Combination of Satellite, Altimetric and Terrestrial Gravity Data

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1.0 INTRODUCTION

The first gravity measurements on the earth were made with pendulums. In the 1930's gravimeters were developed that led to measurements of gravity differences over large areas. These differences led to gravity values and then gravity anomalies over broad land and ocean areas. The gravity anomaly values are stored as point or mean values with coordinates (ϕ, λ, H) given in some specified datum (both horizontal and vertical). The mean anomaly value are computed for a variety of compartment sizes ranging from $6' \times 10'$ to $5° \times 5°$. Although this data set is far from uniform in terms of areal coverage and accuracy it provides substantial information on the earth's gravity field.

In the late 1950's analysis of the perturbations of artificial satellite also led to information on the earth's gravitational potential as represented, for the most part, by a finite series of spherical harmonic coefficients. The maximum degree of the expansion has continually increased as data accuracy has increased and as the availability of satellites to analyze has increased. In the early 1960's models to degree 4 were being computed. In the early 1980's the models were extended to degree 20 (with additional terms) and today's solutions are being developed to degree 50. From these models we can determine gravity (or gravity anomalies) at the surface of the earth. These anomalies are band limited in the sense they have only the frequency content of the harmonics being solved for from the satellite data.

Other data types are available for the determination of the gravity field. Satellite altimetry is a key data type because of it's ocean wide coverage. Satellite to satellite tracking was carried out on a limited basis in the 1970's. And topography (elevations) is also a source of information once some hypothesis are made on the relationship between gravity anomalies and topography.

The problem addressed in this paper is related to the combination of these data types to arrive at an accurate representation of the earth's gravitational potential. The problem is complicated by the different spectral contents of the data. In addition the data may have bias' or correlations that need to be considered for the optimal estimation of the parameters.

For some discussions in this paper we will assume that potential coefficient analysis from satellite data has been carried out. Such solutions are called satellite alone fields. Such solutions, in the past have been, GEM9, GEML2, and recently GEMT1 (Marsh et al, 1988a). We also assume that the error covariances matrix of the parameters (potential coefficients) is also available.

With the above information we will first examine the role of terrestrial gravity data in the combination models. We will then consider the altimeter data and then other data types. Our results will include the methods to obtain models to the same degree as the satellite field or to estimate high (360) degree expansions.

2.0 Representation of the Gravitational Potential

We need to adopt a representation for the potential. The spherical harmonic representation has been most useful in satellite geodesy. It can also be used with surface gravity data but increasing care must be taken as accuracy requirements increase. Recent suggestions focus on the use of ellipsoidal harmonics which reduces some of the problems that arise with spherical harmonics. We therefore consider both representations.

2.1 Spherical Harmonics

We define the polar coordinates of a point as r, θ, and λ where r is the geocentric distance; θ is the geocentric co-latitude ($90°-\theta$, where θ is the geocentric latitude); and λ is longitude. The θ and λ values must be given with respect to a precisely defined rectangular coordinate system. Such a system may be the Conventional Terrestrial System, but most likely is a system defined by a specific group for their satellite analysis. In this system the Newtonian potential is:

$$V(r,\theta,\lambda) = \frac{kM}{r}\left[1 + \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(\cos\theta)\right] \tag{1}$$

where:

- kM geocentric gravitational constant
- a scaling parameter associated with the potential coefficients
- C,S fully normalized potential coefficients
- P_{nm} fully normalized associated Legendre functions

A more compact form for (1) may be obtained with the following substitution:

$$C^{\alpha}_{nm} = \begin{cases} C_{nm}, & \alpha=0 \\ S_{nm}, & \alpha=1 \end{cases} \tag{2}$$

$$Y^{\alpha}_{nm}(\theta,\lambda) = \begin{cases} P_{nm}(\cos\theta)\cos m\lambda, & \alpha=0 \\ P_{nm}(\cos\theta)\sin m\lambda, & \alpha=1 \end{cases} \tag{3}$$

Other forms are possible. Define Y_{nm} as follows:

$$Y_{nm}(\theta,\lambda) = P_{n|m|}(\cos\theta) \begin{cases} \cos m\lambda, & m\geq 0 \\ \sin |m|\lambda, & m<0 \end{cases} \tag{3A}$$

Let $C_{n|m|} = \begin{cases} C_{nm}, & m \geq o \\ S_{n|m|}, & m < o \end{cases}$

then

$$V(r,\theta,\lambda) = \frac{kM}{r}\left[1 + \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=-n}^{n} C_{nm}\, Y_{nm}\,(\theta,\lambda)\right] \tag{4A}$$

We now can write (1) as:

$$V(r,\theta,\lambda) = \frac{kM}{r}\left[1 + \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=o}^{n} \sum_{\alpha=o}^{1} C_{nm}^{\alpha}\, Y_{nm}^{\alpha}\,(\theta,\lambda)\right] \tag{4}$$

We will use both forms (i.e. 4 and 4A) in this paper.

We now can define the disturbing potential T at the point r,θ,λ:

$$T(r,\theta,\lambda) = V(r,\theta,\lambda) - U(r,\theta,\lambda) \tag{5}$$

where U is a reference potential, that is usually implied by a rotationally symmetric, equipotential ellipsoid. Assuming the mass of the reference ellipsoid and the earth are the same we have:

$$T(r,\theta,\lambda) = \frac{kM}{r} \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=o}^{n} \sum_{\alpha=o}^{1} C_{nm}^{\alpha}\, Y_{nm}^{\alpha}\,(\theta,\lambda) \tag{6}$$

where the C_{2i}° coefficients have the reference field coefficients removed.

2.1.1 Spherical Potential Coefficients and Gravity Anomalies

Numerous discussions have taken place in the literature, in the past few years, on the fundamental boundary conditions expressing anomalies as a function of the disturbing potential coefficients. In this discussion we follow Pavlis (1988).

We define the "surface free-air anomaly" as:

$$\Delta g = |\vec{g}_p| - |\vec{\gamma}_q| \tag{7}$$

where $|\vec{g}_p|$ is the magnitude of observed gravity and $|\vec{\gamma}_q|$ is the magnitude of theoretical gravity at the point on the telluroid. In terms of the disturbing potential we can write:

$$\Delta g = -\left[\frac{\partial T}{\partial h}\right]_Q + \frac{1}{\gamma_Q}\left[\frac{\partial \gamma}{\partial h}\right]_Q T_Q - \frac{1}{\gamma_Q}\left[\frac{\partial \gamma}{\partial h}\right] \Delta W + \varepsilon_p \tag{8}$$

where :

$$\Delta W = W_p - U_q = W_o - U_o \tag{9}$$

W_o = geoid potential

U_o = surface potential of the ellipsoid

and:

$$\varepsilon_p = \left[(1 - \cos\theta)\frac{\partial W}{\partial h} - \frac{\xi}{M}\frac{\partial W}{\partial \phi} - \frac{\eta}{N\cos\Phi}\frac{\partial W}{\partial \lambda}\right]_P \tag{10}$$

where θ is the total deflection of the vertical and ξ and η are the usual merìdean and prime vertical components. The height h, is in the direction of the ellipsoidal normal. The geometry associated with the boundary condition is shown in Figure 1 (Pavlis, ibid, Figure 3).

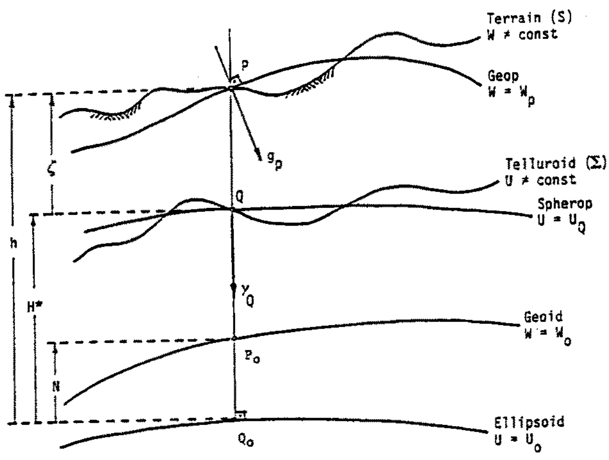

Figure 1. The Geometry Associated with Boundary Condition (10) (Pavlis, 1988)

Taking into account the definition of the normal potential of the ellipsoid one can show that (8) can be written in the following form:

$$\Delta g = -\left[\frac{\partial T}{\partial r}\right]_Q - \frac{2}{r_Q} T_Q + (\varepsilon_h)_Q + (\varepsilon_\gamma)_Q + \frac{2}{r_Q}\Delta W - \left[6J_2 \frac{a^2}{r^3} P_2(\cos\theta_Q) - \frac{3\omega^2 r_Q^2}{kM}\sin^2\theta_Q\right]\Delta W + \varepsilon_p \tag{11}$$

where:

$$\varepsilon_h = e^2 \sin\theta\cos\theta\left[\frac{1}{r}\frac{\partial T}{\partial\theta}\right] \tag{12}$$

$$\varepsilon_\gamma = \left[6J_2\frac{a^2}{r^3}P_2(\cos\theta) - \frac{3\omega^2 r_Q^2}{kM}\sin^2\theta\right]T \tag{13}$$

If $\Delta W(W_0 - U_0)$ were zero and we were to neglect ε_h, ε_γ, ε_p the resultant equation would be the usual spherical approximation of the boundary condition. If we substitute (6) into (11) we can write:

$$\Delta g = -\frac{k\delta M}{r_Q^2} + \frac{kM}{r_Q^2}\sum_{n=2}^{\infty}(n-1)\left(\frac{a}{r_Q}\right)^n \sum_{m=-n}^{n} C_{nm} Y_{nm}(\theta,\lambda)$$

$$+ (\varepsilon_h)_Q + (\varepsilon_\gamma)_Q + \frac{2}{r}\Delta W - \left[6J_2\frac{a^2}{r^3}P_2(\cos\theta) - \frac{3\omega^2 r^2}{kM}\sin^2\theta\right]_Q \Delta W + \varepsilon_p$$

where $\delta kM = k(M-M')$ with M = mass of the actual earth and M' is the mass of the reference ellipsoid.

It is beyond this discussion to examine the small terms given in (14) (i.e. ε_h, ε_γ, ε_p). Pavlis (ibid) and Cruz (1986) have shown how these terms can be represented in a spherical harmonic series based on approximate values of the potential coefficients. Since these quantities are small and long wave length in nature they will be regarded as known for further discussion.

For many applications the anomaly given as the observable is an area mean anomaly usually defined in the following sense:

$$\overline{\Delta g}_{ij} = \frac{1}{\Delta\sigma_{ij}} \iint_{\sigma_{ij}} \Delta g \sin\theta \, d\theta d\lambda \tag{15}$$

Applying (15) to (14) we have (Pavlis, 2.104):

$$\overline{\Delta g}_{ij} = \frac{1}{\Delta\sigma_{ij}} \frac{GM}{\overline{r}_{ij}^2} \sum_{n=2}^{\infty} (n-1) \left(\frac{a}{\overline{r}_{ij}}\right)^n \sum_{m=-n}^{n} C_{nm} \, IY_{nm}$$

$$+ \hat{IE}_{\ell}^{ij} + \hat{IE}_{\gamma}^{ij} + a_1 \, G\delta M + a_2 \, \Delta W + IE_p^{ij} \tag{16}$$

where:

$$\Delta\sigma_{ij} = \int_{\theta_i}^{\theta_{i+1}} \sin\theta d\theta \int_{\lambda_j}^{\lambda_{j+1}} d\lambda$$

$$IY_{nm} = \int_{\lambda_j}^{\lambda_{j+1}} \begin{Bmatrix} \cos m\lambda, & m \geq 0 \\ \sin m\lambda, & m < 0 \end{Bmatrix} d\lambda \int_{\theta_i}^{\theta_{i+1}} P_{nm}(\cos\theta)\sin\theta d\theta$$

$$a_1 = -\frac{1}{\overline{r}_{ij}^{\,2}} \tag{17}$$

$$a_2 = \frac{2}{\overline{r}_{ij}} - \left[3J_2 \frac{a^2}{\overline{r}_{ij}^{\,3}} + \frac{\omega^2 \overline{r}_{ij}^{\,2}}{kM}\right](\cos^2\theta_i + \cos\theta_i \cos\theta_{i+1}$$

$$+ \cos^2\theta_{i+1}) + 3\left(J_2 \frac{a^2}{\overline{r}_{ij}^{\,3}} + \frac{\omega^2 \overline{r}_{ij}^{\,2}}{kM}\right)$$

The values for the other terms represent the integrated expressions for the correction terms. In writing (16), the variation of r_{ij} within the $\Delta\sigma$ block is neglected.

Two points need to be made before the final expression is written. First, the $a_1 \; k\delta M + a_2\Delta W$ term acts essentially as a constant which can be written in the following form:

$$a_1 k\delta M + a_2 \Delta W = -\frac{k\delta M}{\overline{r}_{ij}^{\,2}} + \frac{2}{\overline{r}_{ij}} \Delta W \tag{18}$$

Second, the anomalies computed from (16) refer to an earth with no atmosphere. Since the atmosphere is implicitly considered in the computation of the terrestrial anomaly a suitable correction must be applied to (10) or to (7). If we wish (16) to yield the anomaly comparable to the terrestrial anomaly we must <u>add</u> the atmospheric correction, δg_A, to the terrestrial anomaly. Using (17) we now can write (16) as (Pavlis, 2.142):

$$\overline{\Delta g}_{ij} + \overline{\delta g}_{ij}^{c} = \frac{1}{\Delta\sigma_{ij}} \frac{kM}{\overline{r}_{ij}^2} \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} (n-1) \left(\frac{a}{\overline{r}_{ij}}\right)^n \sum_{m=-n}^{n} C_{nm} \, IY_{n|m|} + a_3 \, \Delta W$$

where:

$$\overline{\delta g}_{ij}^{c} = \delta g_A^{ij} - (\overset{n}{IE}_A + \overset{n}{IE}_{\gamma} + \overset{n}{IE}_p)^{ij} \tag{20}$$

We note that the left hand side of (20) is regarded as known and that (20) represents a precise formulation of the boundary condition in terms of the observed anomaly and potential coefficients. Also note that it is possible to have different ΔW values for different vertical datums.

2.1.2 Spherical Harmonics and Orthogonality Relationships

Consider a form of (14) where all the terms on the right-hand side, except the summation term, are brought to the left-hand side to define a modified anomaly:

$$\Delta g + \delta g_{ij} = \Delta g^* = \frac{kM}{r_Q^2} \sum_{n=2}^{\infty} (n-1) \left[\frac{a}{r_Q}\right]^n \sum_{m=-n}^{n} C_{nm} Y_{n|m|} (\theta,\lambda) \tag{21}$$

Now let $r_Q = a$ so that we regard the anomalies Δg^* given on a sphere of radius a. Then (21) becomes:

$$\Delta g^* = \frac{kM}{a^2} \sum_{n=2}^{\infty} (n-1) \sum_{m=-n}^{n} C_{nm} Y_{n|m|} (\theta,\lambda) \tag{22}$$

We can apply the usual spherical harmonic orthogonality relationship (Heiskanen and Moritz, 1967, page 31) to calculate the potential coefficients given Δg^*. We have:

$$C_{nm} = \frac{1}{4\pi \frac{kM}{a^2} (n-1)} \iint_\sigma \Delta g^* Y_{nm}(\theta,\lambda) d\sigma \tag{23}$$

where σ is now the sphere of radius a. The evaluation of (23) requires Δg^* to be given in a continuous fashion. In practice mean Δg^* values are used which requires (23) to be approximated by a discrete form such as:

$$C_{nm} = \frac{1}{4\pi \frac{kM}{a^2} (n-1) q_n} \sum_{i=0}^{N-1} \sum_{j=0}^{2N-1} \overline{\Delta g}^*_{ij} \iint_{\sigma_{ij}} Y_{n|m|} (\theta,\lambda) d\sigma \tag{24}$$

where q_n is a quantity dependent on n and the block size (σ_{ij}), that is designed to reduce the approximation in going from (23) to (24). N is the number of latitude belts in which the equiangular anomalies are given. Various techniques have been designed to obtain optimum values of q_n. Colombo (1981, p.76) suggested that the quadrature weights q_n be set as follows:

$$q_n = \beta_n^2; \quad 0 \leq n \leq N/3$$

$$q_n = \beta_n; \quad N/3 < n < N$$

$$q_n = 1; \quad n > N \tag{25}$$

where $N = 180^\circ/\theta^\circ$, θ is the block size, and β_n is the Pellinen smoothing operator given by:

$$\beta_n = \frac{1}{1-\cos\psi_0} \frac{1}{\sqrt{2n+1}} \left[P_{n-1}(\cos\psi_0) - P_{n+1}(\cos\psi_0)\right] \tag{26}$$

where ψ_0 is the radius of a spherical cap whose area is the same as that of a block whose sides are θ°. Specifically:

$$\sin \left[\frac{\psi_0}{2}\right] = \left[\frac{\theta \sin\theta}{4\pi}\right]^{1/2} \tag{27}$$

Pavlis (ibid) described some assumptions that lead to an alternate determination of q_n. Specifically he (eq.4.46) suggested q_n is degree and order dependent:

$$q_{nm} = \left[\sum_{i=0}^{N-1} \sum_{j=0}^{2N-1} \frac{(Y_{nm}^{ij})^2}{\Delta\sigma_{ij}} \right]^{-1} \tag{28}$$

The calculation of q_{nm} is more complex than the q_n computation but need be done only once after the block size and maximum degree of expansion is given. Numerical tests by Pavlis suggest that (28) gives slightly better results than (25). The best results, in terms of consistency between given potential coefficients and anomalies is to regard the $\overline{\Delta g}^*$ values as referring to the center point of the cell and expressing (24) in the following form:

$$C_{nm} = \frac{1}{4\pi \frac{kM}{a^2}(n-1)\ \beta_n} \sum_{i=0}^{N-1} \sum_{j=0}^{2N-1} Y_{nm}\ (\theta,\lambda)\ \Delta g^* \sigma_{ij} \tag{29}$$

Additional studies are needed to understand proper ways to evaluate quadrature type formulas using block data and that contains contributions from all parts of the frequency spectra and that has noise (errors) associated with it.

2.2 Ellipsoidal Harmonics

The earth is closer in shape to an ellipsoid than a sphere. Consequently one must ask if ellipsoidal harmonics are more advantageous to use than spherical harmonics. For potential representations for satellite geodesy (orbit calculations), spherical harmonics are still the obvious choice. However, when data is given on an ellipsoid (such as some interpretations of terrestrial anomalies), ellipsoidal harmonics may play a more appropriate role. Discussions on the use of ellipsoidal harmonics may be found in Jekeli (1981, 1988) and Gleason (1988).

We start with the ellipsoidal coordinates u, δ, and λ where:

u is the semi-minor axis of a confocal ellipsoid;

δ is the reduced colatitude;

λ is the longitude.

These coordinates are related to the rectangular coordinates by:

$$\begin{aligned} x &= \sqrt{u^2+E^2}\ \sin\delta\ \cos\lambda \\ y &= \sqrt{u^2+E^2}\ \sin\delta\ \sin\lambda \\ z &= u\ \cos\delta \end{aligned} \tag{30}$$

where E = ae. A function, F, satisfying Laplace's equation, in this ellipsoidal coordinate system can be written as (Gleason, ibid, eq(1.12)):

$$F(u,\ \delta,\ \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\bar{f}}^e_{nm} \frac{\bar{Q}_{n|m|}\left(\frac{iu}{E}\right)}{\bar{Q}_{n|m|}\left(\frac{ib}{E}\right)}\ \bar{Y}_{n,m}\ (\delta,\lambda) \tag{31}$$

where:

$$\overline{Y}_{n,m}(\delta,\lambda) = \overline{P}_{n|m|}\ (\cos\ \delta) \begin{Bmatrix} \cos\ m\lambda \text{ if } m\geq 0 \\ \sin\ m\lambda \text{ if } m<0 \end{Bmatrix} \tag{32}$$

The $\overline{Q}_{n,m}$ are the fully normalized (in the usual sense as $\overline{P}_{nm}$) Legendre function of the second kind and $\bar{\bar{f}}^{e}_{n,m}$ are <u>real</u> fully normalized ellipsoidal harmonic coefficients. Given the function F on the surface of the ellipsoid, and using the standard orthogonality relationships for $\overline{Y}_{nm}$ we can write:

$$\bar{\bar{f}}^{e}_{n,m} = \frac{1}{4\pi} \iint_{\sigma} F(u{=}b,\ \delta,\ \lambda)\ \overline{Y}_{n,m}\ (\delta,\lambda) d\sigma \tag{33}$$

where $d\sigma = \sin\ \delta\ d\delta d\lambda$.

Using spherical harmonic coefficients the solution to Laplace's equation is:

$$F(r_p,\ \theta_p,\ \lambda_p) = \sum_{n=o}^{\infty} \sum_{m=-n}^{n} \left(\frac{a}{r_p}\right)^{n+1} \overline{f}^{s}_{n,m}\ \overline{Y}_{n,m}\ (\theta_p,\ \lambda_p) \tag{34}$$

If the potential function is given on the sphere of radius a the coefficients can be found using the usual orthogonality relationships:

$$\overline{f}^{s}_{n,m} = \frac{1}{4\pi} \iint_{\sigma} F(r{=}a,\ \theta,\ \lambda)\ \overline{Y}_{n,m}\ (\theta,\lambda) d\sigma \tag{35}$$

where $d\sigma = \sin\ \theta d\theta d\lambda$. The problem examined by Jekeli, Gleason and others was to relate the spherical and ellipsoidal coefficients recognizing the uniqueness of the potential outside some boundary surface. An operational conversion between $\overline{f}^{s}_{n,m}$ and $\bar{\bar{f}}^{e}_{n,m}$ is derived by Gleason. He shows that (ibid, 1.22):

$$\overline{f}^{s}_{n,m} = \sum_{k=o}^{s'} \frac{1}{\overline{S}_{n-2k,|m|}\left(\frac{b}{E}\right)} L_{n,m,k}\ \bar{\bar{f}}^{e}_{n-2k,m} \tag{36}$$

All terms are defined in Gleason with the L term computed by a recurssive relationship. Equation (36) is important to us because it will enable us to convert $\bar{\bar{f}}^{e}$ values found from (33) to $\overline{f}^{s}$ values which are compatible with the satellite derived potential coefficients.

2.2.1 Ellipsoidal Harmonics and Gravity Anomalies

Let the gravity anomaly of interest be the radial component. From (14), considering (16) we have:

$$\Delta g^{*}_{E} = \Delta g_s + \delta g_A - \varepsilon_h - \varepsilon_\gamma - \varepsilon_p - \frac{d\Delta g}{dh} h \tag{37}$$

where the last term reduces the surface anomaly to the ellipsoid. If Δg^{*}_{E} were upward continued to the sphere of radius a we would have the Δg^{*} values given by (22) with the spherical potential coefficients given by (23).

Now write (21) in the following form:

$$\Delta g^{*}(r_p,\ \theta_p,\ \lambda_p) = \sum_{n=2}^{\infty} \left(\frac{a}{r_p}\right)^{n+2} \sum_{m=-n}^{n} \overline{g}^{s}_{n,m}\ \overline{Y}_{nm}\ (\theta,\lambda) \tag{38}$$

where

$$\overline{g}^{s}_{n,m} = \frac{kM}{a^2}\ (n-1)\ \overline{C}^{s}_{n,m} \tag{39}$$

Multiplying (38) by r_p we have:

$$r_p \, \Delta g^*(r_p,\theta_p,\lambda_p) = a \sum_{n=0}^{\infty} \left(\frac{a}{r_p}\right)^{n+1} \sum_{m=-n}^{n} \bar{g}^s_{n,m} \, \bar{Y}_{n,m} \, (\theta,\lambda) \tag{40}$$

The left hand side is a harmonic function which can be expressed in a spherical harmonic series (as it is now) or in an ellipsoidal harmonic series. The latter can be written as (Gleason, ibid, 2.7):

$$r_p \, \Delta g^*(u_p,\delta_p,\lambda_p) = a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{\bar{S}_{n|m|}\left(\frac{u_p}{E}\right)}{\bar{S}_{n|m|}\left(\frac{b}{E}\right)} \bar{g}^e_{n,m} \, \bar{Y}_{n,m} \, (\delta_p,\lambda_p) \tag{41}$$

The coefficients "a $\bar{g}^e_{n,m}$" represent ellipsoidal harmonic coefficients related to the bounding ellipsoid. To calculate these coefficients we can use (33) where F is the left-hand side of (41) evaluated on the surface of the ellipsoid divided by a. We then can write:

$$\bar{g}^e_{n,m} = \frac{1}{4\pi a} \iint_{\sigma} r\Delta g^*(u{=}b,\delta,\lambda) \; \bar{Y}_{n,m} \; (\delta,\lambda) d\sigma \tag{42}$$

These ellipsoidal coefficients can be converted to the spherical coefficients using (36). The spherical potential coefficients can then be obtained from (39).

The technique discussed in this section represents an improvement over the spherical analysis where the anomaly on the ellipsoid was analytically upward continued to the bounding sphere of radius a. Although this procedure was done in terms of potential coefficient correction terms, the studies by Gleason indicate the procedure using ellipsoidal harmonics is more correct.

3.0 Data Definition

3.1 Satellite Data

As pointed out in the introduction we primarily view the satellite data in terms of the resultant normal equations formed from such data. The typical data types that one can find in a satellite alone solution include: optical tracking data; laser tracking data; Doppler observations, radar data, satellite to satellite tracking etc. In addition solutions may be carried out with satellite altimeter data but these solutions require methodology not needed in the solutions using the data previously described. The accuracy of these satellite solutions depends on many items including the distribution of satellite inclinations; data density, distribution and accuracy; observation models; and the general parametrization of the problem.

3.2 Terrestrial Gravity Data

The primary terrestrial data type we will use will be free-air gravity anomalies. To this point the anomalies have been defined by (7). We now consider a more precise formulation. Let H* be the normal height of the point P and H the orthometric height. Since H*-H is small we will approximate H* by H. Then normal gravity at the telluroid is:

$$\gamma_Q = \gamma_{Q_0} + \left(\frac{d\gamma}{dh}\right)_{Q_0} H + \frac{1}{2!}\left(\frac{d^2\gamma}{dh^2}\right)_{Q_0} H^2 + - - - \tag{43}$$

where γ_{Q_0} is normal gravity on the ellipsoid. The surface anomaly is then:

$$\Delta g = g_P - \left[\gamma_{Q_0} + \left(\frac{d\gamma}{dh}\right)_{Q_0} H\right] - \frac{1}{2!}\left[\frac{d^2\gamma}{dh^2}\right]_{Q_0} H^2 \tag{44}$$

In practice almost all organizations neglect the H^2 term. Pavlis (ibid) has shown that the neglect of this term can cause systematic errors in the potential coefficient solutions which imply geoid errors on the order of 2 cm with a maximum error of 1.8 m in the Himalaya region. Fortunately it is possible to correct the given anomaly by the last term in (44).

The terrestrial data is usually given in 1°x1° mean values or 30′x30′ values. A variety of techniques, some complex, some simple, are used to evaluate the mean anomalies and their accuracy. The data are usually regarded as independent although it has been shown (Weber and Wenzel, 1982) that in fact there is an error correlation between the anomalies. This correlation is neglected in all combination solutions so far. Systematic errors due to vertical datum inconsistencies also cause errors, some of which can be modeled, but usually are not.

The location of 48,955 1°x1° anomalies based on terrestrial measurements only is shown in Figure 2 (Despotakis, 1986). The accuracy estimates range from ± 1 mgal to ± 62 mgal. There are an additional set of 5,684 anomalies that have been estimated in the early 1960's using geophysical correlation techniques. The location of these anomalies is shown in Figure 3. There is some evidence, to be discussed later, that there are long wavelength errors present in this geophysically predicted anomalies. This is why combination solutions are usually made with and without such anomalies so evaluations can be done to see which solutions provide the best overall results.

Computations have also started for 30′ data files (Despotakis, 1986). Land data is available for North America, Europe, Australia, India, New Zealand, portions of South America and Africa, Japan, and other regions. The coverage of such data will not be as complete as that of 1°x1° data. Coverage in the ocean areas will be fairly good due to the use of altimeter data to derive the anomalies.

Another data type not used so far in combination solutions is topographic heights. Improved elevations in 1°x1° and 30′x30′ cells have recently become available. Rummel et al (1988) has shown there is a good (greater than 0.6) correlation between the observed potential and that implied by a topographic/isostatic potential estimated with an Airy isostatic hypothesis with D = 30Km. Such information may be used to calculate gravity anomalies that could be used to replace the geophysically predicted anomalies and to infer anomalies in areas where no other information is available. Tests are now underway to evaluate the value of such data recognizing that the longwave length information will be missing from the topographic data but the short wavelenth may prove valuable.

4.0 Data Combination

4.1 General Principles

We derive a model where all parameters are estimated taking into account all data with correct accuracies and error correlations considered. This ideal statement is not followed in practice but various attempts are made to achieve it. The basic principle followed in combination solutions to date has been the least squares principle with some prior information taken into account on selected parameters such as potential coefficients. Other techniques are possible and a number are now under development and testing.

4.2 Least Squares Principles

We define in this section our basic mathematical models that will be applied in a subsequent section in several forms. We start with some definitions:

- F a set of functions relating observations and parameters;
- L_ℓ a set of observations,
- L_{x^o} a given set (approximate or observed) of parameters;
- V_ℓ a set of residuals to be added to L_ℓ to obtain the adjusted observations, $L_{\ell a}$;
- V_x a set of residuals to be added to L_{x^o} to obtain the adjusted parameters L_{xa};
- W the misclosure vector, $W=F(L_\ell, L_{x^o})$.

The mathematical model is written as:

$$F = F(L_{\ell a},\ L_{x^o}) = 0 \tag{45}$$

which is linearized to yield the observation equation:

$$B_\ell V_\ell + B_x V_x + W = 0 \tag{46}$$

where

$$B_\ell = \frac{dF}{dL_\ell};\ B_x = \frac{dF}{dL_x} \tag{47}$$

If we designate P_ℓ and P_x as the weight matrices for the observations and parameters respectively, the weighted least squares condition for solutions is:

$$V_\ell^T P_\ell V_\ell + V_x^T P_x V_x = \text{a minimum} \tag{48}$$

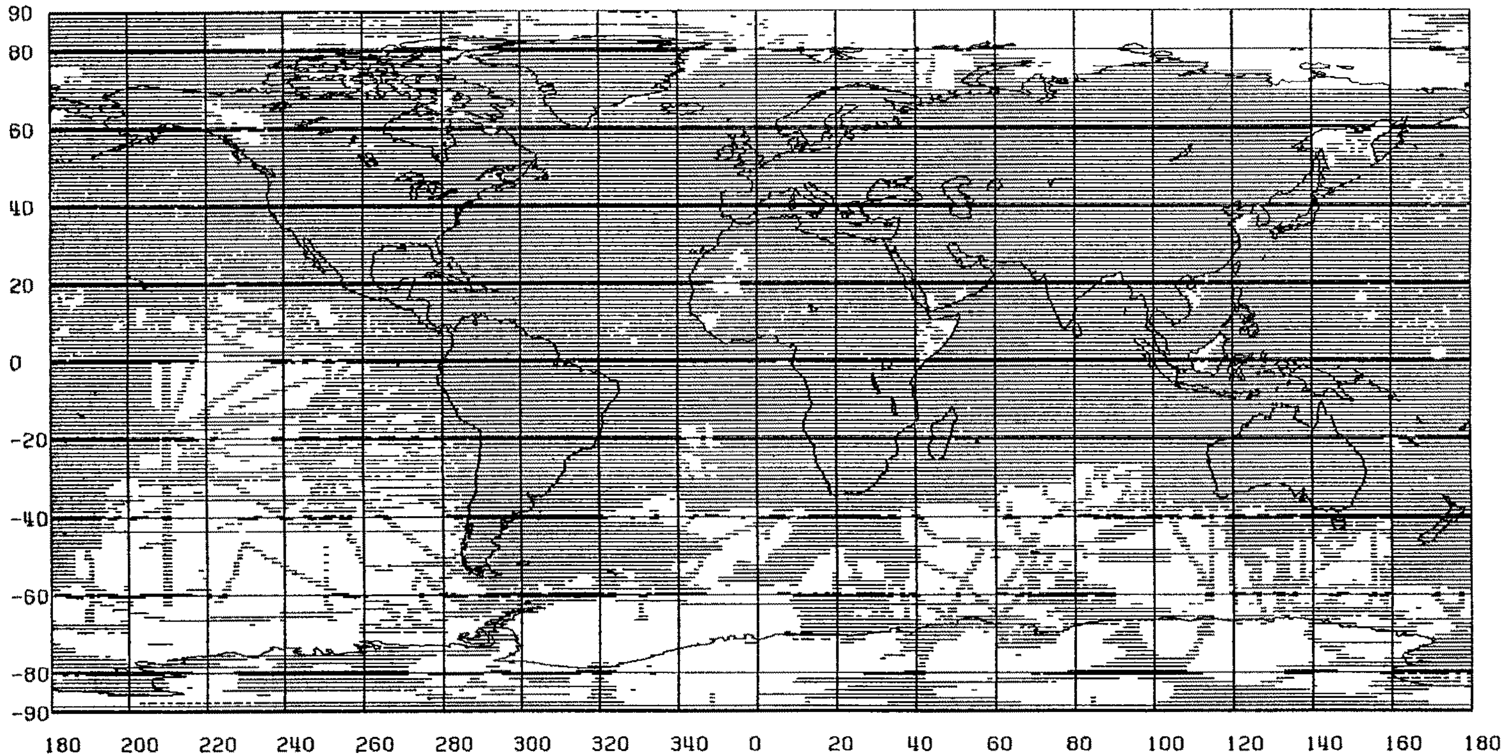

Figure 2. Location of 48955 1° x 1° Anomalies in the June 1986 Data Set.

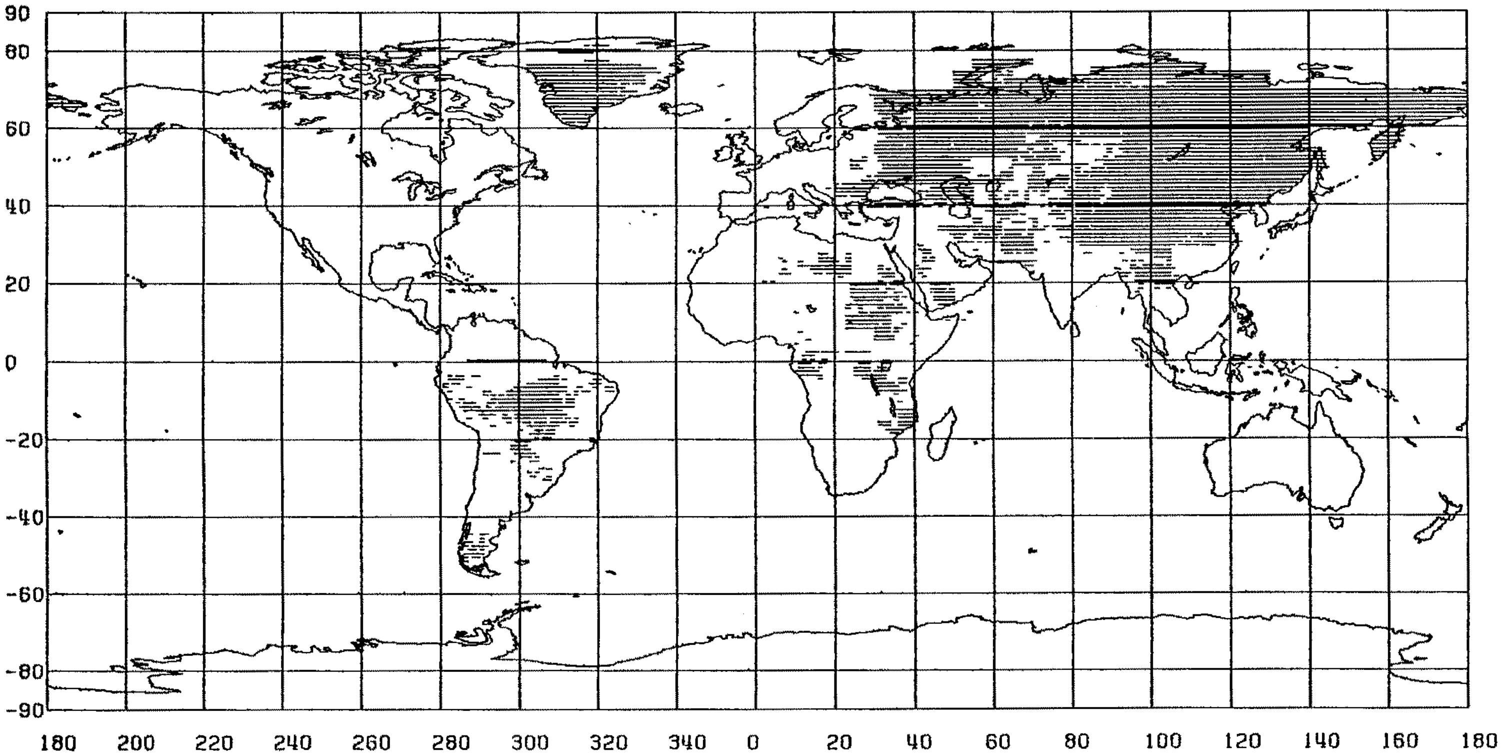

Figure 3. Location of 5684 1°x 1° Geophysically Predicted Anomalies in the June 1986 Data Set.

The solution for V_x is:

$$V_x = -(B_x^T M^{-1} B_x + P_x)^{-1} B_x^T M^{-1} W \tag{49}$$

where

$$M = B_\ell P_\ell^{-1} B_\ell^T \tag{50}$$

The observation residuals are:

$$V_\ell = -P_\ell^{-1} B_\ell^T M^{-1} (B_x V_x + W) \tag{51}$$

The error variance-covariance matrix for the solutions vector would be:

$$\Sigma_{xx} = m_0^2 (B_x^T M^{-1} B_x + P_x)^{-1} \tag{52}$$

where m_0^2 is the variance of unit weight.

These equations will be applied to two specific combination procedures shortly. However it is instructive to consider an alternate combination solution where the normal equations from various data sets are combined. Let N_i be the i th normal equations from a specified data type. Assuming that the variance of unit weight is the same for all data we can write the combined normal equations in the following form:

$$X = (\sum_i N_i + P_{xi})^{-1} (\sum_i U_i) \tag{53}$$

where P_{xi} is the a priori weight matrix for the parameters. In this case the normal equations for certain data types may not include the same parameters found in another data type. For example N_i terms from orbital analysis may include as parameters potential coefficients; station coordinates; tidal parameters; etc., while N_i from terrestrial gravity data may include potential coefficient parameters only. The merger of the individual normal equations must be done carefully and must be done with factors that would allow different weighting for different data types. The advantage of (53) is the inclusion in one simultaneous adjustment of all possible data. Alternate solutions are possible where all but common parameters are eliminated from the individual normal equations.

4.3 Optimal Estimation

Our concept of optimal estimation is analogous to least squares collocation estimation where we recognize pre-existing information on the primary gravity field parameters to be determined, and the covariances between the signals that constitute the observations. The usual collocation solutions require the solution of systems of equations whose size is equal to the number of observations. Colombo (1981) has shown that proper ordering of the parameters and observations can lead to structured covariances matrices that can be solved in an efficient way. Still substantial computational effort is needed to implement these optimal estimation solutions. In some cases it has been found that such solutions smooth the resultant gravity field parameters too much. Additional study is needed to define more precisely the best way to implement optimal estimation procedures in combination solution.

5.0 Observation Equation Formation

In this section we will examine two models that can be used for the combination of satellite and gravity data. Each model will lead to a set of observation equations that will lead to a set of normal equations.

5.1 Combination Procedure- Method A

The mathematical structure of this method is established by forming the difference between a set of spherical potential coefficients, L_{x^o}, and an estimated, $L_{x_c^*}$: computed through a global set of gravity anomalies. In our discussions in section 2.2.1 we defined a procedure of anomaly correction (eq. (37)); calculation of ellipsoidal coefficients (eq. (42)); the calculation of the spherical coefficients (eq. (36)) and finally the spherical potential coefficients (eq. (39)). We write for the adjusted case:

$$F = L_{x^a} - L_{x_c^a} \tag{54}$$

We let L_{x^o} be the spherical potential coefficients estimated from satellite data. We then calculate L_{x^e} using the procedure described above. The misclosure vector would then be:

$$W = L_{x^o} - L_{x^c} \tag{55}$$

We also have:

$$B_x = \frac{dF}{dL_{x^o}} = I; \quad B_\ell = -\frac{dL_{x^c}}{dL_\ell} \tag{56}$$

The L_ℓ values are the gravity anomalies Δg^* and the L_{x^c} are the spherical potential coefficients. The evaluation of B_ℓ is complicated by the conversion from ellipsoidal harmonics to spherical harmonics. An approximate form (sufficient for observation equation coefficients) can be formed by recognizing $L_{n,m,k}$ is $0(e^2)$ so we will consider only the care of k=0. From (36) we would have:

$$\bar{f}^s_{n,m} \approx \frac{\bar{\bar{f}}^e_{n,m}}{\bar{S}_{n,|m|}\left(\frac{b}{E}\right)} \tag{57}$$

We now evaluate (56) using (42), (57), and (39). We have as an element of B_ℓ:

$$[B_e] = \frac{-1}{4\pi\frac{kM}{a^2}(n-1)} \; \frac{1}{\bar{S}_{nm}\left(\frac{b}{E}\right)} \left(\frac{r}{a}\right) \bar{Y}_{nm}(\delta,\lambda)d\sigma \tag{58}$$

One may evaluate the coefficients in (58) by integrating over the block σ_{ij} or using the center point evaluation discussed earlier.

The solution vector is now found from (49):

$$V_x = -\left((B_\ell P_\ell^{-1} B_\ell^T)^{-1} + P_x\right)^{-1} (B_\ell P_\ell^{-1} B_\ell^T)W \tag{59}$$

These V_x are added to the initial satellite implied, spherical potential coefficients to obtain the adjusted potential coefficients. The anomaly residuals may be computed from (51) or the equivalent expression:

$$V_\ell = P_\ell^{-1} B_\ell^T P_x V_x \tag{60}$$

These values are added to the initial Δg^* values to obtain the adjusted Δg^* values on the ellipsoid.

In this procedure we assume that the surface anomalies are reduced to the ellipsoid using the following approximation:

$$\Delta g_E = \Delta g_S - \frac{d\Delta g}{dh} h \tag{61}$$

where h is the ellipsoidal height of the anomaly, or mean compartment if, as would be expected, mean anomalies are used. Other correction terms as shown in (37) must also be applied. Wang (1988) has carried out a global computation of the correction the term working with the Molodensky g_1 term writing:

$$\Delta g_E = \Delta g_S + g_1 \tag{62}$$

Wang used 5′ elevation data and Fourier techniques to calculate g_1 terms for both 30′ and 1° anomalies on a global basis. The root mean square g_1 value is ± 1.5 mgals for 30′ cells and ± 1.2 mgals for 1° cells.

5.2 Combination Procedure– Method B

We first express (18) in the following form:

$$\overline{\Delta g}^*_{ij} = \frac{1}{\Delta\sigma_{ij}} \frac{kM}{\bar{r}^2_{ij}} \sum_{\substack{n=0\\n\neq 1}}^{\infty} (n-1) \left(\frac{a}{\bar{r}_{ij}}\right)^n \sum_{m=-n}^{n} C_{nm}\, IY_{nm} \tag{63}$$

The $\overline{\Delta g}^*$ anomalies represent the mean anomalies after correction for the atmospheric correction and other terms related to a precise interpretation of the boundary condition. (See eq. 20). Note that (63) involves a summation to ∞ which clearly is not possible. It is necessary to truncate the series to some N_{max}. One choice of N_{max} would be to have it compatible with the frequency content of the $\overline{\Delta g}$ values. If such anomalies are given in $\theta°$ cells, N_{max} could roughly be given as $180°/\theta°$. For anomaly blocks given in 1° cells, for example, the number of parameters to be solved for would be too large for most computers. (For example, if N_{max} = 180, there would be 32761 potential coefficients to be estimated). However such solutions are possible if the parameters are ordered in an optimum way and only near diagonal terms are considered in the solution of the system of equations.

An alternative selection of N_{max} is one that makes the number of potential coefficients to be solved for consistent with that found in the analysis of the satellite data. To do this one needs to remove the signal from N_{max} + 1 to ∞ with the latter being replaced by some existing high degree potential coefficient model. Let dg_{hf} be defined as follows:

$$(\overline{dg}_{hf})_{ij} = \frac{1}{\Delta\sigma_{ij}} \frac{kM}{\overline{r}^2_{ij}} \sum_{n=N_{max}+1}^{\overline{N}} (n-1) \left(\frac{a}{\overline{r}_{ij}}\right)^n \sum_{m=-n}^{n} C_{nm}\ IY_{nm} \tag{64}$$

We then use (63) with ∞ replaced by N_{max} and $\overline{\Delta g}*$ replaced by:

$$dg_{ij} = \overline{\Delta g}^*_{ij} - (\overline{\delta g}_{hf})_{ij} \tag{65}$$

The advantage of this procedure is the reduction of the effect of the high frequency values in $\Delta g*$ on the estimated parameters.

Our general mathematical model (see Section 4.2) now becomes:

$$F = L_\ell - f(L_x) \tag{66}$$

where L_ℓ is the observed gravity anomaly (with suitable corrections (see 65)) and L_x are the potential coefficients derived from a satellite analysis. An element of B_x (for degree n and order m would be):

$$[B_r] = -\frac{1}{\Delta\sigma_{ij}} \frac{kM}{\overline{r}^2_{ij}} (n-1) \left(\frac{a}{\overline{r}_{ij}}\right)^n IY_{nm} \tag{67}$$

Since the observable is the anomaly we have B_ℓ=I. The solution vector follows from (49):

$$V_x = -(B_x^T\ P_\ell\ B_x + P_x)^{-1}\ B_x\ P_\ell\ W \tag{68}$$

where W= $[dg_{ij} - dg^c_{ij}]$ where the C designates the anomaly computed from the a priori potential coefficients. Note that the residuals on the anomalies can be computed from (51). However they will reflect only the changes in degrees 0 to N_{max} in the potential coefficients. To obtain the adjusted $\overline{\Delta g}*$ value the high frequency effects must be added back in. Unfortunately such effects may be incorrect and the $\Delta g*$ values will not reflect an adjusted value in all frequency bands.

A modification of the above procedure takes place when (53) is used for the normal equations. In this case a simultaneous solution is made for all parameters of interest for satellite data and for terrestrial data. In this case the normal equation contribution from terrestrial gravity data would be:

$$(B_x^T\ P_\ell\ B_x)X_g = -\ B_x\ P_\ell\ W_a \tag{69}$$

where X_g represents the potential coefficient parameters.

Pavlis (ibid) has studied the form of the normal equations generated from the terrestrial data (with and without the geophysically predicted anomalies) considering different parameter ordering schemes. Some schemes minimize the band width and should be used if a rigorous inversion is not being done. As the terrestrial data increases the normal equations become increasingly diagonal dominant. However the current real world data coverage yields a situation where the coefficients recovered from a terrestrial solutioon only may be highly correlated. Additional discussion on the structure of the normal equations for specified data coverage may be found in Bosch (1987).

5.3 Comment

Other combination solutions have been described in the literature. Sjoberg (1981) has discussed the spectral combination of the coefficients from satellite and terrestrial data. Schaffrin and Middel (1987) have discussed robust estimation procedures. All the combination methods suffer common problems of approximate weighting. All solutions made to date regard the anomalies to be independently estimated. As noted earlier. there are error correlatioons that cause problems with the combination solution. One ad hoc method to reduce the problem was used by Rapp and Cruz (1986) where they multiplied the given anomaly standard deviations by 2.5 and scaled the resultant standard deviation to fall within the interval 8 to 38 mgal. Such scaling seems to have worked in practice.

Another point relates to the computer time needed for the formation of the normal equations and the subsequent inversion. If we include the geophysically predicted anomalies there are 48955 values in the June 86 data set. The calculation of the normals can take a substantial time on scalar processors and only vector processors will yield reasonable times for N_{max} values of interest for current solutions. Pavlis has provided the computer times on a CRAY XMP 2/4 computer (Table 1) when forming the normal equations using 1°x1° anomalies in the formation of the anomaly only normal equations of the Method B combination solution.

Table 1. Cray Computer Times Related to Method B For Various Degrees of Expansion Using 1°x1° Anomalies.

N_{max}	Number of Unknowns	CPU Time (seconds) Normals	Inversion
24	622	432	1.86
36	1366	2119	16.40
50	2598	8644	168.70

The times in Table 1 need to be multiplied by about 60 to obtain the equivalent times on a scalar, IBM 3081D, machine. It is clear that such processing can only be done on supercomputers.

6.0 The Development of High Degree Potential Coefficient Models

To this point we have discussed methods that can be used to combine satellite information and terrestrial gravity data. But the rigorous solutions have been limited to the highest degree present in the satellite derived fields. We now examine the computation of high degree fields whose frequency content is compatible with that of the original data.

Consider combination Method A described in Section 5.1 where the correction to the coefficients are given by (59) and the corrections to the "observed" anomalies are given by (60). We have

$$L_{x^a} = L_{x^o} + V_x \tag{70}$$

where L_{x^a} are the adjusted spherical potential coefficients and L_{x^o} are the approximate (a priori) values. The adjusted anomalies, on the ellipsoid, are found by adding the residuals from (60) to the corrected anomalies on the ellipsoid. Using (20) and (62) we have for the adjusted anomalies:

$$\overline{\Delta g}^{*}_{a} = \overline{\Delta g}_{ij} + \delta g_A^{ij} - (I\hat{E}_A + I\hat{E}_\gamma + I\hat{E}_p)^{ij} + g_1 + [V_\ell] \tag{71}$$

or

$$\overline{\Delta g}^{*}_{a} = \overline{\Delta g}_{ij} + \overline{\delta g}^{s}_{ij} + g_1 + [V_\ell] \tag{72}$$

We now use these values in (42) to find $\bar{g}^{e}_{n,m}$ which are then used in (36) to find the corresponding spherical coefficients. These values can then be used in (39) to find the corresponding spherical harmonic potential coefficients. These coefficients should be the same as the adjusted coefficients from (70). This procedure removes the need for the ellipsoidal correction terms proposed by Cruz (1986) and used in Rapp and Cruz (1986, ibid).

The key equation in this evaluation is (42). The actual evaluation is done in a manner analogous to that used in (24) or (29). Decisions must be made on the q_n factors and the integration of the Y_{nm} functions. The HARMIN type of calculation suggested by Colombo (1981) can be used provided the integration in (42) is done in the (δ,λ) coordinate system. Using 1°x1° anomalies the expansion can be taken to degree 180 while 30′ data can yield expansions to degree 360. In both cases one finds that lack of data in some areas will mean that the expansions will not, in reality, have the same spectral content in all regions.

These methods have the unfortunate attribute of having no accuracy estimates except for those coefficients that were part of the original adjustment. Studies need to be made to show efficient, and realistic ways to obtain the accuracies of the potential coefficients at high degree.

7.0 The Role of Satellite Altimeter Data

Satellite altimeter data represents a measurement from a satellite to the instantaneous ocean surface. This data can be used in orbit determination procedures or by itself as a means of estimating the gravity anomaly field in the ocean areas. A dominant limitation on the use of these altimeter measurements is in the accuracy of the original satellite orbit, or in the ability of the orbit determination process to recover accurate orbits. We now consider, in general terms, the use of altimeter data for gravity field improvement.

We first consider the use of altimeter data in the orbit determination process following Marsh et al. (1988b). Let A be the observed range from the satellite to the ocean surface. The altimeter observation equation is then represented in the following form:

$$R - (R_e + N + \zeta + T) + B \cong A + v \tag{73}$$

where:

- R is the radial distance of the satellite to the center of mass of the Earth. This distance is a function of the usual orbit parametization including station coordinates and gravity field parameters which are adjusted in the solution;
- R_e is the distance from the center of mass to the sub-satellite point;
- N is the geoid undulation computed from the potential coefficients which are adjusted in the solution;
- ζ is the sea surface topography as parameterized by (e.q.) a low degree spherical harmonic expansion whose coefficients are estimated;
- T is the tidal correction;
- B is the altimeter bias, including an equatorial radius correction, which is adjusted;
- A is the altimeter measurement corrected for various environmental effects with high degree geoid information removed so that the frequency content of A is the same as in the potential coefficient parametization.

The observation equations from (73) yield a set of normal equations that are combined with normal equations from other data (both satellite and terrestrial gravity) that then leads to a general parameter estimation using (53).

An alternate method for using the altimeter data is to first remove as much remaining orbital error as possible. Typically this has been done using cross-over methods solving for bias' and tilts in altimeter arcs. This procedure is not completely satisfactory because it does not completely remove geographically correlated errors (Engelis, 1987). After the adjustment the sea surface heights can be corrected, in only an approximate way, to geoid undulations. This conversion is approximate only because sea surface topography is inadequately known. The resultant heights can then be used to derive gravity anomalies with a spacing consistent (in some reasonable way) with the altimeter track spacing. Rapp (1986) describes such conversions that yielded anomalies on a 0.125 grid. These anomalies can be used to fill in empty areas in the oceans or to completely replace existing terrestrial data in the oceans by the more reliable altimeter data. In this context more reliable refers to mean anomalies (such as 1°x1° or 30′x30′) and not necessarily point values acquired from modern ship data. However the accuracy of point anomalies from altimeter data increases with data density, but the spectral content of the altimeter data will be more limited than that of ship data.

The use of altimeter derived anomalies is especially useful in developing an anomaly data base for the Method A Combination Solution (Section 5.1). For the Method B (Section 5.2) solution the geoid heights could be directly used provided the spectral content is made compatible with the potential parametization. This procedure is similar to the method used by Wenzel (1985) in developing the GPM2 model.

In the use of altimeter data one must be careful not to include duplicate information. This would happen if one used altimeter data in orbit estimation in combination with anomalies calculated from the altimeter data.

8.0 Comparison of Satellite and Terrestrial Gravity Anomaly Fields

In Section 1 we described the very general ideas of gravity determinations from satellite and terrestrial measurements. It is instructive to compare such measurements before carring out any combination solution to detect data or model errors. Since satellite gravity data is more band limited than terrestrial data the most informative comparisons can be made in the spectral domain. The most recent such comparisons have been described by Pavlis (1988) where the satellite data was GEMT1 (complete to degree 36) (Marsh et al, 1988a) and the terrestrial data was the June 1986 1°x1° data set.

Pavlis took the 1°x1° data and estimated a set of potential coefficients using the method described in Section 5.2. The anomalies implied by the coefficients were then compared to the anomalies implied by GEMT1. Figure 4 shows the location where differences exceeded 10 mgals when geophysical anomalies are included in the solutions, and Figure 5 shows the comparison when the geophysical anomalies are included. Comparison of Figure 4 with the location of the geophysically predicted anomalies (Figure 3) shows a moderately good correlation indicating some inconsistency between the geophysically predicted anomalies and the satellite implied anomalies.

These anomaly discrepencies are also reflected in undulation discrepencies. These discrepencies reach 20 m in east central Africa and 10 m in the Himalaya Mountain regions. Tests described by Pavlis for Africa shed strong doubt on the credibility of the terrestrial gravity data in portions of Africa and Asia.

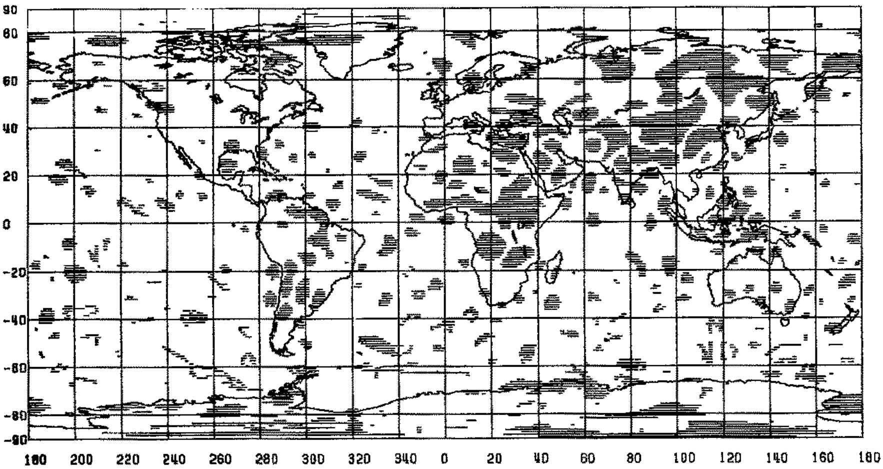

Figure 4. Location of 9745 1°x1° Blocks Where |Δg(GEMT1) - Δg(A20)| Exceeds 10 mgals. (Locations Corresponding to Geophysically Predicted Anomalies are included in the Comparison).

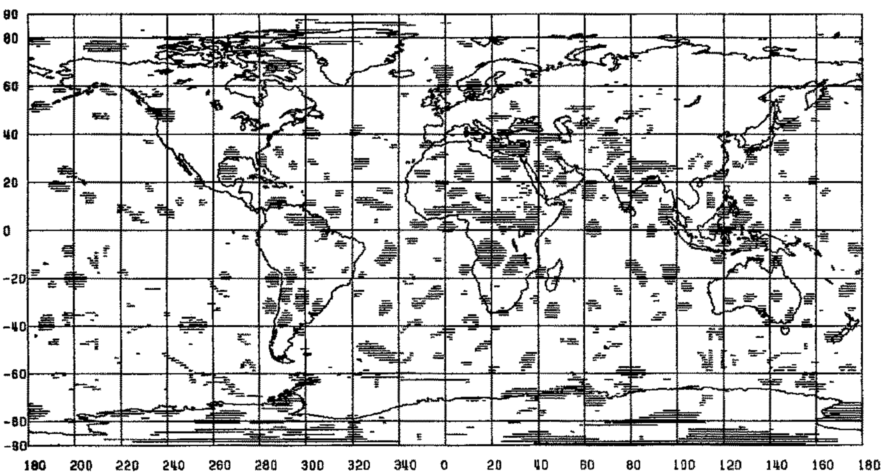

Figure 5. Location of 7545 1°x1° Blocks Where |Δg(GEMT1) - Δg(A40)| Exceed 10 mgals. (Locations Corresponding to Geophysically Predicted Anomalies are Excluded from the Comparison).

9.0 Conclusion

The purpose of this paper has been to describe the method that might be used in the combination of satellite and terrestrial data. The emphasis has been on theory. Several high degree fields are in use today. These include OSU81 (180); OSU86 e/D (250); OSU86 E/F (360) GEM10C (180); GEM2 (200) where the numbers in parenthesis give the highest degree in the expansion. A comparison between some of these models may be found in Wenzel (1985), Rapp (1986) and Rapp and Cruz (1986a, 1986b).

Hopefully it is clear from the discussions in this paper that there are many choices to be made in carrying out combination solutions. Some of these choices lead to approximate solutions but no other alternatives may exist. The success of existing high degree fields implies that more fields will be developed using new satellite models and improved terrestrial data. Careful evaluation of these models is needed to assess their accuracy and the assumptions made in developing the model. Although computer resources continually increase it is doubtful if expansions above degree 360 will prove of value but this type of prediction has proved unreliable in the past.

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