

A least-squares collocation procedure to merge local geoids with the aid of satellite-only gravity models: the Italian/Swiss geoids case study

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(Received: April 15, 2013; accepted: July 16, 2013)

ABSTRACT Neighbouring countries often have national geoids that do not fit to each other, typically showing a discontinuity along the border. Among other effects, this discontinuity is mainly due to the different height datum, producing biased local geoids which can also have different accuracies and spatial resolutions. In some applications, for instance in case of international civil engineering works, a merging between two neighbouring geoids can be necessary. Obviously this procedure cannot be done by simply averaging overlapping areas completely disregarding biases. This paper deals with this problem in connection to the availability of data from satellite gravity missions. In contrast to terrestrial gravity anomalies, gravity and geoid models derived from satellite gravity missions, and in particular from GRACE and GOCE, do not suffer from those inconsistencies. These models in fact are not affected by local biases (local reference systems) since they do not make use of any ground gravity data or levelling. Basically this means that these models can provide the long wavelengths of the resulting merged geoid, in this way removing national biases or other systematic effects. On the other hand, the short wavelengths will directly come from a combination of the available local geoids. This article proposes a least-squares collocation procedure to merge local geoids with the help of these satellite-only gravity models. Even if the correct approach to produce a unique unbiased geoid is to start from the original terrestrial gravity data together with satellite data, the presented procedure can be helpful to merge already available local models. After a review of the mathematical formulation of the problem, the paper illustrates the case of the merging of the Italian and Swiss geoids, more specifically the Swiss CHGeo2004 and the Italian ITALGEO2005 pure gravimetric local models. A constant bias with respect to the GOCE reference (WGS84 ellipsoid) of about 100 cm for the Italian local geoid and of about 80 cm for the Swiss one have been estimated and removed. After that a unique geoid with an accuracy of few centimetres has been computed by collocation. A first application of this new geoid, named GISgeo2012 (GOCE, Italian and Swiss geoid) will be within the interreg project Helidem (HELVetia-Italy Digital Elevation Model) to create a new unified digital elevation model in orthometric height.

Key words: height datum, GOCE, EGM2008, local geoids.

1. Introduction

As it is well known in literature, local geoid models are generally affected by biases (Colombo, 1980; Rummel and Teunissen, 1988; Xu, 2007). These biases are mainly due to the fact that each national or regional levelling network has a reference point defined by the mean sea level observed at a given tide gauge, in a certain period of time. Since the sea level varies from place to place, up to 2 m, due to global variations in the mean dynamic topography (see: Pugh, 2004), there can be jumps between national height systems. This is usually referred as the height datum problem.

The biased heights enter into the computation of terrestrial gravity anomalies which in turn are used to determine the geoid. For this reason a national geoid can be inconsistent with the one of the neighbouring country presenting a jump along the border. Note that these inconsistencies can have very complicated features determined not only by the height datum problem but also by the different data manipulation and methods (e.g., Stokes solution rather than collocation) used to compute the geoids, by the effect of a different local reference ellipsoid and by border effects. Obviously all these contributions should be taken into account and properly removed when creating a unique geoid model for more than one country.

In recent years, the need of a worldwide reference system has brought the scientific community to study the various aspects of the establishment of a common reference datum, and consequently a common reference equipotential surface, giving particular attention to the role played by space techniques that are now improving the knowledge of the Earth gravity field. Some literature about this item is reported for example in Sánchez (2008).

In this framework an important contribution has been given by the ESA GOCE (Gravity field and Ocean Circulation Explorer) satellite mission. In fact its data are independent from local height reference systems (Rummel, 2002; Gerlach and Rummel, 2013) and will allow to estimate the geoid with an accuracy of 2 cm at 100 km space resolution. In addition GOCE provides highly accurate geopotential numbers and a consistent way to refer to the same datum all the relevant gravimetric, topographic and oceanographic data (Rummel *et al.*, 2011).

Finally GOCE observations are homogeneous, both for acquisition method and spatial distribution, and give information also in regions where ground gravity data are poor or unavailable [e.g., part of South America, Africa and the Himalayas (Bomfim *et al.*, 2013)]. Apart from satellite-only models, a set of high resolution global combined models, computed from ground, shipborne and satellite gravity observations are available; between them the Earth Gravity Model 2008 (EGM2008) model (Pavlis *et al.*, 2012) is probably the most important and widely used. However these models containing biased ground-gravity data are certainly influenced by height system inconsistencies, even if only above a certain degree (Gatti *et al.*, 2013).

The procedure presented here faces the problem from a local point of view using a GOCE satellite-only gravity model to merge the geoids of two neighbouring countries. In details the algorithm is based on a least-squares estimation of the biases affecting the local models and their removal, followed by a standard collocation to merge the computed unbiased data set into a unique geoid. In each step, particular attention is given to the modelling of the covariance matrices of the involved quantities. The concept is the one of least-squares collocation with parameters (Moritz, 1989) and similar approaches are illustrated in literature, for instance in

Tscherning (2001), Fotopoulos (2005), and in particular in Pail *et al.* (2010). At the base of the implemented methodology there is a frequency analysis of the gravimetric signal: local models, computed on the basis of ground and shipborne gravity data, contain information related to all the frequency spectrum while GOCE models, expressed as a harmonic development truncated at a certain maximum degree, which is implicitly defined by the accuracy and the spatial distribution of the data used to build the model itself, give accurate information just at low degrees. It should be underlined that the cleanest way to produce a unified geoid, also including GOCE data, is definitely to start from the original ground gravity observations instead of the local geoids that can contain many additional systematic effects due to their computation. Nevertheless it should be stressed that the full data set of the ground gravity observations that have been used to compute the local geoids is in general not accessible. Therefore the proposed strategy, though sub-optimal, is a practical answer to the problem of straightening and merging local geoids of neighbouring countries.

The developed procedure has been applied to merge the Italian and the Swiss gravimetric local geoids. The prediction is made on the overlapping area between the Italian and the Swiss geoids. The limits of the considered area are: 6.5° E, 11.0° E, 44.5° N and 47.0° N. The choice of this area is based on the fact that it is where a merged geoid was requested within the interreg Helidem (HELvetia Italy Digital Elevation Model) project which is somehow related to this research.

2. Mathematical formulation of the problem

The methodology presented in this work is divided into two steps: the first one consists in a least-squares adjustment for biases estimation, the second one is a standard collocation procedure to merge local geoids. Note that this is equivalent to apply a kriging solution being the variance-covariance matrices used in the two steps always the same (Sansò and Tscherning, 1980).

The proposed procedure is based on the use of a global satellite-only gravity model, expressed as a truncated series of spherical harmonic coefficients, to merge local grids of geoid undulation. First of all, the anomalous potential spherical harmonic coefficient vector \underline{T} , which is in principle infinite dimensional, can be split into two parts: \underline{T}^L , containing the low frequencies up to a maximum degree L (that can be retrieved by the satellite-only global model) and the remaining high frequencies \underline{T}^H :

$$\underline{T} = \begin{bmatrix} \underline{T}^L \\ \underline{T}^H \end{bmatrix}. \quad (1)$$

The following system can be built:

$$\begin{cases} Y_1^0(\underline{x}_1) = A(\underline{x}_1)\underline{T} + B_1 t_1 + v_1(\underline{x}_1) \\ Y_2^0(\underline{x}_2) = A(\underline{x}_2)\underline{T} + B_2 t_2 + v_2(\underline{x}_2) \\ \underline{T}^{0L} = \underline{T}^L + \underline{e} \end{cases} \quad (2)$$

where Y^0 is the observation vector, in this study geoid undulations, subscripts 1 and 2 refer to the first and to the second local model respectively, \underline{x} is the coordinate vector collecting all the observation points, generally with \underline{x}_1 different from \underline{x}_2 , A is the linear operator going from spherical harmonic coefficients to geoid undulation (i.e., harmonic synthesis), B is the bias deterministic model as a function of the parameter vector \underline{t} and ν is the observation error of the local data set. Finally \underline{e} is the estimated error of the satellite-only spherical harmonic coefficients \underline{T}^{0L} with covariance matrix C_{ee} .

The contribution of the low frequency signal given by the global model is removed from the observations obtaining:

$$\begin{cases} r_1(\underline{x}_1) = Y_1^0(\underline{x}_1) - A^L(\underline{x}_1)\underline{T}^{0L} = A^H(\underline{x}_1)\underline{T}^H - A^L(\underline{x}_1)\underline{e} + B_1\underline{t}_1 + \nu_1(\underline{x}_1) \\ r_2(\underline{x}_2) = Y_2^0(\underline{x}_2) - A^L(\underline{x}_2)\underline{T}^{0L} = A^H(\underline{x}_2)\underline{T}^H - A^L(\underline{x}_2)\underline{e} + B_2\underline{t}_2 + \nu_2(\underline{x}_2) \end{cases} \quad (3)$$

where, analogously to Eq. (1), A^L and A^H represent a splitting of the linear operator A with respect to the low and high frequency part.

Calling s the high frequency signal component and ξ the difference between ν , for the sake of simplicity assumed to be white noise, and the global model low frequency propagated error, the term $A^L(\underline{x})\underline{e}$, Eq. (3) can be rewritten as:

$$\begin{cases} r_1(\underline{x}_1) = B_1\underline{t}_1 + s_1(\underline{x}_1) + \xi_1(\underline{x}_1) \\ r_2(\underline{x}_2) = B_2\underline{t}_2 + s_2(\underline{x}_2) + \xi_2(\underline{x}_2) \end{cases} \quad (4)$$

Eq. (4) represents the least-squares model that is used to estimate the biases of local models, namely the parameter vectors \underline{t}_1 and \underline{t}_2 . In matrix notation it can be written as:

$$\begin{bmatrix} r_1(\underline{x}_1) \\ r_2(\underline{x}_2) \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \underline{t}_1 \\ \underline{t}_2 \end{bmatrix} + \begin{bmatrix} s_1(\underline{x}_1) \\ s_2(\underline{x}_2) \end{bmatrix} + \begin{bmatrix} \xi_1(\underline{x}_1) \\ \xi_2(\underline{x}_2) \end{bmatrix} \quad (5)$$

The bias can be modelled in various forms, e.g., as a constant value or better as a constant plus a component that depends on the position of the point (see: Heiskanen and Moritz, 1967). In this study we have chosen the former solution, therefore the matrices B_1 and B_2 are simply vectors of ones with dimension n_1 and n_2 , where n_1 and n_2 are the number of observations of the first and the second data set respectively.

The cofactor matrix Q of this least-squares model is equal to:

$$Q = \begin{bmatrix} C_{s_1 s_1} & C_{s_1 s_2} \\ C_{s_2 s_1} & C_{s_2 s_2} \end{bmatrix} + \begin{bmatrix} C_{\xi_1 \xi_1} & C_{\xi_1 \xi_2} \\ C_{\xi_2 \xi_1} & C_{\xi_2 \xi_2} \end{bmatrix} \quad (6)$$

where the first term is the covariance matrix of the high frequency signal, while the second one is the sum of two contributions: the covariance of the low frequency error, computed from the

provided coefficient covariance of the global model, plus the covariance matrix of the local geoid which is assumed to be diagonal; as we said before, the error of the local model is in fact assumed to be a white noise in lack of a more precise information. Of course more general, and more realistic, matrices could be taken into account. Therefore we have:

$$\begin{bmatrix} C_{s_1 s_1} & C_{s_1 s_2} \\ C_{s_2 s_1} & C_{s_2 s_2} \end{bmatrix} = \begin{bmatrix} A^H(\mathbf{x}_1) \\ A^H(\mathbf{x}_2) \end{bmatrix} C_{T^H T^H} \begin{bmatrix} A^H(\mathbf{x}_1) \\ A^H(\mathbf{x}_2) \end{bmatrix}^T \quad (7)$$

and

$$\begin{bmatrix} C_{\xi_1 \xi_1} & C_{\xi_1 \xi_2} \\ C_{\xi_2 \xi_1} & C_{\xi_2 \xi_2} \end{bmatrix} = \begin{bmatrix} A^L(\mathbf{x}_1) \\ A^L(\mathbf{x}_2) \end{bmatrix} C_{ee} \begin{bmatrix} A^L(\mathbf{x}_1) \\ A^L(\mathbf{x}_2) \end{bmatrix}^T + \begin{bmatrix} C_{v_1 v_1} & 0 \\ 0 & C_{v_2 v_2} \end{bmatrix} \quad (8)$$

with $C_{v_1 v_1} = \sigma_1^2 I$ and $C_{v_2 v_2} = \sigma_2^2 I$. It is important to underline that this is a strong approximation because a lot of correlations are introduced in computing local geoids with the standard techniques of physical geodesy (Albertella *et al.*, 1994).

The covariance matrix of the high frequency signal C_{ss} can be estimated empirically starting for the residuals r as explained in details in Section 2.1. Note that being Q a full matrix, the estimates of the two bias parameter vectors are correlated.

Applying the least-squares adjustment, the estimated parameters are given by:

$$\hat{\mathbf{f}} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} = (D^T Q^{-1} D)^{-1} D^T Q^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad (9)$$

with

$$D = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}. \quad (10)$$

Once the biases are estimated, they can be subtracted from the residuals of Eq. (5) to obtain “unbiased” residuals:

$$\begin{cases} r_1^U(\mathbf{x}_1) = r_1(\mathbf{x}_1) - B_1 \hat{f}_1 \\ r_2^U(\mathbf{x}_2) = r_2(\mathbf{x}_2) - B_2 \hat{f}_2 \end{cases} \quad (11)$$

that will be the observations of the subsequent collocation approach, as explained in the following, used to estimate a merged unbiased geoid here represented by the vector Y_3 .

Considering that the functional relation between Y_3 and the vector of spherical harmonic coefficients T can be written as:

$$Y_3(\mathbf{x}_3) = \begin{bmatrix} A^L(\mathbf{x}_3) \\ A^H(\mathbf{x}_3) \end{bmatrix}^T \begin{bmatrix} \mathbf{L}^L \\ \mathbf{L}^H \end{bmatrix} \quad (12)$$

one gets the collocation prediction in the form (Pail *et al.*, 2010):

$$\hat{Y}_3(\mathbf{x}_3) = C_{Y_3 r^U} C_{r_1^U r_2^U}^{-1} \begin{bmatrix} r_1^U(\mathbf{x}_1) \\ r_2^U(\mathbf{x}_2) \end{bmatrix} + A^L(\mathbf{x}_3) \mathbf{L}^{0L} \quad (13)$$

where

$$C_{Y_3 r^U} = \begin{bmatrix} A^L(\mathbf{x}_3) C_{ee} (A^L(\mathbf{x}_1))^T + A^H(\mathbf{x}_3) C_{T^H T^H} (A^H(\mathbf{x}_1))^T \\ A^L(\mathbf{x}_3) C_{ee} (A^L(\mathbf{x}_2))^T + A^H(\mathbf{x}_3) C_{T^H T^H} (A^H(\mathbf{x}_2))^T \end{bmatrix} \quad (14)$$

and $C_{r_1^U r_2^U}$ is the same of Eq. (6). The variance covariance matrix of the predicted geoid error $C_{\varepsilon\varepsilon}$ with $\varepsilon = \hat{Y}_3(\mathbf{x}_3) - Y_3(\mathbf{x}_3)$, where $Y_3(\mathbf{x}_3)$ is the true value of the merged geoid computed in the position vector \mathbf{x}_3 , is given by:

$$C_{\varepsilon\varepsilon} = \begin{bmatrix} A^L(\mathbf{x}_3) C_{ee} (A^L(\mathbf{x}_3))^T & 0 \\ 0 & A^H(\mathbf{x}_3) C_{ee} (A^H(\mathbf{x}_3))^T \end{bmatrix} - C_{Y_3 r^U} C_{r_1^U r_2^U}^{-1} C_{r^U Y_3} \quad (15)$$

2.1. Remarks on the estimation of the high frequency signal covariance matrix function

The covariance function of the high frequency signal, C_{ss} , is derived from the empirical variogram of the residuals \mathbf{r}_1 , \mathbf{r}_2 under the assumption that the field has isotropic and homogeneous increments. This assumption practically means that the value of the variogram Γ depends only on spherical distances (ψ) between observation points.

The covariance function is determined from the variogram (Wackernagel, 2003) and not directly estimated because the variogram is independent from constant biases and permits to estimate a unique model for the two geoids. In fact in the overlapping area the variogram is expected to have the same shape if computed by using \mathbf{r}_1 or \mathbf{r}_2 but with different nugget effects. This is because the nugget represents an estimate of the local geoid error variances ($\sigma_{v_1}^2$ or $\sigma_{v_2}^2$) that can be different in the two models.

Summarizing, the procedure is the following: firstly, two empirical variograms are estimated considering the available data sets separately, thus computing two different nugget effects that are removed from the empirical variogram values. Secondly, the resulting empirical variogram clouds are merged and used to estimate a unique variogram model [for further details see: Wackernagel (2003)]. Subsequently the covariance function is computed using the relation:

$$C(\psi) = C(0) - \Gamma(\psi). \quad (16)$$

The obtained covariance function is the sum of two contributions [see: Eq. (6)]: the covariance due to the high frequency signal C_{ss} and the covariance of global model low frequency error $A^L C_{ee} A^L$, assuming these two contributions as uncorrelated. To evaluate the latter term, a homogeneous and isotropic approximation of the global model error covariance function has to be considered. More precisely, first the covariance of GOCE error is computed in the area of interest with an order-wise block diagonal modelling:

$$C_{ee}(\vartheta, \vartheta', \Delta\lambda) = \left(\frac{GM}{R} \right)^2 \sum_m \sum_{l,k} \sigma_{lk,m} \bar{P}_{lm}(\vartheta) \bar{P}_{km}(\vartheta') \cos(m\Delta\lambda), \quad (17)$$

where the constants G , M and R can be found in the GOCE standards document and \bar{P}_{lm} are the fully normalized Legendre functions of degree l and order m . After that the covariance model based on degree variances that best fits the previously determined model is estimated via least-squares adjustment and becomes:

$$\tilde{C}_{ee}(\psi) \equiv \left(\frac{GM}{R} \right)^2 \sum_l \tilde{\sigma}_l^2 P_l(\cos \psi) \quad (18)$$

where P_l are the Legendre polynomials of degree l ,

$$\tilde{\sigma}_l^2 = \frac{A^2}{\left(\frac{GM}{R} \right)^2 \sum_l \sigma_l^2} \sigma_l^2 \quad (19)$$

and $\sigma_l^2 = \sum_m \sigma_{lm}^2$. The constant A^2 represents the mean variance of the model in Eq. (17), computed over the local area under study. In other words the covariance model based on degree variances in Eq. (18) is locally adapted.

Finally the covariance function due to the high frequency signal can be computed removing this contribution, properly propagated through the linear functional A , from $C(\psi)$. Once the theoretical covariance functions are known, the covariance matrices can be easily calculated.

2.2. Remarks on the estimation of the biases

As explained before, the bias estimation is performed by a least-squares solution where the observations are the residuals of Eq. (5). They are obtained removing from the local geoid the contribution of the low frequency signal of the satellite-only global model. Particular attention should be paid when computing this subtraction since local models are usually given in terms of geoid undulation N , while from global models quasi-geoid ζ is usually synthesized. The topographic correction, needed to go from ζ to N , can be evaluated by using the following simple approximated formula (Moritz, 1989):

$$N = \zeta - \frac{2\pi\rho G}{\gamma} H^2 \quad (20)$$

where ρ is the mass density, G is the universal constant, γ the normal gravity and H the orthometric height (m).

If the geoid undulation N is considered, residuals can be computed as:

$$\begin{cases} r_1(\mathbf{x}_1) = N_1(\mathbf{x}_1) - N_{GOCE}^L(\mathbf{x}_1) \\ r_2(\mathbf{x}_2) = N_2(\mathbf{x}_2) - N_{GOCE}^L(\mathbf{x}_2) \end{cases} \quad (21)$$

where N is the geoid value and N_{GOCE}^L the geoid computed making the synthesis of the GOCE model, both quantities evaluated at point \mathbf{x} . There are two considerations that should be done to properly estimate the biases using the presented procedure. The first is that the reasoning holds when the area of interest is sufficiently large, in fact the signal s of Eq. (5) is supposed to be of zero mean and this is verified when the considered region is larger than a certain area that depends on the maximum degree L of the removed global model (e.g., if L is 180, the region should be greater than $1^\circ \times 1^\circ$). The second is that if the variability of the remaining signal is too high, as in the case of the bordering area between Italy and Switzerland, due to the presence of the Alps, the biases cannot be estimated with a sufficient level of accuracy because they are hidden by the variability of the signal itself.

To overcome both problems, a possible solution is to subtract from the local geoid data also the contribution of the high frequency signal that can be taken for example from the EGM2008 global model. Even if EGM2008 is not an unbiased model it can be assumed that the bias contribution at high frequency, from degree 201 on, is globally of the order of 0.5 cm (Gatti *et al.*, 2013) and therefore can be reasonably neglected. In this way the variability of the residual signal decreases significantly (e.g., in the considered area from a standard deviation of 0.88 to 0.27 m and from 1.57 to 0.02 m, for the Italian and Swiss case respectively) and the biases can be estimated with a proper level of accuracy.

To clarify the above approach, a numerical simulation has been performed in the area shown in Fig. 1.

The residuals $r_1(\mathbf{x}_1)$ and $r_2(\mathbf{x}_2)$ have been simulated according to the following equation:

$$\begin{cases} r_1(\mathbf{x}_1) = N_{EGM2008}^{201,2159}(\mathbf{x}_1) + e_{GOCE}^{0,200}(\mathbf{x}_1) + \nu_1(\mathbf{x}_1) + B_1 t_1 \\ r_2(\mathbf{x}_2) = N_{EGM2008}^{201,2159}(\mathbf{x}_2) + e_{GOCE}^{0,200}(\mathbf{x}_2) + \nu_2(\mathbf{x}_2) + B_2 t_2 \end{cases} \quad (22)$$

where $N_{EGM2008}^{201,2159}$ is the high frequency signal of EGM2008 model (practically the omission error of this model after degree 2159 is negligible for the purpose of this simulation), $e_{GOCE}^{0,200}$ is the error of the GOCE contribution at low frequencies, ν is the geoid error (modelled as white noise) and t_1 and t_2 are the scalar biases. In the simulation t_1 is set to zero and t_2 to 7 cm. Such values have been chosen looking at the European Unified Vertical Network (EUVN) website that shows a relative jump of 7 cm between Italy and Switzerland (<http://www.bkg.bund.de>). Two samples of GOCE errors, consistent with $e_{GOCE}^{0,200}$ stochastic characteristics, and geoid white noises have been randomly drawn. Note that the GOCE error samples are correlated even if they are related to different areas (see Fig. 2) and therefore they are drawn as a single sample. More precisely, the sample of $e_{GOCE}^{0,200}$ has been created by applying the Cholesky decomposition of the matrix C_{ee} :

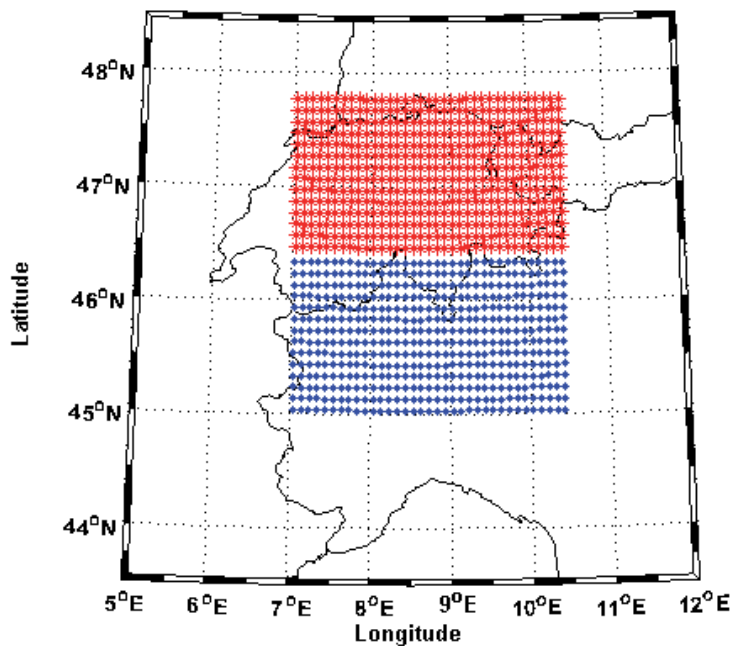


Fig. 1 - Areas considered for the simulation. In red stars and in blue points, the position of the residuals for the Swiss and the Italian local geoid respectively.

$$C_{ee} = L^T L, \quad (23)$$

being L a triangular matrix. Then we have:

$$e_{GOCE}^{0,200} = L^T \underline{\nu} \quad (24)$$

with $\underline{\nu}$ a white noise of unit variance. In this way the variance-covariance matrix of $e_{GOCE}^{0,200}$ is exactly the one provided by the global model considering an order-wise block-diagonal covariance approximation without introducing any further simplification as it was done in Section 2.1. The error sample of the high frequency EGM2008 model $e_{EGM2008}^{201,2159}$ is drawn again applying the Cholesky decomposition (see Fig. 2), but since in this case a detailed description of the stochastic structure of the coefficients error is not available, the variance-covariance matrix is obtained using the coefficient variances and therefore it is a diagonal matrix. Finally the EGM2008 error covariance matrix has been locally adapted by rescaling the variances according to the EGM2008 geoid error map (see: Gilardoni *et al.*, 2013).

The algorithm described in Section 2 has been used to estimate the biases which are -34.69 and 11.91 cm for the Italian and the Swiss regions respectively with a standard deviation in both cases of the order of 260 cm. This high value makes the estimate not significant. In other words the biases cannot be reliably estimated because the variability of the remaining signal is too high, in this way “hiding” the different biases. To numerically prove this concept, we removed from the simulated residuals also the high frequency component obtaining the following observation equation:

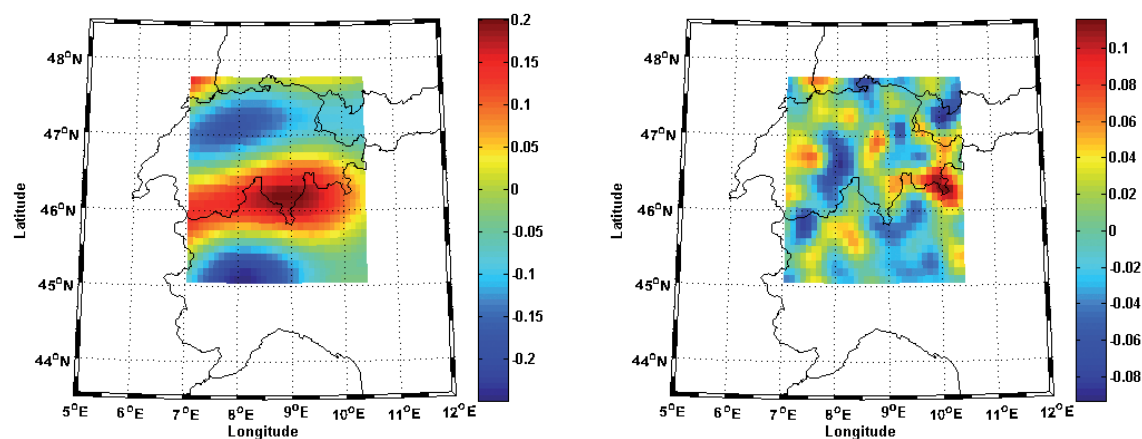


Fig. 2 - The GOCE geoid error sample, in metres, from degree 0 to degree 200 (left image). The EGM2008 geoid error sample, in metres, from degree 201 to degree 2159 (right image).

$$\begin{cases} \tilde{r}_1(x_1) = e_{EGM2008}^{201,2159}(x_1) + e_{GOCE}^{0,200}(x_1) + v_1(x_1) + t_1 \\ \tilde{r}_2(x_2) = e_{EGM2008}^{201,2159}(x_2) + e_{GOCE}^{0,200}(x_2) + v_2(x_2) + t_2 \end{cases} \quad (25)$$

In this case all the quantities of Eq. (25) are the same of Eq. (22) except for the signal of EGM2008 from degree 201 to 2159 that is substituted by its error. Now the biases are estimated with much better accuracy, obtaining biases of 0.99 and 7.04 cm, with a standard deviation of 1.58 and 1.51 cm, for the Italian and Swiss regions respectively. Of course one has to never forget that this estimation could absorb (hopefully in a negligible way) the bias of the EGM2008 global model.

3. Case study

The developed procedure has been applied to merge the Italian and the Swiss gravimetric local geoid. It has been decided not to use the official models, fitted with GPS-levelling data, but the pure gravimetric models to avoid that differences in the orthometric corrections could add further and unmodelled discrepancies between the two geoids.

The Italian gravimetric geoid ITALGEO2005 (Barzaghi *et al.*, 2007) has a resolution of $3' \times 3'$. It is considered within an area with geographic limits: 6.0° E, 11.0° E, 44.5° N and 47.0° N as displayed in Fig. 3.

The Swiss gravimetric geoid CHGeo2004 (Marti, 2007) originally has a resolution of $0.5' \times 0.5'$ and it is shown in Fig. 4. It has been under sampled to have a resolution of $3' \times 3'$ (same as the Italian geoid) and the geographic limits of the considered area are: 5.90° E, 10.50° E, 45.75° N and 47.80° N.

The satellite-only global model used in this work to correct biases and to merge the Italian and the Swiss local geoids is the GOCE SPW R2 model (Migliaccio *et al.*, 2011); more precisely it is a model obtained applying the space-wise approach to the GOCE gravity gradients observations (Migliaccio *et al.*, 2004; Reguzzoni and Tselfes, 2009). The GOCE geoid is shown in Fig. 5. The smoother behaviour with respect to the local geoids is evident.

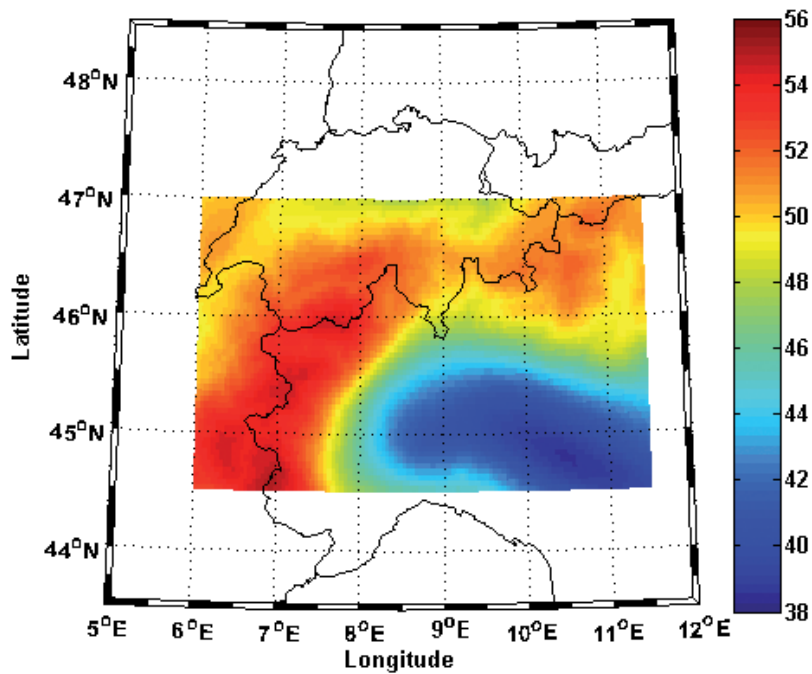


Fig. 3 - The Italian ITALGEO2005 pure gravimetric geoid undulation (in metres).

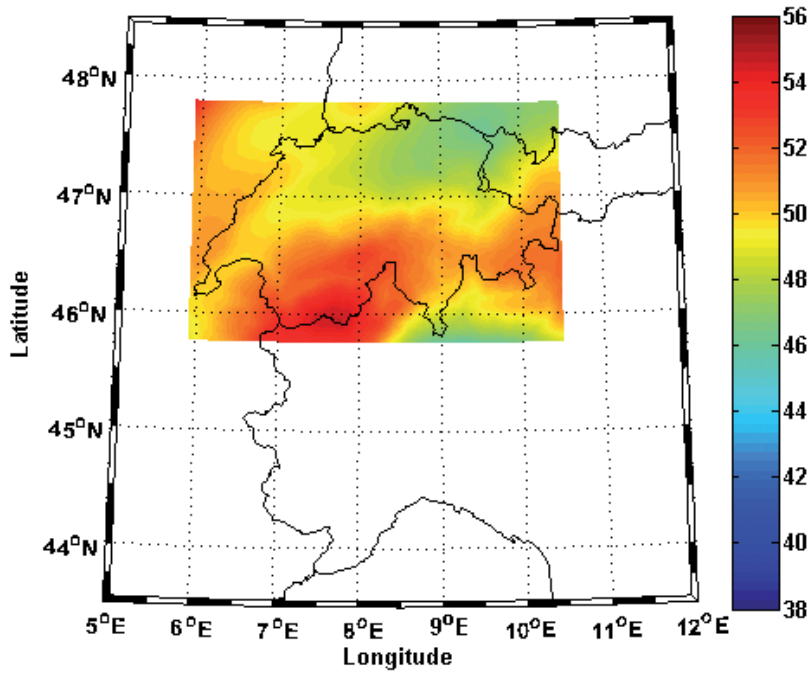


Fig. 4 - The Swiss CHGeo2004 pure gravimetric geoid undulation (in metres).

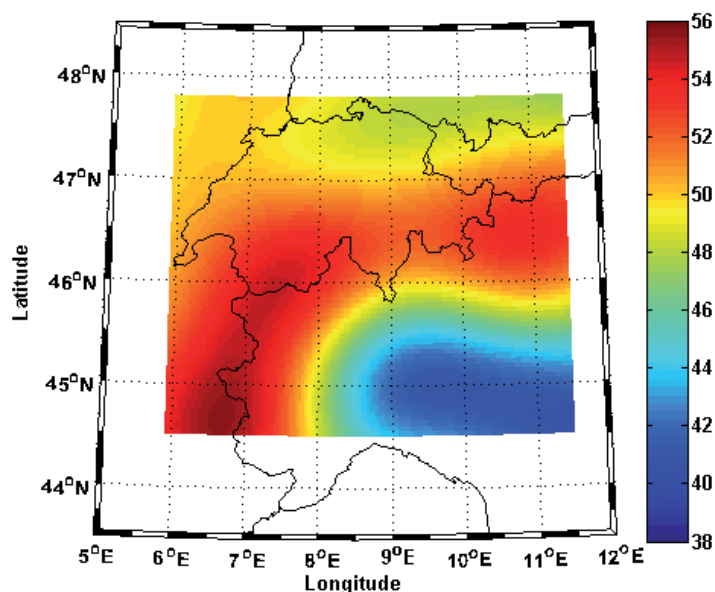


Fig. 5 - The GOCE space-wise R2 gravimetric geoid undulation synthesized from degree 0 to 240 (in metres).

For the biases estimation two cases were considered. In the first case the biases have been estimated considering 200 points for the Italian geoid and 200 points for the Swiss geoid randomly distributed within each national border. In the second case two data sets of gridded data have been used (see Fig. 6). More precisely a grid of 26×117 points for the Italian geoid and a grid of 29×78 points for the Swiss one, both with a resolution of $2' \times 2'$.

Since the quality of each geoid rapidly decreases out of its national border, each geoid does not extend in the other country. According to the simulation described above, residuals have been computed as:

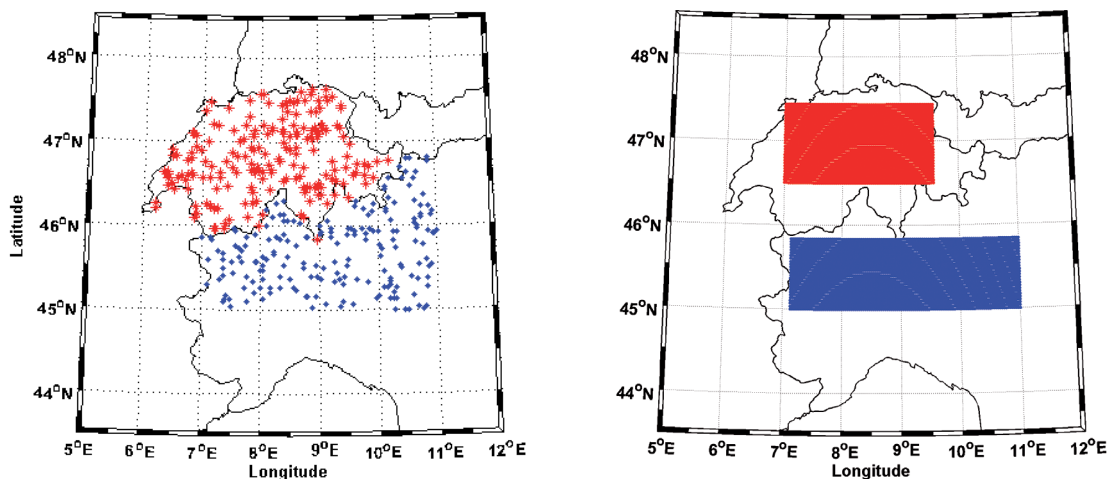


Fig. 6 - Position of the 200 sparse points (left image) and of the gridded data (right image) of the Italian (blue dots) and Swiss gravimetric (red stars) geoids considered to estimate the biases.

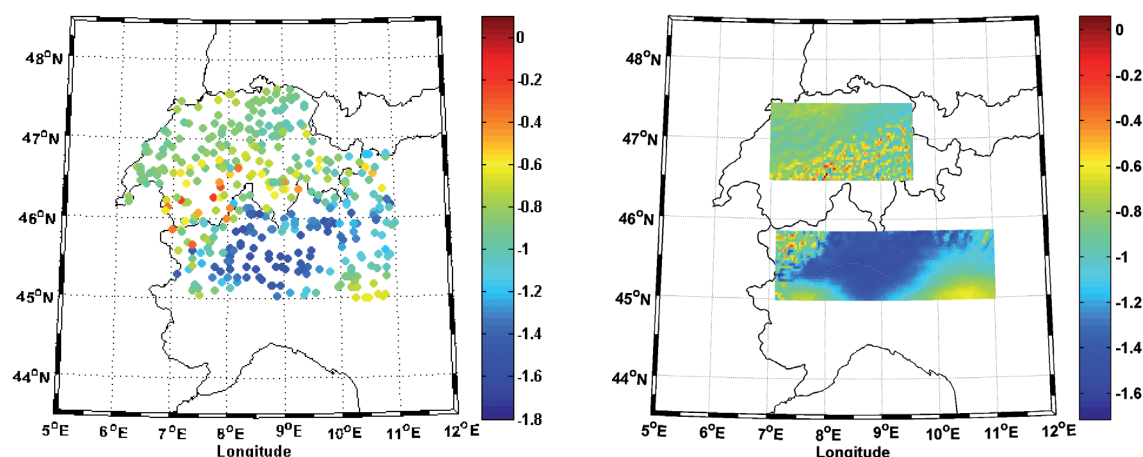


Fig. 7 - Residuals values, in metres, in correspondence to the selected sparse points (left image) and grids (right image).

$$\begin{cases} \tilde{r}_1(x_1) = N_1(x_1) - N_{EGM2008}^{201,2159}(x_1) - N_{GOCE}^{0,200}(x_1) \\ \tilde{r}_2(x_2) = N_2(x_2) - N_{EGM2008}^{201,2159}(x_2) - N_{GOCE}^{0,200}(x_2) \end{cases} \quad (26)$$

and their values are displayed in Fig. 7 for the two cases under study.

The estimated biases are reported in Table 1: it can be noted that the two methodologies bring to the same results (taking into account their error standard deviations), thus confirming the robustness of the solution.

Table 1 - Biases computed using the sparse point data set and the grid data set (in cm).

	Sparse point	Grid data
bias IT	-109.54	-110.89
std bias IT	4.57	5.45
bias CH	-82.98	-85.19
std bias CH	4.19	5.79

As a final step a merged geoid, named GISgeo2012, with its corresponding accuracy is computed by collocation. The estimated biases have been subtracted to the residuals to obtain r_1^U and r_2^U according to Eq. (11). It is important to remind that these residuals contain the high frequency signal (from degree 240 to 2159) because the merging is applied without removing the EGM2008 contribution. The unbiased residuals within the national borders used as observations for the collocation procedure are displayed in Fig. 8. The prediction is made on the overlapping area between the Italian and the Swiss geoid on a grid of 3'×3' resolution as shown in Fig. 9 (limit of the area: 9.0° E, 11.0° E, 44.5° N and 47.0° N). Fig. 10 illustrates the results.

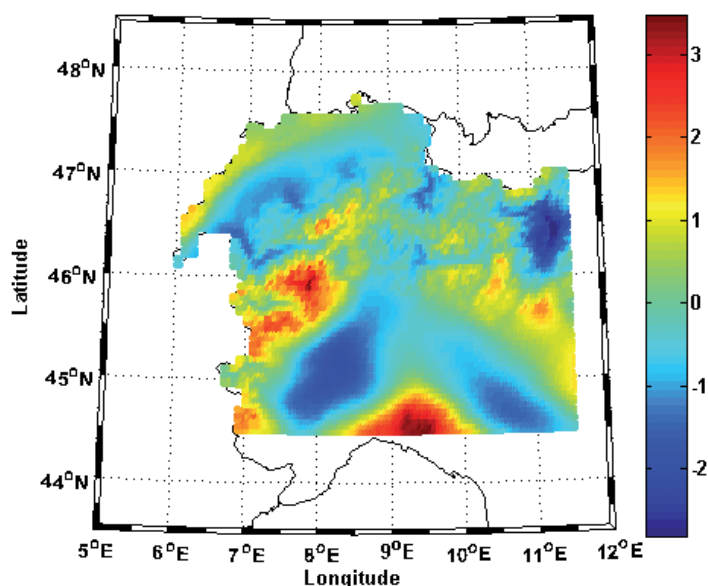


Fig. 8 - Observations used for the collocation procedure. They are the unbiased residuals (in metres) obtained as the local geoid minus the low frequency signal given by GOCE minus the estimated bias.

4. Validation

The quality of the unified geoid GISgeo2012 could be validated with the available GPS-leveling data in Switzerland, kindly provided by Urs Marti (Marti, 2007) and represented in Fig. 11 left image. First of all a value of geoid undulation has been directly derived for each GPS-leveling point, then the gravimetric CHGeo2004 and the GISgeo2012 geoid models have

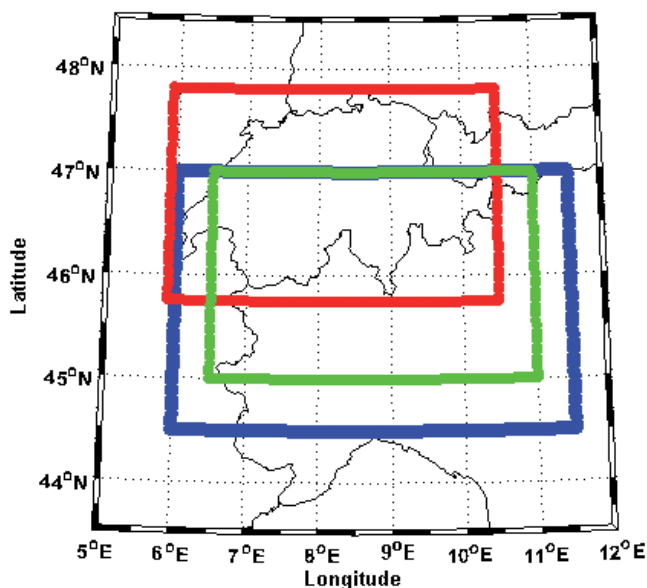


Fig. 9 - Area covered by the Italian geoid (blue rectangle), the Swiss geoid (red rectangle) and the prediction area (green rectangle).

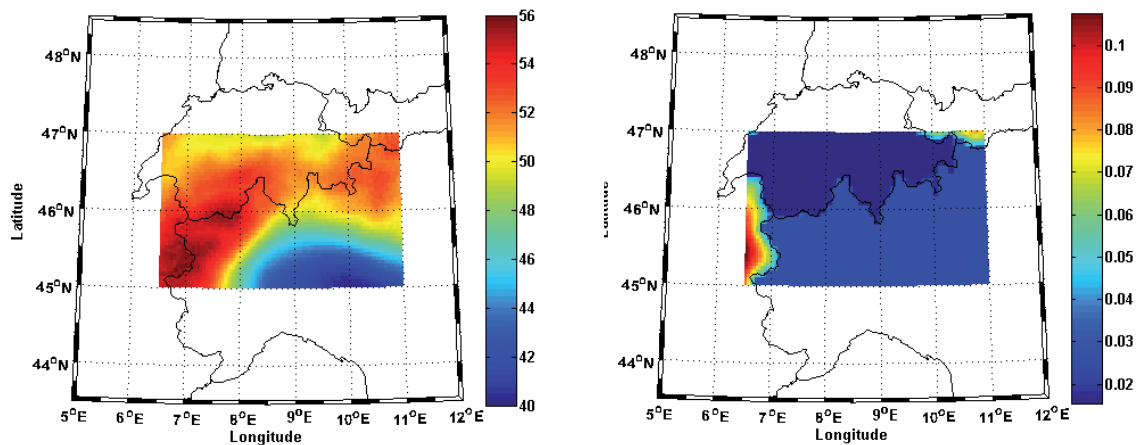


Fig. 10 - GISgeo2012 geoid estimated making a merging of the Italian and Swiss geoid on the borders. The geoid undulation is given in metres (left image). Accuracy, in metres, of the predicted merged geoid GISgeo2012 (right image).

been interpolated on the GPS-levelling points. To compare the data sets the differences of geoid undulation values between couples of points have been considered. In particular for any couple of points P and Q (according to a starlike network structure, shown in Fig. 11, right image), a difference Δ has been computed between the geoid variations obtained from the GPS-leveling and from the local geoid model:

$$\Delta = [N_{GPS}(P) - N_{GPS}(Q)] - [N_{model}(P) - N_{model}(Q)]. \quad (27)$$

In this way the possible presence of constant biases in the models does not affect the statistics. Table 2 reports the results of this comparison showing that the unbiased GISgeo2012 geoid slightly improves the accuracy of the original gravimetric geoid.

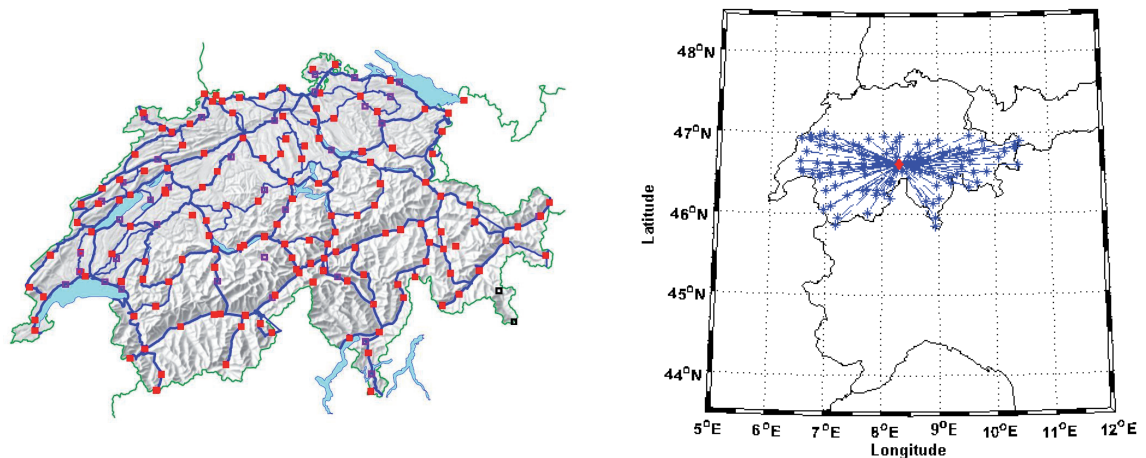


Fig. 11 - The Swiss GPS-levelling network used for the validation (left image). GPS-levelling data used for the validation; the red rhombus indicates the pivot-point of the star network (right image).

Table 2 - Statistics (in centimetres) regarding the differences between the variations from GPS-leveling and the available geoids (the pure gravimetric model CHGeo2004 and the unified GISgeo2012 model).

	Gravimetric model CHGeo2004	Unbiased merged model GISgeo2012
min	-2.67	-19.75
max	16.63	8.13
mean	5.47	2.02
RMSE	0.44	0.28

5. Remarks and conclusions

This study is a first investigation on the way different local geoids can be merged using GOCE data. Even if the cleanest way to produce an unbiased local geoid is to merge GOCE gravity gradients with terrestrial gravity measurements, the explained procedure can be useful to merge already available local geoids without repeating all the computations from the original observations that, by the way, are not always of public domain. The numerical simulation and the case study with real data have shown good results in particular leading to estimates of the biases with an accuracy of the order of few centimetres (4.6 and 4.2 for the Italian and the Swiss region respectively) and a unified unbiased geoid with an accuracy comparable with the one of the original gravimetric models. In other words the height datum problem at the local level has been solved.

Some simplifications have been applied in the present study. For example, the bias model is assumed to be a constant while it is clear that it should have a more complicated shape. Further generalizations have to be considered in further studies.

Finally it has to be stressed that the dimension of the area considered is an important factor in the estimate of the bias. When the area is large enough it is probably not necessary to subtract the high frequency signal contribution of EGM2008, but it is sufficient to subtract a GOCE-only geoid. This at least seems to result from a preliminary analysis that will be the object of future researches.

Acknowledgements. This study has been developed on the framework of the GOCE Italy and Helidem projects. Thanks go to Urs Marti and to Riccardo Barzaghi who provided the Swiss and Italian gravimetric (not official) geoids, respectively.

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