LEAST SQUARES PREDICTION OF GRAVITY ANOMALIES, GEOIDAL UNDULATIONS, AND DEFLECTIONS OF THE VERTICAL WITH MULTIQUADRIC HARMONIC FUNCTIONS

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Abstract. Least squares prediction with MQ (multiquadric) functions is conceptually different from least squares prediction using covariance functions. MQ kernels are based on geometric or physical considerations rather than stochastic processes, and were found to be superior to covariance functions in topographic applications. This may be true also for gravity anomalies or other phenomena which result from marginally stationary, or non-stationary random processes. The MQ harmonic kernel is used to develop a formula for estimating the best depth of point mass anomalies as a function of their number and areal extent on a sphere. Functional relationships between geoidal surface parameters are developed which provide linear equation analogs for the solution of Stokes and Vening Meinesz Integral Formulas as well as for the inversion of these classic problems. These relationships are extended to solutions at exterior equipotential surfaces.

Introduction

The following equation is the general form of a MQ (multiquadric) series as used previously for topography and other irregular surfaces [Hardy, 1970, 1971]

$$\sum_{j=1}^{n} C_j \, [Q(X,Y,X_j,Y_j)] = f(X,Y)=Z \qquad (1)$$

$Q(X,Y,X_j,Y_j)$ is a kernel function defined as a quadric surface (e.g. cone, hyperboloid) in the variables X,Y with its axis of symmetry located at $j=1,\ldots,n$ nodal coordinates X_j,Y_j. The C_j's are the associated, undetermined coefficients.

Inserting $i=1,\ldots,m$ data sets X_i,Y_i,Z_i into (1) results in m equations for the n coefficients. For m=n, the C_j's are computed from a system of n linear equations, while for m>n they are computed by least squares such that

$$\sum_{i=1}^{m} dZ_i^{\,2} = \min \qquad (2)$$

The determined coefficients are substituted into (1) and produce a summation formula for evaluating Z at any X,Y.

It can be shown that (1) is identical, mathematically, with what is referred to as least squares prediction [Kaula, 1963; Moritz, 1965]. A classical text book presentation was given by Heiskanen and Moritz [1967], where the covariance function has been said to be necessary for "optimum least squares prediction." Some authors [Schut, 1974; Rauhala, 1974; Assmus and Kraus, 1974] have assumed that MQ functions belong to the class of covariance functions. These assumptions may have resulted from the fact that multiquadric equations perform well for interpolation, and the mathematical formulation for the solution of the undetermined coefficients is identical.

However, there exists an important conceptual difference in the MQ approach to least squares prediction as compared with other methods. MQ kernels are based on geometric or physical considerations rather than stochastic processes. It is not required that they fit empirical covariances. Moreover some quadric kernels cannot be covariance functions even on a theoretical basis. This can be proven by computing the spectral density, which fails to be non-negative for the cone and hyperboloid kernels. A non-negative spectral density is required for a covariance function, as reported by [Yaglom, 1962]. Yet we have found [Hardy and Göpfert, unpublished data, 1975] that these non-covariance multiquadric kernels are computationally more efficient and produce better interpolations with respect to real topography than procedures involving covariance functions derived from empirical, discrete covariance data. Therefore we are confident that there exists a broader class of functions than covariance functions which are relevant to optimum least squares prediction of topography. A similar statement may apply to prediction of gravity anomalies also. Williamson and Gaposchkin [1973], for example, have shown that gravity is not strictly stationary in the sense of stationary random functions, which is at the heart of the justification for the covariance method. We are currently investigating these implications along with the study of MQ harmonic functions as described below.

The Basic MQ (Multiquadric) Harmonic Function

The MQ concept has been expanded to applications in physical geodesy [Hardy, 1974]. The approach involved an expansion of (1) into a function of three variables X,Y,Z,:

$$f(X,Y,Z) = \sum_{j=1}^{n} C_j \, [Q(X,Y,Z,X_j,Y_j,Z_j)] \qquad (3)$$

It is well-established that the disturbing potential T can be expressed as a summation of discrete point mass anomalies C_j, located at coordinates X_j, Y_j, Z_j. One can interpret this also as the MQ harmonic series

$$T = G \sum_{j=1}^{n} C_j[(X-X_j)^2 + (Y-Y_j)^2 + (Z-Z_j)^2]^{-\frac{1}{2}} \quad (4)$$

where the harmonic kernel is equal to the reciprocal distance function. In (4) we have lost the identity of the kernel function with a classical quadric surface as in (1), but the broader meaning of quadric, i.e. "an expression of the second degree in all its terms," is still applicable. Thus the multiquadric harmonic series may be defined generally as "a series of expressions of the second degree in all its terms, satisfying the Laplace differential equation."

In spherical coordinates (4) becomes

$$T = G \sum_{j=1}^{n} \frac{C_j}{\ell_j} \quad (5)$$

in which

$$\ell_j = (R^2 + r^2 - 2Rr \cos\psi_j)^{\frac{1}{2}} \quad (6)$$

and, in turn

$$\cos\psi_j = \cos\theta\cos\theta_j + \sin\theta\sin\theta_j \cos(\lambda-\lambda_j) \quad (7)$$

In (6), R is the earth's mean radius, and $r<R$ is the radius of the sphere on which the point mass anomalies are located. In (7), ψ_j is the spherical distance involving variable spherical polar distance θ, and longitude λ with respect to discrete nodal coordinates θ_j, λ_j.

Best r-formula

A basic problem associated with the buried point mass concept has been the determination of a useable depth of burial (R-r) as related to the surface area/nodes ratio [Needham, 1970; Balmino, 1972]. Therefore we will discuss a MQ solution to the problem, prior to a discussion of the MQ approach to geodetic problems.

Space does not permit giving the details of the derivation of what we feel justified in calling the best r-formula. Details will be given later in a final report to the National Science Foundation concerning this project. Meanwhile the best r-formula, expressed as a condition equation, is

$$\frac{1}{R-r} + \frac{2}{(R^2 + r^2 - 2Rr \cos\psi_s)^{\frac{1}{2}}} - \frac{3}{(R^2 + r^2 - 2Rr \cos\psi_m)^{\frac{1}{2}}} = 0 \quad (8)$$

This formula was derived from a single equilateral spherical triangle with point mass anomalies at an undetermined radial depth (R-r) below each vertex of the triangle. It has been found that the same condition equation can be used for any number of equally spaced (or nearly equally spaced) data points on a sphere. The formula basically represents the fact that the mean value of the disturbing potential at the vertices of all equilateral spherical triangles on the earth should be equal to the average disturbing potential at the centroids of all such data triangles. This premise is related to the fact that

$$\lim_{n\to\infty} \sum_{j=1}^{n} C_j = 0 \quad (9)$$

which simply implies that the mean value of the disturbing potential, and consequently of the point mass anomalies, integrated over a sphere should be zero. Thus the result in (8) is largely a consequence of the simplicity of the MQ concept. It follows from (8) that it is only the node spacing ψ_s, and the node-to-centroid spacing ψ_m, which effects the interior radius r to be used with the node set. Since the number of non-overlapping triangles t formed by n nodes on a sphere is equal to 2(n-2), the application of spherical trigonometry to an equilateral spherical triangle leads to

$$\psi_s(n) = 2 \tan^{-1} [1-2 \cos\frac{\pi n}{3(n-2)}]^{\frac{1}{2}} \quad (10)$$

by which ψ_s can be computed for any number of equally spaced data points on a sphere. It is noteworthy that

$$\lim_{n\to\infty} \psi_s(n) = 0 \quad (11)$$

as is to be expected, consistent with the premise in (9).

The node-to-centroid spacing ψ_m can be computed from

$$\psi_m = \tan^{-1} [\frac{2^{\frac{1}{2}} (1-\cos\psi_s)}{(\cos\psi_s - \cos2\psi_s)^{\frac{1}{2}}}] \quad (12)$$

which also follows from spherical trigonometry applied to an equilateral triangle.

Consequently the best r can be computed analytically as a function of n nodes. This function is illustrated in Figure 1.

For regional rather than global data problems, equation (10) becomes

$$\psi_s(n,A) = 2 \tan^{-1}[1-2 \cos(\frac{A}{6(n-2)R^2} + \frac{\pi}{3})]^{\frac{1}{2}} \quad (13)$$

in which A is the area of a region on the sphere R, in which the data are located, and n is the number of nodes used.

MQ Harmonic Functions for the Geoid

Starting with the basic equation (5) for the disturbing potential, the geoidal undulation can be expressed (using Bruns' equation $N = T/\gamma$) as

$$N = \frac{G}{\gamma} \sum_{j=1}^{n} \frac{C_j}{\ell_j} \quad (14)$$

Using (5) - (7) and the spherical approximation of the 3rd boundary value problem, i.e., $\Delta g = -\partial T/\partial R - 2T/R$, the geoidal gravity anomalies become [Needham, 1970]

$$\Delta g = G \sum_{j=1}^{n} C_j \left(\frac{R - r \cos\psi_j}{\ell_j^3} - \frac{2}{\ell_j R} \right) \tag{15}$$

Using the definitions of the deflection of the vertical components and partial differentiation of N in (14), with respect to θ and λ, we get

$$\xi = \frac{rG}{\gamma} \sum_{j=1}^{n} C_j \left(\frac{\cos\theta \sin\theta_j \cos(\lambda - \lambda_j) - \sin\theta \cos\theta_j}{\ell_j^3} \right) \tag{16}$$

and

$$\eta = \frac{rG}{\gamma} \sum_{j=1}^{n} C_j \left(\frac{\sin\theta_j \sin(\lambda - \lambda_j)}{\ell_j^3} \right) \tag{17}$$

The anomalous part of the vertical gradient of gravity

$$\frac{\partial g}{\partial R} = \frac{\partial \gamma}{\partial R} + \frac{\partial \Delta g}{\partial R}$$

becomes, in terms of T, [Heiskanen and Moritz, 1967]

$$\frac{\partial \Delta g}{\partial R} = - \frac{\partial^2 T}{\partial R^2} - \frac{2}{R} \frac{\partial T}{\partial R} + \frac{2T}{R^2}$$

By differentiation and substitution we develop the MQ series

$$\frac{\partial \Delta g}{\partial R} = G \sum_{j=1}^{n} \left(\frac{3R - 2r \cos\psi_j}{R \ell_j^3} - \frac{3(R - r\cos\psi_j)^2}{\ell_j^5} + \frac{2}{R^2 \ell_j} \right) C_j \tag{18}$$

Each of the formulas (14) through (17) can be used to develop a linear system of equations, for which either N_i, Δg_i, or the deflection components ξ_i and η_i are treated as observed ordinate data at known $\theta_i \lambda_i$. Thus all problems and solutions are quite analogous to the problem and solution as described for (1) and (2). In every case the system of equations provides a linear equation analog for the solution of a Fredholm integral equation of the first kind.

Since the C_j's (physically the point mass anomalies) are common to all systems, the best r-formula is applicable in all cases. Also, a linear equation solution for the C_j's from any of the systems can be substituted back into its own basic form, or into any other of the given forms (14) through (17). These then are the least squares prediction formulas for geoidal undulations (14), gravity anomalies (15), and deflection of the vertical components, (16) and (17).

One example of the several multiquadric prediction possibilities, involving the best r-formula and other formulas above, is illustrated by the world geoid in Figure 2. A system of equations was first developed from (15), using 101 approximately equally spaced mean gravity anomalies, as given by Arnold [1972] in his Figure 1. The 101 C_j's determined from this system were then substituted into (14) to provide the multiquadric prediction formula for the geoidal undulation N. Upon evaluation as shown (Figure 2) we have the MQ harmonic equivalent of a global solution of Stokes integral using 101 regional anomalies. A determination of coefficients from geoidal data as in (14), followed by substitution of the coefficients in (15) to "predict" gravity anomalies, would be the MQ harmonic inversion of Stokes Problem.

If we had substituted the MQ coefficients, which we determined from (15) with Arnold's gravity data, into (16) and (17) to produce deflection of the vertical charts, the procedure would have been the MQ harmonic equivalent of a solution of the Vening Meinesz integral formulas. Alternatively, a simultaneous determination of coefficients from astro-geodetic data as in (16) and (17), followed by substitution of the coefficients in (15) to predict gravity anomalies, amounts to a MQ harmonic inversion of the Vening Meinesz Problem.

Substitution of the determined C_j's into (18) provides predicted values for the anomalous vertical gradient of gravity at any geoidal point.

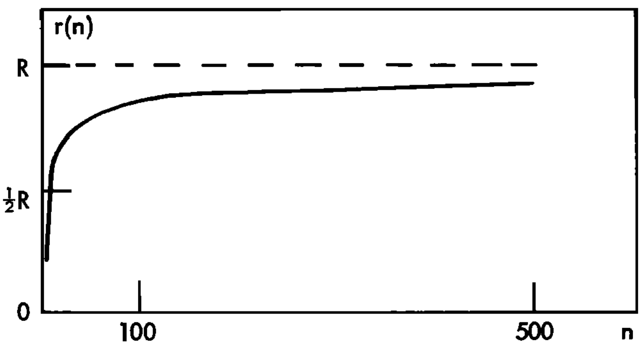

Figure 1. Interior radius r versus node number n.

MQ Harmonic Functions for Exterior Potentials

After the MQ coefficients (point mass anomalies) have been computed from any data source, Δg, N, ξ, η, and $\partial \Delta g/\partial r$ can be readily computed at any exterior equipotential of geocentric radius r_e. We simply substitute r_e for R in equation (14)-(18) to make the evaluation. Also the normal gravity γ in (14)-(17) must now be taken as

$$\gamma(r_e) = \gamma \frac{R^2}{r_e^2}$$

Concentric Superposition of MQ Harmonic Functions

The concept of superposition of the effect

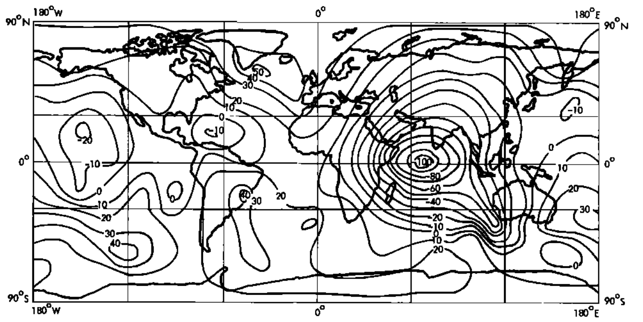

Figure 2. MQ harmonic geoid (contour interval 10m).

of computed point mass disturbing potential upon a global trend of the geoid represented by spherical harmonics was used by [Needham, 1970].

We have expanded this concept to the superposition of regional disturbing potential upon global disturbing potential, both in MQ harmonic form, using the best r-formula. The concept can also be used with the previously discussed generalized MQ harmonic functions. Work in progress indicates greater computational efficiency for concentric superposition of MQ harmonic functions than with a single function approach.

Hybrid Data Systems

Theoretically, MQ harmonic systems of equations as developed from (14) through (17), can be used to solve hybrid data problems. Satellite altimetry data in oceanic areas can be used to form a system of equations from (14). Oceanic and continental gravity data can be used to form a system of equations from (15). Astro-geodetic data, primarily continental in origin, can be used to form systems of equations as in (16) and (17). Using appropriate weights for the observations, there appears to be no theoretical reason why the various data types could not be combined to provide one hybrid data solution for the mass anomalies. Consequently, a solution for any type of geoidal or exterior potential function is virtually assured. Confirming these possibilities with real world data is one of the major aspirations for the future of this project.

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References

Arnold, K. Determination of gravity anomalies by satellite geodesy, The Use of Artificial Satellites for Geodesy, 177-179, Geophysical Monograph 15, AGU, Washington, D.C., 1972.

Assmus, E. and K.Kraus, Die interpolation nach kleinsten Quadraten Prädiktionswerte simulierte Beispiele und ihre Genauigkeiten, Deutsche Geodätische Kommission, Reihe A, Heft Nr 76, 1974.

Balmino, G., Representation of the earth potential by buried masses, The Use of Artificial Satellites for Geodesy, 121-124, Geophysical Monograph 15, AGU, Washington, D.C., 1972.

Hardy, R. L., Multiquadric analysis of topographic and other irregular surfaces, paper presented at the Annual Meeting, AGU, Washington, D.C., April, 1970.

Hardy, R. L., Multiquadric equations of topography and other irregular surfaces, J. Geophys. Res. 76, 1905-1915, 1971.

Hardy, R. L., Geodetic applications of multiquadric harmonic equations, paper presented at the Annual Meeting, AGU, Washington, D.C., April, 1974.

Heiskanen, W. A. and H. Moritz, Physical Geodesy, 251-286, W. H. Freeman, San Francisco, Calif., 1967.

Kaula, W. M., Determination of the earth's gravitational field, Rev. Geophysics 1, 507-551, 1963.

Moritz, H., Schwerevorhersage und Ausgleichungsrechnung, Zeitschift für Vermessungwesen, 90, 181-184, 1965.

Needham, P.E., The formation and evaluation of detailed geopotential models based on point masses (Doctoral dissertation, Ohio State University), Report 149, Dep. of Geod. Sc., Ohio State University, Columbus, 1970.

Rauhala, V. A., Array algebra with applications in photogrammetry and geodesy (Doctoral dissertation, Royal Inst. Tech., Stockholm), Fotogrammetriska Meddelanden, 6, 122-126, 1974.

Schut, G. H., Two interpolation methods, Photogrammetric Engineering, 40, 1447-1453, 1974.

Williamson, M. R. and E. M. Gaposchkin, Estimate of gravity anomalies, 1973 Smithsonian Standard Earth (III), SAO Special Report 353, 1973.

Yaglom, A. M., An Introduction to the Theory of Stationary Random Functions, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

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