Joint BGI/ICET Summer School 2002

TERRESTRIAL GRAVITY DATA ACQUISITION TECHNIQUES

NETWORK ADJUSTMENT

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NETWORK ADJUSTMENT CALCULUS

I. STATISTICS AND THE METHOD OF LEAST SQUARES
II. STATISTICAL DISTRIBUTION FUNCTIONS
III. HYPOTHESIS TESTING AND CONFIDENCE INTERVALS
IV. APPLICATION TO GRAVIMETRY

We will discuss the theory of statistics underlying the concept of network adjustment. We will then focus on the method of least-squares, and the various tests that can be performed to validate the results of the adjustment, including the χ^2 test. Special attention will be given to the notion of confidence intervals.

This document is based on the following sources :

- "Geodesy, the Concepts" by Petr Vaníèek and Edward J. Krakiwsky, Elsevier, Second Edition
- "The method of least squares [Lecture Notes n°18]" by D. E. Wells and E. J. Krakiwsky, Department of Surveying Engineering, University of New Brunswick, Fredericton, Canada and its "External Appendix: Numerical Examples on Interval Estimation and Hypothesis Testing"
- "Introduction to Adjustment Calculus (Third Revised Edition) [Lecture Notes n°35]" by Petr Vaníèek, Department of Surveying Engineering, University of New Brunswick, Fredericton, Canada
- "Notes for SE 2101 Theory of Measurements" by A. Kleusberg, Department of Surveying Engineering, University of New Brunswick, Fredericton, Canada
- "Observations and Least Squares" by Edward M. Mikhail and F. Ackerman, A Dun-Donneley Publisher, New York
- http://www.ensu.ucalgary.ca/~nel-shei/ENGO%20361_Chapter_1.doc

DATA PROCESSING : GRAVITY NETWORK ADJUSTMENT USING GEOLAB

I. THE FRENCH RBF GRAVITY NETWORK
II. PREPARING THE DATA
III. ADJUSTING THE NETWORK
IV. THE RESULTS AND THE STATISTICAL INDICATORS

We will briefly introduce two programs that can be used to adjust gravity networks.
Next we will use data collected by IGN at stations in southern France to illustrate the network adjustment theories. Gravity relative measurements for more than 200 stations will be formatted so as to be used as an input to the Geolab software. The network will then be adjusted using the proper options to obtain estimated values for gravity for all the stations, together with their accuracy estimates. The results will be comprehensively analysed using the indicators provided by the program.

INTRODUCTION

This section is adapted from the "Ashtech Solutions" software User's Guide.

A least-squares adjustment of survey observations is an important step in a gravimetry survey. Properly used, it helps isolate blunders in the observations being adjusted and gives the accuracy and reliability of the gravity values being determined. The primary components of a least-square adjustment are the survey observations (in our case gravity differences) and the uncertainties associated. Due to the measurement limitations of the surveying instruments and the influence of the operators, these observations include some level of error. These errors cause loops not to close perfectly and result in the ability to compute different values for the same station in the network.
The ultimate goal of a least-squares adjustment is to produce a set of observations where all loops close perfectly and only one value can be computed for any point in the network. In order to accomplish this task, the observations going into the adjustment must be changed slightly, i.e. adjusted. Of course we do not want the observations to be modified too much, since this is what was physically observed in the field. But the observations do contain some level of error. Any error associated with an observation is predictable because of the measurement accuracy of the instruments used. A successful adjustment is one where observations are changed as little as possible, and the amount of adjustment to any observation is within expected levels.
Unfortunately there are a number of obstacles that can stand in the way of producing a successful adjustment. Primary on this list are blunders, errors in the observations due to equipment malfunction or operator error (incorrectly measured instrument height, insufficient data, wrong site ID, etc.). Tools exist to assist in overcoming these obstacles, both before and during the adjustment. The analysis tools are mostly statistically based. As a result, it is very important that the uncertainties (error estimates) are realistic. At times, these uncertainties may be a little optimistic (too small) or pessimistic (too large). Methods exist to help identify when uncertainties are unrealistic and to help rectify this situation. Also, adjustment analysis tools cannot function properly without redundancy in the observations. Bear this in mind when designing a survey network. A certain percentage of points should receive multiple observations.

I STATISTICS AND THE METHOD OF LEAST SQUARES

Statistics is the scientific method of collecting, arranging, summarising, presenting, analysing, drawing valid conclusions from DATA. STATISTICAL DATA include numerical facts and measurements or observations of natural phenomena or experiments. A VARIABLE is a quantity which varies, and may assume any one of the values of a specified set. In general, the result of a measurement is a CONTINUOUS VARIABLE, meaning that it can assume any value within some continuous range.

The method of least squares is the standard method used to obtain unique values for physical parameters from redundant measurements of those parameters.

The first use of this method dates back to 1795, and is generally attributed to **Karl Friedrich Gauss**. Gauss invented the method so as to estimate the orbital motion of planets from telescopic measurements. More than 200 years ago, in "Theoria Motus Corporum

Coelestium" published in 1809 (but first drafted in 1795), he presented the concepts that mathematical models may be incomplete, physical measurements are inconsistent, that all that can be expected from such measurements are "estimates" of the "truth", and that redundant measurements will reduce the effects of inconsistencies. In 1823 he published the method of least squares connected with the least possible errors concept.

The method has now come to its full potential owing to the developments from three other fields: the concepts of modern statistical estimation theory, matrix notation and the use of fast digital computers.

Physical quantities can never be measured perfectly. There will always be a limiting measurement precision beyond which either the mathematical model describing the physical quantity, or the resolution of the measuring instrument, or both, will fail. Beyond this limiting precision, redundant measurements will not agree with one another (they will not be consistent).

For example if we measure the length of a table several times with a meter stick and eyeball, the limiting precision is likely to be about one millimeter. If we record our measurements only to the nearest centimeter they will be consistent.

The precision which we desire is often beyond the limiting precision of our measurements. In such a case we can not KNOW the "true" value of our physical quantity. All we can do is make an ESTIMATE of the true value; we want this estimate to be UNIQUE (determined by some standard method which will always yield the same estimate given the same measurements), and we want to have some idea of how "good" the estimate is.

The scientific method for handling inconsistent data is called STATISTICS. The method for determining unique estimates together with how good they are is called STATISTICAL ESTIMATION. The method of least squares is one such method, based on minimizing the sum of the squares of the inconsistencies. It has now become a standard method in many fields.

Of course the problem to be treated has to be first "translated" into the language of mathematics. We need to figure out the relation between the observables (observed quantities) and the parameters (wanted quantities). This relationship is called the MATHEMATICAL MODEL.

We consider the system of linear equations f(X,L)=0, or in matrix form :

$$AX=L \qquad (1)$$

where X is the unknown vector, L is the constant vector, A is the coefficient matrix or DESIGN MATRIX.

Let's assume that the elements of L are the results of physical measurements. L is called the OBSERVATION VECTOR.

In the case where there are no redundant equations (minimum number of measurements), A is square and non singular, and therefore has an inverse. The unique solution is then given by

$$X=A^{-1}L$$

When there are redundant equations, the system is OVERDETERMINED: A is not square, but A^TA is, and the mathematics tell us that

$$X=(A^TA)^{-1}A^TL$$

This is so if and only if the system is consistent. But if there are redundant measurements, they will be inconsistent because physical measurements are never perfect. We have to assume that there were ERRORS in the determination of the observations. This leads us into the THEORY OF ERRORS and STATISTICS. An adjustment becomes necessary when the data available exceed the minimum required for a unique determination.

We must first get rid of SYSTEMATIC ERRORS (for instance the lengthening of the gravimeter spring) and of BLUNDERS (gross errors). The way for accounting for systematic errors and intercepting blunders are numerous and we are not going to venture into this here. Even after eliminating those errors, we still have an unavoidable spread in the observations, and we say that the observations contain RANDOM or ACCIDENTAL ERRORS. To account for this, we have to refer to statistical concepts.

No unique solution will exist, and all we are able to do is make a UNIQUE ESTIMATE of the solution. The most commonly used criterion for the estimate to be UNIQUE is the LEAST SQUARES criterion; **that the sum of the squares of the inconsistencies be a minimum**. Using statistics, we are also usually able to establish the DEGREE OF RELIABILITY of the solution, and thus define the MOST PROBABLE UNIQUE SOLUTION.

To cancel the inconsistencies, we add a vector to the equation (1), which becomes:

$$AX-L=V \qquad (2)$$

where V is usually called the RESIDUAL VECTOR (OBSERVATION ERRORS). The elements of V are not known and must be solved for. So we have to allow some of or all the elements of L to change slightly while solving for X, or to regard L as an approximate value of some other value $\hat{L}$ which yields the unique solution $\hat{X}$. Now the least squares criterion states that the best estimate $\hat{X}$ for X is the estimate which will MINIMIZE the sum of the squares of the residuals (discrepancies between observations and estimated values assigned to

each observable), that is $\hat{V}^T \hat{V}$ is minimum. The estimate $\hat{X}$ so determined is the LEAST SQUARES ESTIMATE[1].

The difference between the observed value and any arbitrarily assumed or computed value is called the MISCLOSURE, which is different from the residual (uniquely determined by the difference between the observation and the sample mean).

The limitation to this method is that we have to assume that the parameters are mutually independent, and to postulate a NORMAL PROBABILITY DISTRIBUTION FUNCTION for the random errors.

Often the physical measurements which make up the elements of L do not all have the same precision (they have been made using different instruments, by different people, under different conditions…). This fact should be reflected in our least squares estimation process, **so we assign to each measurement a known weight** and call P the matrix whose elements are these weights, the WEIGHT MATRIX. We modify the criterion which becomes

$$\hat{V}^T P \hat{V} \text{ is minimum}$$

The resulting estimate is called the weighted least squares estimate, and is given by

$$\hat{X} = \left(A^T P A\right)^{-1} A^T P L \qquad (3)$$

$A^T P A$ **is the normal equation matrix**, and must not be singular for the estimator to be unique.

[1] The symbol ^ denotes an estimated quantity.

Demonstration:

$$\hat{V} = L - A \cdot \hat{X}$$

$$\hat{V}^T = L^T - \hat{X}^T \cdot A^T$$

$$\hat{V}^T \cdot P \cdot \hat{V} = \left(L^T - \hat{X}^T \cdot A^T \right) \cdot P \cdot \left(L - A \cdot \hat{X} \right) = L^T \cdot P \cdot L - \hat{X}^T \cdot A^T \cdot P \cdot L - L^T \cdot P \cdot A \cdot \hat{X} + \hat{X}^T \cdot A^T \cdot P \cdot A \cdot \hat{X}$$

[$\hat{V}^T P \hat{V}$ is minimum] means that $\dfrac{d\left(\hat{V}^T P \hat{V} \right)}{dX} = 0$

$$- A^T \cdot P \cdot L - L^T \cdot P \cdot A + A^T \cdot P \cdot A \cdot \hat{X} + \hat{X}^T \cdot A^T \cdot P \cdot A = 0$$

$$\underbrace{(A^T \cdot P \cdot A \cdot \hat{X} - A^T \cdot P \cdot L)}_{U} + \underbrace{(\hat{X}^T \cdot A^T \cdot P \cdot A - L^T \cdot P \cdot A)}_{U^T} = 0 \quad \Rightarrow U = 0$$

$$A^T \cdot P \cdot A \cdot \hat{X} - A^T \cdot P \cdot L = 0$$

$$\hat{X} = \left(A^T \cdot P \cdot A \right)^{-1} \cdot A^T \cdot P \cdot L$$

Further complications of the mathematical model arise when the functions involved are non-linear, but this problem will not be discussed here.

From a practical point of view, the inversion and multiplication of large matrices require a considerable number of computation steps. The use of fast computers alone allows us to calculate the solutions for large systems of equation where before those machines were widespread in use, the task would have been attempted only when absolutely necessary.

Let's assume there are *n* observations and *u* unknown parameters. The least squares estimation process is applied only WHEN THERE ARE REDUNDANT MEASUREMENTS, that is $n>u$. The number (n-u) is called the REDUNDANCY or NUMBER OF DEGREES OF FREEDOM.

So far, we have not specified how the WEIGHT MATRIX should be chosen. We are going to use the STANDARD DEVIATIONS and the VARIANCES associated with the observations, and the covariances between the observations, to build up this matrix. The variance of an observation is larger when it is less accurately determined. In combining observations, more importance should be attached to those having smaller variances. One reasonable choice for the weight matrix is thus[2]:

[2] The mathematical language is very fond of greek letters; you can refresh your memory about the greek alphabet in Appendix A.

$$P = \Sigma_L^{-1} = \begin{bmatrix} s_1^2 & s_{12} & \dots \\ s_{21} & s_2^2 & \\ \dots & & s_n^2 \end{bmatrix}^{-1}$$

In this case, values must be assigned to the variances and covariances in the matrix BEFORE a least squares estimation can be made. The values arise from a knowledge of the measuring instruments and procedures being used. The covariance of a pair of observations is a measure of STATISTICAL DEPENDENCE of the two values. In practice, the covariances are often assumed to be null in gravimetry because the observation values are uncorrelated, but for instance in GPS networks adjustments, full variance-covariance matrices are often used.

It is often possible only to assign RELATIVE values among the variances and covariances, so that we know Σ_L only to within a scale factor, that is if $\Sigma_L = s_0^2 Q$, we know the relative covariance matrix Q, but not the VARIANCE FACTOR s_0^2. However we can show that in equation (3) the variance factor drops out and either weights Σ_L^{-1} or Q^{-1} result in the SAME ESTIMATOR.

The weight matrix is thus chosen to be proportional to the inverse of the ESTIMATED COVARIANCE MATRIX OF THE OBSERVATIONS. If L is postulated to be uncorrelated, it will be a diagonal matrix. As shown in equation (3), the factor for computing P will not influence the result $\hat{X}$.

The weight matrix accounts for the fact that the data may be of varying quality. Otherwise we would replace the weight matrix with the identity matrix.

It can also be shown that the least squares unbiased estimator $\hat{s}_0^2$ of the variance factor s_0^2 is:

$$\hat{s}_0^2 = \frac{\hat{V}^T P \hat{V}}{n-u}$$

where $P = s_0^2 \Sigma_L^{-1}$, and that the least squares unbiased estimator of the covariance matrix of X is:

$$\hat{\Sigma}_{\hat{X}} = \hat{s}_0^2 \left(A^T P A\right)^{-1} \qquad (4)$$

Here we are dealing with linear mathematical models. Should the model be nonlinear, it must be linearized before the least squares methods can be applied.

II STATISTICAL DISTRIBUTION FUNCTIONS

A random variable is defined by the set of permissible values which it may assume, and by an associated FREQUENCY (or PROBABILITY DENSITY) FUNCTION (PDF) which expresses how often each of these values will appear in the situation under discussion (Example: dice roll). The most important of these functions is the normal (or Gaussian) frequency function. Physical measurements can almost always be assumed to be random variables with a normal frequency function.

This function represents the variation of a given random variable over some domain. The two criteria below must be strictly satisfied in order that a function may be considered as a PDF:

$$f(x) \geq 0$$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

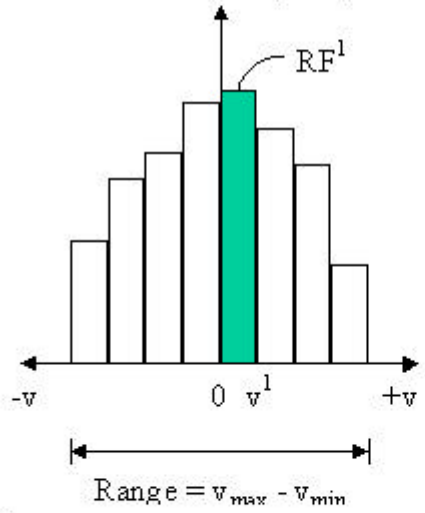
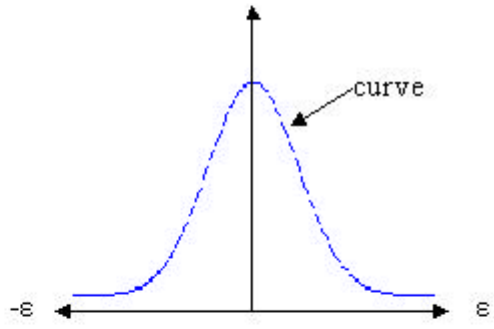
Some kinds of estimates (called interval estimates and hypothesis tests) cannot be made until a particular frequency function is specified.

Remember that in statistics, the population is assumed to be unknown, but a sample is known. We try and make inferences about the population based on the information in the sample. The larger the sample, the better the results.

There are infinitely many families of Probability Distribution Functions, each family being defined by one or more independent parameters whose values characterize its shape.

Frequency distributions have two important characteristics called CENTRAL TENDENCY and DISPERSION. The MEAN is most often used for the central tendency, and the STANDARD DEVIATION is most often used for the measures of dispersion.

The CUMULATIVE FREQUENCY FUNCTION or CUMULATIVE PROBABILITY FUNCTION is the integral of the frequency function. It gives the probability that the value of x is less than or equal to x_0. Probability is represented by an area under a curve.

	Practical Case	*Random Case* *Theoretical Case*
No. of obs.	$n \rightarrow$ finite number	$n \rightarrow \infty$
Variable	Random Sample $L = (\ell_1, \ell_2, \ldots \ell_n)$	Random Variable
Estimate	$\hat{x} = \frac{\Sigma \ell_i}{n}$ (Mean)	$\hat{x} \rightarrow t$ (true value)
Errors	Residuals $v_1 = \hat{x} - \ell_i \quad (\Sigma v_1 = o)$	True Errors $\varepsilon_i = t - \ell_i$
Spread of Errors (Distribution of errors)	Relative Frequency  RF1 is the number of observations which have residuals between o and v1 *Probability Distribution Histogram (PDH)*	Relative Frequency  *Probability Distribution Function (PDF)*

The histogram of the random samples representing observations encountered in practice generally shows a tendency towards being bell-shaped. They obey the Gaussian Probability Distribution Function. As the number of measurements increase, the width of the class interval can be reduced and the histogram will become less and less irregular. With n very large, the relative frequency would appear as a smooth curve.

The arithmetic mean of a set of measurement y_1, y_2, …, y_n is the sum of the measurements divided by n. Sample mean is $\bar{y}$, population mean is μ. The sample mean is an estimator of the true population mean.

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

The theory of statistics also defines the variance of a distribution, and its standard deviation, defined as the positive square root of the variance. The variance (or DISPERSION) of a random variable of mean μ is defined as:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - m)^2 \qquad (5)$$

and the sample variance is

$$s^2 = \frac{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}{n-1}$$

It is sometimes called the "mean square error". Measurements of high precision will have small variance.

The sample variance has a Chi-square distribution with (n-1) degrees of freedom. As a matter of fact, we only have one unknown, and hence need a single observation to determine it : the remainder (n-1) is thus the number of degrees of freedom (or redundancy).

The standard deviation (positive square root of the variance) can be interpreted as a measure of confidence we have in the correctness of the mean.

Several statistical functions serve as a mathematical representation of a given random variable over some domain: the normal, chi-square, student's (t) and F distributions are derived from basic mathematical functions.

The graph of the NORMAL DISTRIBUTION $n(\mu, \sigma^2)$ presented in Appendix C has the following characteristics:

1) symmetric about the vertical axis through μ (mean equal to zero)
2) has a maximum value of $\frac{1}{s\sqrt{2p}}$ at $x=\mu$
3) has the x-axis as the horizontal asymptote
4) has points of inflection at $x=\mu \pm \sigma$

It reflects the GAUSS LAW OF A LARGE SAMPLE OF ERRORS, which states that:

i. smaller errors are more probable than larger errors
ii. positive and negative errors have the same probability.

The term "normal" reflects the trust that people have, or used to have, in the power of the Gaussian law. If the errors behave according to this law, then they are normal. On the other hand, if they are not, then they are abnormal and strange things are suspected to have happened.

Even when observation are not normally distributed, their mean approaches the normal distribution as their number increases.

It is still the most popular PDF for it is relatively simple and contains the least possible number of parameters (the mean and the standard deviation).

If a normal variable x is distributed as n(μ, σ^2), then a new random variable $w = \frac{x - m}{s}$ is distributed as n(0,1). This PDF is called the standard normal PDF, and is particularly suitable for tabulation. This process is called NORMALIZATION.

Pre-computed integrals tabulated in the body of tables allow us to obtain probabilities values. It can be shown that

$$\Pr(x \le c) = N\left(\frac{c - m}{s}\right)$$

In Appendix B you may find the tables for the normal distribution. For instance if we want to know the probability Pr($x \le 4$), x being n(2,16), we have Pr= $N\left(\frac{4-2}{4}\right)$=0.6915.

The CHI-SQUARE DISTRIBUTION is also often used in adjustment calculus. It is defined by the parameter ν which is called the NUMBER OF DEGREES OF FREEDOM, a very practical quantity. A continuous random variable having this probability distribution function is written in abbreviated from as $c^2(n)$. A graph is included in Appendix D.

Our discussion today considers only the case when one random variable is involved, but there exist also MULTIVARIATE NORMAL DISTRIBUTIONS.

III INTERVAL ESTIMATION AND HYPOTHESIS TESTING

Again in this chapter we will restrict our discussions to population distribution functions which involve only one random variable (univariate case).

Related to the estimation is the equally important task of determining the reliability or accuracy with which the estimates are obtained (confidence measures). Statistical testing treats the question of whether the results of estimations are in agreement with the initial assumptions (hypotheses). As a matter of fact, we would like to have an indication of how good the estimation is and how much it can be relied on.

Point estimates can be made without assuming a particular population distribution. However, statistical tests require that a particular population distribution be assumed or specified.

In INTERVAL ESTIMATION we specify a range of values bounded by an upper and lower limit within which the population statistic is estimated to lie. Depending on the probability, we talk about, for instance, a 95% confidence interval. This means that the statement that x lies between two values x_1 and x_2 will be true 95% of the time that such a claim is made.

It involves a probability statement:

$$\Pr(e_1 \le e \le e_2) = 1 - a$$

where ε is a statistic of known probability distribution function, α is a probability value between 0 and 1 which must be specified (often 1-α=0.95), and e_1 and e_2 are abscissa values of the know p.d.f. which are determined by the specified α. The interval $e_1 \le e \le e_2$ is called the CONFIDENCE INTERVAL. The value (1-α) is called the **CONFIDENCE LEVEL**, and α is called the **SIGNIFICANCE LEVEL**.

In HYPOTHESIS TESTING, we make an a priori statement about the population (for example that it is normally distributed with mean μ and variance σ^2). Then, based on the value of the sample statistics, we test whether to accept or reject the hypothesis. The significance level of the test is the percentage α such that the probability that "the hypothesis is true but rejected" is equal to α.

Usually such tests are performed at the 95% confidence level (so α = 5 % or 0.05).

For instance let's consider the hypothesis $H_0 : m = m_H$

We call it the NULL HYPOTHESIS, and read it as "the mean μ is hypothesized to have the particular value μ_H". The ALTERNATIVE HYPOTHESIS is $H_1 : m \neq m_H$.

To test this hypothesis, we may examine the mean in terms of an observation x_i [belonging to a sample with distribution $n(\mu, \sigma^2)$] and variance σ^2. The associated probability statement is

$$\Pr\left(-c \le \frac{x_i - m}{s} \le c\right) = a$$

while the confidence interval for μ is

$$x_i - cs \le m \le x_i + cs$$

The bounds of this interval are estimated from the measurement value x_i, the known value of the variance σ^2, the tabulated table (Appendix B1) value of c corresponding to n(0,1) and α. The above is used to test the null hypothesis.

For instance, using the standard normal PDF table, we want to compute $P(m - s \le e \le m + s)$ where ε has a Gaussian PDF. By using the normalization process, we can write[3]:

$$P(m - s \le e \le m + s) = P\left(\frac{m - s - m}{s} \le t \le \frac{m + s - m}{s}\right) = P(-1 \le t \le 1) = 2P(0 \le t \le 1) = 0.683$$

The idea of the **CONFIDENCE INTERVALS** is based on the assumption of normality of the sample, that is a postulated parent normal PDF for the random sample involved. The interval $P(m - s \le x \le m + s)$ is referred to as the "68% confidence interval". Similarly, one can talk about the "95% confidence interval", etc.

In general the confidence interval is expressed as $[\lambda \pm K.\sigma]$ where K is determined in such a way as to make $P(m - Ks \le x \le m + Ks)$ equal to 0.95, etc. K is sometimes called the **EXPANSION FACTOR**. When K=1, we talk about the STANDARD CONFIDENCE INTERVAL, which corresponds to "one-sigma level".

For a univariate sample, we get K = 1.96 for the 95% confidence interval.

Demonstration:

$P(m - Ks \le x \le m + Ks) = 0.95$ is identical to the probability $P(-K \le t \le K)$ obtained from the standard normal tables, that is $P(t \le K) - P(t \le -K) = 2P(t \le K) - 1 = 0.95$ from which we get $P(t \le K) = 0.975$ and K=1.96.

This notion of confidence interval is one of the most important aspects of the least square adjustments. It is used for control surveys specifications, and for classifying surveys.

It can be shown that the abscissae associated with one-dimensional intervals covering probabilities of 0.90, 0.95 and 0.99 respectively are:

$$P(-1.645 \cdot s_x \le x - m_x \le +1.645 \cdot s_x) = 0.90$$
$$P(-1.960 \cdot s_x \le x - m_x \le +1.960 \cdot s_x) = 0.95$$
$$P(-2.576 \cdot s_x \le x - m_x \le +2.576 \cdot s_x) = 0.99$$

For comparison, K=2.45 for the two-dimensional case and a probability of 90%.

[3] 1-P(t ≤ a)=P(t ≥ a) et P(a ≤ t ≤ b)=P(t ≤ b) – P(t ≤ a) et P(t ≤ a) = 1 – P(t ≤ -a) = 1 – P(t ≥ a)

The standard deviation s is the positive square root of the corresponding variance. It is also called standard error or root mean square (rms) error. It represents a probability (confidence level) of 68% that the difference between the given (calculated) gravity value and the true value is within the interval of -s and +s.

Here we have assumed that the number of degrees of freedom is infinite, but when good estimates of standard deviations of observations are not available, the computed standard deviation for unit weight is used to determine the standard confidence interval, and the factor corresponding to the actual degrees of freedom should be used in the table for computing the 95% confidence interval.

The VARIANCE-COVARIANCE MATRIX of an estimate $\hat{X}$ is obtained through **the law of propagation of variance**.

The law of propagation of variance-covariance states that if z=f(x,y), and if x and y are not correlated, then $s_z^2 = \left(\frac{\partial f}{\partial x}\right)^2 s_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 s_y^2$. Thus if z=x-y, $s_z = \sqrt{s_x^2 + s_y^2}$.

If X is a n-dimensional random variable with covariance matrix Σ_X, and a new n-dimensional random variable Y is computed through a linear relation Y=A.X+B with a constant coefficient matrix A and a constant vector B, then the covariance matrix Σ_Y is given by

$$\Sigma_Y = A \cdot \Sigma_X \cdot A^T$$

For non-linear relationships of the type Y=f(X)+B, A is replaced by the Jacobian matrix J containing the partial derivatives of f with respect to X.

IV APPLICATION TO GRAVIMETRY

Usually one tries to combine advantages of both absolute gravity measurements and relative gravity measurements in the establishment of networks of gravity stations. The stations with absolute gravity determination provide the anchoring point (fixed points) of the network, while the relative measurements provide the ties between the points.

When the absolute and relative observations are made and assessed for accuracy, an adjustment can be carried out using a least-squares adjustment technique. **The adjustment results in the estimated values of gravity for all the stations, together with their accuracy estimates.** The adjustment procedure is practically identical with that of geodetic levelling.

We assume gravity to be a height system, thus the gravity differences can be adjusted as a levelling network. **This is because the summation of gravity differences around a closed loop theoretically goes to zero, and this condition can then be used as the basis for the adjustment.**

Let's assume the following gravity network was observed using relative gravimeters.

Gravity differences (mGals)	Travel time between stations (hours)
Δg_{AB} = 0.143	2
Δg_{BA} = -0.143	2
Δg_{BC} = 2.370	3
Δg_{CD} = 1.437	4
Δg_{DE} = -0.897	2
Δg_{EF} = -1.414	3
Δg_{FC} = 0.880	4
Δg_{CF} = -0.779	4
Δg_{FB} = -1.591	3
Δg_{FA} = -1.635	5
Δg_{FG} = 1.206	6
Δg_{GF} = -1.201	6

Gravity at point A is supposed known and constant (980100.000 mGal). The weight of each observation is inversely proportional to the length of travel of the line.

a) Using the parametric method of least squares, compute the gravity of points B, C, D, E, F, and G. Compute the adjusted observations $\hat{L}$ and the residuals $\hat{V}$.

b) Changing the travel time of the line BC to 6 hours, redo the adjustment. Compare the two different solutions .

c) Set Δg_{BC} = 3.370 mGals and redo the adjustment usign the ORIGINAL weights. Compare the solution with that of (a) and comment on any differences.

a) From the given data, we have a number of observations n=12 and the number of unknowns u=6. Therefore we have 6 redundant observations (=n-u) and 6 degrees of freedom.

The mathematical model is AX=L+V

The 12 independent equations will be:

$g_B = g_A + 0.143 + v1$
$g_A = g_B - 0.143 + v2$
$g_C = g_B + 2.370 + v3$
$g_D = g_C + 1.437 + v4$
$g_E = g_D - 0.897 + v5$
$g_F = g_E - 1.414 + v6$
$g_C = g_F + 0.880 + v7$
$g_F = g_C - 0.779 + v8$
$g_B = g_F - 1.591 + v9$
$g_A = g_F - 1.635 + v10$
$g_G = g_F + 1.206 + v11$
$g_F = g_G - 1.201 + v12$

where g_A has a constant value.

Putting that in matrix form, we get:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} g_B \\ g_C \\ g_D \\ g_E \\ g_F \\ g_G \end{bmatrix}$$

$$L = \begin{bmatrix} 100.143 \\ -100.143 \\ 2.370 \\ 1.437 \\ -0.897 \\ -1.414 \\ 0.880 \\ -0.779 \\ -1.591 \\ -101.635 \\ 1.206 \\ -1.201 \end{bmatrix} \text{ in mGals}$$

The observations are considered uncorrelated with variances proportional to the corresponding travel time between stations. Thus the weight matrix is given as the inverse of the variance-covariance matrix of the observations, that is:

$$P = \text{diag}(1/2,\ 1/2,\ 1/3,\ 1/4,\ 1/2,\ 1/3,\ 1/4,\ 1/4,\ 1/3,\ 1/5,\ 1/6,\ 1/6)$$

The normal equations are $N\hat{X} = U$, yielding the solution $\hat{X} = N^{-1}U$

Spreadsheets and a Fortran program can been used to compute $\hat{X}$ and obtain the estimates for the gravity of points B, C, D, E, F and G. We find:

$$\hat{X} = \begin{bmatrix} 100.133 \\ 102.515 \\ 103.971 \\ 103.083 \\ 101.684 \\ 102.887 \end{bmatrix} \text{ in mGals (980000 mGals to be added)}$$

and

$$\hat{V} = A\hat{X} - L = \begin{bmatrix} -0.010 \\ 0.010 \\ 0.012 \\ 0.019 \\ 0.009 \\ 0.014 \\ -0.049 \\ -0.052 \\ 0.041 \\ -0.049 \\ -0.003 \\ -0.003 \end{bmatrix} \text{ in mGals}$$

The adjusted observations are $\hat{L} = L + \hat{V}$

b) The only change is in the weight matrix P. If we go through the computations again, the adjusted weighted values differ slightly from the first adjustment result. This illustrates the effect of relative weights on the results.

$$\hat{X} = \begin{bmatrix} 100.133 \\ 102.519 \\ 103.974 \\ 103.086 \\ 101.685 \\ 102.889 \end{bmatrix} \text{ in mGals (980000 mGals to be added)}$$

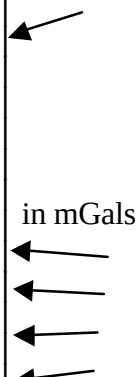
For precise gravity surveys, it is both logical and prudent to perform the weighted adjustment rather than the equal weight adjustment.

c) The only change is in the observation vector L. We have deliberately introduced an error in the observations. This error will be distributed throughout the whole network by the least squares adjustment, and it gives as a result large residuals. The largest residual is related to the third observation. **Thus the residuals allow us to detect errors which are not random errors amongst the observations**.

$$\hat{X} = \begin{bmatrix} 100.083 \\ 103.013 \\ 104.359 \\ 103.417 \\ 101.935 \\ 103.138 \end{bmatrix} \text{ in mGals (980000 mGals to be added)}$$

and

$$\hat{V} = A\hat{X} - L = \begin{bmatrix} -0.060 \\ 0.060 \\ -0.440 \\ -0.091 \\ -0.045 \\ -0.068 \\ 0.198 \\ -0.299 \\ -0.261 \\ -0.300 \\ -0.003 \\ -0.003 \end{bmatrix} \text{ in mGals}$$

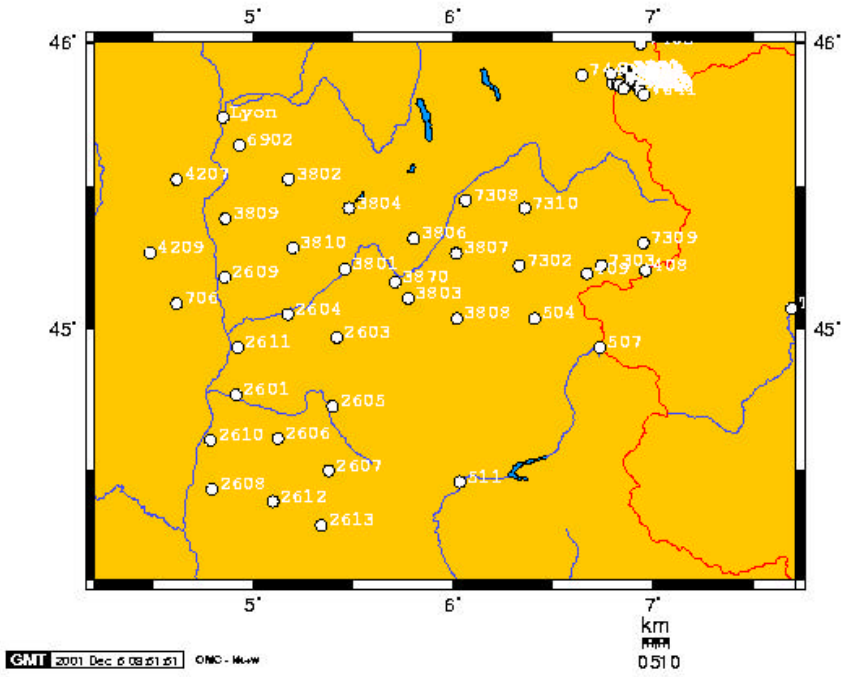


TUTORIAL: THE FRENCH RBF GRAVITY NETWORK (2000-2001)

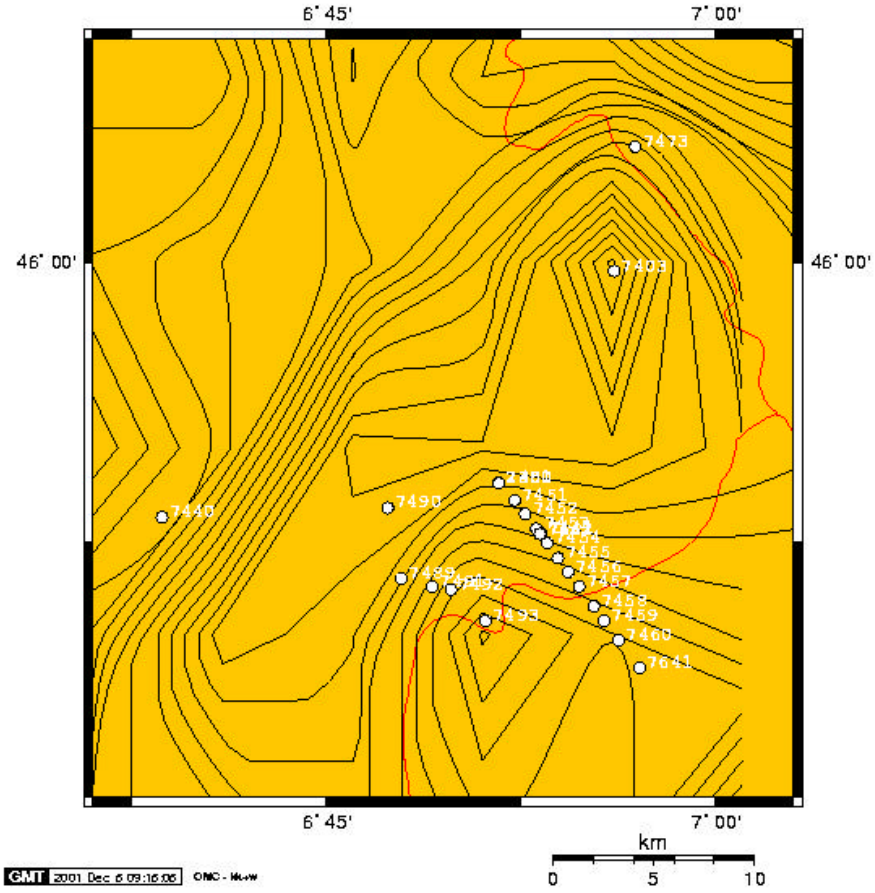
I THE FRENCH IGN GRAVITY NETWORK

In the summers of 2000 and 2001, a number of relative gravity measurements were taken over more than 200 points, mostly from the RBF (French Base Geodetic Network) but also from the NGF-IGN69 levelling network, and RGF83 gravity network. The data was processed using CG3 TOOL, and corrected relative gravity differences at marker level were derived for each leg of the network.

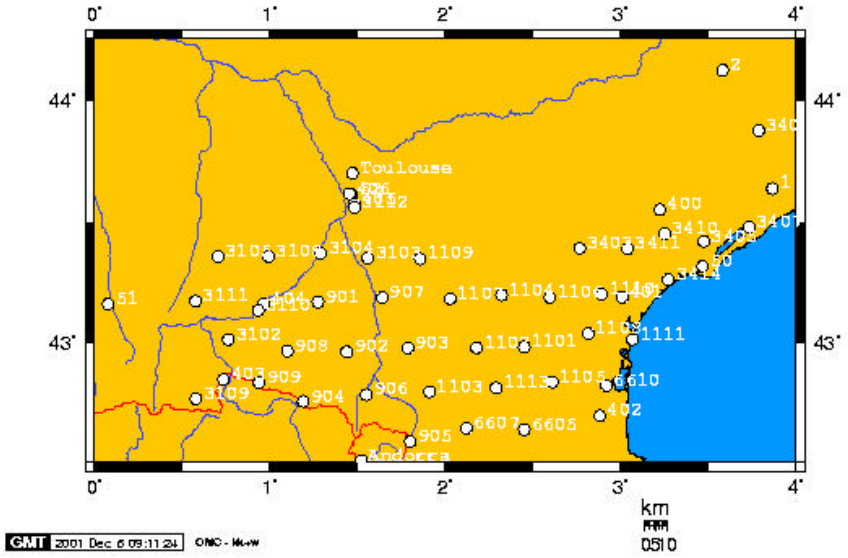
LOCATION OF THE STATIONS (2001)



Northern Alps area.



Mountain and Tunnel of the Mont-Blanc.



Eastern Pyrennean Mountains.

Those maps were produced using the Mercator Projection with the GMT program on the following Online Map Creation Internet Site: http://www.aquarius.geomar.de/omc/make_map.html .

II PREPARATION OF THE DATA

Very often the direct readings from the measuring operation have to be "reduced" or preprocessed before they can be considered appropriate. This reduction is carried out with the CG3TOOL software.

The data was adjusted with the programs **Network** and **Geolab**. Theoretically the results should be very close. In practice, however, the results of the different adjustment models differ significantly due to differences in outlier rejection levels, drift models as well as final determination of the weights. Tests showed that the discrepancies between the two models can reach 45 µGals.

The data contained in **CG3TOOL** result files was run through a makeshift program `CG3TOGLB.EXE` so as to obtain a **Microsearch Geolab 2000** input file, assuming gravity difference observations to be equivalent to levelled height differences, and adjustable as thus.

Each gravity difference is calculated as the difference of the gravity values from one point to the next, multiplied by the calibration of the reference gravimeter. The standard deviation of each observation is calculated using the law of propagation of uncertainty. All values are converted to µGals.

Let's have a closer look at how a set of CG3TOOL result files for a given day must be transformed in order to be used in the Geolab network adjustment:

File `fign4r00.230`

```
# INFO       : CG3TOOL V3.1 (SEPTEMBER 1999) - FIELD RESULTS
# CREATOR    : bonvalot
# DATE       : Wed Nov 22 10:48:10 2000
# GRAVIMETER : CG3 #9601323 (4 in CG3TOOL.init)
# DATA       : Total = 28 / Used = 27 / Station = 6 / Reoccupation (DT>15mn) = 4
# CORRECTION : Earth Tide = Longman / Height = Yes (0.3086 mGal/m) / Pressure = No
# REFERENCE  : Station 2601 (0.000 mGal)
# DRIFT      : Value (mGal/Day) = 0.758 +/- 0.003 / Offset (mGal) = -0.008 +/- 0.001
#            : Standard Deviation (mGal) = 0.011 / Maximum Deviation (mGal) = 0.018
#            : Correlation = 100 %  /  Goodness-of-fit (Q) = 0.000 (Doubtfull)
# FORMAT     : STATION / VALUE (mGal) / ERROR (mGal) / REITERATION / REOCCUPATION
 2601       0.0000  0.0196    9    2
 9997     -88.2581  0.0107    1    0
  705      -4.3641  0.0242    6    1
  708     -54.0860  0.0167    2    0
 1001      32.2738  0.0147    6    1
 2611      13.5857  0.0119    3    0
```

File `fign6r00.230`

```
# INFO       : CG3TOOL V3.1 (SEPTEMBER 1999) - FIELD RESULTS
# CREATOR    : bonvalot
# DATE       : Wed Nov 22 15:09:27 2000
# GRAVIMETER : CG3 #9711408 (6 in CG3TOOL.init)
# DATA       : Total = 35 / Used = 21 / Station = 5 / Reoccupation (DT>15mn) = 3
# CORRECTION : Earth Tide = Longman / Height = Yes (0.3086 mGal/m) / Pressure = No
# REFERENCE  : Station 2601 (0.000 mGal)
# DRIFT      : Value (mGal/Day) = 0.818 +/- 0.004 / Offset (mGal) = -0.002 +/- 0.001
#            : Standard Deviation (mGal) = 0.007 / Maximum Deviation (mGal) = 0.012
#            : Correlation = 100 %  /  Goodness-of-fit (Q) = 0.000 (Doubtfull)
# FORMAT     : STATION / VALUE (mGal) / ERROR (mGal) / REITERATION / REOCCUPATION
 2601       0.0000  0.0100    5    1
  705      -4.4093  0.0103    5    0
  708     -54.0728  0.0156    2    0
 1001      32.2159  0.0135    6    1
 2611      13.5714  0.0119    3    0
```

File `grav0001.iob`

```
* ************************************
* ANNEE 2000
* ************************************
*observations du fichier fign4r00.230
 OHDF     200000002601 200000009997                -88245.7       22.3        50000.000 m
 OHDF     200000009997 200000000705                 83882.2       26.5        50000.000 m
 OHDF     200000000705 200000000708                -49714.9       29.4        50000.000 m
 OHDF     200000000708 200000001001                 86347.6       22.2        50000.000 m
 OHDF     200000001001 200000002611                -18685.5       18.9        50000.000 m
*observations du fichier fign6r00.230
 OHDF     200000002601 200000000705                 -4412.2       14.4        50000.000 m
 OHDF     200000000705 200000000708                -49696.5       18.7        50000.000 m
 OHDF     200000000708 200000001001                 86346.0       20.6        50000.000 m
 OHDF     200000001001 200000002611                -18656.9       18.0        50000.000 m
```

We calculate the gravity difference for two consecutive stations in the CG3TOOL file, and we multiply it by the calibration factor for the reference gravimeter, and possibly the relative calibration factor between the second and first gravimeter for this given day. We calculate the standard deviation for the gravity difference from the standard deviation of the gravity values at each station using the law of propagation of variance. For instance $\sqrt{19.6^2 + 10.7^2}$ will give us the standard deviation for the gravity difference measured with gravimeter number 4 between stations 2601 and 9997.

The input file for Geolab is `grav0001.iob`. The geographic coordinates are in the `gravi01.apx` and `gravi00.apx` files. They are used for plotting the network, but it is also in these latter files that we choose to fix some gravity values (the height value being in fact the gravity value). Lastly, `gravi.gpj` contains the options for processing the network.

III ADJUSTMENT OF THE NETWORK

An adjustment is performed using standard options in Geolab, and the result analysed in the light of the statistical testing performed by the program. We first choose to fix one station per area only, which are Station 1 and Station 15, with the following values 980487121,0 and 980925990,2 µGals respectively.

Comparison may be carried out for stations with previously known gravity values, namely stations 2, 3, 4, 6, 21, 51, 52, 53, 63 for 2000, and 50, 51, 52, 2301 for 2001. We may then identify reliable control points, and hold them fixed in the last run of the adjustment.

IV EXAMINATION OF THE STATISTICAL OUTPUTS : STANDARDIZED RESIDUALS

Even after one has acquired some experience in judging where weakness lies in survey configurations, one should inspect residuals of the adjusted network to detect gross errors because confidence regions do not reveal small pockets of distorsion. If blunders are identified, they must be removed from the data set, and the adjustment must be rerun. Bear in mind that large residuals do not always indicate a blunder since a least-squares adjustment tends to distribute the effects throughout the entire network.

In order to verify if the observations are normally distributed, their residuals obtained from the adjustment may be subjected to the c^2 test of goodness of fit. The residuals have to be standardized since a standardized random variable is normally distributed with a mean zero and variance 1. The standardization may be done using the sample mean and sample variance in place of the population mean and variance, but this is valid only for a large sample.

The standardized residuals take into account for the fact that residuals generated by random errors are somewhat predictable statistically. They are unitless scaled values of the actual residuals. Evaluation of the normalized residuals will reveal one of three things:

a) A value of 1 indicates that the residual is as large as expected based on its standard error. This is usually an indication that the observation contains no blunder
b) A value of less than 1 is usually an indication that the observation contains no blunder
c) A value greater than 1 indicates that it is larger than expected. A normalized residual greater than 3 is either one of the 1% (normal distribution) caused by random errors, or represents an observation containing a blunder. Any normalized residual greater than a threshold value is flagged by Geolab since there is such a small chance that it could belong to a good observation. The actual threshold value is indicated in the Geolab `.lst` file header in the `Residuals` section: `Residuals (critical value = 4.000)`. Such observations can then be examined and a decision about its retention or rejection can be made. We can remove the observation and rerun the adjustment, or modify the a priori uncertainty for this observation.

This test allows to verify the normality of the residuals obtained after the adjustment. It is displayed in the graphical part of the result (`.lst`) file.

IV EXAMINATION OF THE STATISTICAL OUTPUTS : VARIANCE OF UNIT WEIGHT

The variance of unit weight monitor the relationship between the uncertainties assigned to the observations and the magnitude of the change required to each observation (residuals) in the adjustment. Changes to the observations should not be significantly greater than the associated uncertainties. **The variance of unit weight gauges the magnitude of the observation changes compared to the observation uncertainties for the entire network**.

Analysis of the magnitude of the a posteriori variance of unit weight reveals one of the following regarding the quality of the adjustment:

a) A computed value close to 1 is generally an indication of a good adjustment
b) A computed value significantly smaller than 1 indicates an imbalance between the observation residuals and uncertainties. Specifically, the observation uncertainties are too pessimistic (too large).
c) A computed value significantly larger than 1 is also an indication of an imbalance, but this time either one or more blunders exist in the observations, or the uncertainties are too optimistic (too small).

Usually an arbitrary scaled weight matrix is used in an adjustment as we do not know the true covariance matrix of the observation, and $P = s_0^2 C_l^{-1}$ where C_l is the true covariance matrix and s_0^2 is an assumed a priori variance factor, also known as the reference variance or the **variance of unit weight**.

After the adjustment, we can estimate the variance factor by computing

$$\hat{s}_0^2 = \frac{\hat{V}^T P \hat{V}}{n-u}$$

which is called **the a posteriori variance factor or estimated variance factor**. The variance factor, and hence the assumed variance, can be tested using the Chi-Square test:

$$c_r^2 = r\frac{\hat{s}_0^2}{s_0^2} = \frac{\hat{V}^T P \hat{V}}{s_0^2}$$

(the sum of the square weighted residuals, normalized to unit variance) where r, the number of degrees of freedom, is (n-m). For large values of r, $\frac{c_r^2}{r}$ approaches the normal distribution with mean 1 and standard deviation $s = \sqrt{\frac{2}{r}}$. **The sum of the square of n independent normally distributed random variables has a Chi-square probability density function with n degrees of freedom.** The Chi-square distribution function table and typical shapes are given in Appendix D.

Furthermore, s_0^2 is often taken to be unity. Then the statistic $\hat{s}_0^2$ may be compared to unity. If it is significantly different from 1, say by more than 2σ, then probably the assumed a priori weights are incorrect. We may then start looking at the C_l matrix, and even use the a posteriori variance factor from the adjustment to compute more realistic weight.

For instance if $\frac{\hat{s}_0^2}{s_0^2}$ is considerably greater than 1, C_l was postulated too optimistically.

For example if the variance of unit weight is computed to be 4, then the standard error of unit weight is 2, and if there are no blunders it means that the observation uncertainties should be 2 times larger than they are currently stated to be (or that there remains a blunder somewhere in the network)..

IV EXAMINATION OF THE STATISTICAL OUTPUTS : CONFIDENCE INTERVALS

Since the gravity accuracy at each point is related to the reference points that are treated as fixed (errorless) in the process of the network adjustment, the term **absolute gravity confidence interval** is applied. Variances and covariances of the analyzed network are a basis for the calculations of gravity errors. They allow the calculation of the standard deviations of gravity values (or differences in gravity values).

Good estimates of the standard deviations of the measurements are usually available, as in our case. That's why we use the a-priori standard deviation of unit weight to determine the standard confidence interval, and the factor 1.96 to arrive at the 95% confidence interval. However, if good estimates of the measurements are not available, because of inexperience or malfunctioning equipment, then we should use the standard deviation of unit weight computed from the least squares adjustment, provided there is sufficient redundancy to give a reasonable value.

We usually publish the confidence intervals at the 95% confidence level. Here only the VERTICAL confidence interval is relevant. It is expressed in μGals while Geolab will think of it as meters, hence the scale of the confidence interval we had to choose for the graphical output.

GLOSSARY

Observable : physical or geometric quantity to which we assign a number or value with a certain accuracy (angle, distance, mass…)

Observation : value assigned to the observable; the term is often used in practice to refer to both the operation or process itself, as well as the actual outcome of such operation, but with regard to adjustment, only the numerical outcome will be designated as the observations.

Variance (of a coordinate value or an observation) : statistical measure of the reliability of the value (also called mean square error)

CoVariance : describes the mutual variation of two random variables; reflects their interrelationship or mutual correlation

Mathematical model : functional relationship between parameters and observables. It is a theoretical system or an abstract concept by which one describes a physical situation or a set of events. It should conform to the physical reality with sufficient accuracy for the intended purpose.

Parameter : an unknown quantity which we are interested in; they are generally not measured directly

Linear vector space : space where a coordinate system is defined

Metric space : space where a metric is defined

Linear adjustment : observables can be written as some linear function of the parameters

Bias : systematic effect error; the bias of an estimator is defined as the difference between the value of the parameter and the expectation of the estimator

Blunder : outlier, ends up with large residuals in adjustments