

Classical globally reflected gravity field determination in modern locally oriented multiscale framework

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Abstract The purpose of this paper is the canonical connection of classical global gravity field determination following the concept of Stokes (Trans Camb Philos Soc 8:672–712, 1849), Bruns (Die Figur der Erde, Publikation Königl. Preussisch. Geodätisches Institut, P. Stankiewicz Buchdruckerei, Berlin, 1878), and Neumann (Vorlesungen über die Theorie des Potentials und der Kugelfunktionen. Teubner, Leipzig, pp 135–154, 1887) on the one hand and modern locally oriented multiscale computation by use of adaptive locally supported wavelets on the other hand. The essential tools are regularization methods of the Green, Neumann, and Stokes integral representations. The multiscale approximation is guaranteed simply as linear difference scheme by use of Green, Neumann, and Stokes wavelets. As an application, gravity anomalies caused by plumes are investigated for the Hawaiian and Iceland areas.

Keywords Boundary value problems of classical physical geodesy · ‘Zooming-in’ approximation by locally supported wavelets

1 Introduction

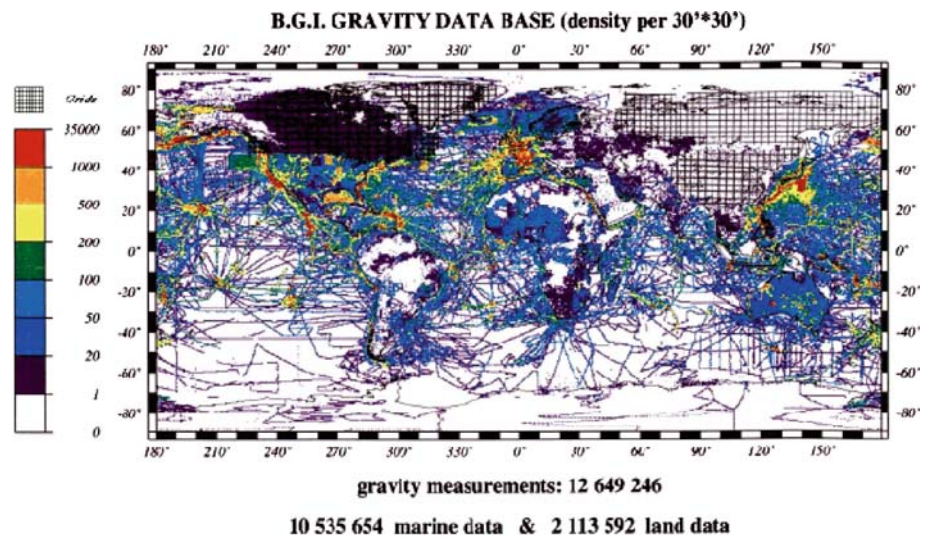
Gravity, as observed on the Earth’s surface is the combined effect of the gravitational mass attraction and the centrifugal force due to the Earth’s rotation. The force of gravity provides

a directional structure to the space above the Earth’s surface. It is tangential to the vertical plumb lines and perpendicular to all level surfaces. Any water surface at rest is part of a level surface. As if the Earth was a homogeneous, spherical body gravity turns out to be constant all over the Earth’s surface, the well-known quantity 9.81 ms^{-2} . The plumb lines are directed towards the Earth’s center of mass, and this implies that all level surfaces are nearly spherical, too. Even for a spherical Earth the gravity varies from the pole to the equator; more precisely, it decreases from the poles to the equator by about 0.05 ms^{-2} . This is caused by the flattening of the Earth’s figure and the negative effect of the centrifugal force, which is maximal at the equator. The level surfaces are ideal reference surfaces, for example, for heights.

The traditional concept of physical geodesy (cf., e.g., Rummel 1992) is based on the assumption that all over the Earth the position (e.g., latitude and longitude) and scalar gravity g are available. Moreover, it is common practice that the gravitational effects of the sun and moon and of the Earth’s atmosphere are accounted for by means of corrections. The gravitational part of the gravity potential can then be regarded as a harmonic function. A classical approach to gravity field modeling was conceived by Stokes (1849). He proposed reducing the given gravity accelerations from the Earth’s surface to the geoid. As the geoid is a level surface, its potential value is constant. The difference between the reduced gravity on the geoid and the reference gravity on the so-called normal ellipsoid is called the gravity anomaly. The disturbing potential, i.e., the difference between the actual and the reference potential, can be obtained from a (third) boundary value problem of potential theory. After spherical and constant radius approximations, its solution is representable in integral form, i.e., by the Stokes integral. The disadvantage of the Stokes approach is that the reduction to the geoid requires the introduction of assumptions

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Fig. 1 Data density and distribution of scalar gravity (due to Rummel et al. (1992))



concerning the unknown mass distribution between the Earth's surface and the geoid.

In this paper, we briefly recapitulate the classical approach to global gravity field determination due to Gauss (1828), Stokes (1849), Bruns (1878), Helmert (1880, 1884), Listing (1873), Neumann (1887), and Wangerin (1921) by formulating the differential/integral relations between gravity disturbance, gravity anomaly, deflections of the vertical on the one hand and the disturbing potential and the geoidal undulations on the other hand. The representation of the disturbing potential in terms of gravity disturbances, gravity anomalies, and deflections of the vertical are written in terms of well-known integral representations over the geoid. For practical purposes the integrals are replaced by approximate cubature formulas using certain integration weights and knots within a spherical framework. Seen from the view of constructive approximation, however, the approximate integration formulas are the essential problem in the spherical framework of determining globally the disturbing potential and the geoidal heights. On the one hand, Weyl's law of equidistribution (cf. Weyl 1916) tells us that numerical integration and equidistribution of the nodal points are mathematically equivalent. To get better and better accuracy in numerical integration procedures we thus need denser and denser globally over the whole sphere equidistributed data sets. On the other hand, observations in sufficient data width and quality are only available for certain parts of the Earth's surface, and there are large areas, particularly at sea, where no data are given at all (cf. Fig. 1). The interested reader is referred to Lambeck (2000). In fact, it should be noted that terrestrial gravity data coverage now and in the foreseeable future is far from being satisfactory and totally inadequate for the purpose of high-precision geoid determination at global scale.

Looking at the key aspects in constructive approximation of gravity field determination we are confronted with the

following situation: Almost 70% of the Earth's gravity field is smooth. Its modeling by polynomial structures, i.e., spherical harmonics, is the appropriate tool. But what about the remaining parts that certainly are of more geophysical and geodetic relevance, today? Because of the lack of terrestrial data, trial functions with global support seem to be an inadequate choice for understanding these geoscientifically significant parts in the 'system Earth'. Even more, globally supported approximation structures such as spherical harmonics applied to a heterogeneous data set, e.g., the Earth's gravity anomalies (see Fig. 1) also influence large areas in negative way, where a sufficiently high accuracy has already been obtained by a fewer equidistributed number of data.

Nowadays, there are two ways out for modeling the Earth's gravity field: the *first one* is to use spaceborne data (for example, collected by the satellites CHAMP, GRACE, and GOCE). They, indeed, give the choice of globally measured and suitably (equi-)distributed data sets (cf. ESA 1996, 1998, 1999; Freeden 1999). The critical point, however, is that satellite data are exponentially smoothed with respect to the altitude; hence, they are of lower quality in comparison with terrestrial ones. Satellite-only models based on global data-sets are reliable only to a maximum level of accuracy. Nevertheless, the determination of the Earth's gravitational field based on spaceborne observations is a great challenge for modern geoscientific research (cf. Rummel et al. 1992). The *second one* is to use high-precision terrestrial data in their actual heterogeneous distribution and to apply a locally adaptive approximation method. From mathematical point of view this requires a careful analysis involving a 'zooming-in' procedure based on locally supported trial structures such as wavelets.

The layout of this paper is as follows: First, we are concerned with the idea of calculating globally reflected integral representations of the disturbing potential by locally oriented

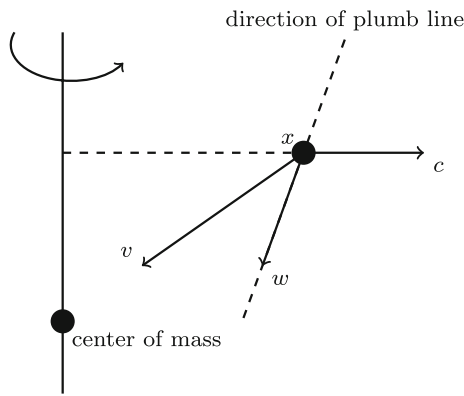


Fig. 2 Gravitation v , centrifugal acceleration c , gravity acceleration w

‘zooming-in’ approximations adaptive to the actual terrestrial data width and distribution. In other words, classical global methods of gravity field determination are formulated in terms of modern local (isotropic) multiscale framework. Essential tools are regularizations of the Green, Neumann, and Stokes integral expressions derived by potential theoretical means. It is shown that the Green, Neumann, and Stokes kernel, respectively, provide us with so-called Green, Neumann, and Stokes scaling functions. Correspondingly, locally supported wavelets are definable for adaptive (local) multiscale approximation. Finally, the efficiency of our methods is illustrated by some test examples, showing the gravity disturbance caused by plumes in the area of Hawaii and Iceland.

2 Gravity anomalies and gravity potential

The *gravity acceleration (gravity)* w is the resultant of gravitation v and centrifugal acceleration c

$$w = v + c. \quad (2.1)$$

The centrifugal force c arises as a result of the rotation of the Earth about its axis. We assume here a rotation of constant angular velocity ω about the rotational axis $\varepsilon^3 = (0, 0, 1)^T$, which is further assumed to be fixed with respect to the Earth. The centrifugal acceleration acting on a unit mass is directed outward perpendicularly to the spin axis (see Fig. 2). If the ε^3 -axis of an Earth’s fixed coordinate system coincides with the axis of rotation, then we have

$$c(x) = -\omega^2 \varepsilon^3 \wedge (\varepsilon^3 \wedge x). \quad (2.2)$$

Using the so-called *centrifugal potential* (note that $\varepsilon^1 = (1, 0, 0)^T$, $\varepsilon^2 = (0, 1, 0)^T$ and $\varepsilon^3 = (0, 0, 1)^T$ denote the

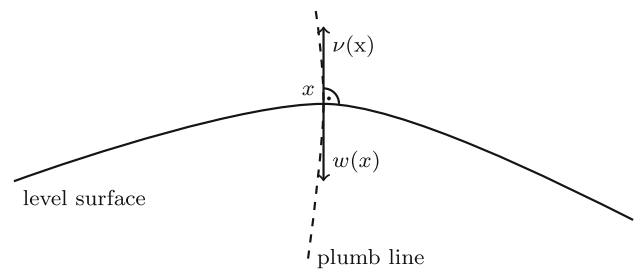


Fig. 3 Level surface and plumb line

Cartesian unit vectors)

$$\begin{aligned} C(x) &= \frac{\omega^2}{2} |\varepsilon^3 \wedge (\varepsilon^3 \wedge x)| \\ &= \frac{\omega^2}{2} \left((x \cdot \varepsilon^1)^2 + (x \cdot \varepsilon^2)^2 \right) \\ &= \frac{\omega^2}{2} (x_1^2 + x_2^2) \end{aligned} \quad (2.3)$$

we can write $c = \nabla C$. Applying the Laplace operator gives us $\Delta C = 2\omega^2$; thus, the function C is *not* harmonic.

The direction of the gravity w is known as the direction of the *plumb line*, the quantity $g = |w|$ is called the *gravity intensity* (often just (scalar) *gravity*). The *gravity potential of the Earth* can be expressed as the sum of the gravitational potential V and the centrifugal potential C , i.e.,

$$W = V + C. \quad (2.4)$$

The gravity acceleration w is given by

$$w = \nabla W = \nabla V + \nabla C. \quad (2.5)$$

The surfaces of constant gravity potential $W(x) = \text{const}$, $x \in \mathbb{R}^3$, are designated as *equipotential (level, or geopotential) surfaces of gravity* (for more details see, e.g., [Groten 1979](#); [Heiskanen and Moritz 1967](#); [Torge 1991](#)).

In Earth’s fixed coordinate system the centrifugal potential C is explicitly known. Hence, the determination of equipotential surfaces of the gravity potential W is strongly related to the knowledge of the gravitational potential V . Equipotential surfaces (see Fig. 3) intuitively express the notion of tangential surfaces, as they are normal to the plumb lines given by the direction of the gravity vector.

According to the classical Newton’s Law of Gravitation (1687), knowing the density distribution ρ of a body, the gravitational potential can be computed everywhere in \mathbb{R}^3 . More explicitly, the gravitational potential V of the Earth’s exterior is given by

$$V(x) = G \int_{\text{Earth}} \frac{\rho(y)}{|x - y|} dV(y), \quad x \in \mathbb{R}^3 \setminus \text{Earth}, \quad (2.6)$$

where G is the gravitational constant [$G = 6.6742 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ (for more details see, e.g., [Groten 1979](#);

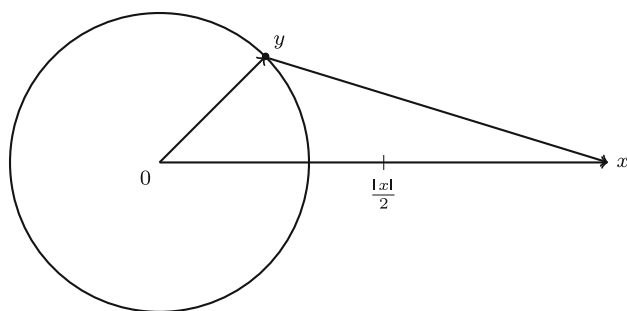


Fig. 4 Regularity at infinity

Heiskanen and Moritz 1967; Torge 1991)]. In consequence, the properties of the gravitational potential (2.6) in the Earth's exterior are easily described as follows:

$$\Delta V(x) = 0, \quad x \in \mathbb{R}^3 \setminus \text{Earth}. \quad (2.7)$$

Moreover, the gravitational potential V is *regular at infinity*, i.e.,

$$|V(x)| = O\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty, \\ |\nabla V(x)| = O\left(\frac{1}{|x|^2}\right), \quad |x| \rightarrow \infty. \quad (2.8)$$

Note that, for suitably large values $|x|$ (see Fig. 4), we have $|y| \leq \frac{1}{2}|x|$; hence, $|x - y| \geq ||x| - |y|| \geq \frac{1}{2}|x|$. Clearly, the gravitational field $v = \nabla V$ fulfills the following identities:

$$L \cdot \nabla V(x) = 0, \quad \nabla \cdot \nabla V(x) = \Delta V(x) = 0, \quad (2.9)$$

where $L \cdot$ denotes the curl operator and $x \in \mathbb{R}^3 \setminus \text{Earth}$. However, the problem is that in reality the density distribution ρ is very irregular and known only for parts of the upper crust of the Earth. It is actually so that we would like to know it from measuring the gravitational field.

Equipotential surfaces of the gravity potential W allow in general no simple representation. This is the reason why a reference surface, in physical geodesy usually an ellipsoid of revolution, is chosen for the (approximate) construction of the geoid. As a matter of fact, the deviations of the gravity field of the Earth from the normal field of such an ellipsoid are small. The remaining parts of the gravity field are gathered in a so-called *disturbing gravity field* ∇T corresponding to the *disturbing potential* T . Knowing the gravity potential, all equipotential surfaces—including the geoid—are given by an equation of the form $W(x) = \text{const}$. By introducing U as the normal gravity potential corresponding to the ellipsoidal field and T as the disturbing potential (for more details see, e.g., Groten 1979; Heiskanen and Moritz 1967; Torge 1991) we are led to a decomposition of the gravity potential in the form

$$W = U + T. \quad (2.10)$$

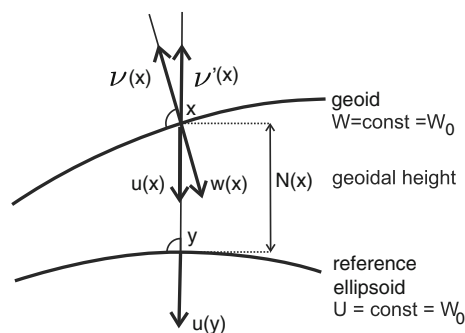


Fig. 5 Illustration of the definition of the gravity anomaly vector $\alpha(x) = w(x) - u(y)$ and the gravity disturbance vector $\delta(x) = w(x) - u(x)$. $v(x)$ denotes the outward unit vector normal to the geoid, $v'(x)$ denotes the outward unit vector normal to the reference ellipsoid

In accordance with the Pizzetti concept (see, e.g., Pizzetti 1910, 1894), we may assume that

- (1) the center of the ellipsoid coincides with the center of the gravity of the Earth,
- (2) the difference of the mass of the Earth and the mass of the ellipsoid is zero.

Consequently, following the classical approach (see, e.g., Heiskanen and Moritz 1967; Meissl 1971), T is given in such a way that

$$\int_{\Omega_R} T(x) \, d\omega(x) = 0, \quad (2.11)$$

$$\int_{\Omega_R} T(x) (\varepsilon^k \cdot x) \, d\omega(x) = 0, \quad k = 1, 2, 3, \quad (2.12)$$

where in spherical nomenclature Ω_R denotes the sphere with mean radius R as defined by, e.g., Heiskanen and Moritz (1967), Hoffmann-Wellendorf and Moritz (2005), i.e. $\Omega_R = \{x \in \mathbb{R}^3 \mid x = R\xi, \quad R = |x|, \quad \xi \in \Omega\}$, $\Omega = \Omega_1$.

A point x of the geoid is projected onto the point y of the ellipsoid by means of the ellipsoidal normal (see Fig. 5). The distance between x and y is called the *geoidal height*, or *geoidal undulation*.

The *gravity anomaly vector* is defined as the difference between the gravity vector $w(x)$ and the normal gravity vector $u(y)$, $u = \nabla U$, i.e.,

$$\alpha(x) = w(x) - u(y). \quad (2.13)$$

It is also possible to form the difference between the vectors w and u at the same point x to get the *gravity disturbance vector*

$$\delta(x) = w(x) - u(x). \quad (2.14)$$

Of course, several basic mathematical relations between the quantities just mentioned are known. In what follows we only

describe heuristically the fundamental relations. We start by observing that the gravity disturbance vector at the point x on the geoid can be written as

$$\delta(x) = w(x) - u(x) = \nabla(W(x) - U(x)) = \nabla T(x). \quad (2.15)$$

Expanding the potential U at x according to Taylor's theorem and truncating the series at the linear term we get

$$U(x) \doteq U(y) + \frac{\partial U}{\partial v'}(y)N(x) \quad (2.16)$$

(\doteq means approximation in linearized sense). Here, $v'(y)$ is the ellipsoidal normal at y , i.e., $v'(y) = -u(y)/\gamma(y)$, $\gamma(y) = |u(y)|$, and the geoid undulation $N(x)$, as indicated in Fig. 5, is the aforementioned distance between x and y , i.e., between the geoid and the reference ellipsoid. Using

$$\begin{aligned} \gamma(y) &= |u(y)| = -v'(y) \cdot u(y) \\ &= -v'(y) \cdot \nabla U(y) = -\frac{\partial U}{\partial v'}(y) \end{aligned} \quad (2.17)$$

we arrive at

$$\begin{aligned} N(x) &= \frac{T(x) - (W(x) - U(y))}{|u(y)|} \\ &= \frac{T(x) - (W(x) - U(y))}{\gamma(y)}. \end{aligned} \quad (2.18)$$

Observing $U(y) = W(x) = \text{const} = W_0$ we obtain the so-called *Bruns' formula* (cf. [Bruns 1878](#))

$$N(x) = \frac{T(x)}{|u(y)|} = \frac{T(x)}{\gamma(y)}. \quad (2.19)$$

It should be noted that Bruns' formula (2.19) relates the physical quantity T to the geometric quantity N .

In what follows we are interested in introducing the deflections of the vertical of the gravity disturbing potential T . For this purpose, let us consider the vector field $v(x) = -w(x)/|w(x)|$. This gives us the identity (with $g(x) = |w(x)|$ and $\gamma(x) = |u(x)|$)

$$w(x) = \nabla W(x) = -|w(x)| v(x) = -g(x)v(x). \quad (2.20)$$

Furthermore, we have

$$u(x) = \nabla U(x) = -|u(x)| v'(x) = -\gamma(x)v'(x). \quad (2.21)$$

The *deflection of the vertical* $\Theta(x)$ at the point x on the geoid is defined to be the angular (i.e., tangential) difference between the directions $v(x)$ and $v'(x)$, i.e., the plumb line and the ellipsoidal normal through the same point:

$$\Theta(x) = v(x) - v'(x) - ((v(x) - v'(x)) \cdot v(x)) v(x). \quad (2.22)$$

Clearly, because of (2.22), $\Theta(x)$ is orthogonal to $v(x)$, i.e., $\Theta(x) \cdot v(x) = 0$. Since the plumb lines are orthogonal to the level surfaces of the geoid and the ellipsoid, respectively, the deflections of the vertical give, briefly spoken, a measure of the gradient of the level surfaces. This aspect will

be described in more detail below: From (2.20) we obtain, in connection with (2.22),

$$\begin{aligned} w(x) &= \nabla W(x) \\ &= -|w(x)|(\Theta(x) + v'(x) \\ &\quad + ((v(x) - v'(x)) \cdot v(x))v(x)). \end{aligned} \quad (2.23)$$

Altogether we get for the gravity disturbance vector

$$\begin{aligned} w(x) - u(x) &= \nabla T(x) \\ &= -|w(x)|(\Theta(x) + ((v(x) - v'(x)) \cdot v(x))v(x) \\ &\quad - (|w(x)| - |u(x)|)v'(x)). \end{aligned} \quad (2.24)$$

The magnitude

$$D(x) = |w(x)| - |u(x)| = g(x) - \gamma(x) \quad (2.25)$$

is called the *gravity disturbance*, while

$$A(x) = |w(x)| - |u(y)| = g(x) - \gamma(y) \quad (2.26)$$

is called the *gravity anomaly*.

Since the vector $v(x) - v'(x)$ is (almost) orthogonal to $v(x)$, it can be neglected in (2.24). Hence, it follows that

$$\begin{aligned} w(x) - u(x) &= \nabla T(x) \\ &\doteq -|w(x)|\Theta(x) - (|w(x)| - |u(x)|)v'(x). \end{aligned} \quad (2.27)$$

The gradient $\nabla T(x)$ can be split into a normal part (pointing into the direction of $v(x)$) and an angular (tangential) part (characterized by the surface gradient ∇^*). It follows that

$$\nabla T(x) = \frac{\partial T}{\partial v}(x)v(x) + \frac{1}{|x|}\nabla^*T(x). \quad (2.28)$$

By comparison of (2.27) and (2.28) we, therefore, obtain

$$D(x) = g(x) - \gamma(x) = |w(x)| - |u(x)| = -\frac{\partial T}{\partial v'}(x), \quad (2.29)$$

i.e., the gravity disturbance, besides being the difference in magnitude of the actual and the normal gravity vector, is also the normal component of the gravity disturbance vector. In addition, we are led to the angular, i.e., (tangential) differential equation

$$\frac{1}{|x|}\nabla^*T(x) = -|w(x)|\Theta(x). \quad (2.30)$$

Since $|\Theta(x)|$ is a small quantity, it may be (without loss of precision) multiplied either by $-|w(x)|$ or by $-|u(x)|$, i.e., by $-g(x)$ or by $-\gamma(x)$.

The reference ellipsoid deviates from a sphere only by quantities of the order of the flattening. Therefore, in numerical calculations, if we treat the reference ellipsoid as a sphere Ω_R , this may cause a relative error of the same order, for more details the reader is referred to standard textbooks of physical geodesy (e.g., [Heiskanen and Moritz 1967](#); [Hoffmann-Wellendorf and Moritz 2005](#)). If this error is permissible, we

are allowed to replace $|u(R\xi)|$ by its spherical approximation GM/R^2 . Let G be the gravitational constant and M the constant of the mass, then by virtue of Bruns' formula we find

$$\nabla_{\xi}^* N(R\xi) = -R \ominus(R\xi), \quad \xi \in \Omega. \quad (2.31)$$

In other words, the knowledge of the geoid undulations allows the determination of the deflections of the vertical by taking the surface gradient on the unit sphere.

From the identity (2.29) it follows that

$$\begin{aligned} -\frac{\partial T}{\partial v'}(x) &= D(x) = |w(x)| - \gamma(x) \\ &\doteq |w(x)| - \gamma(y) - \frac{\partial \gamma}{\partial v'}(y) N(x) \\ &= A(x) - \frac{\partial \gamma}{\partial v'}(y) N(x), \end{aligned} \quad (2.32)$$

where A represents the scalar gravity anomaly as defined by (2.26). Observing Bruns' formula (2.19) we get

$$A(x) = -\frac{\partial T}{\partial v'}(x) + \frac{1}{\gamma(y)} \frac{\partial \gamma}{\partial v'}(y) T(x). \quad (2.33)$$

In well-known *spherical approximation* we have (see, e.g., Heiskanen and Moritz 1967)

$$\gamma(y) = |u(y)| = \frac{GM}{|y|^2}, \quad (2.34)$$

$$\frac{\partial \gamma}{\partial v'}(y) = \frac{y}{|y|} \cdot \nabla \gamma(y) = -2 \frac{GM}{|y|^3}, \quad (2.35)$$

and

$$\frac{1}{\gamma(y)} \frac{\partial \gamma}{\partial v'}(y) = -\frac{2}{|y|}. \quad (2.36)$$

This leads us to the basic relations

$$-D(x) = \frac{x}{|x|} \cdot \nabla T(x), \quad x \in \Omega_R, \quad (2.37)$$

and

$$-A(x) = \frac{x}{|x|} \cdot \nabla T(x) + \frac{2}{|x|} T(x), \quad x \in \Omega_R, \quad (2.38)$$

as so-called *fundamental equations of physical geodesy*.

In the sense of physical geodesy (cf., e.g., Heiskanen and Moritz 1967), the meaning of the spherical approximation should be carefully kept in mind. It is used only for expressions relating to small quantities of the disturbing potential, the geoidal undulations, the gravity disturbances, the gravity anomalies, etc. Actually, in all geodetic approaches, the reference surface will never be understood to be a sphere in any geometrical sense, but it always is an ellipsoid (see Aardalan et al. 2006 for more details). However, as the flattening of this ellipsoid is very small, the ellipsoidal formulas are expandable into Taylor series in terms of the flattening, and then all terms containing higher order expressions of the flattening may be neglected. In this way together with suitable

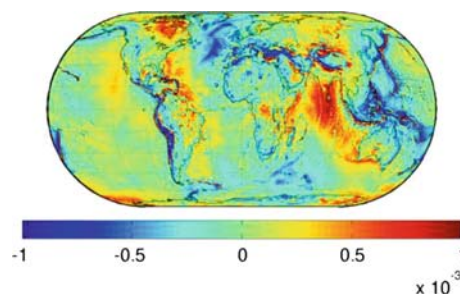


Fig. 6 The gravity disturbance D in $[\frac{m}{s^2}]$

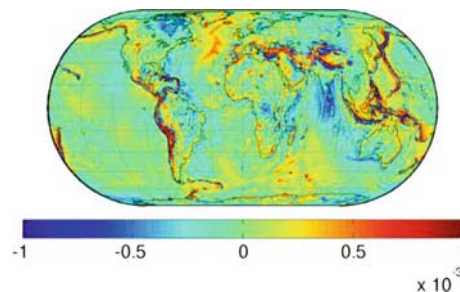


Fig. 7 Gravity anomalies A in $[\frac{m}{s^2}]$

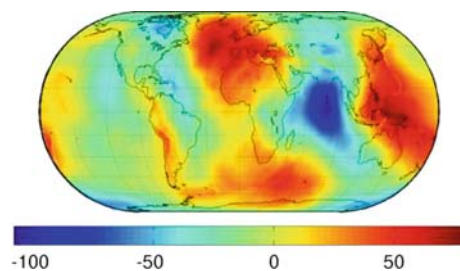


Fig. 8 Geoidal undulations N in $[m]$

pre-reduction processes of gravity formulas are obtained that are rigorously valid for the sphere.

It should be mentioned that, in the gravity disturbances and gravity anomalies all significantly tectonic processes become visible (see Figs. 6 and 7). In accordance with Newton's law the gravity disturbances and gravity anomalies permit the conclusion of an irregular density distribution inside the Earth. Unfortunately, gravity anomalies do not determine uniquely the interior density distribution of the Earth. Geoid undulations are the measure of the perturbations in the hydrostatic equilibrium. They do not show essential correlations to the distribution of the continents (see Fig. 8).

3 Modeling of gravity disturbances, gravity anomalies, and deflections of the vertical by Haar Wavelets

Wavelets enable us to break up functions into many simple pieces at different scales and positions. Thus, three features

are incorporated in this way of thinking about geodetically relevant wavelets: Wavelets are ‘building blocks’ that enable fast decorrelation of gravity data. Whenever we are interested in a multiscale analysis of local ‘fine-structure’ within a smooth(er) global structure, wavelets are the mathematical tools of choice. They are the natural means for investigating signals as, e.g., the gravity disturbance or the gravity anomaly, in small bounded areas out of a global context by the power of a ‘zooming-in’ process.

The wavelets are constructed by use of scaling functions (in the jargon of functional analysis, Dirac sequences) showing the property of tending to the Dirac kernel δ (understood in distributional sense) given by

$$\delta(\xi \cdot \eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta). \quad (3.1)$$

Multiscale modeling can be performed in frequency domain based on specific features within a spherical harmonic context (see, e.g., Fenlger et al. 2004; Freeden 1999; Freeden et al. 1998). But it can also be done exclusively in space domain, and this is the approach that should be presented here.

More formally, let $\{H^{\rho_j}\}_{j \in \mathbb{N}_0}$ be a Dirac family of L^2 -scaling functions such that H^{ρ_j} tends to the Dirac kernel as the ‘scale’ j tends to infinity. Then the convolution

$$F^{\rho_j}(\xi) = \int_{\Omega} H^{\rho_j}(\xi \cdot \eta) F(\eta) d\omega(\eta), \quad F \in L^2(\Omega), \quad (3.2)$$

is a j -level approximation of F , i.e., a low-pass filtered version of F . Moreover, the approximate identity

$$\lim_{j \rightarrow \infty} \|F^{\rho_j} - F\|_{L^2(\Omega)} = 0 \quad (3.3)$$

can be guaranteed. In other words, as the scaling functions H^{ρ_j} converge to the Dirac functional, the functions F^{ρ_j} tend to the limit function

$$F(\xi) = \int_{\Omega} \delta(\xi \cdot \eta) F(\eta) d\omega(\eta), \quad F \in L^2(\Omega), \quad (3.4)$$

(understood in L^2 -topology).

The wavelets $\{(WH)^{\rho_j}\}_{j \in \mathbb{N}_0}$ are nothing else than differences of two consecutive scaling functions,

$$(WH)^{\rho_j} = H^{\rho_{j+1}} - H^{\rho_j}, \quad j \in \mathbb{N}_0, \quad (3.5)$$

such that the detail information in $F^{\rho_{j+1}}$ that is not included in F^{ρ_j} is provided by

$$(WF)^{\rho_j}(\xi) = \int_{\Omega} (WH)^{\rho_j}(\xi \cdot \eta) F(\eta) d\omega(\eta), \quad F \in L^2(\Omega). \quad (3.6)$$

In consequence, the multiscale process is determined by two parameters, viz. the scale j characterizing the ‘zooming-in’

width and the position ξ indicating the ‘zooming-in’ center. With increasing scale j , the wavelets $\{(WH)^{\rho_j}\}_{j \in \mathbb{N}_0}$ extract more and more detailed information out of F .

Clearly, there are millions of scaling functions generating difference wavelet functions. Of particular economy and efficiency for purposes of numerical computation, however, are wavelet kernels given in terms of elementary functions and possessing a local support. But also locally supported trial functions are nothing new, having been discussed already by Haar (1910) in one-dimensional Euclidean theory. The spherical Haar wavelet is zero outside a small spherical cap. In fact, Haar wavelets represent an ideal choice for locally oriented ‘zooming-in’-approximation of geodetic quantities, since signal values outside their spherical support are not taken into account at all. Consequently, local improvements can be guaranteed for smaller and smaller local parts with increase in accuracy, but without any deterioration of the approximated signal in all other parts outside. Note that the size of the local support depends on the scale j of the wavelet H^{ρ_j} , i.e., with increasing scale j as its diameter decreases. This is the reason why the wavelet concept allows an ideal ‘zooming-in’ process to local (high frequency) phenomena.

The apparatus of (smoothed) Haar scaling and wavelet functions (for the one-dimensional approach see Haar 1910) should be concretized:

Let $\{\rho_j\}_{j \in \mathbb{N}_0}$ be a sequence with $\lim_{j \rightarrow \infty} \rho_j = 0$ (for example, $\rho_j = 2^{-j}$ or $\rho_j = 1 - \cos 2^{-j}\pi$). The (smoothed) Haar scaling functions $\{H^{\rho_j}\}_{j \in \mathbb{N}_0}$, are defined by (see also Freeden et al. (1998))

$$H^{\rho_j}(\xi \cdot \eta) = \begin{cases} 0, & \rho_j < 1 - \xi \cdot \eta \leq 2 \\ \left(\frac{k+1}{2\pi}\right) \rho_j^{-(k+1)} (\xi \cdot \eta - 1 + \rho_j)^k, & 0 \leq 1 - \xi \cdot \eta \leq \rho_j \end{cases} \quad (3.7)$$

where ξ, η are elements of the unit sphere Ω and k denotes a (fixed) polynomial degree (Fig. 9). Note, that the smoothed Haar scaling function H^{ρ_j} occurs to be rotationally invariant, i.e., it is a so-called zonal function. Correspondingly, the smoothed Haar wavelets $(WH)^{\rho_j}$, are defined as the difference of two consecutive scaling functions (see (3.5)). Due to its definition as zonal function, the Haar wavelet has a spherical cap as local support on Ω , i.e.,

$$\mathcal{S}^{\rho_j}(\xi) = \text{supp } (WH)^{\rho_j}(\xi \cdot \cdot) = \{\eta \in \Omega \mid 1 - \xi \cdot \eta \leq \rho_j\}. \quad (3.8)$$

Explicitly written out, the Haar wavelets (see Fig. 10) read as follows:

$$(WH)^{\rho_j}(\xi \cdot \eta) = \begin{cases} 0, & \rho_j < 1 - \xi \cdot \eta \leq 2 \\ -\left(\frac{k+1}{2\pi}\right) \rho_j^{-(k+1)} (\xi \cdot \eta - 1 + \rho_j)^k, & \rho_{j+1} < 1 - \xi \cdot \eta \leq \rho_j \\ \left(\frac{k+1}{2\pi}\right) \left(\rho_{j+1}^{-(k+1)} (\xi \cdot \eta - 1 + \rho_{j+1})^k - \rho_j^{-(k+1)} (\xi \cdot \eta - 1 + \rho_j)^k\right), & 0 \leq 1 - \xi \cdot \eta \leq \rho_{j+1}. \end{cases} \quad (3.9)$$

Fig. 9 Illustration of the smoothed Haar wavelets ($\rho_j = 2^{-j}$)

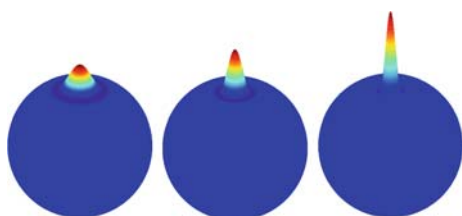
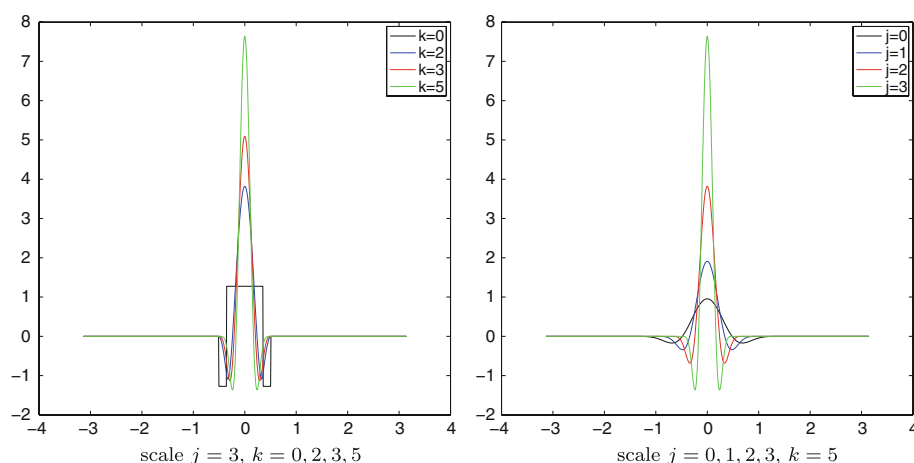


Fig. 10 Illustration of the smoothed Haar wavelet on the sphere ($\rho_j = 2^{-j}$, $j = 2, 3, 4$, $k = 5$)

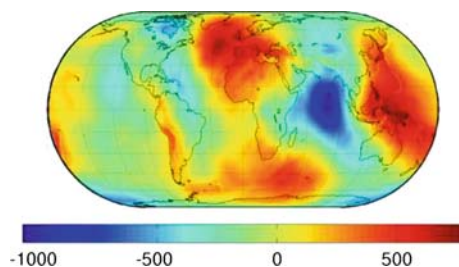


Fig. 11 The disturbing potential T in $[\frac{m^2}{s^2}]$

Note, that the degree k regulates the order of differentiability, i.e., the chosen smoothness property of the Haar kernel.

4 Modeling of the disturbing potential from gravity disturbances by Neumann Wavelets

Since the disturbing potential T (see Fig. 11) is a harmonic function in the exterior Ω_R^{ext} of the sphere Ω_R around the origin with a radius R , we are confronted with a boundary value problem of potential theory to determine T in Ω_R^{ext} from prescribed gravity disturbance D or gravity anomaly A , respectively. Moreover, it should be noted that, at the present state of practice, much more gravity anomalies are available than gravity disturbances. In future, because of GPS-technology, it may be expected that the gravity disturbances become more important than the gravity anomalies (for more details see, e.g., Hoffmann-Wellendorf and Moritz (2005)). This is the reason why both problems will be discussed here.

The determination of the disturbing potential T in the outer space Ω_R^{ext} of Ω_R , from known gravity disturbances D on Ω_R (see Fig. 6), leads us to the Neumann (type) boundary value problem:

(Modified) Exterior Neumann Problem (ENP): We are given $D \in C^{(0)}(\Omega_R)$ with

$$\int_{\Omega_R} D(x) \, d\omega(x) = 0 \quad (4.1)$$

and

$$\int_{\Omega_R} D(x) (\varepsilon^k \cdot x) \, d\omega(x) = 0, \quad k = 1, 2, 3. \quad (4.2)$$

Then the function $T : \overline{\Omega_R^{\text{ext}}} \rightarrow \mathbb{R}$ given by

$$T(x) = \frac{1}{4\pi R} \int_{\Omega_R} N(x, y) D(y) \, d\omega(y) \quad (4.3)$$

with the Neumann kernel function $N : \overline{\Omega_R^{\text{ext}}} \times \overline{\Omega_R^{\text{ext}}} \rightarrow \mathbb{R}$

$$N(x, y) = \frac{2R}{|x - y|} + \ln \left(\frac{|x| + |x - y| - R}{|x| + |x - y| + R} \right), \quad x \neq y, \quad (4.4)$$

is the unique solution of the exterior Neumann boundary value problem:

- (i) T is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in Ω_R^{ext} , i.e.,

$$T \in C^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap C^{(2)}(\Omega_R^{\text{ext}}),$$

- (ii) T is harmonic in Ω_R^{ext} , i.e., $\Delta T = 0$ in Ω_R^{ext} ,
 (iii) T is regular at infinity, i.e.,

$$|T(x)| = O\left(\frac{1}{|x|}\right), |\nabla T(x)| = O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \rightarrow \infty,$$

- (iv) $\int_{\Omega_R} T(x) d\omega(x) = 0$ and $\int_{\Omega_R} T(x)(\varepsilon^k \cdot x) d\omega(x) = 0$, $k = 1, 2, 3$.
 (v) $-\frac{x}{|x|} \cdot \nabla T(x) = D(x)$, $x \in \Omega_R$.

For points $x, y \in \Omega_R$ we (formally) get the so-called *Neumann formula* which is an improper integral over Ω_R (see Kellogg 1929; Neumann 1887; Wangerin 1921)

$$T\left(\frac{Rx}{|x|}\right) = \frac{1}{4\pi R} \int_{\Omega_R} \left(\frac{\sqrt{2}}{\sqrt{1 - \frac{x}{|x|} \cdot \frac{y}{|y|}}} + \ln \left(\frac{\sqrt{2 - 2\frac{x}{|x|} \cdot \frac{y}{|y|}}}{2 + \sqrt{2 - 2\frac{x}{|x|} \cdot \frac{y}{|y|}}} \right) \right) D\left(\frac{Ry}{|y|}\right) d\omega(y). \quad (4.5)$$

Note that the surface integral (4.5) indeed has to be extended over the whole surface. In accordance with our approach it is valid under the following assumptions: (i) the mass within the reference ellipsoid is equal to the mass of the Earth, (ii) the potential with respect to the geoid and the reference ellipsoid is equal, (iii) the center of the reference ellipsoid is coincident with the center of the Earth, (iv) there are no masses outside, and (v) the approximation is simplified in spherical sense.

The identity (4.5) formulated in an equivalent way over the unit sphere Ω yields

$$T(R\xi) = \frac{R}{4\pi} \int_{\Omega} N(\xi \cdot \eta) D(R\eta) d\omega(\eta), \quad \xi \in \Omega, \quad (4.6)$$

where the *Neumann kernel* is given by

$$N(\xi \cdot \eta) = \frac{\sqrt{2}}{\sqrt{1 - \xi \cdot \eta}} - \ln \left(1 + \frac{\sqrt{2}}{\sqrt{1 - \xi \cdot \eta}} \right), \quad 1 - \xi \cdot \eta \neq 0. \quad (4.7)$$

Note that throughout this paper, we write $N(\xi \cdot \eta)$ instead of $N(\xi, \eta)$, $(\xi, \eta) \in \Omega \times \Omega$. This indicates that $N(\xi \cdot \eta)$ depends

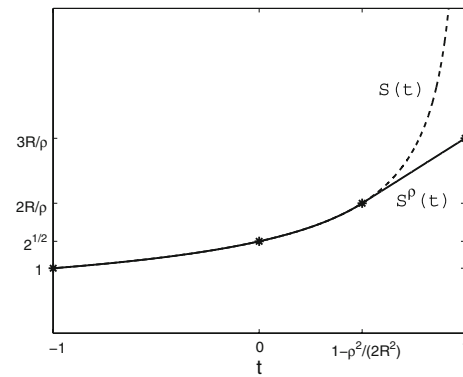


Fig. 12 The functions S and S^ρ on the interval $[-1, 1)$ and $[-1, 1]$, respectively

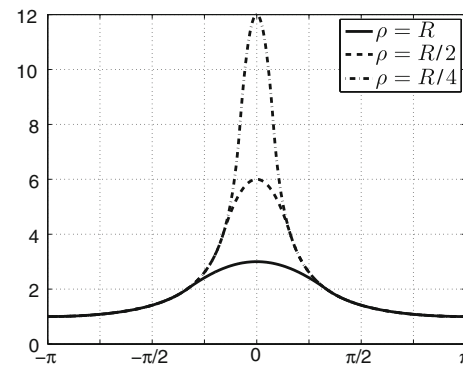


Fig. 13 The kernel $\vartheta \mapsto S^\rho(\cos \vartheta)$ (4.10) for several ρ

only on the scalar product of ξ and η , i.e., N is a zonal function; hence, it may be understood as a function defined on the (one-dimensional) interval $[-1, 1)$. Moreover, note that

$$N(R\xi, R\eta) = N(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \quad (4.8)$$

The essential idea now is illustrated by Fig. 12. It means that the improper integral (4.6) can be regularized, e.g., by replacing the zonal kernel

$$S(\xi \cdot \eta) = \frac{\sqrt{2}}{\sqrt{1 - \xi \cdot \eta}}, \quad 1 - \xi \cdot \eta \neq 0, \quad (4.9)$$

by the space-regularized zonal kernel (see Fig. 13)

$$S^\rho(\xi \cdot \eta) = \begin{cases} \frac{R}{\rho} \left(3 - \frac{2R^2}{\rho^2} (1 - \xi \cdot \eta) \right), & 0 \leq 1 - \xi \cdot \eta \leq \frac{\rho^2}{2R^2} \\ \frac{\sqrt{2}}{\sqrt{1 - \xi \cdot \eta}}, & \frac{\rho^2}{2R^2} < 1 - \xi \cdot \eta \leq 2. \end{cases} \quad (4.10)$$

This leads us to the following space-regularized representation of the disturbing potential (see Fig. 11) in terms of

known gravity disturbances on Ω_R :

$$\begin{aligned} T^\rho(R\xi) &= \frac{R}{4\pi} \int_{1-\xi \cdot \eta > \frac{\rho^2}{2R^2}} \frac{\sqrt{2}}{\sqrt{1-\xi \cdot \eta}} D(R\eta) \, d\omega(\eta) \\ &\quad - \frac{R}{4\pi} \int_{1-\xi \cdot \eta > \frac{\rho^2}{2R^2}} \ln \left(1 + \frac{\sqrt{2}}{\sqrt{1-\xi \cdot \eta}} \right) D(R\eta) \, d\omega(\eta) \\ &\quad + \frac{R}{4\pi} \int_{1-\xi \cdot \eta \leq \frac{\rho^2}{2R^2}} \frac{R}{\rho} \left(3 - \frac{2R^2}{\rho^2} (1 - \xi \cdot \eta) \right) D(R\eta) \, d\omega(\eta) \\ &\quad - \frac{R}{4\pi} \int_{1-\xi \cdot \eta \leq \frac{\rho^2}{2R^2}} \ln \left(1 + \frac{R}{\rho} \right. \\ &\quad \times \left. \left(3 - \frac{2R^2}{\rho^2} (1 - \xi \cdot \eta) \right) \right) D(R\eta) \, d\omega(\eta). \end{aligned} \quad (4.11)$$

In other words, a low-pass filtered version of T is given by

$$T^\rho(R\xi) = \frac{R}{4\pi} \int_{\Omega} N^\rho(\xi \cdot \eta) D(R\eta) \, d\omega(\eta), \quad (4.12)$$

where the regularized Neumann kernel reads as follows:

$$N^\rho(\xi \cdot \eta) = S^\rho(\xi \cdot \eta) - \ln(1 + S^\rho(\xi \cdot \eta)), \quad \xi, \eta \in \Omega. \quad (4.13)$$

Note that $t \mapsto S^\rho(t)$, $t \in [-1, 1]$, given by

$$\begin{aligned} S^\rho(t) &= \begin{cases} \frac{R}{\rho} \left(3 - \frac{2R^2}{\rho^2} (1 - t) \right), & 0 \leq 1 - t \leq \frac{\rho^2}{2R^2} \\ \frac{\sqrt{2}}{\sqrt{1-t}}, & \frac{\rho^2}{2R^2} < 1 - t \leq 2 \end{cases} \end{aligned} \quad (4.14)$$

is continuously differentiable. Moreover, we have (cf. Fig. 12)

$$S \left(1 - \frac{\rho^2}{2R^2} \right) = S^\rho \left(1 - \frac{\rho^2}{2R^2} \right) \quad (4.15)$$

and

$$S' \left(1 - \frac{\rho^2}{2R^2} \right) = (S^\rho)' \left(1 - \frac{\rho^2}{2R^2} \right). \quad (4.16)$$

Furthermore, S and S^ρ are monotonically increasing with

$$S(t) \geq S^\rho(t) \geq 1 \quad (4.17)$$

for all $t \in [-1, 1)$. Furthermore,

$$\begin{aligned} S(t) - S^\rho(t) &= \begin{cases} \frac{\sqrt{2}}{\sqrt{1-t}} - \frac{R}{\rho} \left(3 - \frac{2R^2}{\rho^2} (1 - t) \right), & 0 < 1 - t \leq \frac{\rho^2}{2R^2} \\ 0, & \frac{\rho^2}{2R^2} < 1 - t \leq 2. \end{cases} \end{aligned} \quad (4.18)$$

Elementary calculations give

$$\begin{aligned} \int_{-1}^{+1} |S(t) - S^\rho(t)| \, dt &= \int_{1-\frac{\rho^2}{2R^2}}^1 (S(t) - S^\rho(t)) \, dt \\ &= O(\rho) \end{aligned} \quad (4.19)$$

as $\rho \rightarrow 0$; hence, it follows that

$$\lim_{\rho \rightarrow 0} \int_{-1}^{+1} |S(t) - S^\rho(t)| \, dt = 0. \quad (4.20)$$

Observing the properties of the functions S and S^ρ we find

$$|\ln(S(t)) - \ln(S^\rho(t))| \leq |S(t) - S^\rho(t)| \quad (4.21)$$

and

$$|\ln(1 + S(t)) - \ln(1 + S^\rho(t))| \leq \frac{1}{2} |S(t) - S^\rho(t)|. \quad (4.22)$$

Consequently, we have

$$\int_{\Omega} |\ln(1 + S(\xi \cdot \eta)) - \ln(1 + S^\rho(\xi \cdot \eta))| \, d\omega(\eta) = O(\rho). \quad (4.23)$$

Since $D(R \cdot) : \Omega \rightarrow \mathbb{R}$ is continuous and, therefore, uniformly bounded on Ω_R , we finally obtain in connection with (4.23)

$$\begin{aligned} \lim_{j \rightarrow \infty} \sup_{\xi \in \mathcal{T}} |T(R\xi) - T^{\rho_j}(R\xi)| &= \lim_{j \rightarrow \infty} \sup_{\xi \in \mathcal{T}} \frac{R}{4\pi} \\ &\times \left| \int_{\Omega} (N(\xi \cdot \eta) - N^{\rho_j}(\xi \cdot \eta)) D(R\eta) \, d\omega(\eta) \right| = 0 \end{aligned} \quad (4.24)$$

for all subsets $\mathcal{T} \subset \Omega$, provided that $\{\rho_j\}_{j \in \mathbb{N}_0}$ is a sequence with $\lim_{j \rightarrow \infty} \rho_j = 0$.

Corresponding to the sequence $\{N^{\rho_j}\}_{j \in \mathbb{N}_0}$ of Neumann scaling functions N^{ρ_j} given by

$$\begin{aligned} N^{\rho_j}(\xi \cdot \eta) &= \begin{cases} S^{\rho_j}(\xi \cdot \eta) - \ln(1 + S^{\rho_j}(\xi \cdot \eta)), & 0 \leq 1 - \xi \cdot \eta \leq \frac{\rho_j^2}{2R^2} \\ S(\xi \cdot \eta) - \ln(1 + S(\xi \cdot \eta)), & \frac{\rho_j^2}{2R^2} < 1 - \xi \cdot \eta \leq 2, \end{cases} \end{aligned} \quad (4.25)$$

we introduce the sequence $\{(WN)^{\rho_j}\}_{j \in \mathbb{N}_0}$ of *Neumann wavelets* given by

$$(WN)^{\rho_j}(\xi \cdot \eta) = N^{\rho_{j+1}}(\xi \cdot \eta) - N^{\rho_j}(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \quad (4.26)$$

Written out their representations read as follows:

$$(WN)^{\rho_j}(\xi \cdot \eta) = \begin{cases} S^{\rho_{j+1}}(\xi \cdot \eta) - \ln(1 + S^{\rho_{j+1}}(\xi \cdot \eta)) \\ -S^{\rho_j}(\xi \cdot \eta) + \ln(1 + S^{\rho_j}(\xi \cdot \eta)), & 0 \leq 1 - \xi \cdot \eta \leq \frac{\rho_{j+1}^2}{2R^2} \\ S(\xi \cdot \eta) - \ln(1 + S(\xi \cdot \eta)) \\ -S^{\rho_j}(\xi \cdot \eta) + \ln(1 + S^{\rho_j}(\xi \cdot \eta)), & \frac{\rho_{j+1}^2}{2R^2} < 1 - \xi \cdot \eta \leq \frac{\rho_j^2}{2R^2} \\ 0, & \frac{\rho_j^2}{2R^2} < 1 - \xi \cdot \eta \leq 2. \end{cases} \quad (4.27)$$

In other words, $(\xi, \eta) \mapsto (WN)^{\rho_j}(\xi \cdot \eta)$, $\xi, \eta \in \Omega$, is a zonal function (dependent only on the scalar product $\xi \cdot \eta$ of two unit vectors ξ and η). The function has the local support

$$S^{\rho_j}(\xi) = \text{supp } (WN)^{\rho_j}(\xi \cdot \cdot) = \left\{ \eta \in \Omega \mid 1 - \xi \cdot \eta \leq \frac{\rho_j^2}{2R^2} \right\}. \quad (4.28)$$

Clearly, for increasing j the local support of the Neumann wavelets becomes smaller and smaller such that a ‘zooming-in’ calculation can be assured for locally distributed gravity disturbances (without globally violating Weyl’s law of equidistribution). Due to Weyl’s law of equidistribution, we cannot actually calculate integrals with heterogeneous data. But for reasons of the local support of the Neumann wavelets, only within the scale-dependent spherical caps these kernels differ from zero, i.e., input data located outside these caps have no influence for the computations of the integral. Thus, we can start with a (global) coarse approximation of the disturbing potential and add step by step (local) detailed information adaptive to the available data situation, such that we can ‘zoom’ into such small regions, where a sufficiently dense data set is given (see [Freedon and Wolf 2008](#)).

5 Modeling of the disturbing potential from gravity anomalies by Stokes Wavelets

The determination of the disturbing potential T on Ω_R^{ext} from known gravity anomalies A on Ω_R (see Fig. 7) leads us to the Stokes boundary value problem.

Exterior Stokes Problem (ESP):

We are given $A \in C^{(0)}(\Omega_R)$ with

$$\int_{\Omega_R} A(x) d\omega(x) = 0 \quad (5.1)$$

and

$$\int_{\Omega_R} A(x)(\varepsilon^k \cdot x) d\omega(x) = 0, \quad k = 1, 2, 3. \quad (5.2)$$

Then the function $T : \overline{\Omega_R^{\text{ext}}} \rightarrow \mathbb{R}$ given by

$$T(x) = \frac{1}{4\pi R} \int_{\Omega_R} A(y) St(x, y) d\omega(y) \quad (5.3)$$

with the *Stokes kernel function* (briefly called *Stokes kernel*)

$$St(x, y) = \frac{R}{|x|} + \frac{2R}{|x - y|} - \frac{5R^2}{|x|^2} \frac{x}{|x|} \cdot \frac{y}{|y|} - \frac{3R}{|x|^2} |x - y| \\ - 3 \frac{R^2}{|x|^2} \frac{x}{|x|} \cdot \frac{y}{|y|} \ln \left(\frac{|x| - R \frac{x}{|x|} \cdot \frac{y}{|y|} + |x - y|}{2|x|} \right), \quad x \neq y, \quad (5.4)$$

is the unique solution of the exterior Stokes boundary value problem (see, e.g., [Heiskanen and Moritz 1967](#)):

- (i) T is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in Ω_R^{ext} , i.e.,

$$T \in C^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap C^{(2)}(\Omega_R^{\text{ext}}),$$

- (ii) T is harmonic in Ω_R^{ext} , i.e., $\Delta T = 0$ in Ω_R^{ext} ,
- (iii) T is regular at infinity,
- (iv) $\int_{\Omega_R} T(x) d\omega(x) = 0$,
and $\int_{\Omega_R} T(x)(\varepsilon^k \cdot x) d\omega(x) = 0$, $k = 1, 2, 3$,
- (v) $-\frac{x}{|x|} \cdot \nabla T(x) - \frac{2}{|x|} T(x) = A(x)$, $x \in \Omega_R$.

For points $x = R\xi$, $y = R\eta \in \Omega_R$ we (formally) get an analog to the Neumann formula, called *Stokes’ formula*, which again represents an improper integral over Ω_R

$$T(R\xi) = \frac{1}{4\pi R} \int_{\Omega_R} St(R\xi, R\eta) A(R\eta) d\omega(R\eta). \quad (5.5)$$

Equivalently we have

$$T(R\xi) = \frac{R}{4\pi} \int_{\Omega} St(\xi \cdot \eta) A(R\eta) d\omega(\eta), \quad (5.6)$$

where

$$St(\xi \cdot \eta) = S(\xi \cdot \eta) - 6(S(\xi \cdot \eta))^{-1} + 1 - 5\xi \cdot \eta \\ - 3\xi \cdot \eta \ln \left(\frac{1}{S(\xi \cdot \eta)} + \frac{1}{(S(\xi \cdot \eta))^2} \right), \quad 1 - \xi \cdot \eta \neq 0. \quad (5.7)$$

Note that

$$St(R\xi, R\eta) = St(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \quad (5.8)$$

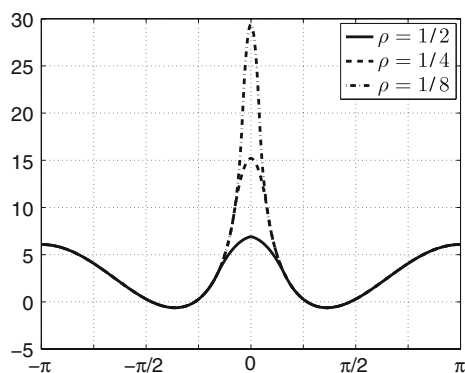


Fig. 14 The regularized Stokes kernel $\vartheta \mapsto St^\rho(\cos \vartheta)$ for several ρ

From Bruns' formula

$$N(R\xi) = T(R\xi) \frac{R^2}{GM} \quad (5.9)$$

we get the geoidal undulations (see Fig. 8). Again, the improper integral (5.6) can be regularized, e.g., by replacing the zonal kernel S (see (4.9)) by the space-regularized zonal kernel S^ρ (see (4.10)).

In fact, the regularization (4.10) leads us to the following regularized global representation of the disturbing potential corresponding to gravity anomalies as boundary data

$$T^\rho(R\xi) = \frac{R}{4\pi} \int_{\Omega} St^\rho(\xi \cdot \eta) A(R\eta) d\omega(\eta) \quad (5.10)$$

with (see Fig. 14)

$$St^\rho(\xi \cdot \eta) = S^\rho(\xi \cdot \eta) - 6(S(\xi \cdot \eta))^{-1} + 1 - 5\xi \cdot \eta - 3\xi \cdot \eta \ln \left(\frac{1}{S^\rho(\xi \cdot \eta)} + \frac{1}{(S^\rho(\xi \cdot \eta))^2} \right). \quad (5.11)$$

The integrands of $T(R\xi)$ and $T^\rho(R\xi)$ differ only on the spherical cap $S^\rho(\xi) = \{\eta \in \Omega | 1 - \xi \cdot \eta \leq \frac{\rho^2}{2R^2}\}$. Here we have

$$\begin{aligned} St(\xi \cdot \eta) - St^\rho(\xi \cdot \eta) &= (S(\xi \cdot \eta) - S^\rho(\xi \cdot \eta)) \\ &\quad - 3\xi \cdot \eta \ln \left(\frac{1}{S(\xi \cdot \eta)} + \frac{1}{(S(\xi \cdot \eta))^2} \right) \\ &\quad + 3\xi \cdot \eta \ln \left(\frac{1}{S^\rho(\xi \cdot \eta)} + \frac{1}{(S^\rho(\xi \cdot \eta))^2} \right). \end{aligned} \quad (5.12)$$

Now it follows that for all $t \in [-1, 1)$ with $1 - t \leq \frac{\rho^2}{2R^2}$

$$\begin{aligned} \ln \left(\frac{1}{S(t)} + \frac{1}{(S(t))^2} \right) - \ln \left(\frac{1}{S^\rho(t)} + \frac{1}{(S^\rho(t))^2} \right) &= \ln(1 + S(t)) - \ln(1 + S^\rho(t)) \\ &\quad - 2(\ln(S(t)) - \ln(S^\rho(t))). \end{aligned} \quad (5.13)$$

Furthermore, by use of the already known properties of the functions S and S^ρ on $[-1, 1)$ we get

$$\begin{aligned} \left| \ln \left(\frac{1}{S(t)} + \frac{1}{(S(t))^2} \right) - \ln \left(\frac{1}{S^\rho(t)} + \frac{1}{(S^\rho(t))^2} \right) \right| &= O(|S(t) - S^\rho(t)|). \end{aligned} \quad (5.14)$$

In connection with (4.18) we, therefore, find

$$\lim_{\rho \rightarrow 0} \sup_{\xi \in T} |T(R\xi) - T^\rho(R\xi)| = 0 \quad (5.15)$$

for every subset $T \subset \Omega$.

Next we consider the *Stokes scaling function* $\{St^{\rho_j}\}_{j \in \mathbb{N}_0}$. Correspondingly, the *Stokes wavelets* $\{(WSt)^{\rho_j}\}_{j \in \mathbb{N}_0}$ are given by

$$(WSt)^{\rho_j}(\xi \cdot \eta) = (St)^{\rho_{j+1}}(\xi \cdot \eta) - (St)^{\rho_j}(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \quad (5.16)$$

In other words, $(\xi, \eta) \mapsto (WSt)^{\rho_j}(\xi \cdot \eta)$, $\xi, \eta \in \Omega$, is a zonal function with local support $S^{\rho_j}(\xi)$. Its explicit representation reads as follows:

$$(WSt)^{\rho_j}(\xi \cdot \eta) = \begin{cases} S^{\rho_{j+1}}(\xi \cdot \eta) - 3\xi \cdot \eta \ln \left(\frac{1}{S^{\rho_{j+1}}(\xi \cdot \eta)} + \frac{1}{(S^{\rho_{j+1}}(\xi \cdot \eta))^2} \right) \\ \quad - S^{\rho_j}(\xi \cdot \eta) + 3\xi \cdot \eta \ln \left(\frac{1}{S^{\rho_j}(\xi \cdot \eta)} + \frac{1}{(S^{\rho_j}(\xi \cdot \eta))^2} \right), & 0 \leq 1 - \xi \cdot \eta \leq \frac{\rho_{j+1}^2}{2R^2} \\ S(\xi \cdot \eta) - 3\xi \cdot \eta \ln \left(\frac{1}{S(\xi \cdot \eta)} + \frac{1}{(S(\xi \cdot \eta))^2} \right) \\ \quad - S^{\rho_j}(\xi \cdot \eta) + 3\xi \cdot \eta \ln \left(\frac{1}{S^{\rho_j}(\xi \cdot \eta)} + \frac{1}{(S^{\rho_j}(\xi \cdot \eta))^2} \right), & \frac{\rho_{j+1}^2}{2R^2} < 1 - \xi \cdot \eta \leq \frac{\rho_j^2}{2R^2} \\ 0, & \frac{\rho_j^2}{2R^2} < 1 - \xi \cdot \eta \leq 2. \end{cases} \quad (5.17)$$

It should be pointed out that all integrals involving Stokes wavelet functions have not to be extended over the whole sphere. Instead, the integral have to be taken over the scale dependent caps $S^{\rho_j}(\xi)$. Again, a 'zooming-in' procedure to more and more local areas is easily implementable.

6 Modeling of the disturbing potential from deflections of the vertical by Green Wavelets

The (unique) solution T of the differential equation for the surface gradient

$$\nabla_\xi^* T(R\xi) = -\frac{GM}{R} \Theta(R\xi), \quad \xi \in \Omega, \quad (6.1)$$

satisfying

$$\int_{\Omega} T(R\xi) d\omega(\xi) = 0, \quad (6.2)$$

$$\int_{\Omega} T(R\xi) (\xi \cdot \varepsilon^k) d\omega(\xi) = 0, \quad k = 1, 2, 3, \quad (6.3)$$

can be formulated in terms of Green's function with respect to the Beltrami operator [Freedon \(1979\)](#), [Freedon et al. \(1998\)](#) given by

$$G(\xi \cdot \eta) = 1 + \ln \left(\frac{1}{(S(\xi \cdot \eta))^2} \right), \quad 1 - \xi \cdot \eta \neq 0, \quad (6.4)$$

as follows:

$$T(R\xi) = \frac{GM}{4\pi R} \int_{\Omega} \nabla_{\eta}^* G(\xi \cdot \eta) \cdot \Theta(R\eta) \, d\omega(\eta), \quad \xi \in \Omega. \quad (6.5)$$

An easy calculation yields

$$\begin{aligned} \nabla_{\eta}^* G(\xi \cdot \eta) &= \nabla_{\eta}^* (1 - 2 \ln(S(\xi \cdot \eta))) \\ &= -2 \frac{S'(\xi \cdot \eta)}{S(\xi \cdot \eta)} (\xi - (\xi \cdot \eta)\eta) \\ &= -\frac{1}{2} (S(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta). \end{aligned} \quad (6.6)$$

Thus, it follows that

$$T(R\xi) = \frac{R}{4\pi} \int_{\Omega} g(\xi, \eta) \cdot \Theta(R\eta) \, d\omega(\eta), \quad (6.7)$$

where

$$g(\xi, \eta) = -\frac{GM}{2R^2} (S(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta), \quad \xi, \eta \in \Omega. \quad (6.8)$$

Replacing S by S^{ρ} we get as regularization T^{ρ} of T corresponding to deflections of the vertical as data set

$$T^{\rho}(R\xi) = \frac{R}{4\pi} \int_{\Omega} g^{\rho}(\xi, \eta) \cdot \Theta(R\eta) \, d\omega(\eta), \quad (6.9)$$

where

$$g^{\rho}(\xi, \eta) = -\frac{GM}{2R^2} (S^{\rho}(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta), \quad \xi, \eta \in \Omega. \quad (6.10)$$

From the properties known for S and S^{ρ} we are able to derive that

$$\begin{aligned} \int_{\Omega} \left((S(\xi \cdot \eta))^2 - (S^{\rho}(\xi \cdot \eta))^2 \right) (\xi - (\xi \cdot \eta)\eta) \\ \cdot \Theta(R\eta) \, d\omega(\eta) &= \int_{1 - \frac{\rho^2}{2R^2} \leq \xi \cdot \eta \leq 1} \left((S(\xi \cdot \eta))^2 - (S^{\rho}(\xi \cdot \eta))^2 \right) \\ &\times \sqrt{1 - (\xi \cdot \eta)^2} \cdot \frac{\xi - (\xi \cdot \eta)\eta}{|\xi - (\xi \cdot \eta)\eta|} \cdot \Theta(R\eta) \, d\omega(\eta) \\ &= O(\rho) \end{aligned} \quad (6.11)$$

provided that $\Theta(\cdot)$ is a continuous vector field on Ω . Consequently, we have

$$\lim_{\rho \rightarrow 0} \sup_{\xi \in \mathcal{T}} |T(R\xi) - T^{\rho}(R\xi)| = 0 \quad (6.12)$$

for all subsets $\mathcal{T} \subset \Omega$.

Corresponding to the *Green (vector) scaling functions* $\{g^{\rho_j}\}_{j \in \mathbb{N}_0}$ we are able to introduce the *Green wavelets* $\{(Wg)^{\rho_j}\}_{j \in \mathbb{N}_0}$ as follows:

$$(Wg)^{\rho_j}(\xi, \eta) = g^{\rho_{j+1}}(\xi, \eta) - g^{\rho_j}(\xi, \eta), \quad \xi, \eta \in \Omega. \quad (6.13)$$

In explicit representation we have

$$(Wg)^{\rho_j}(\xi, \eta) = \begin{cases} -\frac{GM}{2R^2} (S^{\rho_{j+1}}(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta) \\ + \frac{GM}{2R^2} (S^{\rho_j}(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta), & 0 \leq 1 - \xi \cdot \eta \leq \frac{\rho_{j+1}^2}{2R^2} \\ -\frac{GM}{2R^2} (S(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta) \\ + \frac{GM}{2R^2} (S^{\rho_j}(\xi \cdot \eta))^2 (\xi - (\xi \cdot \eta)\eta), & \frac{\rho_{j+1}^2}{2R^2} < 1 - \xi \cdot \eta \leq \frac{\rho_j^2}{2R^2} \\ 0, & \frac{\rho_j^2}{2R^2} < 1 - \xi \cdot \eta \leq 2. \end{cases} \quad (6.14)$$

A reconstruction of the disturbing potential T with Green scaling functions and wavelets is shown in [Fig. 15](#).

7 Multiscale approximation

Let $\{\Phi^{\rho_j}\}_{j \in \mathbb{N}_0}$ be one of the scaling functions introduced in this paper (i.e., the Green, Haar, Neumann, or Stokes scaling function). Let us, as usual, denote by $\{(W\Phi)^{\rho_j}\}_{j \in \mathbb{N}_0}$ the associated wavelet function, i.e.,

$$(W\Phi)^{\rho_j} = \Phi^{\rho_{j+1}} - \Phi^{\rho_j}, \quad j \in \mathbb{N}_0. \quad (7.1)$$

Let $X(R \cdot) : \Omega \rightarrow \mathbb{R}$ denote one of the associated (geodetic) quantities occurring in the integral representations (i.e., in the aforementioned order, deflection of the vertical, (observed) signal, gravity disturbance, gravity anomaly).

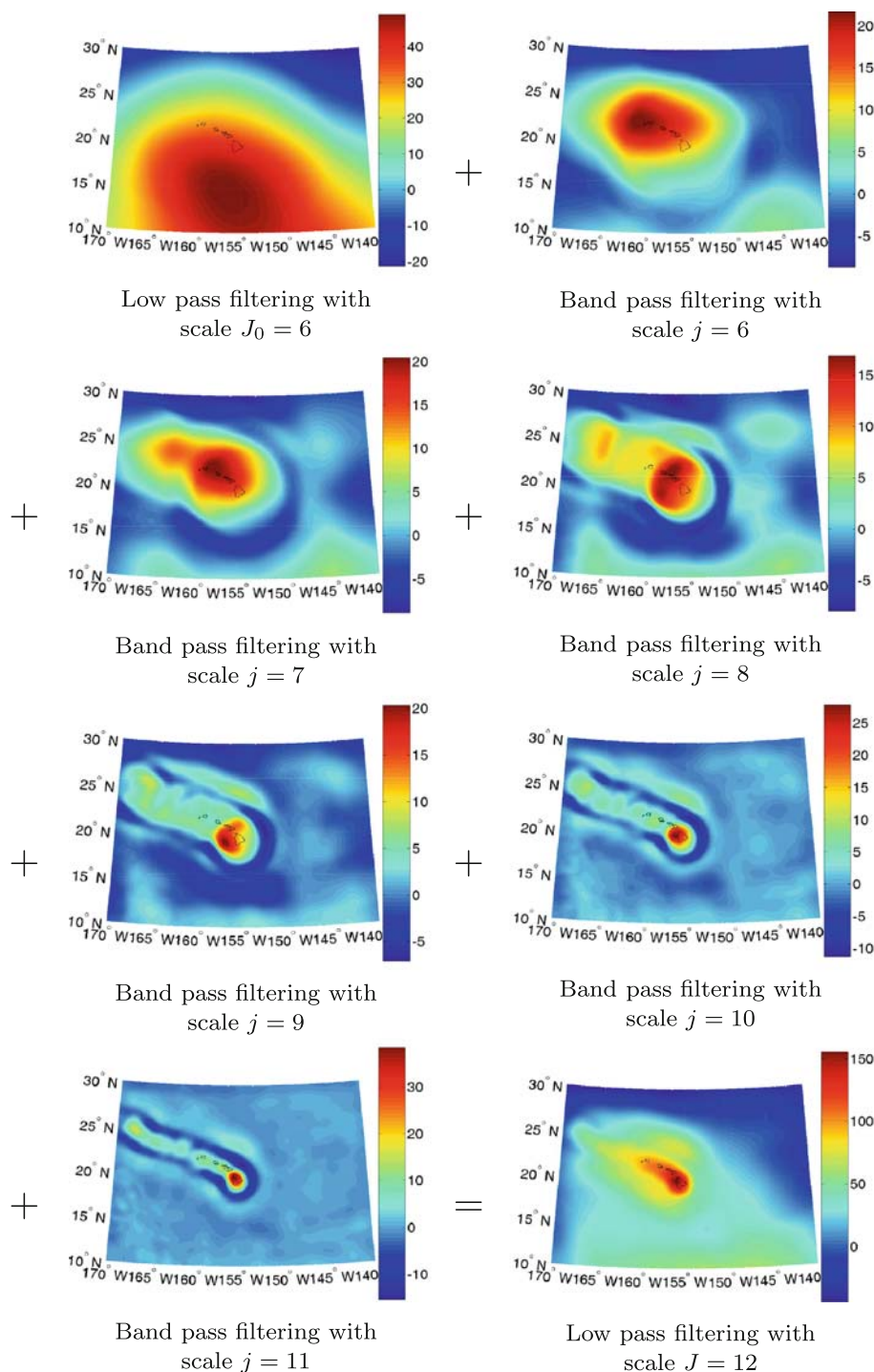
Summarizing our results we are led to the following 'zooming-in' scheme of multiscale approximation:

Step 1: First we guarantee a 'rough' global (initial) approximation by low pass filtering against the kernel $\Phi^{\rho_{J_0}}$ in form of a convolution integral

$$X^{\rho_{J_0}}(R\xi) = \frac{R}{4\pi} \int_{\Omega} \Phi^{\rho_{J_0}}(\xi \cdot \eta) X(R\eta) \, d\omega(\eta). \quad (7.2)$$

Step 2: A first improvement to this rough approximation $X^{\rho_{J_0}}(R\xi)$ by detailed information at scale J_0 is

Fig. 15 Reconstruction of the potential T in $[\frac{m^2}{s^2}]$ from deflections of the vertical in the Hawaiian region using Green wavelets. A rough low pass filtering at scale 6 is improved with several band pass filters of scale 6, ..., 11. The last illustration shows the approximation of T at scale $J = 12$



provided by adding the band-pass filtered version $(WX)^{\rho_{J_0}}$ given by

$$(WX)^{\rho_{J_0}}(R\xi) = \frac{R}{4\pi} \int_{\Omega} (W\Phi)^{\rho_{J_0}}(\xi \cdot \eta) \times X(R\eta) d\omega(\eta). \quad (7.3)$$

This yields the approximation of scale $J_0 + 1$

$$X^{\rho_{J_0+1}}(R\xi) = X^{\rho_{J_0}}(R\xi) + (WX)^{\rho_{J_0}}(R\xi). \quad (7.4)$$

As already pointed out, $(WX)^{\rho_{J_0}}(R\xi)$ is the difference between two smoothings. This expression contains the detailed information of $X^{\rho_{J_0+1}}(R\xi)$, that is not contained in $X^{\rho_{J_0}}(R\xi)$.

In adaptation to the data situation a (local) ‘zooming-in’ process gives us the functions $(WX)^{\rho_{J_0}}(R\xi), \dots, (WX)^{\rho_{J_0+k}}(R\xi)$. This yields the (local) approximation of scale $J_0 + k + 1$

$$X^{\rho_{J_0+k+1}}(R\xi) = X^{\rho_{J_0}}(R\xi) + \sum_{l=0}^k (WX)^{\rho_{J_0+l}}(R\xi). \quad (7.5)$$

Of course, the numerical evaluation requires the application of approximate integration formulas for $j = J_0, \dots, J_0 + k$

$$X^{\rho_j}(R\xi) \simeq \frac{R}{4\pi} \sum_{k=1}^{N_j} w_k^{N_j} \Phi^{\rho_j}(\xi \cdot \eta_k^{N_j}) X(R\eta_k^{N_j}), \quad (7.6)$$

and

$$(WX)^{\rho_j}(R\xi) \simeq \frac{R}{4\pi} \sum_{k=1}^{N_j} w_k^{N_j} (W\Phi)^{\rho_j}(\xi \cdot \eta_k^{N_j}) X(R\eta_k^{N_j}), \quad (7.7)$$

where the summation (7.7) has to be extended over the scale-dependent cap \mathcal{S}^{ρ_j} (the symbol \simeq means that the knots and weights are chosen in such a way that the error between the right and left side can be neglected). Obviously, for every subset $\mathcal{T} \subset \Omega$, the reconstruction formula

$$\sup_{\xi \in \mathcal{T}} \left| X(R\xi) - X^{\rho_{J_0}}(R\xi) - \sum_{l=0}^{\infty} (WX)^{\rho_{J_0+l}}(R\xi) \right| = 0 \quad (7.8)$$

holds true. The reconstruction (‘zooming-in’ procedure) is illustrated by the following scheme:

$$\begin{array}{ccccccc} (WX)^{\rho_{J_0}} & & & (WX)^{\rho_{J_0+1}} & & & \\ & \searrow & & \searrow & & & \\ X^{\rho_{J_0}} & \longrightarrow \oplus \longrightarrow & X^{\rho_{J_0+1}} & \longrightarrow \oplus \longrightarrow & X^{\rho_{J_0+2}} & \dots \end{array}$$

In consequence, an efficient and economical method based on regularization in space domain has been found for determining globally reflected geodetic quantities of classical physical geodesy from locally available data sets.

8 Numerical application: plume anomalies

Finally, the multiscale techniques presented here will be used to investigate gravity anomalies caused by plumes. Due to their local nature we have to use high-resolution gravity data, and the ‘zooming-in’ approximation as presented here is of great significance. Especially, the locally compact (isotropic) smoothed Haar wavelet turns out to be an essential tool of scalar and vectorial multiscale decomposition of gravity disturbances, gravity anomalies, and deflections of the vertical. As case studies, two prototypes of plumes are chosen, namely the oceanic mid-plate Hawaiian plume and the ridge-

centered Iceland plume. They will be discussed in more detail.

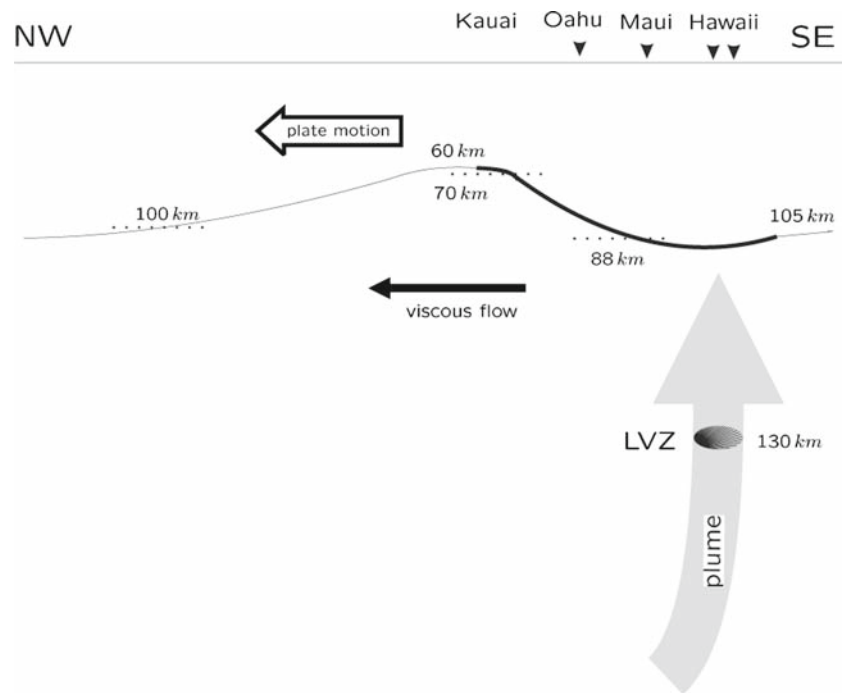
8.1 Hawaiian plume—topographic description

In global plate tectonics, hotspots and seamount chains are understood as fundamental components nowadays. A stationary mantle plume located beneath the Hawaiian islands is believed to be responsible for the creation of the Hawaii-Emperor seamount chain, while the oceanic lithosphere continuously passed over it. The Hawaii-Emperor chain (see Fig. 16) consists of about 100 volcanic islands, atolls, and seamounts that spread nearly 6,000 km from the active volcanic island of Hawaii to the 75–80 Ma (million years) old Emperor seamounts nearby the Aleutian trench. With moving further south-east along the island chain, the geological age decreases. The interesting area is the relatively young south-eastern part of the chain, situated on the Hawaiian swell, a 1,200 km broad anomalously shallow region of the ocean floor, extending from the island of Hawaii to the Midway atoll. Here a distinct geoid anomaly occurs that has its maximum around the youngest island that coincides with the maximum topography and both decrease in north-west direction.

The progressive decrease in terms of the geological age is believed to result from the continuous motion of the underlying plate, or to be more specific, from the motion of the Pacific lithosphere over a stationary mantle plume beneath the youngest island (cf. Morgan 1971; Morgan et al. 1995; Wilson 1963).

This plume is supposed to have its origin in the lower mantle (see Ji and Nataf 1998; Morgan 1971) or even rise from the core mantle boundary (cf. Russel et al. 1998). The excess temperature of the plume is assumed to be more than 200°C and the hot mantle material is expected to run from the origin up to the upper mantle in a narrow stem with a radius of about 100–200 km (see Nataf 2000). With seismic tomography several features of the Hawaiian mantle plume are detected (cf. Ritter and Christensen 2007). The results deliver a so-called low-velocity zone (LVZ) beneath the lithosphere, starting at a depth of about 130–140 km, beneath the central part of the island of Hawaii. This LVZ is interpreted as an indication for a partial melting zone localized inside the plume conduit. The so-called *P* receiver functions suggested a 40-km-large uplift in the mantle transition zone at a depth of 660 km, caused by the rising mantle plume. An in north-west direction increasing shallow of the oceanic lithosphere was obtained with *S* receiver functions, that have been interpreted to suggest a thermal rejuvenation of the lithosphere caused by the Hawaiian mantle plume. The interested reader may refer to Ritter and Christensen (2007) and the references therein.

Fig. 16 Conjectured plume configuration for the Hawaiian area (in accordance with Yuan et al. (2007))



8.2 Dataset and implementation

The gravity data used for our numerical computation represent the gravity anomalies around the islands of Hawaii. Because of the lack of terrestrial-only data, we used the combined gravity model EIGEN-GL04C consisting of satellite data, gravimetry, and altimetry surface data. The dataset used to calculate the gravity anomalies was provided by the GFZ-Potsdam. The sequence $\rho_j = 2^{-j}$ is chosen for gravity anomaly modeling and a smoothed Haar kernel of a polynomial degree of $k = 5$ turned out to be advantageous for our purposes since the kernel has to be at least twice continuously differentiable and well localizing. Since the kernel function has local support we can restrict our considerations to the local area of interest. So the illustrations are calculated on a 350×350 pointgrid around the island of Hawaii, taken from a $5,000 \times 5,000$ longitude-latitude grid (of type Driscoll and Healy 1994). The scalar and vectorial solutions are shown in Figs. 17 and 18.

In the multiscale analysis several interesting observations can be detected. By comparing the different positions ξ and different scales j , we can see that the local maximum of the energy, that corresponds to the wavelet variances (see Freeden and Maier 2002; Mathar 2008), contained in the signal of the gravity anomalies starts in the north-west of the Hawaiian islands for scale 2 and travels in south-east direction with increasing scale. It ends up, for scale $J = 12$, in a position at the geologically youngest island under which the mantle plume is assumed to exist. Again, in the multiresolution property, with increasing scale, more and more local

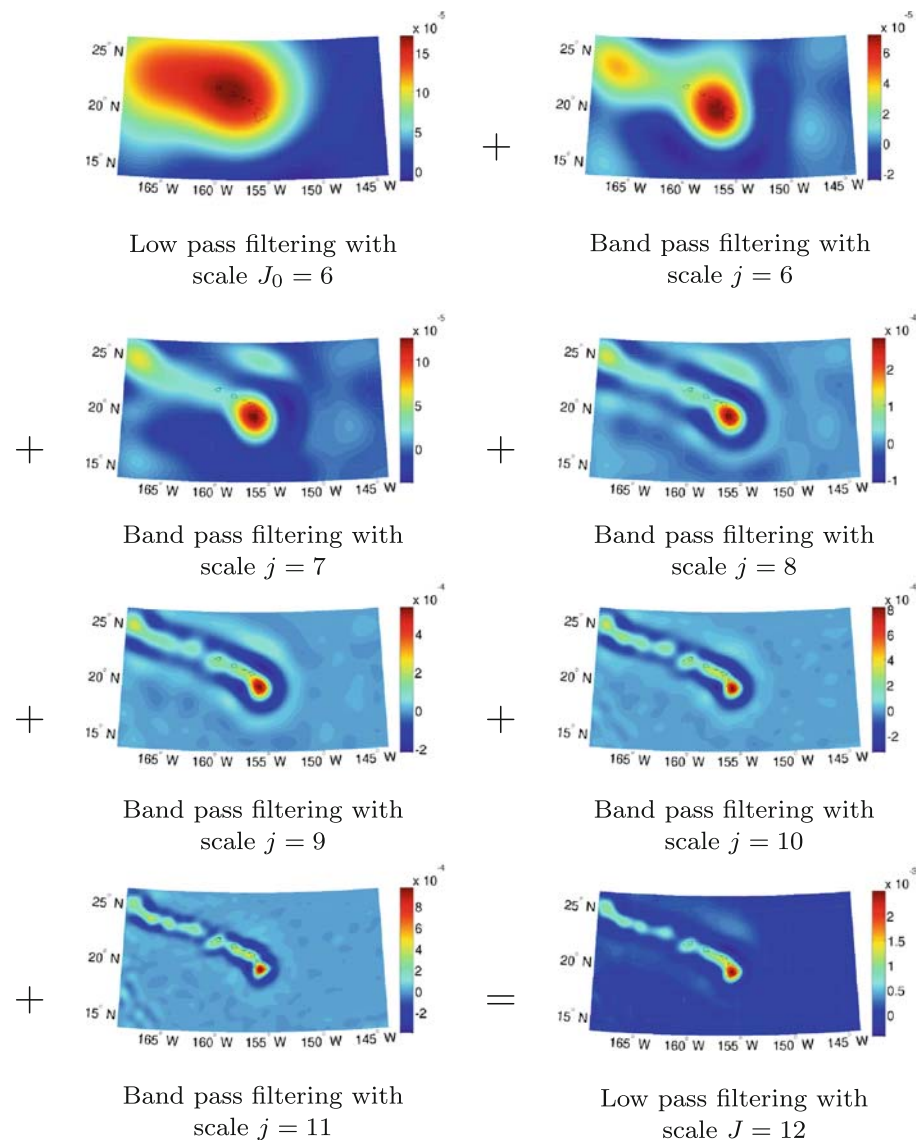
details of the gravity anomalies appear. The structure of the Hawaiian island chain is clearly reflected in the scale and space decomposition, as discovered in the global multiscale analysis. Obviously, the energy peak observed at the youngest island of Hawaii is highly above the energy intensity of the rest of the island chain. This seems to strongly corroborate the belief of a mantle plume just below the main island of Hawaii. Similar results are also developable from deflections of the vertical by using Green wavelets (see Fig. 15).

8.3 Iceland plume—topographic description

The plume beneath Iceland is the prototype of a ridge-centered mantle plume. An interaction between the North Atlantic ridge and the mantle plume is believed to be the reason for the existence of Iceland, resulting in melt production and crust generation since the continental break-up in the late Palaeocene and early Eocene 58 Ma ago. Nevertheless, there is still no agreement on the location of the plume before rifting started in the east. Controversial discussions, whether it was located under central or eastern Greenland about 62–64 Ma ago are still in progress (cf. Ritter and Christensen 2007).

Iceland itself represents the top of a nearly circular rise topography, with the maximum of about 2.8 km above the surrounding seafloor in the South of the glacier ‘Vatnajökull’. This glacier is the largest European one and several active volcanoes are located beneath it, which are supposed to be fed by a mantle plume. The surrounding oceanic crust consists of three different types involving a crust thickness

Fig. 17 Approximation of the gravity anomalies A in $[\frac{m}{s^2}]$ of the Hawaiian region with smoothed Haar wavelets. A rough low pass filtering at scale 6 is improved with several band pass filters of scale 6, \dots , 11. The last picture shows the multiscale approximation at scale $J = 12$



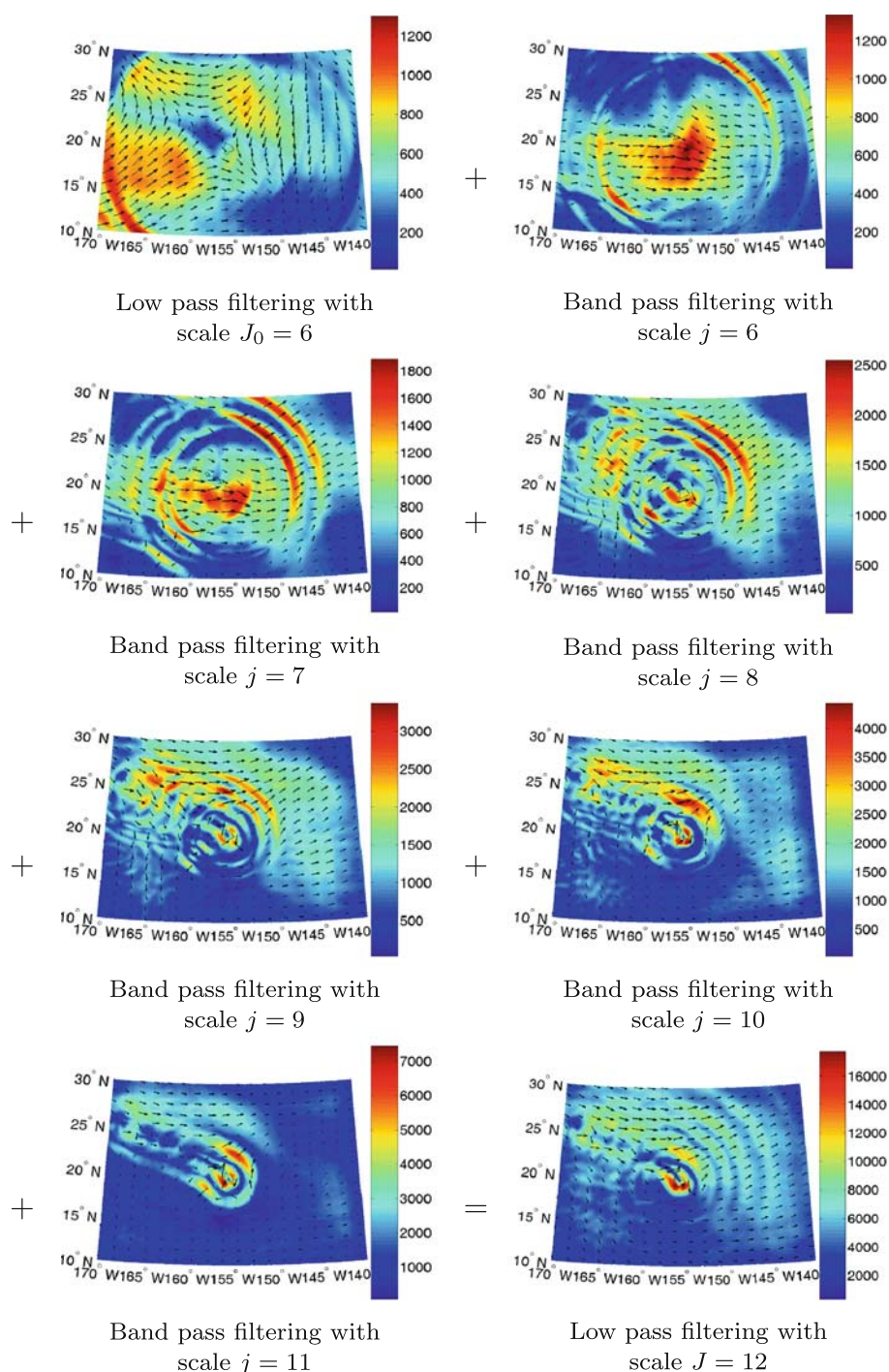
that is more than three times as thick as average oceanic crusts. Seismic tomography provides evidence of the existence of a mantle plume beneath Iceland, resulting in low-velocity zones in the upper mantle and the transition zone, but also hints regarding anomalies in the deeper mantle seem to exist. The low-velocity anomalies have been detected in depths ranging from at least 400 km up to about 150 km in regional P-wave models. Above 150 km, ambiguous seismic velocity structures were obtained involving regions of low velocities covered by regions of high-seismic-velocities. Furthermore, low S-wave velocities at a depth of about 80 km were indicated by special surface wave dispersion data, as well (cf. [Marquart and Schmeling 2004](#)). For a deeper access into the theory of the Iceland plume the, interested reader may refer to [Marquart and Schmeling \(2004\)](#), [Schmeling and](#)

[Marquart \(2004\)](#), [Ritter and Christensen \(2007\)](#) and the references therein.

8.4 Dataset and implementation

The gravity data decomposed here represent the gravity anomalies around Iceland. The dataset used to calculate the gravity anomalies has the same high-resolution gravity model as used in the Hawaiian case study. Again the sequence $\rho_j = 2^{-j}$ is chosen and the smoothed Haar kernel has a polynomial degree of $k = 5$. Since the kernel function has local support we can restrict our considerations to the local area of interest. To avoid a boundary influence by the surrounding topography, which was negligible in the Hawaiian case study, we calculate our multiscale decomposition on a larger point

Fig. 18 Approximation of the vector valued vertical deflections Θ of the Hawaiian region with smoothed Haar wavelets. A rough low pass filtering at scale 6 is improved with several band pass filters of scale $6, \dots, 11$. The last picture shows the multiscale approximation at scale $J = 12$

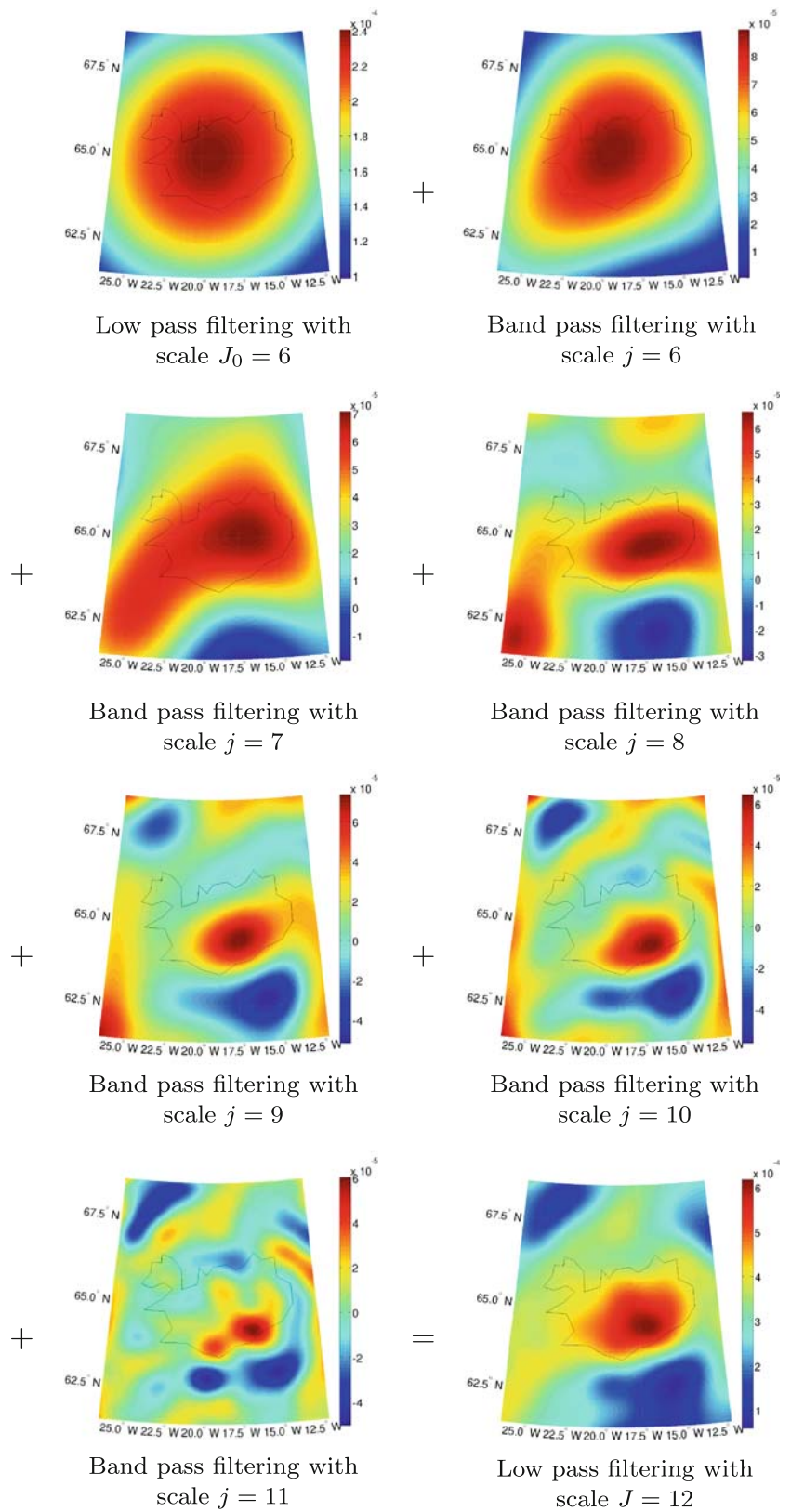


grid (300×300) around the Iceland hotspot. So the illustrations are restricted to a grid of 200×200 gridpoints around Iceland, taken from a $5,000 \times 5,000$ longitude-latitude grid of type [Driscoll and Healy \(1994\)](#). The scalar solutions are shown in Fig. 19.

Some quite interesting observations can be made by looking at the multiscale analysis. In the lower scales (not shown here) the shape and the center of the signal energy does not

vary remarkably. However, from scale 6 on, the energy analyzed in the signal travels toward the south-west along the North Atlantic ridge and the center of the ‘peak’ situated on Iceland moves south-eastward along the Greenland-Iceland-Faeroer ridge. In the high scales 9, 10, and 11, the anomaly in the south-east dominates significantly. This anomaly located in the south-east of Iceland may be interpreted as the signal of the mantle plume beneath Iceland. Since this anomaly

Fig. 19 Approximation of the gravity anomalies A in $[\frac{m}{s^2}]$ of Iceland with smoothed Haar wavelets. A rough low pass filtering at scale 6 is improved with several band pass filters of scale 6, \dots , 11. The last illustration shows the multiscale approximation at scale $J = 12$



clearly manifests in the high scales, its spatial extension has to be rather small, which coincides with seismic observations, limiting the radial diameter of the upper part of mantle plumes to about 100–150 km.

9 Conclusions

The purpose of this paper is to demonstrate that locally (space) supported wavelets provide a powerful approximation technique for the investigation of, local fine-structured features such as those caused by plumes. The illustrations show that the presented multiscale procedure allows a scale- and space-dependent characterization of this geophysical phenomenon. As approximate integration rules fast longitude-latitude grids (as proposed by [Driscoll and Healy 1994](#); [Wolf 2006](#)) have been used; especially in the case study of the Hawaii plume, which has also been discussed in several papers involving seismic tomography (see, e.g., [Yuan et al. 2007](#)), we can achieve comparable results for the spatial position of the anomaly caused by the plume. In fact, the wavelet coefficients can be interpreted as spatial measures of certain frequency bands contained in the signal. Thereby, the wavelet theory offers an applicable physical approach for detecting plume features.

Concerning the Hawaii plume in more detail, the numerical results show us that the anomaly is located on a north-west to south-east line below the Hawaiian islands. As the wavelength part of a signal is associated with depth, we can also conclude that the deep parts of the plume, which correspond to the lower scales, are centered below the islands Maui and Oahu. The higher scales, corresponding to the upper part of the plume, can be localized directly below Hawaii.

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