Simple Layer Model of the Geopotential in Satellite Geodesy

KARL-RUDOLF KOCH

Institut für Theoretische Geodäsie der Universität Bonn
Bonn, Germany

Abstract. The method of satellite geodesy that represents the earth's gravity field by the potential of a simple layer is reviewed. This model of the geopotential, which has been applied to optical satellite observations and Doppler data, is compared with different representations of the gravity field.

The representation of the earth's gravitational field in satellite geodesy by means of the potential of a simple layer has now been applied to optical satellite observations [*Koch and Morrison,* 1970] and to Doppler data [*Koch and Witte,* 1971]. Since the first proposals of this model of the geopotential [*Koch,* 1968*a*, *b*], the method of its application has been slightly modified because of the experience gained with practical computations. A short outline of the method as it is used now is therefore given here.

REPRESENTATION OF THE GRAVITY FIELD

The potential of gravity of the earth W is divided into the potential U, which is known and is expressed by an expansion into spherical harmonics, and into the potential T, which is unknown and is represented by a potential of a simple layer distributed over the surface of the earth. Hence

$$W = U + T \tag{1}$$

where

$$U = \frac{kM}{r}\left[1 + \sum_{n=2}^{n_c} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\phi) \cdot(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\right] + (\omega^2 r^2 \cos^2\phi)/2 \tag{2}$$

Equation 2 is the well-known expansion into spherical harmonics whose coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are taken from available results. The term with the angular velocity ω of the earth stems from the centrifugal force and equals zero outside the earth; n_c is an integer greater than 2; and r, ϕ, λ are geocentric coordinates.

The potential T is given by

$$T = \iint_\Sigma \frac{\Phi}{l}\, d\Sigma \tag{3}$$

where Φ is the unknown density of the simple layer distributed over the surface Σ of the earth, and l denotes the distance between the fixed point and the moving point at Σ. We introduce the auxiliary density χ by

$$\chi = \Phi/\cos(f, r) \tag{4}$$

where (f, r) denotes the angle between the normal f to the earth's surface Σ and r. With the projection dE of the surface element $d\Sigma$

$$dE = d\Sigma \cos(f, r) = r^2 \cos\phi\, d\phi\, d\lambda \tag{5}$$

we obtain, instead of (3),

$$T = \iint_E \frac{\chi}{l}\, dE \tag{6}$$

The integral in (6) is now replaced by a summation over p surface elements ΔE_i of approximately equal size bordered by meridians and parallels. We assume constant densities χ_i for ΔE_i and obtain

$$T = \sum_{i=1}^{p} \chi_i \iint_{\Delta E_i} \frac{dE}{l} \tag{7}$$

The density values χ_i for the surface elements ΔE_i are the unknown parameters of the geopotential. They are determined from satellite observations by a numerical orbit theory based on a differential correction process. This means that the equation of motion of a satellite is integrated numerically, together with the variational equations.

COMPUTATIONAL PROCEDURE

To reduce the influence of errors due to the approximation of the integral (6) by the sum (7), the potential T should be small. Hence, the expansion 2 has to be extended to a high degree and order. However, an upper limit of the degree of the expansion is given by the shortest wavelength of the harmonics in (2), which should be compatible with the sidelength of the surface elements ΔE_i. In addition to these coefficients, the resonant terms of the satellites being analyzed have to be included, in order to get optimal orbit fits.

The integral over the surface element ΔE_i in (7) is solved numerically by subdividing ΔE_i into four elements for whose midpoints the kernel of the integral is assumed to be constant. This method of numerical integration can be interpreted as using point masses instead of a surface layer [*Morrison*, 1971]. This integration procedure is correct if, depending on the position of the satellite, the right point at the surface element is chosen. Since the points are fixed, errors in the numerical integration of (7) cannot be avoided. However, they only enter the variational equations because, by introducing zero densities as preliminary values into (7), the potential U in (2) alone is used to compute the nominal orbits.

For the numerical integration, the surface of the earth has to be defined. This is done by means of (2). An approximate expression for the geopotential U_0 at sea level is obtained from the equation that holds for the potential at the surface of a level ellipsoid

$$U_0 = \frac{kM}{(a^2 - b^2)^{1/2}} \text{ arc tan } \frac{(a^2 - b^2)^{1/2}}{b} + (\omega^2 a^2)/3 \qquad (8)$$

The values for kM, a, and ω are taken from (2), and the semiminor axis b is computed from the value of $\bar{C}_{20}$ used in (2). Setting $U = U_0$ in (2) and solving for r, the radius vector of the equipotential surface at sea level is obtained. To these values are added the topographic heights, which are published for 5° by 5° elements by *Kaula et al.* [1966].

To convert the density values into normalized harmonic coefficients, one uses [*Koch*, 1968*b*]

$$\bar{C}_{nm} = \bar{C}_{nmu} + \frac{1}{(2n+1)kMa^n} \sum_{i=1}^{P} \chi_i \cdot \iint_{\Delta E_i} r^n \bar{P}_{nm}(\sin \phi) \cos m\lambda \, dE$$

$$\bar{S}_{nm} = \bar{S}_{nmu} + \frac{1}{(2n+1)kMa^n} \sum_{i=1}^{P} \chi_i \cdot \iint_{\Delta E_i} r^n \bar{P}_{nm}(\sin \phi) \sin m\lambda \, dE \qquad (9)$$

where $\bar{C}_{nmu}$ and $\bar{S}_{nmu}$ are the harmonic coefficients of (2). The integral over ΔE_i is solved numerically by subdividing ΔE_i into nine elements. If the coordinate system of the orbit integration is a geocentric one, the density values χ_i have to be determined under the constraint that the first-degree harmonics equal zero.

The product kM of the gravitational constant and the mass of the earth is well known from space probes. Hence the transformation 9 should give a harmonic coefficient of zero degree that equals zero. If the coefficient differs from zero, it does so because the equatorial radius of the earth, and therefore U_0, are only approximately known. With a relative error of one part in 300, we can write instead of (8)

$$U_0 = kM/a$$

so that the correction Δa of a is obtained from

$$\Delta U_0 = -(kM/a^2) \, \Delta a \qquad (10)$$

Detailed information about the computational procedures for determining density values from Doppler data is given by *Witte* [1971].

COMBINATION WITH GRAVITY ANOMALIES

Knowledge of the earth's gravity field comes not only from satellite orbits but also from gravity measurements. If Δg are the gravity anomalies referred to the potential U in (2), the density values χ are obtained by [*Koch*, 1968*b*]

$$\chi = \frac{\Delta g - G}{2\pi} + \frac{3}{(4\pi)^2} \cdot \iint (\Delta g - G) S(\psi) \cos \phi \, d\phi \, d\lambda \qquad (11)$$

where

$$G = \frac{1}{4\pi} \int\int \Delta g \cos \phi \, d\phi \, d\lambda$$

if the surface of the earth is approximated by the surface of a sphere. $S(\psi)$ is Stokes' function. For the combination with the satellite results, the covariances of χ have to be computed according to (11) from the given variances of Δg [*Koch,* 1970].

COMPARISON WITH DIFFERENT MODELS

A similar model of the geopotential is obtained if, instead of density values, gravity anomalies are used as parameters of the gravity field [*Arnold,* 1968; *Obenson,* 1970]. With such a representation, the reciprocal distance between the fixed point and the moving point in (7) is replaced by Stokes' function, which contains sine, cosine, and logarithmic functions. The evaluation of the integral corresponding to (7) is therefore more time consuming than (7), which is especially disadvantageous when satellite orbits are integrated numerically. Another point is that the surface of the earth is approximated by the surface of a sphere. This approximation is avoided with (7). However, when combining satellite and gravimetric results, the gravity anomalies can be used directly, so that transformation 11 and the computation of covariances are unnecessary.

Vinti [1968] suggested representing the potential T by the potential of a simple layer distributed over a sphere completely enclosing the earth. This proposal causes some difficulties when gravity measurements are combined with satellite results. Either the gravity anomalies have to be continued upward from the surface of the earth to the surface of the sphere or the density values have to be transferred downward from the surface of the sphere to the surface of the earth.

To determine the gravity field of the moon, *Wong et al.* [1969] expressed the potential T by point masses buried under the lunar surface. This model tries to account for the lunar mass concentrations, the so-called mascons. If applied to the earth as has been proposed in the past [*Weightman,* 1967], this model again would be difficult to use for the combination of gravity anomalies with satellite observations, since point masses close to the surface of the earth will produce singularities at the surface.

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