CLOSED-FORM TRANSFORMATION BETWEEN GEODETIC AND ELLIPSOIDAL COORDINATES

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ABSTRACT

We present formulas for direct closed-form transformation between geodetic coordinates (ϕ, λ, h) and ellipsoidal coordinates (β, λ, u) for any oblate ellipsoid of revolution. These will be useful for those dealing with ellipsoidal representations of the Earth's gravity field or other oblate ellipsoidal figures. The numerical stability of the transformations for near-polar and near-equatorial regions is also considered.

Key words: geodetic coordinates, ellipsoidal coordinates, coordinate transformations, ellipsoidal geodesy

1. INTRODUCTION

1.1. Generic Notation

a .... semi-major axis length of an ellipsoid
$b = a(1-f)$ ...... semi-minor axis length of an ellipsoid
$f = (a-b)/a$ .... flattening of an ellipsoid (dimensionless)
$e = \sqrt{2f - f^2}$ ... first numerical eccentricity of an ellipsoid (dimensionless)
$E = ae$ .... linear eccentricity (or focal length) of an ellipsoid
(ϕ, λ, h) .... geodetic coordinates: latitude, longitude and height
(β, λ, u) .... ellipsoidal coordinates: ellipsoidal co-latitude, longitude and ellipsoidal parameter
(X, Y, Z) .... 3D Cartesian coordinates aligned with the ellipsoids' axes

1.2. Geodetic Coordinates (ϕ, λ, h)

Transformation between geodetic coordinates (geodetic latitude ϕ, longitude λ and ellipsoidal/geodetic height h) and 3D Cartesian coordinates (X, Y, Z) has been studied repeatedly over the years (Table 1). It has attracted the attention of not only geodesists, but also aerospace engineers and others. Curiously, there seems to have been little cross-fertilization of ideas, with some being repeated in the literature.

Table 1. A list of studies on the $(\phi, \lambda, h) \leftrightarrow (X, Y, Z)$ coordinate transformation, with closed-form solutions identified by **CF**. Studies published in the Russian and Eastern European literature are not cited; instead see *Lapaine (1990)*.

Barbee (1982)	*Bartelme and Meissl (1975)*
Bencini (1968)	*Benning (1974,1987)*
Berger and Ricupito (1960)	*Bopp and Krauss (1976)*
Borkowsky (1987,1989) **CF**	*Bowring (1976, 1985)*
Carlson (1980)	*Churchyard (1986)*
Crocetto (1993)	*Deprit and Deprit-Bartholome (1975)*
Ecker (1967) **CF**	*Feltens (2008)*
Fotiou (1998) **CF**	*Frölich and Hansen (1976)* **CF**
Fukushima (1999,2006)	*Gen (1981)*
Gersten (1961)	*Grafarend (2001)* **CF**
Grafarend and Lohse (1991)	*Guo (2001)*
Heck (1987)	*Hedgely (1976)* **CF**
Hedman (1970)	*Heikkinen (1982)* **CF**
Heindl (1997)	*Heiskanen and Moritz (1967)*
Hekimoğlu (1995)	*Hirvonen (1964)*
Hirvonen and Moritz (1963)	*Hsu (1992)* **CF**
Jones (2002) **CF**	*Keeler and Nievergelt (1998)*
Lapaine (1990) **CF**	*Laskowski (1991)*
Lester Jones (1922)	*Levin (1988)*
Lin and Wang (1995)	*Long (1975)*
Lupash (1985)	*Mayer (1978)*
Morrison and Pines (1961)	*Nautiyal (1988)*
Nievergelt and Keeler (2000)	*Olson (1988)*
Olson (1996)	*Ozone (1985)* **CF**
Paul (1973) **CF**	*Pavlov (1967)*
Penev (1978) **CF**	*Pick (1967,1985)* **CF**
Pollard (2002,2005)	*Purcell and Cowan (1961)*
Rinner (1978)	*Sjöberg (1999)*
Sofair (1997,2000)	*Soler and Hothem (1989)*
Sudano (1997)	*Sugai (1967)* **CF**
Sünkel (1999) **CF**	*Toms (1995,1996,1998)*
Vermeille (2002,2004) **CF**	*Vincenty (1976,1978,1980)*
Wu et al. (2003)	*You (2000)*
Zhang et al. (2005) **CF**	*Zhu (1993,1994)*

It is first instructive to point out that while several of the studies cited in Table 1 term this $(\phi, \lambda, h) \leftrightarrow (X, Y, Z)$ coordinate transformation 'geocentric from/to Cartesian', it need not be. It is equally applicable to non-geocentric geodetic ellipsoids, such as those used in conjunction with local geodetic datums, which is needed for Helmert-type horizontal datum transformations. Likewise, the geodetic from/to ellipsoidal coordinate (β, λ, u) transformations presented here do not have to be geocentric.

Referring to Fig. 1, the origin of the Cartesian (X, Y, Z) coordinate system must coincide exactly with the geometrical centre (not the focii) of the geodetic ellipsoid used as the reference surface for the geodetic (ϕ, λ, h) coordinate system. The Cartesian Z axis must lie along the minor axis of the geodetic ellipsoid, pointing positively towards the north pole. Not shown in Fig. 1, the Cartesian X axis must lie in the equatorial plane of the

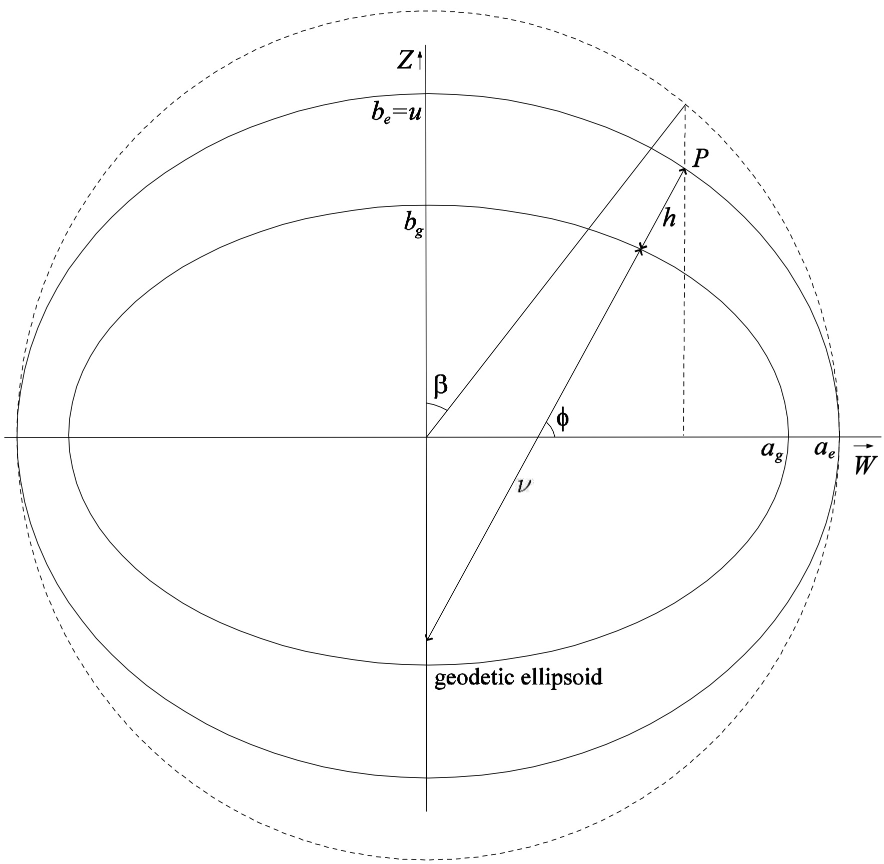

Fig. 1. Meridional cross section (because of rotational symmetry) of the relationships among 2D Cartesian (Z, W), geodetic (ϕ, h) and (oblate) ellipsoidal (β, u) coordinates for an arbitrary point $\boldsymbol{P}$.

geodetic ellipsoid, pointing positively towards the Greenwich meridian, by convention. The Cartesian Y axis completes a right-handed coordinate system, thus pointing positively towards 90°E in the equatorial plane. In Fig. 1, $W^2 = X^2 + Y^2$.

The geodetic ellipsoid is the geometrical figure mapped by an oblate (i.e., flattened towards the poles) ellipse rotated about its minor axis. It is sometimes synonymously called a spheroid in the geodetic literature (e.g., *Bomford, 1980*). The geodetic coordinate system is two-parametric, requiring the a priori selection of the numerical values of the semi-major axis a_g and first numerical eccentricity e_g (or the latter's equivalents: flattening f_g or semi-minor axis b_g). The subscript g is used to distinguish the parameters for the geodetic ellipsoid, so as to maintain generality throughout.

Again referring to Fig. 1, the geodetic latitude ($-\pi/2 \le \phi \le \pi/2$) is the angle between the equatorial plane (X-Y plane or W axis in Fig. 1) and the normal to the surface of the geodetic ellipsoid that passes through the point of interest $\boldsymbol{P}$, measured positively northwards in the meridional (Z-W) plane. The ellipsodial/geodetic height h is the distance along the surface normal from the geodetic ellipsoid to the point of interest $\boldsymbol{P}$, measured positively outwards. Not shown in Fig. 1, the geodetic longitude λ ($-\pi \le \lambda \le \pi$) is the angle between the Greenwich meridian (Z-X plane) and the meridian through the point of interest $\boldsymbol{P}$ (Z-W plane), often measured positively eastwards and negatively westwards in the equatorial plane.

Under these conditions, the geodetic coordinates (ϕ, λ, h) are related to the *corresponding* Cartesian coordinates (X, Y, Z) by (e.g., *Heiskanen and Moritz, 1967; Bowring, 1976; Pollard, 2002*); cf. Fig. 1:

$$X = \left(\nu_g + h\right)\cos\phi\cos\lambda\,, \tag{1}$$

$$Y = \left(\nu_g + h\right)\cos\phi\sin\lambda\,, \tag{2}$$

$$Z = \left\{\nu_g\left(1 - e_g^2\right) + h\right\}\sin\phi\,, \tag{3}$$

where the radius of curvature in the prime vertical of the surface of the geodetic ellipsoid is (e.g., *Bomford, 1980; Vaníček and Kleusberg, 1982*)

$$\nu_g = \frac{a_g}{\sqrt{1 - e_g^2\sin^2\phi}} = \frac{a_g^2}{\sqrt{a_g^2\cos^2\phi + b_g^2\sin^2\phi}}\,. \tag{4}$$

This forward transformation $(\phi, \lambda, h) \to (X, Y, Z)$ is both straightforward and numerically stable, but the reverse $(X, Y, Z) \to (\phi, \lambda, h)$ is not so. This has led to a plethora of iterative, approximative and closed-form transformation equations that have been presented (Table 1). However, the user must be careful when interpreting some authors' claims of a closed-form or non-iterative equation, because most of them really only hold this property for $h = 0$.

Numerical comparisons among some of the methods in Table 1 have been conducted by, e.g., *Laskowski (1991), Barrio and Riaguas (1993), Gerdan and Deakin (1999),* *Seemkooei (2002), Fok and Bâki Iz (2003)* and *Burtch (2006)*, in addition to comparisons made by the proposers of some of the approaches cited in Table 1. As far as we are aware, no comprehensive comparison of all the different methods has yet been made, but that is beyond the scope of this paper. However, the 2003 IERS Standards (*McCarthy and Petit 2004*; *http://www.iers.org/documents/publications/tn/tn32/tn32.pdf*) still endorse *Borkowski's (1989)* method.

1.3. Ellipsoidal Coordinates (β, λ, u)

A relatively less-well-known coordinate system, but which is well-suited to describing positions on, in and above the Earth's surface, is the (oblate) ellipsoidal coordinate system, comprising the ellipsoidal co-latitude β, ellipsoidal longitude λ and ellipsoidal parameter u (Fig. 1). This is a special degenerate case of the spheroidal (i.e., triaxial ellipsoid) coordinate system (e.g., *Hobson, 1931*) but which is of much more geodetic relevance.

One notable benefit is that the ellipsoidal coordinate system is triply orthogonal and thus permits a separation of variables for the solution of Laplace's equation, resulting in ellipsoidal harmonics (e.g., *Darwin, 1901a,b; Hobson, 1931; Byerly, 1959; Heiskanen and Moritz, 1967; Moon and Spencer, 1988*). As such, it is particularly well-suited to representing the Earth's gravity field (e.g., *Jekeli, 1988; Thông and Grafarend, 1989; Sansò and Sona, 2001; Finn and Grafarend, 2004; Bölling and Grafarend, 2005; Grafarend et al., 2006*).

While a complete separation of variables can be achieved in ellipsoidal, spherical-polar or Cartesian coordinates, it cannot be achieved for the geodetic coordinates, apart from *on* the surface (i.e., $h = 0$) of the geodetic ellipsoid (*Grafarend, 1988*). This inseparability problem arises because h and ϕ are closely related in the geodetic coordinate system (see Eqs.(1)–(4)). However, this has not prevented geodetic coordinates from uniquely locating points on or near the Earth's surface, as hundreds of years of geodetic practice do testify.

Due to this inseparability, full 3D harmonic analysis of the Earth's gravity field is not possible in geodetic coordinates. Instead, spherical or ellipsoidal harmonics have to be used, with spherical harmonics still being the most popular for gravity field synthesis (e.g. see the list of spherical harmonic Earth gravity models at the International Centre for Global Earth Models (ICGEM), *http://icgem.gfz-potsdam.de/ICGEM/ICGEM.html*).

Also in this case, the origin of the Cartesian coordinate system must coincide exactly with the geometrical centre (not the focii) of the confocal ellipsoids comprising the ellipsoidal coordinate system; the Cartesian X axis lies in the equatorial plane towards the Greenwich meridian; the Cartesian Z axis lies along the minor axis of the confocal ellipsoids forming the u coordinate in the ellipsoidal coordinate system; the Cartesian Y axis completes a right-handed rectangular system.

Therefore, the Cartesian origin and axes are perfectly aligned for both the geodetic and ellipsoidal coordinate systems. If not, they can easily be aligned using a six-parameter Helmert transformation. There is no need to employ a scale change when dealing with a rigid coordinate system; see *Featherstone and Vaníček (1999)* or *Vaníček et al. (2002)*.

The ellipsoidal co-latitude ($0 \leq \beta \leq \pi$) is the angle, measured positively southwards in the meridional (Z-W) plane, between the minor axis of the ellipsoidal coordinates (or the

Cartesian Z axis) and the radius vector from the origin of the confocal ellipsoids to a point on a circumscribing sphere (of radius $\sqrt{u^2+E^2}$) above the point of interest $\boldsymbol{P}$ on a line parallel to the minor axis of the confocal ellipsoids (Fig. 1). The ellipsoidal parameter u defines the length of the semi-minor axis of each confocal ellipsoid. Coupled with the linear eccentricity parameter E, this causes a unique confocal ellipsoid (u = constant) to pass through the point of interest. The ellipsoidal longitude λ is identical to the geodetic longitude when the corresponding Cartesian axes are aligned.

Under these conditions, the ellipsoidal coordinates (β, λ, u) are related to Cartesian coordinates (X, Y, Z) by (e.g., *Hobson, 1931; Heiskanen and Moritz, 1967, Eq.1-103; Vaníček and Krakiwsky, 1982; Gleason, 1988*), cf. Fig. 1:

$$X = \sqrt{u^2 + E_e^2}\,\sin\beta\cos\lambda\,, \tag{5}$$

$$Y = \sqrt{u^2 + E_e^2}\,\sin\beta\sin\lambda\,, \tag{6}$$

$$Z = u\cos\beta\,, \tag{7}$$

where, in this deliberately generic case, the linear eccentricity is

$$E_e^2 = a_e^2 e_e^2 = a_e^2 - b_e^2\,, \tag{8}$$

where the subscript e is used to distinguish the ellipsoidal coordinate system. Finally, the ellipsoidal coordinate system is one-parametric and triply orthogonal.

Throughout this paper, only the geodetically relevant case of oblate ellipsoidal coordinates will be used. The special degenerate case when E_e and u together yield the form of the geodetic ellipsoid will be given later. Prolate ellipsoids are not considered because they are of lesser geodetic relevance, but the equations here can be adapted to that case if needed.

In this paper, we will present direct closed-form forward and reverse transformations between geodetic and ellipsoidal coordinates. As well as being of curiosity-driven interest (we are currently unaware of any previously presented work explicitly on this, apart from a passing mention in *Vaníček and Krakiwsky (1982)*), this transformation will be of use, e.g., to those who want to directly display the results from an ellipsoidal harmonic synthesis on a map, among other applications.

The motivation is to give a one-stage closed-form transformation between 3D geodetic and 3D ellipsoidal coordinates, thus eliminating the arbitrary error introduced by using iterative techniques. We acknowledge that this transformation could also be achieved by staging it via Cartesian coordinates (e.g., geodetic to Cartesian then Cartesian to ellipsoidal and *vice versa*), but a one-stage transformation is attractive for reasons of simplicity, and it will run faster if coded well (cf. *Fukushima, 2006*).

2. GEODETIC TO ELLIPSOIDAL TRANSFORMATION: (ϕ, λ, h) $\rightarrow$ (β, λ, u)

By definition, the longitudes in the geodetic and ellipsoidal coordinate systems are identical when the corresponding Cartesian axes are aligned, which is evident by inspection of Eqs.(1) and (2) and Eqs.(5) and (6). Dividing Eqs.(5) and (6) yields

$$\lambda = \arctan\frac{Y}{X}, \tag{9}$$

where the signs of X and Y are used to select the appropriate quadrant for the longitude λ. However, there are some limitations on the numerical stability of Eq.(9). *Vaníček and Krakiwsky (1982)* give a form that is numerically more stable near the poles, which is

$$\lambda = 2\arctan\frac{Y}{X + \sqrt{X^2 + Y^2}}. \tag{10}$$

Vermeille (2004) points out that Eq.(10) is still problematic, giving a more numerically stable variant

$$\lambda = \frac{\pi}{2} - 2\arctan\frac{X}{\sqrt{X^2 + Y^2} + Y} \qquad \text{for } Y \geq 0, \tag{11}$$

$$\lambda = -\frac{\pi}{2} + 2\arctan\frac{X}{\sqrt{X^2 + Y^2} - Y} \qquad \text{for } Y < 0. \tag{12}$$

However, this coordinate will not be considered further here because the focus of this paper is not on this transformation as the geodetic and ellipsoidal longitudes are identical.

The geodetic to ellipsoidal coordinate transformation thus reduces to $(\phi, h) \rightarrow (\beta, u)$. First, we consider the most general case for an arbitrary geodetic ellipsoid (a_g, e_g) and an arbitrary (oblate) ellipsoidal coordinate system.

In order to find the ellipsoidal parameter u from the geodetic ϕ and h, we first sum the squares of Eqs.(1) and (2) and equate this to the sum of the squares of Eqs.(5) and (6)

$$\left(\nu_g + h\right)^2 \cos^2\phi = \left(u^2 + E_e^2\right)\sin^2\beta. \tag{13}$$

Rearranging gives

$$\sin^2\beta = \frac{\left(\nu_g + h\right)^2 \cos^2\phi}{\left(u^2 + E_e^2\right)}. \tag{14}$$

Next, we equate the squares of Eqs.(7) and (3)

$$\left(\nu_g\left(1 - e_g^2\right) + h\right)^2 \sin^2\phi = u^2\cos^2\beta \tag{15}$$

and rearranging gives

$$\cos^2\beta = \frac{\left(\nu_g\left(1 - e_g^2\right) + h\right)^2 \sin^2\phi}{u^2}. \tag{16}$$

Eliminating the ellipsoidal co-latitude β by taking the sum of Eqs.(14) and (16) gives

$$\frac{\left(\nu_g+h\right)^2\cos^2\phi}{\left(u^2+E_e^2\right)}+\frac{\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi}{u^2}=1 \tag{17}$$

which reduces to a biquadratic equation in u (quadratic equation in u^2), according to

$$\begin{gathered}u^4+u^2\left[E_e^2-\left(\nu_g+h\right)^2\cos^2\phi-\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi\right]\\ -E_e^2\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi=0\,,\end{gathered} \tag{18}$$

which has the solution

$$\begin{aligned}u=&\left(\frac{1}{2}\left[\left(\nu_g+h\right)^2\cos^2\phi+\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi-E_e^2\right]\right.\\ &\left.+\sqrt{\frac{1}{4}\left[E_e^2-\left(\nu_g+h\right)^2\cos^2\phi-\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi\right]^2+E_e^2\left(\nu_g\left(1-e_g^2\right)+h\right)^2\sin^2\phi}\right)^{\frac{1}{2}},\end{aligned} \tag{19}$$

where the positive square root has been taken for the second term on the right-hand side of Eq.(19) in order to have a real square root for the whole of the right-hand side.

Once the numerical value of the geodetic parameter u has been found from Eq.(19), the ellipsoidal co-latitude β is given by either the square root of Eq.(14)

$$\sin\beta=\frac{\left(\nu_g+h\right)\cos\phi}{\sqrt{u^2+E_e^2}} \tag{20}$$

or the square root of Eq. (16)

$$\cos\beta=\frac{\left(\nu_g\left(1-e_g^2\right)+h\right)\sin\phi}{u}\,, \tag{21}$$

one of which is selected depending on the proximity to the poles or equator so as to avoid roundoff errors in their computation (through the sine and cosine terms). An alternative formula for non-polar regions is given by division of Eqs.(20) and (21)

$$\tan\beta=\frac{u}{\sqrt{u^2+E_e^2}}\frac{\left(\nu_g+h\right)}{\left(\nu_g\left(1-e_g^2\right)+h\right)}\cot\phi\,. \tag{22}$$

Finally, we consider the degenerate case when the geodetic parameter u and linear eccentricity E_e in the oblate ellipsoidal coordinate system are chosen to coincide exactly with the geodetic ellipsoid, or *vice versa*. Accordingly, h must equal zero in the geodetic coordinate system. We also note that for any ellipsoid

$$a^2 = E^2 + u^2 \,. \tag{23}$$

Therefore, since $u = b_g = b_e$, then $a_g^2 = a_e^2 = \left(E_g^2 + b_g^2\right) = \left(E_e^2 + b_e^2\right)$. Inserting these degenerating conditions in Eq.(22) gives, on the surface of the geodetic ellipsoid,

$$\tan\beta = \frac{a_g}{b_g}\cot\phi = \frac{a_e}{b_e}\cot\phi \,, \tag{24}$$

which is a variant of the well-known relation between the geodetic and reduced latitudes for points *on* the geodetic ellipsoid (e.g., *Bomford, 1980, Eq.A.52*). The difference here is because we are dealing with the reduced co-latitude β (cf. Fig. 1).

3. ELLIPSOIDAL TO GEODETIC TRANSFORMATION: $(\beta, \lambda, u) \rightarrow (\phi, \lambda, h)$

As stated in the Introduction, one option to achieve this transformation is to stage the computations via Cartesian coordinates, i.e., $(\beta, \lambda, u) \rightarrow (X, Y, Z) \rightarrow (\phi, \lambda, h)$. However, the latter transformation is cumbersome, needing accuracy criteria if one selects an approximative or iterative solution (cf. Table 1).

Therefore, we seek a direct closed-form solution for $(\beta, \lambda, u) \rightarrow (\phi, \lambda, h)$, actually $(\beta, u) \rightarrow (\phi, h)$, that does not require these intermediate transformation stages. This offers savings in computational cost (cf. *Fukushima, 2006*) and is not subject to considerations of accuracy versus number of iterations (cf. *Keeler and Nievergelt, 1998*).

3.1. Special Case for $u = b_g$ (Equivalently $h = 0$)

First, we show the special case for points already known to be located on the surface of the geodetic ellipsoid, i.e. $h = 0$ in the geodetic coordinate system. Thus, this transformation simplifies to $(\beta, u) \rightarrow (\phi)$. Dividing the square root of Eq.(13) by the square root of Eq.(15), noting that $h = 0$, gives

$$\tan\phi = \frac{u}{\left(1 - e_g^2\right)\sqrt{u^2 + E_e^2}}\cot\beta = \frac{u a_g^2}{b_g^2\sqrt{u^2 + E_e^2}}\cot\beta \,. \tag{25}$$

A further specific degeneration can be made when the ellipsoidal coordinate system is chosen such that it coincides with the surface of the geodetic ellipsoid, or *vice versa*, to give

$$\tan\phi = \frac{a_g}{b_g}\cot\beta = \frac{a_e}{b_e}\cot\beta \,. \tag{26}$$

Equations (25) and (26) are closed-form because of the a priori condition of $h = 0$. This is expected because closed-form equations exist for the transformation from Cartesian to geodetic coordinates when $h = 0$ (cf. *Bowring, 1976*). It is also a by-product of the ability to separate geodetic coordinates for Laplace's equation on the surface of the geodetic ellipsoid (cf. *Grafarend, 1988*).

3.2. General Case for $u \neq b_g$ (Equivalently All h)

Equating Eqs.(1) and (5) or, identically, equating Eqs.(2) and (6), gives

$$\left(\nu_g + h\right)\cos\phi = \sqrt{u^2 + E_e^2}\,\sin\beta\,, \tag{27}$$

then rearranging gives

$$h = \frac{\sqrt{u^2 + E_e^2}\,\sin\beta}{\cos\phi} - \nu_g\,. \tag{28}$$

Likewise, equating Eqs.(3) and (7) gives

$$\left(\nu_g\left(1 - e_g^2\right) + h\right)\sin\phi = u\cos\beta \tag{29}$$

and rearranging gives

$$h = \frac{u\cos\beta}{\sin\phi} - \nu_g\left(1 - e_g^2\right). \tag{30}$$

Equations (28) and (30) require the geodetic latitude ϕ be known, which is our next aim. Dividing them leaves (cf. Eq.(22))

$$\tan\phi = \left(\frac{u}{\sqrt{u^2 + E_e^2}}\right)\left(\frac{\nu_g + h}{\nu_g\left(1 - e_g^2\right) + h}\right)\cot\beta\,, \tag{31}$$

which is an iterative formula because ν_g is a function of ϕ. A starting value for the iteration is given by Eq.(25), but the rate of convergence will depend upon the magnitude of the ellipsoidal height h. *Bowring's (1976)* formula could also be used as a starting value. However, the problem with iterative solutions is that users have to run numerical tests to determine the desired convergence criterion, which is of concern for accuracy-critical applications (cf. *Keeler and Nievergelt, 1998*), and approximative solutions are naturally subject to the approximation errors.

Instead, here we prefer a closed-form solution. Essentially this is a minimum-distance mapping (Helmert projection; cf. *Grafarend and Lohse, 1991; Crocetto, 1993*) problem for a point with respect to the geodetic ellipsoid, which can be solved using quartic equations (e.g., *Paul, 1973; Vaníček and Krakiwsky, 1982; Borkowski, 1987,1988,1989; Ozone, 1995*), Lagrange parameters (e.g., *Zhang et al., 2005*) or vector methods (e.g., *Feltens, 2008*).

3.2.1. *Paul's (1973)* Closed-Form Solution

Here, we choose *Paul's (1973)* quartic solution for the sake of clarity. From a few computational tests, we found virtually no difference in computation speed between this and *Borkowski's (1989)* method. The computational speed could be accelerated by coding to avoid divisions or the use of transcendental functions as much as possible (cf. *Fukushima, 2006*), but we coded *Paul's (1973)* algorithm as published and used *Borkowski's (1989)* original code, which is replicated in the 2003 IERS Standards

(*McCarthy and Petit, 2004*; *http://www.iers.org/documents/publications/tn/tn32/tn32.pdf*). We leave such optimisation to those who are interested, noting that low-cost computer speed increases almost as fast as the time that additional coding and testing for the optimisation would take.

Eliminating h from Eqs.(28) and (30) gives

$$\frac{\sqrt{u^2+E_e^2}\sin\beta}{\cos\phi}=\frac{u\cos\beta}{\sin\phi}+e_e^2\nu_g\,. \tag{32}$$

Inserting the second form of Eq.(4) in Eq.(32) and simplifying yields

$$\frac{a_g^2e_g^2\sin\phi}{\sqrt{a_g^2\cos^2\phi+b_g^2\sin^2\phi}}=\sqrt{u^2+E_e^2}\sin\beta\tan\phi-u\cos\beta\,. \tag{33}$$

Dividing both numerator and denominator of the left-hand side by $\cos\phi$, then squaring both sides and simplifying gives

$$a_g^4e_g^4\tan^2\phi=\left(a_g^2+b_g^2\tan^2\phi\right)\left(\sqrt{u^2+E_e^2}\sin\beta\tan\phi-u\cos\beta\right)^2, \tag{34}$$

then multiplying out and collecting like terms in $\tan\phi$ gives the quartic equation

$$\begin{aligned}&\left(b_g^2\left(u^2+E_e^2\right)\sin^2\beta\right)\tan^4\phi-\left(2b_g^2\sqrt{u^2+E_e^2}\sin\beta u\cos\beta\right)\tan^3\phi\\&+\left(a_g^2\left(u^2+E_e^2\right)\sin^2\beta+u^2b_g^2\cos^2\beta-a_g^4e_g^4\right)\tan^2\phi\\&-\left(2a_g^2\sqrt{u^2+E_e^2}\sin\beta u\cos\beta\right)\tan\phi-a_g^2u^2\cos^2\beta=0\,.\end{aligned} \tag{35}$$

A quartic equation can be solved using Ferrari's method (e.g. *Weisstein 1999, pp. 1489–1491*), which achieves its solution by eliminating the cubic term. This approach was used by *Borkowski (1989)*, through a particular substitution, but requires some selection of the appropriate root. Instead, here we apply the result from *Paul (1973)*, which relies on the solution of a quartic equation through elimination of the quadratic term. This approach was applied to Eq.(35) to give a solution for $\tan\phi$

$$\tan\phi=\frac{\dfrac{u\cos\beta}{2}+\sqrt{D}+\sqrt{\dfrac{u^2\cos^2\beta}{4}-\dfrac{B}{2}+\dfrac{a_gu\cos\beta}{4\sqrt{D}}}}{\sqrt{u^2+E_e^2}\sin\beta}, \tag{36}$$

where

$$A=\frac{\left(u^2+E_e^2\right)\sin^2\beta+a_g^2e_g^4}{1-e_g^2}, \tag{37}$$

$$B = \frac{\left(u^2 + E_e^2\right)\sin^2 \beta - a_g^2 e_g^4}{1 - e_g^2} , \tag{38}$$

$$C = 1 + \frac{27u^2 \cos^2 \beta \left(A^2 - B^2\right)}{2\left(u^2 \cos^2 \beta + B\right)^3} , \tag{39}$$

$$D = \frac{u^2 \cos^2 \beta + B}{12}\left[\left(C + \sqrt{C^2 - 1}\right)^{\frac{1}{3}} + \left(C + \sqrt{C^2 - 1}\right)^{-\frac{1}{3}}\right] - \frac{B}{6} + \frac{u^2 \cos^2 \beta}{12} . \tag{40}$$

Both *Borkowski's (1989)* and *Paul's (1973)* methods become numerically unstable near the equator. Therefore, an additional IF STATEMENT is needed. Our few numerical tests also showed that there was no significant timing difference when this was added (cf. *Fukushima, 2006*). As such, *Paul's (1973, Eq.(9'))* should be used close (say < 5°) to the equator.

3.2.2. *Vermeille's (2002;2004)* Closed-Form Solution

However, *Vermeille (2002;2004)* has presented closed form equations, also based on the solution of a quartic equation, for the $(\phi, \lambda, h) \leftrightarrow (X, Y, Z)$ transformation that are numerically stable near the equator and poles. Therefore, users may prefer these algorithms so as to avoid the use of IF STATEMENTS in *Borkowski's (1989)* and *Paul's (1973)* solutions.

Adapting the algorithm given in *Vermeille (2002, Section 3)* and *Vermeille (2004, Section 2)* to the notation used in this paper gives

$$\phi = 2\arctan \frac{u\cos\beta}{F + \sqrt{F^2 + u^2 \cos^2 \beta}} , \tag{41}$$

where

$$F = \frac{G\sqrt{u^2 + E_e^2}\sin\beta}{G + e_g^2} , \tag{42}$$

$$G = \sqrt{H + I + J^2} - J , \tag{43}$$

$$H = K\left(1 + L + L^{-1}\right), \tag{44}$$

$$I = \sqrt{H^2 + e_g^4 M} , \tag{45}$$

$$J = \frac{e_g^2}{2I}\left(H + I - M\right), \tag{46}$$

$$K = \frac{M + N - e_g^4}{6}, \tag{47}$$

$$L = \sqrt[3]{1 + O + \sqrt{2O + O^2}} , \tag{48}$$

$$M = \frac{\left(1 - e_g^2\right) u^2 \cos^2 \beta}{a_g^2}, \tag{49}$$

$$N = \frac{\left(u^2 + E_e^2\right) \sin^2 \beta}{a_g^2}, \tag{50}$$

$$O = \frac{e_g^4 MN}{4K^3} . \tag{51}$$

Since Eqs.(41)−(51) contain fewer transcendental functions than Eqs.(36)−(40), *Vermeille's (2002)* algorithm, if coded properly, will also run faster than *Paul's (1973)*; cf. *Fukushima (2006)*.

4. CONCLUDING REMARKS

We have presented closed-form equations (plus one iterative equation) to transform between geodetic coordinates (ϕ, λ, h) and ellipsoidal coordinates (β, λ, u). These will be of use, e.g., for those dealing with ellipsoidal harmonic expressions of the Earth's external gravity field, among other applications. The benefit of closed-form equations is that the numerical errors are controllable (e.g., by careful computer coding), rather than having to run tedious numerical experiments for all likely (and perhaps unlikely) scenarios (cf. *Keeler and Nievergelt, 1998*). *Toms (SRI International, Toms R.M., personal communication, 2006)* advised us that a new ISO/IEC 18026 Spatial Reference Model standard on this issue was released as an International Standard in 2006 (*http://standards.sedris.org/18026/index.htm*). This may be where closed-form equations, such as ours, may be beneficial.

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