A SYSTEMATIC APPROACH TO MODELING THE GEOPOTENTIAL WITH POINT MASS ANOMALIES

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Abstract. A point mass model is used to predict oceanic gravity anomalies from simulated altimeter data. Gravity anomalies from land areas are also combined with altimetry as hybrid data inputs to the mathematical model. The geocentric radii of the point mass anomaly sets are determined by use of the 'best r-formula' of Hardy and Göpfert. Test control data, consisting of $1^\circ \times 1^\circ$ mean free air gravity anomalies in a 30 x 30 block, are recovered with a standard deviation of about 20 mGal., using the hybrid data observations as well as altimeter observations alone. Highly significant computational advantages are evident in comparison with other methods.

Introduction

The potential field of the earth, W, can be expressed as the sum of a normal potential, U, and a disturbing potential, T: W = U + T. The normal potential is, by definition, generated by a best-fitting ellipsoid that has the same center of mass, angular velocity, and mass as the earth and the same bounding potential as the geoid.

The problem is to find a mass distribution enclosed by the ellipsoidal surface that will generate the disturbing potential on that surface. From Stokes' theorem it is known that there is only one harmonic function that assumes given boundary values on a surface that encloses all mass, while the principle of Dirichlet assures the existence of such a function.

Since the normal potential U is analytically defined, it can be removed. The removal of the normal potential requires the appropriate reduction of the other quantities involved. After this reduction the parameters that remain are the disturbing potential, geoidal undulations, and an anomalous mass distribution. In the development that follows, this mass distribution will be approximated by point mass anomalies. It should also be noted that all of the derivations and equations that follow, with the exception of the normal gravity, are based on a spherical approximation to the ellipsoid.

Basic Model

The observation equations used in this investigation, expressed in matrix form, are

$$[N] = \frac{G}{\gamma} = [\ell^{-1}]\ [C] \tag{1}$$

$$[\Delta g] = G\left(\frac{R - r\cos\psi}{\ell^3} - \frac{2}{\ell R}\right) [C] \tag{2}$$

These are the expressions for geoidal undulations N and for gravity anomalies Δg. The element ℓ_{ij}^{-1} of the matrix $[\ell^{-1}]$ is the reciprocal of the distance from the point of the ith undulation observation to the jth point mass anomaly. R is the mean radius of the earth, r is the radius of the sphere on which point mass anomalies are located, G is Newton's gravitational constant, and C represents coefficients (point mass anomalies).

Figure 1 shows the geometrical relationship between a computation point and a point mass anomaly in Cartesian and spherical coordinates. The XY plane coincides with the earth's equator, and the Z axis coincides with the earth's axis of rotation. The angle ψ_{ij} is defined as the angle between the radius vector to the ith observation or computation point and the radius vector to the jth point mass anomaly. From the spherical cosine law, ψ_{ij} can be written as

$$\psi_{ij} = \cos^{-1}[\sin\phi_i \sin\phi_j + \cos\phi_i \cos\phi_j \cos(\lambda_i - \lambda_j)] \tag{3}$$

The distance function, ℓ_{ij}, can be formed by using the planar law of cosines as

$$\ell_{ij} = (R^2 + r^2 - 2Rr\cos\psi_{ij})^{1/2} \tag{4}$$

Assuming the locations of the point mass anomalies are known, a least squares adjustment can be used to determine the size of the point mass anomalies. If the observations are geoidal undulations, (1) is the observation equation. If gravity anomalies are the observed quantities (2) is the observation equation. This method can be used to predict gravity anomalies using measured geoid heights or to predict geoid heights using observed gravity anomalies.

The use of both measured geoid heights and observed gravity anomalies can facilitate a combined system using both (1) and (2) to determine geoid heights on land and gravity anomalies on the ocean. Denoting the combined system as

$$[A]\ [C] = [L] \tag{5}$$

the solution is

$$[C] = ([A]^T\ [P]\ [A])^{-1}\ [A]^T\ [P]\ [L] \tag{6}$$

where [P] is the weight matrix and the superscript T indicates a transposed matrix. The

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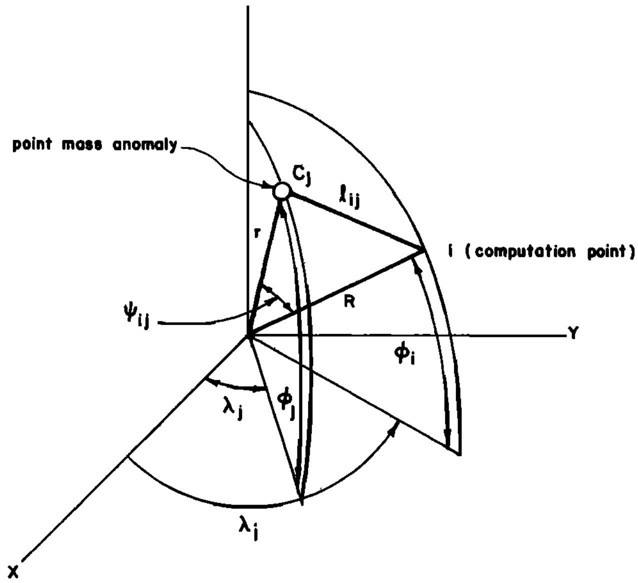

Fig. 1. Point mass geometry and notation.

solution vector, [C], can now be substituted into (1) to predict undulations or into (2) to predict gravity anomalies.

The linear solution of (6) requires that the radius r of the sphere on which the point mass anomalies are located to be known a priori. This radius cannot be entirely arbitrary because, just as there are an infinite number of point mass anomalies that will generate a given disturbing potential, there are also an infinite number that will not give reasonable results for prediction even though discrete observational data are fitted exactly. With reference to Figure 2, assume that A, B, and C are the vertices of an equilateral spherical triangle. Let R denote the radius of the sphere, ψ the arc length of the sides, and ψ_M the arc length from any of the vertices to the centroid of the triangle. Radially below the vertices are point mass anomalies m_1, m_2, and m_3 on a sphere whose radius is less than R. Let r denote this radius. A method of determining r, called the 'best r-formula,' was developed by Hardy and Göpfert [1975]. With its use, the disturbing potential function exhibits reasonable values over the entire surface. The basic premise of the solution is that the values of predicted anomalies should be highly correlated with the nearest observational data, considered as an average over the sphere. The best r-formula is

$$(R - r)^{-1} + 2(R^2 + r^2 - 2Rr \cos \psi)^{-1/2} - 3(R^2 + r^2 - 2Rr \cos \psi_m)^{-1/2} = 0 \qquad (7)$$

Supplemental formulas are available in the publication by Hardy and Göpfert [1975] to assist in the solution of (7) for either global or regional applications. For a complete derivation, see Herbrechtsmeier [1975]. This equation demonstrates that the optimum interior radius is a function only of the spacing of the point mass anomalies and is independent of their size. The derivation of the best r-formula is based on equally spaced point mass anomalies. Although this condition cannot be rigorously enforced (except for a few special cases), the point mass anomalies should be regularly distributed throughout the area in which the functions are used. Gridded distributions may be used without significant loss of accuracy.

Concentric Superposition of Point Mass Anomalies

The mathematical model provides an opportunity to perform what could be called 'trend partitioning.' This means that a first set of point mass anomalies located at a relatively large depth (R - r), consisting of a small number of relatively large point mass anomalies, will model the general trend of the disturbing potential.

As more numerous point mass anomaly sets are added in concentric shells at progressively lesser depths of (R - r), the more localized higher-frequency portions of the disturbing potential field are modeled by the system. This is analogous to increasing the degree and order of a spherical harmonic expansion.

The concentric superposition method can be applied to the disturbing potential, geoidal undulation, and gravity anomalies, separately or in combination, but it will be described here for a disturbing potential only since all other cases are analogous.

Assume that the disturbing potential, $[T_0]$, is known at n_0 points. Initially, a least squares fit for n_1 point mass anomalies is computed as

$$[C_1] = \frac{1}{G} ([\ell_1^{-1}]^T [P_1] [\ell_1^{-1}])^{-1} [\ell_1^{-1}]^T [P_1] [T_0] \qquad (8)$$

where $n_1 < n_0$.

Then at all points n_0, the potential $[T_1]$ generated by $[C_1]$ is computed as

$$[T_1] = G[\ell_1^{-1}] [C_1] \qquad (9)$$

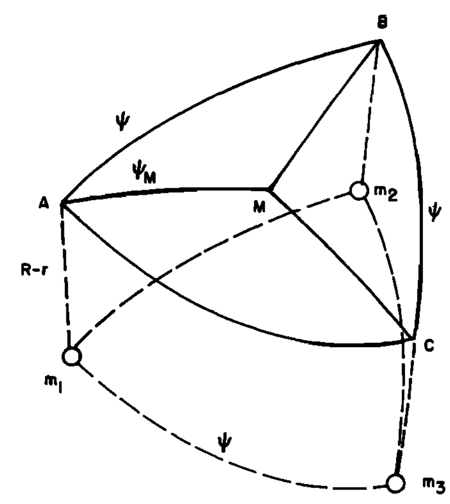

Fig. 2. Geometry and notation for the r-formula.

TABLE 1. Test 1 Mean Anomaly Recovery, mGal

Number of Point Mass Anomalies	$1^{o} \times 1^{o}$		$2^{o} \times 2^{o}$		$5^{o} \times 5^{o}$	
	Mean Deviation	Standard Deviation	Mean Deviation	Standard Deviation	Mean Deviation	Standard Deviation
16	3.05	25.69	3.1	18.8	3.1	1.7
16 + 49	2.48	21.36	2.5	13.3	2.5	0.5
16 + 49 + 100	2.43	20.36	2.4	11.8	2.4	0.1

Test 1 is a simulation of an oceanic area with only undulation data available.

The discrepancy between T_0 and T_1 is now

$$[dT_1] = [T_0] - [T_1] \tag{10}$$

Another set of n_2 point mass anomalies is now fitted to $[dT_1]$

$$[C_2] = \frac{1}{G} ([\ell_2^{-1}]^T [P_2] [\ell_2^{-1}])^{-1} [\ell_2^{-1}]^T [P_2] [dT_1] \tag{11}$$

where $n_1 < n_2 < n_0$.

The potential $[T_2]$ is evaluated, and the discrepancy vector $[dT_2]$ is formed for the n_0 points as

$$[dT_2] = [T_0] - [T_1] - [T_2] = [dT_1] - [T_2] \tag{12}$$

If the residuals, $[dT_2]$, are acceptable, the procedure is halted; if they are not acceptable additional mass anomaly sets are added until the desired detail is obtained.

It should be emphasized that the best r-formula is used at each step of the concentric development to determine the proper (R - r) for the number of data points used in that step.

Experimental Concentric Superposition Simulation Study

In order to perform the simulation it was necessary to have undulations and gravity anomalies from the same geographical area. The undulation data were derived by Gopalapillai [1974] from a set of fully normalized spherical harmonic potential coefficients complete to degree and order (16, 16) from the Smithsonian Astrophysical Observatory Standard Earth II Solution [Gaposchkin and Lambeck, 1971]. The gravity anomaly data were provided by R. H. Rapp of the Department of Geodetic Science, Ohio State University. The data are in the form of $1^{o} \times 1^{o}$ mean free air anomalies. All data are referred to the 1967 Geodetic Reference System (International Association of Geodesy, 1967). Both data sets represent a $30^{o} \times 30^{o}$ block from 20^{o} to 50^{o} north latitude and from 265^{o} to 295^{o} east longitude.

TABLE 2. Test 1 Undulation Modeling Errors, m

Number of Point Mass Anomalies	Mean Deviation	Standard Deviation
16	0.01	2.84
16 + 49	0.00	1.30
16 + 49 + 100	0.00	1.04

Test 1 is a simulation of an oceanic area with only undulation data available.

The model was tested for two situations. For the first test it was assumed that only undulation measurements were available in the $30^{o} \times 30^{o}$ block. This test simulated an area that is entirely oceanic. For the second test it was assumed that undulations were measured in the western $30^{o} \times 15^{o}$ area and that gravity anomalies were measured in the eastern $30^{o} \times 15^{o}$ area. Thus the second test simulated an area that has both land and oceanic areas. In both tests it was assumed that the undulation measurements had a standard error of 1 m. In the second test it was assumed that the mean anomaly values had a standard error of 30 mGal. These estimates may be somewhat conservative, since observed altimeter data appear to have a precision better than 1 m and some of the mean anomalies may have a standard error as low as 10 mGal. A total of 165 undetermined point mass anomalies, regularly distributed throughout the block, was divided into three sets of 16, 49, and 100 point mass anomalies (pma). The best r-formula was used to determine the radii of the three different spheres on which the point mass anomalies were located, with the following results:

pma	r	R - r
16	0.923R	0.077
49	0.969R	0.031
100	0.978R	0.022

where the radius R is assumed to be 1. The determined coefficients (point mass anomalies) were then used to predict gravity anomalies and geoidal undulations at over 900 1^{o} grid intersections.

TABLE 3. Test 2 Mean Anomaly Recovery, mGal

Number of Point Mass Anomalies	$1^{\circ} \times 1^{\circ}$ Mean Deviation	$1^{\circ} \times 1^{\circ}$ Standard Deviation	$2^{\circ} \times 2^{\circ}$ Mean Deviation	$2^{\circ} \times 2^{\circ}$ Standard Deviation	$5^{\circ} \times 5^{\circ}$ Mean Deviation	$5^{\circ} \times 5^{\circ}$ Standard Deviation
16	3.48	27.60	3.5	21.3	3.5	3.8
16 + 49	2.54	21.25	3.5	13.3	3.5	1.0
16 + 49 + 100	2.46	19.94	2.5	11.1	2.5	0.2

Test 2 is a simulation of an area that is half oceanic and half land.

Results

The results of the anomaly recovery experiment from undulation observations only are shown in Table 1. The $2^{\circ} \times 2^{\circ}$ and $5^{\circ} \times 5^{\circ}$ anomalies were estimated as the mean of $1^{\circ} \times 1^{\circ}$ anomalies in the respective areas. The lower recovery error for these anomalies is not indicative of the goodness of fit of the model, since the lower error is statistically assured. Table 2 shows the error in the modeling of the undulation measurements. It can be seen that the standard error of the undulations does not differ significantly from the assumed standard deviation (1 m). The results from test 2, the area that is half oceanic and half land, are shown in Tables 3 and 4.

The results from both test 1 and test 2 indicate that the model converges slowly. For example, in test 1 a tenfold increase in the number of point mass anomalies (from 16 to 165) reduced the undulation modeling error by about 63%, while the anomaly recovery improved by only 28%. However, this is to be expected because of the nature of gravity anomalies and geoidal undulations. The anomalous gravity field is characterized by relatively high-frequency large-amplitude variations, whereas the geoid exhibits lower frequencies and amplitudes. This means that a moderate change in a gravity anomaly will have only a small effect on the geoid; conversely, the anomaly recovery is quite sensitive to the accuracy of the undulation measurements.

The results of both test 1 and test 2 show that $1^{\circ} \times 1^{\circ}$ mean anomalies were recovered with a standard deviation of approximately 20 mGal. As a comparison, Gopalapillai [1974] used a numerical approximation of Stokes equation and recovered the mean anomalies with a standard deviation of about 19 mGal from the same data that were used in the point mass anomaly model

TABLE 4. Test 2 Undulation Modeling Errors, m

Number of Point Mass Anomalies	Mean Deviation	Standard Deviation
16	0.03	3.18
16 + 49	0.00	1.30
16 + 49 + 100	0.00	1.01

Test 2 is a simulation of an area that is half oceanic and half land.

experiments described herein. Gopalapillai approximated Stokes formula,

$$N = \frac{R}{4\pi\gamma} \iint \Delta g S(\psi) d\sigma$$

with

$$N = \frac{R}{4\pi\gamma} \sum_{i} \sum_{j} \Delta g_{ij} S_{ij}(\psi) \Delta\sigma$$

which he solved directly for gravity anomalies. Gopalapillai's method required the solution of a system of 961 unknowns to determine the $1^{\circ} \times 1^{\circ}$ mean anomalies. The superposition of point mass anomalies method required the solution of three systems of 16, 49, and 100 unknowns, respectively. While both methods exhibit comparable accuracy, the point mass anomaly method appears to have a significant computational advantage.

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References

Gaposchkin, E. M., and K. Lambeck, Earth's gravity field to the sixteenth degree and station coordinates from satellite and terrestrial data, J. Geophys. Res., 76(20), 4883-4895, 1971.

Gopalapillai, S., Non-global recovery of gravity anomalies from a combination of terrestrial and satellite altimetry data, Rep. 210, Dep. of Geod. Sci., Ohio State Univ., Columbus, 1974.

Hardy, R. L., and W. M. Göpfert, Least squares prediction of gravity anomalies, geoidal undulations, and deflections of the vertical with multiquadric harmonic functions, Geophys. Res. Lett., 2(10), 423-426, 1975.

Herbrechtsmeier, E. H., A point mass model for estimating gravity anomalies from measured geoid heights, M.S. thesis, Iowa State Univ., Ames, 1975.

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