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ORIGINAL ARTICLE

The height datum problem and the role of satellite gravity models

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Abstract Regional height systems do not refer to a common equipotential surface, such as the geoid. They are usually referred to the mean sea level at a reference tide gauge. As mean sea level varies (by ± 1 to 2 m) from place to place and from continent to continent each tide gauge has an unknown bias with respect to a common reference surface, whose determination is what the height datum problem is concerned with. This paper deals with this problem, in connection to the availability of satellite gravity missions data. Since biased heights enter into the computation of terrestrial gravity anomalies, which in turn are used for geoid determination, the biases enter as secondary or indirect effect also in such a geoid model. In contrast to terrestrial gravity anomalies, gravity and geoid models derived from satellite gravity missions, and in particular GRACE and GOCE, do not suffer from those inconsistencies. Those models can be regarded as unbiased. After a review of the mathematical formulation of the problem, the paper examines two alternative approaches to its solution. The first one compares the gravity potential coefficients in the range of degrees from 100 to 200 of an unbiased gravity field from GOCE with those of the combined model EGM2008, that in this range is affected by the height biases. This first proposal yields a solution too inaccurate to be useful. The second approach compares height anomalies derived from GNSS ellipsoidal heights and biased normal heights, with anomalies derived from an anomalous potential which combines a satellite-only model up to degree 200 and a high-resolution global model above 200. The point is to show that in this last combination the indirect effects of the height biases are negligible. To this aim, an error budget analysis is performed. The biases of the high frequency part are proved to be irrelevant, so that an accuracy of 5 cm per individual GNSS station is found. This seems to be a promising practical method to solve the problem.

Keywords Height datum · GOCE · EGM2008

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1 Introduction

Height systems are connected to the Earth gravity field through its equipotential surfaces. An absolute evaluation of the potential at one point is not possible, we are just able to observe increments of this value with respect to another point chosen as benchmark. Even assuming that all the points can be connected to a single benchmark, which is not the case, we have to face a height datum problem, consisting in the evaluation of the gravity potential value of this benchmark. If more than one benchmark is introduced, and if it is not possible to determine the difference in potential between them, then the height datum problem involves the determination of those offsets. More specifically, one could choose to fix the geoid potential W_0 to that of the normal gravity on the reference ellipsoid U_0. This value in turn can be determined from observable quantities as GM (G being the universal gravitational constant and M the actual mass of the Earth), the angular velocity ω and two geometric parameters a and e, the latter related to the J_2 coefficient. Although the mean sea level is a good approximation of the geoid, still it does not coincide with it; therefore, the regional height systems,

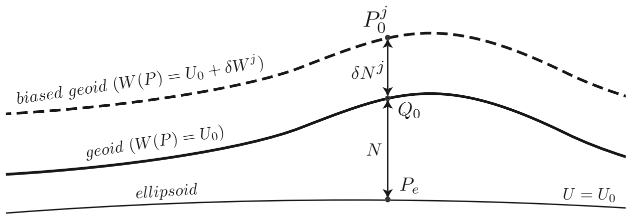

Fig. 1 Geometry of the geoid and the biased geoid

usually referred to the mean sea level at local tide gauges, are not related to the same reference surface, namely to the same W_0. Notice that, more recently, they are related to one or more geodetic satellite observatories, though in this case these reference surfaces are always close (in terms of few centimeters) to the previous ones for historical reasons. If we call W_0^j the value at a local benchmark P_0^j, we can introduce a set of unknown biases

$$\delta W^j = W_0^j - W_0 = W_0^j - U_0. \tag{1}$$

To find out those unknowns one has to compare quantities which are in some way affected by these biases with quantities which are not. Gravity field models derived from satellite missions can be considered as unbiased. The idea is to exploit these models in the solution of the height datum problem. Two strategies are explored in this paper, to assess their feasibility in terms of accuracy of the estimated biases. The approach is that of an error budget.

The equipotential surface through the benchmark P_0^j is called local geoid (see Jekeli 2000). We will refer to this local geoid as "biased" geoid to underline the fact that the potential of this surface is affected by the bias δW^j with respect to the geoid potential $W_0^j = W_0 + \delta W_0^j = U_0 + \delta W^j$. The vertical distance between P_0^j and the geoid, i.e. the distance between P_0^j and the point Q_0 where the vertical through P_0^j intersects the geoid, is shown in Fig. 1 and is given by

$$\frac{|W(P_0^j) - W(Q_0)|}{\bar{g}} = \frac{|\delta W^j|}{\bar{g}} = |\delta N^j| \tag{2}$$

where $\bar{g}$ is the average gravity on the vertical line from P_0^j to Q_0. This value changes from point to point on the biased geoid, due to the variation of $\bar{g}$, that is, the biased geoid, as any other equipotential surface, is not parallel to the geoid. Since we put an a priori bound on δW,

$$|\delta W| < 20\,\frac{\mathrm{m}^2}{\mathrm{s}^2}, \tag{3}$$

and since $\frac{\mathrm{d}\bar{g}}{\bar{g}} \sim 5 \cdot 10^{-3}$, the surfaces are practically parallel, the variation of the bias from point to point being of the order of 1 cm, i.e.

$$O(|\mathrm{d}(\delta N^j)|) \approx \frac{|\delta W|}{\bar{g}}\frac{|\mathrm{d}\bar{g}|}{\bar{g}} < 1\,\mathrm{cm}. \tag{4}$$

For the present analysis this approximation is acceptable and we will use a constant value of gravity ($\frac{GM}{a^2} = 9.7983\,\mathrm{m\,s^{-2}}$) that we will generically call γ without any other specification. With this approximation in mind, we will talk about a constant bias in height among local models. Nevertheless, changing the observation equations to take into account the variation of γ leads to quite elementary formulas, without any change in our conclusions.

The determination of a unique constant bias δN from different kinds of geodetic observations is the height datum problem in its most elementary form, i.e. when a unique benchmark P_0 is introduced. This idealized problem is classical and is already solved in Heiskanen and Moritz (1967), Sect. 2–19, by a suitable modification of Stokes' theory and assuming that an arc length reduced to the ellipsoid is known. We observe that nowadays, much more realistically, the arc length can be substituted with the knowledge of the ellipsoidal height h_{P_0}, easily derived from space geodetic positioning techniques and, in particular, with a permanent GNSS station.

Actually, one cannot avoid having at least one benchmark per continental block and, in practice, even more. In addition, on the oceanic surface the direct stationary shape of the sea surface topography can be derived from radar altimetric observations and it is very much in the nature of such observations to leave at least one bias per ocean, if we assume that the relative dynamic height is already known. Therefore, one could say that the whole surface of the Earth is covered by patches, each with its own origin of the orthometric or normal heights and that the biases have to be estimated to pass from one to the other.

When, on the ocean too, a solution of the so-called inverse Stokes' problem (Andersen and Knudsen 1998) is applied, we arrive at a formulation in which the Earth surface is covered by patches of gravity anomalies. Then the solution of the corresponding boundary value problem (BVP), with the addition of ellipsoidal heights at selected points, can solve the problem of homogenizing the height datum. The solution is described in a number of works, such as Colombo (1980); Rapp (1980); Rummel and Teunissen (1988); Heck (1989); Xu and Rummel (1991); Rapp and Balasubramania (1992); Sansò and Usai (1995); Lehmann (2000); Rummel (2002). A good summary of the existing literature is in Sánchez (2008).

One more point is worth mentioning here, namely that the same problem can be viewed in a local or in a global context. In a local context, the biases are estimated only for a part of the globe. This is possible because the solver $\Delta g \rightarrow T$ can be localized by an approach like collocation (cf. Forsberg 2000; Sansò and Venuti 2002) or by an approach like the one proposed by Kotsakis et al. (2012), where the geometrical aspects of the problem are considered. In a global context the biases are estimated all together in a joint procedure with

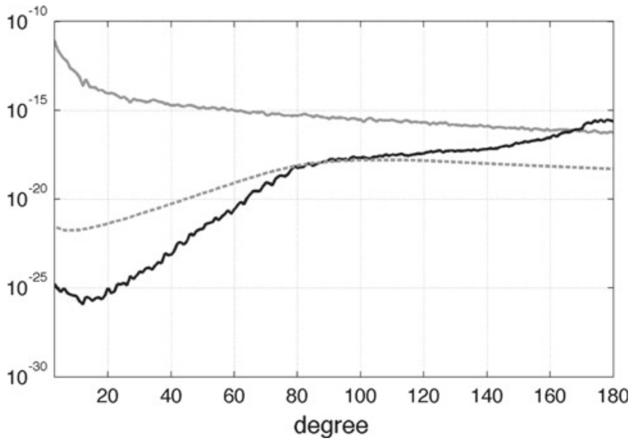

Fig. 2 Degree variances of the differences between EGM2008 and ITG-GRACE03S coefficients (*solid black line*); estimated error degree variances of EGM2008 (*dashed grey line*); signal degree variances from EGM2008 (*solid grey line*). The two models can be considered equivalent when their differences are smaller than the estimated error of EGM2008

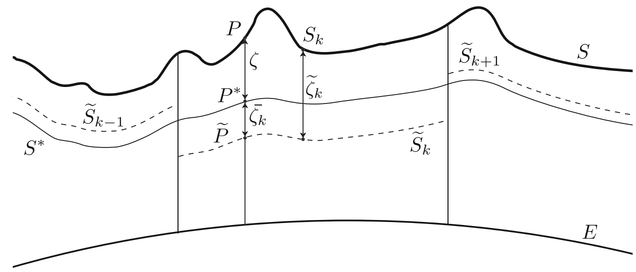

Fig. 3 Geometry of the problem: S Earth surface, S^* telluroid, $\widetilde{S}$ biased telluroid, $\widetilde{S}_k$ patch of $\widetilde{S}$, E ellipsoid, ζ height anomaly, $\widetilde{\zeta}_k$ biased height anomaly, $\bar{\zeta}_k$ bias of the patch $\widetilde{S}_k$

the solution of a boundary value problem. The paper is conceived along this last line, trying to clarify a point which has been overlooked up to now. All the past solutions assume that a fully biased geoid model is computed and available, which is not the present situation. What is available today is a set of satellite-only models that, including for example the GRACE model ITG-GRACE2010S (Mayer-Gürr et al 2010), goes up to degree 180 (though with a large commission error in the highest degrees), and a set of high resolution global combined models, among which probably the most important and used is EGM2008 (Pavlis et al. 2008).

The satellite-only models are not affected by the biases of the height datum; with the advent of the GOCE mission we expect that they will give a geoid up to degree 200, with 1 or 2 cm of commission error (Drinkwater et al. 2003). On the other hand, EGM2008 is certainly influenced by the height datum problem, but only above a certain degree. Roughly, comparing the commission errors of EGM2008 and ITG-GRACE03S (Mayer-Gürr 2006), which is the GRACE model included in EGM2008, one can reasonably guess that above degree 100, the least squares adjustment is basically dominated by the terrestrial gravity input (see Fig. 2).

In this paper we try to show how, by combining the two pieces of information, a satellite-only model up to degree 200 and a high-resolution combined global model above degree 200, one can find a realistic solution of the height datum problem with an overall error budget in the range of centimeters.

After recalling the mathematical formulation of the height datum problem (Sect. 2), the paper explores two possible approaches to its solution. The first (Sect. 3.1) directly compares the anomalous potential coefficients of the unbiased satellite-only models with those of the EGM2008 model. The second (Sect. 3.2) compares biased height anomalies derived from GNSS and normal heights with corresponding values derived from a proper combination of the anomalous potential coefficient of the GRACE, GOCE and EGM2008 models. Comments on the results and future work (Sect. 4) conclude the paper.

2 Recalling the mathematical formulation of the problem

In one of his famous letters, Krarup (2006) formulated a linearized form of Molodensky's problem, in which the unknowns are both the anomalous potential T and the vector surface anomaly $\widetilde{\boldsymbol{\zeta}}$ leading from the points $\widetilde{P}$ of some approximate surface of the Earth $\widetilde{S}$ to the corresponding points P on the actual surface of the Earth S, at which the quantities $W(P)$ (actual gravity potential) and $\boldsymbol{g}(P)$ (actual gravity vector) are given. The same problem can be turned to a scalar version, where the two points P and $\widetilde{P}$ are constrained to stay on the same normal to the ellipsoid so that the separation between $\widetilde{S}$ and S can be described by a scalar function only, that we will call $\widetilde{\zeta}$ or "biased" height anomaly. The term biased is used here, because it is only if the approximate surface $\widetilde{S}$ is the telluroid S^*, defined by the Marussi condition

$$W(P) = U(P^*), \tag{5}$$

that $\widetilde{\zeta}$ becomes ζ, the classical height anomaly (see Fig. 3).

In its scalar form, Molodensky's theory provides, after linearization, two relations: one is the generalized Bruns equation, the other constitutes the equation on the boundary $\widetilde{S}$ of the BVP. They are written, respectively (cf. Sansò 1989)

$$\widetilde{\zeta} = h_P - h_{\widetilde{P}} = \frac{T(P) - \widetilde{\Delta} W}{\gamma}, \tag{6}$$

$$M(T) \equiv -T' + \frac{\gamma'}{\gamma} T \bigg|_{\widetilde{S}} = \widetilde{\Delta} g + \frac{\gamma'}{\gamma} \widetilde{\Delta} W, \tag{7}$$

where we use a prime " $'$ " to represent the vertical derivative (along the normal to the ellipsoid or along the radius, when the spherical approximation is adopted) and we have defined the biased gravity anomaly and the biased potential anomaly as

$$\widetilde{\Delta} g = g(P) - \gamma(\widetilde{P}) \tag{8}$$

$$\widetilde{\Delta} W = W(P) - U(\widetilde{P}). \tag{9}$$

Note that if we assume to know $g(P)$, $W(P)$ and the ellipsoidal height of $\widetilde{P}$, then the two anomalies can be computed and the right-hand side expression in Eq. (7) is a known term.

Let us now come to our specific case. The key issue is how to define $\widetilde{S}$. Call S^j that patch of the Earth surface S whose height system is referred to the benchmark $\{P_0^j, j = 1, 2 \ldots J\}$, with gravity potential

$$W(P_0^j) = W_0^j = W_0 + \delta W^j = U_0 + \delta W^j, \tag{10}$$

δW^j being small in the sense of the inequality (3). Moreover, call P^j a point on S^j.

Now, following leveling lines, one can compute potential differences from leveling increments and absolute gravity; therefore, in each patch S^j, we have an approximated potential given by

$$\begin{aligned}\widetilde{W}(P^j) &= U_0 + [W(P^j) - W_0^j] \\ &= W(P^j) - \delta W^j.\end{aligned} \tag{11}$$

Now, mimicking the Marussi condition (5), we choose patch by patch, as approximated surface, a biased telluroid $\widetilde{S}^j$ by

$$\widetilde{W}(P^j) = U(\widetilde{P}^j). \tag{12}$$

We observe that, with the same argument as in the introduction, each $\widetilde{S}^j$ is practically parallel to the telluroid S^*. In fact, inserting Eqs. (11) and (5) into Eq. (12), we have

$$U(\widetilde{P}^j) = \widetilde{W}(P^j) = W(P^j) - \delta W^j, \tag{13}$$

i.e.

$$U(P^*) = U(\widetilde{P}^j) + \delta W^j, \tag{14}$$

which used into

$$U(P^j) = U(\widetilde{P}^j) - \gamma\widetilde{\zeta} = U(P^*) - \gamma\zeta, \tag{15}$$

gives the equation

$$\widetilde{\zeta} = \zeta - \frac{\delta W^j}{\gamma}. \tag{16}$$

Since δW^j is constant on $\widetilde{S}^j$, and γ is almost constant there, we can write Eq. (16) in the form

$$\widetilde{\zeta}(P^j) = \zeta(P^j) + \overline{\zeta}^j, \tag{17}$$

showing that $\overline{\zeta}^j$ is almost the bias of $\widetilde{S}^j$ with respect to S^*.

To summarize, we can write Eq. (11) in the form

$$W(P) = \widetilde{W}(P) + \sum_{j=1}^{J} \delta W^j \chi_j(P) \tag{18}$$

where $\widetilde{W}(P)$ is known and

$$\chi_j(P) = \begin{cases} 1 & P \in S^j \\ 0 & \text{elsewhere.} \end{cases} \tag{19}$$

From Eqs. (9) and (12), we find then that

$$\widetilde{\Delta} W = \sum_{j=1}^{J} \delta W^j \chi_j(P). \tag{20}$$

In principle, we can return to Eq. (7) and solve the BVP. Since by a classical reasoning the anomalous potential T is void of zero and first degree harmonics, Eq. (7) can be solved in a space of harmonic functions, where such harmonics are eliminated. Let us call M^{-1} this (reduced) inverse operator (Sansò and Venuti 2002), then from Eq. (7) we can write

$$T(P) = \widetilde{T}(P) + \sum_{j=1}^{J} \delta W^j F^j(P) \tag{21}$$

where

$$\widetilde{T} = M^{-1}\left(\widetilde{\Delta} g\right), \tag{22}$$

and

$$F^j(P) = M^{-1}\left(\frac{\gamma'}{\gamma}\chi_j\right). \tag{23}$$

Observe that contrary to χ_j, F^j are non-zero functions over the whole surface $\widetilde{S}$. Now assume that in each S^j there are points $\{P_\ell^j\}$ where the ellipsoidal heights $h_{P_\ell^j}$ are known, so that $\widetilde{\zeta}_\ell^j$ are known too because

$$\widetilde{\zeta}_\ell^j = \widetilde{\zeta}(P_\ell^j) = h_{P_\ell^j} - h_{\widetilde{P}_\ell^j}. \tag{24}$$

Then, recalling Eq. (6), one can write observation equations

$$\widetilde{\zeta}_\ell^j = \frac{\widetilde{T}(P_\ell^j)}{\gamma} + \frac{1}{\gamma}\sum_{h=1}^{J} \delta W^h F^h(P_\ell^j) - \frac{\delta W^j}{\gamma}, \tag{25}$$

that can be subsequently solved by least squares adjustment, providing estimates of the parameters δW^j. Let us observe once more that in Eq. (25), right-hand side, the third term represents the direct effect of the potential biases, while the second term represents the indirect effect.

This approach is certainly correct but it is based on the availability of the solution $\widetilde{T}$, namely of a fully biased global model, which is not the case, as explained in the introduction. In the next section, we will propose two different solutions, which will make use of existing satellite-only global models and EGM2008.

3 New proposals: a feasibility study

In this section, we will present two new proposals for the solution of the height datum problem, trying to assess their usefulness and feasibility. We will see that the first proposal yields a solution too inaccurate to be useful, while the second one provides a simple and accurate solution.

Since our purpose is just to verify the feasibility of the solutions, we shall work here in full spherical approximation, assuming that data are reduced or may be considered as given on a mean Earth sphere, as for the Stokes problem.

In the frame of this approximation every harmonic function (e.g. T, $\widetilde{T}$, F^j) has a harmonic series development, convergent outside the mean Earth sphere of radius R. For instance, we have

$$T(P) = \sum_{n=2}^{+\infty} \sum_{m=-n}^{n} T_{nm} S_{nm}(P) \tag{26}$$

with

$$S_{nm}(P) = \left(\frac{R}{r}\right)^{n+1} Y_{nm}(\theta, \lambda) \tag{27}$$

and

$$Y_{nm}(\theta, \lambda) = \overline{P}_{nm}(\vartheta) \begin{cases} \cos m\lambda & m \geq 0 \\ \sin |m|\lambda & m < 0 \end{cases} \tag{28}$$

where $P = P(\theta, \lambda, r)$. Moreover the operator M^{-1} now has the form of the Stokes operator S, so that $\gamma'/\gamma = -\frac{2}{R}$,

$$\widetilde{T} = S(\widetilde{\Delta} g), \quad F^j = -\frac{2}{R} S(\chi_j) \tag{29}$$

and

$$\widetilde{T}_{nm} = \frac{R}{n-1} \widetilde{\Delta} g_{nm}, \quad F^j_{nm} = -\frac{2}{n-1} \chi_{j,nm}. \tag{30}$$

In particular, Eq. (21) can now be written in spectral form as

$$n \geq 2, \ |m| \leq n, \ T_{nm} = \widetilde{T}_{nm} - 2 \sum_{j=1}^{J} \frac{\chi_{j,nm}}{n-1} \delta W^j. \tag{31}$$

Let us explicitly observe that, though neither in $\widetilde{\Delta} g$ nor in χ_j the zero and first degree harmonics are zero, nevertheless in $\widetilde{T}$ and in F^j such components are automatically filtered out by the Stokes operator.

3.1 Proposal 1

The first idea we have examined is to directly use Eq. (31) as observation equation in a suitable degree range.

In particular, if n is in the range between 100 and 200, we could assume that the biases in EGM2008, caused by the incorporated terrestrial gravity anomalies, dominate the unbiased contribution coming from the GRACE-only model ITG-GRACE03S, as discussed in Sect. 1. So they will contain the effects of the height datum biases and we could use them as $\widetilde{T}_{nm}$ in Eq. (31). On the other hand, in the same range we know with a useful approximation the unbiased coefficients from the new GOCE mission, which provide us with T_{nm} in Eq. (31). This gives us some 30,000 observation equations for which at least the variances of T_{nm} and $\widetilde{T}_{nm}$ are known. By splitting the globe in J areas, we have correspondingly J parameters δW^j to be estimated. A numerical test has been conducted along this line, choosing $J = 11$, as shown in Fig. 4. Let us observe that the design matrix in Eq. (31) has a couple of indices, (n, m), for ordering the observations and an index j (and k) corresponding to the unknown parameters. Accordingly, the entries of the normal matrix of the least squares system are

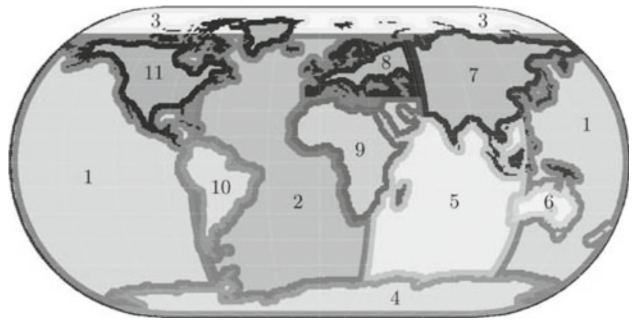

Fig. 4 The eleven patches of the simulation

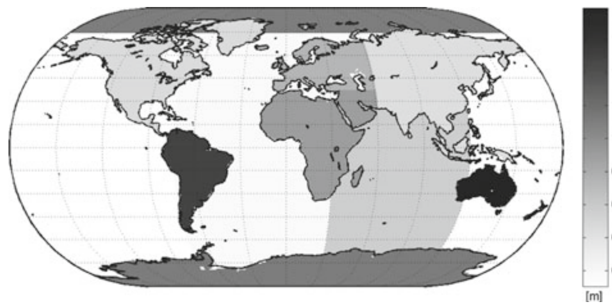

Fig. 5 Estimated standard deviations of the eleven biases

$$N^{jk} = \sum_{n=100}^{200} \sum_{m=-n}^{n} \frac{\chi_{j,nm} \chi_{k,nm}}{(n-1)^2} (\sigma^2_{nm})^{-1} \tag{32}$$

where

$$\sigma^2_{nm} = \frac{1}{4} [\sigma^2(T_{nm}) + \sigma^2(\tilde{T}_{nm})]. \tag{33}$$

As already mentioned, the unitless variances of T_{nm} are taken from the satellite-only model GOCO02S (http://www.itsg.tugraz.at/research/goco, Pail et al. 2010), scaling that result in such a way that the overall rms of the error in geoid is about 3 cm, which we believe to be a conservative final goal for the GOCE mission. Here only variances have been propagated when evaluating the geoid error. Moreover, the unitless variances of $\tilde{T}_{nm}$ are taken from the coefficients file of EGM2008 model (http://earth-info.nima.mil/GandG/wgs84/gravitymod/egm2008/index.html, Pavlis et al. 2008).

Let us observe that the obtained normal matrix N shows a large condition number leading to numerically unstable results; in particular there is an eigenvalue very close to zero (of the order of 10^{-8}) corresponding to an eigenvector of ones. This can be reconnected to the fact that the mean value of the biases cannot be estimated without considering the zero degree coefficient. This rank deficiency problem has been solved by adding a pseudo observation forcing the mean value of the biases to be equal to zero. In this way the condition number of N decreases significantly, thus allowing a stable computation of N^{-1} and in particular of the error standard deviations of the estimated biases. The error standard deviations shown in Fig. 5 are of the order of 1 m, i.e. useless for the solution of the height datum problem. Let us remark that such a negative result can be attributed to the functional relation (31), displaying a fainting influence of the parameters δW^j on the "observables" T_{nm}, when n is greater than the figure of 100. As a consequence, we deemed it useless to continue this numerical investigation and we turned to another approach, where the improvement of the information coming from the GNSS height determination can lead to better results.

3.2 Proposal 2

In this case we assume that at a number of permanent GNSS stations $\{P_\ell^j\}$, the value of $\widetilde{\zeta}(P_\ell^j)$ can be directly computed from the known $h_{P_\ell^j}$ and $h_{\widetilde{P}_\ell^j}$, cf. Eq. (24). We also observe that in oceanic areas $\widetilde{\zeta}$ becomes just the biased geoid which can be directly derived from altimetry, once the dynamic height is assumed to be known from independent sources.

At the same time we know that (cf. Eqs. (6), (16) and (17))

$$\widetilde{\zeta}_\ell^j = \overline{\zeta}^j + \frac{T(P_\ell^j)}{\gamma}, \tag{34}$$

with $\overline{\zeta}^j = -\frac{\delta W^j}{\gamma}$, γ being fixed to a constant mean value.

Let us split the anomalous potential into two parts

$$\begin{aligned} T(P) &= T_L(P) + T_H(P) \\ &= \sum_{n=2}^{200} \sum_{m=-n}^{n} T_{nm} S_{nm}(P) + \sum_{n=201}^{+\infty} \sum_{m=-n}^{n} T_{nm} S_{nm}(P). \end{aligned} \tag{35}$$

Then Eq. (34) can be written as

$$\overline{\zeta}^j = \widetilde{\zeta}_\ell^j - \frac{T_L(P_\ell^j)}{\gamma} - \frac{T_H(P_\ell^j)}{\gamma}. \tag{36}$$

On the other hand, we can assume that T_L will be known from a GOCE-only model, possibly integrated with a GRACE-only model, with a commission error in geoid in the range of some centimeters (Pail et al. 2011).

At the same time we claim that, although biased, T_H can be computed from EGM2008 with a negligible error. In fact, recalling Eq. (31), we have

$$\begin{aligned} T_{\mathrm{EGM2008},H} &= T_H + 2\sum_{j=1}^{J} \delta W^j \sum_{n=201}^{+\infty} \sum_{m=-n}^{n} \frac{\chi_{j,nm}}{n-1} Y_{nm} \\ &= T_H + \widetilde{\Delta} W. \end{aligned} \tag{37}$$

Indeed Eq. (37) is totally disregarding the omission and commission error of EGM2008. The former is negligible (a guess computed by continuing the actual error curve of EGM2008 up to degree 10,000 is 0.6 cm), while the latter, in the degree range 201–2,160, is approximatively given in terms of geoid by 4 cm, as directly computed from the EGM2008 error degree variances.

A good measure of the error due to the biases implicit in the relation

$$T_{\mathrm{EGM2008},H} \cong T_H \tag{38}$$

is given by

$$\begin{aligned} \varepsilon_L^J(T_H)^2 &= \frac{1}{4\pi} \int (\widetilde{\Delta} W_H)^2 \mathrm{d}\sigma \\ &= 4 \sum_{n=L}^{+\infty} \sum_{m=-n}^{n} \frac{\left(\sum_{j=1}^{J} \delta W^j \chi_{j,nm} \right)^2}{(n-1)^2}. \end{aligned} \tag{39}$$

Now, the evaluation of Eq. (39) could be done in different ways, yet a certainly conservative choice is

$$\varepsilon_L^J(T_H)^2 \leq 4 \left(\sum_{j=1}^{J} (\delta W^j)^2 \right) \sum_{j=1}^{J} \sum_{n=L}^{+\infty} \sum_{m=-n}^{n} \frac{\chi_{j,nm}^2}{(n-1)^2}. \tag{40}$$

If we assume Eq. (3) to hold, we have a fortiori,

$$\varepsilon_L^J \left(\frac{T_H}{\gamma} \right) \leq 2\sqrt{4J} \sqrt{ \sum_{j=1}^{J} \sum_{n=L}^{+\infty} \sum_{m=-n}^{n} \frac{\chi_{j,nm}^2}{(n-1)^2} } = B_L^J \quad [\mathrm{m}]. \tag{41}$$

Once the splitting of the globe in patches is performed, Eq. (41) can be easily computed. So, taking an upper limit at degree 1,600 and $J = 11$ (beyond which the contribution of the series becomes negligible), Eq. (41) provides the estimate

$$\varepsilon_{201}^{11} \left(\frac{T_H}{\gamma} \right) \leq 0.49\,\mathrm{cm} \tag{42}$$

clearly showing that the approximation (38) is fully acceptable in the accuracy range in which we are working. Similar results are displayed in Tables 1 and 2 for different values of L and J. On the other hand, the formal commission error of EGM2008 between degrees 201 and 2,160 is 4 cm in terms of

Table 1 Error budget of $\frac{1}{\gamma} T_{\mathrm{EGM2008}}$ ($L = 101$, rms in cm)

	C_{2-100}	$C_{101-2,160}$	B^J_{2-100}	$B^J_{101-2,160}$	Total
$J = 5$	3.7	7.4	–	0.75	8.3
$J = 11$	3.7	7.4	–	1.20	8.4
$J = 23$	3.7	7.4	–	1.93	8.5

Table 2 Error budget of $\frac{1}{\gamma}[T_{L,\mathrm{GOCO}} + T_{H,\mathrm{EGM2008}}]$ ($L = 201$, rms in cm)

	C_{2-200}	$C_{201-2,160}$	B^J_{2-200}	$B^J_{201-2,160}$	Total
$J = 5$	3.0	4.0	–	0.30	5.0
$J = 11$	3.0	4.0	–	0.49	5.0
$J = 23$	3.0	4.0	–	0.76	5.0

geoid, so that, putting everything together, we could say that Eq. (36) can be taken as direct observation equation of $\overline{\zeta}^j$, with errors with a standard deviation of the order of 5 cm. Naturally, averaging on many observation points can only have a beneficial effect on the estimate of $\overline{\zeta}^j$, although a more precise assessment of the final error would require a more thorough study of the covariances of T_L and T_H. This is beyond the scope of the present work.

To be more precise let us compare the error budget that we expect from the use of the GOCO model combined with the high degrees of EGM2008 model, with respect to the pure EGM2008. In Tables 1 and 2, $C_{L_1-L_2}$ stands for the commission error between degrees L_1 and L_2, while $B^J_{L_1-L_2}$ stands for the mean square error of the bias term between degrees L_1 and L_2 when considering J patches.

The choice of degree 200 to cut the low with respect to the high degree part of T has been performed by roughly equating the error of GOCO (or, better, what we expect the error of GOCO to be at the end of the GOCE mission) with that of EGM2008. The neat result is that the use of the GOCE model yields a reduction in the point-wise error by a factor of almost two.

4 Conclusions

The problem of estimating the height datum biases is re-discussed in the paper with the aim of finding a feasible and simple solution by exploiting existing models (specifically EGM2008) and new satellite-only gravity models as those derived from the GOCE mission. Two proposals have been investigated. The first consists in using the anomalous potential coefficients only, in a degree range (e.g. $100 \le n \le 200$) where the global combined model is certainly biased, while unbiased coefficients are available from the GOCE mission. Such an approach is proved to be too poor to give reliable results. Already a classical book like Heiskanen and Moritz (1967) suggests the use of an independent additional geometrical observation. This is done in the second approach, where biased height anomalies are derived from GNSS ellipsoidal heights and biased normal heights. This was considered long ago, e.g. in Rummel and Teunissen (1988). Nevertheless, the remark here is that if in the relevant equations the anomalous potential is split into unbiased low degree part ($n \le 200$), e.g. derived from the GOCO model, and a biased high degree part derived from EGM2008 ($n \ge 201$), then the total error committed is in the range of 5 cm which can be even reduced by exploiting several GNSS stations in each patch. This leads to the new simplified observation Eq. (36), with which further experiments, simulated and realistic, will be performed in future.

A final comment is in order here, summarizing the existing discussion on the matter. The approach outlined above is for the moment only aiming at answering to the question whether with the existing data, including a GOCE global model, one can hope to solve the high datum problem with convenient accuracy. The approach devised is not the final proposal on how to perform the computations, which will probably require a better integration of GOCE, GRACE and e.g. EGM2008 into a new combined global model, but just an a priori evaluation of the accuracy that such an operation can provide. Moreover we have to underline that the whole line of thought in which we are working here is that of BVP solutions with additional parameters. In particular this means that possibly other bias sources can intervene changing the values of the average of Δg in some areas. Such different biases cannot be discriminated in this approach where they are cumulatively estimated. It is also worth recalling that indeed the bias estimation can be applied also on a local level, as described for instance in Sansò and Venuti (2002) and more recently widely shown in Kotsakis et al. (2012).

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