \tag{282}$$

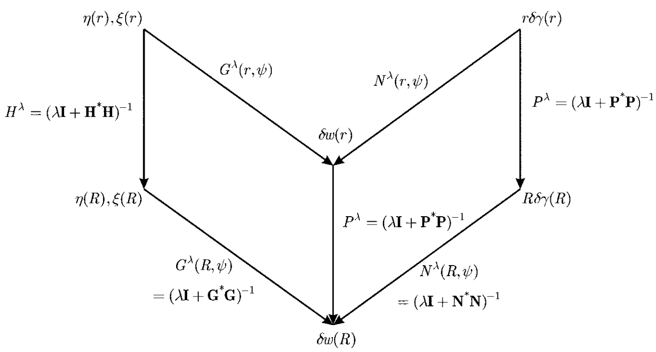

Fig. 18. Regularized downward continuation of the horizontal and vertical derivative (spherical, spatial domain) in terms of HAPS-inverse Green functions

potential in the space domain. In contrast, Fig. 18 illustrates as a commutative diagram the regularized downward continuation of the horizontal and vertical derivative of the incremental potential in the space domain, namely in terms of the HAPS inverse Green functions. These inversions are based upon an inversion process which we outline by means of Table 29 for spherical vertical deflections and Table 30 for incremental gravity following Schaffrin et al. (1977); see also Keller and Hirsch (1994), Xu (1992) and Xu and Rummel (1992, 1994a, b).

Appendix A

Transformation $(\lambda, \phi) \mapsto (\alpha^*, \psi^*)$

Here we collect those transformation formulae which we need switch *from* spherical coordinates {longitude λ, latitude ϕ} with respect to the equator and the North pole *to* {meta-longitude α^*, meta-colatitude ψ^*} with respect to the meta-equator and the meta-North pole P^*, illustrated by Figs. A1 and A2. Meta-longitude α^*, is also called North (left-handed) azimuth, meta-colatitude ψ^*, negative spherical distance P^*P. According to Table A1 we represent by means of Eq. (A1) the position vector $\mathbf{x}$ both in the orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ and in the orthonormal frame of reference $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ at the origin $\mathbf{0}$. The base vectors $\{\mathbf{e}_1, \mathbf{e}_2 \mid \mathbf{0}\}$ span the equator, $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*} \mid \mathbf{0}\}$ the meta-equator ('oblique equator'). In contrast, $\{\mathbf{e}_3 \mid \mathbf{0}\}$ is directed towards the North pole of sphere $\mathbb{S}^2$, $\{\mathbf{e}_{3^*} \mid \mathbf{0}\}$ towards the meta-North pole, the direction to the moving point P^*.

How is the oblique frame of reference $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ at the origin $\mathbf{0}$ generated?

We will follow a path which we can use when we consider more general geodetic coordinates (such as ellipsoidal longitude, ellipsoidal latitude, ellipsoidal height) with respect to the international reference ellipsoid $\mathbb{E}^2_{A,A,B}$ which replaces the international reference sphere $\mathbb{S}^2_R$. First we compute the tangent vectors $\{\mathbf{e}_{\lambda^*}, \mathbf{e}_{\phi^*}, \mathbf{e}_{r^*} \mid P^*\}$ which build up an orthonormal frame of reference at $P^* \sim (\lambda^*, \phi^*, r^*)$. Under the postulate of Euclidean parallelism (Euclid's axiom five) we transport this frame of reference ('moving frame') from the moving point P^* to the origin 0. In this way we identify by means of Eq. (A2) $\mathbf{e}_{1^*} = \mathbf{e}_{\lambda^*}, \mathbf{e}_{2^*} = \mathbf{e}_{\phi^*}, \mathbf{e}_{3^*} = \mathbf{e}_{r^*}$, namely we generate the 'star orthonormal triad'. As shown in Sect. 2, these frames of reference are related by means of an orthogonal matrix [Eqs. (A3), (A4), (A5), (A6)].

Second, we express the Cartesian 'star coordinates' (x^*, y^*, z^*) as well as the Cartesian 'initial coordinates' (x, y, z) in terms of spherical coordinates, namely by means of Eqs. (A7), (A8), (A9) and (A10). (α_*, β^*, r^*) will be called meta-longitude α_*, meta-latitude β^*, radius r^*. Meta-longitude α_* is also referred to as 'azimuth with respect to East', being right-handed. The representation of the placement vector $\mathbf{x}$ [Eqs. (A11) and (A12)] in terms of Cartesian initial coordinates (x, y, z) as well as spherical initial coordinates (λ, ϕ, r) is transformed via Eq. (A5) from the old basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ to the new basis $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ for the case $r = r^*$; Eq. (A13) contains the placement vector [Eq. (A1)] in the new basis. At this point we switch from meta-latitude β^* to meta-colatitude $\psi^* = \frac{\pi}{2} - \beta^*$ and from meta-longitude/right-handed East azimuth α^* to left-handed North azimuth $\alpha^* = \frac{\pi}{2} - \alpha_*$, namely by using Eqs. (A15) and (A16), as is conventionally done in geometric and physical geodesy. Consult also Fig. A3! Note Eq. (A14), where we have transformed meta-colatitude ψ^* (meta-polar distance ψ^*) to the spherical distance ψ, namely PP^*, with opposite sign. This sign correction will have some further consequences, outlined later. Equation (A17) highlights the new meta-spherical coordinate representation in the 'star' frame of reference $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$.

Third, we compare Eqs. (A13) and (A17) in order to compute [Eqs. (A18), (A21), (A24)] $\cos\alpha^*$, [Eqs. (A19), (A22), (A25)] $\sin\alpha^*$ and [Eqs. (A20), (A23)] $\cos\psi$. We could have derived these formulae by means of spherical trigonometry, namely $\sin\alpha^*$ by the law of sine, $\cos\alpha^*$ by the law of sine–cosine and $\cos\psi$ by the law of side cosine.

Table A1. Meta-spherical coordinates

$P^* \sim (\lambda^*, \phi^*)$ meta-North pole

P moving point

$$\mathbf{e}_1 x + \mathbf{e}_2 y + \mathbf{e}_3 z = \mathbf{x} = \mathbf{e}_{1^*} x^* + \mathbf{e}_{2^*} y^* + \mathbf{e}_{3^*} z^* \tag{A1}$$

reference frames

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ orthonormal (unimodular) frame of reference at $\mathbf{0}$

$\{\mathbf{e}_{\lambda^*}, \mathbf{e}_{\phi^*}, \mathbf{e}_{r^*} \mid P^*\}$ orthonormal (unimodular) frame of reference at $P^* \sim (\lambda^*, \phi^*, r^*)$

$$\mathbf{e}_{1^*} = \mathbf{e}_{\lambda^*}, \quad \mathbf{e}_{2^*} = \mathbf{e}_{\phi^*}, \quad \mathbf{e}_{3^*} = \mathbf{e}_{r^*} \tag{A2}$$

East, North, Vertical $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ orthonormal (unimodular) frame of reference at $\mathbf{0}$

(oblique orthonormal frame of reference)

$$\begin{bmatrix} \mathbf{e}_{1^*} \\ \mathbf{e}_{2^*} \\ \mathbf{e}_{3^*} \end{bmatrix} = \begin{bmatrix} -\sin\lambda^* & \cos\lambda^* & 0 \\ -\sin\phi^*\cos\lambda^* & -\sin\phi^*\sin\lambda^* & \cos\phi^* \\ \cos\phi^*\cos\lambda^* & \cos\phi^*\sin\lambda^* & \sin\phi^* \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{A3}$$

$$\mathbf{R}(\lambda^*, \phi^*) := \begin{bmatrix} -\sin\lambda^* & \cos\lambda^* & 0 \\ -\sin\phi^*\cos\lambda^* & -\sin\phi^*\sin\lambda^* & \cos\phi^* \\ \cos\phi^*\cos\lambda^* & \cos\phi^*\sin\lambda^* & \sin\phi^* \end{bmatrix}$$

$$\in SO(3) := \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid R'R = \mathbf{I}_3, |R| = +1\} \tag{A4}$$

$$\mathbf{R}^{-1} = \mathbf{R}' \tag{A5}$$

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} -\sin\lambda^* & -\sin\phi^*\cos\lambda^* & \cos\phi^*\cos\lambda^* \\ \cos\lambda^* & -\sin\phi^*\sin\lambda^* & \cos\phi^*\sin\lambda^* \\ 0 & \cos\phi^* & \sin\phi^* \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1^*} \\ \mathbf{e}_{2^*} \\ \mathbf{e}_{3^*} \end{bmatrix} \tag{A6}$$

$$\begin{aligned} x^* &= r^*\cos\beta^*\cos\alpha_* \\ y^* &= r^*\cos\beta^*\sin\alpha_* \\ z^* &= r^*\sin\beta^* \end{aligned} \tag{A7}$$

versus

$$\begin{aligned} x &= r\cos\phi\cos\lambda \\ y &= r\cos\phi\sin\lambda \\ z &= r\sin\phi \end{aligned} \tag{A8}$$

$$\mathbf{e}_{1^*} x^* + \mathbf{e}_{2^*} y^* + \mathbf{e}_{3^*} z^* = \mathbf{x} \tag{A9}$$

$$\mathbf{x} = \mathbf{e}_{1^*} r^*\cos\beta^*\cos\alpha_* + \mathbf{e}_{2^*} r^*\cos\beta^*\sin\alpha_* + \mathbf{e}_{3^*}\sin\beta^* \tag{A10}$$

$$r = r^*$$

$$P : \mathbf{e}_1 x + \mathbf{e}_2 y + \mathbf{e}_3 z = \mathbf{x} \tag{A11}$$

$$\begin{aligned} \mathbf{x} &= \mathbf{e}_1 r\cos\phi\cos\lambda + \mathbf{e}_2 r\cos\phi\sin\lambda + \mathbf{e}\sin\phi \\ &= r\cos\phi\cos\lambda[-\mathbf{e}_{1^*}\sin\lambda^* - \mathbf{e}_{2^*}\sin\phi^*\cos\lambda^* + \mathbf{e}_{3^*}\cos\phi^*\cos\lambda^*] \\ &\quad + r\cos\phi\sin\lambda[\mathbf{e}_{1^*}\cos\lambda^* - \mathbf{e}_{2^*}\sin\phi^*\sin\lambda^* + \mathbf{e}_{3^*}\cos\phi^*\sin\lambda^*] \\ &\quad + r\sin\phi[\mathbf{e}_{2^*}\cos\phi^* + \mathbf{e}_{3^*}\sin\phi^*] \end{aligned} \tag{A12}$$

$$\begin{aligned} \mathbf{x} &= r\mathbf{e}_{1^*}(-\cos\phi\cos\lambda\sin\lambda^* + \cos\phi\sin\lambda\cos\lambda^*) \\ &\quad + r\mathbf{e}_{2^*}(-\cos\phi\cos\lambda\sin\phi^*\cos\lambda^* - \cos\phi\sin\lambda\sin\phi^*\sin\lambda^* \\ &\quad + \sin\phi\cos\phi^*) + r\mathbf{e}_{3^*}(\cos\phi\cos\lambda\cos\phi^*\cos\lambda^* \\ &\quad + \cos\phi\sin\lambda\cos\phi^*\sin\lambda^* + \sin\phi\sin\phi^*) \end{aligned} \tag{A13}$$

$$\psi^* = -\psi, \quad \psi = -\psi^* \tag{A14}$$

$$\psi^* := \frac{\pi}{2} - \beta^*, \quad \alpha^* = \frac{\pi}{2} - \alpha_* \tag{A15}$$

$$\Leftrightarrow$$

$$\beta = \frac{\pi}{2} - \psi^*, \quad \alpha_* = \frac{\pi}{2} - \alpha^* \tag{A16}$$

$$\begin{aligned} x^* &= r\cos\beta^*\cos\alpha_* = r\sin\psi\sin\alpha^* \\ y^* &= r\cos\beta^*\sin\alpha_* = r\sin\psi\cos\alpha^* \\ z^* &= r\sin\beta^* = r\cos\psi^* \end{aligned} \tag{A17}$$

$$\Leftrightarrow$$

$$\sin\alpha_* = \cos\alpha^* = \frac{y^*}{r\cos\beta^*} = \frac{y^*}{r\sin\psi^*} \tag{A18}$$

$$\cos\alpha_* = \sin\alpha^* = \frac{x^*}{r\cos\beta^*} = \frac{y^*}{r\sin\psi^*} \tag{A19}$$

$$\sin\beta^* = \cos\psi^* = \cos\psi = \frac{z^*}{r} \tag{A20}$$

$$\begin{aligned} \sin\alpha_* &= \cos\alpha^* \\ &= \frac{1}{\sin\psi^*} * (-\cos\phi\cos\lambda\sin\phi^*\cos\lambda^* \\ &\qquad - \cos\phi\sin\lambda\sin\phi^*\sin\lambda^* + \sin\phi\cos\phi^*) \end{aligned} \tag{A21}$$

Table A1. Contd.

$$\cos\alpha_* = \sin\alpha^* = \frac{1}{\sin\psi^*} * (-\cos\phi\cos\lambda\sin\lambda^* + \cos\phi\sin\lambda\cos\lambda^*) \tag{A22}$$

$$\sin\beta^* = \cos\psi = \cos\phi\cos\lambda\cos\phi^*\cos\lambda^* + \cos\phi\sin\lambda\cos\phi^*\sin\lambda^* + \sin\phi\sin\phi^* \tag{A23}$$

$$\Leftrightarrow$$

$$\boxed{\begin{aligned} \sin\alpha_* = \cos\alpha^* &= \frac{\sin\phi\cos\phi^* - \cos\phi\sin\phi^*\cos(\lambda-\lambda^*)}{\sin\psi^*} \\ &= \frac{\cos\phi\sin\phi^*\cos(\lambda-\lambda^*) - \sin\phi\cos\phi^*}{\sin\psi^*} && \text{(A24)} \\ \cos\alpha_* = \sin\alpha^* &= \frac{\cos\phi\sin(\lambda-\lambda^*)}{\sin\psi^*} = -\frac{\cos\phi\sin(\lambda-\lambda^*)}{\sin\psi} && \text{(A25)} \\ \sin\beta^* = \cos\psi^* &= \cos\psi = \cos\phi\cos\phi^*\cos(\lambda-\lambda^*) \\ &\quad + \sin\phi\sin\phi^* && \text{(A26)} \end{aligned}}$$

spherical trigonometry

$\sin\alpha^*$	spherical sine law
$\cos\alpha^*$	spherical sine–cosine law
$\cos\psi$	spherical side cosine law

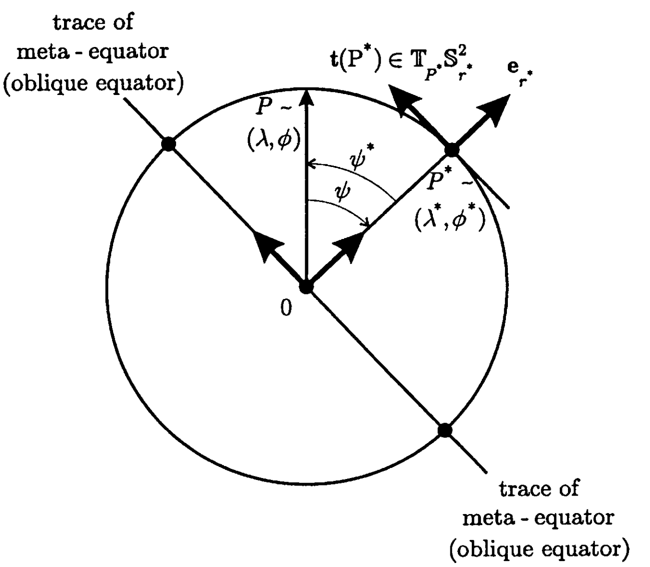

Fig. A1. Meta-spherical (oblique spherical) coordinates, $(\lambda^*, \phi^*) \sim P^*$ meta-North pole, parallel transport (Euclid's axiom five) of $\{\mathbf{e}_{\lambda^*}, \mathbf{e}_{\phi^*} \mid P^*\}$ to $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*} \mid 0\}$ from P^* to $0 : \mathbf{e}_{\lambda^*} = \mathbf{e}_{1^*}, \mathbf{e}_{\phi^*} = \mathbf{e}_{2^*}, \{\mathbf{e}_{1^*}, \mathbf{e}_{2^*} \mid 0\}$ span the meta-equator (oblique equator), $\psi = -\psi^*, \psi^* = -\psi$

Appendix B

Transformation $(\lambda^*, \phi^*) \mapsto (\alpha, \psi)$

Here we collect those transformation formulae which are needed to switch *from* spherical coordinates {longitude λ^*, latitude ϕ^*} with respect to the equator and the North pole *to* {meta-longitude α, meta-*colatitude* ψ} with respect to the meta-equator and the meta-North pole P, illustrated by Figs. B1 and B2. Meta-longitude α^* is also called North (left-handed) azimuth, meta-colatitude ψ spherical distance $\overset{\frown}{PP^*}$. According to Table B1 we represent by means of

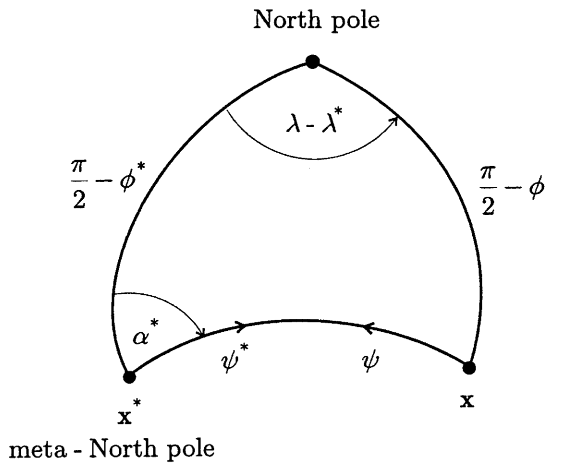

Fig. A2. Meta-spherical coordinates, $(\alpha^*, \psi^*), \mathbf{x}^* \sim (\lambda^*, \phi^*, r^*)$ meta-North pole, $\psi = -\psi^*, \psi^* = -\psi$

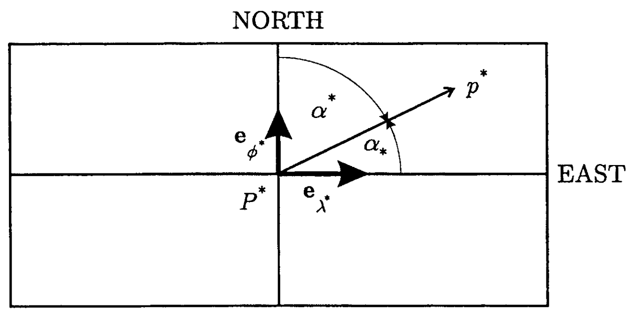

Fig. A3. Tangent space $\mathbb{T}_{(\lambda^*,\phi^*)}\mathbb{S}^2_r$ (tangent plane) at $\mathbf{x}^* \sim (\lambda^*, \phi^*, r^*), p^* = \pi(P^*)$ azimuthal projection of the point $P^* \sim \mathbf{x}^*$ onto $\mathbb{T}_{(\lambda^*,\phi^*)}\mathbb{S}^2_{r^*}$, Northern (left-hand) azimuth α^*

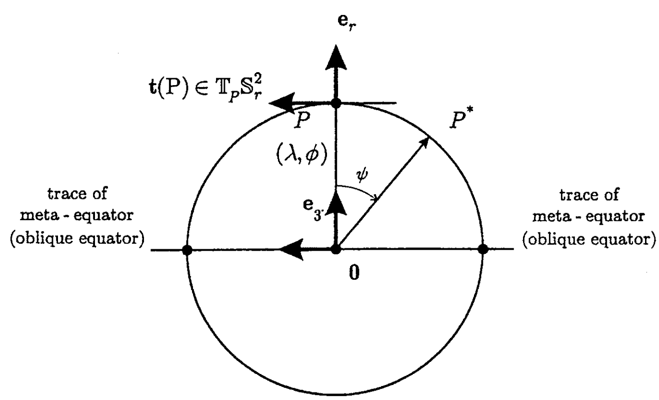

Fig. B1. Meta-spherical (oblique spherical) coordinates, $(\lambda, \phi) \sim P$ meta-North pole, parallel transport (Euclid's axiom five) of $\{\mathbf{e}_\lambda, \mathbf{e}_\phi \mid P\}$ to $\{\mathbf{e}_{1^\cdot}, \mathbf{e}_{2^\cdot} \mid \mathbf{0}\}$ from P to $\mathbf{0} : \mathbf{e}_\lambda = \mathbf{e}_{1^\bullet}, \mathbf{e}_\phi = \mathbf{e}_{2^\bullet}$, $\{\mathbf{e}_{1^\bullet}, \mathbf{e}_{2^\bullet} \mid \mathbf{0}\}$ span the meta-equator (oblique equator)

Eq. (B1) the position vector $\mathbf{x}^*$ both in the orthonormal frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ and in the orthonormal frame of reference $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ at the origin $\mathbf{0}$. The base vectors $\{\mathbf{e}_1, \mathbf{e}_2 \mid \mathbf{0}\}$ span the equator, $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*} \mid \mathbf{0}\}$ the meta-equator ('oblique equator'). In contrast, $\{\mathbf{e}_3 \mid \mathbf{0}\}$ is directed towards the North pole of sphere

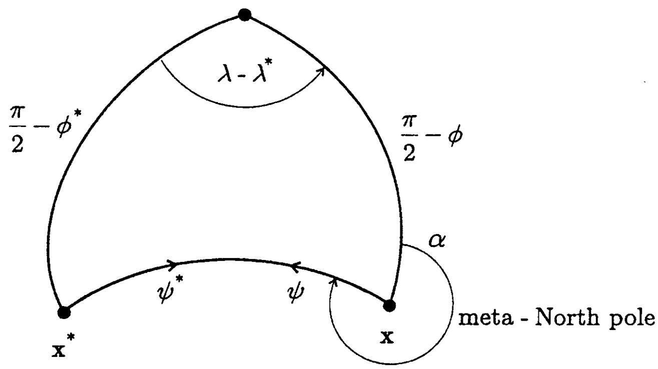

Fig. B2. Meta-spherical coordinates, (α, ψ), $\mathbf{x} \sim (\lambda, \phi, r)$ meta-North pole

Table B1. Meta-spherical coordinates

$P \sim (\lambda, \phi)$ met-North pole

P^* moving point

$$\mathbf{e}_1 x + \mathbf{e}_2 y + \mathbf{e}_3 z = \mathbf{x}^* = \mathbf{e}_{1\bullet} x^\bullet + \mathbf{e}_{2\bullet} y^\bullet + \mathbf{e}_{3\bullet} z^\bullet \tag{B1}$$

reference frames

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ orthonormal (unimodular) frame of reference at $\mathbf{0}$

$\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r \mid P\}$ orthonormal (unimodular) frame of reference at $P \sim (\lambda, \phi, r)$

$$\mathbf{e}_{1\bullet} = \mathbf{e}_\lambda, \quad \mathbf{e}_{2\bullet} = \mathbf{e}_\phi, \quad \mathbf{e}_{3\bullet} = \mathbf{e}_r \tag{B2}$$

East, North, Vertical $\{\mathbf{e}_{1\bullet}, \mathbf{e}_{2\bullet}, \mathbf{e}_{3\bullet} \mid \mathbf{0}\}$ orthonormal (unimodular) frame of reference at $\mathbf{0}$

(oblique orthonormal frame of reference)

$$\begin{bmatrix} \mathbf{e}_{1\bullet} \\ \mathbf{e}_{2\bullet} \\ \mathbf{e}_{3\bullet} \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{B3}$$

$$\mathbf{R}(\lambda, \phi) := \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix}$$
$$\in SO(3) := \{\mathbf{R} \in \mathbb{R}^{3\times 3} | R'R = \mathbf{I}_3, |\mathbf{R}| = +1\} \tag{B4}$$

$$\mathbf{R}^{-1} = \mathbf{R}' \tag{B5}$$

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1\bullet} \\ \mathbf{e}_{2\bullet} \\ \mathbf{e}_{3\bullet} \end{bmatrix} \tag{B6}$$

$$\begin{aligned} \mathbf{x}^\bullet &= r^\bullet \cos\beta^\bullet \cos\alpha^\bullet \\ y^\bullet &= r^\bullet \cos\beta^\bullet \sin\alpha^\bullet \\ z^* &= r^\bullet \sin\beta^\bullet \end{aligned} \tag{B7}$$

versus

$$\begin{aligned} \mathbf{x}^* &= r^* \cos\phi^* \cos\lambda^* \\ y^* &= r^* \cos\phi^* \sin\lambda^* \\ z^* &= r^* \sin\phi^* \end{aligned} \tag{B8}$$

$$\mathbf{e}_{1\bullet} x^\bullet + \mathbf{e}_{2\bullet} y^\bullet + \mathbf{e}_{3\bullet} z^\bullet = \mathbf{x}^* \tag{B9}$$

$$\mathbf{x}^* = \mathbf{e}_{1\bullet} r^\bullet \cos\beta^\bullet \cos\alpha^\bullet + \mathbf{e}_{2\bullet} r^\bullet \cos\beta^\bullet \sin\alpha^\bullet + \mathbf{e}_{3\bullet} \sin\beta^\bullet \tag{B10}$$
$$r^* = r^\bullet$$

$$P^* : \mathbf{e}_1 x^* + \mathbf{e}_2 y^* + \mathbf{e}_3 z^* = \mathbf{x}^* \tag{B11}$$

$$\begin{aligned} \mathbf{x}^* &= \mathbf{e}_1 r^* \cos\phi^* \cos\lambda^* + \mathbf{e}_2 r^* \cos\phi^* \sin\lambda^* + \mathbf{e}_{3^*} r^* \sin\phi^* \\ &= r^\bullet \cos\phi^* \cos\lambda^* [-\mathbf{e}_{1\bullet} \sin\lambda - \mathbf{e}_{2\bullet} \sin\phi\cos\lambda + \mathbf{e}_{3\bullet} \cos\phi\cos\lambda] \\ &\quad + r^\bullet \cos\phi^* \sin\lambda^* [\mathbf{e}_{1\bullet} \cos\lambda - \mathbf{e}_{2\bullet} \sin\phi\sin\lambda + \mathbf{e}_{3\bullet} \cos\phi\sin\lambda] \\ &\quad + r^\bullet \sin\phi^* [\mathbf{e}_{2\bullet} \cos\phi + \mathbf{e}_{3\bullet} \sin\phi] \end{aligned} \tag{B12}$$

$$\begin{aligned} P^* : \mathbf{x} &= r^\bullet \mathbf{e}_{1\bullet} (-\cos\phi^* \cos\lambda^* \sin\lambda + \cos\phi^* \sin\lambda^* \cos\lambda) \\ &\quad + r^\bullet \mathbf{e}_{2\bullet} (-\cos\phi^* \cos\lambda^* \sin\phi\cos\lambda - \cos\phi^* \sin\lambda^* \sin\phi\sin\lambda \\ &\quad + \sin\phi^* \cos\phi) + r^\bullet \mathbf{e}_{3\bullet} (-\cos\phi^* \cos\lambda^* \cos\phi\cos\lambda \\ &\quad + \cos\phi^* \sin\lambda^* \cos\phi\sin\lambda + \sin\phi^* \sin\phi) \end{aligned} \tag{B13}$$

$$\psi := \frac{\pi}{2} - \beta^\bullet \tag{B14}$$

Table B1. Contd.

$$\alpha^\bullet = \frac{\pi}{2} - \alpha \tag{B15}$$

$$\beta^\bullet := \frac{\pi}{2} - \psi, \quad \alpha = \frac{\pi}{2} - \alpha^\bullet \tag{B16}$$

$$\begin{aligned} \mathbf{x}^\bullet &= r^\bullet \cos\beta^\bullet \cos\alpha^\bullet = r^\bullet \sin\psi \sin\alpha \\ y^\bullet &= r^\bullet \cos\beta^\bullet \sin\alpha^\bullet = r^\bullet \sin\psi \cos\alpha \\ z^\bullet &= r^\bullet \sin\beta^\bullet = r^\bullet \cos\psi \end{aligned} \tag{B17}$$

$$\Leftrightarrow$$

$$\sin\alpha^\bullet = \cos\alpha = \frac{y^\bullet}{r^\bullet \cos\beta^\bullet} = \frac{y^\bullet}{r^\bullet \sin\psi} \tag{B18}$$

$$\cos\alpha^\bullet = \sin\alpha = \frac{x^\bullet}{r^\bullet \cos\beta^\bullet} = \frac{y^\bullet}{r^\bullet \sin\psi} \tag{B19}$$

$$\sin\beta^\bullet = \cos\psi = \frac{z^*}{r} \tag{B20}$$

$$\begin{aligned} \sin\alpha^\bullet &= \cos\alpha \\ &= \frac{1}{\sin\psi} * (-\cos\phi^* \cos\lambda^* \sin\phi\cos\lambda \\ &\quad - \cos\phi^* \sin\lambda^* \sin\phi\sin\lambda + \sin\phi^* \sin\phi) \end{aligned} \tag{B21}$$

$$\cos\alpha^\bullet = \sin\alpha = \frac{1}{\sin\psi} * (-\cos\phi^* \cos\lambda^* \sin\lambda + \cos\phi^* \sin\lambda^* \cos\lambda) \tag{B22}$$

$$\begin{aligned} \sin\beta^\bullet = \cos\psi &= \cos\phi^* \cos\lambda^* \cos\phi\cos\lambda + \cos\phi^* \sin\lambda^* \cos\phi\sin\lambda \\ &\quad + \sin\phi^* \sin\phi \end{aligned} \tag{B23}$$

$$\Leftrightarrow$$

$$\boxed{\begin{aligned} \sin\alpha^\bullet &= \cos\alpha = \frac{\sin\phi^* \cos\phi - \cos\phi^* \sin\phi\cos(\lambda - \lambda^*)}{\sin\psi} && \text{(B24)} \\ \cos\alpha^\bullet &= \sin\alpha = \frac{\cos\phi^* \sin(\lambda^* - \lambda)}{\sin\psi} = -\frac{\cos\phi^* \sin(\lambda - \lambda^*)}{\sin\psi} && \text{(B25)} \\ \sin\beta^\bullet &= \cos\psi = \cos\phi\cos\phi^* \cos(\lambda - \lambda^*) + \sin\phi\sin\phi^* && \text{(B26)} \end{aligned}}$$

spherical trigonometry

$\sin\alpha$	spherical sine lemma
$\cos\alpha$	spherical sine–cosine lemma
$\cos\psi$	spherical side cosine lemma

$\mathbb{S}^2$ while $\{\mathbf{e}_{3^*} \mid \mathbf{0}\}$ towards the meta-North pole, the direction to the moving point P.

How is the oblique frame of reference $\{\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*} \mid \mathbf{0}\}$ at the origin $\mathbf{0}$ generated?

Again we will follow a path which we can use when we consider more general geodetic coordinates (such as ellipsoidal longitude, ellipsoidal latitude, ellipsoidal height) with respect to the internaionl reference ellipsoid $\mathbb{E}^2_{A,A,B}$ which may replace the international reference sphere $\mathbb{S}^2_R$. First we compute the tangent vectors $\{\mathbf{e}_\lambda, \mathbf{e}_\phi, \mathbf{e}_r \mid P\}$ which build up an orthonormal frame of reference at $P \sim (\lambda, \phi, r)$. Under the postulate of Euclidean parallelism (Euclid's axiom five) we transport this frame of reference ('moving frame') from the moving point P^* to the origin $\mathbf{0}$. In this way we identify by means of Eq. (A2) $\mathbf{e}_{1\bullet} = \mathbf{e}_{\lambda^\bullet}, \mathbf{e}_{2\bullet} = \mathbf{e}_{\phi^\bullet}, \mathbf{e}_{3\bullet} = \mathbf{e}_{r^\bullet}$, namely we generate the 'dot orthonormal triad'. As shown in Sect. 2, these frames of reference are related by means of an orthonormal matrix [Eqs. (B3), (B4), (B5) and (B6)].

Second, we express the Cartesian 'dot coordinates' $(x^\bullet, y^\bullet, z^\bullet)$ as well as the Cartesian 'initial coordinates' (x^*, y^*, z^*) in terms of spherical coordinates, namely by means of Eqs. (B7), (B8), (B9) and (B10). $(\alpha^\cdot, \beta^\cdot, r^\cdot)$ will be called (meta-longitude $\alpha^\cdot$, meta-latitude $\beta^\cdot$, radius $r^\cdot$). Meta-longitude $\alpha^\cdot$ is also referred to as 'azimuth with

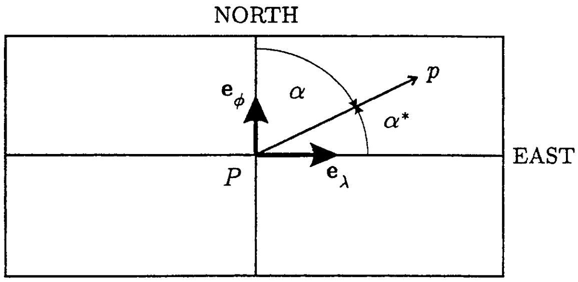

Fig. B3. Tangent space $\mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_r$ (tangent plane) at $\mathbf{x} \sim (\lambda, \phi, r)$, $p = \pi(P)$ azimuthal projection of the point $P \sim \mathbf{x}$ onto $\mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_r$, Northern (left-hand) azimuth α^*

respect to East', being right-handed. The representation of the placement vector $\mathbf{x}^*$ [Eqs. (B11), (B12)] in terms of Cartesian initial coordinates (x^*, y^*, z^*) as well as spherical initial coordinates (λ^*, ϕ^*, r^*) is transformed via Eq. (B5) from the old basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \mid \mathbf{0}\}$ to the new basis $\{\mathbf{e}_{1^\bullet}, \mathbf{e}_{2^\bullet}, \mathbf{e}_{3^\bullet} \mid \mathbf{0}\}$. For the case $r = r^*$, Eq. (B13) contains the placement vector [Eq. (B1)] in the new basis. At this point we switch from meta-latitude $\beta^\bullet$ to meta-colatitude $\psi = \frac{\pi}{2} - \beta^\bullet$ and from meta-longitude/right-handed East azimuth $\alpha^\bullet$ to left-handed North azimuth $\alpha = \frac{\pi}{2} - \alpha^\bullet$, namely by using Eqs. (B15) and (B16), as is conventionally done in geometric and physical geodesy. Consult also Fig. B3! Note Eq. (B14), where we have transformed meta-colatitude ψ (meta-polar distance ψ), now with positive sign and representing the arc PP^*.

Third, we compare Eqs. (B13) and (B17) in order to compute [Eqs. (B18), (B21), (B24)] $\cos\alpha$, [Eqs. (B19), (B22), (B25)] $\sin\alpha$ and [Eqs. (B20), (B23)] $\cos\psi$. We could have derived these formulae by means of spherical trigonometry, namely $\sin\alpha$ by the law of sine–cosine and $\cos\psi$ by the law of side cosine.

Appendix C

Transformation $(\lambda^*, \phi^*) \mapsto (\alpha, \psi)$

In order to perform the computation of the incremental gravitational potential from vertical deflection data or, via upward–downward continuation, vertical deflections in space from vertical deflections on $\mathbb{S}^2_R$ or $\mathbb{E}^2_{A,B}$, we are confronted from spherical integration via quadrature, in particular of kernels which contain (ψ^*, α^*). Most integrations are performed in a local tangent space $\mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_r$ at the evaluation point (λ, ϕ, r), taking advantage of a 'fast approximation', namely a properly chosen map projection (equidistant, equiareal or conformal) of moving points onto the tangent space $\mathbb{T}_{(\lambda,\phi)}\mathbb{S}^2_r$ at (λ, ϕ, r). Here we meet the problem of dealing with $(\psi^*, \cos\alpha^*, \sin\alpha^*)$, which relate as meta-polar distance ψ^* and North (left-handed) azimuth α^* to the moving point P^*. It is therefore reasonable to transform (ψ^*, α^*) at P^* to (ψ, α) at P.

According to Table C1, Eq. (C1), we have already used the transformation $\psi^* \mapsto \psi = -\psi^*$ or $\psi^* = -\psi$. For the derivation of the transformation $\alpha \mapsto \alpha^*$ we recollected the surface gradient operator at P^* (meta-North pole) by means of Eqs. (C2), (C3), (C4) and (C5) and at P (meta-North pole) by means of Eqs. (C6), (C7), (C8) and (C9). From Appendix A we transplanted via Eqs. (C10) and (C11) the transformation formulae $(\cos\alpha^*, \sin\alpha^*)$ and from Appendix B via Eqs. (C12) and (C14) the transformation formulae $(\cos\alpha, \sin\alpha)$. In dividing the terms in Eqs. (C11) and (C13) we find the spherical sine law [Eq. (C14)] which is solved by means of Eq. (C15) to be $\sin\alpha^*(\sin\alpha)$. In order to derive $\cos\alpha^*(\cos\alpha)$ we have two choices. *Either* we solve Eqs. (C10) and (C12) for $\cos\alpha^*$ [Eq. (C16)] *or*, by means of $\sqrt{1 - \sin^2\alpha^*} = \cos\alpha^*$ with given $\sin\alpha^*$, Eq. (C15) for $\cos\alpha^*$ [Eq. (C17)]. The transformation formulae, Eq. (C1), $\psi^* = -\psi$, and Eqs. (C15) and (C17), $\sin\alpha^*(\sin\alpha)$ and $\cos\alpha^*(\cos\alpha)$, finally summarize our results.

Since the upward-continuation operators of vertical deflections contain products of types (1) $\sin\alpha^*(\sin\alpha)$, (2)

Table C1. Transformation $(\lambda^*, \phi^*) \mapsto (\alpha, \psi)$

meta-North pole P^*

$$\psi^* = -\psi \text{ versus } \psi = -\psi^* \tag{C1}$$

$$D_{\lambda^*} f(\psi) = -\frac{1}{\sin\psi}\cos\phi\cos\phi^*\sin(\lambda - \lambda^*) D_\psi f(\psi) \tag{C2}$$

$$D_{\phi^*} f(\psi) = -\frac{1}{\sin\psi}[\sin\phi\cos\phi^* - \cos\phi\sin\phi^*\cos(\lambda - \lambda^*)] D_\psi f(\psi) \tag{C3}$$

$$D_{\lambda^*} f(\psi) = \cos\phi^*\sin\alpha^* D_\psi f(\psi) \tag{C4}$$

$$D_{\phi^*} f(\psi) = \cos\alpha^* D_\psi f(\psi) \tag{C5}$$

meta-North pole P

$$D_\lambda f(\psi) = \frac{1}{\sin\psi}\cos\phi\cos\phi^*\sin(\lambda - \lambda^*) D_\psi f(\psi) \tag{C6}$$

$$D_\phi f(\psi) = -\frac{1}{\sin\psi}[\cos\phi\sin\phi^* - \sin\phi\cos\phi^*\cos(\lambda - \lambda^*)] D_\psi f(\psi) \tag{C7}$$

$$D_\lambda f(\psi) = -\cos\phi\sin\alpha D_\psi f(\psi) \tag{C8}$$

$$D_\phi f(\psi) = -\cos\alpha D_\psi f(\psi) \tag{C9}$$

$$\cos\alpha^* = \frac{1}{\sin\psi}[\cos\phi\sin\phi^*\cos(\lambda - \lambda^*) - \sin\phi\cos\phi^*] \tag{C10}$$

$$\sin\alpha^* = \frac{1}{\sin\psi}\sin(\lambda - \lambda^*) \tag{C11}$$

versus

$$\cos\alpha = \frac{1}{\sin\psi}[\cos\phi\sin\phi^* - \sin\phi\cos\phi^*\cos(\lambda - \lambda^*)] \tag{C12}$$

$$\sin\alpha = -\frac{1}{\sin\psi}\cos\phi^*\sin(\lambda - \lambda^*) \tag{C13}$$

$$\frac{\sin\alpha^*}{\cos\phi} = \frac{\sin\alpha}{\cos\phi^*} \tag{C14}$$

$$\boxed{\sin\alpha^* = \tfrac{\cos\phi}{\cos\phi^*}\sin\alpha} \tag{C15}$$

$$\cos\alpha^* = -\frac{\sin\phi\cos\phi^*}{\sin\psi} + \cot\phi\tan\phi^*\cos\alpha + \frac{\cot\phi\cos\phi\tan\phi^*\sin\phi^*}{\sin\psi} \tag{C16}$$

$$\boxed{\cos\alpha^* = \tfrac{1}{\cos\phi^*}\sqrt{\cos^2\phi^* - \cos^2\phi + \cos^2\phi\cos^2\alpha}} \tag{C17}$$

$$\sin\alpha\sin\alpha^* = \frac{\cos\phi}{\cos\phi^*}\sin^2\alpha \tag{C18}$$

$$\sin\alpha\cos\alpha^* = \frac{\sin\alpha}{\cos\phi^*}\sqrt{\cos^2\phi^* - \cos^2\phi + \cos^2\phi\cos^2\alpha} \tag{C19}$$

$$\cos\alpha\sin\alpha^* = \frac{\cos\phi}{\cos\phi^*}\cos\alpha\sin\alpha \tag{C20}$$

$$\cos\alpha\cos\alpha^* = \frac{1}{\cos\phi^*}\cos\alpha\sqrt{\cos^2\phi^* - \cos^2\phi + \cos^2\phi\cos^2\alpha} \tag{C21}$$

$\sin\alpha\cos\alpha^*$, (3) $\cos\alpha\sin\alpha^*$ and (4) $\cos\alpha\cos\alpha^*$, they have been collected through Eqs. (C15) and (C17) within Eqs. (C18)–(C21) as functions of types (1) $\sin\alpha\sin\alpha$, (2) $\sin\alpha\sqrt{\cdots\cos^2\alpha}$, (3) $\cos\alpha\sin\alpha$ and (4) $\cos\alpha\sqrt{\cdots\cos^2\alpha}$.

Appendix D

Transformation of the surface gradient

In deriving the upward continuation operator of vertical deflections (the surface gradient) we had to relate the surface gradient at an evaluation point in space external to the reference sphere to the surface gradient at the moving point on the reference sphere. Here we derive such a transformation in a more systematic way.

First, by means of Eqs. (D1) and (D2) of Table D we take the longitudinal derivative of a function $f(\psi)$ which depends on 'the spherical distance' ψ between an evaluation point and a moving point, namely with respect to λ or λ^*. Equation (D3) relates D_{λ^*} to D_λ and vice versa. Second, by means of Eqs. (D4) and (D5), as well as Eqs. (D6), (D7), (D8), (D9) and (D10), we consider the lateral derivative of a function $f(\psi)$ with respect to ϕ and ϕ^*. Equation (D11) highlights the transformation of $D_\phi f(\psi)$ into $D_{\phi^*} f(\psi)$ with respect to an additive term which is proportional to $D_\psi f(\psi)$. In contrast, via Eq. (D12) we arrive at the inverse transformation of $D_{\phi^*} f(\psi)$ into $D_\phi f(\psi)$ with respect to another additive term which is proportional to $D_\psi f(\psi)$. Finally, by means of Eqs. (D14) and (D15) we define the symbols ∇_S and ∇_S^* for the surface gradient with respect to (λ, ϕ) and (λ^*, ϕ^*).

Table D. Transformation of the surface gradient

$$D_{\lambda^*} f(\psi) = -\frac{1}{\sin\psi}\cos\phi\cos\phi^*\sin(\lambda-\lambda^*)D_\psi f(\psi) \quad \text{(D1)}$$

$$D_\lambda f(\psi) = +\frac{1}{\sin\psi}\cos\phi\cos\phi^*\sin(\lambda-\lambda^*)D_\psi f(\psi) \quad \text{(D2)}$$

$$D_{\lambda^*} f(\psi) = -D_\lambda f(\psi) \quad \text{versus} \quad D_\lambda f(\psi) = -D_{\lambda^*} f(\psi) \quad \text{(D3)}$$

$$D_{\phi^*} f(\psi) = \frac{1}{\sin\psi}[\cos\phi\sin\phi^*\cos(\lambda-\lambda^*) - \sin\phi\cos\phi^*]D_\psi f(\psi) \quad \text{(D4)}$$

$$D_\phi f(\psi) = \frac{1}{\sin\psi}[\sin\phi\cos\phi^*\cos(\lambda-\lambda^*) - \cos\phi\sin\phi^*]D_\psi f(\psi) \quad \text{(D5)}$$

$$\sin\psi D_{\phi^*} = \cos\phi\sin\phi^*\cos(\lambda-\lambda^*) - \sin\phi\cos\phi^* D_\psi \quad \text{(D6)}$$

$$\sin\psi D_\phi = \sin\phi\cos\phi^*\cos(\lambda-\lambda^*) - \cos\phi\sin\phi^* D_\psi \quad \text{(D7)}$$

$$\cos\phi\sin\phi^*\cos(\lambda-\lambda^*)D_\psi = \sin\psi D_{\phi^*} + \sin\phi\cos\phi^* D_\psi \quad \text{(D8)}$$

$$\sin\phi\cos\phi^*\cos(\lambda-\lambda^*)D_\psi = \sin\psi D_\phi + \cos\phi\sin\phi^* D_\psi \quad \text{(D9)}$$

Eq. (D9) multiplied by $\cot\phi\tan\phi^*$

$$\cos\phi\sin\phi^*\cos(\lambda-\lambda^*)D_\psi = \sin\psi\cot\phi\tan\phi^* D_\phi + \cot\phi\cos\phi\tan\phi^*\sin\phi^* D_\psi \quad \text{(D10)}$$

$$\boxed{D_{\phi^*} f(\psi) = \cot\phi\tan\phi^* D_\phi f(\psi) + \frac{\cos^2\phi\sin^2\phi^* - \sin^2\phi\cos^2\phi}{\sin\psi\sin\phi\cos\phi^*}D_\psi f(\psi)} \quad \text{(D11)}$$

Eq. D(8) multiplied by $\tan\phi\cot\phi^*$

$$\sin\phi\cos\phi^*\cos(\lambda-\lambda^*)D_\psi = \sin\psi\tan\phi\cot\phi^* D_{\phi^*} + \tan\phi\sin\phi\cot\phi^*\cos\phi^* D_\psi \quad \text{(D12)}$$

$$\boxed{D_\phi f(\psi) = \tan\phi\cot\phi^* D_{\phi^*} f(\psi) + \frac{\sin^2\phi\cos^2\phi^* - \cos^2\phi\sin^2\phi^*}{\cos\phi\sin\phi^*}D_\psi f(\psi)} \quad \text{(D13)}$$

$$\nabla_S := \mathbf{e}_\lambda\frac{1}{r\cos\phi}D_\lambda + \mathbf{e}_\phi\frac{1}{r}D_\phi \quad \text{(D14)}$$

$$\nabla_S^* := \mathbf{e}_{\lambda^*} + \frac{1}{r\cos\phi^*}D_{\lambda^*} + \mathbf{e}_{\phi^*}\frac{1}{r}D_{\phi^*} \quad \text{(D15)}$$

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