GEOPHYSICAL RESEARCH LETTERS, VOL. 25, NO. 11, PAGES 2003-2006, JUNE 1, 1998

Equivalent Source Magnetic Dipoles Revisited

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Abstract. Equivalent point source inversion in the rectangular coordinate system has been widely used to reduce satellite magnetic data collected at different altitudes to a common elevation over small areas. This method is based on the expression of the magnetic anomaly caused by a magnetic dipole. Such an expression derived in a spherical coordinate system by *von Frese et al.* [1981] is found erroneous. We point out the errors in *von Frese et al.'s* [1981] formulas and present the correct expression for the magnetic field of a magnetic dipole in a spherical coordinate system.

Introduction

A major difficulty with satellite magnetic data, aside from the effects of the external magnetic field, is the significant variation in the altitude of measurement due to the elliptical orbits of the satellites. POGO's altitude varied by as much as 1100 km [*Langel,* 1990], and Magsat's by about 300 km [*Langel et al.,* 1982], which are comparable with the mean altitude of the satellites. Similar variations are expected for future magnetic satellites such as Ørsted [*Friis-Christensen and Skøtt,* 1997]. A simple but crude way of dealing with the altitude variations is to ignore them and produce a magnetic anomaly map by collapsing the entire data set onto a mid-altitude spherical surface. A usual practice to reduce the effect of these variations is to select data from a narrower range of altitudes which includes the highest density of data, for example between 330 and 500 km altitude for Magsat data [*Arkani-Hamed and Strangway,* 1986]. However, due to the strong attenuation with altitude of small scale magnetic anomalies of crustal origin, the juxtaposition of data collected at high and low altitudes introduces a significant amount of noise, generally comparable in wavelength and amplitude to the actual crustal anomalies. This contaminates the short wavelength component of the satellite anomaly maps, which is the most crucial for delineation of regional tectonics. In the polar regions, the relatively higher density of data makes it possible to derive maps at different altitudes by collapsing data within several narrow altitude ranges [*Alsdorf et al.,* 1994]. The final map is derived by combining the maps produced at different altitudes through a continuation procedure. The higher altitude maps are usually downward continued to retain a significant short wavelength component in the final map. Although this process further reduces the effects of altitude variations, it strongly enhances the high frequency component of external origin.

A better way of deriving a magnetic anomaly map using satellite data with highly varying altitude is through equivalent source inversion [*Mayhew,* 1979]. This method determines the magnetic moments of a set of dipoles on the Earth's surface such that their magnetic anomalies best fit the data. By avoiding the instability of the solution through a proper choice of point source distribution, it is possible to construct a magnetic anomaly map at the mid altitude of the data. As a by product, this process also provides a preliminary information about the lateral variations in the crustal magnetization. *von Frese et al.* [1981] proposed an equivalent source inversion algorithm in the spherical coordinate system and applied it to convert the satellite scalar magnetic anomaly map of the conterminous United States to a radially polarized magnetic anomaly map.

The equivalent source inversion method is based on the expression of the magnetic field at point $\mathbf{r}$ caused by a magnetic dipole located at point $\mathbf{r}_1$. Such an expression can be used as a kernel function in computing the magnetic field of an arbitrary crustal magnetization model. In an attempt to determine the contribution of lithospheric remanent magnetization to satellite magnetic anomalies over the World's oceans, *Dyment and Arkani-Hamed* [1997] encountered serious problems with *von Frese et al.'s* [1981] formulas. These are, to our knowledge, the only explicitly expressed formulas in the literature, and the paper has been cited by many authors (34, based on citation index to date). Since a large amount of satellite magnetic data are expected to become available at different altitudes in the next decade or so (Ørsted project [*Friis-Christensen and Skøtt,* 1997]; Champ project [*Reigber et al.,* 1996]), it is worthwhile to clarify these problems. In this paper

Paper number 98GL51331.
0094-8534/98/98GL-51331$05.00

we first point out the problems with *von Frese et al.'s* [1981] formulas and then derive a correct expression for the magnetic field of a magnetic dipole in a spherical coordinate system.

Problems with previous expressions

According to *von Frese et al.* [1981], the magnetic field $\mathbf{F}(\mathbf{r})$ at point $\mathbf{r}(r, \theta, \phi)$ caused by a magnetic point dipole located at $\mathbf{r}_1(r_1, \theta_1, \phi_1)$ is

$$\mathbf{F}(\mathbf{r}) = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi, \tag{1}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ denote the unit vectors of the spherical coordinate system at the observation point $\mathbf{r}$ and

$$F_r = \frac{-A}{R^4}\mathcal{J}\Delta j, \quad F_\theta = \frac{-B}{R^4}\mathcal{J}\Delta j, \quad F_\phi = \frac{-C}{R^4}\mathcal{J}\Delta j, \tag{2}$$

with

$$\mathcal{J} = \left(\tfrac{R}{A} - \tfrac{3A}{R}\right)\sin I_1 + \left(\tfrac{Rr_1\cos\delta}{B} - \tfrac{3B}{R}\right)\cos I_1 \cos D_1 + \left(\tfrac{Rr_1 \sin\theta_1 \cos(\phi-\phi_1)}{C} - \tfrac{3C}{R}\right)\cos I_1 \sin D_1, \tag{3}$$

and Δj is the magnitude of the magnetic moment, I_1 and D_1 are the inclination and declination of the dipole in the source coordinate, R is the distance and δ is the angle between $\mathbf{r}$ and $\mathbf{r_1}$,

$$\begin{aligned} R &= (r^2 + r_1^2 - 2rr_1\cos\delta)^{1/2}, \\ \cos\delta &= \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1), \\ A &= r - r_1\cos\delta, \\ B &= r_1\left(\sin\theta\cos\theta_1 - \cos\theta\sin\theta_1\cos(\phi-\phi_1)\right), \\ C &= r_1\sin\theta_1\sin(\phi-\phi_1). \end{aligned} \tag{4}$$

A and R are never zero for observation points outside the Earth, but B and C may vanish at certain locations and cause (3) to diverge over these locations.

The scalar magnetic anomaly T at point $\mathbf{r}$ caused by a magnetic point dipole located at $\mathbf{r_1}$ is related to the vector magnetic field $\mathbf{F}$ of this dipole through

$$T = \hat{\mathbf{b}} \cdot \mathbf{F} \tag{5}$$

where $\hat{\mathbf{b}}$ is the unit vector of the core field at $\mathbf{r}$. The scalar magnetic anomalies computed by this method at an altitude of 400 km for five dipoles located at various latitudes and with magnetic moments aligned with the core field are shown in Figure 1-A (see the figure caption for details).

The singularities arise because of errors made in deriving the vector magnetic field (Equation 24 of *von Frese et al.* [1981]) from magnetic potential (Equation 23 of *von Frese et al.* [1981]). These errors have not been detected by *von Frese et al.* [1981], whose results do not show any singularity (their Figure 6-B). This is because they applied their formulas to a very special case by using their reduced to the pole Equation 26, as clearly stated on page 82 of their paper. The derivation of the reduced to the pole Equation 26 of *von Frese et al.* [1981] from their incorrect Equation 24 is obtained by assuming that $I_1 = 90°$ and $D_1 = 0°$, which eliminates the second and third terms in the right hand side of the expression for $\mathcal{J}$ (see (3)). The singularities in Figure 1-A arise directly from these two terms.

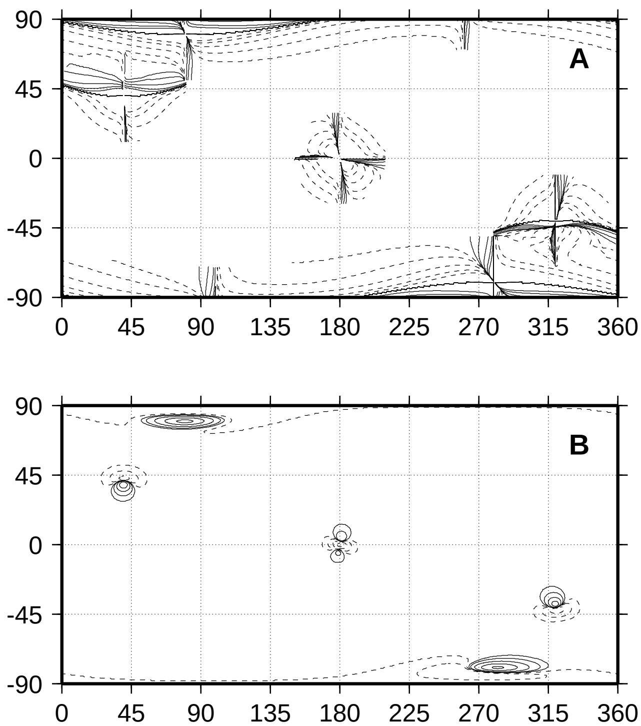

Figure 1. Magnetic anomalies computed for five dipoles located at 80°N, 80°E; 40°N, 40°E; 0°N,180°E; 40°S, 40°W; and 80°S, 80°W on the Earth surface. The strength of the dipole moments is $50\,10^{12}\,\mathrm{u\,m^2}$ where u is the unit in which the resulting anomalies are expressed. The direction of the dipole moments is aligned with the DGRF 1980 core field model [*IAGA Division V Working Group 8,* 1996]. Anomalies are computed at an altitude of 400 km and within a radius of 3300 km around the point sources. A: anomalies computed using (2) [*von Frese et al.,* 1981]; B: anomalies computed using (12). Solid contours are positive and dashed ones negative. Displayed contours correspond to 1, 3, 10, 30, and 70 units.

Another difficulty with *von Frese et al.'s* [1981] derivations arises from the fact that they used the same unit vectors of the coordinate system at $\mathbf{r}$ and $\mathbf{r}_1$. The unit vectors of a rectangular coordinate system are fixed, but those of the spherical coordinate system depends on the position. This error is introduced in their Equation 21 where the amplitudes of the magnetic moment components are expressed at $\mathbf{r}_1$ but the unit vectors at $\mathbf{r}$. This error is propagated to Equation 23, where the scalar product of the magnetic moment and the gradient of $1/R$ is performed in the third expression of column 1 of page 74. This may not introduce a significant error over a small area where the direction of the unit vectors of the spherical coordinate system show minor variations, but the error would be quite appreciable over a large area or on a global scale.

Magnetic field of a magnetic dipole

Here we derive the explicit formula for the magnetic field created by a magnetic point dipole. The magnetic potential $V(\mathbf{r})$ at observation point $\mathbf{r}$ due to a dipole of magnetic moment $\mathbf{J}(J_{r_1}, J_{\theta_1}, J_{\phi_1})$ located at $\mathbf{r}_1$ (Figure 2) is defined as

$$V(\mathbf{r}) = \mathbf{J} \cdot \nabla_1 (\frac{1}{R}), \tag{6}$$

where ∇_1 is the gradient operator at $\mathbf{r}_1$,

$$\nabla_1 = \mathbf{e}_{r_1}\frac{\partial}{\partial r_1} + \mathbf{e}_{\theta_1}\frac{\partial}{r_1 \partial\theta_1} + \mathbf{e}_{\phi_1}\frac{\partial}{r_1 \sin\theta_1 \partial\phi_1}, \tag{7}$$

$\mathbf{e}_{r_1}$, $\mathbf{e}_{\theta_1}$ and $\mathbf{e}_{\phi_1}$ are the unit vectors of the coordinate system at $\mathbf{r}_1$. Putting (4) and (7) into (6) yields

$$V = -\frac{(J_{r_1}A_1 + J_{\theta_1}B_1 + J_{\phi_1}C_1)}{R^3} \tag{8}$$

where

$$\begin{aligned} A_1 &= r_1 - r\cos\delta, \\ B_1 &= r\,(\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1\cos(\phi-\phi_1)) \\ \text{and } \mathrm{C}_1 &= -r\sin\theta\sin(\phi-\phi_1). \end{aligned} \tag{9}$$

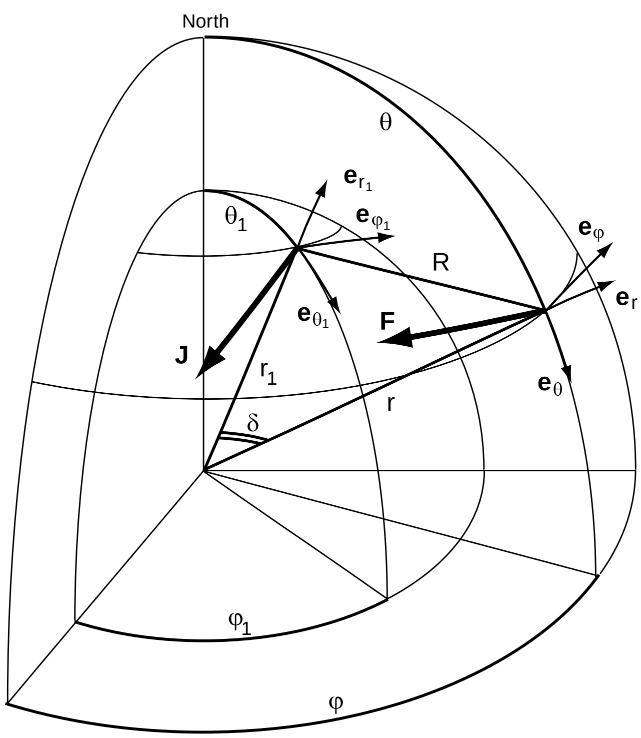

Figure 2. Geometry of the source point, observation point, and the associated unit vectors in spherical coordinates (see text for details). We have adopted the most conventional spherical coordinate system with $\mathbf{e}_r$ pointing outward, from the origin to the point, $\mathbf{e}_\theta$ pointing southward, and $\mathbf{e}_\phi$ eastward. *von Frese et al.* [1981] have used a spherical coordinate system with $\mathbf{e}_r$ pointing inward and $\mathbf{e}_\theta$ northward (see their Figures 1 and 2).

The magnetic field $\mathbf{F}$ at observation point $\mathbf{r}$ due to a dipole located at $\mathbf{r}_1$ is defined by

$$\mathbf{F} = -\nabla V, \tag{10}$$

where ∇ is the gradient operator at point $\mathbf{r}$

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{\partial}{r\partial\theta} + \mathbf{e}_\phi\frac{\partial}{r\sin\theta\partial\phi}. \tag{11}$$

The explicit expressions for the magnetic field components in $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ directions are

$$\begin{aligned} F_r = \tfrac{-1}{R^3}\Big[&\big(\tfrac{3AA_1}{R^2} + \cos\delta\big)J_{r_1} + \big(\tfrac{3AB_1}{R^2} - \tfrac{B_1}{r}\big)J_{\theta_1} \\ &+ \big(\tfrac{3AC_1}{R^2} - \tfrac{C_1}{r}\big)J_{\phi_1}\Big], \\ F_\theta = \tfrac{-1}{R^3}\Big[&\Big(\tfrac{3BA_1}{R^2} - \tfrac{B}{r_1}\Big)J_{r_1} + \big(\tfrac{3BB_1}{R^2} + D\big)J_{\theta_1} \\ &+ \big(\tfrac{3BC_1}{R^2} + E\big)J_{\phi_1}\Big], \\ F_\phi = \tfrac{-1}{R^3}\Big[&\Big(\tfrac{3CA_1}{R^2} - \tfrac{C}{r_1}\Big)J_{r_1} + \big(\tfrac{3CB_1}{R^2} - F\big)J_{\theta_1} \\ &+ \big(\tfrac{3CC_1}{R^2} + G\big)J_{\phi_1}\Big], \end{aligned} \tag{12}$$

where A, B, and C are given in (4) and

$$\begin{aligned} D &= \sin\theta\sin\theta_1 + \cos\theta\cos\theta_1\cos(\phi-\phi_1), \\ E &= \cos\theta\sin(\phi-\phi_1), \\ F &= \cos\theta_1\sin(\phi-\phi_1), \\ \text{and } \mathrm{G} &= \cos(\phi-\phi_1). \end{aligned} \tag{13}$$

Note that there is no singularity in (12).

Figure 1-B shows scalar magnetic anomalies computed by this method for the same five dipoles. Errors introduced by the use of the same unit vector at $\mathbf{r}$ and $\mathbf{r}_1$ amount to about 5% of the anomalies, indicating that the approximation leads to significant errors.

Conclusion

The expression of the magnetic anomaly of a magnetic dipole given by *von Frese et al.* [1981] is found erroneous because of 1) errors made in deriving the vector magnetic field from magnetic potential, and 2) using the same unit vectors of the coordinate system at the source and observation points. The later may not introduce a significant error over a small area, but the error is quite appreciable over a large area or on a global scale. Our expression of the magnetic field of a magnetic dipole is valid for any observation point outside the Earth and requires the components of the magnetization vector to be expressed in its natural coordinate of the point source.

Acknowledgments. This research is supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada Grant # OGP0041245 to JAH and by Programme National de Télédétection Spatiale (PNTS), Centre National de la Recherche Scientifique (CNRS) and Centre National d'Etudes Spatiales (CNES) of France to JD.

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(Received June 24, 1997; revised February 16, 1998; accepted February 18, 1998.)