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THEORY OF THE EARTH'S SHAPE

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Preface

Among the branches of applied science in which modern possibilities of observational methods and calculation techniques have contributed to improvements in the theories, Higher Geodesy occupies a privileged position. Having as object the determination of the Earth's shape, it has been required to and has indeed succeeded in incorporating into the solutions of its basic problems new theoretical approaches which make the best possible use of the present accuracy and abundance of experimental data, as well as of the immense processing and solving capabilities of the electronic computer.

In their present treatise *Theory of the Earth's Shape*, the authors attempt to put at the disposal of the audience a general comprehensive view of the physical-mathematical problems raised by the determination of the form of our planet, thereby making a significant contribution to the technological scientific literature in this field.

By making a critical and selective presentation of the theoretical foundations and their practical consequences, by combining the physical aspects with the geometrical ones and by including the most modern solutions available thanks to the data provided by artificial satellites, the authors present the current state of the art, thus shaping the general framework and indicating the most promising directions for present and future research,

The treatise comprises six parts. The first part, entitled *Physical Geodesy* presents the theory of the determination of the gravitational field, in the definition of which preference was given to the modern method of expansion in spherical harmonics recommended by the International Union of Geodesy and Geophysics in establishing the international "Geodetic Reference System 1967". Also presented are the main methods of determining the geoid, quasi-geoid, deflection of the vertical and the most important results achieved by Romanian workers in this field.

The second part deals with the principal aspects of *Ellipsoidal Geodesy*, i.e. with the methods of solving the geodetic problems on the reference ellipsoid.

A synthesis of the main problems associated with *Astro-geodetic Triangulation* is to be found in the third part of the treatise, concerning as much the conception of materialization and the necessary measurements as the required adjustment procedures. We also in this part note some new approaches regarding the controlled analysis of angular measurements and the description of some original calculation and measurement methods as well as the presentation — for the first time in the post-war geodetic literature — of the 1st-order astro-geodetic network of Romania.

The fourth part of the treatise concerns one of the most modern methods of determining the spatial coordinates of the geodetic points in a unitary system, viz. *Three-Dimensional Geodesy* which has had more concrete applications since the launching of the Earth's first artificial satellites.

Methods for Determining the Terrestrial Ellipsoid and the Geoid constitute the subject of the fifth part, in which as well as conventional methods, the most up-to-date methods of Dynamical Geodesy are also presented.

The last part of the treatise covers the geodetic methods for the *Determination of the Recent Movements of the Earth's Crust.* Along with an overall examination of the theoretical and practical aspects which in principle constitute the object of such activities, the authors present the most significant results obtained up to now in this field within the international framework, as well as in Romania. It is a feature of the work to point out the possibilities of interdisciplinary collaboration in the field of the determination of recent crustal movements, of concentrating the efforts of Geodesy, Geophysics, Oceanography and other disciplines interested in pursuing this phenomenon and in utilizing results already obtained.

The high standing of the authors, well-known workers from either the production and research sector or the technological higher education sector, has ensured a text of a high scientific level regarding both the organization and the presentation of the material.

Among other features to the authors' credit one should mention the consultation of a vast amount of bibliographical material, which has endowed the work with the requisite theoretical depth and made possible the presentation of the most significant results currently available in the field of the study and determination of the Earth's shape.

Academician **Sabba Stefănescu**

President
Romanian National Committee
of Geodesy and Geophysics

1

Origin, Structure and Form of the Earth

1.1 The Earth as a Planet of the Solar System

What are the stars, what is the Sun, what is the Earth's form and what is its position with respect to the neighbouring celestial bodies, are just a few of the questions that man has asked himself from the first moment of his arrival on this planet. Some of these questions have only been answered during the last century, while others still remain unsettled, despite the progress of contemporary science and technology, even to this day.

It has been agreed to denote by METAGALAXY that part of cosmic space which has been investigated so far. It is certain that the Metagalaxy occupies a very restricted part of the Universe, which would leave room for the hypothesis according to which it itself lies within the field of influence of other still unknown Metagalaxies. The matter which exists in the Metagalaxy space is organized into a great variety of forms, being nevertheless subject to the same general laws: the laws of universal gravitation and of the conservation of matter and energy.

One form of organization of the cosmic matter and of the stars is the *Galaxy*, representing a huge association of stars which are rotating round its centre. The Galaxy density is very variable on the scale of Galaxy swarms. There are agglomerations of hundreds and thousands of Galaxies in swarms but it is not known whether there are swarms of swarms of Galaxies. The Galaxy to which the solar system belongs, called *THE MILKY WAY*, has the form of a gigantic lens with a diameter of about one hundred thousand light years. Calculations have led to the conclusion that there are within the *Milky Way* about one hundred thousand milliard stars organized as local groups which are incessantly turning round the Galaxy nucleus. A full rotation of a local star group at the distance of the *Sun* from the Galaxy centre is performed in about two hundred and fifty million years with a velocity of 250 km/s.

The Sun, one of *the Milky Way*'s stars, finds itself at a distance of 33,000 light years from the Galaxy centre. It represents the only energy source of the Solar System, its mass constituting nearly 99% of the mass of the latter.

Round the Sun 9 planets whose characteristics are given in Table 1.1 are in orbit. It is supposed that the *GIGANTIC PLANET X* exists, being situated at the periphery of the Solar System.

Table 1.1 *Characteristics of the Solar System*

Solar System Member	Number of satellites	Mean Distance from the Sun in millions of km	Mean Velocity in Orbit in km/s	Revolution Period in years	Rotation Period	Mass Relative to Earth	Equatorial Diameter Relative to the Earth's Diameter	Mean Gravity in Gal	Temperature in degrees absolute (°K)
Sun	12	—	230.0	—	25^d to 29^d	333.40	109.10	273.16	6000
Mercury	—	37.9	46.9	0.24	58^d	0.05	0.39	4.00	445
Venus	—	108.2	35.0	0.61	$230^d \pm 25^d$	0.81	0.96	8.75	327
Earth	1	149.6	29.8	1.00	$23^h56^m04^s$	1.00	1.00	9.82	277
Mars	2	227.9	24.1	1.88	$23^h36^m23^s$	0.11	0.53	3.63	222
Jupiter	12	778.3	13.0	11.86	9^h50^m	317.38	10.94	25.99	253
Saturn	10	1427.1	9.6	29.46	10^h14^m	95.03	9.06	11.08	90
Uranus	5	2896.1	6.8	84.01	10^h42^m	14.57	3.78	9.89	63
Neptune	2	4498.0	5.4	164.78	15^h48^m	17.25	3.59	10.99	51
Pluto	—	5900.0	4.7	248.42	$6^d9^h16^m56^s$	0.9	0.50	4.70	—

There exist in addition a large number of small planets, the asteroids, most of them between the orbits of the planets *Mars* and *Jupiter*, of which 2,000 are catalogued.

The answers concerning the formation of Metagalaxies, Galaxies and stars are based on a series of cosmogonic hypotheses, among which the most frequently used are the following:

(1) The hypothesis of the Standing Universe and of the continuous creation of matter, which assumes the absolute infinity of the Universe, which has been proved to be unconformable to observational data.

(2) The hypothesis of the Expanding Universe. According to this hypothesis, the entire mass of the Universe was concentrated into a so-called super atom. 15 milliard years ago, the explosion of this super atom expelled into space the matter out of which were subsequently formed Metagalaxies, Galaxies and stars.

(3) The Expansion — Contraction hypothesis. According to this hypothesis, after the super atom explosion, the expansion velocity slows down

with the distance from the explosion centre up to a moment when the expansion is stopped by mutual gravitation. From that moment a contraction begins which continues up to the formation of another super atom. Thenceforth through a new explosion the cycle of producing cosmic formations repeats itself.

The above-mentioned cosmogonic hypotheses are based on the bulk of information available to mankind at present as well as on the present level of development of contemporary science and technology. They are of a schematical and abstract character and do not succeed in giving sufficiently well-founded explanations to all the phenomena which have been observed in outer space.

1.2 The Earth's Physical Structure

The gaseous envelope of the Earth — *the atmosphere* — is one of the main factors to have determined the appearance of life on the Earth. Without solar energy, the only energy of the Solar System, and without the atmosphere the appearance of life in its pristine forms would not have been possible, nor its evolution up to the stage reached at present.

As the physical-chemical properties and the composition of the atmosphere change with height, it has been divided by specialists into 5 zones: *the troposphere* (from 0 km to 10—12 km), *the stratosphere* (from 10—12 km to 55 km), *the mesosphere* (from 55 km to 80—100 km), *the thermosphere* (between 80 and 1,200 km) and the last envelope *the magnetosphere* which extends up to about 90,000 km from the terrestrial crust towards *the Sun*. Due to the solar wind pressure, the magnetosphere does not have a spherical form but that of an ellipsoid, much elongated behind the terrestrial globe and opposite to the direction of the solar radiation.

As well as its role of purveyor of the main chemical elements needed by life, the atmosphere also has the no less important role of filter, stopping the solar ultraviolet radiations which would destroy any form of life if they arrived on the Earth undiminished.

The solid envelope of the Earth is composed of chemical elements common to the Solar System. The time elapsed since the beginnings of the Earth's formation up to now has been divided by specialists into two large stages: *pregeological* and *geological*. The first stage corresponds to the time elapsed since the formation of the Earth as a planet up to the appearance of an external envelope or the Earth's crust. Once the crust formed, the Earth entered the second stage in which it still remains.

By means of geological prospecting, of analyzing the propagation of seismic waves and through other specific research methods, geophysicists have put forward the hypothesis according to which the Earth consists of envelopes covering one another and having distinct physical-chemical properties. These envelopes of approximately spherical form are separated by the so-called *discontinuity surfaces* (surfaces separating two envelopes whose physical-chemical properties are essentially different from one another).

In a first stage the Earth was divided into three major geospheres: *crust, mantle* and *core,* which were called 1st-order envelopes. Subsequent geophysical studies have led to a subdivision of these geospheres into a series of 2nd-order envelopes (Fig. 1.1).

The Earth's crust rests on a surface of varying depth called the *Mohorovičič discontinuity.* Below this discontinuity there begins the Earth's mantle which is bounded, at about 2,900 km, by the *Gutenberg-Wiechert discontinuity* which the transverse waves, used in seismic research in order to determine the structure of the terrestrial globe, were unable to cross. The central part of our planet, called the core is characterized by a very high density. As the transverse waves don't propagate through liquid media, the hypothesis was put forward that the central zone of the Earth is in a liquid state.

The other discontinuities represented in Fig. 1.2 are designated 2nd-order, a unanimous agreement of the investigators in connexion with the depths and the values of the characteristic densities not having been reached.

The accurate determination of the mean density of the entire crust of the Earth as well as the determination of the density of the Earth's crust on more restricted territories (a state, a group of states, or a continent) is particularly important for Physical Geodesy as it is on this that depends the validity of the methods of gravity reduction (Subchapt. 5.4). Although the Earth's crust is characterized by a complicated structure and composition, some generalizing hypotheses may be presented as, e.g., that due to *H. Jeffreys* considering that *the crust is formed by two layers,* viz.:

(1) *The granitic layer* with a thickness of 10—20 km and a density of 2.7 g/cm^3 (for more accurate calculations one has to consider $\delta = 2.67$ g/cm^3

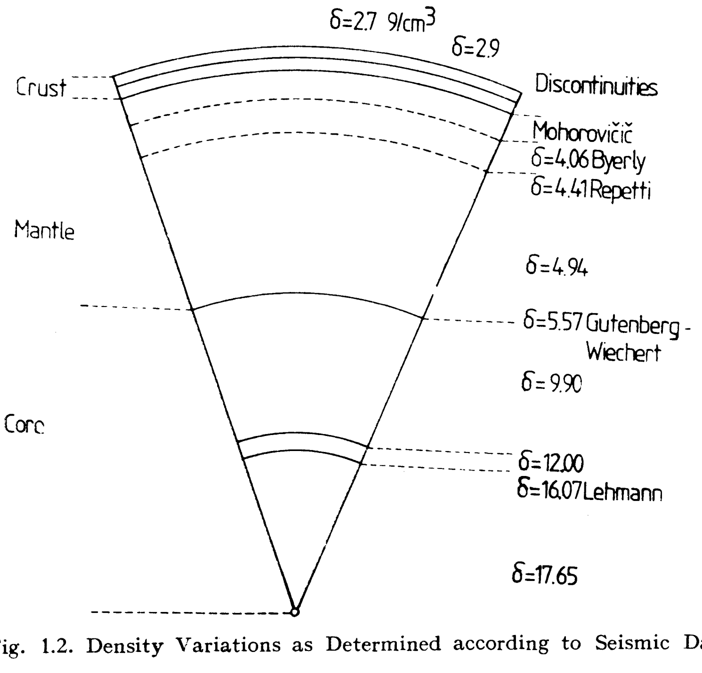
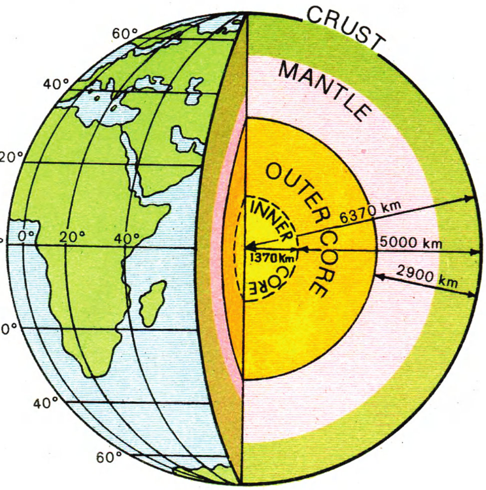
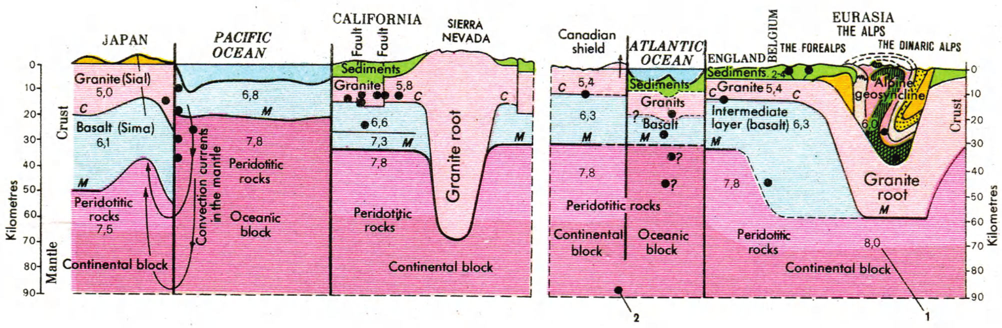


Fig. 1.2. Density Variations as Determined according to Seismic Data



1.1. Internal Structure of the Earth



1.3. Schematic Section through the Earth's Crust

for this crustal layer). Below high mountains the granitic layer may reach thicknesses of over 30 km, whereas under oceans it may thin out to vanishing point.

(2) *The intermediate layer* with an average thickness of about 17 km and density of 2.9 g/cm^3.

As mean density of the Earth the value $\delta = 5.52$ g/cm^3 is accepted in various calculations occurring in the Physical Geodesy.

1.3 Brief Historical Survey of the Ideas Concerning the Earth's Form and Size

Together with the first forms of collaboration between men at an economic and social level, the need to know the Earth's form and size and to chart it arose. The conceptions concerning the Earth's form, its place in space, as well as the data referring to its size have evolved in time with the progress of science, especially of mathematics and astronomy.

The first written statements referring to the Earth's form (imagined as a disk) are associated with *Mesopotamia*'s history (the XXXth century B.C.), the disk form of the Earth being encountered in the Homeric poems as well (the Xth century B.C.). The hypothesis of the Earth's spherical form was put forward, according to some historians, by *Thales of Miletus* (year 600 B.C.) or, according to others, by *Pythagoras* (year 550 B.C.), but the first determination of the radius of a sphere approximating the Earth's figure was made much later by *Eratosthenes* (276—195 B.C.). The latter determined the radius of the terrestrial sphere according to *the principle of arc measurements.*

(1) *The following measurements were made:*

— the length of the meridian arc between *Alexandria* and *Assuan* (then computed to be 5,000 stadia or 925 km);

— the latitude difference between these localities: 7°12′.

(2) *The Earth's radius was calculated* to be 7,350 km (against today's value of approximately 6,370 km thus resulting in an error of about 16%) and consequently the length of the quarter of meridian turned out to be 63,000 stadia, i.e. 11,560 km.

Among the first accurate results one may mention those obtained by the Arabs (year 827 A. D.), who through measuring directly a meridian arc of over 2° (at the NW of *Bagdad*) obtained for the length of the quarter of meridian the value 11,016 km.

Later on, *Tycho Brahe* (in the year 1589) and *W. Snellius* (in the year 1617) lay the foundations of the triangulation method which continues to play a special role within the framework of the methods for determining the Earth's form and dimensions. The first determinations in this direction belong to *W. Snellius* who, from the latitude difference (1°11′30″ as astronomically measured) between *Bergen op Zoom* and *Alkmaar (The Netherlands)* and the corresponding distance (calculated within a triangulation chain) deduces the length of the quarter of meridian to de 9,660 km.

The year 1660 saw the emergence of the *Academy of Sciences in Paris* which, among other primary objectives, applied itself to the determination of the magnitude of the Earth's radius, a task which it entrusted to the astronomer *Jean Picard*. He determined on the basis of observations within a triangulation network, spread along the meridian of *Paris* between *Malvoisine* and *Amiens*, the length of the quarter of meridian to be 10,009 km. This result is regarded as the first determination which may be compared with present results thanks to the measurement accuracy and to the method used.

In this epoch there began a famous scientific debate between the followers of the conceptions of *I. Newton* and *J. D. Cassini* concerning the Earth's form. *In the year 1687 Newton laid the foundations of his theory of universal attraction on whose basis he deduced, among other things, for the Earth as an homogeneous body in rotation that:*

(1) *Its equilibrium form is represented by an oblate ellipsoid flattened at the poles (the flattening given by Newton: 1/231).*

(2) *The mean gravity increases from the equator towards the pole with* $\sin^2 B$ (B is the latitude of the point concerned).

Newton based his studies on numerous previous researches, among which we here mention those of *Galileo Galilei* (1564—1642), concerning the free fall of bodies and the laws of pendulum motions which have been particularly important for the development of geodetic gravimetry.

On the other hand, the measurements carried out by *J. Cassini* (1683—1718) through the prolongation of the arc of meridian measured by *Picard* up to *Dunkerque* and down to *Collioure* led to a surprise result for the scientific world of those days: the Earth has the form of an ellipsoid, but a prolate ellipsoid, i.e. of negative flattening: $f = -(1/95)$.

These results were due to systematic measurement errors and to the imperfection of the methods for processing the observational data.

This debate was cleared up much later by carrying out new measurements, also undertaken as a task set by the Academy of Sciences in *Paris*, within the framework of *two great astro-geodetic expeditions:*

(1) *Measurements of meridian arc* ($B \approx 1°$) *in Lapland* ($B_{av} = 66°20'$), *led by Clairaut and Maupertuis* (in the years 1736—1737).

(2) *Measurements of meridian arc* ($B \approx 3°$) *in Peru* ($B_{av} = -1°31'$), *led by Bouguer, Godin, La Condamine* (in the years 1735—1744). The consequence of these expeditions was the confirmation of *Newton*'s statements, the flattening then determined having the value 1/210.

In this period the foundation for the dynamic methods of Geodesy was laid by *Clairaut* who published in the year 1743, his work *"Theory of the Earth's Figure"* in which he presents, among other things, his theorem referring to the connexion between the geometric flattening f and the gravimetric flattening f^* and that of the gravity variation with latitude.

It follows from this theorem that physical determinations lead to conclusions of a geometrical nature, referring to the form and dimensions of the Earth, which constitutes the essence of the basic ideas of Physical Geodesy.

The arc measurements continued through the XVIIIth—XXth centuries by determining, directly or from triangulation networks, arcs of meridians, of parallels or any arcs, of large dimensions. Among these we mention here the following due to their historical importance:

(1) *The re-measurement of the Paris meridian* ($B_{av} = 46°12'$; $\Delta B = 9°40'$) *between Dunkerque and Barcelona by Delambre and Méchain* (1792—1798) *aimed at establishing the first definition of the metre, given in 1792* as being "the ten millionth part of the quarter of meridian passing through *Paris*". The flattening, determined through a combination with the *Peru* measurements, was 1/334.

(2) *The arc measurements led by F. G. W. Struve, spread over a very great length: from Hammerfest (in the North of Norway) to the mouth of the Danube* (between the years 1816—1852), *as a result of which was deduced the flattening* 1/289.6. This value deserves special attention as it is extremely near to the value recommended in the year 1967 by the *International Association of Geodesy* (1/298.25), showing a quite remarkable measuring and processing technique.

The arc measurements carried out in the XVIIIth—XXth centuries have led to the determination of some reference surfaces (rotation ellipsoids flattened at the poles) of various dimensions, to which we shall return in the second part of the book.

At the beginning of the XIXth century renowned scientists like *P. S. Laplace, C. F. Gauss, F. W. Bessel* developed theories which distinguish the Earth's figure from that of an ellipsoid. There appear the notions of level surface *(Clairaut, Laplace)* and later on that of *geoid* as a surface of zero level (*J. B. Listing,* 1873).

In the year 1849 *G.G. Stokes* issued his work *"On the Variation of Gravity and the Surface of the Earth"*, in which there is to be found his famous formula through which one can determine the distance between the geoid and the ellipsoid in terms of gravity anomalies.

H. Bruns puts forward in his work *"The Earth's Figure"* (1878) a theory in which the planimetric level as well as the astrogravimetric measurements would have to be processed together in a unified system, thus laying the foundations of so-called *Three-Dimensional Geodesy.*

In the year 1927 *F. A. Vening Meinesz* published the calculation formulae of the deflection of the vertical, in terms of gravity anomalies, by means of which one can calculate the corrections for reducing the geodetic observations to the ellipsoid surface.

In the year 1945 *M. S. Molodenski,* in his work *"Basic Problems of the Geodetic Gravimetry"* laid the foundations of a new system of altitudes, *the normal system,* and of a new reference surface called *quasi-geoid.*

Since 1957, geodesy used the results obtained by means of artificial satellites of the Earth for determining the form and the dimensions of our planet.

2
Elements of Potential Theory

All geodetic measurements are carried out on the Earth's physical surface, i.e. in the domain of action of terrestrial gravity. Due to this fact, the study of the physical properties of the gravity field of the Earth, which constitutes the object of Physical Geodesy, represents an important part of the geodetical sciences.

At the basis of the fundamental problems of Physical Geodesy lie certain branches of mathematics, mechanics and mathematical Physics adapted to the case of the gravity field. This is why, within the framework of the present chapter, we recall and summarize those ideas which constitute the basis for treating the problems of Physical Geodesy; for further information the reader may consult specialist works in the field of mathematics and physics.

2.1 Principles of Field Theory

If with every point P of any domain one associates a scalar, then the correspondence between the points of the respective domain and the corresponding scalars is given by a scalar function $f(P)$ called a *scalar field.*

The set of points of three-dimensional space in which the scalar field has the same value is called a *level surface*.

From the definition of the level surface it follows that if a material point is moving along a level surface the value of its scalar field will be constant, whereas in the case of a motion outside the surface the scalar field will change in value. The motion outside the level surface may take place along any direction of the unit vector $\vec{\mathbf{s}}$. From the physical standpoint, however, special interest attaches to the motion along the direction of the normal $\vec{\mathbf{n}}$ to the level surface. In order to define the variation of the scalar field along the normal direction one introduces the concept of *gradient* as being the vector which has the magnitude equal to the value of the scalar field derivative along the normal, the same direction as the normal and as sense of direction, that of the increasing field.

The gradient is denoted by:

$$\text{grad } f = \frac{\mathrm{d}f}{\mathrm{d}n} \vec{\mathbf{n}}.$$

In the Cartesian coordinate system the gradient may be expressed by the relation:

$$\text{grad } f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}. \tag{2.1}$$

On the basis of the concept of scalar-field gradient we may define a series of notions very important for Physical Geodesy.

The function $f(P)$, having the gradient at every point equal to a given vector $\vec{\mathbf{F}}$ is called *the force function* of the vector $\vec{\mathbf{F}}$.

The function $V(P)$, equal and of opposite sign to the force function, bears the name of *potential* of the vector $\vec{\mathbf{F}}$.

From the two last definitions we may write:

$$\text{grad } V(P) = \vec{\mathbf{F}},$$

or

$$\frac{\partial V}{\partial x} \vec{i} + \frac{\partial V}{\partial y} \vec{j} + \frac{\partial V}{\partial z} \vec{k} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}. \tag{2.2}$$

This relation expresses the main property of the potential, viz. that its derivatives with respect to the coordinate axes are equal to the force projections onto these axes.

In a vector field, one defines the divergence of a vector $\vec{\mathbf{V}}$ at the point $P_0(x, y, z)$ as:

$$\text{div } \vec{\mathbf{V}} = \frac{\partial \vec{\mathbf{V}}}{\partial x} \vec{i} + \frac{\partial \vec{\mathbf{V}}}{\partial y} \vec{j} + \frac{\partial \vec{\mathbf{V}}}{\partial z} \vec{k}. \tag{2.3}$$

For case of writing in the relations (2.1) and (2.3) we will use the differential operator ∇, defined as:

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}.$$

By means of this operator the gradient and the divergence may be written in the form:

$$\text{grad } f = \triangle f; \text{ div } \vec{\mathbf{V}} = \nabla \vec{\mathbf{V}}.$$

By applying the operator ∇ to these relation again one gets:

$$\Delta \text{ (grad } f) = \text{div grad } F; \ \Delta \text{ (div } \vec{\mathbf{V}}) = \text{grad div } \vec{\mathbf{V}},$$

which represent the so-called 2nd-order differential operators. Of these two, the operator:

$$\operatorname{div} \operatorname{grad} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f, \tag{2.4}$$

is of particular importance.

The Δf operator is known as the *Laplacian* of the function f, due to the fact that the equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called *Laplace's* equation.

It is useful for subsequent consideration to know the divergence of the product $U\vec{\mathbf{P}}$:

$$\operatorname{div} (U\vec{\mathbf{P}}) = \frac{\partial}{\partial x} (U \vec{\mathbf{P}})\vec{i} + \frac{\partial}{\partial y} (U \vec{\mathbf{P}})\vec{j} + \frac{\partial}{\partial z} (U \vec{\mathbf{P}})\vec{k},$$

in which U is any function and $\vec{\mathbf{P}} = \operatorname{grad} V$. As:

$$\frac{\partial}{\partial x} (U \vec{\mathbf{P}})\vec{i} = \vec{\mathbf{P}}\left(\frac{\partial U}{\partial x}\vec{i}\right) + U\left(\frac{\partial \vec{\mathbf{P}}}{\partial x}\vec{i}\right);$$

$$\frac{\partial}{\partial y} (U \vec{\mathbf{P}})\vec{j} = \vec{\mathbf{P}}\left(\frac{\partial U}{\partial y}\vec{j}\right) + U\left(\frac{\partial \vec{\mathbf{P}}}{\partial y}\vec{j}\right);$$

$$\frac{\partial}{\partial z} (U \vec{\mathbf{P}})\vec{k} = \vec{\mathbf{P}}\left(\frac{\partial U}{\partial z}\vec{k}\right) + U\left(\frac{\partial \vec{\mathbf{P}}}{\partial z}\vec{k}\right),$$

one gets:

$$\operatorname{div} (U \vec{\mathbf{P}}) = \vec{\mathbf{P}} \operatorname{grad} U + U \operatorname{div} \vec{\mathbf{P}}.$$

As $\vec{\mathbf{P}} = \operatorname{grad} V$:

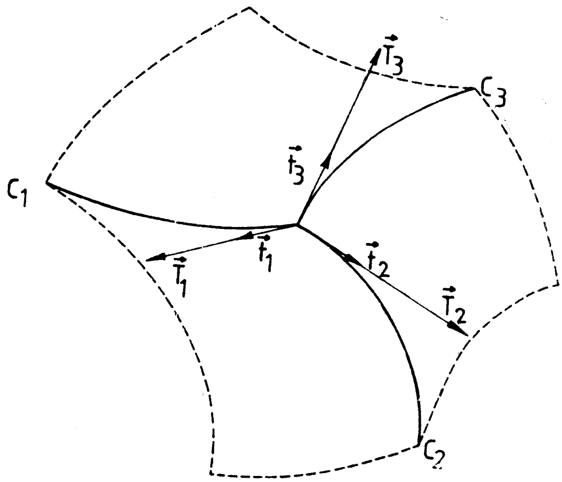
$$\operatorname{div} (U \vec{\mathbf{P}}) = \operatorname{grad} V \operatorname{grad} U + U \operatorname{div} \operatorname{grad} V,$$

or, according to (2.4):

$$\operatorname{div} (U \vec{\mathbf{P}}) = \operatorname{grad} V \operatorname{grad} U + U\Delta V. \tag{2.5}$$

Differential operators in orthogonal curvilinear coordinates. In solving many geodetical problems it is preferable to use a curvilinear coordinate system instead of the Cartesian one. The position of a point P_0 in such a system is obtained by the intersection of three curved surfaces. By intersecting the coordinate surfaces pairwise one gets the coordinate curves C_1, C_2, C_3.

Fig. 2.1. The System of Curvilinear Coordinates C_1, C_2, C_3



The vectors $\vec{\mathbf{T}}_1, \vec{\mathbf{T}}_2, \vec{\mathbf{T}}_3$ (Fig. 2.1) tangent to the coordinate curves may be expressed through the unit vectors $\vec{\mathbf{t}}_1, \vec{\mathbf{t}}_2, \vec{\mathbf{t}}_3$ as:

$$\vec{\mathbf{T}}_1 = R_1\vec{\mathbf{t}}_1 ; \ \vec{\mathbf{T}}_2 = R_2\vec{\mathbf{t}}_2 ; \ \vec{\mathbf{T}}_3 = R_3\vec{\mathbf{t}}_3,$$

in which R_1, R_2, R_3 are the so-called *Lamé* parameters.

By using the *Lamé* parameters, the Laplacian of a function $V(x_1, x_2, x_3)$ in a curvilinear coordinate system may be expressed as:

$$\Delta V = \frac{1}{R_1 \cdot R_2 \cdot R_3}\left[\frac{\partial}{\partial x}\left(\frac{R_2 \cdot R_3}{R_1}\frac{\partial V}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{R_3 \cdot R_1}{R_2}\frac{\partial V}{\partial x_2}\right) + \right.$$
$$\left. + \frac{\partial}{\partial x_3}\left(\frac{R_1 \cdot R_2}{R_3}\frac{\partial V}{\partial x_3}\right)\right]. \tag{2.6}$$

A very useful application of (2.6) is represented by the Laplacian of a function $V(x_1, x_2, x_3)$ in the system of spherical coordinates ρ, θ, λ. In this system, the *Lamé* parameters are:

$$R_1 = 1 ; \ R_2 = \rho ; \ R_3 = \rho \sin \theta,$$

and (2.6) becomes:

$$\Delta V \equiv \rho^2 \frac{\partial^2 V}{\partial \rho^2} + 2\rho \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \theta^2} + \cot \theta \frac{\partial V}{\partial \theta} + \frac{1}{\sin^2 \theta}\frac{\partial^2 V}{\partial \lambda^2} = 0, \tag{2.7}$$

or, after some transformations:

$$\rho \frac{\partial^2 (V)}{\partial \rho^2} + \frac{1}{\sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{\sin^2 \theta}\frac{\partial^2 V}{\partial \lambda^2} = 0. \tag{2.8}$$

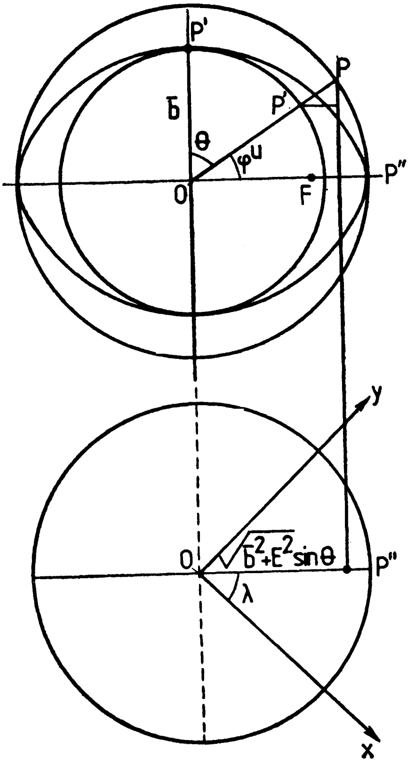


Fig. 2.2. The Ellipsoidal Coordinate System

Given the fact that one of the basic surfaces used in Geodesy is the rotation ellipsoid, in what follows we will determine the Laplacian form in ellipsoidal coordinates. This can be obtained as an application of (2.6) to the coordinate system:

$$x_1 = \sqrt{\bar{b}^2 + E^2}\, \sin\theta \cos\lambda;$$

$$x_2 = \sqrt{\bar{b}^2 + E^2}\, \sin\theta \sin\lambda; \quad x_3 = \bar{b}\cos\theta.$$

The signification of the notations $\theta, \lambda, \bar{b}$ is given in Fig. 2.2, where $E^2 = a^2 - b^2$ represents the square of the linear eccentricity.

For this coordinate system the *Lamé* parameters are (*Heiskanen* and *Moritz* 1967):

$$R_1^2 = \frac{\bar{b}^2 + E^2 \cos^2\theta}{\bar{b}^2 + E^2}; \quad R_2^2 = \bar{b}^2 + E^2\cos^2\theta;$$

$$R_3^2 = (\bar{b}^2 + E^2)\sin^2\theta,$$

and *Laplace's* equation in ellipsoidal coordinates turns out to be:

$$(\bar{b}^2 + E^2)\frac{\partial^2 V}{\partial \bar{b}^2} + 2\bar{b}\frac{\partial V}{\partial \bar{b}} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta\frac{\partial V}{\partial \theta} + \frac{\bar{b}^2 + E^2\cos^2\theta}{(\bar{b}^2 + E^2)\sin^2\theta}\frac{\partial^2 V}{\partial \lambda^2} = 0. \qquad (2.9)$$

At the limit, for the case $E \to 0$, (2.9) becomes (2.8).

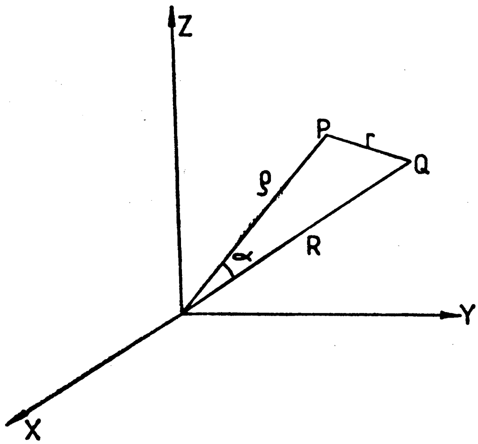
2.2 Harmonic Functions

The function having continuous 2nd-order derivatives on a domain D and satisfying *Laplace's* equation on this domain is called an *harmonic function.*

One can show that any harmonic function may be expanded in a series of powers, in the neighbourhood of any point of the domain D. This will be shown here only for the harmonic function:

$$\frac{1}{r} = (\rho^2 + R^2 - 2\rho R\cos\alpha)^{-(1/2)},$$

Fig. 2.3. Expansion of the Function $1/r$ in Spherical Harmonics



in which ρ, R and α have the meaning given in Fig. 2.3.

Making the change of variable:

$$\cos\alpha = u,$$

after some transformations the harmonic function may be written in the form:

$$\frac{1}{r} = \frac{1}{R}\left(1 - \frac{2\rho}{R}u + \frac{\rho^2}{R^2}\right)^{-1/2} = \frac{1}{R}(1-q)^{-1/2}, \tag{2.10}$$

in which we used the notation:

$$q = \frac{2\rho}{R}u - \frac{\rho^2}{R^2}.$$

By expanding $(1 - q)^{-1/2}$ in series, one gets:

$$(1-q)^{-1/2} = 1 + \frac{1}{2}q + \frac{1\cdot 3}{2\cdot 4}q^2 + \ldots = \sum_{m=0}^{\infty} C_m q^m, \tag{2.11}$$

where C_m denoted the general term's coefficient in the form:

$$C_m = \frac{1\cdot 3 \ldots (2m-1)}{2\cdot 4 \ldots 2m} = \frac{2m\,!}{2^{2m}(m\,!)^2}.$$

The term q^m in (2.11) may be expanded in series in the form:

$$q^m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\frac{2\rho}{R}u\right)^{m-k}\left(\frac{\rho}{R}\right)^{2k},$$

in which we used the notation:

$$\binom{m}{k} = \frac{m\,!}{k\,!(m-k)\,!}. \tag{2.12}$$

The relation (2.11) can now be expressed as:

$$(1-q)^{-1/2}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(2m)\,!}{2^{2m}(m\,!)^2}(-1)^k\binom{m}{k}2^{m-k}u^{m-k}\left(\frac{\rho}{R}\right)^{m+k}\cdot \tag{2.13}$$

Denoting by $P_n\ (u)$ the coefficient of the term $(\rho/R)^n$ and using $m+k=n$, one arrives at the expansion of $1/r$ in the form:

$$\frac{1}{r}=\sum_{n=0}^{\infty}\frac{\rho^n}{R^{n+1}}P_n(u), \tag{2.14}$$

a relation which will be frequently used in the analysis of the gravity potential.

2.2.1 Legendre Polynomials

In the relation (2.14), $P_n(u)$ denoted polynomials in terms of u which, taking into account (2.13) and the notation $n=m+k$, have the form:

$$P_n(u)=\sum_{k=0}^{n/2}\frac{[2(n-k)]\,!}{2^{2n-2k}(n-k)\,!(n-k)\,!}(-1)^k\frac{(n-k)\,!}{k\,!(n-2k)\,!}2^{n-2k}u^{n-2k}, \tag{2.15}$$

where $n/2$ means the largest integer not exceeding $n/2$.

By restricting the terms in (2.15) one gets:

$$P_n(u)=\sum_{k=0}^{n/2}\frac{(-1)^k(2n-2k)\,!}{2^n k\,!(n-k)\,!(n-2k)\,!}u^{n-2k}. \tag{2.16}$$

The relation (2.16) expresses the set of the n-degree polynomials, in terms of $\cos\alpha$, known as *Legendre* polynomials.

The first *Legendre* polynomials have the form:

$$\begin{aligned}
&P_0=1;\\
&P_1(u)=u=\cos\alpha;\\
&P_2(u)=\frac{3}{2}u^2-\frac{1}{2}=\frac{3}{2}\cos^2\alpha-\frac{1}{2};\\
&P_3(u)=\frac{5}{2}u^3-\frac{3}{2}u=\frac{5}{2}\cos^3\alpha-\frac{3}{2}\cos\alpha;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{2.17}$$

The generating function of these polynomials has the form:

$$I(u)=(1+t^2-2tu)^{-1/2}. \tag{2.18}$$

Indeed, expanding this function in series one gets:

$$I(u) = 1 - \frac{1}{2}(t^2 - 2tu) + \frac{3}{8}(t^2 - 2tu)^2 + \ldots =$$

$$= 1 + tu + t^2\left(\frac{3}{2}u^2 - \frac{1}{2}\right) + t^3\left(\frac{5}{2}u^3 - \frac{3}{2}u\right) + \ldots.$$

Taking into account (2.17), this expansion may be written as:

$$I(u) = \sum_{n=0}^{\infty} t^n P_n(u), \tag{2.19}$$

in which $P_n(u)$ represents the *Legendre* polynomials.

In the specialist literature it is demonstrated that the *Legendre* polynomials are solutions of *Legendre*'s differential equation:

$$(1 - u^2)P_n''(u) - 2uP_n'(u) + n(n + 1)P_n(u) = 0. \tag{2.20}$$

It is also known that these constitute a series of orthogonal functions on the interval $[-1, +1]$, i.e. they satisfy the relation:

$$\int_{-1}^{+1} P_n(u) \quad P_m(u)\, du = 0; \quad m \neq n. \tag{2.21}$$

The *Legendre* polynomials are not normalized functions and consequently:

$$\int_{-1}^{+1} P_n^2(u)\, du = k,$$

where k denotes the normalizing constant, the meaning of which is that by dividing the non-normalized orthogonal function by $\sqrt{k}$, one gets a normalized function. The normalizing constant of the *Legendre* polynomials is obtained by integrating the square of the generating function (2.18):

$$\int_{-1}^{+1} I^2(u)\, du = 2 + \frac{2t^2}{3} + \frac{2t^4}{5} + \ldots = \sum_{n=0}^{\infty} \frac{2}{2n + 1} t^{2n}.$$

On the basis of the last equality and of (2.19) one gets:

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{+1} P_n^2(u)\, du + \sum_{\substack{n=0, m=0\\ n \neq m}}^{\infty} 2t^n t^m \int_{-1}^{+1} P_n(u)P_m(u)\, du = \sum_{n=0}^{\infty} \frac{2}{2n + 1} t^{2n}.$$

As the *Legendre* polynomials are orthogonal, the second terms of the previous equality vanishes and, as a consequence, for each term of the sum we have the equality:

$$t^{2n}\int_{-1}^{+1} P_n^2(u)\,\mathrm{d}u = t^{2n}\frac{2}{2n+1},$$

whence one can derive the normalizing constant:

$$k = \int_{-1}^{+1} P_n^2(u)\,\mathrm{d}u = \frac{2}{2n+1}\cdot \tag{2.22}$$

In Physical Geodesy, besides the *Legendre* polynomials, of great use are *the associated Legendre polynomials,* whose defining relation is:

$$P_{nm}(u) = (1-u^2)^{m/2}\frac{\mathrm{d}^{n+m}P_n(u)}{\mathrm{d}u^{n+m}}\cdot \tag{2.23}$$

By introducing into (2.23) the expanded form of *Legendre* polyncmials one gets:

$$P_{nm}(u) = \frac{1}{2^n n!}(1-u^2)^{m/2}\frac{\mathrm{d}^{n+m}}{\mathrm{d}u^{n+m}}(u^2-1)^n,$$

and since,

$$(1-u^2)^n = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!\,(n-k)!}\,u^{2n-2k},$$

then

$$P_{nm}(u) = \frac{1}{2^n}(1-u^2)^{m/2}\sum_{k=0}^{n}\frac{1}{n!}(-1)^k\frac{n!}{k!\,(n-k)!}\frac{\mathrm{d}^{m+n}}{\mathrm{d}u^{m+n}}u^{2n-2k}.$$

Using the fact that:

$$\frac{\mathrm{d}^{m+n}}{\mathrm{d}u^{m+n}}u^{2n-2k} = \frac{(2n-2k)!}{(n-m-2k)!}\,u^{n-m-2k},$$

the associated *Legendre* polynomials may also be written in the form:

$$P_{nm}(u) = \frac{1}{2^n}(1-u^2)^{m/2}\sum_{k=0}^{n}(-1)^k\frac{(2n-2k)!}{k!\,(n-k)!\,(n-m-2k)!}\,u^{n-m-2k}. \tag{2.24}$$

It may equally be shown that the associated *Legendre* polynomials are solutions of the differential equation:

$$(1-u^2)g''(u) - 2ug'(u) + \left[n(n+1) - \frac{m^2}{(1-u^2)}\right]g(u) = 0. \qquad (2.25)$$

They constitute a set of orthogonal functions within the interval $[+1, -1]$, i.e. they satisfy the equality:

$$\int_{-1}^{+1} P_{lm}(u)P_{nm}(u)\,\mathrm{d}u = 0; \quad l \neq m.$$

The normalizing constant of these polynomials is obtained in a similar way as the normalizing constant of the *Legendre* polynomials, in the form:

$$k = \int_{-1}^{+1} P^2_{nm}(u)\,\mathrm{d}u = \frac{(n+m)!}{(n-m)!}\,\frac{2}{2n+1}\,. \qquad (2.26)$$

A property of the associated *Legendre* polynomials which is frequently used in Physical Geodesy is expressed by the identity:

$$P_n(\cos\psi) = P_n(\cos\theta) + P_n(\cos\theta') +$$
$$+ 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\lambda - \lambda'), \qquad (2.27)$$

where θ, θ', λ and λ' are the angles of the spherical triangles, between which there exists the relation:

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda - \lambda').$$

2.2.2 Legendre Polynomials of the Second Kind

As well as the associated *Legendre* polynomials, the differential equation (2.25) has also as solution the so-called *Legendre* polynomials of the second kind, denoted by Q_{nm}, having the following defining relation:

$$Q_{nm}(u) = (1-u^2)^{m/2}\frac{\mathrm{d}^m Q_n(u)}{\mathrm{d}u^m}; \qquad (2.28)$$

$$Q_n(u) = \frac{1}{2}P_n(u)\ln\frac{1+u}{1-u} - \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(u)\,P_{n-k}(u). \qquad (2.29)$$

The equation (2.28) is analogous to (2.23) but the functions $Q_{nm}(u)$ and $P_{nm}(u)$ are of a quite different nature.

From (2.29) and (2.16) one can obtain the first *Legendre* polynomials of the second kind in the form:

$$Q_0(u) = \frac{1}{2}\ln\frac{1+u}{1-u} = \tanh^{-1} u$$

$$Q_1(u) = \frac{u}{2}\ln\frac{1+u}{1-u} - 1 = u\tanh^{-1} u - 1 \tag{2.30}$$

$$Q_2(u) = \left(\frac{3}{4}u^2 - \frac{1}{4}\right)\ln\frac{1+u}{1-u} - \frac{3}{2}u = \left(\frac{3}{2}u^2 - \frac{1}{2}\right)\tanh^{-1} u - \frac{3}{2}u;$$

....

Because in what follows the *Legendre* polynomials of the second kind will be used as functions of a complex argument, we add here that, substituting the real argument u by the complex argument z, one gets as the only change the replacement of:

$$\frac{1}{2}\ln\frac{1+u}{1-u} = \text{arc tanh } u,$$

by:

$$\frac{1}{2}\ln\frac{1+z}{z-1} = \text{arc coth } z,$$

and then:

$$Q_0(z) = \text{arc coth } z;$$

$$Q_1(z) = z \text{ arc coth } z - 1; \tag{2.31}$$

$$Q_2(z) = \left(\frac{3}{2}z^2 - \frac{1}{2}\right)\text{arc coth } z - \frac{3}{2}z;$$

....

2.2.3 Spherical Harmonic Functions

Among the known harmonic functions, the most important for Physical Geodesy are the spherical harmonic functions. These are solutions of *Laplace's* equation, as expressed in spherical coordinates, whose form was given in (2.6) and (2.7). In order to obtain these solutions, one makes the following change of variable:

$$V(\rho, \theta, \lambda) = f(\rho)\, Y(\theta, \lambda),$$

in which f denotes a function of ρ only and Y is a function of θ and λ. Making this substitution in (2.6) and dividing by fY one gets:

$$\frac{1}{f}(\rho^2 f'' + 2\rho f') = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial\theta^2} + \cot\ \theta\ \frac{\partial Y}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\lambda^2}\right).$$

In order that this equality may hold, it is necessary that both sides be equal to the same constant. If this constant is $n(n+1)$, then:

$$\rho^2 f''(\rho) + 2\rho f'(\rho) - n(n+1)f(\rho) = 0;$$

$$\frac{\partial^2 Y(\theta,\lambda)}{\partial\theta^2} + \cot\ \theta\ \frac{\partial Y(\theta,\lambda)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\lambda)}{\partial\lambda^2} + n(n+1)\,Y(\theta,\lambda) = 0. \quad (2.32)$$

The solutions of the first equation in (2.32) are given by the functions: $f(\rho) = \rho^n$ and $f(\rho) = \rho^{-(n+1)}$, which can be verified by direct substitution. Denoting by $Y_n(\theta, \lambda)$ the solutions of equation (2.32), *Laplaces*'s equation (2.6) has as solutions:

$$V = \rho^n Y_n(\theta,\lambda); \quad V = \frac{Y_n(\theta,\lambda)}{\rho^{n+1}}. \quad (2.33)$$

The functions defined by (2.33) are known as solid spherical harmonics and the functions $Y_n(\theta, \lambda)$ as surface spherical harmonics.

The general form of the solutions of *Laplace*'s equation in spherical coordinates is given by a linear combination of the particular solutions (2.33). In this way one gets:

$$V_i = \sum_{n=0}^{\infty} \rho^n Y_n(\theta,\lambda); \quad V_e = \sum_{n=0}^{\infty} \frac{Y_n(\theta,\lambda)}{\rho^{n+1}}. \quad (2.34)$$

Analysing the form of the surface spherical harmonic, we note that this is a function of θ and λ, which suggests a new change of variable:

$$Y_n(\theta,\lambda) = g(\theta)\,h(\lambda),$$

in which g depends on θ only and h on λ only. By taking the partial derivatives of the function $Y(\theta, \lambda)$, introducing them in the second equation in (2.32), and regrouping the terms, one arrives at:

$$\frac{\sin\ \theta}{g(\theta)}[\sin\ \theta g''(\theta) + \cos\ \theta\, g'(\theta) + n(n+1)\,g(\theta)\sin\ \theta] = \frac{-h''(\lambda)}{h(\lambda)}.$$

For this equality to hold, it is necessary that both sides be equal to the same arbitrary constant. By setting this constant equal to m^2, one gets:

$$h''(\lambda) + m^2 h(\lambda) = 0;$$

$$g''(\theta) + g'(\theta)\ \cot\ \theta + n(n+1)\,g(\theta) - \frac{m^2}{\sin^2\theta}g(\theta) = 0. \quad (2.35)$$

The first equation in (2.35) has as solutions:

$$h(\lambda) = \cos m\lambda;\ h(\lambda) = \sin m\lambda, \tag{2.36}$$

which may be verified by direct substitution. The second equation in (2.35) can be put, by using the change of variable $u = \cos\theta$, in the form:

$$(1 - u^2)\, g''(u) - 2u\, g'(u) + \left[n(n+1) - \frac{m^2}{1 - u^2}\right] g(u) = 0.$$

This equation has as solutions the associated *Legendre* polynomials and, consequently, taking into account the relations (2.36) as well, the solutions of the differential equation (2.32) may be written as:

$$Y_n(\theta, \lambda) = P_{nm}(\cos\theta) \cos m\lambda; \quad Y_n(\theta, \lambda) = P_{nm}(\cos\theta) \sin m\lambda.$$

If one knows certain solutions of a linear differential equation, then their sum is also a solution. Consequently the general solution of this equation may be written as:

$$Y_n(\theta, \lambda) = \sum_{m=0}^{n} [a_{nm} P_{nm}(\cos\theta) \cos m\lambda + b_{nm} P_{nm}(\cos\theta) \sin m\lambda] \tag{2.37}$$

and, taking into consideration (2.34), the general solution of *Laplace*'s equation is:

$$V_i = \sum_{n=0}^{\infty} \rho^n \sum_{m=0}^{n} [a_{nm} P_{nm}(\cos\theta) \cos m\lambda + b_{nm} P_{nm}(\cos\theta) \sin m\lambda]; \tag{2.38}$$

$$V_e = \sum_{n=0}^{\infty} \frac{1}{\rho^{n+1}} \sum_{m=0}^{n} [a_{nm} P_{nm}(\cos\theta) \cos m\lambda + b_{nm} P_{nm}(\cos\theta) \sin m\lambda]. \tag{2.39}$$

If the function V is a potential, then the relation (2.38) gives the form of the expansion into series of spherical harmonics of the attraction potential for the case of the point interior to the sphere, whereas (2.39) will be for the case of an exterior point.

To determine the coefficients a_{nm} and b_{nm} one appeals to the orthogonality relation of *Legendre* polynomials. Since the function $f(\theta, \lambda)$ is given on the surface of a sphere, it may be presented in the form of an expansion in terms of spherical harmonics as:

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} Y_n(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{nm} R_{nm} + b_{nm} S_{nm},$$

in which:

$$R_{nm} = P_{nm}(\cos\theta) \cos m\lambda;\ S_{nm} = P_{nm}(\cos\theta) \sin m\lambda. \tag{2.40}$$

Multiplying both sides of the previous equations by any function R_{sr} of the type (2.40), integrating over the surface of the sphere and using the following orthogonality relations:

$$\iint_\sigma R_{nm}(\theta, \lambda)\, R_{sr}(\theta\, \lambda)\, d\sigma = 0; \quad n \neq s; \quad m \neq r;$$

$$\iint_\sigma S_{nm}(\theta, \lambda)\, S_{sr}(\theta, \lambda)\, d\sigma = 0, \quad n \neq s; \quad m \neq r;$$

$$\iint_\sigma R_{nm}(\theta, \lambda)\, S_{sr}(\theta, \lambda)\, d\sigma = 0;$$

$$\iint_\sigma [R_{no}(\theta, \lambda)]^2\, d\sigma = \frac{4\pi}{2n+1};$$

$$\iint_\sigma [R_{nm}(\theta, \lambda)]^2\, d\sigma = \iint_\sigma [S_{nm}(\theta, \lambda)]^2\, d\sigma = \frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}; \; m \neq 0,$$

one gets:

$$a_{no} = \frac{2n+1}{4\pi}\iint_\sigma f(\theta', \lambda')\, P_n(\cos\, \theta)\, d\sigma;$$

$$a_{nm} = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\iint_\sigma f(\theta', \lambda')\, R_{nm}(\theta, \lambda)\, d\sigma; \quad m \neq 0;$$

$$b_{nm} = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\iint_\sigma f(\theta', \lambda')\, S_{nm}(\theta, \lambda)\, d\sigma; \quad m \neq 0.$$

These relations settle the problem of determining the constants a_{nm} and b_{nm}.

In addition to the relation (2.37), the surface spherical harmonics can also be expressed with the aid of the function $f(\theta, \lambda)$ given on the surface of the sphere in the form:

$$Y_n(\theta, \lambda) = \frac{2n+1}{4\pi}\iint_\omega f(\theta', \lambda')\, P_n(\cos\, \psi)\, d\omega. \tag{2.41}$$

Remark. This relation, in which $d\omega$ denotes the solid angle element, may be verified by substituting P_n (cos ψ) given by (2.27).

2.2.4 Ellipsoidal Harmonic Functions

The ellipsoidal harmonic functions are solutions of *Laplace*'s equation as expressed in ellipsoidal coordinates (2.9). In order to determine them, one utilizes a procedure analogous to that used when deriving the spherical harmonic functions.

By making the change of variable:

$$V(\bar{b}, \theta, \lambda) = f(\bar{b})\, g(\theta)\, h(\lambda),$$

in (2.9), one obtains the following three differential equations:

$$(\bar{b}^2 + E^2) f''(\bar{b}) + 2\bar{b} f'(\bar{b}) - \left[n(n+1) - \frac{E^2}{\bar{b}^2 + E^2} m^2\right] f(\bar{b}) = 0;$$

$$\text{sing}\ \theta\, g''(\theta) + \cos\theta\, g'(\theta) + \left[n(n+1)\sin\theta - \frac{m^2}{\sin\theta}\right] g(\theta) = 0;$$

$$h''(\lambda) + m^2 h(\lambda) = 0.$$

The changes of variable:

$$\tau = i\bar{b}/E \text{ and } u = \cos\theta \ (i = \pm\sqrt{-1}),$$

transform the three equations into:

$$(1 - \tau^2) f''(\tau) - 2\pi f'(\tau) + \left[n(n+1) - \frac{m^2}{1 - \tau^2}\right] f(\tau) = 0;$$

$$(1 - u^2)\, g''(u) - 2u g'(u) + \left[n(n+1) - \frac{m^2}{1 - u^2}\right] g(u) = 0; \tag{2.42}$$

$$h''(\lambda) + m^2 h(\lambda) = 0.$$

The first two equations are of the type (2.25) and consequently may have two solutions: the associated *Legendre*'s polynomials $P_{nm}(\cos\theta)$ and the *Legendre* polynomials of the second kind $Q_{nm}(\cos\theta)$. For the first equation in (2.42) both solutions are possible, whereas the second equation can admit as solution only the associated *Legendre* polynomials (the *Legendre* polynomials of the second kind are excluded as solutions because for $u = 1$, $Q_{nm} = \infty$). The third equation has the solutions given in (2.36).

Summing up, the solutions of Eqs (2.42) are:

$$f(\bar{b}) = P_{nm}\left(i\frac{\bar{b}}{E}\right) \quad \text{or} \quad Q_{nm}\left(i\frac{\bar{b}}{E}\right);$$

$$g(\theta) = P_{nm}(\cos\theta);\ h(\lambda) = \cos m\lambda \text{ or } \sin m\lambda,$$

and the general solutins of *Laplace's* equation in ellipsoidal coordinates are:

$$V_i(\bar{b}, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{P_{nm}\left(\frac{\bar{b}}{E} i\right)}{P_{nm}\left(\frac{b}{E} i\right)} [a_{nm} P_{nm}(\cos\theta) \cos m\lambda +$$

$$+ b_{nm} P_{nm}(\cos\theta) \sin m\lambda]; \qquad (2.43)$$

$$V_e(\bar{b}, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Q_{nm}\left(\frac{\bar{b}}{E} i\right)}{Q_{nm}\left(\frac{b}{E} i\right)} [a_{nm} P_{nm}(\cos\theta) \cos m\lambda +$$

$$+ b_{nm} P_{nm}(\cos\theta) \sin m\lambda].$$

In (2.43), b denotes the minor semi-axis of a reference ellipsoid arbitrarily chosen.

2.3 Newtonian Potential

The fundamental principles of Physical Geodesy are based on the law of universal attraction, according to which two material points situated at a distance r apart attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Regarding $A(x', y', z')$ of mass m_1 as the attractive point, $B(x, y, z)$ the attracted point of mass m_2 and r the distance between them (Fig. 2.4), then the force of attraction of the point B by A is:

$$F = -G\frac{m_1 m_2}{r^2},$$

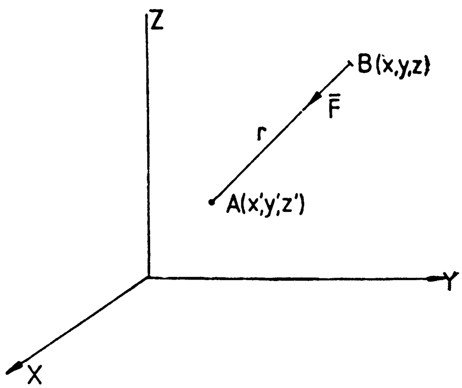


Fig. 2.4. Attraction of Two Material Points

in which G denotes the universal attraction constant. Considering the mass of the attracted point as equal to unity and the mass of the attractive point as equal to m the components of the attraction force along the coordinate axes may be written in the form:

$$F_x = -F\cos\alpha = -Gm\frac{x-x'}{r^3};$$

$$F_y = -F\cos\beta = -Gm\frac{y-y'}{r^3};$$

$$F_z = -F\cos\lambda = -Gm\frac{z-z'}{r^3}.$$

These three components may also be expressed by means of the potential function, whose definition was given in Subchapt. 2.1.

If F is the force of attraction exerted by the Earth's mass, the corresponding potential V is called the gravitational potential and is expressed as follows:

$$V = \frac{GM}{r}. \tag{2.44}$$

According to (2.2), we have the equalities:

$$F_x = \frac{\partial V}{\partial x}; \quad F_y = \frac{\partial V}{\partial y}; \quad F_z = \frac{\partial V}{\partial z},$$

or:

$$\vec{\mathbf{F}} = \operatorname{grad} V.$$

Considering the attractive point as being continuously distributed in a volume τ with the density:

$$\delta = \frac{\mathrm{d}m}{\mathrm{d}\tau},$$

then the attraction potential of the mass point $\mathrm{d}m$ in τ will be, according to (2.44):

$$\mathrm{d}V = G\frac{\mathrm{d}m}{r}.$$

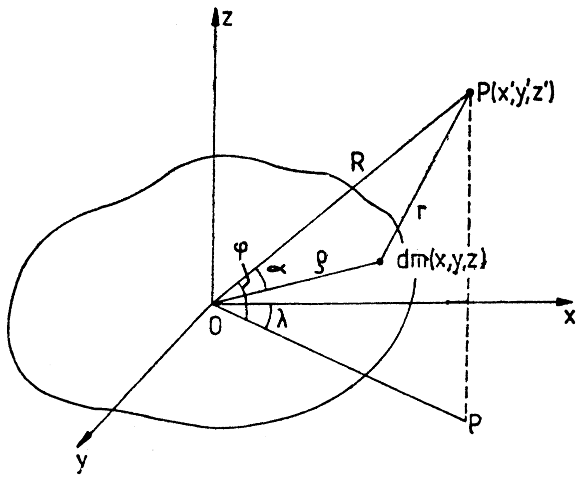
The attraction potential of the entire volume τ is:

$$V = G\iiint\limits_{\tau}\frac{\delta(x', y', z')}{r}\,\mathrm{d}\tau, \tag{2.45}$$

or, using (2.14), the volume potential may be written in the following form:

$$V = \sum_{n=0}^{\infty}\frac{1}{R^{n+1}}G\iiint\limits_{\tau}\rho^n P_n(\cos\alpha)\,\delta\,\mathrm{d}\tau, \tag{2.46}$$

Fig. 2.5. Attraction of a Material Point by a Body



in which R denote the position vector of the point P, at which the potential V is determined; ρ is the position vector of the variable point of mass dm and α is the angle between these position vectors.

The relation (2.46) represents the *volume potential* as expressed in spherical harmonics.

2.3.1 The Single Layer Potential

It is assumed that the attractive masses are distributed on a surface σ, in the form of a layer of thickness h and density δ. If dσ denotes the surface element, then the elementary volume of the layer will be:

$$\mathrm{d}\tau = h\,\mathrm{d}\sigma.$$

If the layer thins out indefinitely, while maintaining its mass unchanged, the density of the single layer may be defined by the ratio:

$$\mu = \frac{\mathrm{d}m}{\mathrm{d}\tau},$$

and the attraction potential of the single layer is given by:

$$V = G\iint\limits_{\sigma} \frac{\mu}{r}\,\mathrm{d}\sigma. \tag{2.47}$$

The single layer potential is continuous, but this is not true of its first derivatives with respect to the direction of the normal to the layer, which have different values depending on the direction of approaching the layer.

When the approach takes place from the exterior, the derivative of the potential along the normal is:

$$\left(\frac{\partial V}{\partial n}\right)_e = -\,2\pi G\mu + G\iint\limits_{\sigma}\mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}\sigma, \tag{2.48}$$

and in the case of the approach from the interior:

$$\left(\frac{\partial V}{\partial n}\right)_i = +\,2\pi G\mu + G\iint\limits_{\sigma}\mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}\sigma. \tag{2.49}$$

We thus note that when crossing the layer the first derivatives of the single-layer potential undergo the discontinuity:

$$\left(\frac{\partial V}{\partial n}\right)_e - \left(\frac{\partial V}{\partial n}\right)_i = -\,4\pi G\mu. \tag{2.50}$$

In (2.48) and (2.49), n denoted the unit vector of the normal to the layer at the attracted point P when this is on the layer.

2.3.2 The Potential of the Homogeneous Spherical Layer

One considers an homogeneous spherical layer of radii R_1 and R_2, and density δ. Adopting the spherical coordinate system and the notations in Fig. 2.6, the mass element of the layer may be expressed in the form:

$$\mathrm{d}m = \delta \cdot \rho^2 \cdot \cos\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\lambda.$$

When the attracted point is situated outside the layer (Fig. 2.6), the attraction potential is expressed by:

$$V_e = G\iiint\limits_{v}\frac{\mathrm{d}m}{r} = G\delta\int\limits_{R_1}^{R_2}\rho^2\,\mathrm{d}\rho\int\limits_{-\pi/2}^{+\pi/2}\frac{\cos\varphi\,\mathrm{d}\varphi}{r}\int\limits_{0}^{2\pi}\mathrm{d}\lambda,$$

or, putting r in the form:

$$r^2 = l^2 + \rho^2 - 2l\rho\sin\varphi$$

one arrives at:

$$V_e = \frac{4\pi G\delta}{l}\int\limits_{R_1}^{R_2}\rho^2\,\mathrm{d}\rho = \frac{4\pi G\delta}{3l}(R_2^3 - R_1^3). \tag{2.51}$$

The relation (2.51) expresses the attraction potential of the homogeneous spherical layer on an exterior point.

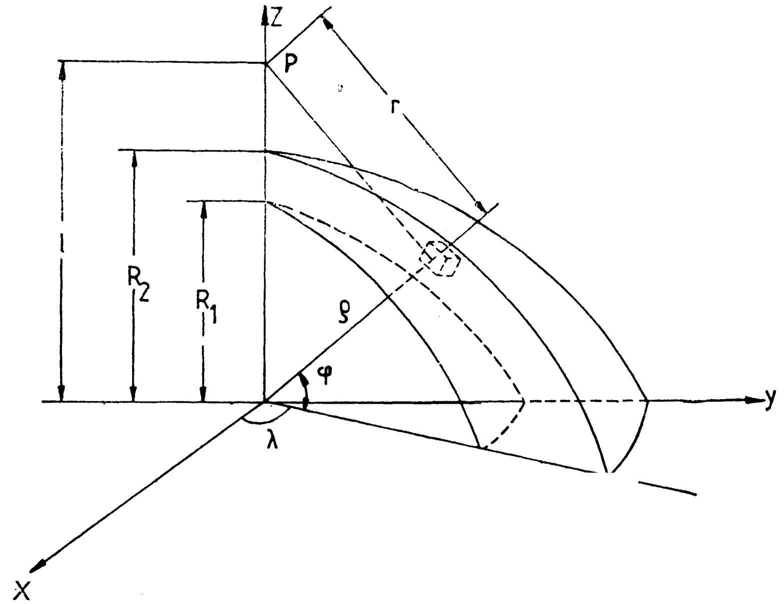


Fig. 2.6. Attraction Potential of the Spherical Layer

As the mass of the spherical layer is:

$$M = \frac{4}{3}\pi\delta\,(R_2^3 - R_1^3),$$

it follows that

$$V = G\frac{M}{l},$$

which shows that the attraction potential of the spherical layer on the exterior point equals the potential of a point of mass M situated at the centre of the sphere. If the attracted point is placed within the spherical layer, then $l = \rho$ and the expression of the attraction potential is obtained from (2.51) in the form:

$$V_i = 2\pi G\delta(R_2^2 - R_1^2). \tag{2.52}$$

From the relations expressing the potential of the homogeneous spherical layer one can derive, in a relatively simple way, *the attraction potential of the sphere on the interior point or on a point situated on the sphere*. Thus, one gets:

(1) *For a point situated within the sphere:*

$$V_i = \frac{2\pi G\delta}{3}(3R^2 - \rho^2), \tag{2.53}$$

where R denotes the radius of the sphere and ρ the distance from its centre to the attracted point.

(2) *For a point situated on the sphere:*

$$V_{sph} = G\frac{M}{R},$$

in which M denotes the mass of the sphere of radius R.

2.3.3 The Double Layer Potential

Two parallel single layers, situated, one with respect to the other, at the infinitely small distance h having layer densities equal and of opposite sign constitute a double layer.

The attraction potential of the double layer on the point $A(x, y, z)$ results as the sum of the potentials of the two constituent layers:

$$V = G\iint_{\sigma}\frac{\delta\,\mathrm{d}\sigma}{r} - G\iint_{\sigma'}\frac{\delta\,\mathrm{d}\sigma'}{r'}.$$

Because the layers are infinitely close, their surface elements can be considered as equal. Equally, one may make the approximation:

$$\frac{1}{r} - \frac{1}{r'} = \frac{r' - r}{rr'} \approx -\frac{h\cos(\vec{\mathbf{n}}, \vec{\mathbf{r}})}{r^2} = h\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right),$$

and then the calculation relation of the double layer's potential, becomes:

$$V = G\iint_{\sigma} h\cdot\delta\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma.$$

When $h \to 0$, the product δh tends to the finite limit ν, and the double layer potential becomes:

$$V = G\iint \nu\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma. \tag{2.54}$$

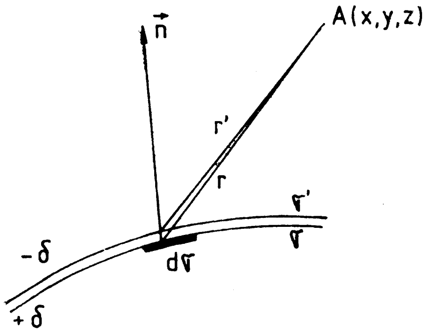


Fig. 2.7. Attraction Potential of the Double Layer

This type of potential is continuous throughout the entire space, except on the surface of the double layer, where one gets two values of the potential, depending on the direction of approaching the layer;

(1) *In the case of approaching the layer from the outside:*

$$V_e = 2\pi G\nu + G\iint\limits_{\sigma} \nu \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right) \mathrm{d}\sigma. \tag{2.55}$$

(2) *In the case of the approach from the inside:*

$$V_i = -\, 2\pi\, G\nu + G\iint\limits_{\sigma} \nu \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right) \mathrm{d}\sigma. \tag{2.56}$$

It follows that on passing through the layer the potential undergoes the discontinuity:

$$V_e - V_i = 4\pi G\nu. \tag{2.57}$$

The relations (2.55) and (2.56), although of the same type as the relations (2.48) and (2.49), are essentially different due to the fact that in the former ones $\vec{\mathbf{n}}$ means the normal to the layer at the attractive point, whereas in the latter ones $\vec{\mathbf{n}}$ denotes the normal to the layer at the attracted point.

2.3.4 Poisson's Equation

As was shown in Subchapt. 2.3, the attraction potential of the volume masses on an exterior point is:

$$V_e = G\iiint\limits_{\tau} \frac{\mathrm{d}m}{r},$$

in which r denotes the distance from the mass element $\mathrm{d}m$ (x', y', z') to the attracted point $P(x, y, z)$.

The lst-order derivatives of the potential equal the projections of the attraction force onto the respective coordinate axes, i.e.:

$$\frac{\partial V_e}{\partial x} = F_x = -\,G\iiint\limits_{\tau} \frac{x - x'}{r^3}\,\mathrm{d}m;$$

$$\frac{\partial V_e}{\partial y} = F_y = -\,G\iiint\limits_{\tau} \frac{y - y'}{r^3}\,\mathrm{d}m;$$

$$\frac{\partial V_e}{\partial z} = F_z = -\,G\iiint\limits_{\tau} \frac{z - z'}{r^3}\,\mathrm{d}m,$$

and the 2nd-order derivatives are given by:

$$\frac{\partial^2 V_e}{\partial x^2} = -G \iiint_\tau \left[\frac{1}{r^3} - \frac{3(x-x')^2}{r^5}\right] dm;$$

$$\frac{\partial^2 V_e}{\partial y^2} = -G \iiint_\tau \left[\frac{1}{r^3} - \frac{3(y-y')^2}{r^5}\right] dm;$$

$$\frac{\partial^2 V_e}{\partial z^2} = -G \iiint_\tau \left[\frac{1}{r^3} - \frac{3(z-z')^2}{r^5}\right] dm.$$

An immediate consequence of these relations is:

$$\Delta V_e = \frac{\partial^2 V_e}{\partial x^2} + \frac{\partial^2 V_e}{\partial y^2} + \frac{\partial^2 V_e}{\partial z^2} = 0,$$

which shows that the potential of the volume masses is an harmonic function outside them.

This is not the case with the Laplacian of the attraction potential of the volume masses on the point situated inside. In this situation the potential is given by (2.53) and the 2nd-order derivatives are equal to:

$$\frac{\partial^2 V_i}{\partial x^2} = -\frac{4\pi G\delta}{3}; \quad \frac{\partial^2 V_i}{\partial y^2} = -\frac{4\pi G\delta}{3}; \quad \frac{\partial^2 V_i}{\partial z^2} = -\frac{4\pi G\delta}{3},$$

so that the Laplacian has the value:

$$\Delta V_i = -4\pi G\delta. \tag{2.58}$$

The relation (2.58) represents *Poisson's* equation, which shows that the attraction potential of the volume masses on the interior point is not an harmonic function. From (2.58) one may also see that *Laplace*'s equation is a particular case of *Poisson*'s equation, for the case of zero density.

2.3.5 The Earth's Attraction Potential

As was shown in Chapter. 1, the forces acting on a point situated on the Earth's surface are mainly two: the gravitation and the centrifugal force, to which correspond the gravitational potential, and the potential of the centrifugal force, respectively. The gravity potential will equal the sum of these two potentials.

Assuming that the material point of unit mass, of coordinates x, y, z, is rotating round the axis z with the constant angular velocity ω, then the magnitude of the centrifugal force acting on the point is:

$$F_c = \omega^2\sqrt{x^2 + y^2},$$

and the vector $\vec{\mathbf{F}}_c$ of this force is given by the relation:

$$\vec{\mathbf{F}}_c = \omega^2(x\vec{\mathrm{i}} + y\vec{\mathrm{j}}).$$

The potential of this force is:

$$Q = \frac{1}{2}\,\omega^2(x^2 + y^2).$$

The potential of the centrifugal force may also be expressed in spherical coordinates in the form:

$$Q = \frac{1}{2}\,R^2\omega^2 \cos^2\varphi.$$

The *Laplace* operator for the potential of the centrifugal force is:

$$\Delta Q = 2\omega^2.$$

The gravity potential can be written as the sum of the gravitational potential and of that of the centrifugal force and using (2.45), we have:

$$W = V + Q = G\iiint \frac{\delta}{r}\,\mathrm{d}\tau + \frac{\omega^2}{2}\,(x^2 + y^2),$$

and its Laplacian is:

$$\Delta W = -\,4\pi G\delta + 2\omega^2.$$

Nor in the case of the exterior point, when $\delta = 0$, is the gravity potential an harmonic function. While the potential of the centrifugal force is an analytic function, the gravitational potential as an infinite harmonic series can be determined only approximately, the degree of approximation being given by the number of terms of the harmonic series taken into consideration. There will subsequently be derived the form of the gravitational potential for the case in which one takes into consideration only the first three terms of the harmonic series.

Considering the Earth as a solid body whose density δ is a point function, the gravitational potential may be calculated with the aid of (2.46). This relation may be written, taking into account (2.17), in the form:

$$V = \sum_{n=0}^{\infty} \frac{1}{R^{n+1}}\,G\iiint \rho^n P_n(u)\,\delta\,\mathrm{d}\tau =$$

$$= G\,\frac{M}{R} + \frac{G}{R^2}\iiint_\tau \rho\cos\alpha\,\mathrm{d}m + \frac{G}{2R^2}\iiint_\tau \rho^2(3\cos^2\alpha - 1)\,\mathrm{d}m + \ldots =$$

$$= V_0 + V_1 + V_2 + \ldots$$

According to the notations in Fig. 2.5, the quantities R, ρ, $\cos\alpha$, can be expressed in the form:

$$R^2 = x'^2 + y'^2 + z'^2; \quad \rho^2 = x^2 + y^2 + z^2; \quad \cos\alpha = \frac{xx' + yy' + zz'}{R\rho}.$$

The first three terms in the expansion of the attraction potential of the Earth may be written as:

$$V_0 = \frac{GM}{R};$$

$$V_1 = \frac{G}{R^3}\left[x\iiint_M x'\,\mathrm{d}m + y\iiint_M y'\,\mathrm{d}m + z\iiint_M z'\,\mathrm{d}m\right];$$

$$V_2 = \frac{G}{2R^5}\left[x^2\iiint_M (2x'^2 - y'^2 - z'^2)\,\mathrm{d}m + y^2\iiint_M (2y'^2 - z'^2 - x'^2)\,\mathrm{d}m +\right.$$

$$+ z^2\iiint_M (2z^{2\prime} - x'^2 - y'^2)\,\mathrm{d}m + 6xy\iiint_M x'y'\,\mathrm{d}m + 6yz\iiint_M y'z'\,\mathrm{d}m +$$

$$\left. + 6zx\iiint_M z'x'\mathrm{d}m\right].$$

The integrals in the expression of V_1 in fact represent the coordinates of the mass centre of the Earth. In the case when the origin of the chosen coordinate system coincides with the mass centre of the Earth, all the terms making up the quantity V_1 are zero. Consequently in such a case:

$$V_1 = 0.$$

Using the notations:

$$A = \iiint_M (y'^2 + z'^2)\,\mathrm{d}m; \qquad D = \iiint_M y'z'\,\mathrm{d}m;$$

$$B = \iiint_M (x'^2 + z'^2)\,\mathrm{d}m; \qquad E = \iiint_M z'x'\,\mathrm{d}m;$$

$$C = \iiint_M (x'^2 + y'^2)\,\mathrm{d}m; \qquad F = \iiint_M x'y'\,\mathrm{d}m,$$

the expression for V_2 becomes:

$$V_2 = \frac{G}{2R^5}[x^2(B + C - 2A) + y^2(C + A - 2B) + z^2(A + B - 2C) +$$

$$+ 6Dyz + 6Ezx + 6Fxy.$$

The quantities A, B, C, D, E, F are the so-called moments of inertia of the Earth.

Introducing the spherical coordinate system:

$$x = R \cdot \cos \varphi \cdot \cos \lambda; \ y = R \cdot \cos \varphi \cdot \sin \lambda; \ z = R \cdot \sin \varphi,$$

the term V_2 may be written in the form:

$$V_2 = \frac{G}{R^3}\left[\frac{1 - 3\sin^2 \varphi}{2}\left(C - \frac{A + B}{2}\right) + (3E \cos \lambda + 3D \sin \lambda) \sin \varphi \cdot \cos \varphi + \right.$$
$$\left. + \left(\frac{3}{4}(B - A) \cos 2\lambda + \frac{3}{2} F \cdot \sin 2\lambda\right) \cos^2 \varphi\right]\cdot \tag{2.60}$$

When the z-axis coincides exactly with the instantaneous rotation axis of the Earth, the D and E moments are zero and the term V_2 takes the form:

$$V_2 = \frac{G}{R^3}\left[\frac{1 - 3 \sin^2 \varphi}{2}\left(C - \frac{A + B}{2}\right) + \frac{3}{4}(B - A)\cos^2 \varphi \cos 2\lambda + \right.$$
$$\left. + \frac{3}{2} F \cos^2 \varphi \sin 2\lambda\right]\cdot$$

Admitting also the hypothesis that the x-and y-axes actually coincide with the principal inertia axes of the Earth (which in reality does not happen), then $F = 0$ and the term V_2 becomes:

$$V_2 = \frac{G}{R^3}\left[\frac{1 - 3 \sin^2 \varphi}{2}\left(C - \frac{A + B}{2}\right) + \frac{3}{4}(B - A) \cos^2 \varphi \cos 2\lambda\right]\cdot$$

If we take into consideration only the first three terms in the expansion of the attraction potential of the Earth, this can be written:

$$V \approx V_0 + V_1 + V_2 = \frac{GM}{R} + \frac{G}{R^3}\left[\frac{1 - 3\sin^2 \varphi}{2}\left(C - \frac{A + B}{2}\right) + \right.$$
$$\left. + \frac{3}{4}(B - A) \cos^2 \varphi \cos 2\lambda\right]\cdot \tag{2.61}$$

The relation (2.61) is known under the name of *MacCullagh*'s theorem. This represents, however, only an approximate form of the attraction potential of the Earth, whose full expression may be written in the form of an expansion in terms of *Legendre* polynomials $P_{nm}(u)$ ($u = \cos \theta = \sin \varphi$):

$$V = \sum_{n=0}^{\infty} \frac{a_e^{n+1}}{R^{n+1}} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm}(u) =$$

$$= \frac{GM}{R} + \frac{G}{R^3}\left[\frac{1 - 3\sin^2 \varphi}{2}\left(C - \frac{A + B}{2}\right) - \frac{3}{4}(A - B) \cos^2 \varphi \cos 2\lambda\right] +$$
$$+ \sum_{n=3}^{\infty} \frac{a_e^{n+1}}{R^{n+1}} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm}(u). \tag{2.62}$$

From (2.62) one can derive the equalities:

$$V_0 = \frac{a_e}{R} a_{00} P_{00}(u) = \frac{a_e}{R} a_{00} = \frac{GM}{R}$$

$$V_1 = \left(\frac{a_e}{R}\right)^2 [a_{10} P_{10}(u) + (a_{11} \cos \lambda + b_{11} \sin \lambda)\, P_{11}(u)] =$$

$$= \left(\frac{a_e}{R}\right)^2 [a_{10} \sin \varphi + (a_{11} \cos \lambda + b_{11} \sin \lambda) \cos \varphi] = 0$$

$$V_2 = \left(\frac{a_e}{R}\right)^3 \left[a_{20} \frac{1}{2} (3 \sin^2 \varphi - 1) + (a_{21} \cos \lambda + b_{21} \sin \lambda)\, 3 \sin \varphi \cdot \cos \varphi + \right.$$

$$\left. + (a_{22} \cos 2\lambda + b_{22} \sin 2\lambda) \cdot 3 \cos^2 \varphi \right].$$

On the basis of these equalities and of the relation (2.60), one may obtain the correspondence between the physical constants of the Earth and the coefficients a_{nm} and b_{nm}:

$$a_{00} = \frac{GM}{a_e};$$

$$a_{10} = a_{11} = b_{11} = 0;$$

$$a_{20} = -\frac{G}{a_e^3}\left(c - \frac{A+B}{2}\right);$$

$$a_{21} = \frac{G}{a_e^3} E; \qquad b_{21} = \frac{G}{a_e^3} D;$$

$$a_{22} = \frac{G}{a_e^3} \frac{B-A}{4}; \qquad b_{22} = \frac{G}{a_e^3} \frac{F}{2}.$$

If one admits the hypothesis of the existence of a rotation symmetry with respect to the z-axis and the x- and y-axes coincide with the principal inertia axes of the Earth, then $A = B$, $D = E = F = 0$ and:

$$a_{00} = \frac{GM}{a_e};$$

$$a_{20} = -\frac{G}{a_e^3}(C - A).$$

All the other 2nd-order coefficients are zero.

In Dynamic Geodesy one utilizes various notations for the terms' coefficients in the harmonic series of expansion of the gravitational potential.

Quite frequently, one uses the dimensionless coefficients C_{nm} and S_{nm} which may be expressed as functions of the coefficients a_{nm} and b_{nm} by means of the relations:

$$C_{nm} = a_{nm} \frac{a_e}{GM}; \qquad S_{nm} = b_{nm} \frac{a_e}{GM}.$$

With these coefficients the exterior potential of the Earth can be expressed as:

$$V = \frac{GM}{R}\left[1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{R}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(u)\right].$$

Another currently used notation is:

$$J_n = -C_{no};$$
$$J_{nm} = -C_{nm};$$
$$K_{nm} = -S_{nm}.$$

With these notations the extorior potential is written as:

$$V = \frac{GM}{R}\left\{1 - \sum_{n=2}^{\infty} \left(\frac{a_e}{R}\right)^n \left[J_n P_n(u) + \sum_{m=1}^{n} (J_{nm} \cos m\lambda + K_{nm} \sin m\lambda)\, P_{nm}(u)\right]\right\},$$

or, when $A = B$ and $D = E = F = 0$:

$$V = \frac{GM}{R}\left[1 - \sum_{n=2}^{\infty} \left(\frac{a_e}{R}\right)^n J_n P_n(u)\right].$$

One equally utilizes the coefficients A_{nm} and B_{nm} in expansions in series having the form:

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{R^{n+1}} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, P_{nm}(u).$$

The relations expressing the connexions between all these coefficients, used in particular in Dynamic Geodesy, are:

$$A_{nm} = -GMa_e^n J_{nm} = GMa_e^n C_{nm} = a_e^{n+1} a_{nm};$$
$$B_{nm} = -GMa_e^n K_{nm} = GMa_e^n S_{nm} = a_e^{n+1} b_{nm}.$$

When one utilizes the associated *Legendre* polynomials normalized or completely normalized, the coefficients are divided through by p_{nm}, and r_{nm} respectively, quantities whose values are given by:

$$p_{nm} = \sqrt{\frac{t_{nm}(n-m)!}{(n+m)!}}; \qquad t_{nm} = \begin{cases} 1 \text{ for } m = 0 \\ 2 \text{ for } m \neq 0 \end{cases}$$
$$r_{nm} = p_{nm}\sqrt{2n+1}\,.$$

With these notations the normalized and completely normalized coefficients of the expansion in spherical harmonics are:

$$\bar{A}_{nm} = \frac{A_{nm}}{p_{nm}}\,; \quad \bar{\bar{A}}_{nm} = \frac{A_{nm}}{r_{nm}}\,;$$

$$\bar{B}_{nm} = \frac{B_{nm}}{p_{nm}}\,; \quad \bar{\bar{B}}_{nm} = \frac{B_{nm}}{r_{nm}}\,.$$

2.4 Boundary-Value Problems of Potential Theory

The determination of the exterior gravitational potential constitutes one of the main problems of Physical Geodesy. This potential can be determined through integrating the differential equation of *Laplace* under the condition that the potential take certain values on the surface σ bounding the attractive masses. This manner of determining the potential is called a boundary-value problem of potential theory.

Depending on the values assumed as being known on the surface σ, one distinguishes the following boundary-value problems in Physical Geodesy:

(1) *The first boundary-value problem of the potential theory or Dirichlet's problem* consists in determining a potential function V, which outside the surface σ should satisfy *Laplace*'s equation and on the surface σ itself should take a known value $f(\theta, \lambda)$.

(2) *The second boundary-value problem or Neumann's problem* consists in determining the harmonic function V knowing its derivatives on the surface σ with respect to the normal to this surface.

(3) *The third boundary-value problem of the potential* consists in finding the harmonic function V when one knows on the surface σ the linear combination:

$$hV + g\frac{\mathrm{d}V}{\mathrm{d}n}.$$

As boundary of the masses one usually takes a closed surface σ and the domain in which the boundary-value problems are solved is considered either as the outside of the surface or as its inside. Depending on this, the boundary-value problems are called exterior or interior.

In the following paragraphs there will be analysed, on the basis of *Gauss-Ostrogradski*'s and *Green*'s formulae respectively, the main methods of solving the boundary-value problems.

2.4.1 Gauss-Ostrogradski's and Green's Formulae

The study of the properties of the spatial functions satisfying *Laplace*'s equation is based on *Gauss-Ostrogradski*'s formula, also known as the integral divergence formula. This formula has the form:

$$\iint\limits_{\sigma} \vec{\mathbf{n}}\,\vec{\boldsymbol{\varphi}}\,\mathrm{d}\sigma = \iiint\limits_{\Omega} \operatorname{div}\vec{\boldsymbol{\varphi}}\,\mathrm{d}\omega,$$

in which Ω denotes a three-dimensional domain bounded by the surface σ; $\vec{\boldsymbol{\varphi}}$ is a vector function with first-order partial derivatives continuous on σ and $\vec{\mathbf{n}}$ is the exterior normal to the surface σ.

If in *Gauss-Ostrogradski*'s formula one makes the change of variable:

$$\vec{\boldsymbol{\varphi}} = V \operatorname{grad} U,$$

whence:

$$\vec{\mathbf{n}}\,\vec{\boldsymbol{\varphi}} = V\vec{\mathbf{n}} \operatorname{grad} U = V\frac{\mathrm{d}U}{\mathrm{d}n},$$

one gets:

$$\oiint V\frac{\mathrm{d}U}{\mathrm{d}n} = \iiint\limits_{\Omega} \operatorname{div}\,(V \operatorname{grad} U),$$

or, according to (2.4) and (2.5), one obtains:

$$\oiint\limits_{\sigma} V\frac{\mathrm{d}U}{\mathrm{d}n}\,\mathrm{d}\sigma = \iiint\limits_{\Omega} (\operatorname{grad} U \operatorname{grad} V + V\,\Delta U)\,\mathrm{d}\omega =$$

$$= \iiint\limits_{\Omega}\left[\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right) + V\,\Delta U\right]\mathrm{d}\omega. \tag{2.63}$$

The relation (2.63) represents *the first formula of Green.*

If in this formula one interchanges the functions U and V one gets a new equation, which subtracted from (2.63) leads to the equality:

$$\oiint\limits_{\sigma}\left(U\frac{\mathrm{d}V}{\mathrm{d}n} - V\frac{\mathrm{d}U}{\mathrm{d}n}\right)\mathrm{d}\sigma = \iiint\limits_{\Omega} U\,\Delta V\,\mathrm{d}\omega - \iiint\limits_{\Omega} V\,\Delta U\,\mathrm{d}\omega, \tag{2.64}$$

representing *the second formula of Green.*

For the particular case $U = 1/r$ one obtains *the third formula of Green:*

$$\oiint\limits_{\sigma}\left(\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}n} - V\frac{\mathrm{d}\left(\frac{1}{r}\right)}{\mathrm{d}n}\right)\mathrm{d}\sigma = KV + \iiint\limits_{\Omega}\frac{1}{r}\,\Delta V\,\mathrm{d}\omega, \tag{2.65}$$

in which K is a constant whose magnitude depends as follows on the position of the attracted point P with respect to the attractive mass:

$$K = \begin{cases} 4\pi & \text{for } P \text{ situated inside } \sigma \\ 2\pi & \text{for } P \text{ situated on } \sigma \\ 0 & \text{for } P \text{ situated outside } \sigma. \end{cases}$$

Of great importance in Physical Geodesy is the application of the third formula of *Green* to the case of the gravity potential W (gravitation plus centrifugal force), considering the attracted point as situated on the Earth's surface. In this case $K = 2\pi$ and (2.65) becomes:

$$-\oiint_S \left[\frac{1}{r}\frac{\partial W}{\partial n} - W\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds = 2\pi W + \iiint_\Omega \frac{1}{r}\,\Delta W\,d\omega. \tag{2.66}$$

As was shown in the preceding paragraph, the gravity potential and its Laplacian may be expressed in the form:

$$W = \iiint_\Omega \frac{\delta}{r}\,d\omega + \frac{1}{2}\,\omega^2(x^2 + y^2); \quad \Delta W = -\,4\pi G\delta + 2\omega^2,$$

and the derivative of this potential along the normal is:

$$\frac{\partial W}{\partial n} = -\,g_n.$$

Replacing the values of W, ΔW and $\partial W/\partial n$ in (2.66) one gets:

$$-\,2\pi W + \iint_S \left[W\frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{g_n}{r}\right] ds + 2\pi\,\omega^2(x^2 + y^2) + 2\omega^2\iiint_\Omega \frac{d\omega}{r} = 0. \tag{2.67}$$

All the quantities of this equation refer to the Earth's physical surface and consequently the relation (2.67) may be considered as a fundamental relation for determining the physical surface of the Earth when one knows the potential W and the measured gravity g.

2.4.2 The Solution of Dirichlet's Problem for the Sphere and the Ellipsoid

We consider the sphere Σ of radius R, on whose surface one knows the function $f(\theta, \lambda)$; (ρ, θ, and λ constitute a spherical coordinate system). *Dirichlet*'s problem for the exterior of the sphere can be formulated as follows: to determine the function V_e (ρ, θ, λ) satisfying the following conditions:

$$\Delta V_e = 0; \quad \lim_{\rho\to\infty} V_e = 0; \quad \lim_{\rho\to R} V_e = f(\theta, \lambda). \tag{2.68}$$

The problem is solved by regarding V_e as the sum of a potential of a single layer and of another of a double layer, distributed on the unit sphere Σ. In view of (2.47) and (2.54), V_e can be written as:

$$V_e(\rho, \theta, \lambda) = \iint_\Sigma \frac{\mu}{r}\,\mathrm{d}\sigma + \iint_\Sigma \nu \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}\sigma. \tag{2.69}$$

The first two conditions in (2.68) are satisfied by $V_e(\rho, \theta, \lambda)$ because both the single layer potential and that of the double layer are harmonic functions, since they vanish at infinity. The problem is thus reduced to determining the densities μ and ν in such a manner that the third condition in (2.68) is also fulfilled. This boundary condition can be written, according to (2.47) and (2.55) in the form:

$$\lim_{\rho\to\infty} V_e(\rho, \theta, \lambda) = \iint_\Sigma \frac{\mu}{r_0}\,\mathrm{d}\sigma + \iint_\Sigma \nu \frac{\partial}{\partial n}\left(\frac{1}{r_0}\right)\mathrm{d}\sigma + 2\pi\nu = f(\theta, \lambda). \tag{2.70}$$

The direct value of the double-layer potential may be expressed in the form:

$$\iint_\Sigma \nu \frac{\partial}{\partial n}\left(\frac{1}{r_0}\right)\mathrm{d}\sigma = \iint_\Sigma \frac{\nu \cos(r_0, R)}{r_0^2}\,\mathrm{d}\sigma = -\iint_\sigma \frac{\nu}{2R\, r_0}\,\mathrm{d}\sigma,$$

because, according to the notations in Fig. 2.8:

$$\cos(r_0, n) = \cos(r_0, R) = -\frac{r_0}{2R}.$$

The relation (2.70) can consequently be written in the form:

$$\lim_{\rho\to\infty} V_e(\rho, \theta, \lambda) = \iint_\Sigma \left[\mu - \frac{\nu}{2R}\right]\frac{1}{r_0}\,\mathrm{d}\sigma + 2\pi\nu = f(\theta, \lambda). \tag{2.71}$$

By choosing the densities μ and ν in such a manner that the double integral vanishes, from (2.71) one gets:

$$\nu(\theta, \lambda) = \frac{1}{2\pi} f(\theta, \lambda) \text{ and}$$

$$\mu(\theta, \lambda) = \frac{1}{4\pi R} f(\theta, \lambda).$$

Substituting these values in (2.69) one gets:

$$V_e = \frac{1}{4\pi}\iint_\Sigma f(\theta', \lambda')\left[\frac{1}{rR} + 2\frac{\partial\left(\frac{1}{r}\right)}{\partial R}\right]\mathrm{d}\sigma, \tag{2.72}$$

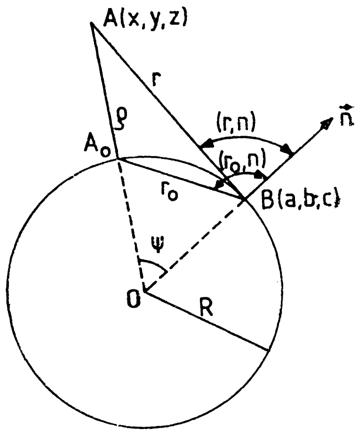


Fig. 2.8. Dirichlet's Exterior Problem for the Sphere

or, since one may deduce from Fig. 2.8 that:

$$\frac{1}{rR}+2\frac{\partial}{\partial R}\left(\frac{1}{r}\right)=\frac{\rho^2-R^2}{Rr^3},$$

one obtains the value of the function V_e, which solves *Dirichlet*'s exterior problem for the sphere in the form:

$$V_e(\rho, \theta, \lambda)=\frac{1}{4\pi}\iint_{\Sigma} f(\theta', \lambda')\frac{\rho^2-R^2}{Rr^3}\,d\sigma. \tag{2.73}$$

The relation (2.73) is known as *Poisson*'s integral for the exterior space.

For solving the interior problem of *Dirichlet*, one can obtain in an analogous manner:

$$V_i(\rho, \theta, \lambda)=\frac{1}{4\pi}\iint_{\Sigma} f(\theta', \lambda')\frac{\rho^2-R^2}{Rr^3}\,d\sigma. \tag{2.74}$$

According to (2.41), $1/r$, may be written as:

$$\frac{1}{r}=\sum_{n=0}^{\infty}\frac{R^n}{\rho^{n+1}}P_n(\cos\psi),$$

in which R, r and ψ have the meanings given in Fig. 2.8.

Taking the derivative of this relation, the square bracket in (2.72) becomes:

$$\frac{1}{Rr}+2\frac{\partial\left(\frac{1}{r}\right)}{\partial R}=\sum_{n=0}^{\infty}\frac{R^{n-1}}{\rho^{n+1}}(2n+1)P(\cos\psi).$$

Substituting this value in (2.72) one gets:

$$V_e=\sum_{n=0}^{\infty}\frac{R^{n-1}}{\rho^{n+1}}\frac{2n+1}{4\pi}\iint_{\sigma} f(\theta', \lambda')P_n(\cos\psi)R^2\sin\theta'\,d\theta' d\lambda',$$

or, using (2.41):

$$V_e=\sum_{n=0}^{\infty}\frac{R^{n+1}}{\rho^{n+1}}Y_n(\theta, \lambda). \tag{2.75}$$

In an analogous manner one obtains *Poisson*'s integral for the interior space in the form:

$$V_i=\sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Y_n(\theta, \lambda). \tag{2.76}$$

The relations (2.75) and (2.76) solve *Dirichlet*'s problem for the sphere.

In Physical Geodesy the sphere is used as a first approximation to the Earth's form; for a more accurate approximation one utilizes the rotation ellipsoid. To this end, we will analyse the solving of *Dirichlet*'s problem for the ellipsoid in the form given by *Heiskanen* and *Moritz* (1967).

One considers the ellipsoid of surface S_0:

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1,$$

whose minor semi-axis is b.

The potential of an ellipsoid is given by (2.43). As the ellipsoid has a rotation symmetry, its potential will not depend on λ, which reduces to considering in (2.43) $m = 0$. In this case, taking $\varphi^U = 90 - \theta$ in (2.43), one gets:

$$V=\sum_{n=0}^{\infty}\frac{Q_n\left(i\,\dfrac{\bar{b}}{E}\right)}{Q_n\left(i\,\dfrac{b}{E}\right)}A_nP_n(\sin\varphi^U).$$

By adding to the gravitational potential of the ellipsoid the potential of the centrifugal force:

$$Q=\frac{1}{2}\,\omega^2(E^2+\bar{b}^2)\cos^2\varphi^U,$$

one gets the gravity potential of the ellipsoid in the form:

$$U(\bar{b},\varphi^U)=\sum_{n=0}^{\infty}\frac{Q_n\left(i\,\dfrac{\bar{b}}{E}\right)}{Q_n\left(i\,\dfrac{b}{E}\right)}\,A_nP_n(\sin\varphi^U)+\frac{1}{2}\,\omega^2(\bar{b}^2+E^2)\cos^2\varphi^U. \qquad (2.77)$$

Solving *Dirichlet*'s problem for any rotation ellipsoid reduces to determining the coefficients A_n in (2.77). To this end, one specializes (2.77) to the case of the ellipsoid S_0 in which $U = U_0$ and $\bar{b} = b$. One obtains:

$$U_0(b,\varphi^U)=\sum_{n=0}^{\infty}A_nP_n\;(\sin\varphi^U)+\frac{1}{2}\,\omega^2(b^2+E^2)\cos^2\varphi^U, \qquad (2.78)$$

in which

$$b^2+E^2=a^2,$$

and according to (2.17) by putting $\varphi^U = 90 - \theta$, one gets:

$$\cos^2\varphi^U=\frac{2}{3}\,[1-P_2(\sin\varphi^U)].$$

Introducing these values into (2.78) one arrives at:

$$\left(A_0+\frac{1}{3}\,\omega^2a^2-U_0\right)P_0(\sin\varphi^U)+A_1P_1(\sin\varphi^U)+$$

$$+\left(A_2-\frac{1}{3}\,\omega^2a^2\right)P_2(\sin\varphi^U)+\sum_{n=3}^{\infty}A_nP_n(\sin\varphi^U)=0.$$

As S_0 is an equipotential surface, the potential U_0 must not depend on φ^U (it must have the same value at the pole and at the equator). This happens only in the case when the coefficients of $P_n(\sin \varphi^U)$ vanish. From this condition one derives the values of the coefficients A_n:

$$A_0 = U_0 - \frac{1}{3}\omega^2 a^2; \; A_1 = 0; \quad A_2 = \frac{1}{3}\omega^2 a^2; \quad A_3 = A_4 = \ldots = 0.$$

Introducing these values in (2.77) one gets:

$$U(\bar{b}, \varphi^U) = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{Q_0\left(i\dfrac{\bar{b}}{E}\right)}{Q_0\left(i\dfrac{b}{E}\right)} + \frac{1}{3}\omega^2 a^2 \frac{Q_2\left(i\dfrac{\bar{b}}{E}\right)}{Q_2\left(i\dfrac{b}{E}\right)} P_2(\sin \varphi^U). \tag{2.79}$$

The relation (2.79) solves *Dirichlet*'s problem for the ellipsoid.

2.4.3 The Solutions of Neumann's and Stokes' Boundary-Value Problems

Neumann's problem consists of determining the exterior potential for the case in which on the limiting surface one knows the potential derivative along the normal to the surface.

In order to solve the problem one considers a sphere of radius R, on whose surface one knows the value of the derivative of the exterior potential with respect to the normal. Expanding this function in surface spherical harmonics one gets:

$$\lim_{\rho \to R}\left(\frac{\partial V}{\partial n}\right) = \sum_{n=0}^{\infty} Y_n(\theta, \lambda).$$

The harmonic function which solves the exterior problem of *Neumann* for the sphere is:

$$V_e(\rho, \theta, \lambda) = -R \sum_{n=0}^{\infty} \left(\frac{R}{\rho}\right)^{n+1} \frac{Y_n(\theta, \lambda)}{n+1}, \tag{2.80}$$

which may be verified by differentiating (2.80) with respect to $n \equiv R$ and passing to the limit for $\rho \to R$.

The third boundary-value problem of potential theory consists in determining the harmonic function V_e for the exterior of the sphere of radius R in the case when on this sphere one knows the following linear combination of the harmonic function V_e and its derivative with respect to the normal:

$$\lim_{\rho \to R}\left[hV_e(\rho, \theta, \lambda) + g\frac{\partial V_e(\rho, \theta, \lambda)}{\partial \rho}\right] = f(\theta, \lambda) = \sum_{n=0}^{\infty} Y_n(\theta, \lambda). \tag{2.81}$$

The function V_e (ρ, θ, λ) solving the third boundary-value problem is:

$$V_e(\rho, \theta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{\rho}\right)^{n+1} \frac{Y_n(\theta, \lambda)}{h - \frac{g}{R}(n+1)}, \tag{2.82}$$

which can be verified by differentiating (2.82) with respect to ρ, for $\rho \rightarrow R$ and substituting this into (2.81).

The third boundary-value problem has a special importance for Physical Geodesy as it constitutes the mathematical background in determining the undulations of the geoid. To this end one chooses $h = 2/R$ and $g = 1$, the relation (2.82) becoming:

$$V_e(\rho, \theta, \lambda) = R \sum_{n=0}^{\infty} \left(\frac{R}{\rho}\right)^{n+1} \frac{Y_n(\theta, \lambda)}{1 - n}. \tag{2.83}$$

The function V_e thus defined solves the so-called boundary-value problem of Physical Geodesy. In the same way as *Dirichlet*'s problem has as solution *Poisson*'s integral, so the boundary-value problem of Physical Geodesy is solved by *Stokes'* integral. Moreover, the series defined by (2.83) bears the name of *Stokes'* series.

In *Stokes'* series the denominator vanishes for $n = 1$, which shows that the 1st-order spherical harmonic cannot be determined in this problem. Denoting by $X_1(\theta, \lambda)$ the 1st-order arbitrary spherical harmonic, the series (2.82) can be written as:

$$V_e(\rho, \theta, \lambda) = + R^2 \frac{Y_0(\theta, \lambda)}{\rho} - R \sum_{n=2}^{\infty} \left(\frac{R}{\rho}\right)^{n+1} \frac{Y_n(\theta, \lambda)}{n - 1} + \frac{R^2 X_1(\theta, \lambda)}{\rho^2}. \tag{2.84}$$

Introducing into (2.82) the value of $Y_n(\theta, \lambda)$ given by (2.41), one gets:

$$V_e(\rho, \theta, \lambda) = + \frac{R^2}{4\pi\rho} \iint_{\omega} f(\theta', \lambda')\, d\omega + \frac{R^2}{\rho^2} X_1(\theta, \lambda) -$$

$$- \frac{R}{4\pi} \iint_{\omega} f(\theta', \lambda') \left[\sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R}{\rho}\right)^{n+1} P_n(\cos\psi)\right] d\omega,$$

or writing

$$S(\rho, \psi) = \frac{1}{R} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R}{\rho}\right)^{n+1} P_n(\cos\psi), \tag{2.85}$$

one obtains *Stokes'* generalized integral formula in the form:

$$V_e(\rho, \theta, \lambda) = - \frac{R^2}{4\pi} \iint_{\omega} f(\theta', \lambda') \left[S(\rho, \psi) - \frac{1}{\rho}\right] d\omega + \frac{R^2}{\rho^2} X_1(\theta, \lambda), \tag{2.86}$$

in which the function $S(\rho, \psi)$, also called *Stokes'* function, has the explicit form:

$$S(\rho, \psi) = \frac{2}{r} + \frac{1}{\rho} - \frac{3r}{\rho^2} - \frac{5R}{\rho^2} \cos\psi \ln \frac{r + \rho - R\cos\psi}{2\rho} - \frac{5R}{\rho^2} \cos\psi. \quad (2.87)$$

The transition from the form (2.85) of *Stokes'* function to the explicit form (2.87) was not given here as it may be found in detail in the technical literature (e.g. *Brovar* et al. 1961).

For the case of the geoid, when $\rho = R$ one gets:

$$V_e(R, \theta, \lambda) = -\frac{R}{4\pi} \iint_\omega f(\theta', \lambda')[S(\psi) - 1]\, d\omega + X_1(\theta, \lambda), \quad (2.88)$$

in which:

$$S(\psi) = RS(R, \psi) = \operatorname{cosec}\frac{\psi}{2} - 6 \sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right), \quad (2.89)$$

represents the form of *Stokes'* function for the geoid.

3

Reference Surfaces. Coordinate Systems

In its early days, Geodesy utilized the level surfaces only for determining the altitudes of points, by defining them, appropriately for this purpose, as the locus of points of equal altitude. Such a definition is incomplete as long as the reference surface of the altitude system is not defined. The incomplete and relative character of this definition is avoided if in the level-surface study one appeals to their dynamic characteristics, viz. to the potential.

3.1 Level Surfaces

To every point situated within the field of action of gravity corresponds a value of the gravity potential $W(P)$. The locus of points having the same value of the gravity potential is called a *level surface*. The equation of the level-surface family is:

$$W = W_0 = \text{const.},$$

for which reason these are called equipotential surfaces.

The derivative of the potential W with respect to any direction s, expressed with the aid of the gradient, is:

$$\frac{\mathrm{d}W}{\mathrm{d}s} = \vec{s} \operatorname{grad} W = \vec{s}\,\vec{g} = s\,g \cos(\vec{s}, \vec{g}). \tag{3.1}$$

If $\vec{s}$ is taken along an equipotential surface, then $\mathrm{d}W = 0$ and (3.1) becomes:

$$s\,g \cos(\vec{s}, \vec{g}) = 0,$$

whence it follows that the gravity vector at a point is always perpendicular to the level surface passing through that point.

The level surfaces are functions of the gravity potential or, according to (2.45), functions of the density of the Earth's interior. For points situated outside the attractive masses, the potential can be expressed mathematically through very complicated formulae; the same characteristics are also possessed by the level surfaces situated completely outside the Earth. The level surfaces which pass partly or totally through the Earth's interior are no longer of an analytical character, because the potential does not have an analytical character inside the attractive masses. One can show that the level surfaces are continuous both inside and outside the Earth but their curvature radius varies discontinuously along with the density.

One of the level surfaces, of very great importance in Geodesy, is the surface of the seas and oceans in a quiet state. This surface was proposed by *Gauss* as *"mathematical figure of the Earth"* and introduced into Geodesy under the name of *geoid* by *Listing* in the year 1873.

The geoid surface is considered in classical Geodesy[1] as an intermediate surface of zero altitude and is used for reducing the geodetic measurements on the geoid. Although this surface has a very complicated form, whose analytical representation is impossible, it remains, thanks to the possibility of intuitive representation, the fundamental surface of Geodesy.

In connexion with the representation of the level surfaces as equipotential surfaces, one may define *the vertical* as being the space curve which intersects normally the family of the level surfaces of the gravity potential. In the technical literature one also meets this curve under the name of *plumb-line direction* or *gravity-vector direction*. The vertical represents the direction of maximum gradient of the gravity potential. Indeed, from (3.1) one sees that if $\vec{s}$ coincides with the normal $\vec{n}$ to the level surface, then $\cos(\vec{g}, \vec{n}) = 1$ and the gradient achieves its maximum value.

By assuming the vector dH to have the same direction as $\vec{n}$ but taken in the sense of direction of increasing altitudes (consequently opposite to $\vec{n}$), then from (3.1) there result two relations of very great importance for Physical Geodesy:

$$g = -\frac{\mathrm{d}W}{\mathrm{d}H}; \ \mathrm{d}W = -g\,\mathrm{d}H. \tag{3.2}$$

The first relation in (3.2) shows that the gravity is the negative vertical gradient of the gravity potential. The second relation constitutes the basis for defining the altitude systems. Both relations can be regarded as illustrating an inseparable connexion between the gravimetric and the dynamic conceptions of Geodesy. This connexion is also illustrated by the fact that the geodetic measurements refer almost entirely to the level surfaces and to the vertical direction, which are quantities of a dynamic character. This is the reason for which *H. Bruns* has formulated the aim of Geodesy as *"the determination of the level surfaces of the gravity field"*.

[1] In the technical literature one doesn't use a division of Geodesy into "classical" and "modern". The notion of classical geodesy was used here in order to distinguish the methods of determining the Earth with the aid of the geoid from the methods of the "geodesy without geoid" *(Bjerhammar 1964; Levallois 1963)*.

3.1.1 The Curvature of the Level Surfaces

Let $W(x, y, z) = \text{const.}$ be a level surface and P a point situated on this surface. At P one chooses the coordinate system X, Y, Z with the Z-axis oriented along the tangent to the vertical direction (Fig. 3.1).

By intersecting the level surface with the plane XPZ, one obtains the plane curve S_1, with the PX axis tangent at P.

The relation for calculating the curvature of S_1 is:

$$k = \frac{1}{R} = \frac{\dfrac{\partial^2 z}{\partial x^2}}{\left[1 + \left(\dfrac{\partial z}{\partial x}\right)^2\right]^{3/2}},$$

and the equation of this curve is

$$W(x, 0, z) = W_0,$$

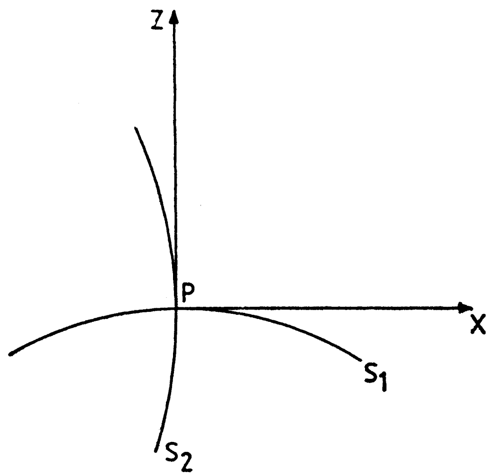


Fig. 3.1. Curvature of the Level Surfaces and of the Vertical

because, S_1 being in the ZPX plane, at any point of it $y = 0$. Considering z as a function of x, the partial derivative of the function $W(x, 0, z)$ is:

$$\frac{\partial W}{\partial x} + \frac{\partial W}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} = 0,$$

and taking the second derivative:

$$\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial^2 W}{\partial x \partial z}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\partial^2 W}{\partial z^2}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + \frac{\partial W}{\partial z}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

As the PX axis is tangent to S_1, it follows that:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0 \tag{3.3}$$

and then the 2nd-order derivative is obtained as:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -\frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial z}},$$

or, according to the first relation in (3.2), the expression for the curvature k_1 in the XPZ plane becomes:

$$k_1 = \frac{\partial^2 z}{\partial x^2} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{g},$$

or, denoting the 2nd-order derivative with respect to x by W_{xx}, one gets:

$$k_1 = \frac{W_{xx}}{g}. \tag{3.4}$$

In an analogous manner one obtains the projection of the level-surface curvature onto the YPZ plane in the form:

$$k_2 = \frac{W_{yy}}{g}, \tag{3.5}$$

and the average level-surface curvature will be:

$$k = -\frac{W_{xx} + W_{yy}}{2g}, \tag{3.6}$$

in which the minus sign is adopted by convention.

In order to express the average curvature in terms of known elements, one recalls *Poisson*'s formula:

$$W_{xx} + W_{yy} + W_{zz} = -4\pi G\delta + 2\omega^2,$$

whence, taking into account (3.6) and (3.2), one gets:

$$\frac{\partial g}{\partial H} = -2gk + 4\pi G\delta - 2\omega^2, \tag{3.7}$$

from which one can derive the average curvature in the form:

$$k = \frac{-\dfrac{\partial g}{\partial H} + 4G\delta - 2\omega^2}{2g}. \tag{3.8}$$

For the case in which the level surface is the ellipsoid surface itself, outside which no masses exist ($\delta = 0$), (3.7) will express the vertical gradient of normal gravity:

$$\frac{\partial \gamma}{\partial H} = -2(\gamma k_0 + \omega^2), \tag{3.9}$$

where k_0 denotes the ellipsoid curvature.

From (3.8) one sees that the magnitude of the average curvature depends on the vertical gradient of gravity, on the mean density δ and on the gravity, which leads to the conclusion that the level surfaces don't all have the same curvature, and *consequently are not parallel.*

3.1.2 The Curvature of the Vertical

As is known, the curvature k may also be expressed using the direction cosines α, β, γ of the tangent to the curve.

Considering the PZ-axis of direction cosines α, β, γ (Fig. 3.1) as tangent to the vertical S_2 at P, then the curvature of the vertical can be expressed by the relation:

$$k = \sqrt{\left(\frac{\partial\alpha}{\partial z}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2}. \tag{3.10}$$

The tangent to the field line and the gravity vector will have the same direction cosines, i.e.:

$$\alpha = \frac{1}{g} W_x; \quad \beta = \frac{1}{g} W_y; \quad \gamma = \frac{1}{g} W_z,$$

in which W_x, W_y and W_z denote the derivatives of the potential with respect to the coordinates x, y, and z. With these notations the derivatives of the direction cosines will be:

$$\frac{\partial\alpha}{\partial z} = \frac{g\dfrac{\partial W_x}{\partial z} - W_x\dfrac{\partial g}{\partial z}}{g^2};$$

$$\frac{\partial\beta}{\beta z} = \frac{g\dfrac{\partial W_y}{\partial z} - W_y\dfrac{\partial g}{\partial z}}{g^2};$$

$$\frac{\partial\gamma}{\partial z} = \frac{g\dfrac{\partial W_z}{\partial z} - W_z\dfrac{\partial g}{\partial z}}{g^2}.$$

Because in the coordinate system chosen in Fig. 3.1 the direction of the Z-axis coincides with the direction of the gravity vector, the components of the latter along the PX and PY-axes vanish, i.e.:

$$W_x = W_y = 0; \quad W_z = g,$$

which transforms the derivatives of the direction cosines into:

$$\frac{\partial \alpha}{\partial z} = \frac{1}{g}\frac{\partial W_x}{\partial z} = \frac{1}{g}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial x}\right) = \frac{1}{g}\frac{\partial^2 W}{\partial x\,\partial z} = \frac{1}{g}\frac{\partial}{\partial x}(g),$$

or

$$\frac{\partial \alpha}{\partial z} = \frac{1}{g} g_x.$$

As the curve S_2 is the projection of the vertical onto the PXZ plane, the partial derivative of gravity with respect to the y coordinate vanishes and then:

$$\frac{\partial \beta}{\partial z} = \frac{1}{g} g_y.$$

From the equality $W_z = g$ it is equally seen that:

$$\frac{\partial \gamma}{\partial z} = 0.$$

Consequently, the curvature radius of the projection of the vertical onto the XPZ plane is given by:

$$k_1 = \frac{1}{g} g_x$$

and the curvature radius of the projection of the vertical onto XPZ is:

$$k_2 = \frac{1}{g} g_y.$$

According to (3.10) the average curvature will be:

$$k = \frac{1}{g}\sqrt{g_x^2 + g_y^2}.$$

3.2 The Level Spheroid

Determining the family of level surfaces of the Earth is tantamount to determining the function of the gravity potential. Indeed, if this potential is known as a function of coordinates, then all the Earth's equipotential surfaces, including the geoid, can be determined.

The gravity potential as sum of the gravitational potential (2.46) and of the centrifugal-force potential may be expressed as follows:

$$W = G\sum_{n=0}^{\infty}\frac{1}{R^{n+1}}\iiint_{\tau} a^n P_n(\cos\theta)\ \delta d\tau + \frac{1}{2}R^2\omega^2\cos^2\varphi. \tag{3.11}$$

The relation (3.11) represents an infinite series, which leads, at once, to the conclusion that the exact evaluation of the gravity potential is not possible. The problem can be solved only approximately, the degree of approximation being given by the number of terms taken into consideration and by the additional conditions imposed.

If one considers only the first terms of the series (3.11), one obtains the spherical approximation, whose intuitive value is acknowledged but which from the practical point of view doesn't satisfy the accuracy requirements of Geodesy.

Depending on the number of terms taken into account in the series (3.11), one gets various forms of the gravity potential, corresponding to bodies called *level spheroids*. Consequently, in a second approximation, the Earth's surface may be represented by that of a level spheroid.

Considering only the first terms of the series (3.11) the gravity potential can be written, according to (2.61) in the form:

$$W = \frac{GM}{R} + \frac{G}{R^3}\left[\frac{1 - 3\sin^2\varphi}{2}\left(C - \frac{A+B}{2}\right) - \right.$$

$$\left. - \frac{3}{4}(A - B)\cos^2\varphi\cos 2\lambda\right] + \frac{1}{2}R^2\omega^2\cos^2\varphi.$$

For the case of bodies with rotational symmetry ($A = B$), this relation becomes:

$$W = \frac{GM}{R} + \frac{G}{2R^3}(1 - 3\sin^2\varphi)(C - A) + \frac{1}{2}R^2\omega^2\cos^2\varphi. \qquad (3.12)$$

The relation (3.12) expresses the gravity potential of *Bruns' spheroid.*

In the paragraph 2.3.5 it was shown that the gravity potential may be expressed in the form:

$$V = \frac{GM}{R}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{a_e}{R}\right)^n\left[J_n P_n(u) + \sum_{m=1}^{n}(J_{nm}\cos m\lambda + K_{nm}\sin m\lambda)P_{nm}(u)\right]\right\}.$$

If one takes into consideration only the first three terms of the expansion ($n = 0, 1, 2$), the gravity potential can be approximated by:

$$V \approx \frac{GM}{R}\left\{1 - \left(\frac{a_e}{R}\right)^2[J_{20}P_{20}(u) + (J_{21}\cos\lambda + K_{21}\sin\lambda)P_{21}(u) + \right.$$

$$\left. + (J_{22}\cos 2\lambda + K_{22}\sin 2\lambda)P_{22}(u)]\right\}.$$

In the case of the ellipsoid with rotational symmetry the moments of inertia are $A = B$, $D = E = F = 0$ and consequently the coefficients $a_{21} = b_{21} = b_{22} = 0$ and thus $K_{21} = K_{22} = J_{21} = 0$. In this case the gravity potential becomes:

$$V = \frac{GM}{a_e}\left[\frac{a_e}{R} - \left(\frac{a_e}{R}\right)^3 J_{20}\, P_{20}(u)\right].$$

The meaning of the coefficients of this series is that given in the paragraph 2.3.5, i.e.:

$$\begin{aligned} J_2 &= - a_{20}\frac{a_e}{GM}; \\ J_{21} &= a_{21}\frac{a_e}{GM}; \quad K_{21} = - b_{21}\frac{a_e}{GM}; \\ J_{22} &= a_{22}\frac{a_e}{GM}; \quad K_{22} = - b_{22}\frac{a_e}{GM}. \end{aligned} \tag{3.13}$$

For the case of rotational symmetry, when the z-axis coincides with the instantaneous axis of rotation of the Earth and the x- and y-axes coincide with the principal axes of inertia, the only one of the coefficients a_{nm} and b_{nm} in the potential expansion which is different from zero is (paragraph 2.3.5):

$$a_{20} = - \frac{G}{a_e^3}(C - A)$$

and consequently:

$$J_{20} = J_2 = \frac{1}{Ma_e^2}(C - A). \tag{3.14}$$

In this case the gravity potential takes the form:

$$V \approx \frac{GM}{R}\left[1 - \left(\frac{a_e}{R}\right)^2 J_2 P_2(u)\right].$$

By introducing the potential of the centrifugal force as well, one can obtain an approximation of the Earth's gravity potential in the form:

$$W \approx \frac{GM}{a_e}\left[\left(\frac{a_e}{R}\right) - \left(\frac{a_e}{R}\right)^3 J_2 P_2(u)\right] + \frac{R^2}{2}\,\omega^2 \cos^2\varphi. \tag{3.15}$$

The relation (3.15) represents the potential of *Bessel*'s spheroid. It can be accepted as a particular case of the potential of the level spheroids expressed (*Levallois* 1971) in the form:

$$W = \frac{GM}{a}\left[\left(\frac{a}{R}\right) - \left(\frac{a}{R}\right)^3 J_2 P_2(\cos\theta) - \left(\frac{a}{R}\right)^5 J_4 P_4(\cos\theta) \ldots\right] + \frac{R^2}{2}\,\omega^2\cos^2\varphi. \tag{3.16}$$

As a consequence of the hypothesis of rotational symmetry, the relation (3.16) contains only zonal terms and, as a consequence of the hypothesis of symmetry with respect to the equatorial plane, only even zonal harmonics.

In (3.16) the odd terms are missing because the spheroid is considered as a body having a symmetry equatorial plane. Indeed, in this case the potential must remain the same when changing θ into $(\pi - \theta)$, i.e. we always have the equality $P_n(\cos\theta) = P_n(\cos(\pi - \theta))$; this equality is valid only in the case of the even *Legendre* polynomials.

As the level spheroid is an equipotential surface, the potential W given by (3.16) must not depend on θ, or, in other words, the potential at the poles must equal that at the equator. For *Bruns'* spheroid, the potential at the equator where $R = a$ and $\theta = 90°$, is:

$$W_E = G\left[\frac{M}{a} + \frac{C - A}{2a^3}\right] + \frac{1}{2}\,\omega^2 a^2,$$

and at the pole, because $b = 0$ and $R = b$, one gets:

$$W_P = G\left[\frac{M}{b} - \frac{C - A}{b^3}\right].$$

As *Bruns'* spheroid is equipotential, we have the equality:

$$G\left[\frac{M}{a} + \frac{C - A}{2a^3}\right] + \frac{1}{2}\,\omega^2 a^2 - G\left[\frac{M}{b} - \frac{C - A}{b^3}\right] = 0,$$

or considering $1/a^3 \approx 1/b^3$:

$$GM\left(\frac{b - a}{ab}\right) + \frac{3}{2}\,\frac{G}{a^3}\,(C - A) + \frac{1}{2}\,\omega^2 a^2 = 0.$$

Dividing throughout by GM/b one obtains:

$$\frac{a - b}{a} = \frac{3}{2}\,\frac{b}{a}\,\frac{(C - A)}{Ma^2} + \frac{1}{2}\,\frac{\omega^2 a^2 b}{GM}. \tag{3.17}$$

Denoting:

$$f = \frac{a - b}{a}; \quad m = \frac{\omega^2 a^2 b}{GM} \tag{3.18}$$

and considering for the level spheroid $a/b \approx 1$, and taking also into account the relation (3.14), one gets:

$$f = \frac{3}{2}\,J_2 + \frac{1}{2}\,m, \tag{3.19}$$

a relation known as *Clairaut's first formula*. This relation expresses the connexion between the flattening f of the spheroid, the principal moments of inertia and the centrifugal force. It represents another illustrative example of linking the geometrical and the dynamic notions in the determination of the Earth's figure.

We have here adopted for m the notation in (3.16) suggested by *H. Jeffreys* (1962) in the form:

$$m = \frac{\omega^2 a^3}{GM}(1 - f)$$

and adopted by *Caputo* (1968) and by *Heiskanen* and *Moritz* (1967). It differs from that used by *Levallois* and *Kovalevski* (1971):

$$m = \frac{\omega^2 a^3}{GM}.$$

If the solution of the problem is restricted to the 1st-order approximations only, as is here the case, then $a \approx b$ and the notations may be considered as identical.

From the gravity potential of *Bruns'* spheroid (3.12) one can determine the gravity g as the derivative along the normal to the spheroid. As the angle between the radius of the spheroid at the point concerned and the normal to the spheroid at this point is small, instead of the derivative along the normal one may take the derivative along the radius R:

$$\frac{\mathrm{d}W}{\mathrm{d}R} = G\left[-\frac{M}{R^2} - \frac{3}{2}\frac{(C-A)}{R^4}(1 - 3\sin^2\varphi)\right] + \omega^2 R\cos^2\varphi,$$

whence, taking particular values for φ, one determines the gravity at the pole and at the equator as:

$$g_E = \frac{GM}{a^2}\left[1 + \frac{3}{2}\frac{C-A}{Ma^2}\right] + \omega^2 a; \quad g_P = \frac{GM}{b^2}\left[1 - \frac{3(C-A)}{Mb^2}\right]. \tag{3.20}$$

One can now determine the gravimetric flattening f of the spheroid defined as:

$$f^* = \frac{g_P - g_E}{g_E}. \tag{3.21}$$

Using the relations (3.20), after transformation, the relation (3.21) can be written in the form:

$$f^* = 2f - \frac{9}{2}J_2 + m, \tag{3.22}$$

which represents *Clairaut's second formula.*

If from (3.22) and (3.19) one removes J_2, one gets *Clairaut's third formula:*

$$f + f^* = \frac{5}{2}m. \tag{3.23}$$

Clairaut's formulae are important because they give the possibility of determining the flattening of the Earth's ellipsoid. Before using the Earth's artificial satellites, this was one of the best known methods for determining the flattening f. It should be emphasized that in this paragraph *Clairaut's* formulae were derived only in a first approximation. A more accurate form of these formulae will be given in the following paragraphs.

3.3 The Level Ellipsoid

The level spheroid represents a good approximation of the Earth's form but has the disadvantage that the mathematical relation through which it is defined is complicated.

It is for this reason that another method for solving the theoretical and practical problems of Geodesy has been adopted. One chooses a surface which under good conditions should approximate the Earth's form but which might be represented mathematically through a closed formula and for which one could determine the potential as well as the gravity. Such a closed surface is considered a normal form of the Earth and the corresponding potential and gravity are considered as *normal potential* (denoted by U) and *normal gravity* (denoted by γ). The difference between the Earth's actual potential and the normal potential is called the *perturbing potential.*

Dividing the gravity potential into normal potential and perturbing potential greatly facilitates the task of Physical Geodesy, which reduces to the determination through measurements of only small values of the latter, because the normal potential can be uniquely and completely determined in the case in which the Earth's mass M is given.

We will determine both the normal potential U and the normal gravity γ of the level ellipsoid.

3.3.1 The Normal Potential

For the rotation ellipsoid:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

the normal potential is mathematically defined by the relation (2.79) which solves *Dirichlet*'s problem for the ellipsoid. As the normal potential will be used for determining the normal gravity by the method followed by *Heiskanen* and *Moritz* (1967), we will subsequently transform (2.79) suitably so that the normal potential is a function of the ellipsoid's mass M.

To this end, writing in (2.31):

$$Z = i\,\frac{\bar{b}}{E}\,; \quad i = \pm\sqrt{-1}$$

and taking into account that:

$$\coth^{-1} i \frac{\bar{b}}{E} \approx \frac{1}{i} \cot^{-1} \frac{\bar{b}}{E} = - i \tan^{-1} \frac{E}{\bar{b}},$$

Legendre's functions of the second kind Q_0 and Q_2 become:

$$Q_0 = - i \tan^{-1} \frac{E}{\bar{b}}; \quad Q_2 = \frac{i}{2} \left[\left(1 + 3 \frac{\bar{b}^2}{E^2} \right) \tan^{-1} \frac{E}{\bar{b}} - 3 \frac{\bar{b}}{E} \right].$$

By introducing these values into (2.79) one gets:

$$U = \left(U_0 - \frac{1}{3} \omega^2 a^2 \right) \frac{\tan^{-1} \frac{E}{\bar{b}}}{\tan^{-1} \frac{E}{b}} + \frac{1}{3} \omega^2 a^2 \frac{q}{q_0} P_2(\sin \varphi^U), \tag{3.24}$$

in which the notations:

$$q = \frac{1}{2} \left[\left(1 + \frac{3\bar{b}^2}{E^2} \right) \tan^{-1} \frac{E}{\bar{b}} - \frac{3\bar{b}}{E} \right]; \tag{3.25}$$

$$q_0 = \frac{1}{2} \left[\left(1 + \frac{3b^2}{E^2} \tan^{-1} \frac{E}{b} - \frac{3b}{E} \right] \tag{3.26}$$

appeared.

For large values of $\bar{b}$ one can make the approximation:

$$\tan^{-1} \frac{E}{\bar{b}} \approx \frac{E}{\bar{b}} = \frac{E}{R}$$

and, because in (3.24) the first terms is much greater than the second one, it is possible to write:

$$U \approx \left(U_0 - \frac{1}{3} \omega^2 a^2 \right) \frac{1}{R} \frac{E}{\tan^{-1} \frac{E}{b}}.$$

On the basis of the same reasoning, (2.62) can be written as follows:

$$U \approx \frac{GM}{R}.$$

From the two last relations one may deduce:

$$GM = \left(U_0 - \frac{1}{3} \omega^2 a^2 \right) \frac{E}{\tan^{-1} \frac{E}{b}}, \tag{3.27}$$

or:

$$U_0 = \frac{GM}{E} \tan^{-1} \frac{E}{b} + \frac{1}{3} \omega^2 a^2. \tag{3.28}$$

Introducing into (3.24) the value of U_0 given by (3.28), by expressing the second-order *Legendre* polynomials in the form:

$$P_2(\sin \varphi^U) = \frac{3}{2} \sin^2 \varphi^U - \frac{1}{2}$$

and by adding the potential of the centrifugal force, one gets:

$$U(\bar{b}, \varphi^U) = \frac{GM}{E} \tan^{-1} \frac{E}{\bar{b}} + \frac{1}{2} \omega^2 a^2 \frac{q}{q_0} \left(\sin^2 \varphi^U - \frac{1}{3} \right) + \\ + \frac{1}{2} \omega^2 (\bar{b}^2 + E^2) \cos^2 \varphi^U. \tag{3.29}$$

The relation (3.29) represents the expression of the normal potential of the level ellipsoid as a function of *Stokes'* constants: a, b, GM and ω. If in (3.28) one introduces the value of the second eccentricity (Section 8.1):

$$e' = \frac{E}{b},$$

one gets:

$$U_0 = \frac{GM}{be'} \tan^{-1} e' + \frac{1}{3} \omega^2 a^2.$$

Expanding the term $\tan^{-1} e'$ in a series:

$$\tan^{-1} e' = e' - \frac{1}{3} e'^3 + \frac{1}{5} e'^5 \ldots, \tag{3.30}$$

the potential of the ellipsoid S_0 becomes:

$$U_0 = \frac{GM}{b} \left(1 - \frac{1}{3} e'^2 + \frac{1}{5} e'^4 \right) + \frac{1}{3} \omega^2 a^2. \tag{3.31}$$

The relation (3.31) expresses the 2nd-order expansion of the normal potential of the level ellipsoid.

3.3.2 The Normal Gravity

The normal gravity can be expressed as a function of the normal potential as follows:

$$\vec{\gamma} = \text{grad } U,$$

in which U is given by (3.29). The distance element in the ellipsoidal coordinate system $\bar{b}$, $\theta = 90 - \varphi^U$, is:

$$ds^2 = ds_{\bar{b}}^2 + ds_U^2 + ds_\lambda^2;$$

$$ds_b = \sqrt{R_1}d\bar{b}; \quad ds_\varphi{}^U = \sqrt{R_2}d\varphi^U; \quad ds_\lambda = \sqrt{R_3}d\lambda.$$

Here R_1, R_2 and R_3 denote the *Lamé* parameters. Taking into consideration the equality $E^2 = a^2 - b^2$, the *Lamé* parameters, defined in Section 2.1, may be written in the form:

$$\begin{aligned} R_1 &= \frac{\bar{b}^2 + E^2 \sin^2 \varphi^U}{\bar{b}^2 + E^2} = \frac{1}{a^2}(a^2 \sin^2 \varphi^U + \bar{b}^2 \cos^2 \varphi^U); \\ R_2 &= \bar{b}^2 + E^2 \sin^2 \varphi^U = R_1(\bar{b}^2 + E^2); \\ R_3 &= (\bar{b}^2 + E^2)\cos^2 \varphi^U. \end{aligned} \tag{3.32}$$

The increases along the coordinate curves will be:

$$ds_b = \sqrt{R_1}d\bar{b}; \quad ds_\varphi{}^U = \sqrt{R_1(\bar{b}^2 + E^2)}\, d\varphi^U; \quad ds_\lambda = R_3 d\lambda.$$

One can now determine the components of the normal gravity along the ellipsoidal coordinates in the form:

$$\begin{aligned} \gamma_{\bar{b}} &= \frac{\partial U}{\partial s_{\bar{b}}} = -\frac{1}{\sqrt{R_1}}\left[\frac{GM}{\bar{b}^2 + E^2} + \frac{\omega^2 a^2 E}{\bar{b}^2 + E^2}\frac{q'}{q_0}\left(\frac{1}{2}\sin^2 \varphi^U - \frac{1}{6}\right) - \right. \\ &\qquad \left. - \omega^2 \bar{b} \cos^2 \varphi^U\right]; \\ \gamma_{\varphi^U} &= \frac{\partial U}{\partial s_\varphi{}^U} = -\frac{\sin \varphi^U \cos \varphi^U}{\sqrt{R_1}}\left[-\frac{\omega^2 a^2}{\sqrt{\bar{b}^2 + E^2}}\frac{q}{q_0} + \omega^2 \sqrt{\bar{b}^2 + E^2}\right]; \\ \gamma_\lambda &= \frac{\partial U}{\partial s_\lambda} = 0. \end{aligned} \tag{3.33}$$

In (3.33) we used the notation of *Heiskanen* and *Moritz* (1967):

$$q' = 3\left(1 + \frac{\bar{b}^2}{E^2}\right)\left(1 - \frac{\bar{b}}{E} \text{ arc tang } \frac{E}{\bar{b}}\right) - 1.$$

For the ellipsoid $\bar{b} = b$ we have the equalities $q = q_0$ and:

$$\sqrt{\bar{b}^2 + E^2} = \sqrt{b^2 + E^2} = a,$$

and consequently the square bracket in the second relation in (3.33) will vanish. Thus the normal gravity on the ellipsoid $\bar{b} = b$, which is denoted by γ, may be written, by taking into consideration (3.32), in the form:

$$\gamma = -\frac{GM}{a\sqrt{a^2 \sin^2 \varphi^U + b^2 \cos^2 \varphi^U}}\left[1 + \frac{\omega^2 a^2}{GM} E \frac{q_0'}{q_0}\left(\frac{1}{2}\sin^2\varphi^U - \frac{1}{6}\right) - \right.$$
$$\left. - \frac{\omega^2 a^2}{GM} b \cos^2 \varphi^U\right], \tag{3.34}$$

in which we used the notation:

$$q_0' = 3\left(1 + \frac{b^2}{E^2}\right)\left(1 - \frac{b}{E}\arctan\frac{E}{b}\right) - 1.$$

Taking into account the notation adopted in (3.18) and considering in (3.34):

$$1 = \cos^2 \varphi^U + \sin^2 \varphi^U,$$

this relation becomes:

$$\gamma = \frac{GM}{a\sqrt{a^2 \sin^2 \varphi^U + b^2 \cos^2 \varphi^U}}\left[\left(1 + \frac{m}{3}\frac{e' q_0'}{q_0}\right)\sin^2 \varphi^U + \right.$$
$$\left. + \left(1 - m - \frac{m}{6}\frac{e' q_0'}{q_0}\right)\cos^2 \varphi^U\right]. \tag{3.35}$$

Assigning particular values to φ^U in (3.53) one can obtain the normal gravity at the equator ($\varphi^U = 0°$):

$$\gamma_E = \frac{GM}{ab}\left(1 - m - \frac{m}{6}\frac{e' q_0'}{q_0}\right) \tag{3.36}$$

and the normal gravity at the pole ($\varphi^U = \pm 90°$):

$$\gamma_P = \frac{GM}{a^2}\left(1 + \frac{m}{3}\frac{e' q_0'}{q_0}\right). \tag{3.37}$$

If one introduces into the expressions of q_0 and q_0' the value of the expansion in series of $\tan^{-1} e'$ given by (3.30) one gets the ratio:

$$\frac{e' q_0'}{q_0} = 3\left(1 + \frac{3}{7} e'^2 \ldots\right). \tag{3.38}$$

The relations (3.36) and (3.37) may then be written in the form:

$$\gamma_E = \frac{GM}{ab}\left(1 - \frac{3}{2} m - \frac{3}{14} e'^2 m\right); \tag{3.39}$$

$$\gamma_P = \frac{GM}{a^2}\left(1 + m + \frac{3}{7} e'^2 m\right), \tag{3.40}$$

which represent the gravity at the pole and at the equator in the 2nd-order approximation, very frequently used in Physical Geodesy.

From (3.39) one can obtain the product GM in the form:

$$GM = \gamma_E ab\left(1 - \frac{3}{2}\,m - \frac{3}{14}\,e'^2 m\right]^{-1} =$$

$$= \gamma_E ab\left[1 + \frac{3}{2}\,m + \frac{3}{14}\,e'^2 m + \frac{9}{4}\,m^2\right]. \tag{3.41}$$

On the basis of (3.41) the potential U_0 defined by (3.31) can be expressed as a function of the normal gravity at the equator. If in (3.31), in view of the notation (3.18), one considers:

$$\frac{1}{3}\,\omega^2 a^2 = \frac{1}{3}\,\frac{GM}{b},$$

the potential U_0 may be written as:

$$U_0 = \frac{GM}{b}\left(1 - \frac{1}{3}\,e'^2 + \frac{1}{5}\,e'^4 + \frac{m}{3}\right).$$

Introducing into this relation the value GM given by (3.41) and limiting ourselves to the 4 th-order terms in e' one gets another expression of the normal potential in the form:

$$U_0 = a\gamma_E\left(1 - \frac{1}{3}\,e'^2 + \frac{1}{5}\,e'^4 + \frac{11}{6}\,m - \frac{2}{7}\,e'^2 m + \frac{11}{4}\,m^2\right). \tag{3.42}$$

Similarly, from (3.39) and (3.40) one can calculate the sum:

$$\frac{a-b}{a} + \frac{\gamma_P - \gamma_E}{\gamma_E} = \frac{a\gamma_P - b\gamma_E}{a\gamma_E}. \tag{3.43}$$

As the ellipsoid is a particular case of the spheroid, the expression for the gravimetric flattening (3.21) becomes:

$$f^* = \frac{\gamma_P - \gamma_E}{\gamma_E}. \tag{3.44}$$

Taking into account (3.18), (3.36) and (3.37) the relation (3.43) may be written in the form:

$$f + f^* = \frac{\omega^2 b}{\gamma_E}\left(1 + \frac{e' q_0'}{2q_0}\right), \tag{3.45}$$

which represents the rigorous expression of *Clairaut's* formula. If in this relation one expresses q_0 and q_0' in terms of e' (3.38), one gets:

$$f + f^* = \frac{5}{2}\,\frac{\omega^2 b}{\gamma_E}\left(1 + \frac{9}{35}\,e'^2\right), \tag{3.46}$$

which represents the expression of *Clairaut*'s formula in terms of the second eccentricity.

At the conclusion of this paragraph, the normal gravity γ will be expressed in terms of the gravity at the pole and at the equator. Taking into account (3.39) and (3.40), the relation (3.34) becomes:

$$\gamma = \frac{1}{\sqrt{a^2 \sin^2 \varphi^U + b^2 \cos^2 \varphi^U}} \left(a\gamma_P \sin^2 \varphi^U + b\gamma_E \cos^2 \varphi^U\right).$$

From ellipsoidal Geodesy one knows the connexion relation between the reduced latitude φ^U and the geodetic latitude B (relation (8.24)).

Introducing this value into the expression of γ one gets:

$$\gamma = \frac{a\gamma_P \cos^2 B + b\gamma_E \sin^2 B}{\sqrt{a^2 \cos^2 B + b^2 \sin^2 B}} \tag{3.47}$$

which represents the well-known formula of *Somigliana*. After some transformations, and also taking into consideration (3.39) and (3.40), this relation can be written in the form:

$$\gamma = \gamma_E \left[1 + \left(-\frac{1}{2} e'^2 + \frac{5}{2} m + \frac{1}{2} e'^4 - \frac{13}{7} e'^2 m + \frac{15}{4} m^2\right) \sin^2 B + \right.$$
$$\left. + \left(-\frac{1}{8} e'^4 + \frac{5}{4} e'^2 m\right) \sin^4 B\right]. \tag{3.48}$$

By expressing the second eccentricity in terms of the flattening f, one can obtain another expression for the normal gravity, viz. (*Heiskanen* and *Moritz* 1967):

$$\gamma = \gamma_E \left[1 + \left(-f + \frac{5}{2} m + \frac{1}{2} f^2 - \frac{26}{7} fm + \frac{15}{4} m^2\right) \sin^2 B + \right.$$
$$\left. + \left(-\frac{1}{2} f^2 + \frac{5}{2} fm\right) \sin^4 B\right]. \tag{349}$$

This relation is frequently presented in the form:

$$\gamma = \gamma_E \left(1 + f^* \sin^2 B - \frac{1}{4} f_4 \sin^2 2B\right), \tag{3.50}$$

in which:

$$f^* = \frac{\gamma_P - \gamma_E}{\gamma_E} = f_2 + f_4;$$
$$f_2 = -f + \frac{5m}{2} + \frac{f^2}{2} - \frac{26\,fm}{7} + \frac{15m^2}{4}; \quad f_4 = -\frac{f^2}{2} + \frac{5fm}{2}. \tag{3.51}$$

3.3.3 The Expansion of the Normal Potential in Spherical Harmonics

In the preceding paragraphs the normal potential of the ellipsoid was expressed in terms of ellipsoidal harmonics. For Physical and Dynamic Geodesy it is, however, useful to express this potential in terms of spherical harmonics.

The problem can be solved either by starting from the potential of *MacLaurin*'s homogeneous ellipsoid (*Levallois* and *Kovalevsky* 1971) or by specializing the general theory for the bi-axial ellipsoid (*Caputo* 1968) or, finally, by the procedure of identifying the coefficients (*Heiskanen* and *Moritz* 1967). In what follows we will use the latter procedure, which is more advantageous thanks to the fact that it avoids the laborious task of transforming the ellipsoidal coordinates into spherical coordinates.

The gravity potential of the ellipsoid, assumed to have symmetry both of rotation and with respect to the equatorial plane, can be expressed through a relation of the type (3.16):

$$V = \frac{GM}{R} + A_2 \frac{P_2 \cos(\theta)}{R^3} + A_4 \frac{P_4(\cos\theta)}{R^5} + \dots \tag{3.52}$$

The solution of the problem is reduced to determining the coefficients A_{2n}. To this end, by using the expansion in series:

$$\tan^{-1}\frac{E}{b} = \frac{E}{b} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}\left(\frac{E}{b}\right)^{2n+1},$$

the quantity q, given by (3.25) will have the form:

$$q = -\sum_{n=1}^{\infty} (-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{b}\right)^{2n+1}.$$

Introducing these quantities into the expression of the gravitational potential of the ellipsoid (expressed as a function of ellipsoidal harmonics) which according to (3.29) is:

$$V = \frac{GM}{E}\,\text{arc}\tan\frac{E}{b} + \frac{1}{3}\,\omega^2 a^2 \frac{q}{q_0} P_2(\sin\varphi^U),$$

one gets:

$$V = \frac{GM}{b} + \frac{GM}{E}\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)}\left(\frac{E}{b}\right)^{2n+1} -$$

$$- \frac{\omega^2 a^2}{3q_0}\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{b}\right)^{2n+1} P_2(\sin\varphi^U).$$

Introducing the value of m defined by (3.18) and recalling the expression of the second eccentricity $e' = E/b$ one arrives at:

$$V = \frac{GM}{b} + \sum_{n=1}^{\infty}(-1)^n \frac{GM}{(2n+1)E}\left(\frac{E}{b}\right)^{2n+1}\left[1 - \frac{me'}{3q_0}\,\frac{2n}{2n+3}\,P_2(\sin\varphi^U)\right].$$

For a point situated outside the ellipsoid, on the rotation axis ($\varphi^U = 90°$), we have the equalities $P_n(1) = 1$ and $\bar{b} = b = R$, the previous relation becoming:

$$V = \frac{GM}{R} + \sum_{n=1}^{\infty} (-1)^n \frac{GME^{2n}}{2n+1}\left(1 - \frac{2n}{2n+3}\frac{me'}{3q_0}\right)\frac{1}{R^{2n+1}}. \quad (3.53)$$

For the same point situated outside the ellipsoid, with $\varphi^U = 90°$ and $\theta = 90° - \varphi^U = 0$, as $P_n(\cos\theta) = P_n(1) = 1$, the relation (3.52) takes the form:

$$V = \frac{GM}{R} + \sum_{n=1}^{\infty} A_{2n}\frac{1}{R^{2n+1}}.$$

Comparing this relation with (3.53) one gets:

$$A_{2n} = (-1)^n \frac{GME^{2n}}{2n+1}\left(1 - \frac{2n}{2n+3}\frac{me'}{3q_0}\right), \quad (3.54)$$

a relation which solves the proposed problem.

One can further obtain the expressions of the coefficients J_{2n} in terms of the ellipsoid's elements. To this end one specializes (3.54) for $n = 1$ and obtains:

$$A_2 = -\frac{1}{3}GME^2\left(1 - \frac{2}{15}\frac{me'}{q_0}\right), \quad (3.55)$$

On the other hand it was shown in § 2.3.5 that:

$$A_{nm} = -GM\ a^n J_{nm},$$

and consequently:

$$A_{20} = -GMa^2 J_{20}.$$

Taking into account (3.14), the coefficient A_{20} becomes:

$$A_2 = G(A - C).$$

Introducing this expression into (3.55) one gets:

$$G(C - A) = \frac{1}{3}GME^2\left(1 - \frac{2}{15}\frac{me'}{q_0}\right). \quad (3.56)$$

From (3.54) and (3.56), removing q_0 and considering $E = ea$, it follows that:

$$A_{2n} = (-1)^n \frac{3GMa^{2n}e^{2n}}{(2n+1)(2n+3)}\left(1 - n + 5n\frac{C-A}{ME^2}\right) = a^{2n}\bar{A}_{2n}GM. \quad (3.57)$$

By introducing (3.57) into (3.52) one gets:

$$V=\frac{GM}{R}+\frac{GMa^2}{R^3}\bar{A}_2P_2(\cos\theta)+\frac{GMa^4}{R^5}\bar{A}_4P_4(\cos\theta)+\ldots=$$
$$=\frac{GM}{a}\left[\frac{a}{R}-\left(\frac{a}{R}\right)^3\bar{A}_2P_2(\cos\theta)-\left(\frac{a}{R}\right)^5A_4P_4(\cos\theta)\ldots\right]. \tag{3.58}$$

From comparing (3.58) and (3.16) it follows that:

$$J_{2n}=(-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\left(1-n+5n\frac{C-A}{ME^2}\right). \tag{3.59}$$

The last term of the bracket in the above relation can be written according to (3.56) in the form:

$$\frac{C-A}{ME^2}=\frac{1}{3}\left(1-\frac{2}{15}\frac{me'}{q_0}\right).$$

If in the relation (3.26) defining the quantity q_0 one considers:

$$\frac{E}{b}=e';$$

$$\tan^{-1}e'=e'-\frac{1}{3}e'^3+\frac{1}{5}e'^5-\frac{1}{7}e'^7+\ldots,$$

the quantity q_0 becomes:

$$q_0=\frac{2}{15}e'^3\left(1-\frac{6}{7}e'^2+\ldots\right).$$

With these specifications one may write:

$$\frac{C-A}{ME^2}=\frac{1}{3}-\frac{2}{45}\frac{me'}{q_0}=\frac{1}{3}-\frac{2}{45}\frac{me'}{\frac{2}{15}e'^3\left(1-\frac{6}{7}e'^2\right)}=$$

$$=\frac{1}{3}-\frac{1}{3}\frac{me'}{e'^3}\left(1+\frac{6}{7}e'^2\ldots\right)=\frac{1}{e'^2}\left(\frac{1}{3}e'^2-\frac{1}{3}m-\frac{2}{7}me'^2\right).$$

Taking into consideration that:

$$b=\frac{a}{\sqrt{1+e'^2}},$$

the above relation takes the form:

$$\frac{C-A}{Mb^2e'^2}=\frac{(C-A)(1+e'^2)}{Ma^2e'^2}=\frac{1}{e'^2}\left(\frac{1}{3}e'^2-\frac{1}{3}m-\frac{2}{7}me'^2\right).$$

From the above relation one can derive the value of the coefficient J_2, which according to (3.14) is:

$$J_2 = \frac{C-A}{Ma^2} = (1-e'^2)\left(\frac{1}{3}e'^2 - \frac{1}{3}m - \frac{2}{7}e'^2 m\right) =$$

$$= \frac{1}{3}e'^2 - \frac{1}{3}m - \frac{1}{3}e'^4 + \frac{1}{21}e'^4 m. \tag{3.60}$$

Analogously, from (3.59) one gets:

$$J_4 = -\frac{1}{5}e'^4 + \frac{2}{7}e'^2 m.$$

The gravitational potential of the ellipsoid may consequently be expressed by a series of spherical harmonics by means of (3.52) in which the coefficients are given by the relation (3.54), or with the aid of a series of the form (3.16) in which the coefficients are given by the formula (3.59).

The level ellipsoid and the gravitational field can be determined by means of 4 constants: U_0, M, $C - \dfrac{A+B}{2}$ and ω. From among the family of the level ellipsoids which can be defined in this way, there exists only one having the potential and mass equal to the potential W_0 and mass M of the geoid and the difference of the moments of inertia and the angular velocity identical with those of the Earth. In very many respects this ellipsoid can be considered as the best representation of the Earth for which reason it is frequently encountered in the technical literature under the name of *mean terrestrial ellipsoid.*

The adoption of the ellipsoid as reference surface in Physical Geodesy may be supported by the following three reasons:

(1) *The ellipsoid is used as reference surface for processing the triangulation; the same ellipsoid can also be used as physical reference surface in Physical Geodesy.*

(2) *The utilization of the closed formulae allows the accurate determination of the normal field of gravity.*

(3) *The level ellipsoid represents the first approximation in the expansion of the Earth's actual potential W in ellipsoidal harmonics.*

3.4 Coordinate Systems

Together with the diversification of the investigation methods of Geodesy, there have appeared a multitude of coordinate systems, each method adopting that system which better serves its declared aims.

The coordinate systems utilized in Geodesy can be classified on the following criteria *(Zhongolovich* 1962):

1) **Classification according to destination:**

1.1) *Depending on the observer's position:*
1.1.1) Geocentric coordinate system.
1.1.2) Quasi-geocentric (rectangular, geodetic, astronomical, with reduced latitude) coordinate system.
1.2) *Coordinate systems defining the position of a moving body (artificial satellite) and its motion:*
1.2.1) Sideral geocentric coordinate system.
1.2.2) Orbital coordinate system.
1.2.3) Topocentric coordinate system.
1.2.4) Satellite-centric coordinate system.

2) **Classification according to the form of the coordinate system:**

2.1) *Rectangular coordinate system* (X, Y, Z).
2.2) *Spherical coordinate system* (ρ, θ, λ).
2.3) *Ellipsoidal coordinate system* $(\bar{b}, \theta, \lambda)$.
2.4) *Natural coordinate system:*
2.4.1) Geodetic coordinate system (B, L, H).
2.4.2) Astronomical coordinate system (Φ, Λ, W).

3) **Classification according to the disposition of the coordinate system's origin:**

3.1) *Geocentric coordinate system.*
3.2) *Quasi-geocentric coordinate system (geodetic reference system).*
3.3) *Topocentric coordinate system.*
3.4) *Satellite-centric coordinate system.*

In what follows there will be given a brief description only of those coordinate systems which are widely used in Geodesy.

The rectangular coordinate system X, Y, Z

The origin of this system is situated at the ellipsoid's centre, the Z-axis coincides with the polar axis and the X- and Y-axes are disposed in the ellipsoid's equatorial plane (Fig. 3.2, a).

Generally, the X-axis is assumed to be situated in the plane of the origin meridian *(Greenwich)*.

The geodetic coordinate system B, L, H

The geodetic latitude B of any point P (Fig. 3.2, *a*) is defined as the angle formed by the normal to the ellipsoid at P and the equator's plane. *The geodetic longitude* L is defined as the dihedral angle between the origin-meridian plane and the geodetic-meridian plane of the point P. The geodetic-meridian plane is defined as the plane containing the polar axis and the normal at P to the ellipsoid.

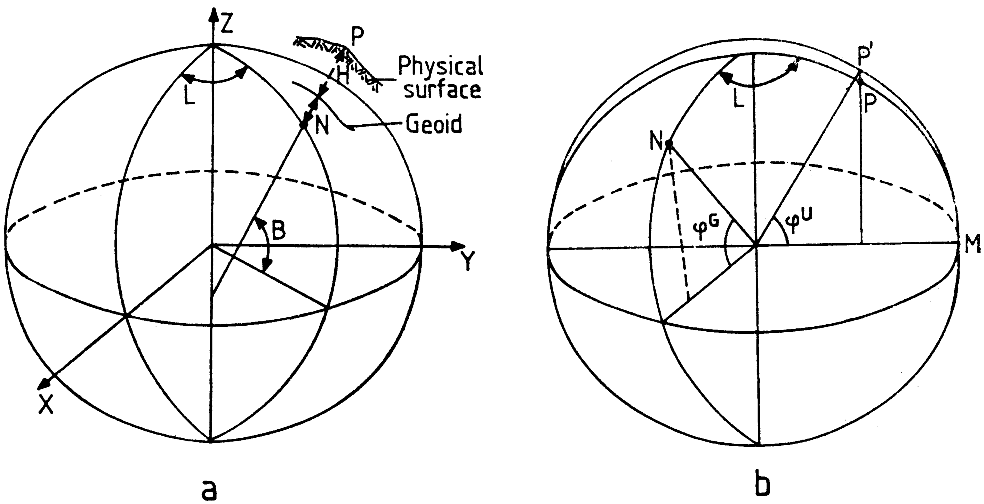


Fig. 3.2. Coordinate Systems: a — geodetic; b — with reduced latitude

As third coordinate of this system, we take *the altitude of the point above the ellipsoid,* which has two components:

(1) *The altitude* H defined as the difference between the geoid and the point P of the physical surface measured along the vertical.

(2) *The geoid undulation* N defined as the difference between the geoid and the ellipsoid.

In other words, the altitude of a point with respect to the ellipsoid is defined as:

$$H^E = H + N. \tag{3.61}$$

The astronomical coordinate system Φ, Λ, H

The astronomical latitude Φ of the point P is the angle formed by the vertical at P and the equator's plane.

The astronomical longitude Λ is the dihedral angle formed by the astronomical meridian of the point P and the origin meridian. The astronomical meridian of the point P is defined as the plane containing the polar axis and the vertical through P.

As third coordinate one similarly considers the altitude with respect to the ellipsoid:

$$H^E = H^{OR} + N,$$

in which *the orthometric altitude* is expressed with the aid of the actual potential $W(P)$. According to (3.2):

$$H^{OR} = \int_{W_0}^{W(P)} \frac{\mathrm{d}W}{g}, \tag{3.62}$$

in which W_0 denotes the potential of the geoid and $W(P)$ denotes the gravity potential at the point P.

The coordinate system with reduced latitude φ^U, L

The longitude of this coordinate system is identical with the geodetic longitude L.

The reduced latitude φ^U is obtained with the aid of the following auxiliary construction: one describes a semi-circle of radius a with the centre O of the ellipse (Fig. 3.2, *b*). Prolonging the ordinate of the point P one obtains the point P' as the intersection of this ordinate with the semi-circle of radius a. The angle POM will be the reduced latitude φ^U of the point P.

As this coordinate system is generally utilized for defining the position of points situated on the ellipsoid, the third coordinate is taken as a rule equal to be zero.

The geocentric coordinate system φ^G, L

One of the coordinates of this system is the geodetic longitude L. The second one, the geocentric latitude φ^G, is defined by the angle between the radius vector of the point and the plane of the equator (Fig. 3.2, *b*). This coordinate system is frequently used in Astronomy and in Satellite Geodesy.

3.5 Altitude Systems

The two components of the coordinate H^E (3.61) differ from one another inasmuch as the former — the altitude of a point with respect to the geoid — is a quantity which is determined by measurements with respect to a material reference surface whereas the latter — the geoid undulation — is determined by calculation with respect to an imaginary reference surface.

In the category of altitude systems there, we will only include those altitudes whose principal part can be determined through measurements. For this reason, the second quantity in (3.61) will be called *geoid undulation* and not geoid altitude.

All levelling measurements have as origin point a point O situated at the mean sea-level, i.e. on the geoid. Let $W(O)$ be the potential of this point. Let also A be a point situated on the Earth's physical surface through which passes the equipotential surface $W(A) = \text{const}$. The difference of potential between the two points is, according to (3.2):

$$\mathrm{d}W = W(O) - W(A) = -g\,\mathrm{d}H,$$

in which:

$$H_A - H_0 = -\int_0^A \frac{\mathrm{d}W}{g}. \tag{3.63}$$

The potential difference $W(O) - W(A)$ bears the name of geopotential number and is denoted by C, being defined as follows:

$$C = W(O) - W(A) = \int_0^A g\,\mathrm{d}H. \tag{3.64}$$

The geopotential number, a notion adopted at the *I. A. G. Subcommission meeting* at *Florence* in the year 1955, has as units the so-called *geopotential units,* denoted by g.p.u.:

1 g.p.u. = 1 kgal meter = 1,000 gal meter.

The relation (3.63) can be written in terms of geopotential numbers in the form:

$$H_A = -\frac{1}{\bar{g}}\,[W(A) - W(O)] = \frac{C}{\bar{g}},$$

in which $\bar{g}$ denotes the mean gravity along the vertical passing through A.

The preceding relation may be regarded as a physical definition of the altitude, preferable to the geometrical definition for reasons to be analysed shortly.

Generically, the altitude can consequently be defined as follows:

$$H = C/F, \tag{3.65}$$

in which F denotes any dynamic quantity, which transforms (3.65) into the definition of a whole altitude system. Depending on the manner in which one defines the quantity F one obtains various kinds of altitudes.

Thus, for $F = \gamma^\circ$, in which γ° denotes the normal gravity at any latitude, one gets the *dynamic altitude* of the point A:

$$H_A^D = \frac{C(A)}{\gamma^\circ}. \tag{3.66}$$

For $F = \bar{g}$ one obtains the *orthometric altitude:*

$$H_A^{OR} = \frac{C(A)}{\bar{g}}, \tag{3.67}$$

where $\bar{g}$ represents the mean gravity along the vertical at A.

For $F = \bar{\gamma}_A$, one obtains the *normal altitude:*

$$H_A^N = \frac{C(A)}{\bar{\gamma}_A}, \tag{3.68}$$

in which $\bar{\gamma}_A$ denotes the mean normal gravity along the normal at A to the ellipsoid.

3.5.1 The Dynamic Altitude

The altitudes of this kind have the great advantage that they do not depend on the line followed in levelling. Indeed, from (3.66) one sees that these altitudes only differ from the geopotential numbers by a proportionality factor and the geopotential numbers which represent a difference of potential do not depend on the line followed in levelling. This is the great advantage of defining the altitudes by means of dynamic quantities.

The dynamic altitude has no geometric meaning but only a physical one. The dividing of the geopotential numbers by γ (kgal) hides this physical meaning, leading to a dimensional equation of the form:

$$[H^D] = \frac{\text{gal} \cdot \text{meter}}{\text{gal}} = [\text{m}],$$

which, however, has no geometric meaning.

It would be more correct, from the dimensional point of view, for the dynamic altitude to be expressed through the following measure unit:

$$\frac{\text{g} \cdot \text{p} \cdot \text{u}}{\text{kgal}}.$$

The main quantity of the dynamic altitude is constituted by the altitude provided by the measurements of geometric levelling, to which is joined the so-called *dynamic correction* whose form will be defined in the sequel.

According to (3.66) the difference of two dynamic elevations is:

$$H_B^D - H_A^D = \frac{1}{\gamma_0}(C_B - C_A) = \frac{1}{\gamma_0}\int_A^B g\,\mathrm{d}h. \tag{3.69}$$

Adding and subtracting γ_0 in the preceding integral, one gets:

$$\Delta H_{AB}^D = \int_A^B \mathrm{d}h + \int_A^B \frac{g - \gamma_0}{\gamma_0}\,\mathrm{d}h = \frac{1}{\gamma_0}(C_B - C_A) \tag{3.70}$$

in which the first integral represents the level difference measured by geometric levelling and the second integral represents the dynamic correction:

$$\varepsilon_{AB}^D = \int_A^B \frac{g - \gamma_0}{\gamma_0}\,\mathrm{d}h. \tag{3.71}$$

Consequently, the difference of dynamic altitude is:

$$\Delta H_{AB}^D = \Delta H_{AB}^{m\breve{a}s} + \varepsilon_{AB}^D.$$

The relation (3.71) may be used for calculating the difference of the geo-potential numbers. Indeed, from (3.69), (3.70) and (3.71) it follows that:

$$C_B - C_A = \gamma_0 \Delta h_{AB}^{m\breve{a}s} + \gamma_0\, \varepsilon_{AB}^{D}.$$

In this relation all the quantities on the right side can be determined, and consequently the difference of two geopotentiat numbers can be determined.

3.5.2 The Orthometric Altitude

The orthometric altitude of the point A (Fig. 3.3) is defined by (3.67) in which the mean value of the gravity along the normal AA' is defined by:

$$\bar{g} = \frac{1}{H}\int_{0}^{H} g(z)\,\mathrm{d}z, \tag{3.72}$$

with $g(z)$ denoting the gravity at the running point situated on the vertical of the point A at the orthometric altitude z. The orthometric altitude has a clear geometric interpretation, as it represents the distance between the geoid and the point A on the Earth's physical surface as measured along the vertical at A. From this interpretation and from the relation (3.67) defining this kind of altitude it follows that we need to know the value of the gravity along the vertical at P, from which to calculate the mean gravity $\bar{g}$. To determine this quantity one utilizes *Poincaré* and *Prey*'s reduction, whose principle will be given later.

Let M be the running point on the vertical AA'. Then:

$$g = g_A - \int_{M}^{A} \frac{\partial g}{\partial H}\,\mathrm{d}H. \tag{3.73}$$

It is difficult to express correctly the vertical gradient of the gravity through (3.7), because one doesn't know the curvature k of the level surface so then one admits the hypothesis according to which the ratio of the curvature

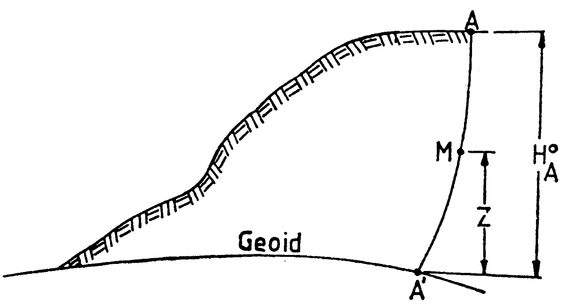


Fig. 3.3. The Orthometric Altitude

k of any level surface to the curvature k_0 of the ellipsoid is approximately equal to the ratio of the normal gravity to the gravity g at A, i.e.:

$$gk \approx \gamma k_0,$$

or:

$$k = \frac{\gamma}{g} k_0.$$

Introducing this relation into(3.7) and using the formula (3.9), one gets:

$$\frac{\partial g}{\partial H} = \frac{\partial \gamma}{\partial H} + 4\pi G\delta. \tag{3.74}$$

The ratio $\partial\gamma/\partial H$ is the vertical gradient of the normal gravity, which as will be seen in Chapter 5, has the value:

$$\frac{\partial \gamma}{\partial H} = -\ 0.3086 \text{ gal/km}. \tag{3.75}$$

The direction of application of this gradient is from A' towards A, i.e. opposite to the direction of the gravity vector; this explains the minus sign in (3.75). Introducing into (3.74) the numerical values for $G = 6.67 \times 10^{-8}$ $\text{cm}^3\text{g}^{-1}\text{s}^{-2}$ and $\delta = 2.67$ g/cm^3 and taking into account (3.75), one gets:

$$\frac{\partial g}{\partial H} = -\ 0.0848 \text{ gal/km}.$$

Therewith, (3.73) takes the form of *Poincaré* and *Prey*'s reduction:

$$g_M(z) = g_A + 0.0848 \int_{H_M}^{H_A} \mathrm{d}h = g_A + 0.0848(H_A - z). \tag{3.76}$$

One can now obtain, with the aid of (3.72), the mean value of the gravity along the vertical at A:

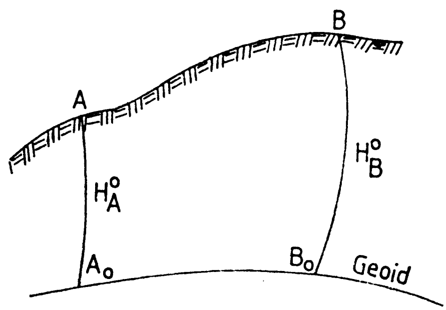
$$\bar{g} = \frac{1}{H} \int_{0}^{A} [g_A + 0.0848\ (H - z)]\mathrm{d}z = g_A + 0.0424\ H.$$

Hence, from (3.67) one gets:

$$H^{OR} = \frac{C}{g + 0.0424\ H}. \tag{3.77}$$

This relation is utilized more as a definition of the orthometric altitude than as a calculation relation, because, generally, the geopotential numbers are not accurately known. For the practical calculation of these altitudes one prefers the method of correcting the results of the geometric levelling with the so-called orthometric correction.

Fig. 3.4. The Orthometritic Correction



To derive this correction, one uses the following device:

$$\Delta H^{OR}_{AB} = H^{OR}_B - H^{OR}_A = H^{OR}_B - H^{OR}_A - H^D_B + H^D_A + (H^D_B - H^D_A) =$$
$$= \Delta H^D_{AB} + (H^{OR}_B - H^D_B) - (H^{OR}_A - H^D_A). \tag{3.78}$$

The dynamic altitude of the points A and B (Fig. 3.4) is:

$$H^D_A = H^{OR}_A + \varepsilon^D_{A_0A};\ H^D_B = H^{OR}_B + \varepsilon^D_{B_0B},$$

in which H_A and H_B denote the altitudes obtained from the geometric levelling.

For view of the preceding relation, (3.78) becomes:

$$\Delta H^{OR}_{AB} = \Delta H^{meas}_{AB} + \varepsilon^D_{AB} + \varepsilon^D_{A_0A} - \varepsilon^D_{B_0B},$$

in which ΔH^{meas}_{AB} represents the level difference obtained from the geometric levelling and:

$$\varepsilon^{OR}_{AB} = \varepsilon^E_{AB} + \varepsilon^D_{A_0A} - \varepsilon^D_{B_0B}, \tag{3.79}$$

is the orthometric correction.

Using the relations (3.71) the orthometric correction may be written in a form utilizable in practice:

$$\varepsilon^{OR}_{AB} = \sum_A^B \frac{g - \gamma_0}{\gamma_0}\, dh + \frac{\bar{g}_A - \gamma_0}{\gamma_0} H_A - \frac{g_A - \gamma_0}{\gamma_0} H_B, \tag{3.80}$$

in which g denotes the mean value of the gravity along the distance between two consecutive bench marks A and B, $\bar{g}_A$ and $\bar{g}_B$ denote the mean gravity along the verticals $\overline{AA_0}$ and $\overline{BB_0}$ and γ_0 represents the normal gravity at any latitude (e.g. $\gamma_{45°}$).

3.5.3 The Normal Altitude

The normal altitude, a notion introduced by *M. S. Molodenski* in the year 1945, is mathematically defined by (3.68). This kind of altitude has no clear geometric meaning, being more easily accessible to intuition through the medium of some dynamic quantities.

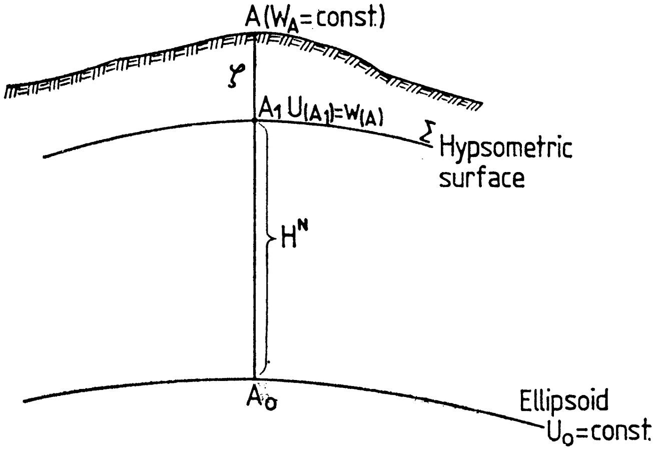


Fig. 3.5. The Normal Altitude

Let A be a point on the Earth's physical surface, situated on the equipotential surface $W(A) = \text{const.}$ (Fig. 3.5).

On the normal to the ellipsoid $U_0 = \text{const.}$ one chooses the point A_1 whose normal potential $U(A_1)$ is equal to the actual potential $W(A)$. The distance A_0A_1 measured along the normal to the ellipsoid defines the normal altitude of the point A.

If for every point M of the Earth's physical surface one defines the corresponding point M_1, as was previously shown, the points M_1 will lie on a surface Σ called the *hypsometric surface (Molodenski* et al. 1960*)*.

The distance AA_1 between the Earth's physical surface and the hypsometric surface is called the *height anomaly* and is denoted by ξ.

The practical determination of the normal altitude according to the scheme presented in Fig. 3.5 is not possible because the reference surface — the ellipsoid — is not a material surface. To solve the problem, *M. S. Molodenski* has introduced an auxiliary surface called *quasi-geoid*. To every point M_i of the Earth's physical surface there correspond a normal altitude $H^N_{M_i}$ and a height anomaly ξ_{M_i}. If on the normals $M_iM_i^0$ (Fig. 3.6) one takes the segments $M_i^0M_i' = \xi_{M_i}$, one gets the quasi-geoid's surface.

From the definition of the quasi-geoid there follow two very important properties of it:

(1) *The quasi-geoid is not an equipotential surface,* because the hypsometric surface itself (by means of which one determines the quasi-geoid) is not an equipotential surface.

(2) *On the surface of the seas and oceans the quasi-geoid coincides with the geoid,* because in this case the geopotential numbers of the points M_i will be zero and according to (3.68) the normal altitudes too will be equal to zero. It follows that the altitude of the points situated on the sea surface, measured from the ellipsoid, will equal the height anomaly or, in other words, the geoid corresponds here to the quasi-geoid.

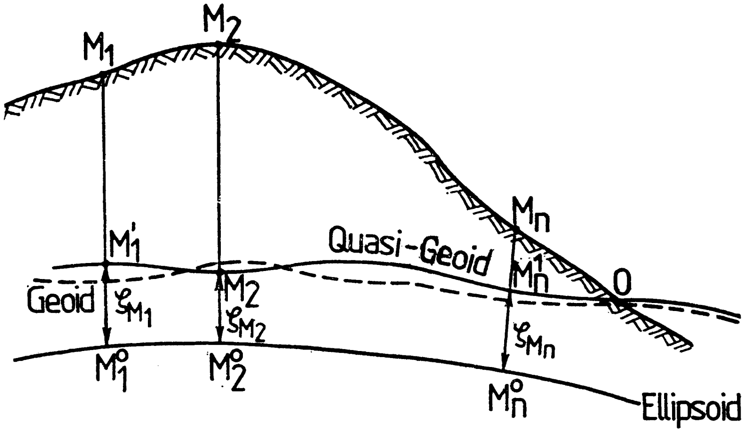


Fig. 3.6. The Quasi-Geoid

The height anomaly ζ can now be defined as the distance between the level surface of the actual potential $W(M)$:

$$W(M) = W(0) + \int_{0}^{M} g \, \mathrm{d}h, \tag{3.81}$$

and the level surface:

$$U(M) = U(O) + \int_{0}^{M} g \, \mathrm{d}h, \tag{3.82}$$

in which O is the initial point of the levelling situated at the sea-level, i.e. there where the geoid and the quasi-geoid coincide.

The introduction of the quasi-geoid offers the possibility of obtaining the normal altitude as the sum of two components:

$$H^N = H^{meas} + \varepsilon^N, \tag{3.83}$$

where H^{meas} denotes the level difference as measured by geometric levelling and ε^N denotes the normal correction.

The defining relation of the normal altitude (3.68) may be written according to (3.64) in the form:

$$H_A^N = \frac{1}{\bar{\gamma}_A} \int_{A_0}^{A} g \mathrm{d}h, \tag{3.84}$$

in which A is the point situated on the Earth's physical surface and A_0 its projection onto the ellipsoid. The meaning of $\bar{\gamma}$ as mean value of the normal gravity along the normal between A and A_0 (Fig. 3.5) must be stressed here.

This value is calculated from the relation:

$$\bar{\gamma}_A = \frac{1}{H_A^N} \int_0^{H_A^N} \gamma \mathrm{d}H^N, \tag{3.85}$$

in which γ is given by (*Eremeev* 1951):

$$\gamma = \gamma_0^A - k_1 H + k_2 H^2. \tag{3.86}$$

Here γ_0^A is calculated in terms of the parameters of the chosen ellipsoid, according to (3.50). The coefficients k_1 and k_2 are functions both of the parameters of the chosen reference ellipsoid and of the latitude of the point where one calculates the mean value of the normal gravity.

For *Krasovski*'s ellipsoid, considering $\gamma_E = 981.030$ gal, after *Helmert*'s normal formula, one gets the following values:

$$k_1 = 0.30856(1 + 0.000691 \cos 2B); \; k_2 = +\, 0.0723 \cdot 10^{-3}.$$

The numerical value of k_1 corresponds to H in metres and that of the coefficient k_2 to the altitude H in km.

Introducing (3.85) into (3.86) and carrying out the integration, one obtains the mean gravity as:

$$\bar{\gamma}^A = \gamma_0^A - \frac{1}{2} k_1 H_A^N + \frac{1}{3} k_2 (H_A^N)^2. \tag{3.87}$$

It is necessary to make clear the meaning of $\bar{\gamma}$ in order to distinguish this value from that used in the altitude system adopted in *France* (*Vignal* and *Simonsen* 1962):

$$H^{N*} = \frac{1}{\frac{1}{2}(\gamma_A - \gamma_a)} \int_0^A g \mathrm{d}h, \tag{3.88}$$

where a denotes the point at which the vertical through A intersects the geoid.

The denominators of (3.84) and (3.88) are different from one another, which leads to a certain difference between the altitudes calculated by means of the two formulae.

3.5.4 Transferring the Measured Level Differences into the System of Normal Altitudes

The coming of precise inter-continental levelling has necessitated the adoption of a unique system of altitudes. Several countries, especially the European ones, have adopted as a unique system that of the normal altitudes.

In order to calculate the differences of normal altitudes between two consecutive bench marks n and $n+1$ of the levelling line, *V.F. Eremeev* has derived the following formula:

$$H^N_{n+1} - H^N_n = \Delta H^{meas}_{n,n+1} - \frac{1}{\bar{\gamma}}(\gamma_0^{n+1} - \gamma_0^n)H_m + \frac{1}{\bar{\gamma}}(g-\gamma)_m \Delta H_{n,n+1}, \quad (3.89)$$

in which $\Delta H^{meas}_{n,n+1}$ is the altitude difference between the n and $n+1$ bench marks as obtained from measurements; H_m is the arithmetical average of the elevations of the n and $n+1$ bench marks; $(g-\gamma)_m$ is the mean gravity anomaly between the two bench marks n and $n+1$; $\bar{\gamma} = (\bar{\gamma}^{n+1} + \bar{\gamma}^n)/2$, in which $\bar{\gamma}^{n+1}$ and $\bar{\gamma}^n$ are calculated by means of (3.87).

For obtaining normal altitudes, other formulae have also been worked out for practical calculations. We will mention here only that which is based on the hypothesis that the actual potential equals the normal one, i.e. $W = U$, $g = \gamma$, $T = 0$ (*Heiskanen* and *Moritz* 1967). In this case (3.85) becomes

$$\bar{\gamma} = \frac{1}{H^N}\int_0^{H^N} \gamma(z)\,dz, \quad (3.90)$$

where $\gamma(z)$ is the normal gravity at the altitude (z), which is expressed in terms of γ through the relation:

$$\gamma(z) = \gamma\left[1 - \frac{2}{a}(1 + f + m - 2f\sin^2 B)z + \frac{3}{a^2}z^2\right]. \quad (3.91)$$

Here γ denotes the normal gravity on the ellipsoid which depends only on B (and not on z), while f and m have the meaning which they had in (3.18).

Introducing (3.91) into (3.90) and integrating one gets:

$$\bar{\gamma} = \gamma\left[1 - (1 + f + m - 2f\sin^2 B)\frac{H^N}{a} + \left(\frac{H^N}{a}\right)^2\right]. \quad (3.92)$$

From (3.92) and (3.68) one obtains the practical formula for calculating the normal altitude in the form:

$$H^N = \frac{C}{\gamma}\left[1 + (1 + f + m - 2f \sin^2 B)\frac{C}{a\gamma} + \left(\frac{C}{a\gamma}\right)^2\right]. \tag{3.93}$$

The relation (3.93) achieves the calculation accuracy necessary for practical purposes. The geopotential numbers in (3.93) can be calculated in terms of the gravity anomalies and of the mean gravity by means of the relation:

$$C_A = \overline{\gamma}^A \int_O^A \mathrm{d}h + \int_O^A (\gamma_0 - \gamma_0^B)\, \mathrm{d}h + \int_O^A (g - \gamma)\, \mathrm{d}h, \tag{3.94}$$

in which the point O is situated on the origin level-surface ($C = 0$).

For the closed levelling lines, (3.89) takes the form:

$$\sum_A^A H_{n,n+1} - \sum_A^A (\gamma_0^{n+1} - \gamma_0^n)\frac{H_m}{\overline{\gamma}} + \sum_A^A \frac{(g-\gamma)_m}{\overline{\gamma}}\,\Delta H_{n,n+1} =$$

$$= \sum_A^A \Delta H_{n,n+1} + \varepsilon^T = 0, \tag{3.95}$$

in which the quantity ε^T defined as:

$$\varepsilon^T = \sum_A^A \frac{\gamma_0^{n+1} - \gamma_0^n}{\overline{\gamma}} H_m - \sum_A^A \frac{(g-\gamma)_m}{\overline{\gamma}}\,\Delta H_{n,n+1}, \tag{3.96}$$

is the theoretical misclosure of the levelling polygon. The theoretical misclosure can be calculated in terms of the geopotential elevations too, with the relation:

$$\varepsilon^T = \Sigma \frac{C_{n+1} - C_n}{\overline{\gamma}^n} - \Sigma \frac{\overline{\gamma}^{n+1} - \overline{\gamma}^n}{\overline{\gamma}^n} H^N_{n+1}.$$

In order to exemplify the magnitude of the normal corrections and of the theoretical misclosures we present in Fig. 3.7 the levelling network of the *Socialist Republic of Romania.*

To obtain the normal correction with an accuracy of ± 0.25 mm/km of levelling line, the elements of (3.95) must be known with a precision of:

$$m_{\Delta g} = \pm\, 2.8 \text{ mgal};\ m_{\Delta\gamma} = \pm\, 0.3 \text{ mgal};\ m_{\bar{\gamma}} = \pm\, 10^4 \text{ mgal};\ m_H = \pm\, 18 \text{ m}. \tag{3.97}$$

In general, the accuracy parameters imposed by (3.97) can easily be achieved in practice.

The gravity anomalies are determined from gravity measurements carried out on the lst-order levelling lines with a constant density on the entire territory of the country of 1 point to 4—6 km. The density adopted for the gravity measurements is close to that proposed by *K. Ramsayer,* as that proposed by *G. Bomford* (0.3—1.5 km in mountainous areas, 1.0—2.0 in hilly areas and 2.0—3.0 in plain zones) was considered to represent too exacting a requirement for practical levelling necessities. The distance of 4—6 km between the gravity stations was chosen on the basis of the requirement that the gravity anomaly be determined through measurement at each bench mark of the lst-order levelling line. This requirement is a consequence of the fact that by means of (3.89) one calculates the normal correction between two consecutive bench marks n and $n + 1$. The mean gravity anomaly is calculated in this case as an arithmetical average of the anomalies determined with the aid of gravity measurements at the bench marks n and $n + 1$.

To obtain the difference of normal gravity with the accuracy given in (3.97) it is necessary to know the latitude of the levelling bench mark with a precision of $\pm\, 8''$, which can easily be achieved using topographical maps on the scale 1/25,000 *(Mihăilescu* et al. 1967*)*.

Taking into account the accuracy requirements for the mean normal gravity $\bar{\gamma}$, it was considered sufficient to adopt for the whole levelling network of the *Socialist Republic of Romania* a single value viz. *(Rotaru* et al. 1972*)*:

$$\bar{\gamma} = 978\,600 \text{ mgal};\ \frac{1}{\bar{\gamma}} = 102 \cdot 10^{-8} \text{ mgal}^{-1}.$$

As the lst-order levelling polygons have a perimeter of 200—400 km, practical considerations led to the adoption of the method of calculating the normal corrections on separate levelling lines, their sum on each polygon yielding the theoretical misclosure on polygons.

The value of the normal correction between two consecutive bench marks doesn't exceed $\pm\, 5$ mm for the lst-order levelling network of the *Socialist Republic of Romania.* The theoretical misclosure on these polygons may obtain, however, great values, of over 80 mm (Fig. 3.7).

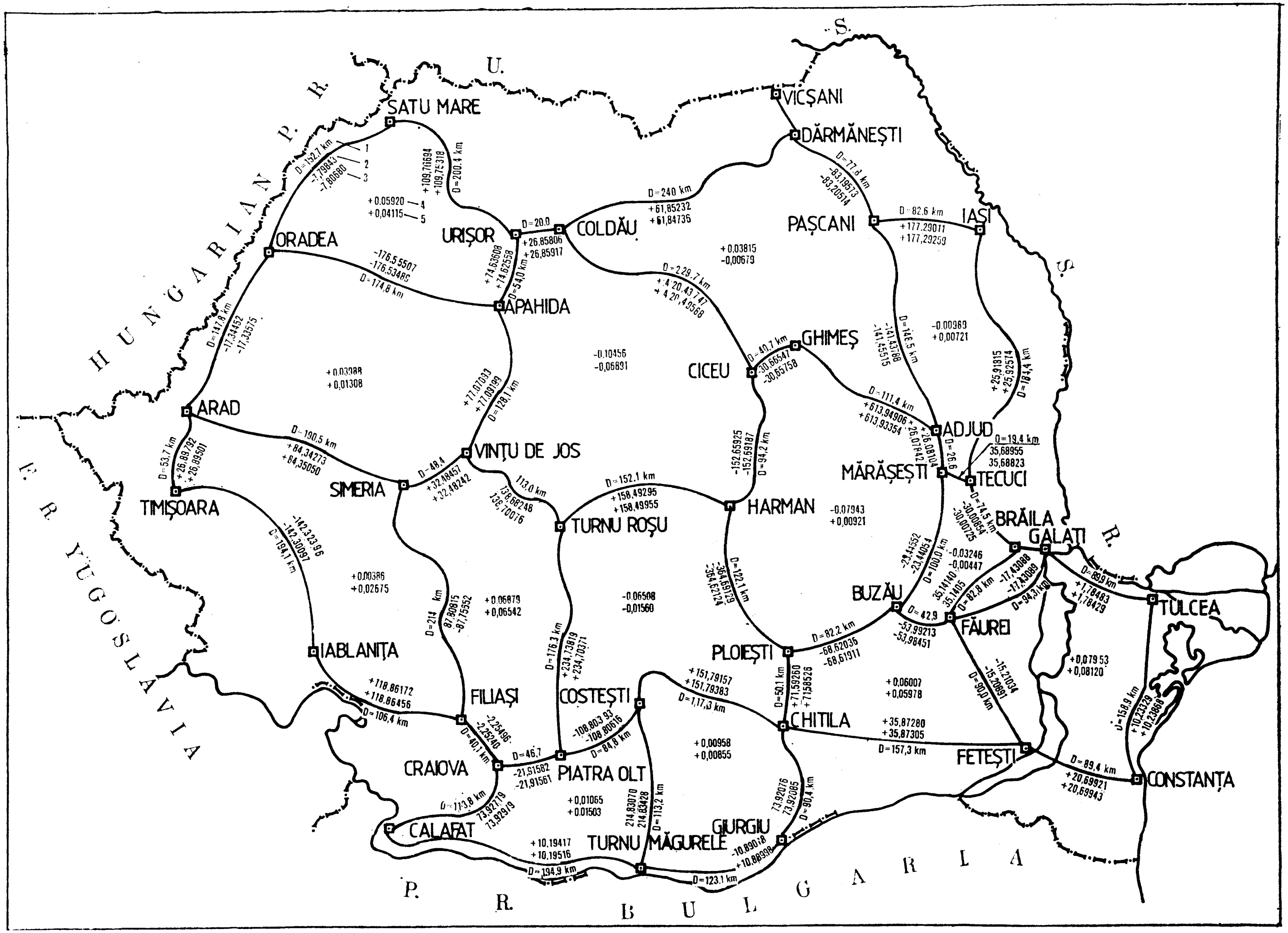


Fig. 3.7. The Precision Levelling Network of the Socialist Republic of Romania

1 — Length of the levelling line, in km; 2 — Measured level difference; 3 — Level difference in the normal-altitude system; 4 — Polygon misclosure of the measured level differences; 5 — Polygon misclosure of the level differences in the normal system

4
The Perturbing Potential

In order to determine the Earth's actual potential $W(B, L, H)$ it is most convenient to split it up into a normal potential $U(B, H)$ of the form (3.29) and a so-called perturbing potential $T(B, L, H)$. These three forms of potential are interrelated by the expression:

$$W(B, L, H) = U(B, H) + T(B, L, H). \tag{4.1}$$

The perturbing potential has a low value compared with the actual potential. If the reference ellipsoid is suitably chosen, the ratio of these two potentials reaches values of the order:

$$\frac{T(B, L, H)}{W(B, L, H)} \leqslant 1{,}6 \cdot 10^{-5}.$$

In such a case, when expanding the perturbing potential in series its lst-order derivatives and the terms in T^2 may be neglected. Also, the angle made by the normal to the ellipsoid with the vertical is very small and this is the reason why, when tackling certain problems, one may assume the equality:

$$\frac{\mathrm{d}W}{\mathrm{d}n}\vec{\mathbf{n}}_0 = \frac{\mathrm{d}W}{\mathrm{d}H}\vec{\mathbf{H}}_0,$$

in which $\vec{\mathbf{n}}_0$ represents the unit vector of the normal to the ellipsoid and $\vec{\mathbf{H}}_0$, the unit vector of the vertical.

4.1 Bruns' Formula. The Third Boundary-Value Problem for the Geoid

Assuming the Earth's actual potential to be equal to the potential of the geoid, the relation (4.1) means that to any point P on the Earth's physical surface there correspond an actual potential $W = W(P)$ and a normal potential $U = U(P)$, the difference between them being the perturbing potential T.

Let P be any point on the surface of the geoid $W = W_0$ and let $U =$ const. be the family of surfaces of the normal field, called *spherops*. Generally, the spherops are not ellipsoids except for the reference surface, i.e. the reference ellipsoids. The potential of the spherops is called *spheropotential* (*Hirvonen* 1960).

From among the set of spherops one chooses the equipotential surface $U = W_0$ (Fig. 4.1), i.e. the surface of the normal ellipsoid.

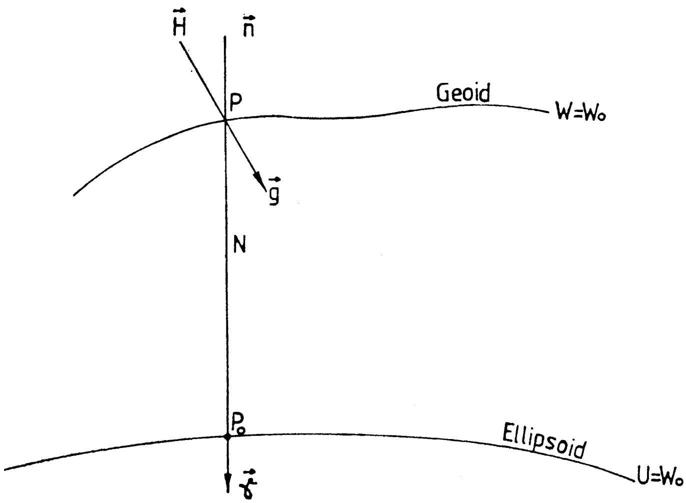


Fig. 4.1. The Vectors of Actual and Normal Gravity

The normal at P intersects the ellipsoid at the point P_0, whose potential will be:

$$U(P_0) = W_0. \tag{4.2}$$

The distance PP_0 represents the geoid undulation N.

Two potentials are associated with the point P, viz. the actual one W_0 and the normal potential:

$$U(P) = U(P_0) - \left(\frac{\partial U}{\partial n}\right)_{P_0} N = U(P_0) - \gamma N. \tag{4.3}$$

According to (4.1) and (4.3):

$$W_0 = U(P) + T = U(P_0) - \gamma N + T.$$

Taking into account (4.2) as well, the preceding relation becomes:

$$W_0 = W_0 - \gamma N + T,$$

whence:

$$T = \gamma N, \tag{4.4}$$

or:

$$N = T/\gamma. \tag{4.5}$$

The relation (4.5) known as *Bruns'* formula is particularly important for Physical Geodesy as it underlies the theory of the determination of the geoid undulations.

With the aid of the representations in Fig. 4.1 one can define several basic elements of Physical Geodesy. The distance PP_0 between the geoid and the ellipsoid is the *geoid undulation* N, whose significance was made clear in Chapt. 3. The vectors of the actual gravity $\vec{\mathbf{g}}$ and of the normal gravity $\vec{\boldsymbol{\gamma}}$ have different points of action and different directions. The difference in magnitude between the two vectors is known as the *gravity anomaly.*

Because the two vectors have different points of action, this quantity is also encountered under the name of *combined gravity anomaly* (*Brovar* et al. 1961).

The difference in direction between the vectors $\vec{\mathbf{g}}$ and $\vec{\boldsymbol{\gamma}}$ represents the deflection of the vertical u. As a rule, in Geodesy one seldom utilizes the deflection of the vertical; more frequently one uses its two components: the ξ component representing the projection of the angle u onto the meridian and the η component, the projection onto the parallel.

It is also possible to compare the vectors $\vec{\mathbf{g}}$ and $\vec{\boldsymbol{\gamma}}$ at the same point P, the result being called *the gravity perturbation* $\overrightarrow{\delta \mathbf{g}}$:

$$\overrightarrow{\delta \mathbf{g}} = \vec{\mathbf{g}}_P - \vec{\boldsymbol{\gamma}}_P. \tag{4.6}$$

In this case, the difference in magnitude between the two vectors represents *the gravity perturbation:*

$$\delta g = g_P - \gamma_P, \tag{4.7}$$

while the difference in direction remains the same angle u of deflection of the vertical, since the directions of $\vec{\mathbf{g}}_P$ and $\vec{\boldsymbol{\gamma}}_P$ coincide. The gravity perturbation is seldom used in Physical Geodesy, because it has not the same importance as that of the gravity anomaly.

Taking into consideration that the angle $(\vec{\mathbf{n}}, \vec{\mathbf{H}})$ is small, from the relation (4.7) one gets:

$$\delta g = -\left(\frac{\partial W}{\partial H} - \frac{\partial U}{\partial n}\right) \approx -\left(\frac{\partial W}{\partial H} - \frac{\partial U}{\partial H}\right),$$

or, in view of (4.1):

$$\delta g = -\frac{\partial T}{\partial H}, \tag{4.8}$$

relation which expresses the gravity perturbation as a function of the perturbing potential. For Physical Geodesy it is more important to express the gravity anomaly as a function of the perturbing potential. To this end, the normal gravity at the point P is expressed as a function of the gravity at P_0 by the relation:

$$\gamma_P = \gamma_{P_0} + \frac{\partial \gamma}{\partial H} N,$$

by using which, (4.8) becomes:

$$-\frac{\partial T}{\partial H} = g_P - \gamma_P = g_P - \gamma_{P0} - \frac{\partial \gamma}{\partial H} N,$$

or:

$$-\frac{\partial T}{\partial H} + \frac{\partial \gamma}{\partial H} N = \Delta g. \tag{4.9}$$

The relation (4.9) represents the *fundamental equation of Physical Geodesy*. As in this equation the Δg value can only be known on the geoid (and not within the entire exterior space), the relation (4.9) becomes a boundary condition from which one can determine the perturbing potential. The potential T to be determined must obey the conditions (2.68). The first two conditions are satisfied on the assumption that outside the geoid no attractive masses exist. In this case, the density δ outside the geoid is everywhere zero and the perturbing potential satisfies *Laplace*'s equation:

$$\Delta T = 0. \tag{4.10}$$

The last condition in (2.68) is the boundary condition for the geoid as given in (4.9) which as was shown in Section 2.4.3 has the form of the third boundary-value problem.

On the basis of (4.9) and (4.10) one can determine the perturbing potential as the third boundary-value problem for the geoid and then, from (4.5), the geoid undulation may be obtained.

One can thus conclude that the main problem of Physical Geodesy — the geoid determination — is essentially a third-type boundary-value problem. The solution of this problem is given by *Stokes*' integral.

4.2 The Third Boundary-Value Problem for the Earth's Physical Surface

Solving the boundary-value problem for the geoid assumes knowledge of the gravity anomalies on its surface or, according to (4.6), knowledge of the gravity on the geoid. The gravity on the geoid is obtained by reducing the gravity as measured on the physical surface of the Earth to the geoid level. The gravity reduction procedures, which will be analysed in Chapter 5, are based on the knowledge of the density of the mass layers contained between the physical surface and the geoid. If this density is not accurately known, the value obtained for the gravity anomaly will be erroneous and, consequently, the boundary condition (4.9) will also be erroneous.

In order to avoid this shortcoming, *M. S. Molodenski* proposed to solve the boundary-value problem directly for the Earth's physical surface, thus circumventing the error source introduced by the gravity reduction.

Since outside the Earth's physical surface there are no attractive masses, the perturbing potential is a harmonic function, it consequently obeys *Laplace*'s equation:

$$\Delta T = 0.$$

If this differential equation is supplemented by a boundary condition on the physical surface, the perturbing potential can be determined on this surface; then, with the aid of (4.1) one may determine the actual potential W.

Let P be the point situated on the physical surface and Q its corresponding point on the hypsometric surface Σ (Fig. 4.2). From the definition of the hypsometric surface as given in Section 3.5.3, it follows that the two points have the same potential:

$$U(Q) = W(P). \tag{4.11}$$

According to (4.1), the perturbing potential at P is:

$$T(P) = W(P) - U(P),$$

or, in view of (4.11):

$$T(P) = UQ) - U(P). \tag{4.12}$$

If in *Bruns'* formula (4.5) one introduces the perturbing potential given by (4.12) in the left-hand side of this relation, one will obtain the height anomaly ζ. *Bruns'* formula will thus become:

$$\zeta = \frac{U(Q) - U(P)}{\gamma_Q}, \tag{4.13}$$

or, using (4.1):

$$\zeta = \frac{U(Q) - [W(P) - T(P)]}{\gamma_Q}. \tag{4.14}$$

Expressing the actual potential $W(P)$ and the spheropotential $U(Q)$ by (3.81) and (3.82), for the general case $U_0 \neq W_0$ the relation (4.14) becomes:

$$\zeta = \frac{1}{\gamma_Q}[T(P) + U_0 - W_0], \tag{4.15}$$

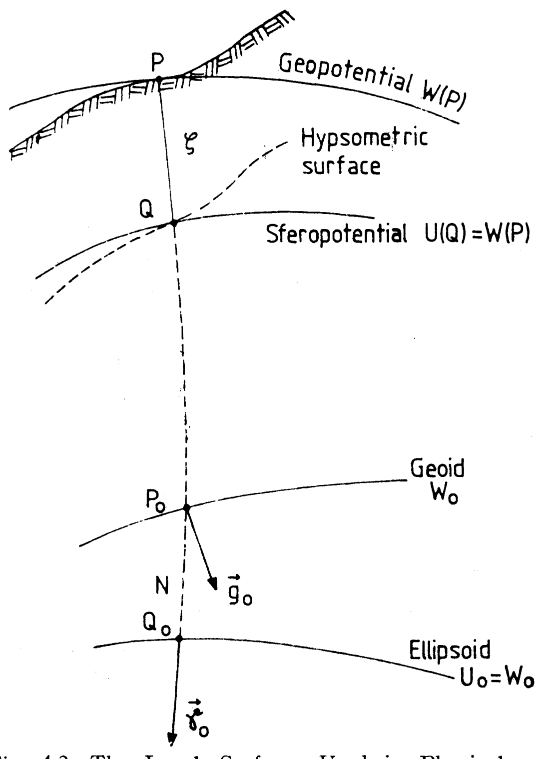


Fig. 4.2. The Level Surfaces Used in Physical Geodesy

where;

$$\gamma_Q = \gamma_E\left(1 + f^* \sin^2 B - \frac{1}{4} f_4 \sin^2 2B\right) + \frac{\partial\gamma}{\partial H} H_P^N. \tag{4.16}$$

The meaning of the terms used in (4.16) is the same as in the formulae (3.50) and (3.51).

Writing

$$\Delta W = U_0 - W_0, \tag{4.17}$$

the relation (4.15) becomes:

$$\zeta = \frac{1}{\gamma_Q}[T(P) + \Delta W], \tag{4.18}$$

which is called *Bruns' generalized formula.* Introducing this relation into (4.9) one gets:

$$\left(\frac{\partial T(P)}{\partial H}\right)_P = - g_M + \gamma_Q + \left(\frac{\partial\gamma}{\partial H}\right)_Q \left[\frac{T(P)}{\gamma_Q} - \frac{\Delta W}{\gamma_Q}\right]. \tag{4.19}$$

Here the derivative of the perturbing potential refers to the point P of the Earth's physical surface. Since the distance PQ generally has a small value, the boundary condition (4.19) can be written as a boundary condition on Σ to an accuracy of the square of the perturbing potential. In this case, (4.19) becomes:

$$\left(\frac{\partial T}{\partial H}\right)_\Sigma - \left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial H}T\right)_\Sigma = - g_P + \gamma_\Sigma - \left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial H}\right)_\Sigma \Delta W. \tag{4.20}$$

The relation (4.20) represents the boundary condition on the hypsometric surface as derived by *M. S. Molodenski* in the year 1945 *(Molodenski* 1945*)*.

In a general form the relation (4.20) may be written *(Brovar* et al. 1961*)* as follows:

$$\left(\frac{\partial T}{\partial H}\right)_\Sigma - (AT)_\Sigma = p, \tag{4.21}$$

in which:

$$A = \frac{1}{\gamma}\frac{\partial\gamma}{\partial H}\,;\; p = - g_P + \gamma_\Sigma - \left(\frac{1}{\gamma}\frac{\partial\gamma}{\partial H}\right)_\Sigma \Delta W.$$

In the relation (4.21) the derivative of the perturbing potential should be taken in the direction $\vec{\mathbf{H}}$ of the vertical and not along the normal $\vec{\mathbf{\Sigma}}$ to the hypsometric surface. Since the angle $(\vec{\mathbf{H}}, \vec{\mathbf{\Sigma}})$ between the two normals, which, in fact, represents the inclination of the physical surface may reach very high values, the derivatives of the perturbing potential, with respect to the normal to $\mathbf{\Sigma}$ and with respect to the vertical are quite different. This difference constitutes the main complication of the boundary condition (4.20) as compared with the boundary condition for the geoid.

4.3 Expansion of the Perturbing Potential in Spherical Harmonics. The Spherical Approximation of the Boundary Condition for the Geoid

If, in deriving the formulae associated with the perturbing potential, the ellipsoid is approximated by a sphere, the errors thus introduced are of the order of magnitude of the flattening $f = 3 \times 10^{-3}$.

In the spherical approximation, the normal gravity and its derivative have the form:

$$\gamma = \frac{GM}{\rho^2}; \quad \frac{\partial \gamma}{\partial H} = \frac{\partial \gamma}{\partial \rho} = -\frac{2\gamma}{\rho} \tag{4.22}$$

in which ρ denotes the radius vector of an external point P.

If the Earth is approximated by a sphere of radius R and if one assumes a mean value g_m of the gravity for the entire Earth, the relations (4.22) become:

$$g_m = \frac{GM}{R^2}; \quad \frac{\partial g_m}{\partial H} = -\frac{2 g_m}{R}. \tag{4.23}$$

In this case, (4.9) may be written in the form:

$$-\frac{\partial T}{\partial H} = \Delta g + \frac{2 g_m}{R} N, \tag{4.24}$$

and the fundamental equation of Physical Geodesy in the spherical approximation becomes:

$$\Delta g = -\frac{\partial T}{\partial \rho} - \frac{2 g_m}{R} N, \tag{4.25}$$

or, in terms of the perturbing potential only:

$$\Delta g = -\frac{\partial T}{\partial \rho} - \frac{2T}{R}. \tag{4.26}$$

The relation (4.26) represents the spherical approximation of the boundary condition for the geoid.

In view of (4.28) and (4.24), the perturbation of the gravity may be represented in the spherical approximation in the form:

$$\delta g = \Delta g + \frac{2 g_m}{R} N, \tag{4.27}$$

or, in view of *Bruns'* formula:

$$\delta g = \Delta g + \frac{2}{R} T. \tag{4.28}$$

While the relations (4.25), (4.26), (4.27) and (4.28) hold rigorously for the sphere, they only constitute a first approximation for the reference ellipsoid. When using these relations for the ellipsoid, the value of γ, which plays a part in determining the gravity anomaly, must be calculated to a high degree of accuracy.

Outside the attractive masses, the perturbing potential used in (4.25), and (4.26) is a harmonic function and for this reason can be written in the form of a harmonic series:

$$T(\rho, \theta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{\rho}\right)^{n+1} T_n(\theta, \lambda), \tag{4.29}$$

in which $T_n(\theta, \lambda)$ is the surface harmonic of the nth degree. On the geoid surface $\rho \rightarrow R$ and (4.29) becomes:

$$T(R, \theta, \lambda) = \sum_{n=0}^{\infty} T_n(\theta, \lambda). \tag{4.30}$$

The gravity perturbation (4.8) is obtained from (4.29) in the form:

$$\delta g = -\frac{\partial T}{\partial \rho} = \frac{1}{\rho} \sum_{n=0}^{\infty} (n+1) \left(\frac{R}{\rho}\right)^{n+1} T_n(\theta, \lambda), \tag{4.31}$$

and for the geoid, where $\rho \rightarrow R$, one gets:

$$\delta g = -\frac{\partial T}{\partial \rho} = \frac{1}{R} \sum_{n=0}^{\infty} (n+1) T_n(\theta, \lambda). \tag{4.32}$$

For an external point situated at a distance ρ from the centre of the sphere (Fig. 4.3), the boundary condition takes the form:

$$\Delta g = -\frac{\partial T}{\partial \rho} - \frac{2T}{\rho}. \tag{4.33}$$

Introducing (4.29) and (4.31) into (4.33) yields:

$$\Delta g = \frac{1}{\rho} \sum_{n=0}^{\infty} (n-1) \left(\frac{R}{\rho}\right)^{n+1} T_n(\theta, \lambda), \tag{4.34}$$

and for the geoid, where $R = \rho$:

$$\Delta g = \frac{1}{R} \sum_{n=0}^{\infty} (n-1) T_n(\theta, \lambda). \tag{4.35}$$

The relation (4.35) shows that in the expansion in series of the gravity anomaly, the spherical harmonic of the 1st degree will always be missing, even though it is present for the perturbing potential.

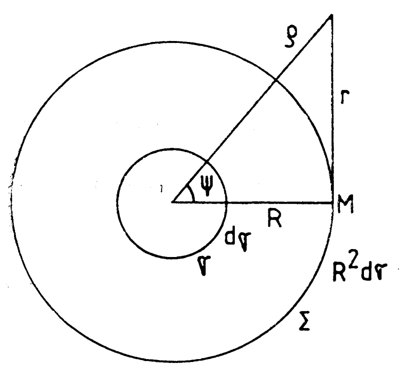


Fig. 4.3. The Perturbing Potential and the Boundary Condition in the Spherical Approximation

Expressing the anomaly by means of the harmonic series:

$$\Delta g = \sum_{n=0}^{\infty} \Delta g_n,$$

and comparing the latter with (4.35) one gets:

$$T_n = \frac{R}{n-1} \Delta g_n. \tag{4.36}$$

In this case the relation (4.30) may be written in the form:

$$T = R \sum_{n=0}^{\infty} \frac{\Delta g}{n-1} \cdot \tag{4.37}$$

This relation shows that the lst order harmonic of Δg_n must be absent. Otherwise the term $\Delta g_n/(n-1)$ becomes infinity for $n = 1$.

4.4 The Gravity Anomaly in External Space

The knowledge of the gravity anomaly and perturbation constitutes a relatively recent concern of the geodetic specialists; it has appeared as a result of the use of artificial satellites for geodetic purposes.

The determination of the gravity anomaly in the external space can be formulated as a *Dirichlet's* problem for the case of the spherical approximation: given the value of the anomaly on the unit sphere, let the gravity anomaly be determined at the altitude H as a harmonic function.

According to Section 2.4.2, the *Dirichlet's* solution for the sphere is given by *Poisson*'s integral (2.73). Applying this relation to determine the gravity anomaly at the point P situated at a distance ρ from the Earth's centre one gets:

$$\Delta g_P = \frac{(\rho^2 - R^2)}{4\pi R} \iint_\sigma \frac{\Delta g}{r^3} d\sigma, \tag{4.38}$$

in which R, r, ρ and σ have the meaning given in Fig. 4.3.

In general, the formulae used in Physical Geodesy are simpler if the zero and lst-order harmonics are missing in the expansion in spherical harmonics. To this end, it is useful to solve *Dirichlet*'s problem for the function:

$$\Delta' g = \Delta g - (\Delta g_0 + \Delta g_1) = \sum_{n=2}^{\infty} \Delta g_n,$$

in which Δg_0 and Δg_1 denote the zero and lst-order spherical harmonics.

To solve this problem, *Poisson*'s integral shall be suitably transformed (*Heiskanen* et al. 1967). Let H be a harmonic function whose value is known on the terrestrial sphere of radius R (Fig. 4.3). If in (2.73) the surface element of the terrestrial sphere is expressed in terms of the surface element of the unit sphere $d\Sigma = R^2\,d\sigma$ one gets:

$$H_P = \frac{R}{4\pi}\iint_\sigma \frac{\rho^2 - R^2}{r^2} H d\sigma,$$

where H_P is the value of the function at the external point P.

The harmonic function H can be written in the form of a harmonic series:

$$H = \frac{R}{\rho} H_0 + \left(\frac{R}{\rho}\right)^2 H_1 + \sum_{n=2}^{\infty}\left(\frac{R}{\rho}\right)^{n+1} H_n,$$

in which $H_0, H_1, \ldots, H_n$ denote the surface harmonics. By separating out in the previous relation the zero and 1st-degree harmonics one gets:

$$H' = H - \frac{R}{\rho} H_0 - \left(\frac{R}{\rho}\right)^2 H_1 = \sum_{n=2}^{\infty}\left(\frac{R}{\rho}\right)^{n+1} H_n. \tag{4.39}$$

According to (2.41) and (2.17), the zero and 1st-degree harmonics have the form:

$$H_0 = \frac{1}{4\pi}\iint_\sigma H\,d\sigma; \quad H_1 = \frac{3}{4\pi}\iint_\sigma H \cos\psi\,d\sigma. \tag{4.40}$$

Expressing H by *Poisson*'s integral and H_0 and H_1 by the relations (4.40), the value of the function H' at the external point becomes:

$$H'_P = \frac{R}{4\pi}\iint_\sigma\left(\frac{\rho^2 - R^2}{r^3} - \frac{1}{\rho} - \frac{3R}{\rho^2}\cos\psi\right) H d\sigma. \tag{4.41}$$

This relation can be used for determining the gravity anomaly outside the terrestrial sphere. To this end, (4.34) is written as:

$$\rho\Delta g_P = \sum_{n=0}^{\infty}(n-1)\left(\frac{R}{\rho}\right)^{n+1} T_n(\theta, \lambda). \tag{4.42}$$

The right-hand side of (4.42) shows that the function $\rho\Delta g_P$ can be expressed as a harmonic series, which allows the application of (4.41) to this function. One obtains:

$$\Delta g_P = \frac{R}{4\pi\rho}\iint\left(\frac{\rho^2 - R^2}{r^3} - \frac{1}{\rho} - \frac{3R}{\rho^2}\cos\psi\right)(R\Delta g)\mathrm{d}\sigma. \qquad (4.43)$$

The relation (4.43) permits the computation of the gravity anomaly outside the Earth; it is known as the calculation formula of the *upward continuation* of the gravity anomaly. The difference between (4.38) and (4.43) lies in the fact that the first takes into cosideration the zero and 1st-order harmonics of the expansion in series of the anomaly, while the second does not.

The concepts described in this chapter as well as the mathematical relations defining them will underlie the methods for determining the geoid and the Earth's physical surface.

5
The Determination of the Geoid by Gravity Methods

The determination of the geoid by gravity methods is based on the formula (4.5) published by *H. Bruns* in the year 1878. Indeed, if the general terrestrial ellipsoid is assumed known, then by means of *Somigliana*'s formula (3.47) one can calculate accurately the value of normal gravity and the perturbing potential T may be obtained by means of the third boundary, value problem of Physical Geodesy for the geoid (Section 4.1).

5.1 Stokes' Formula

In Section 2.4.3 it was demonstrated that the direct solution of the third boundary-value problem of Physical Geodesy is given by *Stokes'* integral (2.83) In deriving this relation, the function $f(\theta', \lambda')$ was considered as arbitrary and for this reason the relation (2.86) may be viewed as a general form of the solution of the third boundary-value problem. In order to determine the perturbing potential, it is necessary to specialize the form of the boundary function $f(\theta', \lambda')$ viz. to consider it as a limiting value of the gravity on the geoid.

Starting from the boundary condition (4.33), one can determine the perturbing potential as an harmonic function outside the geoid. The integration of the boundary condition (4.33) is possible only to the extent to which the gravity anomaly can be known outside the Earth's surface. This can be achieved through the medium of (4.43) so that this relation may be viewed as a differential equation which can be integrated with respect to ρ.

Multiplying (4.33) by $(-\rho^2)$ one gets:

$$-\rho^2 \Delta g(\rho) = \frac{\partial}{\partial \rho}(\rho^2 T),$$

in which $\Delta g(\rho)$ represents the gravity anomaly outside the Earth's surface, which can be calculated with (4.43).

Integrating this relation between ∞ and ρ yields:

$$-\int_{\infty}^{\rho} \rho^2 \Delta g(\rho)\,d\rho = \rho^2 T \Big|_{\infty}^{\rho} = \rho^2 T - \lim_{\rho\to\infty} \rho^2 T. \tag{5.1}$$

The perturbing potential determined from the third boundary-value problem must satisfy the conditions (2.68). As a consequence, the last term of (5.1) vanishes so that:

$$\rho^2 T = -\int_{\infty}^{\rho} \rho^2\, \Delta g\,(\rho)\,d\rho. \tag{5.2}$$

Introducing the value of $\Delta g(\rho)$ given by (4.43) into (5.2) one obtains:

$$\rho^2 T = \frac{R^2}{4\pi} \int_{\infty}^{\rho} \left[\iint_{\tau} \left(- \frac{\rho^3 - R^2 \rho}{r^3} + 1 + \frac{3R}{\rho} \cos\psi \right) \Delta g\, d\sigma \right] d\rho.$$

Carrying out the integration, one obtains the perturbing potential for a point exterior to the geoid in the form *(Heiskanen and Moritz 1967)*:

$$T(\rho, \theta, \lambda) = \frac{R}{4\pi} \iint_{\sigma} S(\rho, \psi)\, dg\, d\sigma, \tag{5.3}$$

where:

$$S(\rho, \psi) = \frac{2R}{r} + \frac{R}{\rho} - \frac{3Rr}{\rho^2} - \frac{R^2}{\rho^2} \cos\psi \left(5 + 3 \ln \frac{\rho - R\cos\psi + 1}{2\rho}\right). \tag{5.4}$$

For the surface of the geoid $\rho = R$ and the perturbing potential is:

$$T = \frac{R}{4\pi} \iint_{\sigma} \Delta g S(\psi)\, d\sigma, \tag{5.5}$$

where the function $S(\psi)$ is *Stokes'* function (2.89).

By using *Bruns'* formula (4.5) one can determine the undulation of the geoid:

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} \Delta g S(\psi)\, d\sigma. \tag{5.6}$$

This is the famous formula of *Stokes*, published by him in the year 1849.

The perturbing potential derived using (5.5) does not contain spherical harmonics of zero and first order. This fact can be established through analysis of the relation (4.35) which expresses the connexion between the perturbing potential and the gravity anomaly.

Stokes' formula was derived on the basis of (4.33), a relation which represents the spherical approximation of the boundary condition. As a consequence, the calculation of the geoid's undulations by means of (5.6) is carried out with an error of $3 \times 10^{-3}\ N$, which can be disregarded for many practical purposes. For a more accurate determination of the geoid's undulations, *Stokes'* formula was derived by taking into consideration the flattening of the reference ellipsoid as well *(Vening Meinesz* 1928; *Molodenski* 1945; *Heiskanen* and *Vening Meinesz* 1958*)*.

5.2 Generalization of Stokes' Formula

In determining the geoid with the help of *Stoke's* formula, we have used as reference surface **the ellipsoid which fulfils the following conditions**:

(1) *It has its centre at the Earth's centre.*

(2) *Its mass equals the Earth's mass.*

(3) *Its potential* $U(0)$ *equals the geoid's potential* $W(0)$.

In general, these three conditions imposed on the reference ellipsoid cannot be satisfied in practice, for which reason *Stokes'* formula may be generalized to an ellipsoid which should fulfil a sole condition: to be so close to the geoid that the difference between the two surfaces be reasonably considered as having a linear variation. In this case the perturbing potential can be written as an infinite series in which, however, the zero- and first-order terms will no longer be absent.

Indeed, the zero-order spherical harmonic in the expansion of a potential is equal to GM/R. When the potential represents a difference of potentials, as is the case of the perturbing potential $T = W - U$, the zero-order harmonic will be:

$$T_0 = \frac{G\Delta M}{R}, \tag{5.7}$$

in which ΔM represents the difference between the Earth's mass and the mass of the reference ellipsoid.

As the Earth's mass is not known exactly, the zero-order spherical harmonic does not vanish and the form of (5.5) becomes in this case:

$$T = \frac{G\Delta M}{R} + \frac{R}{4\pi}\iint_\sigma \Delta g S(\psi)\, d\sigma. \tag{5.8}$$

The first-order spherical harmonic will be absent in the expansion in series of the perturbing potential only if the centre of the reference ellipsoid coincides with the Earth's centre of mass. Denoting, in a coordinate system with the origin at the centre of the reference ellipsoid, by x_0, y_0 and z_0 the coordinates of the Earth's centre of mass, the first-order spherical harmonic of the perturbing potential is:

$$T_1 = \frac{GM}{R^2}[(x_0 \cos \lambda + y_0 \sin \lambda) \sin \theta + z_0 \cos \theta]. \tag{5.9}$$

Thus, the most general form of the expansion of the perturbing potential contains three arbitrary constants: x_0, y_0 and z_0. In practice, by placing the ellipsoid's centre at the Earth's centre of mass, these constants vanish and as a consequence the first-order spherical harmonic vanishes too.

In the case when the potentials $W(0)$ and $U(0)$ of the geoid and of the reference ellipsoid are different, the perturbing potential will be given, by analogy with (4.18), by:

$$T = \gamma N - \Delta W, \tag{5.10}$$

in which ΔW has the meaning given by (4.17).

From (5.8) and (5.10) follows *Stokes'* generalized formula in the form:

$$N = \frac{G\Delta M}{R\gamma} + \frac{\Delta W}{\gamma} + \frac{R}{4\pi\gamma}\iint_\sigma \Delta g\, S(\psi)\, d\sigma. \tag{5.11}$$

The relation (5.11) shows that *Stokes'* formula (5.6) holds only in the case when the reference ellipsoid has the same mass as the Earth and the same potential as the geoid ($\Delta W = \Delta M = 0$).

In the case when the normal potential and the actual one are equal ($\Delta W = 0$), but the Earth's mass is different from the mass of the level ellipsoid ($\Delta M \neq 0$), one applies to the calculation of the geoid's undulations, the formula derived by *R. Hirvonen:*

$$N = \frac{R}{4\pi\gamma}\iint_\sigma \Delta g[S(\psi) - 1]d\sigma. \tag{5.12}$$

This relation has a more general character than *Stokes'* formula (5.6) due to the fact that it assumes only one of the two conditions:

$$M = M' \text{ and } W_0 = U_0.$$

5.3 Determination of the Gravity Deflection of the Vertical

For orienting the reference ellipsoid and for reducing the geodetic measurements on the ellipsoid it is necessary to know the vertical at all points of the first-order triangulation network. The components of the vertical can be determined from astronomical and geodetic measurements, from (10.4) and (10.5):

$$\xi = \Phi - B; \; \eta = (\Lambda - L) \cos \Phi,$$

or by using the gravity anomalies. In the first case, for determining the components of the deflection of the vertical, astronomical determinations are necessary at all the points of the triangulation network. The ξ and η components are thus determined in the geodetic coordinate system, i.e. are referred to the chosen reference ellipsoid.

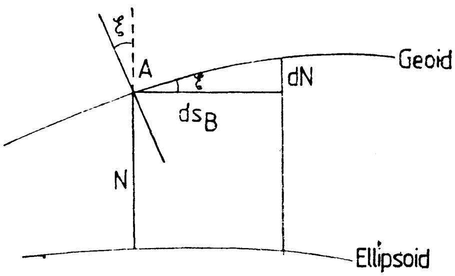


Fig. 5.1. The Connexion between the Geoid Undulation and the Component of the Deflection of the Vertical

A second possibility for determining the deflection of the vertical is based on the formulae derived by *Vening Meinesz* in the year 1928. These formulae are based on the connexion between the deflection of the vertical and the geoid undulation as illustrated in Fig. 5.1. Considering a section in the meridian plane through the surface of the geoid and that of the general terrestrial ellipsoid, it follows from Fig. 5.1 that:

$$\tan \xi = -\frac{\mathrm{d}N}{\mathrm{d}s_B}.$$

In an analogous manner, for the η component one obtains:

$$\tan \eta = -\frac{\mathrm{d}N}{\mathrm{d}s_L}.$$

The minus sign is a convention to which we shall return later.

As the components of the deflection of the vertical have values of the order of some tens of seconds, one can make the approximation $\tan \xi \approx \xi$; $\tan \eta \approx \eta$ and then:

$$\xi = -\frac{\mathrm{d}N}{\mathrm{d}s_B}; \quad \eta = -\frac{\mathrm{d}N}{\mathrm{d}s_L}. \tag{5.13}$$

The element of arc of meridian and of parallel are defined (Chapter. 9) as follows:

$$\mathrm{d}s_B = R\,\mathrm{d}B_0; \ \mathrm{d}s_L = R \cos B_0 \,\mathrm{d}L_0,$$

in which R denotes the average radius of the general terrestrial ellipsoid and B_0 and L_0 denote the geodetic coordinates of the point A (Fig. 5.1).

On the basis of the last relations, from (5.13) one derives:

$$\xi = -\frac{1}{R}\frac{\partial N}{\partial B_0}; \quad \eta = -\frac{1}{R \cos B_0}\frac{\partial N}{\partial L_0}. \tag{5.14}$$

The partial derivatives in (5.14) may be obtained by using (5.6) in the form:

$$\frac{\partial N}{\partial B_0} = \frac{R}{4\pi\gamma}\iint_\sigma \Delta g \frac{\partial S(\psi)}{\partial B_0}\,\mathrm{d}\sigma = \frac{R}{4\pi\gamma}\,\Delta g \iint_\sigma \frac{\mathrm{d}S(\psi)}{\mathrm{d}\psi}\,\frac{\partial \psi}{\partial B_0}\,\mathrm{d}\sigma;$$

$$\frac{\partial N}{\partial L_0} = \frac{R}{4\pi\gamma}\iint_\sigma \Delta g \frac{\partial S(\psi)}{\partial L_0}\,\mathrm{d}\sigma = \frac{R}{4\pi\gamma}\,\Delta g \iint_\sigma \frac{\mathrm{d}S(\psi)}{\mathrm{d}\psi}\,\frac{\partial \psi}{\partial L_0}\,\mathrm{d}\sigma. \tag{5.15}$$

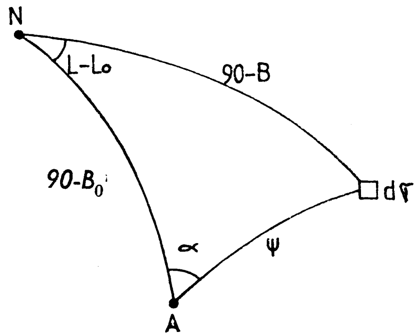


Fig. 5.2. The Position Spherical Triangle

In order to work out the partial derivatives of the spherical distance with respect to B_0 and L_0 one utilizes the position spherical triangle (Fig. 5.2) formed by the point A at which one determines the components of the deflection of the vertical, the north pole and the surface element dσ. Using the elements of this spherical triangle one can write the following relations:

$$\begin{aligned}
&\cos\psi = \sin B_0 \sin B + \cos B_0 \cos B \cos (L - L_0);\\
&\sin\psi \cos\alpha = \cos B_0 \sin B - \sin B_0 \cos B \cos (L - L_0);\\
&\sin\psi \sin\alpha = \cos B \sin(L - L_0).
\end{aligned} \tag{5.16}$$

The partial derivatives in the first relation in (5.16) with respect to B_0 and L_0 are:

$$\begin{aligned}
&-\sin\psi \frac{\mathrm{d}\psi}{\mathrm{d}B_0} = \cos B_0 \sin B - \sin B_0 \cos B \cos (L - L_0);\\
&-\sin\psi \frac{\mathrm{d}\psi}{\mathrm{d}L_0} = \cos B_0 \cos B \sin (L - L_0).
\end{aligned} \tag{5.17}$$

From (5.16) and (5.7) we obtain:

$$\frac{\partial\psi}{\partial B_0} = -\cos\alpha; \quad \frac{\partial\psi}{\partial L_0} = -\sin\alpha \cos B_0. \tag{5.18}$$

Introducing the relations (5.18) into (5.15) one obtains the partial derivatives of the geoid undulations with respect to B_0 and L_0:

$$\frac{\partial N}{\partial B_0} = -\frac{R}{4\pi\gamma}\iint_\sigma \Delta g \frac{dS(\psi)}{d\psi} \cos\alpha \, d\sigma;$$

$$\frac{\partial N}{\partial L_0} = -\frac{R\cos B_0}{4\pi\gamma}\iint_\sigma \Delta g \frac{dS(\psi)}{d\psi} \sin\alpha \, d\sigma. \tag{5.19}$$

On the basis of (5.14) and (5.19) one obtains the *Vening Meinesz* formulae for calculating the components of the gravity deflection of the vertical:

$$\xi = \frac{1}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g \sin\psi \frac{dS(\psi)}{d\psi} \cos\alpha \, d\psi \, d\alpha;$$

$$\eta = \frac{1}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi} \Delta g \sin\psi \frac{dS(\psi)}{d\psi} \sin\alpha \, d\psi \, d\alpha, \tag{5.20}$$

in which $d\sigma = \sin\psi \, d\psi \, d\alpha$.

The function:

$$V(\psi) = \frac{1}{2}\sin\psi \frac{dS(\psi)}{d\psi} = -\frac{1}{2}\left[\operatorname{cosec}\frac{\psi}{2} + 12\sin\frac{\psi}{2} - 32\sin^2\frac{\psi}{2} + \right.$$

$$\left. + \frac{3}{1+\sin\frac{\psi}{2}} - 12\sin^2\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right] \cdot \cos^2\frac{\psi}{2}, \tag{5.21}$$

is called the *Vening Meinesz* function. On the basis of (5.21) the formulae for calculating the components of the gravity deflection of the vertical become:

$$\xi = \frac{1}{2\pi\gamma}\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g \, V(\psi) \cos\alpha \, d\psi \, d\alpha;$$

$$\eta = \frac{1}{2\pi\gamma}\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g \, V(\psi) \sin\alpha \, d\psi \, d\alpha. \tag{5.22}$$

The ξ and η components as found from (5.22) have the same sign as the astro-geodetic ones defined by (10.4) and (10.5) (i.e. the component ξ is considered positive towards North and the η component positive towards East). This is the reason for the sign convention considered in (5.13)

The *Vening Meinesz* formulae hold even for the case of an arbitrary ellipsoid (for which *Stokes'* formula has the form given in (5.11)). Indeed, on differentiating formula (5.11), the first two terms vanish and as the final result one will obtain the same formula (5.20).

5.4 The Gravity Reduction

The determination of the geoid undulations by means of *Stokes'* formula had as initial premise the fact that the perturbing potential is an harmonic function in the space outside the geoid. As a consequence it was assumed that outside the geoid there are no attraction masses. Similarly, the boundary condition (4.9) assumes knowledge of the gravity on the geoid's surface. In practice, however, the gravity measurement is carried out on the physical surface of the Earth and between this surface and the geoid there exist attraction masses whose density δ is only approximately known. Henceforth these masses will be called *topographic masses*. In order to bring the gravity value from the physical surface onto the geoid and to evaluate the influence of the topographic masses on it, one applies to the g value a correction which is called the gravimetric reduction. Depending on the manner in which one takes into consideration the attraction of the topographic masses, one distinguishes several types of gravimetric reductions.

5.4.1 The Free-Air Reduction

In this type of reduction the topographic masses are removed, while the point where the gravity measurements were carried out remains suspended in "*free air*" (Fig. 5.3) at the altitude H.

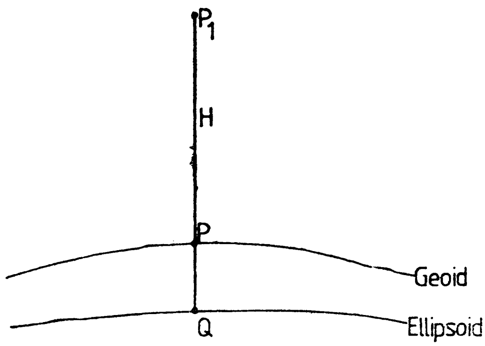


Fig. 5.3. The Free-Air Reduction

Considering the vertical gradient of gravity as being $\partial g/\partial H$, the gravity value at P will be given by:

$$g(P) = g(P_1) - \frac{\partial g}{\partial H} H. \tag{5.23}$$

The vertical gradient of gravity is determined starting from (3.7) in which the curvatures k_1 and k_2 are replaced by the reciprocals of the curvature radii R_1 and R_2. This yields:

$$\frac{\partial^2 W}{\partial H^2} = g\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + 2\omega^2 - 4\pi G\delta. \tag{5.24}$$

Considering the coordinate system chosen in such a manner that the X-axis is tangent to the meridian and the Y-axis tangent to the first vertical, the curvature radii R_1 and R_2 will be equal to the principal curvature-radii of the ellipsoid, whose values are determined by means of the relations (Chapt. 9, relations (9.5) and (9.50)):

$$M = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 B)^{3/2}}; \quad N = \frac{a}{(1 - e^2 \sin^2 B)^{1/2}}.$$

In this case, the relation (5.24) is applied to the reference ellipsoid of normal potential U outside which there are no longer attraction masses ($\delta = 0$). This relation becomes:

$$\frac{\partial^2 U}{\partial H^2} = -\frac{\partial \gamma}{\partial H} = \gamma\left(\frac{1}{M} + \frac{1}{N}\right) + 2\omega^2.$$

Introducing into this relation the necessary numerical values, one gets:

$$-\frac{\partial \gamma}{\partial H} = 0.000\,003086\ (1 + 0.00071\ \cos\ 2B)\ \text{gal/km}. \tag{5.25}$$

For practical purposes the second term in the bracket is neglected and one assumes with sufficient accuracy:

$$-\frac{\partial \gamma}{\partial H} = 0{,}3086\ \text{mgal/m}. \tag{5.26}$$

With sufficient accuracy the vertical gradients of the free-air and of the normal gravity may be considered as equal and then the free-air reduction can be written, taking into account (5.23) and (5.26) in the form:

$$g(P) = g(P_1) + 0{,}3086\ H. \tag{5.27}$$

Taking account of (4.6) one can now write the free-air gravity anomaly in terms of the gravity, observed on the physical surface, in the form:

$$\Delta g_a = [g(P_1) + 0{,}3086 H] - \gamma_Q. \tag{5.28}$$

5.4.2 The Bouguer Reduction

The principle of this reduction is based on the hypothesis that the difference between the gravity measured on the physical surface at P_1 and the gravity at P at the geoid level is due to the following causes: to the attraction Δg_r of the topographic masses and to the fact that the point P_1 is located at the distance H above the geoid. The second cause can be eliminated through the free-air reduction, so that the *Bouguer* reduction may be written as follows:

$$g(P) = g(P_1) + 0.3086\, H + \Delta g_T. \tag{5.29}$$

In order to evaluate the influence of the topographic masses, one assumes a mass element dm situated inside these masses, which acts upon a point P_1 on the physical surface (Fig. 5.4) with an attraction force $\vec{\mathbf{F}}$. The vertical component of the attraction force exerted by the mass dm on the point A is:

$$\mathrm{d}g_T = F \cos(\vec{\mathbf{F}}, \mathrm{d}\vec{\mathbf{g}})$$

in which:

$$F = \frac{G\mathrm{d}m}{s^2 + (h + H - z)^2}\,; \quad \cos(\vec{\mathbf{F}}, \mathrm{d}\vec{\mathbf{g}}) = \frac{h + H - z}{\sqrt{s^2 + (h + H - z)^2}}.$$

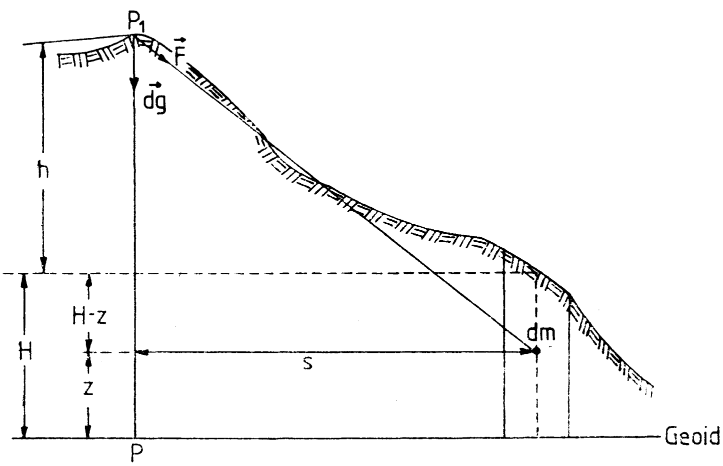


Fig. 5.4. The Attraction of the Topographic Masses on a Point P_1 of the Physical Surface

The vertical component of the effect of the entire topographic mass will be given by:

$$\Delta g_T = G\delta \int_0^{2\pi}\int_0^{a}\int_0^{H} \frac{s(h + H - z)\mathrm{d}r\, \mathrm{d}a\, \mathrm{d}z}{[s^2 + (h + H - z)^2]^{3/2}}, \tag{5.30}$$

where δ is the density of the topographic masses, assumed to be constant, and a is the radius of a circular domain inside which the influence of the topographic masses is taken into consideration.

Considering the topographic surface round the point P_1 as being plane up to a distance a from the point, the topographic masses will have the form of a cylinder of radius a and height H. In this case, the relation (5.30) will express the attraction of a cylinder on the point situated at the centre of the upper base:

$$\Delta g_T = 2\pi G \cdot \delta \cdot H\left(1 - \frac{H}{2a}\right).$$

In the case when $a > 10H$, the last term in the bracket is disregarded and the correction due to the influences of the topographic masses becomes:

$$\Delta g_T = 2\pi G \delta H. \tag{5.31}$$

One can show (*Brovar* et al. 1961) that the term $2\pi G$ may be expressed in terms of the Earth's mean density δ_0 in the form:

$$2\pi G = \frac{3\gamma}{2R\delta_0} = 0.0418\,\frac{\text{mgal} \cdot \text{cm}^3}{\text{m} \cdot \text{g}}.$$

Thus (5.31) becomes:

$$\Delta g_T = 0.0418 \cdot \delta \cdot H[\text{mgal}]. \tag{5.32}$$

The effect of the attraction of the layer of topographic masses is an increase in the observed gravity. In order to cancel this effect, the Δg_T correction must be subtracted from the value of the observed gravity.

In this manner, on the basis of (5.29) and (5.32) one can calculate the complete *Bouguer* reduction in the form:

$$g(P) = g(P_1) + (0.3086 - 0.0418\delta)H. \tag{5.33}$$

On the basis of (4.6) one can also write the *Bouguer* anomaly in the form:

$$\Delta g_B = g(P_1) + (0.3086 - 0.0418\delta)H - \gamma_Q. \tag{5.34}$$

When calculating the influence of the topographic masses using (5.32), the physical surface was considered as plane round the point concerned up to a distance $s = a$; the influence of the topographic masses situated at distances $s > a$ and the influence of the curvature of the terrestrial surface was also neglected. For this reason, in the case when a greater accuracy is required in calculating the influence of the topographic masses, one utilizes other procedures and calculation relations.

5.4.3 The Terrain Reduction

When calculating the topographic correction it was assumed that the terrain round the point P_1 is perfectly plane, which doesn't happen in reality. The masses M_1 situated above the level surface passing through P_1 (Fig. 5.5)

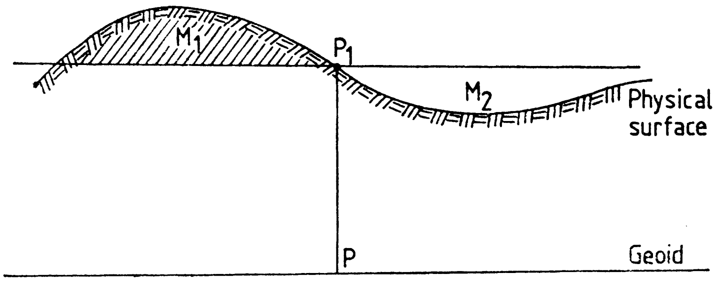


Fig 5.5. The Terrain Correction

as well as the mass deficit in the zone M_2 influence the gravity value $g(P_1)$. The masses situated in the zone M_1 reduce the gravity value at the point P_1, as they exert an attraction in a direction opposite to that of $\vec{g}(P_1)$. The correction which must be applied in order to remove this influence is consequently positive.

The mass deficit in the zone M_2 causes the measured gravity $g(P_1)$ to have a smaller value than it would have had if in this zone there existed attraction masses too.

Consequently, also in this case the terrain correction is positive.

The practical calculation of this correction is carried out by dividing the terrain round the gravity station into circular sectors with the aid of suitable templates (Fig. 5.6).

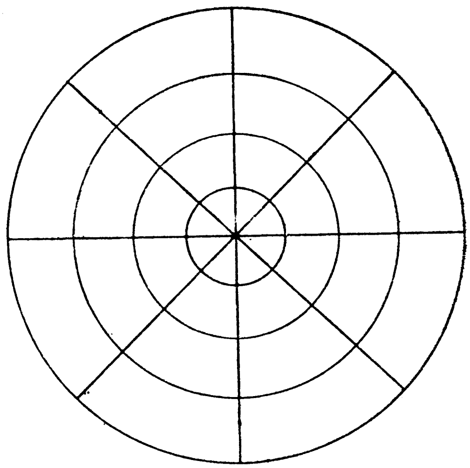


Fig. 5.6. Template for Calculating the Terrain Correction

For every circular sector of the template one determines the mean value of the height of the physical surface, on the basis of which one calculates the influence of the topographic mass of each sector on the total value of the terrain correction. The terrain correction is obtained as sum of the partial corrections using the relation:

$$\Delta g_r = \frac{0.0418\ \delta}{n} \Sigma(\sqrt{r_i^2 + h^2} - \sqrt{r_{i+1}^2 + h^2} + r_{r+1} - r_i), \tag{5.35}$$

in which n represents the number of circular sectors while r_i and r_{i+1}

represent the radii of the circular rings. The radii of the circular sectors as well as the detailed description of the technology of calculating the terrain correction may be found in the technical literature (e.g. *Mudretsova* 1940). Other more practical methods, which use automated calculation are also to be found (*Schleusener* 1940; *Pick* 1943). The terrain correction is applied to the measured value of the gravity, resulting in a levelling through calcula tion of the topographic masses. After applying this correction, the hypothesis admitted in deriving the formula of the *Bouguer* reduction, concerning the plane character of the topographic masses round the station is fulfilled.

5.4.4 The Isostatic Reduction

The principle of the *Bouguer* reduction is based on the hypothesis that the gravity anomalies are caused by the topographic masses. If this were the case, the *Bouguer* anomaly, which remains after removing the influence of these masses, should have values very close to zero. Equally, if this hypothesis were valid, the *Bouguer* anomaly should be positive in mountainous areas and the magnitude of the deflection of the vertical should be much influenced by the existence of topographic massifs. În reality, things do not happen like this but, on the contrary, the *Bouguer* anomaly is strongly negative in the mountainous zones and the deflection of the vertical is much less influenced by the topographic massifs than was to be expected. Thus, the conclusion is reached that the topographic masses are not superimposed on an homogeneous crust of density approximately constant but in the Earth's interior there exists a density variation which compensates the mass of the topographic massifs.

In order to explain this compensation two theories, whose principles will be presented in what follows, have been developed.

The Pratt-Hayford isostatic theory assumes the existence of a crust of constant density, located at a distance D below the sea-level. The distance D is called *compensation level*. Between this crust and the sea-level are masses whose density varies as a function of the volume of topographic masses existing between the sea surface and the Earth's physical surface.

Dividing the mass existing above the compensation level in columns, as in Fig. 5.7 and imposing the condition of the equality of the mass of each column, one obtains for each column of height D a variable density which is the smaller, the greater the topographic mass above.

Denoting the height of the topographic mass of the i column by h_i, the density of the column of height D ($h_i = 0$) by δ_0, and the density of the column of height $D + h_i$ by δ_i, the condition of equality of the masses of all the columns is written in the form:

$$(D + h_i)\,\delta_i = D\delta_0. \tag{5.36}$$

The topographic mass above the sea-level is compensated by a mass deficit which, according to (5.36), is

$$\delta_0 - \delta_i = \frac{h_i}{D + h_i}\,\delta_0, \tag{5.37}$$

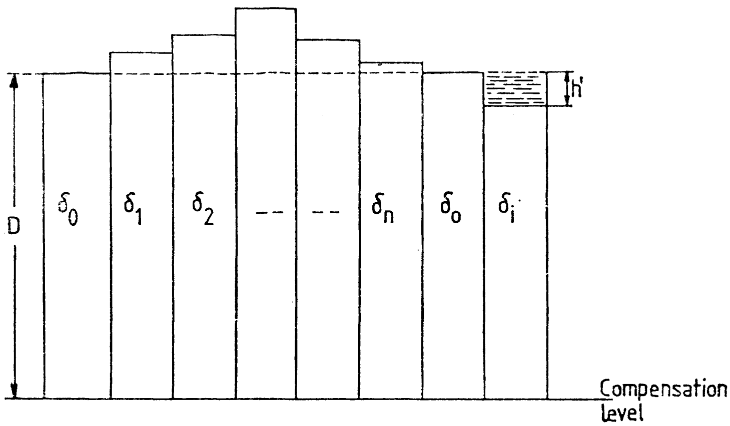


Fig. 5.7. The *Pratt-Hayford* Isostatic Hypothesis

and for the sea surface:

$$\delta_i - \delta_0 = \frac{h_i'}{D - h_i'} (\delta_0 - \delta_w), \tag{5.38}$$

in which h_i' is shown in Fig. 5.7.

The Airy-Heiskanen isostatic theory assumes that the topographic masses of constant density $\delta_0 = 2.67$ g/cm^3 are floating on a magma of greater density $\delta_1 = 3.27$ g/cm^3. The greater the topographic mass, the more it sinks into the magma (Fig. 5.8).

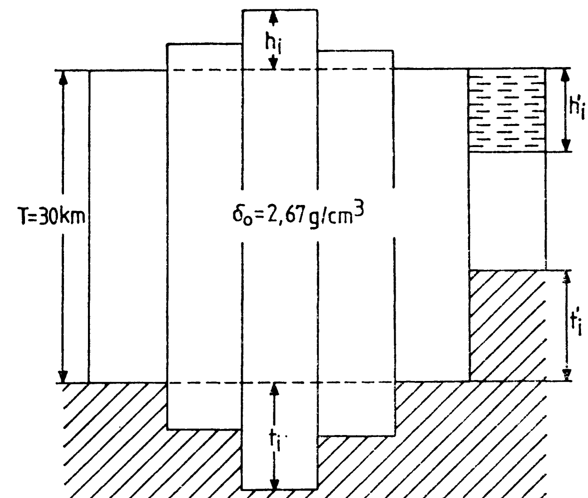


Fig. 5.8. The *Airy-Heiskanen* Isostatic Hypothesis

With the notations in Fig. 5.8, the floating condition of the topographic masses is:

$$t_i(\delta_1 - \delta_0) = h_i\delta_0, \tag{5.39}$$

whence one deduces the sinking depth of each column:

$$t_i = \frac{\delta_0}{\delta_1 - \delta_0} h_i, \tag{5.40}$$

and for the sea surface:

$$t_i' = \frac{\delta_0 - \delta_w}{\delta_1 - \delta_0} h_i'. \tag{5.41}$$

The theories of isostasy constitute geophysical models of the distribution of the masses situated below the Earth's physical surface down to a compensation depth. On their basis the gravity reductions have a more complete meaning. Thus, the topographic masses are not totally removed as in the *Bouguer* reduction but are transported to the interior so that a mass compensation should be achieved (in order to obtain for each column of height D or T the same mass).

The isostatic reduction is thus carried out in the following three stages:

(1) *Through the Bouguer corrections one removes the topographic masses.*

(2) *Through a compensation correction the topographic masses are introduced inside the columns of height D (Pratt-Hayford) or T (Airy-Heiskanen).*

(3) *Through the free-air correction one introduces the measured value to the geoid.*

The calculation of the isostatic reduction is caried out by dividing the topographic masses into columns with the aid of templates such as those in Fig. 5.6. The isostatic reduction has the form:

$$g(P) = g(P_1) - \Sigma A_T + \Sigma A_C + F, \tag{5.42}$$

where (*Heiskanen* and *Moritz* 1967):

A_T is the attraction of the topographic masses;

A_C — attraction of the compensation masses;

F — free-air correction.

Details concerning the calculation of the isostatic reduction may be found in the technical literature (*Graf* et al. 1966; *Heiskanen* 1931; *Kukkamäki* 1973).

5.5 Types of Reductions Used in Physical Geodesy

In addition to the gravity reductions mentioned in § 5.3, other reduction methods of less applicability in Physical Geodesy are also known: the *Rudzki* reduction, the condensation *Helmert* reduction, the *Poincaré-Prey* reduction.

All the reductions have however a common characteristic: through a chosen geophysical model, peculiar to each reduction type, the topographic masses are removed or transported inside the geoid. This changes the geoid potential and consequently even the geoid form. This change of the geoid constitutes the so-called indirect effect of the gravity reductions. The utilization of the gravity anomalies in *Stokes'* formula leads, as a consequence of the indirect effect, to the determination of a surface slightly different from the geoid, called the *co-geoid.*

For the *Bouguer* anomaly the indirect effect is very great, reaching values ten times larger than the geoid undulation. For the free-air and isostatic reductions the indirect effect is much smaller and does not exceed, in extreme cases, 10 m.

It is therefore necessary that when calculating the geoid undulations one takes into account the indirect effect of the gravity reductions. But this is not the only restriction imposed on gravity anomalies when they are used for calculating the geoid undulations and the deflection of the vertical. These must be easily calculated, to allow an interpolation as accurate as possible and to correspond to a geophysical model as close to reality as possible.

From among the known gravity anomalies, those having the smallest indirect effect are the free-air and the isostatic anomalies. The free-air anomalies depend however very much on the terrain topography and therefore their interpolation is carried out with considerable errors. They have nevertheless the great advantage of being easily calculated. As for the isostatic anomalies the situation is reversed: they allow a good interpolation but are much more difficult to calculate.

5.6 Practical Determination of the Components of the Deflection of the Vertical and of the Geoid Undulations

For the practical calculation of the geoid undulations and of the deflection of the vertical, the relations (5.6), respectively (5.21) are transformed, through an approximation procedure, into the so-called practical calculation formulae. For ease of calculation one's goal is that these formulae be derived in such a manner that they allow the simultaneous calculation of the geoid undulations and of the deflection of the vertical.

The principle of deriving the practical calculation formulae is the folowing: the integration domain σ, which in (5.6) and (5.21) represents the entire surface of the Earth, is divided into a series of subdomains σ_i, obeying the condition:

$$\sum_{i=1}^{n} \sigma_i = \sigma. \tag{5.43}$$

One then calculates the influence of each subdomain σ_i on the total value of the geoid undulations, and respectively of the deflection of the vertical at the point concerned. Denoting these influences by N_{σ_i}, ξ_{σ_i}, η_{σ_i}, the total values of the geoid undulation and of the deflection of the vertical at any point P are obtained thus:

$$N_P = \sum_{i=1}^{n} N_{\sigma_i}; \quad \xi_P = \sum_{i=1}^{n} \xi_{\sigma_i}; \quad \eta_P = \sum_{i=1}^{n} \eta_{\sigma_i}. \tag{5.44}$$

According to the manner in which the surfaces σ_i are chosen, up to now two calculation procedures have been worked out : using circular templates and using a network of rectangular surfaces respectively.

In the first procedure the subdomains σ_i have a fixed relative position with respect to the studied point, the surface of these subdomains varying with their distance from the point at which one determines the values ξ, η and N. In the second procedure the subdomains σ_i of equal surface have variable relative positions with respect to the point concerned.

5.6.1 The Determination of the Components of the Gravity Deflection of the Vertical and of the Geoid Undulation by Using Circular Templates

In order to obtain the practical formulae for calculating the components of the gravimetric deflection of the vertical, the relations (5.22) are presented in the form *(Eremeev* 1957*):*

$$\begin{aligned} \xi'' &= +\frac{1}{2\pi}\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g Q(\psi)\cos\alpha \, d\psi \, d\alpha; \\ \eta'' &= +\frac{1}{2\pi}\int_{\psi=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g\, Q(\psi)\sin\alpha \, d\psi \, d\alpha, \end{aligned} \tag{5.45}$$

in which $Q(\psi)$ denotes the expression:

$$Q(\psi) = -\frac{\rho''}{\gamma} V(\psi) = \frac{\rho''}{2\gamma}\sin\psi \frac{dS(\psi)}{d\psi}. \tag{5.46}$$

On the basis of (5.21) and (5.46), the function $Q(\psi)$ may be presented in the form:

$$\begin{aligned} Q(\psi) = -\frac{\rho''}{2\gamma}\cos^2\frac{\psi}{2}\Bigg[&\operatorname{cosec}\frac{\psi}{2} + 12\sin\frac{\psi}{2} - 32\sin^2\frac{\psi}{2} + \\ &+ \frac{3}{1+\sin\frac{\psi}{2}} - 12\sin^2\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\Bigg]. \end{aligned} \tag{5.47}$$

The variation of the function $Q(\psi)$ with respect to the spherical distance ψ is given in Table 5.1.

Table 5.1. *The Value of the Function* $Q(\Psi)$ ($\gamma = 979\,770$ mgal)

Ψ	$Q(\Psi)$	Ψ	$Q(\Psi)$	Ψ	$Q(\Psi)$
0°	∞	50	+0.43	110	−0.41
1	+12.35	60	+0.22	120	−0.40
10	+ 1.59	70	+0.03	130	−0.34
20	+ 1.02	80	−0.15	150	−0.16
30	+ 0.79	90	−0.29	170	−0.02
40	+ 0.61	100	−0.38	180	0.00

From Table 5.1 one can observe that the function $Q(\psi)$ varies very rapidly for small values of the argument, arriving at the value ∞ for $\psi = 0$. The general value of the integrals (5.45) remains however finite, except for the central point, where $\psi = 0$ and $Q(\psi) = \infty$. At this point, as the free-air anomaly has a constant value, it can be taken out from under the integral sign and since

$$\int_0^{2\pi} \cos\alpha \, d\alpha = \int_0^{2\pi} \sin\alpha \, d\alpha = 0,$$

one arrives at the indeterminate form $0 \times \infty$. In the rest of the integration domain, the integral remains finite.

For the special reasons discussed above, the calculation of the integral (5.45) will be carried out by dividing the integration domain with respect to ψ into zones as follows:

— first zone $0 \leqslant \psi \leqslant \psi_0$;

— second zone $\psi_0 \leqslant \psi \leqslant \psi_1$;

— third zone $\psi_1 \leqslant \psi \leqslant 2\pi$,

where $2' \leqslant \psi_0 \leqslant 5'$ and $1° \leqslant \psi_1 \leqslant 2°$. For ease of calculation in calculating the parameters of the circular templates, the spherical distances ψ are substituted by linear distances r (for ψ small) with the relation *(Eremeev* 1957; *Molodenski* et al. 1960*)*:

$$r = 2R \sin\frac{\psi}{2}\cdot$$

By expanding in series the trigonometric functions in the expression of $Q(\psi)$ and neglecting the 2nd-order terms, one gets:

$$Q(\psi) = -\frac{\rho''}{2\gamma}\left(\frac{2}{\psi} + \frac{\psi}{12} + 6\psi - \frac{3\psi}{2} - 3\right)\left(1 - \frac{\psi^2}{4}\right)\cdot$$

Putting now $\psi = \frac{r}{R}$ (for ψ small):

$$Q_1(r) = -\frac{\rho''}{2\gamma}\left(\frac{2R}{r} + \frac{49}{12}\frac{r}{R} + 3\right). \tag{5.48}$$

The utilization of the function $Q_1(r)$ instead of $Q(\psi)$ does not introduce perceptible errors into the calculation of the deflection of the vertical. Now introducing into $Q_1(r)$ the values $\rho'' = 206{,}265$, $R = 6378.1$ km and $\gamma = 980{,}600$ mgal, one gets the following expression:

$$Q_1(r) = -\frac{1333}{r} - 0.000067\,r - 0.314. \tag{5.49}$$

This form of the *Vening Meinesz* function is used in calculating the influence of relatively small domains (under 300 km round the point) i.e. for domains in which the difference $Q(\psi) - Q_1(r)$ does not introduce perceptible errors (these errors do not exceed $0''.05$ *(Makarov* 1968*)*).

In this case, noting that $d\psi = dr/R$ and considering $Q(\psi) = Q_1(r)$, the relatiins (4.45) become:

$$\xi = -\frac{1333}{2\pi R}\int_0^{r_0}\int_0^{2\pi}\frac{\Delta g\cos\alpha}{r}\,dr\,d\alpha - \frac{0.000067}{2\pi R}\int_0^{r_0}\int_0^{2\pi}\Delta g\, r\cos\alpha\,dr\,d\alpha -$$

$$-\frac{0.314}{2\pi R}\int_0^{r_0}\int_0^{2\pi}\Delta g\cos\alpha\,dr\,d\alpha = I_1 + I_2 + I_3. \tag{5.50}$$

For η one obtains an analogous formula.

We note that the two last terms on the right-hand side are negligible for a zone $r_0 = 10$ km and even larger (up to $r_0 = 30$ km). Indeed, if we consider the product $\Delta g\cos\alpha = \Delta = \text{const.}$, where $\Delta = \max|\Delta g\cos\alpha|$ (the worst case which could practically occur), then these two terms will have the form:

$$I_2 + I_3 = -\frac{0.000067}{2\pi R}\Delta\int_0^{r_0}\int_0^{2\pi} r\,dr\,d\alpha - \frac{0.314\,\Delta}{2\pi R}\int_0^{r_0}\int_0^{2\pi} dr\,d\alpha =$$

$$= K_1\Delta\pi r_0^2 - K_2\Delta 2\pi r_0 = K_1\Delta S_0 - K_2\Delta L_0,$$

in which:

$$\Delta = \Delta g\cos\alpha;$$

$$K_1 = \frac{0.000067}{2\pi R} \approx 10^{-10};$$

$$K_2 = \frac{0.314}{2\pi R} \approx 5\times 10^{-7};$$

$S_0 =$ the surface of the central zone of radius r_0;

$L_0 =$ the length of the circle limiting the central zone.

Considering $\Delta = 100$ mgal, we obtain for the central zone:

$$I_2 + I_3 = -10^{-10}100 \cdot 314 - 5 \cdot 10^{-8}100 \cdot 62{,}8 =$$

$$= -314 \cdot 10^{-8} - 314 \cdot 10^{-5} = -0''.0031.$$

Consequently, for the most unfavourable case the value of the last two terms doesn't exceed one hundredth of a second. In practice, these values are completely negligible.

The demonstration was made for $\Delta = \Delta g \cos \alpha$, i.e. for the value of ξ. The same demonstration holds also for η in which $\Delta = \Delta g \sin \alpha$.

Consequently, for the central zone one takes into consideration only the first term in (5.50) and the relations (5.45) for calculating the gravimetric deflection of the vertical have the following form:

$$\xi = -\frac{1333}{2\pi R}\int_0^{r_0}\int_0^{2\pi}\frac{\Delta g}{r}\cos\alpha\, dr\, d\alpha - \frac{1}{2\pi R}\int_{r_0}^{r_1}\int_0^{2\pi}\Delta g\, Q_1(r)\cos\alpha\, dr\, d\alpha -$$
$$-\frac{1}{2\pi}\int_{\psi_1}^{\pi}\int_0^{2\pi}\Delta g\, Q(\psi)\cos\alpha\, dr\, d\alpha;$$

$$\eta = -\frac{1333}{2\pi R}\int_0^{r_0}\int_0^{2\pi}\frac{\Delta g}{r}\sin\alpha\, dr\, d\alpha - \frac{1}{2\pi R}\int_{r_0}^{r_1}\int_0^{2\pi}\Delta g Q_1(r)\sin\alpha\, dr -$$
$$-\frac{1}{2\pi}\int_{\psi_1}^{\pi}\int_0^{2\pi}\Delta g\, Q(\psi)\sin\alpha\, dr\, d\alpha. \tag{5.51}$$

The formulae are not suitable for practical work and they are therefore transformed by approximating the integrals by sums. The domains chosen for summation are the circular sectors (Fig. 5.9) of the specially constructed templates. For each integration domain in formulae (5.51) ($[0, r_0]$; $[r_0, r_1]$; $[\psi_1, \pi]$) a template has been constructed and separate calculation formulae developed. The construction principle of the templates assumes the division of the whole domain into circular sectors of variable size, so that for the same value of the anomaly the influences of the sectors on the point under study, situated at the template centre, are equal. As this influence decreases with the increase of the radius r, it follows that in proportion to the distance from the centre the size of the circular sectors will increase.

For the influence of the circular zone of radius $r_0 = 5$ km the following calculation formulae have been established *(Eremeev* 1957*):*

$$\xi''_{0-5} = -0.03\sum_{k=1}^{8} \Delta g_k(r_1)\cos\alpha_x - 0.03\sum_{k=1}^{8} \Delta g_k(r_2)\cos\alpha_k - 0.003\sum_{k=1}^{8} \Delta g(r_3)\cos\alpha_k;$$

$$\eta''_{0-5} = -0.03\sum_{k=1}^{8} \Delta g_k(r_1)\sin\alpha_k - 0.03\sum_{k=1}^{8} \Delta g_k(r_2)\sin\alpha_k - 0.003\sum_{k=1}^{8} \Delta g_k(r_3)\sin\alpha_k,$$

in which $r_1 = 1.05$ km; $r_2 = 2.88$ km; $r_3 = 4.52$ km and $\Delta g_k(r_i)$ represents the anomalies at the point k situated on the circle of radius r_i $(i = 1, 2, 3)$ (Fig. 5.9, *a*).

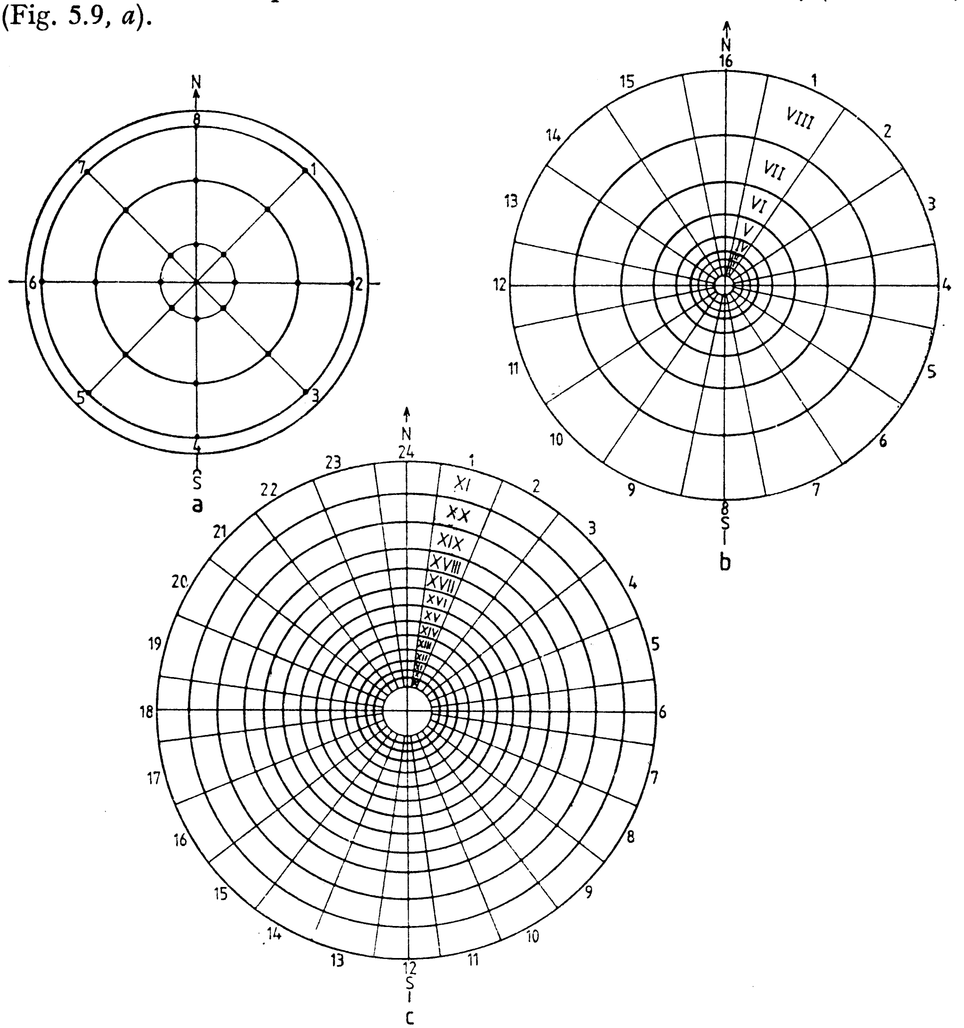


Fig. 5.9. The Templates for Calculating the Gravimetric Deflection of the Vertical
a — template for calculating the influence of the central zone; b — template for calculating the influence of the near zone; c — template for calculating the influence of the remote zone.

The derivation of the previous relations was based on the approximation of the double integrals by sums by means of a *Gauss* quadrature with three knots.

For this zone some more accurate formulae have also been established:

$$\xi''_{0-5} = -\sum_{i=1}^{6} C'_i \sum_{k=1}^{8} \Delta g_k(r_i) \cos \alpha_k ;$$

$$\eta''_{0-5} = -\sum_{i=1}^{6} C''_i \sum_{k=1}^{8} \Delta g_k(r_i) \sin \alpha_k , \tag{5.52}$$

in which: $r_1 = 0.511$ km; $r_2 = 0.748$ km; $r_3 = 1.094$ km; $r_4 = 1.599$ km; $r_5 = 2.338$ km; $r_6 = 3.419$ km. The numerical coefficients C' and C'' are calculated for each circular zone separately.

For the influence of the circular zones situated between 5 and 2,000 km the practical formula for calculating the deflection of the vertical is *(Eremeev, 1957):*

$$\xi''_{5-2000} = -0.005 \sum_{i=1}^{8} \sum_{k=1}^{16} \Delta g_{ik} \cos \alpha_k - 0.002 \sum_{i=9}^{13} \sum_{k=1}^{24} \Delta g_{ik} \cos \alpha_k -$$

$$- 0.0015 \sum_{i=14}^{121} \sum_{k=2}^{24} \Delta g_{ik} \cos \alpha_k - 0.000871 \sum_{i=22}^{26} \sum_{k=1}^{48} \Delta g_{ik} \cos \alpha_k ;$$

$$\eta''_{5-2000} = -0.005 \sum_{i=1}^{8} \sum_{k=1}^{16} \Delta g_{ik} \sin \alpha_k - 0.002 \sum_{i=9}^{13} \sum_{k=1}^{24} \Delta g_{ik} \sin \alpha_k - \tag{5.53}$$

$$- 0.0015 \sum_{i=14}^{21} \sum_{k=1}^{24} \Delta g_{ik} \sin \alpha_k - 0.000871 \sum_{i=22}^{26} \sum_{k=1}^{24} \Delta g_{ik} \sin \alpha_k ,$$

in which Δg_{ik} represents the average anomaly on the circular sector *ik*.

The radii of the circles r_i and the number of sectors in each ring zone are given in Table 5.2.

Additional details in connexion with the calculation technology for the components of the deflection of the vertical may be found in the technical literature *(Grushinski* 1963, *Makarov* 1969, *Eremeev* 1957).

Based on the same principle, *D. Rice* has worked out calculation templates for the deflection of the vertical, for a much more detailed division of the integration domain. The values of the radii and the number of the circular sectors of these templates are given in Table 5.3. *(Rice* 1952*)*.

The templates for calculating the gravimetric deflection of the vertical were constructed in such a manner that they can also be utilized for calculating the geoid undulations. One must however take into consideration the fact that the formula for calculating the geoid undulations does not depend on

Table 5.2. *The Radii of the Circular Templates for Calculating the Gravimetric Deflection of the Vertical*

Zone number	Number of zone sectors	Zone radia	Zone number	Number of zone sectors	Zone radia
1	16	5.0	14	24	305.4
2	16	7.3	15	24	357.8
3	16	10.7	16	24	418.1
4	16	15.7	17	24	487.4
5	16	22.8	18	24	566.6
6	16	33.3	19	24	656.6
7	16	48.5	20	24	758.0
8	16	70.6	21	24	872.0
9	24	102.6	22	48	1000.0
10	24	128.0	23	48	1163.7
11	24	159.6	24	48	1345.5
12	24	198.6	25	48	1545.6
13	24	246.7	26	48	1763.9
		305.4			2000.0

Table 5.3. *The Radii of Rice's Templates for Calculating the Gravimetric Deflection of the Vertical*

Zone	Inner radia	Zone	Inner radia	Zone	Inner radia	Zone	Inner radia
1	0.119	15	1.304	29	14.29	43	151.9
2	0.141	16	1.547	30	16.94	44	179.1
3	0.167	17	1.836	31	20.09	45	210.9
4	0.198	18	2.179	32	23.83	46	248.0
5	0.235	19	2.586	33	28.25	47	291.2
6	0.279	20	3.068	34	33.48	48	341.2
7	0.331	21	3.641	35	39.67	49	299.0
8	0.393	22	4.320	36	47.00	50	465.5
9	0.467	23	5.125	37	55.66	51	541.5
10	0.554	24	6.081	38	65.90	52	628.1
11	0.657	25	7.216	39	67.97	53	725.0
12	0.780	26	8.560	40	92.22	54	835.9
13	0.926	27	10.15	41	109.00	55	958.5
14	1.099	28	12.05	42	128.7	56	1094.3

azimuth and consequently the coefficients of the practical calculation formula will be a function only of the radii of the ring zones.

For calculating the geoid undulations one utilizes a similar calculation procedure. In order to obtain some formulae suitable for calculation, one introduces into (5.6) the surface element $d\sigma = \sin\psi \, d\psi \, d\alpha$, arriving at:

$$N = \frac{R}{2\pi\gamma} \int_0^{\pi} \int_0^{2\pi} \Delta g \, F(\psi) \, d\psi \, d\alpha, \tag{5.54}$$

in which

$$F(\psi) = \frac{dS(\psi)}{d\psi} \sin \psi.$$

Making the change of variable $\psi = r/R$, in which r is the radius of the circular zone and R the mean radius of the Earth considered as a sphere and considering $F(r_0) = 1$ for $0 \leqslant r_0 \leqslant 5$ km, (5.54) can be written as:

$$N = \frac{1}{2\pi\gamma} \int_0^{r_0} \int_0^{2\pi} \Delta g \, dr \, d\alpha + \frac{1}{2\pi\gamma} \int_{r_1}^{r_0} \int_0^{2\pi} \Delta g \, F(r) \, dr \, d\alpha + \frac{R}{2\pi\gamma} \int_{\psi_1}^{\pi} \int_0^{2\pi} \Delta g F(\psi) \, d\psi \, d\alpha. \tag{5.55}$$

Approximating the integrals by sums, in a manner analogous to the procedure used for establishing the practical formulae for calculating the gravimetric deflection of the vertical, one gets:

$$\begin{aligned} N = {} & 0.00255\,\Delta g_0 + 0.00032 \sum_{k=1}^{8} \Delta g_{r_0, k} + 10^{-5} \Bigg[15 \sum_{k=1}^{16} \Delta g_{1,k} + \\ & + 22 \sum_{k=1}^{16} \Delta g_{2,k} + 32 \sum_{k=1}^{16} \Delta g_{3,k} + 47 \sum_{k=1}^{16} \Delta g_{4,k} + 69 \sum_{k=1}^{16} \Delta g_{5,k} + \\ & + 101 \sum_{k=1}^{16} \Delta g_{6,k} + 147 \sum_{k=1}^{16} \Delta g_{7,k} + 219 \sum_{k=1}^{16} \Delta g_{8,k} + 118 \sum_{k=1}^{24} \Delta g_{9,k} + \\ & + 148 \sum_{k=1}^{24} \Delta g_{10,k} + 186 \sum_{k=1}^{24} \Delta g_{11,k} + 232 \sum_{k=1}^{24} \Delta g_{12,k} + 288 \sum_{k=1}^{24} \Delta g_{13,k} + \\ & + 262 \sum_{k=1}^{24} \Delta g_{14,k} + 304 \sum_{k=1}^{24} \Delta g_{15,k} + 353 \sum_{k=1}^{24} \Delta g_{16,k} + 408 \sum_{k=1}^{24} \Delta g_{17,k} + \\ & + 467 \sum_{k=1}^{24} \Delta g_{18,k} + 529 \sum_{k=1}^{24} \Delta g_{19,k} + 596 \sum_{k=1}^{24} \Delta g_{20,k} + 668 \sum_{k=1}^{24} \Delta g_{21,k} + \\ & + 424 \sum_{k=1}^{48} \Delta g_{22,k} + 462 \sum_{k=1}^{48} \Delta g_{23,k} + 499 \sum_{k=1}^{48} \Delta g_{24,k} + 518 \sum_{k=1}^{48} \Delta g_{25,k} + \\ & + 528 \sum_{k=1}^{48} \Delta g_{26,k} \Bigg]. \end{aligned} \tag{5.56}$$

In (5.56), the indices of the gravity anomaly represent the order number of the circular ring of the templates in Fig. 5.9 and the azimuth of the circular sector.

Using the formulae (5.56) the geoid undulation can be calculated by means of the same templates (with the exception of the central zone and consequently of the template in Fig. 5.9, *a* as those utilized for calculating the gravimetric deflection of the vertical.

Once calculated, the mean values of the gravity anomalies can be used both for calculating the ξ and η components and for calculating the geoid undulation N.

5.6.2 Calculating ξ, η and N by means of the Standard Surfaces

Dividing the entire Earth's surface into a series of *standard surfaces* q_k ($k = 1, 2, \ldots, n$) by means of a network of meridians and parallels (Fig. 5.10), and taking the mean anomaly of each standard surface k to be constant, the formulae for calculating the deflection of the vertical may be presented according to (5.45) and (5.48), in the form:

$$\begin{aligned} \xi'' &= -\frac{1}{2\pi} \sum_{k=1}^{n} \Delta g_{q_k} \iint_{q_k} Q_1(r) \cos\alpha \, dr \, d\alpha; \\ \eta'' &= -\frac{1}{2\pi} \sum_{k=1}^{n} \Delta g_{q_k} \iint_{q_k} Q_1(r) \sin\alpha \, dr \, d\alpha. \end{aligned} \tag{5.57}$$

If the points at which one determines the gravimetric deflection of the vertical are chosen at the intersection of meridian and parallels (e.g. the point P_1, Fig. 5.10), the calculation of the deflection of the vertical is solved

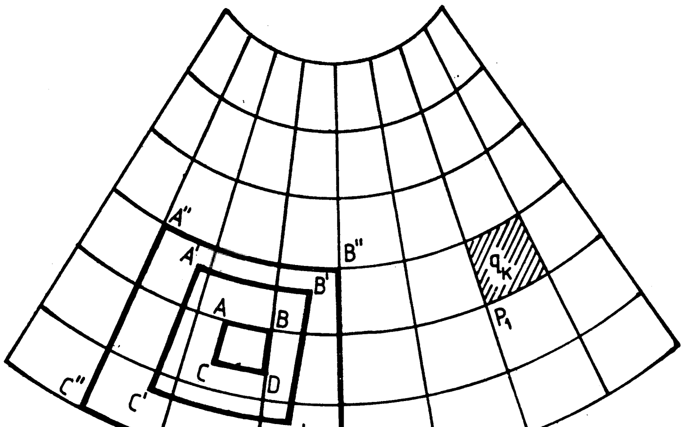


Fig. 5.10. Dividing the Integration Surface into Standard Surfaces

in two stages: firstly one calculates the influence of the anomaly in the four standard surfaces adjacet to the point P_1 and then one calculates the influence of the gravity anomalies from the other standard surfaces on the deflection of the vertical at P_1. This stage execution of the calculation is imposed by the indetermination which appears at the point P_1 *(Fischer* 1966*)*.

When one pursues the calculation of the gravimetric deflection of the vertical at a triangulation point this will no longer be at the intersection of the template lines but may have any position (as e.g. the point P_2, Fig. 5.10). In this situation the calculation will be carried out following another technique *(Mihăilescu* 1970, 1974*)*. For the practical calculation the Earth's surface is divided into three zones: a central zone (*ABCD*, Fig. 5.10); the near zone, limited inside by the central zone and outside by the polygonal line *A'B'C'D'*; the remote zone defined as the outside of the near zone.

The central zone is devoted to settling the indetermination which appears at the point P_2. The near zone was introduced in order to take into consideration in detail the influence of the anomaly in the neighbourhood of the point. As the remote zone is extended only up to 300 km round the point, one can introduce a rectangular coordinate system with the centre at the point under study and the X-axis pointing North. In this system the formulae for calculating the components of the deflection of the vertical were deduced in such a manner that the influences of the central and near zones are calculated by approximating the integrals by sums and the influence of the remote zone is calculated by using the mean value theorem for the double integrals in (5.51) *(Mihăilescu* 1970*)*.

The relations for the practical calculation of the components of the deflection of the vertical will take the form:

$$
\begin{aligned}
\xi'' = {} & \frac{\rho''}{2\pi\gamma}\sum_{l=1}^{m} C_l'\,\Delta g_l - 0{,}0333\sum_{i=1}^{n}\sum_{l=1}^{n} Cx_i y_j\,\Delta g(x_l, y_j)\,\frac{x_i}{(x_j^2+y_j^2)^{3/2}} - \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n} 0{,}0333\ln\frac{y_j+\sqrt{x_{i+1}^2+y_j^2}}{y_{j+1}+\sqrt{x_{j+2}^2+y_{j+1}^2}}\;\frac{y_{j+1}+\sqrt{x_i^2+y_{j+1}^2}}{y_j+\sqrt{x_i^2+y_j^2}}\,\Delta g(i,j) - \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{0{,}314}{4\pi R}\left[y_{j+1}\ln\frac{x_{i+1}^2+y_{j+1}^2}{x_i^2+y_{j+1}^2}+y_j\ln\frac{x_i^2+y_j^2}{x_i^2+y_j^{2+}{}_1}+\right. \\
& \left.+\,2x_{i+1}\tan^{-1}\frac{x_{i+1}(y_{j+1}-y_j)}{x_{i+1}^2+y_p y_{j+1}}+2x_i\tan^{-1}\frac{x_i(y_j-y_{j+1})}{x_i^2+y_j y_{j+1}}\right]\Delta g(i,j) - \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{0{,}000066}{4\pi R}\left[y_{j+1}\left(\sqrt{x_{i+1}^2+y_{j+1}^2}-\sqrt{x_i^2+y_{j+1}^2}\right)+y_j\left(\sqrt{x_i^2+y_j^2}-\right.\right. \\
& \left.-\sqrt{x_{i+1}^2+y_j^2}\right)+x_{i+1}^2\ln\frac{y_{j+1}+\sqrt{x_{i+1}^2+y_{i+1}^2}}{y_j+\sqrt{x_{i+1}^2+y_j^2}}+ \\
& \left.+\,x_i^2\ln\frac{y_j+\sqrt{x_i^2+y_j^2}}{y_{j+1}+\sqrt{x_i^2+y_{j+1}^2}}\right]\Delta g(i,j).
\end{aligned}
\tag{5.58}
$$

For the η component the formula will be analogous. In (5.58) the coefficients C_i are numerical coefficients obtained using the least squares method for the case of the approximation of the gravity anomaly by a regression function *(Diaconu* 1974*)*. The first term of (5.58) solves the problem of the influence of the central zone (the zone $ABCD$, Fig. 5.10). The second term of (5.58) solves the problem of the influence of the near zone ($A'B'C'D'$, Fig. 5.10). $C_{x_i y_i}$ are numerical coefficients of the *Gauss* quadrature used in approximating the integrals by sums, x_i and y_i are the coordinates of the quadrature points and i represents the number of knots of the *Gauss* quadrature ($i = 3.5$ or 7; for $i > 7$ the results no longer benefit from a perceptible increase in accuracy). The last three sums in (5.58) represent the influence of the remote zone. Here x_i and y_i represent the coordinates of the standard surfaces corners q_k (Fig. 5.10). When calculating the influence of the remote zone one must take into account the contribution of the zone lying between the polygonal lines $A'B'C'D'$ and $A''B''C''D''$ which is calculated with formulae of the same type as (5.58) *(Mihăilescu* 1974*)*.

The calculation methodology described in the preceding paragraph was worked out bearing in mind the utilization of electronic computers. In the method using the circular templates (§ 5.5.1), for each point at which one calculates the ξ and η components, one repeats the calculation of the mean anomalies on each circular sector of the template. In the procedure put forward here the calculation of the mean anomalies on the standard surfaces q_k is carried out only once, these values being utilized for obtaining the ξ and η components at every point of the domain under study. This is the main advantage of the method.

The formulae (5.58) are deduced in such a manner that on the basis of the same principle the geoid undulation can be determined as well. Because in the practical calculation, the gravity anomaly is not available on the entire surface of the Earth but only within a domain of radius $\psi_0$round the point under study, *Stokes'* formula may be presented in the form:

$$\mathrm{d}N = \frac{R}{4\pi\gamma}\int_0^{2\pi}\int_0^{\psi 0} \Delta g[S(\psi) - S(\psi_0)] \sin\psi \,\mathrm{d}\psi\, \mathrm{d}\alpha, \tag{5.59}$$

where dN means that part of the geoid undulations which is induced by the anomalies in the domain of radius ψ_0. Considering $\psi_0 = 285$ km, on the same principle as that by which the equations (5.58) were deduced, the following practical formulae for calculating the truncated value dN of the geoid undulations have been obtained *(Diaconu* and *Gulie* 1969*)*:

$$\begin{aligned}\mathrm{d}N = {} & \frac{1}{6160}\sum_{l=1}^{m} C_l''\,\Delta g_l + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{6160} \times \\ & \times \left(0.001329 + \frac{1}{\bar{r}_{ij}} - \frac{\bar{x}}{285\bar{r}_{ij}} - 0.000235 \ln x_i\right)\Delta g_{ij};\ \bar{x}_i \geqslant \bar{y}_j\end{aligned} \tag{5.60}$$

$$dN = \frac{1}{6160}\sum_{l=1}^{m} C_l^{\text{III}}\,\Delta g_l + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{6160}\times$$
$$\times\left(0.001329+\frac{1}{\bar{r}_{ij}}-\frac{\bar{y}_j}{285\,\bar{r}_{ij}}-0.000235\ln\bar{y}_j\right)\Delta g_{ij};\quad \bar{x}_i \geqslant y_j, \tag{5.60}$$

in which: C_1''' are numerical coefficients which do not depend on the quadrature chosen for approximating the integrals by sums; x_i, x_{i+1}, y_i, y_{i+1} are the coordinates of the corners of any standard surface k; $\bar{x}_i = (x_i + x_{i+1})/2$; $\bar{y}_i = (y_i + y_{i+1})/2$; r_{ij} is the distance from the point at which one determines the value dN to the centre of the standard surface concerned; Δg_{ij} is the mean anomaly on the standard surface k.

6

Determination of the Geoid by Astro-Gravimetric Methods

The geoid, as a mathematical form of the Earth, can be determined by gravity methods on the basis of the third boundary-value problem of Physical Geodesy. This determination assumes knowledge of the gravity anomalies, as boundary values at the geoid level, on the whole surface of the Earth. **This condition leads to the main drawbacks of the gravity methods.**

(1) *The existing gravity determinations don't provide knowledge of the gravity anomalies throughout the Earth's surface.* Consequently, the double integral in *Stokes'* formula cannot be extended to the entire surface of the Earth but is limited to a domain of radius $r = 1{,}000$—$2{,}000$ km, which leads to the appearance of a so-called *truncation error.*

(2) *The gravity anomalies have a discrete distribution on the Earth's surface, which necessitates the approximation of the integrals by sums.* This approximation process also introduces rather large errors.

(3) *According to the boundary condition imposed on the gravity anomalies these should be known on the geoid surface.* The methods of gravimetric reduction (Chapt. 5), by means of which one can obtain the gravity values at the geoid level, depend on knowledge of the density of topographic masses. As this value is unknown, it follows that the boundary value is erroneous and as a consequence the geoid undulation itself will be wrong.

All these disadvantages, to which should be added the fact that *Stokes'* formula (5.6) can be applied only to the case of an ellipsoid whose normal potential U_0 equals the actual potential W_0 of the geoid, limit the application of the gravimetric methods.

Neither can the geometric methods whose aim is to determine that ellipsoid which best approximates the Earth's surface, solve the problem independently, due to the fact that in the angle and distance measurements one uses the vertical which, being the direction of the gravity vector, is a quantity defined in the field of Physical Geodesy.

A combination of geodetic, astronomical and gravimetric measurements seems to be a better solution for the determination of the geoid.

6.1 The Astro-Geodetic Levelling

If at every point of a triangulation network astronomical determinations are carried out as well, one can determine the difference of the geoid undulations at two neighbouring points by using the relation:

$$dN = - u \, ds, \tag{6.1}$$

in which dN is the relative geoid undulation; ds — the elementary distance between two infinitely near points; u — the total deflection of the vertical.

Along an arbitrary direction of azimuth α, the deflection of the vertical is determined by means of the relation:

$$u_a = \xi \cos \alpha + \eta \sin \alpha. \tag{6.2}$$

Integrating the equation (6.1) one obtains the relative undulation of the geoid between two points A and B of the triangulation network:

$$N_B - N_A = - \int_A^B u \, ds. \tag{6.3}$$

The relation (6.1) represents the basic formula of the method of astro-geodetic levelling, whose name comes from the fact that the ξ and η components of the deflection of the vertical are determined on the basis of the relations (5.12) from astronomical and geodetic measurements.

The integral (6.3) may be evaluated by graphic or numerical integration. In practice, one utilizes two methods: either one applies the mean value theorem and then (6.3) becomes:

$$N_B - N_A = - u_m \frac{D_{AB}}{\rho''},$$

or one uses, for determining the value of u_m, maps of the deflection of the vertical if such maps are available.

The determinations of the relative undulations are carried out on closed polygons, after which one adjusts these polygons.

For the practical execution of astro-geodetic levelling there were recommended several procedures, out of which two will be analysed in the sequel.

6.1.1 The Utilization of n-Degree Polynomials in Solving the Problem of Astro-Geodetic Levelling

The principle of this method is based on the hypothesis that the geoid surface may be approximated by polynomials of degree n, whose coefficients c_{ij} are unknown.

If the coefficients c_{ij} can be determined by any method, the geoid undulation at a point $P(x_P, y_P)$ will be obtained in the form:

$$N(x_P, y_P) = \sum_{i,j=0}^{n} c_{ij} x^i y^j. \tag{6.4}$$

The coefficients c_{ij} are determined by the least squares method, starting from the relations (5.13) which may be written as error equations for a number L of triangulation points at which the astronomical determinations are also carried out:

$$\begin{aligned} &\frac{\mathrm{d}N(x_l, y_l)}{\mathrm{d}x} + \xi_l = v_{\xi_l}; \\ &\frac{\mathrm{d}N(x_l, y_l)}{\mathrm{d}y} + \eta_l = v_{\eta_l}; \ (l = 1, 2, \ldots, L). \end{aligned} \tag{6.5}$$

Denoting by p_ξ and p_η the weights of the determinations of the ξ and η components of the astro-geodetic deflection of the vertical, the condition of minimum for the system (6.5) may be written as follows:

$$[p_\xi \, v_\xi \, v_\xi] + [p_\eta \, v_\eta \, v_\eta] = \text{minimum}. \tag{6.6}$$

Introducing the matricial notations:

$$\boldsymbol{\eta} = \begin{Vmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_L \end{Vmatrix}; \ \boldsymbol{\xi} = \begin{Vmatrix} \xi \\ \xi_2 \\ \vdots \\ \xi_L \end{Vmatrix}; \quad \mathbf{N} = \begin{Vmatrix} N_1 \\ N_2 \\ \vdots \\ N_L \end{Vmatrix}; \quad \mathbf{V}_\xi = \begin{Vmatrix} v_{\xi_1} \\ v_{\xi_2} \\ \vdots \\ v_{\xi_L} \end{Vmatrix}; \quad \mathbf{V}_\eta = \begin{Vmatrix} v_{\eta_1} \\ v_{\eta_2} \\ \vdots \\ v_{\eta_L} \end{Vmatrix};$$

$$\mathbf{P}_\xi = \begin{Vmatrix} p_{\xi_1} & 0 & 0 \ldots 0 \\ 0 & p_{\xi_2} & 0 \ldots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \quad p_{\xi_L} \end{Vmatrix}; \quad \mathbf{P}_\eta = \begin{Vmatrix} p_{\eta_1} & 0 & 0 \ldots 0 \\ 0 & p_{\eta_2} & 0 \ldots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \quad p_{\eta_L} \end{Vmatrix}. \tag{6.7}$$

$$\mathbf{C} = \left\| \begin{matrix} c_{10} \\ \cdot \\ \cdot \\ c_{n0} \\ c_{01} \\ \cdot \\ \cdot \\ \cdot \\ c_{0n} \\ c_{11} \\ \cdot \\ \cdot \\ c_{n1} \\ \cdot \\ \cdot \\ \cdot \\ c_{1n} \\ \cdot \\ \cdot \\ \cdot \\ c_{nn} \end{matrix} \right\| ; \ \mathbf{A} = \left\| \begin{matrix} x_1 & x_1^2 \ldots x_1^n & y_1 \ldots y_1^n & x_1 y_1 \ldots & x_1^n y_1 \ldots & x_1 y_1^n \ldots & x_1^n y_1^n \\ x_2 & x_2^2 \ldots x_2^n & y_2 \ldots y_2^n & x_2 y_2 \ldots & x_2^n y_2 \ldots & x_2 y_2^n \ldots & x_2^n y_2^n \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_L & x_L^2 \quad x_L^n & y_L \quad y_L^n & x_L y_L & x_L^n y_L & x_L y_L^n & x_L^n y_L^n \end{matrix} \right\| , \tag{6.7}$$

the equation (6.4) can be written for L support points in the form:

$$\mathbf{N} = \mathbf{C}_{00} + \mathbf{A}\,\mathbf{C}, \tag{6.8}$$

in which $\mathbf{A}$ is a rectangular $L \times q$ matrix, q being the number of terms of the n-degree polynomial, without the free term $\mathbf{C}_{00}$ $(q = (n+1)^2 - 1)$. The free term of the polynomial represents a scale term which may be determined on the basis of the known value of the geoid undulation at one of the support points.

The equations (5.13) can be written, for the L support points, in the form:

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\,\mathbf{C} = -\,\boldsymbol{\xi}\,; \quad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}y} = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\,\mathbf{C} = -\,\boldsymbol{\eta}, \tag{6.9}$$

and the error equations (6.5) become:

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\,\mathbf{C} + \boldsymbol{\xi} = \mathbf{V}_\xi\,; \quad \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\,\mathbf{C} + \boldsymbol{\eta} = \mathbf{V}_\eta. \tag{6.10}$$

The coefficients c_{ij} are determined from the condition of minimum:

$$\begin{aligned} \boldsymbol{\Phi} &= \mathbf{V}_\xi^T \mathbf{P}_\xi \mathbf{V}_\xi + \mathbf{V}_\eta^T \mathbf{P}_\eta \mathbf{V}_\eta = \mathbf{C}^T \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right)^T \mathbf{P}_\xi \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right) \mathbf{C} + \mathbf{C}^T \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right)^T \mathbf{P}_\xi \boldsymbol{\xi} + \\ &+ \boldsymbol{\xi}^T \mathbf{P}_\xi \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right) \mathbf{C} + \boldsymbol{\xi}^T \mathbf{P}_\xi \boldsymbol{\xi} + \mathbf{C}^T \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\right)^T \mathbf{P}_\eta \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\right) \mathbf{C} + \mathbf{C}^T \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\right)^T \mathbf{P}_\eta \boldsymbol{\eta} + \\ &+ \boldsymbol{\eta}^T \mathbf{P}_\eta \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}y}\right) \mathbf{C} + \boldsymbol{\eta}^T \mathbf{P}_\eta \boldsymbol{\eta} = \min. \end{aligned} \tag{6.11}$$

The condition of minimum is achieved as follows:

$$\frac{d\mathbf{\Phi}}{dc}=\left[\left(\frac{d\mathbf{A}}{dx}\right)^T\mathbf{P}_\xi\left(\frac{d\mathbf{A}}{dx}\right)+\left(\frac{d\mathbf{A}}{dy}\right)^T\mathbf{P}_\eta\left(\frac{d\mathbf{A}}{dy}\right)\right]\mathbf{C}+ \\ +\left(\frac{d\mathbf{A}}{dx}\right)^T\mathbf{P}_\xi\boldsymbol{\xi}+\left(\frac{d\mathbf{A}}{dy}\right)^T\mathbf{P}_\eta\boldsymbol{\eta}=0. \tag{6.12}$$

The relation (6.12) represents the system of normal equations whose solution yields the matrix $\mathbf{C}$ of the coefficients of the polynomial (6.4), in the form:

$$\mathbf{C}=-\left[\left(\frac{d\mathbf{A}}{dx}\right)^T\mathbf{P}_\xi\left(\frac{d\mathbf{A}}{dx}\right)+\left(\frac{d\mathbf{A}}{dy}\right)^T\mathbf{P}_\eta\left(\frac{d\mathbf{A}}{dy}\right)\right]^{-1}\left[\left(\frac{d\mathbf{A}}{dx}\right)^T\mathbf{P}_\xi\boldsymbol{\xi}+ \\ +\left(\frac{d\mathbf{A}}{dy}\right)^T\mathbf{P}_\eta\boldsymbol{\eta}.\right]. \tag{6.13}$$

After determining the coefficient matrix, one can, by means of (6.4), determine the geoid undulation at any point of known coordinates. In this case, instead of the matrix **A**, whose elements are given by the coordinates of the support points, one utilizes a matrix **B**, of the same type as the matrix **A** whose elements involve the coordinates of the points at which the geoid undulation is to be calculated. The order of the matrix **B** is $r \times ((n+1)^2-1)$, in which r denotes the numbes of points to be interpolated. Consequently, for determining the coefficients of the nth-order polynomial one uses the matricial equation (6.8) and for interpolating the geoid undulations at any point one utilizes the matricial equation:

$$\mathbf{N}=\mathbf{C}_{00}+\mathbf{BC}. \tag{6.14}$$

With the aid of the matrix **C**, as determined according to (6.13), one can calculate the function $\mathbf{\Phi}$ (6.11) by means of which one may calculate the standard deviation of the weight unit by means of the relation:

$$\mathbf{m}_0=\frac{\mathbf{\Phi}}{L-(n+1)^2+1}. \tag{6.15}$$

The cofactors of the coefficients c_{ij} are given by the inverse of the matrix of the normal equations in (6.13):

$$\mathbf{Q}_{cc}=\left[\left(\frac{d\mathbf{A}}{dx}\right)^T\mathbf{P}_\xi\left(\frac{d\mathbf{A}}{dx}\right)+\left(\frac{d\mathbf{A}}{dy}\right)^T\mathbf{P}_\eta\left(\frac{d\mathbf{A}}{dy}\right)\right]^{-1}. \tag{6.16}$$

Considering the geoid undulations found using (6.14) in terms of adjusted elements, one can determine, with the aid of the law of cofactor propagation, the matrix of the geoid-undulation cofactors:

$$\mathbf{Q}_{NN}=\mathbf{B}\mathbf{Q}_{CC}\mathbf{B}^T. \tag{6.17}$$

On the basis of (6.15) and (6.17) one can now determine the interpolation accuracy for any point $P(x_i, y_i)$ $(i = 1, 2, \ldots, r)$:

$$m_{N_i}^2 = m_0^2 q_{ii}, \tag{6.18}$$

in which q_{ii} denotes the ith-order term on the main diagonal of the matrix $\mathbf{Q}_{NN}$.

In connexion with the accuracy parameters established before, some remarks should be made. The astro-geodetic levelling procedure described is in fact an interpolation procedure, by means of which the geoid surface is determined as a regression surface which, within certain limits, differs from the actual surface of the geoid. The estimation of the accuracy in obtaining the regression surface with respect to the actual surface of the geoid is achieved through the standard deviation of the weight unit. The value m_{N_i} expressed by (6.18) represents the accuracy in determining the geoid undulations with respect to the regression surface and not with respect to the actual geoid surface.

6.1.2 The Utilization of the Least Squares Collocation in Solving the Problem of Astro-Geodetic Levelling

Collocation is a procedure of mathematical processing of measurements, combining into one single method: adjustment, filtering and prediction (*Moritz* 1969, *Krarup* 1969, *Heitz* 1968).

Let $L_i (i = 1, 2, \ldots, q)$ be a field of q measurements, an error n_i corresponding to each measured value, and a number of m $(m < q)$ unknown parameters $X_r (r = 1, 2, \ldots, m)$. The connexion between the measurements L_i and the parameters X_j is achieved by means of a transformation matrix $\mathbf{A}$, in the form:

$$\mathbf{L} = \mathbf{AX} + \mathbf{n}, \tag{6.19}$$

in which the following matricial notations were used:

$$\mathbf{L} = \begin{Vmatrix} L_1 \\ L_2 \\ \vdots \\ L_q \end{Vmatrix}; \quad \mathbf{A} = \begin{Vmatrix} a_{11} a_{12} \ldots a_{1m} \\ a_{21} a_{22} \ldots a_{2m} \\ \vdots \quad \vdots \quad\quad \vdots \\ a_{q1} a_{q2} \ldots a_{qm} \end{Vmatrix}; \quad \mathbf{X} = \begin{Vmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{Vmatrix}; \quad \mathbf{n} = \begin{Vmatrix} n_1 \\ n_2 \\ \vdots \\ n_q \end{Vmatrix}. \tag{6.20}$$

The relation (6.19) has the form of an error equation. Here we have retained the notation n_i for errors as it was considered that in the collocation problem the errors have a wider meaning. On the basis of the q points at which the values L_i, affected by the errors n_i are known, one can determine the regression curve, represented by a continuous line in Fig. 6.1, *a*.

As well as the measure errors n_i two further types of errors affect the egression curve: a systematic error $\mathbf{AX}$ represented in Fig. 6.1, *b*, and a

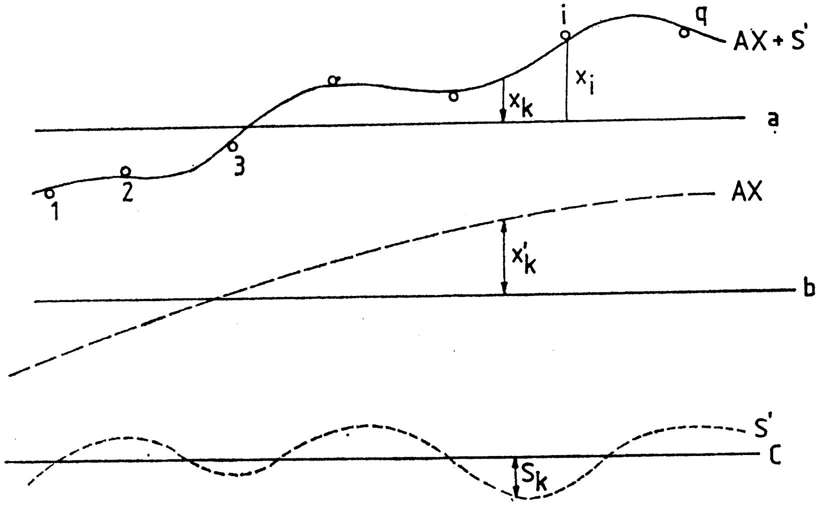


Fig. 6.1. Geometrical Representation of the Problem of Least-Squares Interpolation

random error $\mathbf{S}'$, represented in Fig. 6.1, *c*, also called a signal. The measurement value L_i at any point *i*, consequently consists of the systematic part $\sum_{i=1}^{m} a_{ij}X_i$ and the random values n_i and S'_1. Thus, more generally, (6.19) may be written in the form:

$$\mathbf{L} = \mathbf{AX} + \mathbf{n} + \mathbf{S}', \tag{6.21}$$

in which $\mathbf{S}'$ denotes the column vector:

$$\mathbf{S}' = \begin{Vmatrix} S'_1 \\ S'_2 \\ \vdots \\ S'_q \end{Vmatrix}. \tag{6.22}$$

Writing (*Moritz* 1973):

$$\mathbf{Z} = \mathbf{n} + \mathbf{S}' = \begin{Vmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_q \end{Vmatrix}. \tag{6.23}$$

(6.21) may be written as:

$$\mathbf{L} = \mathbf{AX} + \mathbf{z}. \tag{6.24}$$

The curve S' in Fig. 6.1, *c* is regarded as continuously variable, even though the L_i values are measured at discrete points. This allows one to interpolate the values L_K at K points ($K = 1. 2, ..., p$) at which no measurements were carried out.

The value L_K in Fig. 6.1, *a* is obtained from the systematic value L'_K in Fig. 6.1, *b* and the signal value S.

In what follows, for the quantities referring to the points at which one carries out the interpolation the indices k and l will be used, and for the quantities referring to the data points the indices i and j will be used. Also, for the points at which the interpolation is to be carried out the signal will be denoted by S.

The interpolation problem reduces to calculating the signal S_k at a number $k = 1, 2, ..., p$ of interpolation points, or, in other words, to determining the vector $\mathbf{S}$.

$$\mathbf{S} = \left\| \begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_p \end{matrix} \right\|. \tag{6.25}$$

If the vector $\mathbf{S}$ (6.25) is combined with the vector $\mathbf{Z}$ (6.24) into the vector:

$$\mathbf{V} = \left\| \begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_p \\ Z_1 \\ Z_2 \\ \vdots \\ Z_q \end{matrix} \right\|, \tag{6.26}$$

then (6.24) can be generalized in the form:

$$\mathbf{AX} + \mathbf{BV} - \mathbf{L} = 0, \tag{6.27}$$

where $\mathbf{B}$ is a matrix of the form:

$$\mathbf{B} = \left\| \begin{matrix} 0 & 0 \ldots 0 & 1 & 0 \ldots & 0 \\ 0 & 0 \ldots 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \quad 0 & 0 & 0 \ldots & 1 \end{matrix} \right\| = \| \mathbf{OE} \|. \tag{6.28}$$

The matrix $\mathbf{O}$ is a zero matrix of order $q \times p$ and $\mathbf{E}$ is the unit $q \times q$ matrix.

The relation (6.27) represents the general form of the standard problem V of compensation: condition equation with unknowns.

The solving of the system (6.27) is carried using the condition of minimum:

$$\mathbf{V}^T\mathbf{P}\mathbf{V} = \min,$$

in which $\mathbf{P} = \mathbf{Q}^{-1}$ is the covariance matrix:

$$\mathbf{Q}^{-1} = \begin{Vmatrix} \mathbf{C}_{SS} & \mathbf{C}_{SL} \\ \mathbf{C}_{LS} & \mathbf{C}_{LL} \end{Vmatrix}. \tag{6.29}$$

The elements of the covariance matrix:

$$\begin{aligned} \mathbf{C}_{SS} &= M[\mathbf{S}\mathbf{S}^T]; \\ \mathbf{C}_{SL} &= M[\mathbf{S}(\mathbf{L} - \mathbf{A}\mathbf{X})^T] = M[\mathbf{S}\mathbf{Z}^T]; \\ \mathbf{C}_{SL} &= \mathbf{C}_{LS}^T; \\ \mathbf{C}_{LL} &= M[(\mathbf{L} - \mathbf{A}\mathbf{X})(\mathbf{L} - \mathbf{A}\mathbf{X})^T] = M[\mathbf{Z}\mathbf{Z}^T], \end{aligned} \tag{6.30}$$

are assumed known.

From the condition of minimum, applied to the function:

$$\Phi = \frac{1}{2}\mathbf{V}^T\mathbf{Q}^{-1}\mathbf{V} - \mathbf{K}^T(\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{V} - \mathbf{L}), \tag{6.31}$$

where $\mathbf{K}$ denotes the vector of the correlates, one gets:

$$\mathbf{V} = \mathbf{Q}\,\mathbf{B}^T\mathbf{K}; \tag{6.32}$$

$$\mathbf{A}^T\,\mathbf{K} = \mathbf{0}. \tag{6.33}$$

Substituting (6.32) into (6.27) one obtains, together with (6.33), the system of normal equations of the standard problem V:

$$\begin{aligned} \mathbf{B}\,\mathbf{Q}\,\mathbf{B}^T\,\mathbf{K} + \mathbf{A}\mathbf{X} &= \mathbf{L}; \\ \mathbf{A}\,\mathbf{K} &= \mathbf{0}. \end{aligned} \tag{6.34}$$

From the equations (6.34) one deduces:

$$\mathbf{A}^T(\mathbf{B}\,\mathbf{Q}\,\mathbf{B}^T)\,\mathbf{A}\mathbf{X} = \mathbf{A}^T(\mathbf{B}\,\mathbf{Q}\,\mathbf{B}^T)^{-1}\,\mathbf{L}, \tag{6.35}$$

whence one can find the value of $\mathbf{X}$. Indeed, taking into consideration (6.28) and (6.29), one can determine the value of $\mathbf{X}$ in the form:

$$\mathbf{X} = (\mathbf{A}^T\,\mathbf{C}_{LL}^{-1}\,\mathbf{A})^{-1}\,\mathbf{A}^T\,\mathbf{C}_{LL}^{-1}\,\mathbf{L}. \tag{6.36}$$

From the equation (6.32) and from the first equation in (6.34) one finds the vector $\mathbf{V}$:

$$\mathbf{V} = \mathbf{Q}\,\mathbf{B}^T(\mathbf{B}\,\mathbf{Q}\,\mathbf{B}^T)^{-1}(\mathbf{L} - \mathbf{A}\mathbf{X}), \tag{6.37}$$

which, taking account of (6.26), (6.28) and (6.29) can be written in the form:

$$\begin{Vmatrix} \mathbf{S} \\ \mathbf{Z} \end{Vmatrix} = \begin{Vmatrix} \mathbf{C}_{SS} & \mathbf{C}_{SL} \\ \mathbf{C}_{LS} & \mathbf{C}_{LL} \end{Vmatrix} \cdot \begin{Vmatrix} \mathbf{O} \\ \mathbf{E} \end{Vmatrix} \mathbf{C}_{LL}^{-1}(\mathbf{L} - \mathbf{AX}). \tag{6.38}$$

From (6.38) one can obtain the value of the signal **S**:

$$\mathbf{S} = \mathbf{C}_{SL}\mathbf{C}_{LL}^{-1}(\mathbf{L} - \mathbf{AX}). \tag{6.39}$$

The relations (6.36) and (6.39) solve the complex collocation problem: the determination of the parameters **X** and then, on the basis of (6.39), one can carry out the prediction from whose outcome one gets the values of the signal **S**.

The relation (6.39) has formed the basis of the astro-geodetic levelling procedure applied by *S. Heitz* for the local determination of the geoid in the Federal Republic of Germany. To this end, the signal was considered equal to the geoid undulation at p data points, the matrix **A** was considered equal to zero and the covariance function was approximated *(Heitz* 1968*)* by:

$$C_{ij} = p_1 e^{-q_1 l_{ij}^2} + p_2 e^{-q_2 l_{ij}^2}, \tag{6.40}$$

in which the coefficients p_1, p_2, q_1, q_2 were empirically chosen:

$$p_1 = 7.50;\ q_1 = 0.2239;\ p_2 = 10.50;\ q_2 = 6.2134. \tag{6.41}$$

Instead of the matrix **L** the matrix **N** was chosen:

$$\mathbf{N} = \begin{Vmatrix} N_1 \\ N_2 \\ \vdots \\ N_q \end{Vmatrix},$$

whose elements being unknown have to be determined. With these transformations, (6.39) becomes

$$\bar{\mathbf{N}} = \mathbf{C}_{\bar{N}N}\mathbf{C}_{NN}^{-1}\,\mathbf{N}. \tag{6.42}$$

In (6.42) $\bar{\mathbf{N}}$ denotes the matrix of the geoid undulations at the support points:

$$\bar{\mathbf{N}} = \begin{Vmatrix} \bar{N}_2 \\ \bar{N}_2 \\ \vdots \\ \bar{N}_p \end{Vmatrix}, \tag{6.43}$$

which is also unknown. Consequently (6.42) represents a system of m equations with $2m$ unknowns which have no unique solutions. In order to solve the problem, one takes the derivative of (6.42) with respect to the rectangular coordinates x and y, taking account of the fact that for a given point configuration the matrix $\mathbf{C}_{\bar{N}N}$ is constant and the matrix $\mathbf{C}_{NN}$ is a function of the square of the distance l_{ik} between the data points P_i and the points to be interpolated P_K (s. (6.40)):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N} &= \frac{\mathrm{d}}{\mathrm{d}(l^2)}\,\mathbf{C}_{\bar{N}N}\,\frac{\mathrm{d}(l^2)}{\mathrm{d}x}\,\mathbf{C}_{NN}^{-1}\,\mathbf{N}\,; \\ \frac{\mathrm{d}}{\mathrm{d}y}\bar{\mathbf{N}} &= \frac{\mathrm{d}}{\mathrm{d}(l^2)}\,\mathbf{C}_{\bar{N}N}\,\frac{\mathrm{d}(l^2)}{\mathrm{d}y}\,\mathbf{C}_{NN}^{-1}\,\mathbf{N}. \end{aligned} \tag{6.44}$$

Setting

$$\mathbf{A} = \frac{\mathrm{d}}{\mathrm{d}(l^2)}\,\mathbf{C}_{\bar{N}N}\,\frac{\mathrm{d}(l^2)}{\mathrm{d}x}\,\mathbf{C}_{NN}^{-1}\,; \quad \mathbf{B} = \frac{\mathrm{d}}{(\mathrm{d}l^2)}\,\mathbf{C}_{\bar{N}N}\,\frac{\mathrm{d}(l^2)}{\mathrm{d}y}\,\mathbf{C}_{NN}^{-1}$$

and using (5.13), the relations (6.44) may be written as:

$$\boldsymbol{\xi} = \mathbf{A}\,\mathbf{N}\,; \;\; \boldsymbol{\eta} = \mathbf{B}\,\mathbf{N}, \tag{6.45}$$

in which $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ denote the column matrices of the components of the deflection of the vertical at the data points, which this time are known values. If one chooses a number q of data points for determining a number of q values of the geoid undulations, from (6.45) one gets $2q$ error equations unknowns:

$$\mathbf{V}_\xi = \mathbf{A}\,\mathbf{N} - \bar{\boldsymbol{\xi}}\,; \;\; \mathbf{V}_\eta = \mathbf{B}\,\mathbf{N} - \bar{\boldsymbol{\eta}}. \tag{6.46}$$

These equations are solved under the condition:

$$\mathbf{V}_\xi^T\,\mathbf{P}_\xi\,\mathbf{V}_\xi + \mathbf{V}_\eta^T\,\mathbf{P}_\eta\,\mathbf{V}_\eta = \min,$$

in which $\mathbf{P}_\xi$ and $\mathbf{P}_\eta$ denote the matrices of the weights of the components of the deflection of the vertical:

$$\mathbf{P}_\xi = \begin{Vmatrix} p_{\xi_1} & 0 & 0 \ldots 0 \\ 0 & p_{\eta_2} & 0 \ldots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \ldots p_{\xi_q} \end{Vmatrix}; \quad \mathbf{P}_\eta = \begin{Vmatrix} p_{\eta_1} & 0 & 0 \ldots 0 \\ 0 & p_{\eta_2} & 0 \ldots 0 \\ \vdots & \ldots\ldots & \vdots \\ 0 & 0 & 0 \ldots p_{\eta_q} \end{Vmatrix}$$

From the condition of minimum one arrives at the system of normal equations:

$$(\mathbf{A}^T\,\mathbf{P}_\xi\,\mathbf{A} + \mathbf{B}^T\mathbf{P}_\eta\,\mathbf{B})\,\mathbf{N} = \mathbf{A}^T\mathbf{P}_\xi\,\bar{\boldsymbol{\xi}} + \mathbf{B}^T\mathbf{P}_\eta\,\bar{\boldsymbol{\eta}}, \tag{6.47}$$

which immediately yield the matrix of the geoid undulations:

$$\mathbf{N} = (\mathbf{A}^T\,\mathbf{P}_\xi\,\mathbf{A} + \mathbf{B}^T\,\mathbf{P}_\eta\,\mathbf{B})^{-1}(\mathbf{A}^T\,\mathbf{P}\,\bar{\boldsymbol{\xi}} + \mathbf{B}^T\,\mathbf{P}_\eta\,\bar{\boldsymbol{\eta}}). \tag{6.48}$$

The system of normal equations is generally weakly shaped, the more so if the correlation of the geoid undulations is very strong. In order to eliminate this drawback, *S. Heitz* has proposed that the system of the normal equations be supplemented by the condition:

$$\sum_{k=1}^{q} N_q = 0. \tag{6.49}$$

The condition (6.49) is equivalent to referring the geoid to a surface parallel to the reference ellipsoid. For the determination of the absolute position of the geoid one needs knowledge of at least one absolute value of N.

6.2 The Astro-gravimetric Levelling

The methods of astro-geodetic levelling assume the existence of a field with a large density of astronomical determinations. The accurate determination of the coefficients of the polynomial (6.4) or of the covariances expressed through (6.30) is possible only in the case of the existence of a large number of values of the components of the astro-geodetic deflection of the vertical. In the technical literature, opinions concerning the necessary density of astronomical determinations differ somewhat, this density extending from 1 point to 25 km in plain areas and 1 point to 10 km in mountainous areas *(Heiskanen* and *Moritz* 1967*)* up to 1 point to 15 km in plain zones and 1 point to 3 km in mountainous zones (*Brovar* et al. 1961).

Such a density of astronomical determinations is difficult to achieve for which reason *M. S. Molodenski* has proposed a combined method for determining the relative geoid undulations, utilizing astronomical and gravimetric data. This method bears the name of astro-gravimetric levelling.

The principle of the method is based on the interpolation of the astro-geodetic deflections from gravimetric deflections of the vertical. The difference between these two types of deflections is due to the following factors:

(1) *To the difference between the general terrestrial ellipsoid to which refer the gravimetric deflections of the vertical and the reference ellipsoid to which are referred the astro-geodetic deflections.*

(2) *To the truncation error (when calculating the gravimetric deflection of the vertical, with the aid of the Vening Meinesz formula, one doesn't take into consideration the anomalies on the entire Earth's surface but the integration extends only over a limited domain* Σ*).*

Denoting by $\Delta\xi$ (respectively by $\Delta\eta$) the influence of the first abovementioned factor and the influence of the second one by $\xi_{(S-\Sigma)}$ (respectively $\eta_{(S-\Sigma)}$), in which S denotes the entire Earth's surface, the difference between the astro-geodetic and the gravimetric deflections of the vertical can be expressed in the following manner:

$$\xi^{ag} - \xi^{gr}_{\Sigma} = \xi_{S-\Sigma} + \Delta\xi\,; \quad \eta^{ag} - \eta^{gr}_{\Sigma} = \eta_{S-\Sigma} + \Delta\eta. \tag{6.50}$$

In (6.50), $\xi^{ag}(\eta^{ag})$ denotes the components of the astro-geodetic deflection of the vertical and $\xi^{(gr)}(\eta^{gr})$ denotes the components of the gravimetric deflection of the vertical calculated by means of the *Vening Meinesz* formula for which the integration extended over the domain Σ.

For the total deflection of the vertical on any azimuth α, one can write:

$$u_\alpha^{ag} - u_\Sigma^{gr} = u_{S-\Sigma} + \Delta u. \tag{6.51}$$

The variation of the astro-geodetic or gravimetric deflections of the vertical can be considered as linear only for very small distances, of 5—10 km. For areas in which the geoid relief is very rough these distances are even further reduced — down to 2—3 km. A linear interpolation of these deflections on greater distances will be affected by considerable errors and as a result the direct determination of the relative geoid undulations with the help of the method of the astro-geodetic levelling is not appropriate.

On the other hand, the difference between the two types of deflection of the vertical due to the two above-mentioned factors may be considered as varying linearly, even for much greater distances.

In the case when the surface Σ extends up to 2,000 km, the differences between the two types of deflection of the vertical may be considered as having a linear variation over a distance which increases up to 100—200 km. This hypothesis constitutes the basis of astro-gravimetric levelling.

In order to establish the formula of astro-gravimetric levelling one starts from the relations (6.3) and (6.51) from which one deduces:

$$N_B - N_A = -\int_A^B u_\Sigma^{gr}\,\mathrm{d}l - \int_A^B (u_{S-\Sigma} + \Delta u)\,\mathrm{d}l,$$

or, using (6.3):

$$N_B - N_A = N_{B(\Sigma)} - N_{A(\Sigma)} - \int_A^B (u_{S-\Sigma} + \Delta u)\,\mathrm{d}l, \tag{6.52}$$

in which $N_{B(\Sigma)}$ and $N_{A(\Sigma)}$ denote the geoid undulations at the points A and B as determined by means of *Stokes'* formula, the integration being extended over the domain Σ.

In order to evaluate the second integral in (6.52) one projects the points A and B onto the ellipsoid as A_0 and B_0. One approximates the ellipsoid by a plane and one introduces the system of plane coordinates X, Y as in Fig. 6.2.

The distance A_0B_0 in denoted by $2l$. On the basis of the hypothesis concerning the linearity of the function under the integral (6.52) one may write:

$$u_{S-\Sigma} + \Delta u = a + bx, \tag{6.53}$$

whence one can obtain the value of the sum $u_{S-\Sigma} + \Delta u$ at the point A:

$$(u_{S-\Sigma} + \Delta u)_A = a - bl,$$

and at the point B:

$$(u_{S-\Sigma} + \Delta u)_B = a + bl.$$

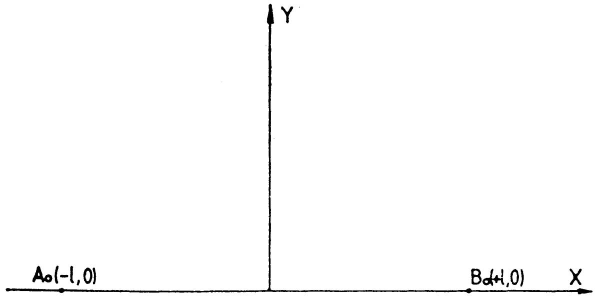


Fig. 6.2. The Coordinate System Used in Deriving the Calculating Formulae of Astro-gravimetric Levelling

From the two last relations one can get the value of the coefficient a:

$$a = \frac{(u_{S-\Sigma} + \Delta u)_A + (u_{S-\Sigma} + \Delta u)_B}{2},$$

or, taking account of (6.51), one obtains:

$$a = \frac{u_A^{ag} + u_{A(\Sigma)}^{gr} - u_{B(\Sigma)}^{gr} - u_B^{ag}}{2}. \tag{6.54}$$

On the other hand, introducing (6.53) into the last integral in (6.52), one gets:

$$\int_A^B (u_{S-\Sigma} + \Delta u) \mathrm{d}l = \int_{-l}^{+l} (a + bx)\, \mathrm{d}x = 2al. \tag{6.55}$$

From (6.54) and (6.55) we find:

$$\int_A^B (u_{S-\Sigma} + \Delta u)\, \mathrm{d}l = (u_A^{ag} + u_B^{ag})\, l - (u_{B(\Sigma)}^{gr} + u_{A(\Sigma)}^{gr})\, l. \tag{6.56}$$

On the basis of the formula (6.56), the relation (6.52) becomes:

$$N_B - N_A = -\,(u_B^{ag} + u_A^{ag})\, l + [(N_{B(\Sigma)} - N_{A(\Sigma)}) + (u_{B(\Sigma)}^{gr} + u_{A(\Sigma)}^{gr})]. \tag{6.57}$$

The relation (6.57) represents the formula for astro-gravimetric levelling. The term contained within the square bracket represents the gravimetric correction of the astronomical levelling which takes into account the non-linearity of the variation of the astro-geodetic deflection of the vertical between the points A and B. The great advantage of this procedure lies in the fact that the necessary density of astronomical determinations is much smaller (1 astronomical point to 100—200 km).

7

Determination of the Earth's Physical Surface

The geoid determination from gravity anomalies, as the third boundary-value problem of potential theory, assumes the perturbing potential to satisfy *Laplace*'s equation:

$$\Delta T = 0.$$

This assumes that outside the geoid there exist no attractive masses. With a view to achieving this desideratum and the condition according to which the gravity anomaly be known at the geoid level, one utilizes the gravity reductions which by removing the topographic masses or locating them inside the geoid, bring about the so-called regularization of the geoid. This regularization leads to the appearance of the indirect effect and, as a consequence, to the change of the geoid into co-geoid. To calculate the geoid undulations correctly, one must take into consideration the indirect effect δN, so that the geoid undulation is obtained in the following manner:

$$N = N_C + \delta N, \tag{7.1}$$

in which N_C is the geoid undulation.

The gravity reductions utilized for calculating the geoid undulations assume knowledge of the density of the topographic masses. As this density is unknown, in practice one makes certain hypotheses according to which the density of the topographic masses is constant $\delta = 2.67$ g/cm^3. In addition, in practical calculations one considers the vertical gradient of the gravity as equal to the vertical gradient of the normal gravity:

$$\frac{\delta\gamma}{\partial H} = -\ 0.3086 \text{ mgal/m}.$$

These two hypotheses falsify the results of the calculation of the geoid undulations and even of other geodetic quantities. Thus, when calculating the orthometric altitudes it is necessary to use the *Prey* reduction, which also assumes known the interior density of the topographic masses. The astronomical coordinates Φ and Λ are determined on the Earth's physical surface and the ξ and η components of the deflection of the vertical can be calculated on the geoid by means of *Vening Meinesz*'s formula, taking into consideration the indirect effect. In order that these two categories of quantities (astronomical coordinates and deflections of the vertical) may be compared it is necessary that either the former ones be brought onto the geoid or the quantities ξ and η be brought onto the physical surface; in both cases knowledge of the density of the topographic masses is required.

One can consequently conclude that the utilization of the conventional methods of Physical Geodesy assumes knowledge of the density of the topographic masses. In order to avoid this drawback, *M. S. Molodenski* has proposed a new approach to this problem *(Molodenski* et al. 1960*)*.

7.1 Molodenski's Problem

Molodenski's problem consists of determining the Earth's physical surface by using gravity quantities measured on the physical surface and not reduced on the geoid. Abandoning the geoid and the gravity reductions has the advantage of eliminating the error sources caused by the hypotheses concerning the density of the topographic masses.

If on the Earth's physical surface one knows the astronomical coordinates Φ and Λ for a series of points M_i, as well as their normal altitudes H^N one can construct the hypsometric surface Σ as was shown in § 3.5.3.

In order to obtain the accurate geodetic coordinates of the points M_i on the physical surface, the astronomical coordinates Φ and Λ as well as the altitudes H^N must be corrected using the quantities:

$$\delta\varphi_{M_i} = B_{M_i} - \Phi_{M_i}\,;\; \delta\lambda_{M_i} = L_{M_i} - \Lambda_{M_i}\,;\; H_{M_i} - H^N_{M_i} = \zeta_{M_i} \tag{7.2}$$

The corrections $\delta\varphi$, $\delta\lambda$ and ζ may be calculated if the perturbing potential T at the point M_i is known.

Indeed, according to (4.18) the height anomaly can be determined if one knows the potential T at the point M, assuming the difference $\Delta W = U_0 - W_0$ given by (4.17) to be known. In this case:

$$H_M = H^N_M + \frac{T_M}{\gamma_N} + \frac{U_0 - W_0}{\gamma_N}, \tag{7.3}$$

in which N represents the point on the hypsometric surface corresponding to the point M on the physical surface.

The determination of the corrections $\delta\varphi$ and $\delta\lambda$ is based on the known relations:

$$\delta\varphi_M = B_M - \Phi_M = -\xi_M\;;\; \delta\lambda_M = L_M - \Lambda_M = -\eta_M \sec B_M. \tag{7.4}$$

But according to (5.13):

$$\xi_M = -\frac{\partial N}{\partial x}\;;\; \eta_M = -\frac{\partial N}{\partial y},$$

or, in view of the relation (4.5), we have:

$$\xi_M = -\frac{1}{\gamma}\frac{\partial T}{\partial x} = -\frac{1}{\gamma\rho}\frac{\partial T}{\partial B}\;;\; \eta_M = -\frac{1}{\gamma}\frac{\partial T}{\partial y} = -\frac{1}{\gamma\rho\cos B}\frac{\partial T}{\partial L}, \tag{7.5}$$

in which ρ denotes the vector radius of the point M. On the basis of the formulae (7.4) and (7.5) one can write:

$$L_M = \Lambda_M + \frac{1}{\gamma\rho \cos B \sin 1''} \frac{\partial T_M}{\partial L}; \quad B_M = \Phi_M + \frac{1}{\gamma\rho \sin 1''} \frac{\partial T_M}{\partial B}. \tag{7.6}$$

Taking account of the curvature of the vertical as well the last relation is written as:

$$B_M = \Phi_M + \frac{1}{\gamma\rho \sin 1''} \frac{\partial T_M}{\partial B} - 0'',171\ H_M^N \sin 2B. \tag{7.7}$$

The relations (7.3), (7.6) and (7.7) solve the problem of determining the Earth's physical surface in the case when one knows the value of the perturbing potential T on this surface. One of the fundamental aspects of *Molodenski*'s theory is indeed the determination of this potential.

7.2 Determination of the Perturbing Potential by Using Green's Formulae

By applying to the actual potential *Green*'s third identity, we obtained in Chapt. 2 the relation (2.64). Taking (3.2) into consideration as well we get:

$$-2\pi W + \iint\limits_S \left[W \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial W}{\partial n} \right] dS + 2\pi\omega^2(x^2+y^2) + 2\omega^2 \iint\limits_S \frac{d\omega}{r'} = 0, \tag{7.8}$$

in which r, r' and n have the meanings given in Fig. 7.1.

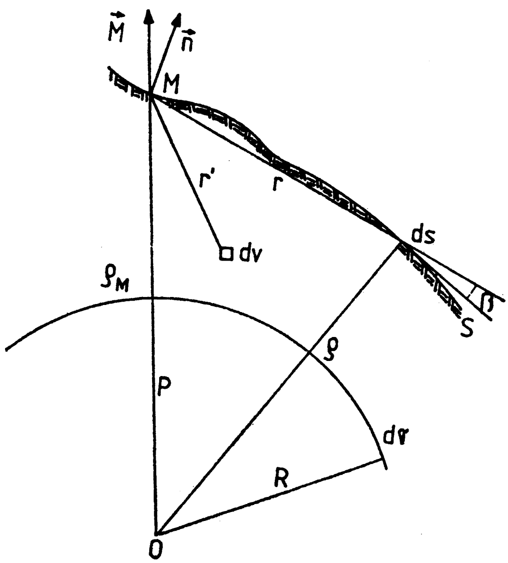


Fig. 7.1. Physical Signification of *Green*'s Formula as Applied to the Actual Potential

This relation establishes the connexion between the Earth's surface S, the actual potential W and its normal derivative $\partial W/\partial n$. It represents the most direct mathematical formulation of *Molodenski*'s problem.

By applying the relation (7.8) to the normal potential U, one gets:

$$-2\pi U + \iint\limits_S \left[U \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S + 2\pi\omega^2(x^2 + y^2) +$$

$$+ 2\omega^2 \iint\limits_S \frac{\mathrm{d}\omega}{r'} = 0.$$

Subtracting this relation from (7.8) and using (4.1) one gets:

$$-2\pi T + \iint\limits_S \left[T \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial T}{\partial n}\right] \mathrm{d}S = 0.$$

The integration over the unknown surface S can be replaced by integration over the hypsometric surface Σ, because:

$$T\,\mathrm{d}\Sigma \approx T\,\mathrm{d}S \text{ sau } \mathrm{d}S \approx \mathrm{d}\Sigma.$$

The previous relation may be written in the following manner:

$$-2\pi T + \iint\limits_\Sigma \left[T \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial T}{\partial n}\right] \mathrm{d}\Sigma = 0, \tag{7.9}$$

in which n is the normal to the Earth's physical surface which differs from the direction of the vertical by the angle β which may reach values of the order of degrees. For this reason the drivative of the normal potential along the $\vec{n}$ direction is different from the derivative along the direction of the vertical given by the boundary condition (4.20):

$$\frac{\partial T}{\partial n} \neq \frac{\partial T}{\partial H} = \left(\frac{1}{\gamma}\frac{\partial \gamma}{\partial H}T\right)_\Sigma - \left[\Delta g_\Sigma + \left(\frac{1}{\gamma}\frac{\partial \gamma}{\partial H}\right)_\Sigma \Delta W\right].$$

The value of the partial derivative of the perturbing potential along the normal to the physical surface depends on the components of the deflection of the vertical and on the inclinations β_1 and β_2 of the $N - S$ (respectively $E - W$) profiles of the terrain.

This derivative has been established (*Molodenski* et al. 1960) in the form:

$$\frac{\partial T}{\partial n} = \left[-\Delta g_\Sigma + \frac{1}{\gamma}\frac{\partial \gamma}{\partial H}T + \gamma(\xi \tan \beta_1 + \eta \tan \beta_2)\right] \cos \beta, \tag{7.10}$$

so that (7.9) can be written in the form:

$$T - \frac{1}{2\pi}\iint\limits_{\Sigma}\left[\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{\gamma}\frac{\partial \gamma}{\partial H}\frac{\cos\beta}{r}\right] T\, \mathrm{d}\Sigma =$$
$$= \frac{1}{2\pi}\iint\limits_{\Sigma}\frac{1}{r}\left[\Delta g_{\Sigma} - \gamma(\xi \tan \beta_1 + \eta \tan \beta_2)\right] \cos\beta\, \mathrm{d}\Sigma. \tag{7.11}$$

Solving the integral equation (7.11) is difficult, for which reason *M. S. Molodenski* has given another much more convenient way of solving it.

To this end, in order to formulate the boundary-value problem one uses another integral equation called the main integral equation. In order to obtain this equation, one expresses the perturbing potential T as the potential of a single layer distributed on the Earth's surface (with the same order of precision the single layer can be looked on as distributed on the hypsometric surface).

According to (2.44), denoting the density of the single layer by μ, the perturbing potential may be written in the following way:

$$T = \iint\limits_{\Sigma}\frac{\varphi}{r}\, \mathrm{d}\Sigma, \tag{7.12}$$

where $\varphi = G\mu$.

Substituting the perturbing potential given by (7.12) into the boundary condition (4.20), one gets:

$$\left(\frac{\partial T}{\partial H}\right)_{\Sigma} - \iint\limits_{\Sigma}\frac{1}{\gamma}\frac{\partial \gamma}{\partial H}\frac{\varphi}{r} = -\left[\Delta g + \frac{\Delta W}{\gamma}\left(\frac{\partial \gamma}{\partial H}\right)\right]_{\Sigma}. \tag{7.13}$$

It should be recalled that the derivatives of the potential of single layer undergo a discontinuity when passing through the layer. The first term in (7.13) represents the derivative in the direction of the vertical taken towards the outside of the single layer. If one denotes by β the angle between the direction of the vertical and the normal to the hypsometric surface, the first term in (7.13) can be obtained by using (2.47) in the form:

$$\left(\frac{\partial T}{\partial H}\right)_{\Sigma} = -\,2\pi\,\varphi\cos\beta + \iint \varphi\frac{\partial}{\partial H}\left(\frac{1}{r}\right)\mathrm{d}\Sigma =$$
$$= -\,2\pi\,\varphi\cos\beta + \iint \varphi\frac{\cos(r, \rho_M)}{r^2}\,\mathrm{d}\Sigma. \tag{7.14}$$

Introducing (7.14) into (7.13) and expressing $\cos(r, \rho_M)$ by (Fig. 7.1):

$$\cos(r, \rho_M) = -\frac{\rho^2 - \rho_M^2}{2\rho_M r} + \frac{r}{2\rho_M},$$

one gets:

$$-2\pi\varphi\cos\beta + \iint\left[\frac{\rho^2 - \rho_M^2}{2\rho_M r^3}\right]\varphi\, d\Sigma - \iint\frac{\varphi}{2r\rho_M}\, d\Sigma - \iint\frac{1}{\gamma}\frac{\partial\gamma}{\partial H}\frac{\varphi}{r} =$$

$$= -\left[\Delta g + \frac{\Delta W}{\gamma}\left(\frac{\partial\gamma}{\partial H}\right)\right]_\Sigma .$$

Taking (4.22), into consideration as well, the previous equality becomes:

$$2\pi\varphi\cos\beta = \left(\Delta g + \frac{2\,\Delta W}{\rho_M}\right)_\Sigma + \frac{1}{2\rho_M}\iint_\Sigma\frac{\rho^2 - \rho_M^2}{r^3}\varphi\, d\Sigma + \frac{3}{2\rho_M}\iint_\Sigma\frac{\varphi}{r}\, d\Sigma. \quad (7.15)$$

If the reference surface is approximated by a sphere, the surface element $d\Sigma$ is replaced by:

$$d\Sigma = \rho^2 \sec\beta\, d\sigma,$$

and (7.15) takes the form *(Heiskanen* and *Moritz* 1967*)*:

$$2\pi\,\varphi\cos\beta - \iint_\sigma\left(\frac{3}{2r} + \frac{\rho^2 - \rho_M^2}{2r^3}\right)\frac{\rho^2}{\rho_M}\sec\beta\,\varphi\, d\sigma = \Delta g, \quad (7.16)$$

which in fact represents the spherical approximation of the main integral equation in which one has taken $\Delta W = 0$.

7.3 Solving the Main Integral Equation

By introducing the spherical approximation into (7.16) one has admitted an error of the order of 3×10^{-3} in the perturbing potential. With the same approximation one can consider:

$$\frac{\rho^2 - \rho_M^2}{2\rho} \approx \frac{(R + H)^2 - (R + H_M)^2}{2R} \approx H^N - H_M^N,$$

in which H_M^N is the normal height of the point M and H^N the normal height of the running point of the Earth's physical surface; in addition we have considered $\rho \approx R$ and $\rho^2/\rho_M = R$.

In this case (7.16) can be written as:

$$2\pi\,\varphi\cos\beta = \Delta g + \frac{3R}{2}\iint_\sigma\frac{\varphi\sec\beta}{r}d\sigma + R^2\iint_\sigma\frac{H - H_M}{r^3}\varphi\sec\beta\, d\sigma.$$

Introducing a new density of the single layer:

$$\chi = \varphi\sec\beta,$$

the previous relation becomes:

$$2\pi \chi \cos^2 \beta = \Delta g + \frac{3R}{2} \iint_\sigma \frac{\chi \, d\sigma}{r} + R^2 \iint_\sigma \frac{H - H_M}{r^3} \chi \, d\sigma. \tag{7.17}$$

In (7.17) only changes of variable have been made, the integration being nevertheless carried out on the surface Σ. If one introduces a new boundary surface $\bar{\Sigma}$ for which one considers the vector radius:

$$\bar{\rho} = R + KH, \tag{7.18}$$

in which R is any constant and K is a coefficient whose numerical value lies between zero and one and if the anomalies $\Delta g_{\bar{\Sigma}}$ are given on this surface, then one gets the new perturbing potential:

$$\bar{T} = R^2 \iint_\sigma \frac{\bar{\chi}}{\bar{r}} \, d\sigma, \tag{7.19}$$

in which $\bar{\chi}$ is the new density on the surface $\bar{\Sigma}$. This density must satisfy an equation of the type (7.17):

$$2\pi \bar{\chi} \cos^2 \bar{\beta} = \Delta g + \frac{3R}{2} \iint_\sigma \frac{\bar{\chi}}{\bar{r}} \, d\sigma + R^2 \iint_\sigma \frac{H - H_M}{r^3} \chi \, d\sigma, \tag{7.20}$$

in which r denotes the distance between the running point with the vector radius $R + K H_i^N$ and the fixed point on the same surface with the vector radius $R + K H^N$. $\bar{\beta}$ denotes the angle of inclination of the new surface $\bar{\Sigma}$, which is connected through the following relation with the inclination β of the initial surface Σ:

$$\tan \bar{\beta} = K \tan \beta.$$

From this relation, for $\beta < 45°$, one can obtain the following convergent series:

$$\cos^2 \beta = (1 + K^2 \tan^2 \beta)^{-1} = 1 - K^2 \tan^2 \beta + K^4 \tan^4 \beta - \ldots \tag{7.21}$$

The functions $\bar{T}$ and $\bar{\chi}$ may be expressed in terms of K by the series:

$$\bar{T} = \sum_{n=0}^{\infty} K^n T_n ; \quad \bar{\chi} = \sum_{n=0}^{\infty} K^n \chi_n, \tag{7.22}$$

in which T_n and χ_n are unknown functions which don't depend on K. For $K = 1$, the relation (7.22) become:

$$\bar{T} = \sum_{n=0}^{\infty} T_n ; \quad \bar{\chi} = \sum_{n=0}^{\infty} \chi_n .$$

From Fig. 7.1 one can deduce the relation:

$$r^2 = r_0^2 \frac{\rho\, \rho_M}{R^2} + (\rho - \rho_M)^2,$$

or, on the basis of the approximation already introduced, the radius r may be expressed in the form:

$$r^2 = r_0^2 + (\rho - \rho_M)^2 = r_0 + [(R + H) - (R + H_M)]^2 = r_0^2 + (H - H_M)^2.$$

For the surface $\overline{\Sigma}$, the distance r between the point under study and the running point of the surface can be expressed by means of a similar expresssion:

$$\bar{r}^2 = r_0^2 + K^2(H - H_M)^2 = r_0^2(1 - K^2x^2), \tag{7.23}$$

in which the notation:

$$x = \frac{H - H_M}{r_0},$$

has been used.

On the basis of the relations (7.23) one can write the expansions in series:

$$\begin{aligned} \frac{1}{\bar{r}} &= \frac{1}{r_0}(1 + K^2x^2)^{-1/2} = \frac{1}{r_0}\left(1 - \frac{1}{2}K^2x^2 + \frac{3}{8}K^4x^4 - \ldots\right); \\ \frac{1}{\bar{r}^3} &= \frac{1}{r_0^3}(1 + K^2x^2)^{-3/2} = \frac{1}{r_0^3}\left(1 - \frac{3}{2}K^2x^2 + \frac{15}{8}K^4x^4 - \ldots\right). \end{aligned} \tag{7.24}$$

In view of (7.21), (7.22) and (7.24) the relation (7.20) can be written in the form:

$$\begin{aligned} 2\pi\sum_{n=0}^{\infty} K^n\chi_n(1 + K^2\tan^2\beta)^{-1} &= \Delta g_n + \frac{3R}{2}\iint \frac{1}{r_0}\sum_{n=0}^{\infty} K^n\chi_u\,(1 + \\ &+ K^2x^2)^{-1/2}\, d\sigma + R^2\iint_\sigma \frac{Kx}{r_0^2}(1 + K^2x^2)^{-3/2}\sum_{n=0}^{\infty} K^n\chi_n\Big)\, d\sigma. \end{aligned} \tag{7.25}$$

From the analysis of (7.25), taking into consideration the relations (7.24) as well, one sees that the left-hand side and the first integral on the right-hand side contain, for $n = s$, only values χ_n in which $n \leqslant s$, whereas the second integral on the right-hand side containes for $n = s$ only values χ_n in which $n \neq s$. Thus, for $n = 0$ one gets:

$$2\pi\chi_0 = \Delta g_0 + \frac{3R}{2}\iint_\sigma \frac{\chi_0}{r_0}\, d\sigma, \tag{7.26}$$

a relation from which one can determine the value χ_0. For $n = 1$, the left-hand side and the first integral on the right-hand side will contain terms in χ_0 and χ_1; furthermore the second integral in (7.25) will contain only terms in χ_0. In this manner, as χ_0 is known from (7.26), one can determine the value χ_1.

Generalizing, one can say that the terms in K^n will contain $\chi_n, \chi_{n-1}, \chi_{n-2}, \ldots$ with values already previously determined and which will intervene as constant terms in the integral equation of χ_n. This fact allows to write the general equation in the form:

$$2\pi\chi_n = \Delta g_n + \frac{3R}{2}\iint_\sigma \frac{\chi_n}{r_0}\, d\sigma, \tag{7.27}$$

in which the terms Δg_n contain, according to what was previously shown, values χ_{n-s} originating from the second integral in (7.25). By Δg_n one understands here the expansion in spherical harmonics of the gravity anomaly given by (4.35) which, as was shown in Chapt. 4, doesn't contain terms $p = 1$.

Expanding the quantity χ_n in spherical harmonics, the relation (7.27) can be written as:

$$2\pi\chi_{n,p}(M) = \Delta g_{n,p} + \frac{3}{2}\iint_\sigma \chi_{n,p} P_{n,p}\, d\sigma, \tag{7.28}$$

in which $\chi_{n,p}$ and $\Delta g_{n,p}$ denote the pth-order harmonic functions in the expansions of χ_n and Δg_n and P_n the *Legendre* polynomials. From (2.40) one gets:

$$\chi_{n,p}(M) = \frac{2p+1}{4\pi}\iint_\sigma \chi_{n,p} P_{n,p}\, d\sigma,$$

whence, in view of the relation (7.28) as well

$$2\pi\chi_{n,p}(M) = \Delta g_{n,p} + \frac{3}{2}\,\frac{4\pi}{2p+1}\,\chi_{n,p}(M),$$

or:

$$\Delta g_{n,p} = \chi_{n,p}\left[2\pi - \frac{3}{2}\,\frac{4\pi}{2p+1}\right] = \chi_{n,p}\,\frac{4\pi(p-1)}{2p+1},$$

leading for χ_{np}, to the value:

$$\chi_{n,p} = \frac{\Delta g_{np}}{4\pi}\,\frac{2p+1}{p-1}. \tag{7.29}$$

Introducing (7.29) into (7.27) and carrying out the expansion in spherical harmonics, one gets for the general term:

$$2\pi\chi_n = \Delta g_n + \frac{3}{2}\iint_\sigma \frac{\Delta g_n}{4\pi}\sum_{p=0}^{\infty}\frac{2p+1}{p+1}P_p(\cos\psi)\,d\sigma. \tag{7.30}$$

The relation (7.30) doesn't contain terms $p = 1$, because, as was shown in (4.35), the expansion of the anomaly in spherical functions doesn't contain such terms. The relation (7.30) can consequently be written in the form:

$$2\pi\chi_n = \Delta g_n + \frac{3}{2}\iint_\sigma \frac{\Delta g_n}{4\pi}\sum_{n=2}^{\infty}\frac{2p+1}{p-1}P_p(\cos\psi)\,d\sigma +$$

$$+\frac{3}{2}\iint_\sigma \frac{\Delta g_n}{4\pi}\left(\frac{2p+1}{p-1}\right)_{p=0}P_p(\cos\psi)\,d\sigma.$$

Noting the sum in the first integral is nothing else than *Stokes'* function $S(\psi)$ and the coefficient in the second integral equals -1 (for $p = 0$), it follows that:

$$\chi_n = \frac{\Delta g_n}{2\pi} + \frac{3}{(4\pi)^2}\iint_\sigma \Delta g_n\,[S(\psi) - 1]\,d\sigma. \tag{7.31}$$

Using (7.31) one can determine the values χ_n, if the quantities Δg_n are known. For their calculation, one introduces (7.21) and (7.24) into the relation (7.25). Equating then the coefficients of K_n on the right-hand and left-hand sides of the relation thus obtained, one obtains:

$$\begin{aligned}
&\Delta g_0 = g_M - \gamma_M;\\
&\Delta g_1 = R^2\iint \chi_0\frac{H - H_M}{r_0^3}\,d\sigma;\\
&\Delta g_2 = R^2\iint_\sigma \chi_1\frac{H - H_M}{r_0^3}\,d\sigma - \frac{3R}{4}\iint \chi_0\frac{(H - H_M)^2}{r_0^3}\,d\sigma + 2\pi\chi_0\tan^2\beta_0;\\
&\quad\ldots\qquad\qquad\ldots\qquad\qquad\ldots
\end{aligned} \tag{7.32}$$

The relations (7.32) offer the possibility of determining the density χ_n, through a process of successive approximation. This in turn makes possible the determination of the perturbing potential. Thus, on the basis of the relations (7.19), (7.22) and the first one in (7.24), one can write the equality:

$$\sum_{n=0}^{\infty} K^n T_n = R^2\iint_\sigma (\chi_0 + K\chi_1 + K^2\chi_2)\,\frac{1 - \frac{1}{2}K^2x^2 + \frac{3}{8}K^4x^4\ldots}{r_0}\,d\sigma.$$

This equality holds for each of the coefficients of K^n. Consequently, one gets:

$$T_0 = R^2 \iint_\sigma \frac{\chi_0}{r_0}\, d\sigma;$$

$$T_1 = R^2 \iint_\sigma \frac{\chi_1}{r_0}\, d\sigma; \tag{7.33}$$

$$T_2 = R^2 \iint_\sigma \frac{\chi_2}{r_0}\, d\sigma - \frac{R^2}{2} \iint_\sigma \chi_0 \frac{(H - H_M)^2}{r_0^2}\, d\sigma;$$

$$\dots \qquad \dots \qquad \dots$$

Thus, the perturbing potential is given in the form of a series:

$$T = T_0 + T_1 + T_2 + \dots .$$

For practical purposes one considers it sufficient to take into consideration only the first three terms. The main part of the perturbing potential is given by the term T_0 which has the same appearance as the formula published by *Stokes* in the year 1849.

On the basis of the relations (7.31), the relations (7.33) can be written as:

$$T_0 = \frac{R}{4\pi} \iint_\sigma \Delta g_0 [S\psi) - 1]\, d\sigma;$$

$$T_1 = \frac{R}{4\pi} \iint_\sigma \Delta g_1 [S(\psi) - 1]\, d\sigma; \tag{7.34}$$

$$T_2 = \iint_\sigma \frac{R}{4\pi} \Delta g_2 [S(\psi) - 1]\, d\sigma - \frac{R^2}{2} \iint_\sigma \frac{(H - H_M)^2}{r_0^3} \chi_0\, d\sigma;$$

$$\dots \qquad \dots \qquad \dots$$

Utilizing now *Bruns'* formula one can deduce the *height anomaly* (the difference between the hypsometric surface and the Earth's physical surface) in the form:

$$\zeta = \zeta_0 + \zeta_1 + \dots = \frac{R}{4\pi\gamma} \iint_\sigma \Delta g_0 [S(\psi) - 1]\, d\sigma +$$

$$+ \frac{R}{4\pi\gamma} \iint_\sigma \Delta g_1 [S(\psi) - 1]\, d\sigma + \dots \tag{7.35}$$

For practical purposes the first two approximations are considered sufficient.

In establishing the relations (7.35) one has assumed known the difference $W_0 - U_0$ between the actual potential and the normal one but no hypothesis was made concerning the difference between the Earth's actual mass M_P and the mass of the general terrestrial ellipsoid M_E. In other words, the relations (7.35) were deduced in the hypotheses: $\Delta W = 0$; $\Delta M \neq 0$, for which reason the form of the relations (7.35) is similar to the formula (5.12) deduced by *R. Hirvonen* for determining the geoid undulations.

As far as satisfying the conditions $\Delta W = \Delta M = 0$ is concerned, the relations (7.34) become:

$$
\begin{aligned}
T_0 &= \frac{R}{4\pi}\iint_\sigma \Delta g_0 S(\psi)\, d\sigma;\\
T_1 &= \frac{R}{4\pi}\iint_\sigma \Delta g_1 S(\psi)\, d\sigma;\\
T_2 &= \frac{R}{4\pi}\iint_\sigma \Delta g_2 S(\psi)\, d\sigma - \frac{R^2}{2}\iint_\sigma \frac{(H - H_M)^2}{r_0^3}\chi_0\, d\sigma;\\
&\ldots \qquad \ldots \qquad \ldots
\end{aligned}
\tag{7.36}
$$

In this case the first approximations of the height anomaly become:

$$
\begin{aligned}
\zeta_0 &= \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_0 S(\psi)\, d\sigma;\\
\zeta_1 &= \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_1 S(\psi)\, d\sigma;\\
\zeta_2 &= \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_2 S(\psi)\, d\sigma - \frac{R^2}{2\gamma}\iint_\sigma \frac{(H - H_M)^2}{r_0^3}\chi_0\, d\sigma;
\end{aligned}
\tag{7.37}
$$

Because only the first two approximations have a significant effect on the total height anomaly, the formula for determining the latter can be written as follows:

$$
\begin{aligned}
\zeta &= \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_0 S(\psi)\, d\sigma + \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_1 S(\psi)\, d\sigma =\\
&= \frac{R}{4\pi\gamma}\iint_\sigma (\Delta g_0 + \Delta g_1) S(\psi)\, d\sigma.
\end{aligned}
\tag{7.38}
$$

Thus, ζ is found from a relation which can be accepted as a *Stokes'* formula supplemented with higher-order approximations.

7.4 Geometrical Interpretation of the Solution of Molodenski's Equation

The approximate solution of *Molodenski*'s equation has a very intuitive geometrical interpretation which will be discussed in what follows *(Heiskanen* and *Moritz* 1967*)*.

For the case of the geoid ($\beta = 0$), the relation (7.25) takes the special form:

$$2\pi\chi_0 - \frac{3R}{2}\iint\limits_\sigma \frac{\chi_0}{r_0}\, d\sigma = \Delta g, \tag{7.39}$$

and for the same case, taking *Bruns'* formula (4.5) into consideration too, the relation (7.19) becomes:

$$T = g_m\zeta_0 = R^2\iint\limits_\sigma \frac{\chi_0}{r_0}\, d\sigma, \tag{7.40}$$

where g_m denotes the mean value of the gravity on the entire Earth. From (7.39) and (7.40) one deduces:

$$\chi_0 = \frac{1}{2\pi}\left(\Delta g + \frac{3g_m}{2R}\zeta_0\right). \tag{7.41}$$

Using for χ_0 the value deduced from (7.41), the correction in (7.32) can be written as follows:

$$\Delta g_1 = \frac{R^2}{2\pi}\iint\limits_\sigma \frac{H - H_M}{r_0^3}\,\mu\, d\sigma, \tag{7.42}$$

in which one has utilized the notation:

$$\mu = \Delta g + \frac{3g_m}{2R}\zeta_0 = \Delta g + \frac{3T_0}{2R}. \tag{7.43}$$

Using the transformation *(Molodenski* et al. 1960*)*:

$$(H^N - H_M^N)\mu = -H_M^N(\mu - \mu_M) + (\mu H^N - \mu_M H_M^N),$$

(7.42) becomes:

$$\Delta g_1 = -H^N\frac{R^2}{2\pi}\iint\limits_\sigma \frac{\mu - \mu_M}{r_0^3}\, d\sigma + \frac{R^2}{2\pi}\iint\limits_\sigma \frac{\mu H^N - \mu_M H_M^N}{r_0^3}\, d\sigma. \tag{7.44}$$

Considering the equality *(Heiskanen* and *Moritz* 1967*)*:

$$\frac{R^2}{2\pi}\int\limits_{\lambda=0}^{2\pi}\int\limits_{\theta=0}^{\pi} \frac{V - V_M}{r_0^3}\sin\theta'\, d\theta'\, d\lambda' = -\frac{1}{R}\sum_{n=0}^{\infty} n\, Y_n(\theta, \lambda), \tag{7.45}$$

in which V represents a differentiable harmonic function and V_M represents the value of the function at a point M situated on the sphere:

$$V_M = \sum_{n=0}^{\infty} Y_n(\theta, \lambda)$$

and in view of the expansion in spherical harmonics of the functions μ and $H^N \mu$:

$$\mu = \sum_{n=0}^{\infty} \mu_n; \quad \mu H = \sum_{n=0}^{\infty} (\mu H)_n, \tag{7.46}$$

(7.44) may be written in the form:

$$\Delta g_1 = \frac{H_M}{R} \sum_{n=0}^{\infty} n \mu_n - \frac{1}{R} \sum_{n=0}^{\infty} n(\mu H)_n. \tag{7.47}$$

From (7.46) one deduces:

$$H^N \sum_{n=0}^{\infty} \mu_n = \mu H^N = \sum_{n=0}^{\infty} (\mu H^N)_n,$$

whence one can write the equality:

$$\frac{1}{R} \sum_{n=0}^{\infty} (\mu H)_n = \frac{H^N}{R} \sum_{n=0}^{\infty} \mu_n. \tag{7.48}$$

On the basis of (7.48) one can write (7.47) in the form:

$$\Delta g_1 = \frac{H}{R} \sum_{n=0}^{\infty} n \mu_n - \frac{H}{R} \sum_{n=0}^{\infty} \mu_n - \frac{1}{R} \sum_{n=0}^{\infty} n(\mu H)_n + \frac{1}{R} \sum_{n=0}^{\infty} (\mu H)_n. \tag{7.49}$$

Setting:

$$\begin{aligned} \Delta g_{11} &= \frac{H_M}{R} \sum_{n=0}^{\infty} n \mu_n - \frac{H_M}{R} \sum_{n=0}^{\infty} \mu_u; \\ \Delta g_{12} &= - \frac{1}{R} \sum_{n=0}^{\infty} n(\mu H)_n + \frac{1}{R} \sum_{n=0}^{\infty} (\mu H)_n, \end{aligned} \tag{7.50}$$

the relation (7.48) becomes:

$$\Delta g_1 = \Delta g_{11} + \Delta g_{12}.$$

On the basis of this relation, the 1st-order approximation in the calculation formula for the height anomaly becomes:

$$\zeta_1 = \frac{R}{4\pi\gamma} \iint_\sigma \Delta g_{11} S(\psi) d\sigma + \frac{R}{4\pi\gamma} \iint_\sigma \Delta g_{12} S(\psi)\, d\sigma. \tag{7.51}$$

Considering the expansions in series $\Delta g = \Sigma \Delta g_n$ and $T_0 = \Sigma T_n$, in which T_0 denotes the zero-order approximation and not the zero-degree spherical harmonic, on the basis of (7.43) one can write:

$$\mu_n = \Delta g_n + \frac{3}{2R} T_n.$$

In this case, the first equality in (7.50) becomes:

$$\Delta g_{11} = \frac{H_M}{R} \sum_{n=0}^{\infty} (n+2)\Delta g_n - \frac{3H_M}{R} \Delta g + \frac{3H_M}{2R^2} \sum_{n=0}^{\infty} (n-1) T_n. \tag{7.52}$$

From (4.34) and (4.37) one deduces:

$$\Delta g = \sum_{n=0}^{\infty} (R/\rho)^{n+2} \Delta g_n.$$

If one differentiates this relation with respect to ρ and if one takes $\rho = R$, one obtains at the geoid level:

$$\frac{\partial \Delta g}{\partial \rho} = -\frac{1}{R} \sum_{n=0}^{\infty} (n+2) \Delta g_n. \tag{7.53}$$

In view of (7.53) and (4.35), the relation (7.52) becomes:

$$\Delta g_{11} = -H_M \frac{\partial \Delta g}{\partial H} - \frac{3H_M}{2R} \Delta g. \tag{7.54}$$

The last term in (7.54) can be disregarded, as it introduces into Δg_{11} an error of at most $10^{-3}\,\Delta g$, so that this relation reduces to:

$$\Delta g_{11} = -H_M \frac{\partial \Delta g}{\partial H}. \tag{7.55}$$

The relation (7.44) shows that the value Δg_{11} corresponds to the reduction of the free-air anomaly from the level of the Earth's physical surface to the geoid level.

For the geometrical interpretation of the term on the right-hand side of (7.55), one starts from the fact that the height anomaly, being determined from a relation of the same type as the geoid undulation, can be expanded in spherical harmonics using the same relation which is valid for *Stokes'* formula as well:

$$\zeta_n = \frac{R}{(n-1)\gamma} \Delta g_n.$$

Specializing this relation for the case:

$$\zeta_n = (\zeta_{12})_n; \quad \Delta g_n = (\Delta g_{12})_n,$$

one gets:

$$(\zeta_{12})_n = \frac{R}{(n-1)\gamma}(\Delta g_{12})_n;$$

or, in view of the second relation in (7.50), one arrives at:

$$(\zeta_{12})_n = -\frac{1}{\gamma}(\mu H_M)_n.$$

Summing up, on the basis of (7.46), one finds:

$$\zeta_{12} = -\frac{\mu H_M}{\gamma}; \tag{7.56}$$

or, in view of (7.43) and removing $g_m \approx \gamma$:

$$\zeta_{12} = -\frac{H_M \Delta g}{\gamma} - \frac{3H_M}{2R}\zeta_0 \approx -\frac{H_M\,\Delta g}{\gamma}. \tag{7.57}$$

On the other hand, taking (4.9) into consideration, the derivative of the height anomaly with respect to the altitude H is:

$$\frac{\partial\zeta}{\partial H} = \frac{\partial}{\partial H}\left(\frac{T}{\gamma}\right) = -\frac{1}{\gamma}\left(-\frac{\partial T}{\partial H} + \frac{1}{\gamma}\frac{\partial\gamma}{\partial H}T\right) \approx -\frac{\Delta g}{\gamma}. \tag{7.58}$$

Comparing (7.57) and (7.58) one deduces:

$$\zeta_{12} = \frac{\partial\zeta}{\partial H}H_M, \tag{7.59}$$

which shows that the term ζ_{12} corresponds to the reduction of the height anomaly from the geoid level to the level of the Earth's physical surface.

On the basis of (7.55) and (7.59), the relation (7.38) can be written as follows:

$$\zeta = \frac{R}{4\pi\gamma}\iint\limits_{\sigma}\left(\Delta g_0 - H\frac{\partial\Delta g}{\partial H}\right)S(\psi)\,d\sigma + \frac{\partial\zeta}{\partial H}H_M, \tag{7.60}$$

which admits the following geometrical interpretation: the free-air ancmaly at the level of the physical surface is reduced to the geoid level, becoming:

$$\overline{\Delta g} = \Delta g_0 - H_M\frac{\partial\Delta g}{\partial H},$$

and then, with the aid of *Stokes'* formula, one calculates the height anomaly at the geoid level. This latter anomaly is reduced to the level of the physical surface by means of the second term of (7.60).

In establishing (7.60) no hypotheses were made concerning the setting up of a certain reference surface with respect to which the height H should be measured. If instead of the sea-level one takes another reference surface whose altitude with respect to the sea-level is H_0, the relation (7.60) becomes:

$$\zeta = \frac{R}{4\pi\gamma}\iint_{\sigma}\left[\Delta g_0 - (H_M - H_0)\frac{\partial \Delta g}{\partial H}\right] S(\psi)\, d\sigma + \frac{\partial \zeta}{\partial H}(H_M - H_0).$$

If one takes as reference surface the level surface itself of the point M, then $H_0 = H_M$ and the previous relation becomes:

$$\zeta = \frac{R}{4\pi\gamma}\iint_{\sigma}\left[\Delta g_0 - (H - H_M)\frac{\partial \Delta g}{\partial H}\right] S(\psi)\, d\sigma. \tag{7.61}$$

When using (7.61) one must take into account the fact that the reference level is different for each point at which one calculates the height anomaly.

7.5 The Deflection of the Vertical on the Earth's Physical Surface

The basic relation for calculating the components of the gravimetric deflection of the vertical (5.13) was deduced in the year 1928 by *Vening Meinesz*. This relation may be put in the form:

$$\gamma\xi = -\frac{1}{R}\frac{\partial T}{\partial B}; \quad \gamma\eta = -\frac{1}{R\cos B}\frac{\partial T}{\partial L}. \tag{7.62}$$

The relations (7.62) were deduced for the case of the perturbing potential T known as a harmonic function at the geoid surface and as a result the deflection of the vertical is given at the geoid surface.

For determining the components of the deflection of the vertical on the Earth's physical surface, one must take into consideration the fact that the perturbing potential is known on this surface in the form of a series:

$$T = T_0 + T_1 + \ldots$$

and that this varies not only as a function of the latitude B_M and of the longitude L_M but also of the altitude H_M of the point M at which these components are determined.

In this case the relations (7.62) become:

$$\begin{aligned} \gamma\xi &= -\frac{1}{R}\left(\frac{\partial T}{\partial B}\right)_{L=\text{const.}} + \frac{1}{R}\frac{\partial T}{\partial H}\frac{\partial H}{\partial B}; \\ \gamma\eta &= -\frac{1}{R\cos B}\left(\frac{\partial T}{\partial L}\right)_{B=\text{const.}} + \frac{1}{R\cos B}\frac{\partial T}{\partial H}\frac{\partial H}{\partial L}. \end{aligned} \tag{7.63}$$

If one takes into consideration only the first two approximations in the expansion of the perturbing potential, the quantity T in (7.63) is:

$$T = \frac{R}{4\pi}\iint_\sigma (\Delta g_0 + \Delta g_1)\, S(\psi) d\sigma. \tag{7.64}$$

On the basis of the same reasoning used in Section 5.3 in deducing *Vening Meinesz*'s formule, (7.63) can be written in the form:

$$\begin{aligned} \xi &= \frac{1}{4\pi\gamma}\iint_\sigma (\Delta g_0 + \Delta g_1) \frac{dS(\psi)}{d\psi} \cos \alpha\, d\sigma + \frac{1}{R\gamma} \frac{\partial T}{\partial H} \frac{\partial H}{\partial B}; \\ \eta &= \frac{1}{4\pi\gamma}\iint_\sigma (\Delta g_0 + \Delta g_1 \frac{ds(\psi)}{d\psi} \sin \alpha\, d\sigma + \frac{1}{R\gamma \cos B} \frac{\partial T}{\partial H} \frac{\partial H}{\partial L}. \end{aligned} \tag{7.65}$$

The last terms of (7.65) may be transformed as follows:

$$\frac{1}{R} \frac{\partial H}{\partial S_B} = \tan \beta_1; \frac{1}{R \cos B} \frac{\partial H}{\partial L} = \frac{\partial H}{\partial S_L} = \tan \beta_2 \tag{7.66}$$

in which β_1 and β_2 denote the angles of inclination of the physical surface on the profiles $N - S$ (respectively $E - W$). In view of the relation (4.24) as well, the final terms of the relations (7.65) can be written in the form:

$$\begin{aligned} \frac{1}{R\gamma} \frac{\partial T}{\partial H} \frac{\partial H}{\partial B} &= -\left(\frac{\Delta g}{\gamma} + \frac{2\zeta}{R}\right) \tan \beta_1 \approx -\frac{\Delta g}{\gamma} \tan \beta_1; \\ \frac{1}{R\gamma \cos B} \frac{\partial T}{\partial H} \cdot \frac{\partial H}{\partial L} &= -\left(\frac{\Delta g}{\gamma} + \frac{2\zeta}{R}\right) \tan \beta_2 \approx -\frac{\Delta g}{\gamma} \tan \beta_2. \end{aligned} \tag{7.67}$$

On the basis of the relations (7.67), the relations (7.65) become:

$$\begin{aligned} \xi &= \frac{1}{4\pi\gamma}\iint_\sigma \Delta g_0 \frac{dS(\psi)}{d\psi} \cos \alpha\, d\sigma + \frac{1}{4\pi\gamma}\iint_\sigma \Delta g_1 \frac{dS(\psi) \cos \alpha}{d\psi} d\sigma - \frac{\Delta g}{\gamma} \tan \beta_1; \\ \eta &= \frac{1}{4\pi\gamma}\iint \Delta g_0 \frac{dS(\psi)}{d\psi} \sin \alpha\, d\sigma + \frac{1}{4\pi\gamma}\iint_\sigma \Delta g_1 \frac{dS(\psi) \sin \alpha}{d\psi} d\sigma - \frac{\Delta g}{\gamma} \tan \beta_2. \end{aligned} \tag{7.68}$$

The relations (7.68) represent the calculation formulae for the deflection of the vertical on the physical surface. Comparing (7.68) and (5.21) and in view also of (7.32), one sees that *Vening Meinesz*'s formulae represent the first approximation of the calculation formulae of the gravimetric deflection of the vertical on the Earth's physical surface.

For practical applications the relations (7.68) can be conveniently transformed. Thus, on the basis of (7.55) and (7.59), the last two terms in (7.68) can be written as:

$$\frac{1}{4\pi\gamma}\iint_\sigma \Delta g_1 \frac{dS(\psi)}{d\psi}\cos\alpha \, d\sigma - \frac{\Delta g}{\gamma}\tan\beta_1 =$$

$$-\frac{1}{4\pi\gamma}\iint_\sigma H\frac{\partial \Delta g}{\partial H}\frac{dS(\psi)}{\partial\psi}\cos\alpha \, d\sigma - \frac{1}{R}\frac{\partial\zeta_{12}}{\partial B} - \frac{\Delta h}{\gamma}\tan\beta_1 =$$

$$-\frac{1}{4\pi\gamma}\iint_\sigma H\frac{\partial \Delta g}{\partial H}\frac{dS(\psi)}{\partial\psi}\cos\alpha \, d\sigma - \frac{1}{R\gamma}\frac{\partial(H\Delta g)}{\partial B} - \frac{\Delta g}{\gamma}\tan\beta_1.$$

In view of (7.67), the previous relation becomes:

$$\frac{1}{4\pi\gamma}\iint_\sigma \Delta g_1 \frac{dS(\psi)}{d\psi}\cos\alpha \, d\sigma - \frac{\Delta g}{\gamma}\tan\beta_1 = \\ = -\frac{1}{4\pi\gamma}\iint_\sigma H\frac{\partial \Delta g}{\partial H}\frac{dS(\psi)}{d\psi}\cos\alpha \, d\sigma. \tag{7.69}$$

An analogous relation can also be established for the second relation in (7.68), so that these relations may be presented in the form (*Heiskanen* and *Moritz* 1967):

$$\xi = \frac{1}{4\pi\gamma}\iint_\sigma \left(\Delta g_0 - \frac{\partial \Delta g}{\partial H}H\right)\frac{dS(\psi)}{d\psi}\cos\alpha \, d\sigma; \\ \eta = \frac{1}{4\pi\gamma}\iint_\sigma \left(\Delta g_0 - \frac{\partial \Delta g}{\Delta H}H\right)\frac{dS(\psi)}{d\psi}\sin\alpha \, d\sigma. \tag{7.70}$$

Because in establishing the relations (7.60) no limiting hypotheses were made concerning the zero level surface, with respect to which the altitude H is determined, this surface can be taken as any level surface, e.g. the level surface passing through the point M at which the components of the gravimetric deflection of the vertical are calculated. In this case, the relations (7.70) can be presented in the form:

$$\xi = \frac{1}{4\pi\gamma}\iint_\sigma \left[\Delta g_0 - \frac{\partial \Delta g}{\partial H}(H - H_M)\right]\frac{dS(\psi)}{d\psi}\cos\alpha \, d\sigma; \\ \eta = \frac{1}{4\pi\gamma}\iint_\sigma \left[\Delta g_0 - \frac{\partial \Delta g}{\partial H}(H - H_M)\right]\frac{dS(\psi)}{d\psi}\sin\alpha \, d\sigma. \tag{7.71}$$

The relations (7.71) represent the calculation formulae for the deflection of the vertical on the Earth's physical surface, for the case when one takes into consideration only the first two approximations of *Molodenski*'s formulae.

According to the geometrical interpretation which was put forward in the preceding paragraph, the quantity:

$$\Delta g^* = \Delta g_0 - \frac{\partial \Delta g}{\partial H}(H - H_M), \tag{7.72}$$

may be considered as a gravity anomaly reduced in the free air from the sea-level to the level of the station point. By using such an anomaly in *Vening Meinesz*'s formula, one gets the deflection of the vertical on the Earths' physical surface.

Second part

Ellipsoidal Geodesy

Ellipsoidal Geodesy studies the methods for solving geodetic problems on the surface of the reference ellipsoid.

This doesn't include, but merely presupposes the general study of the ellipsoid, as a mathematical surface, of the methods of reducing geodetic observations on the reference ellipsoid, as well as of other topics having a direct connexion or deriving from the main object of study of Ellipsoidal Geodesy.

In various publications in other countries this part of Geodesy is known by other names, such as e.g: *Spheroidal Geodesy (Krasovski* 1942, *Bagratuni* 1962, *Grushinski* 1963, *Zakatov* 1958, 1964, 1976 et al.), *Mathematical Geodesy (Jordan* 1932, 1958, *Torge* 1975, *Schnädelbach* 1974 et al.), *Geometrical Geodesy* (*Heiskanen* 1967) etc., the object of study remaining, however, the same. In Romania the name of Ellipsoidal Geodesy has been adopted although other views have been expressed.

As was shown in the first part of this work (Chapter 3.3), the ellipsoid appears in Geodesy as one of the surfaces approximating the Earth's surface sufficiently well, and at the same time leads to simpler practical possibilities of calculation in comparison with other surfaces – the geoid or the level spheroids – which may well represent the Earth's surface more faithfully but which introduce remarkable difficulties into the solutions of practical computational nature. For the same reasons, namely practical ones, the rotation ellipsoid of small flattening at the poles has hitherto been used almost exclusively in Geodesy. The tri-axial ellipsoid, which would be a more adequate geometrical figure in representing the Earth's form and size, has hitherto been of limited purely theoretical applicability, which influences us not to tackle its detailed study in this work of relatively restricted size.

To represent the Earth's surface as nearly as possible as well as to allow its utilization as a reference surface of wide application in Geodesy, the ellipsoid must have an optimum orientation within the Earth's surface (the fifth part of this work). From this point of view it is necessary to distinguish between the following notions:

(1) *The reference ellipsoid* is the ellipsoid which is used at a given moment, in a country or in several countries, for solving geodetic problems. Its rotation axis doesn't coincide perfectly, being only paralell with the Earth's axis of rotation and its geometrical centre is located in the vicinity of the Earth's mass centre.

As a result of the improvement and of the variety of methods for determining the reference ellipsoids, there have been established in the course of time different values for the parameters defining these surfaces. This determination process continues today, being influenced by the improvement of the conventional methods, by the utilization of modern ways and means, based on improved measurement and calculation technologies, as well as by a more judicious distribution of the observation points over the entire Earth's surface.

(2) *The general terrestrial ellipsoid* is a theoretical notion, being in principle the limit to which the actual determinations of the reference ellipsoid tend. The rotation axis and the geometrical centre of the terrestrial ellipsoid coincide with the rotation axis, and the mass centre of the Earth respectively, its dimensions and orientation with respect to the geoid being determined in such a manner that there should be minimum differences between these two surfaces.

As is known, geodetic measurements (lengths, horizontal angles, zenithal angles etc.) are carried out either on the Earth's physical surface or in its immediate neighbourhood or towards/from objects located at great heights, i.e. in real three-dimensional space. It is natural at the beginning to ask the question as to why Geodesy solves some of its important problems on the surface of the reference ellipsoid and doesn't tackle solutions of a purely three-dimensional character. Although this problem will be dealt with in the fourth part of this work, we should show here that Geodesy was unable to use such methods because it was and it is still faced with problems of great difficulty concerning the determination of the zenithal angles (even for the usual geodetic distances, of the order of tens of kilometres) with an accuracy comparable with that of the other measured elements. Therefore, although the method of solving geodetic problems within a global three-dimensional system has been known for a long time (*Bruns* 1878) "classical" Geodesy has resorted to a solution with many advantages of a practical nature distinguishing the two big problems of Geodesy:

(a) *The "position" problem* (determining the coordinates of the position of the geodetic points on a certain reference surface: reference ellipsoid or projection plane).

(b) *The "altitude" problem* (determining the altitude with respect to a reference surface other than that for "position": geoid or quasi-geoid).

In this way, although every geodetic point has always been determined by means of three coordinates, these were obtained separately and referred to two different surfaces. This approach led to the necessity of reducing the geodetic observations, carried out in three-dimensional space, onto the two reference surfaces on which one then calculates the coordinates of the geodetic points. The determination of the reductions is also accompanied by errors and additional terrain and calculation activities, but these have been more easily accepted in comparison with the risks inherent in the solutions of a strictly three-dimensional character.

Neither now nor in the immediate future is it likely that conventional Geodesy — of which Ellipsoidal Geodesy is one component — is to be completely substituted by three-dimensional Geodesy. For example, one can show that on the principle of the separation of the "position" problem from the "altitude" problem one calculates the national and international triangulation and trilateration networks as well as numerous other geodetic networks (parts three and four of the book). Despite the limits enforced by this separation, it is absolutely necessary to emphasize the fact that the results of conventional Geodesy have been useful not only to the national geodetic surveys but were and are still widely utilized in determining the parameters of the various reference ellipsoids, making in this way an important contribution to the solution of the basic problem of Geodesy: the determination of the form and of the dimensions of the Earth. Therefore, the treatment of the main problems appearing in Ellipsoidal Geodesy is the duty of every technical work in this field, so that the chapters in the second part of the present book will deal with this topic.

8
The Rotation Ellipsoid as Reference Surface in Geodesy

8.1 The Parameters of the Reference Ellipsoid

Geometrically, the rotation ellipsoid can be defined by two of the parameters which will be mentioned in what follows of which one must be a linear one (Fig. 8.1):

$a = OE = OW$ — semi major axis (equatorial radius);

$b = OP = OP'$ — semi minor axis;

$f = \dfrac{a-b}{a}$ — (geometrical) flattening;

$e = \sqrt{\dfrac{a^2-b^2}{a^2}}$ — first (numerical) eccentricity;

$e' = \sqrt{\dfrac{a^2-b^2}{b^2}}$ — second (numerical) eccentricity;

$E = \sqrt{a^2-b^2}$ — linear eccentricity;

$c = \dfrac{a^2}{b}$ — polar curvature radius.

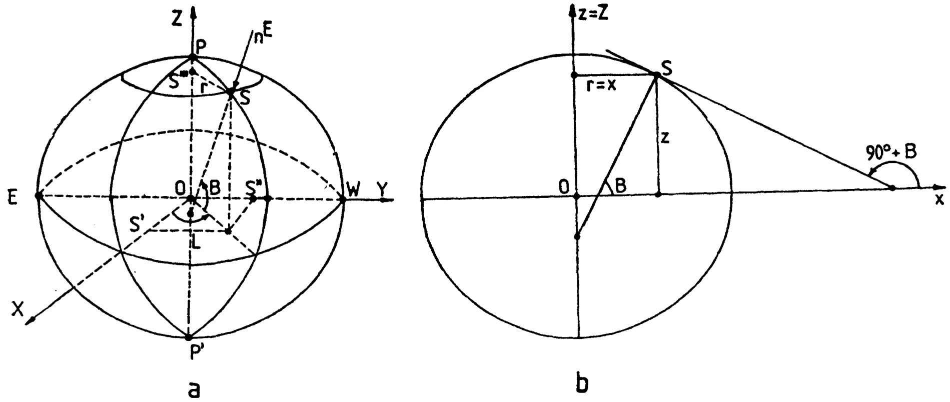


Fig. 8.1. The Rotation Ellipsoid of Small Flattening at the Poles

The first three parameters are also called the main geometrical parameters. In some calculations the following constants:

$$m' = \frac{a^2 - b^2}{a^2 + b^2}; \quad n' = \frac{a - b}{a + b},$$

appear.

Unlike Physical Geodesy, where the complete definition of the terrestrial ellipsoid is given by two physical parameters (e.g., the mass and the angular rotation velocity) and two geometrical parameters, in Ellipsoidal Geodesy one operates only with geometrical parameters.

Between the geometrical parameters of the rotation ellipsoid one can easily establish the following *main connexion relations:*

$$b = a(1 - f) = a\frac{1 - n'}{1 + n'}; \tag{8.1}$$

$$b^2 = a^2(1 - e^2); \tag{8.2}$$

$$f = 1 - \sqrt{1 - e^2} = 1 - \sqrt{\frac{1}{1 + e'^2}} = \frac{2n}{1 + n} = 1 - \sqrt{\frac{1 - m}{1 + m}}; \tag{8.3}$$

$$e^2 = 2f - f^2 = \frac{e'^2}{1 + e'^2} = \frac{4n'}{(1 + n')^2} = \frac{2m'}{1 + m'}; \tag{8.4}$$

$$e'^2 = \frac{2f - f^2}{(1 - f)^2} = \frac{e^2}{1 - e^2} = \frac{4n'}{(1 - n')^2} = \frac{2m'}{1 - m'}; \tag{8.5}$$

$$c = \frac{a}{1 - f} = \frac{a}{\sqrt{1 - e^2}} = \frac{b}{1 - e^2} = b(1 + e'^2). \tag{8.6}$$

Using expansions in series of the preceding relations one also obtains:

$$f = \frac{e^2}{2} + \frac{e^4}{8} + \frac{e^6}{16} + \frac{5e^8}{128} + \ldots = \frac{e'^2}{2} - \frac{3e'^4}{8} + \frac{5e'^6}{16} - \frac{35e'^8}{128}. \tag{8.7}$$

The temporal evolution of the magnitude of the main geometrical parameters of the reference ellipsoids used in Geodesy in certain periods is selectively illustrated in Table 8.1 by the presentation of some better-known reference ellipsoids.

Table 8.1 *Reference Ellipsoids*

Ellipsoid name	Year of determination	a (m)	f
		Using clasical methods	
Bessel	1841	6 377 397.115	1: 299.1528
Clarke	1880	6 378 243.000	1: 293.5
Helmert	1906	6 378 140.000	1: 298.3
Hayford	1909	6 378 388.000	1: 297.0
Krasovski	1940	6 378 245.000	1 : 298.3
		Using artificial sattelits	
Geodetic reference system 1967	1967	6 378 160.000	1:298.247
Geodetic reference system 1980	1980	6 378 137.000	1:298.257
O'Keefe	1958	—	1: 298.28
Kaula	1961	6 378 163	1: 298.24
Veis	1965	6 378 142	1: 298.25
Lambeck	1971	6 378 140	1: 298.25
Rapp	1973	6 378 142.8	1: 298.256
Khan	1973	6 378 142	1: 298.255
Gaposchkin	1973	6 378 140.4	1: 298.256

8.2 The Parametric Equations of the Rotation Ellipsoid

The general equation of a rotation ellipsoid, expressed in implicit form:

$$\frac{X^2 + Y^2}{a^2} + \frac{Z^2}{b^2} - 1 = 0, \tag{8.8}$$

is little used in Ellipsoidal Geodesy. Frequently one operates with the parametric equations in terms of the geodetic coordinates B and L, i.e. $X=X(B,L)$; $Y = Y(B, L)$; $Z = Z(B)$. For their derivation it is useful to determine, first of all, the parametric equations of the meridian ellipse: $x = x(B)$; $z = z(B)$, because the connexion between the coordinates X, Y, Z and x, z (Fig. 8.1) is immediate:

$$X = x \cos L; \quad Y = x \sin L; \quad Z = z. \tag{8.9}$$

The equation of the meridian ellipse in implicit form is:

$$f(x, z) = \frac{x^2}{a^2} + \frac{z^2}{b^2} - 1 = 0;$$

or:

$$f(x, z) = x^2 + \frac{z^2}{1 - e^2} - a^2 = 0. \tag{8.10}$$

Expressing the angular coefficient of the tangent at the point S (Fig. 8.1,b) in the following forms:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\cot B\,;\; \frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{f'_x}{f'_z} = -\frac{(1-e^2)\,x}{z},$$

it follows that

$$z = (1-e^2)\,x \tan B, \tag{8.11}$$

so that from (8.10) one gets:

$$x = \frac{a\cos B}{\sqrt{1-e^2\sin^2 B}}, \tag{8.12}$$

and, then, from (8.11):

$$z = \frac{a(1-e^2)\sin B}{\sqrt{1-e^2\sin^2 B}}. \tag{8.13}$$

The last two relations represent the parametric equations of the meridian ellipse in terms of the geodetic latitude B. In order to write these equations more compactly and to facilitate the practical calculations, one frequently utilizes the following *auxiliary functions:*

$$W = \sqrt{1-e^2\sin^2 B}; \tag{8.14}$$

$$V = \sqrt{1+e'^2\cos^2 B} = \sqrt{1+\eta^2}, \tag{8.15}$$

where:

$$\eta = e'\cos B. \tag{8.16}$$

The auxiliary functions W and V have been expanded in series and tabulated *(Khristov* 1950, *Tárczi-Hornoch* and *Khristov* 1959 et al.).

From such tables one can extract, with the necessary accuracy, the natural values as well as the logarithmic values for W and V as functions of the geodetic latitude B.

Thus, for the function V one obtains the following expansion in series:

$$\begin{aligned} V = {} & 1 + \frac{1}{4}e'^2 - \frac{2}{64}e'^4 + \frac{30}{1536}e'^6 - \frac{525}{49152}e'^8 + \\ & + \left(\frac{1}{4}e'^2 - \frac{4}{64}e'^4 + \frac{45}{1536}e'^6 - \frac{840}{49152}e'^8\right)\cos 2B - \\ & - \left(\frac{1}{64}e'^4 - \frac{18}{1536}e'^6 + \frac{420}{49152}e'^8\right)\cos 4B + \\ & + \left(\frac{3}{1536}e'^6 - \frac{120}{49152}e'^8\right)\cos 6B - \\ & - \frac{15}{49152}e'^8\cos 8B. \end{aligned} \tag{8.17}$$

For the *Krasovski* ellipsoid, used at present in *Romania* as reference ellipsoid, the relation (8.17) becomes:

$$V = 1.001\,682\,508\,82 + 0.001\,681\,802\,30 \cdot \cos 2B - 0.000\,000\,705\,93 \cdot \cos 4B + 0.000\,000\,000\,59 \cdot \cos 6B. \tag{8.18}$$

We may similarly express the auxiliary function W and the logarithms of these functions $\log W$ and $\log V$.

Using the connexion relations (8.1) — (8.6) between the parameters of the reference ellipsoid, one finds:

$$W^2 = \frac{1}{1 + e'^2} V^2 = (1 - e^2)V^2; \tag{8.19}$$

$$\frac{a}{W} = \frac{c}{V}. \tag{8.20}$$

In this manner, the parametric equations of the meridian ellipse (8.12 and (8.13) can be expressed in the form:

$$x = \frac{a \cos B}{W} = \frac{a \cos B}{V}\,; \; z = \frac{a\,(1-e^2) \sin B}{W} = \frac{c(1-e^2) \sin B}{V}. \tag{8.21}$$

The relations (8.9) yield the parametric equations of the rotation ellipsoid:

$$X = \frac{a \cos B \cos L}{W} = \frac{c \cos B \cos L}{V};$$
$$Y = \frac{a \cos B \sin L}{W} = \frac{c \cos B \sin L}{V}; \tag{8.22}$$
$$Z = \frac{a(1 - e^2) \sin B}{W} = \frac{c(1 - e^2) \sin B}{V}.$$

***Remarks*:**

(1) In some calculations in Ellipsoidal Geodesy one also works with the reduced latitude and the geocentric latitude, defined in Section 3.4. From Fig. 8.2 one remarks that:

$$x = a \cos \varphi^U; \quad z = b \sin \varphi^U. \tag{8.23}$$

One can establish several connexion relations between the reduced latitude and the geodetic latitude. Thus, from (8.23) and (8.21) on the one hand and (8.2) on the other, it folows that:

$$\frac{z}{x} = \frac{b}{a} \tan \varphi^U = (1 - e^2) \tan B = \frac{b^2}{a^2} \tan B;$$

i.e.:

$$\tan \varphi^U = \frac{b}{a} \tan B; \tag{8.24}$$

$$\frac{\tan \varphi^U}{\tan B} = \sqrt{1 - e^2} = \frac{1}{\sqrt{1 + e'^2}} = \frac{1 - n'}{1 + n'}. \tag{8.25}$$

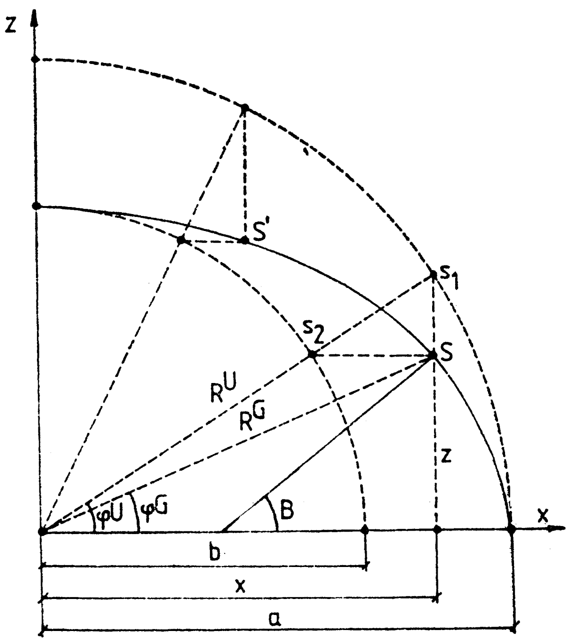


Fig. 8.2. Representation of the Geodetic B, Reduced φ^U and Geocentric Latitudes

For some practical purposes the following relation established by *F. R. Helmert* is also useful:

$$(\varphi^U - B)'' = n' \sin 2B - \frac{n'^2}{2} \sin 4B + \frac{n'^3}{3} \sin 6B - \frac{n'^4}{4} \sin 8\,B + \ldots \tag{8.26}$$

(2) *Conventional Ellipsoidal Geodesy includes, in addition to direct solutions of the kind of the relations (8.24) or (8.25), solutions obtained from the latter, usually through expansions in series.* In the past, this manner of working was determined by the necessity of lessening the calculation errors because both the tables of trigonometric functions being used and, especially, the available computers were operating to a relatively small number of decimal places.

At present, when highly developed computing methods have led to the disappearance of these drawbacks, the direct formulae, having the advantage of an obvious simplicity, become much more utilizable. This is why, in the present book, we will generally stick to the direct method of demonstration and calculation while the solutions based on expansions in series of the trigonometrical functions will be presented only as final results. This manner of working is fully justified because expressions seeming complicated at first sight can now be programmed for calculation by electronic computers, by direct calculation or by the method of successive approximations.

(3) From Fig. 8.2 it follows that:

$$x = R^G \cos \varphi^G; \quad z = R^G \sin \varphi^G; \quad \frac{z}{x} = \tan \varphi^G, \tag{8.27}$$

so that one obtains the connexion between the geodetic latitude B and the geocentric latitude φ^G:

$$\frac{\tan \varphi^G}{\tan B} = 1 - e^2 = \frac{1}{1 + e'^2} = \frac{1 - m'}{1 + m'}. \tag{8.28}$$

One can also establish (*Krasovski* 1942, *Khristov* 1950 et al.):

$$(B - \varphi^G)'' = \rho'' \left(\frac{e^2}{2 - e^2} \sin 2B - \frac{e^4}{2(2 - e^2)^2} \sin 4B + \frac{e^6}{3(2 - e^6)^3} \sin 6B - \ldots \right); \quad (8.29)$$

$$(B - \varphi^G)'' = \rho'' \left(m' \sin 2B - \frac{m'^2}{2} \sin 4B + \frac{m'^3}{3} \sin 6B - \ldots \right); \quad (8.30)$$

$$R^G = a \left(1 - \frac{e^2}{2} \sin^2 B + \frac{e^4}{2} \sin^2 B - \frac{5}{8} e^4 \sin^4 B + \ldots \right). \quad (8.31)$$

The equation of the meridian ellipse, expressed in geocentric coordinates φ^G, R^G, in terms of the flattening f results from the formulae (8.10), (8.1)—(8.3) and (8.27):

$$R^G = a(1 - f) \left(1 + \frac{f^2}{2} \cos^2 \varphi^G - f \cos^2 \varphi^G \right)^{-1/2}. \quad (8.32)$$

If one ignores the terms containing $f^2, f^3, \ldots$ one gets an approximate solution:

$$R^G \approx a\,(1 - f \sin^2 \varphi^G), \quad (8.33)$$

which is identical, within the accepted approximation, with the corresponding equation of a level spheroid of rotation. This conclusion represents one of the justifications for replacing the complex figure of a level spheroid by the rotation ellipsoid.

9
Curves on the Surface of the Reference Ellipsoid

9.1 The Coordinate Lines

The curvilinear coordinate lines on the surface of the reference ellipsoid are represented by the families of meridians ($L = \text{const.}$) and parallels ($B = \text{const.}$). With respect to the coordinate lines one defines certain quantities with which one frequently works in Geodesy (certain coordinate systems, the geodetic azimuth etc.).

The angle of intersection of the coordinate lines is a right angle and, thus, in certain calculations leads to simplifications, in comparison with the general situation which one meets in studying surfaces, where this angle can have any value.

9.1.1 The Radius of Curvature of the Meridian Ellipse

Let S_1 and S_2 be two points situated on the same meridian ellipse, at a latitude difference ΔB (Fig. 9.1). The curvature radius M of the meridian ellipse may be defined by the relation:

$$M = \lim_{\Delta B \to 0} \frac{\Delta s}{\Delta B} = \frac{\mathrm{d}s}{\mathrm{d}B}, \tag{9.1}$$

in which $\mathrm{d}s$ is the element of ellipse arc:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}z^2. \tag{9.2}$$

Thus:

$$M = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}B}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}B}\right)^2}. \tag{9.3}$$

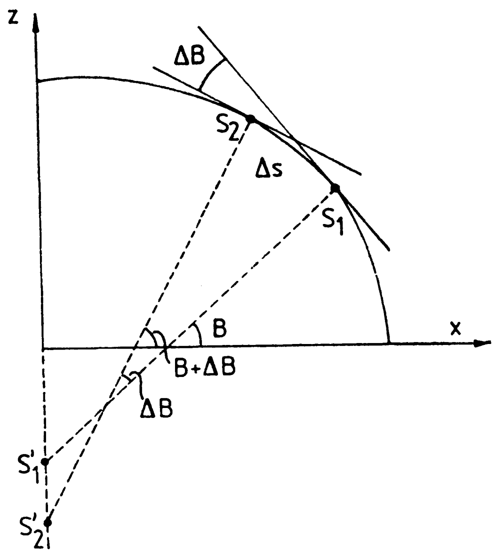


Fig. 9.1. Curvature Radius of the Meridian Ellipse

Utilizing the relations (8.21), we obtain:

$$\frac{\mathrm{d}x}{\mathrm{d}B} = -\frac{a(1-e^2)\sin B}{W^{3/2}};$$
$$\frac{\mathrm{d}z}{\mathrm{d}B} = \frac{a(1-e^2)\cos B}{W^{3/2}}, \tag{9.4}$$

i.e.:

$$M = \frac{a(1-e^2)}{W^3} = \frac{c}{V^3}. \tag{9.5}$$

We note that the curvature radius of the meridian ellipse increases along with the variation of the geodetic latitude B from the equator to the pole:

$$M_{0^\circ} = a(1-e^2); \quad M_{90^\circ} = \frac{a}{(1-e^2)^{1/2}} = c. \tag{9.6}$$

The magnitude of the radius of curvature of the meridian ellipse can be extracted from tables depending on the geodetic latitude of the point concerned and on the reference ellipsoid being used.

Table 9.1. *Extract from the Tables for Determining the Quantites V and M* (Krasovski's Ellipsoid).

B	V	d	M m	d m
46°		–		+
0′	1.001 624 5190		6 368 610.665	
		9778		18.653
1′	1.001 623 5412		6 368 629.318	

9.1.2 The Length of the Arc of Meridian

The length of the meridian arc between the points S_1 and S_2, of latitude, B_1 and respectively B_2, can be found by integrating a relation of the form (9.1).

$$s_{1-2} = \int_{S_1}^{S_2} \mathrm{d}s = \int_{B_1}^{B_2} M\,\mathrm{d}B, \tag{9.7}$$

or, in view of (9.5):

$$s_{1-2} = a(1-e^2)\int_{B_1}^{B_2}(1 - e^2\sin^2 B)^{-3/2}\,\mathrm{d}B = c\int_{B_1}^{B_2}(1 + e'^2\cos^2 B)^{-3/2}\mathrm{d}B. \tag{9.8}$$

To calculate the above elliptic integrals one first carries out the expansion in series of the expressions $(1 - e^2 \sin^2 B)^{-3/2}$ and $(1 + e'^2 \cos^2 B)^{-3/2}$ respectively, following which the result thus obtained can be integrated term by term. Thus:

$$(1 - e^2\cos^2 B)^{-3/2} = A' - B' \cos 2B + C' \cos 4B - D' \cos 6B + + E'\cos 8B - \dots \tag{9.9}$$

where:

$$\begin{aligned}
A' &= 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11025}{16384} e^8 + \dots; \\
B' &= \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2205}{2048} e^8 + \dots; \\
C' &= \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2205}{4096} e^8 + \dots; \\
D' &= \frac{35}{512} e^6 + \frac{315}{2048} e^8 + \dots; \\
E' &= \frac{315}{16384} e^8 + \dots .
\end{aligned} \tag{9.10}$$

For *Krasovski*'s reference ellipsoid the coefficients $A', \dots, E'$ have the following values:

$$\begin{aligned}
&A' = 1.005\,051\,773\,90; \quad D' = 0.000\,000\,020\,81; \\
&B' = 0.005\,062\,377\,64; \quad E' = 0.000\,000\,000\,04. \\
&C' = 0.000\,010\,624\,51;
\end{aligned} \tag{9.11}$$

Within the limits of the accepted approximations, the integral (9.8) becomes:

$$s_{1-2} = a(1 - e^2) \left[A' \frac{(B_2 - B_1)^\circ}{\rho^\circ} - \frac{B'}{2} (\sin 2B_2 - \sin 2B_1) + + \frac{C'}{4} (\sin 4B_2 - \sin 4B_1) - \frac{D'}{6} (\sin 6B_2 - \sin 6B_1) - - \frac{E'}{8} (\sin 8B_2 - \sin 8B_1) \right]. \tag{9.12}$$

Using the values of the parameters of the *Krasovski*'s reference ellipsoid and of the coefficients $A', \dots, E'$, previously presented, one gets the following calculation relation:

$$s_{1-2} = 111\,134.861\,083\,803\ (B_2 - B_1)^\circ - 16\,036.480268 (\sin 2B_2 - - \sin 2B_1) + 16.828067 (\sin 4B_2 - \sin 4B_1) - 0.021975 (\sin 6B_2 - - \sin 6B_1) + 0.000\,031\ (\sin 8B_2 - \sin 8B_1). \tag{9.13}$$

The length of the arc of meridian from the equator to the point concerned, denoted by β, can be determined by means of (9.13), by setting $B_1 = 0°$. From special tables one can interpolate this quantity, depending on the latitude B of the point concerned, with calculation errors of the order of $\pm$ 0.001 mm (Table 9.2).

Remarks:

(1) *The length of the arc of meridian is a function of the parameters of the reference ellipsoid being used and of the geodetic latitudes B_1 and B_2 situated at the ends of the arc.* Assuming as known the geodetic latitudes B_1 and B_2, as well as the length of the arc of meridian s_{1-2}, there is the possibility of determining the parameters of the reference ellipsoid, a and e^2, as unknowns in Eq. (9.12). If more than two measurements of arcs of meridian and latitude determinations at the ends of the arcs are available, one can carry out a solution by the least squares method. This method, described only in principle here, was used as a basic method in determining the dimensions of the various reference ellipsoids.

(2) *For certain approximate calculations one can deduce from* (9.13) *the approximate values of the arcs of meridian of* $1°$; $1'$; $1''$ *for the sexagesimal gradation and* 1^g; 1^c; 1^{cc} *for the centesimal gradation, respectively*:

$$\begin{array}{llll} \text{arc } 1° & 111 \text{ km}; & \text{arc } 1^g & 100 \text{ km}; \\ \text{arc } 1' & 1852 \text{ m}; & \text{arc } 1^c & 1000 \text{ m}; \\ \text{arc } 1'' & 31 \text{ m}; & \text{arc } 1^{cc} & 10 \text{ m}. \end{array} \tag{9.14}$$

(3) *In reality, the length of the arc of meridian is a function of the latitude B at which the point concerned is located, an increase taking place from the equator to the pole, just as for the radius of curvature of the meridian ellipse M*:

$$(\text{arc } 1°)_{0°} \approx 110\,576.3 \text{ m}; \; (\text{arc } 1°)_{90°} \approx 111\,695.78 \text{ m}. \tag{9.15}$$

(4) *For small lengths of arcs of meridian, $s \leqslant 45$ km, one can also utilize the following calculation formula depending on the mean latitude B_m and the latitude difference ΔB* (*Botez* 1969):

$$B_m = \frac{1}{2}(B_1 + B_2); \quad \Delta B = B_2 - B_1; \tag{9.16}$$

$$s_{1-2} \approx M_m \frac{\Delta B''}{\rho''}\left[1 + \frac{1}{8}\left(\frac{\Delta B''}{\rho''}\right)^2 e^2 \cos 2B_m\right]. \tag{9.17}$$

(5) *The element of arc of meridian is given by the relation*:

$$\mathrm{d}s_m = M\mathrm{d}B. \tag{9.18}$$

9.1.3 The Radius of Curvature of the Parallel

The radius of curvature of the parallel is equal to the coordinate x in Fig. 8.1:

$$r = x = \frac{a \cos B}{W}, \tag{9.19}$$

with a variation, depending on the geodetic latitude, from the equator to the pole:

$$r_{0°} = a; \quad r_{90°} = 0. \tag{9.20}$$

The magnitude of the radius of curvature of the parallel can be determined with the aid of tables, depending on the value actually tabulated: r arc 1′ (Table 8.2) in the knowledge that:

$$\rho' = 1/\text{arc } 1' = 3\,437.746\,770\,784\,939.$$

9.1.4 The Length of the Arc of Parallel

Let S_1 and S_2 be two points situated on the parallel of radius r (latitude B) at the longitudes L_1 and $L_2 = L_1 + \mathrm{d}L$ respectively. The length of the arc of parallel $\mathrm{d}s_p$ between the two points will be:

$$\mathrm{d}s_p = r\,\mathrm{d}L. \tag{9.21}$$

The above expression may be integrated immediately because $r =$ const. for a given parallel:

$$s_p = r(L_2 - L_1)'\,\text{arc } 1'. \tag{9.22}$$

9.1.5 The Geodetic Azimuth of a Curve Situated on the Reference Ellipsoid

One of the quantities frequently used, *the geodetic azimuth A* of a curve (c), is the angle formed by the element of arc $\mathrm{d}s$ of this curve with the positive direction of the coordinate line $L =$ const. (Fig. 9.2). In order to derive a calculation expression of the azimuth one can start from the general relation:

$$\cos A = \alpha\alpha_L + \beta\beta_L + \gamma\gamma_L. \tag{9.23}$$

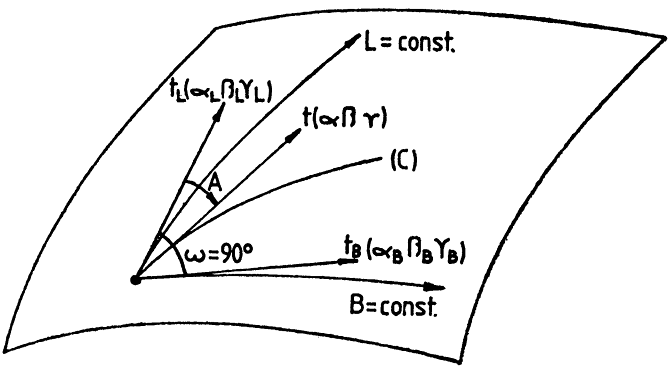


Fig. 9.2. Geodetic Azimuth of a Curve Situated on the Reference Ellipsoid

in which α, β, γ are the direction cosines of the tangent to the curve (c):

$$\alpha = \frac{dx}{ds} = \frac{\partial x}{\partial B}\frac{dB}{ds} + \frac{\partial x}{\partial L}\frac{dL}{ds};$$
$$\beta = \frac{dy}{ds} = \frac{\partial y}{\partial B}\frac{dB}{ds} + \frac{\partial y}{\partial L}\frac{dL}{ds}; \tag{9.24}$$
$$\gamma = \frac{dz}{ds} = \frac{\partial z}{\partial B}\frac{dB}{ds} + \frac{\partial z}{\partial L}\frac{dL}{ds},$$

and $\alpha_L, \beta_L, \gamma_L$ are the direction cosines of the tangent to the coordinate line $L = \text{const}$.

The arc element ds of a curve on any surface may be expressed in the form:

$$ds^2 = dX^2 + dY^2 + dZ^2; \tag{9.25}$$

or:

$$ds^2 = E\,dB^2 + 2F\,dB\,dL + G\,dL^2, \tag{9.26}$$

an expression known as the first quadratic fundamental form, where:

$$E = \left(\frac{\partial X}{\partial B}\right)^2 + \left(\frac{\partial Y}{\partial B}\right)^2 + \left(\frac{\partial Z}{\partial B}\right)^2;$$
$$F = \left(\frac{\partial X}{\partial B}\right)\left(\frac{\partial X}{\partial L}\right) + \left(\frac{\partial Y}{\partial B}\right)\left(\frac{\partial Y}{\partial L}\right) + \left(\frac{\partial Z}{\partial B}\right)\left(\frac{\partial Z}{\partial L}\right); \tag{9.27}$$
$$G = \left(\frac{\partial X}{\partial L}\right)^2 + \left(\frac{\partial Y}{\partial L}\right)^2 + \left(\frac{\partial Z}{\partial L}\right)^2.$$

In the case of the rotation ellipsoid, the partial derivatives appearing in the defining equation (9.27) are obtained from the relations (8.22):

$$\frac{\partial X}{\partial B} = -\frac{a\cos L\sin B(1-e^2)}{W^{3/2}}; \quad \frac{\partial X}{\partial L} = -\frac{a\cos B\sin L}{W};$$
$$\frac{\partial Y}{\partial B} = -\frac{a\sin L\sin B(1-e^2)}{W^{3/2}}; \quad \frac{\partial Y}{\partial L} = \frac{a\cos B\cos L}{W}; \tag{9.28}$$
$$\frac{\partial Z}{\partial B} = \frac{a(1-e^2)\cos B}{W^{3/2}}; \quad \frac{\partial Z}{\partial L} = 0;$$

with the following ways of expressing the coefficients E, F, G:

$$E = \frac{a^2(1-e^2)^2}{W^3} = M^2; \quad F = 0; \quad G = \frac{a^2\cos^2 B}{W^2} = r^2. \tag{9.29}$$

Remark. The equation $F = 0$ is generally valid, when the coordinate system is orthogonal.

The direction cosines of the tangent to the coordinate line $L = \text{const.}$ can be deduced from (8.26) by introducing the conditions: $\mathrm{d}L = 0$ and $\mathrm{d}s \equiv \mathrm{d}s_m$:

$$\mathrm{d}s_m^2 = E\,\mathrm{d}B^2, \tag{9.30}$$

whence:

$$\alpha_L = \frac{1}{\sqrt{E}}\frac{\partial X}{\partial B}; \qquad \beta_L = \frac{1}{\sqrt{E}}\frac{\partial Y}{\partial B}; \qquad \gamma_L = \frac{1}{\sqrt{E}}\frac{\partial Z}{\partial B}. \tag{9.31}$$

In this manner, all the ingredients necessary for calculating the azimuth of the curve (c) by means of (9.23) on the rotation ellipsoid are available:

$$\cos A = \frac{\sqrt{E}\,\mathrm{d}B}{\mathrm{d}s}; \tag{9.32}$$

$$\sin A = \sqrt{1 - \cos^2 A} = \frac{\sqrt{G}\,\mathrm{d}L}{\mathrm{d}s}; \tag{9.33}$$

$$\tan A = \sqrt{\frac{G}{E}}\frac{\mathrm{d}L}{\mathrm{d}B}. \tag{9.34}$$

9.1.6 The Area Element of the Surface of the Reference Ellipsoid

The area element $\mathrm{d}S$ of the surface lying between two meridians, situated at a longitude difference $\mathrm{d}L$, and between two parallels, situated at a latitude difference $\mathrm{d}B$ respectively, can be expressed as follows:

$$\mathrm{d}S = \mathrm{d}s_m\,\mathrm{d}s_p = \sqrt{EG}\,\mathrm{d}B\,\mathrm{d}L = Mr\,\mathrm{d}B\,\mathrm{d}L. \tag{9.35}$$

For practical calculations, one specializes the formula (9.35), taking $\mathrm{d}L = 1'$. Thus, one determines the area of the surface lying between the equator and the parallel of the point concerned, of latitude B, on the longitude interval of $1'$:

$$\Delta S_{\Delta L = 1'} = \operatorname{arc} 1'\, a^2(1 - e^2)\int_0^B \frac{\cos B}{W^{5/2}}\,\mathrm{d}B. \tag{9.36}$$

Just as with the preceding cases, this expression can be expanded in series, yielding:

$$\Delta S_{\Delta L = 1'} = A^* \sin B - B^* \sin 3B + C^* \sin 5B - D^* \sin 7B + \dots, \tag{9.37}$$

in which:

$$
\begin{aligned}
A^* &= \frac{1}{\rho'} a^2(1-e^2)\left(1+\frac{1}{2}e^2+\frac{3}{8}e^4+\frac{5}{16}e^6+\frac{35}{128}e^8\right);\\
B^* &= \frac{1}{\rho'} a^2(1-e^2)\left(\frac{1}{6}e^2+\frac{3}{16}e^4+\frac{3}{16}e^6+\frac{35}{192}e^8\right);\\
C^* &= \frac{1}{\rho'} a^2(1-e^2)\left(\frac{3}{80}e^4+\frac{1}{16}e^6+\frac{5}{64}e^8\right);\\
D^* &= \frac{1}{\rho'} a^2(1-e^2)\left(\frac{1}{112}e^6+\frac{5}{156}e^8\right).
\end{aligned}
\tag{9.38}
$$

If one considers the parameters of *Krasovski*'s reference ellipsoid, one obtains the following calculation formula for the elementary area, in km²:

$$
\begin{aligned}
\Delta S_{\Delta L=1'} = 11\,794.24\,561 \sin B - 13.21261 \sin 3B + 0.01997 \sin 5B -\\
- 0.00003 \sin 7B,
\end{aligned}
\tag{9.39}
$$

a quantity which may be extracted from tables depending on the geodetic latitude B.

Table 9.2. *Extract from the Tables for Determining the Qauntities* β, r*arc* 1′ *and* $S_{\Delta L=1'}$

B	β m	d m	r arc 1′ m	d m	$\Delta S_{\Delta L=1'}$ ha	d ha
46°		+		−		+
0′	5 096 175.747		1 291.07653		847 521.41	
		1852.556		0.38770		239.14
1′	5 098 028.303		1 290.68883		847 760.5 5	

The area S of the surface lying between the parallels of latitudes B_i, B_j and the meridians of longitudes L_m, L_n can be determined with the help of quantities extracted from tables by utilizing the following calculation formula:

$$
S = [\Delta S(B_j) - \Delta S(B_i)](L_m - L_n)'. \tag{9.40}
$$

9.2 Normal Sections

The intersection between a normal plane (a plane containing the normal to the ellipsoid at a point $S(X, Y, Z)$, Fig. 9.3, *a*) and the surface of the ellipsoid is called a ***normal section***. For studying the normal sections it is necessary to use some concepts from differential geometry.

The curve (c) situated on any surface (Fig. 9.3, a) has the curvature:

$$\frac{1}{R}\cos\psi = \frac{D\,\mathrm{d}B^2 + 2D'\,\mathrm{d}B\,\mathrm{d}L + D''\,\mathrm{d}L^2}{E\,\mathrm{d}B^2 + 2F\,\mathrm{d}B\,\mathrm{d}L + G\,\mathrm{d}L^2}, \tag{9.41}$$

in which:

R = the radius of curvature of the curve (c);
ψ = the angle between the normal n_s to the surface and the principal normal n_c to the curve;

$D\,\mathrm{d}B^2 + 2D'\mathrm{d}B\,\mathrm{d}L + D''\,\mathrm{d}L^2$ = the second quadratic fundamental form.

When the system of coordinate lines on the surface is orthogonal, it follows that:

$$F = D' = 0, \tag{9.42}$$

a situation which was already encountered in the case of the reference ellipsoid.

The other coefficients, D and D'', are determined from the relations:

$$\begin{aligned} D &= -\frac{\partial x}{\partial B}\cdot\frac{\partial X}{\partial B} - \frac{\partial y}{\partial B}\cdot\frac{\partial Y}{\partial B} - \frac{\partial z}{\partial B}\cdot\frac{\partial Z}{\partial B}; \\ D'' &= -\frac{\partial x}{\partial L}\cdot\frac{\partial X}{\partial L} - \frac{\partial y}{\partial L}\cdot\frac{\partial Y}{\partial L} - \frac{\partial z}{\partial L}\cdot\frac{\partial Z}{\partial L}, \end{aligned} \tag{9.43}$$

yielding, for the rotation ellipsoid:

$$D = M; \quad D'' = r\cos B. \tag{9.44}$$

The curvature of a normal section, denoted by $1/R_n$, is obtained from the general relation (9.41) under the condition $\psi = 0°$;

$$\frac{1}{R_n} = \frac{D\,\mathrm{d}B^2 + 2D'\,\mathrm{d}B\,\mathrm{d}L + D''\,\mathrm{d}L^2}{E\,\mathrm{d}B^2 + 2F\,\mathrm{d}B\,\mathrm{d}L + G\,\mathrm{d}L^2}. \tag{9.45}$$

One remarks that:

$$R = R_n\cos\psi, \tag{9.46}$$

a relation representing *Meusnier*'s *theorem*.

The quantity $1/R_g$:

$$\frac{1}{R_g} = \frac{1}{R}\sin\psi, \tag{9.47}$$

is called the *geodetic curvature*.

9.2.1 The Radii of Curvature of the Principal Normal Sections

From the infinity of normal sections passing through a point S (Fig. 9.3), situated on the ellipsoid surface, particular interest attaches to *the principal normal sections,* whose curvature radii, denoted in differential geometry by R_1 and R_2, have extreme values (minimum and maximum, respectively). R_1 and R_2 are also called *principal radii of curvature.*

From the formulae (9.45) and (9.42) and (9.44) respectively, one gets:

$$\frac{1}{R_n} = M\left(\frac{\mathrm{d}B}{\mathrm{d}s}\right)^2 + r\cos B\left(\frac{\mathrm{d}L}{\mathrm{d}s}\right)^2.$$

If one considers the relations (9.32), (9.33) as well, it follows, from comparison with *Euler's formula* that:

$$\frac{1}{R_n} = \frac{\cos^2 A}{R_1} + \frac{\sin^2 A}{R_2}, \tag{9.48}$$

and in the case of the reference ellipsoid:

$$R_1 = M; \quad R_2 = r/\cos\ B. \tag{9.49}$$

The positions of the principal normal sections, which pass through the point S (Fig. 9.3, *b*) can be deduced from (9.48) by using the minimum (maximum) condition:

$$\frac{\partial}{\partial A}\left(\frac{1}{R_n}\right) = \frac{R_1 + R_2}{R_1 R_2}\sin 2A = 0,$$

i.e.: $A = 0^g$ and $A = 100^g$, respectively.

Consequently, the principal normal sections on the reference ellipsoid are perpendicular to one another.

The principal normal section of radius R_1 and azimuth $A = 0^g$ is the meridian section and has been previously examined.

The principal normal section of radius R_2 and azimuth $A = 100^g$ is the section of the first vertical. Its radius of curvature, denoted by N in the case of the reference ellipsoid, may be deduced from (9.49):

$$N = \frac{r}{\cos B} = \frac{a}{W} = \frac{c}{V}. \tag{9.50}$$

One remarks that the radius of curvature of the first vertical varies from the equator to the pole:

$$N_{0^\circ} = a; \quad N_{90^\circ} = \frac{a}{(1 - e^2)^{1/2}} = c; \tag{9.51}$$

the exact values can be extracted from tables, depending on the geodetic latitude of the point concerned (Table 9.3).

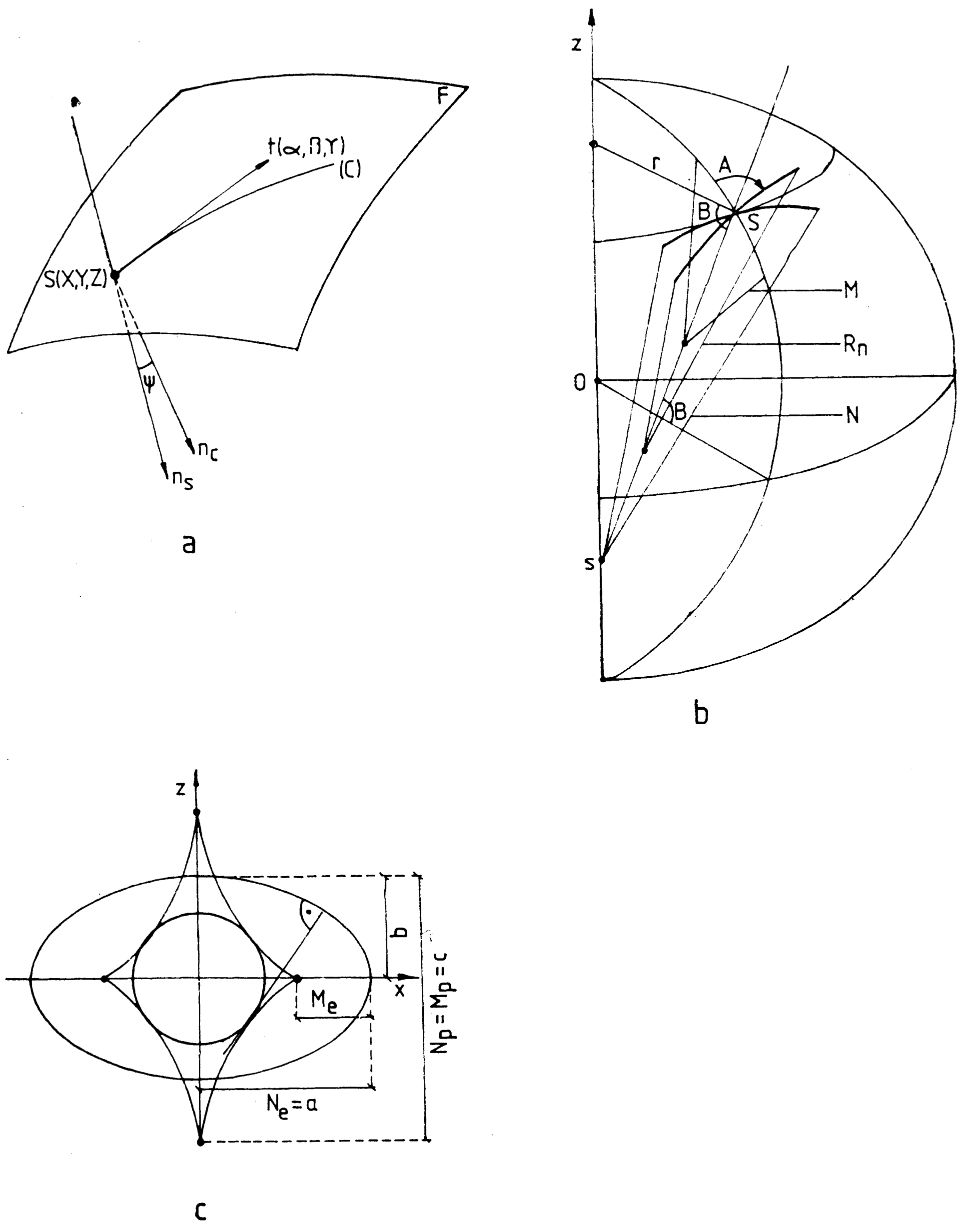


Fig 9.3. Normal Sections on the Reference Ellipsoid

The ratio between the radii of curvature of the principal normal sections being:

$$N/M = V^2 = 1 + \eta^2, \tag{9.52}$$

it follows that: $N \geqslant M$, for which reason the radius of curvature of the first vertical is also called *the major normal.*

In order to simplify the writing of some more complex expressions one also employs the following symbols:

$$\rho^{cc}/M = [1]; \quad \rho^{cc}/N = [2]. \tag{9.53}$$

Expanding (9.5) and (9.50) in series one gets:

$$M = c\left(1 - \frac{3}{2}\eta^2 + \frac{15}{8}\eta^4 - \frac{35}{16}\eta^6 + \frac{315}{128}\eta^8 - \frac{693}{256}\eta^{10} + \ldots\right); \tag{9.54}$$

$$N = c\left(1 - \frac{1}{2}\eta^2 + \frac{3}{8}\eta^4 - \frac{5}{16}\eta^6 + \frac{35}{128}\eta^8 - \frac{63}{256}\eta^{10} + \ldots\right). \tag{9.55}$$

Remark. Geometrically, the major normal is represented by the segment Ss (Fig. 9.3, *b*). In order to have a picture of the relationships existing between M, N and R_n, in Fig. 9.3, b an approximate representation of all of these three quantities is illustrated. To represent rigorously the magnitude of the radius of curvature of the meridian ellipse at the point S, it is necessary to construct *the evolute of the ellipse* (Fig. 9.3, *c*). The radii of curvature at any of the points of the meridian ellipse are tangent, inside, to the ellipse evolute, determining in this way the respective centres of curvature.

The equation of the ellipse evolute is:

$$(ax)^{2/3} + (bz)^{2/3} = (a^2 - b^2)^{2/3},$$

the curve being tangent, outside, to the circle of radius $a - b$. In the case of the ellipses of small flattening, the evolute is very thin; at the limit, in the case of the circle, the evolute degenerates into the centre of the latter.

9.2.2 The Mean Radius of Curvature (Gauss' Mean Radius)

By taking the special case of *Euler's formula* for the case of the rotation ellipsoid one gets:

$$R_n = R_A = \frac{MN}{N\cos^2 A + M\sin^2 A}. \tag{9.56}$$

The arithmetical average of the radii of curvature of the normal sections passing through a point situated on the ellipsoid, when the number of these sections tends to infinity, is called the *mean radius of curvature* or *Gauss' mean radius,* denoted by R:

$$R = \lim_{\Delta A \to 0} \frac{\displaystyle\sum_{A=0}^{A=2\pi-\Delta A} \frac{NM}{N\cos^2 A + M\sin^2 A}}{\dfrac{2\pi}{\Delta A}}. \tag{9.57}$$

At the limit, the expression (9.57) can be replaced by:

$$R = \frac{2}{\pi}\int_0^{\pi/2} \frac{MN}{N\cos^2 A + M\sin^2 A}\,\mathrm{d}A = \frac{2\sqrt{MN}}{\pi}\int_0^{\pi/2} \frac{\sqrt{\dfrac{M}{N}}\dfrac{1}{\cos^2 A}}{1+\left(\sqrt{\dfrac{M}{N}}\tan A\right)^2}\,\mathrm{d}A.$$

If one introduces the change of varables: $\sqrt{M/N}\tan A = t$, simple calculations yield:

$$R = \sqrt{MN}; \tag{9.58}$$

i.e.:

$$R = \frac{a(1-e^2)^{1/2}}{W^2} = \frac{b}{W^2} = \frac{c}{V^2}; \tag{9.59}$$

or:

$$R = c(1 - \eta^2 + \eta^4 - \eta^6 + \eta^8 - \eta^{10} + \ldots). \tag{9.60}$$

The mean radius of curvature varies with the geodetic latitude B:

$$R_{0^\circ} = b;\ R_{90^\circ} = \frac{a}{(1-e^2)^{1/2}} = c, \tag{9.61}$$

and depending on this can be extracted from tables (Table 9.3).

Table 9.3. *Extract from the Tables for Determining the Curvature Radii N and R*

B	N m	d m	R m	d m
46°		+		+
0′	6 389 319.331		6 378 956.594	
		6.238		12.456
1′	6 389 325.569		6 378 969.050	

The expressions:

$$K = \frac{1}{MN} = \frac{1}{R^2} = \frac{1+\eta^2}{N^2};\quad H = \frac{1}{2}\left(\frac{1}{M}+\frac{1}{N}\right) = \frac{M+N}{2R^2} \tag{9.62}$$

are called *total curvature* or *Gauss' curvature* and *mean curvature*, respectively.

9.2.3 The Radius of Curvature of Any Normal Section of Azimuth A

From the definition of the main radii of curvature, it follows that: $M \leqslant R_A \leqslant N$ (Fig. 9.3, b). The radius of curvature of a section of any azimuth A my be calculated by means of the relation (9.56), which can also be written in the following forms:

$$R_A = \frac{N}{\sin^2 A + V \cos^2 A} = \frac{N}{1 + \eta^2 \cos^2 A}, \tag{9.63}$$

it also being possible to carry out a series expansion as well:

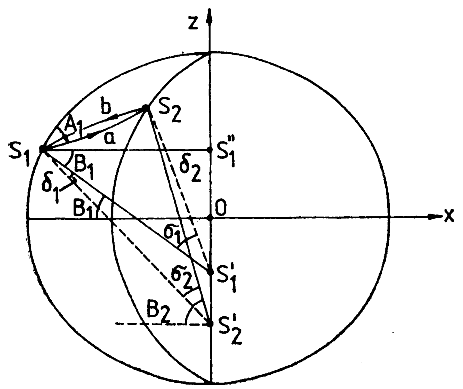
$$R_A = N(1 - \eta^2 \cos^2 A + \eta^4 \cos^4 A - \eta^6 \cos^6 A + \ldots). \tag{9.64}$$

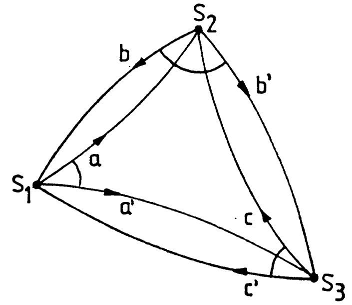
Frequently one also makes use of a calculating formula depending on *Gauss'* mean radius, which is obtained by expanding in series and conveniently transforming the relation (9.56):

$$R_A \approx R\left(1 - \frac{1}{2} e^2 \cos^2 B \cos 2A\right). \tag{9.65}$$

9.2.4 Reciprocal Normal Sections

Let S_1 and S_2 be points situated on the ellipsoid (Fig. 9.4, *a*) of different latitudes and longitudes ($B_1 \neq B_2$; $L_1 \neq L_2$). The planes determined by the normals to the ellipsoid at the points S_1 and S_2 and by the points S_2 and S_1 respectively, will intersect the ellipsoid along the curves denoted by *a* and *b*, which are called reciprocal normal section. The sections *a* is also called the





a b

Fig. 9.4. Reciprocal Normal Sections

direct normal section with respect to the point S_1 and inverse normal section with respect to the point S_2. Analogous labels clearly apply to the section b as well.

Remarks:

(1) One remarks (Fig. 9.4, *a*) that in the case when the two points S_1 and S_2 are on the same parallel ($B_1 = B_2$) or on the same meridian ($L_1 = L_2$) the reciprocal normal sections coincide.

(2) Concerning the notion of normal section, one can also imagine a connexion with practical geodetic activities. If one accepts that the main axis of a theodolite, installed at the point S_1, is oriented along the normal to the ellipsoid, then the plane pointing towards the point S_2 (determined by the main axis of the theodolite and the point aimed at) will intersect the ellipsoid along the direct normal section *a*. Consequently, the observations carried out in a geodetic triangle $S_1S_2S_3$ (Fig. 9.4, *b*) will be represented by the six corresponding sections. The angles which are formed between the direct normal sections do not, however, lie in the interior of a continuous and closed figure. It is therefore necessary for the geodetic observations to be processed (Chapter. 10) before being utilized in subsequent calculations.

For certain calculations, which will be carried out in due course, it is useful to determine the magnitude of the segment $\overline{S_1'S_2'}$. To this end one remarks that:

$$\overline{OS_1'} = \overline{S_1''S_1'} - \overline{S_1''O} = N_1 \sin B_1 - z_1 = e^2 N_1 \sin B_1$$

and similarly:

$$\overline{OS_2'} = e^2 N_2 \sin B_2,$$

so that:

$$\overline{S_1'S_2'} = \overline{OS_2'} - \overline{OS_1'} = e^2 c \left(\frac{\sin B_2}{V_2} - \frac{\sin B_1}{V_1} \right). \tag{9.66}$$

Note. The subscripts 1, 2 or m on the quantities N, V, η etc. indicate the fact that these are calculated in terms of the geodetic latitudes B_1, B_2 or of the mean latitude B_m.

For approximate calculations one can accept:

$$\overline{S_1'S_2'} \approx ae^2(B_2 - B_1)^{\mathrm{rad}} \cos B_m. \tag{9.67}$$

9.2.5 The Length of the Arc of the Normal Section

One assumes known the geodetic coordinates of the point $S_1(B_1, L_1)$; $S_2(B_2, L_2)$ and, consequently, the geodetic azimuth A_1 at the point S_1 (Fig. 9.4, *a*). To determine a calculation formula for the length of the arc of normal section between these points it is first necessary to determine some intermediate quantities: the angles δ_1, δ_2, σ_1 and the distances $S_1S_2' = N_1'$, $S_2S_1' = N_2'$.

From Fig. 9.4, *a* we have:

$$\tan \delta_1 = \frac{\overline{S_1'S_2'} \cos B_1}{N_1 + \overline{S_1'S_2'} \sin B_1} = \frac{e^2\left(\frac{V_1}{V_2} \sin B_2 - \sin B_1\right) \cos B_1}{1 + e^2\left(\frac{V_1}{V_2} \sin B_2 - \sin B_1\right) \sin B_1}; \quad (9.68)$$

$$\tan \delta_2 = \frac{e^2\left(\sin B_2 - \sin B_1 \frac{V_2}{V_1}\right) \cos B_2}{1 - e^2\left(\sin B_2 - \sin B_1 \frac{V_2}{V_1}\right) \sin B_2} \cdot \quad (9.69)$$

The relations for determining the angles δ_1 and δ_2 which were presented previously don't contain calculation approximations so that they can always be utilized, regardless of the latitude difference $\Delta B = B_2 - B_1$.

From Fig. 9.4, *a* we also obtain:

$$N_1'^2 = N_1^2 + \overline{(s_1s_2)}^2 + 2N_1\overline{s_1s_2} \sin B_1,$$

in which:

$$\left(\frac{N_1'}{N_1}\right)^2 = 1 + 2e^2\left(\sin B_2 \frac{V_1}{V_2} - \sin B_1\right)\sin B_1 + + e^4\left(\sin B_2 \frac{V_1}{V_2} - \sin B_1\right)^2; \quad (9.70)$$

$$\left(\frac{N_2'}{N_2}\right)^2 = 1 - 2e^2\left(\sin B_2 - \sin B_1 \frac{V_2}{V_1}\right)\sin B_2 + e^4\left(\sin B_2 - \sin B_1 \frac{V_2}{V_1}\right)^2 \cdot \quad (9.71)$$

From the last two relations, which do not contain approximations, one can calculate the segments N_1' and N_2'. Using expansions in series of these relations one gets:

$$\begin{aligned} N_1' &= N_1\left[1 + \Delta B \frac{\eta^2}{V^2} t - \frac{1}{2}\left(\frac{\Delta B}{V^2}\right)^2 \eta^2(1 - \eta^2 t^2) - \right. \\ &\quad \left. - \frac{1}{6}\left(\frac{\Delta B}{V^2}\right)^3 \eta^2 t + \frac{1}{24}\left(\frac{\Delta B}{V^2}\right)^4 \eta^2\right]; \\ N_2' &= N_2\left[1 - \Delta B \frac{\eta^2}{V^2} t - \frac{1}{2}\left(\frac{\Delta B}{V^2}\right)^2 \eta^2 (1 - \eta^2 t^2) + \right. \\ &\quad \left. + \frac{1}{6}\left(\frac{\Delta B}{V^2}\right)^3 \eta^2 t + \frac{1}{24}\left(\frac{\Delta B}{V^2}\right)^4 \eta^2\right], \end{aligned} \quad (9.72)$$

in which:

$$t = \tan B. \tag{9.73}$$

In order to deduce a dependence relation between the latitude difference ΔB and the angle σ_1 one utilizes an auxiliary sphere which has its centre at the point S_1' and passes through the points S_1 and S_2 (Fig. 9.4, *a*). The spherical latitude of the point S_2 will be $B_2' = B_2 - \delta_2 = B_1 + \Delta B - \delta_2$. In the auxiliary spherical triangle (Fig. 9.5, *a*) one writes the formula for the cosine of an element:

$$\sin B_2' = \sin B_1 \cos \sigma_1 + \cos B_1 \sin \sigma_1 \cos A_1,$$

from which one obtains by expansion in series:

$$\sin B_1 + \Delta B \cos B_1 - \frac{1}{2} \Delta B^2 \sin B_1 - \frac{1}{6} \Delta B^3 \cos B_1 - \delta_2 \cos B_1 =$$

$$= \sin B_1 \left(1 - \frac{1}{2} \sigma_1^2 + \ldots\right) + \cos B_1 \cos A_1 \left(\sigma_1 - \frac{1}{2} \sigma_1^3 + \ldots\right).$$

From a succession of transformations of the last relation, into which one introduces for δ_2 only the main term as obtained from expanding (9.68) in series, it follows that:

$$\Delta B \approx \sigma_1 \cos A_1 - \frac{1}{2} \sigma_1^2 \sin^2 A_1 \cdot t_1. \tag{9.74}$$

The practical determination of the angle σ, in terms of the known elements ($\Delta B = B_2 - B_1$; A_1) by means of (9.74) can now be achieved rather easily, even with desk electronic computers, using the successive-approximation method.

From the expressions (9.73) one can form the following ratio:

$$\frac{N_2'}{N_1} = 1 - \frac{1}{2} \Delta B^2 \eta_1^2 + \frac{1}{2} \Delta B^3 \eta_1^2 t_1,$$

which can also be expressed in terms of the angle σ_1 using (9.74):

$$\frac{N_2'}{N_1} = 1 - \frac{1}{2} \sigma_1^2 \eta_1^2 \cos^2 A_1 + \frac{1}{2} \sigma_1^3 \eta_1^2 t_1 \cos A_1. \tag{9.75}$$

With the help of the auxiliary elements which have now been determined, one can proceed with the calculation of the length of the arc of normal section. From Fig. 9.5, *b* it follows that:

$$ds_1^2 = (N_2' \, d\sigma)^2 + (dN_2')^2.$$

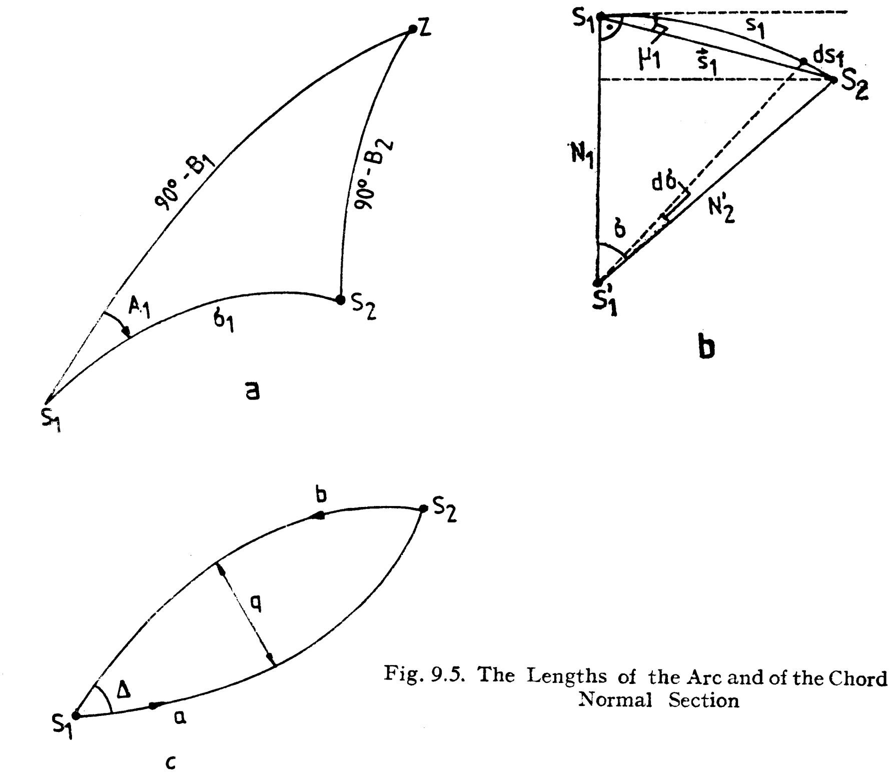


Fig. 9.5. The Lengths of the Arc and of the Chord of a Normal Section

Because the last term in the preceding relation is insignificant compared with the first term:

$$dN_2' = -N_1\,\sigma_1\,\eta_1^2\cos A_1\left(\cos A_1 - \frac{3}{2}\,\sigma_1 t_1\right)d\sigma,$$

one can approximate:

$$ds_1 \approx N_2'\,d\sigma = N_1\left(1 - \frac{1}{2}\,\sigma_1^2\eta_1^2\cos^2 A_1 + \frac{1}{2}\,\sigma_1^3\,\eta_1^2\,t_1\cos A_1\right)d\sigma,$$

whence integration yields:

$$s_1 = N_1\sigma_1\left(1 - \frac{1}{6}\,\sigma_1^2\eta_1^2\cos^2 A_1 + \frac{1}{8}\,\sigma_1^3\eta_1^2\,t_1\cos A_1\right) \tag{9.76}$$

and then inverting the series:

$$\sigma_1 = \frac{s_1}{N_1}\left(1 + \frac{1}{6}\,\frac{s_1^2}{N_1^2}\,\eta_1^2\cos^2 A_1 - \frac{1}{8}\,\frac{s_1^3}{N_1^3}\,\eta_1^2 t_1^2\cos A_1\right). \tag{9.77}$$

Thus, the length of the arc of the normal section may be calculated by means of (9.76) using as an intermediate element the angle σ_1 as determined from (9.77).

9.2.6 The Length of the Chord of the Normal Section

For deducing a calculation formula for the length of the chord of the normal section, it is necessary first of all that the angle μ_1 called ***angle of horizon depression*** (Fig. 9.5, *b*) be determined:

$$\tan(90° - \mu_1) = \frac{N_2' \sin\sigma_1}{N_1 - N_2' \cos\sigma_1},$$

i.e.:

$$\tan\mu_1 \sin\sigma_1 = \frac{N_1}{N_2'} - \cos\sigma_1. \tag{9.78}$$

The elements which are necessary for the direct computation of the angle μ_1 using (9.78) have already been determined from the formulae (9.75 and (9.77). Of course, one can here utilize an expansion in series too, from which it follows that:

$$\mu_1 = \frac{1}{2}\sigma_1(1 + \eta_1^2 \cos^2 A_1) - \frac{1}{2}\sigma_1^2\eta_1^2 t_1 \cos A_1$$

or, in view of (9.77):

$$\mu_1 = \frac{s_1}{2N_1}(1 + \eta_1^2 \cos^2 A_1) - \frac{s_1^2}{2N_1^2}\eta_1^2 t_1 \cos A_1. \tag{9.79}$$

From Fig. 9.5, *b* we obtain a first method for directly calculating the length of the chord of the normal section $\overline{S_1S_2} = \bar{s}_1$:

$$\frac{\bar{s}_1}{N_1'} = \frac{\sin\sigma_1}{\cos(\sigma_1 - \mu_1)}, \tag{9.80}$$

because the calculation formulae for N_1', σ_1 and μ_1 are already established. Expanding (9.80) in series and using sequence of transformations one gets:

$$\bar{s}_1 = N_1\sigma_1\left[1 - \frac{\sigma_1^2}{24}(1 + 6\,\eta_1^2\cos^2 A_1) + \frac{\sigma_1^3}{4}\eta_1^2 t_1 \cos A_1 + \frac{\sigma_1^4}{1920}\right] \tag{9.81}$$

or:

$$\bar{s}_1 = s_1\left[1 - \frac{1}{24}\frac{s_1^2}{N_1^2}(1 + 2\eta_1^2 \cos^2 A_1) + \frac{1}{8}\frac{s_1^3}{N_1^3}\eta_1^2 t_1 \cos A_1 + \frac{1}{1920}\frac{s_1^4}{N_1^4}\right]. \tag{9.82}$$

Remark. In the previous calculation formulae one assumed known the geodetic azimuth A_1 at the point S_1. The latter can be determined in terms of the geodetic coordinates of the points S_1 and S_2 by means of the following relation (*Bagratuni* 1962):

$$\tan A_1 = \frac{\sin \Delta L}{\left(\cos \Delta L - \frac{r_1}{r_2}\right)\sin B_1 - (1 - e^2)(\tan B_2 - \frac{r_1}{r_2}\tan B_1)\cos B_1} \tag{9.83}$$

in which $L = L_2 - L_1$ and r_1 and r_2 are the radii of the parallel circles passing through the points S_1 and S_2.

9.2.7 The Relative Position of the Reciprocal Normal Sections

The angle between the reciprocal normal sections, denoted by Δ, as well as the maximum distance between the latter denoted by q (Fig. 9.5, *c*) may be calculated using the following formulae (*Bagratuni* 1962, *Jordan* 1959 et al.):

$$\Delta^{cc} = \rho^{cc}\frac{s_1^2\,\eta_m^2}{4N_m^2}\sin 2A_1; \tag{9.84}$$

$$q = \frac{\eta_m^2 s_1^3}{16N_m^2}\sin 2A_1. \tag{9.85}$$

The last formula also furnishes the analytical motivation for the first remark in § 9.2.4: for $A_1 = 0^g$ and $A_1 = 100^g$ respectively, $\Delta = q = 0$. The maximum values for Δ and q are recorded when $A_1 = 50^g$, but even in such a case these are very small. E.g., when $B_m = 46°$ and $A_1 = 50^g$ one gets for $s_1 = 30$ km: $\Delta = 0^{cc}.011$ and $q = 0.1$ mm, and for $s_1 = 100$ km: $\Delta = 0^{cc}.127$ and $q = 5.0$ mm.

9.3 The Geodetic Line

9.3.1 Definition. Properties

The curve which is so constructed that at each of its points the osculating plane contains the normal n_s to the surface is called the *geodetic line*. As the osculating plane is formed by the tangent t and the principal normal n_c to the curve (Fig. 9.3, *a*), it follows that at each point of the geodetic line the normal to the surface coincides with the principal normal to the curve and, consequently, the geodetic curvature $1/R_g$ vanishes ($\psi = 0$).

On the rotation ellipsoid, the meridians and the equator, and on a sphere all the great circles are geodetic lines. This property is not enjoyed by the circles of parallel, which never contain the normal to the surface. In the projection plane, the geodetic lines are straight lines. One must draw attention to the fact that the geodetic lines do not have an equivalent within the framework of the field geodetic operations but only intervene in the computing processes. **To get an additional pieture of the geometrical meaning of these curves, one can imagine the following procedure for tracing a geodetic line in the field** (Fig. 9.6, *a*):

(1) One takes up position at the point A (it is assumed that the main axis of the theodolite is oriented along the normal to the reference ellipsoid) and a bearing is made to the point B (direct normal section AaB).

(2) The operation is repeated at the point B, where initially one points towards the point A (normal section BbA) and then the point C (normal section BbC), after rotating the theodolite by exactly 200^g.

(3) A similar procedure is followed at the points $C, D, \ldots$

If the distances between the station points are very small (tend to zero), it is acceptable to assume coincidence of the direct normal section and the inverse one respectively (e.g. AaB with BbA). In such a case, the plane normal to the surface of the reference ellipsoid at the point B will also contain two tangents to the curve which was constructed in the previously described

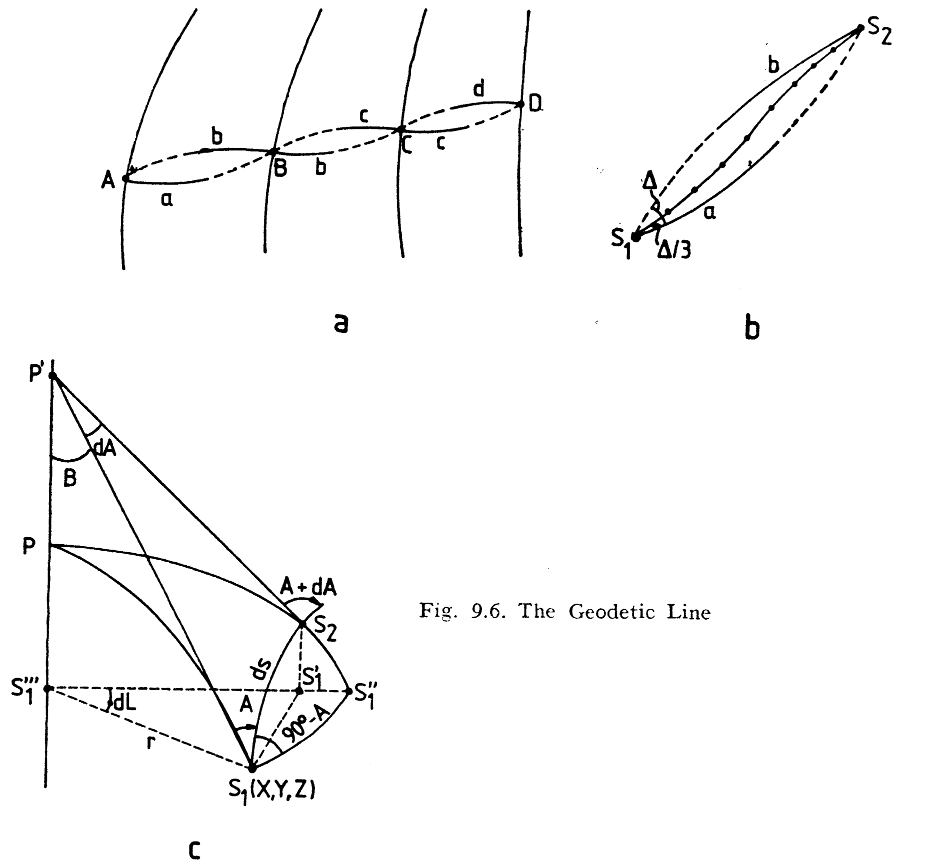


Fig. 9.6. The Geodetic Line

manner (the tangents to the arcs $\widehat{BA}$ and $\widehat{BC}$), thus also being an osculating plane to the curve. It follows that a geodetic line was indeed traced in the field.

Between two points S_1 and S_2 situated on the surface of the reference ellipsoid one can draw just one geodetic line. Its position with respect to the normal sections may be studied in Fig. 9.6, *b* and will be examined in greater detail in § 9.3.4. In this manner, unlike the situation presented in Fig. 9.4, *b*, continuous and closed figures will be available even in the case of very long sights, by passing from the normal sections to the corresponding geodetic lines.

The geodetic line is the minimum-length curve which can be drawn through two points situated on the surface of the reference ellipsoid. Indeed, because the normals n_C and n_S coincide at the two points concerned and $\psi = 0$, the radii of curvature of the geodetic line calculated at these points (e.g. by means of *Meusnier's* formula (9.46)) will take the maximum values attainable.

9.3.2 Differential and Finite-Term Equations of the Geodetic Line

Let F be any surface, defined in implicit form:

$$F(X, Y, Z) = 0,$$

and the geodetic line defined in parametric form:

$$X = X(s); \quad Y = Y(s); \quad Z = Z(s),$$

depending on the arc length s, at a running point of coordinates X, Y, Z. At this point, the equation of the normal to the surface is:

$$\frac{a - X}{\dfrac{\partial F}{\partial X}} = \frac{b - Y}{\dfrac{\partial F}{\partial Y}} = \frac{c - Z}{\dfrac{\partial F}{\partial Z}},$$

where a, b, c denote the coordinates of a point situated on this normal. As the normal to the surface coincides, in the case of the geodetic lines, with the principal normal to the curve, of equation:

$$\frac{a - X}{\dfrac{\mathrm{d}^2X}{\mathrm{d}s^2}} = \frac{b - Y}{\dfrac{\mathrm{d}^2Y}{\mathrm{d}s^2}} = \frac{c - Z}{\dfrac{\mathrm{d}^2Z}{\mathrm{d}s^2}},$$

we have the following differential equations of the geodetic lines on the surface F:

$$\frac{\dfrac{\partial F}{\partial X}}{\dfrac{d^2X}{ds^2}} = \frac{\dfrac{\partial F}{\partial Y}}{\dfrac{d^2Y}{ds^2}} = \frac{\dfrac{\partial F}{\partial Z}}{\dfrac{d^2Z}{ds^2}}. \tag{9.86}$$

In the case of the reference ellipsoid, as defined by the equation (8.8) written in the form:

$$X^2 + Y^2 + f(Z) = 0,$$

one gets:

$$\frac{\partial F}{\partial X} = 2X; \quad \frac{\partial F}{\partial Y} = 2Y; \quad \frac{\partial F}{\partial Z} = f'(Z), \tag{9.87}$$

so that the differential equations of the geodetic lines will be:

$$\frac{2X}{\dfrac{d^2X}{ds^2}} = \frac{2Y}{\dfrac{d^2Y}{ds^2}} = \frac{f'(Z)}{\dfrac{d^2Z}{ds^2}}. \tag{9.88}$$

Thence it follows that:

$$Y\frac{d^2X}{dx^2} - X\frac{dY}{ds^2} = 0$$

and integrating:

$$YdX - XdY = C\,ds, \tag{9.89}$$

where C is an integration constant.

Let ds be the element of geodetic line lying between the points $S_1(X, Y, Z)$ and $S_2(X + dX;\ Y + dY;\ Z + dZ)$ in Fig. 9.6, *c*, whose geodetic azimuth was denoted by A. One can show that the areas of the triangle $S_1'S_1''S_1'''$ and of the sector $S_1'S_1''S_1'''$ are $(YdX - XdY)/2$ and $(r\ ds\,\sin A)/2$ respectively. In the limit, when these areas are equal we shall have:

$$\frac{1}{2}(Y\,dX - X\,dY) = \frac{1}{2}\,r\,ds\sin A,$$

and by comparison with (9.79) we will obtain the equation in finite terms of the geodetic line (*Clairaut*'s theorem — 1735):

$$r\sin A = \text{const.} \tag{9.90}$$

By using the formulae (9.19), (8.23) and (8.27) as well, one can equally write:

$$\cos\varphi^U\sin A = \text{const};\ \cos\varphi^G\sin A = \text{const.} \tag{9.91}$$

If one considers the plane tangent to the reference ellipsoid at the point S_1 (Fig. 9.6, *c*), the length of the infinitely small arc of parallel $\widehat{S_1'S_1''}$ may be evaluated in two ways:

$$\widehat{S_1S_1''} = \mathrm{r}\,\mathrm{d}L = \frac{r}{\sin B}\,\mathrm{d}A,$$

leading to *Clairaut*'s differential relation of the geodetic line:

$$\mathrm{d}A = \mathrm{d}L \sin B. \tag{9.92}$$

9.3.3 Parametric Equations of Puiseaux-Weingarten-Gauss of the Geodetic Line on the Reference Ellipsoid

In order to deduce the parametric equations of the geodetic line on the reference ellipsoid:

$$\begin{aligned} X &= X(s) = Y_0 + \frac{s}{1\,!}\left(\frac{\mathrm{d}X}{\mathrm{d}s}\right)_0 + \frac{s^2}{2\,!}\left(\frac{\mathrm{d}^2X}{\mathrm{d}s^2}\right)_0 + \frac{s^3}{3\,!}\left(\frac{\mathrm{d}^3X}{\mathrm{d}s^3}\right)_0 + \cdots;\\ Y &= Y(s) = Y_0 + \frac{s}{1\,!}\left(\frac{\mathrm{d}Y}{\mathrm{d}s}\right)_0 + \frac{s^2}{2\,!}\left(\frac{\mathrm{d}^2Y}{\mathrm{d}s^2}\right)_0 + \frac{s^3}{3\,!}\left(\frac{\mathrm{d}^3Y}{ds^3}\right)_0 + \cdots;\\ Z &= Z(s) = Z_0 + \frac{s}{1\,!}\left(\frac{\mathrm{d}Z}{\mathrm{d}s}\right)_0 + \frac{s^2}{2\,!}\left(\frac{\mathrm{d}^2Z}{\mathrm{d}s^2}\right)_0 + \frac{s^3}{3\,!}\left(\frac{\mathrm{d}^3Z}{\mathrm{d}s^3}\right)_0 + \cdots, \end{aligned} \tag{9.93}$$

we will use *Euler*'s system (Fig. 9.7) constructed in the following way: the axes X and Y are represented by the tangents to the principal curvature lines and the Z axis is oriented towards the interior of the surface, along the

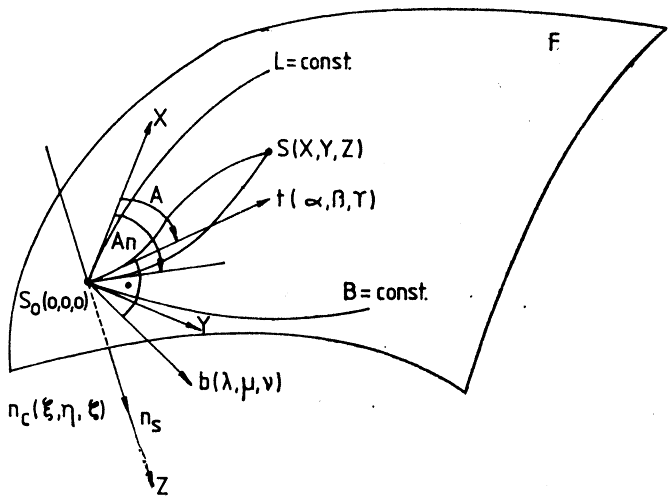


Fig. 9.7. The Parametric Equations of the Geodetic Line

normal to the reference ellipsoid. In addition, the origin of the system has the coordinates:

$$X_0 = Y_0 = Z_0 = 0. \tag{9.94}$$

The calculation of the derivatives necessary in the formulae (9.93) is made by applying *Frênet*'s formulae to the present situation:

$$\left(\frac{dX}{ds}\right)_0 = \alpha_0 = \cos A\,; \quad \left(\frac{dY}{ds}\right)_0 = \beta_0 = \sin A\,; \quad \left(\frac{dZ}{ds}\right)_0 = \gamma_0 = 0\,; \tag{9.95}$$

$$\left(\frac{d^2X}{ds^2}\right)_0 = \left(\frac{\xi}{R}\right)_0 = 0; \quad \left(\frac{d^2Y}{ds^2}\right)_0 = \left(\frac{\eta}{R}\right)_0 = 0; \quad \left(\frac{d^2Z}{ds^2}\right)_0 = \left(\frac{\zeta}{R}\right)_0 = \frac{1}{R_A}; \tag{9.96}$$

$$\begin{aligned}
\left(\frac{d^3X}{ds^3}\right)_0 &= -\left(\frac{\alpha}{R^2} + \frac{\lambda}{R\tau} + \frac{\xi}{R^2}\frac{dR}{ds}\right)_0 = -\frac{\cos A}{R_A^2} + \frac{\sin A}{R_A\tau_A}\,;\\
\left(\frac{d^3Y}{ds^3}\right)_0 &= -\left(\frac{\beta}{R^2} + \frac{\mu}{R\tau} + \frac{\eta}{R^2}\frac{dR}{ds}\right)_0 = -\frac{\sin A}{R_A^2} - \frac{\cos A}{R_A\tau_A}\,;\\
\left(\frac{d^3Z}{ds^3}\right)_0 &= -\left(\frac{\gamma}{R^2} + \frac{\nu}{R\tau} + \frac{\zeta}{R^2}\frac{dR}{ds}\right)_0 = -\frac{1}{R_A^2}\frac{dR}{ds} \approx 0.
\end{aligned} \tag{9.97}$$

In the preceding formulae $1/\tau$ denotes the, torsion of the geodetic line which can be calculated in terms of the curvatures of the principal normal sections and of the azimuth of the geodetic line:

$$\frac{1}{\tau_A} = -\left(\frac{1}{M} - \frac{1}{N}\right)\sin A \cos A. \tag{9.98}$$

After transformations which are omitted here, one obtains the parametric equations of *Puiseaux-Weingarten* of the geodetic line on the rotation ellipsoid:

$$X = s\cos A\left(1 - \frac{s^2}{6MR_A} + \cdots\right); \quad Y = s\sin A\left(1 - \frac{s^2}{6NR_A} + \cdots\right);$$

$$Z = \frac{s^2}{2R_A} + \cdots \tag{9.99}$$

The greater approximation which is accepted for the Z coordinate is justified by the practical method of deducing the altitudes, at large distances, by trigonometric levelling, where the errors due to atmospheric refraction do not allows computations as precise as those necessary for deducing the X and Y coordinates.

The original form of *Weingarten*'s formulae is *(Torge* 1973*):*

$$X = s\cos A - \frac{s^3}{6N_0^2}\cos A(1 + \eta_0^2 + \eta_0^2\cos^2 A) + \ldots;$$

$$Y = s\sin A - \frac{s^3}{6N_0^2}\sin A(1 + \eta_0^2 + \cos^2 A) + \ldots; \tag{9.100}$$

$$Z = \frac{s^2}{2N_0}(1 + \eta_0^2\cos^2 A) - \frac{s^3}{2N_0}\eta_0^2 t_0\cos A + \ldots .$$

For practical purposes it is useful now to apply *Gauss'* transformations, in considering some relations derived from the formula (9.56):

$$\frac{1}{R_A} = \frac{1}{M}\left(1 - \frac{e^2\cos^2 B}{1 - e^2\sin^2 B}\sin^2 A\right); \tag{9.101}$$

$$\frac{1}{R_A} = \frac{1}{N}\left(1 + \frac{e^2}{1 - e^2}\cos^2 B\cos^2 A\right), \tag{9.102}$$

there being obtained the parametric equations of *Puiseaux-Weingarten-Gauss* of the geodetic line on the reference ellipsoid, in the approximation in which the 3rd and higher-order terms are neglected:

$$X = s\cos A\left(1 - \frac{s^2}{6R^2} - e^2\frac{s^2}{6R^2}\cos^2 B\cos^2 A + \ldots\right);$$

$$Y = s\sin A\left(1 - \frac{s^2}{6R^2} + e^2\frac{s^2}{6R^2}\cos^2 B\sin^2 A + \ldots\right); \tag{9.103}$$

$$Z = \frac{s^2}{2R}\left(1 + \frac{1}{2}e^2\cos^2 B\cos 2A + \ldots\right).$$

9.3.4 The Position of the Geodetic Line with respect to the Direct Normal Section

The azimuth of the normal section determined by the normal to the reference ellipsoid at the point S_0 and by the point S (Fig. 9.7) denoted by A_n may be determined by means of the relation:

$$\tan A_n = \frac{X}{Y},$$

where the coordinates X and Y are given by the relations (9.100) so that one obtains:

$$\tan A_n = \tan A\left(1 + e^2\frac{s^2}{6N^2}\cos^2 B + \ldots\right).$$

From the equality $A = A_n - (A_n - A)$ expansion in series yields:

$$\tan A = \tan A_n - \frac{A_n - A}{\cos^2 A_n} + \ldots,$$

i.e.:

$$(A - A_n)^{cc} = -\rho^{cc} \frac{\eta^2}{12N^2} s^2 \sin 2A + \ldots . \tag{9.104}$$

The formula (9.104) is sufficiently exact for the calculations which occur in the usual geodetic operations, as e.g. those in lst-order triangulation. The difference between the azimuth of the direct normal section and the azimuth of the geodetic line can be expressed still more accurately (*Jordan* 1958):

$$(A_n - A)^{cc} = \rho^{cc} \left(\frac{1}{12} \frac{\eta^2}{N^2} s^2 \sin 2A - \frac{1}{24} \frac{s^3}{N^3} \eta^2 t \sin A \right) . \tag{9.105}$$

The difference between the azimuths of the geodetic line and of the direct normal section respectively is very small. For instance, for the latitude $B = 45°$ and the geodetic azimuth $A = 50^g$, to the distances of 30, 60 and 100 km correspond the following differences: $0^{cc}.003$; $0^{cc}.018$ and $0^{cc}.044$. For practical calculations one utilizes the formula (9.106):

$$(A - A_n)^{cc} = - K_1 (s_{km})^2 \sin 2A, \tag{9.106}$$

where the coefficient K_1:

$$K_1 = \rho^{cc} \frac{\eta^2}{12N^2}, \tag{9.107}$$

can be extracted from Table 9.4 as a function of the latitude B, when one considers the centesimal gradation for the difference $(A - A_n)^{cc}$.

Comparing the formulae (9.48) and (9.104), one notes that the geodetic line is located at a distance of $\Delta/3$ from the direct normal section (Fig.9.6,*b*).

The difference δ_s between the lengths of the geodetic line and of the direct normal section may be determined using the relation (*Bagratuni* 1962):

$$\delta s = \frac{s^5 \eta^4 \sin^2 2A}{360 N^4} . \tag{9.108}$$

Table 9.4. *The Value of the Coefficients K_1 an K_2* (Krasovski's *Ellipsoid*)

B / K_1, K_2	44°	44°30′	45°	45°30′	46°	46°30′	47°	47°30′	48°	48°30′	49°
$K_1 \cdot 10^6$	4.51	4.44	4.36	4.28	4.21	4.13	4.05	3.98	3,90	3.82	3.75
$K_2 \cdot 10^4$	1.73	1.70	1.67	1.64	1.61	1.58	1.55	1.52	1.49	1.46	1.43

Even for greater lengths: $s = 1{,}000$ km, the difference δs is very small: $\delta \approx 0.07$ mm, which may be disregarded in the calculations appearing in Applied Geodesy.

9.3.5 The Reduced Length of the Geodetic Line

Let P be a point, situated on a surface F, through which are drawn an infinity of geodetic lines of equal lengths. The terminal points S_1, S_2, ... of these geodetic lines lie on a closed curve called a *geodetic circle* (Fig. 9.8, *a*). The latter is orthogonal to the geodetic lines passing through the point P (in this case also called *geodetic radii*). Of course, the geodetic circles are not geodetic lines. For instance, on the reference ellipsoid the meridians play the role of geodetic radii and the circles of parallel are geodetic circles too.

The elementary arc of geodetic circle $\mathrm{d}s_C$ may be defined by the following general relation:

$$\mathrm{d}s_c = m\,\mathrm{d}A, \tag{9.109}$$

where $\mathrm{d}A$ is the angle between two geodetic lines infinitely close together and m is called *the reduced lenth of the geodetic line*. In the example considered above, the reduced lengths of the geodetic lines are represented by the radii of the circles of parallel taken into consideration.

In Fig. 9.8, *b* one considers the general case when the points S and Q have any position on the surface of the reference ellipsoid and one wishes to determine the reduced length of the geodetic line between these points. To a rotation $\mathrm{d}A$ of the geodetic line round the point S corresponds a rotation $\mathrm{d}A_n$ of the direct normal section at the point towards the point Q. As before, we denote by $\bar{s}$ the length of the chord of the normal section and by μ the angle of horizon depression (Fig. 9.5, *b*). Consequently, the distance between the point Q and the vertical passing through the point S is equal to $\bar{s} \cos \mu$, so that the elementary arc of geodetic circle can be expressed not only by the relation (9.109) but also using the formula:

$$\mathrm{d}s_c = \bar{s} \cos \mu \cdot \mathrm{d}A_n. \tag{9.110}$$

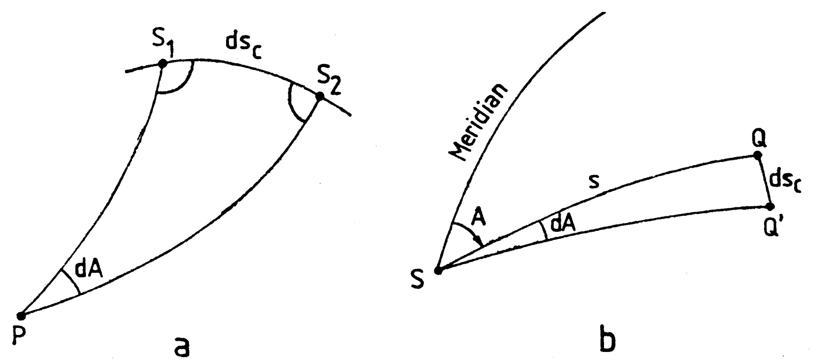


Fig. 9.8. The Reduced Length of the Geodetic Line

From (9.109) and (9.110) follows the differential relation for determining the reduced length of the geodetic line:

$$m = \bar{s} \cos \mu \frac{\mathrm{d}A_n}{\mathrm{d}A} \cdot \tag{9.111}$$

The product $s \cos \mu$ can be calculated with the aid of (9.82) and (9.79), yielding:

$$\bar{s} \cos \mu = s\left[1 - \frac{1}{6}\frac{s^2}{N^2}(1 + 2\eta^2 \cos^2 A) + \right.$$
$$\left. + \frac{3}{8}\frac{s^3}{N^3}\eta^2 t \cos A + \frac{1}{120}\frac{s^4}{N^4}\right]. \tag{9.112}$$

From (9.105) we obtain:

$$\mathrm{d}A_n = \mathrm{d}A\left(1 + \frac{1}{6}\frac{s^2}{N^2}\eta^2 \cos 2A - \frac{1}{24}\frac{s^3}{N^3}\eta^2 t \cos A\right), \tag{9.113}$$

so that the formula for actually computing the reduced length of the geodetic line results from utilizing the last three relations:

$$m = s\left[1 - \frac{1}{6}\frac{s^2}{N^2}(1 + \eta^2) + \frac{1}{3}\frac{s^3}{N^3}\eta^2 t \cos A + \frac{1}{120}\frac{s^4}{N^4}\right] \cdot \tag{9.114}$$

Thanks to the connexion relation (9.62) one can also write, within the limits of the same approximations as before:

$$m \approx s\left(1 - \frac{1}{6}\frac{s^2}{R^2} + \frac{1}{120}\frac{s^4}{R^4} + \cdots\right), \tag{9.115}$$

i.e.:

$$\frac{m}{R} = \sin \frac{s}{R}, \tag{9.116}$$

where R is *Gauss'* mean radius.

10

The Reduction of the Geodetic Observations on the Reference Ellipsoid's Surface

Before being used in calculations, the geodetic observations are reduced to the surface of the reference ellipsoid. This operation is necessary because the field observations are referred to the vertical to the geoid (with respect to which adjustment of the geodetic instruments is always carried out, i.e. their lay-out in the working position) whereas the calculations carried out in Geodesy make use of the normal to the reference ellipsoid.

In order to calculate the corrections for reducing the geodetic observations to the ellipsoid it is necessary to determine first of all the geoid undulations N and the components ξ, η of the deflection of the vertical by means of the methods which were presented in the first part of the book.

Bringing the support geodetic networks, which exist on the Earth's physical surface, onto the surface of the reference ellipsoid can be achieved by several methods, among which the most important will be mentioned here.

The Development Method. The elements which are measured on the Earth's physical surface are reduced to the geoid surface (to sea-level) and are to be subjected to an adjustment, depending on the geometry of the triangulation network.

To this end one chooses a *fundamental point*, at which one considers that the reference ellipsoid is tangent to the geoid. Thus one assumes at this point identity between the astronomical coordinates and the geodetic ones, as well as coincidence of the normal to the ellipsoid with the vertical to the geoid. We thus have *the initial data of the triangulation:* the geodetic coordinates of the fundamental point, the length and the azimuth of a side which joins this point to any other point of the triangulation network. The calculations are carried out in turn by development: starting from the fundamental point, one determines the coordinates of all the points of the network, using the elements reduced to the geoid surface, without passing to the surface of the reference ellipsoid.

The development method introduces systematic calculation errors which increase as one moves away from the fundamental point and, as a consequence, can be utilized only for relatively small territories, for which one can assume that the reference ellipsoid envelops the geoid surface best. In this manner the calculation of the primordial 1st-order triangulation in *Romania* before World War I was undertaken, on *Hayford's* international ellipsoid, as oriented at the military astronomical observatory at *Dealul Piscului*.

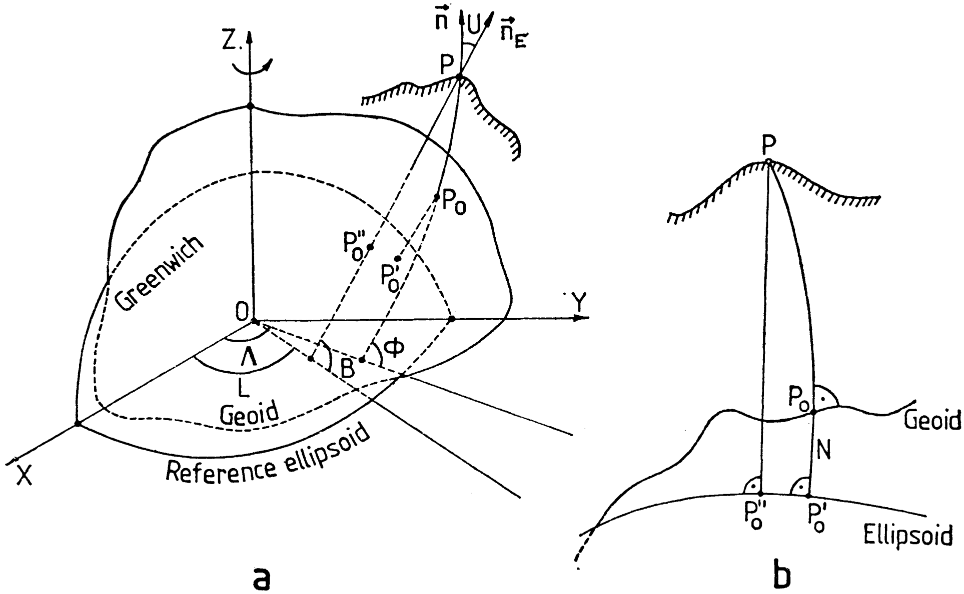


Fig. 10.1. The Projection Method

The Projection Method. In the projection method one proceeds to bringing the measured elements (angles, directions, lengths etc.) onto the ellipsoid surface, by applying certain corrections which will be presented in this Chapter. There are two possibilities for doing this, which may be summed up as follows:

Pizzetti's Method. The point P on the Earth's physical surface (Fig. 10.1, *a*) is first projected, with the aid of the vertical, onto the geoid surface at P_0 and then, with the aid of the normal to the ellipsoid it is to be projected at P'_0, on the surface of the reference ellipsoid. This method introduces great complications through the fact that it assumes knowledge of the curvatures of the verticals, necessary for establishing the corrections in the first stage of designing and has hitherto therefore not been of particularly practical applicability.

Bruns-Helmert's Method. The point P on the Earth's physical surface is projected at P''_0 on the ellipsoid surface, with the help of the normal to this surface. This method is much more practical and has been applied more widely in achieving the great national and international triangulation networks.

The coordinates of all the state triangulation points in *Romania* are determined by *Bruns-Helmert*'s projection method.

The differences which are obtained by utilizing the methods of Pizzetti and respectively Bruns-Helmert are small *(Heiskanen* 1967*):*

(1) *The ellipsoidal altitude, denoted by* H^E, *obtained by adding the geoid undulations to the orthometric altitudes* H^{OR}:

$$H^E = H^{OR} + N \tag{10.1}$$

is practically the same, in the case of both methods, the differences being of the order of fractions of millimetres.

The preceding relation also expresses the transition from the altitude systems which don't have the geoid as reference surface to the so-called *"ellipsoidal-altitude system"*, which is used in Three-dimensional Geodesy. In principle, the relation also holds for the normal-altitude system, where the reference surface is the quasi-geoid, with the difference that the geoid undulations N must be replaced by the height anomalies, denoted by ζ:

$$H^E = H^N + \zeta. \tag{10.2}$$

(2) *The differences between the geodetic coordinates are also small:*

$$\begin{aligned} B_{Bruns\text{-}Helmert} &= B_{Pizzetti} + \frac{H}{R}\,\xi\,; \\ L_{Bruns\text{-}Helmert} &= L_{Pizzetti} + \frac{H}{R}\,\eta \sec B, \end{aligned} \tag{10.3}$$

where H is the approximate altitude of the point P. As the ratio H/R is, for *Romania*, less than 1/3,000 and ξ, η have magnitudes of the order of the seconds of arc, it is obvious that the differences in the geodetic coordinates are trifling, in comparison with the position errors of the points of the support networks. Therefore, although *Pizzetti*'s method corresponds better to the actual situation, in practice one uses, almost without exception, *Bruns-Helmert*'s method which is much easier to apply.

10.1 The Reduction of the Astronomical Observations

One considers a point S_1 (Fig. 10.2) for which one knows, from calculations, the geodetic coordinates (B, L) and the geodetic azimuth A towards a point S_2, as well as the astronomical coordinates (Φ, Λ) and the astronomical azimuth α towards the same point S_2.

We construct an auxiliary sphere with centre at the point S_1 and radius equal to unity. The normal to the reference ellipsoid passing through this point intersects the auxiliary sphere at the point Z_g (projection of the geodetic zenith). Similarly, the local vertical intersects this sphere at the point Z_a (projection of the astronomical zenith).

From the point S_1 one draws a parallel to the Earth's rotation axis (to the *World axis*), which will intersect the auxiliary sphere at the point P (projection of the *World pole*). One readily notes that the arcs Z_gPQ and Z_aP represent the projections of the geodetic meridian, and of the astronomical meridian respectively, on the auxiliary sphere. The angle between these latter

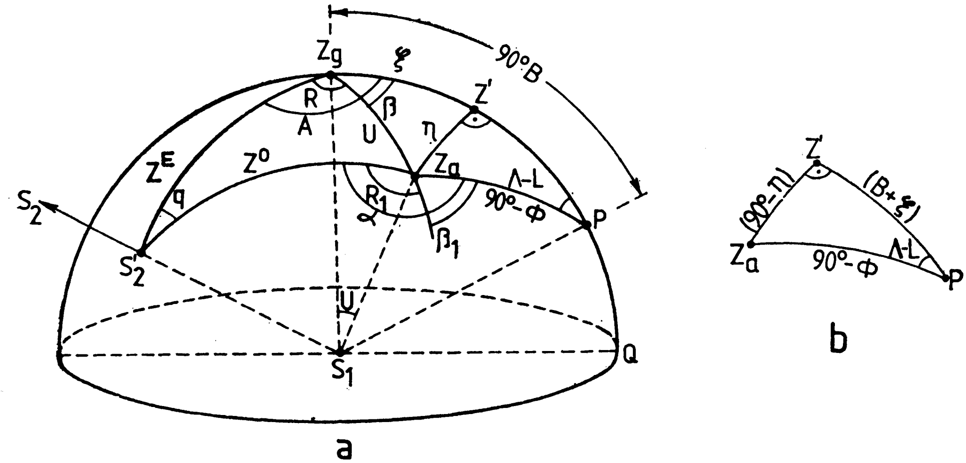


Fig. 10.2. The Reduction of the Astronomical Determinations on the Surface of the Reference Ellipsoid

equals the difference between the astronomical longitude and the geodetic longitude ($\Lambda - L$). As is known, the pole's height above the horizon is the latitude, so that on the auxiliary sphere one obtains the arcs ($90° - B$) and ($90° - \Phi$).

S_2' denotes the projection on the auxiliary sphere of the sighted point S_2. Now one can define other measurable elements (α = the astronomical azimuth, $z°$ = the apparent zenithal angle) as well as the corresponding elements on the ellipsoid (A = the geodetic azimuth, z^E = the zenithal angle reduced to the normal to the ellipsoid).

10.1.1 The Reduction of the Astronomical Coordinates

From the right-angle spherical triangle $Z_a Z' P$, one gets:

$$\frac{\sin \eta}{\sin (\Lambda - L)} = \cos \Phi.$$

Due to the smallness of the terms on the left-hand side of the previous relation one may write:

$$L = \Lambda - \eta \sec \Phi. \tag{10.4}$$

From the same triangle, first transformed according to *Neper*'s rule, separately represented in Fig. 10.2, *b*, we obtain:

$$\cos (\Lambda - L) = \tan \Phi \cot (B + \xi),$$

i.e.:

$$B = \Phi + \xi. \tag{10.5}$$

Remark. For certain solutions reductions of the astronomical observations to the geoid surface are necessary (Chapter. 6) to which end one utilizes the following relations:

$$\Lambda_{geoid} = \Lambda; \; \Phi_{geoid} = \Phi - 0{,}171 \; H_{\mathrm{km}} \sin 2B, \tag{10.5 a}$$

where H_{km} denotes the approximate altitude of the point concerned, in km.

The previous relations are used for many geodetic purposes: determining the parameters of the reference ellipsoid (the surface method), determining the coordinates of the fundamental point, computing the astro-geodetic components of the deflection of the vertical etc. It is well known that the astronomical coordinates are characterized by position errors much greater than the errors imposed on the geodetic coordinates, so that the relations (10.4) and (10.5) can be used for reducing the astronomical coordinates only in the case of control points, located at great distances, of the order of hundreds of kilometres. Between these points the support geodetic networks are developed.

10.1.2 The Reduction of the Astronomical Azimuths

From Fig. 10.2, *a* one notes that:

$$A = R + \beta; \quad \alpha = R_1 + \beta_1,$$

so that:

$$A = \alpha + (\beta - \beta_1) + (R - R_1). \tag{10.6}$$

For determining the difference $(\beta - \beta_1)$ as a function of known elements, we write the formula of the cosine of an angle in the spherical triangle $Z_g Z_a P$:

$$\cos \beta = \cos \beta_1 \cos (\Lambda - L) + \sin \beta_1 \sin (\Lambda - L) \sin \Phi,$$

which, within the limits of the approximations being used, becomes:

$$\beta - \beta_1 = -(\Lambda - L) \sin \Phi = -\eta \tan \Phi. \tag{10.7}$$

Similarly, from the spherical triangle $Z_g Z_a S_2'$ one gets:

$$R - R_1 = -\sin q \cos z^\circ.$$

The quantity q may be expressed in terms of determinable quantities:

$$\sin q = \frac{u \sin R}{\sin z^\circ},$$

yielding:

$$R - R_1 = (\eta \cos A - \xi \sin A) \cot z^\circ. \tag{10.8}$$

Introducing (10.8) and (10.7) into (10.6) yields the formula looked for:

$$A = \alpha - (\Lambda - L) \sin \Phi + (\eta \cos A - \xi \sin A) \cot z^\circ. \tag{10.9}$$

The azimuth determinations are, generally, carried out at the points of the development networks of the geodetic bases, between which there exist small level differences. Under these conditions $z^\circ \to 90^\circ$ and consequently $\operatorname{ctg} z^\circ \approx 0$ with the result that the magnitude of the last term is smaller relative to that of the last term but one in (10.9) so that in a sufficiently good approximation one can write:

$$A \approx \alpha - (\Lambda - L) \sin \Phi = \alpha - \eta \tan \Phi. \tag{10.10}$$

This is the well-known equation of *Laplace,* utilized in calculations for processing the higher-order triangulation networks.

For a greater accuracy, the relations (10.9) and (10.10) must be completed with the azimuthal correction due to the altitude of the sighted point, which will be examined later.

Example. In *Romania's* 1st-order triangulation, for the direction *Măgura Slătiorului – Dealul lui Bucur,* the astronomical azimuth $\alpha = 0^\circ\, 02'\, 23''.741$ and the geodetic azimuth determined with (10.9) is $A = 0^\circ\, 02'\, 19''.868$.

10.2 The Reduction of the Azimuthal Observations

In order to reduce the directions and the azimuthal angles on the reference ellipsoid one applies a total of three corrections.

10.2.1. The Correction of Reducing from the Direct Normal Section to the Geodetic Line

This correction is currently called "geodetic line correction" and can be calculated by means of (9.106):

$$c_1^{cc} = - K_1 (s_{\text{km}})^2 \sin 2A. \tag{10.11}$$

The order of magnitude of the correction c_1 was illustrated in § 9.3.4.

10.2.2 The Correction Due to the Altitude of the Sighted Point

From the point S_1, assumed to be projected onto the reference ellipsoid with the aid of the normal S_1S_1', one takes a sight at the point S_2 situated at the ellipsoidal altitude H. The sighting plane (under the assumption that the normal to the ellipsoid coincides with the local vertical), determined by the normal S_1S_1' and the sighted point, intersects the reference ellipsoid yielding the section S_1S_2'' of azimuth A_m. The projection of the point S_2 onto the reference ellipsoid being S_2''', there will ensue a difference between the section

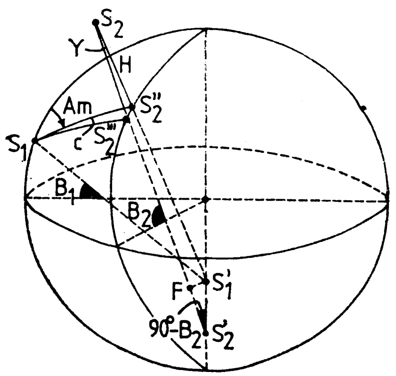


Fig. 10.3. The Correction Due to the Altitude of the Sighted Point

S_1S_1'' and the curve S_1S_2''' (along which one carries out the geodetic computations) represented by the correction c_2. As this correction is sufficiently small, one can accept certain approximations in determining it:

$$c_1 \approx \frac{S_2''' S_2''}{s} \sin A = \frac{H\gamma^{\mathrm{rad}}}{s} \sin A.$$

Further:

$$\gamma^{\mathrm{rad}} = \frac{FS_1'}{S_2F} \approx \frac{FS_1'}{a} = \frac{\overline{S_1'S_2'} \cos B_1}{a}$$

and together with formula (9.67):

$$\gamma^{\mathrm{rad}} \approx e^2(B_2 - B_1)^{\mathrm{rad}} \cos^2 B_2 \approx e^2 \frac{s}{M_m} \cos A \cos^2 B_2.$$

Consequently, the correction due to the altitude of the sighted point will be determined by the formula:

$$c_2^{cc} = K_2 H \sin 2A, \tag{10.12}$$

where the coefficient:

$$K_2 = \frac{e^2}{2} [1]_m \cos^2 B_2, \tag{10.13}$$

can be extracted from Table 9.4 as depending on the latitude of the sighted point. From formula (10.12) as well as from Fig. 10.3 one remarks that this correction does not depend on the altitude of the station point but only on the altitude of the sighted point, whence also the name of the correction. In the case of the points situated on the same parallel ($A = 100^g$ and respectively $A = 0^g$) the correction vanishes.

As an example, let us take $B_2 = 45°$, $A = 50^g$; if the sighted points have altitudes of 10, 100, and 1,000 m there will be obtained the corrections: $0^{cc}.002$; $0^{cc}.017$ and respectively $0^{cc}.167$.

In *Romania*'s 1st-order triangulation the maximum value of the sum of the corrections $c_1^{cc} + c_2^{cc}$ for the direction *Chicera Hamba—Negoiu* yielded: $0^{cc}.389$.

10.2.3 The Correction Due to the Deflection of the Vertical

In examining (10.9) one finds out that the total corrective term in the azimuthal measurements in a station is made up of the following two component parts:

$$-(\Lambda - L) \sin \Phi;\ (\eta \cos A - \xi \sin A) \cot z°.$$

The first component is a function of the position of the station point alone, having the same contribution for all the azimuths and azimuthal directions which are measured at the point concerned.

On the other hand, the second component comprises both the influence of the deflection of the vertical at the station point (ξ, η) and the influence of the azimuth and of the zenithal angle of the direction towards the sighted point. Thus, this component, variable from one azimuthal direction to another, constitutes the third correction which must be applied to the horizontal angular observations:

$$c_3^{cc} = (\eta \cos^{cc} A - \xi^{cc} \sin A) \cot z^{\circ}. \tag{10.14}$$

By studying the error propagation in (10.14) one may easily recognize that in order to obtain for the correction c_3 calculation errors comparable, in order of magnitude, with those obtained for other corrections (e.g. $\pm 0''.01$) it is necessary that the errors of the component elements be: $m_A < 90''$, $m_z < 17''$, $m_\xi = m_\eta < 0''.01$. The special claims are made concerning the zenithal angle z and the components of the deflection of the vertical. Therefore, the zenithal angles necessary in (10.14) are calculated as functions of the level difference ΔH existing between the station point and the sighted point:

$$\cot z = \frac{\Delta H}{D}.$$

It is known that the level difference is determined either by geometric levelling or by trigonometric levelling, but at small distances (sides of 1—4 km) in special networks. Thus, the errors in the heights of the support points in the state network are, generally, of order about $\pm$ 10 cm.

The magnitude of the correction c_3 is limited, depending on the actual conditions of carrying out the triangulation network, in particular due to the fact that the zenithal angles z° do not take arbitrarely small values.

Example. For the direction *Mîndra — Piatra Gemeni* in *Romania's* 1st-order triangulation network: $\Delta H = 1{,}065.7$ m; $D = 50{,}291$ m; $\xi = -25''.58$; $\eta = -3''.83$; $\cos A = -0.423$; $\sin A = -0.906$. Calculations yield: $c_3 = -1^{cc}.411$.

In comparison with the other two corrections previously examined the correction due to the deflection of the vertical can reach much more significant magnitudes so that taking account of its influence is absolutely justified, even in view of the additional costs of the necessary operations.

Remark. As an angle is a difference between two directions, its reduction to the surface of the ellipsoid is represented by the difference of the corrections c_1, c_2, c_3, corresponding to the two directions concerned.

10.3 The Reduction of the Zenithal Observations to the Normal to the Ellipsoid

One must draw the attention to the fact that this reduction is necessary only in the case of the operations carried out in Three-Dimensional Geodesy, where all of the three coordinates (X, Y, H^E of B, L, H^E) are expressed with respect to the surface of the reference ellipsoid. As is known, in the higher-order support geodetic networks one makes a distinction between the reference surface for the coordinates X, Y or B, L (using the projection plane or the reference ellipsoid) and the reference surface for altitudes — the geoid or the quasi-geoid ((10.1) and (10.2)). In such a case, the reduction of the zenithal observations to the normal to the ellipsoid is meaningless.

Due to atmospheric refraction, in the field one doesn't measure the true zenithal angles z_1 and z_2 (Fig. 10.4) but the apparent zenithal angles z_1^0 and z_2^0:

$$z_1 = z_1^0 + \rho_1;\ z_2 = z_2^0 + \rho_2. \tag{10.15}$$

Under conditions of small distances (up to 6 km) one can approximate the refraction curve by an arc of a circle and, consequently, one assumes equality of the refraction angles $\rho_1 \approx \rho_2$.

From Fig. 10.4 one notes that the connexion between the measured zenithal angles and those reduced to the normal to the ellipsoid is given by the relations:

$$z_1^E = z_1^0 + u_{\alpha_1};\ z_2^E = z_2^0 - u_{\alpha_2}. \tag{10.16}$$

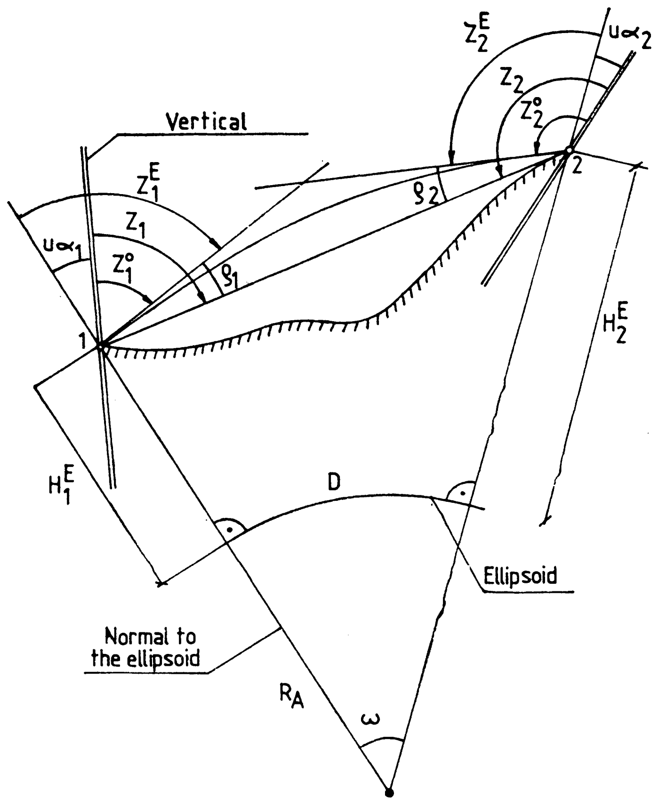


Fig. 10.4. The Reduction of the Zenithal Angles to the Normal to the Ellipsoid

The angles u_{α_1} and u_{α_2} represent *the components of the deflection of the vertical on the direction 1 — 2*, of azimuth α, determined at the points 1 and 2.

Because:

$$u_\alpha = \xi \cos\alpha + \eta \sin\alpha,$$

it follows that:

$$\begin{aligned} z_1^E &= z_1^0 + \xi_1 \cos\alpha + \eta_1 \sin\alpha; \\ z_2^E &= z_2^0 - \xi_2 \cos\alpha - \eta_2 \sin\alpha. \end{aligned} \tag{10.17}$$

Without repeating here the demonstration already known from doing the geodetic trigonometric levelling, we have in the present case:

$$\Delta H_{12}^E = H_2^E - H_1^E = D\left(1 + \frac{H_m}{R_A}\right)\tan\frac{z_2^E - z_1^E}{2} + \frac{I_1 - I_2}{2} + \frac{S_2 - S_1}{2}, \tag{10.18}$$

in which: H_m is the approximate mean altitude of the points 1 and 2 = $= (1/2)(H_1 + H_2)$. The altitudes H_1 and H_2 can be extracted from maps.

R_A — radius of the normal section of azimuth A, calculable by means of (9.56) by utilizing the mean latitude B_m of the two points concerned;

I, S — height of the instrument, and of the signal respectively at the point concerned.

Remark. It is well known that formula (10.18) is applied in the case of the reciprocal trigonometric levelling, for which the influence of the errors due to the refraction is a minimum. In the case of unilateral (in one direction) trigonometric levelling it is not justifiable to take into account relations of this form because the refraction errors are much greater as compared with the corrections of reduction to the normal to the ellipsoid. This is why this case will not be dealt with in detail.

Example. Using the elements of the previous example yields the influence of the deflection of the vertical on the zenithal observations on the direction *Mîndra-Piatra Gemeni* as: $u_\alpha = 4^{cc}.105$.

10.4 The Reduction of the Distances on the Ellipsoid

As is known, the technological processes for obtaining the sides needed in geodetic networks differ depending on the category of instruments being used:

— measurement of the geodetic bases with invar wire. The necessary sides are obtained by means of a supplementary network, one of base development;

— direct measurements of the necessary sides, utilizing modern electromagnetic instruments.

As a consequence, the results obtained by preliminary-processing calculations have different meanings so that in what follows they have to be examined separately.

10.4.1 The Reduction of the Geodetic Bases Measured with Invar Wire

The length of the geodetic base $d°$, measured with invar wire and initially processed following the known methodology, results in being reduced to a mean level surface, situated at the altitude $H_m = (1/2)(H_1 + H_2)$, where H_1 and H_2 are the altitudes of the ends of the base.

In order to further reduce the base to the ellipsoid one can proceed as follows:

(1) *One reduces, to begin with, the distance $d°$ to a surface parallel to the ellipsoid, situated at the ellipsoidal altitude $H_m^E = (1/2)(H_1^E + H_2^E)$*. As in the preceding paragraph one can replace the ellipsoid surface by a spherical surface, of radius R_A, which is calculated by means of (9.56), by utilizing the mean latitude at the ends of the base. From Fig. 10.5, *a*, it follows that:

$$d' = d° \cos u_\alpha, \tag{10.19}$$

where:

$$u_\alpha = \xi_m \cos\alpha + \eta_m \sin\alpha;$$

$$\xi_m = \frac{1}{2}(\xi_1 + \xi_2); \; \eta_m = \frac{1}{2}(\eta_1 + \eta_2). \tag{10.20}$$

(2) *The bases reduced to the ellipsoid level is then given by the relation:*

$$\frac{s}{R_A} = \frac{d'}{R_A + H_m^E}. \tag{10.21}$$

The needed ellipsoidal altitudes H_1^E and H_2^E are obtained using (10.1) or (10.2) as required knowledge of the geoid undulations at the ends of the bases being needed here.

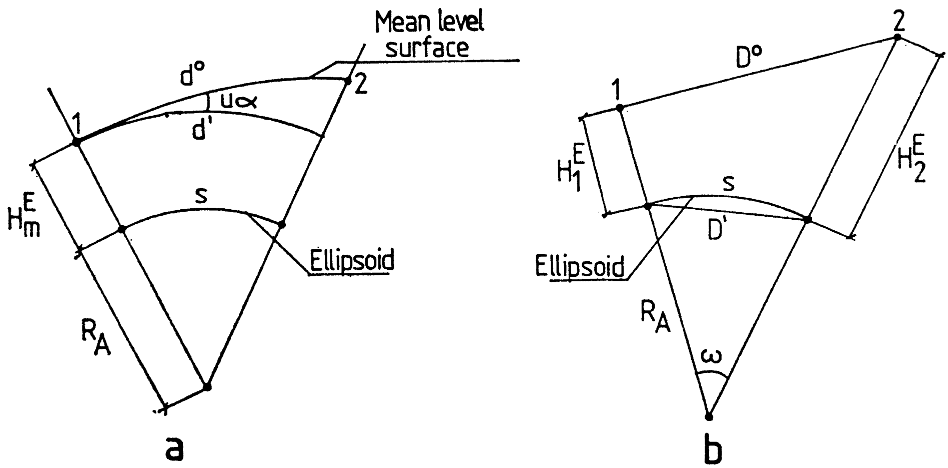


Fig. 10.5. The Reduction of the Distances on the Ellipsoid

Expanding the previous relation in series yields the formula sought:

$$s = d'\left[1 - \frac{H_m^E}{R_A} + \left(\frac{H_m^E}{R_A}\right)^2 - \cdots\right]. \tag{10.22}$$

Consequently, for the rigorous reduction of the bases measured with invar wire to the surface of the reference ellipsoid, both the components of the deflection of the vertical and the geoid undulations at the points situated at the ends of the bases are reeded.

10.4.2 The Reduction of the Sides Measured with Electromagnetic Devices

The sides measured with the aid of electromagnetic devices, such as geodimetres, tellurometres etc., are also processed first of all by applying the so-called *physical corrections*. The distance D° yielded by this operation is situated at the level of the two points between which the measurement has been carried out (Fig. 10.5, *b*). It follows that it is necessary to know the ellipsoidal altitudes H_1^E and H_2^E and, consequently, the geoid undulations at these points.

From the figure it follows that:

$$(D^\circ)^2 = (R_A + H_1^E)^2 + (R_A + H_2^E)^2 - 2(R_A + H_1^E)\,(R_A + H_2^E)\cos\omega. \tag{10.23}$$

If one introduces:

$$\cos\omega = 1 - 2\sin^2\frac{\omega}{2}\ ;\ D' = 2R_m \sin\frac{\omega}{2}\ ;\ \Delta H_{12}^E = H_2^E - H_1^E, \tag{10.24}$$

straightforward developments yield:

$$(D^0)^2 = (\Delta H_{12}^E)^2 + \left(1 + \frac{H_1^E}{R_A}\right)\left(1 + \frac{H_2^E}{R_A}\right)(D')^2. \tag{10.25}$$

The length of the chord D':

$$D' = \sqrt{\frac{(D^0)^2 - (\Delta H_{12}^E)^2}{\left(1 + \frac{H_1^E}{R_A}\right)\left(1 + \frac{H_2^E}{R_A}\right)}} \tag{10.26}$$

is then obtained as an intermediate result.

The length of the arc s representing the length of the measured side reduced to the ellipsoid surface is obtained using the formula:

$$s = 2R_m \arcsin\frac{D'}{2R_A}. \tag{10.27}$$

By examining the final formulae (10.26) and (10.27) one notes that, in this case, it is not necessary to know the deflections of the verticals at the points 1 and 2, as was the case when reducing the bases measured with the invar wire, but only the geoid undulations at these points.

11

Solving the Geodetic Triangles on the Reference Ellipsoid

After reducing the azimuthal observations and the distances mesured in the support geodetic network on the reference ellipsoid, one can proceed with the solving of the geodetic triangles on this surface.

Within the framework of the present treatise, particular attention will be paid to the situations currently to be found in the higher-order geodetic networks:

— the geodetic triangles generally have sides less than 60 km and are therefore also called *small ellipsoidal triangles;*

— the angles of the geodetic triangles oscillate round the optimum value of 60°, fixed limits being imposed by technical regulations, less than their values may not be (e.g., in *Romania*'s 1st-order triangulation the admissible limit for a network triangle is 40°);

— in a geodetic triangle of the support network one knows more elements than strictly necessary: almost without exception all of the three angles (from measurements) and at least one side (either from previous calculations or from measurements).

The practical solving of a small ellipsoidal triangle is based on *Gauss*' theorem, regarding as located on a mean *Gauss*' sphere of radius R_{B_i}, where B_i is the geodetic latitude of the mass centre of the triangle concerned. Even in such cases one does not resort to the direct utilization of formulae of spherical trigonometry but applies calculation methods specific to Geodesy, which will be presented in the sequel.

More special situations, apt to occur sometimes in Geodesy, viz. when one of the sides of the triangle is much smaller or, on the other hand, much larger than the other sides of the triangle (and, consequently, the value of the opposite angle tends either to 0° or to 180°) are examined by *Ehlert* (1978), who also derives the general formulae which may be applied both in the usual situations and in the above-mentioned special cases.

11.1 The Spherical Excess

The sum of the angles α, β, γ in a spherical triangle (assumed free from measurement errors) is always larger than 200^g: the difference is called the spherical excess:

$$\varepsilon = \alpha + \beta + \gamma - 200^g. \tag{11.1}$$

Between the angles measured and reduced on the surface of the reference ellipsoid, α°, β°, γ° and the angles adjusted by the least squares method, α, β, γ, there hold the relations:

$$\alpha = \alpha^0 + v_\alpha;\ \beta = \beta^0 + v_\beta;\ \gamma = \gamma^0 + v_\gamma, \tag{11.2}$$

in which v_α, v_β, v_γ, denote the corrections obtained from adjustment on the basis of condition equations of the form:

$$v_\alpha + v_\beta + v_\gamma + w = 0. \tag{11.3}$$

Thus, the sum of the angles measured and reduced on the ellipsoid is different from 200^g not only by the spherical excess but also by a quantity w, due to measurement errors:

$$\alpha^0 + \beta^0 + \gamma^0 - 200^g = \varepsilon + w. \tag{11.4}$$

When isolated triangles are solved, one considers:

$$v_\alpha = v_\beta = v_\gamma = -\, w/3, \tag{11.5}$$

which is not possible in the case of the rigorous adjustment of a geodetic network.

In both situations, however, it is necessary to know the spherical excess ε in order to be able to carry out the calculations for adjusting and solving the geodetic triangles.

From Fig. 11.1 one notes that the surfaces of the spherical sectors $(\alpha\alpha)$, $(\beta\beta)$ and $(\gamma\gamma)$ corresponding to the angles α, β, γ concerned may be expressed as functions of the surface of the spherical triangle ABC, denoted by F:

$$(\alpha\alpha) = F + BCA';\ (\beta\beta) = F + ACB';\ (\gamma\gamma) = F + ABC',$$

so that:

$$(\alpha\alpha) + (\beta\beta) + (\gamma\gamma) = 2(F + \pi R^2). \tag{11.6}$$

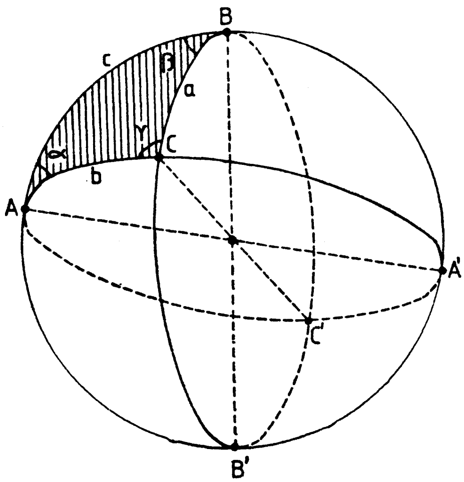


Fig. 11.1. The Spherical Excess

On the other hand:

$$(\alpha\alpha) = \frac{\alpha}{400^g} 4\pi R^2; \; (\beta\beta) = \frac{\beta}{400^g} 4\pi R^2; \; (\gamma\gamma) = \frac{\gamma}{400^g} 4\pi R^2,$$

i.e.:

$$(\alpha\alpha) + (\beta\beta) + (\gamma\gamma) = \frac{2\pi R^2}{200^g} (\alpha + \beta + \gamma). \tag{11.7}$$

Since $200^g/R = \rho^g$, it follows from (11.6) and (11.7):

$$\varepsilon'' = \frac{F}{R^2} \rho''. \tag{11.8}$$

For calculations in small geodetic triangles, the spherical surface F may be replaced by the surface of the corresponding plane triangle $A'B'C'$, denoted by F':

$$\varepsilon'' \approx \frac{F'}{R^2} \rho'' = \rho'' \frac{ab \sin \gamma'}{2R^2} = \rho'' \frac{ac \sin \beta'}{2R^2} = \rho'' \frac{bc \sin \alpha'}{2R^2}, \tag{11.9}$$

where α', β', γ' are the angles of the plane triangle.

From the tables of *Krasovski*'s reference ellipsoid (*Khristov* 1950, *Tárczi Hornoch* and *Khristov*, 1959 et al.) one can extract the coefficient:

$$f = \frac{\rho''}{2R^2}, \tag{11.10}$$

valid for the sexagesimal gradation, as a function of the mean latitude of the vertices of the triangle ABC, so that:

$$\varepsilon'' \approx f\, a\, b \sin \gamma' = f\, a\, c \sin \beta' = f\, b\, c \sin \alpha'. \tag{11.11}$$

For sides larger than 60 km, the spherical excess can be calculated wiht the formula:

$$\varepsilon'' = \rho'' \frac{F'}{R^2}\left(1 + \frac{m^2}{8R^2}\right), \tag{11.12}$$

Table 11.1. *Examples of Spherical Excesses*

SIDE LENGTH $(s)_{km} =$	10	20	30	40	50	60
SPHERICAL EXCESS $\varepsilon'' =$	0.219	0.878	1.976	3.5119	5.4873	7.9018

where with m^2 was denoted:

$$m^2 = \frac{a^2 + b^2 + c^2}{3}. \tag{11.13}$$

Examples concerning the order of magnitude which can be attained by the spherical excess as a function of the side length s are given in Table 11.1. In the same table one may study the influence corresponding to formula (11.12) as compared with that of formula (11.9) for geodetic triangles with sides larger than 60 km. Equilateral triangles were considered and the mean latitude of the triangle vertices was taken as 46°.

11.2 Solving Small Geodetic Triangles by means of Legendre's Theorem

One of the methods widely used for solving small geodetic triangles is based on the utilization of *Legendre*'s theorem, published in the year 1787:

"A small spherical triangle can be solved as a plane triangle if one maintains the equality of the sides of the two triangles while the angles of the plane triangle are obtained by subtracting from each of the spherical angles one third of the spherical excess." In order to demonstrate this theorem we write the cosine formula in the spherical triangle ABC:

$$\cos \frac{a}{R} = \cos \frac{b}{R} \cos \frac{c}{R} + \sin \frac{b}{R} \sin \frac{c}{R} \cos \alpha,$$

which can be expanded in series by using relations of the form:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots; \ \sin x = x - \frac{x^3}{6} + \dots, \tag{11.14}$$

leading to:

$$\cos \alpha = \frac{-a^2 + b^2 + c^2}{2bc} + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{24\ R^2bc}. \tag{11.15}$$

WITH (11.9)			WITH (11.12)		
80	100	150	80	100	150
14.047 62	21.949 41	49.38618	14.047 90	21.950 08	49.389 59

In the plane triangle $A'B'C'$, with the angles α', β', γ' and the same sides a, b, c, it follows from *Pythagoras*'s generalized theorem that:

$$\cos \alpha' = \frac{b^2 + c^2 - a^2}{2bc} \tag{11.16}$$

and consequently:

$$\sin^2 \alpha' = -\frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{4b^2c^2}. \tag{11.17}$$

From the last three relations it follows that:

$$\cos \alpha = \cos \alpha' - \frac{bc}{6R^2} \sin^2 \alpha'. \tag{11.18}$$

Expanding in series the equality yields:

$$\cos \alpha \approx \cos \alpha' - (\alpha - \alpha') \sin \alpha' \tag{11.19}$$

i.e.:

$$\alpha - \alpha' = \frac{bc \sin \alpha'}{6R^2} = \frac{1}{3} \frac{F'}{R^2} \approx \frac{\varepsilon}{3}. \tag{11.20}$$

Analogous relations may also be obtained for the differences $\beta - \beta'$; $\gamma - \gamma'$, which constitutes the demonstration of *Legendre*'s theorem.

The approximations generated by the series expansions (11.14), (11.19) and by the relation (11.20) as well as the approximation of the ellipsoid's surface by an average sphere limit the applicability domain of the solutions of the geodetic triangles by *Legendre*'s theorem. **In the case of geodetic triangles with sides larger than 60 km the computation algorithm is the following:**

(1) *One calculates the spherical surface F as a function of the plane surface F'*:

$$2F' = ab \sin \gamma' = ac \sin \beta' = bc \sin \alpha'; \; F = F'\left(1 + \frac{K}{8} m^2\right),$$

where $K = 1/R^2$ is the total curvature, calculated for the mean latitude of the triangle concerned.

(2) *One determines the spherical excess by utilizing the spherical surface, leading to formula* (11.12):

$$\varepsilon^{cc} = \rho^{cc} FK.$$

(3) *One calculates the differences between the angles of the spherical and plane triangles respectively by means of the relations:*

$$\alpha - \alpha' = \frac{\varepsilon}{3} + \frac{\varepsilon}{12}\left(\frac{K_A - K}{K}\right) + \frac{\varepsilon K}{60}(m^2 - a^2);$$

$$\beta - \beta' = \frac{\varepsilon}{3} + \frac{\varepsilon}{12}\left(\frac{K_B - K}{K}\right) + \frac{\varepsilon K}{60}(m^2 - b^2); \tag{11.21}$$

$$\gamma - \gamma' = \frac{\varepsilon}{3} + \frac{\varepsilon}{12}\left(\frac{K_C - K}{K}\right) + \frac{\varepsilon K}{60}(m^2 - c^2),$$

in which K_A, K_B, K_C represent the total curvatures calculated at the points A, B, C.

An example of solving geodetic triangles by means of *Legendre*'s theorem is given in Table 11.2.

11.3 Solving Small Geodetic Triangles by the Additament Method

By means of the additament method, introduced by *J. G. Soldner* in the year 1810, any small geodetic triangle (approximated as a spherical triangle) is solved using a plane triangle, whose angles equal the angles of the geodetic triangle (reduced on the ellipsoid), while modifying, however, the sides of the latter.

The additament method has enjoyed a more limited application in comparison with *Legendre*'s method but a change is to be expected in this respect as a consequence of the present possibilities of directly measurig distances in the support geodetic networks.

The sine theorem applied in the spherical triangle ABC and in the plane triangle $A'B'C'$, formed as previously mentioned, yields:

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\frac{a}{R}}{\sin\frac{b}{R}} \quad \text{or} \quad \frac{\sin\alpha}{\sin\beta} = \frac{a'}{b'},$$

whence

$$\frac{a'}{b'} = \frac{a - \dfrac{a^3}{6R^2} + \cdots}{b - \dfrac{b^3}{6R^2} + \cdots}. \tag{11.22}$$

Tabele 11.2. *Example of Solving a Small Geodetic Triangle by Legendre's Method*

Spherical excess calculation	Vertex	Angle measured and reduced on the ellipsoid	Correction	Adjusted angle in the spherical triangle	$-\frac{\varepsilon}{3}$ cc	Adjusted angle in the plane triangle	Angle's sine in the plane triangle	Side's length in the plane and spherical triangles	Side
a = 39 248,880	A	70 97 07.901	+0.244	70 97 08,145	−3.217	70 97 04.298	0.897 823 57	39 248.880	a
B = 45°.0	B	62 24 51.441	+0.244	62 24 51.685	−3.217	62 24 48.468	0.829 3625	36 250.545	b
$f=25.351.10^{-10}$	C	66 78 49.577	+0.244	66 78 49.821	−3.217	66 78 46.604	0.866 950 64	37 899.252	c
$\varepsilon = 9^{cc},651$		200 00 08.919	+0.732	200 00 09.651	−9.651	200 00 00.000	MODULUS = $a/\sin\alpha'$ = = 43 715.5822 m		
		$\varepsilon + w = +$ $+\ 8^{cc}.919$							

This equation is fulfilled only if:

$$a' = a - \frac{a^3}{6R^2} \quad \text{and} \quad b' = b - \frac{b^3}{6R^2}. \tag{11.23}$$

or generally

$$s' = s - \frac{s^3}{6R^2}. \tag{11.24}$$

The quantity:

$$A_s = \frac{s^3}{6R^2}, \tag{11.25}$$

by which the side s of the spherical triangle must be modified in order to obtain the corresponding length s' in the intermediate plane triangle is called the *linear additament.*

One can show that for calculating the additaments in the case of small geodetic triangles ($s < 60$ km) it suffices to utilize a mean value for the total curvature $K = 1/R^2$ for territories covering 5° in latitude northward and southward (about 1,000 km in all in the north-south direction). Thus for *Romania* one can consider $R = R_{46°} = 6{,}378{,}957$ m, for which $(1/6R^2) = 4.0959 \times 10^{-15}$.

For logarithmic calculations we will use the relation:

$$\lg s' = \lg s - \frac{\text{Mod } s^2}{6R^2}, \tag{11.26}$$

in which:

$$Mod = \lg e = 0.434\ 294\ 481$$

$$\frac{\text{Mod } s^2}{6R^2} = \text{logarithmic additament.} \tag{11.27}$$

In the case of distances s larger than 60 km one may extend the initial series expansion, so that:

$$s' = s - \frac{s^3}{6R^2} + \frac{s^5}{120\ R^4} - \cdots; \tag{11.28}$$

$$\lg s' = \lg s - \frac{\text{Mod } s^2}{6R^2} - \frac{\text{Mod } s^4}{180\ R^4}. \tag{11.29}$$

The corrective terms introduced into the last relations don't make any essential contributins, as is also to be seen from Table 11.3;

$$\frac{1}{120\ R^4} = 5.0329 \times 10^{-30}.$$

In order to illustrate the solution of a small geodetic triangle by means of the additament method, the example in Table 11.2 was repeated, the calculations being carried out in Table 11.4.

Table 11.3. *Examples of Additaments*

Side length $(s)_{km}$ =							With the Formulae (11.25) – (11.27)			With the formulae (11.28), (11.29)		
	10	20	30	40	50	60	80	100	150	80	100	150
Linear additament (mm) =	4.1	32.8	110.6	262.1	512.0	884.7	2 097.1	4 095.9	13 823.6	2 097.1	4 095.9	13 823.2
Logarithmic additament × $\times\ 10^8$ =	17.7	71.1	160.0	284.6	444.7	640.4	1 138.45	1 778.82	4 002.36	1 138.45	1 778.83	1 778.89

Table 11.4. *Solving Small Geodetic Triangles by the Additament Method*

Spherical excess and additament calculation	Vertex	Angle measured and reduced on the ellipsoid	Correction	Adjusted angle in the spherical triangle	Angle's sine in the spherical triangle	Side in the plane triangle	Additament	Side in the spherical triangle	Side
$a = 39248.880$m	A	70 97 07.901	+0.244	70 97 08.145	0.987 825 79	39 248.6323	0.247	39 248.880	a
$B = 45°.0$	B	62 24 51.441	+0.244	62 24 51.685	0.829 23907	36 250.350	0.195	36 250.545	b
$f = 25.351.10^{10}$	C	66 78 49.577	+0.244	66 78 49.821	0.866 95315	37 899.029	0.223	27 899.252	c
$\varepsilon^{cc} = 9^{cc}.651$		200 00 08.919	+0.732	200 00 09.651		MODULUS = $a'/\sin\alpha'$ = 43 715,1980 m			
$A_a = 0.2477$ m		$\varepsilon + w =$							
$A_b = 0.195$ m		$= 8^{cc}.919$							
$A_c = 0.223$ m									

12

Calculation of the Geodetic Coordinates on the Reference Ellipsoid

The final aim of the calculations carried out on the reference ellipsoid is to determine the geodetic coordinates, i.e. the latitude B and the longitude L, of the points in the support geodetic networks. The operations of rigorous processing of the astro-geodetic determinations require the computation of the geodetic coordinates in several stages: computation of the provisional coordinates, which are needed in the stage preliminary to the rigorous processing, and the computation of the final coordinates, after completing the adjustment proper. One can consequently appreciate that these kind of calculations play a particularly important role for which reason they are also known in the technical literature under the name of *solutions of the basic geodetic problems.*

The first basic geodetic problem, also called the *direct geodetic problem,* consists of determining the geodetic coordinates B_2, L_2 of the point S_2 (Fig. 12.1,*a*) and the geodetic azimuth A_2 (also called *inverse geodetic azimuth*) as functions of the coordinates B_1, L_1 of the point S_1, the geodetic azimuth A_1 (also called *direct geodetic azimuth*) and the length of the geodetic line s between the points S_1 and S_2.

The successive utilization of the direct geodetic problem is also known under the name of *coordinate transport.*

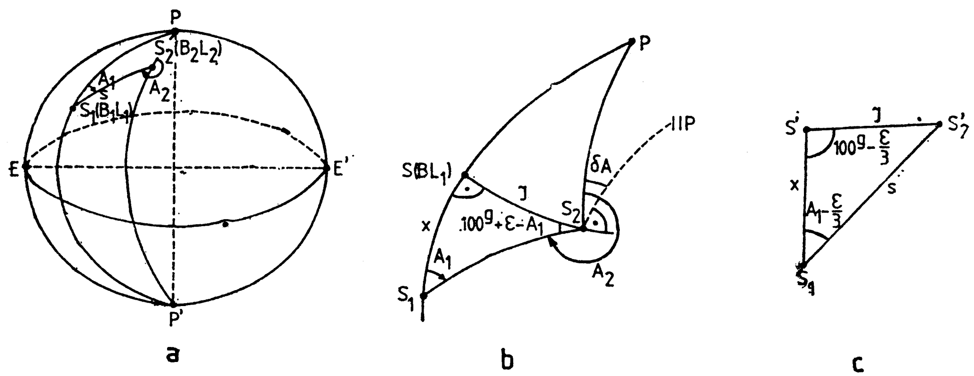


Fig. 12.1. The Basic Geodetic Problems

The second basic geodetic problem, also called the *inverse geodetic problem,* consists of determining the length of the geodetic line s, the direct geodetic azimuth A_1 and the inverse one A_2 when the geodetic coordinates of the points S_1 and S_2 are known.

The inverse geodetic problem is used for deducing the initial elements (geodetic distances and azimuths) which are needed when carrying out the computations for determining the coordinates of all of the points in a geodetic network, as well as a control method for checking the calculations carried out in the case of the direct geodetic problem.

Over 50 procedures are known for solving the basic geodetic problems. This variety has arisen due to the continuing necessity of reducing the computation volume, of increasing the accuracy of the final results, even under the conditions of large geodetic distances, as well as by the available methods of computation.

Some of these procedures at present enjoy a somewhat reduced applicability, e.g. those based on the utilization of logarithms, and these will be studied no further in the present book.

There are several criteria for classifying the methods and procedures for computing the geodetic coordinates on the reference ellipsoid, depending on the element being considered as most important within the framework of these computations. One of the classification criteria which is currently used, also adopted within the framework of this work, considers as main element the length of the geodetic line s. From this point of view one can distinguish: methods of solving for small geodetic distances ($s < 60$ km), for medium ones ($60 \leqslant s < 600$ km) and for large ones ($s \geqslant 600$ km). Within the framework of the three categories of methods, we will present those solving procedures which have an historical priority or are currently being used in geodetic practice.

Since the formulae needed in solving the inverse geodetic problem are obtained, in principle, from the formulae used for solving the direct geodetic problems, by applying suitable mathematical transformations, e.g. inversions of series expansions, we will present in what follows only the principles for deriving the calculation formulae for the direct geodetic problem, but the calculation formulae and the numerical examples will be given for the solution of both basic geodetic problems.

The numerical examples which will be presented in the following section have been solved by using non-programmable desk computers and their purpose is to facilitate the understanding by the reader of the procedure put forward theoretically. It is obvious that for current operations, occurring within the framework of national geodetic services, as well as in wide-range operations of an international character, one can carry out the programming of the solution of the basic geodetic problems on electronic computers of large capacity. As in the other chapters of the treatise, where such laborious calculation problems crop up, neither the algorithming nor the programming of these problems are described, for reasons determined by the scope of the book.

From among the fields in which solutions of the basic geodetic problems appear one can mention: the construction of the lst-order support geodetic networks (in which small geodetic distances, on the average 30—40 km occur), of the continental and intercontinental geodetic networks (in which medium geodetic distances intervene), problems in air-navigation, rocket technology etc. (large geodetic distances). In principle, one may pose the problem of calculating the geodetic distance and azimuth between points of known coordinates, situated at very great distances, of the order of tens of thousands of kilometres, so that the applicability of the solutions of the basic geodetic problems is practically unlimited.

Another aspect which must be taken into consideration in the actual solutions concerns the accuracy in calculating the geodetic coordinates, which leads to distinguishing exact and approximate methods. Gradually as the geodetic distances increase, the calculation accuracy has different meanings. As in other geodetic computations, within the framework of the basic geodetic problems also, one's goal is that the computation errors be about ten times less than the average errors characterizing the field operations. Thus, one can show that in the lst-order geodetic triangulation it is necessary that the calculation approximation for the geodetic coordinates B and L be to $\pm 0\overset{''}{.}0001$, for the geodetic azimuths A to $\pm 0^{cc}\cdot 001$ and for the geodetic distances s to ± 0.001 m. These calculation approximations would certainly be exaggerated in the case of distances of the order of hundreds of kilometers, determined by radar or similar techniques, whose field-determination errors are much greater. Gradually as the laser technique of distance determination improves, accompanied by suitable processing methods (we are referring in particular to accurate reductions on the ellipsoid surface, based on the knowledge of the geoid profile), the basic geodetic problems will be solved with high accuracy in the case of medium and large geodetic distances as well.

12.1 Solving the Basic Geodetic Problems for Small Geodetic Distances

In the case of small geodetic distances ($s < 60$ km) which occur in the lst-order support networks one applies methods of selving the basic geodetic problems in which one accepts some approximations such as series expansions with respect to the point S_1 considered as origin point, replacement of the geoid surface by *Gauss*' mean sphere etc., approximations which cannot be accepted in the case of medium and large geodetic distances. The computation formulae obtained in the case of some of the procedures have a somewhat wider range of applicability, up to 100—200 km.

12.1.1 The Method of Expanding in Series (Legendre's Method)

Since the differences in latitude, longitude and azimuth between the initial point S_1 and any current point S_i depend on the length of the geodetic line s, *one can accept the following Maclaurin series expansions:*

$$B_i = B_1 + s\left(\frac{\mathrm{d}B}{\mathrm{d}s}\right)_1 + \frac{s^2}{2}\left(\frac{\mathrm{d}^2B}{\mathrm{d}s^2}\right)_1 + \frac{s^3}{6}\left(\frac{\mathrm{d}^3B}{\mathrm{d}s^3}\right)_1 + \frac{s^4}{24}\left(\frac{\mathrm{d}^4B}{\mathrm{d}s^4}\right)_1 + \ldots;$$

$$L_i = L_1 + s\left(\frac{\mathrm{d}L}{\mathrm{d}s}\right)_1 + \frac{s^2}{2}\left(\frac{\mathrm{d}^2L}{\mathrm{d}s^2}\right)_1 + \frac{s^3}{6}\left(\frac{\mathrm{d}^3L}{\mathrm{d}s^3}\right)_1 + \frac{s^4}{24}\left(\frac{\mathrm{d}^4L}{\mathrm{d}s^4}\right)_1 + \ldots; \quad (12.1)$$

$$A_i = A_1 + 200^{\mathrm{g}} + s\left(\frac{\mathrm{d}A}{\mathrm{d}s}\right)_1 + \frac{s^2}{2}\left(\frac{\mathrm{d}^2A}{\mathrm{d}s^2}\right)_1 + \frac{s^2}{6}\left(\frac{\mathrm{d}^3A}{\mathrm{d}s^3}\right)_1 + \frac{s^4}{24}\left(\frac{\mathrm{d}^4A}{\mathrm{d}s^4}\right)_1 + \ldots.$$

The first series expansions of the form (12.1), up to terms containing s^3, were deduced by *F. M. Legendre* in the year 1806.

The derivatives needed in formulae (12.1) are to be calculated at the known geodetic point S_1, so that the direct geodetic problem becomes solvable. **The lst-order derivatives are obtained from** (9.32), (9.33), (9.29) **and** (9.92) **as:**

$$\frac{\mathrm{d}B}{\mathrm{d}s} = \frac{\cos A}{M} = \frac{V^3}{c}\cos A;$$

$$\frac{\mathrm{d}L}{\mathrm{d}s} = \frac{\sin A}{N\cos B} = \frac{V}{c}\cdot\frac{\sin A}{\cos B}; \quad (12.2)$$

$$\frac{\mathrm{d}A}{\mathrm{d}s} = \frac{\mathrm{d}A}{\mathrm{d}L}\cdot\frac{\mathrm{d}L}{\mathrm{d}s} = \frac{1}{N}\tan B\sin A = \frac{V}{c}\tan B\sin A$$

The higher-order derivatives may be calculated by means of the following general formulae:

$$B^{(n)} = \frac{\partial}{\partial B}B^{(n-1)}\frac{\mathrm{d}B}{\mathrm{d}s} + \frac{\partial}{\partial A}B^{(n-1)}\frac{\mathrm{d}A}{\mathrm{d}s};$$

$$L^{(n)} = \frac{\partial}{\partial B}L^{(n-1)}\frac{\mathrm{d}B}{\mathrm{d}s} + \frac{\partial}{\partial A}L^{(n-1)}\frac{\mathrm{d}A}{\mathrm{d}s}; \quad (12.3)$$

$$A^{(n)} = \frac{\partial}{\partial B}A^{(n-1)}\frac{\mathrm{d}B}{\mathrm{d}s} + \frac{\partial}{\partial A}A^{(n-1)}\frac{\mathrm{d}A}{\mathrm{d}s}.$$

Depending on the number of terms included in the series expansions (12.1) one achieves the degree of approximation needed in the calculations. Improvements are, however, limited by the main approximation, generated by the actual computation of the necessary derivatives with respect to the latitude of the initial point S_1.

Complete solutions of the problem under examination, up to higher-order derivatives, 4th and 5the order, intended for practical purposes, are associated

with some well-known geodesists: *F. R. Helmert* (1880); *O. Schreiber* (1878); *H. Boltz* (1942); *G. V. Bagratuni* (1952); *V. K. Khristov* (1950, 1957) etc., who also compiled the tables necessary for these solutions.

Thus, the formulae (12.1) **may be written in fhe form** *(Tárczi Hornoch* and *Khristov* 1959):

$$B_i = B_1 + b_{10}u + b_{20}u^2 + b_{02}v^2 + b_{30}u^3 + b_{12}uv^2 + b_{22}u^2v^2 + b_{04}v^4 + \dots;$$

$$L_i = L_1 + l_{01}v + l_{11}uv + l_{21}u^2v + l_{03}v^3 + l_{31}u^3v + l_{13}uv^3 + \dots; \tag{12.4}$$

$$A_i = A_1 \pm 200^g + a_{01}v + a_{11}uv + a_{21}u^2v + a_{03}v^3 + a_{31}u^3v + a_{13}uv^3 + \dots,$$

in which:

$$u = 10^{-5}\, s \cos A_1;\ v = 10^{-5}\, s \sin A_1; \tag{12.5}$$

$$b_{10} = \frac{10^5}{N_1}\rho''(1 + \eta_1^2);$$

$$b_{20} = \frac{3 \cdot 10^{10}}{2N_1^2}\rho'' t_1(-\eta_1^2 - \eta_1^4);$$

$$b_{02} = \frac{10^{10}}{2N_1^2}\rho'' t_1(-1 - \eta_1^2);$$

$$b_{30} = \frac{10^{15}}{2N_1^3}\rho''(-\eta_1^2 + t_1^2\eta_1^2 - 2\eta_1^4 + 6t_1^2\eta_1^4);$$

$$b_{12} = \frac{10^{15}}{6N_1^3}\rho''(-1 - 3t_1^2 - 2\eta_1^2 + 6t_1^2\eta_1^2 - \eta_1^4 + 9t_1^2\eta_1^4);$$

$$b_{22} = \frac{10^{20}}{12N_1^4}\rho'' t_1(-4 - 6t_1^2 + 9\eta_1^2 + 3t_1^2\eta_1^2); \tag{12.6}$$

$$b_{04} = \frac{10^{20}}{24N_1^4}\rho'' t_1(1 + 3t_1^2 + 2\eta_1^2 - 6t_1^2\eta_1^2);$$

$$l_{01} = \frac{10^5\,\rho''}{N_1 \cos B_1};$$

$$l_{11} = \frac{10^{10}\,\rho''}{N_1^2 \cos B_1} t_1;$$

$$l_{21} = \frac{10^{15}\,\rho''}{3N_1^3 \cos B_1}(1 + 3t_1^2 + \eta_1^2);$$

$$l_{03} = \frac{10^{15}\,\rho''}{3N_1^3 \cos B_1}(-t_1^2);$$

$$
\begin{aligned}
l_{31} &= \frac{10^{20}\rho''}{3N_1^4 \cos B_1} t_1(2 + 3t_1^2 + \eta_1^2);\\
l_{13} &= \frac{10^{20}\rho''}{3N_1^4 \cos B_1} t_1(-1 - 3t_1^2 - \eta_1^2);\\
a_{01} &= \frac{10^{5}\rho''}{N_1} t_1;\\
a_{11} &= \frac{10^{10}\rho''}{2N_1^2} (1 + 2t_1^2 + \eta_1^2);\\
a_{21} &= \frac{10^{15}\rho''}{6N_1^3} t_1(5 + 6t_1^2 + \eta_1^2 - 4\eta_1^4);\\
a_{03} &= \frac{10^{15}\rho''}{6N_1^3} t_1(-1 - 2t_1^2 - \eta_1^2);\\
a_{31} &= \frac{10^{20}\rho''}{24N_1^4} (5 + 24t_1^2 + 24t_1^4);\\
a_{13} &= \frac{10^{20}\rho''}{24N_1^4} (-1 - 20t_1^2 - 24t_1^4).
\end{aligned}
\tag{12.6}
$$

For *Krasovski*'s reference ellipsoid the coefficients needed in calculations are extracted from tables *(Tárczi Hornoch* and *Khristov* 1959*)* as functions of the geodetic latitude B_1 of the initial point.

The formulae which are necessary for solving the inverse geodetic problem are derived from formulae (12.1) by series inversion, because the latitude and longitude differences as well as the mean latitude are known:

$$b = 10^{-4}(B_i - B_1)''; \; l = 10^{-4}(L_i - L_1)''; \; B_m = \frac{1}{2}(B_1 + B_i). \tag{12.7}$$

To begin with, one calculates:

$$
\begin{aligned}
s \sin A_m &= r_{01}l + r_{21}b^2l + r_{03}l^3 + [r_{41}b^4l + r_{23}b^2l^3 + r_{05}l^5] + \dots;\\
s \cos A_m &= s_{10}b + s_{30}b^3 + s_{12}bl^2 + [s_{32}b^3l^2 + s_{14}bl^4] + \dots;\\
\Delta A'' &= t_{01}l + t_{21}b^2l + t_{03}l^3 + [t_{41}b^4l + t_{23}b^2l^3 + t_{05}l^5] + \dots,
\end{aligned}
\tag{12.8}
$$

whence follows the length of the geodetic line s by means of double (control) calculation and then the direct and inverse geodetic azimuths:

$$A_1 = A_m - \frac{1}{2}\Delta A; \; A_2 = (A_m \pm 200^g) + \frac{1}{2}\Delta A. \tag{12.9}$$

The necessary coefficients:

$$r_{01} = \frac{10^4}{\rho''} N_m \cos B_m;$$

$$r_{21} = \frac{10^{12}}{24\rho^3} N_m \cos B_m(1 - \eta_m^2 - 9t_m^2\eta_m^2 + \eta_m^4 + 18t_m^2\eta_m^2);$$

$$r_{03} = \frac{10^{12}}{24\rho^3} N_m \cos^3 B_m(-t_m^2);$$

$$r_{41} = \frac{7 \cdot 10^{20}}{5760\rho^5} N_m \cos B_m;$$

$$r_{23} = \frac{10^{20}}{2\,880\rho^5} N_m \cos^3 B_m(-8 - 35t_m^2);$$

$$r_{05} = \frac{10^{20}}{1\,920\rho^5} N_m \cos^5 B_m(-8t_m^2 + t_m^4);$$

$$s_{10} = \frac{10^4}{\rho''} M_m;$$

$$s_{30} = \frac{10^{12}}{8\rho^3} N_m (\eta_m^2 - t_m^2\eta_m^2 - 2\eta_m^2 + 7t_m^2\eta_m^4); \qquad (12.10)$$

$$s_{12} = \frac{10^{12}}{24\rho^3} N_m \cos^2 B_m(-2 - 3t_m^2 + 3t_m^2\eta_m^2 - 3t_m^2\eta_m^4);$$

$$s_{32} = \frac{10^{20}}{1\,440\rho^5} N_m \cos^2 B_m(-4 - 15t_m^2);$$

$$s_{14} = \frac{10^{20}}{5\,760\rho^5} N_m \cos^4 B_m(-8 - 20t_m^2 + 15t_m^4);$$

$$t_{01} = 10^4 \sin B_m;$$

$$t_{21} = \frac{10^{12}}{24\rho^2} \cos B_m t_m(3 + 2\,\eta_m^2 - 2\eta_m^4);$$

$$t_{03} = \frac{10^{12}}{12\rho^2} \cos^3 B_m t_m(1 + \eta_m^2);$$

$$t_{41} = \frac{5 \cdot 10^{20}}{384\rho^4} \cos B_m t_m;$$

$$t_{23} = \frac{10^{20}}{96\rho^4} \cos^3 B_m t_m (1 - 2t_m^2);$$

$$t_{05} = \frac{10^{20}}{240\rho^4} \cos^5 B_m t_m (2 - t_m^2),$$

are also extracted from the tables of *Krasovski*'s reference ellipsoid as functions of the mean geodetic latitude B_m.

Table 12.1 *Example of Solving the Indirect Geodetic Problem by Legendre Series Expansions*

Initial elements	Initial-point designation Gălbiori		
	$B_1 = 45°37'05''.3037$;	$L_1 = 21°59'42''.5164$;	$A_1 = 104\ 58\ 96\ .665$;
			$s = 22\,930.196$ m
Auxiliary elements	$u = -0.016\ 517\ 05$	$v^2 = +0.052\ 306\ 58$	$v^3 = +0.011\ 962\ 85$
	$v = +0.228\ 706\ 31$	$v^3 = -0.000\ 004\ 51$	$u^3v = -0.000\ 001\ 03$
	$u^2 = +0.000\ 272\ 81$	$u^2v = +0.000\ 062\ 39$	$u^2v^2 = +0.000\ 014\ 27$
	$uv = -0.003777\ 55$	$uv^2 = -0.000\ 863\ 94$	$uv^3 = -0.000\ 197\ 59$
			$v^4 = +0.002\ 735\ 98$
Coefficients from tables	$b_{10} = +3\,238.9899$	$l_{01} = +4\,615.6364$	$a_{01} = +3\,298.768$
	$b_{20} = -\ 0.2561$	$l_{11} = +\ 73.8173$	$a_{11} = +\ 78.104$
	$b_{02} = -\ 25.9004$	$l_{21} = +\ 1.5587$	$a_{21} = +\ 1.517$
	$b_{30} = +\ 0.0001$	$l_{03} = -\ 0.3935$	$a_{03} = -\ 0.416$
	$b_{12} = -\ 0.5428$	$l_{31} = +\ 0.0310$	$a_{31} = +\ 0.031$
	$b_{22} = -\ 0.0108$	$l_{13} = -\ 0.0249$	$a_{13} = -\ 0.025$
	$b_{04} = +\ 0.0022$		
Intermediate calculation	$b_{10}u = -53.49856$	$l_{01} = +1\,055.62517$	$a_{01}v = +754.44906$
	$b_{20}u^2 = -\ 0.00007$	$l_{11}uv = -\ 0.27885$	$a_{11}uv = -\ 0.29504$
	$b_{02}v^2 = -\ 1.35476$	$l_{21}u^2v = +\ 0.00010$	$a_{21}u^2v = +\ 0.00010$
	$b_{30}u^3 = \ 0.00000$	$l_{03}v^3 = -\ 0.00471$	$a_{03}v^3 = -\ 0.00498$
	$b_{12}uv^2 = +\ 0.00047$	$l_{31}u^3v = \ 0.00000$	$a_{31}u^3v = \ 0.00000$
	$b_{22}u^2v^2 = \ 0.00000$	$l_{13}uv^3 = \ 0.00000$	$a_{13}uv^3 = \ 0.00000$
	$b_{04}v^4 = +\ 0.00001$		
Final results	$\Delta B = -54''.85291$	$\Delta L = +1\,055''.34171$	$\Delta A'' = +754''.14914$
			$\Delta A'' = +2327^{cc}.620$
	Final-Point designation Vîrfuşor		
	$B_2 = 45°36'10''.4508$;	$L_2 = 22°17'17''.8581$;	$A_2 = 304°82'24''.285$

Since the computations are carried out with respect to the mean latitude (also Section 12.2) the series expansions are useful for a wider utilization field and can be used up to $s \leqslant 200$ km without the terms in the square brackets and with these terms up to distances $s \leqslant 400$ km.

An example of the calculations for solving the direct geodetic problem is presented in Table 12.1.

The control of the calculations, performed by solving the inverse geodetic problem, is presented in Table 12.2.

Table 12.2 *Example of Solving the Direct Geodetic Problem by Legendre Series Expansions*

Initial elements	Initial-point designation Gălbiori $B_1 = 45°37'05''.3037$; $L_1 = 21°59'42''.5164$ Final-point designation Vîrfuşor $B_2 = 45°36'10''.4508$; $L_2 = 22°17'17|''.8581$		
Auxiliary elements	$B_m = 45°36'.631\,287\,76$ $b = -0.005\,485\,529$ $l = +0.105\,534\,17$	$b^2 = +0.000\,030\,09$ $bl = -0.000\,578\,90$ $l^2 = +0.011\,137\,46$	$b^3 = 0.000\,000$ $b^2l = +0.000\,003$ $bl^2 = -0.000\,061$ $l^3 = +0.001\,175$
Coefficients from tables	$r_{01} = +216\,684.183$ $r_{21} = +\ 20.498$ $r_{03} = -\ 10.837$	$s_{10} = +308\,737.822$ $s_{30} = -\ 0.008$ $s_{12} = -\ 76.011$	$t_{01} = +7\,146.012$ $t_{21} = +\ 2.104$ $t_{03} = +\ 0.687$
Intermediate calculations	$r_{01}l = +22\,867.5854$ $r_{21}b^2l = +\ 0.0001$ $r_{03}l^3 = -\ 0.0127$	$s_{10}b = -1\,693.5165$ $s_{30}b^3 = 0.000$ $s_{12}bl^2 = +\ 0.0046$	$t_{01}l = +754.1484$ $t_{21}b^2l = 0.0000$ $t_{03}l^3 = +\ 0.0008$
	$s \sin A_m = +22\,867.5728$ $\cot A_m = -0.074\,05735$ $\sin A_m = +0.997\,26898$ $\cos A_m = -0.073\,85510$	$s \cos A_m = -\ 1\,693.5119$ $A_m = 104^g\,60^c\,60^c\ ^c.476$	$\Delta A'' = 754.1492$ $\Delta A^{cc} = 2\,327.621$
Final results	$s = 22\,930.195$ *m*;	$A_1 = 104^g\,58^c\,96^{cc}.665$;	$A_2 = 304^g\,82^c\,24^{cc}.286$

12.1.2 The Method of the Auxiliary Point

For distances somewhat greater (60 km $< s <$ 150 km) one can apply the method of the auxiliary point, which secures the necessary accuracy with simultaneously a more rapid convergence in comparison with the series (12.1). The origin of the procedure is ascribed in some works to *O. Schreiber* and in others to *F. N. Krasovski*, the principles which underlie the working-formulae derivation being, however, the same.

The auxiliary point $S(B, L_1)$ (Fig. 12.1, *b*) is chosen in such a manner that the geodetic line SS_2 have azimuth at S equal to 100^g.

Without presenting in its entirety the demonstration of the derivation of the formulae, we will examine in what follows the necessary computation stages:

(1) *Solving the small ellipsoidal triangle S_1SS_2 with the aid of Legendre's method, by means of the plane triangle $S_1'S'S_2'$:*

$$x = s\,\frac{\sin\left[100^g - \left(A_1 - \frac{2}{3}\,\varepsilon\right)\right]}{\cos\frac{\varepsilon}{3}}\,;\quad y = s\,\frac{\sin\left(A_1 - \frac{\varepsilon}{3}\right)}{\cos\frac{\varepsilon}{3}}, \tag{12.11}$$

where the spherical excess ε may be determined from the relation:

$$\varepsilon^{cc} = \rho^{cc}\,\frac{s^2 \sin A_1 \cos A_1}{2R^2}\,. \tag{12.12}$$

With the notations (12.5) and using series expansions one gets:

$$x = u\left(1 + \frac{v^2}{3R^2}\right);\; y = v\left(1 - \frac{u^2}{6R^2}\right);\; \varepsilon = uv/2R^2. \tag{12.13}$$

(2) *The determination of the latitude difference between the initial point and the auxiliary point can be achieved using* (9.18) *which yields*:

$$\delta B_1 = B - B_1 = \frac{x}{M_m}\,\rho'', \tag{12.14}$$

where the mean radius of curvature of the meridian ellipse M_m is, however, unknown. The determination of the radius of curvature M_m can be made in several ways, among which the method of successive approximations suggests itself here due to the available methods of calculation.

When the quantity x is small, as is the case in the example presented in Table 12.3, the convergence of the procedure is particularly rapid (only two iterations were necessary in the case under examination).

After this stage the auxiliary point is determined, because the latitude $B = B_1 + \delta B_1$ and the longitude $L = L_1$ are now known.

(3) *Determining the latitude and longitude differences between the auxiliary point S and the final point S_2.*

The calculations are taken further with respect to the auxiliary point, by applying formulae of the type (12.4) with the following special values: $s = y$; $A = 100^g$. By transformations and adaptations which we skip here, we obtain:

$$\Delta L = L_2 - L_1 = \nu_1\left[1 - \frac{\tau_1^2}{3\rho^2} + \frac{\tau_1^4}{15\rho^4}\,(1 + 3t_1^2)\right]; \tag{12.15}$$

$$\delta B_2 = B - B_2 = \frac{1}{2}\,\frac{V_1^2}{\rho}\,k_1\,\tau_1\left[1 - \frac{\nu_1^2}{12(1 - e^2)\,\rho^2} - \frac{\tau_1^2}{6(1 - e^2)\,\rho^2} + \right.$$
$$\left. + \frac{e^2\tau_1^2}{12(1 - e^2)\,\rho^2}\,(13 - 10\sin^2 B_1)\right], \tag{12.16}$$

in which:

$$k = \frac{y}{N}\rho''; \; \nu = k \sec B; \; \tau = k \tan B. \tag{12.17}$$

Thus, the geodetic latitude of the point S_2 is determined by means of the relation:

$$B_2 = B_1 + \delta B_1 - \delta B_2, \tag{12.18}$$

and the geodetic longitude by formula (12.15).

(4) *Determining the inverse azimuth* A_2 *at the point* S_2. In Fig. 12.1, *b* one notes that: $A_2 + (100^g + \varepsilon - A_1) + 100^g - \delta A = 400^g$ so that:

$$A_2 = A_1 + 200^g - \varepsilon + \delta A, \tag{12.19}$$

where δA is the meridian convergence, calculable by means of the relation *(Jordan* 1958*)*:

$$\delta A = \tau_1 \left[1 - \frac{\nu_1^2}{6(1-e^2)\rho^2} - \frac{\tau_1^2}{6(1-e^2)\rho^2} + \right.$$
$$\left. + \frac{\tau_1^2}{6(1-e^2)\rho^2} e^2 (3 - \sin^2 B_1) \right] \cdot \tag{12.20}$$

An example of organizing the calculations with the formulae previously shown, taking into consideration the possibility of utilizing the tables of *Krasovski*'s reference ellipsoid and the desk electronic computer *Hewlett-Packard*, is presented in Table 12.1, concerning the initial data contained in Table 12.1. The results are identical with those obtained by means of the series-expansion method *(Legendre*'s Method).

Remark. The relation (12.19) is not valid for all situations; it holds only for $A_1 < 100^g$. In all other possible situations the following formulae are to be used:

For:	$A_1 = 100^g$	we have	$A_2 = 300^g + \delta A$;
	$100^g < A_1 < 200^g$,		$A_2 = A_1 + 200^g + \varepsilon + \delta A$;
	$A_1 = 200^g$,		$A_2 = 0^g$
	$200^g < A_1 < 300^g$,		$A_2 = A_1 - 200^g - \varepsilon - \delta A$:
	$A_1 = 300^g$,		$A_2 = 100^g - \delta A$;
	$300^g < A_1 < 400^g$.		$A_2 = A_1 - 200^g + \varepsilon - \delta A$;
	$A_1 = 0^g$,		$A_2 = 200^g$.

12.2 Gauss' Method (Method of the Mean Arguments) for Solving the Basic Geodetic Problems for Medium Geodetic Distances

The procedure worked out by *K. F. Gauss* in the year 1846 for solving the basic geodetic problems may be applied for small and medium distances ($s < 600$ km). The method is also called "method of the mean arguments"

Table 12.3 *Example of Solving the Direct Geodetic Problem by the Auxiliary-point Method*

Initial elements	Initial-point designation Gălbiori				
	$B_1 = 45°37'05''.3037$;	$L_1 = 21°59'42'',5164$;	$A_1 = 104^g58^c96^{cc}\,.\,665$;	$s = 22\,930.196$ m.	
Solving the triangle S_1SS_2	$B_1 = 45°37'.088\,395$;	$R_1 = 6\,378\,671.170$ m	$\varepsilon = 0^{cc}\,.\,2955$		
	$x = -1\,651{,}6980$ m;	$y = 22\,870.6312$ m			
Auxiliary-point latitude determination	Iteration	1st iteration	2nd iteration	3rd iteration	4th iteration
	B_m	45°37′	45°36′.6426	—	—
	M_m	6 368 181 m	6 368 174.910	—	—
	δ_B	−53″.4983	−53″.4984	—	—
	$B = 45°36'11''.8053 = 45°36'.196\,753 = 45°.603\,279\,21$				
Intermediate calculations	$N_1 = 6\,389\,170.822$;	$k_1 = 738.344\,058$;	$\nu_1 = 1\,055.346\,438$;	$\tau_1 = 754.058\,456$	
	$\mathrm{tg}\,B_1 = 1.021\,283\,30$;	$\tau_1^2/\rho^2 = 1.336\,472 \times 10^{-15}$	$\nu_1^2/\rho^2 = 2.617\,820 \times 10^{-5}$		
	$V_1^2 = 1.003\,298\,315$;	$-5(1-e^2) = 5{,}959\,8395$			
	$\Delta L = 1\,055''.3417$;	$\delta B_2 = +1''.3545$;	$\delta A = 754''.053\,48 = 2\,327^{cc}.3255$		
Final results	Final-point designation Vîrfuşor				
	$B_2 = 45°36'10''.4508$;	$L_2 = 22°17'17''.8581$;	$A_2 + 304^g82^c24^{cc}.2855$		

inasmuch as the solution is based on the utilization of the following mean values:

$$B_m = \frac{B_1 + B_2}{2}; \quad L_m = \frac{L_1 + L_2}{2}; \quad A_m = \frac{A_1 \pm 200^g + A_2}{2}. \tag{12.21}$$

Within the framework of the direct geodetic problem, the mean arguments are unknown, whence follows the necessity of applying an iterative solution. To begin with, in a first iteration, the necessary elements can be determined by expeditious methods (graphically or by using approximate relations, as e.g. (9.32), (9.33)). In the case of small geodetic distances the convergence of the procedure is very rapid.

Let $C(B, L)$ be the point situated in the middle of the geodetic line s joining the points $S_1(B_1, L_1)$ and $S_2(B_2, L_2)$. The latitudes B_1 and B_2 can be expressed in terms of the latitude B by *Maclaurin* series expansions of the form (12.1):

$$\begin{aligned} B_2 &= B + \frac{s}{2}\left(\frac{dB}{ds}\right)_C + \frac{s^2}{8}\left(\frac{d^2B}{ds^2}\right)_C + \frac{s^3}{48}\left(\frac{d^3B}{ds^3}\right)_C + \ldots; \\ B_1 &= B - \frac{s}{2}\left(\frac{dB}{ds}\right)_C + \frac{s^2}{8}\left(\frac{d^2B}{ds^2}\right)_C - \frac{s^3}{48}\left(\frac{d^3B}{ds^3}\right)_C + \ldots \end{aligned} \tag{12.22}$$

Analogous expressions may be written for the longitudes L_2 and L_1 and for the azimuths A_2 and A_1 respectively. To save space, in what follows we will only present the main stages in deriving the calculation formulae for the geodetic latitudes, since for the geodetic longitudes and azimuths one can proceed in a similar way. From (12.22) follows a first possibility for calculating the latitude difference with respect to the point C:

$$b = B_2 - B_1 = s\left(\frac{dB}{ds}\right)_C + \frac{s^3}{24}\left(\frac{d^3B}{ds^3}\right)_C + \frac{s^5}{1\,920}\left(\frac{d^5B}{ds^5}\right)_C + \ldots. \tag{12.23}$$

Since we seek a series expansion in terms of the mean arguments (12.21) it is useful to determine the difference between the mean latitude and the latitude of the point C:

$$B_m = B_C + \frac{s^2}{8}\left(\frac{d^2B}{ds^2}\right)_C + \frac{s^4}{384}\left(\frac{d^4B}{ds^4}\right)_C + \ldots. \tag{12.24}$$

As was already shown (12.2), the derivatives in the expressions (12.22) —(12.24) are functions of the latitude B and the azimuth A, so that their variations from the point C to the point of mean latitude can be expressed in the form of series expansions of a function of two variables:

$$\left(\frac{dB}{ds}\right)_C=\left(\frac{dB}{ds}\right)_m+(B-B_m)\frac{\partial}{\partial B}\left(\frac{dB}{ds}\right)_m+(A-A_m)\frac{\partial}{\partial A}\left(\frac{dB}{ds}\right)_m. \qquad (12.25)$$

In a similar way one can also calculate the other differences which exist between the higher-order derivatives, calculated at the point C and at the point of mean latitude respectively, so that the formulae (12.22) — (12.23) can be expressed in terms of the mean elements (12.21). The detailed computations may be studied in more extensive works (*Jordan* 1958, *Bagratuni* 1962 et al.), the final results being those presented below:

Formulae for the direct geodetic problem:

$$\Delta L=L_2-L_1=[2]_m\frac{s\sin A_m}{\cos B_m}[1+[3]\,\Delta L^2\sin B_m-[4]_m\,\Delta B^2-$$

$$-([9]\Delta B^4-[10]_m\,\Delta B^2\,\Delta L^2-[11]_m\,\Delta L^4\,];$$

$$\Delta B=B_2-B_1=[1]_m\frac{s\cos A_m}{\cos\dfrac{\Delta L}{2}}[1-[5]_m\Delta L^2\cos^2 B_m+ \qquad (12.26)$$

$$+[6]_m\Delta B^2+([10]_m\Delta B^2\Delta L^2+[12]_m\Delta L^4)];$$

$$\Delta A=A_2-A_1=\Delta L\sin B_m[1+[7]_m\Delta L^2\cos^2 B_m+[8]_m\Delta B^2+$$

$$+([13]\,\Delta B^2-[14]_m\Delta B^2\,\Delta L^2+[15]_m\Delta L^4)].$$

For small geodetic distances, $s<60$ km, one can ignore the terms contained within the round brackets in (12.26).

The coefficients denoted symbolically in square brackets have the following meanings:

$$[1]_m=\frac{\rho''}{M_m};\quad [2]_m=\frac{\rho''}{N};\quad [3]=\frac{1}{24\rho''^2};$$

$$[4]_m=[3]\frac{1+\eta_m^2-9\eta_m^2t_m^2}{V_m^4};\quad [5]_m=[3](1-2\eta_m^2);$$

$$[6]_m=3\;[3]\frac{\eta_m^2(t_m^2-1-\eta_m^2-4\eta_m^2t_m^2)}{V_m^4};\quad [7]_m=2\;[3]V_m^2; \qquad (12.27)$$

$$[8]_m=[3]\;\frac{3+8\eta_m^2+5\eta_m^4}{V_m^4};\quad [9]=\frac{1}{2\,880\rho''^4};$$

$$[10]_m = \frac{1}{1\,440\ \rho''^4}(4 + 15\,t_m^2)\cos^2 B_m;\ [11]_m = [9](12t_m^2 + t_m^4)\cos^4 B_m;$$

$$[12]_m = [9]_m(14 + 40t_m^2 + 15t_m^4)\cos^4 B_m;\ [13] = \frac{1}{192\ \rho''^4}; \tag{12.27}$$

$$[14]_m = \frac{1}{48\,\rho^4}\sin^2 B_m;\ [15]_m = \frac{1}{1\,440\,\rho^4}(7 - 6t^2)\cos^4 B_m.$$

Formulae for the inverse geodetic problem:

$$s \sin A_m = \frac{\Delta L \cos B_m}{[2]_m}[1 - [3]\ \Delta L^2 \sin^2 B_m + [4]_m \Delta B^2 +$$
$$+ ([9]\Delta B^4 - [10]\ \Delta B^2\,\Delta L^2 - [11]_m \Delta L)^4]; \tag{12.28}$$

$$s \cos A_m = \frac{\Delta B \cos \dfrac{\Delta L}{2}}{[1]_m}[1 + [5]_m \Delta L^2 \cos^2 B_m - [6]_m \Delta B^2 -$$
$$- ([10]_m \Delta B^2 \Delta L^2 + [12]_m \Delta L^4)].$$

From the preceding expressions follows the mean azimuth A_m and the length of the geodetic line s. The direct geodetic azimuth A_1 and the inverse one A_2 are calculated by means of (12.9), using the last relation in the group (12.26) for calculating the azimuth difference ΔA.

An illustrative example of solving the basic geodetic problems by *Gauss*' method is presented in Tables 12.4 and 12.5. The following approximate values were used in the first iteration for the direct geodetic problem: $B_m = 47°$; $A_m = 46°20'$. The coefficients $[1]_m$ and $[2]_m$ must be calculated in all the iterations. The other coefficients $[3]$ — $[15]_m$ are determined in a stage of the computations in which one establishes a certain stability of the results obtained (in the example concerned — in the third iteration).

Because the variations of these coefficients are insignificant with respect to small variations (of the order of seconds of arc) of the mean latitudes obtained iteratively, their values may be kept constant in all the iterations.

Inasmuch as the method assumes the carrying out of a certain number of successive iterations, with number having many digits, a relatively large volume of calculations is involved, which represents an obstacle in its utilization for solving the basic geodetic problems with the aid of the non-programmable desk computers. By resorting to a very small number of input data, the method may, however, be readily programmed in order to be used with large-capacity computers.

The method worked out by *Gauss* presents obvious advantages in solving the inverse geodetic problem, where the applicability field too is wider, exceeding even the limit of the medium geodetic distances. From the above examples it is clear that the method secures a high accuracy in the geodetic computations.

Table 12.4. *Example of Solving the Direct Geodetic Problem by Gauss' Method in the Case of Medium Geodetic Distances*

Initial elements	The name of initial point S_1 $B_1 = 46°$; $L_1 = 25°$; $A_1 = 45°$; $s = 400$ km		
Iterations elements of formula	1	2	3
M_m	6 369 606 m	6 369 884.3 m	6 369 874.60 m
N_m	6 389 652 m	6 389 745.2 m	6 389 741.98 m
ΔB	7 200″	9 002″.9	8 939″.9
ΔL	10 800″	13 619″,4	13 771″.6
B_m	47°	47°.250 402 78	47°.241 652 78
ΔA	7 200″	9 960″.6	10 111″.43
A_m	46°	46°.383 416 67	46°.404 365 28
$[1]_m$	0.032 382 694	0.032 381 248	0.032 381 298
$[2]_m$	0.032 281 101	0.032 280 600	0.032 280 616
η_m^2			$3.105\,8921 . 10^{-3}$
V_m^2			1.003 094 644
t_m^2			1.169 594 178
$[3]_m$			$9.793\,513 \cdot 10^{-13}$
$[4]_m$			$9.446\,238 \cdot 10^{-13}$
$[5]_m$			$9.732\,898 \cdot 10^{-13}$
$[6]_m$			$1.373\,7 \cdot 10^{-15}$
$[7]_m$			$1.964\,764 \cdot 10^{-12}$
$[8]_m$			$2.944\,097 \cdot 10^{-12}$
$[9]_m$			$1.9 \cdot 10^{-25}$
$[10]_m$			$3.8 \cdot 10^{-24}$
$[11]_m$			$6.3 \cdot 10^{-25}$
$[12]_m$			$3.3 \cdot 10^{-24}$
$[13]_m$			$2.9 \cdot 10^{-24}$
$[14]_m$			$6.0 \cdot 10^{-24}$
$[15]_m$			$4.0 \cdot 10^{-26}$
$1 + [3]\,\Delta L^2 \sin^2 B_m - [4]_m\,\Delta B^2 - [9]_m\,\Delta B^4 + [10]_m\,\Delta B^2\,\Delta L^2 + [11]_m\,\Delta L^4 =$			1.000 024 692
$1 - [5]_m\,\Delta L^2 \cos^2 B_m + [6]_m\,\Delta B^2 + [10]_m\,\Delta B^2\,\Delta L^2 + [12]_m\,\Delta L^4 =$			0.999 915 1063
$1 + [7]_m\,\Delta L^2 \cos^2 B_m + [8]_m\,\Delta B^2 + [13]_m\,\Delta B^4 - [14]_m\,\Delta B^2\,\Delta L^2 + [15]_m\,\Delta L^2 =$			1.000 406 979
Final result	The name of final point S_2 $B_2 = 48°28'55''.50562$; $L_2 = 28°49'34''.41871$; $A_2 = 47°48'37''.5181$		

4	5	6	7
6 369 999.079 m	6 369 999.841 m	6 369 999.840 m	
6 389 783.602 m	6 389 783.386 m	6 389 783.379 m	
8 935″.64	8 935″.505	8 935″.505 67	8 935″.505 62
13.774″.38	13.774″.415	13 774″.418 67	13 774″.418 72
47°.241 061 61	47°.241 042 49	47°241042 45	47°.241 042 45
10,117″.50	10,117″.515	10,117″.5181	10,117″.5181
46°.405 208 33	46°.405 210 54	46°.405 210 85	
0.032 380 665	0.032 380 668	0.032 380 668	
0.032 280 406	0.032 280 407	0.032 280 407	
1.000 024 824	1.000 024 827	1.000 024 827	
0.999 915 0702	0.999 915 0697	0.999 915 0696	
1.000 406 826	1.000 406 820	1.000 406 820	

Table 12.5. *Example of Solving the Inverse Geodetic Problem by Gauss' Method in the Case of Medium Geodetic Distances*

Initial elements	The name of initial point S_1 $B_1 = 46°$; $L_1 = 25°$ The name of final point S_2 $B_2 = 48°28'55''.50562$; $L_2 = 28°49'34''.41871$
Auxiliary elements	$B_m = 47°14'27''.75282$; $\Delta B = 8935''.50562$; $\Delta L = 13774''.41872$ $[1]_m = 0.032\,280\,407$; $[2]_m = 0.032\,280\,407$; $\eta^2_{1m} = 3.1059636 \cdot 10^{-3}$; $V^2_m = 1.003\,105\,964$; $t^2_m = 1.169\,544\,190$ $[3] = 9.793513 \cdot 10^{-13}$; $[4]_m = 9.444988 \cdot 10^{-13}$; $[5]_m = 9.732676 \cdot 10^{-13}$ $[6]_m = 1.3776 \cdot 10^{-15}$; $[7]_m = 1.964786 \cdot 10^{-12}$; $[8]_m = 2.944118 \cdot 10^{-12}$ $[9] = 1.92 \cdot 10^{-25}$; $[10]_m = 3.8 \cdot 10^{-24}$; $[11]_m = 6.3 \cdot 10^{22}$ $[12]_m = 3.3 \cdot 10^{-24}$; $[13] = 2.9 \cdot 10^{-24}$; $[14]_m = 6.0 \cdot 10^{22}$ $[15]_m = 4.0 \cdot 10^{-26}$; $1 - [3]\,\Delta L^2 \sin^2 B_m + [4]_m\,\Delta B^2 + [9]\,\Delta B^4 - [10]_m\,\Delta B^2\,\Delta L^2 - [11]_m\,\Delta L^2 =$ $= 0.999\,751\,736.$ $1 + [5]_m\,\Delta L^2\,\mathrm{ocs}^2 B_m - [6]\ \Delta B^2 - [10]_m\,\Delta B^2\,\Delta L^2 - [12]_m\,\Delta L^2 = 1.000\,084\,938$ $1 + [7]_m\,\Delta L^2 \cos^2 B_m + [8]_m\,\Delta B^2 + [13]\,\Delta B^4 - [14]_m\,\Delta B^2\,\Delta L^2 + [15]_m\,\Delta L^2 =$ $= 1.000\,406\,820$
Intermediate calculus	$s \sin A_m = 289\,693.8307$ m; $s \cos A = 275\,821.4723$ $\cot A_m = 0.952\,113\,725$; $A_m = 46°24'18''.7589$ $\sin A_m = 0.724\,234\,5766$; $\cos A_m = 0.689\,553\,6804$ $\Delta A_m = 10\,117''.5181$; $\Delta A/2 = 1°24'18''.7591$
Final results	$s = 400\,000.0002$ m (Double calculus-check) $A_1 = 44°59'59''.9999$; $A_2 = 47°48'37''.5180$.

12.3 Solving the Basic Geodetic Problems for Large Geodetic Distances

The methods for solving the basic geodetic problems in the case of large geodetic distances, (600 km $< s <$ 20,000 km) have been developed on the basis of principles worked out by *W. Bessel* in the year 1828. The central idea of these methods lies in projecting the geodetic line from the reference ellipsoid onto an auxiliary sphere, as a great-circle element, so that the reduced latitude φ^U of a point situated on the geodetic line be equal to the spherical latitude φ of the corresponding point on the great circle of the auxiliary sphere.

Let the geodetic line passing through the points $S_1(B_1, L_1)$ and $S_2(B_2, L_2)$ be continued until it meets the equator (Fig. 12.2, *a*). B_M denotes the maximum latitude reached by the geodetic line. Under these conditions *Clairaut*'s theorem (9.90) can be written in the form:

$$r \sin A = a \sin A_0 = r_M = \text{const.} \tag{12.29}$$

or, in view of (9.91):

$$\cos \varphi^U \sin A = \sin A_0 = \cos \varphi^U_M. \tag{12.30}$$

From the last relation it follows that:

$$A_0 \pm \varphi^U_M = \frac{\pi}{2}, \tag{12.31}$$

the + sign having to be used when after the intersection with the equator there follows to the east a maximum of the geodetic line and the — sign when a minimum follows.

As *Clairaut*'s theorem holds for any rotation surface, one can write in the case of an auxiliary sphere of radius equal to unity a relation similar to (12.30):

$$\cos \varphi \sin A' = \sin A'_0 = \cos \varphi_M, \tag{12.32}$$

where A' denotes, provisionally, the azimuth of the great circle on the auxiliary sphere. Because the projection of the geodetic line on the sphere is made under the condition $\varphi = \varphi^U$, it follows that:

$$\cos \varphi^U \sin A' = \sin A'_0 = \cos \varphi^U_M, \tag{12.33}$$

and by comparison with (12.30):

$$A = A'.$$

Thus, at any of the corresponding points, located on the geodetic line and on the great circle of the auxiliary sphere respectively, the azimuths are equal.

Inasmuch as the connexion relation between the azimuths and the latitudes on the ellipsoid and those of the corresponding points on the auxiliary sphere respectively are established, it is next necessary to determine the respective relations between the lengths s and σ, on the one hand, and the longitude differences l and l' on the other. To this end one can write:

Ellipsoid	*Auxiliary sphere of radius equal to unity*
$\mathrm{d}s \cos A = M\,\mathrm{d}B$;	$\mathrm{d}\sigma \cos A = \mathrm{d}\varphi^U$;
$\mathrm{d}s \sin A = N \cos B\,\mathrm{d}L$;	$\mathrm{d}\sigma \sin A = \cos \varphi^U \mathrm{d}L'$,

whence it results:

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma} = M \frac{\mathrm{d}B}{\mathrm{d}\varphi^U} = \frac{a}{V},$$

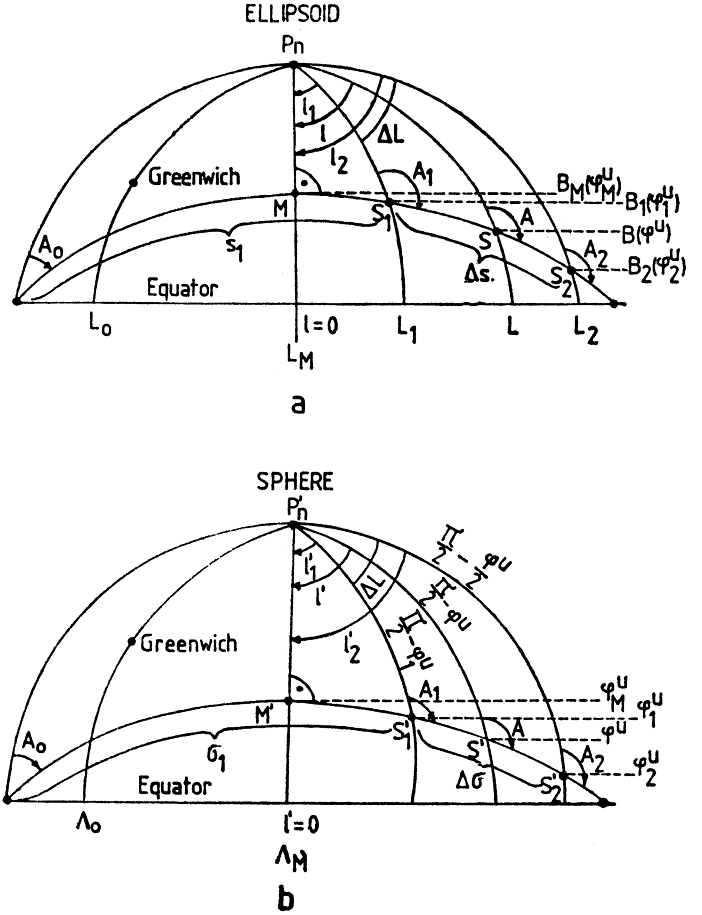


Fig. 12.2. Solving the Basic Geodetic Problems in the Case of Large Geodetic Distances

so that:

$$ds = \frac{a}{V}\, d\sigma = a\sqrt{1 - e^2 \cos^2 \varphi^U}\, d\sigma.$$

In a similar way:

$$dL = \frac{dL'}{V} = \sqrt{1 - e^2 \cos^2 \varphi^U}\, dL'.$$

These are the fundamental relations starting from which various authors have arrived, in more or less different ways, at solutions for the two basic geodetic problems. The development of the last two relations leads, in principle, to the necessity of calculating some elliptical integrals, the solutions offered by *Bessel* (1882), *Legendre, Helmert* or, more recently, *Levallois* and *Dupuis* (1952) etc. differing in the main in the specific method of calculating these integrals.

Solving in principle the basic geodetic problems is possible by means of the right-angled spherical triangle $P_n'M'S'$, applied on the real triangles $P_n'M'S_1'$ and $P_n'M'S_2'$ (Fig. 12.2) in which, according to different cases, one recognizes a number of elements necessary to the solution. One can write the following relations of spherical trigonometry:

$$
\begin{aligned}
&1)\ \cos A = -\tan\sigma\tan\varphi^U = \sin l'\sin\varphi_M^U;\\
&2)\ \cos l' = \cos\sigma\sin A = \cot\varphi_M^U\tan\varphi_M^U;\\
&3)\ \sin\sigma = -\cot\varphi_M^U\cot A = \sin l'\cos\varphi^U;\\
&4)\ \sin\varphi^U = \sin\varphi_M^U\cos\sigma = -\cot l'\cot A;\\
&5)\ \cos\varphi_M^U\cot l'\tan\sigma = \cos\varphi^U\sin A.
\end{aligned}
\tag{12.34}
$$

These equations may be combined, leading to other formulae of use in computations, as e.g.:

$$\cos\varphi^U\cos A = \cos\varphi_M^U\cot A;\ \cos\varphi^U\cos l' = \cos\varphi_M^U\cos\sigma \text{ etc.} \tag{12.35}$$

One can mention the preoccupation of the previously mentioned authors with compiling special auxiliary tables intended to facilitate the intervening laborious computations. These tend to lose their importance somewhat under conditions in which desk electronic computers are viable calculation aids, eliminating in this way intricate interpolation operations, which increase the necessary amount of calculation. This is the reason why in what follows we shall present the calculation algorithm based on *Bessel-Helmert*'s method, adapted to the utilization of such calculation means, without presenting in detail the demonstration for deriving the formulae which are used.

The direct geodetic problem:

One knows:

$$S_1(B_1, L_1),\ \Delta s,\ A_1$$

One determines:

$$S_2(B_2, L_2),\ A_2$$

(1) *One determines the reduced latitude of the point* S_1 (see (8.25)):

$$\tan\varphi_1^U = \sqrt{1-e^2}\tan B_1 \tag{12.36}$$

(2) *Computing the auxiliary elements* φ_M^U, σ_1, l_1 (see (12.34)):

$$
\begin{aligned}
&\cos\varphi_M^U = \cos\varphi_1^U\cdot\sin A_1;\ \tan\sigma_1 = -\frac{\cos A_1}{\tan\varphi_1^U};\\
&\cot l_1' = -\sin\varphi_1^U\tan A_1.
\end{aligned}
\tag{12.37}
$$

Check:

$$\sin\varphi_M^U = \frac{\sin\varphi_1^U}{\cos\sigma_1};\ \cos l_1' = \frac{\tan\varphi_1^U}{\tan\varphi_M^U} \tag{12.38}$$

(3) *Computing the geodetic-line lengths* s_1 and s_2:

$$s_1 = b\,\frac{1 + \frac{1}{4}K_1^2}{1 - K_1}\left[\sigma_1^{rad} + \left(\frac{1}{2}K_1 - \frac{3}{16}K_1^3\right)\sin 2\sigma_1 - \right.$$

$$\left. - \frac{1}{16}K_1^2 \sin 4\sigma_1 + \frac{1}{48}K_1^3 \sin 6\sigma + \cdots\right]; \qquad (12.39)$$

$$s_2 = s_1 + \Delta s,$$

where: b = minor semi-axis of the reference ellipsoid:

$$K_1 = \tan^2\frac{E}{2}\ ;\ \tan E = e' \sin\varphi_M^U \qquad (12.40)$$

(4) *Calculating the length of the great-circle arc* σ_2:

$$\sigma_2^{rad} = s_2' - \left(\frac{1}{2}K_1 - \frac{9}{32}K_1^3\right)\sin 2s_2' +$$

$$+ \frac{5}{16}K_1^2 \sin 4s_2' - \frac{29}{96}K_1^3 \sin 6s_1' + \ldots, \qquad (12.41)$$

where:

$$s_2' = \frac{s_2(1 - K_1)}{b\left(1 + \frac{1}{4}K_1^2\right)}\ . \qquad (12.42)$$

(5) *Determining the unknowns* φ_2^U, l_2', A_2 (s. (12.34)):

$$\sin\varphi_2^U = \sin\varphi_M^U \cdot \cos\sigma_2;\ \tan l_2' = \frac{\tan\sigma_2}{\cos\varphi_M^U};\ \cos A_2 = -\tan\sigma_2 \cdot \tan\varphi_2^U \qquad (12.43)$$

Check:

$$\cos\varphi_2^U = \frac{\sin\sigma_2}{\sin l_2'};\ \cos\varphi_M^U = \cos\varphi_2^U \sin A_2. \qquad (12.44)$$

(6) *Final calculations:*

— geodetic latitude of the point S_2:

$$\tan B_2 = \frac{\tan\varphi_2^U}{\sqrt{1 - e^2}}; \qquad (12.45)$$

— geodetic longitude of the point S_2:

$$L_2 = L_1 + \Delta L;$$

$$\Delta L = \Delta L' - \frac{e^2}{2}\cos\varphi_M^U\left[\left(1 + n' - \frac{1}{2}K_1 - \frac{1}{4}K_1^2 + \ldots\right)\Delta\sigma - \right.$$

$$\left. - \frac{1}{2}K_1 \cos 2\sigma \sin\Delta\sigma + \frac{1}{8}K_1^2 \cos 4\sigma \sin 2\Delta\sigma + \ldots\right], \qquad (12.46)$$

where:

$$\Delta L' = l_2' - l_1'; \; n' = \frac{a-b}{a+b}; \quad \sigma = \frac{1}{2}(\sigma_1 + \sigma_2). \tag{12.47}$$

The inverse geodetic azimuth A_2 is calculated using (12.43).

The inverse geodetic problem:
One knows:

$$S_1(B_1, L_1) \text{ and } S_2(B_2, L_2).$$

One determines: $\Delta s, A_1$ and A_2.

(1) *Calculating the reduced latitudes* φ_1^U *and* φ_2^U.

(2) *In order to deduce the auxiliary elements* φ_M^U, σ_1 *and* σ_2 *Helmert utilizes an iterative procedure, which was subsequently improved by other authors:*

$$\Delta L' \approx \Delta L;$$

$$\cot \frac{l_1' + l_2'}{2} = -\tan \frac{\Delta L'}{2} \frac{\sin(\varphi_1^U + \varphi_2^U)}{\sin(\varphi_2^U - \varphi_1^U)} \tag{12.48}$$

whence follows in a first approximation, l_1' and l_2'. The reduced latitude φ_M^U may be calculated in two ways:

$$\tan \varphi_M^U = \frac{\tan \varphi_1^U}{\cos l_1'} = \frac{\tan \varphi_2^U}{\cos l_2'}. \tag{12.49}$$

The great circle elements σ_1 and σ_2 are calculated by means of the following relations (12.34, d):

$$\cos \sigma_1 = \frac{\sin \varphi_1^U}{\sin \varphi_M^U}; \; \cos \sigma_2 = \frac{\sin \varphi_2^U}{\sin \varphi_M^U}. \tag{12.50}$$

The following iterations utilize (12.46) for calculating the difference of spherical longitudes $\Delta L'$ (for the first iteration there will be used only the first corrective term, for the second iteration the next corrective term as well etc.) and for the rest of the necessary elements one makes use of the formulae (12.48) — (12.50).

(3) *Calculating the geodetic azimuths* A_1 *and* A_2*:*

$$\sin A_1 = \frac{\cos \varphi_M^U}{\cos \varphi_1^U}; \; \sin A_2 = \frac{\cos \varphi_M^U}{\cos \varphi_2^U}, \tag{12.51}$$

with the following checks:

$$\cot A_1 = -\sin \varphi_1^U \cdot \tan l_1' = -\tan \varphi_M^U \sin \sigma_1; \tag{12.52}$$

$$\cot A_2 = -\sin \varphi_2^U \tan l_2' = -\tan \varphi_M^U \sin \sigma_2. \tag{12.53}$$

(4) *Calculating the distance* Δs:

$$\Delta s = b\,\frac{1+\dfrac{1}{4}K_1^2}{1-K_1}\left[\Delta\sigma + \left(K_1 - \frac{3}{8}K_1^3\right)\cos 2\sigma \sin \Delta\sigma - \right.$$

$$\left. - \frac{1}{8}K_1^2 \cos 4\sigma \sin 2\Delta\sigma + \frac{1}{24}K_1^3 \cos 6\sigma \sin 3\Delta\sigma + \ldots\right]. \qquad (12.54)$$

Calculation examples for solving the basic geodetic problems by means of *Bessel-Helmert*'s improved algorithm, in the case of large geodetic distances, are presented in Tables 12.6 and 12.7. It was worked with a *Hewlett-Packard* desk electronic computer which allowed the calculation time to be substantially reduced and a concentration of the intermediate results by comparison with the solutions based on the utilization of logarithms or of auxiliary tables. This showed up the high accuracy of the formulae contained in *Bessel-Helmert*'s algorithm: for a geodetic line of 10,000,000.000 m in length, the calculation error in azimuth was of $-0''.00002$ and in distance of -2 mm.

Table 12.6. *Example of Solving the Direct Geodetic Problem by means of Bessel-Helmert's Improved Algorithm in the Case of Large Geodetic Distances*

Initial elements	Initial-point designation S_1 $B^1 = 45°$; $L^1 = 27°$; $A^1 = 135°$; $\Delta s = 10\,000$ km	
Auxiliary elements	$1 - e^2 = 0.996\,647\,67013$ $\varphi_M^U = 59°.94449102$ $l_1' = 54°78102215$ $E = 4°.064\,089\,780$ $s_1 = 3\,931\,315.3415$ m $s_2' = 125°.407\,8318$	$\varphi_2^U = 44°903\,801\,67$ $\sigma^1 = 35°.355\,13716$ The check (11.38) is fulfilled $K_1 = 1.258\,883\,715 \cdot 10^{-3}$ $s_2 = 13\,931\,315.3415$ m $\sigma_2 = 125°.441\,9109$
Interme-diate un-knowns	$\varphi_2^U = -30°.126\,267\,28$ $A_2 = 144°.6157569 = 144°36'56''.66436$ The check (11.44) is fulfilled $\sigma = 80°.398\,524\,05$ $n' = 0.001\,678\,97\ 181$ $\Delta L = 54°.687\,859\,17$	$l_1' = 109°.620\,0969$ $\Delta\sigma = 90°.086\,773\,74$ $\Delta L' = 54.839\,074\,75$
Final results	$B_2 = -30°.2098\,5833 = -30°12'35''.489\,988$ $L_2 = 81°.687\,859\,17 = 81°41'16''.293\,012$ $A_2 = 144°.615\,7569 = 144°36'56''.66436$	

Table 12.7. *Example of Solving the Inverse Geodetic Problem by means of Bessel-Helmert's Improved Algorithm in the Case of Large Geodetic Distances*

Initial elements	$B_1 =$ 45° $B_2 = -30°.209\ 858\ 33$ $= -30°12'35''.489988$		$L_1 = 27°$ $L_2 = 81°.687\ 859\ 17 =$ $= 81°41'16''.293012$	
Auxiliary elements	$\varphi_1^U = 44°.903\ 801\ 67$ $n' = 0.001\ 678\ 979\ 181$		$\varphi_2^U = -30°.126\ 267\ 28$	
Iteration elements	1	2	3	4
$\Delta L'$	54°.687 859 17	54°.838 704 09	54°.839 0737	54°.839 074 75
$l'_1 + l'_2$	164°.450 8251	164°.401 2409	164°.401 1194	164°.401 1191
l'_1	54°.881 482 95	54°.781 2684	54°.781 022 85	54°.781 022 15
φ_M^U	60°.006 259 43	59°.944 642 23	59°.944 491 43	59°.944 491 03
σ_1	35°405 404 97	35°.355 260 51	35°.355 1375	35°.355 137 15
σ_2	125°.416 5146	125°.441 8485	125°.441 9107	125°.44 19109
K_1		$1.2588\ 875\ 47 \cdot 10^{-3}$	$1.258\ 883722 \cdot 10^{-3}$	$1.258\ 883\ 715 \cdot 10^{-3}$
$\Delta\sigma$	90°.011 109 63	90°.086587 99	90°.086 7732	90°.086 773 75
σ	80°.410 9598	80°.398 5545	80°.398 5241	80°.398 524 05
Final results	$A_1 = 135°,\ 00000006 = 135°00'00'',\ 0006$ $A_2 = 144°.6157\ 569 = 144°36'56''.66436$ $\Delta s = 9\ 999\ 999.998$ m			

12.4 Differential Formulae

Under the name of *differential formulae* one understands in Ellipsoidal Geodesy the formulae by which one expresses the variation of the geodetic coordinates of a point S_i in terms of the quantities dB_i and dL_i, as well as of the azimuth A_i of a geodetic line, passing through this point, in terms of the quantity dA as a consequence of modifications having appeared in the initial data in the support network (modification of the geodetic coordinates of the initial point S_1, of the initial geodetic azimuth A_1 and of the initial side s_1 by the amounts dB_1, dL_1, dA_1 and ds_1) or of the parameters of the reference ellipsoid a and f by the amounts da and df respectively. Several actual methods for solving such problems, due to *F. W. Bessel* (1826), *F. R. Helmert* (1886) *W. K. Khristov* (1957), *F. N. Krasovski* (1953) etc. are known, which differ amongst themselves in the accepted degree of approximation and, therefore, in their applicability.

Inasmuch as the coordinates B_i and L_i of the point S_i and the azimuth A_i are functions of the initial data of the support network and the parameters of the reference ellipsoid:

$$B_i = B_1 + \Delta B;\ L_i = L_1 + \Delta L;\ A_i = A_1 + \Delta A \pm 200^g,$$

the differential formulae can be expressed in a general form as total differentials with respect to the previously mentioned variables:

$$\mathrm{d}B_i = \mathrm{d}B_1 + \frac{\partial \Delta B}{\partial B_1}\mathrm{d}B_1 + s_1\frac{\partial \Delta B}{\partial s_1}\frac{\mathrm{d}s_1}{s_1} + \frac{\partial \Delta B}{\partial A_1}\mathrm{d}A_1 + a\frac{\partial \Delta B}{\partial a}\frac{\mathrm{d}a}{a} + \frac{\partial \Delta B}{\partial f}\mathrm{d}f;$$

$$\mathrm{d}L_i = \mathrm{d}L_1 + \frac{\partial \Delta L}{\partial B_1}\mathrm{d}B_1 + s_1\frac{\partial \Delta L}{\partial s_1}\frac{\mathrm{d}s_1}{s_1} + \frac{\partial \Delta L}{\partial A_1}\mathrm{d}A_1 + a\frac{\partial \Delta L}{\partial a}\frac{\mathrm{d}a}{a} + \frac{\partial \Delta L}{\partial f}\mathrm{d}f; \quad (12.55)$$

$$\mathrm{d}A_i = \mathrm{d}A_1 + \frac{\partial \Delta A}{\partial B_1}\mathrm{d}B_1 + s_1\frac{\partial \Delta A}{\partial s_1}\frac{\mathrm{d}s_1}{s_1} + \frac{\partial \Delta A}{\partial A_1}\mathrm{d}A_1 + a\frac{\partial \Delta A}{\partial a}\frac{\mathrm{d}a}{a} + \frac{\partial \Delta A}{\partial f}\mathrm{d}f.$$

For the sake of simplifying the writing of subsequent developments it is useful to rewrite the preceding relations in the form:

$$\mathrm{d}B_i = \quad + p_1\,\mathrm{d}B_1 + p_3\,\mathrm{d}s_1 + p_4\,\mathrm{d}A_1 + p_5\frac{\mathrm{d}a}{a} + p_6\,\mathrm{d}f;$$

$$\cos B_i\,\mathrm{d}L_i = \mathrm{d}L_1 + q_1\,\mathrm{d}B_1 + q_3\,\mathrm{d}s_1 + q_4\,\mathrm{d}A_1 + q_5\frac{\mathrm{d}a}{a} + q_6\,\mathrm{d}f; \quad (12.56)$$

$$\cot B_i\,\mathrm{d}A_i = \quad + r_1\,\mathrm{d}B_1 + r_3\,\mathrm{d}s_1 + r_4\,\mathrm{d}A_1 + r_5\frac{\mathrm{d}a}{a} + r_6\mathrm{d}f.$$

Specializing the previous general relations yields **differential formulae of two types** viz.:

(1) When one takes into consideration only the modifications in the initial data in the support network ($\mathrm{d}B_1$, $\mathrm{d}L_1$, $\mathrm{d}A_1$ and $\mathrm{d}s_1$) one gets *differential formulae of the first category.*

(2) When one considers only modifications in the parameters of the reference ellipsoid ($\mathrm{d}f$ and $\mathrm{d}a$) one obtains *differential formulae of the second category.*

The partial derivatives required in (12.55) may be deduced in several ways. As an example we present in what follows the determination of the variation of the latitude of the point S_i with respect to the modification of

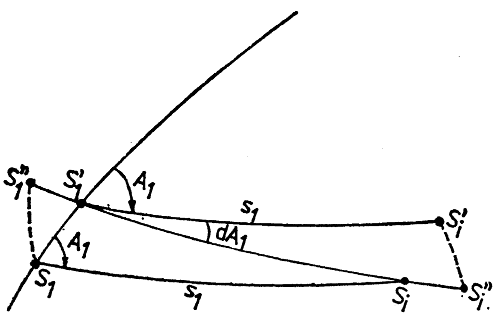


Fig. 12.3. Variation of the Geodetic Latitude as a Function of the Modification of the Initial Data in the Support Geodetic Network

the latitude of the point S_1, viz. of the term p_1, on the assumption that the other possible modifications are zero.

In Fig. 12.3 the point S_1 is subject to a variation in latitude denoted by dB_1, arriving at S_1'. One rotates the geodetic line round the point S_i until it passes through the point S_1', so that the point S_1 arrives at S_1''. According to assumption, one considers that the length of the geodetic line is not modified so that to its prolongation is added the arc $\widehat{S_iS_i''} = \widehat{S_1''S_1'}$. Inasmuch as by hypothesis one also disregards the modification of the geodetic azimuth, it is necessary to rotate the geodetic line $S_1'S_i''$ round the point S_1' until one again obtains the azimuth A_1 at this point. Thus, the final position of the geodetic line S_1S_i due to the modification of the latitude of the initial point S_1 will be $S_1'S_i'$. In this way the difference between the latitudes of the points S_i' and S_i is the influence sought. From Fig. 12.3 it follows that:

$$S_iS_i'' = S_1'S_1'' = M_1\,dB_1 \cos A_1;\ S_1S_1'' = M_1\,dB_1 \sin A_1.$$

The latitude difference between the points S_i and S_i'' can be calculated with the formula known from the direct geodetic problem (in the present case we will use only the first term):

$$-\frac{M_1\,dB_1 \cos A_1}{M_i}\cos A_i, \tag{12.57}$$

and between the points S_i'' and S_i':

$$-\frac{M_1\,dB_1 \sin A_1}{M_i}\sin A\ , \tag{12.58}$$

so that:

$$p_1 = -\frac{M_1}{M_i}\,dB_1\,(\cos A_1 \cos A_i + \sin A_1 \sin A_i). \tag{12.59}$$

In an analogous way one may also deduce the other partial derivatives appearing in the relations (12.55)—(12.56). This kind of solution is found in the works of N. *F. Helmert* (1886), *F. N. Krasovski* (1942) etc. The resulting formulae are to be applied from point to point, starting from the initial point of the support geodetic network. **The field of utilization is relatively limited** ($s < 200$ km):

$$\begin{aligned}
p_1 &= -\frac{M_1}{M_i}\left[\cos A_1 \cos A_i + \left(\frac{dm}{ds}\right)_i \sin A_1 \sin A_i\right];\\
p_3 &= -\rho''\frac{\cos A_i}{M_i};\\
p_4 &= \frac{m \sin A_i}{M_i};\\
p_5 &= \rho''\frac{s_1}{a}\frac{W_i^3}{1-e^2}\cos A_i;
\end{aligned} \tag{12.60}$$

$$p_6 = 2\Delta B - \left(3\ \Delta B - \frac{\Delta L^2}{2\rho''} \sin\ B_1 \cos B_1\right) \sin^2\ B_1;$$

$$q_1 = \frac{M_1}{N_i} \sin\ \Delta L \sin\ B_i;$$

$$q_3 = -\ \rho'' \frac{\sin\ A_i}{N_i};$$

$$q_4 = W_i \sin\ \Delta L \cos\ B_1 \cot\ A_i;$$

$$q_5 = \rho'' \frac{s_1}{a} W_i \sin\ A_i;$$

$$q_6 = -\ \Delta L \sin^2\ B_1 \cos\ B_1; \tag{12.60}$$

$$r_1 = \frac{\sin\ \Delta L}{\sin\ B_2} (1 - e^2 \sin^2\ B_1 \cos^2\ B_1);$$

$$r_3 = q_3;$$

$$r_4 = q_4 + \cos \frac{s}{a} \cot\ B_i;$$

$$r_5 = \rho'' \frac{s_1}{a} W_i \sin\ A_i;$$

$$r_6 = q_6 + \frac{\Delta B\ \Delta L}{\rho''} \frac{\cos^4\ B_1}{\sin\ B_1}.$$

In the preceding formulae m denoted the reduced length of the geodetic line which can be determined using (9.114) and:

$$\left(\frac{dm}{ds}\right)_i = -\frac{p_2 \cos\ A_i}{p_1 \cos\ A_1} + m \frac{\tan\ B_1}{N_1 \cos A_1}. \tag{12.61}$$

If one considers small geodetic distances ($s < 60$ km) **the formulae** (12.60) **can be further simplified and may be applied in the lst-order triangulation**:

$$p_1 = \frac{M_1}{M_i} \cos\ \Delta L;$$

$$p_3 = -\ \rho'' \frac{\cos\ A_i}{M_i};$$

$$p_4 = -\ \frac{a}{M_i} \sin\ \Delta L \cos\ B_1; \tag{12.62}$$

$$p_5 = \rho'' \frac{\cos\ A_i}{M_i} s_1;$$

$$
\begin{aligned}
p_6 &= 2\Delta B \cos^2 B_1 + p_5 \sin^2 B_1; \\
q_1 &= \frac{M_1}{N_2} \sin B_2 \sin \Delta L; \\
q_3 &= -\rho'' \frac{\sin A_i}{N_2}; \\
q_4 &= -p_5; \\
q_5 &= -\cos B_1; \\
q_6 &= q_5 \sin^2 B_1; \\
r_1 &= \frac{\sin \Delta L}{\sin B_2}(1 - e^2 \sin^2 B_1 \cos^2 B_1); \\
r_3 &= q_3; \\
r_4 &= -\frac{\cos \Delta L \cos B_1}{\sin B_2} - \frac{e^2 \Delta B}{\rho''}\cos^2 B; \\
r_5 &= q_5; \\
r_6 &= q_6 + \frac{\Delta B\, \Delta L}{\rho''}\,\frac{\cos^4 B}{\sin B}.
\end{aligned}
\tag{12.62}
$$

For large domains, in the case of geodetic lines of up to 1,000 km in length, one can utilize the formulae of *W. Khristov* (1942) obtained on the basis of the general formulae (12.55). The necessary partial derivatives were calculated by using the series expansions (12.5) and the coefficients (12.6):

$$
\begin{aligned}
\mathrm{d}B_i = &\left[1 - 3t(\eta^2 - \eta^4)\,\Delta B - \frac{3}{2}(\eta^2 - t^2\eta^2)\,\Delta B^2 - \frac{1}{2}\cos^2 B(1 + t^2 + \eta^2)\Delta L^2\right]\mathrm{d}B_1 + \\
&+\left[\Delta B - \frac{3}{2}\,t\eta^2\Delta B^2 - \frac{1}{2}\cos^2 Bt(1+\eta^2)\Delta L^2 - \frac{1}{3}\cos^2 B\Delta B\Delta L^2\right]\frac{\mathrm{d}s}{s} + \\
&+\left[-\cos B(1+\eta^2)\,\Delta L + 3\cos Bt\eta^2\,\Delta B\,\Delta L + \frac{1}{6}\cos^3 B(1+t^2)\,\Delta L^3\right]\mathrm{d}A_1 + \\
&+\left[-\Delta B + \frac{3}{2}\,t\eta^2\Delta B^2 + \frac{1}{2}\cos^2 Bt(1+\eta^2)\,\Delta L^2 + \frac{1}{3}\cos^2 B\Delta B\Delta L^2\right]\frac{\mathrm{d}a}{a} + \\
&+\Bigg[\cos^2 B\bigg(2 - t^2 + \eta^2 + \frac{7}{2}t^2\eta^2 - \frac{1}{2}t^4\eta^2 - \frac{1}{4}\eta^4 - \frac{15}{8}t^2\eta^4 + \\
&+\frac{3}{2}t^4\eta^4 + \frac{1}{8}t^6\eta^4\bigg)\Delta B - \frac{3}{2}\cos^2 Bt(2 - 3\eta^2 + 2t^2\eta^2)\Delta B^2 +
\end{aligned}
\tag{12.63}
$$

$$+\frac{1}{2}\cos^4 Bt\left(t^2+\frac{1}{2}t^2\eta^2+\frac{1}{2}t^4\eta^2\right)\Delta L^2+\frac{1}{3}\cos^4 B(1-5t^2)\,\Delta B\,\Delta L^2\Bigg]\,\mathrm{d}f.$$

$$\begin{aligned}
\mathrm{d}L_i = \mathrm{d}L_1 &+ \Big[t(1-\eta^2+\eta^4)\,\Delta L + (1+t^2-\eta^2-2t^2\eta^2)\,\Delta B\,\Delta L + \\
&+ t(1+t^2)\,\Delta B^2\,\Delta L - \frac{1}{6}\cos^2 Bt(1+t^2)\Delta L^3\Big]\,\Delta B_1 + \\
&+ \Big[\Delta L + t(1-\eta^2)\,\Delta B\,\Delta L + \frac{1}{3}(2+3t^2)\,\Delta B^2\,\Delta L - \frac{1}{6}\cos^2 Bt^2\,\Delta L^3\Big]\frac{\mathrm{d}s}{s} + \\
&+ \Big[\frac{1}{\cos B}(1-\eta^2+\eta^4)\,\Delta B + \frac{1}{\cos B}t\left(1-\frac{1}{2}\eta^2\right)\Delta B^2 - \frac{1}{2}\cos Bt\,\Delta L^2 + \\
&+ \frac{1}{3\cos B}(1+3t^2)\,\Delta B^3 - \frac{1}{2}\cos B\,(1+t^2)\,\Delta B\,\Delta L^2\Big]\,\mathrm{d}A_1 + \qquad (12.64)\\
&+ \Big[-\Delta L - t(1-\eta^2)\,\Delta B\,\Delta L - \frac{1}{3}(2+3t^2)\,\Delta B^2\,\Delta L + \frac{1}{6}\cos^2 Bt^2\,\Delta L^3\Big]\frac{\mathrm{d}a}{a} + \\
&+ \Big[-\cos^2 B(t^2 - \frac{1}{2}t^2\eta^2 + \frac{1}{2}t^4\eta^2 + \frac{3}{8}t^2\eta^4 - \frac{3}{4}t^4\eta^4 - \frac{1}{8}t^6\eta^4\,\Delta L - \\
&- \cos^2 Bt\left(t^2 - \frac{3}{2}t^2\eta^2 + \frac{1}{2}t^4\eta^2\right)\Delta B\,\Delta L - \\
&- \frac{1}{3}\cos^2 B(2t^2+3t^4)\,\Delta B^2\,\Delta L + \frac{1}{6}\cos^4 Bt^4\,\Delta L^3\Big]\,\mathrm{d}f.
\end{aligned}$$

$$\begin{aligned}
\Delta A_i = &\Big[\cos B(1+t^2-t^2\eta^2+t^2\eta^4)\,\Delta L + \cos Bt(1+t^2-3\eta^2-2t^2\eta^2)\,\Delta B\,\Delta L + \\
&+ \frac{1}{2}\cos B(1+3t^2+2t^4)\,\Delta B^2\,\Delta L - \frac{1}{6}\cos^3 B(1+2t^2+t^4)\,\Delta L^3\Big]\,\mathrm{d}B_1 + \\
&+ \Big[\cos Bt\Delta L + \cos B(1+t^2-t^2\eta^2)\,\Delta B\,\Delta L + \frac{1}{6}\cos Bt(7+6t^2)\,\Delta B^2\,\Delta L - \\
&- \frac{1}{6}\cos^3 Bt^3\Delta L^3\Big]\frac{\mathrm{d}s}{s} + \Big[1 + t(1-\eta^2+\eta^4)\,\Delta B + \qquad (12.65)\\
&+ \frac{1}{2}(1+2t^2-\eta^2-t^2\eta^2)\,\Delta B^2 - \frac{1}{2}\cos^2 B(1+t^2+\eta^2)\,\Delta L^2 + \\
&+ \frac{1}{6}t(5+6t^2)\,\Delta B^3 - \frac{1}{2}\cos^2 Bt(1+t^2)\,\Delta B\,\Delta L^2\Big]\,\mathrm{d}A_1 + \\
&+ \Big[-\cos Bt\,\Delta L - \cos B(1+t^2-t^2\eta^2)\,\Delta B\,\Delta L -
\end{aligned}$$

$$- \frac{1}{6} \cos Bt(7 + 6t^2)\, \Delta B^2\, \Delta L + \frac{1}{6} \cos^3 Bt^3\, \Delta L^3 \Big] \frac{d_s}{a} +$$

$$+ \Big[- \cos^3 Bt \Big(t^2 - \frac{1}{2} t^2\eta^2 + \frac{1}{2} t^4\eta^2 + \frac{3}{8} t^2\eta^4 - \frac{3}{4} t^4\eta^4 - \frac{1}{8} t^6\eta^4\, \Delta L +$$

$$+ \cos^3 B \Big(1 - t^2 + t^4 + \frac{1}{2} \eta^2 + 2t^2\eta^2 + t^4\eta^2 - \frac{1}{2} t^6\eta^2 \Big) \Delta B \Delta L -$$

$$\frac{1}{6} \cos^3 Bt(6 + 7t^2 + 6t^4)\, \Delta B^2\, \Delta L + \frac{1}{6} \cos^5 Bt(3 + 7t^4)\, \Delta L^3 \Big] \mathrm{d}f.$$

The differential formulae are also useful for framing the basic equations intervening in the surface method (*Helmert*'s method) of determining the parameters of the reference ellipsoid (Fifth part of the treatise) and are therefore of primary scientific importance.

13
The Notion of Performing Astro-Geodetic Triangulation

13.1 General Considerations

Among the methods used in Geodesy for geometrical determinations of the Earth's figure (Fifth Part of the treatise) are the astro-geodetic methods, among which the most frequently utilized has been *the triangulaton method*, sometimes also called *conventional geometrical geodesy.*

Triangulation is a method of determining the position on the Earth's surface of the points which form a system of triangles (chains, compact network). The position of these points is uniquely determined by the geodetic coordinates B and L of the projections of the points from the Earth's physical surface onto the reference ellipsoid and by their altitudes H. Frequently, the altitudes H of the geodetic points refer to other reference surfaces (geoid, quasi-geoid), the reference ellipsoid being used to this end only in the case of Three-Dimensional Geodesy (Fourth part of the book).

A triangulation network can be defined by the following characteristic elements:

(1) *The form of the network* which, irrespective of the fundamental geometrical figure being used (triangle, quadrilateral, central system) is eventually determined by triangles with measured angles.

(2) *The scale of the network,* which is established by direct distance measurements or distances calculated from coordinates of old geodetic points (considered as fixed).

(3) *The orientation and position of the network,* which are established by determinations of geodetic astronomy or, by calculation, by using the coordinates of old geodetic points.

Before the utilization for geodetic purposes of the Earth's artificial satellites, a two-fold function was ascribed to astro-geodetic triangulation *(Wolf* 1977*):*

(1) Establishment of a detailed field of points with fixed position on the Earth's physical surface as support for subsequent technical engineering and mapping operations, aiming at achieving, by means of the catalogue of the definitive coordinates of these points, a putting together of reality on a 1: 1 scale out of which one should afterwards be able to get all the distances, angles and surfaces with the established, necessary accuracy.

(2) An auxiliary method for arc measurements and the determination of the Earth's form and size.

As to the second function, the parameters of the Earth's figure can now be determined much more accurately by observations on the Earth's artificial satellites (Chapter. 19), conventional triangulation being used for establishing the scale of the spatial geodetic network, such as e.g., in *Europe,* the traverses *Tromsö—Catania* (Section 21.3) and *Malvren—Graz,* as well as for obtaining the continental systems of deflections of the vertical by comparing the astronomical coordinates with those resulting from triangulation.

A general view of the procedures used within the framework of astro-geodetic triangulation may be taken from Fig. 13.1. One notes that the final

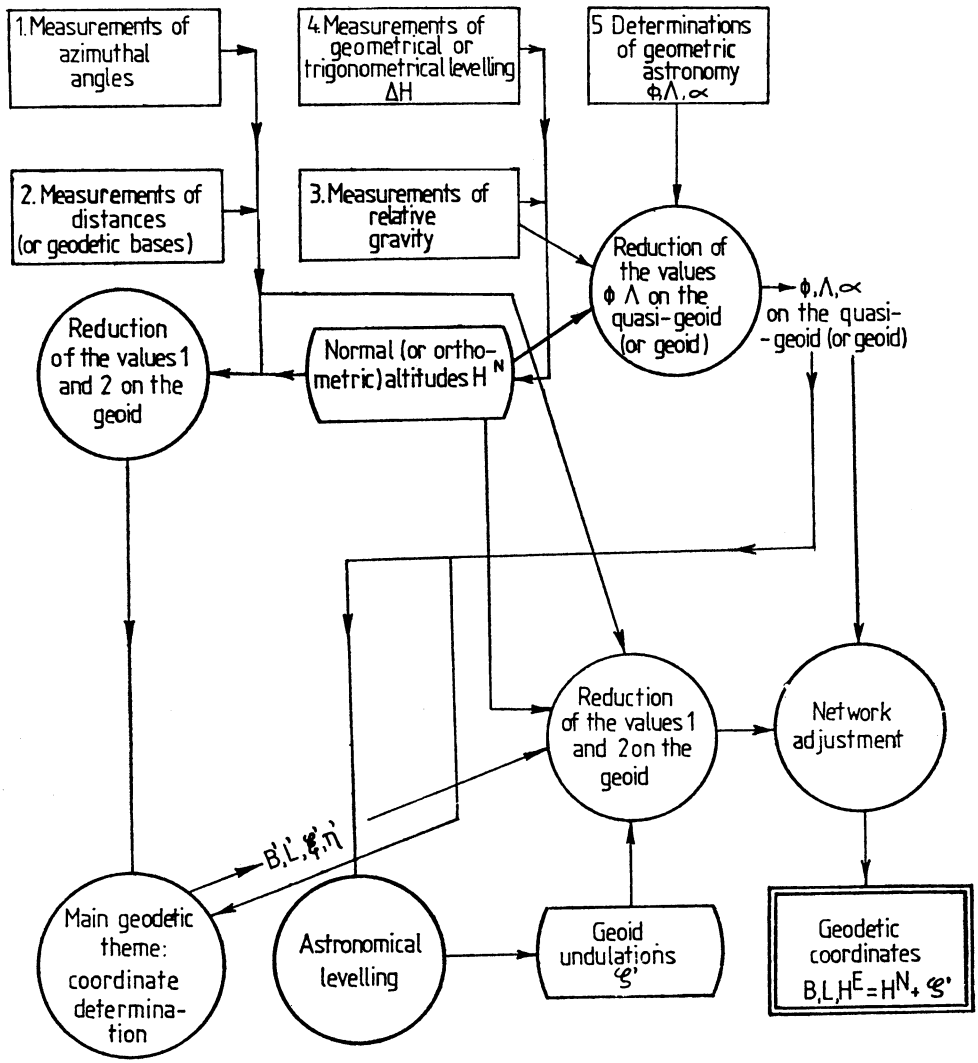


Fig. 13.1. Conventional Triangulation

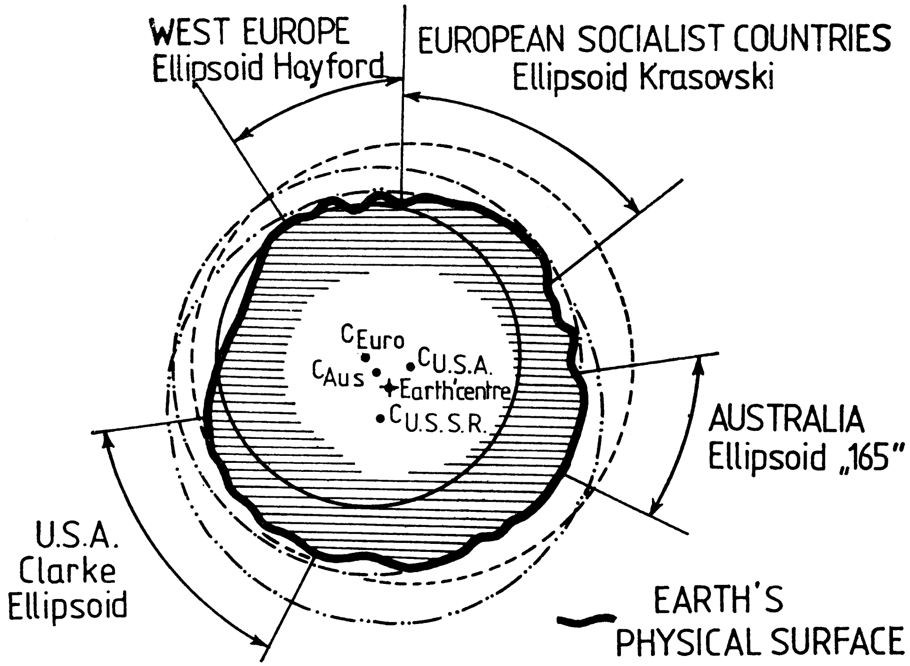


Fig. 13.2. Reference Surfaces for Various Triangulations

triangulation product is finally represented by the coordinates B, L, H of the points of the regional or national network, coordinates referring to a certain reference ellipsoid. Generally, for the various triangulations so far carried out, this ellipsoid differs from one state to another or, more precisely, from one group of states to another (Fig. 13.2; *Torge* 1973 a).

Principle of the triangulation method. Assuming that the triangulation network is projected onto the reference ellipsoid (Fig. 13.3) in the sense that

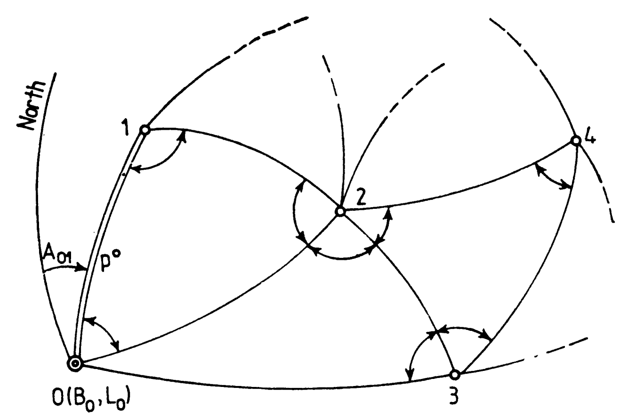


Fig. 13.3. Measured Elements of the Triangulation

all the measured elements[1] have been reduced on the reference surface (Chapter 10) and that these are not affected by measurement errors, the computations for determining the position of the geodetic points on the ellipsoid are carried out in the following order:

— solving the geodetic triangles on the reference ellipsoid (Chapter 11);

— successively computing the geodetic coordinates on the reference ellipsoid (Chapter 12).

Since, in all practical cases, the measured elements are affected by measurement errors, the determination mentioned as to be carried out in principle is always accompanied by adjustment calculations based on the least squares method, which will be presented in Chapter 15.

In the last three decades another idea concerning the principle of constructing the 1st-order national astro-geodetic network seems to have interested the specialists, viz. the solution of the derived networks given by *E. Regöczi* (1952). This idea may be sketched out as follows (Fig. 13.4):

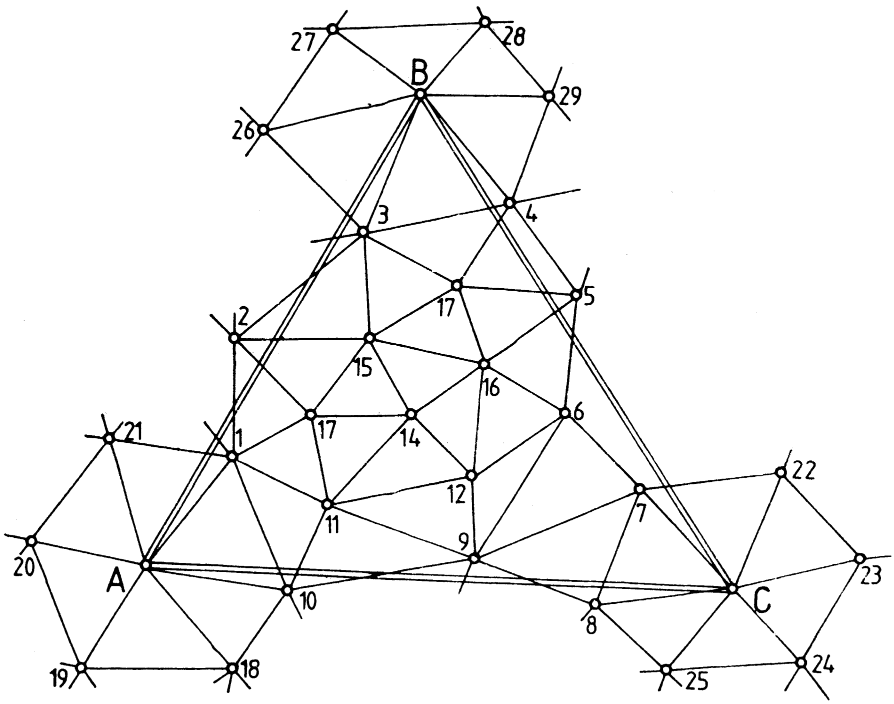


Fig. 13.4. Principle of Constructing the Derived Network;
A, *B*, *C* — dominant points; 1; 2;, ..., ;10 — 3rd-order connexion points; 11; 12; .... ; 17 — interior 3rd-order connexion points; 18; 19;...; 29 — points of the central systems at the vertices of the derived triangle

[1] As measured elements must be understood: the length of any side (s_0) in the network of ellipsoidal triangles, the horizontal angles of these triangles (0, 1, 2, 3, ...), the azimuth A_{01} and the geodetic coordinates B_0 and L_0 of one of the ends of the side s_0; A_{01}, B_0 and L_0 result from the reduction on the ellipsoid of the astronomical coordinates φ_0, Λ_0, and of the azimuth α_{01} as determined astronomically.

One builds up a homogeneous network, as well framed as possible, having side lengths of 7—8 km. In the author's conception and generally in that of the Hungarian geodetic school, this network is a simple triangle network, without diagonals and superimposed sides and without side measurements and determinations of geodetic astronomy [1]. This network, which in what follows will be called the *direct network* (corresponding to the 3rd-order network in the classical conception) is the only one in which the angles (or directions) are directly measured, in which, consequently, one has to do with actual measurements and which is connected with the lst-order chains — also immediately measured.

From among the points of the direct network one chooses a series of points — called *dominant points* — in such a manner that these should form between them nearly equilateral triangles with sides of about 30 km, thus resulting in the so-called *derived network* [2] (corresponding to the lst-order network in the classical conception). The direct network is divided into groups of triangles inside every derived triangle, so that the borderline between groups should generally follow the track of the derived sides. For a determination corresponding to the vertices of the derived triangles, which are found by means of functions of the direct-network measurements, these are located at the centre of central systems of triangles in the direct network. Thus, inside the derived triangles are contained a number, say 14—25, triangles of the direct network (Fig. 13.4).

Afterwards one goes through the following calculation stages:

f1) *One adjusts separately the direct network within the framework of the sections bounded by the derived triangles.* This adjustment is carried out as a free network, after which one obtains the coordinates of the points of the derived triangulation (lst-order one). By solving the inverse geodetic problem one calculates the orientations (or geodetic azimuths) of the derived-triangulation sides, with the aid of which one obtains, by subtraction, the angles of the derived triangles.

(2) *One adjusts the derived network, taking into account all the conditions following from measurements or computations.*

As final results one gets the definitive coordinates of the lst-order points.

This manner of conceiving the national geodetic triangulation differs from the conventional conception (as a rule with four successive triangulation orders I—IV) in the following ways:

1) In the conventional conception, starting from higher to lower, every point of higher order is also included in the lower-order network, so that the higher-order network serves as support for the subsequent networks (lower-order ones) which, in turn, being directly measured, are affected both by the influence of their own measurement errors and by that of the errors in the networks on which they are leaning.

In the case of the derived network, the dominant points are deduced by computation from the direct network, somehow from lower to higher, a fact

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[1] In the conception of the specialists of the *German Democratic Republic, Polish People's Republic* and *Soviet Union* side measurements and determinations of geodetic astronomy can also be carried out.

[2] Also called the *fictitious network.*

which has to be taken into account, particularly in estimating the accuracy of the elements determined in this network.

2) The measurement processing within the framework of the conventional conception is made separately for each triangulation order, whereas in the case of the derived triangulation, the direct network undergoes two processings: one as a free network and a second one as a network depending on the derived network which is its own product.

3) The dependence of the values of the elements determined from the derived network represents an aspect which must be taken into acccount, whereas with the conventional conception this can for the most part be disregarded.

By comparison with the conventional conception of achieving the directly measured astro-geodetic triangulation, **the method of the derived network offers important advantages,** viz.:

1) *It reduces the great consumption of time and materials and, implicitly, the cost of the field operations.*

2) *The measurements carried out in the direct network with the same type of instrument, by the same method and with the same weight are more accurate,* because, as the sights are much shorter (7—8 km as compared with 30 km), the influence of the lateral refraction is in this case smaller in absolute value and sufficiently variable in sign for it to be considered a random error.

3) *A much greater operativity.*

The above-mentioned advantages dictate use of the method only in the case when the accuracy in measuring the angles (or directions) in the direct network is sufficiently high, so that for the derived network standard errors smaller than, or at most close to, those accepted for the lst-order triangulation in the conventional conception are obtained; it is also necessary to utilize side measurements and determinations of geodetic astronomy (e.g. *Dragomir* 1975).

13.2 Error Propagation in the Astro-Geodetic Triangulation

A study of the problem of error propagation in triangulation emphasizes the influence of the form of the triangulation network on some errors characteristic to the latter, such as the standard deviation of any side of the network, the standard deviation of the azimuth of this side, the longitudinal and the transverse errors of a triangulation chain etc.

Concerning the optimum configuration of the triangulation network, various answers have been given, which have not yet been unanimously accepted by specialists, beginning with the doctoral thesis of *F. R. Helmert* (1868) and continuing with numerous other contributions (*Krasovski* 1953—1956, *Provorov* 1956, *Sigl* 1956, *Wolf* 1970, *Schädlich* 1971 etc.). One fact is, however, unanimously accepted, viz. that the probable error propagation in a triangulation network can only be determined concomitantly with its adjustment, calculating the standard deviation of every adjusted element in the network, after which one can take steps to improve the network configuration by carrying out new measurements and a new adjustment etc. In fact, the present functioning of the high-precision astro-geodetic networks is based on just such a cycle of successive measurements and analyses. For immediate practical purposes, the problem of the optimum configuration of the triangulation network may be reduced to the a priori possibility of establishing the characteristic errors of a network consisting of triangulation chains, since, as has been shown (*Sigl* 1956), the transmission errors and the accuracy of the position of the points of this network differ slightly from those resulting from the compact triangulation network.

Let us continue by explaining what is understood by an *optimum configuration*. Any additional measurement introduced in the triangulation network influences both the external aspect of the network and the final results of its adjustment. Consequently, only a certain configuration of the network, with a certain number of measurements laid out in a certain manner, facilitates an optimum error distribution. Since the network adjustment on the basis of the least squares method represents a typical optimization problem (linear programming), the optimum configuration of the triangulation network can be introduced in this process by adding supplementary conditions of minimum (or maximum) corresponding to the new technical requirements (e.g. *Wolf* 1970): optimum length of the polygons forming a network consisting of triangulation chains; optimum diagonal connexions over several elementary figures in proportion to the increase of the standard deviation of the weighting unit; optimum distribution of the astronomical determinations and of the distance measurements in the network etc.) and to the economic ones (e.g.: minimum volume of field operations and computations; minimum expanses etc.). A solution of such complexity, i.e. the construction of a triangulation network on the basis of all possible conditions of minimum (or maximum) which would lead to excellent scientific and practical results, has not hitherto been carried out. Taking into account that electronic computers of big capacity are now available, such a solution is, however, to be expected in the not too remote future.

13.2.1 Error Propagation in a Simple Triangulation Chain

Since the derivation of the calculation formulae for the various errors characteristic of the simple triangulation chain as well as comments concerning them are to be found in several technical works (*Krasovski* 1953—1956, *Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Ghițău* 1972 etc.), the presentation here will confine itself merely to reviewing the most important final calculation formulae and some conclusions of a practical nature concerning applications.

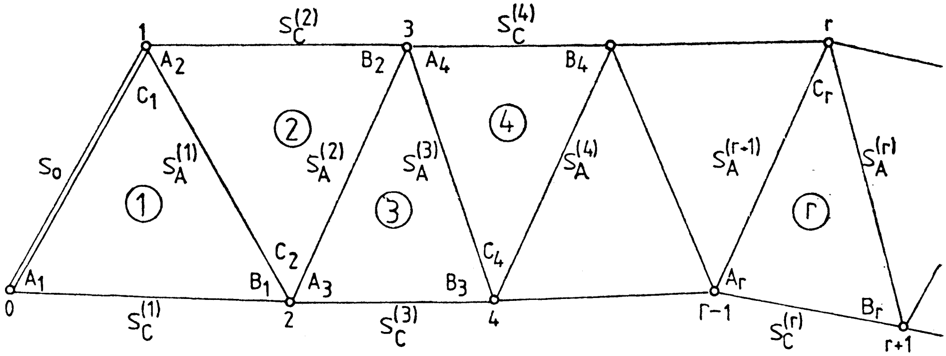

Fig. 13.5. Single Triangle-Chain

Standard Deviation of Any Connexion Side $s_A^{(r)}$ (Fig. 13.5 [1]):

$$\frac{\mu_{s_A^{(r)}}}{s_A^{(r)}} = \frac{\mu''}{\rho''}\sqrt{q\{[\cot^2 A^\circ]_1^r + [\cot^2 B^\circ]_1^r + [\cot A^\circ \cot B^\circ]_1^r\}} \qquad (13.1)^2$$

where A_i°, B_i°, C_i° ($i = 1, 2, \ldots,$ r) represent the values of the angles of the triangles measured with the standard deviation μ'' and:

$$q = \begin{cases} 2/3 \text{ (for the chain formed only out of triangles)} \\ 0.5 \text{ (when in the chain there also appear geodetic quadrangles)} \end{cases} \qquad (13.2)$$

or:

$$\mu_{\log s_A^{(r)}} = \mu'' \sqrt{q[R]_1^r}, \qquad (13.3)$$

in which:

$$R_i = \delta^2_{A_i^\circ} + \delta^2_{B_i^\circ} + \delta_{A_i^\circ}\,\delta_{B_i^\circ} \qquad (13.4)$$

and δ_{A°, δ_{B° — tabular differences of log sin A° and log sin B° respectively, when the argument varies by $1''$.

Remarks:

(1) In some works *(Krasovski 1953 – 1956, Lörinczi 1959, Ghițău 1972)* in (13.1) under the radical sign, there also appears a term $(\mu_{so}/s_o)^2$ due to the accuracy in determining the starting side, which is generally neglected when actually calculating, since one assumes that for the astro-geodetic triangulation s_0 is determined with a high accuracy, practically without errors by comparison with the errors in the angle measurement.

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[1] The notations in Fig. 13.5 have the following meanings: $s_0 = \overline{01}$ — the known length of the initial (starting) side; A_i, $s_A^{(i)}$ — angles, connexion sides respectively; C_i, $s_C^{(i)}$ — angles, intermediate side respectively; i represents the triangle number.

[2] The expression under the radical for any angle i ($i = 1, 2, \ldots, r$) is also called the erorr of the geometrical connexion of the triangle.

(2) In order to evaluate the accuracy in determining a side from r "geometrical figures" (e.g.: geodetic quadrilaterals or central systems), in the *U.S.A.*, the formula (*Krasovski 1953–1956*):

$$\mu_{\log s_A^{(r)}} = \frac{\mu_d''}{\rho''}\sqrt{\frac{4}{3}\left[\frac{r'-2-r}{r'}\right]_1^r [R]_1^r}, \tag{13.5}$$

was used, in which r' is the number of all the directions measured (with the standard deviation μ_d'') in the geometrical figure and r the number of the figure conditions.

The exprssioen $(r' - 2 - r)/r'$ has the values given in Table 13.1 for various geometrical figures.

Table 13.1. *Value of the Expression* $\dfrac{r'-2-r}{r'}$

Geometrical figure	$r' - 2$	r	$\dfrac{r'-2-r}{r'}$
Triangle	4	1	0.75
Geodetic quadrilateral	10	4	0.60
Central system formed out of 6 triangles	22	7	0.68

Conclusions:

1) $\mu_{s_A^{(r)}}/s_A^{(r)}$ *depends both on the accuracy in measuring the angles and on the triangle configuration.*

2) *The accuracy of the side* $s_A^{(r)}$ *decreases proportionally with the square root of the number of triangles, counted from the starting side.*

Optimum configuration of the Triangles. For a simple triangulation chain this problem is indeterminate. It can be solved only by choosing some additional criteria which should show the point of view from which this optimum configuration is interesting. These may be the following:

(1) In order that when subsequently developing the triangulation network both the connexion sides $s_A^{(r)}$ and the intermediate ones $s_C^{(r)}$ *should be used as support sides, these must have the same accuracy, i.e.:*

$$\frac{\mu_{s_C^{(r)}}}{s_C^{(r)}} = \frac{\mu_{s_A^{(r)}}}{s_A^{(r)}}. \tag{13.6}$$

(2) *The value of the error of the geometrical connexion in every triangle must be minimum.*

(3) *On a given surface, the number of triangles must be minimum, while satisfying the criteria (1) and (2).*

Conclusions:

1) *The optimum configuration of the triangle in a simple triangulation chain is the equilateral triangle.*

2) *Mostly, in practice, one cannot rigorously achieve the optimum configuration of the triangles and therefore in the working instructions one provides the minimum values admissible for the angles A_i, B_i, C_i, departing from the value of 60°.* For instance (*Direcţia topografică militară* 1962): the angles of the triangles should not be less than 40° and in the quadrilaterals the connexion angles should not be less than 30°.

The longitudinal error of the chain

$$\mu_D = D\sqrt{\left(\frac{\mu_{s_0}}{s_0}\right)^2 + \left(\frac{4m^2 - 3m + 5}{9m} - E\right)\left(\frac{\mu''}{\rho''}\right)^2}, \tag{13.7}$$

where $m = D/s$ and

$$E = \begin{cases} 0 \text{ (for measured angles)} \\ \dfrac{10m^2 - 7m - 9}{300\ m^2} \approx 0{,}04\ \mu_D/m \text{ (for measured directions)} \end{cases} \tag{13.8}$$

or:

$$\mu_D = \sqrt{\mu_s^2 + \mu_u^2}, \tag{13.9}$$

in which:

$$\mu_s = \frac{D}{\sqrt{2}}\frac{\mu_{s_0}}{s_0} \text{ and } \mu_u = \frac{D}{\sqrt{2}}\sqrt{\frac{2m^2 - 3m + 10}{9m}}\,\frac{\mu''}{\rho''}. \tag{13.10)—(13.11}$$

Conclusions:

1) *Utilizing two initial sides (at the two terminals) in a triangulation chain leads to lessening μ_D twice compared with the case when there is only one initial side.*

2) *The value of the influence of the error of the starting side (μ_s) on μ_D is approximately 50% of the value of the influence of the measurement errors of*

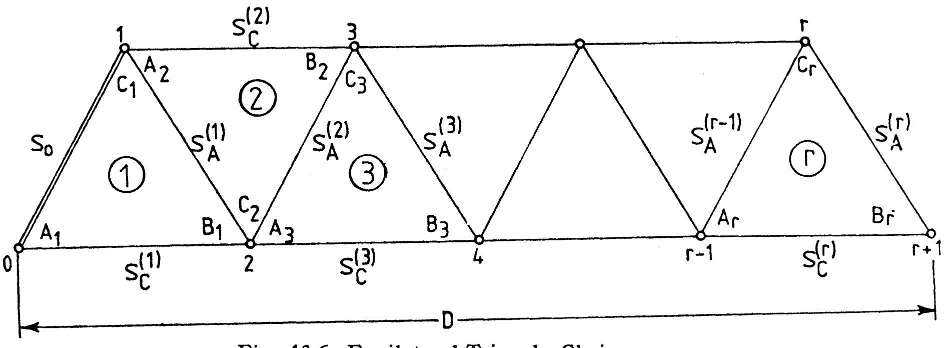

Fig. 13.6. Equilateral-Triangle Chain

the angles (μ_u). Therefore, in astro-geodetic triangulation, the lengths of the initial sides must be known with a rather high accuracy (1/400,000) and the chain angles with a high precision ($\pm 0''.7$), which assumes the use of theodolites of the highest precision.

3) *In the triangulation chains formed out of geodetic quadrilaterals with both diagonals observed, μ_D is greater than in the case of simple triangle chains of the same length (Lörinczi* 1959*).*

The standard deviation of transmitting the azimuth (Fig. 13.7):

$$\mu''_{\alpha_r} = \mu'' \sqrt{\frac{2}{3} r} = 0{,}8165\, \mu'' \sqrt{r}, \tag{13.12}$$

where μ'' represents the standard deviation of the measured intermediate triangles of the chain, through which the transmission takes place (dashed ine in Fig. 13.7).

Conclusions:

1) *μ_{α_r} does not depend on the configuration of the triangles through which the transmission is made, nor on their reciprocal disposition.* In the case when subsequent adjustment is to be made on directions, then the reciprocal disposition of the triangles within the chain plays a role which has to be taken into consideration *(Jordan/Eggert/Kneissl* 1958*).*

2) *μ_{α_r} increases proportionally with the square root of the number of triangles used for transmitting the azimuth.*

3) *The triangulation chains formed from geodetic quadrilaterals favour the transmission of azimuths;* this is best done in the case when the corresponding geodetic quadrilaterals are formed from right-angled triangles *(Lörinczi* 1959*).*

The transverse error of the chain (Fig. 13.8):

$$\mu_T = \frac{D}{\rho''} \sqrt{\mu''^2_{\alpha_0} + \frac{2}{15}\left(\frac{m^2 + m + 3}{m}\right) \mu''^2}, \tag{13.13}$$

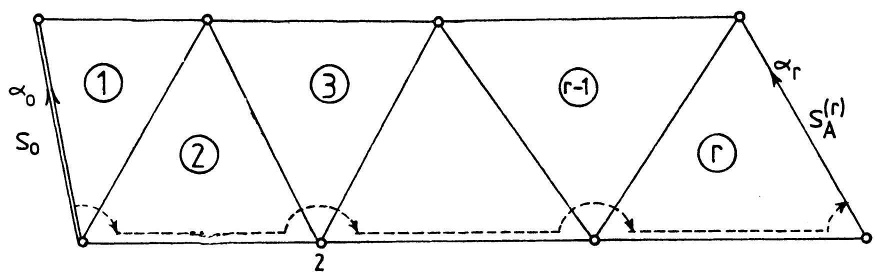

Fig. 13.7. Triangulation Chain

where μ_{α_0} represents the standard deviation of the azimuth of the starting side or if in the triangulation chain singly determined astronomical azimuths are available at both terminals:

$$\mu_T = \frac{D}{\rho''\sqrt{2}}\sqrt{\mu''^2_{\alpha_0} + \frac{m^2 + 1{,}9\,m + 12}{15m}\,\mu''^2}. \tag{13.14}$$

Conclusions:

1) μ_T *defines the rotation of the triangulation chain at the point O* (Fig. 13.8).

2) *By determining one astronomical azimuth at each of the two terminals of a triangulation chain,* μ_T *is decreasing by 35—45% compared with the case when there is only one azimuth determined in one of the terminals.* For instance, if $D = 300$ km; $m = 10$; $\mu'' = \pm\, 0''.7$; $\mu_{\alpha_0} = \pm\, 0''.5$ and $\rho'' \approx 2'' \times 10^5$, then it follows that: $\mu_T = \pm\, 1.5$ m (by using (13.13)) and ± 0.83 m respectively (by using (13.14)). Consequently, one obtains a decrease in the value of μ_T by 45%.

3) μ_T *decreases still more by introducing the side condition equations.*

4) *For chains formed from geodetic quadrilaterals* μ_T *is smaller than for chains formed from triangles, for the same chain length.*

The standard deviation of the orientation of a connexion side (Fig. 13.5):

$$\mu''_{\theta_r} = \sqrt{\mu''^2_{\theta_0} + \frac{2}{3}\,r\mu''^2}, \tag{13.15}$$

where μ''_{θ_0} represents the standard deviation of the orientation of the starting side.

Conclusions:

1) μ_{θ_r} *does not depend on the configuration of the triangles, nor on their reciprocal disposition within the framework of the chain.*

2) *The value* μ_{θ_0} *increases proportionally with the square root of the number of the triangles being used for transmitting the orientation* θ_0 *of the starting side.*

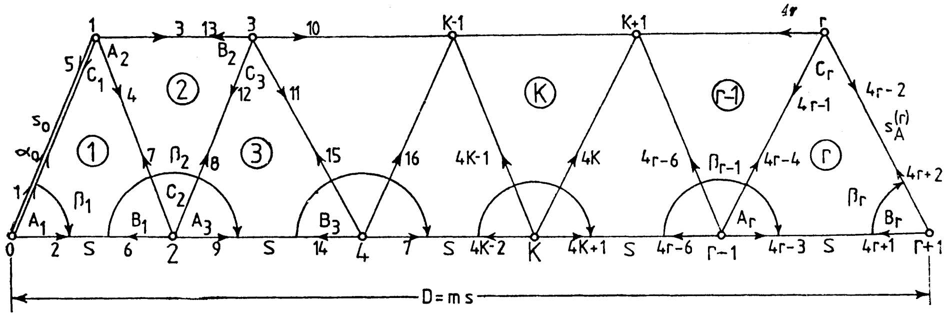

Fig. 13.8. Break Angles of the Chain

13.2.2 Error Propagation in a Compact Triangulation Network

Since in the technical literature the problem of error propagation in compact (surface) triangulation networks has been tackled by many authors some of them working in great detail *(Provorov* 1956, *Sigl* 1956*)* and others (e.g. *Ghiţău* 1972) even presenting a synthesis of the results obtained, we shall limit ourselves here to only a brief survey of the formulae for determining the longitudinal and transverse errors. Reference will be made in particular to these two errors since they serve in general as a criterion of the error propagation. Thus, the longitudinal error μ_D of one of the ends of a diagonal of the network formed by equilateral triangles is given with respect to its other end by the relation *(Provorov* 1956*):*

$$\mu_D = \frac{1}{\rho''}\sqrt{\frac{1}{2}\left(\rho''\frac{\mu_s}{s}\right)^2 + \mu''^2\left(\frac{\delta_0}{\delta_{60^\circ}}\right)^2\frac{r+15}{100}},$$

in which μ_s/s represents the relative error in measuring the sides; μ — the standard deviation of a measured angle; r — the number of triangles contained between the two ends of the diagonal (two *Laplace* points); δ_{60° — the variation of the logarithm of the sine of the angle of 60° for a variation in the angle equal to 1″ (tabular difference); δ_0 — the arithmetic mean of the tabular differences of the angles in the given network.

The transverse error μ_T of one of the ends of the diagonals with respect to the other end is given by the relation *(Provorov* 1956*):*

$$\mu_T = \frac{1}{\rho''}\sqrt{\frac{1}{2}\,\mu_\alpha''^2 + \mu''^2\,\frac{r+15}{100}},$$

in which μ_α is the standard deviation of the determination of the *Laplace* azimuth.

As to the optimum form of the elementary figures out of which is formed the compact triangulation network, one can say that this is again, as in the case of the triangulation chains, *the equilateral triangle.*

The utilization of the geodetic quadrilateral as elementary figure of the network leads to an increase of the accuracy in its determination of approximately 10—15%, there appearing, however, some technical-economic implications, inasmuch as the use of the geodetic quadrilateral involves supplementary field operations and calculations. As *W. Baarda* (1967) has shown, the form of the triangulation network will be *"now, and probably always, determined by the limited availability of time, money and material means".*

Continuing, we shall dwell in more detail on the mechanism of error propagation in extensive astro-geodetic networks in which traverses measured with electromagnetic equipment are also used, viz. on the optimum configuration of the traverses in these networks, as well as on the modification of the longitudinal and transverse mean displacement of a geodetic line belonging to the network, by modifying the acuray rate between the distance measurements and the azimuthal ones.

The present fundamental figure of the extensive continental astro-geodetic networks is the quadrilateral. The network diagonals (single sides) obtained from triangulation chains are functions, as to their length and direction, of the azimuthal directions and the measured distances, as well as of the *Laplace* azimuths. If in these triangulation chains, the *Laplace* azimuths are suitably disposed, then the neighbouring diagonals of the network are as good as independent (connexionless)

As has been shown *(Schädlich* 1971*)*, the vector character of the network diagonals is still more obvious if the *Laplace* points are connected in the future by electromagnetically measured traverses. In connexion with the application of this effective and exact method of distance measurement for improving the continental astro-geodetic networks with a view to determining the Earth's form and size, there also appears, among others, the problem of error propagation in its various aspects previously subjected to analysis.

In what follows we will show a way of dealing with this question, indicated by *M. Schädlich* (1971). In his contribution, the author analyses the details of error propagation in the following three configurations of combined networks (Fig. 13.9):

I) *vector network of quadrilaterals RV 1 (b);*

II) *vector network of quadrilaterals RV 2 (b);*

III) *vector network of quadrilaterals RV 3 (b).*

The parameter b represents the number of chains of single quadrilaterals of length $L = D(N - 1)$, where N — the number of points $P_i(i = 1, 2, ..., N)$ along the chain's length and D — the length of the intermediate sides of the chain of quadrilaterals (which in fact constitute the lengths of the triangulation chains from which they arise).

The corresponding analysis is made by adopting the following criteria:

(1) *The longitudinal mean displacement and the transverse one of the final point P_N of the base polygon serves as criterion of error propagation.*

(2) *One assumes that the measurements are affected only by unavoidable random errors.*

(3) *The calculations refer to a plane reference surface.*

(4) *Expressing the inverse of the weight of a function $F = F(l_i)$ of the adjusted observations l_i,* i.e. *Q_{FF}, is difficult for combinations of networks with various measured elements (directions α_i and distances D_i) and weight ratios q^2.* Here q means the relation:

$$q = \frac{\mu_D}{D} : \frac{\mu''_\alpha}{\rho''}, \tag{13.16}$$

in which μ_D/D defines the accuracy of the distance measurements and μ''_α/ρ'' — the accuracy of the azimuth measurements. One can express Q_{FF} only for certain simple cases, by finding out the laws of formation of the separate terms which might lead to an approximate formula. For the case of the vector networks of quadrilaterals these difficulties may be removed by choosing the function F suitably, while giving up the condition of complete rigour.

(5) *Relatively simple but sufficiently precise approximations for the longitudinal mean displacement and for the transverse one of the point P_N (*i.e. *the*

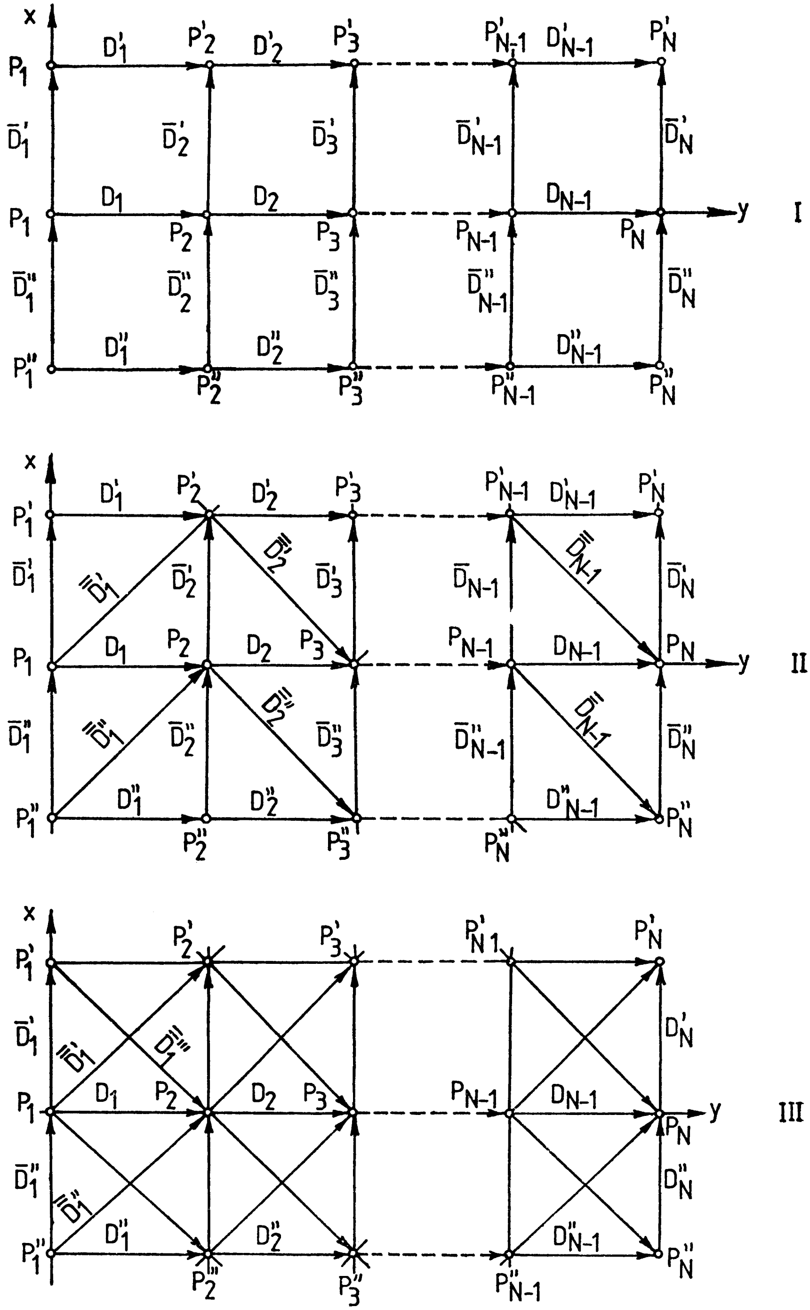

Fig. 13.9. Scheme of the $RV(b)$ Vector Network:
I —vector network of the $RV\ 1\,(b)$ type; II — vector network of the $RV\ 2\,(b)$ type; III — vector network of the $RV\ 3\,(b)$ type

sign, direction and upper limit of the error) may be obtained as functions of the proportionality parameter q of the chains and of the number N of points P_i, by utilizing the so-called multi-way procedure (Wolf 1949), based on the principle of the arithmetic mean.

For the analysis of the network of the type $RV1$ (b) one considers initially the case of the single chain ($b = 1$). For this case the following formulae are applicable for the longitudinal displacement (μ_L) and for the transverse one (μ_T) of the final chain point P_N:

$$(\mu_L)_{RV1(1)} \approx \mu_D \sqrt{(N-1)\frac{1+q^2}{1+2q^2}}\,; \quad (\mu_T)_{RV1(1)} \approx \frac{\mu_D}{q}\sqrt{(N-1)\frac{1+q^2}{1+2q^2}}, \tag{13.17}$$

where $N = 2, 3, \ldots$ and $0 < q \neq \infty$.

In the special case $q = 1$, the relations (13.17) become:

$$(\mu_L)_{RV1(1)} = (\mu_T)_{RV1(1)} \approx \mu_D \sqrt{\frac{2}{3}(N-1)}. \tag{13.18}$$

For a sufficiently large number N of points, one can make the approximation $N \approx N - 1$ in (13.18).

The longitudinal displacement and the transverse one may also be expressed by standard relative errors, viz.:

$$\left(\frac{\mu_L}{L}\right)_{RV1(1)} \approx \frac{\mu_D}{D}\,\frac{1}{\sqrt{(N-1)\dfrac{1+2q^2}{1+q^2}}}\,;$$
$$\left(\frac{\mu_T}{L}\right)_{RV1(1)} \approx \frac{\mu_D}{qD}\,\frac{1}{\sqrt{(N-1)\dfrac{2+q^2}{1+q^2}}}, \tag{13.19}$$

which serve for several general formulations as degree of error propagation.

For the case of the double chain of quadrilaterals ($b = 2$), the relations (13.19) take the form:

$$\left(\frac{\mu_L}{L}\right)_{RV1(2)} \approx \frac{\mu_D}{D}\,\frac{1}{\sqrt{(N-1)\dfrac{1+3q^2}{1+q^2}}}\,;$$
$$\left(\frac{\mu_T}{L}\right)_{RV1(2)} \approx \frac{\mu}{qD}\,\frac{1}{\sqrt{(N-1)\dfrac{3+q^2}{1+q^2}}}\,; \tag{13.20}$$

and for the general case $b = 0, 1, 2, \ldots, N-1$, these relations become:

$$\left(\frac{\mu_L}{L}\right)_{RV1(b)} \approx \frac{\mu_D}{D} \frac{1}{\sqrt{(N-1)F^{RV1}_{\mu_L}(b,q)}};$$
$$\left(\frac{\mu_T}{L}\right)_{RV1(b)} \approx \frac{\mu_D}{qD} \frac{1}{\sqrt{(N-1)\,F^{RV1}_{\mu_T}(b,q)}}, \tag{13.21}$$

in which:

$$F^{RV1}_{\mu_L}(b,q) = 1 + F_{\mu_L}(b,q);\; F^{RV1}_{\mu_T}(b,q) = 1 + F_{\mu_T}(b,q), \tag{13.22'}$$

where:

$$F_{\mu_L}(b,q) = \frac{bq^2}{1+q^2}\;;\quad F_{\mu_T}(b,q) = \frac{b^2}{1+q^2}\cdot \tag{13.22''}$$

In the special case $q = 1$, one gets:

$$F^{RV1}_{\mu_L}(b,q) = F^{RV1}_{\mu_T}(b,q) = \frac{1}{2}(b+2). \tag{13.23}$$

For the case of the vector polygon ($b = 0$), the relations (13.23) take the particular form:

$$F^{RV1}_{\mu_L}(b,q) = F^{RV1}_{\mu_T}(b,q) \equiv 1. \tag{13.24}$$

The functions $F^{RV1}_{\mu_L,\mu_T}(b,q) > 1$ which represent the influence of the distance and azimuth measurements outside the "main traverse" $P_1, P_2, \ldots, P_N$ on the longitudinal and transverse displacements of the point P_N are called *network factors (Baarda* 1967, *Schädlich* 1971*)*.

Remarks:

(1) If $b \leqslant N-1$, then the network factors yield an error $d\mu_L = d\mu_L(N, b, q)$ and $d\mu_T = d\mu_T(N, b, q)$ respectively, where the sign is independent of the network type and is always negative.

(2) If $b > N-1$, the azimuth transmission is inadequate and the parameters b and N must be interchanged.

In the case of the networks of the type $RV\,2(b)$, in which additionally with respect to the type $RV1(b)$ the diagonal vectors D_i' were measured, the analysed standard relative errors are given, in a first approximation, by the following formulae:

$$\left(\frac{\mu_L}{L}\right)_{RV2(b)} \approx \frac{\mu_D}{D} \frac{1}{\sqrt{(N-1)F^{RV2}_{\mu_L}(b,q)}};$$
$$\left(\frac{\mu_T}{L}\right)_{RV2(b)} \approx \frac{\mu_D}{qD} \frac{1}{\sqrt{(N-1)\,F^{RV2}_{\mu_T}(b,q)}}, \tag{13.25}$$

in which:

$$F_{\mu_L}^{RV2}(b,q) = 1 + 2F_{\mu_L}(b,q);\; F_{\mu_T}^{RV2}(b,q) = 1 + 2F_{\mu_T}(b,q),$$

where F_{μ_L} and F_{μ_T} are given by (13.22'').

For the networks of the type $RV3\ (b)$, including, in every single quadrilateral chain, diagonal vector polygons in twos and in which the error propagation is similar to that of the types $RV1\ (b)$ and $RV2\ (b)$, the calculation formulae of the analysed standard relative errors are as follows:

$$\left(\frac{\mu_L}{L}\right)_{RV3(b)} \approx \frac{\mu_D}{D}\frac{1}{\sqrt{(N-1)\,F_{\mu_L}^{RV3}(b,q)}};$$
$$\left(\frac{\mu_T}{L}\right)_{RV3(b)} \approx \frac{\mu_D}{qD}\frac{1}{\sqrt{(N-1)\,F_{\mu_T}^{RV3}(b,q)}}, \qquad (13.26)$$

with the network factors:

$$F_{\mu_L}^{RV3}(b,q) = 1 + 3F_{\mu_L}(b,q);\; F_{\mu_T}^{RV3}(b,q) = 1 + 3F_{\mu_T}(b,q),$$

where $F_{\mu_L}(b,q)$ and $F_{\mu_T}(b,q)$ are given by (13.22'').

For the special case $b = q = 1$, the relations (13.26) become:

$$\left(\frac{\mu_L}{L}\right)_{RV3} = \left(\frac{\mu_T}{L}\right)_{RV3} \approx \frac{\mu_D}{D}\frac{1}{\sqrt{\frac{2}{5}(N-1)}}.$$

From the analysis of the calculation formulae presented for the three network types, one can draw the following conclusions:

(1) *Taking into consideration that the basic theoretical principles were common for all network types, the resulting approximate calculation formulae prove, however, with sufficient certainty, that the error propagation is different in the three analysed network types.*

(2) *The single chain $RV2(1)$ is as efficient as the double chain $RV1(2)$, despite the fact that it has less measured elements.*

(3) *In the networks of the type $RV2(b)$ and $RV3(b)$ there takes place, independently of N, an increase of the accuracy in determining the point P_N as compared with the network of the type $RV1(b)$, if $q = 1$ and $b \leqslant N-1$.* Thus, e.g. for $b = 1$ this increase is of 13% in the type $RV\ 2\ (b)$ network and of 23% in that of the type $RV\ 3\ (b)$, while for $b = 50$ one gets 29% and 42% respectively.

(4) *If $q = 1$, the increase of the accuracy in the determination of the point P_N compared with the vector chain $RV0\,(b = 0)$, achieved by the three analysed*

types of network (RV1, RV2, RV3) is defined by the ratio of the following standard relative errors:

$$\left(\frac{\mu_L}{L}\right)_{RV0} : \left(\frac{\mu_L}{L}\right)_{RV1} : \left(\frac{\mu_L}{L}\right)_{RV2} : \left(\frac{\mu_L}{L}\right)_{RV3} =$$

$$= \left(\frac{\mu_T}{L}\right)_{RV0} : \left(\frac{\mu_T}{L}\right)_{RV1} : \left(\frac{\mu_T}{L}\right)_{RV2} : \left(\frac{\mu_T}{L}\right)_{RV3} \approx 1 : 0{,}816 : 0{,}707 : 0{,}632.$$

As regards the relation between the accuracies of the angular measurements and of those of distances for the astro-geodetic networks, one used to take it as $q = 1$, i.e.:

$$\frac{\mu_D}{D} = \frac{\mu''_\alpha}{\rho''}, \tag{13.27}$$

More recent investigations *(Trofimov* 1965*)* have shown that the relation (13.27) has to be taken in the form:

$$\frac{\mu''_\alpha}{k\rho''} = \frac{\mu_D}{D}, \tag{13.28}$$

where k is a coefficient depending on the network form.

For the astro-geodetic networks, which are compact networks or networks formed out of triangle chains, the optimum value of the coeficient k may vary between the limits 1.21—1.33 *(Trofimov* 1965*)* depending on the character of the errors of the measured distances.

14

Measurements in Astro-Geodetic Triangulation

In astro-geodetic triangulation one carries out several kinds of measurements, depending on the quantities needed. Thus, one distinguishes (Fig. 13.1): (1) *angular measurements;* (2) *distance (or base) measurements* (3) *levelling measurements;* (4) *measurements concerning gravity;* (5) *determinations of geodetic astronomy.*

The obtainable precision to be expected in these measurements is limited in the main by:

(1) *The precision in manufacturing the axes and in dividing the circles of the measuring instruments.*

(2) *The refraction phenomenon:* the propagation of light happens in the atmosphere — a heterogeneous and variable medium — which replaces a rectilinear geometrical propagation by a propagation presenting some uncertainty of a random character, i.e. local variations of the value of the refractive index along the sight.

Actually, the precision obtained so far world-wide in the best measurement of astro-geodetic triangulation do not seem to go beyond the limits shown in Table 14.1.

Table 14.1. *Measurement Precision in Astro-Geodetic Triangulation*

Measurement of ...	Precision (which corresponds to ...)	Remarks
Azimuthal angles	$\pm 0''.2$ (± 0.05 m/50 km)	
Zenithal angles	$\pm 1''$(± 0.1 m/20 km)	
Distances	$\pm 1 \ldots 5 \cdot 10^{-6}$ ($\pm 0.05 \ldots 0.25$ m/50 km)	
Level differences by geometrical levelling	± 0.3 mm/km	
Gravity (relative measurements)	$\pm 0.01 \ldots 0.1$ mgal ± 1 mgal $\pm 5 \ldots 10$ mgal	On land At sea for air measurements
Astronomical latitude and longitude	$\pm 0''.03 \ldots 0''.3$($\pm 1 \ldots 10$ m)	
Astronomical azimuth	$\pm 0''.4$ (± 0.1 m/50 km)	

In what follows we will present some considerations dealing with all kinds of measurements in astro-geodetic triangulation, more emphasis being laid on the measurements of azimuthal angles, which, in fact, take first place from the point of view of the volume of field operations.

14.1 Angle Measurements

In order to determine the point positions in astro-geodetic triangulation, among other kinds of measurements, one also carries out measurements of azimuthal angles and of zenithal or vertical angles.

For these measurements one utilizes in astro-geodetic triangulation high-precision theodolites, which secure in measurements a standard deviation of an observed direction, in the two positions of the telescope, less than, or at most equal to $\pm 2''$ *(Deumlich* 1977*)*; examples are the theodolites *Wild T3* and *Wild T4* or *DKM 3* and the theodolites with photographic recording of the readings (type *Wild T3*, type *Askania Tpr*).

Assuming that the divisions of the horizontal circle of a theodolite are drawn with a precision of 1 μ, one can suppose that, if the circle has a diameter of 6 cm and the readings are made simultaneously at two extremities diametrically opposed, then the precision of the reading of one direction is:

$$\frac{1}{\sqrt{2}} \frac{1}{60\,000} \approx \frac{1}{100\,000}$$

This error may be further reduced down to 1/400,000 or 1/500,000, i.e. $m_a = \pm 1''.5$ (m_a — the standard deviation of one direction), if the required direction is calculated as a mean from 16 or 25 measurements. Increasing the number of measurements will not lead to any substantial improvement since, as has been shown *(Levallois* 1969), the light propagation uncertainties are of the order of approximately $\pm 1''$. As one notes in Table 14.1, the precision obtained in the best measurements in astro-geodetic triangulation (lst-order points) has not exceeded $\pm 0''.2 - 0''.3$ and this fact has caused the lst-order points observed by the optical method to be determined with a relative precision in position of ± 5 or 6 cm, because these values represent parallax of $1''$ at a distance of 35 km. Taking also into consideration that the precision in the direct distance-measurements by electromagnetic methods is of about the same order of magnitude, we note that the homogeneity of the triangles of the triangulation cannot exceed this limit. If one wishes a precision of ± 1 cm, then the general geodetic framework does not suffice. Consequently, the entire array of measuring methods and of reading and correction accessories of the theodolites must be used in such a way that they secure a reading precision superior to the second decimal place (which represents on the circle a position variation of ± 0.1 μ, i.e. approximately 5—10 times smaller than the precision in drawing the divisions). This very high precision can only be attained by eliminating from the measurement results all systematic errors (errors of reading on the circles, errors due to

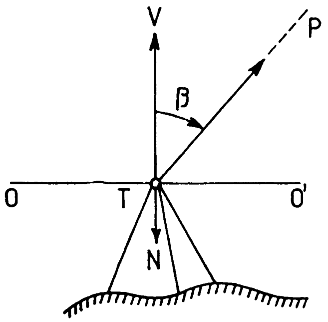

Fig. 14.1. Zenithal Angle β
T — theodolite stationing point; V — zenith; N — nadir; OO' — theodolite's horizon; P — sighted point; VN — direction of the local vertical at T; TP — sighting direction

the telescope, the stand torsion, position errors of the theodolite's axes, graduating errors of circles, run error etc.). Therefore, the angular measurements are executed in series, in which each of the two semi-series is measured in opposite direction, utilizing theodolites provided with two devices diametrically opposed and are uniformly distributed along the circumference of the circle.

The *zenithal angle* (frequently also called *zenithal distance*), denoted by β, means the angle formed by the vertical of the station place with the corresponding sighting direction (Fig. 14.1). The measurement proper of a zenithal angle consists of sighting and reading the value of the direction TP for two positions of the telescope, after having first of all placed the vertical rotation axis of the theodolite in the vertical position by means of a precision cylindrical level.

The calculation formulae of the zenithal angle β differ depending on the theodolite type utilized in the measurement and on the layout of the inscription of the division values on its vertical circle. The actual form of the various formulae met in astro-geodetic triangulation practice, together with other details, are to be found in the technical literature (e.g. *Lörinczi* 1959, *Oprescu et al.* 1973). The programme for measuring the zenithal angles is generally simpler than for the azimuthal angles, this simplicity due mainly to the fact that the precision of these measurements is smaller ($\pm 1''$), due to the more telling influence of the atmospheric refraction and to the fact that one cannot measure the zenithal angles with different origins.

As regards the azimuthal measurements, two categories of measuring metods are known, depending on the quantity which is to be measured, viz.:

(1) *Methods for measuring directions,*

(2) *Methods for measuring angles.*

If at a station point in astro-geodetic triangulation, in which the number of points to be observed is K, all (or nearly all) the points $K > 2$ lie in the measured series, the measurement is called *directional* and if $K = 2$ the measurement is called *angular*.

Obviously, the measurement of angles represents a special case of the measurement of directions. Strictly speaking, in all cases of angular measurements one measures, in fact, directions whose numerical value must be understood as resulting from means of values of the two semi-series in which these are measured. Although in the second category of measurements each angle is obtained as the difference of two measured directions (the right-hand direction minus the left-hand one), this may be regarded, however, as an independent quantity (*Gaidaev* 1960, *Jordan/Eggert/Kneissl* 1958, *Pelzer* 1974).

For measuring the azimuthal angles, various methods have been used in astro-geodetic-triangulation practice which the most important ones are shown in Table 14.2. Any one of these methods involves three working stages:

Table 14.2. *Main Methods Used for Measuring Azimuthal Angles*

Measuring ...	Designation of the measuring method
Directions	Method of the horizon rounds (reiteration method) in 3 variants: 1. Method of complete series 2. Method of incomplete series 3. Method of direction series in all combinations
Angles	1. Method of measuring angles in all combinations 2. Method of measuring neighbouring angles 3. Method of the reference torques 4. Method of the sectors

A. **Preparing the measurement programme,** which aims at establishing the origins of the series in which the directions (or angles) at the corresponding station point are to be measured.

B. **The measurement proper,** according to the method utilized.

C. **The station adjustment,** by which one performs:

1) *The determination of the probable values* in the sense of the method of least squares, for the measured directions or angles.

2) *Assessment of the measurement precision* (individual errors, mean eror).

3) *Obtaining the correlation matrix of the station-adjusted elements.*

The station adjustment presents some important peculiarities, viz.:

(1) *It constitutes a preliminary stage in the process of obtaining the final adjusted values of the angles (or directions) at the station points of the network and, consequently, leads to a first approximation to these values.*

(2) *The weights of the measured angles are assumed to be equal to the number of measurement series and the directions measured in series — of the same precision.* This reasoning leads to a departure to a greater or lesser extent, from the rigorous adjustment, inasmuch as one does not take into account the diversity of the conditions of measuring (external medium, instrument, operator) the various angles (or directions).

(3) *In the modern astro-geodetic triangulations the values of the standard deviation of the measured angles, calculated from the station adjustment, are obtained, of the average 1.5—2 times smaller (Gaidaev 1960) than the values of the same quantities calculated with Ferrero's formula (also called the international formula), i.e. by using the misclosures of the network's triangles.* This difference is explained by the influence of systematic errors (in particular lateral refraction) on the value of each measured value, which being nearly constant in all the series of measurement of the angles (directions) at the various

station points but varying between them does not act significantly on the results of the station adjustment. Consequently, the precision of the azimuthal measurements is not fully characterized by the standard deviation of a measured angle (direction) as obtained from the station adjustment but only to a certain degree of approximation.

(4) *It is necessary that for every adjustment procedure be established the degree of dependence of the results obtained (Wolf 1968, Höpcke 1969).* In order not to affect the rigour of the network's adjustment, depending on this result, the station-adjusted values of the angles (or directions) may be subsequently considered either as independent values if it turns out that they are orthogonal functions of the measured elements (i.e. all mixed weight-coefficients of the adjusted elements vanish), or as dependent values if the mixed weight-coefficients turn out to be different from zero. In this latter case there will be used, for adjusting the network, the theory of the adjustment of dependent observations (*Tienstra* 1956, *Wolf* 1968, *Botez* et al. 1971).

Within the framework of the measurements, as the sighting line passes close to the Earth's surface it is not only refracted in the vertical plane but also undergoes a certain systematic horizontal deviation (lateral refraction). Up to now no procedure has been found which might lead to the complete elimination (at least from the practical point of view) of the influence of the lateral refraction from the results of the measurements of azimuthal angles but the practice of the geodetic operations at least tries to diminish this influence as much as possible. In this respect, two ways for diminishing the influence of the lateral refraction have been indicated in the technical literature *(Dufour 1952, Jordan/Eggert/Kneissl 1958, Rotaru 1969, Mayer 1978,* etc.):

(1) *By computation, using meteorological data;*

(2) *By choosing suitable measuring methods and conditions.*

Since so far no calculation formula has been unanimously accepted for the correction due to the influence of the lateral refraction, the only reasonable way which remains is the choice of suitable measuring conditions of the kind of those displayed in the technical literature (*Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Gaidaev* 1960, *Rotaru* 1969, *Mayer* 1978 etc.), on which we shall not insist here. We mention, however, that what is essential for solving this problem will be the realization in the future of optical instruments for measuring angles [1] utilizing several wave-lengths in order to enable the influence of the atmospheric refraction to be corrected, instruments which may well be realized in the 1980—1990 decade (*Committee on geodesy: Trends and perspectives, 1978*).

Such developments of instruments, based on a principle similar to that used at present for the electromagnetic measurement of distances, are already under way at the *University of Uppsala—Sweden* (*Tengström* 1978) and at the *National Physical Laboratory in England* (*Williams* 1977).

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[1] For horizontal angles as well as for zenithal ones.

14.1.1 The Method of the Horizon Rounds

This method, which is also called the *reiteration method*, consists of the successive measurements, by a given number n of series, of the directions at a station point, starting from a direction considered as origin (usually the direction towards the most remote point which can be sighted under optimum conditions) and coming back to the origin direction.

The method has been particularly used in *France* (as a basic method) and in the *Federal Republic of Germany*, with $n = 15 \ldots 20$ series, and also in the astro-geodetic network of the *Socialist Republic of Romania* (but to a lesser extent).

Depending on the number of sighted directions, one distinguishes within the framework of the horizon round, 3 variants:

(1) *When in the horizon round one measures all the directions from the station point concerned.* In this case the method also bears the name of *method of the complete series (Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Grossmann* 1969*).*

(2) *When only a part of the n measurement series contains all the directions from the station point.* In this case the method is also called the *method of the incomplete series. (Jordan/Eggert/Kneissl* 1958, *Lörinczi 1959, Wolf 1968, Grossmann 1969).*

(3) *When from the K directions of a station point there are taken over in each case $K_1 < K$ directions and one measures a complete direction series so that there are $C_K^{K_1}$ series of different directions, which are measured n times.* This procedure which was given by *A. Vogler* (1885) *and F. R. Helmert* (1885) also bears the designation of *method of the direction series in all combinations (Wolf* 1968*).*

A. Preparing the measuring programme

If n is the number of the series, the angular interval I between two successive series will be:

$$I = \frac{180°}{n} \quad \text{or} \quad \frac{200^g}{n},$$

according as the horizontal circle of the theodolite used in the measurement has sexagesimal or centesimal divisions. Generally, the value I is rounded to the nearest minute. The origins of the n series will be $0, I, 2I, \ldots, (n-1)I$.

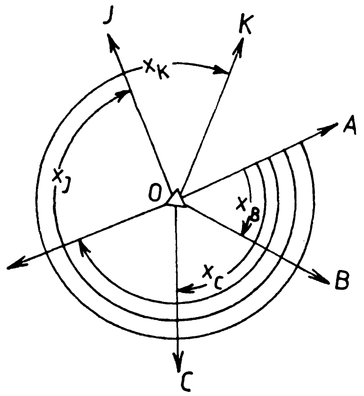

Fig. 14.2. Horizon-Rounds Method

B. Measuring the directions (Fig 14.2)

Sighting first of all the origin direction A one then sights successively all the other directions of the horizon round in the order in which they present themselves $B, C, \ldots, J, K$ and one stops on the direction A.

Continuing, starting again from the direction A one sights successively in the opposite order, i.e. $A, K, J, \ldots, C, B, A$, in order to eliminate the influence of the stand's torsion (or of the pillar, in the case of height geodetic signals). In a similar way the other $(n-1)$ series are measured. To be correct, a measurement must not contain more than 8—10 new directions in addition to the origin direction. The measurements are distributed over at least 3 days.

C. **The station adjustment**

Performing the measurements yields n series of values $L_i^o (i = A, B, \ldots, K)$ of the measured directions:

Series 1: $L_A^{o\prime}, L_B^{o\prime}, L_C^{o\prime}, \ldots, L_J^{o\prime}, L_K^{o\prime}$

Series 2: $L_A^{o\prime\prime}, L_B^{o\prime\prime}, L_C^{o\prime\prime}, \ldots, L_J^{o\prime\prime}, L_K^{o\prime\prime}$

. . . .

Series n: $L_A^{o(n)}, L_B^{o(n)}, L_C^{o(n)}, \ldots, L_J^{o(n)}, L_K^{o(n)}$.

As a rule: $L_A^{o\prime} = L_A^{o\prime\prime} = \ldots = L_A^{o(n)} = 0$.

By using the procedure of the indirect observations, in which one chooses as unknowns $x_B, x_C, \ldots, x_K$ (Fig. 14.2) in consequence of the station adjustment, it follows that: *(Lörinczi* 1959, *Ghițău* 1972*):*

$$x_o = \frac{[L_i^o]}{n} \tag{14.1}$$

and the standard deviation m_0 of a measured direction is given by:

$$m_0 = \sqrt{\frac{[v_i\, v_i]}{(n-1)(K-1)}},$$

where v_i — the direction corrections, and the standard deviation m of an adjusted direction is:

$$m = \frac{m_0}{\sqrt{n}}$$

The values x_i of the adjusted directions obtained from (14.1) are independent values, since the mixed weight-coefficients Q_{ij} $(i \neq j = B, C, \ldots, K)$ vanish, so that in adjusting the triangulation network they may be used as such.

Remarks:

(1) In the case of the method of the incomplete series, the station adjustment is carried out considering the respective measurement series as complete series in which to every direction measurement there belongs a weight, which is taken as equal to zero for all the directions which have not been measured. Several procedures of station adjustment for the incomplete series are known: *Bessel's procedure* (as unknowns one considers the angles of the adjusted

directions with respect to a zero direction), *Hansen's procedure* (as unknowns one utilizes the free directions x_B, x_C, ..., x_K), *Clarke's iterative procedure* (the unknowns are chosen as in *Hansen's* procedure), *Litschauer and Kneissl's composing procedure* (the unknowns the adjustment in the station are determined from the condition $[v] = 0$, for each series). The results of the station adjustment generally do not possess in the case of the incomplete series the characteristics of free functions (*Wolf* 1968). Consequently, in adjusting the triangulation network they will be used as dependent observations.

(2) In the case of the method of the direction series in all combinations, the station adjustment is in fact a special application of station adjustment of incomplete series. For observations of the same precision, the directions adjusted in the station represent free functions (*Helmert* 1885).

The method of the horizon rounds exhibits various desirable features such as: the measuring programme and the adjustment in the station are very simple, the influence of the graduating errors of the theodolite's horizontal circle is considerably reduced, the torsion of the stanid or pillar is exposed, in the case of good visibility along all observed directons the measurement results present high technical-economic indices; however there are also some shortcomings: for station points with a large number of directions (8—10 directions) the time required for completing a series is large (10—15 minutes), stringent requirements are imposed as regards the rigidity and stability of the geodetic signals, it is difficult to achieve the condition of good visibility, approximately the same, on all observed directions, differing measuring accuracy for the origin direction with respect to the other directions.

Note. The difference between the values of the origin direction in one series, called *misclosure of the series,* is due both to the unavoidable random measurement-errors and to contingent small twists of the stand or pillar. Concerning the utilization of the series misclosure one distinguishes three situations:

(a) If there are sound reasons that the series misclosure is due exclusively to unavoidable random measurement-errors, the one takes then average of the two values obtained for the origin direction in each series and later on one operates with this average, which must, however, receive the weight 2.

(b) If it is certain that the misclosure is due only to contingent twists, then either this is uniformly distributed on the various directions or, better, one makes this distribution on the basis of a quadratic law, for which *E. Ackerl* (*Wolf* 1968) has made available ready-calculated tables.

(c) The closing sight is used only for checking. This method of use is applied in Romania (Direcţia topografică militară 1962).

14.1.2 The Method of Measuring the Angles in All Combinations

This method was tackled for the first time by *P. A. Hansen* in the year 1871, but had already been considered in anticipation by *C. F. Gauss* in 1821 and by *C. L. Gerling* in 1839. Later on, the German geodesist *O. Schreiber* utilized the method in *Prussia's* 1st-order triangulation (1864—1874), publishing a contribution in this direction (1878). The principles established by *Schreiber* for applying this method have been maintained almost un modified up to now in the various national astro-geodetic triangulations, including that of *Romania (Direcţia topografică militară 1962)*. Another contribution to

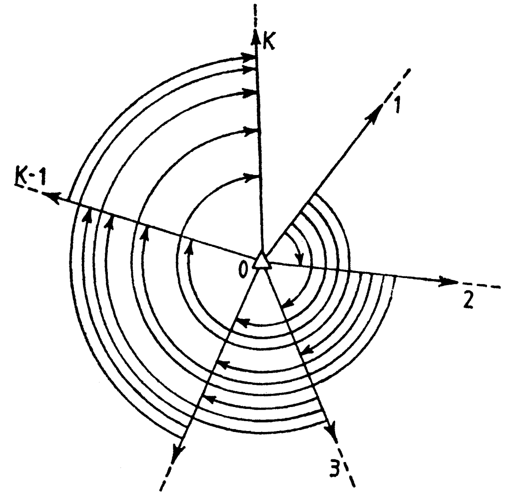

Fig. 14.3. Angle Measurement in All Combinations

this situation is the fact that *Schreiber* has developed the method of measuring the angles in all combinations establishing a unitary scheme of station adjustment. Therefore, generally, the method is also designated by his name. The method of measuring the angles in all combinations consists of measuring separately, by a given number n of series, all the angles formed by the K directions of the station point, which do not amount to 360° (Fig. 14.3). The total number of measured angles will, consequently, equal the number of possible combinations of the $K = 1, 2, \ldots$ directions taken two by two, i.e. $C_K^2 = K(K - 1)/2$ angles. Thence also comes the name of the method. The number of series is determined by the relation:

$$n = 2p/K, \tag{14.2}$$

in which p represents the weight of a station-adjusted angle. For the astro-geodetic triangulation of the *Socialist Republic of Romania* where one takes $p = 18$, the relation (14.2) becomes $n = 36/K$, whose values for the number K of directions most frequently met with in practice are shown in Table 14.3.

Table 14.3. *Series Number* $n = f(K)$

K	2	3	4	5	6	7	8
n	18	12	9	8	6	6	5

A. Preparing the measuring programme

Aiming at reducing to a minimum the influence of the graduating errors of the theodolite's horizontal circle, the groups of angles not having common directions are measured with the same initial origin. The angular interval or i' between the various initial positions of the horizontal circle is given by the relation:

$$i = \frac{180°}{nK} \quad \text{or} \quad \frac{200^g}{nK} \text{(for odd } K\text{)};$$

$$i' = \frac{180°}{n(K-1)} \quad \text{or} \quad \frac{200^g}{n(K-1)} \text{(for even } K\text{)},$$

and for determining the angular interval I between the series of the same angle, within the framework of one initial position, one utilizes the formula:

$$I = 180°/n \text{ or } 200^g/n$$

For establishing the measuring origins one can use the general quite intuitive principle scheme shown in Table 14.4 (for odd K).

Table 14.4. *Distribution of Measurement Origins*

Initial positions / Series	0^g	i^g	$2i^g$	...	$(K-1)i_g$
1	0	i	$2i$	...	$(K-1)i$
2	I	$i+I$	$2i+I$	...	$(K-1)i+I$
3	$2I$	$i+2I$	$2i+2I$	...	$(K-1)i+2I$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	$(n-1)I$	$i+(n-1)I$	$2i+(n-1)I$	...	$(K-1)i+(n-1)I$

For the case in which K is an even number, the scheme follows immediately on replacing i with i' in Table 14.4 and the last initial position will no longer be $(K = 1)i$ but $(K-1)i'$.

The distribution of the origins in such a manner that in no direction more than one of the initial positions of the horizontal circle appears, can be made in various ways (*Jordan/Eggert/Kneissl* 1958, *Ghiţău* 1972, *Rotaru* 1975).

B. Measuring the angles

The measurement proper of the azimuthal angles is carried out on the basis of the previously described programme. One aims to measure those angles in whose directions there is the best visibility at a given moment. Therefore, the sequence of measurements is established according to this factor. The corresponding measurements are extended over a certain time interval, both by day and by night, using optical signalling by means of heliotropes or reflectors.

Other details in connexion with the measurement of the azimuthal angles by *Schreiber*'s method are to be found in many technical contributions (e.g., *Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Ghițău* 1972, *Oprescu* et al. 1973).

C. **Station adjustment**

The equal weights and the symmetrical disposition of the measurements make the station adjustment in the case of *Schreiber*'s method easily adaptable and clear. This adjustment may be carried out on directions or on angles, using the least squares method, by indirect observations or by conditional observations. From the practical point of view, the station adjustment on angles is somewhat simpler than that on directions.

The station adjustment on angles. There are numerous contributions in the technical literature which tackle the station adjustment on angles within the framework of *Schreiber*'s method in various ways, their common upshot being able to be summerized in the following: the value of each station-adjusted angle $L_{ij}(i \neq j = 1, 2, ..., K)$ results as a weighted arithmetic mean from the value L^0_{ij} of the angle directly measured, to which one ascribes the weight 2 and from $(K - 2)$ values $L^{(m)}_{ij}$ of the same angle, obtained as sum or difference of other angles directly measured, to which one ascribes the weight 1. This result can be expressed by the general formula:

$$L_{ij} = \frac{1}{K}\left(2L^o_{ij} + \sum_{m=1}^{K-2} L^{(m)}_{ij}\right).$$

In matricial form the results of the station adjustment on angles are presented as follows (*Rotaru* 1968, 1975):

(1) *By indirect observations:*

$$\mathbf{L} = \mathbf{C}\mathbf{L}^o;\ \mathbf{V} = \mathbf{C}_1\mathbf{L}^o;$$

$$\mathbf{Q}_{L(K-1)\times(K-1)} = \frac{1}{p^\circ K}\left\|\begin{array}{cccccccc} 2 & -1 & 0 & 0 & 0 \ldots 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \ldots 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \ldots 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \ldots 0 & -1 & 2 \end{array}\right\|,$$

or

$$\mathbf{Q}_{L(K-1)\times(K-1)} = \frac{1}{p^\circ K}\left\|\begin{array}{cccccccc} 2 & 1 & 1 & 1 & 1 \ldots 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \ldots 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \ldots 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1 \ldots 1 & 1 & 2 \end{array}\right\|;$$

$$m_0^2 = \frac{p^\circ \mathbf{V}^T\mathbf{V}}{\frac{1}{2}(K-1)(K-2)};\ \mathbf{V}^T\mathbf{V} = -\mathbf{L}^{oT}\mathbf{C}_1\mathbf{L}^\circ;\ m_{ij} = m_0\sqrt{\frac{2}{p^\circ K}}, \tag{14.3}$$

where $\mathbf{L}$ represents the vector of the unknowns; $\mathbf{L}^\circ$ — the vector of the values of the measured angles; $\mathbf{C}$ and $\mathbf{C}_1$ — standard matrices (fixed matrices) for a certain given number K of directions; $\mathbf{V}$ — the vector of the corrections of the angles; $\mathbf{Q}_L$ — the matrix of cofactors of the unknowns of the adjustment; $p^\circ = n$ — the weight of the measured angles; m_0 — the standard deviation of an angle measured in one series (standard deviation of the weight unit) and m_{ij} — the standard deviation of a station-adjusted angle. In the relations (14.3), the first formula for $\mathbf{Q}_L$ holds in the case in which one chooses as unknowns $L_{12}, L_{23}, \ldots, L_{(K-1)K}$ (Fig. 14.4, a) and the second one when one chooses as unknowns $L_{12}, L_{13}, .., L_{1K}$ (Fig. 14.4, *b*).

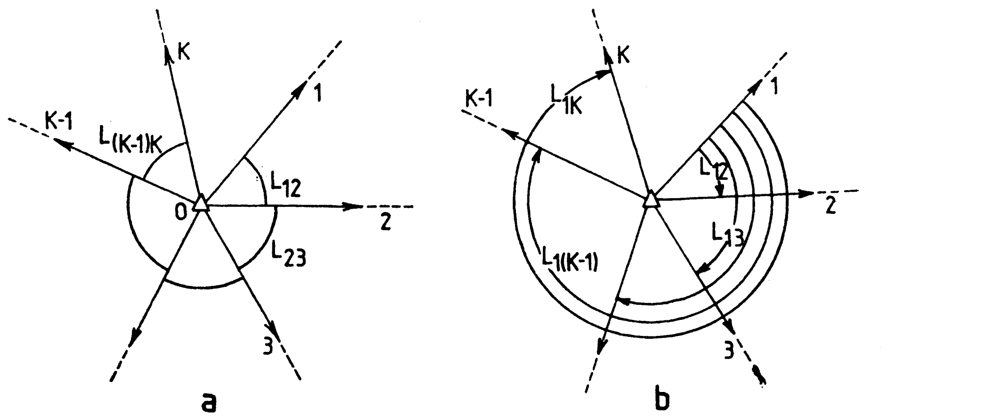

Fig. 14.4. Variants in Choosing the Unknowns of the Station-Adjustment on Angles by Indirect Observations

(2) *By conditioned observations:*

$$\mathbf{L} = \mathbf{L}^0 + \mathbf{V}; \; \mathbf{V} = \mathbf{DW}; \; m_0^2 = \frac{p^\circ \mathbf{V}^T\mathbf{V}}{\frac{1}{2}(K-1)(K-2)};$$

$$m_{ij} = m_0\sqrt{\frac{2}{p^\circ K}}, \tag{14.4}$$

where $\mathbf{D}$ represents a standard matrix (fixed matrix) for a certain given number K of directions.

$$
Q_{L\,\frac{1}{2}K(K-1)\times\frac{1}{2}K(K-1)} =
\left\|\begin{array}{ccccccc|cccccc|cccccc|c|c}
2 & 1 & 1 & 1 & \cdots & 1 & 1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 2 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & 0 \\
1 & 1 & 2 & 1 & \cdots & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 2 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & -1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 2 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\hline
-1 & 1 & 0 & 0 & \cdots & 0 & 0 & 2 & 1 & 1 & \cdots & 1 & 1 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & 0 \\
-1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & 1 & 2 & \cdots & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\hline
0 & -1 & 1 & 0 & \cdots & 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 2 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 \\
0 & -1 & 0 & 1 & \cdots & 0 & 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots \\
0 & -1 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 & \cdots & 1 \\
\hline
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \cdots & \vdots \\
\hline
0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & -1 & 1 & \cdots & 2
\end{array}\right\|
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\}(K\text{-}1)\ \textit{rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}(K\text{-}2)\ \textit{rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}(K\text{-}3)\ \textit{rows} \\
\\
1\ \textit{row}
\end{array}
\qquad (14.4)
$$

$$
(K\text{-}1)\ \textit{columns} \qquad (K\text{-}2)\ \textit{columns} \qquad \underbrace{\qquad\qquad}_{(K\text{-}3)\ \textit{columns}} \qquad \underbrace{\quad}_{1\ \textit{columns}}
$$

Remarks:

(1) In the case of the station adjustment by indirect observations as well as by conditioned ones, the values of the adjusted angles cannot be considered as independent observations for the subsequent adjustment of the triangulation network. To ensure correctness of the observations it is necessary to introduce the correlation matrix $\mathbf{Q}_L$.

(2) The fact that the standard matrices $\mathbf{C}, \mathbf{C}_1$ and $\mathbf{D}$ represent fixed matrices for a certain given number K of directions (e.g., for the astro-geodetic triangulation of the *Socialist Republic of Romania* the K values most frequently encountered are $K = 3; 4; 5; 6; 7;$ and 8) offers the possibility of standardizing the computations of the station adjustment, since if these matrices, established in advance, are available by simply multiplying by the corresponding vector $\mathbf{L}^\circ$, according to (14.3) and (14.4), the data of the station adjustment follow at once (*Rotaru 1969*).

Transition from the station-adjusted angles to a direction set. The station-adjusted angles being available, since their utilization as dependent values in adjusting the network complicates the computations very much, it is important to know whether there exists a valid procedure passing from them to a row of directions. The computation proper of the directions is simple — from angle differences — but complications do appear in establishing the weight system of the directions. The problem has a correct solution only in the case of the choice of the unknowns as in Fig. 14.4, *b*. Ascribing to one of the directions the value zero, the other directions become equal to the values of the adjusted angles considered form this direction.

For computing the weights of these values of the directions one may utilize:

(1) *Gaidaev's formulae* (*Gaidaev* 1960):

$$\begin{aligned} q_1 &= Q; \\ q_2 &= Q_{12,12} - Q; \\ &\dots\dots\dots \\ q_K &= Q_{1(K-1),\,1(K-1)} - Q, \end{aligned} \tag{14.5}$$

in which $q_i (i = 1, 2, \dots, K)$ represent the inverses of the weights of the directions; $Q_{12,\,12}, \dots, Q_{1(K-1),\,1(K-1)}$ — quadratic weight coefficients in the matrix $\mathbf{Q}_L$($\mathbf{Q}_L$ in (14.3)) for Fig. 14.4, *b*, whence it follows that $Q_{12,\,12} = \dots = Q_{1(K-1),\,1(K-1)} = 2/p^\circ K$ and Q — the value of the mixed weight-coefficients, which in the situation in Fig. 14.4, *b* all equal $1/p^\circ K$.

(2) *Urmaev's formulae* (*Gaidaev* 1960):

$$p_{ij}^{0\,\prime} = \frac{p_i p_j}{[p]}; \quad \frac{p_{1i}^0 p_{1j}^0}{p_{ij}^0} = \frac{p_1^2}{[p]} = \text{const.}, \tag{14.6}$$

in which $[p] = p_1 + p_2 + \dots + p_K$;

(3) *Helmert's formulae* (*Wolf* 1968):

$$q_j = \frac{1}{K-2}\left(S_j - \frac{[S]}{2(K-1)}\right), \tag{14.7}$$

in which $j = 1, 2, \dots, K$; $S_j = \Sigma q_{ij}$ — sum of the inverses of the weights of all the angles between the direction j and the other directions $i = j + 1, j + 2, \dots, K$; $[S]$ — sum of the inverses of the weights of all the angles at the station point, which were measured.

Station adjustment on directions. The transition from the $K(K-1)/2$ measured angles at the station point, by *Schreiber*'s method to a row of K directions may be accomplished directly by the station adjustment. To this end, one utilizes the theory of indirect observations, *Jordan's procedure*, which also has the advantage of allowing one to demonstrate the fact that in *Schreiber*'s method the values of the station-adjusted directions are independent of one another.

One can also use *Schreiber*'s procedure, which differs from that of *Jordan* in the fact that *Schreiber* starts from the semi-separate series and connects every angle's average with the station adjustment.

We shall not dwell any longer on these procedures here, since they are fully discussed in various technical contributions (*Schreiber* 1878 *a Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Wolf* 1968, *Ghițău* 1972, *Rotaru* 1975 etc.).

Remarks:

(1) *Schreiber's* method has some shortcomings such as:

— it assesses the weights of measurements on the basis of the number of series, without making allowances for the various physical correlations taking place within the measurement process;

— the great amount of time required for carrying out all measurements at the station point (and not only those of the angles directly necessary), sometimes even taking weeks, is not justified in view of the "very doubtful" gain (*Jordan/Eggert/Kneissl 1958*) as regards the accuracy of the results;

— one measures a large number of angles which are not actually needed in the subsequent network adjustment whereas the angle K 1, which is necessary, is not measured. For intance, at a station point with $K = 6$ directions, out of 15 distinct angles which are measured 10 angles (viz. 1 3; 1 4; 1 5; 1 6; 2 4; 2 5; 2 6; 3 5; 3 6; 4 6) are not required by the adjustment of the triangulation network.

(2) Due to the shortcomings of *Schreiber's* method, objections have been made to it or some time now and several proposals for improvement have been made, e.g. by *Levasseur* (*1942*), *Wolf* (*1942*) and *Danial* (*1967*).

14.1.3 The Method of Measuring the Neighbouring Angles

This method, also designated in the technical literature by other names (*measurement of the neighbouring angles covering the horizon* (*Jordan/Eggert/Kneissl* 1958), *measurement of the neighbouring angles with horizon closure* (*Wolf* 1968, *Grossmann* 1969, *Höpcke* 1969)) consists of separately measuring all the angles which do not cover each other, from among the successive directions of the station point. Unlike *Schreiber*'s method, one measures here only K angles, viz. 1 2, 2 3, ..., [(K — 1) K, K1 (Fig. 14.5), each of them with the same number n_1 of series.

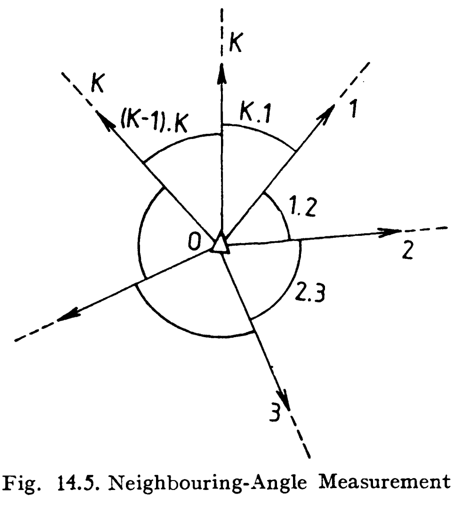

Fig. 14.5. Neighbouring-Angle Measurement

The method of measuring the neighbouring angles was applied in the *Czechoslovak Socialist Republic* for the observation of 58 station points in the new lst-order astro-geodetic triangulation (about 1/3). In the year 1975, the method was experimentally applied in *Romania* as well, in the national geodetic polygon, for 7 1st-order station points.

A. **Preparing the measuring programme** *(Rotaru* 1975*)*

First of all, one establishes the relation for determining the number n_1 of series in which the neighbouring angles are to be measured. To this end, the condition is imposed that in the method of measuring the neighbouring angles the same weight (p_1) of the adjusted elements be achieved, after the station adjustment, as in *Schreiber*'s method (p), i.e.:

$$p_1 = p. \tag{14.8}$$

As will be shown when discussing the station adjustment, the weight p_1 of a station-adjusted neighbouring angle is given by:

$$p_1 = n_1 \frac{K}{K-1}. \tag{14.9}$$

If in (14.9) one takes account of the condition (14.8), then one may thence deduce the relation for determining the number of series:

$$n_1 = p \frac{K-1}{K},$$

or, in view of the instructions in *Romania (Direcția topografică militară 1962)*,

$$n_1 = 18 \frac{K-1}{K}. \tag{14.10}$$

Table 14.5. *Series Number* $n_1 = f_1(K)$

K	2	3	4	5	6	7	8
n_1	9	12	14	15	15	16	16

After having established, as a function of K, the number n_1 of series, it is still necessary for ultimately shaping the measuring programme to know the origins of the n_1 series (i.e. the n_1 different positions of the theodolite's horizontal circle). These origins are calculated by means of the relation:

$$i = \frac{180^\circ}{n_1 - 1} \quad \text{or} \quad \frac{200^g}{n_1 - 1}, \tag{14.11}$$

in which i represents the angular interval between any two successive positions from among the n_1 initial positions. Consequently, the n_1 initial origins will be:

$$0, i, 2i, \ldots, (n_1 - 1)i, \tag{14.12}$$

which in fact also represent the origins for the angle 1 2.

For the other angles, the origins are established according to the following rule: *for the angle jl the origins of the direction j will be those resulting from the measurement of the preceding angle ij for the direction j.* As was shown practically *(Böhm* 1971*)*, proceeding in this manner almost completely eliminates from the sum of the neighbouring angles the influence of the errors in dividing the horizontal circle.

Example. *Let the measuring programme be established for a station with* $K = 6$ *directions.* First of all, from Table 14.5 it follows, in accordance with $K = 6, n_1 = 15$ series. Then, according to (14.11) one gets $i = 14^g\,28^c$.

According to (14.12), the n_1 initial origins of the angle 12 are shown in Table 14.6.

The working scheme (origin distribution) for all of the 6 neighbouring angles to be measured is shown in Table 14.7 *(Rotaru 1975)*.

The notations in Table 14.7 have the following meanings.

DS = left-hand direction of the neighbouring angle (actually representing the origins);

DD = right-hand direction of the neighbouring angle;

L° – value of the reading in the theodolite's position with the vertical circle on the left (face left); the superscript of L° represents the series number in brackets; the subscript of L° represents the number of the corresponding series.

B. Measuring the neighbouring angles

The measurement proper of the neighbouring angles is carried out on the basis of the programme previously established (point A). The neighbouring angles are measured, each of them in n_1 series, a series being performed as for any angle in *Schreiber*'s method.

Example. Using (14.10) depending on the various values K occurring most frequently in the astro-geodetic triangulation, follows the number n_1 of series as given in Table 14.5.

C. Station adjustment

By station adjustment one understands, in the case of the method of measuring the neighbouring angles, obtaining by means of the values L° of the neighbouring angles, measured at the station point, the most probable values of these angles (in the sense of the least squares method) which satisfy the condition of horizon round $[L] - 400^g = 0$, their precision, as well as the dependence character of the adjusted values.

The station adjustment may be performed on angles (by direct observations, by conditioned observations or by indirect ones) or on directions.

Table 14.6. *Initial Origins of the Angle 12*

ORIGIN	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV
VALUE g c	0 00	14 28	28 56	42 84	57 12	71 40	85 68	99 96	114 24	128 52	142 80	157 08	171 36	185 64	199 92

Table 14.7. *Origin Distribution for the Case* $K = 6$

SERIES / ANGLE	1		2		3		4		5		6		7	
	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD
12	I	$L_2^{O(1)}$	II	$L_2^{O(2)}$	III	$L_2^{O(3)}$	IV	$L_2^{O(4)}$	V	$L_2^{O(5)}$	VI	$L_2^{O(6)}$	VII	$L_2^{O(7)}$
23	$L_2^{O(1)}$	$L_3^{O(1)}$	$L_2^{O(2)}$	$L_3^{O(2)}$	$L_2^{O(3)}$	$L_3^{O(3)}$	$L_2^{O(4)}$	$L_3^{O(4)}$	$L_2^{O(5)}$	$L_3^{O(5)}$	$L_2^{O(6)}$	$L_3^{O(6)}$	$L_2^{O(7)}$	$L_3^{O(7)}$
34	$L_3^{O(1)}$	$L_4^{O(1)}$	$L_3^{O(2)}$	$L_4^{O(2)}$	$L_3^{O(3)}$	$L_4^{O(3)}$	$L_3^{O(4)}$	$L_4^{O(4)}$	$L_3^{O(5)}$	$L_4^{O(5)}$	$L_3^{O(6)}$	$L_4^{O(6)}$	$L_3^{O(7)}$	$L_4^{O(7)}$
45	$L_4^{O(1)}$	$L_5^{O(1)}$	$L_4^{O(2)}$	$L_5^{O(2)}$	$L_4^{O(3)}$	$L_5^{O(3)}$	$L_4^{O(4)}$	$L_5^{O(4)}$	$L_4^{O(5)}$	$L_5^{O(5)}$	$L_4^{O(6)}$	$L_5^{O(6)}$	$L_4^{O(7)}$	$L_5^{O(7)}$
56	$L_5^{O(1)}$	$L_6^{O(1)}$	$L_5^{O(2)}$	$L_6^{O(2)}$	$L_5^{O(3)}$	$L_6^{O(3)}$	$L_5^{O(4)}$	$L_6^{O(4)}$	$L_5^{O(5)}$	$L_6^{O(5)}$	$L_5^{O(6)}$	$L_6^{O(6)}$	$L_5^{O(7)}$	$L_6^{O(7)}$
61	$L_6^{O(1)}$	$L_1^{O(1)}$	$L_6^{O(2)}$	$L_1^{O(2)}$	$L_6^{O(3)}$	$L_1^{O(3)}$	$L_6^{O(4)}$	$L_1^{O(4)}$	$L_6^{O(5)}$	$L_1^{O(5)}$	$L_6^{O(6)}$	$L_1^{O(6)}$	$L_6^{O(7)}$	$L_1^{O(7)}$

8		9		10		11		12		13		14		15	
DS	DD	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD	DS	DD
VIII	$L_2^{O(8)}$	IX	$L_2^{O(9)}$	X	$L_2^{O(10)}$	XI	$L_2^{O(11)}$	XII	$L_2^{O(12)}$	XIII	$L_2^{O(13)}$	XIV	$L_2^{O(14)}$	XV	$L_2^{O(15)}$
$L_2^{O(8)}$	$L_3^{O(8)}$	$L_2^{O(9)}$	$L_3^{O(9)}$	$L_2^{O(10)}$	$L_3^{O(10)}$	$L_2^{O(11)}$	$L_3^{O(11)}$	$L_2^{O(12)}$	$L_3^{O(12)}$	$L_2^{O(13)}$	$L_3^{O(13)}$	$L_2^{O(14)}$	$L_3^{O(14)}$	$L_2^{O(15)}$	$L_3^{O(15)}$
$L_3^{O(8)}$	$L_4^{O(8)}$	$L_3^{O(9)}$	$L_4^{O(9)}$	$L_3^{O(10)}$	$L_4^{O(10)}$	$L_3^{O(11)}$	$L_4^{O(11)}$	$L_3^{O(12)}$	$L_4^{O(12)}$	$L_3^{O(13)}$	$L_4^{O(13)}$	$L_3^{O(14)}$	$L_4^{O(14)}$	$L_3^{O(15)}$	$L_4^{O(15)}$
$L_4^{O(8)}$	$L_5^{O(8)}$	$L_4^{O(9)}$	$L_5^{O(9)}$	$L_4^{O(10)}$	$L_5^{O(10)}$	$L_4^{O(11)}$	$L_5^{O(11)}$	$L_4^{O(12)}$	$L_5^{O(12)}$	$L_4^{O(13)}$	$L_5^{O(13)}$	$L_4^{O(14)}$	$L_5^{O(14)}$	$L_4^{O(15)}$	$L_5^{O(15)}$
$L_5^{O(8)}$	$L_6^{O(8)}$	$L_5^{O(9)}$	$L_6^{O(9)}$	$L_5^{O(10)}$	$L_6^{O(10)}$	$L_5^{O(11)}$	$L_6^{O(11)}$	$L_5^{O(12)}$	$L_6^{O(12)}$	$L_5^{O(13)}$	$L_6^{O(13)}$	$L_5^{O(14)}$	$L_6^{O(14)}$	$L_5^{O(15)}$	$L_6^{O(15)}$
$L_6^{O(8)}$	$L_1^{O(8)}$	$L_6^{O(9)}$	$L_1^{O(9)}$	$L_6^{O(10)}$	$L_1^{O(10)}$	$L_6^{O(11)}$	$L_1^{O(11)}$	$L_6^{O(12)}$	$L_1^{O(12)}$	$L_6^{O(13)}$	$L_1^{O(13)}$	$L_6^{O(14)}$	$L_1^{O(14)}$	$L_6^{O(15)}$	$L_1^{O(15)}$

Inasmuch as the adjustment on angles by direct and by conditioned observations has been rather amply dealt with already in other technical contributions *(Jordan/Eggert/Kneissl* 1958, *Wolf* 1968, *Grossmann* 1969*)*, these problems will not be approached in the present book.

We shall present only the station adjustment on angles by indirect observations, in a matricial treatment, which offers the possibility of drawing the necessary conclusions as well as some specifications concerning the station adjustment on directions.

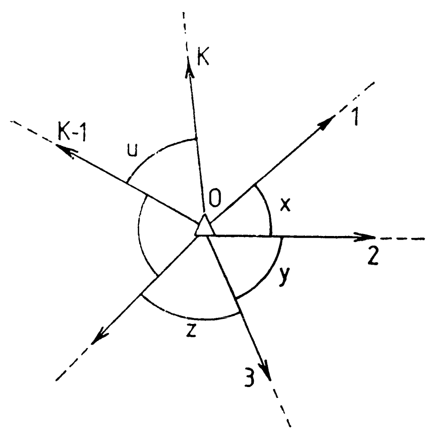

Fig. 14.6. Choosing the Unknowns of the Station Adjustment by Indirect Observations

The station adjustment on angles by indirect observations *(Rotaru* 1975*)*. Let us assume that at any station point with K directions there were measured, by the method of measuring the neighbouring angles, the angles 1 2, 2 3, ..., $(K - 1)$ K, K 1, each of them with n_1 series (i.e. $p_1^0 = n_1$). In order to be able to carry out the adjustment by indirect observations, let us choose $(K - 1)$ unknowns considering as such (Fig. 14.6)

$$x, y, z, \ldots, u. \tag{14.13}$$

Obviously, we could equally have chosen any other combination of $(K - 1)$ unknowns out of the K existing in the case under analysis. The values L of the K station-adjusted neighbouring angles are, generally, given by various functions of the unknowns (14.13), i.e.:

$$L = F(x, y, z, \ldots, u). \tag{14.14}$$

Considering that in the case under analysis F represents various linear functions for the most probable values L_{ij} $(i = 1, 2, \ldots, K; j = i + 1$, except for the angle K 1, where $j = 1)$ of the measurements L_{ij}^0, according to the adjustment by indirect observations, the following expressions ensue from (14.14):

$$L_{ij} = L_{ij}^0 + v_{ij} = F_{ij}(x, y, z, \ldots, u),$$

where L_{ij}^0 denotes the mean value of the angle ij measured in n_1 series and v_{ij} the correction to be applied to L_{ij}^0.

The system of the equations of corrections will have the form:

$$\begin{aligned} L_{12} &= L_{12}^0 + v_{12} = x; \\ L_{23} &= L_{23}^0 + v_{23} = y; \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ L_{(K-1)K} &= L_{(K-1)K}^0 + v_{(K-1)K} = u; \\ L_{K1} &= L_{K1}^0 + v_{K1} = 400^g - (x + y + \ldots + u). \end{aligned} \tag{14.15}$$

The system (14.15) may also be written in the form:

$$\begin{array}{llll}
v_{12} = L_{12} & & -L^0_{12}, & \text{with weight } p^0_1; \\
v_{23} = & L_{23} & -L^0_{23}, & p^0_1; \\
\multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
v_{(K-1)K} = & & L_{(K-1)K} - L^0_{(K-1)K}, & p^0_1; \\
v_{K1} = & -L_{12} - L_{23} - \dots - L_{(K-1)K} - L^0_{K1} + 400^g, & & p^0_1,
\end{array}$$

which matricially is expressed as:

$$\mathbf{V} = \mathbf{BL} - \mathbf{L}^\circ, \text{ with the weight matrix } \mathbf{P}^0_1, \tag{14.16}$$

where:

$\mathbf{V}^T_{1\times K} = \| v_{12}\ \ v_{23} \dots v_{(K-1)K}\ \ v_{K1} \|$ — the transposed vector of the corrections;

$$\mathbf{B}_{K\times(K-1)} = \begin{Vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & -1 & -1 & \dots & -1 \end{Vmatrix}$$ = matrix of the coefficients of the unknowns;

$\mathbf{L}^T_{1\times(K-1)} = \| L_{12}\ \ L_{23} \dots L_{(K-1)K} \|$ = transposed vector of the unknowns;

$\mathbf{L}^{0T}_{1\times K} = \| L^0_{12}\ \ L^0_{23} \dots L^0_{(K-1)K}\ \ L^0_{K1} - 400^g \|$ = transposed vector of the observations;

$\mathbf{P}^0_1 = p^0_1 \mathbf{E}_{K\times K}$; $\mathbf{E}$ — unit matrix.

Since only $(K-1)$ unknowns are sought, the system (14.16) contains a supplementary determination which can be removed only by *Legendre*'s condition: $\mathbf{V}^T\mathbf{P}^0_1\mathbf{V} = p^0_1\mathbf{V}^T\mathbf{V} =$ minimum. Substituting for $\mathbf{V}$ its expression given by (14.16) in this condition and performing some simple operations with matrices, one gets:

$$\mathbf{V}^T\mathbf{V} = \mathbf{L}^T\mathbf{B}^T\mathbf{B}\mathbf{L} - 2\mathbf{L}^{0T}\mathbf{B}\mathbf{L} + \mathbf{L}^{0T}\mathbf{L}^0. \tag{14.17}$$

In order that *Legendre*'s condition be fulfilled, it is necessary that $\dfrac{\partial(\mathbf{V}^T\mathbf{V})}{\partial L_{i(i+1)}} = 0$ $(i = 1, 2, \dots, K$, with $L_{K(K+1)} = L_{K1})$, resulting from formula (14.17):

$$\mathbf{L} = \mathbf{C}\mathbf{L}^0, \tag{14.18}$$

where:

$$\mathbf{C} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T, \tag{14.19}$$

represents a standard matrix for a given number K of directions. Introducing (14.18) into (14.16) yields:

$$\mathbf{V} = \mathbf{C}_1\mathbf{L}^0, \tag{14.20}$$

where:

$$\mathbf{C}_1 = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T - \mathbf{E}, \tag{14.21}$$

also represents a standard matrix.

Computing the products appearing in (14.19) and (14.21), the standard matrices $\mathbf{C}$ and $\mathbf{C}_1$ finally turn out as follows:

$$\mathbf{C}_{(K-1)\times K} = \frac{1}{K}\left\|\begin{matrix} K-1 & -1 \ldots & -1 & -1 & -1 \\ -1 & K-1 \ldots & -1 & -1 & -1 \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ -1 & -1 \ldots & K-1 & -1 & -1 \\ -1 & -1 \ldots & -1 & K-1 & -1 \end{matrix}\right\|; \quad \mathbf{C}_{1K\times K} = \frac{1}{K}\left\|\begin{matrix} 1 & 1 \ldots & 1 \\ 1 & 1 \ldots & 1 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 1 & 1 \ldots & 1 \end{matrix}\right\|. \tag{14.22}$$

Introducing (14.22) into (14.18) and into (14.20) respectively finally yields the following results of the station adjustment:

$$\mathbf{L}^T = \| L_{12} \quad L_{23} \ldots L_{(K-1)K} \| = \left\| L_{12}^0 - \frac{w}{K} \quad L_{23}^0 - \frac{w}{K} \quad \ldots L_{(K-1)K}^0 - \frac{w}{K} \right\|; \tag{14.23}$$

$$\mathbf{V}^T = \| v_{12} \quad v_{23} \ldots v_{(K-1)K} \quad v_{K1} \| = \left\| -\frac{w}{K} \; -\frac{w}{K}, \; \ldots, \; -\frac{w}{K} \right\|,$$

where $w = [L^0] - 400^g$.

For calculating the precision, by applying the general formula of the adjustment by indirect observations *(Lörinczi* 1966, *Wolf* 1968*)*, which in the case under analysis takes the form $m_0^2 = p_1^0\mathbf{V}^T\mathbf{V}$ (m_0 being the standard deviation of a neighbouring angle measured in one series), and taking into consideration the second relation in (14.23), one finally gets:

$$m_0 = \pm w \sqrt{\frac{n_1}{K}}. \tag{14.23'}$$

Continuing, in order to be able to decide whether or not the results of the station adjustment are independent values, one resorts to the matrix of the cofactors (correlation matrix) $\mathbf{Q}_L$ *(Wolf* 1968*)*, which for the present case has the form:

$$\mathbf{Q}_L = \frac{1}{p_1^0} (\mathbf{B}^T\mathbf{B})^{-1}.$$

Working out the product $\mathbf{B}^T\mathbf{B}$ and then its inverse, we end up with:

$$\mathbf{Q}_{L(K-1)\times(K-1)} = \frac{1}{n_1 K} \left\| \begin{array}{cccc} K-1 & -1 & \ldots & -1 \\ -1 & K-1 & \ldots & -1 \\ \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \ldots & K-1 \end{array} \right| . \qquad (14.24)$$

From the analysis of formula (14.24) one may draw the following conclusions:

(1) *After the station adjustment, all the values L_{ij} have the same weight*

$$p_1 = n_1 \frac{K}{K-1}$$

since the quadratic weight coefficients are equal. This result confirms the correctness of the established measuring programme.

(2) *The values L_{ij} are dependent values, as all mixed weight coefficients Q_{lh} ($l \neq h$; $l, h = L_{12}, L_{23}, \ldots, L_{(K-1)K}$) differ from zero.* In order to evaluate this dependence one can proceed in the following way: one calculates the correlation coefficient r_{lh} with the relation *(Grossmann* 1969*)*:

$$r_{lh} = \frac{Q_{lh}}{\sqrt{Q_{ll}Q_{hh}}}, \qquad (14.25)$$

Table 14.8. *Various Values $r = f(K)$*

K	2	3	4	5	6	7	8
r	−1	−0.50	−0.33	−0.25	−0.20	−0.17	−0.14

and one discovers that $r_{ll} = r_{hh} = 1$ while all $r_{lh} = -1/(K-1)$. These results suggest a very important remark: gradually as the number K of directions at the station point increases, the correlation between the values of the adjusted neighbouring angles decreases. In Table 14.8 the coefficients r as depending on the various values K encountered in the practice of astro-geodetic triangulation are calculated. From Table 14.8 it follows that for the station points with $K > 4$ directions, this correlation is already weak ($|r| \leqslant 0.25$).

Remark. Inasmuch as the angle $K1$ has been measured according to the same programme as the $(K-1)$ angles (which represent here the unknowns of the station adjustment) as well as due to the fact that as a result of the station adjustment results v_{K1}, which does not differ at all from the other corrections, one may consider that its adjusted value, i. e. $L_{K1} = L^0_{K1} + v_{K1}$, is characterized by the same weight coefficients (quadratic and mixed) as the unknowns of the adjustment, viz.:

$$Q_{L_{K1}L_{K1}} = \frac{K-1}{n_1 K} \text{ and } Q_{L_{K1}L_{12}} = Q_{L_{K1}L_{23}} = \dots = Q_{L_{K1}L_{(K-1)K}} = -\frac{1}{n_1 K}.$$

(3) *In the subsequent adjustment of the triangulation network, the values L_{ij} will be introduced as dependent elements, according to the correlation matrix $\mathbf{Q}_L$, of order $K \times K$.*

Knowing the matrix $\mathbf{Q}_L$, one may now calculate the standard deviation of a station-adjusted neighbouring angle, by means of the general formula *(Wolf* 1968*)*:

$$m_i = m_0 \sqrt{Q_{ll}},$$

in which, taking account of (14.23′) and the fact that:

$$Q_{L_{ij}L_{ij}} = \frac{K-1}{n_1 K},$$

one gets:

$$m_{ij} = \pm \frac{w}{K} \sqrt{K-1}. \tag{14.26}$$

The station adjustment on directions. This problem is completely treated *(Wolf* 1957*)* starting from the consideration that the station-adjusted directions $L_h (h = 1, 2, \dots, K)$ can be represented as functions deduced from the values L^0_{ij} of the measured neighbouring angles and taking the direction 1 as equal to zero, one finds the collection of directions L'_h. Then, giving up the hypothesis $L'_1 = 0$ (in order to obtain the coefficients of the values L_{ij} in the form of symmetrical ratios) and orienting the values L'_h, by adding an orientation angle $Z = -[L']/K$, so that the value of any direction L_h is determined by means of the relation $L_h = L'_h + Z$, one then arrives finally at the following practical formulae for calculating the station-adjusted directions *(Rotaru* 1975*)*:

$$\begin{aligned}
L_1 &= \frac{1}{2K}\{-(K-1)L^0_{12} - (K-3)L^0_{23} - (K-5)L^0_{34} - \dots + \\
&\quad + (K-3)\,L^0_{(K-1)K} + (K-1)L^0_{K1} - (K-1)400^g\}; \\
L_2 &= \frac{1}{2K}\{+(K-1)L^0_{12} - (K-1)L^0_{23} - (K-3)\,L^0_{34} - \dots + \\
&\quad + (K-5)\,L^0_{(K-1)K} + (K-3)L^0_{K1} - (K-3)\,400^g\}; \qquad (14.26) \\
L_3 &= \frac{1}{2K}\{+(K-3)L^0_{12} + (K-1)L^0_{23} - (K-1)L^0_{34} - \dots + \\
&\quad + (K-7)\,L^0_{(K-1)K} + (K-5)L^0_{K1} - (K-5)400^g\};
\end{aligned}$$

. . . .

$$L_{K-1} = \frac{1}{2K}\{-(K-5)L^0_{12} - (K-7)L^0_{23} - (K-9)L^0_{34} - \ldots -$$

$$- (K-1)\,L^0_{(K-1)K} - (K-3)L^0_{K1} + (K-3)400^g\};$$

$$L_K = \frac{1}{2K}\{-(K-3)L^0_{12} - (K-5)L^0_{23} - (K-7)L^0_{34} - \ldots +$$

$$+ (K-1)\,L^0_{(K-1)K} - (K-1)L^0_{K1} + (K-1)\,400^g\},$$

with the correlation matrix:

$$\mathbf{Q}_{L(K\times K)} =$$

$$\frac{1}{12K}\left\|\begin{matrix} K^2-1 & K^2-6K+5 & K^2-12K+23 \ldots & K^2-12K+23 & K^2-6K+5 \\ K^2-6K+5 & K^2-1 & K^2-6K+5 \ldots & K^2-18K+53 & K^2-12K+23 \\ K^2-12K+23 & K^2-6K+5 & K^2-1 \ldots & K^2-24K+95 & K^2-18K+53 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K^2-12K+23 & K^2-18K+53 & K^2-24K+95 \ldots & K^2-1 & K^2-6K+5 \\ K^2-6K+5 & K^2-12K+23 & K^2-18K+53 \ldots & K^2-6K+5 & K^2-1 \end{matrix}\right\|.$$

Remark. With respect to the relations given by *Wolf* (1957, (9)), the relations (14.26) have an additional term, that involving 400^g. This result, slightly different, but important from the point of view of the practical calculation, is due to the form of the initial condition from which the author starts, viz. (3):

$$v_{12} + v_{23} + \cdots + v_{K1} + w = 0,$$

where $w = [L^\circ]$.

Analysing the matrix Q_L leads to the following conclusions:

(1) *All the values L_h have the same weight $p_h = 1/Q_{hh} = 12K/(K^2-1)$.* The standard deviation of a station-adjusted direction will consequently be calculated using the formula $m_h = m_0\sqrt{Q_{hh}}$ where m_0 represents the standard deviation of a neighbouring angle measured in n_1 series.

(2) *Despite the symmetrical order of the measurements, the values of the station-adjusted directions do not possess the character of orthogonal functions, since all mixed weight coefficients are different from zero.*

(3) *When subsequently adjusting the triangulation network, the values L_h will be introduced as dependent elements, according to the matrix $\mathbf{Q}_L$.*

The method of measuring the neighbouring angles offers various advantages, important as compared with *Schreiber*'s method, viz. *(Rotaru* 1975*):*

(1) *The method is simpler and more convenient, since for every station point there appears only one condition equation and the measurement proper is carried out over a shorter time interval.*

(2) *One only measures the angles strictly needed by the subsequent adjustment of the triangulation network, the angle K1 included.*

(3) *It facilitates a better analysis, checked in the field itself, of the observations with the aid of statistical tests, inasmuch as here the number of additional observations of an angle is larger.*

(4) *The errors due to a poorer visibility in some directions, where greater variations of the lateral refraction have taken place during the measurements, remain isolated.*

(5) *Thanks to the established measuring programme, the influence of the errors in dividing the theodolite's horizontal circle ar ealmost completely eliminated from the sum of the neighbouring angles.*

(6) *For the station points with $K \geqslant 5$ directions the total number of measurements is smaller.*

Of course, two objections could be made to the method of measuring the neighbouring angles:

(1) *As only one condition equation is available, the standard deviation of a station-adjusted angle has no practical meaning.*

(2) *After the station adjustment, the values of the adjusted angles or directions are not independent, which complicates the subsequent calculations in adjusting the triangulation network.*

Both objections, which are valid in principle, are completely removed (*Wolf* 1957, *Rotaru* 1975) by the fact that, as a single condition equation is available, it is possible and convenient to give up the station adjustment, the corresponding equation being introduced along with the other condition equations of the network, as an initial condition equation when adjusting the triangulation network, the adjustment being carried out by conditioned observations with angle corrections. In the case of *Schreiber*'s method, however, one cannot give up the station adjustment, which is strictly necessary.

Even if one does not give up the station adjustment, the fact that to the row of neighbouring angles or directions, resulting from this adjustment, one adds the corresponding correlation matrix $\mathbf{Q}_L$ — as the adjustment of the triangulation network utilizes the theory of the dependent observations (*Tienstra* 1956, *Wolf* 1968, *Botez* et al. 1971) — does not pose a difficult problem under conditions where electronic computers are used.

14.1.4 The Method of the Reference Torques

This method has been widely used in *France* (*Levallois* 1969) and, to a lesser extent, in the astro-geodetic triangulation of *Romania*. The method of the reference torques consists of bringing every direction $(2, 3, \ldots, K)$ from the corresponding station point to the origin 1 (which can be the direction 1 of the horizon round of the station or another common fixed reference direction), by carrying out two succesive measurements, i.e., e.g. for the direction 2:

1 2 circle right-hand direct rotation (reverse rotation);
2 1 circle left-hand inverse rotation (direct rotation).

The operation is then repeated as many times as necessary so that the disposition "circle right-hand", "circle left-hand" and the rotation course of the instrument should alternate. In this manner one measures one by one the angles formed by each of the K directions and the reference direction 1. Assuming the reference direction 1 to be equal to zero, one arrives at the case of a complete series of directions.

One considers *(Jordan/Eggert/Kneissl* 1958*)* an advantage of the method the fact that one can measure the angles in respect of visibility of reference 1, so that in this way one can obtain measurements of the same accuracy. However, the method also has the disadvantage that among all pointings of the reference direction 1, only half of them are used for the triangulation network.

In the technical literature, the method is also encountered under the name *measuring the angles with an auxiliary signal* (e.g. *Jordan/Eggert/Kneissl* 1958*)*.

For *France*'s 1st-order astro-geodetic triangulation, for example approximately 16—20 couples at the repeated origins were used, the observations being distributed over several days, every direction measurement corresponding to several readings on the horizontal circle, pointing at the signal.

14.1.5 The Method of the Sectors

The method of the sectors was worked out at the beginning of this century by *Heinrich Wild* and was first used, with very good results, from the year 1904 in *Switzerland* for 2nd and 3rd-order angular measurements and only since 1910 for the 1st-order triangulation in the region of *the Alps* and for revising the angular measurements in the *Swiss Plateau*. Therefore, the method is sometimes also called the *Swiss method of the sectors (Wolf* 1968*)*.

Although within the framework of the above-mentioned operations theodolites of *Hildebrand* type with a diameter of the horizontal circle of only 21 cm were used for the measurements and the observations were made, as a rule, on stone pillars or on tripods, an excellent standard deviation of a measured angle resulted, calculated from *Ferrero*'s formula, viz. $\pm 0''.3$ *(Jordan/Eggert/Kneissl* 1958*)*.

A. **Preparing the measuring programme**

At every station point with K directions one first chooses a few directions as *main directions* (which offer the best visibility and have the longest sights), dividing the horizon round into several angles or sectors (Fig. 14.7), as a rule 3 or 4 main directions. These main directions are distributed as symmetrically as possible on the entire horizon round and contain between them the other directions of the station point, which are called *intermediate directions* (e.g.: 2; 4; ... K).

The sectors (angles between the main directions) cover the horizon round with neighbouring angles whose sum must equal 2π. These angles are measured several times as separate angles in various positions (origins) of the theodolite's horizontal circle.

The intermediate directions form between them angles designated as intermediate angles which are also measured as separate angles. These directions must be continuously connected, by measuring the intermediate

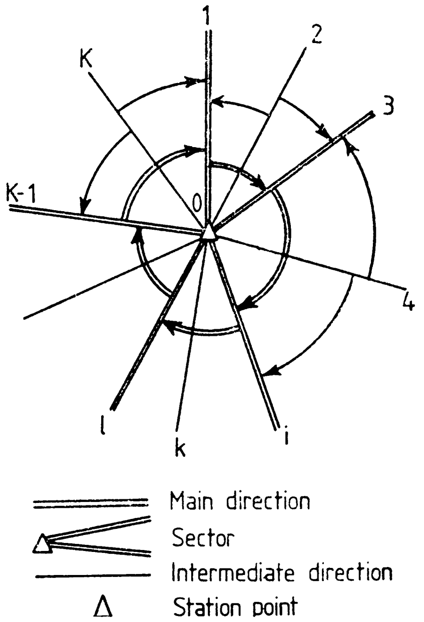

Fig. 14.7. Sectors' Method

angles, with both main directions including them, viz. in such a manner that the sum of the intermediate angles give the value of the angle of the corresponding sector. If within one sector there are several intermediate directions, then that sector may be divided so that an intermediate direction with a good visibility is used as a main direction.

The weight of an angle directly measured equals the number of measured series, a series usually meaning the mean of the values of the both positions of the theodolite.

B. **Measuring the angles**

The measurement of separate angles (sector angles or intermediate angles) is carried out within the framework of the method of sectors in the same way as for any angle in *Schreiber*'s method or in the method of measuring the neighbouring angles.

C. **Station adjustment**

In the case of the method of sectors the station adjustment involves three stages *(Jordan/Eggert/Kneissl* 1958*)*:

(1) *Calculating the weighted arithmetic average for every angle of the corresponding sectors.*

(2) *Adjusting the angles of the sectors according to the horizon-round condition.*

(3) *Calculating and adjusting the intermediate angles according to the fixed-angle condition of the corresponding sectors.*

Within the framework of the station adjustment all measured angular values are considered as observations independent of one another. The results of the station adjustment are not orthogonal functions *(Wolf* 1968*)*.

From the practical works carried out so far (particularly in *Switzerland*), it follows that the sector method scores over *Schreiber*'s method in several ways, viz. *(Jordan/Eggert/Kneissl* 1958*)*:

(1) *It makes it easier to take into consideration the atmospheric and topographic conditions concerning both the measurement proper and the station adjustment.*

(2) *By measuring separately the directions of good visibility one very rapidly obtains the angles of the sectors, which may be considered as being of the same precision.* These angles facilitate a provisional closure of the horizon round and thereby a reliable assessment of the intermediate angles.

(3) *The measuring programme, which is simple and clear, can be most simply developed during the measurement by inserting additional intermediate as well as main directions.* Therefore, it is advantageous to use the sector

method for modernizing the 1st-order triangulation networks, inasmuch as it is frequently necessary to improve the geometrical configuration of the network by transforming 2nd-order and even 3rd order points into 1st-order points and, consequently, new measurements must be carried out at these points.

(4) *The total number of angles to be measured is smaller.*

(5) *The method of establishing the weights of both observations and definitive values gives the operator enough freedom of action, so that he can measure an angle as many times as necessary until he is convinced that the observations are correct.*

(6) *The method leads to better results, while being undoubtedly superior as far as economy is concerned.*

Remarks:

(1) The qualities (1) . . . (6) show up in particular in the case of measurements carried out under special conditions (mountainous terrain, areas of difficult access) in which the geodetic signals can be maintained in the field for only a short time and the measurements at the corresponding station points cannot be carried out several times.

(2) Measuring the angles by the method of the sectors may be interpreted as a variant of the method of measuring the neighbouring angles, inasmuch as here too the horizon round is covered, albeit twice, by the angles of the sectors and then by the intermediate angles.

14.1.6 Checking the Field Angle-Observations by means of Statistical Significance Tests

One of the present problems of geodesists and specialized institutions consists of the modernization of the existing national 1st-order astro-geodetic network. Among the criteria which must underlie this qualitative improvement, is the analysis of field observations before being adjusted. Such a judicious analysis, which should allow a framing of the qualitative value of the performed observations and the establishment of a plan to repeat measurements, is impossible without resorting to mathematical statistics. As a matter of fact, the utilization of mathematical statistics has greatly modified the view taken by specialists of Geodesy, as well as the understanding of certain notions, the main benefit being especially in the process of checking the hypotheses, viz. in two respects *(Wolf* 1977*)*:

a) It has allowed one to pass from a general qualitative valuation (in the sense of "good" or "bad")using the comparison of empirical frequencies with theoretical ones, to a quantitative more exact estimation using as estimator the chi-square number (χ^2) in order to trace the results of measurements significantly affected by systematic errors.

b) It has offered a much more precise and objective means than the treble of the standard deviation viz. the confidence interval (which depends on a probability of given certainty), for checking whether one can rely upon adjustment results when one assumes a normal distribution.

(a) **Some general considerations on statistical tests.** Geodetic practice has shown that when repeating the measurement of one and the same quantity, the results continuously vary among themselves. The differences are explained by the presence within the framework of the measurements of so-called unavoidable random errors. In the case of a very large number of such errors, they follow *Gauss'* normal distribution. As is well known *(Bőhm* 1967, *Göhler* 1968*)*, the theoretical distribution, constituting *the population* [1] distribution, can be characterized by a few parameters, among which the most important ones are *the actual value (true value)* ξ of the quantity which is being measured and *the precision (standard deviation)* σ^2. For instance, the measurement of the azimuthal angles at a station point continuously aims at determining the actual, generally unknown, parameters ξ and σ of a normal-distribution population. From the measurement proper one can in practice obtain only estimation values for ξ and σ, viz. the mean L^0 and the standard deviation m_0 from a limited number of repeated measurements — *the sounding sample* [3]. E.g., in the case of *Schreiber*'s method $n_{max} = 12$ for $K = 3$ directions and with the method of measuring the neighbouring angles $n_{max} = 16$, for the case $K = 7$; 8 directions. The basic estimators L^0 and m_0 of a sounding sample are random values and are called *functions of the sounding sample (Göhler* 1968*)*, whose distribution laws are known (e.g.: *Fisher*'s distribution, *"Student"*'s distribution, *Helmert-Pearson*'s distribution etc.). The values L^0 and m_0 obtained from angle measurement at a station point lead to the most probable values L and m which, in a first approximation, best estimate the unknown parameters ξ and σ.

Mathematical statistics at present offers methods permitting the estimation of the parameters of the population by using the basic estimators of the sounding sample.

Remark. The methods of mathematical statistics do not yield an absolutely certain conclusion but only a probability-theoretical one.

In principle, one distinguishes two possibilities: statistical estimation methods and statistical checking procedures *(statistical tests)*. With the help of the statistical tests one checks whether or not the statistical estimators of the sounding sample are admissible with regard to a hypothesis initially taken into consideration, while by statistical hypothesis is understood any assumption concerning the frequency distribution of random variable capable of being measured. If the form of the distribution function is known, then one considers a hypothesis and one checks it only as far as the unknown parameters are concerned, the test being called in this case a *parametric test.* If the statistical hypothesis also requires knowledge of the distribution form,

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[1] *The population* represents here all possible imaginable observations which could be obtained by continuing to infinity the measurements carried out under identical means, conditions and methods, on any one and the same angle at the station point.

[2] σ represents the square root of the variance σ^2 of the population, which is given by the relation $\sigma^2 = (1/N) \sum_{i=1}^{N} \Delta_i^2$ in which $\Delta_i = L_i - \xi$ $(i = 1, 2, \ldots, N)$, N — the population volume $(N \rightarrow \infty)$. E.g., the rigorous relation between the measured angles (A, B, C) of a geodetic triangle: $W = L_A^0 + L_B^0 + L_C^0 - (180^\circ + \varepsilon)$, where W is the triangle's misclosure, L_A^0, L_B^0, L_C^0 — the values of the angles A, B, C as found from measurement; ε — the spherical excess. The correspondence with the relation of Δ_i is as follows: $W = \Delta_i$, $L_A^0 + L_B^0 + L_C^0 = L_i$, $180 + \varepsilon = \xi$.

[3] *The sounding sample* represents the row of observations obtained from n series of independent measurements carried out on any one and the same angle at the station point; n is called *the sounding-sample volume.*

then the checking procedure is called a *non-parametric test* or *distribution-free test (Göhler* 1968*)*.

A decision on admitting or rejecting a statistical hypothesis assumes the consideration of a *null hypothesis* through which the population of all the results of the possible sounding samples is divided into two precisely defined regions: *the admittance (acceptance) region and the rejection region (critical region)*. The boundary between these two regions is established with the help of the occurrence probability of significant errors which could result from the hypothesis being considered.

The principle of the statistical tests has foreseen that the decision should be made on the basis of the sounding samples, but the estimators of the latter are random values. This circumstance favours the occurrence of errors in estimating the population parameters. Thus, considering it very improbable that from a single measurement there should result a value best estimating the true value of any horizontal angle, something which occurs quite seldom—only in $\alpha\%$ of the cases—, one rejects the hypothesis that this value could belong to the established population, even if this does not happen. Consequently, a wrong decision is made and the error which is made in this case is called an *error of the first kind (Mihoc* and *Firescu* 1966, *Wolf* 1968, *Göhler* 1968*)*, to which corresponds the *risk of the first kind* (or *producer's risk*) α[1] *(Mihoc* and *Firescu* 1966, *Wolf* 1968*)*. There exists a second situation, opposite to the first one, in which a hypothesis may be accepted although it does not agree with the population, from which a good sounding sample is by chance available. Consequently, one makes the error of admitting a wrong hypothesis, which is called an *error of the second kind (Mihoc* and *Firescu* 1966, *Wolf* 1968, *Göhler* 1968), to which corresponds the *risk of the second kind (consumer's risk)* β *(Mihoc* and *Firescu* 1966, *Wolf* 1968*)*. Both errors are random events and occur with definite probabilities.

In the case of statistical tests, verifying a certain hypothesis, viz. if this hypothesis is not wrong, while at the same time one does not consider any other hypothesis either, i.e. for the so-called *significance tests (Göhler* 1968, *Giurgiu* 1972*)*, only the probability α of the error of the first kind is fixed, the error β not being considered here. The difference $1 - \alpha = P$ is called *statistical reliability (Wolf* 1968*)*. With the aid of α one establishes the so-called *reliability limit (critical limit)* between the acceptance and rejection regions. Inasmuch as for the choice of α no single mathematical motivation exists, it is necessary to carry out a thorough analysis bearing on possible and random wrong decisions, on measuring conditions and on the importance of the corresponding measurement. In the technical literature the trend is that the risk of the first kind, which in the following will be simply called *"risk"*, be kept as low as possible. Generally, $0.001 \leqslant \alpha \leqslant 0.1$ and quite frequently $\alpha = 0.05$.

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[1] In the technical literature one finds other designations various for α: *error probability* (*Mihoc* and *Firescu 1966*, *Wolf 1968*), *level of significance* (*Böhm* 1967, *Wolf 1968*). *reliability threshold* (*Lörinczi* 1966, *Smirnov* and *Belugin* 1969), *test level* (*Wolf 1968*), *power of the test* (*Mihoc* and *Firescu 1966*, *Baarda 1967*).

The statistical significance tests, which lend themselves to checking the angular observations, may be divided into two groups [1]:

(1) *Tests assuming as known the standard deviation* σ (e.g.: the W test, the u test, the τ test).

(2) *Tests assuming the standard deviation* σ *as unknown* (e.g.: the F test, the t test).

Applying any one of the significance tests implies going through the following stages (*Göhler* 1968):

1) *Assuming a null hypothesis.*

2) *Calculating the test quantity;* at this stage one chooses a function of the sounding samples whose distribution is known and whose parameters are checked.

3) *Establishing the risk* α *and the statistical reliability* P *respectively.*

4) *Extracting the critical limit corresponding to* α, *respectively to* P *from statistical tables* (*e.g.: Fisher and Yates* 1957, *Bolishev and Smirnov* 1965, *Graf et al.* 1966, *Wilks* 1967, *Böhm* 1967/1968, *Wolf* 1968, *Grossmann* 1969, *Smirnov and Belugin* 1969, *Rotaru* 1975) according to the number of degrees of freedom ν [2].

5) *Making the decision;* here one distinguishes two situations:

5.1) If the value of the test quantity lies within the region defined by the critical limit (the acceptance region) or equals the value of this limit, then there are no reasons for rejecting the hypothesis taken into consideration; the hypothesis is accepted.

5.2) If the value of the test quantity lies outside the region defined by the critical limit (the rejection region) then the assumed hypothesis is not accepted, it is rejected.

Remarks:

(1) The use in Geodesy of all parametric tests known so far is based on the assumption that the population probability law is a normal distribution. Thence follows the conclusion that the statistical estimators of one or several sounding samples should have their origin in a normally distributed population. If this is not true, the result of the statistical test is wrong.

(2) In the practice of triangulation operations, one attempts to eliminate systematic measurement errors by various procedures or by introducing corrections which should neutralize the action of systematic factors. However, a complete removal from the measurement results is not possible from the practical point of view. The observations contain, as a rule, systematic errors but by convention one assumes that these are eliminated if their magnitude is not significant compared with the value of the random component of the measurement error. It is this very thing which is best checked with the help of statistical significance tests.

In the following we shall limit ourselves to presenting three statistical significance tests (all of them belonging to group 1) concerning the principle and the method of application to analysing angular measurements. The presentation of other statistical significance tests is to be found in the technical literature (*Böhm* 1968, *Rotaru* 1973 *a*).

b) **The** W **test.** This is a test for checking the amplitude of the sounding sample $L_{ij}^{0(n)} - L_{ij}^{0(1)}$, where by $L_{ij}^{0(n)}$ and $L_{ij}^{0(1)}$ one understands the largest and

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[1] The tests are designated here by the notation for the test quantity to be checked.

[2] In fact ν represents the number of supplementary observations.

smallest values of the angle ij respectively, as obtained in the n measurement series.

Remark. When using this test, the observations must consequently be arranged according to their size.

The W test is given by the probability that the test quantity W exceed the critical limit W_α, i.e. (*Böhm* 1968, *Rotaru* 1973 *a*):

$$P(W > W_\alpha) = \alpha,$$

where $W = \dfrac{L_{ij}^{0(n)} - L_{ij}^{0(1)}}{\sigma}$; σ — the standard deviation, α — the risk.

The critical limit W_α is given in statistical tables as a function of the arguments α and n. For instance, for the values n and n_1 which may occur in *Schreiber*'s method and in the method of measuring the neighbouring angles respectively, the critical limit $W_\alpha(\alpha = 5\%)$ is given in Table 14.9. These values W_α have been taken from existing tables (*Böhm* 1968, No. 4), some of them directly, others by interpolation.

Table 14.9. *Values of the critical limit* $W_\alpha(\alpha = 5\%)$

Series number n or n_1	5	6	8	9	12	14	15	16	18
W	3.68	4.03	4.29	4.38	4.62	4.74	4.80	4.85	4.93

Example. An azimuthal angle at any station point of the astro-geodetic triangulation was measured by *Schreiber*'s method in 6 series, thence leading to the sounding sample of volume $n = 6$ (Table 14.10). Let us apply the W test, going through the stages shown in 14.1.6. a:

1) *Assuming the null hypothesis*: the sounding sample $L^{0(n)}$ ($n = 1, 2, ..., 6$) belongs to the population with the standard deviation $\sigma = \pm 2^{cc}.16$.

2) *Calculating the test quantity*:

$$W = \frac{L^{0(6)} - L^{0(1)}}{\sigma} = \frac{8.900}{2.16} = 4.12.$$

Table 14.10. *Sounding Sample*

No. of series n	Value of the angle measured in one series ($L^{0(n)}$)	$v = L^{0(n)} - L^0$
1	$168^g\ 79^c\ 39^{cc}.525$	$+2^{cc}.675$
2	36. 000	−0. 850
3	37. 150	+0. 300
4	33. 625	−3. 225
5	32. 950	−3. 900
6	41. 850	+5. 000
(Mean) L^0	$168^g\ 79^c\ 36^{cc}.850$	$[v] = 0.000$

3) *One establishes the risk* $\alpha = 5\%$ and correspondingly the statistical reliability $P = 1 - \alpha = 95\%$.

4) *From Table 14.9*, as a function of $n = 6$ and $\alpha = 5\%$ *one gets the critical limit* $W_\alpha = 4.03$.

5) *Decision making*: One notes that $W > W_\alpha$. Consequently, at the risk of being mistaken by 5% (or, in other words, with a statistical reliability of 95%) one can state that the null hypothesis is to be rejected. This means that, with a risk of being in error by 5%, one can state that the sounding sample is affected by gross errors or by significant systematic errors. Since $W > W_\alpha$, one then excludes further from the respective sounding sample the largest and most asymmetrical value with respect to the remainder of the values, i.e. $L^{0(6)} = 41.850$ and one again applies the W test to the remaining sounding sample ($n = 5$). It follows that $W = \{L^{0(5)} - L^{0(1)}\}/\sigma = 3.04$; $W_\alpha = 3.86$ and consequently $W < W_\alpha$. With a statistical reliability of 95% one can state now that the analysed sounding sample, without $L^{0(6)}$, is not affected by gross errors or significant systematic errors.

Remark. If we had again got $W > W_\alpha$, then the hypothesis of the sounding sample belonging to the population with the standard deviation σ would have been rejected.

c) **The** u **test.** *A. T. McKay* (1935) and *K. R. Nair* (1948) worked out the distribution of the extreme values $L^{0(1)}$ and $L^{0(n)}$ of a sounding sample of volume equal to n and discovered the test which is designated here as the u test. This test is given by the probability that the value of the test quantity u exceed the critical limit u_α, i. e. *(Böhm* 1966, *Rotaru* 1973 *a)*:

$$P(u > u_\alpha) = \alpha,$$

where

$$u = \frac{L^{0(n)} - L^0}{\sigma} \quad \text{or} \quad \frac{L^0 - L^{0(1)}}{\sigma}\,; \quad L^0 = \frac{1}{n}\sum_{i=1}^{n} L^{0(i)}.$$

The critical limit u_α is to be found in statistical tables as a function of α and n. For instance, for the values n or n_1 which may occur in *Schreiber*'s method and in the method of measuring the neighbouring angles respectively, $u_\alpha(\alpha = 5\%)$ is given in Table 14.11. These values have been taken from tables *(Böhm* 1968, No. 4*)*, some of them directly, others by interpolation.

If we get $u > u_\alpha$, then with a risk of being mistaken by $\alpha\%$ one can consider that the observation $L^{0(n)}$ or $L^{0(1)}$ is affected by a gross error or a significant systematic error. After excluding the corresponding observation, one calculates a new mean value L^0 and one repeats the checking of the new extreme values of the sounding sample. If again it is found that $u > u_\alpha$, then one proceeds just as before etc.

Table 14.11. *Values of the Critical Limit* u_α ($\alpha = 5\%$)

Series number n or n_1	5	6	8	9	12	14	15	16	18
u_α	2.08	2.18	2.33	2.39	2.52	2.59	2.62	2.64	2.69

In the case of a sounding sample of restricted volume, with one or several values affected by gross errors or significant systematic errors, it is possible that the u test should not lead to the expected result, because one arrives at a greater displacement of the mean value L^0.

Therefore, in such cases it is always necessary to check the standard deviation of the sounding sample:

$$m = \sqrt{\frac{[v\,v]}{n-1}}, \tag{14.27}$$

with the aid of the critical limit of the latter, e.g.:

$$m_M = \sigma\left(1 + \sqrt{\frac{2}{n-1}}\right). \tag{14.28}$$

If we find $m \approx m_M$, then the corresponding value can be considered as doubtful and if, after removing the corresponding value $L^{0(i)}$, one again gets $u > u_\alpha$, then for checking it is preferable to carry out supplementary measurements.

Example. Let us apply the u test to the sounding sample in Table 14.10, going through the stages mentioned in § 14.1.6.

1) *Assuming the null hypothesis*: the sounding sample $L^{0(i)}(i = 1, 2, ..., 6)$ in Table 14.10 belongs to the population with the standard deviation $\sigma = \pm 2^{cc}.16$.

2) *Calculating the test quantity*:

– for $L^{0(i)}$:

$$u_6 = \frac{L^{0(6)} - L^0}{\sigma} = \frac{5.000}{2.16} = 2.31;$$

– for $L^{0(1)}$:

$$u_1 = \frac{L^0 - L^{0(1)}}{\sigma} = \frac{3.900}{2.16} = 1.81.$$

3) *One establishes the risk* $\alpha = 5\%$ and the statistical reliability $P = 1 - \alpha = 95\%$ respectively.

4) From Table 14.11, as a function of $n = 6$ and $\alpha = 5\%$ *one gets the critical limit* $u_\alpha = 2.18$.

5) *Decision making*: One notes that for $L^{0(6)}$ one obtains $u > u_\alpha$ and for $L^{0(1)}$ – $u < u_\alpha$. Consequently, at the risk of being in error by 5% one can say that the observation $L^{0(6)}$ is affected by gross errors or significant systematic errors, whereas $L^{0(1)}$ is a good observation.

Now calculating m according to formula (14.27) and m_M according to (14.28), one gets:

$$m = 3.42 = \frac{3.42}{\sigma}\sigma = \frac{3.42}{2.16}\sigma = 1.58\,\sigma$$

and $m_M \approx 1.67\sigma$ respectively. Consequently, $m_M > m$ which shows that most certainly the observation $L^{0(6)}$ is doubtful. Indeed excluding $L^{0(6)}$ from the sounding sample and applying again the u test it follows for $L^{0(5)}$ (which has now become an extreme value) that: $u < u_\alpha$.

d) **The τ test.** Actually this test represents the χ^2 test of mathematical statistics, adapted to the special situation of geodetic activities and worked out by *J. Böhm* (1968, No. 1) as being the most complete test from the point of view of precision for any sounding sample $L^{0(i)}(i = 1, 2, ..., n)$. As a significance test, the τ test is given by the probability that the value of the test quantity τ should excceed the critical limit τ_α, i.e. *(Böhm* 1968):

$$P(\tau > \tau_\alpha) = \alpha,$$

where
$$\tau = m/\sigma;\ m = \sqrt{\frac{[v\,v]}{\nu}}\ ;\ \nu = n - 1\,;$$

consequently, one checks whether the difference $|m - \sigma|$ is the result of a random grouping of errors (affecting the population σ) or is due to a less accurate measurement (carried out carelessly or under unfavourable conditions).

The critical limit τ_α is to be found in tables calculated by *J. Böhm*, as a function of ν and α. For instance, for the values n or n_1 (respectively $\nu = n - 1$ or $n_1 - 1$) which may occur in *Schreiber*'s method and in the method of measuring the neighbouring angles respectively, the critical limit τ_α ($\alpha = 5\%$) is given in Table 14.12. These values have been taken from *Böhm*'s contribution (1968, No. 1), some of them directly and the rest by interpolation.

Table 14.12. *Values of the critical limit* $\tau_\alpha(\alpha = 5\%)$

Series number n or n_1	5	6	8	9	12	14	15	16	18
ν	4	5	7	8	11	13	14	15	17
τ_α	1.54	1.49	1.42	1.39	1.34	1.31	1.30	1.29	1.28

Example. Applying the τ test to the sounding sample from Table 14.10, going through the stages mentioned in § 14.1.6. a yields:

1) *Assuming the null hypothesis*: the sounding sample $L^{0(i)}$ with the standard deviation $m = 3^{cc}.42$ belongs to the population with the standard deviation $\sigma = 2^{cc}.16$.

2) *Calculating the test Puantity*: $\tau = m/\sigma = 1.60$.

3. *One establishes the risk* $\alpha = 5\%$ and, correspondingly, the statistical reliability $P = 1 - \alpha = 95\%$.

4) From Table 14.12, for $n = 6$ and $\alpha = 5\%$ *one obtains the critical limit* $\tau_\alpha = 1.49$.

5) *Decision making*: Inasmuch as $\tau > \tau_\alpha$, one can state, at a risk of 5% of being mistaken, that the analysed sounding sample does not belong, in its present composition, to the population σ, a result which, on the other hand, was also confirmed by the application of the W and u tests. In practice, however, it is very important and necessary to be able to say which in particular of the observations $L^{0(i)}$ of the sounding sample are mistaken or are affected by significant systematic errors so that one can remove them from the sounding sample under analysis. The problem of establishing which in particular of the observations $L^{0(i)}$ has to be eliminated is completely solved by subsequently applying after the τ test, the u test (or the W test) (*Rotaru* 1973 *a*). Consequently, this solution is actually reached through a scheme of tests: the τ test and the u test (or the W test), to be applied in the indicated sequence. For example, in Table 14.10 it follows, according to the u test (or to the W test), that the observation $L^{0(6)}$ is not satisfactory and must be removed from the sounding sample being analysed. Indeed, eliminating the observation $L^{0(6)}$ from the corresponding sounding sample and applying again the test to the remaining sounding sample yields $\tau < \tau_\alpha$, which shows that the values of the sounding sample in its last assembly belong to the population characterized by the parameter $\sigma = \pm 2^{cc}.16$.

Remark. As has already been shown (*Rotaru* 1973 *a*), in order to check the field angular observations obtained at every station point in the astro-geodetic triangulation — where one knows the value which one wishes to obtain —, the best results with a view to the reliability of the decision are obtained if one applies the test scheme, which in practice involves two working stages:

(1) One applies as significance test the τ test. Two situations are to be distinguished here:

a) If it happens that $\tau < \tau_\alpha$, the application of the scheme stops at this stage.

b) If the result is $\tau > \tau_\alpha$, then one continues by applying the u test (or the W test) for pinpointing the mistaken (or doubtful) observation (or observations).

(2) For the case b) one removes the unsatisfactory observation (or observations) and again applies the τ test to the remaining sounding sample in order to confirm that now $\tau < \tau_\alpha$.

Example (*Rotaru* 1975). At a 1st-order station point in *Moldavia*, with $K = 6$ directions the corresponding angles were measured by the method of measuring the neighbouring angles (§ 14.1.3). The check on the quality of the results of the corresponding measurements, by utilizing the test scheme (the τ test plus the u test), is shown in Table 14.13. Also in this table (column 12) the conclusions of this check are shown.

Remark. For the angle 4.5 the u test was no longer applied except for the observation $\bar{L}_9$, because, as was mentiond in §14.1.6, here one has to do with a sounding sample with several values affected by significant systematic errors ($\bar{L}_3$, $\bar{L}_7$, $\bar{L}_{13}$, $\bar{L}_1$) of about the same size. Indeed, calculating s_M by means of the relation $s_M = \sigma\left(1+\sqrt{\frac{2}{(n-1)}}\right)$ for $n = 15$ and $\sigma = 2^{cc}.16$ it follows that:

$$m_M = 2^{cc}.68 = 1.39\sigma;$$

and:

$$m = 3^{cc}.77 = 1.75\sigma;$$

consequently:

$$m > m_M,$$

which shows that the u test must not be applied, in this situation, after each elimination of individual observations but only after removing all the values having magnitudes close to the first value considered as mistaken.

14.2 Distance Measurements

Irrespective of the particular method of achieving the astro-geodetic triangulation (chains or compact network), the lengths of certain initial sides must be determined, either by the older procedure of developing the geodetic base or by direct measurement with the aid of electromagnetic instruments. The aim of these determinations is to evaluate all these lengths in the system of the standard metre, defined as "the length equal to 1,650,763.73 wavelengths in vacuum of the radiation corresponding to the transition between the $2p_{10}$ and $5d_5$ levels of the krypton-86 atom". Such a standardization is absolutely necessary as the lengths used in astro-geodetic triangulation are measured with different instruments and sometimes at great time intervals, which imposes a definition of the length unit which permits correlating the various measurements and their associated uses. For the case of a network of triangulation chains, the initial sides are, as a rule, those at the intersection of the chains (situated at distances of the order of 150—200 km). For the case of the compact triangulation network, the density of the initial sides, which must be established only in relation to the density of the measured *Laplace* azimuths, is from 10 to 10 triangles after some authors (*Krasovski* 1953—1956, *Trăistaru and Rotaru* 1968) and, according to others, from 7 to 7 triangles (*Khaimov* 1968) and even from 3—5 to 3—5 triangles (*Sigl* 1956). The conclusion accepted at present is that it is better to have a greater number of sides and azimuths measured in the network (even without

Table 14.13. *Checking the Angular Observations by Means of the Test Scheme*

Series number i	Designation of the sighted directions forming the angle	Value of the angle obtained in one series $\bar{L}_i$ deg min sec	Value of the angle obtained from n series $\bar{L}$ deg min sec	$v_i = \bar{L}_i - \bar{L}$	$m = \sqrt{\frac{[vv]}{n-1=\nu}}$	$\tau = \frac{m}{\sigma} = \frac{m}{2.16}$	τ_α	τ Test	$u = v_n/\sigma$ or v_1/σ	u_α	u Test
1	2	3	4	5	6	7	8	9	10	11	12
1	1 2	$35^g\ 46^c 32^{cc}.750$		$-3^{cc}.692$	For $\bar{L}_1$, $\bar{L}_2, \bar{L}_3, \dots,$ $\bar{L}_{15}$: $3^{cc}.62$	1.68	1.30	$\tau > \tau_\alpha$	$u_5 = \frac{v_5}{\sigma} = 3,29$ $u_{14} = \frac{v_{14}}{\sigma} = 0.19$	2.59	$u_5 > u_\alpha$ $u_{14} \ll u_\alpha$ Conclusion: $\bar{L}_5$ must be eliminated
2		29. 600		$-6\ .842$							
3		31. 950		$-4\ .492$							
4		40. 025		$+3.\ 583$	For $\bar{L}_1, \bar{L}_2, \bar{L}_3,$ $\bar{L}_4, \bar{L}_6, \bar{L}_7, \dots,$ $\bar{L}_{15}$: $3^{cc}.24$	1.50	1.31	$\tau > \tau_\alpha$	$u_2 = \frac{v_2}{\sigma} = 3.17$	2.55	$u_2 > u_\alpha$ Conclusion: $\bar{L}_2$ must be eliminated
5		43. 550		$+7.\ 108$							
6		39. 475		$+3.\ 033$							
7		34. 275		$-2.\ 167$	For $\bar{L}_1, \bar{L}_3,$ $\bar{L}_4, \bar{L}_6, \bar{L}_7, \dots,$ $\bar{L}_{15}$: $2^{cc}.74$	1.27	1.32	$\tau < \tau_\alpha$			
8		37. 950	$35^g 46^c 36^{cc}.442$	$+1.\ 508$							
9		35. 175		$-1.\ 267$							
10		39. 675		$+3.\ 233$							
11		36. 000		$-0.\ 442$							
12		34. 075		$-2.\ 367$							
13		38. 150		$+1.\ 708$							
14		36. 025		$-0.\ 417$							
15		37. 950		$+1.\ 508$							
				$[v] = -0,005$							

1	2 3	100 18 07. 000		−2. 920	For $\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4, \dots, \bar{L}_{15}$: $2^{cc}.61$	1.21	**1.30**	$\tau<\tau_\alpha$	$u_7=\frac{v_7}{\sigma}=$ $=2.36$	2.59	$u_7<u_\alpha$ **Conclusion:** **$\bar{L}_7$ must not be eliminated**
2		06. 525		−3. 395							
3		08. 025		−1. 895							
4		08. 025		−1. 895							
5		13. 050		+3. 130							
6		12. 200		+2. 280							
7		15. 025		+5. 105							
8		06. 950	**100 18 09. 920**	−2. 970							
9		09. 750		−0. 170							
10		12. 650		+2. 730							
11		09. 100		−0. 820							
12		08. 050		−1. 870							
13		12. 150		+2. 230							
14		09. 250		−0. 670							
15		11. 050		+1. 130 $[v]=0$							
1	3 4	81 63 49. 900		+2. 810	For $\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4, \dots, \bar{L}_{15}$: $1^{cc}.71$	0.79	1.30	$\tau<\tau_\alpha$			
2		48. 350		+1. 260							
3		45. 150		−1. 940							
4		46. 500		−0. 590							
5		43. 975		−3. 115							

1	2	3	4	5	6	7	8	9	10	11	12
6		44. 850		−2. 240							
7		48. 025		+0. 935							
8		49. 075	**81 63 47.090**	+1. 985							
9		46. 825		−0. 265							
10		48. 250		+1. 160							
11		47. 150		+0. 060							
12		45. 050		−2. 040							
13		47. 075		−0. 015							
14		48. 050		+0. 960							
15		48. 125		+1. 035 $[v]=0$							
1	4 5	87 15 89. 625		−1. 753	For $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\bar{L}_4$, ..., $\bar{L}_{15}$: $3^{cc}.77$	1.75	1.30	$\tau>\tau_\alpha$	$u_9=\dfrac{v_9}{\sigma}=2.51$	2.59	$u_9 \approx u_\alpha$ conclusion: $\bar{L}_9$ must be eliminated
2		87 .650		−3. 728							
3		92 .000		+0. 622							
4		87 .125		−4. 253	For $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\bar{L}_4$, ..., $\bar{L}_8$, L_{10}, ..., $\bar{L}_{15}$: $3^{cc}.48$	1.61	1.31	$\tau>\tau_\alpha$			We must also eliminate the observation $\bar{L}_7$
5		87 .975		−3. 403							
6		96 .475		+5. 097							
7		96 .600		+5. 222	For $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\bar{L}_4$, $\bar{L}_5$, $\bar{L}_6$, $\bar{L}_8$, L_{10}, ..., $\bar{L}_{15}$: $3^{cc}.19$	1.48	1.32	$\tau>\tau_\alpha$			We must also eliminate the observation $\bar{L}_{13}$
8		92 .050		+0. 672							
9		96 .800		+5. 422							

10		87 .475	**87 15 91,378**	−3. 903	For $\bar{L}_1, \bar{L}_2, .., \bar{L}_5, \bar{L}_6, \bar{L}_8$ $\bar{L}_{10}, \ldots, \bar{L}_{12}, \bar{L}_{14}, \bar{L}_{15}$: $2^{cc}.91$	1.35	1.34	$\tau > \tau_\alpha$			We must also eliminate $\bar{L}_4$
11		88 .150		−3. 228							
12		90 .250		−1. 128							
13		96 .300		+4. 922	For $\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_5, \bar{L}_6, \bar{L}_8$ $\bar{L}_{10}, \ldots, \bar{L}_{12}, \bar{L}_{14}, \bar{L}_{15}$: $2.^{cc}67$	1.24	1.35	$\tau < \tau_\alpha$			Conclusion: must be eliminated $\bar{L}_4, \bar{L}_7, \bar{L}_9$ and $\bar{L}_{13}$ containing significant systematic errors
14		88 .125		−3. 253							
15		94 .075		+2. 697 $[v] = +0.005$							
1	**5** **6**	39 56 43 .200		−0. 820	For $\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4, \ldots, \bar{L}_{15}$: $2^{cc}.09$	0.97	1.30	$\tau < \tau_\alpha$			
2		38 .000		−6. 020							
3		46 .125		+2. 105							
4		42 .375		−1. 645							
5		43 .225		−0. 795							
6		45 .200		+1. 180							
7		46 .325		+2. 305							
8		45 .900	**39 56 44.020**	+1. 880							
9		45 .950		+1. 930							
10		43 .900		−0. 120							
11		44 .000		−0. 020							
12		43 .300		−0. 720							
13		45 .225		+1. 205							
14		43 .525		−0. 495							
15		44 .050		+0. 030 $[v] = 0$							

1	2	3	4	5	6	7	8	9	10	11	12
1	**6** **1**	55 99 69. 250		−3. 528	For $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\bar{L}_4$, .., $\bar{L}_{15}$: $2^{cc}.12$	0.98	1.30	$\tau<\tau_\alpha$			
2		69. 900		−2. 878							
3		76. 025		+3. 247							
4		70. 300		−2. 478							
5		72. 150		−0. 628							
6		75. 625		+2. 847							
7		73. 750		+0. 972							
8		71. 950	**55 99 72. 778**	−0. 828							
9		74. 700		+1. 922							
10		74. 125		+1. 347							
11		72. 100		−0. 678							
12		74. 125		+1. 347							
13		70. 425		−2. 353							
14		73. 150		+0. 372							
15		74. 100		+1. 322 $[v]=+0.005$							

Remark. For the angle 4.5 the u test was no longer applied except for the observation $\bar{L}_9$, because, as was mentioned in § 14.1.6, here one has to do with a sounding sample with several values affected by significant systematic errors ($\bar{L}_9$, $\bar{L}_7$, $\bar{L}_{13}$, $\bar{L}_4$) of about the same size. Indeed, calculating m_M by means of the relation $m_M = \sigma\left(1+\sqrt{\frac{2}{(n-1)}}\right)$ for $n = 15$ and $= 2^{cc}.16$ it follows that:

$$m_M = 2^{cc}.68 = 1.39\,\sigma;$$

and:

$$m = 3^{cc}.77 = 1.75\sigma;$$

consequently:

$$m > m_M,$$

which shows that the u test must not be applied, in this situation, after each elimination of individual observations but only after removing all the values having magnitudes close to the first value considered as mistaken.

having a very high accuracy), which should be introduced into adjustment as measured elements and not as fixed elements, for which one attempts to find the most probable value. More recent investigations *(Khaimov* 1968*)* have confirmed that measuring initial sides without determining the *Laplace* azimuths at their ends is meaningless from the practical point of view, since these lead only to an insignificant increase in the accuracy of the sides and coordinates of the network points.

Measuring geodetic bases or initial sides is an essential operation, which has to be very accurate, since then when one discovers a closure error in a triangulation chain, on an initial side, or in a compact network, one must be certain that this error is due to the propagation of the angular errors and not to any error in measuring the geodetic base (or the initial sides).

The measurement of geodetic bases is carried out with invar wires [1] which ensure an accuracy of 10^{-6} or 2×10^{-6}. Thus, the length of the initial side is determined by the procedure of developing the geodetic base with an accuracy of the order of $3 \times 10^{-5} - 4 \times 10^{-5}$, which is sufficient for astro-geodetic triangulation. As a rule, 10^{-6} represents 1 cm for a geodetical base of 10 km. This measurement is an awkward and very expensive operation.

An example of a base-development network is shown in Fig. 14.8, viz. that of the geodetic base *Constanța*, measured with invar wire to obtain the initial side *Constanța—Ciocîrlia* in the 1st-order astro-geodetic triangulation of the *Socialist Republic of Romania*. The *Constanța* geodetic base, of about 6 km in length, was measured, within the interval 10—25 September 1940, by means of the *Carpentier* invar set (3 invar wires of 24 m, 1 invar wire of 8 m and 1 invar tape of 4 m) — calibrated in February 1939 at *Sèvres (France)*. The base was divided into 4 segments, each segment being measured simultaneously in both directions by means of two invar wires. Ten readings were carried out with the naked eye, at both divided scales, every 10 bays. The

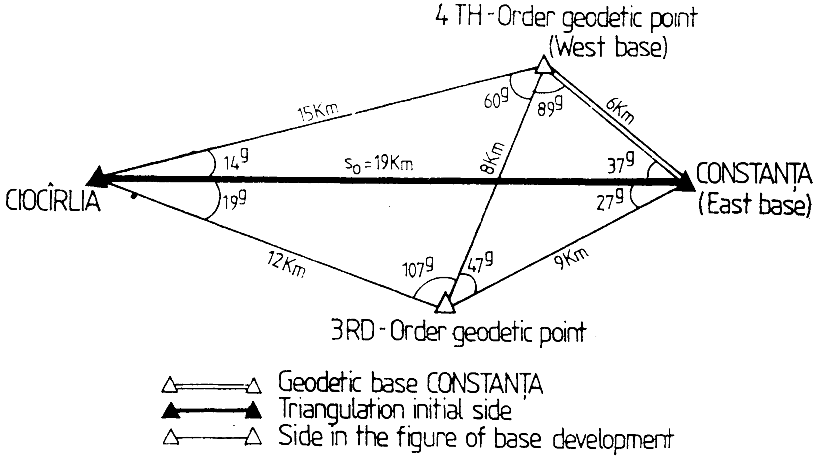

Fig. 14.8. Developing Network of the Geodetic Base of Constanța

[1] For instance, those manufactured by the firms *Carpentier* (*France*), *Aerogeopribor* (*USSR*), *Askania* (*F.R. of Germany*) etc.

temperature was measured for every two bays. The corrections applied to the measured value (s_{Base}= 6,148.312870 m) were as follows: calibration correction for each invar wire, correction for reduction to horizon, temperature correction, correction for catenary deformation, correction for scale slope and correction for reduction to *Krasovski*'s ellipsoid. The definitive value of the length of the *Constanța* geodetic base turned out to be: 6,148.227972 m, characterized by a precision of 1/1,663,054.

The corresponding horizontal angles in the development figure of the base were measured with an accuracy characterized by a standard deviation of $\pm 3^{cc}.161$. In the end, the length of the initial side turned out to be equal to 19,923.715 m, a value characterized by a relative error of 1/687,025.

In the years 1970 and 1971 the *Constanța* geodetic base was measured again with two NASM-A geodimeters, the following values being obtained: 6,148.3083 m and 6,148.3308 m respectively.

Inasmuch as the advent of electromagnetic instruments for distance measurements has brought about the possibility of directly obtaining the lengths of the triangulation initial sides, measuring geodetic bases was no longer necessary and nor obviously performing a network for their development either. Another contributory factor has been that these instruments ensure an accuracy of at least 1/300,000 ... 1/400,000 for sides of up to 30—40 km and allow the measuring of sides in the triangulation network, irrespective of the roughness of the terrain where they are located, by a very restricted field party in a much shorter time interval than when using the procedure of the geodetic base development.

With the help of the electromagnetic instruments (operating on the principle of measuring the propagation time of a light beam or of a very high frequency electromagnetic wave between two geodetic points and deducing the distance between them taking the velocity of light as known, one determines the distance s to be measured by means of the relation:

$$s = ct/2,$$

where $c = 299{,}792.5 \pm 0.3$ km/s *(Levallois* 1969*)* represents the propagation velocity of the electromagnetic waves in the air and t — the time taken for the electromagnetic waves to travel back and forth along the distance s. All these instruments utilize modulated electromagnetic waves, i.e. whose intensity varies between zero and a maximum, with an exactly known frequency.

For astro-geodetic triangulation one generally uses geodimeters of the type *AGA 8* (a laser telemeter which has begun to be successfully utilized also for measuring very large distances up to artificial satellites *(Levallois* 1969*)*, *NASM-2A*, *EOD-1*, *Rangemaster* etc.[1]

These geodimeters achieve an excellent accuracy, which has allowed the velocity of light to be determined by successive comparisons of their data with geodetic bases measured with calibrated invar wires.

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[1] There are at present about 30 types of distance-measuring electromagnetic instruments *(Committee on Geodesy* 1978).

According to *Fermat*'s principle, what one obtains by measuring a distance with the geodimeter is:

$$\int_A^B n \, ds$$

and what one desires to obtain is *(Levallois* 1969*)*:

$$s = \int_A^B ds = \int_A^B n \, ds - \int_A^B (n - 1) ds.$$

Consequently, one notes that it is necessary to know the refractive index of the air under the measurement conditions in order to be able to evaluate the integral:

$$\int_A^B (n - 1) \, ds$$

giving the appropriate correction. To meet this requirement, one assesses along the distance s the meteorological data, either by measuring them at its ends or by interpolation along the track. For measuring the meteorological elements, one utilizes in *Romania* electronic thermometers with transistors, mounted on a metallic pillar which is fixed at the end points of the distance and at its middle.

In this respect there is a particularly favourable situation, because, taking into account the fact that close to the ground the variation of the refractive index is small, one can also use less rigorous formulae, such as *Gladstone's formula (Levallois* 1969*)*:

$$(n - 1) = (n_0 - 1) \frac{p}{p_0} \frac{T_0}{T},$$

in which T is the absolute temperature and p — the pressure in mm Hg (the subscript o indicating quantities obtained in vacuum).

In this formula the influence of the temperature and pressure variations is rather small, since the term $(n - 1)$ is of the order of 300×10^{-6} and the atmospheric refraction may intervene only by values of the order of 10^{-5} or 10^{-6}.

Another method for determining the influence of the atmospheric refraction is that of using two lights (blue and red) and comparing the phase in both cases; the results appear to be promising *(Levallois* 1969*)*.

Generally, the precision reached by geodimeters is of the order of 2×10^{-6} for distances of 10—20 km but their range may reach up to distances of a maximum 25—30 km in fine weather, the measurements being carried out preferably in the twilight or by night.

Besides their advantages, electromagnetic instruments also present some drawbacks in that they are sensitive to sudden temperature and humidity variations and in particular to shocks which are unavoidable during transport and manipulation. Therefore, for checking the accuracy of the results obtained there arises the necessity of periodically calibrating these instruments on a *standard base* measured with an accuracy capable of conferring on the latter the attribute of a comparison standard. In order to achieve this aim, in various countries standard geodetic bases whose length varies between 0.4 and 11 km have been introduced. In *Romania*, for example there is a standard base located in the *Prahova* department, between the town of *Mizil* and the *Loloiasca* village, having a length of about 10.4 km which was divided by placing reinforced-concrete pillars every kilometre and within the first kilometre every 100 metres as well. The corresponding distances were measured with invar wire, an operation which is periodically repeated, as a rule twice a year.

The precision in measuring the *Mizil* base was of 1/4,500,000.

This notion of constructing the base allows both the calibration of electromagnetic instruments and the investigation of their behaviour under various conditions, in measuring distances of different lengths.

An essential feature of the progress of Geodesy in the coming decades as regards distance measurements will be the realization of electromagnetic instruments which, using three wave-lengths, should offer a measuring accuracy of 5×10^{-8} for 50 km *(Committee on geodesy: Trends and perspectives* 1978*)*

Other details concerning electromagnetic apparatuses are to be found in the technical literature (e.g.: *Ursea* and *Petruţă* 1968, *Levallois* 1969, *Oprescu* et al. 1973, *Tomlinson* and *Burger* 1977, *Committee on geodesy: Trends and perspectives* 1978*)*.

14.3 Relative Measurements of Gravity

With a view to obtaining the altitudes of the points as well as the components of the deflection of the vertical (ξ, η) and the geoid undulation (N), for astro-geodetic triangulation, one also carries out relative measurements of gravity.

The relative determinations of gravity are based on measuring the difference of gravity δg at the given point with respect to any other point, where the value of the gravity is known (called a *gravity base station*). These determinations can be made by pendular observations or by means of gravimeters. The principle of the relative determination with the pendulum is based on the assumption that the pendulum length l remains constant when observing at the base station and at the points to be determined.

In this situation, the gravity variation between two points will be given by the relation *(Oprescu* et al. 1973*)*:

$$\delta g = -\frac{2g_0}{T_0}\,\delta T,$$

in which g_0 and T_0 represent the value of gravity and the oscillation period of the pendulum respectively, at the base point (reference point) and δT — the difference between the period values of the pendulum oscillations at the point to be determined and at the base station.

The accuracy of the relative determination of the gravity depends only on the random errors in determining the oscillation period of the pendulum, on the variations of the systematic errors, as well as on the constants of the apparatus (which can be determined by observations at the base point). For checking and characterizing the systematic influences one as a rule utilizes periodically repeated observations at the base point.

Inasmuch as the modern pendulum apparatuses consist of several pendulums (2—6 pendulums), by comparing the results of the observations obtained at different pendulums one can check the regularity of the apparatus operation. At present one utilizes several types of pendulum apparatuses which ensure a precision of ± 0.2 mgal in the relative determination of the gravity, such as e.g. the pendulum apparatuses of the types *Half-Wisconsin* and *TsNIIGAIK* (both with quartz pendulums) or *Cambridge* and *Askania* (both with invar pendulums) or *Italian Society of Geodesy* (molybdenum pendulums). In *Romania* the *Askania* four-pendulum apparatus has been used.

It is possible that in the future portable pendulum apparatuses may be manufactured, capable of securing an accuracy of $\pm$ 0.001 mgal or ± 0.002 mgal in the relative determination of the gravity by repeating the measurements at one or two hour intervals *(Committee on geodesy: Trends and perspectives* 1978).

The relative determination of the gravity by means of gravimeters is based on the principle of compensating the force $F = mg$ acting on the mass m in the gravity field of intensity g by a spring elastic force. This is in fact an indirect measurement of the gravity. The gravimeters do not measure but only compare the gravity values at various selected locations. These determinations are carried out in a short time interval and with a high accuracy. For instance, the measuring operation with the *Worden* gravimeter lasts 3.5 minutes whereas when using the pendulum it lasts 6 or even 12 hours.

At present, for relative gravity measurements in astro-geodetic triangulation, one utilizes various types of gravimeters, as a rule gravimeters with a wide range (geodetic gravimeters) of the *Worden* type (ensuring a measuring accuracy of ± 0.5 ... ± 0.07 mgal), *La Coste-Romberg, Geodynamics or GAK-4M* (ensuring a measuring accuracy of ± 0.1 ... 0.4 mgal). With discrete measurements on land, the accuracy limitation is now due to technologically-conditioned instrumental error-sources, as well as to disturbing external effects (earthquakes, temperature variations, tide influence). In measurements carried out at sea and in the air one obtains mean values of the gravity, the accuracy being limited here too but this time by tremors of the means of transport, i.e. ship, submarine or airplane, respectively.

It is expected that in the future portable improved gravimeters will be realized which should secure an accuracy of ± 0.003 mgal in the relative determination of the gravity on land *(Committee on geodesy: Trends and perspectives 1978)*.

Every gravimeter measurement provides, as immediate data, scale-division numbers, corresponding to the equilibrium of the gravimetric system of the instrument under the conditions in which it was placed. Therefore, both before the actual start of the measurements and in the course of the measuring process one carries out the gravimeter *calibration*[1] — an operation leading to knowledge of the value C of the scale division given by the relation:

[1] For instance, in the *Socialist Republic of Romania* one utilizes as calibration base the base *POIANA BRAŞOV* which is formed by three points (P_1, P_2, P_3) and was determined with the aid of a group of gravimeters of different types (*GAK, Worden, Nörgaard*), the following differences between the gravity values at the three extreme stations being obtained:

$$g_{P1} - g_{P2} = 49.94 \pm 0.06 \text{ mgal};$$

$$g_{P1} - g_{P3} = 78.51 \pm 0.10 \text{ mgal}.$$

$$C = \frac{g_1 - g_2}{n_1 - n_2} = \frac{\delta g}{\delta n},$$

in which g_1, g_2 represent the gravity known at two stations and n_1, n_2 — the readings on the gravimeter scale at the two stations.

Details concerning the equipment and the measuring and data-processing methodology are to be found in specialist contributions (*Socolescu* and *Bişir* 1956, *Graf* 1961, *Constantinescu* 1961, *Shokin* 1969, *Oprescu* et al. 1973, *Committee on geodesy: Trends and perspectives* 1978).

The relative gravity values are referred to the *International Gravity Standardization Net—1971* (acronym: *IGSN-71*), which is a network extended over the entire terrestrial globe whose gravity station points were determined during the period 1962—1971 by the cooperation of an international working group, organized within the framework of the Study Group No. 5 of the *International Association of Geodesy*.

This network was adopted as absolute reference system for gravity at the *XVth General Assembly of the IUGG (Moscow, August* 1971) and has replaced the former reference system — the *Potsdam* system [1].

The *IGSN-71* system is formed by 1854 adjusted station points, obtained by a least-squares block-adjustment of pendulum absolute data and gravimeter data. The standard errors of this system for the gravity values are smaller than ± 0.1 mgal.

The observation points for gravimetric measurements are, generally, distributed in networks of various types covering variable surfaces, depending on necessities. This way of knowing the distribution of the gravity field at discrete points is imposed by the impossibility of analytic knowledge of this distribution. The scheme of the gravity surveys is based on the same principles as those underlying the construction of the state astro-geodetic networks.

The gravity support networks may be fundamental networks of stations measured with the pendulum or with the gravimeter which represent national gravity networks for the territory of a country. From the geodetic point of view, these networks must ensure the possibility of obtaining quantitative data on the form of the geoid and the reduction of the measurements within the framework of the astro-geodetic triangulation and of the general levelling of the territory, by using gravity data tied to the basis of the support network.

The national gravity network can be achieved according to several working schemes which, generally, involve two stages, viz.:

(a) *The 1st-order network, made up of stations situated at large distances and uniformly distributed over the corresponding territory, grouped into one or several triangulations or polygonations.*

Of late, gravimeters (for 1st order, 3—5 gravimeters simultaneously), transported by airplane, have generally been used for realizing the gravity

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[1] At whose basis lies the value of the gravity determined in absolute value at the *Potsdam* point:

$$g_{Potsdam} = 981{,}260 \text{ mgal},$$

the value recommended by the *XIVth General Assembly of the IUGG — Zürich,* 1967 (*IVth Section of the International Association of Geodesy*).

support networks. **Utilizing several gravimeters simultaneously offers two important advantages:**

1) *Reliability of results, in the sense that the results are not affected by systematic errors appropriate to the instruments being used — or that they emphasize such errors.*

2) *Increase in the determination accuracy.*

(b) *The 2nd-order and, if such is the case, 3rd-order network, with closer gravity stations grouped along profiling polygons, with checking at higher-order stations.*

While complying with the general scheme, the solutions adopted so far in various countries are different, depending on the existing gravity data, the details of the territory and the present technical possibilities. *In the Socialist Republic of Romania* the working scheme chosen was that based on triangulations with a central station — the pendulum station at the *Surlari* Geophysical Observatory.

14.4 Levelling Measurements

For completely determining the position of the points on the Earth's physical surface one needs to know their altitude as well. The methods so far known for obtaining the points altitude consists of the determination of the altitude difference between two points, one of which has a known altitude and the other the unknown altitude to be determined.

These methods also assume previous knowledge of the altitude of a given slope point with respect to the geoid, called the *fundamental zero point.* The altitude of the fundamental zero point follows from its measurement with respect to the mean sea-level over a long lapse of time (30—50 years), by means of a tide gauge or of a tide-gauge system. The fundamental zero point will be dealt with in the sixth part of this treatise (§ 29.2.1).

The methods for determining the altitude, also called levelling methods, may be divided into:

(1) *Physical methods (hydrostatic levelling* — using the physical phenomenon which occurs in communicating pipes, viz. a liquid placed in communicating pipes has its free equilibrium surface at the same level surface; *barometric levelling* — using the physical phenomenon represented by the fact that, under the same external conditions, the atmospheric pressure varies as a function of the altitude of the corresponding points above sea-level.

(2) *Geometrical methods (direct levelling* or *precision geometrical levelling* and *indirect levelling* or *trigonometric levelling* [1]*)*.

In astro-geodetic triangulation up to now geometrical methods have been used almost exclusively, *Romania*'s case included. Geometrical levelling is a very precise method for determining the altitudes of geodetic points, which will be described in Section 29.2. Trigonometric levelling leads to deducing the altitude difference between the sighted point and the station point using

[1] In some technical works it is also found under the name of *geodetic levelling* (*Levallois* 1969, *Oprescu et al.* 1973).

a trigonometric calculation: at a point i of known altitude H_i one measures the zenithal angle Z_{ij} of the point j whose altitude is to be determined and, by a simple calculation, one deduces its level difference h_{ij}, knowing the side length ij ($= s_{ij}$) given by triangulation, i.e.:

$$H_j - H_i = h_{ij} = s_{ij} \cot Z_{ij} + l_i - I_j$$

where l_i, I_j represent the height of the instrument at the point i and the height above the landmark of the sighted signal at the point j respectively. This calculating relation is valid for so-called *unilateral* or *single* trigonometric levelling (when one measures Z only at the point i or at the point j). In the case when one carries out a *bilateral* or *reciprocal* trigonometric levelling, i.e. one measures Z both at the point i as well as at the point j, and then the calculating formula is:

$$(H_j - H_i)_{av.} = h_{ij\,av.} = s_{ij} \tan \frac{Z_{ji} - Z_{ij}}{2} + \frac{l_i + I_i}{2} - \frac{l_j + I_j}{2}$$

To the value h_{ij} (respectively $h_{ij\,av}$) one also adds a series of corrections, viz.: the correction due to the average of the altitudes of the points i and j, the correction due to the influence of the deflection of the vertical, the correction due to the Earth's curvature and to atmospheric refraction, as well as the correction of transition from the measured altitude difference to the difference of normal altitudes. The calculating formulae for these corrections are to be found in the technical literature (e.g. *Oprescu* et al. 1973, pp. 560—562).

The accuracy of the trigonometric-levelling method is lower than that of the geometrical-levelling method (*Levallois* 1969): of the order of a few decimetres as compared to a few centimetres, this being mainly due to limited knowledge of the coefficient of atmospheric refraction. However, trigonometric levelling has important advantages from the practical point of view, being utilized, as a rule, for determining the altitude of the geodetic points located in areas were it is impossible to carry out geometrical levelling or for transmitting the altitude from a precision-levelling landmark to a triangulation point. Since these are the situations most frequently met with in astro-geodetic-triangulation practice, trigonometric levelling represents the basic method.

14.5 Determinations of Geodetic Astronomy

As its name implies, within the framework of astro-geodetic triangulation one also utilizes determinations of geodetic astronomy, by which one must understand the determination of astronomical coordinates — latitude Φ and longitude Λ — for points of this triangulation, as well as of the azimuth α of some directions between these points by using astronomical measurements. By their very nature, these astronomical measurements assume knowledge, from fundamental astronomical determinations, from star catalogues, from the sidereal time/civil time relationship of all the quantities which are given by the conventional astronomical ephemerides: time ephemeris data, apparent position of fundamental stars etc. These measurements also assume that the observation stations avail themselves of all the equipment capable of receiving the time signals of the various transmitters, as well as quartz chronometers measuring the time with an accuracy of at least $0^s.01$.

The geodetic-astronomy determinations are used for establishing the vertical at the corresponding geodetic points in a fixed reference system, as well as for calculating the coordinates of an origin point of a triangulation network.

By *astronomical latitude* one understands the pole height above the horizon, and by *astronomical longitude* — the dihedral angle between the origin astronomical meridian (as a rule the *Greenwich* meridian) and the astronomical meridian of the station point.

By *astronomical azimuth* one must understand the dihedral angle between the astronomical meridian of the station location and the star vertical.

The values Φ, Λ and α are usually approximately known, either from a map or as previously determined by an approximate method.

For the astronomical determinations one needs a uniform time scale, to which the observations should be referred. In Astronomical Geodesy one utilizes the true sidereal time, by which one understands the hour angle [1] of the true γ point[2] (additionally one also takes account of nutation [3] corrections).

The astronomical longitude Λ is indirectly obtained by determination of the chronometer correction u, by using the relation $\Lambda = \theta_G - \theta_\Lambda$, where θ_G is the time at *Greenwich*, $\theta_\Lambda = T + u$ — the local time and T — the chronometer reading (indication). Usually, Λ is taken as negative eastwards from the origin astronomical meridian, i.e. opposite to the practice in Geodesy.

For instance, the fundamental astronomical point *Dealul Piscului* of the *Romanian* astro-geodetic network has the astronomical longitude:

$$\Lambda = -\ 1^h\ 44^m\ 27^s \cdot 084.$$

For geodetic-astronomy determinations one utilizes the following basic differential formulae, connecting the elements of the position triangle (Fig. 14.9) and in which the differential quantities are assimilated with errors *(Oprescu et al.* 1973*)*:

$$\Delta z = \cos\alpha\ \Delta\Phi + \sin\alpha\cos\Phi\Delta u; \tag{14.29}$$

$$\sin z\ \Delta\alpha = -\cos z\sin\alpha\ \Delta\Phi + \cos q\cos\delta^*\ \Delta u; \tag{14.30}$$

$$\Delta\alpha = -\operatorname{ctg} z\sin\alpha\ \Delta\Phi + \cos q\cos\delta^*\operatorname{cosec} z\ \Delta u; \tag{14.31}$$

$$\cos\Phi\sin t\ \Delta\alpha - \cos q\ \Delta z + \cos t\ \Delta\Phi = 0, \tag{14.32}$$

---

[1] *The hour angle* is the dihedral angle formed by the astronomical meridian of the station location and the hour angle of the star — the great circle passing through the Earth's rotation axis and the corresponding star.

[2] *The γ point* is the place on the celestial sphere where the ecliptic intersects the plane of the celestial equator, i.e. the place where the *Sun* is at the vernal equinox. This point is also designated as *First Point of Aries.*

[3] By *nutation* one understands the undulatory movement of the Earth's pole axis which is due to the variable attraction exerted by the *Sun* on the various positions of the equatorial bulge of the Earth, with respect to the celestial equator.

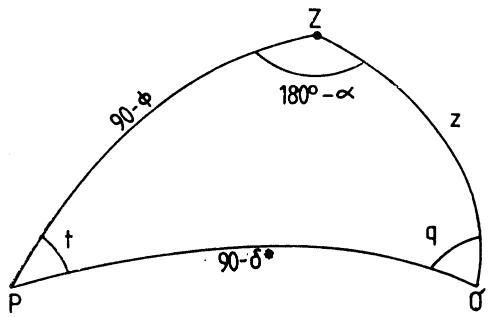

Fig. 14.9. The Position Triangle
τ — star's projection on the celestial sphere; P — pole of the sphere; Z — local zenith

in which z represents the zenithal distance of the star; q — the parallactic angle; t — the hour angle of the star[1] and δ^* the declination of the star.

From these formulae it follows that for determining the latitude Φ it is necessary to known the chronometer correction u and conversely. It is equally necessary when determining the azimuth α to know Φ and u or at least one of them.

The precise calculation of the corresponding elements is, however, possible if the *favourable conditions* of observation are satisfied, i.e. if the observations are carried out at those moments of the diurnal movement when the influence of the errors of these elements is null or minimum. These favourable conditions are established by analysing the formulae (14.29)—(14.32).

Remark. Generally, in Geodetic Astronomy one doesn't take into consideration the errors in the star positions from the star catalogue, and consequently $\Delta\delta^* = 0$ and $\Delta t = \Delta u$.

In geodetic-astronomy determinations one utilizes theodolites which have certain additional accessories and, generally, larger dimensions, e.g. the theodolites *Wild T4, OT002, Zeiss 02* or *DKM—3A*, also designated as *universal instruments*. These, along with the *transit telescope* (a portable instrument which is used in particular for determining the chronometer correction), represent the basic astronomical instruments for geodetic-astronomy determinations at the points of the astro-geodetic triangulation. One can also utilize *astrolabes*, e.g. the pendulum astrolabe in whose case the sight slope is simply obtained by gravity action (an instrument much used in the polar regions), or the prism astrolabe (an instrument formerly used, even in Romania, which is no longer utilized nowadays).

For determining the latitude Φ one may apply several accurate methods, among which the most frequently used are: the latitude determination from meridian z (15 series); *Pevtsov*'s method (12 star pairs); *Talcott*'s method (12—15 star pairs) and the latitude determination from pairs of stars in the prime vertical. The first three methods are based on the following favourable conditions which result from analysing formula (14.29): $\Delta\Phi_{\Delta z} =$ minimum and $\Delta\Phi_{\Delta u} = 0$ in the meridian. The fourth method is based on the favourable conditions resulting from the analysis of formula (14.30), viz.: $\Delta\Phi_{\Delta\alpha} =$ $=$ minimum, $\Delta\Phi_{\Delta u} =$ minimum and z small, in the prime vertical. The

[1] *The hour angle* is the dihedral angle formed by the astronomical meridian of the station site and the hour circle of the star.

Pevtsov method consists of observing two stars on the same side of the meridian, one of them southward and the other northward, the stars being symmetrical with respect to the prime vertical and having the same zenithal distance. In the case of the *Talcott* method, for which one utilizes a universal instrument, provided with a *Talcott* spirit level and with a mobile-wire micrometer, or the transit instrument, one observes two stars in the meridian, one of them northward from the zenith and the other southward, approximately symmetrical with respect to the zenith. For determining the chronometer correction u (respectively the longitude Λ) one usually utilizes two methods: the *Zinger* method (16—20 pairs of stars) and the determination of the correction u from the star observation in the meridian. The first method relies on the following favourable conditions resulting from the analysis of formula (14.29): $\Delta u_{\Delta z} =$ = minimum and $\Delta u_{\Delta\Phi} = 0$ in the prime vertical, and for the second method from analysing formula (14.30): $\Delta u_{\Delta\alpha} =$ minimum and $\Delta u_{\Delta\Phi} = 0$, in the meridian.

The *Zinger* method consists of observing two stars, one of them eastward and the other westward, close to the prime vertical, at the same zenithal distance. In the method of determining the correction u from observing the stars in the meridian, by using the transit instrument, one observes a group of about 15 stars in the meridian, on one side and the other of the zenith, having a zenithal distance of less than 15°, as well as 1—2 polar stars.

As regards the measurement accuracy for the astronomical coordinates Φ and Λ, one can make the following comments *(Levallois* 1969*)*: From practice it follows that the determination of the zenithal distance of a star with an instrument possessing a good spirit level is characterized by an accuracy of the order of $\pm 1''$. A group of 15 stars uniformly distributed with respect to the azimuth and readily observable within one hour, lead to the station-point coordinates Φ and Λ with an accuracy of $\pm 1''$ or even better. Consequently, in two consecutive observation sittings one can achieve $\pm 0''.7$.

In order to determine the azimuth α, one utilizes in astro-geodetic triangulation two methods in particular: the determination of the azimuth from the hour angle of *Polaris* (18 series or programmes with universal instruments of $1''$ and $2''$) and the determination of the azimuth directly from pairs of stars in the vertical of the signal (for each pair of stars, one star is positioned eastward and the other westward relative to the signal vertical).

The first method is based on the following favourable conditions resulting from analysis of formula (14.31): $\Delta\alpha_{\Delta\Phi} = 0$ in the meridian, $\Delta\alpha_{\Delta u} = 0$ at the maximum digression [1] and when $\delta^* =$ maximum; for polar stars the maximum digression is near the meridian. The second method rellies on the favourable conditions provided by analysis of formula (14.32): $\Delta\alpha_{\Delta z} = 0$ at the maximum digression and $\Delta\alpha_{\Delta\Phi} =$ minimum when $t =$ maximum; if the star does not pass at the maximum digression, then this is observed near the prime vertical.

The method of determining the azimuth from the *Polaris* hour angle is a very good method for measuring the astronomical azimuth, consisting of sighting *Polaris* near the maximum digression, a position in which for

[1] In the technical literature it is also found under the designation of *star elongation* (*Oprescu et al.* 1973).

some time the azimuth of a circumpolar star varies only slightly with respect to time. The observations are carried out with a theodolite of the type *Wild T3* or *Wild T4* by the method of reference torques with a weight at least equal to that of the 1st-order horizontal angles in the triangulation network. As a rule, these observations are carried out about 3/4 hour before and after the moment of the maximum digression and must be distributed over several series, as for the 1st-order angular measurements. The reference terrestrial signal is either a reference mark located on the vertical of maximum digression or a signal from the horizon round of the station point which is being connected to *Polaris*. It suffices to know the moment of pointing with an accuracy of about a few tenths of a second. By means of this method the precision of a measured azimuth is of the order of $\pm 0''.5$.

A contribution to determining the azimuth α by the second method has been made by the *Romanian* astronomer *I. Stamatin* (1941) who has worked out *the method of the direct astronomical azimuth for* the case when a transit instrument is available. The method consists of orienting and fixing the transit instrument in the vertical of the terrestrial signal whose azimuth is sought and then in observing the star passage in this vertical, on one side and then the other of the zenith. The azimuth calculation is carried out using the formula:

$$\alpha_{signal} = \alpha_{star} \pm (1/2)K\,(L_W - L_E)\ \mathrm{cosec}\, z \pm i \cot\, z,$$

in which α_{star} represents the star measured azimuth corrected for the error of inclination of the horizontal rotation axis; K — the value of a complete rotation of the recording micrometer drum; L_W and L_E — the means of the readings at the terrestrial signal made for the two positions of the telescope and i — the inclination of the instrument rotation axis. The upper sign is valid for the terrestrial signal situated southward with respect to the prime vertical and the lower sign — for the signal situated northward; for the latter situation one has to add 180° to the azimuth α_{star}.

For approximately 10 years, particularly as a consequence of investigations by *H. Dufour*, another method has been used for observing the azimuth, viz. the so-called *method of determining the azimuth on low stars*. This method, which assumes knowledge of the astronomical coordinates of the station, measures the azimuth by reference to stars observed in the neighbourhood of large zenithal distances (75°, 80°). The azimuth calculation is again calculated by means of the formula *(Levallois* 1969*)*:

$$\tan z = \frac{\sin \alpha}{\sin \Phi \cos \alpha - \cos \Phi \tan \delta^*}$$

Other details in connexion with the instruments and methods utilized in geodetic-astronomy determinations are to be found in the technical literature *(Wolf* 1968, *Levallois* 1969, *Oprescu* et al. 1973*)*.

At present it is difficult to achieve an increase in the determination shown in Table 14.1 for Φ, Λ and α, due to errors in star position ($\Delta\alpha^*$, $\Delta\delta^*$, α^* — star right ascension), to a star's own movements and to atmospheric refraction. The determining procedures are rather expensive, which limits their wider utilization although this would be desirable.

15
Adjustment of the Astro-Geodetic Triangulation

15.1 General Considerations

The point coordinates in astro-geodetic triangulation are calculated, as a rule, on the surface of the reference ellipsoid (rather seldom on the projection plane) and one obtains the coordinates B, L and H^E. The calculations are carried out with a view to the results of the measurements performed according to Chapter. 14 and reduced to the surface of the reference ellipsoid (Chapter. 10). Inasmuch as astro-geodetic triangulation always involves additional measurements, in order to obtain single values of the point coordinates one also needs rigorous processings, based on the theory of the measuring errors and on the least squares method.

As measured elements one takes the angles (or directions), as well as the length and the azimuth of the various triangulation sides.

If one also pursues the determination of the position and orientation of the triangulation network, the notion of *astro-geodetic adjustment* also appears.

A supplementary number of geodetic bases (or of directly measured sides) and of azimuths of the different sides not only helps to determine the network scale and position but, by adjustment, they also exert an influence on the form of the network.

As a rule, redundancy measurements already appear at every separate station point. This fact makes it possible to carry out station adjustments, prior to the network general adjustment. As was shown in Section 14.1, the results of the station adjustment, in the sense of adjustment calculations, usually represent functions of the measured elements which, in most cases, can no longer be regarded as independent values. In certain cases only, viz. for angular observations disposed absolutely symmetrically at the station point (e.g.: *Schreiber's* method or the method of the complete direction series), the station-adjusted directions may be regarded as independently measured elements. In all other cases the results of the station adjustment have to be considered as dependent observations within the framework of the astro-geodetic-triangulation adjustment.

Whether the adjustment of the astro-geodetic triangulation is to be carried out on angles or on directions, this is established according to the rule (*Jordan/Eggert/Kneissl* 1969, *Wolf* 1973): *"in the case of direction* *measurements one adjusts directions and in the case of angle measurements one adjusts angles".*

As regards the adjustment procedure, one has to distinguish, in principle, adjustment by conditioned observations and that by indirect observations. It is advisable to use the indirect-observation adjustment for large astro-geodetic-triangulation networks (even of continental extent), since in this case it is superior to the conditioned-observation adjustment as regards organizing calculations in a very large volume and, consequently, it allows the advantageous use of the electronic computers (one has to introduce into the computer, in addition to the results of the observations proper, only the necessary approximate coordinates; all subsequent computations are carried out independently by means of the computer according to appropriate programme). The idea of the indirect-observation adjustment of the large networks of astro-geodetic triangulation belongs to *F. R. Helmert.*

15.2 Adjustment of the Triangulation Network by means of the Method of Conditioned Observations

The method of conditioned observations solves the problem of finding out the most probable correction system by using the geometrical conditions of astro-geodetic triangulation; for every supplementary measurement one must make up a condition equation. Generally, for a certain given astro-geodetic triangulation there exists a necessary and sufficient number of equations which must be fulfilled. If one sets down more conditions, then the equation system will not be linearly independent and by introducing the unknowns (the correlates) one finds the non-determined quantities of the zero-to-zero type. If, on the contrary, one sets down sewer condition equations, then not all geometrical conditions of the network are obeyed, a fact which usually appears even when calculating sides and coordinates.

When the triangulation network is not connected to a neighbouring (older) network, such a network is designated as a *free network;* in the opposite case it is called a *connected network (Oprescu* et al. 1973*).*

If in adjusting a free network one deals only with the azimuthal directions (respectively angles), then one speaks of a *geometrical adjustment* which includes only the network's own conditions, also called *internal conditions* [1] *(Ghițău* 1972*).* If within the framework of the network there are also included the sides obtained from the extension of the base or the sides directly measured, the measured *Laplace* azimuths or the supplementary connexions from an older network, then the so-called *external conditions* [2] appear *(Ghițău* 1972*).*

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[1] In the technical literature they are also found under the designation of network's *own conditions* and *conditions extraneous to the network* respectively (*Wolf* 1968).

[2] They are also found in the technical literature under the designation of *compelling conditions* (*Wolf* 1968).

If the various quantities to be measured or functions of these quantities are regarded as error-free, i.e. their corrections are from the outset considered as null, then one speaks of *constraint conditions* [2] and of a *constraint adjustment* respectively.

15.2.1 Type and Number of the Condition Equations

As is known *(Lörinczi* 1966*)*, two methods exist to establish the type and number of necessary condition equations correctly, viz.: *the numerical method* (which works with formulae) and *the graphical method* (which utilizes an auxiliary graphical construction referring to the network sketch). Both methods are based on establishing the necessary data for completely building up the given astro-geodetic network.

Inasmuch as in the practice of the triangulation adjustments the numerical method is that which is more frequently used, we shall deal with it in what follows. We first introduce the following notations:

- N is the total number of triangulation points (stationed and non-stationed);
- N_1 — the total number of non-stationed triangulation points which are determined by forward intersection;
- l — the total number of (unilaterally or reciprocally) sighted sides;
- l_1 — the number of sides only sighted unilaterally;
- U — the number of all measured azimuthal angles;
- D — the total number of measured directions;
- N_2 — the number of station points round which all angles were measured;
- R — the number of partial networks into which the entire network is divided, if one removes the unilaterally sighted sides;
- A — the number of *joints*, i.e. the number of missing connexions between the collections of measured directions at the same station point.

In any astro-geodetic network, in which azimuthal angles were measured, the following types of condition equations may appear:

(a) **Internal conditions:**

1) *Horizon-round conditions* [1]. The horizon-round condition refers to the fact that the sum of all neighbouring angles round a station point must equal 400^g (or 360°).

The number C_T of the equations of horizon-round condition equals the number of points round which all angles were measured, i.e.:

$$C_T = N_2 \tag{15.1}$$

Remark. When the adjustment is carried out on directions, this kind of equation does not intervene.

[1] In the technical literature they are also found under other designations, e. g. *station conditions* or *horizon conditions (Jordan/Eggert/Kneissl* 1958, *Wolf* 1968).

2) *Figural conditions* [1]. The figural condition refers to the fact that in a polygon with i sides the sum of all interior angles equals $i \cdot 200^g$ (or 180°) plus the corresponding spherical excess.

One deduces the relation for determining the number C_F of the figural-condition equations by the following reasoning *(Wolf* 1968*)*:

One considers the closed polygonal path formed by all $(N - N_1)$ station points, i.e. a polygon with $(N - N_1)$ sides. The total number of reciprocal sights is $l - l_1$. Consequently, in addition to the above-mentioned polygonal path there also exist $(l - l_1) - (N - N_1)$ connexions. With every one of them one can intersect a point of the path and one can thus form a new figural equation; hence there are $(l - l_1 - N + N_1) + 1$ equations of this type too. From these one subtracts, however, the number of joints A. If a network formed by R partial networks is available, then instead of an initial polygonal path one extracts from it R paths; substituting R for l yields the general relation for determining the number C_F of figural-condition equations:

$$C_F = l - l_1 - N + N_1 - A + R. \tag{15.2}$$

3) *Pole conditions* [2]. For building up a triangulation network, starting with an initial side, one needs a number $2(N - 2)$ of sides; consequently, there are $(1 + 2N - 4)$ sides for the entire network.

If the network contains l sides, then $l - (1 + 2N - 4) = l - 2N + 3$ of these are supplementary. For every supplementary side one must form a pole-condition equation. If the network is split-up into A partial networks, then for each of the $(A - 1)$ partial networks one may introduce one new initial side into the computation from which it follows that the number C_P of the pole-condition equations is given by the relation:

$$C_P = l - 2N + 3 - (A - 1) = l - 2N - A + 4. \tag{15.3}$$

In principle, the number of pole-condition equations equals the number of supplementary sides, i.e. the number of the sides which are not unconditionally necessary for uniquely defining the relative position of the network points.

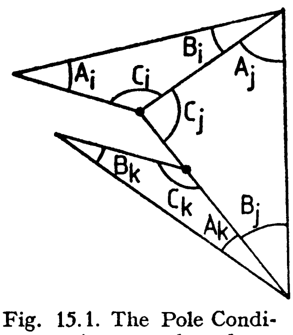

Fig. 15.1. The Pole Condition not obeyed.

The necessity of the pole condition can be clearly seen from Fig. 15.1 (plane case) where, although the horizon-round conditions and the figural ones are fulfilled, this does not involve satisfying the pole condition as well.

The pole condition can be stated as follows: if, starting from one side of the network, it is possible to determine another network side in at least two ways [3], then the two results must be consistent with one another.

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[1] Or *angle-sum conditions*.

[2] Also found in the technical literature (e. g. *Wolf* 1968) under the designation of *side conditions* or, in view of their form, *sine conditions*.

[3] A *way* means here a succession of triangles (inside the triangle network) and not simply a sequence of distances.

Remarks:

(1) The total number C_a of internal-condition equations follows from this reasoning: starting from two fixed points, every other point of the network is determined from two station points, i.e. for the whole network it is necessary for $2(N-2)$ angles to be measured. If U is the total number of measured azimuthal angles, then there is a number:

$$U - 2(N - 2)$$

of supplementary determinations and, consequently, the total number C_a of internal-condition equations is:

$$C_a = U - 2N + 4. \tag{15.4}$$

As a check one utilizes the relation:

$$C_a = C_T + C_F + C_P. \tag{15.5}$$

(2) In the case when azimuthal distances are measured, then the relations (15.1) and (15.2) retain their form while $C_T = 0$ and:

$$C_a = 2l - l_1 - 3N + N_1 - A + 4. \tag{15.6}$$

4) *Coordinate conditions.* The coordinate conditions refer to the fact that if, starting from the coordinates of any point, it is possible to calculate the coordinates of another station point in at least two different ways [1], then the same values for the coordinates must result.

These conditions always occur in pairs, viz. by twos for one station point.

When one actually works out such conditions, it is necessary to eliminate an equally large number of other conditions (in most cases pole conditions).

(b) **External conditions** [2]

In an astro-geodetic network the following types of external conditions may occur:

1) *Conditions of horizon-round agreement.* This condition, which represents a particular case of condition (a) 1), occurs when at a station point one measures only the angles (or directions) between two fixed directions arising from an old network. Here the sum of the measured angles must equal the fixed angle. Therefore, in this case the condition is also designated as *fixed-angle condition.*

2) *Conditions of angle-sum agreement,* which represent a particular case of the condition (a) 2); they occur when one or several angles in the polygonal figure having i sides arise from an old network.

3) *Conditions of starting-sides agreement,* which represent a particular case of the condition (a) 3); they occur when there exist two or several sides arising from an old network of the same order or when one has measured more than one starting side in the network.

4) *Conditions of coordinates agreement,* representing a particular case of the conditions (a) 4); they occur when there exist 3 or several station points

[1] See preceding foot-note.

[2] In the technical literature they also occur under the name of *constraint conditions* (e.g. *Lörinczi* 1966).

in the network — also called *connexion* or *junction points* — of known coordinates, which arise from an old network.

5) *Conditions of azimuths agreement* (or *Laplace conditions*), occurring when several astronomical azimuths (*Laplace* azimuths) were measured in the network, instead of the various *Laplace* azimuths, connexion azimuths from older neighbouring networks also possibly occurring.

Remarks:

(1) The condition equations of different types are mutually interchangeable.

(2) As mentioned by *Wolf* (1968), the working out of special rules aimed at finding the number of external conditions is practically not worthwhile; if the form, scale, position and orientation of the network are uniquely determined, then it is necessary that for every network element which becomes additionally known, one works out an external condition. In the technical literature (e.g. *Lörinczi* 1966) a calculating relation of the total number C_b of constraint conditions is given, viz.:

$$C_b = 2N_0 - 4,$$

where N_0 represents the number of initial station points (regarded as fixed points).

15.2.2 Forming the Condition Equations

The horizon-round condition. If at a station point (Fig. 15.2) the angles $A_K(K = 1, 2, ..., n)$ which must make up 400^g (360°) were measured, then, denoting by v_K the corrections associated with these angles after adjustment, the following condition must be fulfilled:

$$[A_K^0 + v_K]_1^n = 400^g;$$

or:

$$[v_K]_1^n + w_T = 0, \tag{15.7}$$

where:

$$w_T = [A_K^0]_1^n - 400^g \tag{15.8}$$

is designated as the *horizon-round misclosure*.

The maximum admissible value for w_T depends on the accuracy in measuring the angles A_K^0, i.e. on the standard deviation m of the measured angles. Applying the error-propagation law to (15.8) yields:

$$m_{w_T} = m\sqrt{n}.$$

Generally, for each condition equation taken separately, the maximum admissible value of the free term w^{max} is established by means of the relation $w^{max} = tm_w$, in which t represents the factor of the limit error which in the triangulation adjustment is taken as equal to 2 (*Lörinczi* 1966) and m_w — the standard

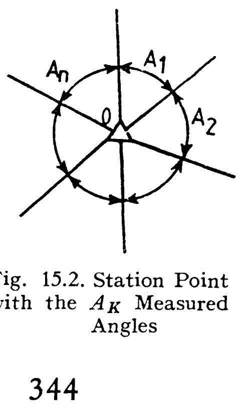

Fig. 15.2. Station Point with the A_K Measured Angles

deviation of the free term, calculated according to the error-propagation formula.

For the free term of the horizon-round condition equation we have

$$w_T^{max} = 2m\sqrt{n}. \tag{15.9}$$

Remark. The tolerance w_T^{max} of the horizon-round misclosure depends on the number of the neighbouring angles around the station point.

In the case of *Romania*'s 1 st-order triangulation where $n^{max} = 8$ and $m = \pm 0''.7$, we have $w_T^{max} \approx 4''$.

The figural condition. For a closed polygon with i internal measured sides $\beta_K (K = 1, 2, ..., i)$, whose spherical excess is ε, the following condition must be fulfilled:

$$[\beta_K^0 + v_K]_1^i = (i - 2)\, 200^g + \varepsilon,$$

or:

$$[v_K]_1^i + w_F = 0, \tag{15.10}$$

where:

$$w_F = [\beta_K^0]_1^i - \{(i - 2)\, 200^g + \varepsilon\} \tag{15.11}$$

is called *the misclosure* for the i angles.

If S is the area of the closed polygon, with i internal angles, then (11.1):

$$\varepsilon = \rho \frac{S}{R^2}.$$

The tolerance w_F^{max} for the misclosure w_F is got by applying the error-propagation law to (15.11), viz.:

$$m_{w_F} = \sqrt{i}\, m$$

and, consequently, considering that one is dealing with high-precision measurements:

$$w_F^{max} = 2\sqrt{i}\, m, \tag{15.12}$$

where m is the standard deviation of an angle measurement, calculated by means of *Ferrero's international formula.*

For the case of the triangle, (15.12) becomes:

$$w_F^{max} = 2\sqrt{3}\, m,$$

which is valid for a network in which the number of triangles is less than 200. If the number of triangles is very large (over 200), then:

$$w_F^{max} = 3\sqrt{3}\, m.$$

For *Romania*'s 1 st-order triangulation, where $m = 0''.7$, we have $w_F^{max} = 3''.6$.

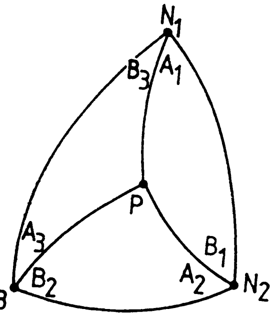

Fig. 15.3. Central System with 3 Bundles

The pole condition. For the case of the reference ellipsoid, the calculation of the sides may be carried out by both *Legendre*'s theorem (Section 11.2) and the additament method (Section 11.3).

By applying, e.g., *Legendre*'s theorem, one gets by successively solving the triangles considered[1] (Fig. 15.3):

$$\frac{\sin(A_1' + v_{A_1'}) \sin(A_2' + v_{A_2'}) \sin(A_3' + v_{A_3'})}{\sin(B_1' + v_{B_1'}) \sin(B_2' + v_{B_2'}) \sin(B_3' + v_{B_3'})} = 1.$$

Taking the logarithm of this last relation and generalizing one finds:

$$[\log \sin (A' + v_{A'})]_1^r - [\log \sin (B' + v_{B'})]_1^r = 0, \quad (15.13)$$

a relation which holds for a central system with r collections.

Noting that $v_{A'} \ll A'$ and $v_{B'} \ll B'$, if one expands in *Taylor* series the corresponding log sin terms from (15.13), one finally gets the pole-condition equation in the form:

$$[\cot A' v_{A'}] - [\cot B' v_{B'}] + w_P'' = 0, \quad (15.14)$$

in which:

$$w_P'' = \frac{\rho''}{\mathrm{Mod}} ([\log \sin A'] - [\log \sin B')]. \quad (15.15)$$

is called the *side misclosure*.

Now applying the additament method, the demonstration follows the method for plane triangles and one thus gets the relation:

$$[\cot A \ v_A] - [\cot B \ v_B] + w_P'' = 0, \quad (15.14')$$

in which:

$$w_P'' = \frac{\rho''}{\mathrm{Mod}} ([\log \sin A] - [\log \sin B], \quad (15.15')$$

Remarks:

(1) In view of the fact that the results obtained by means of (15.15) and (15.15′) must be identical, one gets the checking relation for the misclosure calculation:

$$w_P^{(\log)} = [\log \sin A] - (\log \sin B] = \left[\log \sin \left(A - \frac{\varepsilon}{3}\right)\right] - \left[\log \sin \left(B - \frac{\varepsilon}{3}\right)\right].$$

---

[1] We have here:

$$A_K' = A_K^0 - \frac{\varepsilon_K}{3} \text{ and } B_K' = B_K^0 - \frac{\varepsilon_K}{3}.$$

(2) Instead of utilizing the values cot (as coefficients for v) in (15.14) or (15.14′), one frequently uses the tabular differenes Δ_A of the function log sin A and Δ_B of the function log sin B respectively. In this case one utilizes, e.g. for the angle A, the relation:

$$\frac{\mathrm{d}(\log \sin A)}{\mathrm{d}A} = \Delta(\log \sin A) = \Delta_A = \frac{\mathrm{Mod}}{\rho''} \cot A.$$

Taking account of this last relation, as well as the corresponding one for the angle B formula (15.14′) becomes:

$$[\Delta_A \cdot v_A] - [\Delta_B \cdot v_B] + w_P^{(\log)} = 0.$$

As well as the triangle, another geometrical figure frequently met in astro-geodetic networks is the geodetic quadrilateral with both diagonals reciprocally observed (Fig. 15.4). For this quadrilateral there exist 7 ways of writing the pole equation as a function of the position of the chosen central point (pole), viz. (Fig. 15.4): 1, 2, 3, 4, M, O_1 and O_2.

The choice of the optimum pole is facilitated by using *Zachariä*'s theorem extended by *V. Jordan*, viz.: the most favourable central point is that which is opposite to the triangle of largestarea *(Jordan/Eggert/Kneissl* 1958*)*.

From this point of view, the pole M is the most advantageous inasmuch as it corresponds to the entire area of the quadrilateral. Regarding M as a pole, the condition equation will be:

$$\frac{\sin A_1 \sin A_3 \sin A_5 \sin A_7}{\sin A_2 \sin A_4 \sin A_6 \sin A_8} = 1,$$

which by linearization finally takes the following form:

$$[\pm v_A \cot A^\circ]_1^8 + w_P'' = 0,$$

where the + sign is taken for all the terms referring to the angles A_i in which i is an odd number, while the − sign is taken when i is an even number;

$$w_P'' = \frac{\rho''}{\mathrm{Mod}} [\pm \log \sin A^\circ]_1^8,$$

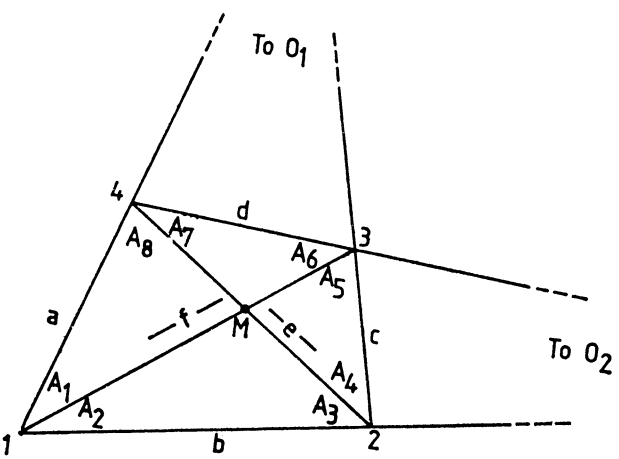

Fig. 15.4. The Geodetic Quadrilateral with Both Diagonals Reciprocally Observed

or:

$$[\pm \Delta_A v_A]_1^8 + w_P^{(\log)} = 0, \tag{15.16}$$

where:

$$w_P^{(\log)} = [\pm \log \sin A^\circ]_1^8.$$

In order to find the tolerance w_P of the free term, one proceeds as follows: tackling (15.16) as a separate equation and forming the corresponding correlate normal equation $[\Delta\Delta]k + w_P = 0$, one then calculates the corresponding corrections v_i by means of the relation $v_i = \Delta_i k = -\Delta_i \cdot \frac{w_P}{[\Delta\Delta]}$. Inasmuch as $k = \text{const.}$, we shall have $v_i = max$ when $\Delta_i = max$, i.e.:

$$|v_i^{max}| = \left| \Delta_i^{max} \frac{w_P}{[\Delta\Delta]} \right|.$$

In the case when the measurements have been carefully carried out one may impose the condition:

$$|v_i^{max}| \leqslant 2m; \text{ so that: } \Delta_i^{max} \frac{w_P}{[\Delta\Delta]} \leqslant 2m.$$

It then follows that:

$$w_P^{max} \leqslant \frac{2m[\Delta\Delta]}{\Delta_i^{max}}.$$

The condition of starting-sides agreement. This condition occurs when there exist at least two initial sides in the triangulation network (obtained from extending the geodetic base measured with invar wire, or directly measured with electromagnetic apparatuses) of known length (s_0 and s_n).

For the case of the adjustment on the reference ellipsoid, the condition equation will have the form (Fig. 15.6) *(Jordan/Eggert/Kneissl* 1958, *Ghițău* 1972*):*

1) *When utilizing Legendre's theorem:*

$$\log (s_0 + v_0) - \left[\log \sin \left(A^\circ - \frac{\varepsilon}{3} + v_A\right)\right] + \left[\log \sin \left(B^\circ - \frac{\varepsilon}{3} + v_B\right)\right] = \\ = \log (s_n + v_n). \tag{15.17}$$

2) *When utilizing the additament method:*

$$\log (s_0 + v_0) + A_0 - [\log \sin (A^\circ + v_A)] + [\log (B^\circ + v_B)] - \\ - \log (s_n + v_n) - A_n = 0 \tag{15.18}$$

in which A_0 and A_n are the additaments of the sides s_0 and s_n respectively (11.3).

Linearizing (15.17) and (15.18) leads finally to the equation:

$$[v_A \cot A^0] - [v_B \cot B^0] - \rho''(\sigma_0 - \sigma_n) + w_L'' = 0$$

in which:

$$\sigma_0 = \frac{v_0}{s_0}; \quad \sigma_n = \frac{v_n}{s_n},$$

and:

$$w_L'' = \frac{\rho''}{Mod}\left(\left[\log\sin\left(A^0 - \frac{\varepsilon}{3}\right)\right] - \left[\log\sin\left(B^0 - \frac{\varepsilon}{3}\right)\right] + \log s_n - \log s_0\right),$$

for the relation (15.17) or:

$$w_L'' = \frac{\rho''}{Mod}([\log\sin A^0] - [\log\sin B^0] + \log s_n - \log s_0 + A_n - A_0),$$

for the relation (15.18).

Remarks:

(1) For checking the calculations one may use the relation:

$$[\varepsilon(\cot B^0 - \cot A^0)] = 3\rho''(A_n - A_0)\frac{1}{Mod}.$$

(2) If s_0 and s_n arise from an old network, then one takes $\sigma_0 = \sigma_n = 0$.

For the tolerance of the free term of the equation of the condition of starting-sides agreement one gets, utilizing the linearized relation (15.17) or (15.18) and the fact that $w_L^{max} \leqslant 2m_{w_L}$:

$$w_L^{max} \leqslant 2\sqrt{\left.\begin{matrix}0\\1\\2\end{matrix}\right\} m_{\log s}^2 + m^2[\Delta\Delta]},$$

where the coefficient of the standard-deviation component $m_{\log s}$ is 0, 1 or 2, according as both, one or none of the starting sides are fixed elements, which are not corrected by adjustment, and Δ — the logarithmic tabular differences appearing in the considered equation of starting-sides agreement.

The condition of Laplace-azimuths agreement. This condition occurs when the astronomical azimuth is measured and the corresponding *Laplace* azimuths A_0 and A_n respectively (Fig. 15.5) [1] determined at two station points within the framework of the network.

[1] In Fig. 15.5 the angles A_i were denoted by $\bar{A}_i$ in order to keep to the conventional notation used for the *Laplace* azimuth.

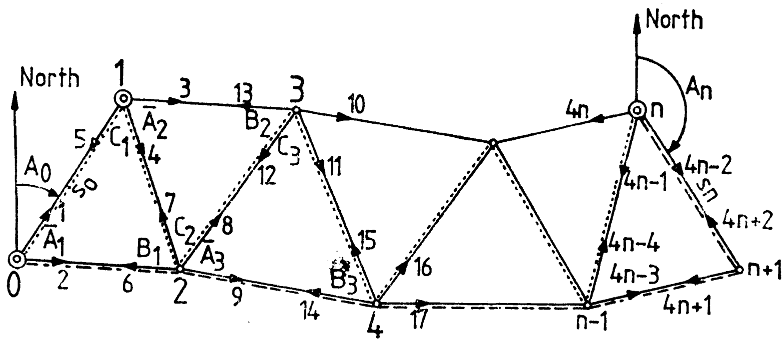

Fig. 15.5. Triangulation Chain with Two Determined Laplace Azimuths and Two Known Starting Sides

In modern adjustments the *Laplace* azimuths are introduced as elements affected by errors, to be themselves adjusted, finally yielding the corrections:

$$A_o = A_o^o + v_{A_o}; \quad A_n = A_n^o + v_{A_n}.$$

We denote by ΔA the difference between the direct azimuth (A_{ij}^o) and the inverse one (A_{ji}^o), viz.:

$$\Delta A_{ij} = A_{ji}^o - A_{ij}^o.$$

For the case in which in the triangulation chain in Fig. 15.5 the horizontal directions $L_1^o, L_0^2, \ldots, L_{4n+2}^o$ were measured, the transmission of the azimuth A_0 along the path $0, 2, 4, \ldots, n-1, n+1$ (the dashed line in Fig. 15.5) leads to the following condition *(Jordan/Eggert/Kneissl* 1958*)*:

$$A_o^o + v_{A_o} + \Delta A_{01} + L_2^o + v_2 - (L_1^o + v_1) + \Delta A_{26} + L_9^o + v_9 -$$
$$- (L_6^o + v_6) + \Delta A_{914} + \ldots + L_{4n+2}^o + v_{4n+2} - (L_{4n+1}^o + v_{4n+1}) +$$
$$+ \Delta A_{(4n+2)(4n-2)} = A_n^o + v_{A_n},$$

where v_i $(i = 1, 2, \ldots, 4n+2)$ represents the direction corrections or, put another way:

$$-v_1 + v_2 - v_6 + v_9 - \ldots - v_{4n+1} + v_{4n+2} + v_{A_o} - v_{A_n} + w_A = 0, \quad (15.19)$$

where *the Laplace misclosure* w_A is calculated from the relation:

$$w_A = (L_2^o - L_1^o) + (L_9^o - L_6^o) + \ldots + (L_{4n+2}^o - L_{4n+1}^o) + [\Delta A] - (A_n^o - A_o^o). \quad (15.20)$$

For the case in which in the chain in Fig. 15.5 the horizontal angles $\bar{A}_i^o$, B_i^o, C_i^o $(i = 1, 2, \ldots, n)$ were measured and the azimuth transmission is made along the path $0, 1, 2, 3, 4, \ldots, n-1, n$ (the dotted line in Fig. 15.5), the condition equation will have the following form *(Ghiţău* 1972*)*:

$$A_o^o + v_{A_o} - (C_1^o + v_{C_1}) + (C_2^o + v_{C_2}) - \ldots$$
$$\ldots + (-1)^n(C_n^o + v_{C_n}) + [\Delta A] \pm 200^g = A_n^o + v_{A_n},$$

or:

$$v_{A_0} - v_{C_1} + v_{C_2} - \dots + (-1)^n v_{C_n} - v_{A_n} + w_A^{cc} = 0, \qquad (15.19')$$

where:

$$w_A^{cc} = A_0^\circ - C_1^\circ + C_2^\circ - \dots + (-1)^n C_n^\circ + [\Delta\alpha] \pm 200^g - A_n^\circ. \quad (15.20')$$

Remarks:

(1) If one wishes to take account of the very slight dependence of $[\Delta A]$ on "v"'s, depending on the corresponding astronomical longitudes, then this is achieved by the fact the that coefficients of the "v"'s are not exactly -1 or $+1$, which, however, in most cases is not of practical importance (*Roelofs* 1947).

(2) The values ΔA can be obtained with enough precision (along with the astronomical longitudes needed for the *Laplace* azimuths) from a calculation of the astronomical coordinates which is carried out before the adjustment (*Wolf* 1948).

(3) If the azimuths A_0 and A_n do not result from astronomical measurements but from an old network, then one must take, from the outset $v_{A_0} = v_{A_n} = 0$ (constraint condition on azimuth).

Inasmuch as the free term w_A is formed (e.g. (15.20')) from the azimuths A_0° and A_n° as well as the n angles C° of transmission of the azimuth, we have the following expression for calculating the tolerance of this term:

$$w_A^{max} = \pm 2 \sqrt{\left.\begin{matrix}0\\1\\3\end{matrix}\right\} m_A^2 + nm^2},$$

where the coefficient of the standard-deviation component m_A of determining the *Laplace* azimuths is 0; 1 or 2, according as both, one or none of the azimuth are fixed elements, which are not corrected by adjustment.

Example. For a chain in the astro-geodetic triangulation where one has $m_A = \pm 0''.5$; $m = \pm 0''.7$; $n = 10$, we obtain $w_A^{max} = 4''.6$ (in the case when neither *Laplace* azimuth is a fixed element).

The condition of coordinates agreement. This condition is always expressed by two condition equations, which are obtained with the aid of the first basic geodetic problem by way of differentiating *(Prondzynski* 1868*)* with respect to the length of the distance s and the azimuth A (both referred to the reference ellipsoid). The simplest formulae, in which only the linear terms are taken into consideration, have been worked out by *O. S. Adams* (1915). These formulae are obtained from the relations valid for the projection plane (e.g. *Jordan /Eggert/Kneissl* 1958*)*, the ellipsoid surface being identified with the plane of the tangent, i.e. the x axis corresponds to the meridian as linear element and the y axis — to the arc of parallel. Considering M, N, B as referred to the intermediate latitude corresponding to the contact point of the tangent plane, we then have:

$$x_i = \frac{B_i - B_o}{\rho''} M; \quad y_i = \frac{L_i - L_o}{\rho''} N \cos B.$$

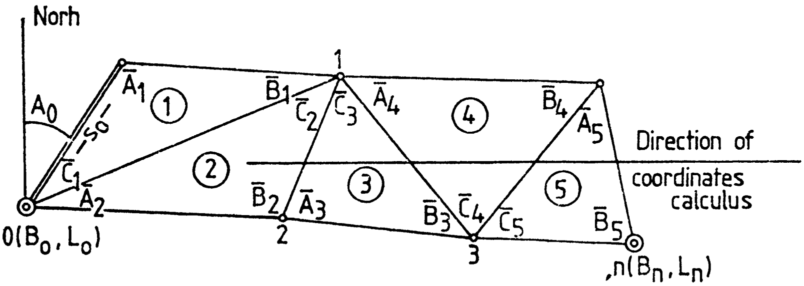

Fig. 15.6. Triangulation Chain Formed out of 5 Triangles

With these considerations, one gets finally, from the relations for x and y in the plane, the following equations of agreement of coordinates on the ellipsoid (Fig. 15.6):

The equation of agreement of latitudes:

$$\rho'' \frac{\sigma_o}{s_o}(B_n^o - B_o^o) + c_1(L_n^o - L_o^o)v_{A_o} +$$
$$+ (B_n^o - B_o^o) \cot \bar{A}_1^o v_{\bar{A}_1} - (B_n^o - B_o^o) \cot \bar{B}_1^o v_{\bar{B}_1} + c_1(L_n^o - L_o^o)v_{\bar{C}_1} + \dots +$$
$$+ (B_n^o - B_3^o) \cot \bar{A}_5^o v_{\bar{A}_5} - (B_n^o - B_3^o) \cot \bar{B}_5^o v_{\bar{B}_5} + c_1(L_n^o - L_3^o)v_{\bar{C}_5} + w_B = 0, \tag{15.21}$$

in which $w_B = \rho''\{\Delta B_{on} - (B_n^o - B_o^o)\}$; $\Delta B_{on} = [\Delta B]_o^n$ — the difference between the latitude of the initial point O and that of the final point n, calculated by translation of coordinates, using the measured angles; — $c_1 = \frac{N_m}{M_m} \cos B_m$; $\sigma_0 = v_{s_0}/s_o$; $\bar{A}, \bar{B}, \bar{C}$ — the triangle angles.

The equation of agreement of longitudes:

$$\rho'' \frac{\sigma_0}{s_0}(L_n^o - L_o^o) + c_2(B_n^o - B_o^o)v_{A_o} +$$
$$+ (L_n^o - L_o^o) \cot \bar{A}_1^o v_{\bar{A}_1} - (L_n^o - L_o^o) \cot \bar{B}_1^o v_{\bar{B}_1} + \tag{15.22}$$
$$+ c_2(B_n^o - B_o^o)v_{\bar{C}_1} + \dots + (L_n^o - L_3^o) \cot \bar{A}_5^o v_{\bar{A}_5} -$$
$$- (L_n^o - L_3^o) \cot \bar{B}_5^o v_{\bar{B}_5} + c_2(B_n^o - B_3^o)v_{\bar{C}_5} + w_L = 0,$$

in which $w_L = \rho''\{\Delta L_{on} - (L_n^o - L_o^o)\}$; $\Delta L_{on} = [\Delta L]_o^n$ — the difference between the longitude of the initial point O and that of the final point n, calculated by translation of coordinates utilizing the measured angles;

$$c_2 = \frac{M_m}{N_m \cos B_m} = -\frac{1}{c_1}.$$

The formulae (15.21) and (15.22) hold for sides on the ellipsoid of up to 60 km. For larger sides one utilizes second-order formulae for the correction coefficients, viz. *(Sander* 1952, *Jordan/Eggert/Kneissl* 1958*)*:

$$\rho'' \frac{\sigma_o}{s_o}\left\{B_n^o - B_o^o - \frac{\sin 2B_o^o}{4\rho''}(L_n^o - L_o^o)^2\right\} + c_1(L_n^o - L_o^o)v_{A_o} +$$

$$+\left\{B_n^o - B_o^o - \frac{\sin 2B_o^o}{4\rho''}(L_n^o - L_o^o)^2\right\}\cot \bar{A}_1^o v_{\bar{A}_1} -$$

$$-\left\{B_n^o - B_o^o - \frac{\sin 2\,B_o^o}{4\rho''}(L_n^o - L_o^o)^2\right\}\cot \bar{B}_1^o v_{\bar{B}_1} + c_1(L_n^o - L_o^o)^2 v_{\bar{C}_1} + \ldots +$$

$$+\left\{B_n^o - B_3^o - \frac{\sin 2B_3^o}{4\rho''}(L_n^o - L_o^o)^2\right\}\cot \bar{A}_5^o v_{\bar{A}_5} -$$

$$-\left\{B_n^o - B_3^o - \frac{\sin 2B_3^o}{4\rho''}(L_n^o - L_o^o)^2\right\}\cot \bar{B}_5^o v_{\bar{B}_5} + c_1(L_n^o - L_o^o)v_{\bar{C}_5} + w_B = 0$$

and, correspondingly:

$$\rho'' \frac{\sigma_o}{s_o}(L_n^o - L_o^o)\left(1 + \tan B_o^o \frac{B_n^o - B_o^o}{\rho''}\right) +$$

$$+ c_2(B_n^o - B_o^o)\left(1 + \tan B_o^o \frac{B_n^o - B_o^o}{\rho''}\right) v_{A_o} +$$

$$+ (L_n^o - L_o^o)\left(1 + \tan B_o^o \frac{B_n^o - B_o^o}{\rho''}\right)\cot \bar{A}_1^o v_{\bar{A}_1} -$$

$$- (L_n^o - L_o^o)\left(1 + \tan B_o^o \frac{B_n^o - B_o^o}{\rho''}\right)\cot \bar{B}_1^o v_{\bar{B}_1} +$$

$$+ c_2\left(B_n^o - B_o^o + \frac{\tan B_o^o(B_n^o - B_o^o)^2}{\rho''} -\right.$$

$$\left.- \frac{\sin B_o^o \cos B_o^o}{2}\,\frac{(L_n^o - L_o^o)^2}{\rho''}\right)v_{\bar{C}_1} + \ldots +$$

$$+ (L_n^o - L_3^o)\left(1 + \tan B_3^o \frac{B_n^o - B_3^o}{\rho''}\right)\cot \bar{A}_5^o v_{\bar{A}_5} -$$

$$- (L_n^o - L_3^o)\left(1 + \tan B_3^o \frac{B_n^o - B_3^o}{\rho''}\right)\cot \bar{B}_5^o v_{\bar{B}_5} +$$

$$+ c_2\left(B_n^o - B_3^o + \frac{\tan B_3^o(B_n^o - B_3^o)^2}{\rho''} -\right.$$

$$- \frac{\sin B_3^o \cos B_8^o}{2} \frac{(L_n^o - L_8^o)^2}{\rho''} \Bigg) v_{\overline{C}_s} + w_L = 0.$$

Remark. For long triangulation chains other terms are additionally needed with respect to the previous formulae and in this case one utilizes the formulae given by *Adams* (*Jordan/Eggert/Kneiss/1958*).

15.2.3 The Block Adjustment

For the most general possible case, i.e. that of the correlated measurements as well as for the special case of independent measurements, the basic formulae for the block adjustment of an astro-geodetic-triangulation network by the method of conditioned observations are presented in Table 15.1, their demonstration to be found in numerous technical works (e.g. *Lörinczi* 1966, *Wolf* 1968, *Grossmann* 1969, *Botez/Ghițău/Fotescu* 1971, *Ghițău* 1972).

In Table 15.1 the corresponding notations have the following meanings: $\mathbf{A}'^T = (\mathbf{AF}^{-1})^T$; $\mathbf{A}^T$ — transposed matrices of the condition-equation coefficients; $\mathbf{F}$ — the matrix of transforming the original, independent observation

Table 15.1. *Basic Formulae of the Adjustment by Conditioned Observations*

Measurement type / Adjustment-process stage	Measurements	
	correlated (dependent)	independent
Linearized condition equations of the corrections	$\mathbf{A}'^T\mathbf{e} + \mathbf{W} = 0$	$\mathbf{A}^T\mathbf{V} + \mathbf{W} = 0$
Normal-equation system of the correlates	$\mathbf{A}'^T\mathbf{Q}_U\mathbf{A}'\mathbf{K} + \mathbf{W} = 0$	$\mathbf{A}^T\mathbf{P}^{-1}\mathbf{A}\mathbf{K} + \mathbf{W} = 0$
Relations for determining the corrections	$\mathbf{e} = \mathbf{Q}_U\mathbf{A}'\mathbf{K}$	$\mathbf{V} = \mathbf{P}^{-1}\mathbf{A}\mathbf{K}$
Condition of minimum	$\mathbf{e}^T\mathbf{Q}_U^{-1}\mathbf{e} = \min$	$\mathbf{V}^T\mathbf{P}\mathbf{V} = \min$
Specific-check relation of the conditioned observations	$\mathbf{e}^T\mathbf{O}_U^{-1}\mathbf{e} = -\mathbf{W}^T\mathbf{K}$	$\mathbf{V}^T\mathbf{P}\mathbf{V} = -\mathbf{W}^T\mathbf{K}$
Weight-unit standard deviation	$m_0^2 = \dfrac{\mathbf{e}^T\mathbf{Q}_U^{-1}\mathbf{e}}{r}$	$m_0^2 = \dfrac{\mathbf{V}^T\mathbf{P}\mathbf{V}}{r}$

L_i into a row of dependent elements $U_i (i = 1, 2, ..., n)$, i.e. $U_i = \mathbf{F}L_i$; $\mathbf{V}^T = \| v_1 v_2 \ldots v_n \|$ — the transposed vector of the unknown corrections of the independent measurements L_i^0 and $\mathbf{e}^T = \| e_1 e_2 \ldots e_n \|$ respectively, the transposed vector of the corresponding corrections to be applied to the dependent elements U_i^0 in order to obtain the same results from the two adjustments (for dependent and independent measurements respectively), i.e.:

$$\mathbf{L}^o + \mathbf{V} = \mathbf{F}^{-1}(\mathbf{U}^o + \mathbf{e}).$$

Consequently: $\mathbf{V} = \mathbf{F}^{-1}\mathbf{e}$; $\mathbf{W}^T = \| W_1 W_2 \ldots W_r \|$ — the transposed vector of the free terms in the condition equations; $\mathbf{K}^T = \| K_1 K_2 \ldots K_r \|$ — the transposed vector of the correlations (or *Lagrange*'s multipliers); n — the number of the measurements for which corrections are sought ; r — the number of condition equations ($r < n$); $\mathbf{P}$ — the diagonal matrix of the weights of the independent observations; $\mathbf{Q}_U = \mathbf{F}\mathbf{P}^{-1}\mathbf{F}^T$ — the matrix of the weights of the dependent elements.

In what follows, references will be made to the case of adjusting the independent measurements.

Putting $\mathbf{N} = \mathbf{A}^T\mathbf{P}^{-1}\mathbf{A}$, it follows that $\mathbf{N}$ is always a square symmetric matrix, for which $|\mathbf{A}| \neq 0$, so that the system of the normal equations of the correlates may also be written in the form:

$$\mathbf{N}\mathbf{K} + \mathbf{W} = 0. \tag{15.23}$$

The matricial equation (15.23) can be solved by exact methods or by iterative methods. Among the exact methods, in the practice of triangulation adjustment two methods are most frequently used: *Gauss*' method[1] (or the method of elimination) and *Cholesky-Banachiewicz*'s method (or the method of the square roots) and (more seldom) *Helmert*'s method (or the method of the undetermined coefficients). The first two methods are based on the splitting up of the matrix of coefficients of the normal equations $\mathbf{N}$ into a corresponding number of divisors, so that their matricial product is equal to the matrix $\mathbf{N}$. Thus, in *Gauss*' method the matrix $\mathbf{N}$ is split up as follows *(Lörinczi 1966)*:

$$\mathbf{N} = \mathbf{G}^T\mathbf{D}\,\mathbf{G},$$

where $\mathbf{G}$ represents upper triangular square matrix and $\mathbf{D}$ — a diagonal matrix; while in *Cholesky-Banachiewicz*'s method:

$$\mathbf{N} = \mathbf{T}^T\mathbf{T},$$

where $\mathbf{T}$ is upper triangular matrix.

In these two exact methods the solution of the system (15.23) follows from a finite number of operations without calculating $\mathbf{N}^{-1}$ and therefore they are actually direct methods, whereas in *Helmert*'s method one utilizes the solution $\mathbf{K} = -\mathbf{N}^{-1}\mathbf{W}$, which assumes calculation of the inverse $\mathbf{N}^{-1}$ so that this method is an indirect exact method.

[1] Sometimes also called the *successive-substitution method.*

Remark. The elements of the matrix $\mathbf{N}^{-1}$ represent the weight coefficients themselves of the adjusted network. Therefore it is advisable that *Helmert*'s method be utilized when it is required to assess the accuracy of the adjusted elements and of their functions.

The iterative methods, also designated as *approximate methods* or *successive-approximation methods (Lörinczi* 1966*)* lead to the solution of the system (15.23) in the limit form of a vector set, in which the first element is the vector of an approximate solution, the rest of the elements being deduced from the first one by successive approximations. Among these methods one distinguishe: *Jacoby*'s procedure, *Seidel*'s method, the relaxation method, the super-relaxation method, the method of the conjugated gradients and others.

The iterative methods present a series of advantages as: (1) *the calculating algorithm is somehow simpler;* (2) *they are very useful in particular when the diagonal coefficients are greater in absolute value than the non-diagonal ones or if the system matrix is symmetric and positive definite;* (3) *they lead more rapidly than the exact methods to the desired result.*

Nevertheless, the iterative methods have an important shortcoming too, viz.: *sometimes the convergence of the successive approximations is very slow;* this convergence depends mainly on the elements of the matrix $\mathbf{N}$ of the system, as well as on a good choice of the initial approximation $\mathbf{K}^{(1)}$

Inasmuch as the the matrix $\mathbf{N}$ is a symmetric positive definite matrix, the iterative methods are recommended for application, in particular the method of the conjugated gradients, which seems to be gaining more and more importance internationally especially for the adjustment of vast triangulation networks. This is due firstly to the possibilities of utilizing electronic computers and secondly to the fact that here the matrix $\mathbf{N}$ has many zero elements and the number of unknowns is very large and, consequently, the time required for computations will be reduced. Subsequently, we shall present the method of the conjugated gradients, without dwelling on the other methods, as they have already been dealt with in detail in numerous works (e.g. *Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1966, *Wolf* 1968, *Ghiţău* 1972*)*.

The method of the conjugated gradients was worked out by *E. Stiefel* and *M. Hestenes* (1952) for solving systems of symmetrically defined equations. *Stiefel* (1952/1953) has then showed that by using the method of the conjugated gradients in solving the normal-equation system (15.23), one may replace the matrix $\mathbf{N}$ by the matrix of the coefficients of the correction (condition) equations $\mathbf{A}$. The way of applying the method for the case of adjustment by both conditioned observations and indirect observations has subsequently been demonstrated *(Schwarz* 1970*)*.

Let us briefly describe the method of the conjugated gradients as applied to the case of adjustment by conditioned observations *(Schwarz* 1970*)*.

Let $\mathbf{K}^{(j)}$ be the j-th approximation for the vector of the correlates $\mathbf{K}$ from (15.23); the approximation $\mathbf{V}^{(j)}$ corresponding to the vector of the corrections sought will be in this case:

$$\mathbf{V}^{(j)} = \mathbf{A}\mathbf{K}^{(j)}. \tag{15.24}$$

From (15.23) there follows, using (15.24), the residue vector $\mathbf{r}^{(j)}$, viz.:

$$\mathbf{r}^{(j)} = \mathbf{N}\mathbf{K}^{(j)} + \mathbf{W} = \mathbf{A}^T\mathbf{V}^{(j)} + \mathbf{W}. \tag{15.25}$$

By means of (15.25) the link is made between the approximation $\mathbf{V}^{(j)}$ and the residue vector $\mathbf{r}^{(j)}$. Utilizing the recurrence relation:

$$\mathbf{K}^{(j)} = \mathbf{K}^{(j-1)} + \lambda_j \mathbf{h}^{(j)}, \tag{15.26}$$

in which:

$$\mathbf{h}^{(j)} = \begin{cases} -\,\mathbf{r}^{(0)} & (j = 1), \\ -\,\mathbf{r}^{(j-1)} + \varepsilon_{j-1}\,\mathbf{h}^{(j-1)} & (j \geqslant 2); \end{cases} \tag{15.27}$$

$$\varepsilon_{j-1} = \frac{\mathbf{r}^{(j-1)T}\,\mathbf{r}^{(j-1)}}{\mathbf{r}^{(j-2)T}\,\mathbf{r}^{(j-2)}} \qquad (j \geqslant 2); \tag{15.28}$$

$$\lambda_j = \frac{\mathbf{r}^{(j-1)T}\,\mathbf{r}^{(j-1)}}{\mathbf{h}^{(j)T}\,\mathbf{N}\mathbf{h}^{(j)}}, \tag{15.29}$$

after multiplying its left-hand side by the matrix $\mathbf{A}$ and in view of (15.24), one finds:

$$\mathbf{V}^{(j)} = \mathbf{V}^{(j-1)} + \lambda_j(\mathbf{A}\mathbf{h}^{(j)}). \tag{15.30}$$

One notes that in this way the matrix $\mathbf{N}$ has been eliminated from the calculations. After this preparation, one chooses any starting vector $\mathbf{K}^{(0)}$ and with it one calculates:

$$\mathbf{V}^{(0)} = \mathbf{A}\,\mathbf{K}^{(0)} \tag{15.31}$$

For the case when the correction values of the vector $\mathbf{V}$ are small, a reasonable initial choice is $\mathbf{K}^{(0)} = \mathbf{0}$ and consequently $\mathbf{V}^{(0)} = \mathbf{0}$.

The general relaxation step ($j = 1, 2, \ldots,$) is described by the following formulae *(Schwarz 1970)*:

$$\left.\begin{array}{l} \mathbf{r}^{(j-1)} = \mathbf{A}^T\,\mathbf{V}^{(j-1)} + \mathbf{W} \\ \text{formula (15.28)} \\ \text{formula (15.27)} \\ \lambda_j = \dfrac{\mathbf{r}^{(j-1)T}\,\mathbf{r}^{(j-1)}}{(\mathbf{A}\,\mathbf{h}^{(j)})^T(\mathbf{A}\,\mathbf{h}^{(j)})} \\ \text{formula (15.26)} \\ \mathbf{V}^{(j)} = \mathbf{V}^{(j-1)} + \lambda_j(\mathbf{A}\,\mathbf{h}^{(j)}) \end{array}\right\} \tag{15.32}$$

The vectors $\mathbf{V}^{(j)}$ and $\mathbf{A}\mathbf{h}^{(j)}$ are of order n and the vectors $\mathbf{r}^{(j)}$, $\mathbf{W}, \mathbf{h}^{(j)}$ and $\mathbf{K}^{(j)}$ — of order r (the number of condition equations).

In the computing instructions (15.32), the approximations $\mathbf{K}^{(j)}$ of the vector of the correlates may be omitted without the calculating procedure being influenced by this omission *(Schwarz 1970)*. Thus, for the direct determination of the vector $\mathbf{V}$ there appears an iterative procedure in which both the matrix $\mathbf{N}$ and the vector $\mathbf{K}$ are eliminated.

A series of important rules for the method of the conjugated gradients has been indicated *(Schwarz* 1970*)* viz.:

(1) *The relaxation directions* $\mathbf{h}^{(j)}$ *constitute a system of conjugate direction-pairs,* i.e.:

$$\mathbf{h}^{(j)^T} \mathbf{N} \, \mathbf{h}^{(i)} = \mathbf{0}, \quad (i \neq j).$$

(2) *The residue vectors* $\mathbf{r}^{(j)}$ $(j = 0, 1, 2, \ldots)$ *are pair wise orthogonal,* i.e.:

$$\mathbf{r}^{(i)^T} \mathbf{r}^{(j)} = \mathbf{0}(i \neq j).$$

(3) *The method of the conjugate gradients provides the solution* $\mathbf{V}$ *for adjustment by conditioned observations after at most r iterations.*

(4) *If the problem of adjustment by conditioned observations is solved by the method of the conjugate gradients, then the value of the correction vector* $\mathbf{V}^{(j)}$ *increases monotonically.*

The practical importance of rule (4) *lies in the following*:

1) *The iterative calculating process on the basis of the square of the correction vector can be stopped as soon as its value becomes less than a given/imposed limit and in this case we declare ourselves to be satisfied with a corresponding approximate solution* $\mathbf{V}^{(j)}$.

2) *In the case of the solution of large systems of correction equations — the case of the astro-geodetic-triangulation networks — the value of the square of the correction vector frequently reaches a minimum after only relatively few iteration steps.* Even if the residue vector $\mathbf{r}^{(j)^T} \mathbf{r}^{(j)}$ has not reached zero in the sense of the computer accuracy, the computing process may be stopped and the obtained value $\mathbf{V}^{(j)}$ regarded as a definitive solution.

3) *The fact that in practice, in applying the method of the conjugate gradients, for fewer iteration steps then r are needed leads to an important saving of time in computations.*

15.2.4 The Group Adjustment

In the case of adjusting large astro-geodetic-triangulation networks one is confronted with computation difficulties even when utilizing the electronic computer. Therefore the group adjustment is still valuable, under certain given conditions. Thus, if the network is large, then from considerations of clarity and computation organization it is necessary that the condition-equation system be divided into groups, which are to be separately or consecutively processed. The group adjustment also presents the advantage of simplifying the calculations arising from the utilization of partial results which are already tabulated on the basis of normal-equation systems, obtained quite simply out of figural conditions.

Dividing the condition-equation system into groups may be carried out depending on either the network structure (consequently on regional considerations), or on the basis of formal instructions depending on the condition-equation type (e.g.: 1st group: horizon-round conditions and figural conditions; 2nd group: pole conditions and coordinates conditions; 3rd group: conditions external to the network).

One distinguishes several group-adjustment procedures, viz.: *Gauss-Krüger*'s procedure of two-group adjustment, *Boltz*'s procedures (substitution procedure and successive-development procedure respectively), the buffer-network procedure of *Friedrich* and *Asplund, Bjerhammar*'s procedure, *Strobel*'s procedure, *Tienstra*'s procedure, *Bessel*'s procedure, *Helmert-Pranis Pranievich*'s procedure etc., as well as some iterative procedures (*Gauss*' procedure, *Bamberger*'s method etc.). Among these procedures, the most advisable to be used, under the conditions of using electronic computers and on the assumption that measured directions are being adjusted, are those of *Asplund* and *Helmert-Pranis Pranievich.* Since these procedures are minutely described in the technical literature (e.g. *Jordan/Eggert/Kneissl* 1958, *Wolf* 1968, *Ghițău* 1970 *b*, 1972), no reference will be made to them in this work except for *Helmert-Pranis Pranievich*'s procedure, which will be succinctly presented in § 15.3.3. Here, we make only a few remarks on the rigorous integration of the already adjusted astro-geodetic networks. This problem may occur in the case when one's goal, on the basis of international agreements, is the integration of national networks into a unique network or when one wishes to integrate already adjusted zonal/regional networks. Thus, from among these separate partial adjustments, constituting the 1st group, one selects certain functions F on the basis of geometrical considerations and these functions are then subjected to a main adjustment (forming the 2nd group) in which all the conditions resulting from the geometry of the complete (unique) network must be satisfied. In the following, some examples are given of such group adjustments (*Wolf* 1968).

Adjusting the frame networks. Here the partial systems are usually triangle chains (which actually represent *connexion chains*) out of which one builds the frame network (Fig. 15.7). If each of these triangulation chains is replaced with the corresponding geodetic line connecting the terminal

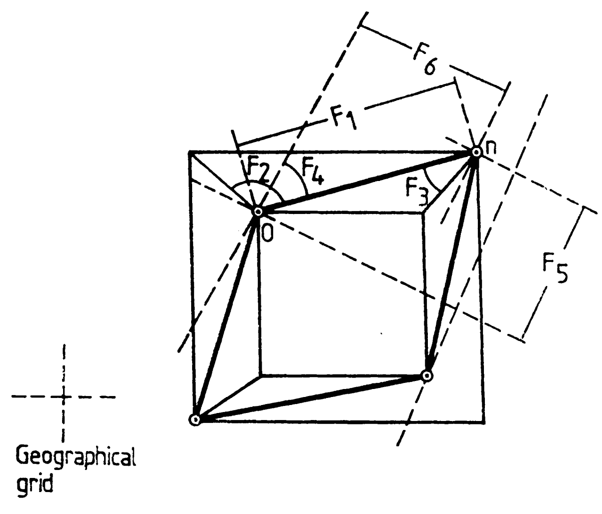

Fig. 15.7. Frame Network

Laplace points of the chain (e.g. O and n in Fig. 15.7), then **as functions F we may use:**

(1) *The length of the geodetic line F_1 and the angles F_2 and F_3 formed by the geodetic line at the points O and n with certain sides of the triangles* (Fig. 15.7). This is the manner adopted by e.g., *O. Eggert* (1937), *G. Lehmann* (1937), *F. N. Krasovski* (1953—1956) etc.

(2) *The length of the geodetic line F_1 and its azimuth F_4 with respect to one of the terminal points of the chain.* This was, e.g., the method of *V.R. Ölander* (1949).

(3) *The differences of latitude F_5 and of longitude F_6 between the two terminal points O and n of the triangulation chain,* as created by *K. Arnold* (1956), *H. Wolf* (1950 *b*) etc.

Adjusting the derived networks. In the case of derived networks (Fig. 13.4), the partial systems are surface partial systems which are replaced with one large (fictitious) triangle each. As functions F we here choose the sides (respectively the angles) of the large triangle.

Within the framework of the main adjustment, the derived network formed out of several large triangles, is subjected to a special adjustment. This is the method of *St. Hazay* (1953) and *A. Tárczi-Hornoch* (1959) for the adjustment of continental astro-geodetic-triangulation networks.

Adjusting the compact triangulation networks by means of their connexion-polygon [1] elements. By connexion polygon one must understand the common polygonal track between two independently adjusted triangulation networks. In this case, the partial systems are represented, e.g. (Fig. 15.8) by the networks I, II and III.

As functions F one chooses here the breaking angles of the connexion polygon arising from the adjustment of the partial network, as well as the values of the ratio between two consecutive sides, i.e. F_1, F_2, F_3, F_4, ... (Fig. 15.8).

Within the framework of the main adjustment the condition is imposed that the values of the functions F be identical, regardless of whether these

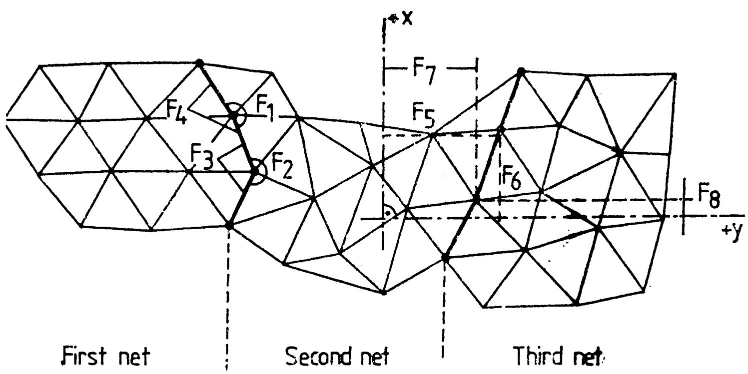

Fig. 15.8. Integration Polygons and Their Elements

[1] In the technical literature one also finds the designation of *boundary polygons* — in German *"Grenz Polygone"* (*Wolf* 1968).

arise from one or other of the networks (of the two neighbouring networks). The integrated triangulation networks is, consequently, formed by the system of connexion polygons (*Wolf* 1950 *c*).

Adjusting the compact triangulation networks by means of the coordinates of the common connexion-points[1]. Here, from the independent partial networks one takes over, as functions F, the astronomical (or plane rectangular) coordinates of the common points of connexion between networks, e.g. for the case of the plane coordinates F_5, F_6, as well as F_7, F_8 (Fig. 15.8).

Within the framework of the main adjustment, the condition is imposed that the definitive coordinates of the connexion points should be the same irrespective of the network from which they arise (i.e. the differences between the point coordinates calculated from one network and those from the neighbouring network should vanish). This manner of approaching the problem actually amounts to *Helmert*'s procedure of block adjustment (*Wolf* 1968).

Remarks:

(1) In all cases of group adjustment which have presented, the functions F represent dependent observations, resulting from independent partial adjustments. Consequently, for each case one must calculate the corresponding correlation matrix Q_F. Afterwards, for the main adjustment, one establishes the condition equations in the functions F (arising from the network connexion) and one immediately obtains then the corresponding correlates, the corrections e of the functions F, the sum of the squares of the errors and, possibly, the weigths of the functions.

(2) One must avoid having the observations located on the connexion line between the partial networks entering the main adjustment twice. This can be achieved by the corresponding observation, being:

a) either assigned from the beginning only to one partial network (this is the manner of e.g., *K. Arnold* (1965), or

b) introduced with a partial weight into several networks which are adjusted in common (e.g. in the contributions of *Eggert* (1937) and *Arnold* (1955)); for instance, in two neighbouring partial networks the corresponding weight is taken as equal to 0.5 (*Wolf* 1968). In this case it is necessary that as well as the supplementary condition cquations there should also be worked out supplementary condition equations of identity of the corrections resulting from the partial adjustments (*Eggert* 1937, *Wolf* 1950 *c*).

15.3 Adjustment of the Triangulation Network by means of the Method of Indirect Observations [2]

In the method of indirect observations, the problem of finding the system of the most probable corrections is solved using the given mathematical conditions of the network, expressing the connexions between the direct measurements and the quantities to be calculated as functions of them. Each mathematical condition is expressed in the form of a correction equation.

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[1] Or *boundary points* (or *connexion* ones) — in German "*Grenz* (*Verbindungs* —) *Punkte*" (*Wolf* 1968).

[2] This is usually also called *point-group adjustment* or is alternatively known under the designation of *method of variation of coordinates* (*Lörinczi* 1959, *Wolf* 1968, *Ghiţău* 1972).

The number of correction equations equals the total number of measurements which have been carried out; consequently, to every geodetic observation corresponds such an equation, which represents a real advantage in adjusting complex triangulation networks.

As a rule, in adjusting the triangulation network by means of the method of indirect observations, one chooses as unknowns the coordinates of the separate station points.

In what follows references will be made to the adjustment on the reference ellipsoid, i.e. the case of the large astro-geodetic networks, by coordinates of the network points being, consequently, understood the geodetic coordinates B and L.

15.3.1 The Form of the Correction Equations

The correction equations occurring in the adjustment of the astro-geodetic triangulation network have various forms, depending on the nature of the measurements carried out. Firstly, we shall present the form of the correction equations for the azimuthal observations, viz. for directions. One assumes that at any station point P_i of the network (Fig. 15.9) were measured the directions $1, 2, \ldots, K$, the values:

$$l_{i1}^0, l_{i2}^0, \ldots, l_{ij}^0, \ldots, l_{ik}^0,$$

being obtained,
and one looks for the corrections:

$$v_{i1}, v_{i2}, \ldots, v_{ij}, \ldots, v_{ik},$$

which are to be determined by the network adjustment.

One also assumes that the observations l_{ij}^0 were centred and reduced on the ellipsoid.

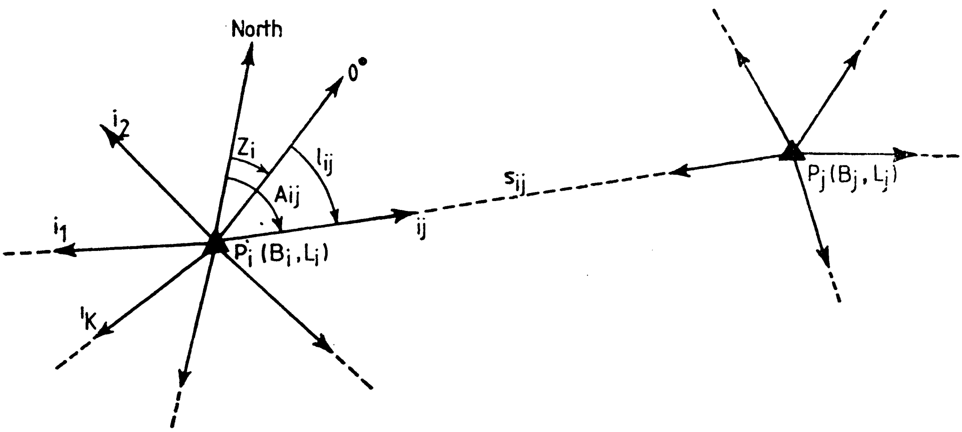

Fig. 15.9. Orienting the Station Point P_i

Consequently, the adjusted directions will be:

$$l_{ij} = l^0_{ij} + v_{ij}, \quad (j = 1, 2, \ldots, K) \tag{15.33}$$

From Fig. 15.9 it follows that:

$$A^*_{ij} = z_i + l_{ij}, \tag{15.34}$$

where A^*_{ij} is the azimuth of the direction ij; z_i — orientation angle of the station P_i (i.e. the angle between the north direction and the direction of the zero division of the theodolite's horizontal circle). The mathematical condition will consequently be:

$$l_{ij} = - z_i + A^*_{ij}. \tag{15.35}$$

Comparing (15.35) with (15.33) yields the following correction equation:

$$v_{ij} = - z_i + A^*_{ij} - l^o_{ij}. \tag{15.36}$$

Since the azimuth A^*_{ij} is a function of the geodetic coordinates of the two points i and j, i. e. of B_i, L_i, B_j and L_j, in order to be able to carry out the adjustment it is necessary to linearize the correction equation (15.36) by expanding the azimuth A^*_{ij} in a *Taylor* series. To this end it is, however, necessary to know the provisional (approximate) coordinates of the points i and j, i.e. B^0_i, L^0_i, B^0_j and L^0_j:

$$\begin{aligned} B_i &= B^o_i + \mathrm{d}B_i; & B_j &= B^o_j + \mathrm{d}B_j; \\ L_i &= L^o_i + \mathrm{d}L_i; & L_j &= L^o_j + \mathrm{d}L_j \\ A^*_{ij} &= A^{*o}_{ij} + \mathrm{d}A^*_{ij}; & A^i_i &= A^{*o}_{ji} + \mathrm{d}A^*_{ji}, \end{aligned} \tag{15.37}$$

where $\mathrm{d}B, \mathrm{d}L, \mathrm{d}A^*$ represent the variations (increases) of the geodetic coordinates (respectively the azimuth variation) at the corresponding station point i, (respectively j).

One also assumes that one knows the provisional value z^o_i of the orientation angle, which is to be corrected by adjustment, i.e.:

$$z_i = z^o_i + \mathrm{d}z_i. \tag{15.38}$$

Expanding A^*_{ij} in a *Taylor* series, limited to the 1st-order terms, and taking account of (15.37) and (15.38), then the correction equation (15.36) becomes:

$$v_{ij} = - \mathrm{d}z_i + \frac{\partial A^*_{ij}}{\partial B_i} \mathrm{d}B_i + \frac{\partial A^*_{ij}}{\partial L_i} \mathrm{d}L_i + \frac{\partial A^*_{ij}}{\partial B_j} \mathrm{d}B_j + \frac{\partial A^*_{ij}}{\partial L_j} \mathrm{d}L_j - t_{ij}, \tag{15.39}$$

where $- t_{ij} = A^{*o}_{ij} - (l^o_{ij} + z^o_i)$. The azimuth $A^{*o}_{ij} = A^{*o}_{ij}\ (B^o_i,\ L^o_i,\ B^o_j,\ L^o_j)$ is obtained by solving the inverse geodetic problem. The corresponding partial derivates in (15.39) were deduced by *F. R. Helmert* (1962), the correction equation (15.39) taking the following form suitable for calculations (*Jordan/Eggert/Kneissl* 1958):

$$v_{ij} = -\mathrm{d}z_i + \frac{M_i \sin A_{ij}^{*o}}{a \tan \frac{s_{ij}}{a}} \mathrm{d}B_i + \frac{M_j \sin A_{ji}^{*o}}{a \sin \frac{s_{ij}}{a}} \mathrm{d}B_j +$$

$$+ \frac{N_j \cos B_j^o \cos A_{ji}^{*o}}{a \sin \frac{s_{ij}}{a}} (\mathrm{d}L_i - \mathrm{d}L_j) - l_{ij}, \qquad (15.40)$$

in which $A_{ji}^{*o} \approx A_{ij}^{*o} + \frac{s_{ij} \sin A_{ij}^{*o}}{N_i} \tan B_i^o \pm 180°$; M, N — radius of curvature of the meridian, and of the prime vertical respectively; a — the equatorial radius.

Remarks. In calculations on the ellipsoid, the geodetic coordinates B and L are expressed in sexagesimal graduation, the azimuth A^* in centesimal graduation and s_{ij} and a in km.

Inasmuch as every angle may be expressed as a difference of two directions, in the case when the azimuthal angles are centred and reduced to the reference ellipsoid, the correction equations can be deduced from the difference of equations of type (15.40).

Remarks:

(1) The equations of the azimuthal-angle corrections no longer contain the unknowns dz.

(2) The seemingly formal expressing of angles as differences of two corresponding directions modifies both the functional model[1] and the stochastic model[2], accepted models of the network adjustment, introducing correlations between observations. Thus, if one adopts such a manner of expressing the angles, one needs to take into consideration the corresponding correlations.

(3) If angles as well as measured azimuthal directions are available for a station point, then the correction equations are formed in the previously described manner, with the difference that the equations corresponding to directions will have a twofold weight compared with those corresponding to angles.

We will next give the form of the correction equations in the case in which at the station point P_i the *Laplace* azimuth A_{ij} was determined from astronomical measurements. In this case one can write:

$$A_{ij} + v_A = A_{ij}^{*o} + \mathrm{d}A_{ij}^{*}. \qquad (15.41)$$

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[1] *The functional model* provides the geometrical-physical description of the adjustment program to be solved, by means of functions, constant quantities or variable ones.

[2] *The stochastic model* brings out the stochastic qualities and the connexions of stochastic interdependence existing between the elements by means of which the functional model was formed. One must mention here, in particular, the condition of minimum and the correlation matrix of the observations having been carried out.

where v_A is the correction of the *Laplace* azimuth and A^{*o}_{ij} — the provisional value of the geodetic-line azimuth ij (calculared from the provisional coordinates of the points i and j). We get:

$$v_A = \mathrm{d}A^*_{ij} - t^A_{ij}, \tag{15.42}$$

where:

$$- t^A_{ij} = A^{*o}_{ij} - \{\alpha^o_{ij} - (\Lambda^o_i - L^o_i) \sin B^o_i\};$$

$$\mathrm{d}A^*_{ij} = + \frac{M_i \sin A^{*o}_{ij}}{a \tan (s_{ij}/a)} \mathrm{d}B_i + \frac{M_j \sin A^{*o}_{ji}}{a \sin (s_{ij}/a)} \mathrm{d}B_j +$$

$$+ \frac{N_j \cos B^o_j \cos A^{*o}_{ji}}{a \sin (s_{ij}/a)} (\mathrm{d}L_i - \mathrm{d}L_j).$$

Here α^o_{ij} and Λ^o_i represent the value of the azimuth and of the longitude respectively, as obtained from geodetic-astronomy determinations.

Remarks:

(1) The equation (15.42) doesn't contain the unknown dz.

(2) Thorough preliminary investigations are generally needed as regards the weight $p\alpha$ which must be assigned to the *Laplace* azimuths with respect to the weight of the azimuthal directions/angles; at present, one currently assigns the same weight to the *Laplace* azimuth as to the definitive values of the adjustment (*Jordan/Eggert/Kneissl 1958*).

Eventually, if within the framework of the astro-geodetic-triangulation network there occur distances directly measured by means of electromagnetic instruments or, less frequently nowadays, distances arising from networks of base development, such lengths must be dealt with just as any one of the geodetic observations. Thus, in this case there will appear a correction equation of the form (*Jordan/Eggert/Kneissl* 1958):

$$v^s_{ij} = - M_i \cos A^{*o}_{ij} \mathrm{d}B_i - N_i \sin A^{*o}_{ij} \cos B^o_i \,\mathrm{d}L_i - M_i \cos A^{*o}_{ji} \,\mathrm{d}B_j -$$

$$- N_j \sin A^{*o}_{ji} \cos B^o_j \,\mathrm{d}L_j - t^s_{ij}, \tag{15.43}$$

where $- t^s_{ij} = s^o_{ij} - s^*_{ij}$; A^{*o}_{ij}, s^o_{ij} are the azimuth and the distance respectively, calculated from the provisional geodetic coordinates of the triangulation-network points (by using the inverse geodetic problem) and s^*_{ij} — the measured distance reduced to the ellipsoid.

Remark. As a rule, preliminary investigations are needed in order to evaluate the weight p_s of the distances s with respect to the weight of the azimuthal directions/angles. It is much more correct to evaluate, even approximately, the weights under the given conditions of the adjustment program than to take $p_s = 0$ or $p_s = \infty$. If one takes $p_s = 0$, then, generally, the distance variations are not taken into consideration by adjustment and if $p_s = \infty$ then in (15.43) there appears $v^s_{ij} = 0$ and consequently a constraint condition. Generally, for distances arising from networks of geodetic-base development, one assumes $p_s = \infty$ so that these are introduced into the adjustment as fixed elements.

15.3.2 The Block Adjustment

The basic formulae of the block adjustment of an astro-geodetic-triangulation network by the method of indirect observations are presented in Table 15.2, for the most general case, i.e. that of the correlated measurements, as well as for the special case of the independent measurements; their demonstration is to be found in numerous technical works (e.g. *Lõrinczi* 1966, *Wolf* 1968, *Grossmann* 1969, *Botez* et al. 1971, *Ghițău* 1972).

In table 15.2 the corresponding notations have the following meanings: $\mathbf{B}' = \mathbf{FB}$, $\mathbf{B}$ — the matrices of the correction-equation coefficients; $\mathbf{X}$— the vector of the unknowns in the correction equations; $\mathbf{l}, \mathbf{l}' = \mathbf{Fl}$ — the vectors of the free terms of the correction equations; n — the number of observations; u — the number of the unknowns; $n > u$ and $n - u = r$ (checking relation) in which r — the number of the correction equations; $\mathbf{F}$, $\mathbf{V}$, $\mathbf{e}$, $\mathbf{P}$ and $\mathbf{Q}_U$ have the same meanings as in Table 15.1.

Table 15.2. *Basic Formulae of the Adjustment by Indirect Observations*

Measurement type / Adjustment-process stage	Measurements	
	correlated (dependent)	independent
Correction equations	$\mathbf{e} = \mathbf{B}'\mathbf{X} + \mathbf{l}'$ with the matrix $\mathbf{Q}_U$	$\mathbf{V} = \mathbf{BX} + \mathbf{l}$ with the weight matrix $\mathbf{P}$
Condition of minimum	$\mathbf{e}^T\mathbf{Q}_U^{-1}\mathbf{e} = \min$	$\mathbf{V}^T\mathbf{PV} = \min$
Normal-equation system	$\mathbf{B}'^P\mathbf{Q}_U^{-1}\mathbf{BX} + \mathbf{B}'^T\mathbf{Q}_U\mathbf{l}' = \mathbf{0}$	$\mathbf{B}^T\mathbf{PBX} + \mathbf{B}^T\mathbf{Pl} = \mathbf{0}$
Weight-unit standard deviation	$m_o^2 = \dfrac{\mathbf{e}^T\mathbf{Q}_U^{-1}\mathbf{e}}{n - u}$	$m_o^2 = \dfrac{\mathbf{V}^T\mathbf{PV}}{n - u}$

After working out the correction equations (according to § 15.3.1), one forms the corresponding normal equations for whose solution one can utilize *Cholesky-Banachiewichz*'s method, *Gauss'* method etc. One also knows (*Jordan/Eggert/Kneissl* 1958) approximate procedures for adjusting the triangulation networks by means of the indirect-observation method as: *Bowie*'s procedure (*Adams* 1915, *Kneissl* 1943), *Urmaev*'s procedure (*Urmaev* 1943), *Marussi*'s procedure (*Marussi* 1947) etc. In the last 10 years, the adjustment of networks of considerable size has been handled with great efficiency, by utilizing electronic computers of great capacity on the basis of adequate programmes (e.g. *Ghițău* 1972): *Mugica*'s programme allowing the determination of about 350 new points on an *IBM 360* computer; *Ehlert*'s programme permitting solutions of up to 1,500 new points on a *Telefunken* computer and having been applied to adjusting the *West-European* triangulation network etc.).

Such complex programmes provide for the complete automation of the computing process, including the evaluation of the accuracy of the final results and of the corresponding correlations.

Details concerning the methods of block adjustment of the triangulation network are to be found in many technical contributions (e.g.: *Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959, *Wolf* 1968, *Ghițău* 1970 *b*, and 1972 and *Botez* et al. 1971). In the present work we will only present the method of conjugate gradients which has lately received greater attention from specialists and which lends itself very well both to utilizing electronic computers and to the problem of adjusting large-size triangulation networks.

One notes in Table 15.2 that, in the case of the block adjustment by means of the method of indirect observations, the correction equations have the form:

$$\mathbf{BX} + \mathbf{l} = \mathbf{V},$$

in which $\mathbf{B}$ represents a *superior* rectangular matrix with r rows and n columns of maximum order n. Assuming equal weights for the correction equations, which can always be achieved by a suitable charge of variable (e.g., *Gotthardt* 1968), the corresponding normal equations are:

$$\mathbf{B}^T\mathbf{BX} + \mathbf{B}^T\mathbf{l} = \mathbf{0} \text{ or } \mathbf{NX} + \mathbf{B}^T\mathbf{l} = \mathbf{0}, \tag{15.44}$$

where $\mathbf{N} = \mathbf{B}^T\mathbf{B}$. The $\mathbf{X}$ solution of the symmetrically defined normal equations (15.44) can be determined with the aid of the method of conjugate gradients. One assumes that, for the $\mathbf{X}$ solution sought, there exists an approximating vector $\mathbf{X}^{(j)}$ and — consequently a vector $\mathbf{V}^{(j)}$ of the corresponding corrections which is determined by the relation:

$$\mathbf{V}^{(j)} = \mathbf{BX}^{(j)} + \mathbf{l}, \tag{15.45}$$

and the residue vector $\mathbf{r}^{(j)}$ of the normal equations will correspondingly be given by the relation:

$$\mathbf{NX}^{(j)} - \mathbf{B}^T\mathbf{l} = \mathbf{r}^{(j)}. \tag{15.46}$$

According to (15.45) and (15.46), the relation between the vectors $\mathbf{V}^{(j)}$ and $\mathbf{r}^{(j)}$ is:

$$\mathbf{r}^{(j)} = \mathbf{B}^T\mathbf{V}^{(j)}. \tag{15.47}$$

In accordance with the idea expressed by *Stiefel* (1952/1953), in the method of the conjugate gradients for solving the system of normal equations (15.44), the matrix $\mathbf{N}$ can be eliminated from the calculating algorithm of the method by means of the transformation:

$$\mathbf{h}^{(j)T}\mathbf{Nh}^{(j)} = \mathbf{h}^{(j)T}\mathbf{B}^T\mathbf{Bh}^{(j)} = (\mathbf{Bh}^{(j)})^T(\mathbf{Bh}^{(j)}),$$

where $\mathbf{h}^{(j)}$ is given by (15.27).

From the recurrence formula for the residue vector:

$$\mathbf{r}^{(j)} = \mathbf{r}^{(j-1)} + \lambda_j(\mathbf{Nh}^{(j)}),$$

in which λ_j is given by (15.29), one eliminates $\mathbf{N}$ with the help of (15.47). Consequently, this recurrence formula may be written as:

$$\mathbf{B}^T\mathbf{V}^{(j)} = \mathbf{B}^T\mathbf{V}^{(j-1)} + \lambda_j(\mathbf{B}^T\mathbf{B}\mathbf{h}^{(j)}) \text{ or } \mathbf{V}^{(j)} = \mathbf{V}^{(j-1)} + \lambda_j(\mathbf{B}\mathbf{h}^{(j)}). \qquad (15.48)$$

The calculating algorithm for solving the correction equations by means of the method of conjugate gradients is as follows: one chooses any starting vector $\mathbf{X}^{(o)}$ and with it one calculates the correction vector:

$$\mathbf{V}^{(o)} = \mathbf{B}\mathbf{X}^{(o)} + \mathbf{l}.$$

Remark. Due to the special choice of $\mathbf{X}^{(o)} = \mathbf{0}$, we have $\mathbf{V}^{(o)} = \mathbf{l}$.

The general relaxation step ($j = 1, 2, \ldots$) is described by the successive use of the following formulae (*Schwarz* 1970):

$$\begin{aligned}
\mathbf{r}^{(j-1)} &= \mathbf{B}^T\,\mathbf{V}^{(j-1)} \\
\varepsilon_{(j-1)} &= \frac{\mathbf{r}^{(j-1)T}\,\mathbf{r}^{(j-1)}}{\mathbf{r}^{(j-2)T}\,\mathbf{r}^{(j-2)}} && (j \geqslant 2) \\
\mathbf{h}^{(j)} &= \begin{cases} -\,\mathbf{r}^{(o)} & (j = 1) \\ -\,\mathbf{r}^{(j-1)} + \varepsilon_{j-1}\mathbf{h}^{(j-1)} & (j \geqslant 2) \end{cases} \\
\lambda_j &= \frac{\mathbf{r}^{(j-1)T}\,\mathbf{r}^{(j-1)}}{(\mathbf{B}\mathbf{h}^{(j)})^T\,(\mathbf{B}\mathbf{h}^{(j)})} \\
\mathbf{X}^{(j)} &= \mathbf{X}^{(j-1)} + \lambda_j\mathbf{h}^{(j)} \\
\mathbf{V}^{(j)} &= \mathbf{V}^{(j-1)} + \lambda_j(\mathbf{B}\mathbf{h}^{(j)})
\end{aligned} \qquad (15.49)$$

In the formulae (15.49), the vectors $\mathbf{V}^{(j)}$ and $\mathbf{B}\mathbf{h}^{(j)}$ have the same dimension r, whereas $\mathbf{r}^{(j)}$, $\mathbf{h}^{(j)}$ and $\mathbf{X}^{(j)}$ represent vectors of dimension n.

Remarks:

(1) The rules shown in presenting the method of conjugate gradients for the case of adjustment by conditioned observations (§ 15.2.2) hold here too.

(2) One makes as many relaxation steps as needed by the point-coordinate variations to remain less than 2 cm (*Wolf 1968*).

15.3.3 The Group Adjustment

We shall mention within the context of this paragraph, while keeping in mind the general considerations presented in § 15.2.3, that for the group adjustment of the triangulation networks by the method of indirect observations one may also use some of the procedures which were indicated in the case of the conditioned observations (§ 15.2.3). Examples of such (rigorous) procedures are: *Boltz*'s procedures and the buffer-network procedure (whose use, however, cannot be recommended unreservedly since in this case one cannot use the pre-calculated inverse matrices); *Helmert-Pranis Pranievich*'s procedure; *Levallois*'s procedure (*Levallois* 1947, p. 47—82 and 239—249) (which differs from the previous procedure by the fact that as unknowns one introduces here not the coordinates but the azimuths of the sides), and as approximate procedures: *Finsterwalder*'s method of fields *(Wolf* 1941*)* etc. One of the best-known and most used in the practice of triangulation-network adjustment is *Helmert-Pranis Pranievich*'s procedure. References will be made in the present treatise only to this procedure; details concerning the other procedures are to be found in the technical literature *(Adams* 1930, *Wolf* 1941 and 1968, *Kneissl* 1943, *Urmaev* 1943, *Marussi* 1947, *Levallois* 1947, *Jordan/Eggert/Kneissl* 1958, *Lörinczi* 1959*)*. The procedure was applied on a wide scale to adjusting the triangulation of the *U.S.S.R. (Pranis Pranievich* 1956*)* as well as to adjusting the *Finnish* triangulation ring *(Ölander* 1949*)*; some contributions were made in *Romania* regarding the application of the procedure to adjusting the triangulation networks *(Lörinczi* 1964 and 1966*)*.

Let us assume that the astro-geodetic-triangulation network is divided into two sectors and the general system of the correction equations $\mathbf{V} = \mathbf{BX} + \mathbf{l}$, with the weight matrix $\mathbf{P}$, is correspondingly also divided into two groups (I and II):

$$\begin{aligned} &\mathbf{V}_{\mathrm{I}} = \mathbf{B}_{\mathrm{I}}\mathbf{X}_{\mathrm{I}} + \mathbf{C}_{\mathrm{I}}\mathbf{X}_{\mathrm{I-II}} + \mathbf{l}_{\mathrm{I}}, \text{ with the weight matrix } \mathbf{P}_{\mathrm{I}}; \\ &\mathbf{V}_{\mathrm{II}} = \mathbf{B}_{\mathrm{II}}\mathbf{X}_{\mathrm{II}} + \mathbf{C}_{\mathrm{II}}\mathbf{X}_{\mathrm{I-II}} + \mathbf{l}_{\mathrm{II}}, \text{ with the weight matrix } \mathbf{P}_{\mathrm{II}}, \end{aligned} \tag{15.50}$$

where the internal elements appearing in only one of the two groups were denoted by the index I (respectively II) and the common (boundary) elements by the index I—II.

Remark. When making this division one takes into consideration the basic criterion (*Lörinczi* 1959 and *Ghițău 1972*): every geodetic observation belongs to exactly one sector.

In order that the system (15.50) be block-solved, the relations specific to the method of indirect observations of transition from the correction equations to the normal equations must be satisfied, viz. *(Ghițău* 1970 and 1972*)*:

$$\mathbf{B}_{\mathrm{I}}^T\mathbf{P}_{\mathrm{I}}\mathbf{V}_{\mathrm{I}} = \mathbf{0}; \ \mathbf{B}_{\mathrm{II}}^T\mathbf{P}_{\mathrm{II}}\mathbf{V}_{\mathrm{II}} = \mathbf{0}; \ \mathbf{C}_{\mathrm{I}}^T\mathbf{P}_{\mathrm{I}}\mathbf{V}_{\mathrm{I}} + \mathbf{C}_{\mathrm{II}}^T\mathbf{P}_{\mathrm{II}}\mathbf{V}_{\mathrm{II}} = 0. \tag{15.51}$$

Substituting the relations (15.50) into (15.51) and introducing the notations $\mathbf{N}_{ij}^h = \mathbf{i}_h^T\mathbf{P}_h\mathbf{j}_h$, where $i, j = \mathbf{B}, \mathbf{C}, \mathbf{l}$ (the case $\mathbf{i} = \mathbf{j}$ is also included, except for $\mathbf{l}$) and $h = \mathrm{I}, \mathrm{II}$, the system of the normal equations then takes the form:

$$\begin{aligned} &\mathbf{N}_{BB}^{\mathrm{I}}\mathbf{X}_{\mathrm{I}} + \mathbf{N}_{BC}^{\mathrm{I}}\mathbf{X}_{\mathrm{I-II}} + \mathbf{N}_{Bl}^{\mathrm{I}} = \mathbf{0}; \\ &\mathbf{N}_{BB}^{\mathrm{II}}\mathbf{X}_{\mathrm{II}} + \mathbf{N}_{BC}^{\mathrm{II}}\mathbf{X}_{\mathrm{I-II}} + \mathbf{N}_{Bl}^{\mathrm{II}} = \mathbf{0}; \\ &(\mathbf{N}_{BC}^{\mathrm{I}})^T\mathbf{X}_{\mathrm{I}} + (\mathbf{N}_{BC}^{\mathrm{II}})^T\mathbf{X}_{\mathrm{II}} + (\mathbf{N}_{CC}^{\mathrm{I}} + N_{CC}^{\mathrm{II}})\mathbf{X}_{\mathrm{I-II}} + (\mathbf{N}_{Cl}^{\mathrm{I}} + \mathbf{N}_{Cl}^{\mathrm{II}}) = \mathbf{0}. \end{aligned} \tag{15.52}$$

Since the matrices $\mathbf{N}^{\mathrm{I}}_{BB}$ and $\mathbf{N}^{\mathrm{II}}_{BB}$ are square and symmetric, they may be used for eliminating the unknowns $\mathbf{X}_{\mathrm{I}}$ and $\mathbf{X}_{\mathrm{II}}$ by means of (15.52), this leading in the end to:

$$\{\mathbf{N}^{\mathrm{I}}_{CC} - (\mathbf{N}^{\mathrm{I}}_{BC})^T(\mathbf{N}^{\mathrm{I}}_{BB})^{-1}\,\mathbf{N}^{\mathrm{I}}_{BC}\}\mathbf{X}_{\mathrm{I-II}} + \{\mathbf{N}^{\mathrm{II}}_{CC} - (\mathbf{N}^{\mathrm{II}}_{BC})^T(\mathbf{N}^{\mathrm{II}}_{BB})^{-1}\mathbf{N}^{\mathrm{II}}_{BC}\}\,\mathbf{X}_{\mathrm{I-II}} + \\ + \{\mathbf{N}^{\mathrm{I}}_{Cl} - (\mathbf{N}^{\mathrm{I}}_{BC})^T(\mathbf{N}^{\mathrm{I}}_{BB})^{-1}\mathbf{N}^{\mathrm{I}}_{Bl}\} + \{\mathbf{N}^{\mathrm{II}}_{Cl} - (\mathbf{N}^{\mathrm{II}}_{BC})^T(\mathbf{N}^{\mathrm{II}}_{BB})^{-1}\mathbf{N}^{\mathrm{II}}_{Bl}\} = \mathbf{0}. \quad (15.53)$$

The brackets { } may be worked out by separately tackling the two groups of correction equations in the system (15.50). Thus, the system of the normal equations corresponding *only to the first group* in (15.50) is:

$$\begin{aligned} &\mathbf{N}^{\mathrm{I}}_{BB}\mathbf{X}^{(\mathrm{I})}_{\mathrm{I}} + \mathbf{N}^{\mathrm{I}}_{BC}\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}} + \mathbf{N}^{\mathrm{I}}_{Bl} = \mathbf{0}\,; \\ &(\mathbf{N}^{\mathrm{I}}_{BC})^T\mathbf{X}^{(\mathrm{I})}_{\mathrm{I}} + \mathbf{N}^{\mathrm{I}}_{CC}\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}} + \mathbf{N}^{\mathrm{I}}_{Cl} = \mathbf{0}, \end{aligned} \quad (15.54)$$

where $\mathbf{X}^{(\mathrm{I})}_{\mathrm{I}}$ and $\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}}$ denote the values of the unknowns $\mathbf{X}_{\mathrm{I}}$ and $\mathbf{X}_{\mathrm{I-II}}$ respectively, obtained from this partial, incomplete solution.

If one eliminates $\mathbf{X}^{(\mathrm{I})}_{\mathrm{I}}$ from the system (15.54), one gets:

$$\{\mathbf{N}^{\mathrm{I}}_{CC} - (\mathbf{N}^{\mathrm{I}}_{BC})^T(\mathbf{N}^{\mathrm{I}}_{BB})^{-1}\mathbf{N}^{\mathrm{I}}_{BC}\}\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}} + \{\mathbf{N}^{\mathrm{I}}_{Cl} - (\mathbf{N}^{\mathrm{I}}_{BC})^T\,(\mathbf{N}^{\mathrm{I}}_{BB})^{-1}\mathbf{N}^{\mathrm{I}}_{Bl}\} = \mathbf{0}. \quad (15.55)$$

Remarks:

(1) The matrix of the coefficients of the unknowns $\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}}$ is identical with the corresponding matrix in the first term of equation (15.53). Similarly, the free term of (15.55) is identical with the first of the two parts of the free term of equation (15.53).

(2) Tackling also the IInd group of correction equations in the system (15.50) in the same way, one obtains results resembling (15.55).

The centralized results of the two partial solvings of the system (15.50) presented in Gauss-algorithmic form, will be *(Ghiţău* 1972*)*:

$$\begin{aligned} &[\mathbf{N}^{\mathrm{I}}_{CC}\cdot u_{\mathrm{I}}]\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}} + [\mathbf{N}^{\mathrm{I}}_{Cl}\,u_{\mathrm{I}}] = \mathbf{0}\,; \\ &[\mathbf{N}^{\mathrm{II}}_{CC}\cdot u_{\mathrm{II}}]\mathbf{X}^{(\mathrm{II})}_{\mathrm{I-II}} + [\mathbf{N}^{\mathrm{II}}_{Cl}\cdot u_{\mathrm{II}}] = \mathbf{0}, \end{aligned} \quad (15.56)$$

where u_{I} and u_{II} represent the number of the internal unknowns existing in the group I and II respectively.

Consequently, **the stages of the principle** of *Helmert-Pranis Pranievich*'s **procedure** can be stated as follows:

a) *Forming the systems of correction equations and then the systems of normal equations, separately for each of the groups corresponding to the sectors into which the triangulation network has been divided.* One must take care to ensure that the boundary unknowns occupy, grouped together, the last position in these systems.

b) *Reducing separately the partial systems of normal equations, corresponding to the two groups, until all the Gauss-algorithmic coefficients for the boundary unknowns as well as for the corresponding free terms in* (15.56) *are obtained.*

c) *Adding correspondingly the coefficients, obtained,* i.e. $[\mathbf{N}^{\mathrm{I}}_{CC} \cdot u_{\mathrm{I}}] + [\mathbf{N}^{\mathrm{II}}_{CC} \cdot u_{\mathrm{II}}]$ *and separately the free terms* $[\mathbf{N}^{\mathrm{I}}_{Ci} \cdot u_{\mathrm{I}}] + [\mathbf{N}^{\mathrm{II}}_{Ci} \cdot u_{\mathrm{II}}]$, *thus obtaining the relation* (15.53), *from which one then determines the vectors of the boundary-unknowns* $\mathbf{X}_{\mathrm{I-II}}$.

d) *Considering* $\mathbf{X}^{(\mathrm{I})}_{\mathrm{I-II}} \equiv \mathbf{X}^{(\mathrm{II})}_{\mathrm{I-II}} \equiv \mathbf{X}_{\mathrm{I-II}}$, *using an inverse calculation one obtains, with the aid of* (15.52), *the internal unknowns* $\mathbf{X}_{\mathrm{I}}$ *and* $\mathbf{X}_{\mathrm{II}}$ *respectively.*

Remarks:

(1) The triangulation network may be divided into more than two sectors, the solving methodology remaining the same as in the case of two groups.

(2) If the adjustment is carried out on angles, then, since an angle is either only in sector I or only in sector II, the angular quantities belong only to one sector. In the case of the adjustment on directions, the directions situated on the line separating the sectors belong at the same time both to sector I and to sector II and consequently the separation into groups presents more idiosyncracies, which are analysed in the technical literature (e.g. *Wolf 1968, Ghiţău 1972*) and which will not be further dwelt on here.

(3) Within the framework of international adjustments, when one pursues the integration of national networks into a unitary general network, an approach to carrying out the adjustment of the general network consists of the following. The general solution of the adjustment may be presented matricially in the form (Table 15.2):

$$\mathbf{X} = -\mathbf{N}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{l},$$

in which $\mathbf{N} = \mathbf{B}^T\mathbf{P}\mathbf{B}$ represents the matrix of the coefficients of the normal equations. Since the matricial product $-\mathbf{N}^{-1}\mathbf{B}^T\mathbf{P}$ does not depend on the measurement results, the solution $\mathbf{X}$ can be expressed as:

$$\mathbf{X} = \mathbf{C}\mathbf{l},$$

where $\mathbf{C} = -\mathbf{N}^{-1}\mathbf{B}^T\mathbf{P}$ is a standard matrix for a given general network or, explicitly:

$$\begin{Vmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{X}_R \end{Vmatrix} = \begin{Vmatrix} \mathbf{C}_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \mathbf{C}_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \mathbf{C}_R \end{Vmatrix} \cdot \begin{Vmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{l}_R \end{Vmatrix},$$

in which the indices $1, 2, \ldots, R$ represent the number of separate networks submitted to the general adjustment. After the computation of the matrix $\mathbf{C}$, every state assumes the corresponding submatrix $\mathbf{C}_i$ ($i = 1, 2, \ldots, R$), by means of which it continues the computation for determining the coordinates of the points in its own network. This computing methodology presents the advantage of providing the possibility of carrying out separately, on isolated networks, the adjusting operations, except for the computation of the standard matrix $\mathbf{C}$. Obviously, in applying this adjusting idea, one can no longer utilize the method of the conjugate gradients but instead the method of direct solving by inverting the matrix $\mathbf{N}$.

15.4 Weight Determination after Adjustment

Analysing the manner of establishing the weights before adjustment, *F. R. Helmert* was the first to sense the necessity of checking the initially accepted hypotheses after adjustment, particularly as regards the weights of the observation groups. In the case of the astro-geodetic-triangulation networks, one may consider as observation groups: azimuthal directions (d), distances (s) and azimuths (A). For these groups, it is necessary to determine lump wieght-coefficients P_d, P_s, P_A (by which one should correspondingly multiply the initial weights adopted for the observations included in these groups). The values of these coefficients must remain approximately the same, before and after adjustment respectively.

F. R. Helmert (1924) has worked out a direct and rigorous procedure for evaluating the ratio between the weights of the different observation groups which are included into a single lump adjustment. More recently, the same results have been obtained in other ways (*Kubik* 1967, *Samuil* 1971 etc.), as well as by other procedures (*Wolf* 1968 etc.).

The application of *Helmert*'s procedure, as well as of other direct procedures, presents, however, some difficulties, in particular when the number of the groups being considered is relatively large. It is therefore preferable (*Kubik* 1967, *Wolf* 1968, *Ghițău* 1972 etc.) that the ratios P_i $(i = d, s, A, \ldots)$ be determined by successive approximations. An approximate variant of *Helmert*'s procedure is obtained (*Wolf* 1968) by using the 1st-order terms from an expansion in which the number n of the observation corrections is considered as very large. For the case of astro-geodetic triangulation the relation for evaluating the weights is (*Ghițău* 1972):

$$P_d : P_s : P_A = \frac{n_d}{[pvv]_d} : \frac{n_s}{[pvv]_s} : \frac{n_A}{[pvv]_A}, \tag{15.57}$$

where n_d, n_s and n_A denote the numbers of azimuthal observations, distances and azimuths respectively, which were measured.

The successive-approximation procedure involves the following calculation stages:

1) *After each adjustment one calculates the values of the ratios* $P_d : P_s : P_A$ *by means of formula* (15.57); *for the group having the smallest ratio* $n/[pvv]$ *one chooses* $P = 1$.

2) *One multiplies the initial weights in the separate groups by the coefficient* P *corresponding to the appropriate group.*

3) *With the new weights one carries out a new adjustment.*

The computations are resumed in the above-described cycle until each of the coefficients P has the same value before and after adjustment (maximum departures of 5—10%).

In general, after 3—4 iterations one obtains adequate stability of the results.

Remarks:

(1) In the group adjustments of large triangulation networks or in the rigorous integration of already adjusted networks/zones, there occur common observations carried out in both networks/zones. For each of the observations contained in the separate adjustments of these networks one adopts (*Eggert 1936*, *Wolf 1950* etc.) a certain weight. For instance, in the case of two networks, the relation:

$$p_{ij}^{(1)} + p_{ij}^{(2)} = p_{ij}.$$

must be obeyed.

(2) For more complex triangulation networks, in which one has carried out azimuthal measurements (d), distance (s) and azimuth (A) measurements, the condition $[pvv]$ = minimum may be written in the form:

$$[pvv]_d + [pvv]_s + [pvv]_A = \text{minimum};$$

or:

$$[\tilde{v}\tilde{v}]_d + [\tilde{v}\tilde{v}]_s + [\tilde{v}\tilde{v}]_A = \text{minimum},$$

where $\tilde{v} = v/m$ and m — the standard deviation of the measurement of the corresponding kind of observations. *The corrections* $\tilde{v}$, also designated (*Ghiţău 1972*) as *standardized corrections*, possess a series of features which in fact constitute advantages for carrying out a more complex adjustment, viz.:

— *they are dimensionless*, which leads to simplification of the subsequent computations, by removing the need to utilize transformation coefficients in certain computing relations in which elements of various dimensions (meters, seconds etc.) intervene.

— *all of them have the same weight*, which represents a simplification of the adjustment computations; this advantage manifests itself especially in the group adjustment of national or zonal triangulation networks of different accuracies.

15.5 Processing of the Observations in Free Geodetic Networks

Under the name of free network or constraintless network or connexionless network one understands a geodetic network in which there do not exist accepted elements or elements (coordinates, lengths, orientations etc.) considered as fixed.

In the observation processing within the framework of the free networks by means of the method of indirect observations, the matrix of the coefficients of the normal system is singular.

In order to remove the singularity, it is necessary to impose certain supplementary conditions.

One approach consists of imposing the condition *(Mittermayer 1972):*

$$\mathbf{X}^T\mathbf{X} = \text{minimum}, \tag{15.58}$$

which leads to:

$$trace\ \mathbf{Q} = \text{minimum}. \tag{15.59}$$

It should be noted that this result may be interpreted as a condition for obtaining a maximum determination-accuracy of the network points.

The covariance matrix with the minimum trace may be found by utilizing the generalized inverse matrix.

Starting from the initial system of the observation equations *(Bjerhammar 1973):*

$$\mathbf{BX} + \mathbf{l} = \mathbf{V}$$

the condition:

$$\mathbf{V}^T\mathbf{PV} = \text{min}$$

leads to the system of the normal equations:

$$\mathbf{B}^T\mathbf{PBX} + \mathbf{B}^T\mathbf{Pl} = 0, \tag{15.60}$$

or, in a shorter form:

$$\mathbf{NX} = \mathbf{L}, \tag{15.61}$$

but with:

$$\det \|\mathbf{N}\| = 0. \tag{15.62}$$

Here the notations are:

$$\begin{aligned} \mathbf{N} &= \mathbf{B}^T\mathbf{P}\mathbf{B}, \\ \mathbf{L} &= -\mathbf{B}^T\mathbf{P}. \end{aligned} \tag{15.63}$$

Using the generalized inverse matrix *(Mittermayer* 1972*)* yields the solution:

$$\mathbf{X} = \mathbf{N}(\mathbf{N}\mathbf{N})^{-1}\ \mathbf{L}, \tag{15.64}$$

which, considered as a function of the observations **L**, leads to the covariance matrix:

$$\mathbf{Q} = \mathbf{N}(\mathbf{N}\mathbf{N})^{-1}\ \mathbf{N}\ (\mathbf{N}\mathbf{N})^{-1}\ \mathbf{N}. \tag{15.65}$$

The solutions may also be obtained with:

$$\mathbf{X} = \mathbf{Q}\mathbf{L}. \tag{15.66}$$

Another method *(Wolf* 1972 *b)* consists of writing the vector of the unkn owns in two distinct parts:

$$\mathbf{X} = \begin{vmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{vmatrix} \tag{15.67}$$

and imposing the condition (15.58) in the form:

$$\mathbf{X}_1^T\mathbf{X}_1 = \text{minimum}. \tag{15.68}$$

The $\mathbf{X}_1$ vector contains the unknowns which usually occur in the observation processing in the geodetic networks for which one takes a number strictly necessary in order to avoid singularity of the determinant of the normal-equation system. The $\mathbf{X}_2$ vector contains the other unknowns which are used in processing the observations in the free geodetic networks. Thus one notes that in processing the levelling observations the $\mathbf{X}_2$ vector will contain one unknown, in processing the trilateration or triangulation networks, 3 and 4 unknowns respectively will appear, while when adjusting three-dimensional geodetic networks, the $\mathbf{X}_2$ vector contains 7 unknowns.

The system of the observation equations may then be written as:

$$\mathbf{V} = \mathbf{A}\mathbf{X}_1 + \mathbf{B}\mathbf{X}_2 + \mathbf{l}. \tag{15.69}$$

Using the notation:

$$\mathbf{B}\mathbf{X}_2 + \mathbf{l} = -\mathbf{t}, \tag{15.70}$$

the above system becomes:

$$\mathbf{V} = \mathbf{A}\mathbf{X}_1 - \mathbf{t}, \tag{15.71}$$

which, under the condition:

$$\mathbf{V}^T\mathbf{P}\mathbf{V} = \text{min.},$$

leads to:

$$\mathbf{A}^T\mathbf{PAX}_1 = \mathbf{A}^T\mathbf{Pt},$$

whence:

$$\mathbf{X}_1 = (\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T\mathbf{Pt}. \tag{15.72}$$

Replacing $\mathbf{t}$ with (15.70) yields:

$$\mathbf{X}_1 = (\mathbf{A}^T\mathbf{PA})^{-1}\,\mathbf{A}^T\mathbf{PBX}_2 + (\mathbf{A}^T\mathbf{PA})^{-1}\,\mathbf{A}^T\mathbf{Pl} \tag{15.73}$$

to which may also be added:

$$\mathbf{X}_2 = \mathbf{EX}_2, \tag{15.74}$$

in which $\mathbf{E}$ is the unit matrix.

The system (15.73) and (15.74) regarded as *a system of observation equations* leads to the system of normal equations:

$$(\mathbf{B}^T\mathbf{PA}(\mathbf{A}^T\mathbf{PA})^{-1}\,(\mathbf{A}^T\mathbf{PA})^{-1}\,\mathbf{A}^T\mathbf{PB} + \mathbf{E})\mathbf{X}_2 + \mathbf{B}^T\mathbf{PA}(\mathbf{A}^T\mathbf{PA})^{-1}\,(\mathbf{A}^T\mathbf{PA})^{-1}\,\mathbf{A}^T\mathbf{Pl} = 0, \tag{15.75}$$

from which one calculates $\mathbf{X}_2$ and then from (15.73) there results $\mathbf{X}_1$.

H. Wolf has demonstrated that the two solving methods presented above lead to identical results.

Another way of removing the singularity of the normal-system matrix consists *(Rinner* 1969*)* of imposing conditions which should express the fact that the solution to be obtained must not lead to translations, rotations or scale modifications.

Denoting by $\mathbf{T}$ the linear-transformation matrix, the above conditions may be written in the form:

$$\mathbf{TX} = 0 \tag{15.76}$$

which, added to the initial correction equations, leads to the situation known in the theory of observation processing under the name of *indirect observations with condition equations between unknowns*. Although rigorous, this method of solving is laborious. Therefore, the equations (15.76) can be tackled as representing equations of pseudo-observations, to which one assigns a very large weight.

In the practical application of the above-mentioned methods it is necessary to take into consideration a series of specific aspects. Thus, unlike the case of the constraint-network processing, the most probable values of the unknowns are not independent of their provisional values. This fact follows from the condition:

$$\mathbf{X}^T\mathbf{X} = \text{minimum},$$

which is equivalent to an optimum integration of the network within the scope of the provisional values of the unknowns.

The stochastic model adopted within the framework of *Wolf*'s procedure expresses more exactly the different nature of the unknowns occuring in the adjustment process, providing the possibility of a dependent manipulation consistent with the actual situation being examined. In several contributions, *H. Wolf* draws attention to some possibilities of making mistakes when all the **X** unknowns are automatically contained in the condition of minimum (15.58).

Due to the fact that the three above-mentioned methods lead to identical results, one can appreciate that, from the practical point of view, important advantages are presented by the pseudo-observation procedure, the application of which needs only an adaptation of the programme already drafted for the processing of the indirect observations in constrained networks. However, in the case of this procedure, the corresponding choice of the weights of the fictitious-observation equations is important. Experience has shown that the order of magnitude of the coefficients in the fictitious-observation equations must be about 100 times greater than that of the coefficients in the other equations.

16

The 1st-Order Astro-Geodetic Triangulation Network of the Socialist Republic of Romania

The idea of how to carry out the 1st-order astro-geodetic-triangulation network in *Romania* has varied from time to time. Up to the year 1956, the prevailling idea was that of building up the 1st-order triangulation in the form of single or double triangles-chains, disposed along meridians and parallels, forming certain polygons inside which there were developed networks equivalent, according to the side length and to measurement weight, sometimes to 1st-order networks and in other cases to 2nd-order complementary networks. Even in the case of the primordial chains, the weight of the angular measurements varied between 18 and 48, while the measurement conditions and the instruments being used did not comply with certain scientific necessities imposed by the requirements of the order worked for. For this reason, before 1956, the 1st-order triangulation was not fulfilling the exigencies imposed by the construction of modern triangulation.

The period from 1956 up to now may be regarded as a second stage, which, although a short period, has witnessed the execution of *Romania*'s modern 1st-order triangulation. Thus, the 1st-order triangulation was achieved in the form of a compact network (Fig. 16.1) of geometrical figures — in most cases triangles but also quadrilaterals with both diagonals reciprocally observed — and was adjusted in a single group by means of the electronic computer.

In what follows, we shall briefly present the layout conception, the manner of constructing and marking the points in the field, the measuring methods utilized and the measurement quality, as well as a few more important results obtained after adjustment, all of them referring to the network built since 1956.

16.1 The Network Layout

From the geomorphological point of view, the territory of the *Socialist Republic of Romania* presents a harmonious synthesis of three great terrain categories (30% mountains, 37% hills and plateaus and 33% plain and marshy areas). The details of this varied relief, as well as the available equipment have determined some peculiarities as regards the layout, the field construction and the choice of methods for measuring the network elements

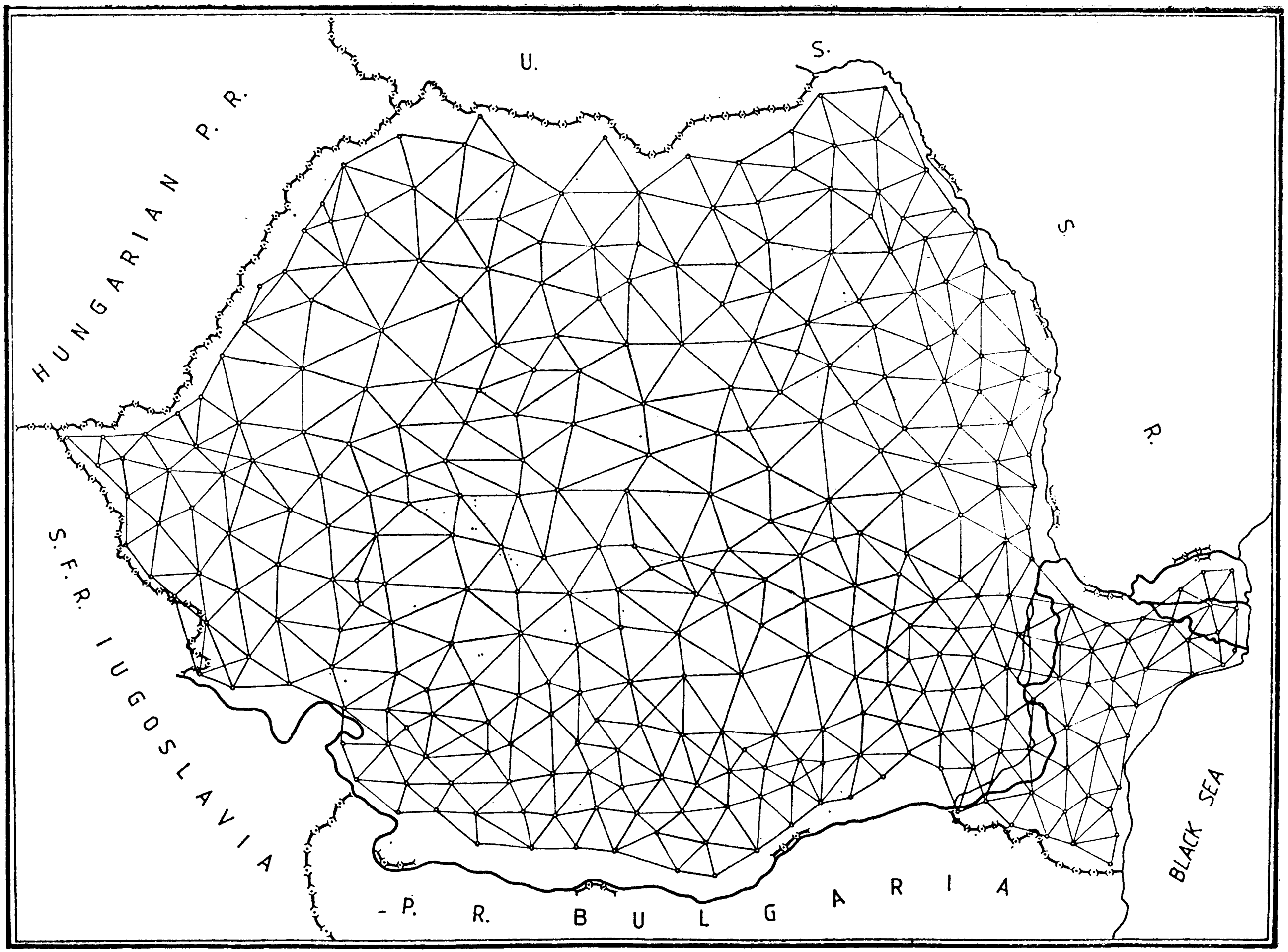

Fig. 16.1. The 1st-order Astro-geodetic-triangulation Network of the Socialist Republic of Romania

(sides, angles etc.). Thus, in the plain areas the side lengths range between 17 and 25 km, in the zone of the sub-Carpathian hills between 20 and 35 km, and in the mountainous regions and in the *Transylvanian Plateau* between 25 and 61.5 km, which has determined for the entire network an average side length of 30.2 km. Under these conditions were laid out 374 1st-order points forming 657 triangles and 6 quadrilaterals with a reciprocal visibility of both diagonals.

In the entire network there exist 8% triangles having one angle under 35^g and 0.5% angles smaller than 25^g and, in view of the fact that in most cases these angles belong to the 6 quadrilaterals with both diagonals reciprocally observed, one can state that in the triangles of the 1st-order network there do not exist angles less than 25^g. Thence it follows that, from the point of view of the geometrical construction, *Romania*'s 1st-order triangulation network meets the requirements of a modern triangulation.

The layout of the 1st-order triangulation has suggested the possibility of its block adjustment, as well as the connexion with the 1st-order networks of the neighbouring countries, with a view to carrying out more extensive scientific and practical studies.

Inasmuch as the sides obtained from the networks developing previously existing bases did not secure the necessary accuracy for the starting sides — 1/400,000 — initially there were designed side measurements, to be carried out with the aid of electro-optical instruments, intended to ensure throughout the network one measured side to at most 10—12 triangles. Subsequently, side measurements using the NASM-2A geodimeter were designed and are being performed which should secure a density per network of one measured side to 5 triangles. Determinations of geodetic astronomy (Φ, Λ, α) were designed to be carried out at the terminals of these sides.

With a view to calculating the corrections due to the deflection of the vertical, 125 astronomical points (= station points within the network) were initially designed at which one should carry out Φ and Λ determinations; subsequently, 103 other such astronomical points were also designed. Equally, 45 points at which the astronomical azimuth α is to be determined have also been designed.

16.2 Marking and Constructing the Station Points of the Network

The 1st-order triangulation network we are referring to was built up between the years 1956 and 1962. The fixing of the station points in the field was made with the aid of reinforced-concrete pillars and subterranean centres, of dimensions determined in accordance with the relief and the soil existing in different areas (mountain, hill, plain, marshy zone; *Oprescu* et al. 1973). In order to measure the azimuthal angles, one geodetic wooden sign was built at each point, above the station point's centre at a height varying from one point to another, depending on the need for reciprocal visibility between them. Generally, the structures have achieved the raising of the sights from the ground by at least 6 m, reducing thereby the influence of the external factors and particularly the effect of lateral refraction on the angular-measurement results. The material utilized for the structures was of a soft kind, in general fir wood, having the smallest torsion coefficient compared with other building materials. Thus, along with some special conditions imposed when actually performing the measurements constant (but as short as possible), time interval between measuring the two directions of any angle; choosing roughly the same time of day for carrying out the measurements and others), one has endeavoured to eliminate as far as as possible, errors due to the pillar torsion from the angular-measurement results.

16.3 Angle and Distance Measurements as well as Astro-Geodetic Determinations Carried out at the Station Points of the Network

16.3.1 Azimuthal Angle Measurements

The azimuthal angles have been measured by *Schreiber*'s method, making use, as a rule, of *Wild T*3 theodolites at all the station points of the 1st-order network, except for a total of 39 points for which older observations were taken (carried out in the period 1927—1956 by means of the horizon-round method or of the reference-torque method). From the total of 374 1st-order points, at 308 points the measurements were carried out with the weight 36—48 and at the other 66 points with the weight ranging between 24 and 26. The standard deviation of a measured azimuthal angle, calculated with *Ferrero*'s formula, was established as having the value of $\pm 0''.72$, which complies with the value of $\pm 0''.7$ imposed by the instructions in force at present.

The accuracy of the angular measurements having been carried out is characterized by the quantities calculated in Table 16.1 (where $N = 669$ represents the total number of the network misclosures w_F).

Table 16.1. *Calculated Quantities Characterising the Angular-measurement Accuracy*

Number of triangles with misclosures		Sum of triangle misclosures cc		Average error	Standard deviation of a measured angle	Average value of errors
Positive N_1	negative N_2	positive $[+w_F]$	negative $[-w_F]$	$\theta'' = \frac{[w_F]}{N}$	$m'' = \sqrt{\frac{[w_F w_F]}{N}}$	$w''_{av} = \frac{[w_F]}{N}$
342	327	1217	1156	± 1.00	± 1.25	-0.03

Analysing Table 16.1 leads to the following conclusions:

(1) The differences $|N_1 - N_2|$ and $|[+w_F]-[-w_F]|$ are also very small (2.24% and 2.57% respectively), which shows that the *distributive principle* does hold.

(2) The fact that the average value of errors tends to zero shows that the *probability principle* of random errors is also obeyed.

Remark. As regards the application of the other two characteristic principles of random errors, known from the classical theory of errors (*Wolf 1968*, *Grossmann 1969*), viz. the *limitation principle* and the *causality principle*, it follows that they themselves are also satisfied. Thus, none of the values of the triangle misclosures (w_F) exceeded the limit value $w_{max} = 3\ m \approx \pm 4''$ (limitation principle), while for the entire network the number of the values $|w_F| > 2.5''$ was 38 as compared with 631 values $|w_F| < 2.5''$, consequently small errors occur more frequently than large ones (causality principle).

In consequence of the analysis carried out according to the classical theory of measurement errors, it followed that in the triangulation network under consideration random and not systematic errors prevail, and the random errors obey the error-distribution laws. This will just be illustrated here by applying *Cornu*'s criterion (*Wolf* 1968):

$$|A| = \left|\frac{\theta}{m}\sqrt{\frac{\pi}{2}}\right| - 1 = 0.001\,; \quad |B| = \frac{\sqrt{2N}-2}{\sqrt{2N}-1}\sqrt{\frac{\pi}{2}-1} = 0.029$$

and consequently $|A|/|B| < 1$. As the ratio is less than 1, this means that the random errors prevail throughout the network.

As regards the analysis of the field angular observations before adjusting the network, this has been carried out in the period 1974—1975, making use of the *test scheme* (τ test plus u test; § 14.1.6 d) for all of the station points of the network measured by means of *Schreiber*'s method. This analysis has allowed a framing of the qualitative value of the corresponding angular observations carried out and establishing a plan for remaking some of the observations.

16.3.2 Distance Measurements

On the whole network were initially measured 8 sides with the *SVV*-1 electro-optical telemeter and subsequently 65 sides by means of the *NASM*-2*A* and *Range master II* geodimeters. As regards the *SVV*-1 measurements, the side length ranged between 17.3 and 24.7 km and for the *NASM*-2*A* and *Range master II* measurements between 15.0 and 40.7 km. In cases when the distance could not be directly measured as a whole, it was cut into two parts (with a dividing point), while maintaining the corresponding alignment at less than 1 km and the breaking angles were measured with a standard deviation of $\pm 0''.3$.

The corresponding distance measurements were characterized by a relative average error of the side length lying between 1/450,000 and 1/1,000,000 for those carried out with the *SVV*-1 telemeter and between 1/400,000 and 1/7,200,000 for those carried out with the *NASM*-2*A* geodimeter.

Remark. Since the accuracy in determining the length of a few starting sides obtained from networks developing previously existing geodetic bases (1/71,000 . . . 1/308,000) was not the accuracy imposed by the instructions in force — 1/400,000, these have not been used in adjusting the network.

16.3.3 Determinations of Geodetic Astronomy

Following the layout, **the geodetic-astronomy determinations were carried out as follows:**

(1) *Laplace determinations were made at 77 station points of the network, utilizing transit instruments of the Bamberg type and universal instruments of the A.U.2, Wild T3 and DKM-3A types.*

(2) *Latitude and longitude determinations were carried out at 257 station points of the network.*

The determination methods and the corresponding standard deviation obtained for each method are shown in Table 16.2.

Table 16.2. *The Determining Method and the Accuracy for* Φ, Λ *and* α

Astronomical coordinate	Determination method used	Determination standard-deviation
Latitude Φ	Talcott's method	$\pm 0''.21$
Longitude Λ	Zinger's method	$\pm 0^s.012$ ... $\pm 0^s.025$
Azimuth α	Azimuth determination from the hour angle of polaris	$\pm 0''.16$ ... $\pm 0''.8$

After calculatin: he azimuth, directly and inversely, an azimuth standard-deviation, calculated as difference between two measurements, equalling $\pm 0''.60$ was obtained at the ends of the measured sides.

The free terms of all of the *Laplace* azimuth-equations did not exceed the tolerance; on the contrary, in most of cases they were twice as small as the latter.

As a general conclusion concerning the geodetic-astronomy determination, one can state that these have covered the entire triangulation network fairly uniformly and that their density was satisfactory.

16.4 The Network Adjustment

The astro-geodetic triangulation network of the *Socialist Republic of Romania* was block-adjusted in the year 1962, by means of the method of coordinate variation on angles, for which a computing programme was worked out for the electronic computer.

Taking the average accuracy in measuring distances and *Laplace* azimuths as equalling 1/500,000 and the accuracy of the angular observations as equal to 1/100,000 (Section 14.1), the correction equations for distances and azimuths, to which one has assigned the weight $P = 5$, have been formed. In consequence of the adjustment, the initial sides and *Laplace* azimuths have undergone small corrections corresponding to the measurement accuracy and consequently the triangulation network as a whole has suffered smaller corrections on angles. Thanks to this fact, the standard deviation after adjustment has increased less than in the case when one considers as fixed the measured distances and *Laplace* azimuths. Considering the latter as fixed, all the errors, which would have occurred when measuring them, would have affected the measured azimuthal angles and thus the corrections of these angles would not have followed a normal distribution with respect to the weight of the measurements carried out. As will subsequently be seen, the results of the adjustment have overwhelmingly confirmed the correctness of the applied adjusting technique. In what follows we will present a few more important results which were obtained following the adjustment.

In order to be able to draw certain conclusions concerning the quality of the triangulation after adjustment, the standard deviation of an azimuthal angle after adjustment by means of the formula:

$$m^* = \pm \sqrt{\frac{[v^2]}{n}}$$

has been computed, in which v is the angle correction after adjustment and n — the number of corrections in the network, and the value $m = \pm 0''.90$ has been obtained. With the help of this value, Table 16.3 has then been compiled, in which is shown the correction distribution existing after adjustment with respect to the theoretical distribution (=*Gauss*' normal distribution).

Analysing Table *16.3* yields the following conclusions:

(1) *The actual correction distribution, on intervals at every 0''.5, closely corresponds to Gauss' theoretical distribution.*

(2) *The correction number existing within the interval 0'' ... 0''.5 is greater, to the detriment of the corrections within the interval 1''.0 ... 2''.0, the rest of the distribution being maintained in accordance with the theoretical distribution.*

(3) *Within the interval from 2''.5 to 4''.0 one theoretically predicts 12 corrections rather than the 24 which actually exist.* The difference of 12 corrections, having rather great values, leads to a considerable increase in the value of m^*. The conclusion has thence been drawn that into the frame of the network have, nevertheless, crept a few significant systematic errors, which have to be removed in the future, by corresponding remeasurements.

(4) *On the whole, random errors, unavoidable in measurements* prevail *in Romania's triangulation network.*

In order to check the correctness of the weighting of the side and *Laplace* azimuth supplementary conditions, as well as of regarding these elements as not being fixed within the adjustment framework, the following procedure has been adopted.

Table 16.3. (*Actual and Theoretical*) *Distribution of the Azimuthal-angle Corrections*

Interval No.	Interval	Number of corrections					
		Positive branch of Gauss' curve			Negative branch of Gauss' curve		
		Actual $n_E^{(-)}$	Theoretical $n_T^{(-)}$	$n_E^{(-)}-n_T^{(-)}$	Actual $n_E^{(+)}$	Theoretical $n_T^{(+)}$	$n_E^{(+)}-n_T^{(+)}$
1	0" –0".50	450	428	+22	454	428	+26
2	0.51–1.00	303	310	– 7	314	310	+ 4
3	1.01–1.50	160	173	–13	146	173	–27
4	1.51–2.00	50	69	–19	69	69	0
5	2.01–2.50	21	21	0	24	21	+ 3
6	2.51–3.00	11	5	+ 6	4	5	– 1
7	3.01–3.50	2	1	+ 1	3	1	+ 2
8	3.51–4.00	1	0	+ 1	0	0	0
9	>4.00	1	0	+ 1	1	0	+ 1
	TOTAL	999	1007	– 8	1015	1007	+ 8

First of all, the standard deviation of the weighting unit has been computed using the formula:

$$m_0 = \pm \sqrt{\frac{[Pvv]}{q-k}},$$

in which $[Pvv]$ represents the sum of the squares of the corrections of the measured azimuthal angles, initial sides and *Laplace* azimuths, multiplied by the corresponding weights assigned to these elements; q — the number of all measurements; k — the number of necessary measurements (which equals twice the number of adjusted station points); the value $m_0 = \pm 1''.06$ has been obtained.

Remark. The standard deviation has increased compared with the standard deviation calculated according to *Ferrero*'s formula with the value $1''.06 - 0''.72 = 0''.34$. This increase is a normal one, which shows that the initial data submitted for adjustment were satisfactory from the point of view of accuracy.

Now calculating the standard deviation of a measured angle with the aid of the azimuthal angle corrections alone, by means of the formula:

$$m_1 = \pm \sqrt{\frac{[v^2]}{q-k}},$$

yielded the value $m_1 = \pm 1''.12$. One remarks that the difference $m_1 - m_0$ is rather small (0''.06), which shows that the initial data, starting sides and *Laplace* azimuths have been correctly taken from the point of view of accuracy and of establishing the weights within the framework of the network adjustment.

In a similar way, calculating the standard deviation of a *Laplace* azimuth after adjustment by means of the relation:

$$m_1' = \pm \frac{m_0}{\sqrt{P}},$$

has led to $m_1' = 0''.47$ and for the standard deviation relative to the starting sides, calculated with the formula:

$$\frac{m_s}{s} = \pm \frac{m_0''}{\sqrt{P}} \frac{1}{s''},$$

one has obtained $m_s/s = 1/439{,}000$. These results confirm the validity of applying the weights within the framework of the network adjustment to both *Laplace* azimuths and starting sides. Table 16.4 shows, in a unified presentation, the results obtained in consequence of the network adjustment, results which confirm the correctness of the applied adjusting notion.

Table 16.4. *Results Obtained in Consequence of the Network Adjustment*

No	Measured quantities	$\lvert w_{max} \rvert$ on network	$\lvert v_{max} \rvert$ on network	Calculated standard deviations	
				Before adjustment	After adjustment
1	Horizontal angles	3''.91	4''.05	$\pm 1''.25$	$\pm 1''.06$
2	Laplace azimuths	—	—	$\pm 0''.60$	$\pm 0''.47$
3	Starting sides	—	—	1: 500 000	1: 439 000

Concluding, one can state, on the basis of the results obtained by analysing *Romania*'s 1st-order triangulation network, both before and after carrying out the adjustment, that, under good conditions, this network fulfils the requirements of a modern network.

For the future, taking into consideration the large number of geodetic-astronomy determinations and of side measurements, as well as of angular remeasurements within the framework of the 1st-order triangulation network of the *Socialist Rupublic of Romania*, it is intended to perform a new adjustment of this network on the ellipsoid, along directions, making use of *Helmert-Pranis Pranievich*'s procedure for indirect observations.

Similarly, as another problem for the future, on the basis of recent studies (*Dragomir* 1975), it is envisaged to proceed to the qualitative improvement of the present astro-geodetic network of the country, utilizing the derived-network method. To this end, **the following, among other things, will be taken into consideration**:

(1) *The angle measurements at the geodetic points situated at about 8 km from one another will be carried out under special conditions*, viz.: with theodolites of the *DKM-3* type, by means of the complete-series method, with a series number equal to 18, the observations carried out by day shall utilize the checking telescope as well, the sighting shall only be carried out on the system mounted on the geodetic-signal pillar, the determination of the centring and reducing elements shall be made at the beginning and at the end of the measuring process (both at the station point and at the points taken sight at), the angular interval between two measurement origins shall be of $11^{g}11^{c}$, the detailled description of the conditions under which the measurements were carried out etc.

(2) *In order to estimate the accuracy of the adjusted elements, e.g. for the azimuthal angles, the standard deviation M_0 of the weight unit shall be calculated by means of the relation (Wolf* 1968, *Dragomir* 1975*):*

$$M_0 = \pm \sqrt{\frac{[(N-u)m_0^2]_1^r}{[(N-u)]_1^r}},$$

in which $N - u$ represents the difference between the total number of measurements and the number of measurements needed for the corresponding stage in the $r(r = 1, 2, \ldots)$ preliminary adjustments and m_0 — the standard deviation of the weight unit characteristic for the corresponding preliminary adjustment.

Fourth part

Three-Dimensional Geodesy

As always happens in a science which has developed a great deal and in which the measurements have entered the province of pure technology, in Geodesy too since World War II the need to overhaul its theoretical notions has become more and more pressing since the profusion and the accuracy of the empirical material has enabled one to consider ever higher-order approximations.

In the year 1956, on the occasion of a symposium organized at Munich, the British geodesist *M. Hotine* [1] put forward a new view of geodesy, which was soon to become *Three-Dimensional Geodesy*. Thus, there was suggested the idea of reconsidering the geodesy of the conventional framework — the space of two dimensions of the reference-ellipso-surface (the geodetic coordinates) to which is added a third completely independent one — the altitude — within the framework of a system with three dimensions, defined by a general (tri)rectangular coordinate trihedron and by a number of auxiliary local trihedra, connected with the general one. In other words, the separate treatment of the problem of determining the geodetic coordinates and the problem of determining the altitudes, as in conventional geodesy, should be replaced by a *common* or *combined* treatment of both problems which, although considering differently defined reference surfaces, should be regarded as a unique central measurement-theme, at the level of the territorial extent of a state and, simultaneously, at the level of measuring the terrestrial Globe as a whole.

One must, however, emphasize the fact that the basic ideas of Three-Dimensional Geodesy had been worked out more than a hundred years previously by *H. Bruns* (1878).

Remark. The origins of the use of coordinates in three-dimensional space go back to *R. Descartes* (1637), *A. Parent* (1700) and *A. C. Clairaut* (1731), the foundations being laid by *L. Euler* (1748) (*Wolf* 1977).

Along with the contributions of *M. Hotine* (*1956, 1959* and others), Three-Dimensional Geodesy has gained a new foundation and in particular a modern development, depending on the new measuring technologies, especially those carried out in extra-terrestrial space, as well as on the new computing possibilities (*Wolf 1963 b, c 1969, Levallois 1963, Grafarend 1975* and others).

Rid, in principle, of the problem of the atmospheric refraction, which, however, unfortunately does actually exist, Geodesy thus solves its fundamental problems much more elegantly and smoothly achieving a much sounder connexion between its components (*Heitz 1975*), giving up numerous auxiliary hypotheses and approximations to which classical Geodesy, and consequently Ellipsoidal Geodesy as well, had to resort in order to solve the specific problems. These modern aspects of Geodesy will be presented in the sequel.

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[1] It seems that about the year 1948 the Soviet scientist *M.S. Molodenski* expressed similar ideas, which, however, were only circulated in the *U.S.S.R.*

17

Basic Equations of Three-Dimensional Geodesy[1]

17.1 Coordinate Systems Used in Three-Dimensional Geodesy

17.1.1 Local Coordinate Systems

For every geodetic point of astro-geodetic network one must define, first of all, a special local coordinate-system in which one should be able to express conveniently the measured elements; a system representing, from the geometrical point of view, the local trihedron.

a) **The local Cartesian-astronomical system** $\bar{x}, \bar{y}, \bar{z}$. The axes of this system are the following (Fig. 17.1), (*Körner* 1968): the $\bar{x}$ axis represents the East direction, the $\bar{y}$ axis — the North direction and the $\bar{z}$ axis — the Zenith direction (i.e. the direction of the vertical axis of a theodolite installed and levelled by means of spirit-levels at the station point P_S).

The $\bar{x}, \bar{y}$ plane coincides, consequently, with the physical horizontal plane.

Remark. E. Grafarend (1975) designates the Cartesian astronomical local system the *natural local system.*

b) **The polar local system** D, α, β'. Within the framework of this system, one considers as polar coordinates of the point P_V (Fig. 17.1): the distance D_{SV}, the astronomical azimuth α_{SV} and the actual zenithal angle β'_{SV} [2]. According to Fig. 17.1, for the transition from the $D_{SV}, \alpha_{SV}, \beta'_{SV}$ system to the $\bar{x}, \bar{y}, \bar{z}$ system one makes use of the following relations:

$$\bar{x}_V = D_{SV} \sin \alpha_{SV} \sin \beta'_{SV}\,;\ \bar{y}_V = D_{SV} \cos \alpha_{SV} \sin \beta'_{SV}\,;\ \bar{z}_V = D_{SV} \cos \beta'_{SV}, \quad (17.1)$$

---

[1] We shall use the classification and designations of the systems as given by *H. Körner (1968)*. Some of these systems constitute an adaptation of the general coordinate-systems (Section 3.4) to Three-Dimensional Geodesy.

[2] It is expressed by the relation $\beta'_{SV} = \beta_{SV} + d\beta_{SV}$, in which $d\beta_{SV}$ represents the apparent zenithal angle, $d\beta_{SV} = \rho'' D_{SV} \dfrac{k}{2R}$ — a quantity describing the influence of refraction on angle, $k = R/r$ — the refraction coefficient, R — the Earth's radius and r is the radius of the circle arc formed by the light ray.

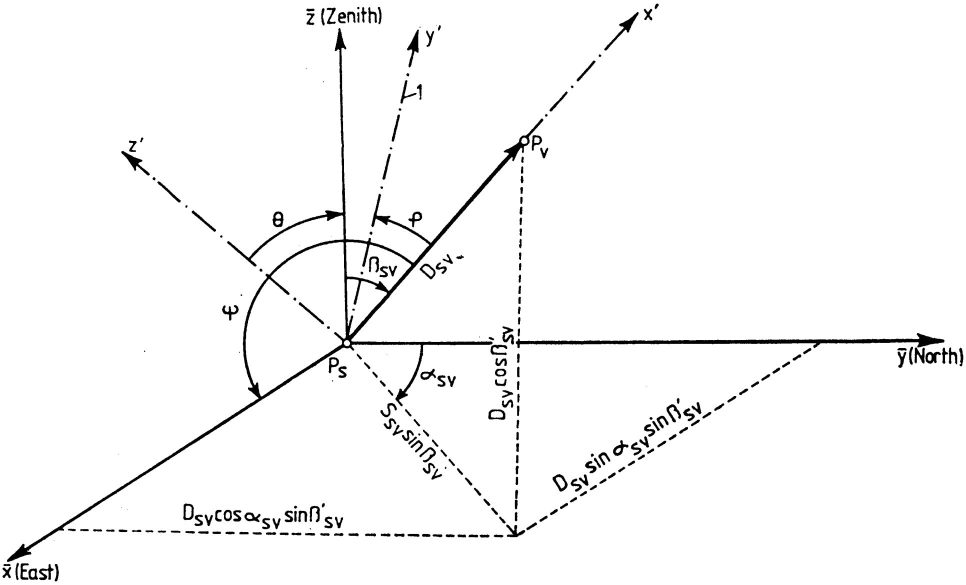

Fig. 17.1. The Local Cartesian-Astronomical and the Instrumental Cartesian Systems: 1 — the nodal line

which yield:

$$\alpha_{SV} = \tan^{-1}\frac{\bar{x}_V}{\bar{y}_V}\,;\; D_{TV} = \sqrt{\bar{x}_V^2 + \bar{y}_V^2 + \bar{z}_V^2}\,;\; \beta'_{VV} = \cos^{-1}\frac{\bar{z}_V}{D}\,. \tag{17.2}$$

c) **The instrumental Cartesian local system** x', y', z'. The instrumental Cartesian local system is defined by the following axes (Fig. 17.1): the x' axis is the sighting line P_SP_V, the y' axis is the rotation axis of the telescope, and z'axis is normal to the x'y' plane. Passing from this system to the $\bar{x}, \bar{y}, \bar{z}$ system made by means of a rotation according to the relation (*Körner* 1968):

$$\bar{\mathbf{x}} = \mathbf{R}\mathbf{x}', \tag{17.3}$$

in which:

$$\bar{\mathbf{x}}^T = [\bar{x}_V\bar{y}_V\bar{z}_V] \text{ and } \bar{\mathbf{x}}'^T = [x'_Vy'_Vz'_V].$$

The elements of the rotation matrix **R** are obtained by utilizing *Euler*'s angles θ, φ and ψ (Fig. 17.1), given by the relations [1]:

$$\theta = \frac{\pi}{2} - \beta'_{SV}\,;\quad \varphi = \frac{\pi}{2}\,;\quad \psi = \pi + \alpha_{SV}. \tag{17.4}$$

[1] Taking into account that the $x'y'$ and $\bar{x}\bar{y}$ planes intersect along the y' axis as the *nodal line* (Fig. 17.1).

This leads in the end to (*Lagally* and *Franz 1956*, *Körner* 1968)[1]:

$$\mathbf{R} = \begin{Vmatrix} \sin\alpha_{SV}\ \sin\beta'_{SV} & \cos\alpha_{SV} & \sin\alpha_{SV}\ \cos\beta'_{SV} \\ \cos\alpha_{SV}\ \sin\beta'_{SV} & \sin\alpha_{SV} & -\cos\alpha_{SV}\ \cos\beta'_{SV} \\ \cos\beta'_{SV} & 0 & \sin\beta'_{SV} \end{Vmatrix}. \tag{17.5}$$

In view of (17.5) and since $\mathbf{x}'^T = \|D_{SV}\ 0\ 0\|$, (17.3) becomes:

$$\bar{\mathbf{x}} = \begin{Vmatrix} D_{SV}\ \sin\alpha_{SV}\ \sin\beta'_{SV} \\ D_{SV}\ \cos\alpha_{SV}\ \sin\beta'_{SV} \\ D_{SV}\ \cos\beta'_{SV} \end{Vmatrix}, \tag{17.6}$$

which is in accordance with (17.1).

Remark. The equations (17.6) correspond to those given by *H. Dufour* (*1959*, p. *5* a. 138; p. 174 (4)), *M. Hotine* (*1959*, p. 6 (18.1)) and *H. Wolf* (*1963 c*, p. 225 (1)).

17.1.2 Global Coordinate Systems

By global coordinate systems one understands those systems by means of which one can conveniently describe the astro-geodetic networks extending over such a vast expanse as that of a group of states, of a continent or even of the entire terrestrial Globe.

In certain technical works these systems are called *general systems* of coordinates (*Dufour* 1962, *Levallois* and *Kovalevsky* 1971) or *spatial systems* of coordinates (*Wolf 1969*, *Ramsayer* 1973).

a) **The Cartesian global system.** Here one distinguishes two types of systems:

(a) *The equatorial Cartesian global system* X, Y, Z (Fig. 17.2; *Körner 1968*). The origin O of this system, designated by *Grafarend* (1975) as the *geocentric global system*, is taken in the neighbourhood of the Earth's mass centre and its axes are as follows: the X axis is parallel to the *Greenwich* local meridian, the Z axis — parallel to the World's axis and the Y axis — perpendicular to the XOZ plane[1]. Consequently, the OX axis will define with the OZ axis a plane parallel to the *Greenwich* meridian and the XOY plane — a plane parallel to the plane of the astronomical equator.

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[1] That is the Y axis is parallel to the plane of the local astronomical meridian of a point situated at 90° East longitude from *Greenwich*.

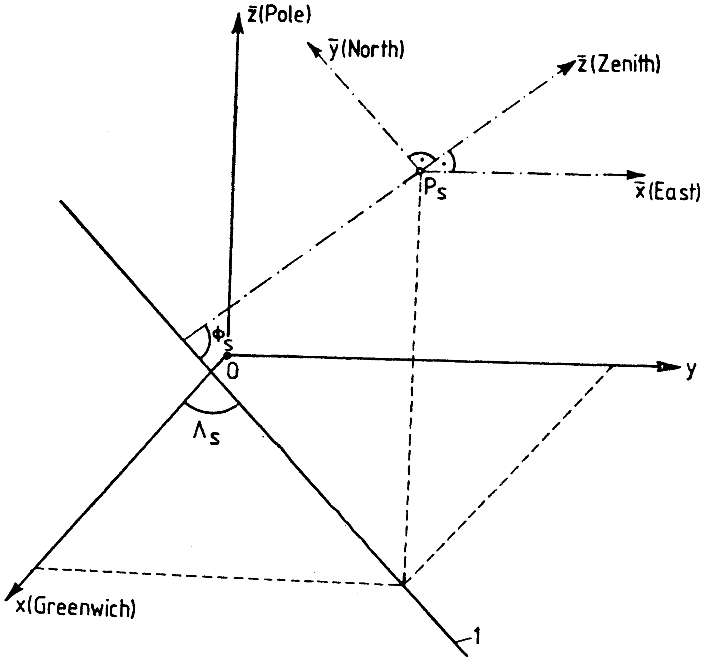


Fig. 17.2. The Equatorial Cartesian Global System: 1 — the trace of the astronomical meridian plane

If one ignores the translation of the point P_S to the origin point O (not of interest here), then the elements of the rotation matrix $\mathbf{R}_1$ for the transformation (of the global system into a local system):

$$\bar{\mathbf{x}} = \mathbf{R}_1\mathbf{X}, \tag{17.7}$$

are obtained by using *Euler*'s angles, which, since the nodal line coincides with the $\bar{x}$ axis (Fig. 17.3; *Körner 1968*), are given by:

$$\theta_1 = \frac{\pi}{2} - \Phi_S; \quad \varphi_1 = \frac{\pi}{2} + \Lambda_S; \; \psi_1 = 0. \tag{17.8}$$

Remark. For transforming the local system into the global system one makes use of the relation:

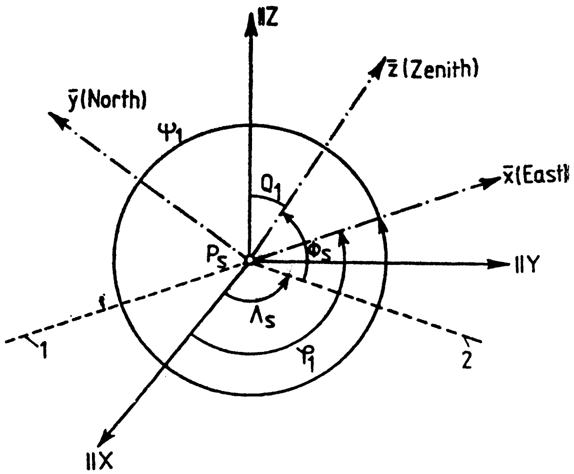
$$\mathbf{X} = \mathbf{R}_1^T\bar{\mathbf{x}}.$$

Using, (17.8), the matrix $\mathbf{R}_1$ is obtained in the form *(Körner 1968):*

$$\mathbf{R}_1 = \begin{Vmatrix} -\sin\Lambda_S & \cos\Lambda_S & 0 \\ -\sin\Phi_S\cos\Lambda_S & -\sin\Phi_S\sin\Lambda_S & \cos\Phi_S \\ \cos\Phi_S\cos\Lambda_S & \cos\Phi_S\sin\Lambda_S & \sin\Phi_S \end{Vmatrix}. \tag{17.9}$$

Remark. The matrix $\mathbf{R}_1$ corresponds to the matrix $\mathbf{R}^*$ given by *Levallois and Kovalevsky* (1971, p. 12 (12—3)) and is designated by *E. Grafarend* (1975) as a *Frobenius matrix of the second kind.*

Fig. 17.3. *Euler'* Angles in the Case of Coincidence of the Nodal Line with the x Axis:
1 — the nodal line; 2 — the meridian plane



In view of (17.9) and since $\mathbf{X}^T = \| X_V - X_S \quad Y_V - Y_S \quad Z_V - Z_S \|$, (17.7) becomes:

$$\bar{\mathrm{x}} = \begin{Vmatrix} \bar{x}_V \\ \bar{y}_V \\ \bar{z}_V \end{Vmatrix} = \begin{Vmatrix} -(X_V - X_S)\sin\Lambda_S + (Y_V - Y_S)\cos\Lambda_S \\ -(X_V - X_S)\sin\Phi_S\cos\Lambda_S - (Y_V - Y_S)\sin\Phi_S\sin\Lambda_S + (Z_V - Z_S)\cos\Phi_S \\ (X_V - X_S)\cos\Phi_S\cos\Lambda_S + (Y_V - Y_S)\cos\Phi_S\sin\Lambda_S + (Z_V - Z_S)\sin\Phi_S \end{Vmatrix}^{1)}. \tag{17.10}$$

Taking account of (17.10), from the relations (17.2) one gets *(Körner 1968)*:

$$\alpha_{SV} = \tan^{-1}\frac{(Y_V - Y_S)\cos\Lambda_S - (X_V - X_S)\sin\Lambda_S}{(Z_V - Z_S)\cos\Phi_S - (X_V - X_S)\sin\Phi_S\cos\Lambda_S - (Y_V - Y_S)\sin\Phi_S\sin\Lambda_S}; \tag{17.11}$$ [2]

$$D_{SV} = \sqrt{(X_V - X_S)^2 + (Y_V - Y_S)^2 + (Z_V - Z_S)^2}; \tag{17.12}$$ [2]

$$\beta'_{SV} = \cos^{-1}\frac{(X_V - X_S)\cos\Phi_S\cos\Lambda_S + (Y_V - Y_S)\cos\Phi_S\cos\Lambda_S + (Z_V - Z_S)\sin\Phi_S}{\sqrt{(X_V - X_S)^2 + (Y_V - Y_S)^2 + (Z_V - Z_S)^2}}. \tag{17.13}$$ [2]

Remarks:

(1) The relations (17.11)—(17.13) represent connexion formulae between the equatorial Cartesian global system and the polar local system.

(2) If the azimuth of the direction from P_S to P_V follows immediately from geodetic-astronomy determinations (α^*_{SV} in P_S), then one can assume (*Wolf 1963 b*): $\alpha_{SV} = \alpha^*_{SV}$. If, on the other hand, the azimuth is found by orienting a measured direction d_{SV} (i.e. one deduced from a series of directions), then one makes use of the relation $\alpha_{SV} = d_{SV} + o_{SV}$ in which o_{SV} represents the unknown of corresponding orientation.

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[1] See *H. Dufour* (*1959*, p. 5; *1969*, p. 174(4)), *M. Hotine* (*1959*, p. 16 (37.2—4)) and *H. Wolf* (1963 *c*, p. 226(2)).

[2] See *H. Wolf* (*1963 c*, p. 226 (3, 4)).

(b) *The horizontal Cartesian global system* $\bar{X}, \bar{X}, \bar{Z}$. In certain geodetic operations (e.g. the calibration of electromagnetic length-measurements), according to a remark of *K. Ramsayer* (*1965*) one may encounter a situation when the origin O of the Cartesian global coordinate system might coincide with a chosen point P_0 of the astro-geodetic network to be calculated. In this case one utilizes the coordinate axes defined in § 17.1.1. a. However, in order to distinguish it from the local system $\bar{x}, \bar{y}, \bar{z}$, this global system, designated as horizontal Cartesian, is denoted by $\bar{X}, \bar{Y}, \bar{Z}$ (Fig. 17.4; *Körner 1968*).

For transforming the $\bar{X}, \bar{Y}, \bar{Z}$ system into the X, Y, Z system, one applies the relation (*Körner 1968*):

$$\bar{\mathbf{X}} = \bar{\mathbf{R}}_1 \mathbf{X}, \tag{17.14}$$

in which $\bar{\mathbf{X}}^T = ||\bar{X}\,\bar{Y}\,\bar{Z}||$, $\mathbf{X}^T = ||\,X\,Y\,Z||$ and the rotation matrix $\bar{\mathbf{R}}_1$ has the form shown in (17.9).

Remark. Before beginning the network calculation, *K. Ramsayer* (*1965*) reduces the observations to the $\bar{X}$, $\bar{Y}$ plane and afterwards carries out simultaneously both the spatial-position adjustment of the geodetic points and the adjustment of their altitudes.

b) **The ellipsoidal global system** B, L, H^E. If one assigns a co-axial rotation ellipsoid to the X, Y, Z system in such a way that the Z axis coincides with the ellipsoid's rotation axis also the ellipsoid's centre with the origin of the X, Y, Z system, and afterwards one projects all the geodetic points from the Earth's surface along the corresponding normals to this ellipsoid (Fig. 17.5), then one utilizes as surface parameters for these points the geodetic (or ellipsoidal) latitude (B) and longitude (L). The distance $P_S P_S' = H^E$ represents the ellipsoidal altitude. The newly resulting

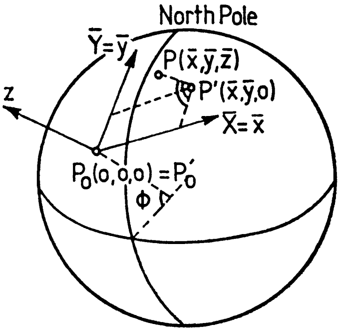

Fig. 17.4. The Horizontal Cartesian Global System

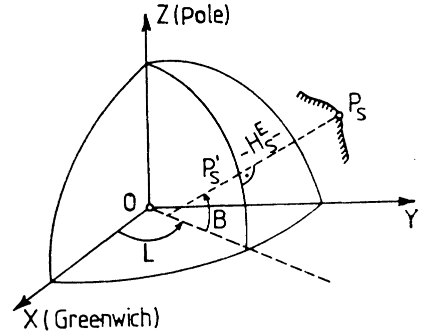

Fig. 17.5. The Ellipsoidal Global System

coordinates B, L, H^E relate to the X, Y, Z system according to the following relations (Fig. 17.5):

$$\begin{aligned} X &= (N + H^E) \cos B \cos L; \\ Y &= (N + H^E) \cos B \sin L; \\ Z &= (N(1 - e^2) + H^E) \sin B. \end{aligned} \tag{17.15}$$

For the inverse transformation, i.e. that of the X, Y, Z coordinates into the B, L, H^E coordinates, several formulae have been used viz.:

(a) *Using series expansions (Rinner 1958, Körner 1968):*

$$L = \tan^{-1}\frac{Y}{X};$$

$$B = \tan^{-1}\frac{CZ}{\sqrt{X^2+Y^2}}; \tag{17.16}$$

$$H^E = D\sqrt{1-\left(\frac{b}{D}\right)^2\frac{(e'^2-\eta^2)\eta^2}{V^2}} - bV,$$

here:

$$C = 1 + e'^2\left(1-\frac{HV}{b}+\frac{H^2V^2}{b^2}-\ldots\right);$$

$$\frac{HV}{b} = \left(\frac{D}{b}-V\right) - \frac{1}{2}\frac{b}{D}(e'^2-\eta^2)\eta^2; \; D = \sqrt{X^2+Y^2+Z^2}.$$

(b) *Using iterative procedures (Laping 1962, Körner 1968):*

$$L = \tan^{-1}\frac{Y}{X};$$

$$B_0 = \tan^{-1}\frac{(1+e'^2)Z}{c};$$

$$B_1 = \tan^{-1}\frac{Z+\Delta_1}{c}; \tag{17.17}$$

$$H^E = \begin{cases} \dfrac{d}{\cos B} - N, \text{ for } |B| < 45^\circ \\[2ex] \dfrac{d}{\sin B} - N(1-e^2), \text{ for } |B| > 45^\circ \end{cases}$$

where:

$$c = \begin{cases} \dfrac{Y}{\sin L}, \text{ for } Y > X \\[2ex] \dfrac{X}{\cos L}, \text{ for } Y < X \end{cases}$$

$$\Delta_1 = Ne^2 \sin B_0;$$

(c) *By differentiating the relations* (17.15) *(Wolf 1963 c, Körner 1968)*

$$\mathbf{dB} = \mathbf{R}_2\,\mathbf{dX}, \tag{17.18}$$

where:

$$\mathbf{dB}^T = ||(M + H^E)\,\mathrm{d}B \quad (N + H^E)\cos B\,\mathrm{d}L \quad \mathrm{d}H^E||;$$

$$\mathbf{R}_2 = \begin{Vmatrix} -\sin B\cos L & -\sin L & \cos B\cos L \\ -\sin B\sin L & \cos L & \cos B\sin L \\ \cos B & 0 & \sin B \end{Vmatrix};$$

$$\mathbf{dX}^T = ||\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z||.$$

Remark. For transforming the coordinates from the ellipsoidal global system into the X, Y, Z system one makes use of the relation $\mathbf{dX} = \mathbf{R}_2^T\,\mathbf{dB}$ (*Körner 1968*).

17.2 Origin, Methodology and Purpose of Three-Dimensional Geodesy

Even before the launching of the first artificial satellites of the Earth, there arose the need to revise and deepen the initial concepts — those of Classical Geodesy. Three-Dimensional Geodesy aims to give a geometrical description of the Earth without resorting to simplifying hypotheses. While in practice making use of all known geodetic techniques, Three-Dimensional Geodesy regards the Earth's surface (the polyhedron Earth) as the only concrete reality, which must be defined by entirely numerical characteristics, both geometric and dynamic.

The measurements on which Three-Dimensional Geodesy is based are the same as in astro-geodetic triangulation (Chapter 14). Consequently, from the measurements' point of view, nothing new has arisen; the combined utilization of all of these means and the manner of making use of them represent the innovating elements.

Three-Dimensional Geodesy aims at determining for every point of the topographic surface:

(1) *The trirectangular coordinates in a Cartesian global system, centred in the neighbourhood of the Earth's mass centre,* (X, Y, Z).

(2) *The direction parameters of the orientation of the astronomical vertical* (Φ, Λ).

(3) *The values* g *of the gravity and* W *of its potential.*

Remark. All these problems are also tackled by Classical Geodesy and, as *J. J. Levallois* and *J. Kovalevsky* (*1971*)) have shown, "we should perhaps see in the Three-Dimensional Geodesy not so much an absolutely new concept but the desire to unify and the intention to do away with the dificulties of the conceptions referring to two dimensions".

A retrospective view of the history of Geodesy shows that this problem was tackled for the first time, in the modern spirit, by *H. Bruns* in the year 1878. The main obstacle to realizing this modern conception lay in the fact that it assumes perfect knowledge of the phenomenon of atmospheric refraction, which until quite recently was merely a desideratum.

In the following we shall briefly present some considerations made by *H. Bruns* (1878) and reappraised by *H. Wolf* (*1963 b*).

One considers the different geodetic points on the Earth's surface as being rectilinearly connected, corresponding to the sighting lines [1] (Fig. 17.6; *Wolf 1963 b*); the totality of these points constitute a polyhedron.

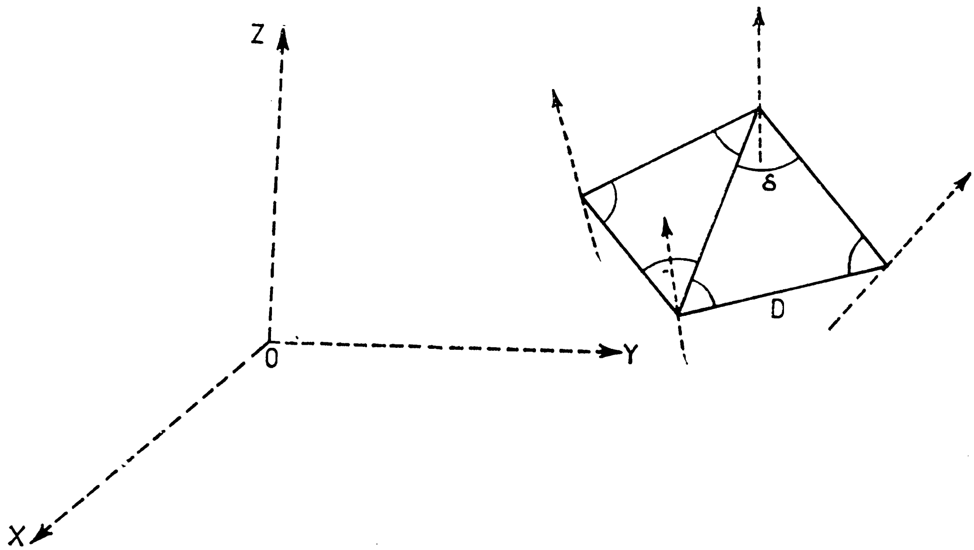

Fig. 17.6. Connecting the Geodetic Points on the Earth's Surface in the X, Y, Z System

At each geodetic point one can additionally imagine a straight line representing the gravity direction or the direction of the vertical. Consequently, in a global coordinate system X, Y, Z, one must determine at every point five values: three values for the spatial position of the point and two values for the direction of the vertical and the zenith's direction respectively.

In the general polyhedron one can measure: the distances D between the geodetic points and the position angles δ[2] between the polyhedron's surfaces.

The triangles bounding the polyhedron are geometrically determined by D and δ.

Remark. If this triangulation system were applicable to the entire terrestrial Globe, the Earth's figure would be uniquely defined to the extent to which it is formed out of geodetic points; with the help of the zenithal angles — figuratively speaking — one could fix the zenith's direction overall from the different points of the polyhedron (Fig. 17.1).

If at any point P_0, called a *central point*, one measures the direction towards the celestial pole by means of two angles, viz. the azimuth α_0 and the latitude Φ_0 (Fig. 17.7; *Wolf 1963 b*), thereby one fixes a parallel to the Earth's rotation axis. Now, one may choose a Cartesian global coordinate

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[1] Assuming perfect knowledge of the atmospheric refraction, one may arrive at a rectilinear path for every geodetic sighting.

[2] These can be determined by combining the azimuthal directions d and the zenithal distances β (*Wolf 1963 b*).

system X, Y, Z, in such a way that the Z axis is parallel to the Earth's rotation axis. The origin O of this system is taken to be in the neighbourhood of the Earth's mass centre and one assigns to the latter certain arbitrary values X_0, Y_0, Z_0 as trirectangular coordinates.

Remark. Of course, it would be very convenient, if somewhat improbable, if the origin O, chosen in such a way, coincided with the Earth's mass centre itself.

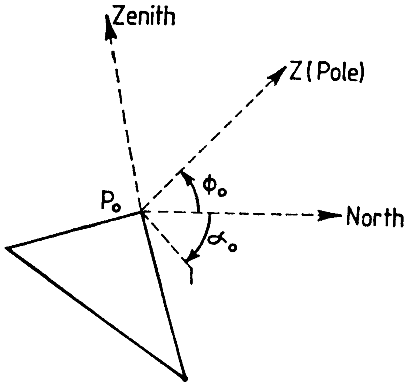

Fig. 17.7. Fixing the Parallel to the Earth's Rotation Axis

As a mathematical reference surface one may consider the sphere of radius equal to unity with its centre at O or, as happens in practice, the rotation ellipsoid of given dimensions. If from any geodetic point P_S one draws a perpendicular to this reference surface (with the projection P'_S), one can calculate, making use of (17.16), (17.17) or (17.18), the exact position of the projection P'_S on the ellipsoid and the distance H^E of the point P_S with respect to this surface (Fig. 17.5), i.e.:

$$B = B(X, Y, Z);\ L = L(X, Y, Z);$$
$$H^E = H^E(X, Y, Z). \tag{17.19}$$

Remarks:

(1) It is, consequently, absolutely immaterial whether one uses in calculations the coordinate X, Y, Z or B, L, H^E. With the aid of (17.19) one can pass from one system to the other.

(2) The coordinate transformation (17.19) faithfully corresponds to *Helmert's* projection, viz. to drawing a perpendicular to the ellipsoid. Therefore, in some technical contributions (e.g. *Wolf 1963 b*) it is mentioned that this projection should be called not *Helmert's* projection but *Bruns's* projection and for this reason one speaks of a *Bruns-Helmert* projection.

As a result, the five values necessary for the complete determination of a geodetic point are: B, L, H^E and two parameters for the direction of the *astronomical* (or *actual*) *vertical*, which is clearly defined by the latitude Φ and longitude Λ obtained from geodetic-astronomy determinations (Section 14.5).

Note. In the following we shall make a notational distinction between the measured values of the astronomical latitude and longitude (Φ^*, Λ^*) and the corresponding adjusted values (Φ, Λ).

If for each of the mentioned unknowns one now introduces an approximate value, (respectively $B^{(0)}$, $L^{(0)}$, $H^{E(0)}$, $\Phi^{(0)}$ and $\Lambda^{(0)}$), then one gets:

$$B = B^{(0)} + \mathrm{d}B;\ L = L^{(0)} + \mathrm{d}L;\ H^E = H^{E(0)} + \mathrm{d}H^E,$$
$$\Phi = \Phi^{(0)} + \mathrm{d}\Phi;\ \ \Lambda = \Lambda^{(0)} + \mathrm{d}\Lambda. \tag{17.20}$$

Remark. Since for every geodetic point the values $B^{(0)}$, $L^{(0)}$ can be extracted from already existing tables, *H. Bruns* suggests the following approximate values:

$$\Phi^{(0)} = B^{(0)};\ \Lambda^{(0)} = L^{(0)};\ H^{E(0)} = \text{altitude from levelling.}$$

The method to be followed in the sequal is the known one *(Wolf 1963 b)*: with the aid of the approximate coordinates one calculates the values (also approximate ones) of all the quantities to be measured, which are then compared with the values actually found from measurement, thus being obtained the free terms of the correction equations for the corresponding quantities. The coefficients of the five unknowns in these equations are obtained by differentiation with the help of expansions in *Taylor*'s series.

In this way are obtained the correction equations for the azimuthal directions (v_d), the zenithal angles (v_β), the space distances (v_D) and for the latitudes, longitudes and astronomical azimuths (v_{Φ^*}, v_{Λ^*} and v_{α^*} respectively). Subsequently solving the system of corresponding normal equations, and using the corresponding weights p_d, p_β, p_D, p_{Λ^*}, p_{Φ^*} and p_{α^*}, yields the unknowns looked for.

Unfortunately, this solution is still only at the "desideratum" stage since, as *H. Wolf* (*1963 b*) has shown "this beautiful plan fails, because of the refraction anomalies, which have such an inauspicious influence on the zenithal angles, especially for large sighting distances". Therefore, *H. Bruns* suggests the idea that v_β (as well as the unknowns associated with them H^E, Φ, Λ) should be relegated to a secondary plane and that one should determine first of all the geodetic coordinates B and L.

Put in modern terms, this solution may be expressed by the fact that the adjusting process should be carried out in two groups:

(1) *In the first group* there should be taken into consideration the correction equations for v_d, v_D, $v_{\alpha^*_{PL}}$ and $v_{\Lambda^*_{PL}}$ (the subscript PL means Laplace point), in which B and L appear as unknowns.

Remark. The astronomical quantities are dealt with as errorless quantities with respect to the zenithal angles.

(2) *The second group*, in which H^E, Φ and Λ occur as unknowns, has already been carried out as a completely independent adjusting process, viz. most recently by *F. Kobold* and *N. Wunderlin* (*1960*). It is true only for the meridian's profiles, so that α^*, Λ^* and Λ did not enter the calculation.

Remark. As is known (*Wolf* 1963 *b*), such a determination is possible only in mountainous areas, where, thanks to the fact that the sighting rays are at high altitudes, the refraction obeys the appropriate law.

Due to the difficulties attributable to inexact knowledge of the refraction phenomenon, the period that followed the publication of *Bruns*'s work, marked by *Helmert*'s activity, put the spatial conception on a secondary footing. Since World War II, this conception has been reconsidered, firstly by *A. Marussi* (*1951*) in its physical aspect, instead of *Bruns*'s "geometrical" unknowns B, L, H^E, the *"physical unknowns"* Φ and Λ being used, to which was added, as a third coordinate, the potential difference $\Delta W = \int g \, \mathrm{d}H$, such as it appears today in the form of geopotential heights. It follows the publication of *M. Hotine's* work (1956) which gives a new expression to the geometrical idea by proposing firstly to calculate the position angles δ between the polyhedron's surfaces as values derived from the azimuthal directions d and the zenithal angles β. Afterwards it would proceed to the adjustment of the values δ by means of the conditioned observation procedure, the replacement of v_δ with v_β and v_d being possible. Referring to this conception, *H. Wolf* (*1963 b*) makes the following remark: even if the weights p_β and p_d are correspondingly approximated, it is not certain that the occuring misclosures (in particular those arising from the zenithal-distance errors) will be correctly distributed between β and d. This therefore means that, under certain circumstances there is the possibility that the precisely measured azimuthal directions will be affected by the zenithal angles, being measured with a much smaller accuracy, even when adopting rather small values for p_β.

In the year 1959, there appears another contribution of *M. Hotine* (*1959*) presenting the adjustment by the indirect-observation procedure with the separation of the correction equations into two groups, which represents nothing but *Bruns*'s solution. *M. Hotine*'s particular merit lies in deriving, in an elegant way, by making use of tensor analysis, the mathematical form of the correction equations, which *H. Bruns* had only sketched. The derivation of the corresponding correction equations, also called basic equations of Three-Dimensional Geodesy, has subsequently been carried out in other ways too (*Dufour 1962, Wolf 1962 c, Levallois and Kovalevsky 1971, Grafarend 1975*).

As regards the removal of the difficulties which appear due to the imprecise knowledge of the refraction phenomenon, one can state that these have somehow been solved due to the advent of two new phenomena:

(a) *On the practical-applied level or carrying out the measurements, viz. the new measuring method, designated as star triangulation or, more recently in the U.S.A., as the rocket flash method* consisting of the following: on the background of the fixed-star sky one photographs one or several light signals ("flashes") simultaneously emitted from two terrestrial stations situated at a distance of several hundred kilometres from one another. In this case one may calculate the zenithal angle and at the same time the azimuth between the two observation points, free of the influence of atmospheric refraction.

The correction equations corresponding to this case have also been worked out by *M. Hotine*, in an equally elegant way.

In fact, besides the star triangulation and the rocket flash method, Three-Dimensional Geodesy also includes *the spatial triangulation method* (sometimes also designated, more generally, as *spatial geodesy*) *by means of the Earth's artificial satellites*, which in fact has given the most important results.

(b) *On the theoretical level, viz. Molodenski's equation*:

$$H_2^E - H_1^E = \int_1^2 \delta z - \int_1^2 u_{12}\, \delta D, \qquad (17.21)$$

where $H_2^E - H_1^E$ represents the difference of the ellipsoidal altitudes between two points P_1 and P_2; $\int \delta z$ — the rough level difference between the two points, as obtained from trigonometric-levelling measurements, and u_{12} are the values of the deflections of the vertical along the path from P_1 to P_2. On the basis of equation (17.21), knowing the ellipsoidal altitudes and the corresponding deflections of the vertical, one gets the zenithal angles β' freed from the influence of atmospheric refraction.

Regarding the removal of the difficulties due to atmospheric refraction, one could also metion the solution given by *M. Hotine*, viz.: abandoning the conventional refraction-model (starting no longer from a constant mean value of the refractive index within a certain domain) and introducing for every station point of the triangulation network a specific refraction quantity, whose value is to be determined by adjustment, which enters as an unknown.

This solution, however, has two important drawbacks *(Dufour 1962, Wolf 1963b)*:

(1) The number of supplementary determinations is considerably reduced.

(2) The number of unknowns and the number of equations to be solved increase to such an extent that even by utilizing large electronic computers the solution is very hard to carry out.

In addition to this procedure of *M. Hotine*, there exists another procedure, proposed by *A. Marussi*, according to which for each station point one introduces a separate refraction vector, which must be described by means of two numerical values. This procedure is, obviously, still more difficult to apply than the first one, where for every point only one value due to refraction was necessary *(Wolf* 1963 *b)*.

More recently, in the year 1969, *H. Körner* has indicated, in his doctoral thesis, 4 refraction models, depending on the following hypotheses: a constant mean value of the refractive index k for the entire triangulation network, different refractive index k for the entire triangulation network, different refractive indices k for station points, constant refractive indices k for separate domains and refractive indices assigned to each sighting.

Returning to the problem of adjusting an astro-geodetic network in the three-dimensional conception, one can say that one may regard as a further contribution the idea already anticipated by *H. Bruns* and afterwards by *M. Hotine*, which was subsequently taken up again by *H. M. Dufour* (1969 a) viz. the idea of the splitting up into adjustment groups. *H. M. Dufour* conceives this procedure as an iterative process, retaining at each stage, alternately varying, the unknowns: "altitude" and "position". *J. J. Levallois* (*1963*) also emphasizes this idea, while additionally showing that, if the position adjustment is carried out separately (at any stage of the successive approximation), it follows that one gets exactly the same values concerning the unknowns B and L as when one makes use of the plain projection of *Helmert* and the computation is carried out two-dimensionally in the projection plane. If in a (two-dimensional) calculation of an astro-geodetic triangulation neither the deflections of the vertical u nor the geoid undulations G (viz. for reducing to the ellipsoid the measured quantities) are taken into consideration, such an approximate calculation is called, according to a proposal made by *M. S. Molodenski*, a *development in the network* (according to which nearly all present triangulations are, in fact, calculated). **Thus, the** *Bruns-*

Dufour's **iterative process would roughly involve the following stages** *(Wolf 1963 b)*:

(1) *The altitude calculation* (rough and orthometric altitudes H^0 and, under certain circumstances, geopotential heights ΔW).

(2) *The development in the network, giving the approximate geodetic coordinates $B^{(0)}$ and $L^{(0)}$.*

(3) *The calculation of the deflection of the vertical* (respectively of its components $\xi = \Phi - B_0$ and $\eta = (\Lambda - L_0) \cos \Phi$ *and of the ellipsoidal altitudes H^E.*

(4) *The projection in the network according to Helmert, yielding B, L and Λ_{PL}.*

(5) *The calculation of the corrected deflection of the vertical and of the altitudes.*

(6) *The projection in the corrected network* etc.

One notices that the reciprocal influence between the values characterizing the position (B, L and Λ_{PL}) and those characterizing the altitude (H^E, ξ, η, ξ_{PL}) is very weak, so that after such an iterative process rapid convergence of the results may certainly follow.

Remark. Using this conception one has tried to meet all the requirements of the measurements carried out on a given territory, viz. one obtains the exact position relations for all the geodetic points, as well as a complete solution of the problem of the ellipsoidal altitudes. In addition to this, at every point one calculates the actual direction of the zenith, the actual direction of the gravity respectively).

Another problem which has to be solved in the context of Three-Dimensional Geodesy is the transfer of the origin of the Cartesian global coordinate system to the Earth's mass centre, i.e. to calculate the geocentric coordinates for all the polyhedron's points. Put another way, it is a question of "centring" *(Wolf 1963b* and *1968, Popovici 1965)* the ellipsoid (an operation closely connected to the system X, Y, Z; (17.19)) at the Earth's mass centre. *H. Bruns* (*1878*) had already indicated "with (an) admirable foresight", as H. *Wolf* (*1963b*) showed, a method which has only now become applicable and which necessitates the carrying out of measurements of a completely different kind — observations concerning the Moon.

Remark. From *H. Bruns*'s work up to the modern procedures of Satellite Dynamic Geodesy (which will be tackled in the fifth part of the book), there is, at least on the ideas' level, only a short step, if at the same time considers the one development of observational technique [1].

The progress of modern technique and particularly that of the satellite technique will provide other possibilities too, in this respect, e.g. calculating the coordinates of the Earth's mass centre from measurements on satellites, initially in any global system X, Y, Z and afterwards, by the necessary transposition, in the sought geocentric system.

In closing this section one can draw the following conclusions:

(1) *The origin of Three-Dimensional Geodesy lies in the work of H. Bruns* (*1878*), in which the basic ideas are to be found (coordinate systems to be

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[1] For instance, *Marcowitz's method* (*1958*), in which with the aid of photographically fixing the Moon on the background of the starry sky one draws a conclusion on the geocentric coordinates of the various observation points.

introduced, *Helmert*'s projection, splitting up into groups of the adjusting process, centring the coordinate system with the aid of Moon observations, formulating the theme of measuring the Earth).

(2) *As regards the measuring methodology* more numerous and improved technical means are available today than in the past, particularly in the field of gravity measurements applied over wide expanses (on land, at sea and in the air), as well as improved procedures (radar [1] procedures, star triangulation, lunar and satellite methods), serving, on the one hand, to remove the difficulties determined by the fact that on the terrestrial Globe there are still spaces lacking fixed points and, on the other, to calculate the coordinates of the Earth's mass centre.

Concerning the calculating methodology, the introduction of electronic computers has brought the possibility that solutions conceived as universal (e.g. *A. Bjerhammar*'s solution (*1967*)) should not remain only at the reasoning stage but achieve their practical realization.

(3) *As to the purpose of Three-Dimensional Geodesy,* this consists of determining the geocentric coordinates of all the points on the Earth's surface and in establishing the potential as a function of any in space point; already according to *H. Bruns*'s conception expressed as early as 1878 this was the necessary aim of any scientific geodetic activity.

(4) *The importance of Three-Dimensional Geodesy consists mainly* of the following *(Wolf 1963 b, Levallois* and *Kovalevsky 1971):*

— *it helps to clarify the problems,* such as are put by the measurement of any territory (at the size of a country or of a group of countries) and of the Earth as a whole;

— *from the practical viewpoint it admits successive approximations,* through which the way is opened for any subsequent development;

— *it allows an easy transition from the measurement of any territory to the Earth's measurement,* by completing the system of correction equations only with a number of supplementary observations and with other unknowns;

— *it presents some theoretical advantages,* as e.g.: the elimination of the reduction to ellipsoid of measured azimuthal angles (because one only takes into consideration the physical vertical); the values of the sides directly measured on the Earth's surface are no longer reduced, being immediately used in calculations; the removal of the non-rigorous simplifying hypotheses on the level of both definitions and applications; the possibility is provided of defining calculable a quasi-geoid and a convenient ellipsoid for whatever extent of the territory being analysed.

17.3 Basic Equations of Three-Dimensional Geodesy

Several methods have been used for deriving the basic equations of Three-Dimensional Geodesy, viz.: by means of tensor analysis *(M. Hotine 1959),* differential geometry *(H. Dufour 1962, J. J. Levallois* and.

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[1] Radio Detection and Ranging.

J. Kovalevsky 1971, E. Grafarend 1975) or in an elementary way *(H. Wolf 1963 c)*. In this section we will present the method used by *H. Wolf*, as being the simplest one, and that utilized by *E. Grafarend* which provides the most general formulae in the 1st-order approximation.

17.3.1 Wolf's Solution

H. Wolf derives the basic equations of Three-Dimensional Geodesy for the coordinates in the equatorial Cartesian global system as well as in the ellipsoidal global system. In both cases the solution idea consists of differentiating the basic transforming relations characteristic to the two systems. For the case of the coordinates in the equatorial Cartesian global system X, Y, Z, the starting formulae are (17.11)—(17.13), whose differentiation yields the basic equations in the form:

$$\begin{aligned} d\alpha_{SV} = {} & \frac{\sin \Phi_S \cos \Lambda_S \sin \alpha_{SV} - \sin \Lambda_S \cos \alpha_{SV}}{D_{SV} \sin \beta'_{SV}} (dX_V - dX_S) + \\ & + \frac{\sin \Phi_S \sin \Lambda_S \sin \alpha_{SV} + \cos \Lambda_S \cos \alpha_{SV}}{D_{SV} \sin \beta'_{SV}} (dY_V - dY_S) - \\ & - \frac{\cos \Phi_S \sin \alpha_{SV}}{D_{SV} \sin \beta'_{SV}} (dZ_V - dZ_S) + \cot \beta'_{SV} \sin \alpha_{SV}\, d\Phi_S + \\ & + (\sin \Phi_S - \cos \alpha_{SV} \cos \Phi_S \cot \beta'_{SV})\, d\Lambda_S; \end{aligned} \tag{17.22}$$

$$\begin{aligned} dD_{SV} = {} & \frac{X_V - X_S}{D_{SV}} (dX_V - dX_S) + \frac{Y_V - Y_S}{D_{SV}} (dY_V - dY_S) + \\ & + \frac{Z_V - Z_S}{D_{SV}} (dZ_V - dZ_S); \end{aligned} \tag{17.23}$$

$$\begin{aligned} d\beta'_{SV} = {} & \frac{(X_V - X_S) \cos \beta'_{SV} - D_{SV} \cos \Phi_S \cos \Lambda_S}{D^2_{SV} \sin \beta'_{SV}} (dX_V - dX_S) + \\ & + \frac{(Y_V - Y_S) \cos \beta'_{SV} - D_{SV} \cos \Phi_S \sin \Lambda_S}{D^2_{SV} \sin \beta'_{SV}} (dY_V - dY_S) + \\ & + \frac{(Z_V - Z_S) \cos \beta'_{SV} - D_{SV} \sin \Phi_S}{D^2_{SV} \sin \beta'_{SV}} (dZ_V - dZ_S) - \cos \alpha_{SV}\, d\Phi_S - \\ & - \cos \Phi_S \sin \alpha_{SV}\, d\Lambda_S. \end{aligned} \tag{17.24}$$

For the case of the coordinates in the ellipsoidal global system B, L, H^E, one starts from the formulae (17.15). Differentiating the first relation in (17.15) one gets:

$$\frac{\partial X}{\partial B}=\left(\frac{\partial N}{\partial B}\cos B-(N+H^E)\sin B\right)\cos L.$$

Inasmuch as $\frac{\partial N}{\partial B}=\frac{N}{1+\eta^2}\eta^2\tan B$ and $N=\frac{c}{1+\eta^2}$, $M=\frac{c}{(1+\eta^2)^3}$,

it follows that:

$$\frac{\partial X}{\partial B}=\cos L\left(\frac{c}{(1+\eta^2)^3}\sin B-H^E\sin B-\frac{c}{1+\eta^2}\sin B\right)=$$

$$=\cos L\left(-H^E\sin B-\frac{c}{(1+\eta^2)^3}\sin B(1+\eta^2-\eta^2)\right)=$$

$$=-\cos L(H^E+M)\sin B,$$

and:

$$\frac{\partial X}{\partial L}=-(N+H^E)\cos B\sin L \text{ and } \frac{\partial X}{\partial H^E}=\cos B\cos L.$$

In view of the first two relations in (17.15) it follows that $Y=X\sin L$, from which by differentiation one obtains:

$$\frac{\partial Y}{\partial B}=\frac{\partial X}{\partial B}\tan L=-(M+H^E)\sin L\sin B;$$

$$\frac{\partial Y}{\partial L}=(N+H^E)\cos B\cos L \text{ and } \frac{\partial Y}{\partial H^E}=\cos B\cos L.$$

Proceeding analogously as for X and L, the third relation in (17.15) yields:

$$\frac{\partial Z}{\partial B}=(N(1-e^2)+H^E)\cos B+\frac{\partial N}{\partial B}\sin B(1-e^2);$$

$$\frac{\partial N}{\partial B}=\frac{c}{(1+\eta^2)^3}\eta^2\tan B=H^E\cos B+\frac{c}{(1+\eta^2)^3}(1-e^2)\left((1+\eta^2)^2\cos B+\right.$$

$$\left.+\eta^2\tan B\sin B\right)=H^E\cos B+\frac{M(1-e^2)}{\cos B}(\cos^2 B+\eta^2(\cos^2 B+\sin^2 B))$$

and, in view of the fact that $\eta^2 = e'^2 \cos^2 B = \dfrac{e^2}{1-e^2} \cos^2 B$, one gets:

$$\frac{\partial Z}{\partial B} = H^E \cos B + M(1 - e^2 + e^2) \cos B = (M + H^E) \cos B,$$

as well as $\dfrac{\partial Z}{\partial L} = 0$ and $\dfrac{\partial Z}{\partial H^E} = \sin B$.

In view of:

$$\mathrm{d}X = \frac{\partial X}{\partial B}\mathrm{d}B + \frac{\partial X}{\partial L}\mathrm{d}L + \frac{\partial X}{\partial H^E}\mathrm{d}H^E \text{ etc.,}$$

we get the following differential formulae for the coordinates:

$$\mathrm{d}X = -(M + H^E) \cos L \sin B \,\mathrm{d}B - (N + H^E) \cos B \sin L \,\mathrm{d}L + \\ + \cos B \cos L \,\mathrm{d}H^E;$$

$$\mathrm{d}Y = -(M + H^E) \sin L \sin B \,\mathrm{d}B + (N + H^E) \cos B \cos L \,\mathrm{d}L + \\ + \cos B \sin L \,\mathrm{d}H^E;$$

$$\mathrm{d}Z = +(M + H^E) \cos B \,\mathrm{d}B + \sin B \,\mathrm{d}L. \tag{17.25}$$

If one substitutes in (17.22)—(17.24) dX, dY and dZ from (17.25) and uses the notations:

$$(M + H^E)\frac{\mathrm{d}B}{\rho''} = \mathrm{d}\bar{B} \text{ and } (N + H^E) \cos B \frac{\mathrm{d}L}{\rho''} = \mathrm{d}\bar{L}, \tag{17.26}$$

we finally get:

$$\mathrm{d}\alpha_{SV} = a_1 \,\mathrm{d}\bar{B}_S + a_2 \,\mathrm{d}\bar{L}_S + a_3 \,\mathrm{d}H^E_S + a_4 \,\mathrm{d}\bar{B}_V + a_5 \,\mathrm{d}\bar{L}_V + \\ + a_6 \,\mathrm{d}H^E_V + a_7 \,\mathrm{d}\Phi_S + a_8 \,\mathrm{d}\Lambda_S, \tag{17.27}$$

where:

$$a_1 = +\rho'' \frac{\sin \alpha_{SV}}{D_{SV} \sin \beta'_{SV}}; \quad a_2 = -\rho'' \frac{\cos \alpha_{SV}}{D_{SV} \sin \beta'_{SV}}; \; a_3 = 0, \tag{17.28}$$

$$a_4 = -\rho'' \frac{\sin \alpha_{SV}}{D_{SV} \sin \beta'_{SV}} \bar{Q}_{SV}; \quad a_5 = +\rho'' \frac{\cos \alpha_{SV}}{D_{SV} \sin \beta'_{SV}} \bar{P}_{SV};$$

$$a_6 = +\rho'' \frac{\cos \alpha_{SV} \cos B_V}{D_{SV} \sin \beta'_{SV}} \bar{R}_{SV}; \tag{17.29}$$

$$a_7 = +\sin \alpha_{SV} \cot \beta'_{SV}; \quad a_8 = \sin \Phi_S - \cos \alpha_{SV} \cos \Phi_S \cot \beta'_{SV}, \tag{17.30}$$

in which:

$$\bar{Q}_{SV} = \cos(B_V - B_S) + \sin B_V \sin(L_V - L_S) \cot \alpha_{SV};$$
$$\bar{P}_{SV} = \cos(L_V - L_S) - \sin B_S \sin(L_V - L_S) \tan \alpha_{SV};$$
$$\bar{R}_{SV} = \sin(L_V - L_S) + (\sin B_S \cos(L_V - L_S) - \tan B_V \cos B_S) \tan \alpha_{SV},$$

or which is often satisfactory in practice *(Wolf 1963c)*:

$$\bar{Q}_{SV} \approx 1 + \sin B_V \sin(L_V - L_S) \cot \alpha_{SV};$$
$$\bar{P}_{SV} \approx 1 - \sin B_S \sin(L_V - L_S) \tan \alpha_{SV};$$
$$\bar{R}_{SV} \approx \sin(L_V - L_S) + (\sin B_S - \tan B_V \cos B_S) \tan \alpha_{SV}.$$

Remark. For checking of the calculations one makes use of the relation:

$$(\bar{Q}_{SV} \sin \alpha_{SV})^2 + (\bar{P}_{SV} \cos \alpha_{SV})^2 + (\bar{R}_{SV} \cos B_V \cos \alpha_{SV})^2 = 1.$$

Correspondingly, one gets for the distances:

$$\begin{aligned} \mathrm{d}D_{SV} = e_1 \mathrm{d}\bar{B}_S + e_2 \mathrm{d}\bar{L}_S + e_3 \mathrm{d}H_S^E + e_4 \mathrm{d}\bar{B}_V + e_5 \mathrm{d}\bar{L}_V + \\ + e_6 \mathrm{d}H_V^E + e_7 \mathrm{d}\Phi_S + e_8 \mathrm{d}\Lambda_S, \end{aligned} \tag{17.31}$$

in which:

$$\begin{aligned} &e_1 = -\sin \beta'_{SV} \cos \alpha_{SV};\; e_2 = -\sin \beta'_{SV} \sin \alpha_{SV};\; e_3 = -\cos \beta'_{SV}; \\ &e_4 = -\sin \beta'_{SV} \cos \alpha_{VS};\; e_5 = -\sin \beta'_{VS} \sin \alpha_{VS};\; e_6 = -\cos \beta'_{VS}; \\ &e_7 = e_8 = 0, \end{aligned} \tag{17.32}$$

and for the zenithal angles:

$$\begin{aligned} \mathrm{d}\beta'_{SV} = f_1 \mathrm{d}\bar{B}_S + f_2 \mathrm{d}\bar{L}_S + f_3 \mathrm{d}H_S^E + f_4 \mathrm{d}\bar{B}_V + f_5 \mathrm{d}\bar{L}_V + \\ + f_6 \mathrm{d}H_V^E + f_7 \mathrm{d}\Phi_S + f_8 \mathrm{d}\Lambda_S, \end{aligned} \tag{17.33}$$

where:

$$\begin{aligned} f_1 = -\rho'' \frac{\cos \beta'_{SV} \cos \alpha_{SV}}{D_{SV}};\; f_2 = -\rho'' \frac{\cos \beta'_{SV} \sin \alpha_{SV}}{D_{SV}}; \\ f_3 = +\rho'' \frac{\sin \beta'_{SV}}{D_{SV}}; \end{aligned} \tag{17.34}$$

$$f_4 \doteq +\rho'' \frac{\cos \beta'_{SV} \sin B_V \cos(L_V - L_S) - \sin B_S \cos B_V - \cos \beta'_{SV} \sin \beta'_{VS} \cos \alpha_{VS}}{D_{SV} \sin \beta'_{SV}}, \tag{17.35}$$

$$f_5 = +\rho'' \frac{\cos \beta'_{SV} \sin(L_V - L_S) - \cos \beta'_{SV} \sin \beta'_{VS} \sin \alpha_{VS}}{D_{SV} \sin \beta'_{SV}}; \tag{17.36}$$

$$f_6 = -\rho'' \frac{\cos\beta'_{SV}\cos\beta'_{VS} + \sin B_V \sin B_S + \cos B_S \cos B_V \cos(L_V - L_S)}{D_{SV}\sin\beta'_{SV}}; \tag{17.37}$$

$$f_7 = -\cos\alpha_{SV} \text{ and } f_8 = -\cos\Phi_S \sin\alpha_{SV}. \tag{17.38}$$

Remark. To practice it is often enough that:

$$f_4 \approx \frac{\rho''}{D_{SV}\sin\beta'_{SV}}(\sin(B_V - B_S) - \cos\beta'_{SV}\sin B_V \cos\alpha_{VS});$$

$$f_6 \approx \frac{\rho''}{D_{SV}\sin\beta'_{SV}}(\cos(B_V - B_S) + \cos\beta'_{SV}\cos\beta'_{VS}).$$

We shall next give an example of the application of the basic equations (17.22) ... (17.24) and (17.27), (17.31), (17.33).

As was shown in Section 17.2, the geodetic points located on the Earth's surface form a spatial network (geometrically a polyhedron). With the aid of the measured elements $d, \beta', D, \alpha^*, \Phi^*$ and Λ^* the polyhedron is over determined, so that an adjustment is necessary. If this adjustment is carried out by means of the procedure of indirect observations, then one must first of all establish the unknowns. For this, there are two equivalent possibilities:

(I) X, Y, Z, Φ, Λ;

(II) B, L, H^E, Φ, Λ.

Frequently, one takes approximate values $X^{(0)}$, $Y^{(0)}$ etc. for the unknowns, so that, e.g. for the case (I):

$$\begin{aligned} &X = X^{(0)} + \mathrm{d}X;\ Y = Y^{(0)} + \mathrm{d}Y;\ Z = Z^{(0)} + \mathrm{d}Z;\\ &\Phi = \Phi^{(0)} + \mathrm{d}\Phi;\ \Lambda = \Lambda^{(0)} + \mathrm{d}\Lambda;\ o^{1} = o^{(0)} + \mathrm{d}o. \end{aligned} \tag{17.39}$$

In what follows we will present the form of the correction equations for each of the two cases I and II, equations which are obtained with the help of the basic equations.

The correction equations in the equatorial Cartesian global system. In case I one gets the following basic equations *(Wolf* 1963*c* and 1975*)*:

(a) *For the horizontal directions d* (equation (17.22)):

$$\begin{aligned} v_{d_{SV}} = &-\mathrm{d}o_{SV} + A_1(\mathrm{d}X_S - \mathrm{d}X_V) + A_2(\mathrm{d}Y_S - \mathrm{d}Y_V) + A_3(\mathrm{d}Z_S - \\ &-\mathrm{d}Z_V) + A_4\mathrm{d}\Phi_S + A_5\mathrm{d}\Lambda_S + \{\alpha^{(0)}_{SV} - (\mathrm{d}^*_{SV} + o^{(0)}_{SV})\}, \text{ the weight } p_d. \end{aligned} \tag{17.40}$$

where:

$$\alpha^{(0)}_{SV} = \tan^{-1}\frac{(Y^{(0)}_V - Y^{(0)}_S)\cos\Lambda^{(0)}_S - (X^{(0)}_V - X^{(0)}_S)\sin\Lambda^{(0)}_S}{(Z^{(0)}_V - Z^{(0)}_S)\cos\Phi^{(0)}_S - (X^{(0)}_V - X^{(0)}_S)\sin\Phi^{(0)}_S\cos\Lambda^{(0)}_S - (Y^{(0)}_V - Y^{(0)}_S)\sin\Phi^{(0)}_S\sin\Lambda^{(0)}_S},$$

[1] o represents the corresponding *orientation unknown* ($\alpha = d + o$).

which is obtained from (17.11) by replacing:

$\alpha_{SV}, X_V, Y_V, Z_V, X_S, Y_S, Z_S, \Phi_S, \Lambda_S$ with their approximate values $\alpha_{SV}^{(0)}$, $X_V^{(0)}$, $Y_V^{(0)}$, $Z_V^{(0)}$ etc.:

$$A_1 = \frac{\sin \Phi_S^{(0)} \cos \Lambda_S^{(0)} \sin \alpha_{SV}^{(0)} - \sin \Lambda_S^{(0)} \cos \alpha_{SV}^{(0)}}{D_{SV}^{(0)} \sin \beta_{SV}^{\prime(0)}};$$

$$A_2 = \frac{\sin \Phi_S^{(0)} \sin \Lambda_S^{(0)} \sin \alpha_{SV}^{(0)} + \cos \Lambda_S^{(0)} \cos \alpha_{SV}^{(0)}}{D_{SV}^{(0)} \sin \beta_{SV}^{\prime(0)}};$$

$$A_3 = \frac{-\cos \Phi_S^{(0)} \sin \alpha_{SV}^{(0)}}{D_{SV}^{(0)} \sin \beta_{SV}^{\prime(0)}}; \; A_4 = + \sin \alpha_{SV}^{(0)} \cot \beta_{SV}^{\prime(0)} = a_7 \qquad [\text{v. } (17.30)];$$

$$A_5 = \sin \Phi_S^{(0)} - \cos \alpha_{SV}^{(0)} \cos \Phi_S^{(0)} \cot \beta_{SV}^{\prime(0)} = a_8 \qquad [(\text{v. } (17.30)];$$

$$D_{SV}^{(0)} = \sqrt{(X_V^{(0)} - X_S^{(0)})^2 + (Y_V^{(0)} - Y_S^{(0)})^2 + (Z_V^{(0)} - Z_S^{(0)})^2} \qquad [\text{v. } (17.12)];$$

d_{SV}^* = the value of the azimuthal direction SV obtained from measurements.

(b) *For the zenithal angles* β' (17.24):

$$v_{\beta'_{SV}} = - D_{SV}^{(0)} \, dk_{SV} + F_1(dX_V - dX_S) + F_2(dY_V - dY_S) + F_3(dZ_V - dZ_S) + F_4 \, d\Phi_S + F_5 \, d\Lambda_S + \{\beta_{SV}^{\prime(0)} - (\beta_{SV}^{\prime *} + D_{SV}^{(0)} k_{SV}^{(0)})\}, \text{ weight } p_{\beta'} \quad (17.41)$$

where:

$$\beta_{SV}^{\prime(0)} = \cos^{-1}\left(\frac{1}{D_{SV}^{(0)}} (X_V^{(0)} - X_S^{(0)}) \cos \Phi_S^{(0)} \cos \Lambda_S^{(0)} + (Y_V^{(0)} - Y_S^{(0)}) \cos \Phi_S^{(0)} \sin \Lambda_S^{(0)} + (Z_V^{(0)} - Z_S^{(0)}) \sin \Phi_S^{(0)}\right),$$

which is obtained from (17.13) by replacing $\beta'_{SV}, X_V, Y_V, Z_V, X_S, Y_S, Z_S, \Phi_S, \Lambda_S, D_{SV}$ with the approximate values $\beta_{SV}^{\prime(0)}$, $X_V^{(0)}$ etc.; $k_{SV}^{(0)}$ is the approximate value of the refractive coefficient, calculated, e.g. with the relation:

$k^{(0)} = 0.13/2\,R$; R = the Earth's radius;

$$F_1 = + \frac{(X_V^{(0)} - X_S^{(0)}) \cos \beta_{SV}^{\prime(0)} - D_{SV}^{(0)} \cos \Phi_S^{(0)} \cos \Lambda_S^{(0)}}{D_{SV}^{(0)2} \sin \beta_{SV}^{\prime(0)}};$$

$$F_2 = + \frac{(Y_V^{(0)} - Y_S^{(0)}) \cos \beta_{SV}^{\prime(0)} - D_{SV}^{(0)} \cos \Phi_S^{(0)} \sin \Lambda_S^{(0)}}{D_{SV}^{(0)2} \sin \beta_{SV}^{\prime(0)}};$$

$$F_3 = + \frac{(Z_V^{(0)} - Z_S^{(0)}) \cos \beta_{SV}^{\prime(0)} - D_{SV}^{(0)} \sin \Phi_S^{(0)}}{D_{SV}^{(0)2} \sin \beta_{SV}^{\prime(0)}};$$

$$F_4 = - \cos \alpha_{SV}^{(0)} = f_7 \; (17.38); \quad F_5 = - \cos \Phi_S^{(0)} \sin \alpha_{SV}^{(0)} = f_8 \; (17.38);$$

$\beta_{SV}^{\prime *}$ — the value of the zenithal angle obtained from measurements.

Remark. The choice of the unknown dk_{SV} from among the following variants:
— calculating a mean value for the whole territory under analysis;
— calculating "regional" unknowns dk_{I}, dk_{II}, dk_{III}, ..., dk_ν (ν — the number of regions taken into consideration) into which one divides the unknown dk,
— introducing an individual unknown

$$dk_j (j = 1, 2, ..., N_s)$$

for each of the N_s station points (*Hotine 1969*), depends on the degree of overdetermination.

(c) *For the inclined distances D* (17.23):

$$v_{D_{SV}} = E_1(dX_V - dX_S) + E_2(dY_V - dY_S) + E_3(dZ_V - dZ_S) + \\ + (D_{SV}^{(0)} - D_{SV}^*), \text{ the weight } p_D, \tag{17.42}$$

where:

$$E_1 = \frac{X_V^{(0)} - X_S^{(0)}}{D_{SV}^{(0)}}; \quad E_2 = \frac{Y_V^{(0)} - Y_S^{(0)}}{D_{SV}^{(0)}}; \quad E_3 = \frac{Z_V^{(0)} - Z_S^{(0)}}{D_{SV}^{(0)}};$$

D_{SV}^* — the value of the distance SV obtained from measurement.

(d) *For the astronomical azimuths* α:

$$v_{\alpha_{SV}} = A_1(dX_S - dX_V) + A_2(dY_S - dY_V) + A_3(dZ_S - dZ_V) + \\ + A_4 d\Phi_S + A_5 d\Lambda_S + (\alpha_{SV}^{(0)} - \alpha_{SV}^*), \text{ the weight } p_\alpha, \tag{17.43}$$

where the coefficients A_i $(i = 1, 2, 3, 4, 5)$ and $\alpha_{SV}^{(0)}$ are calculated according to the same formulae as in the case of equation (a) and α_{SV}^* represents the value of the astronomical azimuth obtained from geodetic-astronomy determinations.

(e) *For the astronomical latitudes* Φ:

$$v_{\Phi_S} = d\Phi_S + (\Phi_S^{(0)} - \Phi_S^*), \text{ the weight } p_\Phi, \tag{17.44}$$

where Φ_S^* represents the astronomical latitude provided by geodetic-astronomy measurements.

(f) *For the astronomical longitudes* Λ:

$$v_{\Lambda_S} = d\Lambda_S + (\Lambda_S^{(0)} - \Lambda_S^*), \text{ the weight } p_\Lambda, \tag{17.45}$$

where Λ_S^* represents the value of the astronomical longitude obtained from geodetic-astronomy determinations.

Remarks:

(1) As regards the approximate values $\Phi_S^{(0)}$ and $\Lambda_S^{(0)}$, these may be taken as equal to the geodetic latitude, respectively longitude (i.e. $\Phi_S^{(0)} = B_S^{(0)}$; $\Lambda_S^{(0)} = L_S^{(0)}$) or equal to the value of the astronomical latitude, respectively longitude, obtained from geodetic-astronomy determinations (i.e. $\Phi_S^{(0)} = \Phi_S^*$; $\Lambda_S^{(0)} = \Lambda_S^*$) or entirely different from them.

(2) If level differences (ΔH^*) were also determined by precision levelling, then, in addition to the equations (17.40)—(17.45), there also appear correction equations, of the form (*Wolf 1975*):

$$v_{\Delta H_{SV}} = G_1 dX_V + G_2 dX_S + G_3 dY_V + G_4 dY_S + G_5 dZ_V + G_6 dZ_S + \\ + \{\Delta H_{SV}^{(0)} - (\Delta H_{SV}^* + \Delta\zeta_{SV})\}, \text{ the weight } p_{\Delta H}, \tag{17.45'}$$

where:

$G_1 = \cos B_V \cos L_V$; $\quad G_2 = -\cos B_S \cos L_S$;

$G_3 = \cos B_V \sin L_V$; $\quad G_4 = -\cos B_S \sin L_S$;

$G_5 = \sin B_V$; $\quad G_6 = -\sin B_S$;

$\Delta\zeta_{SV}$ — the geoid-undulation difference (in the case of utilizing orthometric altitudes), respectively that of the quasi-geoid-undulations (in the case of making use of normal altitudes).

The correction equations in the ellipsoidal global system. For case II one obtains the following basic equations *(Wolf* 1963*c):*

(a) *For the horizontal directions d* (17.27):

$$v_{d_{SV}} = -\,\mathrm{d}o_{SV} + a_1\,\mathrm{d}\bar{B}_S + a_2\,\mathrm{d}\bar{L}_S + a_3\,\mathrm{d}H_S^E + a_4\,\mathrm{d}\bar{B}_V + a_5\,\mathrm{d}\bar{L}_V + \\ + a_6\,\mathrm{d}H_V^E + a_7\,\mathrm{d}\Phi_S + a_8\,\mathrm{d}\Lambda_S + (\alpha_{SV}^{(0)} - (d_{SV}^{*} + o_{SV}^{(0)})), \text{ the weight } p_d, \tag{17.46}$$

where (17.28) ... (17.30):

$$a_1 = +\,\rho''\,\frac{\sin\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}}\,; \quad a_2 = -\,\rho''\,\frac{\cos\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}}\,; \quad a_3 = 0;$$

$$a_4 = -\,\rho''\,\frac{\sin\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}}\,\bar{Q}_{SV}; \quad a_5 = +\,\rho''\,\frac{\cos\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}}\,\bar{P}_{SV};$$

$$a_6 = +\,\rho''\,\frac{\cos\alpha_{SV}^{(0)}\cos B_V^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}}\,\bar{R}_{SV}; \quad a_7 = +\sin\alpha_{SV}^{(0)}\cot\beta_{SV}'^{(0)};$$

$$a_8 = \sin\Phi_S^{(0)} - \cos\alpha_{SV}^{(0)}\cos\Phi_S^{(0)}\cot\beta_{SV}'^{(0)},$$

in which:

$$\bar{Q}_{SV} = \cos(B_V^{(0)} - B_S^{(0)}) + \sin B_V^{(0)}\sin(L_V^{(0)} - L_S^{(0)})\cot\alpha_{SV}^{(0)};$$

$$\bar{P}_{SV} = \cos(L_V^{(0(} - L_S^{(0)}) - \sin B_S^{(0)}\sin(L_V^{(0)} - L_S^{(0)})\tan\alpha_{SV}^{(0)};$$

$$\bar{R}_{SV} = \sin(L_V^{(0)} - L_S^{(0)}) + \{\sin B_S^{(0)}\cos(L_V^{(0)} - L_S^{(0)}) - \tan B_V^{(0)}\cos B_S^{(0)}\}\tan\alpha_{SV}^{(0)}.$$

(b) *For the zenithal angles* β' (17.33):

$$v_{\beta'_{SV}} = -\,D_{SV}^{(0)}\,\mathrm{d}k_{SV} + f_1\,\mathrm{d}\bar{B}_S + f_2\,\mathrm{d}\bar{L}_S + f_3\,\mathrm{d}H_S^E + f_4\,\mathrm{d}\bar{B}_V + f_5\,\mathrm{d}\bar{L}_V + \\ + f_6\,\mathrm{d}H_V^E + f_7\,\mathrm{d}\Phi_S + f_8\,\mathrm{d}\Lambda_S + \{\beta_{SV}'^{(0)} - (\beta_{SV}'^{*} + D_{SV}^{(0)}k_{SV}^{(0)})\}, \text{the weight } p_{\beta'}, \tag{17.47}$$

where (17.34) ... (17.38)):

$$f_1 = -\,\rho''\,\frac{\cos\beta_{SV}'^{(0)}\cos\alpha_{SV}^{(0)}}{D_{SV}^{(0)}}\,; \quad f_2 = -\,\rho''\,\frac{\cos\beta_{SV}'^{(0)}\sin\alpha_{SV}^{(0)}}{D_{SV}^{(0)}}\,; \quad f_3 = +\,\rho''\,\frac{\sin\beta_{SV}'^{0}}{D_{SV}^{(0)}};$$

$$f_4 = +\,\rho''\,\frac{\cos\beta_{SV}'^{0}\sin B_V^{(0)}\cos(L_V^{(0)} - L_S^{(0)}) - \sin B_S^{(0)}\cos B_V^{(0)} - \cos\beta_{SV}'^{(0)}\sin B_V^{(0)}\cos\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}};$$

$$f_5 = +\rho'' \frac{\cos\beta_{SV}'^{(0)}\sin(L_V^{(0)} - L_S^{(0)}) - \cos\beta_{SV}'^{(0)}\sin B_V^{(0)}\sin\alpha_{SV}^{(0)}}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}};$$

$$f_6 = -\rho'' \frac{\cos\beta_{SV}'^{(0)}\cos\beta_{VS}'^{(0)} + \sin B_V^{(0)}\sin B_S^{(0)} + \cos B_S^{(0)}\cos B_V^{(0)}\cos(L_V^{(0)} - L_S^{(0)})}{D_{SV}^{(0)}\sin\beta_{SV}'^{(0)}};$$

$f_7 = -\cos\alpha_{SV}^{(0)}$; $f_8 = -\cos\Phi_S^{(0)}\sin\alpha_{SV}^{(0)}$.

(c) *For the inclined distances D* (17.31):

$$v_{D_{SV}} = e_1\,\mathrm{d}\bar{B}_S + e_2\,\mathrm{d}\bar{L}_S + e_3\,\mathrm{d}H_S^E + e_4\,\mathrm{d}\bar{B}_V + e_5\,\mathrm{d}\bar{L}_V + {} \\ + e_6\,\mathrm{d}H_V^E + (D_{SV}^{(0)} - D_{SV}^{*}),\ \text{the weight } p_D, \tag{17.48}$$

where (17.32):

$$e_1 = -\sin\beta_{SV}'^{(0)}\cos\alpha_{SV}^{(0)};\ e_2 = -\sin\beta_{SV}'^{(0)}\sin\alpha_{SV}^{(0)};$$

$$e_3 = -\cos\beta_{SV}'^{(0)};\ e_4 = -\sin\beta_{VS}'^{(0)}\cos\alpha_{VS}^{(0)};$$

$$e_5 = -\sin\beta_{VS}'^{(0)}\sin\alpha_{VS}^{(0)};\ e_6 = -\cos\beta_{VS}'^{(0)}.$$

(d) *For the astronomical azimuths* α:

$$v_{\alpha_{SV}} = a_1\,\mathrm{d}\bar{B}_S + a_2\,\mathrm{d}\bar{L}_S + a_3\,\mathrm{d}H_S^E + a_4\,\mathrm{d}\bar{B}_V + a_5\,\mathrm{d}\bar{L}_V + a_6\,\mathrm{d}H_V^E + {} \\ + a_7\,\mathrm{d}\Phi_S + a_8\,\mathrm{d}\Lambda_S + (\alpha_{SV}^{(0)} - \alpha_{SV}^{*}),\ \text{the weight } p_\alpha, \tag{17.49}$$

(e) *For the astronomical latitudes* Φ, as in case I — (17.44).

(f) *For the astronomical longitudes* Λ, as in case II — (17.45).

Remark. In (17.46) and (17.49) one can operate a term restriction viz. writing: $a_8 = \sin\Phi_S^{(0)} + \bar{a}_8$ (where $\bar{a}_8 = -\cos\alpha_{SV}^{(0)}\cos\Phi_S^{(0)}\cot\beta_{SV}'^{(0)}$) and $\mathrm{d}\bar{o}_{SV} = \mathrm{d}o_{SV} - \sin\Phi_S^{(0)}\,\mathrm{d}\Lambda_S$, e.g. from (17.46) one finds:

$$v_{d_{SV}} = -\mathrm{d}\bar{o}_{SV} + a_1\,\mathrm{d}\bar{B}_S + a_2\,\mathrm{d}\bar{L}_S + a_3\,\mathrm{d}H_S^E + a_4\,\mathrm{d}\bar{B}_V + a_5\,\mathrm{d}\bar{L}_V + {} \\ + a_6\,\mathrm{d}H_V^E + a_7\,\mathrm{d}\Phi_S + \bar{a}_8\,\mathrm{d}\Lambda_S + \{\alpha_{SV}^{(0)} - (\mathrm{d}_{SV}^{*} + o_{SV}^{(0)})\},$$

and from (17.49) the same equation.

17.3.2 Grafarend's Solution

With a view to establishing the basic equations of Three-Dimensional Geodesy, *E. Grafarend* starts from the fact that any geodetic point is defined, within the framework, of the coordinate system proposed by *A. Marussi*[1]

[1] *E. Grafarend* designates it as a *holonomic geodetic system of the* 1*st kind,* because its coordinates are observable. The notion of holonomy from differential geometry is synonymous with the notion of integrability.

(1951), by the astronomical latitude Φ and longitude Λ, as well as the geopotential W. An intuitive approximation for these physical coordinates is given by the geodetic latitude B and longitude L, respectively the normal potential U, which refer to a level ellipsoid with rotational symmetry of the type *P. Pizetti* and *C. Somigliana* (Section 3.3). These values can be *connected with the system* Φ, Λ, W with the help of only a few corrections δB, δL and T, according to the relations *(Grafarend 1975):*

$\Phi = B + \delta B$, where $\delta B = \xi$, the meridian component of the deflection of the vertical;

$\Lambda = L + \delta L$, where $\delta L = \eta \sec \Phi$;

$$W = U + T, \text{ where } T = \text{the perturbing potential.} \tag{17.50}$$

Remark. By means of the set of three coordinates B, L, U one cannot find the "true" coordinates of the points on the Earth's surface or on a surface external to it.

We now examine the possibility of connecting B, L, U with Φ, Λ, W. To this end, one first of all expands the geodetic coordinates of a point P round the point Q in *Taylor* series:

$$\Phi_P = B_P + \delta B_P = B_Q + \left(\frac{\partial B}{\partial x_i}\right)_Q \Delta x_i + o_1(\Delta x_j \, \Delta x_k) + \delta B_P; \tag{17.51}$$

$$\Lambda_P = L_P + \delta L_P = L_Q + \left(\frac{\partial L}{\partial x_i}\right)_Q \Delta x_i + o_2(\Delta x_j \, \Delta x_k) + \delta L_P; \tag{17.52}$$

$$W_P = U_P + T_P = U_Q + \left(\frac{\partial U}{\partial x_i}\right)_Q \Delta x_i + o_3(\Delta x_j \, \Delta x_k) + T_P,$$

where $i = 1, 2, 3$; x_1, x_2, x_3 represent the set of three coordinates in a general notation; o_1, o_2, o_3 characterize the truncation of the series expansion up to the indicated order and Δx_i — the difference of coordinates between P and Q, e.g.:

$$\Delta x_1 = X_P - X_Q; \; \Delta x_2 = Y_P - Y_Q; \; \Delta x_3 = Z_P - Z_Q.$$

Remark. The gradients $\partial/\partial x_i$ may be obtained in any coordinate system.

One then defines the zero differences by:

$$\Delta\Phi = \Phi_P - B_Q = 0; \; \delta B_P = -\sum_{i=1}^{3} \left(\frac{\partial B}{\partial x_i}\right)_Q \Delta x_i; \tag{17.53}$$

$$\Delta\Lambda = \Lambda_P - L_Q = 0; \; \delta L_P = -\sum_{i=1}^{3} \left(\frac{\partial L}{\partial x_i}\right)_Q \Delta x_i; \tag{17.54}$$

$$\Delta W = W_P - U_Q = 0; \; T_P = -\sum_{i=1}^{3} \left(\frac{\partial U}{\partial x_i}\right)_Q \Delta x_i, \tag{17.55}$$

or, in matricial form, the above relations (17.53) ... (17.55) become:

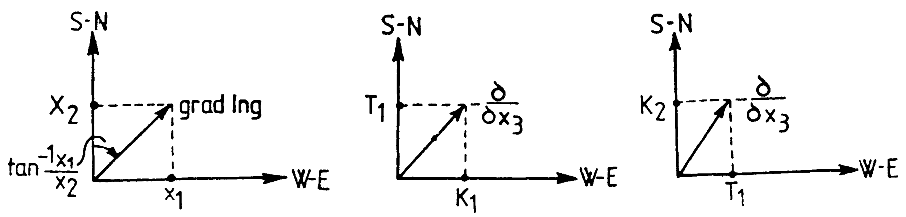
$$\begin{Vmatrix} \delta B_P \\ \\ \partial L_P \\ \\ T_P \end{Vmatrix} = \begin{Vmatrix} -\dfrac{\partial B}{\partial x_1} & -\dfrac{\partial B}{\partial x_2} & -\dfrac{\partial B}{\partial x_3} \\ \\ -\dfrac{\delta L}{\partial x_1} & -\dfrac{\partial L}{\partial x_2} & -\dfrac{\partial L}{\partial x_3} \\ \\ -\dfrac{\partial U}{\partial x_1} & -\dfrac{\partial U}{\partial x_2} & -\dfrac{\partial U}{\partial x_3} \end{Vmatrix} \cdot \begin{Vmatrix} \Delta x_1 \\ \\ \Delta x_2 \\ \\ \Delta x_3 \end{Vmatrix}. \tag{17.56}$$

The relation (17.56) represents a *three-dimensional version* of *Bruns'* equation (4.5), the totality of the points Q[1] defined by (17.53) ... (17.55) is called the *telluroid* and the set $\Phi_P = B_Q$; $\Lambda_P = L_Q$; $W_P = U_Q$ — the *extended isozenithal equation;* the direction of the $\vec{g}_P$ vector of the actual gravity at the point P coincides with that of the $\vec{\gamma}_Q$ vector of the normal gravity at the point Q.

One can write now the general model:

$$\begin{aligned} &K_1 = k_1 + \delta k_1;\ K_2 = k_2 + \delta k_2;\ T_1 = t_1 + \delta t_1; \\ &\varkappa_1 = \varkappa_1^0 + \delta\varkappa_1;\ \varkappa_2 = \varkappa_2^0 + \delta\varkappa_2;\ \Phi = B + \delta B; \\ &\Lambda = L + \delta L;\ W = U + T;\ g = \gamma + \delta g. \end{aligned} \tag{17.57}$$

where K_1 and K_2 are the normal curvatures, in the East and North directions, respectively of the equipotential surface W, which passes through the observation point; T_1 — the geodetic torsion of the equipotential surface in the East direction and χ_1 and χ_2 represent the projections of the gradient grad $\ln g$ (Fig. 17.8, *Grafarend 1975*).



17.8. The K_1, K_2, T_1, $\varkappa_1$, $\varkappa_2$ Projections:
S-N — the South-North direction; *W-E* — the West-East direction;
grad ln $g = g^{-1}$ grad g

In the relations (17.57), due to (17.53) up to (17.55), the curvatures and the geodetic torsion within the framework of the normal of reference field are functions of the geodetic latitude and longitude and of the normal potential of the telluroid surface, e.g.:

$$k_1(\Phi_P, \Lambda_P, W_P) = k_1(B_Q, L_Q, U_Q) \text{ etc.}$$

[1] To every point P on the physical surface of the Earth corresponds another point Q where the normal potential U_Q equals that at the point P.

One now introduces the rotation ellipsoid as level ellipsoid. The geodetic latitudes and longitudes are designated as ellipsoidal latitudes and longitudes. The reference potential is characterized by means of the standard potential of *P. Pizetti* and *C. Somigliana*. The connexion between the point Q on the telluroid and the ellipsoidal base surface can now be achieved — as one usually proceeds — by projecting Q onto the base surface along the corresponding normal. *E. Grafarend* prefers, however, an isozenithal projection, which has also been used by *A. Marussi* (*1950 a* and *b*), viz.: one projects Q onto the reference ellipsoid assuming that $B = \text{const}$, $L = \text{const}$. The point Q and its projection Q' coincide as ellipsoidal latitude and longitude but differ as normal potential. According to *A. Marussi* (*1950 a*), the curvature and the geodetic torsion of a set of points Q', for the base level ellipsoid, may be calculated with the aid of the following relations:

$$\frac{1}{k_1} = -\frac{a}{\sqrt{1 - e^2 \sin^2 B}};$$

$$\frac{1}{k_2} = -\frac{a(1-e^2)}{\sqrt{(1 - e^2 \sin^2 B)^3}}; \; t_1 = \varkappa_1^0 = 0;$$

$$\varkappa_2^0 = -\frac{(1 - e^2 \sin^2 B)\left(\left(a_1 - \frac{a_2^2}{2}\right)\sin 2B + 2\left(\frac{a_1^2}{8} - a_2\right)\sin 4B\right)}{a(1-e^2)}; \tag{17.58}$$

$$\gamma = \gamma_0(1 + a_1 \sin^2 B - a_2 \sin^2 2B),$$

in which a_1, a_2 are constants, a — the semi major-axis of the rotation ellipsoid, e — the first eccentricity, γ_0 — the normal gravity at the equator and B — the ellipsoidal latitude.

Remarks:

(1) The formulae (17.58) do not depend on the ellipsoidal longitude L.

(2) In the relations (17.57) of the general model there will remain the corrections δk_1 and δk_2 of the ellipsoidal curvature, δt_1 — the correction of ellipsoidal geodetic torsion, $\delta\chi_1$, $\delta\chi_2$ — the corrections of the plumb-line direction, δB, δL — the corrections of the ellipsoidal latitude and respectively longitude, the perturbing potential T and the correction δg for normal gravity ("standard" *Grafarend 1975*), being disregarded the influence of the small angle between the vectors $\vec{\mathbf{g}}$ and $\vec{\boldsymbol{\gamma}}$.

Inasmuch as the points of interest are the curvature and the geodetic torsion on the telluroid — characterized by the set of points Q — and not on the ellipsoid — characterized by the set of points Q' — it is necessary for the integration to be carried out from Q' to Q along the chosen isozenithal, e.g.:

$$k_1(B_Q, L_Q, U_Q) = k_1(B_{Q'}, L_{Q'}, U_{Q'}) + \int_{U_{Q'}}^{U_Q} \frac{\partial k_1}{\partial U} \mathrm{d}U.$$

Remark. By definition, there is no dependence between B and L (*Grafarend 1975*).

NOTES

(1) *For transforming the coordinates from the Cartesian astronomical local system* $\bar{x}, \bar{y}, \bar{z}$ *into Marussi's geodetic system,* Φ, Λ, W, *E. Grafarend* (1975, *p.* 276 (4.1)) has discovered the following relation:

$$\left\| \begin{matrix} \mathrm{d}\Lambda \\ \mathrm{d}\Phi \\ \mathrm{d}W \end{matrix} \right\| = \mathbf{F}_1 (K_1, K_2, T_1 \varkappa_1 \varkappa_2, g, \Phi) \left\| \begin{matrix} \mathrm{d}\bar{x} \\ \mathrm{d}\bar{y} \\ \mathrm{d}\bar{z} \end{matrix} \right\|, \tag{17.59}$$

in which:

$$\mathrm{d}\Lambda = \Lambda_V - \Lambda_S;\ \mathrm{d}\Phi = \Phi_V - \Phi_S;\ \mathrm{d}W = W_V - W_S;$$

$$\mathrm{d}\bar{x} = \bar{x}_V - \bar{x}_S;\ \mathrm{d}\bar{y} = \bar{y}_V - \bar{y}_S$$

(V — sighted geodetic point, S — station geodetic point) and $\mathbf{F}_1$ represents the transformation matrix ("*Frobenius* matrix of the 1st kind or integrating-factor matrix", *Grafarend 1975*), which is a function of the curvatures K_1, K_2, T_1, $\varkappa_1$, $\varkappa_2$, the absolute value of gravity and the astronomical latitude, which is of the form:

$$\mathbf{F}_1 = \left\| \begin{matrix} -K_1 \sec\Phi & -T_1 \sec\Phi & \varkappa_1 \sec\Phi \\ -T_1 & -K_2 & \varkappa_2 \\ 0 & 0 & -g \end{matrix} \right\| \tag{17.60}$$

(2) *For transforming the coordinates from the Cartesian astronomical local system* $\bar{x}, \bar{y}, \bar{z}$ *into the Cartesian global system* X, Y, Z, *E. Grafarend* (*1975, p. 276* (4.2)) obtains the following relation:

$$\left\| \begin{matrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{matrix} \right\| = \mathbf{F}_2 (\Phi, \Lambda) \left\| \begin{matrix} \mathrm{d}\bar{x} \\ \mathrm{d}\bar{y} \\ \mathrm{d}\bar{z} \end{matrix} \right\|, \tag{17.61}$$

in which $\mathrm{d}X = X_V - X_S$, $\mathrm{d}Y_V = Y_V - Y_S$, $\mathrm{d}Z = Z_V - Z_S$ and $\mathbf{F}_2$ represents the transformation matrix ("*Frobenius* matrix of the 2nd kind", *Grafarend 1975*), which is a function only of Φ and Λ, which is of the same form as that of the matrix $\mathbf{R}_1$ — relation (17.9).

(1) *For directly*[1] *transforming the coordinates from Marussi's geodetic system* Φ, Λ, W *into the equatorial Cartesian global system* X, Y, Z, one finds from (17.60) and (17.61), using a mere substitution:

$$\left\| \begin{matrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{matrix} \right\| = \mathbf{F}_2 \mathbf{F}_1^{-1} \left\| \begin{matrix} \mathrm{d}\Lambda \\ \mathrm{d}\Phi \\ \mathrm{d}W \end{matrix} \right\|, \tag{17.62}$$

in which:

$$\mathbf{F}_2\mathbf{F}_1^{-1} = \left\| \begin{matrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{matrix} \right\|,$$

[1] Very important for geodetic applications.

where:

$$F_{11} = \frac{K_1}{K} \sin \Lambda \cos \Phi - \frac{T_1}{K} \cos \Lambda \sin \Phi \cos \Phi:$$

$$F_{12} = - \frac{T_1}{K} \sin \Lambda + \frac{K_1}{K} \cos \Lambda \sin \Phi;$$

$$F_{13} = \frac{K_2 \varkappa_1 - T_1 \varkappa_2}{Kg} \sin \Lambda + \frac{K_1 \varkappa_2 - T_1 \varkappa_1}{Kg} \cos \Lambda \sin \Phi - \frac{1}{g} \cos \Lambda \cos \Phi;$$

$$F_{21} = - \frac{K_2}{K} \cos \Lambda \cos \Phi - \frac{T_1}{K} \sin \Lambda \sin \Phi \cos \Phi;$$

$$F_{22} = \frac{T_1}{K} \cos \Lambda + \frac{K_1}{K} \sin \Lambda \sin \Phi; \tag{17.63}$$

$$F_{23} = \frac{T_1 \varkappa_2 - K_2 \varkappa_1}{Kg} \cos \Lambda + \frac{K_1 \varkappa_2 - T_1 \varkappa_2}{Kg} \sin \Lambda \sin \Phi - \frac{1}{g} \sin \Lambda \cos \Phi;$$

$$F_{31} = \frac{T_1}{K} \cos^2 \Phi; F_{32} = - \frac{K_1}{K} \cos \Phi;$$

$$F_{33} = \frac{T_1 \varkappa_1 - K_1 \varkappa_2}{Kg} \cos \Phi - \frac{1}{g} \sin \Phi,$$

and:

$$K = K_1 K_2 - T_1^2.$$

The relation (17.62) has a quite general character. In order to illustrate it, we shall consider the following two examples *(Grafarend* 1975*)*:

1st Example. If one assumes a constant spherical curvature given by the radius of curvature R ($K_1 = K_2 = -1/R$), the geodetic torsion — zero ($T_1 = 0$) ,the physical vertical — "rectilinear" ($\varkappa_1 = \varkappa_2 = 0$) and $dW = -g\,dR$, then from (17.62) and (17.63) we have:

$$dX = -R \sin \Lambda \cos \Phi \, d\Lambda - R \cos \Lambda \sin \Phi \, d\Phi - \frac{1}{g} \cos \Lambda \cos \Phi \, dW;$$

$$dY = R \cos \Lambda \cos \Phi \, d\Lambda - R \sin \Lambda \sin \Phi \, d\Phi - \frac{1}{g} \sin \Lambda \cos \Phi \, dW;$$

$$dZ = R \cos \Phi \, d\Phi - \frac{1}{g} \sin \Phi \, dW,$$

whence, by integration, one gets the formulae:

$$X = R \cos \Lambda \cos \Phi + \text{const};$$

$$Y = R \sin \Lambda \cos \Phi + \text{const};$$

$$Z = R \sin \Phi + \text{const}.$$

which are well-known from spherical geometry.

2nd Example. If one assumes a curvature field with rotational symmetry, i. *e*. an equipotential rotational ellipsoid of semi-major axis a and first eccentricity e, then:

$$T_1 = \varkappa_1 = \varkappa_2 = 0 \text{ (by definition)}; \; g = \gamma;$$

$$\frac{1}{K_1} = -\frac{a}{\sqrt{1-e^2 \sin^2 \Phi}}; \quad \frac{1}{K_2} = -\frac{a(1-e^2)}{\sqrt{(1-e^2 \sin^2 \Phi)^3}}.$$

Taking account of these considerations in (17.62) and (17.63) and integrating, we finally obtain the coordinates:

$$X = \left(\frac{a}{\sqrt{1-e^2 \sin^2 \Phi}} + H^\gamma \right) \cos \Lambda \cos \Phi + \text{const};$$

$$Y = \left(\frac{a}{\sqrt{1-e^2 \sin^2 \Phi}} + H^\gamma \right) \sin \Phi \cos \Lambda + \text{const};$$

$$Z = \left(\frac{a(1-e^2)}{\sqrt{1-e^2 \sin^2 \Phi}} + H^\gamma \right) \sin \Phi + \text{const}.$$

which are well-known from ellipsoidal geodesy (Second Part of this book).

Everything is now on hand to proceed with integrating the relation (17.62). To this end, *E. Grafarend* chooses the following integration path (Fig. 17.9):

from Λ_S to Λ_V: Φ_S, W_S fixed;

from Φ_S to Φ_V: Λ_V, W_S fixed;

from W_S to W_V: Λ_V, Φ_V fixed,

between two points P_S and P_V.

If one adopts the limitation to the lst-order approximation, the partial derivatives:

$$\frac{\partial X}{\partial \Lambda} = \frac{k_2}{k} \sin \Lambda \cos \Phi - \frac{k_2^2}{k^2} \sin \Lambda \cos \Phi \; \delta k_1 - \frac{1}{k} \cos \Lambda \sin \Phi \; \delta t_1;$$

$$\frac{\partial X}{\partial \Phi} = \frac{k_1}{k} \cos \Lambda \sin \Phi - \frac{k_1^2}{k^2} \cos \Lambda \sin \Phi \; \delta k_2 - \frac{1}{k} \sin \Lambda \; \delta t_1;$$

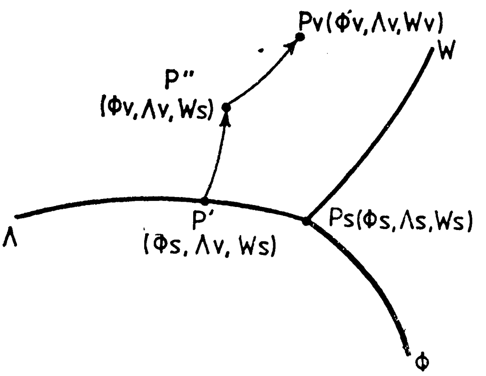


Fig. 17.9. The Integration Path

$$\frac{\partial X}{\partial W} = \frac{\varkappa_2^0}{k_2\gamma}\cos\Lambda\sin\Phi - \frac{1}{\gamma}\cos\Lambda\cos\Phi - \frac{\varkappa_2^0}{k_2^2\gamma}\cos\Lambda\sin\Phi\,\delta k_2 -$$

$$-\frac{\varkappa_2^0}{k\gamma}\sin\Lambda\,\delta t_1 + \frac{1}{k_1\gamma}\sin\Lambda\,\delta\varkappa_1 + \frac{1}{k_2\gamma}\cos\Lambda\sin\Phi\,\delta\varkappa_2 -$$

$$-\left(\frac{\varkappa_2^0}{k_2\gamma^2}\cos\Lambda\sin\Phi - \frac{1}{\gamma^2}\cos\Lambda\cos\Phi\right)\delta g;$$

$$\frac{\partial Y}{\partial\Lambda} = -\frac{1}{k_1}\cos\Lambda\cos\Phi + \frac{1}{k_1^2}\cos\Lambda\cos\Phi\,\delta k_1 - \frac{1}{k}\sin\Lambda\sin\Phi\cos\Phi\,\delta t_1;$$

$$\frac{\partial Y}{\partial\Phi} = \frac{1}{k_2}\sin\Lambda\sin\Phi - \frac{1}{k_2^2}\sin\Lambda\sin\Phi\,\delta k_2 + \frac{1}{k}\cos\Lambda\,\delta t_1;$$

$$\frac{\partial Y}{\partial W} = \frac{\varkappa_2^0}{k_2\gamma}\sin\Lambda\sin\Phi - \frac{1}{\gamma}\sin\Lambda\cos\Phi - \frac{\varkappa_2^0}{k_2\gamma}\sin\Lambda\sin\Phi\,\delta k_2 +$$

$$+\frac{\varkappa_2^0}{k\gamma}\cos\Lambda\,\delta t_1 - \frac{1}{k_1\gamma}\cos\Lambda\,\delta\varkappa_1 + \frac{1}{k_2\gamma}\sin\Lambda\sin\Phi\,\delta\varkappa_2 -$$

$$-\left(\frac{\varkappa_2^0}{k_2\gamma^2}\sin\Lambda\sin\Phi - \frac{1}{\gamma^2}\sin\Lambda\cos\Phi\right)\delta g;$$

$$\frac{\partial Z}{\partial\Lambda} = \frac{1}{k}\cos^2\Phi\,\delta t_1;$$

$$\frac{\partial Z}{\partial\Phi} = -\frac{1}{k_2}\cos\Phi + \frac{1}{k_2^2}\cos\Phi\,\delta k_2;$$

$$\frac{\partial Z}{\partial W} = -\frac{\varkappa_2^0}{k_2\gamma}\cos\Phi - \frac{1}{\gamma}\sin\Phi + \frac{\varkappa_2^0}{k_2}\cos\Phi\,\delta k_2 - \tag{17.64}$$

$$-\frac{1}{k_2\gamma}\cos\Phi\,\delta\varkappa_2 + \left(\frac{\varkappa_2^0}{k_2\gamma^2}\cos\Phi + \frac{1}{\gamma^2}\sin\Phi\right)\delta g,$$

are functions of Φ, Λ and W.

Remark. By means of the relations from (17.53) to (17.55) the partial derivatives may also be expressed as functions of B, L and U.

Integrating the relations:

$$X_V - X_S = \int\limits_{X_S}^{X_V} \mathrm{d}X = \int\limits_{\Lambda_S}^{\Lambda_V} \frac{\partial X}{\partial \Lambda} (\Phi_S, \Lambda, W_S)\, \mathrm{d}\Lambda + \int\limits_{\Phi_S}^{\Phi_V} \frac{\partial X}{\partial \Phi} (\Phi, \Lambda_S, W_S)\, \mathrm{d}\Phi +$$

$$+ \int\limits_{W_S}^{W_V} \frac{\partial X}{\partial W} (\Phi_V, \Lambda_V, W)\, \mathrm{d}W;$$

$$Y_V - Y_S = \int\limits_{Y_S}^{Y_V} \mathrm{d}Y = \int\limits_{\Lambda_S}^{\Lambda_V} \frac{\partial Y}{\partial \Lambda} (\Phi_S, \Lambda, W_S)\, \mathrm{d}\Lambda + \int\limits_{\Phi_S}^{\Phi_V} \frac{\partial Y}{\partial \Phi} (\Phi, \Lambda_V, W_S)\, \mathrm{d}\Phi +$$

$$+ \int\limits_{W_S}^{W_V} \frac{\partial Y}{\partial W} (\Phi_V, \Lambda_S, W)\, \mathrm{d}W; \tag{17.65}$$

$$Z_V - Z_S = \int\limits_{Z_S}^{Z_V} \mathrm{d}Z = \int\limits_{\Lambda_S}^{\Lambda_V} \frac{\partial Z}{\partial \Lambda} (\Phi_S, \Lambda, W_S)\, \mathrm{d}\Lambda + \int\limits_{\Phi_S}^{\Phi_V} \frac{\partial Z}{\partial \Phi} (\Phi, \Lambda_V, W_S)\, \mathrm{d}\Phi +$$

$$+ \int\limits_{W_S}^{W_V} \frac{\partial Z}{\partial W} (\Phi_V, \Lambda_V, W)\, \mathrm{d}W,$$

finally yields:

$$X_V - X_S = \cos (L_V + \delta L_V) \cos (B_V + \delta B_V) \left(\frac{a}{\sqrt{1 - e^2 \sin^2 (B_V + \delta B_V)}} - \right.$$

$$- \int\limits_{US}^{U_V} \frac{1}{\gamma}\left(1 - \frac{\delta g}{\gamma}\right) \mathrm{d}U - \int\limits_{T_S}^{T_V} \frac{1}{\gamma}\, \mathrm{d}T -$$

$$\left. - \frac{a(1 - e^2) \tan (B_V + \delta B_V)}{\sqrt{(1 - e^2 \sin^2 \Phi_V)^3}} \int\limits_{W_S}^{W_V} \varkappa_2^0 \left(\frac{1}{\gamma} - \frac{\delta g}{\gamma^2}\right) \mathrm{d}W \right) -$$

$$- \frac{a}{\sqrt{1 - e^2 \sin (B_S + \delta B_S)}} \cos (L_S + \delta L_S) \cos (B_S + \delta B_S) +$$

$$+ o_1 \left(\delta k_1, \delta k_2, \delta t_1, \delta \varkappa_1, \delta \varkappa_2, \frac{\partial k_1}{\partial U}, \frac{\partial k_2}{\partial U}, \frac{\partial \varkappa_2}{\partial U}, \frac{\partial \gamma}{\partial U} \right);$$

$$Y_V - Y_S = \sin(L_V + \delta L_V)\cos(B_V + \delta B_V)\left(\frac{a}{\sqrt{1 - e^2\sin^2(B_V + \delta B_V)}} -\right.$$

$$- \int_{U_S}^{U_V} \frac{1}{\gamma}\left(1 - \frac{\delta g}{\gamma}\right)\mathrm{d}U - \int_{T_S}^{T_V} \frac{1}{\gamma}\,\mathrm{d}T - \frac{a(1 - e^2)\tan(B_V + \delta B_V)}{\sqrt{(1 - e^2\sin^2\Phi_V)^3}} \int_{W_S}^{W_V} \varkappa_2^0\left(\frac{1}{\gamma} -\right.$$

$$\left.\left. - \frac{\delta g}{\gamma^2}\right)\mathrm{d}W\right) - \frac{a}{\sqrt{1 - e^2\sin^2(B_S + \delta B_S)}}\sin(L_S + \delta L_S)\cos(B_S +$$

$$+ \delta B_S) + o_2\left(\delta k_1, \delta k_2, \delta t_1, \delta\varkappa_1, \delta\varkappa_2, \frac{\partial k_1}{\partial U}, \frac{\partial k_2}{\partial U}, \frac{\partial \varkappa_2}{\partial U}, \frac{\partial \gamma}{\partial U}\right);$$

$$Z_V - Z_S = \sin(B_V + \delta B_V)\left(\frac{a}{\sqrt{1 - e^2\sin^2(B_V + \delta B_V)}} - \int_{U_V}^{U_S} \frac{1}{\gamma}\left(1 -\right.\right.$$

$$\left. - \frac{\delta g}{\gamma}\right)\mathrm{d}U - \int_{T_S}^{T_V} \frac{1}{\gamma}\,\mathrm{d}T + \frac{a(1 - e^2)\tan(B_V + \delta B_V)}{\sqrt{(1 - e^2\sin^2\Phi_V)^3}} \int_{W_S}^{W_V} \varkappa_2^0\left(\frac{1}{\gamma} -\right.$$

$$\left.\left. - \frac{\delta g}{\gamma^2}\right)\mathrm{d}W\right) + o_3\left(\delta k_1, \delta k_2, \partial t_1, \delta\varkappa_1, \delta\varkappa_2, \frac{\partial k_1}{\partial U}, \frac{\partial \varkappa_2}{\partial U}, \frac{\partial \gamma}{\partial U}\right). \qquad (17.66)$$

Remarks:

(1) In (17.66) the higher-order terms o_1, o_2, o_3, are given in implicit form, because the explicit form is very laborious.

(2) Although o_1, o_2 and o_3 appear in implicit form, the formulae (17.66) allow, nevertheless, one to study the influence of the deflection of the vertical, of the gravity anomaly and of the height anomaly (the two latter often being disregarded in the various forms given in the geodetic literature for the formulae of the Cartesian geocentric coordinates.

(3) The formulae (17.66) have a disadvantage, viz. their lack of symmetry as regards the coordinates of the points P_S and P_V, as in fact was mentioned by *T. Krarup* (*1974*) and even by their author *E. Grafarend* (1975).

18
Elementary Calculations in Three-Dimensional Geodesy

18.1 The Calculation Principle in the Local System

In order to further emphasize some characteristic traits of Three-Dimensional Geodesy, imagine the following example *(Levallois* and *Kovalevsky* 1971*)*.

At the geodetic point S (Fig. 18.1), where one knows the astronomical vertical $S\bar{z}$, there were measured the azimuth α_{SV_1} of the direction SV_1 and the azimuthal angle $\hat{S}$ between two directions SV_1 and SV_2 (here V_1 and V_2 are the vertices of a triangle SV_1V_2, whose scale was directly determined by measuring one of its sides—e.g. D_{SV_1}). Further, at every vertex of the triangle concerned, one has determined the zenithal angles, e.g. for the point S: β'_{SV_1} and β'_{SV_2}.

One assumes that the zenithal-angle measurements are error-free, the refraction calculation providing rigorous values for the corresponding zenithal angles.

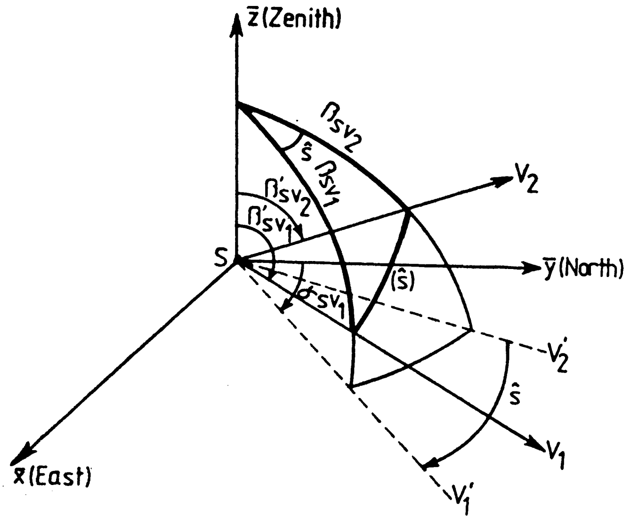
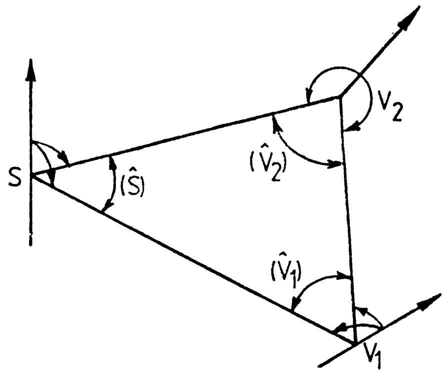


Fig. 18.1. Geodetic Point at which α_{SV_1}, D_{SV_1}, $\hat{S}$, β'_{SV_1} and β'_{SV_2} Were Measured

In order to calculate the elements of the triangle SV_1V_2 and in particular the coordinates of the geodetic points V_1 and V_2 one proceeds as follows: first of all one calculates the angles of the plane triangle (Fig. 18.2) denoted



18.2. The SV_1V_2 Plane Triangle

by $(\hat{S})$, $(\hat{V}_1)$, $(\hat{V}_2)$, having recourse to spherical trigonometry. For instance, from the solid-line triangle in Fig. 18.1, there results for the side $(\hat{S})$, applying the basic formula of spherical trigonometry:

$$(\hat{S}) = \cos \beta'_{SV_2} \cos \beta'_{SV_1} + \sin \beta'_{SV_1} \sin \beta \cos \hat{S}.$$

Afterwards, with the aid of the values $(\hat{S})$, $(\hat{V}_1)$, $(\hat{V}_2)$ and D_{SV_1} (the latter obtained from direct measurements with electromagnetic instruments), one calculates the other two sides of the triangle ($\overline{SV_2}$ and $\overline{V_1V_2}$), using the sine theorem.

Calculating the coordinates of the points V_1 and V_2 is now a polar-coordinate problem, since one knows the azimuths of the sides SV_1 and SV_2 ($\alpha_{SV_2} = \alpha_{SV_1} - \hat{S}$), as well as the corresponding zenithal angles β'_{SV_1}, β'_{SV_2}.

Using the coordinates of the points V_1 and V_2 one then checks whether the corresponding side indeed has the value obtained by the previous direct calculation (using the sine theorem) and whether the angles $(\hat{V}_1)$ and $(\hat{V}_2)$ correspond to the values provided by the scalar product of the corresponding coordinate differences, e.g.:

$$D_{SV_1} D_{V_1V_2} \cos (\hat{V}_1) = (x_{V_2} - x_S)(x_{V_2} - x_{V_1}) +$$
$$+ (y_{V_2} - y_S)(y_{V_2} - y_{V_1}) + (z_{V_2} - z_S)(z_{V_2} - z_{V_1}).$$

Since the zenithal distances (β'_{V_1S}, $\beta'_{V_1V_2}$, β'_{V_2S}, $\beta'_{V_2V_1}$) were measured both at the point V_1 and at V_2, it follows that for the entire triangle SV_1V_2 there exist reciprocal zenithal angles, leading at least from the theoretical point of view, to the realization of the unit vectors of the physical verticals at V_1 and V_2, even if one has not determined astronomical elements: the direction of the physical vertical, e.g. at the point V_2, is the intersection of two revolution cones with the vertex at V_2 and the opening z_{V_2S}, respectively z_{SV_1}.

Consequently, there exist two symmetrical solutions with respect to the SV_1V_2 plane; in the sense of the problem under analysis only one of them is in fact sound.

As a matter of fact, the trihedral is completely determined by its three faces.

Remarks:

(1) One notes the fundamental role played, throughout the above reasoning, by the hypothesis that the refraction is perfectly known. However it was mentioned in Section 17.2 how difficult it is to determine this variable. The assumed hypothesis actually represents, as has already been emphasized, the weak point of the theories of Three-Dimensional Geodesy.

(2) The sides intervene through their values as directly measured, without being reduted to the geoid, which removes all suspicion as to the precise meaning of the calculations being carried out.

(3) If one assumes that the astronomical latitude, longitude and azimuth (or *Laplace* azimuth) have been obtained at the point S, by means of geodetic-astronomy determinations, then between these elements there must be consistency and, in particular, the astronomical azimuth of a side must refer to the astronomical vertical. *Laplace*'s equation does not disappear but it will be automatically satisfied when one knows the definitive position of the local geodetic trihedron at the point concerned.

18.2 Choice of the Local Geodetic Trihedron

As was already mentioned in Chapter. 17, the method of Three-Dimensional Geodesy supposes a local reference system to be defined at every geodetic point. At any geodetic point S one defines the trihedron (S, x, y, z) so that the Sz axis coincides with the vertical, the Syz plane with the meridian plane and the Sx axis is perpendicular to the Syz plane. To this end one may choose among the geodetic elements, the astronomical elements, or others (§ 17.1.1). The ideal would be to choose the astronomical elements (the vertical, the meridian etc.) — i.e. the Cartesian astronomic local system — since this leads to a simplification of the reasoning. However, it is sometimes impossible to do this, because there do not exist, at present, geodetic-astronomy determinations at all geodetic points. Therefore, and not infrequently, it is necessary to make use of a trihedron close to that mentioned before, to which the various observational or computational elements will be referred and which is designated by *H. Dufour* (1962) as the *Laplacian trihedron.* The axes of this trihedron differ very little from the astronomical one.

By definition, the Laplacian trihedron contains in the ySz plane a parallel to the World axis (SP). Consequently, this direction represents a directional *invariant.* The transition from the astronomical to the Laplacian trihedron needs the definition at one of the geodetic points, the values $\Delta\Phi$ and $\Delta\Lambda$ which characterize the direction variation of the z — axis of the *Laplacian trihedron.*

One denotes by ξ and η the variations:

$$(\Phi_2 - \Phi_1) = \xi; \ (\Lambda_2 - \Lambda_1) \cos \Phi = \eta,$$

of the astronomical latitude and longitude, respectively. In a rigorous sense, ξ and η may be the components of the deflection of the vertical or, in an approximate sense, the geodetic-coordinate variations, which for any radius can be assigned to the direction of the chosen close vertical. **The transition from the initial system** (1) **to the system** (2) **involves the following operations** *(Levallois* and *Kovalevsky* 1971*):*

a) A rotation of $\eta \tan \Phi$ (or at least η) round the Sz axis;

b) A rotation of η round the Oy axis;

c) A rotation of ξ round the Ox axis.

Under these conditions, the rotation matrix needed for the transition from the axes of the Laplacian trihedron to the astronomical axes is given by the relation *(Levallois* and *Kovalevskiy* 1971 *):*

$$\mathbf{R} = \underset{(Ox)}{\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\xi \\ 0 & \xi & 1 \end{Vmatrix}} \underset{(Oy)}{\begin{Vmatrix} 1 & 0 & -\eta \\ 0 & 1 & 0 \\ \eta & 0 & 1 \end{Vmatrix}} \underset{(Oz)}{\begin{Vmatrix} 1 & \eta \tan \Phi & 0 \\ -\eta \tan \Phi & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}} = \begin{Vmatrix} 1 & \eta \tan \Phi & -\eta \\ -\eta \tan \Phi & 1 & -\xi \\ \eta & \xi & 1 \end{Vmatrix}. \tag{18.1}$$

Consequently, the relation for transforming the coordinates from the system (1) (x_1, y_1, z_1) to the system (2) (x_2, y_2, z_2) has the form:

$$\begin{Vmatrix} x_2 \\ y_2 \\ z_2 \end{Vmatrix} = \mathbf{R} \begin{Vmatrix} x_1 \\ y_1 \\ z_1 \end{Vmatrix}. \tag{18.2}$$

Remark. The matrix $\mathbf{R}$ in (18.1) only holds for very small angles, of the order of magnitude of the deflections of the vertical; it equally does not hold for differences $\Delta\Phi$ and $\Delta\Lambda$ of the order of tenths of degrees (*Levallois and Kovalevsky 1971*). This result is due to the fact that (18.1) was obtained with the approximations $\sin\xi = \xi$, $\cos\xi = 1$ etc., disregarding the 2nd-order terms in $\xi\eta$, η^2 etc. produced by the product of the three matrices.

Consequently, the knowledge of a Laplacian trihedron at a geodetic point immediately leads to the possibility of passing from the local to the global system, by applying the rotation matrix $\mathbf{R}$ ((18.1) or (17.9) and (18.2)). This $\mathbf{R}$ matrix may be defined a priori by means of the measured elements, e.g. the astronomical ones, or — as happens more frequently — one adopts some approximate, though sufficiently close, values of the astronomical coordinates of the point, whence follows the reassuring availability of an approximating surface, as that of the ellipsoid.

18.3 Calculating the Coordinates of a Point

As has already been mentioned in Section 18.1, the problem of calculating the coordinates of the points V_1 and V_2 (Fig. 18.1) is a problem of polar coordinates, so that they are given by the following formulae (17.1), e.g. for the point V_1:

$$\bar{x}_{V_1} = D_{SV_1} \sin \alpha_{V_1S} \sin \beta'_{V_1S}; \; \bar{y}_{V_1} = D_{SV_1} \cos \alpha_{V_1S} \sin \beta'_{V_1S}; \; \bar{z}_{V_1} = D_{SV_1} \cos \beta'_{V_1S}, \tag{18.3}$$

in which $\beta'_{V_1S} = \beta_{V_1S} - (\xi_{V_1} \cos \alpha_{V_1S} + \eta_{V_1} \sin \alpha_{V_1S})$, where β_{V_1S} is the value of the zenithal angle of the direction V_1S obtained from measurement, ξ_{V_1}, η_{V_1} — the components of the deflection of the vertical at the point V_1 and $\alpha_{V_1S} = \alpha_{SV_1} - \eta \tan \Phi$ — the *Laplace* azimuth of the side SV_1.

The coordinates $\bar{x}_{V_1}$, $\bar{y}_{V_1}$, $\bar{z}_{V_1}$ (calculated by means of the relations (18.3)) being known, one can now calculate the coordinates X_{V_1}, Y_{V_1}, Z_{V_1} of the geodetic point V_1 in the equatorial Cartesian global system, using the transforming formulae (17.7) and (17.9).

18.4 Determining the Local Astronomical Vertical

Let us assume, for the triangle $S\,V_1V_2$ also considered in Section 18.1, that the geodetic point V_1, for exemple, is a new point whose vertical one wishes to determine. This may result either from a direct measurement (the case in which the vertical's direction cosines are known), or by means of the measured zenithal angles (β_{V_1S}, $\beta_{V_1V_2}$) at the points S and V_1 (Fig. 18.3).

One denotes by p the unknown angle (which is to be determined) of the vertical $V_1\bar{z}_{V_1}$ with the $\bar{z}_S$ axis of the local system of the point S and by P the (also unknown) angle of the plane defined by the directions $V_1\bar{z}_{V_1}$ and $S\bar{z}_S$ with the plane $S\bar{z}_S\bar{y}_S$. With these notations, the direction cosines of the vertical $V_1\bar{z}_{V_1}$ will be: $-\sin p \sin P$, $\sin p \cos P$ and $\cos p$.

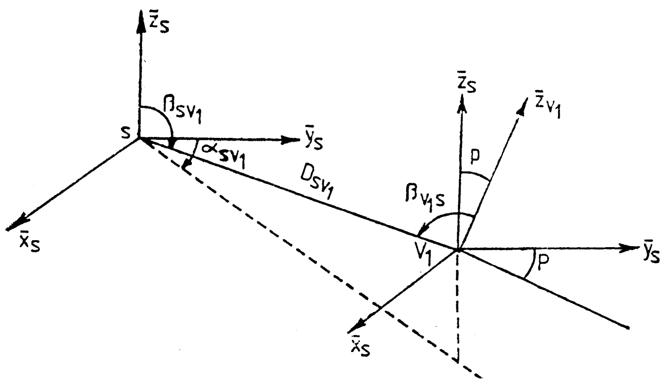


Fig. 18.3. The Astronomical Vertical of the V_1 Point

The following notation will be used here:

$$i_1 = \frac{\bar{x}_S - \bar{z}_{V_1}}{D_{SV_1}}\,; \quad j_1 = \frac{\bar{y}_S - \bar{y}_{V_1}}{D_{SV_1}}\,; \quad k_1 = \frac{\bar{z}_S - \bar{z}_{V_1}}{D_{SV_1}}\,;$$
$$i_2 = \frac{\bar{z}_{V_2} - \bar{z}_{V_1}}{D_{V_1V_2}}\,; \quad j_2 = \frac{\bar{y}_{V_2} - \bar{y}_{V_1}}{D_{V_1V_2}}\,; \quad k_2 = \frac{\bar{z}_{V_2} - \bar{z}_{V_1}}{D_{V_1V_2}}\,. \tag{18.4}$$

Between the quantities (18.4), p and P exist the following relations *(Levallois* and. *Kovalevsky 1971):*

$$\begin{aligned} i_1 \sin p \sin P + j_1 \sin p \cos P + k_1 \cos p - \cos \beta_{SV_1} = 0; \\ i_2 \sin p \sin P + j_2 \sin p \cos P + k_2 \cos p - \cos \beta_{V_1V_2} = 0. \end{aligned} \tag{18.5}$$

Since the angle p is sufficiently small and, consequently, $\cos p \approx 1$ (a lst-order variation in p involves a 2nd-order variation in $\cos p$), the system (18.5) may be solved by successive approximations. Thus: one takes, first of all, $\cos p = 1$ and one solves the system, obtaining the values $\sin p \sin P$ and $\sin p \cos P$, with the help of which, making use of the relation $\sqrt{(\sin p \sin P)^2 + (\sin p \cos P)^2} = \sin p$, one calculates in a first approximation the value $\sin p$ and then $\cos p$, which are now introduced into the constant terms $k_1 \cos p$, $k_2 \cos p$ etc.

The vector of the vertical of the point V_1 thus being known, it is transformed in the global system of coordinates by means of a multiplication by the rotation matrix **R**, which will determine its projections (Φ, Λ), whence, consequently, follows the astronomical trihedron at the new geodetic point V_1.

Remark. The quantity is roughly determined by (18.5), because, practically, the vertical at the point V_1 is nearly parallel to the $S\bar{z}_S$ axis (since β are obtained with uncertainty due to the imperfect knowledge of the refraction corrections).

Consequently, in view of this remark, if only the measured zenithal angles are available, for determining the local astronomical vertical it is better to find it directly in the global coordinate system *(Levallois* and *Kovalevsky* 1971*)*. With the aid of the rotation matrix **R** the local coordinates of the points S, V_1, V_2 are transformed in the global system or, which comes to the same thing, one transforms the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, using to this end, e.g. for the point V_1, the system (18.5), rewritten in the form:

$$\begin{aligned} I_1 \cos \Phi_{V_1} \cos \Lambda_{V_1} + J_1 \cos \Phi_{V_1} \sin \Lambda_{V_1} + K_1 \sin \Phi_{V_1} = \cos \beta_{SV_1}; \\ I_2 \cos \Phi_{V_1} \cos \Lambda_{V_1} + J_2 \cos \Phi_{V_1} \sin \Lambda_{V_1} + K_2 \sin \Phi_{V_1} = \cos \beta_{V_1V_2}. \end{aligned} \tag{18.6}$$

For the system (18.6) one also knows an approximate solution $(\Phi_{V_1}^{(0)}, \Lambda_{V_1}^{(0)})$ coresponding to a close position of the point V_1 (using which one has precisely defined the Laplacian trihedron of the point V_1). As unknowns one will here regard, consequently, the values $\mathrm{d}\Phi$ and $\mathrm{d}\Lambda$, which must satisfy the relations:

$$\Phi_{V_1}^{(0)} = \Phi_{V_1}^{(0)} + d\Phi \text{ and } \Lambda_{V_1} = \Lambda_{V_1}^{(0)} + \mathrm{d}\Lambda.$$

In view of:

$$I_1 \cos \Phi \cos \Lambda + J_1 \cos \Phi \sin \Lambda + K_1 \sin \Phi = \cos \beta_{SO};$$

$$I_2 \cos \Phi \cos \Lambda + J_2 \cos \Phi \sin \Lambda + K_2 \sin \Phi = \cos \beta_{V_1O},$$

the system (18.6) becomes:

$$\begin{aligned} &\mathrm{d}\Phi\,(I_1 \sin\Phi \cos\Lambda + J_1 \sin\Phi \sin\Lambda - K_1 \cos\Phi) + \\ &+ \mathrm{d}\Lambda \cos\Phi\,(I_1 \sin\Lambda - J_1 \cos\Lambda) - (\cos\beta_{SV_1} - \cos\beta_{SO}) = 0; \\ &\mathrm{d}\Phi\,(I_2 \sin\Phi \cos\Lambda + J_2 \sin\Phi \sin\Lambda - K_2 \cos\Phi) + \\ &+ \mathrm{d}\Lambda \cos\Phi\,(I_2 \sin\Lambda - J_2 \cos\Lambda) - (\cos\beta_{V_1V_2} - \cos\beta_{V_1O}) = 0. \end{aligned} \tag{18.7}$$

$\mathrm{d}\Phi$ and $\mathrm{d}\Lambda$ are in fact the values of the deflection of the vertical with respect to the near values of the Laplacian trihedron of the point V_1.

18.5 Orientation Calculation in the Local System

Let us assume that one pursues the orientation of the Laplacian trihedron, or more precisely the orientation of the sights with respect to this trihedron and that the astronomical azimuth of the geodetic point V_1 (or V_2) was not measured. By calculation one knows the coordinates Φ_{V_1}, Λ_{V_1}, of the point V_1 in the global system. One also knows the direction cosines (I_1, J_1, K_1) of the side V_1S, because the point S is the old starting point from which one calculates the point V_1. The azimuth of the side V_1S is given by the relation (Fig. 18.4):

$$\cos\alpha_{V_1S} \sin\beta_{V_1S} = \cos(V_1S, V_1\bar{y}). \tag{18.8}$$

But $\overrightarrow{\mathbf{V_1S}}$ and $\overrightarrow{\mathbf{V_1\bar{y}}}$ are known, their direction cosines being:

$$\frac{X_S - X_{V_1}}{D_{SV_1}} = I_1; \quad \frac{Y_S - Y_{V_1}}{D_{SV_1}} = J_1; \quad \frac{Z_S - Z_{V_1}}{D_{SV_1}} = K_1 \tag{18.9}$$

and $-\sin\Phi_{V_1}\cos\Lambda_{V_1}$, $-\sin\Phi_{V_1}\sin\Lambda_{V_1}$, $\cos\Phi_{V_1}$ respectively.

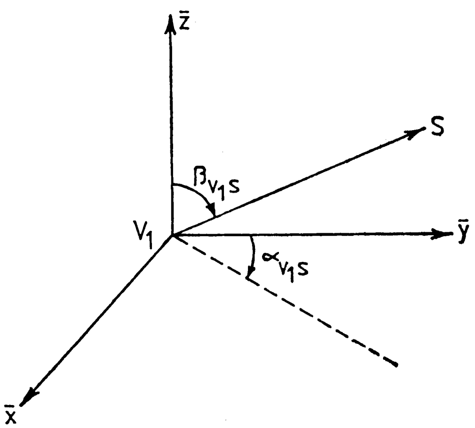


Fig. 18.4. The Local System of the V_1 Point

As a consequence, from (18.8) it follows that

$$\cos \alpha_{V_1S} \sin \beta_{V_1S} = - I_1 \sin \Phi_{V_1} \cos \Lambda_{V_1} - J_1 \sin \Phi_{V_1} \sin \Lambda_{V_1} + K_1 \cos \Phi_{V_1}, \tag{18.10}$$

an expression determining α_{V_1S}.

Remark. In (18.10) the zenithal angle β_{V_1S} must be corrected for the deflection of the vertical, i. e. $-(\xi_{V_1} \cos \alpha_{V_1S} + \eta_{V_1} \sin \alpha_{V_1S})$. Generally, however, since β_{V_1S} is close to 90°, this correction will nearly always be a second-order one.

With the aid of (18.10) one can find the well-known equation of *Laplace*, in another variation. To this end, one imposes on the Laplacian trihedron of the point V_1 a longitude variation $d\Lambda$, which means that one changes the $V_{1\bar{z}}$ axis and implicitly the other two axes. Then, differentiating (18.10) with respect to Λ yields:

$$\begin{aligned} - \sin \beta_{V_1S} \sin \alpha_{V_1S}\, d\alpha &= (I_1 \sin \Phi_{V_1} \sin \Lambda_{V_1} - J_1 \sin \Phi_{V_1} \cos \Lambda_{V_1})\, d\Lambda = \\ &= d\Lambda \sin \Phi_{V_1} (I_1 \sin \Lambda_{V_1} - J_1 \cos \Lambda_{V_1}). \end{aligned} \tag{18.11}$$

Now, in order to express the quantity $(I_1 \sin \Lambda_{V_1} - J_1 \cos \Lambda_{V_1})$ as a function of the local system's variables, one makes use of, for example, the relations (17.7) and (17.9) which correspondingly leads to:

$$\mathbf{R}_1 \begin{Vmatrix} I_1 \\ J_1 \\ K_1 \end{Vmatrix} = \begin{Vmatrix} i_1 \\ j_1 \\ k_1 \end{Vmatrix}$$

or

$$\begin{Vmatrix} -\sin \Lambda_{V_1} & \cos \Lambda_{V_1} & 0 \\ -\sin \Phi_{V_1} \cos \Lambda_{V_1} & -\sin \Phi_{V_1} \sin \Lambda_{V_1} & \cos \Phi_{V_1} \\ \cos \Phi_{V_1} \cos \Lambda_{V_1} & \cos \Phi_{V_1} \sin \Lambda_{V_1} & \sin \Phi_{V_1} \end{Vmatrix} \cdot \begin{Vmatrix} I_1 \\ J_1 \\ K_1 \end{Vmatrix} = \begin{Vmatrix} i_1 \\ j_1 \\ k_1 \end{Vmatrix},$$

whence one gets:

$$\begin{aligned} i_1 &= - I_1 \sin \Lambda_{V_1} + J_1 \cos \Lambda_{V_4}; \\ j_1 &= - I_1 \sin \Phi_{V_1} \cos \Lambda_{V_1} - J_1 \sin \Phi_{V_1} \sin \Lambda_{V_1} + K_1 \cos \Phi_{V_1}; \\ k_1 &= I_1 \cos \Phi_{V_1} \cos \Lambda_{V_1} + J_2 \cos \Phi_{V_1} \sin \Lambda_{V_1} + K_1 \sin \Phi_{V_1}, \end{aligned} \tag{18.12}$$

which represent the components of the vector $\overrightarrow{V_1S}$ in the local system. On the other hand, however:

$$i_1 = \sin \alpha_{V_1S} \sin \beta_{V_1S};\ j_1 = \cos \alpha_{V_1S} \sin \beta_{V_1S};\ k_1 = \cos \beta_{V_4S}. \tag{18.13}$$

If in (18.11) one takes account of the first expression in (18.12) and (18.13) respectively, it follows that

$$d\alpha = d\Lambda \sin \Phi, \tag{18.14}$$

which represents *Laplace*'s equation, established independently of any ellipsoidal hypothesis or any other hypothesis whatever, since only rotations in three-dimensional space intervene here.

Remarks:

(1) The formulae (18.6) and (18.10) solve the problem of calculating the vertical and the orientation in the local systems and, consequently, in the global system. The calculation development is plain, the transitions from one system to the other being carried out by means of matricial computation.

(2) An adjusted spatial figure, or one assumed to be perfectly geometrical, being given, one notes that in order to define its spatial positions one can proceed through a concatenation of local systems; one interconnects the latter arbitrarily or one passes directly to the global system which is particularly interesting for Geodesy.

18.6 Calculation of a Geodetic Traverse

One considers a geodetic traverse, consisting of a set of geodetic points, whose sides have been directly measured with electromagnetic instruments. At every geodetic point the zenithal angles, the azimuthal angles and, as will be seen in due course, other elements as well, were measured, because it is desirable that the respective traverse be *intrinsically calculable* in the three-dimensional optics. One's goal is the determination of the coordinates of the geodetic points calculable without any hypothesis and, in addition to them, the direction of the vertical at every geodetic point.

First of all, one assumes that one knows these data, obtained by any calculation, by successive approximations, at one of the corresponding geodetic points S, whose Laplacian trihedron is perfectly determined. This point shall be designated as *reference point*. For the determination of the next geodetic point, e.g. V_1, with respect to the point S, one needs five parameters: X_{V_1}, Y_{V_1}, Z_{V_1}, Φ_{V_1}, Λ_{V_1}. But one measures: two zenithal angles, an azimuthal angle and a distance; this means that only four data are available for five unknowns. For Classical Geodesy the four observations do suffice, because to them is added the ellipsoid condition (which serves for determining the Laplacian trihedra); this means adding a hypothesis which is actually equivalent to two conditions. Consequently, from the point of view of Classical Geodesy, of the four observations one is useless for strictly defining the problem, e.g. one of the two reciprocal zenithal angles.

From the standpoint of Three-Dimensional Geodesy, the four data are, however, insufficient for determining all of the five unknowns. Here we must add a condition or a supplementary measurement, as for example the Laplace azimuth of the side between the initial point S and the geodetic point V_1 to be determined, or the astronomical azimuth measured at the point V_1, or one of the coordinates Φ_{V_1}, Λ_{V_1} obtained from geodetic-astronomy determinations at the point V_1.

Remark. It would of course be better to measure both coordinates Φ_{V_1}, Λ_{V_1}, rather than only one of them.

We now show the manner of developing the calculation in three-dimensional optics.

For the initial point S, to which one assigns the coordinates $\bar{x}_S = B_S$, $\bar{y}_S = L_S$, $\bar{z}_S = H_S^E$ in its local system and the matrices $\mathbf{R}_1(S)$ and $\mathbf{R}_1^T$ respectively, with the aid of which (17.19) one gets the coordinates X_S, Y_S, Z_S in the equatorial Cartesian global system, one assumes that one knows the astronomical coordinates Φ_S and Λ_S.

Consequently, one knows for the point S all of the five fundamental unknowns.

(1) *One calculates the coordinates of the point V_1 in the local system* (i.e. $\bar{x}_{V_1}$, $\bar{y}_{V_1}$, $\bar{z}_{V_1}$), making use of (18.3).

(2) *One calculates the coordinates of the point V_1 in the equatorial Cartesian global system*, utilizing the matricial transformation (17.7) in the form $\mathbf{X} = \mathbf{R}_1^T\bar{\mathbf{x}}$, with the following result:

$$\begin{Vmatrix} X_{V_1} \\ Y_{V_1} \\ Z_{V_1} \end{Vmatrix} = \begin{Vmatrix} X_S \\ Y_S \\ Z_S \end{Vmatrix} + \begin{Vmatrix} -\sin\Lambda_S & -\sin\Phi_S\cos\Lambda_S & \cos\Phi_S\cos\Lambda_S \\ \cos\Lambda_S & -\sin\Phi_S\sin\Lambda_S & \cos\Phi_S\sin\Lambda_S \\ 0 & \cos\Lambda_S & \sin\Lambda_S \end{Vmatrix} \cdot \begin{Vmatrix} \bar{x}_{V_1} \\ \bar{y}_{V_1} \\ \bar{z}_{V_1} \end{Vmatrix} \cdot \qquad (18.15)$$

Remark. If from the coordinates X_{V_1}, Y_{V_1}, Z_{V_1}, one passes now in the ellipsoidal global system, to B_{V_1}, L_{V_1}, $H_{V_1}^E$, one gets the corresponding rotation matrix R_2^T, respectively R_2 ((17.18)), characteristic of the local Laplacian trihedron at the point V_1.

(3) In the global system X, Y, Z one calculates the direction cosines I_1, J_1, K_1 of the vector $\overrightarrow{\mathbf{V_1S}}$ with the aid of (18.9).

(4) If the astronomical coordinates Φ_{V_1}, Λ_{V_1} have been obtained (by means of geodetic-astronomy determinations at the point V_1), then one can calculate the components of the deflection of the vertical at V_1:

$$\xi_{V_1} = \Phi_{V_1} - B_{V_1} \text{ and } \eta_{V_1} = (\Lambda_{V_1} - L_{V_1}) \cos B_{V_1}.$$

Consequently, in this case we know the Laplace azimuth of the side V_1S by means of (18.10) and one can now deduce that of the side V_1V_2, by adding to α_{V_1S} the measured azimuthal angle, corrected for the deflection of the vertical in azimuth and, especially, for the zenithal angle.

Remark. It follows that there does exist a supplementary measurement (either the zenithal angle β_{V_1S}, or one of the astronomical coordinates Φ_{V_1}, Λ_{V_1}), since the zenithal angle of the $\bar{z}$ axis of the Laplacian trihedron is determined by the vector (I, J, K) and by the geodetic coordinates of the point V_1, which have already been calculated.

18.7 Triangulation Calculations

These calculations derive from those shown in the preceding section, with some differences arising from the fact that the measurement of the zenithal angles at the three vertices of a triangle determine the physical verticals of these points. Consequently, one obtains directly the direction cosines of the verticals with the aid of the relations (18.6) and (18.7).

From the theoretical point of view it is no longer necessary to measure Laplace azimuths, since these are already automatically defined by (18.10), as was shown in Section 18.5. The triangulation calculations are, in consequence, relatively easy as regards their development, but one must, nevertheless, say that this type of triangulation is more complex than the conventional one.

Remark. From what has already been shown, follows the fundamental role allotted to the zenithal angles in Three-Dimensional Geodesy, concerning the definition of the vertical or of the *Laplace* azimuths. As *J. J. Levallois* and *J. Kovalevsky* here shown (*1971*), this fact is the method's "Achilles heel" and explains why few works of Three-Dimensional Geodesy have hitherto been undertaken apart from measurements on the Earth's artificial satellites. On the other hand, most of the astro geodetic triangulation works have been carried out in the light of Classical Geodesy and, unfortunately, we do not yet contemplate retackling them from the point of view of Three-Dimensional Geodesy.

There is, however, a means for obtaining (or nearly so) the same results as those of Three-Dimensional Geodesy, by replacing the zenithal angles with direct levelling and astronomical measurements (i.e. astro-geodetic levelling). This point of view will be analysed in what follows.

18.7.1 The Projection Method. Method of Unfolding

Proceeding by successive approximations, astro-geodetic levelling may replace the three-dimensional methods, i.e. one can reconstitute three-dimensional space by constructing the quasi-geoid or the geodetic profiles, starting from *Molodenski*'s equation (17.21).

In order to illustrate this reasoning, it is first of all necessary to recall the basic principles of Classical Geodesy. According to these, one determines a fundamental point, to which one assigns the initial elements Φ, Λ, H and by means of triangulation measurements (Chapter. 14) one works out the corresponding astro-geodetic network.

The calculation of this network, which is regarded as perfectly geometrical for what follows, proceeds in two approximating stages, as shown below: (*Levallois* and *Kovalevsky* 1971):

In a first approximation: one reduces to the ellipsoid all distance measurements which have been carried out, while the angular ones are assumed to have been performed along the normal to the ellipsoid and one effects the triangulation calculation on the ellipsoid, one determines the altitudes H^E of the geodetic points, one evaluates the components of the deflection of the vertical ξ and η (by comparison with the geodetic-astronomy determinations) of these points and one builds the profiles — $\int u\delta D$ (17.21) or one determines the quasi-geoid's heights.

The second approximation. Knowledge of the values ξ, η and of the profiles in the first approximation enables one to reduce (anew) the measured distances, as well as the azimuthal observations to the ellipsoid.

The calculation cycle may be repeated and by successive approximations the results are improved etc.

According to this method, the basic calculations are carried out on the ellipsoid — which represents the reference surface on which the astro-geodetic network is unfolded. Therefore, *M. S. Molodenski* (1945 and 1960) designates this method as the *method of unfolding*.

As follows from the considerations of this chapter, the Three-Dimensional Geodesy proceeds through projections onto the reference surface — here the physical surface of the Earth —, in order to obtain the geodetic coordinates. Although the method of Classical Geodesy and the three-dimensional method appear to be very similar, they lead to different results, because of the fact of applying them to very wide areas. In order to bring out these different results, one may consider the example used by *M. S. Molodenski* (1945) and by *(J. J. Levallois* and *J. Kovalevsky* 1971*)*, viz.: let a traverse or a meridian triangulation chain be calculated according to Classical Geodesy (the unfolding method) and according to Three-Dimensional Geodesy (the projection method). One assumes that for any network it would be obligatory to consider at the origin (the fundamental point) a total deflection of the vertical u_0 (which in the chosen case equals its meridian component ξ_0) in order to re-orientate the ellipsoid at this point. In Classical Geodesy, where all the calculations are carried out on the ellipsoid, one would, be tempted to remove the point m to m' so that $mm'/R = \xi_0$ (Fig. 18.5; *Levallois* and *Kovalevsky* 1971). Starting from this origin, one calculates the entire triangulation, the result being the modification of all geodetic latitudes by a quantity of the order of ξ_0 (i.e. of approximately 2nd-order). In other words one has carried out a quasi-rotation through ξ_0 of all the geodetic elements and one has thereby induced a variation ξ_0 of the deflection of the vertical at all the points.

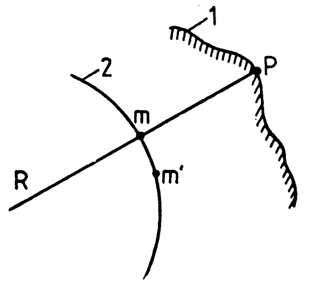


Fig. 18.5. The Removal of the Point m to m' Due to the ξ_0 Variation:

1 — the Earth's physical surface; 2 — the reference ellipsoid

The corresponding general relations of the deflection of the vertical ξ and of the geodetic profile ζ for any triangulation point would, in this case, be the following *(Levallois* and *Kovalevsky* 1971*)*:

$$\xi = \xi_0; \; \zeta = \zeta_0 - \xi_0 R(B - B_0), \tag{18.16}$$

where the values ξ_0, ζ_0, B_0 characterize the origin point.

By the method of Three-Dimensional Geodesy, a translation would be given to the reference ellipsoid, while maintaining the altitude at the fundamental point and modifying the geodetic latitude to the corresponding value.

For any point of the triangulation, the general calculating formulae corresponding to (18.16), in the case of utilizing the three-dimensional method, would be *(Levallois* and *Kovalevsky* 1971*)*:

$$\begin{aligned} \xi &= \xi_0 \,\text{fos}\, (B - B_0) + \frac{\zeta_0}{R} \sin (B - B_0); \\ \zeta &= \zeta_0 \cos (B - B_0) - R\, \xi_0 \sin (B - B_0). \end{aligned} \tag{18.17}$$

Comparing (18.16) and (18.17) shows that these formulae differ from one another; a perfect agreement only exists between them at the origin point ($B = B_0$), which in fact shows that (18.16) is a particular case of the general formulae (18.17). In particular the linear variation ξ as a function of $(B - B_0)$ does not hold beyond a certain limit. Put another way, the astronomical-levelling formulae also can only cautiously be applied within a restricted domain.

18.7.2 The Formulae of Astro-Geodetic Levelling in the Three-Dimensional Conception

Let us suppose two points P_1 and P_2 to be on the topographic surface, the altitude of the point P_1 above the ellipsoid having any value $(H^E_{P_1} + \zeta_0)$, ζ_0 being the quasi-geoid's height with respect to the ellipsoid. If one communicates to the point P_1 an altitude variation $d\zeta_1$ along $H^E_{P_1}$, then, as a consequence of this removal in which the quasi-geoid itself is involved, the entire topographic surface undergoes a translation $P_1P_1' = d\zeta_1$ (Fig. 18.6; *Levallois* and *Kovalevsky 1971*); the point P_1 moves to P_1' and the increase of the ellipsoidal altitude at the point P_2 will equal $P_2P_2' = d\zeta_1 \cos \psi$, where ψ is the angle between the two normals at P_1 and P_2 respectively.

One further assumes that:

— along a given triangulation chain astronomical levelling would be applied;

— the topographic surface obtained in a first approximation would result simply from the summation of the altitude segments along the norms (i.e. without astronomical levelling);

— one communicates an altitude variation ζ_0 to the fundamental point of the chain.

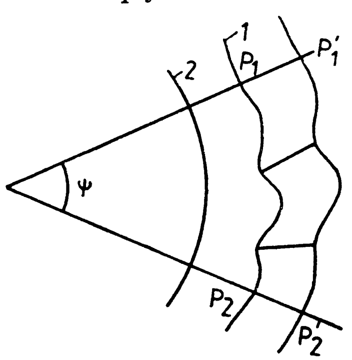


Fig. 18.6. The Translation of the Topographic Surface Due to the $d\zeta_0$ Variation:
1 — the topograhic surface;
2 — ellipsoid

The effect of the variation ζ_0 on the chain's end S will be $\zeta_S = \zeta_0 \cos \widehat{OS}$. If through an astronomical-levelling operation at a small distance one starts from the new position of the point O with a view to determining the altitude of the neighbouring point P, then $\zeta_P = \zeta_0 - u.\ \overline{OP}$ and the influence on the point S will be $u\ \overline{OP} \cdot \cos \widehat{MS}$. One finally gets (*Levallois* and *Kovalevsky 1971*):

$$\zeta_S = \zeta_0 \cos \widehat{OS} - \sum_P^S (u\ PQ) \cos \widehat{PS}, \qquad (18.18)$$

where P is a variable point, PQ — the geodetic side, after which one carries out the calculation of the ellipsoidal-altitude variation, u — the deflection of the vertical with respect to the side PQ and S — the end of the chain, whose total removal is being calculated.

Remark. It is not necessary that the points $O, P, Q, \ldots, S$ be located on a great circle, since the formula (18.18) is general and, as the verticals are practically convergent, corrections are not needed.

If at the fundamental point O there exists a difference of the deflection of the vertical $\Delta\xi_0$ between the method of unfolding and the projection method, then in (18.18) one must take into account an additional term (18.17), viz. $- R\,\Delta\,\xi_0 \sin \widehat{OS}$, i.e.:

$$\zeta_S = \zeta_0 \cos \widehat{OS} - \sum_P^S (u\ PQ) \cos \widehat{PS} - R\Delta\xi_0 \sin \widehat{OS}. \qquad (18.19)$$

Remark. If one compares the relation (18.19) with the corresponding one from the method of unfolding:

$$\zeta_S = \zeta_0 - \sum_P^S (u\ PQ) - \Delta\xi_0 \widehat{OS}, \qquad (18.20)$$

then one notes that actually (18.20) represents nothing else but the principal part of the general formula (18.19).

18.7.3 Three-Dimensional Adjustment of an Astro-Geodetic Network

According to an idea of *H. Bruns* (1878), which is to be found again later on in *H. Dufour*'s contribution (1962), it is advisable, because of the uncertainty in knowing the refraction, to carry out an adjustment on groups of an astro-geodetic network, viz. in the sense that the 1st group should include the unknowns $\mathrm{d}B$, $\mathrm{d}L$, $\mathrm{d}o$ and $\mathrm{d}\Lambda_{PL}$ (PL = *Laplace* point) and the 2nd one all the remaining unknowns ($\mathrm{d}H$, $\mathrm{d}\Phi$, $\mathrm{d}\Phi_{PL}$, $\mathrm{d}\Lambda$, $\mathrm{d}K$). Consequently, the 1st group embraces the "position" problem and the 2nd group the problem of the altitudes and of the deflection of the vertical. Correspondingly, one must group the correction equations:

$$\text{1st group: } v_d,\ v_D,\ v_{\alpha_{PL}};\ v_{\Lambda_{PL}};$$

$$\text{2nd group: } v_\beta,\ v_\Phi,\ v_\Lambda,\ v_\alpha.$$

If the unknowns of the 1st group (position unknowns) are concentrated in the $\mathbf{x}_1$ vector and those of the 2nd group (the unknowns of altitude and of the deflection of the vertical) in $\mathbf{x}_2$, then the system of the corresponding correction-equations may be presented in the form *(Wolf 1963 c)*:

$$\text{1st group: } \mathbf{v}_1 = \mathbf{B}_1\mathbf{x}_1 + \underline{\mathbf{B}}_1\mathbf{x}_2 + \mathbf{w}_1, \text{ the weight } \mathbf{P}_1, \qquad (18.21)$$

$$\text{2nd group: } \mathbf{v}_2 = \underline{\mathbf{B}}_2\mathbf{x}_1 + \mathbf{B}_2\mathbf{x}_2 + \mathbf{w}_2, \text{ the weight } \mathbf{P}_2, \qquad (18.22)$$

where $\mathbf{v}$ and $\mathbf{P}$ represent the vector of the corresponding corrections, and the weight matrix respectively and $\mathbf{B}, \underline{\mathbf{B}}$ — the matrices of the coefficients of the unknowns, coefficients given in detail in the relations from (17.40) up to (17.49).

The general adjustment consists then in tackling the systems (18.21) and (18.22) together:

$$\begin{Vmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{Vmatrix} = \begin{Vmatrix} \mathbf{B}_1 & \underline{\mathbf{B}}_1 \\ \underline{\mathbf{B}}_2 & \mathbf{B}_2 \end{Vmatrix} \begin{Vmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{Vmatrix} + \begin{Vmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{Vmatrix} \quad \text{or } \mathbf{v} = \mathbf{Bx} + \mathbf{w}. \tag{18.23}$$

If, according to the known rules (*Jordan/Eggert/Kneissl* 1958, *Lörinczi 1966, Wolf 1968)*, one establishes from (18.23) the corresponding normal equations, one then gets:

$$(\mathbf{B}_1^T\mathbf{P}_1\mathbf{B}_1 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\underline{\mathbf{B}}_2)\mathbf{x}_1 + (\mathbf{B}_1^T\mathbf{P}_1\underline{\mathbf{B}}_1 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\mathbf{B}_2)\mathbf{x}_2 + \mathbf{B}_1^T\mathbf{P}_1\mathbf{w}_1 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\mathbf{w}_2 = \mathbf{0};$$

$$(\underline{\mathbf{B}}_1^T\mathbf{P}_1\mathbf{B}_1 + \mathbf{B}_2^T\mathbf{P}_2\underline{\mathbf{B}}_2)\mathbf{x}_1 + (\mathbf{B}_1^T\mathbf{P}_1\underline{\mathbf{B}}_1 + \mathbf{B}_2^T\mathbf{P}_2\mathbf{B}_2)\mathbf{x}_2 + \underline{\mathbf{B}}_1^T\mathbf{P}_1\mathbf{w}_1 + \mathbf{B}_2^T\mathbf{P}_2\mathbf{w}_2 = \mathbf{0}. \tag{18.24}$$

Instead of solving the system (18.24) in general, e.g. by mea ns of *Gauss* algorithm, *H. Dufour* (*1962*) writes this system in another form:

$$\mathbf{x}_1 = -(\mathbf{B}_1^T\mathbf{P}_1\mathbf{B}_1)^{-1}(\mathbf{B}_1^T\mathbf{P}_1\mathbf{w}_1 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\mathbf{w}_2 + (\mathbf{B}_1^T\mathbf{P}_1\underline{\mathbf{B}}_1 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\mathbf{B}_2)\mathbf{x}_2 + \underline{\mathbf{B}}_2^T\mathbf{P}_2\underline{\mathbf{B}}_2\mathbf{w}_1);$$

$$\mathbf{x}_2 = -(\mathbf{B}_2^T\mathbf{P}_2\mathbf{B}_2)^{-1}(\underline{\mathbf{B}}_1^T\mathbf{P}_1\mathbf{w}_1 + \mathbf{B}_2^T\mathbf{P}_2\mathbf{w}_2 + (\underline{\mathbf{B}}_1^T\mathbf{P}_1\mathbf{B}_1 + (\mathbf{B}_2^T\mathbf{P}_2\underline{\mathbf{B}}_2^T)\mathbf{x}_1 + \underline{\mathbf{B}}_1^T\mathbf{P}_1\underline{\mathbf{B}}_1\mathbf{x}_2), \tag{18.25}$$

thus producing an iterative solution, viz. in a double sens, in which $\mathbf{x}_1$ does not depend only on $\mathbf{x}_2$ but also on $\mathbf{x}_1$ obtained in a previous step, and similarly $\mathbf{x}_2$. Consequently, the calculations of the "position" and of the altitudes (respectively of the deflections of the vertical) are reciprocally converging in steps. In order that this convergence should be very rapid, it is necessary, as was established by *H. Dufour* (*1962*) that the elements of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$ have very large numerical values and those of the matrices $\underline{\mathbf{B}}_1$, $\underline{\mathbf{B}}_2$, on the contrary, small values.

As *H. Wolf* (*1963 c*) has shown, if the relations (18.21) and (18.22) are written in the form:

$$\mathbf{v}_1 = \mathbf{B}_1\mathbf{X}_1 + \bar{\mathbf{w}}_1; \tag{18.26}$$

$$\mathbf{v}_2 = \mathbf{B}_2\mathbf{X}_2 + \bar{\mathbf{w}}_2, \tag{18.27}$$

where $\bar{\mathbf{w}}_1 = \mathbf{w}_1 + \underline{\mathbf{B}}_1\mathbf{x}_2$ and $\bar{\mathbf{w}}_2 = \mathbf{w}_2 + \underline{\mathbf{B}}_2\mathbf{x}_1$, then a meaning is thereby given to the notation of "rehabilitation of Classical Geodesy" advanced by *J. J. Levallois* (*1963*). Thus, $\bar{\mathbf{w}}_1$ appears as representing those free terms which result if, by a so-called *Helmert* projection, one establishes the value of the necessary reduction ($\underline{\mathbf{B}}_1\mathbf{x}_2$) to the position observations (d, D, α_{PL}) and afterwards, in the sense of Classical Geodesy, i.e. two-dimensionally, one carries out the corresponding calculation. Of course, it is necessary afterwards to take into consideration still another calculation of altitude and of deflection of the vertical, in order to obtain in this way the elements which are needed for reduction.

Remark. If one introduces into the correction equations the identities:

$$\Phi = (\Phi - B) + B \text{ and } \Lambda = (\Lambda - L) + L, \tag{18.28}$$

in which one takes account of the fact that:

$$\Phi - B = \xi \text{ and } \Lambda - L = \eta \sec \Phi \tag{18.29}$$

and as approximate values $\Phi^{(0)}$, $\Lambda^{(0)}$

$$\Phi^{(0)} = B^{(0)} \text{ and } \Lambda^{(0)} = L^{(0)}, \tag{18.30}$$

are considered, then subtracting the relations (18.28) and (18.30) yields:

$$\mathrm{d}\Phi = \mathrm{d}B + \xi \text{ and } \mathrm{d}\Lambda = \mathrm{d}L + \eta \sec B^{(0)}, \tag{18.31}$$

and, consequently, one can now replace everywhere, in the correction equations, $\mathrm{d}\Phi$ and $\mathrm{d}\Lambda$ by the relations (18.31). It then follows (*Wolf 1963 c*) that one must abandon any opinions concerning the uselessness of utilizing the deflection of the vertical (whose components are ξ and η) in the system of Three-Dimensional Geodesy.

19

Basic Methods for Developing the Geodetic Networks by Means of the Earth's Artificial Satellites

19.1 General Considerations

The methods of Three-Dimensional Geodesy have found application in the case of utilizing the Earth's artificial satellites for geodetic purposes. Providing the possibility of sighting a target situated at hundreds of kilometres, visible from every point on the terrestrial Globe and moving in the field of terrestrial gravity, the artificial satellites have brought about incredible and unforeseeable advances in Geodesy, in a very short space of time. In the sequel, we shall present a few of the advantages of making use of the Earth's artificial satellites for geodetic purposes.

Every artificial satellite gravitates round the Earth obeying, in a rough approximation, *Kepler*'s laws. According to the third law of *Kepler*, between the semi-major axis a' of the satellite's orbit and its revolution period T there exists the relation *(Levallois* and *Kovalevsky 1971)*:

$$\frac{4\pi^2 a'^3}{T^2} = GM, \tag{19.1}$$

in which G is the universal constant of gravitation and M — the Earth's mass. If one makes use of the fact that an approximate value of the product GM can be obtained from measuring the gravity g, i.e. (relation (4.23)):

$$g \approx GM/R^2, \text{ whence } GM \approx R^2 g \tag{19.2}$$

and if one considers that the satellite would fly at the distance R (equalling the terrestrial radius) from the Earth's mass centre, then substituting (19.2) into (19.1) yields the relation:

$$T^2 = 4\pi^2 R/g, \text{ for its period} \tag{19.3}$$

which, for $R = 6{,}372$ km and $g = 981$ gal, gives $T = 84^m.3$.

In view of (19.2), from (19.1) one gets:

$$T = \sqrt{\frac{4\pi^2 R}{g}} \sqrt{\frac{a'^3}{R^3}},$$

which shows, if one introduces the T value from (19.3), that the period of a satellite, whose orbital semi-major axis is a', may be calculated using the (still approximate) formula:

$$T^m \approx 84^m{,}3 \sqrt{\frac{a'^3}{R^3}}.$$

The orbital velocity is a function of the distance to the satellite and of the form of the satellite's orbit. If, e.g., one assumes that the satellite has a nearly circular orbit, then its linear velocity will be:

$$V = \frac{2\pi a'}{T}. \tag{19.4}$$

Substituting (19.4) into (19.1) yields $V^2 a' = GM$, from which, taking account of (19.2), one gets:

$$V = R\sqrt{\frac{g}{a'}} \tag{19.5}$$

For the case in which $a' = R$ and with the numerical values adopted for (19.3), one finds a velocity $V = \sqrt{gR} = 7{,}750$ m/s and, if (19.5) is written in the form:

$$V = \sqrt{gR}\sqrt{\frac{R}{a'}},$$

then the velocity for an orbit of radius a' is given by:

$$V = 7750\sqrt{\frac{R}{a'}}\ \text{m/s}.$$

In the case in which one disregards the perturbations due to complex gravitational effects and to other factors, a satellite orbit is an ellipse whose plane may be regarded as fixed with relative to the stars, for an approximate observation, over a short time interval.

It thus follows that, if an artificial satellite is visible at the same place over several revolutions, depending on the Earth's rotation, then the time interval (approximately equalling a period T) — which separates two successive transits (in the same direction) of the satellite at a parallel of a given latitude — will correspond to an apparent westward shift by an amount equal to the angle of terrestrial rotation which has taken place during the period T.

In the course of one and the same night, when the visibility conditions are satisfied during several consecutive revolutions, the satellite will first be observed to the East. The shining duration of an artificial satellite, for example a passive one, i.e. one illuminated by the Sun, depends on the Sun's height and position at the observation time. Consequently, the observational possibilities, at a certain place, vary with the season, being possible to determine the areas of visibility at a given hour and at a given place for a known date.

Let us assume that some artificial satellite has been launched, following a certain equatorial inclination, its orbit's plane making then the angle i with the equator. In the course of its revolution, the vertical of the satellite's centre will fly over the places on the Earth's surface having latitudes lying between zero and i, assuming that the Earth is approximated by a sphere. The latitude of the places flown over is given in this case by the relation (*Levallois* and *Kovalevsky 1971*):

$$\sin \Phi = \sin i \sin \omega,$$

where ω is the satellite's elongation on its orbit, calculated with the orbital ascending node as origin.

Remarks:
(1) Only the polar satellites have the capability of flying over the whole terrestrial Globe.
(2) An artificial satellite launched at a latitude Φ will generally not have an orbit whose equatorial inclination is smaller than Φ.

As regards the spatial triangulation by means of artificial satellites, the rule used in conventional triangulation, viz.: in order that a triangulation be sound, one must not use very acute or very obtuse angles, holds here also. Consequently, in the case of utilizing artificial satellites for a spatial triangulation, one must make use of them under correct sighting conditions, the distance between the terrestrial points from which the satellite is sighted having to be, in a first approximation, of the order of magnitude of the satellite's height. In order to be usable for geodetic purposes, the artificial satellites must not fly too low for two reasons:

1) Their life is short in the low atmosphere.
2) The apparent angular velocity of these satellites is considerable; for instance (*Levallois* and *Kovalevsky 1971*): for a satellite flying at 1,000 km above the ground, at the observer's zenith, the apparent angular velocity is approximately 0.4°/s or 7.1×10^{-3} rad/s.

From the quoted example it follows, that such an artificial satellite cannot be viewed through a telescope, because it runs across the telescope's field of view in about 2s.

Therefore, the only procedure for determining the position of the satellite at a given moment is to photograph it against the stars as background, after which, by determining the spatial directions of these stars, one should deduce the position of the satellite at the moment of the observation. This procedure calls for special photographic equipment known as *ballistic cameras*, which, depending on the possibility or otherwise of following an artificial satellite on its apparent path, may be grouped into: cameras allowing tracking of the satellite (e.g. the Soviet cameras of the type *AFU*-75 and *VAU*, the American camera *Baker Nunn*, the French camera *Antares*, the camera *SBG* of the *G.D.R.* firm *Zeiss* and the camera *BMK* of the firm *F.R.G. Zeiss*) and cameras which do not permit this, which have a fixed optical axis (e.g. the Soviet cameras of the type *NAFA 3s/25*, *NAFA MK-75* and *FAS-3A*, the American cameras of the type *Wild BC-4*, *PC-1000*, *MOTS-40* and *K-50*, the British cameras of the type *Hewitt* and *RRE*, the French camera *IGN* and the Polish camera *Poznan-2*).

Regardless of the camera being used, one always utilizes the stars as witness marks. Therefore, the ballistic cameras embody a fundamental limitation in their utilization for obtaining spatial directions, due to the fact that they depend on the precise positions of the stars, and are at present surpassed for geodetic purposes by the laser distance-determinations *(Committee on Geodesy: Trends and prospects, 1978)*

For the development of the base geodetic networks, the Earth's artificial satellites play only the role of moving sighting target, accessible for synchronous observations from a few terrestrial-station points.

Depending on the composition of the observations at these station points, three basic methods are possible for developing such geodetic networks *(Razumov 1974):* triangulation, trilateration and the vector network. Each of them has its own special features and the choice of one of them depends on the problem being considered and on the actual conditions, as is shown in Sections, 19.5, 19.6 and 19.7.

19.2 Elements of an Artificial Satellite's Orbit

The elements of the orbit of an Earth's artificial satellite are the elements of an orbit in general [1]. With a view to defining these elements, one considers the celestial equatorial coordinate system $Oxyz$ (Fig. 19.1). The axes of this system are as follows: Oz is directed towards the celestial north pole, Ox to the chosen mean equinox and Oy is perpendicular to Ox in the equator's plane. Consequently, the Oxy plane represents the equator's plane.

One can pass from one axis to another through a series of rotations which will be defined in what follows. The orbital plane π intersects the Oxy plane along the straight line $N'ON$.

One chooses the oriented half-line $\overrightarrow{ON}$ in such a way that when the artificial satellite S^* intersects it the height z of the latter increases; i.e. the satellite passes from the southern into the northern hemisphere. The $\overrightarrow{ON}$ direction is termed *the ascending node of the orbit* and is marked out by means of the right ascension [2] Ω (Fig. 19.1).

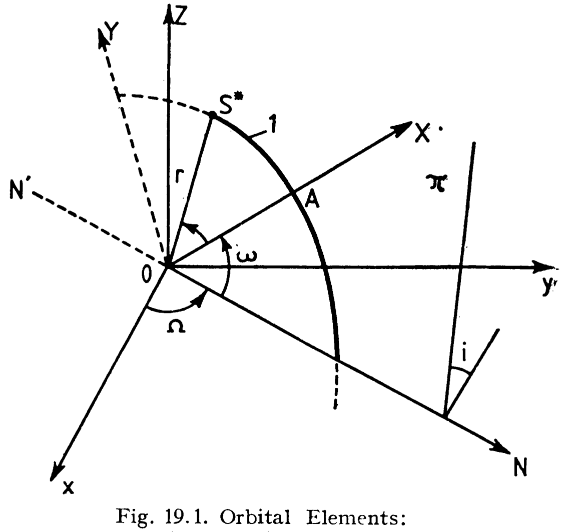


Fig. 19.1. Orbital Elements:
1 — satellite's orbit

[1] also *J. J. Levallois* and *J. Kovalevsky (1971)*. We shall use here the same notations, designations and definitions.

[2] It is the angle between the $\overrightarrow{Ox}$ axis and the orbit's trace π on the Oxy plane. This angle may take any value between 0° and 360°.

The angle i formed by the π plane of the orbit with the Oxy plane is called the *inclination* and is considered positive between 0° and 180°.

1) $0° < i < 90°$: the direction of motion of the satellite on the orbit is projected onto the Oxy plane following the same direction; the motion is said to be *direct.*
2) $90° < i < 180°$: the motion is projected in the opposite direction; the motion is said to be *retrograde.*
3) $i = 90°$: the orbit is designated as *polar.*
4) $i = 0°$: the orbit plane coincides with the Oxy plane; the orbit is called *equatorial.*

In the orbit plane, the direction of the perigee A is defined by *the latitude argument of the perigee,* denoted by ω in Fig. 19.1, representing the angle $(\overrightarrow{ON}, \overrightarrow{OA})$.

Remark. The three angles Ω, i and ω define the position of the orbit in space. The size and form of this ellipse are entirely defined by the knowledge of its semi-major axis a' and eccentricity e.

It now remains for the position of the satellite S^* to be established at every moment. This is done by calculating *the actual anomaly v,* which needs first of all the solution of *Kepler*'s equation:

$$E - e \sin E = M = n(t - \tau_0), \tag{19.6}$$

in which E (termed *eccentric anomaly*) is the angle, in the orbit plane, between the radius vector $\overrightarrow{CS_1^*}$ and the OX axis (Fig. 19.2); M — the mean anomaly; $n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a'^3}}$ — the mean motion: τ_0 — the moment of passage at the perigee, whence arises the eccentric anomaly at any moment t, after which one solves the equation:

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}, \tag{19.7}$$

thus the actual anomaly sought is obtained. Consequently, **the orbit elements**[1] **of an artificial satellite completely defining its motion are as follows:**

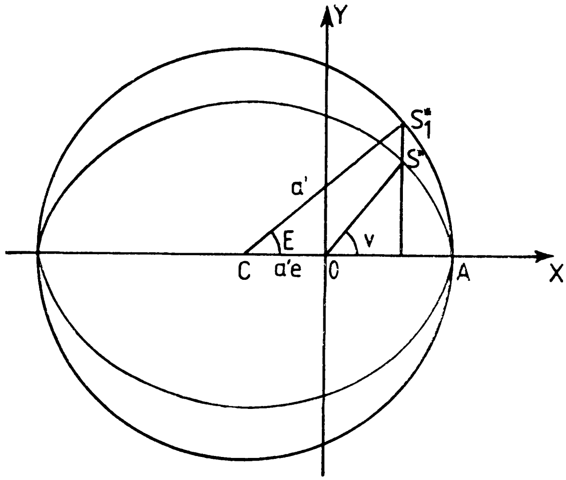


Fig. 19.2. Eccentric Anomaly

a' = semi-major axis;
e = eccentricty;
i = inclinatiion;
Ω = right ascension of the ascending node;

[1] They are also called *elliptical elements* or *keplerian elements.*

ω = latitude argument of the perigee;
τ_0 = moment of passage at the perigee.

Remark. Instead of τ_0 one can utilize the mean anomaly M_0 of a given origin time-moment t_0.

$$M_0 = n(t_0 - \tau_0) = \sqrt{\frac{GM}{a'^3}}\,(t_0 - \tau_0).$$

19.3 Geodetic Artificial Satellites

When, at one of the technical sessions of the XIth General Assembly of the *International Union of Geodesy and Geophysics*, which took place at *Toronto, Canada* in *September 1957*, the possibility of utilizing man-made Earth satellites for solving geodetic problems was being discussed hardly anybody believed in a very early realization of such a project.

Thus, on October 4th, 1957, geodesists, just like the rest of the world, in fact, were surprised at the launching of the first artificial satellite of the Earth, *Sputnik I*. Today, more than 20 years later, it is, however, almost impossible to indicate all the satellites which have been launched in the meantime and to cover all their scientific uses.

According to *NASA*'s data, since 1957, about 7,600 satellites have been launched, of which by the middle of the year 1975, 4,300 have discontinued their activity *(Sigl* 1977*)*. Consequently, the number of flying bodies amounts to about 3,300, among which there are, of course, space probes [1] as well (e.g. *Ranger, Surveyor* etc.), also burnt rockets and launching-stage remnants. At present, more than 1,000 satellites are flying round the Earth, to which an enormous quantity of former-satellite remnants should be added.

A (selected) list of the main satellites utilized for geodetic purposes is shown in Table 19.1. Some of these, e.g. the satellites *Echo I*, *Echo II* and *Pageos* a.o., have ceased their existence.

An artificial satellite may be used either as a sighting target, flying at height, or as a measuring body (sensor) in the Earth's gravitational field (or as both). This imposes a few conditions concerning the establishing of the orbit (particularly as regards the orbit inclination i and the mean height of flight), the dimensioning and instrumental equipping of the satellite, viz.:

(1) For achieving the triangulation by satellites it is necessary for the flight height to be chosen equal to approximately once or twice the mean distance between the terrestrial station points, in order to be able to obtain an optimum configuration of the spatial geodetic networks.

(2) For determining the Earth's gravitational field one needs satellites with different orbital inclinations.

(3) In order to obtain a resolving power as high as possible, the flight height must be as small as possible. But, at small flight heights, the air drag increases, thence the necessity that this atmospheric perturbation be removed by a surface/mass ratio as small as possible and by an as symmetrical as possible form of the satellite.

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[1] They are important for determining the product of the gravitational constant and the Earth's mass.

(4) As regards the instrumental equipment, one must distinguish between the active and passive artificial satellites respectively.

The active satellites are equipped, e.g., with devices for emitting *light signals* or *flashes* [1] for photographically observing the direction, with instruments for distance measurements in the field of microwaves, with a transmitter for a continuous reflection of the electromagnetic waves for Doppler measurements, with quartz clocks for emitting time (hour) signals etc. Such satellites are for example, those of the *Anna* or *Geos* [2] type, which were launched for precise geodetic purposes. The *Anna* satellites were provided with luminous sparkling lamps, which, at a command coming from the ground, were able to emit 5 successive flashes into space, with about one flashe in a second. This operation could be repeated at each revolution, during a few hours every day.

The *Geos* satellites are also equipped with devices emitting very short and strong light-flashes (they emit 7 luminous signals in series, at an interval of 4 s); they have a transmitter of calibre frequency on two wave-lenghsts for *Tranet* operations, a responding station *Secor*, reflective prisms for directly measuring the distances with lasers, and atomic clocks securing the transmission of the light signals for observations at moments established by programming, with a very high precision (up to 5×10^{-2} Hz/s). These satellites (which are good gravimetric satellites) gravitate at a height of about 1,000—1,200 km along nearly circular *(Geos C)*, oblique *(Geos 1)* or polar *(Geos 2)* orbits and are almost invisible to the naked eye.

The passive satellites [3] reflect the visible light of the Sun and may be utilized by an observer finding himself in the Earth's shadow cone, for visually or photographically observing the directions. For instance, the satellites of the *Echo* or *Pageos* type were very light balloons made from aluminium-coated polyester which swelled after launching to a diameter of 30 m and placed in orbit by means of the sublimation of a solid substance, passing under the conditions of space into the gaseous phase. The *Echo 1* and *Echo 2* satellites appeared as stars of magnitude 0 or —1, and *Pageos* as a star of magnitude 2.5 or 3.0. All three satellites were launched especially for Geodesy; the first two of them, which have laid the bases for building the cosmic-triangulations networks, have allowed connexions at distances of the order of 1,500—2,000km and *Pageos* has facilitated its being simultaneously photographed from terrestrial station points located at distances of 5,000—6,000 km from one another. Consequently, these three satellites have made it possible to carry out operations on a continental scale and even on a world scale; in particular, *Pageos* was launched in order to ensure the carrying out of the *USA* project of a *world triangulation*, initially formed out of 42 stations, lying at mean distances between them of about 4,000 km. An important role has been played for geodetic purposes — for making up the world geodetic network — by the 4 French artificial satellites launched up to the year 1974: three of them in the *Diamant 1* series (*D 1 A*, also called *Diapason*, *D 1 C* and *D 1 D*, designated also as *Diadème 1* and respectively *2*) and the fourth: *Péole*.

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[1] Sometimes also designated as *scintillations* or *pulses*.

[2] *Geodetic Earth Orbiting Satellite* — the first flying object from *EOPAP* (*Earth and Ocean Physics Applications Program*) of *NASA*.

[3] Also termed *balloon satellites* (*Levallois* and *Kovalevsky 1971*).

If the active or passive satellites are equipped with reflectors (satellites of the second generation, as e.g. *Geos C, Starlette, Lageos* [1] and, consequently, no longer give out flashes, then distance measurements with lasers, and direction photographic-observations respectively, are possible after the lasers' echo [2].

Anyhow, in order to be able to serve as a dynamic and/or gravimetric satellite and to contribute to geodetic investigations, every spherical satellite should fulfil the following requirements:

— it must gravitate at a great height (about 1,000—1,500 km, in order that the air drag be as small as possible);

— its orbit should have a nearly circular form (for ensuring constancy of the air drag);

— the surface/mass ratio has to be small (with a view to rendering negligible the influence of the radiation pressure — satellite's phase —);

— a luminous flash must be emitted every 5 s (for synchronizing observations and for visibility by night);

— this flash should be strong enough for its photograph to be taken with a ballistic camera.

19.4 Methods of Observing Artificial Satellites

The problem of geodetically observing the Earth's artificial satellites implies taking into consideration the measured quantities obtained for determining the position of the latter in space, in any coordinate system and at a given time moment. Such observations are carried out by means of visual, optical (or photographic) and radiotechnical methods as well as by methods using laser sources *(Razumov 1974)*. These methods are applied independently of one another, either simultaneously or at different moments of time.

Inasmuch as the visual observations don't have a high precision (a direction error is of the order of 5—20″ and the error in recording the observation moment, $0^s.01$), so are unsuitable for geodetic purposes, they will not be presented here. Details concerning visual observations may be found in the literature (e.g. *Izotov et al.* 1974, *Razumov* 1974).

19.4.1 Optical Observations [3]

As was shown in Section 19.1., for determining the satellite's position at a given moment, one takes its photograph against the stellar background, using ballistic cameras to this end.

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[1] *Laser Geodynamic Satellite* — the second *EOPAP* flying object — presenting a great advantage with respect to the previous satellites through the fact that the errors due to radiation-pressure and gravitational perturbations, as well as to the effect of the reflectors' configuration are reduced to minimum values.

[2] Acoustic wave which, reflected from a surface, arrives at a point where it can be perceived distinctly with respect to the direct wave at a time interval greater than, or equal to, 0.1 s.

[3] No detailled description of the observations proper will be given here, but there will be presented only a few elements necessary for understanding what follows.

Table 19.1 *Main Artificial Satellites Usable from the Geodetic Point of View, Launched up to* 1978

Satellite's Designation and Country	Brief Description	Missions	Orbital Elements		Launching Date
			$a'-R$ km	i deg	
1	2	3	4	5	6
Vanguard 1 USA	Sphere, 16 cm diameter, weighing 2 kg, the resistance structure and a minimum of equipment (radio transmitter, magnetometer etc.) inclusive.	First measurement of the Earth's figure; studies on radio wave propagation.	650 — 3968	34	March 17 1958
Vanguard 2 USA	Sphere, 15 cm diameter, weighing 9,8 kg, the resistance structure and a minimum of equipment (radio transmitter, magnetometer etc.) inclusive. First meteorological satellite provided with two television cameras.	Cloudiness studies	559 — 3320	33	February 17 1959
Vanguard 3 USA	Sphere, 51 cm diameter, weighing 25 kg, the resistance structure and a minimum of equipment (radio transmitter, magnetometer etc.) inclusive.	Study of radiadiation temperature and of magnetic fields	512 — 3744	33	September 18 1959
Echo 1 USA	Balloon-type satellite, plastic (Mylar) sphere with aluminium covering, 30 m in diameter in the initial period when sunlit and sun-heated.	Geodetic triangulation; geophysical investigations.	1524 — 1684	47	August 12 1960
Telstar 1 USA	Sphere, 86 cm in diameter and weighing 77 kg.	Radio-technical and television connexions; study of the radiation belts.	952 — 5632	45	July 10 1962

Table 19.1 (continued)

1	2	3	4	5	6
Anna 1 B USA	Spacecraft, 1 m diameter, magnetically stabilized, equipped, with light-signal transmitter, with repeater and Doppler transmitters.	Gravity data; geodetic triangulation; evolution of the distance-measuring equipment	1077—1182	50	October 31 1962
Secor USA		Determining by triangulation the coordinates of stations in America	904—935	70	January 11 1964
Echo 2 USA	Balloon-type satellite, rigidized, built by a superposition of several layers of Mylar and aluminum foils, apt to maintain its form at low temperatures and under the action of the radiation pressure.	Geodetic triangulation; coordinated Soviet-American telecommunications.	1148—1178	82	Janury 25 1964
Beacon-Explorer 22 (BE-B) USA	Spacecraft, 55 kg weight, magnetically orientated, equipped with Doppler and Minitrack beacons and with laser reflectors.	Radar ionospheric studies; gravity data; telemetric-laser experimenting.	889—1081	80	October 10 1964
Beacon-Explorer 27 (BE-C) USA	Spacecraft, 60 kg weight, magnetically orientated, equipped with Doppler and Minitrack beacons and with laser reflectors.	Study of the ionosphere; laser geodetic investigations; measurements concerning the terrestrial gravitational field	914—1314	41	April 29 1965
Explorer 29 Geos 1(A) USA	Spacecraft weighing 193 kg, stabilized with the aid of the gravity gradient, equipped with light-signal transmitter, pulse repeater for Secor-distance measurement, laser reflector, Goddard pulses, pulse repeator and Minitrack beacon	Geodetic triangulation and trilateration; gravity data; laser measurements; direct comparison with the geodetic systems.	1115—2277	59	November 6 1965

Table 19.1 (concluded)

1	2	3	4	5	6
Pageos USA	Sphere, 30 m diameter and 57 kg weight, aluminum-coated.	Geodetic triangulation.	4207—4271	87	June 26 1966
Diadème 1 Diadème 2 France	Geodetic satellites, equipped with a system of Doppler transmitters and with laser reflectors, of cylindrical form, weighing 22.7 kg.	Setting up the world geodetic network; testing solar batteries.	557—1411 590—1890	40 39	February 8 1967 February 15 1967
Geos 2 (B) USA	Similar to Geos 1, in addition radar-signal repeator in the C band.	Ths same as for Geos 1	1100—1500	106	January 11 1968
Péole France	Scientific satellite, in octahedron form, weighing 70 kg, 70 cm in diameter.	Geodetic studies space experiments with scientific equipment.	517—747	15	December 12 1970
Starlette France	Sphere with a diameter of 24 cm and weighing 47.3 kg; it consists of an icosahedral core made up of an alloy of Uranium 238 and 1.5% Molybdenum (density 18.79 g/cm^3 and weight 35.5 kg), over which there are placed 20 spherical-triangular caps (out of Al-Mg alloy); in each cap there are installed 3 laser retroreflectors.	Kinematic studies (pole movement, Earth rotation, crustal movements); study of the Earth's gravity field and of relativity; determining the positions of laser stations.	817—1096	50	February 6 1975
Geos C (3) USA	Spacecraft equipped with reflectors for lasser distance-measurements, a transponder for radar measurements in the C band of the flight altitude and provided with the so-called	Topographic-geodetic measurement of the ocean surface by means of	833—849	115	April 9 1975

Table 19.1 (concluded)

1	2	3	4	5	6
	"Satellite-to-Satellite Tracking" system.[1]	the radar altimeter, securing an accuracy of about 0.5 m. Performing for the first time an integrated observation system.			
Lageos USA	Sphere with a diameter of 60 cm and weighing about 426 kg, covered with 426 reflectors.	Monitoring the movements of the Earth's crust with a high accuracy (of the order of cm) with a view to predicting earthquakes.	5832 — 5945	110	May 4 1976
Seasat A USA	Spacecraft launched at Western Test Range, transmitting on frequencies of 2287.5 and 2265.1 MHz.	Measuring the distance from satellite to the sea surface with an accuracy of about $\pm$ 10 cm	769 — 799	108	Iune 27 1978

[1] It measures the distance and the distance variation between Geos C and the geostationary satellite ATS-F.

Remark. In Table 19.1: R is the Earth's radius; in the column a'-R there is indicated the smallest (at perigee) and the largest (at apogee) flight altitude with respect to the Earth's surface.

Since in most cases passive satellites were utilized, whose illuminations are obviously continuous, their traces on the photographic plate will consequently be continuous. However, these traces cannot be dated as a whole. Therefore, the ballistic camera is equipped with an obturator, synchronized with the observer's time, which divides the satellite's trace into segments which can be dated with a high precision. If one takes into account that the satellite travels from 7 to 8 $m \cdot s^{-1}$ in 10^{-3} s, then this corresponds to a distance of 1,000 to 1,500 km with a precision of 1/200,000. Consequently, it will be necessary to date the places in the neighbourhood of the trace discontinuities with a precision of 10^{-3}, which imposes the use of special obturators (either turning obturators mounted in front of the objective, or plate obturators), which should allow a cadence of the interruptions of at least the order of a second or less. On the other hand, when the satellite itself emits light signals, the assurance is given that, with certain corrections, their image corresponds to a precise spatial position of the satellite, identical for all the station points from which its observation is being carried out.

In order to ensure the observation of the passive satellite from various terrestrial station points at the same moment, it is necessary that the obturators of the different cameras be synchronized with a precision of approximately 10^{-3} s or that they should be connected to a *common time base*, with the same precision.

This requirement is fulfilled either by means of continuous hour signals or with the aid of very accurate clocks (e.g. atomic clocks), which are compared with a mobile clock. Consequently, one can assume that the available timekeeper, which, e.g.. in the ballistic cameras utilized by the French National Geographical Institute (*Levallois* and *Kovalevsky* 1971), consists of a turning obturator, effectively warrants the synchronization at approximately $\pm 10^{-3}$ s, at present even more accurately.

Generally, the observations are planned according to a working scheme (also termed as *working ephemeris*) which is already communicated to the observation stations prior to beginning the observations. This scheme indicates: the observation hour, the terrestrial point flown over at the moment of the observation and the satellite's height.

At the appropiate moment the camera is oriented along the desired direction. Before and after the satellite's pass, the operators carry out the taking of the photograph of the stellar backrgound, on which is being segmented the dotted trace of the satellite, and carefullly note down the observation time.

There are also recorded, to an accuracy of one degree (or half a degree), the azimuth and the zenithal angle of the cameras' optical axis. Afterwards one develops the photographic plate so that the stars' and the satellite's images may appear.

This plate (e.g., Fig. 19.3) contains a curvilinear or nearly rectilinear dotted trace (representing the satellite's trace) and the stars' traces, particularly the dotted round images appearing in an established sequence (which are the witness stellar positions recorded on the plate at the moment of photographing with the accuracy of a few hundredths of second.

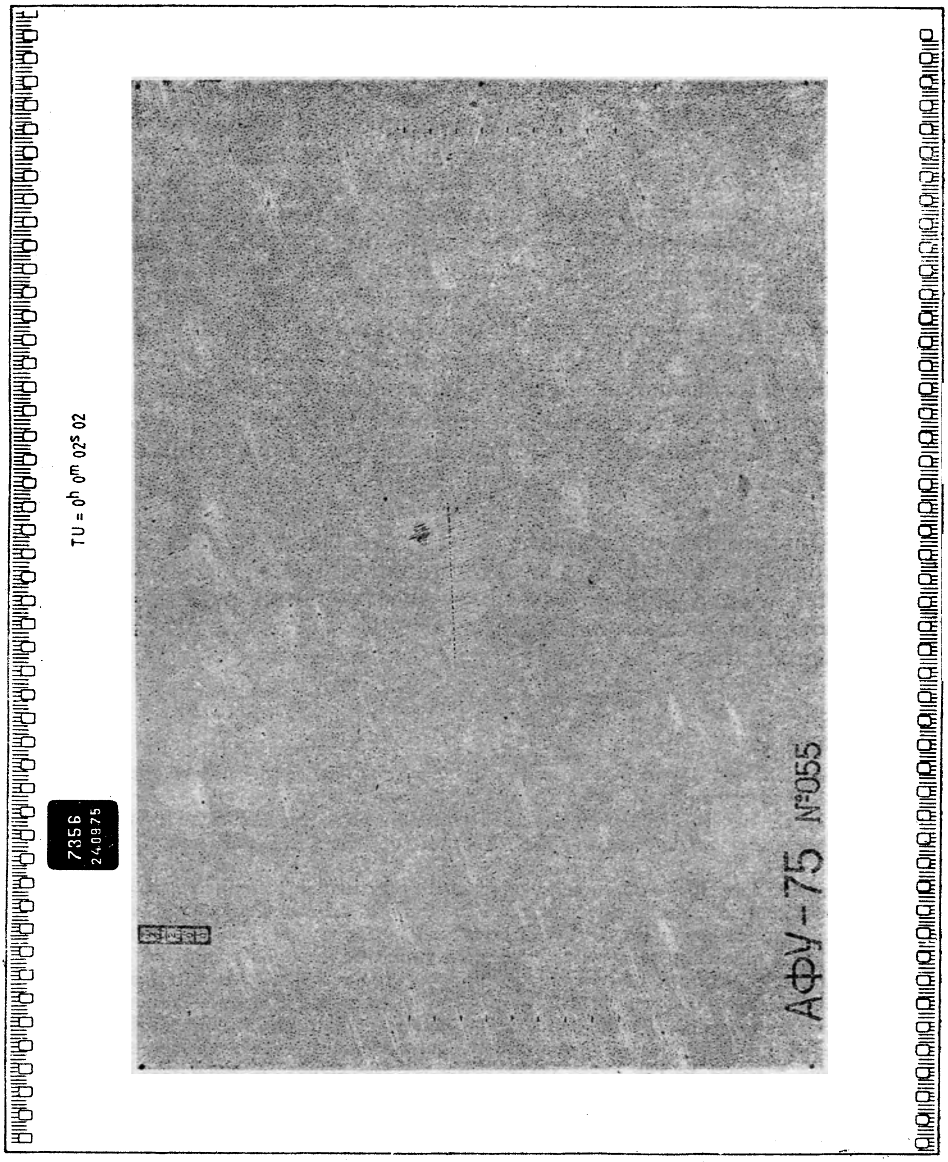
Finally, the plate is preserved and afterwards measured by means of a high-precision comparator (e.g. of the *Ascorecord* type) capable of fixing the position of the stars' and satellite's images with the accuracy of 1 μ.

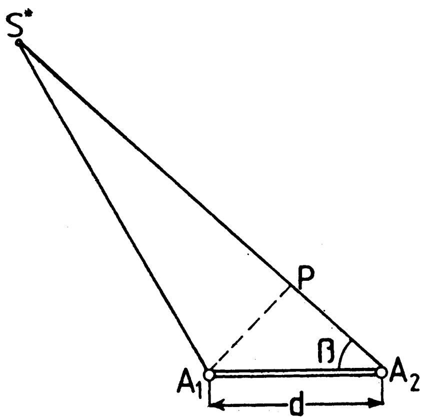
19.4.2 Radio-Technical Observations [1]

Radio-technical observations are used for determining the direction and the distance from the terrestrial station point up to the Earth's artificial satellite, as well as the rate of change of this distance. For carrying out these

[1] See also *Izotov et al. 1974, Razumov 1974.*

19.3. Photographic Plate – Pageos Satellite





19.4. Scheme of the Interference Pursuing-System:
S^* — satellite; A_1, A_2 — antennae

observations one utilizes interferential, radio-telemetric and Doppler [1] equipment.

Compared with optical observations, radio-technical observations present a special qualitly, viz. they may be practically carried out under any conditions, both by day and by night.

In order to determine the directions to the satellite, one utilizes *interferential systems.* As the system's designation itself shows, the direction towards the satellite with respect to the base line d between the antennae is determined with the aid of the interference image of the signals simultaneously received by the two antennae (Fig. 19.4; *Razumov* 1974). If the distances S^*A_1 and S^*A_2 are much greater than d, then the phase difference of the received interferometric signals will be proportional to the segment PA_2 and the direction A_2S^* is determined from the relation $\cos\beta = \overline{PA_2}/d$.

Expressing the distance PA_2 through the measured phase difference, viz.:

$$l = k + a,$$

where k is the number of complete wave cycles which are not recorded by the interferometer and which are determined with the aid of the second interferometer with a shorter base; a — the fractional part of the cycles measured with the interferometer; assuming that λ is the wave-length of the radio signal, then:

$$\cos\beta = l\lambda/d.$$

Remarks:

(1) As a check, one determines a second value of $\cos\beta$ with the aid of another pair of antennae installed at an angle of 90° with respect to the first pair.

(2) It is on the basis of the above-described principle that the American *Minitrack* work systems, whose accuracy in determining the direction towards the satellite is of the order of $\pm 0'.5 \ldots 1'.0$.

(3) During the last 10 years, several experiments on utilizing for geodetic purposes the long-base interferometric techniques (designated in short as *LBI = Long Base Interferometry*) and *very long base interferometry* (in short respectively: *VLBI*) have been made. These techniques are based on the following principle: "Random microwave signals emitted by each of several extragalactic sources are received at two widely separated antennas and beat against monochromatic signals generated locally from high-stability frequency standards (clocks). The beat signals are recorded on wide-band magnetic tape for later processing. If the frequency standards were synchronized, the beat signals at the two antennas would be the same if they were offset by a time τ equal to the extra travel time for the signal to reach the more distant antenna. For non-synchronized frequency standards, a computer search is carried out using data from sources in different directions in the sky to find the baseline length, its orientation at a particular epoch, and the epoch and the rate differences of the clocks" *(Committee on geodesy: Trends and prospects 1978)*

[1] By *Doppler effect* one understands the frequency variation of a signal, when the transmitter and the recording apparatus are moving relative to one another.

Up to now, a great number of *LBI* measurements, utilizing different antennae have been carried out (e.g. those from the *Haystack, Green Bank* and *Owen Valley* observatories, from *Fairbanks-Alaska, Onsala-Sweden* and from the *NASA* stations for the "Deep Space" *DSN* a.o.).

For instance, already in 1972 there were carried out repeated measurements on the 3,900 km base between the *Haystack* Observatory and the *DSN* station at Goldstone with an accuracy of about $\pm$ 10 cm.

The analysis of more recent observations allows precisions of about $\pm$ 3 cm to be expected when an improved catalogue will be used for the radio sources *(Committee on geodesy: Trends and prospects 1978).*

As regards the geodetic applications of the *VLBI* techniques, these have clearly shown an accuracy in determining the base lengths, thanks to repeated determinations, of between approximately $\pm$3 mm for distances of about 1 km (*Rogers et al. 1978*) and of about $\pm$ 3 cm for transcontinental distances. It is to be noted that this distance/precision relationship will shape *VLBI* into a very good technique for monitor displaying the dependence on the time parameter of the regional and continental bases.

Despite all these special advantages, *VLBI* is not yet widely utilized to this end. The main reason is the very high cost of these techniques (antennae of very great diameter, expensive atomic frequency-standards, very wide recording bands and correlation systems).

A system combining the *VLBI* advantages with those of strong satellite signals will open a new era in Geodesy. Some proposals for relatively simpler systems have already been made, such as, for example, that presented by *Counselman III et al.* (*1979*), which utilizes a compact terrestrial system (without any mobile part) and a system of reduced power termed *MITES* (*Miniature Interferometer Terminals for Earth Surveying*).

As regards *the radio-technical systems utilizing the Doppler effect,* these determine the radial velocity v_r of the satellite as a function of the difference between the emitted f_0 and received f frequencies of the radio signals:

$$v_r = \frac{v_m}{f_o}(f - f_o),$$

where v_m is the propagation velocity of the radio-waves.

The *Doppler* systems also allow the determining of the distance to the satellite at the moment of its greatest nearness to the observing station, as well as the difference ΔD of the distances to the satellite in two distant positions corresponding to the time interval Δt. If Δt is not too great, the calculating formula is:

$$\Delta D = \frac{v_m}{f_1}(f_2 - f_1)\,\Delta t.$$

The working accuracy of these systems depends on the working stability of the generators, on the resolving power of the frequency counters and on the possibilities of recording the influences of the external medium, as well as on their accuracy. In order to attenuate the influence of the refraction and of the ionospheric effects on the observational results, the *Doppler* systems work on a few coherent frequencies.

Remarks:

(1) In recent times, using *Doppler* systems has led to an accuracy in determining the radial velocity of about $\pm 1 \div 2$ cm/s (*Razumov* 1974).

(2) Although the *Doppler* systems had already begun to be utilized 17 years ago in navigation, they have only lately found their wide application. The main reason for this situation is to be found in the scientific progress achieved by *DPMS* (*Dahlgren Polar Motion Service*), which, using the observations of the *Doppler TRANET* system and a few other stations distributed in several countries, calculates at present, every other day, the pole position with a standard deviation of $\pm$ 50 cm.

The results are in agreement with the astronomical determinations and within the framework of their accuracy.

A few precision tests concerning the determination of the latitude Φ from Doppler observations at a single station and from the ephemerids calculated by *DPMS* have shown that a station can be obtained with a precision of $\pm$ 1 m after two days of observation. Such independent determinations are now carried out at a few stations.

The *Doppler* technique is also more and more utilized for building and thickening local geodetic networks. The most important example in this respect is represented by the campaign carried out with a view to re-adjusting the North American horizontal data. On this occasion, the *Doppler* satellite-stations were located at 65 terrestrial points, of which 6 were in *Alaska* and 10 on marine-drilling platforms. Other extensive *Doppler* triangulation campaigns were made in the North of *Canada* and in *Australia*.

In recent years, attempts were made to coordinate the activities of the European *Doppler* stations. Thus:

(a) The first campaign of observations, called *EDOC (European Doppler Observation Campaign)—1* was organized in May 1975 and was followed by a second one, *EDOC-2,* in the period April-May 1977. To this second observation campaign have contributed 37 stations from 16 countries in *Western Europe (Austria, Belgium, Denmark, Finland, France, F.R. of Germany, Great Britain, Greece, Ireland, Italy, Norway, Portugal, Spain, Sweden, Switzerland,* and *The Netherlands)* and *Israel,* 4 types of receivers *(CMA 722 B, GEO II, JMR-1* and *Tranet)* being used. As well as achieving a connexion between the national *Doppler* networks of the participant countries, *EDOC-2* has also pursued other objectives, such as, e. g., establishing a zero-order network of the reference points for *Western Europe* concerning a geocentric datum, investigating possible local distortions in the existing European network (*ED-50* and the following *ED-77*), building a data bank for future investigations and others.

A few preliminary results of *EDOC-2* are shown in the literature *(Wilson et al. 1978)*.

(b) Another work refers to the *German-Austrian Doppler observation campaign* (abbreviated to *DÖDOC*) which has involved 21 station points of the 1st-order network, 6 of them in *Austria* and 15 in the *F.R. of Germany*. The objectives of this campaign were the following:

— calculating the accurate parameter transformation for the entire or — if necessary — for parts of the 1st-order triangulation network;

— checking the scale and orientation of this network;

— studying the accessible accuracy of the *NNS (Navy Navigation Satellites) Doppler* measurements by means of 1st-order triangulation networks already defined in *Central Europe*;

— calculating the geocentric coordinates for all the station points of the 1st-order triangulation network of the *F.R. of Germany* and for a certain number of station points of *Austria*'s 1st-order triangulation network.

The observations at the 21 station points were carried out from the middle of the month of May up to the beginning of the month of July 1977, within the framework of 4 observation campaigns, each of 10 days, 9 *Doppler* receivers being simultaneously utilized.

The results which were obtained were presented by *Seeger et al.* (*1979*) at the second international geodetic symposium on the positioning of *Doppler* satellites, which took place at *Austin, Texas (U.S.A.)* in the period January 22—26, 1979.

(c) The Doppler observation campaign *EROS — European Range Observations to Satellites*, in short, *EROS-DOC*, which was achieved by simultaneous *Doppler* measurements immediately after the first *EROS* campaign carried out with laser, at all the station points for observing the artificial satellites in *Western Europe*, having been, or having to be, equipped in the near future with *Ranging laser system*. It was pursued the with aim of finding an independent solution for 8 observatories in Europe, viz.: *Cagliari (Italy), Dionysos (Greece), Grasse (France), Kootwijk (The Netherlands), Metsähovi (Finland), San Fernando (Spain), Wettzell (F.R. of Germany)* and *Zimmerwald (Switzerland)*.

The *EROS-DOC* campaign was performed between the 2nd and 16th of December 1977, using *Marconi CMA 722 B* receivers. Its results were also reported by *Seeger et al.* (1979) at the symposium quoted before at the end of (b).

As regards *the radio-telemetric systems*, these are based on the principle of measuring the time interval Δt during which the radio-waves travel the distance D from the terrestrial station to the satellite, to and from, the calculating formula being:

$$D = \frac{1}{2} v_m \Delta t.$$

Δt is measured either in a direct way (with pulse telemeters) or indirectly — after the results of the comparison of the phase difference between the transmitted and received signals.

The precision of the measured distance (m_D) depends on the precision in knowing the velocity v_m, on the precision in measuring the time $\Delta t (m_{\Delta t})$ and on the precision of the time moment t_0 to which are the observations (m_{t_0}) referred, viz. *(Razumov 1974)*:

$$m_D^2 = \frac{1}{4} v_m^2 m_{\Delta t}^2 + D^2 \frac{m_{v_m}^2}{v_m^2} + v_r^2 m_{t_0}^2.$$

Among the radio-telemetric systems for pursuing the Earth's artificial satellites, the most important development in applications was taken by the phase system consisting of successively comparing the distances, termed *Secor (Sequential Collation of Rangers)*[1].

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[1] For geodetic purposes one also utilizes the combined system termed as *GRARR* (*Goddard Range and Range Rate*—system of measuring the distance and its rate of charge of the *Goddard Space Flight Center*), but on a more reduced scale than the *Secor* system.

Recently, the *GRARR* system was used in the satellite-satellite pursuing experiment of *Geos C* from *ATS-6* (*Applications Technology Satellite-6*), for accurately determining the height of the orbit of *Geos C* in a geometrical way, in a test area, based on laser observations in the *C* band, at the points *Goddard, Bermuda, Grand Turk* and *Cape Kennedy*. The *ATS-6* satellite, launched on May 30, 1974, placed on a geostationary orbit, at approximately 94° *W* long. was removed at 35° *E* long. in the summer of the year 1975.

This system consists of a transmitter-receiver installed on the satellite and a few terrestrial stations equipped with emission-reception devices, quartz clocks and magnetic-tape recording facilities.

The process of distance measuring is based on the fact that from the terrestrial stations one sends out towards the satellite modulated radio signals of 12 milliseconds duration, which after having been received by the satellite station are amplified and re-transmitted. The received signals are compared as to phase with the emission frequency and with the help of the phase difference one determines the distance they have travelled. In order to remove the ambiguity of the results obtained, and to attenuate the influence of refraction, the measurements are carried out on a few modulation frequencies.

Remarks:

(1) The operations carried out until now have shown that the precision of the distance measurements by means of the *Secor* system is of the order of $\pm 5 - \pm 10$ m (*Razumov 1974*).

(2) The *Secor* system was worked out in the year 1960 by the *U.S.A. Army Map Service*, aiming at: building the thickening continental satelite network, incorporating into a single world geodetic system the remote islands and obtaining data concerning the Earth's figure and its gravitational field. The inboard equipment of the system was initially installed on the *Anna 1B* satellite.

The launching of the *Secor* satellites was begun in the year 1964. The system consists of four stations and the transmitter-receiver satellite; three stations are installed at three terrestrial station points with known geodetic coordinates, located at distances of about 4,000 km from one another, and the fourth one at the terrestrial station point whose coordinates are to be determined.

By means of this *Secor* method, the geodetic connexions between the *Hawaii Islands* and *Japan*, and *North America* respectively, as well as between *South America* and *Africa* (*Ramsayer 1968 b, Grafarend 1974*) were carried out.

(3) The working out of a combined system between the *Secor* and *Doppler* systems is in prospect.

19.4.3 Laser Observations[1]

In Geodesy, lasers[2] may be used in two ways: for telemetry and for goniometry.

If one considers a satellite of the *Geos 1* or *Geos 2* or *D1C* type provided with catadiopter prisms[3] with three faces, then this satellite is sighted with a terrestrial laser which carries out an observation in its direction. The light beam is reflected by the catadiopter prisms and re-transmitted in the direction of the source, where it is received by a telescope of large opening — 30 or 40 cm in diameter.

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[1] After *J. J. Levallois* and *J. Kovalevsky* (*1971*).

[2] *Laser* (*Light Amplification by Stimulated Emission of Radiation*) is a source capable of producing a very intensive, coherent, parallel monochromatic light beam.

[3] They have the capability of re-transmitting a light ray exactly in the direction from which it comes.

For telemetry one utilizes an active laser of 1 watt power, the sighting duration being of 20 ns[1] and the receiver focuses the response on a photocell provided with several stages of electronic amplification (photo-multiplication); the propagation duration is deduced by means of a pulse computer and the corresponding distance is obtained by multiplying the propagation duration by the speed of light, serving as length standard (as in geodimeters or tellurometers).

For goniometry one uses a passive laser whose sighting duration is of the order of 10^{-3}, with an energy of about 30 joules; the luminous track of the satellite is photographed as a star in the field of a *Schmidt* telescope. Every successful sighting leads to the photograph of a luminous trace, which may be expressed by the right ascension and the declination in the stellar field.

If one aims at recording both the distance and the spatial direction, two sources must be simultaneously available: an active laser for telemetry and a passive laser for directions. Both lasers are directed by a clock and work alternately. Very rapid progress is expected in the future towards solving this problem.

For observations, the laser is mounted on a *D.C.A.*-type turret [2], which is able to rotate in the horizontal and the vertical planes under a variable-speed command. Its optical axis is parallel to the axis of an astronomical telescope which serves to track the artificial satellite illuminated by the Sun, maintaining its image at the intersection of the reticular wires. In the case of a telemetry observation [3], the receiving telescope is also mounted on a turret and follows the motions of the directing telescope.

In the case of goniometry equipment, one may utilize a wide-field telescope — the *Schmidt* telescope — independent of the sighting turret, mounted e.g. at the equatorial telescope, and carrying out the observation in the part of the sky where the telescope is located (which observes the artificial satellite's passage).

Goniometric lasers may also be used as well as the satellites emitting light flashes; from the terrestrial station point, one photographs the reflection on the satellite for which the starting moment has been determined.

Consequently, one gets the satellite's direction in the stellar field.

It is rather difficult to specify the observation moment with an approximation better than 3×10^{-4} s, which corresponds to a path of the artificial satellite of the order of 2 m. But, as the measurement of the photographic plate doees not allow an accuracy greater than $\pm 0''.5$ for the direction, then it follows that the two precisions (for the observation moment and for the measurement of the direction on the photographic plate) are comparable and reciprocally consistent. Undoubtedly, one can ascribe to the measurements the same corrections and the same errors as in the case of measuring the light signal launched by the artificial satellite, especially as regards the systematic and random errors (anomalous refraction, twinklings),

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[1] 1 ns = 10^{-9} s.

[2] Rotatory platform.

[3] It consists of: the laser radiation is reflected back from the artificial satellite equipped with special reflectors; the measurement of the returning signal may be carried out by determining the propagation time of the radiation from the station to the satellite.

which very much reduce the precision. It should be noted that one needs simultaneously as many lasers as existing station points, because the laser's track can only be photographed at the emission station point. Therefore, the problem arises of a rigorous synchronization of the station points, which is solved today by utilizing atomic clocks or hour transmissions with three frequency-bases. These must ensure the synchronization of the local times with an accuracy of about 10^{-4} s.

Remarks:

(1) The goniometric method is still in the first tentative stage, but the future will see it working in practice.

The experiments carried out in photographing the laser rays against the stellar background have shown the possibility of determining a direction to the satellite with an accuracy of $\pm 1 \div 2''$, i.e. the same as that obtained by photographing the flashes of the lamps installed on the satellite (*Arnold 1970*).

(2) The telemetry lasers utilized in recent years in the *U.S.A.*, *France* and *Japan* (which have an output power of the transmitter of $10 \div 50$ MW, a pulse duration of $10 \div 20$ ns and a resolving capability of the time recorder of up to 1 ns) may work with a cadence of up to 1 s and can provide the distance to the satellite with an accuracy of $\pm 0.8 \div 1.5$ (*Razumov 1974*), regardless of distance, in a time of 10 to 15 min.

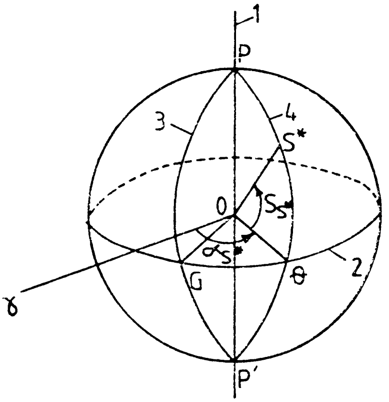
It is to be expected that by reducing the pulse duration down to 1 ns, the measuring precision should reach $\pm 10 \div 15$ cm (*Izotov et al. 1974*) or $\pm 15 \div 25$ cm (*Razumov 1974*).

19.5 Spatial Triangulation

19.5.1 Determination of a Stellar Direction

The artificial-satellite triangulation-method is based on determining the spatial direction of one or several satellite's positions with respect to the stellar background.

One assumes that at a given *universal-time*[1] moment t one observes, from a terrestrial station point, a star S^* known through its equatorial coordinates (Fig. 19.5): *right ascension* α_{S^*}[2] and declinat-



19.5. Equatorial Coordinates of the S^* Star: 1 – the Earth's rotational axis (PP′); 2 – the fundamental plane (the celestial equator); 3 – origin semi-circle (passing through Greenwich); 4 – the star's hour circle

[1] This is the *Greenwich* mean time, adopted by international agreement.

[2] The dihedral angle between the hour angle of the first point of Aries (vernal equinox) and the star's hour circle.

ion δ_{S^*}[1]. With respect to the star system, the direction cosines of the star S^* are: $\cos\delta_{S^*}\cos\alpha_{S^*}$, $\cos\delta_{S^*}\sin\alpha_{S^*}$ and $\sin\delta_{S^*}$.

If one considers at the same given moment t the *Greenwich* meridian, then this forms with the hour circle of the γ point an angle equal to the *Greenwich* sideral time θ_{S^*} so that, with respect to a system connected with the *Greenwich* meridian, the star's direction cosines are:

$$\cos\delta_{S^*}\cos(\alpha_{S^*}-\theta_{S^*})\,;\ \cos\delta_{S^*}\sin(\alpha_{S^*}-\theta_{S^*})\,;\ \sin\delta_{S^*}. \tag{19.8}$$

Remark. In the expressions (19.8) the value θ_{S^*} is completely determined by knowledge of the universal time t (1^h mean time $= 1^h$ sidereal time $\pm 9^s.856$).

It is now assumed that the direction OS^* is realized on the Earth by the axis of a fixed telescope and that one again begins observing at another time moment t', without having moved the telescope's axis, which has remained absolutely fixed with respect to the Earth. This situation corresponds to another star S_1^* having the same declination as S^* ($\delta_{S_1^*}=\delta_{S^*}$) and the right ascension $\alpha_{S_1^*}$ (which will be located at the intersection of the reticular wires) and the corresponding *Greenwich* sideral time will be $\theta_{S_1^*}$. The direction cosines of the star S_1^* will then be:

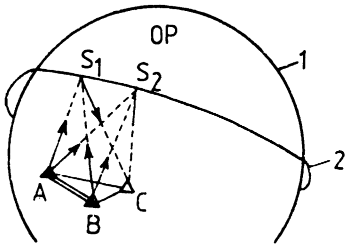
$$\cos\delta_{S^*}\cos(\alpha_{S_1^*}-\theta_{S_1^*})\,;\quad \cos\delta_{S^*}\sin(\alpha_{S_1^*}-\theta_{S_1^*})\,;\quad \sin\delta_{S^*}. \tag{19.9}$$

Remark. Inasmuch as the dihedron (PP', G, θ; Fig. 19.5) is invariant, then the direction cosines (19.9) will be the same as (19.8).

Thence it follows that knowledge of the direction cosines of any spatial direction OS^* in the global system of three-dimensional coordinates, referred to the *Greenwich* astronomical meridian, may be achieved by means of the formulae (19.8), since this direction can be related to known stars at a given time moment.

It is further assumed that the OS^* direction refers to any luminous event appearing on the stellar background, e.g. that in which a light flash emitted by an *Anna*-type artificial satellite was observed, and that, from the ground, one takes the photograph of this flash, *keeping the ballistic camera fixed*; photographing against the stellar background is carried out at a given time moment t which must not be the moment corresponding to the flash's outbreak. Using any interpolation method, one may then identify the position in right ascension and in declination (or in whatever other reference system) of the direction of this flash with respect to the stellar directions at moment t. Finally one obtains the direction cosines of the station-satellite direction in the $OXYZ$ global system of three-dimensional geodesy, since the formulae (19.8) or (19.9) just express the direction cosines in this reference system. If one considers that the launched artificial satellite is capable of sending out very short light signals (with a duration of a few microseconds) within a wide solid angle (over 20°), then from several terrestrial station points A, B, C, ... one photographs the corresponding light signals on the plates of the ballistic cameras installed at these points (Fig. 19.6). Afterwards, one photographs the stellar field which moves past the objective of the balilstic cameras,

[1] The dihedral angle $S^*O\theta$, which is equal to the arc $\widehat{S^*\theta}$.



19.6. Principle of the Spatial Triangulation:
1 — the terrestrial Globe; 2 — the artificial satellite's orbit

for each camera, the instant of taking the stars' images being carefully recorded. Consequently, for every station one will obtain a record plate, on which the satellite's light signal will appear as a point image on the stellar background, variable from one station to another. It will be possible now to interpolate the direction cosines expressed in the global coordinate system and, consequently, to define the spatial station-satellite direction. The intersection of the corresponding spatial directions defines the satellite's position in the $OXYZ$ global system in which the positions of the station points $A, B, C, \ldots$ are equally known.

19.5.2 Principle of the Method

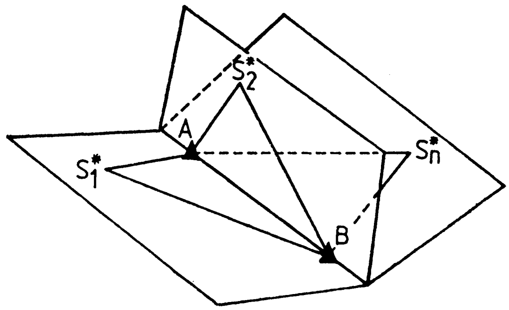
The problem whose solution will now be presented is as follows: one assumes (Fig. 19.6) that from among the station points A, B, $C, \ldots$ at which the satellite was photographed, one knows two points (A and B) in the $OXYZ$ global system of coordinates or only one point (A) and a distance [1] $(\overline{AB})$. One seeks to determine the position of the other points $C, \ldots$ Starting, for example, from the two known points A and B, from the spatial intersection of the known directions AS_1^* and BS_1^*, one determines the position S_1^* of the satellite. If one further assumes that from an unknown station point C the light signals of the satellite at S_1^* and S_2^* were photographed, then the spatial directions CS_1^* and CS_2^* can be calculated without knowing anythig about the position of the point C. Thus, on the other hand, if the satellite's positions S_1^* and S_2^* are known, then the position of the point C will follows from the reverse intersection of the S_1^*C and S_2^*C directions.

Remark. The determination of the position of the point C in the above manner doesn't resort at all to the vertical of the points A, B and C. Consequently, such a determination is independent of any dynamic hypothesis, the only direction witness-datum being the stars' background, whose positions are well known from catalogues compiled on the basis of secular observations of the position astronomy (observations carried out in fixed observatories).

This method of determination represents the principle of the method of spatial triangulation, whose originator is the Finnish geodesist *Y. Väisälä.* In the year 1945, the latter presented the principle of the spatial-triangulation method in a different manner, which is, however, the same in principle. Thus (*Levallois* and *Kovalevsky* 1971), *Y. Väisälä* considers the two stations A and B and proposes photographing the light signals $S_1^*, S_2^*, \ldots, S_n^*$. The planes defined by the points S_i^* $(i = 1, 2, \ldots, n)$, A and B all have the

[1] This defines the length scale.

connexion AB in common (Fig. 19.7). This spatial direction is, consequently, *a priori* defined by the intersections of the AS_i^*B planes in the global coordinate-system. It is thus enough to know the origin — e.g. the point A — and the distance AB (or AS_i^*) for the position of the point B to be perfectly defined. The same reasoning may be applied to the points C, ..., so that one may get, gradually nearer and nearer, a terrestrial triangulation formed by the points A, B, C, ...



19.7. The Common Connexion AB

Remark. This idea, which makes clear the fact that for working out a three-dimensional triangulation it suffices to know only the coordinates of the point A and the distance AB, raises, nevertheless, some delicate problems as regards the observation weights. In the year 1946, there were even carried out experimental determinations of the direction of the chord between two points: *Turku* (A) and *Helsinki* (B); the magnesium flashes of an aerostat were photographed. This was in fact the first experiment of the spatial-triangulation method.

In the conception presented above, the method has not get found a wider utilization, since the sighting heights have been insufficient for providing the possibility of a considerable increase of the triangulation sides. Only after the launching on October 4, 1957, in *U.S.S.R.*, of the first artificial satellite of the Earth, did there arise the actual possibility of building a large-side spatial triangulation.

The fact must nevertheless be emphasized that lately there has been an intensive preoccupation with performing the spatial triangulation by means of *high targets*, other than the Earth's artificial satellites, viz. *balloons* or *aeroplanes*.

Such utilizations are of particular interest, especially for the economic advantages they provide in comparison with the case of using the Earth's artificial satellites.

The balloon spatial triangulation represents a modern procedure for building up and checking the 1st-order regional geodetic networks.

The procedure's efficacy is great, especially from the geodetic point of view, for territories of relatively small extent with good atmospheric conditions. This procedure is already utilized in *Finland* and in *the Mongolian People's Republic* and was tested with notable results in *the German Democratic Republic*.

Generally, the technical equipment being used consists of terrestrial stations for launching the balloon probes emitting light signals, of flash probes and of photographic cameras for accurately positioning the luminous flashes. For instance (*Marek* and *Rehse* 1976), at the *Central Institute of Earth Physics* at *Potsdam* there was built a launching mobile station formed by a motor-vehicle specially laid out for securing, as well as the working room for the field team, also the equipment for launching the balloon probes (balloons, hydrogen, filling equipment, flash probes), in addition to the equipment for recording with accuracy the time moments of luminous signalling (current-providing unit, radio-signal receiver, crystal chronometer, data printer, codifying device, 10 w transmitter, aerials, additional electrical units). The flash probes are lifted up to a height of 20 $\div$ 30 km with the aid of meteorological balloons, whence they send light flashes which are recorded by the observing terrestrial stations.

Among others, such a flash probe, 2.2 kg in weight, contains: an energy source, a dipole aerial, a decodification and emergency circuit, a commutator, a heat insulator, a set of ropes, a parachute and 100 flash cartridges with magnesium powder which are lighted with the help of low-tension lighters.

For the accurate positioning of the luminous flashes one utilizes astrographs with mirror-lenses (with parallactic setting and sidereal rotation), with a focal distance of 1 m and a field-of-view diameter of a few degrees.

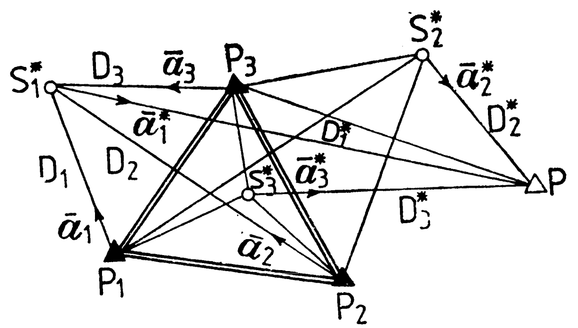
Remark. The ballistic cameras utilized for observing the Earth's artificial satellites are also suitable for the balloon triangulation.

Since the observations are made at large zenithal distances, one utilizes maximum-sensitivity emulsions, such as e.g. the *ORWO NP 27* film.

Remark. The technology for generating the light signals which was worked out at *Potsdam* as well as the mentioned apparatuses of the mobile stations and the flash probes were tested in February-March 1976 in a *test network* in the South of the *German Democratic Republic*, in which the terrestrial geodetic points were located at distances of between 50 and 160 km, with a resulting accuracy of about $\pm$ 1″.5 for a topocentric spatial direction (*Marek* and *Rehse 1976*).

The aircraft spatial triangulation is another alternative for utilizing the high targets in order to perform the geodetic networks and has been tested with promising results in *U.S.S.R.* and in the *Czechoslovak Socialist Republic.*

The determination principle consists, briefly, of the following (Fig. 19.8; *Kabeláč* 1976): If A and B are two terrestrial station points provided with ballistic cameras and laser equipment, located at distances of about 30—150 km, the flight itinerary of the aeroplane, on whose cockpit are mounted the devices for emitting at regular time intervals the flashes and on the upper parts of the tanks — laser reflectors, passes perpendicularly to the line AB, through its centre O. The connexion between the main station A and the flight control is maintained by telephone and between the flight



19.8. Elementary Figure of the Spatial Trilateration control and the pilot — by radio; both connexions are permanent. The observations are carried out by night, within a certain time interval, (e.g. 1 h), several flights being performed, during which the aeroplane passes repeatedly, several times, in the $\overrightarrow{X,Y}$ and $\overrightarrow{Y,X}$ directions. The aeroplane turns are made on the I_1 and I_2 curls. When the aeroplane finds itself near the point O on the vertical, one carries out laser measurements of the distances from the A and B stations to the target and one simultaneously photographs the laser rays sent back by the aeroplane reflectors. In this way, one obtains in the course of the measurements several hundreds of distances, of images of the reflected laser rays and of flash images respectively.

Remarks:

(1) Proceeding in the above-described or in a similar manner, has yielded an error in determining the spatial direction towards the sighted target of about $1''.5$ (*Kabelač 1976*) or one not exceeding $2 \div 3''$ (*Kuzenkov et al. 1977*).

(2) The position of the terrestrial geodetic point is determined either in the horizontal-coordinate system or in the equatorial-coordinate system, depending on the setting of the ballistic camera utilized (an azimuthal or a parallactic setting).

19.6 Spatial Trilateration [1]

In recent years one notices the trend towards using on a much wider scale distance measurements in the spatial geodetic networks, a fact which is mainly due to the increase in accuracy in the operations of the telemetric systems and the possibility of utilizing these systems practically under any atmospheric conditions. Measuring the distances from the observing stations to the Earth's artificial satellite gives conditions for determining the reciprocal positions of the terrestrial-surface points by means of building the trilateration networks.

In practice, one comes across the utilization of the trilateration method in two different forms (*Razumov* 1974):

(1) Building the trilateration network by means of the length of the chords uniting the observation spaces.

(2) Building the trilateration network by direct and indirect linear intersections of the satellite.

As regards (1), the length of chords from 60 up to 400 km may be determined by the method of intersecting the alignment of the chord to be measured with radio-geodetic systems of the type *Hiran*, *Shoran*, *Aerodist* and others. This method presupposes the utilization, as an intermediate point, of a mobile transmission-reception station installed on satellite or aeroplane, which must intersect the alignment of the terrestrial points at a height and under any angle, both constant. In order to determine the chord's length it is necessary that, gradually as the mobile station is nearing the alignment, one should measure, synchronously and at equal time intervals ($1 \div 2$ s), the distances from the terrestrial points to the station, after which adding up the two distances yields the chord's length looked for.

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[1] See also *Izotov et al. 1974*, *Razumov 1974*.

The first operations carried out using the method of intersecting the alignment were of a poor accuracy. However, subsequent experimental work, carried out with *Aerodist* and *Shoran* systems and with the *TsNIIGAIK's* aircraft-telemeter, has shown that sides of 100 km and more in length can be measured, under favourable conditions, with an accuracy of the order of $2 \div 5 \times 10^{-6}\ L$ (L — the chord's length). Lately, one has also carried out such experimental work for determining the length of chords of up to 4,000 km by the alignment-intersection method with the aid of the *Secor* system, with transmission-reception stations installed on the Earth's artificial satellite, the results being characterized by an accuracy of $\pm\ 3 \div 8 \times 10^{-6}\ L$.

The trilateration building by means of the chord method is seldom used, since it is difficult to obtain in the trilateration network (in which the basic element is the chord and the elementary figure, the tetrahedron) high-accuracy results in determining the spatial position of terrestrial points.

The reason is that for average lengths of the chords, between 1,000 and 2,000 km, the angles formed by the tetrahedra's faces are very accute ($1 \div 2°$) and the corresponding faces are very close to the Earth's surface; therefore, one gets errors incomparably larger in the position of the points to be determined in radial directions than the errors in their plane positions. This is why, according to tradition, the spatial-trilateration network obtained by means of the chords' method is reduced and adjusted on the calculation-ellipsoid's surface. In order to determine the spatial rectangular coordinates of the points, one needs here additional data referring to the altitude of these points above the calculation surface.

As regards the second approach to building the trilateration network, the elementary figure is represented by Fig. 19.8 *(Razumov* 1974*)*, where the position of the point to be determined is obtained by solving the spatial linear intersection of three reciprocal positions of the artificial satellite; each of these is located at the intersection in the initial points P_1, P_2 and P_3.

Utilizing this basic elementary figure, one can build the trilateration network on vast territories, even at a global level.

This second form of building the spatial trilateration network has many variants in practice, depending on the reciprocal disposition and on the number of the points P_j and S_j^* ($j = 1, 2, 3, ...$) being used.

As for the error distribution in a spatial trilateration network, this has a complex character. If one considers that the measurement results are rigorously synchronous and affected only by random errors and that the errors of the reciprocal positions of the points P_j have a small influence compared with the measurement errors, then the position error of the point P is determined in an indirect way: first of all one finds out the position errors of the satellite S_j^* and then with their help one determines the accuracy with which one may find the coordinates of the point to be determined.

Solving the intersection proper may be carried out by one of the methods known from the technical literature (e.g. *Rinner* 1958).

In order to determine the accuracy of the position of the Earth's artificial satellite one may use the known relation of the distance D as a function of the coordinates X^*, Y^*, Z^*, whose differentiation yields for the points S_1^*:

$$(X_1^* - X_i)\,dX_1^* + (Y_1^* - Y_i)\,dY_1^* + (Z_1^* - Z_i)\,dZ_1^* = D_j dD_j; \qquad (19.10)$$

$$i = P_1, P_2, P_3 \text{ and } j = 1, 2, 3.$$

Dividing each of these relations with D, the general system will be, in matricial form:

$$\begin{Vmatrix} I_1 & J_1 & K_1 \\ I_2 & J_2 & K_2 \\ I_3 & J_3 & K_3 \end{Vmatrix}_{(S_1^*)} \begin{Vmatrix} dX_1^* \\ dY_1^* \\ dZ_1^* \end{Vmatrix} = \begin{Vmatrix} dD_1 \\ dD_2 \\ dD_3 \end{Vmatrix}_{(S_1^*)} \qquad (19.11)$$

where I_j, J_j, K_j $(j = 1, 2, 3)$ represent the direction cosines of the unit vector $\bar{a}_j$ of the measured topocentric sides D_j.

Solving the system (19.11) leads to the generic solution:

$$\begin{Vmatrix} dX_1^* \\ dY_1^* \\ dZ_1^* \end{Vmatrix} = \frac{1}{|A|} \begin{Vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{Vmatrix}_{(S_1^*)} \begin{Vmatrix} dD_1 \\ dD_2 \\ dD_3 \end{Vmatrix}_{(S_1^*)}, \qquad (19.12)$$

in which $|A|$ — the determinant of the direction-cosine matrix and A_{jm} $(j, m = 1, 2, 3)$ — the adjugate of this matrix.

If one passes, in (19.12), from differentials to finite increases and then to the corresponding variances, one gets *(Razumov 1974)*:

$$\begin{Vmatrix} m^2_{X_1^*} \\ m^2_{Y_1^*} \\ m^2_{Z_1^*} \end{Vmatrix}_{(S_1^*)} = \frac{1}{|A|^2_{(S_1^*)}} \begin{Vmatrix} A^2_{11} & A^2_{21} & A^2_{31} \\ A^2_{12} & A^2_{22} & A^2_{32} \\ A^2_{13} & A^2_{23} & A^2_{33} \end{Vmatrix}_{(S_1^*)} \begin{Vmatrix} m^2_{D_1} \\ m^2_{D_2} \\ m^2_{D_3} \end{Vmatrix}_{(S_1^*)} \qquad (19.13)$$

Remarks:

(1) The adjugate A_{jm} in (19.12) and (19.13) in fact represent the projections of the vector products $(\bar{a}_k \times \bar{a}_j)$ of the unit vectors on the axes of the spatial-coordinate system, viz. *(Razumov 1974)*:

$$\begin{aligned} A_1(1,2,3) &= (\bar{a}_2 \times \bar{a}_3)_{X,Y,Z}; \\ A_2(1,2,3) &= (\bar{a}_3 \times \bar{a}_1)_{X,Y,Z}; \\ A_3(1,2,3) &= (\bar{a}_1 \times \bar{a}_2)_{X,Y,Z} \end{aligned} \qquad (19.14)$$

and, in view of the property $I_j^2 + J_j^2 + K_j^2 = 1$ of these vectors, one gets the following equalities:

$$\begin{aligned} A^2_{11} + A^2_{12} + A^2_{13} &= \sin^2(\bar{a}_2, \bar{a}_3); \\ A^2_{21} + A^2_{22} + A^2_{23} &= \sin^2(\bar{a}_1, \bar{a}_3) \\ A^2_{31} + A^2_{32} + A^2_{33} &= \sin^2(\bar{a}_1, \bar{a}_2). \end{aligned} \qquad (19.15)$$

(2) The determinant $|A|$ equals the volume τ of parallelepiped constructed from the vectors $\bar{a}_j$ $(|A| = \tau_{\bar{a}_1\bar{a}_2\bar{a}_3} = (\bar{a}_1 \times \bar{a}_2)\,\bar{a}_3)$ (*Razumov* 1974).

In view of (19.13) ... (19.15), the calculating formula of the accuracy in determining the position of the satellite S^* takes the form:

$$m_{S*}^2 = m_{X1}^2 + m_{Y1}^2 + m_{Z1}^2 = \frac{1}{\tau^2_{\bar{a}_1 \bar{a}_2 \bar{a}_3}} [m_{D_1}^2 \sin^2(\bar{a}_2, \bar{a}_3) + \\ + m_{D_2}^2 \sin^2(\bar{a}_1, \bar{a}_3) + m_{D_3}^2 \sin^2(\bar{a}_1, \bar{a}_2)], \tag{19.16}$$

or, using:

$$\frac{\tau_{\bar{a}_1 \bar{a}_2 \bar{a}_3}}{\sin(\bar{a}_2, \bar{a}_3)} = \sin \Delta_1; \quad \frac{\tau_{\bar{a}_1 \bar{a}_2 \bar{a}_3}}{\sin(\bar{a}_1, \bar{a}_3)} = \sin \Delta_2; \quad \frac{\tau_{\bar{a}_1 \bar{a}_2 \bar{a}_3}}{\sin(\bar{a}_1, \bar{a}_2)} = \sin \Delta_3,$$

where, e.g. Δ_1 represents the angle between the vector $\bar{a}_1$ and the plane determined by the other two vectors $\bar{a}_2, \bar{a}_3$; :

$$m_{S*}^2 = \frac{m_{D_1}^2}{\sin \Delta_1} + \frac{m_{D_2}^2}{\sin \Delta_2} + \frac{m_{D_3}^2}{\sin \Delta_3}. \tag{19.17}$$

According to (19.13) and taking into consideration, in addition to the errors of the measured distances D_j^* of the linear intersection, the errors in the positions of the artificial satellite also, which are determined independently on the direction of the measured distances, one gets for estimating the accuracy of the position of the point P looked for:

$$\left\| \begin{matrix} m_X^2 \\ m_Y^2 \\ m_Z^2 \end{matrix} \right\|_{(P)} = \frac{1}{|A|^2_{(P)}} \left\| \begin{matrix} A_{11}^2 & A_{21}^2 & A_{31}^2 \\ A_{12}^2 & A_{22}^2 & A_{32}^2 \\ A_{13}^2 & A_{23}^2 & A_{33}^2 \end{matrix} \right\|_{(P)} \left\| \begin{matrix} m_{D1}^2 + m_{S1-P}^2 \\ m_{D2}^2 + m_{S2-P}^2 \\ m_{D3}^2 + m_{S3-P}^2 \end{matrix} \right\|$$

By analogy with (19.16) and (19.17), one finds the total error in the position of the point P:

$$m_P^2 = \frac{1}{\tau_{\bar{a}_1 \bar{a}_2 \bar{a}_3}} [(m_{D1}^2 + m_{S1-P}^2) \sin^2(\bar{a}_2, \bar{a}_3) + (m_{D2}^2 + m_{S1-P}^2) \sin^2(\bar{a}_1, \bar{a}_3) + \\ + (m_{D3}^2 + m_{S1-P}^2) \sin^2(\bar{a}_1, \bar{a}_2)],$$

or:

$$m_P^2 = \frac{m_{D1}^2 + m_{S1-P}^2}{\sin^2 \Delta_1} + \frac{m_{D2}^2 + m_{S2-P}^2}{\sin^2 \Delta_2} + \frac{m_{D3}^2 + m_{S3-P}^2}{\sin^2 \Delta_3}. \tag{19.18}$$

Remarks (*Razumov 1974*):

(1) Analysing (19.17) and (19.18) shows that the maximum value of the denominators is equal to unity (if the angles between the vectors of the intersection equal 90°) and if all the sides of the intersection are measured with the same precision m, then, for the satellite's position: $m_{S_j^*}^2 = 3\, m_D^2$. Consequently, under these conditions, the erorr of the artificial-satellite's position in whatever direction will be equal to m_D.

Assuming that three intersected positions of the satellite correspond to these data and that, in turn, the vectors determining the position of the point P looked for form a triangular pyramid, then, according to (19.18), one finds that $m_P^2 = 6\, m_D^2$ and the error of the point's position in whatever direction will be 1.43 m_D.

Under average conditions, when the angles between the intersection's vectors are around 60°, then the error of the position of the determined point will be from 4 to 5 m_D.

(2) In practice, for determining the positions of the unknown points P, in the trilateration method one utilizes short-side intersections and the position of the initial points is not considered as free from errors.

The adjustment of the trilateration network is made by the least squares method, either by indirect observations or by conditioned observations (e.g., *Wolf 1968, Izotov et al. 1974*).

(3) The spatial-trilateration method also presents some drawbacks concerning its practical application, viz.:

a) The processing of a very large number of measurements. For instance, utilizing the *Secor* system for building the short-side spatial intersection may lead to the necessity of processing a few thousand distance measurements carried out within a short time interval, i.e. during the time when the satellite finds itself within the limits of visibility from the four terrestrial station points.

b) In view of a), if the measurement results are affected by systematic errors or have been obtained with some uncertainty, then their precision is relatively low.

In fact, from the series of observed elementary figures the choice must be made in accordance with their geometrical characteristics. In order to improve these characteristics, for processing there are to be taken the observational results not only for one passage of the satellite but combinations of several passages.

c) The dependence of the orientation of the spatial-trilateration network on the position of the initial points. This represents the main shortcoming of the method, which, however, is not characteristic of other methods (spatial triangulation, vector network).

Other details concerning the spatial-trilateration method are to be found in the technical literature (*Rinner 1958, Wolf 1968, Dinescu 1972, Izotov et al. 1974* etc).

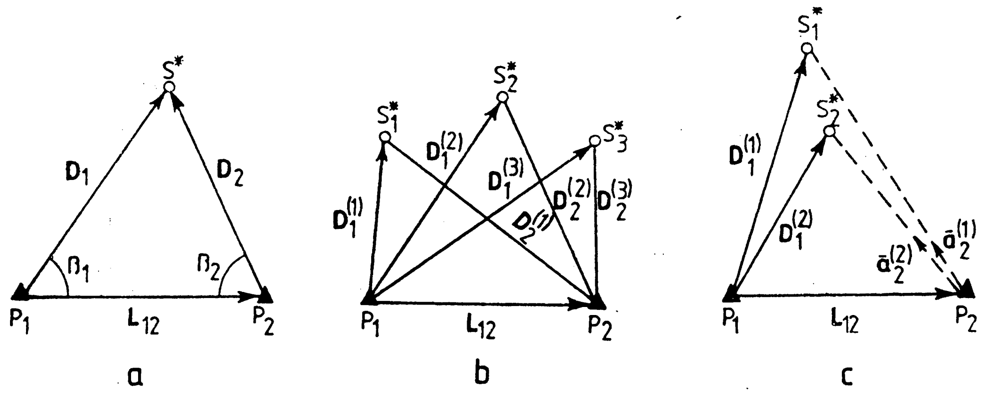
19.7 Vector Network

If from the terrestrial station points one carries out combined observations and one measures both the directions towards the Earth's artificial satellite and the distances up to this, then the stage is set for determining all the elements of the geodetic topocentric vector $\mathbf{L}$ (L, Φ, Λ) joining the neighbouring station points. The set of vectors $\mathbf{L}$ originates in the field a linear-angular network designated as a *vector network* or *cosmic polygonometry* (*Razumov* 1974).

Three basic combinations of the synchronous observations on the satellite are possible for determining the elements of the geodetic vector $\mathbf{L}$ and are represented in Fig. 19.9 (*Razumov 1974*).

In most simple cases, shown in Fig. 19.9, *a*, the geodetic vector $\mathbf{L}_{12}$ is determined from a series of observations as a difference of two astronomical topocentric vectors.

The second combination (Fig. 19.9, b) allows one to determine the geodetic vector $\mathbf{L}_{12}$ by means of the method of the linear intersection of the chord's final point with respect to three positions of the Earth's artificial satellite, given by means of the astronomical topocentric vectors $\mathbf{D}_1^{(i)}(i = 1, 2, 3)$ from the initial point P_1. In the third combination of the observations (Fig. 19.9, c), where instead of the linear intersection one utilizes a reverse angular intersection, for determining the reciprocal position of the station points P_1 and P_2 it suffices to determine two satellite's positions with respect to the chord's final point.



19.9. Basic Combinations of the Synchronous Observations:
a – determining the chord's length and direction from synchronous satellite observations;
b – determining the chord's length and direction from the combined vector linear intersection;
c – determining the chord's length and direction from the combined vector-angular intersection

In adidtion to the three basic schemes of the combined observations mentioned above, in practice there are also possible cases of fragmentary determination of the length and direction of the vector **L** by means of the triangulation or of the spatial-trilateration methods.

The vector network embodies the elements of the spatial trilateration and triangulation but differs from both through mobility in organizing the field operations and freedom in choosing the geometrical configuration.

Rermark. In the year 1969, *I. D. Zhongolovich* presented a project for a geodetic vector-traverse, connecting *Arctica* with *Antarctica*.

This traverse, which joins the points *Barentsburg* (*Spitzbergen Arch.— Arctica*), *Zvenigorod* (*U.S.S.R.*), *Cairo* (*Egypt*), *Mogadiscio* (*Somaliland*), *Reunion Island* and *Mirnyi* (*U.R.S.S.*'s scientific station in *Antarctica*) has a length of 17,000 km. Every vector of an average length of 2,800 km (joining two neighbouring station points) can be obtained from synchronous photographic observations and laser measurements.

The project is in course of accomplishment and enters into the programme of multilateral scientific co-operation *INTERCOSMOS*, in which Romania also takes part (*Bucharest Astronomical Observatory*); it is expected that the reciprocal positions of these station points will be determined with an accuracy of ± 4.8 m and that of the *Mirnyi* station point with respect to the *Barentsburg* station point with about $\pm$ 12 m (*Izotov et al. 1974*).

20
Triangulation Calculations Using Artificial Satellites

These calculations are carried out in the equatorial Cartesian global system. **The development stages of the calculations are as follows** (*Levallois* and *Kovalevsky* 1971):

(1) *Identifying the stars which are photographed on the plate.*

(2) *Calculating the direction cosines of these stars in the global system.*

(3) *Calculating the coordinates of these stars on the photographic plate, using an approximate hypothesis; only after this calculation does one proceed to carrying out the comparator measurements.*

(4) *Comparing the results of the actual measurements on the plate with the star positions calculated at stage (3) ; one determines the coefficients of a formula allowing the passage from the actually observed values to the calculated values.*

(5) *Linearizing, if necessary, the satellite's positions on the plate and establishing a formula which should define these positions on the plate, as a function of time (generally a 2nd-degree polynomial).*

(6) *Applying to these positions the transforming formula obtained at stage (4).*

(7) *Transforming the satellite's coordinates into spatial direction cosines by applying inverse transformations and recovering the direction cosines of a central direction* [1], *at a given moment of time.*

(8) *As the case stands, calculating the position in two approximations (one of which is necessary in order to calculate the differential refraction).*

(9) *Adjusting the spatial triangulation as a whole, in order to determine the positions of the unknown station points.*

These calculation stages will be presented in the sequel.

Remark. Some of the procedures and operations indicated in the calculation stages (1) ... (9) may be carried out on an electronic computer.

20.1 Identification of the Recorded Stars

In order to determine the stars recorded on the photographic plate one makes use of the following initial data: recording moment, the approximate coordinates of the site where the observations were carried out, the zenithal angle and the azimuth of the axis of the ballistic camera used.

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[1] Or of several directions established beforehand.

There are two possibilities for determining the recorded stars:

1) *By graphic constructions made on the sky map, on which the mentioned initial data were noted down.*

2) *By calculating the coordinates of the stars which are to appear on the plate and which may be found in a star catalogue*[1] *(calculated for a certain epoch).*

The determining procedure utilized at the French National Geographical Institute is the following *(Levallois* and *Kovalevsky* 1971*):* by means of an initial transformation, the camera's optical axis is expressed in the coordinate system of the star catalogue, i.e. the α^*, δ^* coordinates established for the epoch 1950. 0, and one then determines the stars which are located within a radius of about 6 cm from the middle of the plate. One transforms the coordinates of appropriate stars (reduced to the observation date) in the equatorial Cartesian global system, by astronomical calculations.

In *Romania* the following procedure has been used: On the photographic plate one chooses as witness stars (i.e. stars against whose background one has taken the satellite's photograph), those stars which have a precise, well-contoured image and as far as possible the same luminosity. They must be uniformly situated round the artificial satellite's trace image and as close as possible. As a rule, the number of these stars is equal to at least 8.

If the stars are 3—4 cm distant from the satellite's trace on the photograph and non-uniformly distributed, then their number is increased up to 25—30. In this case one will take account of the fact that all of the chosen stars must have their coordinates in star catalogues and after the star identification (according to these catalogues) it may happen that the photograph should be checked a second time, using as witness stars those located in a more favourable position.

The identification of the witness stars on the photograph is carried out in the following order:

1) *On the basis of the horizontal coordinates (the zenithal angle and the azimuth of the ballistic camera's axis) of the photograph's centre, one determines, by means of a template (the Wolf network), the corresponding equatorial coordinates* α_0^* *and* δ_0^*.

2) *On the basis of these coordinates one chooses the corresponding sheet or sheets from the "Atlas of the starred sky, with numbered stars".*

3) *By means of a projector or of a copy of the star atlas on the photograph's scale, one identifies the chosen stars on the corresponding sheet.*

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[1] At present the most exact ones are the catalogues of fundamental stars: *N* — 30 (catalogue of standard stars at epoch 1950.0, compiled by *H. R. Morgan*; 5,268 stars of magnitude , $m \leqslant 8.0$); *GC* (*General Catalogue*, for the epoch 1950.0, compiled by *Benjamine Boss*; 33,342 stars of magnitude $m \leqslant 9$); *FK* 4 (*Fourth Fundamental Catalogue*, for the epoch 1950.0, obtained by revising the *FK* 3 catalogue — 1,535 stars — to which was added the *FK* 4 — Supplement; 3,522 stars with magntitude $m \leqslant 7.5$); *KGZ* — 2 (*Catalogue of geodetic stars*, worked out by the *Pulkovo Observatory* for the epoch 1975.0; 2,957 stars chosen from *N*—30, *GC* and *FK* 4, with magnitudes $m \leqslant 6.0$). Inasmuch as these catalogues contain the coordinates α^*, δ^* for a relatively reduced number of stars, one can also utilize other less exact catalogues, such as, e.g., the *Yale University*'s catalogue (approximately 130,000 stars) and the catalogue of the *Smithsonia Astrophysical Observatory* (258,997 stars).

20.2 Calculation of the Coordinates on the Photographic Plate of the Recorded Stars

This is in fact the first calculation operation of the space triangulation proper.

One knows:

1) *The direction cosines of the stars included in the photographic plate, in the equatorial Cartesian global system* (which is here the star coordinate system itself), i.e. (19.9):

$$I = \cos \delta_{S*} \cos (\alpha_{S*} - \theta_{S*})\,; \quad J = \cos \delta_{S*} \sin (\alpha_{S*} - \theta_{S*})\,; \quad K = \sin \delta_{S*}. \tag{20.1}$$

2) *The zenithal angle and the azimuth of the plate's centre, to an approximation of 1 degree*; these elements are regarded as definitive.

3) *The approximate position of the station point*, either from an older geodetic network or as approximate astronomical coordinates provided by an astronomical determination.

For the calculation of the coordinates on the plate of the recorded stars one proceeds as follows. Taking into consideration the initial data of the problem, one can construct at the station point a *Laplacian* trihedron and one calculates the aproximate coordinates of this point in the global system. The transforming matrix will obviously the $\mathbf{R}_1$ matrix, so that (Fig. 17.9):

$$\mathbf{R}_1 = \begin{Vmatrix} -\sin \Lambda_A & \cos \Lambda_A & 0 \\ -\sin \Phi_A \cos \Lambda_A & -\sin \Phi_A \sin \Lambda_A & \cos \Phi_A \\ \cos \Phi_A \cos \Lambda_A & \cos \Phi_A \sin \Lambda_A & \sin \Phi_A \end{Vmatrix},$$

where A is the station point (Fig. 20.1). The global coordinates of the stellar vector are transformed with the help of the $\mathbf{R}_1$ matrix in the Laplacian trihedron.

In order to calculate the approximate coordinates of the stars' traces on the photographic plate, one must now translate this system into the instrumental-coordinate system (§ 17.1.1), i.e. refer to the stellar vectors, expressed in the station's local system, to a system whose z' axis is the camera's optical axis, the x' axis — the direction of the plate's horizontals, the y' axis is the direction of the plate's line of greatest slope (considered as positive downwards). The plane of the plate

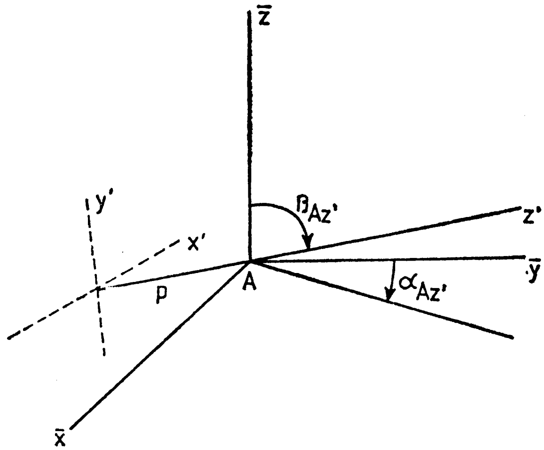

Fig. 20.1. The x', y', z' Instrumental-Coordinaet System

is itself considered higher ($-p$) with respect to the origin (where p is the objective's principal distancc [1], (Fig. 20.1).

In order to find out the corresponding transforming matrix between the station's local and instrumental systems respectively one carries out two rotations, e.g.:

1) *A rotation of the* $\bar{y}$ *axis by an angle equalling* $-\alpha_{Az'}$, *round the* $A\bar{z}$ *axis.*

2) *A rotation of the new position of the* $A\bar{x}$ *axis (after the rotation 1) by an angle equalling*-$\beta'_{Az'}$.

The resulting rotation matrix is then *(Levallois* and *Kovalevsky* 1971 *):*

$$\mathbf{Q}^T = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\ \beta'_{Az'} & -\sin\ \beta'_{Az'} \\ 0 & \sin\ \beta'_{Az'} & \cos\ \beta'_{Az'} \end{Vmatrix} \begin{Vmatrix} \cos\ \alpha_{Az'} & -\sin\alpha_{Az'} & 0 \\ \sin\ \alpha_{Az'} & \cos\alpha_{Az'} & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

whence:

$$\mathbf{Q}^T = \begin{Vmatrix} \cos\ \alpha_{Az'} & -\sin\alpha_{Az'} & 0 \\ \cos\ \beta'_{Az'}\ \sin\ \alpha_{Az'} & \cos\beta_{Az'}\ \cos\ \alpha_{Az'} & -\sin\ \beta'_{Az'} \\ \sin\ \beta'_{Az'}\ \sin\ \alpha_{Az'} & \sin\ \beta'_{Az'}\ \cos\ \alpha_{Az'} & \cos\ \beta'_{Az'} \end{Vmatrix} \tag{20.2}$$

The direction cosines of the stellar vectors are consequently given by the relation:

$$\| \bar{S}_h^* \| = \begin{Vmatrix} I_h \\ J_h \\ K_h \end{Vmatrix} = \mathbf{Q}^T \mathbf{R}_1 \| S_h^{*\prime} \|,$$

in which h ($h = 1, 2, \ldots, n$) represents the number of stars utilized, whence follows the calculating relations of the coordinates on the plate of the recorded stars:

$$\frac{x'_e}{I_h} = \frac{y'_e}{J_h} = \frac{-p}{K_h}, \tag{20.3}$$

where I, K, J are calculated from (20.1).

Remarks:

(1) The $x_c, y_c, -p$ coordinates calculated using formulae (20.3) represent approximate values, since they were calculated by means of an approximate system of the positions of the instrumental axes and don't take any account of the distortions of the light-beam cones due to the imperfections of the objectives, of the plates etc.

[1] This is the length, slightly different from the focal distance, corresponding on the average to the sharpness volume.

(2) Nevertheless, calculating the coordinates by means of (20.3) has the following advantages:

— a solution close to the rigorous one is provided;

— with the aid of (20.3) one gets the plate coordinates in a rigid manner, by means of a calculation which can be carried out to any degree of accuracy:

— the coordinates calculated from (20.3) may serve to identify the stars' positions with the comparator, as well as a nearly rigorous solution in a calculation in which the latter appears as a linearized solution.

20.3 Comparison of the Measured and Calculated Values

By means of measurements carried out at the comparator on the plate, one gets the coordinates x_0, y_0, which are to be compared with the coordinates x'_c, y'_c, calculated using formulae (20.3). We next describe the manner of establishing the relation which connects the two sets of coordinates.

First of all, it is necessary to take into consideration the following important remark: the values (x_0, y_0) and (x'_c, y'_c) are not perfectly comparable as they exhibit systematic discordances, the ones with respect to the others, due to several causes, among which are: the atmospheric refraction having affected the image position on the plate; the distortion proper of the objective; the objective's optical defects, in particular defects in centring the lenses; local defects on the photographic plate etc.

The refraction correction is calculated by means of the usual formulae and may be applied either to the x_0, y_0 values or the x'_c, y'_c with the opposite sign. *J. J. Levallois* and *J. Kovalevsky* (1971) recommend the second way as being the best one. Thus, the matricial operation $\mathbf{R}_1 \, || \overrightarrow{S^{*\prime}} ||$ transforms the stars' coordinates in the local system of the station and gives for each star the direction cosines:

$$i = \sin\beta' \sin\alpha^*; \; j = \sin\beta' \cos\alpha^*; \;\; k = \cos\beta'.$$

On the other hand, calculating the refraction correction with the aid of the *Laplace formula*[1] *(Levallois 1969,* p. 263*)*:

$$d\beta' = 60'', 27 \tan\beta' - 0'',067 \tan^3\beta',$$

in which $\tan\beta' = \sqrt{\dfrac{i^2 + j^2,}{k^2}}$, then yields the final values of the direction cosines in the local system:

$$\sin(\beta' + d\beta') \sin\alpha^*, \sin(\beta' + d\beta')\cos\alpha^*, \cos(\beta' + d\beta').$$

As regards the distortion correction, which is a revolution correction, this has a radial form *(Levallois* and *Kovalevsky 1971)*:

$$x_0 = x_0(a + b\rho^2 + c\rho^4); \; y_0 = y_0(a + b\rho^2 + c\rho^4),$$

where $\rho = \sqrt{x_0^2 + y_0^2}$, and a, b, c, are coefficients which can be extracted from a special study of the objective (either on the background of zenithal stars, or by a study at the photogoniometer).

[1] This is a correct formula, independent of any hypothesis whatever, up to zenithal angles of 80°.

As to other defects of the objective, these may be as described functions of x_0, y_0 by means of a rather difficult study which has to be undertaken for each objective.

Once all the systematic corrections have been applied to the x_0, y_0 values, these should differ from x'_c, y'_c only due to random influences; between the two corresponding collections (x_0, y_0) and (x'_c, y'_c) respectively there exists a homographic relation [1], i.e. one can pass from the (x_0, y_0) frame to the (x'_c, y'_c) frame by means of formulae of the type *(Levallois* and *Kovalevsky* 1971, p. 48 (12 bis)):

$$x'_c = \frac{x_0 + a + bx_0 + cy_0}{1 + Ax_0 + By_0}; \quad y'_c = \frac{y_0 + a' + b'x_0 + c'y_0}{1 + Ax_0 + By_0}, \qquad (20.4)$$

in which a, b, c, a', b', c', A and B are the coefficients to be determined by comparing the calculated values with the measured ones. These 8 unknown coefficients are determined by the least squares method, using all of the stars contained within the usable area of the plate. Thus, the system of observation relations:

$$a + bx_0 + cy_0 - Ax_0x'_c - By_0y'_c + x_0 - x'_c = v;$$
$$a' + b'x_0 + c'y_0 - Ax_0y'_c - By_0y'_c + y_0 - y'_c = v';$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

is solved by known procedures.

Once these coefficients have been determined, one can transform, with the aid of (20.4), all the (x_0, y_0) coordinates, corrected for systematic errors, into (x'_c, y'_c) coordinates, i.e. in the system of coordinates of the satellite's positions on the plate.

20.4 Linearization of the Satellite's Position on the Photographic Plate

If the artificial satellite is an active satellite *(Anna, Geos)*, its positions on the photographic plate are numerous and they are measured in their entirety.

If, however, a passive artificial satellite has been used, since there exist several tens of successive points on the photographic plate, representing the successive positions of the satellite, one doesn't calculate (with very rare exceptions) the spatial coordinates of all these points.

In this section we shall briefly present[2] the method of obtaining a formula which should define, as functions of the time moment, the satellite's successive positions on the photographic plate, i.e. $x_0 = x_0(t)$ and $y_0 = y_0(t)$.

---

[1] Between the straight lines of the two spaces (x_0, y_0) and (x'_c, y'_c) respectively, there exists a one-one correspondence.

[2] For details see the technical literature (e.g. *Levallois* and *Kovalevsky 1971*).

The path of the artificial satellite may be assimilated, on a certain arc, to a 4th-degree curve. Consequently, one can *a priori* consider that:

$$x_0 = A_1 + B_1t + C_1t^2 + D_1t^3 + E_1t^4; \qquad (20.5)$$
$$y_0 = A_2 + B_2t + C_2t^2 + D_2t^3 + E_2t^4.$$

The $A_1, ..., E_1$, $A_2, ..., E_2$ coefficients are obtained using all of the satellite's observed positions as functions of the time moment t, which are known.

Remarks:

(1) If the satellite's photographing is synchronously performed from several simultaneous station points, one must take into account the fact that, as the satellite is not at the same distance from all the stations out of which it is photographed, synchronisation of images on the photographic plate is only achieved if one refers the (x_0, y_0) positions — which correspond to ths actual spatial position of the satellite — to the time moment t.

What is recorded on the photographic plate at the time moment t is in fact the position occupied by the satellite at the moment $(t - D/c)$, where D represents ths distance from the station point to the satellite and c — the speed of light. To the artificial satellite's spatial position at the moment t (the same for all of the station points) there corresponds, consequently, on the photographic plate, the position $x_0(t + D/c)$, $y_0(t + D/c)$, where x_0, y_0 are calculated by means of the formulae (20.5).

(2) From the theoretical point of view, the operation of linearizing the artificial satellite's positions on the photographic plate should be carried out for every station point, but a simple calclation shows that if one operates on 1 time minute, then this can be adopted as an average correction, which corresponds to the central position. In one minute the artificial satellite travels 450 km, which corresponds to a maximum correction $D/c = 0^s.0015$ (or $0^s.00075$ for each of the extremities of the path of 450 km with respect to its middle).

(3) The t moments themselves, corresponding to the observations, must also be corrected with a propagation-time correction for synchronizing with the hour signals.

20.5 Returning to the Global System

As was shown in Section 20.3, after applying the systematic corrections to the (x_0, y_0) coordinates, these are transformed into (x_c', y_c') coordinates, with the help of the formulae (20.4). Afterwards, one applies to the $x_c', y_c', -p$ coordinates the transforming matrix $\mathbf{Q}^T$ (20.2), the corresponding direction cosines being obtained in the local system. Inasmuch as (x_0, y_0) were not applied for the satellite's positions, it is further necessary to apply the refraction correction, using appropriate formulae, e.g. *(Levallois 1969*, p. 267 (3—18) and (3—18 bis)):

$$d\beta_{PS*} = d\beta_\infty \frac{H_0 - H_e}{H_0}; \quad d\beta_{S*P} = d\beta_\infty \frac{H_e}{H_o} \qquad (20.6)$$

or [1]:

$$d\beta_{S*P} = d\beta_\infty \frac{H_e}{H_0}\left(1 - \frac{1}{300 \cos^2 \beta}\right), \qquad (20.7)$$

---

[1] Formula obtained by *H. Dufour*, valid for large zenithal angles.

where (Fig. 20.2) $d\beta_{PS*}$ and $d\beta_{S*P}$ represent the refraction angle at the station point P, and the reciprocal refraction angle respectively; $d\beta_{\infty} = d\beta_{PS*} + d\beta_{S*P}$ is the total astronomical refraction; H_e — the height equivalent to the atmosphere ($H_e = 8.00$ km) [1], $H_0 = D \cos\beta$ — the artificial satellite's altitude above the horizon of the observing station point [2] and D — the geometrical distance to the artificial satellite at the observation moment.

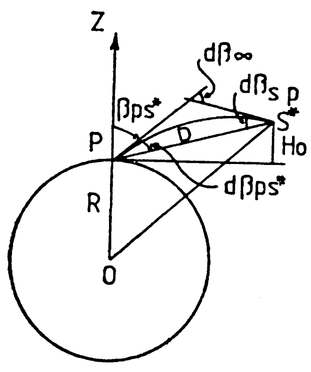

Fig. 20.2. Refraction on An Artificial Satellite:
P — station point; S^* — artificial satellite;

Remark. The new direction cosines, resulting from the application of the refraction correction, now correspond to local spatial directions.

With the help of the $\mathbf{R}_1$ matrix (Section 20.2), these local spatial directions are brought into the global system, leading in the end to the definitive spatial directions, which correspond point by point at all the stations from which the artificial satellite's position was observed at a given time moment t. Afterwards one may pass to calculating the positions of the unknown terrestrial station points.

20.6 General Adjustment of a Spatial Triangulation by means of Artificial Satellites

20.6.1 General Considerations

The problem which arises is to find out the positions of the unknown station points, if possible, the artificial satellite's positions as well, and to express them in the global system of $OXYZ$ coordinates.

As was seen in Chapter 15, in order to carry out the adjustment of an astro-geodetic triangulation on the ellipsoid (or in the plane), one starts from angular measurements or azimuthal-direction measurements and one minimizes the sum of the squares of the differences between their measured values and the adjusted ones. By analogy, in the case of the spatial triangulation by the Earth's artificial satellites, where the measured values are represented by the totality of (x_0, y_0) values observed on the plate, one will minimize the sum of the squares of the differences:

$$\Sigma \frac{1}{p_x^2}(x^* - x_0)^2 + \Sigma \frac{1}{p_y^2}(y^* - y_0)^2,$$

[1] If the station point is at sea level, then $H_e < 8$ km

[2] It is not the actual altitude of the artificial satellite.

where p_x, p_y are the corresponding weights and (x^*, y^*) — the adjusted values, corresponding to a measurement which would be subsequently carried out on the photographic plate (in the instrumental local system) by the operator, assuming that the artificial satellite has been observed in the space of the adjusted station-points network. Consequently, starting from the approximate values calculated in a general network, it is necessary to establish the corresponding observation relations in the instrumental local system. Realizing such a conception is rather complicated. Therefore, one considers *(Levallois* and *Kovalevsky 1971)* a simplifying hypothesis which will now be presented.

Let P be a station point at which one considers the observed spatial direction PS^* and the corresponding adjusted direction $P\bar{S}^*$ (satisfying all conditions) on which is located the definitive position assigned to the artificial satellite $\bar{S}^*$ (Fig. 20.3). Consequently, after carrying out the adjustment, between these two directions there is a small angle Δ. In this case, one may choose as hypothesis $\Sigma\Delta^2 = \text{min.}$, which also means:

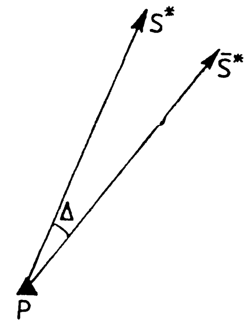

Fig. 20.3. The Angle between the Observed and the Adjusted Directions of An Artificial Satellite

$$\Sigma \frac{1}{p_x^2}(x^* - x_0)^2 + \Sigma \frac{1}{p_y^2}(y^* - y_0)^2 = \text{min.},$$

at the centre of the photographic plate. Concerning this hypothesis, *J.J. Levallois* and *J. Kovalevsky* (1971) show that *"it is not absurd, although one knows that the accuracy in the comparator aiming is correlated with the direction of the daily rotation for stars and with the direction of its photographic path for the satellite aimings".*

If $\bar{i}, \bar{j}, \bar{k}$ are the direction cosines of the adjusted directions and $\hat{i}, \hat{j}, \hat{k}$ — those of the corresponding observed directions, then:

$$\Sigma\Delta^2 \approx \Sigma((\bar{i} - \hat{i})^2 + (\bar{j} - \hat{j})^2 + (\bar{k} - \hat{k})^2). \tag{20.8}$$ [1]

If in (20.8) one expands the brackets and takes into account that $\bar{i}^2 + \bar{j}^2 + \bar{k}^2 = 1$ and $\bar{i}\hat{i} + \bar{j}\hat{j} + \bar{k}\hat{k} = \cos\Delta$, then:

$$\Sigma\Delta^2 \approx 2\Sigma(1 - \cos\Delta) = 4\Sigma\sin^2\Delta/2. \tag{20.9}$$

The formula (20.9) allows to write the observation relations, which lead only to obtaining the direction cosines of the corresponding directions. The calculating method will consist of working out an appropriate three-dimensional network which contains the coordinates of the satellite and of the known and unknown station points, after which one writes the corresponding observation relations[2], as well as the contingent condition equations (conditions of directly-measured-sides agreement etc.).

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[1] The relation holds up to 4th-order approximating terms.

[2] For the satellite-satellite direction three observation relations appear.

20.6.2 Calculating a Position of the Artificial Satellite

If the number of the artificial satellite's sightings is large, then the calculation of one of its positions can be made by spatial intersections. Thus one knows the direction cosines of the sightings P_1S^* and P_2S^* (Fig. 20.4), as well as the coordinates of the station points P_1 and P_2. One attempts to calculate the coordinates of the satellite point S^*. To this end, one resorts to the intersection of the straight lines P_1S^* and P_2S^*:

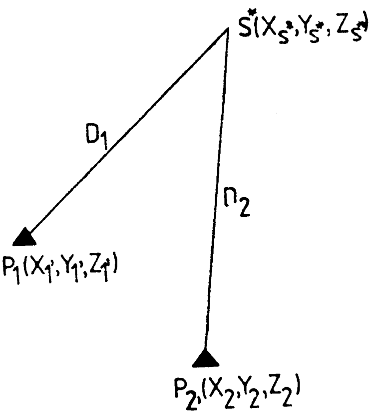

Fig. 20.4. The Spatial Intersection of the S^* Satellite Point

$$\frac{X_{S^*} - X_1}{I_1} = \frac{Y_{S^*} - Y_1}{J_1} = \frac{Z_{S^*} - Z_1}{K_1}$$

and (20.10)

$$\frac{X_{S^*} - X_2}{I_2} = \frac{Y_{S^*} - Y_2}{J_2} = \frac{Z_{S^*} - Z_2}{K_2},$$

where (I_1, J_1, K_1) and (I_2, J_2, K_2) are known from inspection of the photographic plate (20.1). The system (20.10) is compatible inasmuch as there exist four equations with three unknowns; the two space straight lines meet only if they are co-planar. Assuming that the two space straight lines are co-planar and $P_1S^* = D_1$, $P_2S^* = D_2$, the system (20.10) becomes:

$$\begin{aligned} X_{S^*} &= X_1 + I_1D_1; & X_{S^*} &= X_2 + I_2D_2; \\ Y_{S^*} &= Y_1 + J_1D_1; & Y_{S^*} &= Y_2 + J_2D_2; \\ Z_{S^*} &= Z_1 + K_1D_1; & Z_{S^*} &= Z_2 + K_2D_2. \end{aligned} \qquad (20.11)$$

Solving the system (20.11) — consisting of 6 equations with 5 unknowns $(X_{S^*}, Y_{S^*}, Z_{S^*}, D_1, D_2)$ — by the least squares method, yields the coordinates of the artificial satellite S^* looked for.

Remarks:

(1) The X_{S^*}, Y_{S^*}, Z_{S^*} coordinates of the artificial satellite at various moments are needed for calculating the geocentric coordinates of the observing terrestrial station-points, for calculating elements of the satellite's orbit etc.

(2) For determining the geocentric coordinates of the artificial satellite four possible cases may be identified (*Dinescu 1971*):

a) from optical and laser simultaneous observations at one station point;

b) from optical simultaneous observations at two station points;

c) from laser simultaneous observations at three station points;

d) from simultaneous observations, an optical one and a laser one, at two station points.

These cases are tackled in detail in several technical contributions (e.g.: *Popovici 1962 b, Dinescu 1970, 1971, 1972*).

20.6.3 Adjustment of the Spatial Triangulations[1]

If from several station points P there were obtained exactly simultaneously, with the aid of the corresponding photographic plates, the *(topocentric)* coordinates α_{S^*} and δ_{S^*} of the artificial satellite (19.8)), and $\alpha_{S^*} - \theta_{S^*} = \Lambda_{S^*}$ respectively (where θ_{S^*} is the *Greenwich* sidereal time), then n an $OXYZ$ global system for each of the points P (of astronomical coordinates Φ_p and Λ_p)[2] there exists one position vector $\mathbf{X}$, viz.:

$$\mathbf{X}^T = \| X_P, Y_P, Z_P, \| = D \| \cos\Phi_P \cos\Lambda_P, \cos\Phi_P \sin\Lambda_P, \sin\Phi_P \|, \tag{20.12}$$

while the position vector $\mathbf{Y}$ of the satellite point S^* is:

$$\mathbf{Y}^T = \| X_{S^*}, Y_{S^*}, Z_{S^*} \|. \tag{20.13}$$

The vector $\mathbf{a}$ of the observations carried out from P towards S^*, if $PS^* = s$, is:

$$\mathbf{a}^T = s \| \cos\delta_{S^*} \cos\Lambda_{S^*}, \cos\delta_{S^*} \sin\Lambda_{S^*}, \sin\delta_{S^*} \|. \tag{20.14}$$

In adjustment, for $\mathbf{X}$ and $\mathbf{Y}$ one introduces the approximating vectors $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(0)}$ [3] respectively and, because of the unavoidable observation errors, one adds to the $\mathbf{a}$ vector a corrective vector $\mathbf{v}_a$ so that one obtains:

$$\mathbf{a} + \mathbf{v}_a = (\mathbf{Y}^{(0)} + \mathrm{d}\mathbf{Y}) - (\mathbf{X}^{(0)} + \mathrm{d}\mathbf{X}),$$

whence:

$$\mathbf{v}_a = \mathrm{d}\mathbf{Y} - \mathrm{d}\mathbf{X} + (\mathbf{Y}^{(0)} - \mathbf{X}^{(0)} - \mathbf{a}). \tag{20.15}$$

Differentiating (20.12) and (20.13), if $v_{\delta_{S^*}}$ and $v_{\Lambda_{S^*}} = v_{\alpha_{S^*}}$ represent the corrections of δ_{S^*} and Λ_{S^*} (respectively α_{S^*}) and the arguments of $\mathrm{d}\mathbf{X}$ are then $\mathrm{d}\Phi_P$ and $\mathrm{d}\Lambda_P$, yields:

$$\mathbf{v}_a = \mathbf{a}_1 v_{\delta_{S^*}} + \mathbf{a}_2 v_{\Lambda_{S^*}}, \tag{20.16}$$

where:

$$\mathbf{a}_1^T = s \| -\sin\delta_{S^*} \cos\Lambda_{S^*}, -\sin\delta_{S^*} \sin\Lambda_{S^*}, \cos\delta_{S^*} \|;$$

$$a_2^T = s \| -\cos\delta_{S^*} \sin\Lambda_{S^*}, \cos\delta_{S^*} \cos\Lambda_{S^*}, 0 \|,$$

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[1] *Arnold 1961*, *Arnold* and *Schoeps 1964*, *Wolf 1968*.

[2] If the station points P were, e.g., on the surface of a co-axial reference ellipsoid (with the curvature radius of the poles c), then, considering that $V^2 = 1 + e'^2 \cos^2 B$, the $\mathbf{X}$ vector will be:

$$\mathbf{X}^T = \frac{1}{V} \| c \cos B \cos L, c \cos B \sin L, b \sin B \|,$$

where b is the ellipsoid's semi-minor axis and B, L — the geodetic coordinates of the point P.

[3] Calculated from $\mathbf{X}^0 + \mathbf{a}$.

and, if $D \approx$ const.:

$$\mathbf{dX} = \mathbf{b}_1\, d\Phi_P + \mathbf{b}_2\, d\Lambda_P, \tag{20.17}$$

in which:

$$\mathbf{b}_1^T = D\,\|\, -\sin\Phi_P \cos\Lambda_P,\ -\sin\Phi_P \sin\Lambda_P,\ \cos\Phi_P \,\|\,;$$

$$\mathbf{b}_2^T = D\,\|\, -\cos\Phi_P \sin\Lambda_P,\ \cos\Phi_P \cos\Lambda_P,\ 0 \,\|.$$

Introducing (20.16) and (20.17) into (20.15), one gets:

$$\mathbf{a}_1 v_{\delta_{S^*}} + \mathbf{a}_2 v_{\Lambda_{S^*}} = \mathbf{dY} - (\mathbf{b}_1 d\Phi_P + \mathbf{b}_2 d\Lambda_P) + (\mathbf{Y}^{(0)} - \mathbf{X}^{(0)} - \mathbf{a}). \tag{20.18}$$

If the left-hand side of (20.18) is multiplied first by $\mathbf{a}_1^T$ and then by $\mathbf{a}_2^T$ and one considers that:

$$\mathbf{a}_1^T\mathbf{a}_1 = s^2;\ \mathbf{a}_2^T\mathbf{a}_2 = s^2\cos^2\delta_{S^*}\ ;\ \mathbf{a}_1^T\mathbf{a}_2 = \mathbf{0};\ \mathbf{a}_2^T\mathbf{a}_1 = \mathbf{a}_1^T\mathbf{a} = \mathbf{a}_2^T\mathbf{a} = \mathbf{0},$$

then one obtains:

$$s^2 v_{\delta_{S^*}} = \mathbf{a}_1^T\, \mathbf{dY} - \mathbf{a}_1^T\mathbf{b}_1\, d\Phi_P - \mathbf{a}_1^T\mathbf{b}_2\, d\Lambda_P + \mathbf{a}_1^T(\mathbf{Y}^{(0)} - \mathbf{X}^{(0)});$$

$$s^2\cos^2\delta_{S^*}\, v_{\Lambda_{S^*}} = \mathbf{a}_2^T\mathbf{dY} - \mathbf{a}_2^T\mathbf{b}_1 d\Phi_P - \mathbf{a}_2^T\mathbf{b}_2 d\Lambda_P + \mathbf{a}_2^T(\mathbf{Y}^{(0)} - \mathbf{X}^{(0)}), \tag{20.19}$$

or, if one carries out the corresponding products of vectors *(Wolf* 1968*)*:

$$\begin{aligned}
v_{\delta_{S^*}} = \frac{1}{s}\{&-\sin\delta_{S^*}\cos\Lambda_{S^*}\, dX_{S^*} - \sin\delta_{S^*}\sin\Lambda_{S^*}\, dY_{S^*} + \cos\delta_{S^*}\, dZ_{S^*} -\\
&- D(\sin\delta_{S^*}\sin\Phi_P\cos(\Lambda_{S^*} - \Lambda_P) + \cos\delta_{S^*}\cos\Phi_P)\, d\Phi_P +\\
&+ D\sin\delta_{S^*}\cos\Phi_P\sin(\Lambda_{S^*} - \Lambda_P)\, d\Lambda_P + \sin\delta_{S^*}\cos\Lambda_{S^*}\,(X_P^{(0)} -\\
&- X_{S^*}^{(0)}) + \sin\delta_{S^*}\sin\Lambda_{S^*}\,(Y_P^{(0)} - Y_{S^*}^{(0)}) - \cos\delta_{S^*}\,(Z_P^{(0)} - Z_{S^*}^{(0)})\};\\
v_{\Lambda_{S^*}}\cos\delta_{S^*} = \frac{1}{s}\{&-\sin\Lambda_{S^*}\, dX_{S^*} + \cos\Lambda_{S^*}\, dY_{S^*} - D\sin\Phi_P\sin(\Lambda_{S^*} -\\
&- \Lambda_P)\, d\Phi_P - D\cos\Phi_P\cos(\Lambda_{S^*} - \Lambda_P) d\Lambda_P + \sin\Lambda_{S^*}\,(X_P^{(0)} -\\
&- X_{S^*}^{(0)}) - \cos\Lambda_{S^*}\,(Y_P^{(0)} - Y_{S^*}^{(0)})\}.
\end{aligned} \tag{20.20}$$

If from (20.20) one develops the corresponding normal equations, one afterwards determines the unknowns dX_{S^*}, dY_{S^*}, dZ_{S^*}, $d\Phi_P$, $d\Lambda_P$.

Remarks:

(1) For the quantity D, and $X_P^{(0)}$, $Y_P^{(0)}$, $Z_P^{(0)}$ respectively, one assumes that the points P are located on an ellipsoid (D — "the geocentric radius"). If the geoid's undulations are known, then these may be utilized for calculating the quantity D. For the quantity s in (20.20) an approximate value suffices.

(2) As other unknowns one may also include the values θ_{S^*} (for every S^* or P point). The possibility exists (*Wolf 1968*) that the measured time values be regarded as observations and that other relations be expressed for them involving the corrections.

(3) If out of the total number of station points P, some are of known coordinates (fixed points), then for these points, from which one determines by measurements, with the aid of the artificial satellite, the coordinates of the other station points (new points), one takes $d\Phi_P = 0$ and $d\Lambda_P = 0$.

(4) According to (20.12), Φ and Λ are here *geocentric coordinates*, where Λ coincides with the geodetic longitude L on the co-axial rotation ellipsoid. The astronomical longitude Λ^*, as obtained from time determinations, differs, however, from Λ by the value of the deflection of the vertical in longitude ($= \eta \sec B$) — (*Markowitz 1958*).

(5) If instead of the artificial satellite one uses other luminous targets at great altitudes (flashes produced by rocket, luminous bomb etc.), then the adjusment of the corresponding triangulation (termed stellar triangulation) can be carried out by the same procedure as previously described or by other similar procedures (e.g.: *Väisälä* and *Oterman 1960*).

(6) Adopting the approximate solution (*Levallois* and *Kovalevsky 1971*) involves the following. Dividing the relation (20.14) by s yields the corresponding direction cosines, i.e.:

$$\|I,\ J,\ K\| = \frac{\mathbf{a}^T}{s} = \|\cos\delta_{S^\bullet}\cos\Lambda_{S^\bullet},\ \cos\delta_{S^\bullet}\sin\Lambda_{S^\bullet},\ \sin\delta_{S^\bullet}\|;$$

or:

$$\|I,\ J,\ K\| = \frac{1}{s}\,\|\,X_{S^\bullet} - X_P,\ Y_{S^\bullet} - Y_P,\ Z_{S^\bullet} - Z_P\,\|.$$

If one also adds the corrections v_I, v_J, v_K, e.g.:

$$I + v_I = \frac{X_{S^\bullet}^{(0)} + dX_{S^\bullet} - (X_P^{(0)} + dX_P)}{s^{(0)} + ds},$$

and one takes into account that:

$$s^{(0)} = \sqrt{(X_{S^\bullet}^{(0)} - X_P^{(0)})^2 + (Y_{S^\bullet}^{(0)} - Y_P^{(0)})^2 + (Z_{S^\bullet}^{(0)} - Z_P^{(0)})^2}$$

and:

$$I^{(0)} = \frac{X_{S^\bullet}^{(0)} - X_P^{(0)}}{s^{(0)}};\ J^{(0)} = \frac{Y_{S^\bullet}^{(0)} - Y_P^{(0)}}{s^{(0)}};\ K^{(0)} = \frac{Z_{S^\bullet}^{(0)} - Z_P^{(0)}}{s^{(0)}},$$

then, after developing in the corresponding *Taylor* series (retaining only the linear terms), one gets:

$$\begin{aligned} v_I &= D(dX_{S^\bullet} - dX_P) - DI^{(0)}\,ds + (I^{(0)} - I); \\ v_J &= D(dY_{S^\bullet} - dY_P) - DJ^{(0)}\,ds + (J^{(0)} - J); \\ v_K &= D(dZ_{S^\bullet} - dZ_P) - DK^{(0)}\,ds + (K^{(0)} - K), \end{aligned} \tag{20.21}$$

where $D = 1/s^{(0)}$.

From the correction equations (20.21) one develops the corresponding normal equations whose solution then yields the unknowns. Here it is also necessary to include ds in order to meet the condition $I^2 + J^2 + K^2 = 1$. As has been demonstrates (*Levallois* and *Kovalevsky 1971*), the condition $[v_I^2 + v_J^2 + v_K^2] = \text{min.}$, is identical with $[\Delta^2] = \text{min.}$, where Δ has the meaning from Fig. 20.3.

This solving method represents an approximate solution inasmuch as the corrections are not applied to the initial measurement quantities. As was shown by *H. Wolf* (*1968*), this drawback may be avoided if one utilizes the theory of the correlated observations, in which case the condition of minimum will be $\mathbf{\Delta}^T \mathbf{Q}^{-1} \mathbf{\Delta} = \text{min.}$, where $\mathbf{Q}$ is the correlation matrix and the vector $\mathbf{\Delta}$ separates into its constituent parts.

(7) The possibility exists (*Dobaczewska 1966*, *Wolf 1968*) that instead of D, $d\Phi_P$, $d\Lambda_P$, one may use dX_P, dY_P, dZ_P. Thus, calculating the quantity:

$$g = \sqrt{(X_{S^\bullet} - X_P)^2 + (Y_{S^\bullet} - Y_P)^2},$$

and then:

$$\cos\,\delta_{S*} = \frac{g}{s}\,;\quad \sin\,\delta_{S*} = \frac{Z_{S*} - Z_P}{s}\,;\ \cos\,\Lambda_{S*} = \frac{X_{S*} - X_P}{g}\,;$$

$$\sin\,\Lambda_{S*} = \frac{Y_{S*} - Y_P}{g}\,;\ \tan\Lambda_{S*}^{(0)} = \frac{Y_{S*}^{(0)} - Y_P^{(0)}}{X_{S*}^{(0)}}\,;\ \tan^2\delta_{S*}^{(0)} = \frac{(Z_{S*}^{(0)} - Z_P^{(0)})^2}{(X_{S*}^{(0)} - X_P^{(0)})^2 + (Y_{S*}^{(0)} - Y_P^{(0)})^2}\,,$$

from (20.20) one gets the new correction equations (*Wolf* 1968):

$$v_{\delta_{S*}} = m\mathrm{d}X_{S*} + n\mathrm{d}Y_{S*} + o\mathrm{d}Z_{S*} - m\mathrm{d}X_P - n\mathrm{d}Y_P - o\mathrm{d}Z_P + (\delta_{S*}^{(0)} - \delta_{S*})\,;$$

$$v_{\Lambda_{S*}} = p\mathrm{d}X_{S*} + q\mathrm{d}Y_{S*} - p\mathrm{d}X_P - q\mathrm{d}Y_P + (\Lambda_{S*}^{(0)} - \Lambda_{S*}), \tag{20.22}$$

where

$$m = -\frac{(X_{S*} - X_P)\,(Z_{S*} - Z_P)}{gs^2}\,;\ n = -\frac{(Y_{S*} - Y_P)\ (Z_{S*} - Z_{S*})}{gs^2}\,;$$

$$o = \frac{g}{s^2}\,;\ p = -\frac{Y_{S*} - Y_P}{g^2}\,;\ q = \frac{X_{S*} - X_P}{g^2}\,.$$

21
Some Examples of Spatial Triangulation Operations

The first operation of spatial triangulation proper was carried out in *France* in the year 1962, with two main purposes:

(a) Determining the position of the artificial satellite *Echo I* with respect to the *French* astro-geodetic network.

(b) Establishing experimentally the rules for the use of the ballistic cameras and the observing methods.

The results were, on the whole, satisfactory.

Starting from 4 station points *(Lacanau* — near *Bordeaux, Plogonec* — near *Brest, Herlisheim* — near *Colmar* and *Goult* — near *Avignon),* the position of the passive satellite *Echo I* with an accuracy of $\pm$ 20 m has been determined. The coordinates of the calculated station points, with one of these points taken as origin, turned out to differ widely from the values previously known from *France*'s astro-geodetic network.

This first operation of spatial triangulation has led, however, to some important conclusions, viz.:

(1) *The calculating methods may be improved.*

(2) *The prototype ballistic cameras used were not of an appropriate design and construction.*

(3) *The comparators used were not adequate.*

(4) *The experiment carried out was conclusive as regards the observing techniques and the management of such an operation.*

21.1 The France — Algeria Connexion

This operation of spatial triangulation, carried out in May — June 1963, aimed at calculating the astro-geodetic network of *France* and of the *Algerian Republic,* making use to this end exclusively of observations made on passive satellites of the *Echo* type.

For comparison purposes, the connexion already performed by means of conventional methods between the triangulations of *Spain* and *Morocco* (1879) as well as that between *Italy* and *Tunisia* (1876—1878) was used. Everything was included into a single block, within the more recent system of countries of *Western Europe.* A basis for comparison now being available, the relative positions obtained could be checked, the distances could be determined and discussed etc.

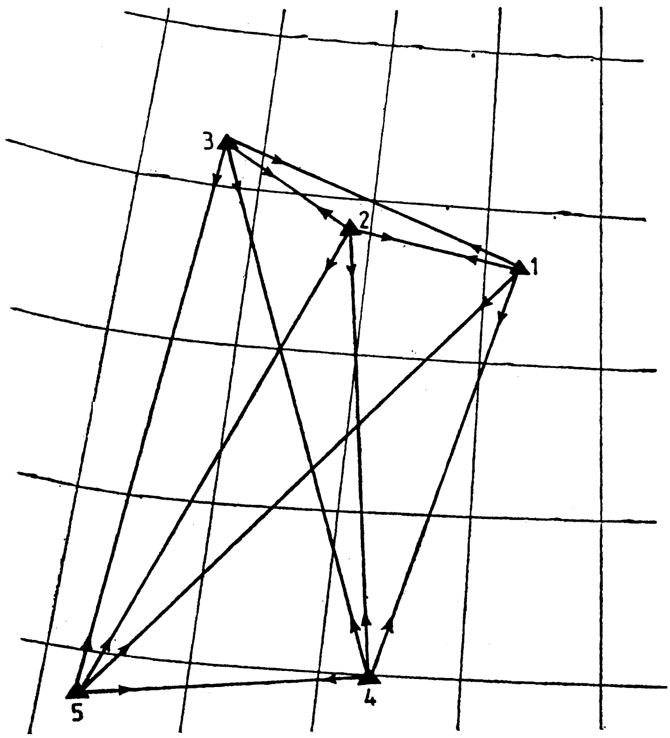
Five station points were chosen (Fig. 21.1): three of them in *France (Lacanau, Agde, Oletta)* and two in *Algeria (Ouargla* and *Hammâguir)*, from which the artificial satellite *Echo I* was photographed, the number of photographic plates indicated in Table 21.1. being obtained.

Remark. With the exception of the *Hammâguir* station point, the weather was favourable during the observations at the other points; this explains the large number of photographic plates obtained at the stations 1—4.

Table 21.1. *The Number of Photographic Plates Realized*

Station point	1. Lacanau	2. Agde	3. Oletta	4. Ouargla	5. Hammâguir
Number of photographic plates	25	35	40	40	15

The five station points were connected to the nearest geodetic point in the 1st-order astro-geodetic network. The coordinates of the station points were calculated in the $OXYZ$ global system as functions of their geodetic coordinates on the international ellipsoid and the geoid's height



21.1. The *France-Algeria* Connexion: 1 — *Oletta* station; 2—*Agde* station; 3 — *Lacanau* station; 4 — *Ouargla* station; 5— *Hammâguir* station was obtained by astro-geodetic levelling (making use to this end of the deflections of the vertical at the geodetic points[1] of *France, Spain, Italy* and *North Africa* triangulations). The processing of the observations carried out at the station points was performed at *France's National Geographical Institute*. Table 21.2 gives the values [2] of the azimuths and of the corresponding distances as obtained both by conventional (first value in the Table) and spatial (second value in the Table) triangulation. It is to be noted that the greatest difference reaches the value $|4^{cc}.72|$ in azimuth — for direction 12 — and $|9.7\ \text{m}|$ in distance — for direction 23.

Remark. No phase correction was applied to the measurements, since it is difficult to know the artificial satellite's exact behaviour as a light source and, in particular, whether it behaves as a perfect reflector [3] or as a transmitter, or further, whether the light flow coming from the satellite is reflected or emitted, or both at the same time and, if so, in what proportion.

Obviously, applying a phase correction can only improve the results.

After the adjustment of the triangulation as a free network, the standard deviation of an observed spatial direction equalled $\pm\ 4^{cc}.77$.

The artificial satellite's mean altitude during the observations was 1,750 km in the area where the work was carried out. Inasmuch as the observation distance varied between 1,800 and 2,500 km, an average value of 2,100 km having been taken for this distance, the direction standard deviation of $\pm\ 4^{cc}.77$ corresponds to a distance standard deviation of $\pm$ 16 m. This departure is due to the many observational errors and to the synchronizing error. Analysing the discrepancies in the reception of the hour signals has confirmed the fact that synchronization is generally correct only for an error of approximately $\pm\ 10^{-3}$ s.

After carrying out all the calculations and checks shown at stages (1)—(8) (Chapter 20), one has proceeded to the general adjustment of the network, adopting as the network's origin the station point *Agde* and imposing as its scale the *Agde-Ouargla* side, known from the conventional triangulation. This adjustment yielded the coordinates X, Y, Z of the corresponding station points, which are shown in Table 21.3 along with the corresponding coordinates provided by the conventional triangulation (the corresponding values have been taken from *Levallois* and *Kovalevsky* (*1971*, p. 71, Table 3)).

21.2 The European Continent — Azores Islands Connexion

This connexion which, exceeding the scope of a mere experimental work, represents the solution of a practical problem was carried out during the period July—August 1965 within the framework of a co-operation between *the National Geographical Institute of France* and *the Geographical and*

[1] Their number was reduced, which implicitly didn't lead to an absolutely correct result.

[2] They were taken from *Levallois and Kovalevsky* (*1971*, p. 70, Table 2).

[3] This assumption seems close to reality and is in agreement with experimental information (*Levallois and Kovalevsky 1971*).

Table 21.2. *The Values of the Azimuths and Distances Provided by Conventional Triangulation and by Spatial Triangulation*

	1 Oletta	2 Agde	3 Lacanau	4 Ouargla	5 Hammáguir	Elements
1 Oletta		$312^g37^c07^{cc}.10$ 02.38 484 184.1 m 182.6	$323^g36^c57^{cc}.20$ 58.19 888 604.5 m 610.6	$219^g67^c21^{cc}.93$ 23.67 1 232 645.8 m 650.5	$249^g21^c69^{cc}.11$ 68.86 1 699 842.0 m 839.3	Azimuths Distances
2 Agde	$107^g93^c08^{cc}.40$ 03.70 484 184.1 m 182.6		$331^g54^c92^{cc}.57$ 96.95 420 407.7 m 417.4	$190^g86^c47^{cc}.20$ 50.20 1 270 247.4 m 247.4	$227^g68^c43^{cc}.86$ 44.59 1 491 208.9 m 204.7	Azimuths Distances
3 Lacanau	$115^g23^c34^{cc}.08$ 34.29 888 604.5 m 610.6	$127^g91^c46^{cc}.67$ 50.16 420 407.7 m 417.4		$173^g55^c05^{cc}.65$ 05.65 1 555 721.3 m 728.0	$207^g21^c40^{cc}.75$ 37.06 1 571 977.8 m 977.9	Azimuths Distances
4 Ouargla	$17^g00^c24^{cc}.70$ 26.06 1 232 645.8 m 650.5	$392^g16^c80^{cc}.24$ 82.82 1 270 247.4 m 247.4	$378^g15^c01^{cc}.61$ 01.89 1 555 721.3 m 728.0		$293^g26^c18^{cc}.82$ 21.82 811 352.5 m 345.2	Azimuths Distances
5 Hammáguir	$40^g90^c96^{cc}.04$ 95.71 1 699 842.0 m 839.3	$23^g27^c34^{cc}.09$ 34.76 1 491 208.9 m 204.7	$5^g94^c37^{cc}.27$ 34.19 1 571 977.8 m 977.9	$88^g36^c11^{cc}.65$ 14.96 811 352.5 m 345.2		Azimuths Distances

Table 21.3. *The x, y, z Coordinates Provided by Conventional Triangulation and by Spatial Triangulation*

Station point	Conventional triangulation m	Spatial triangulation m	Diferences m
1 Oletta	X_1 = 4639113.45 Y_1 = 764588.58 Z_1 = 4296137.67	= 4639104.91 = 764586.61 = 4296134.08	δX = −8.54 δY = −1.97 δZ = −3.59
2. Agde (origin)	X_2 = 4640374.31 Y_2 = 283672.31 Z_2 = 4352282.36		δX = 0 δY = 0 δZ = 0
3. Lacanau	X_3 = 4516066.69 Y_3 = —94246.80 Z_3 = 4488177.22	= 4516055.18 = —94254.37 = 4488175.61	δX = −11.51 δY = − 7.57 δZ = − 1.61
4. Ouargla	X_4 = 5393662.31 Y_4 = 510808.43 Z_4 = 3355038.70	= 5393653.18 = 510801.91 = 3355030.31	δX = −9.13 δY = −6.52 δZ = −8.41
5. Hammâguir	X_5 = 5471590.45 Y_5 = −290633.99 Z_5 = 3255488.81	= 5471575.27 = —290634.06 = 3255483.04	δX = −14.82 δY = − 0.07 δZ = − 5.77

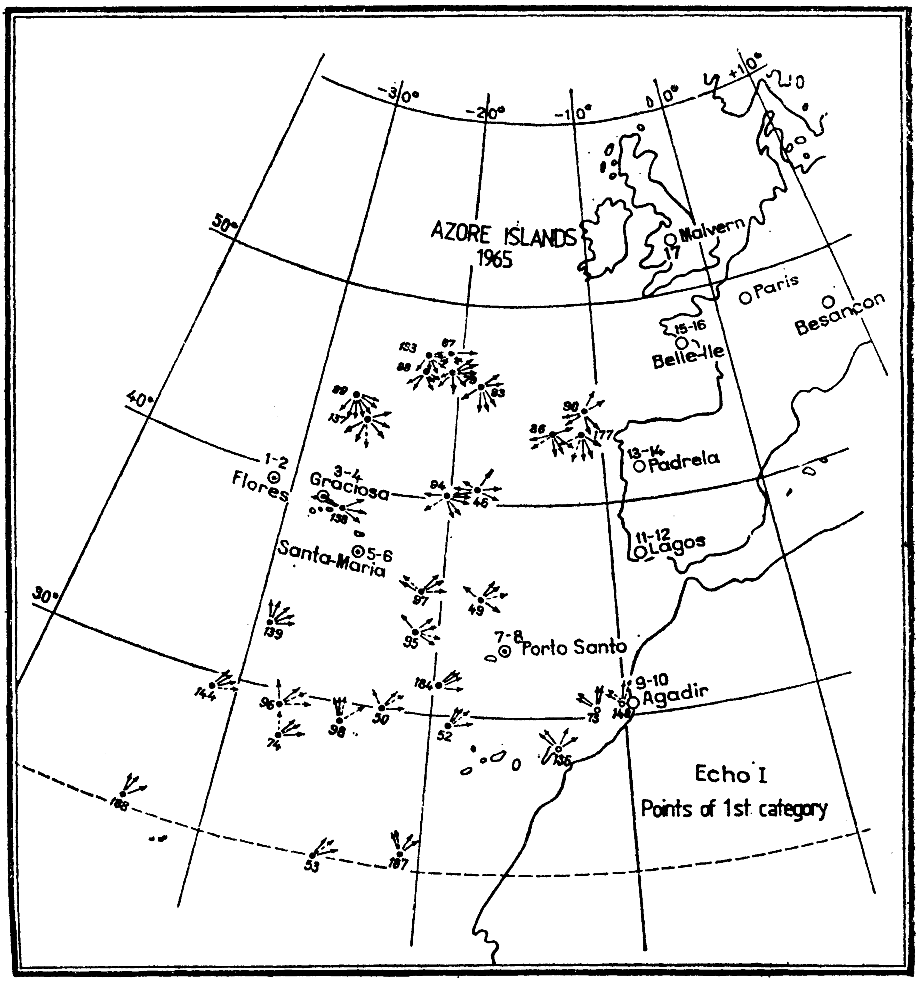
Cadastral-Survey Institute of Portugal. The working scheme (Fig. 21.2; *Levallois* and *Kovalevsky 1971)* was implemented by means of the artificial satellites *Echo I* and *Echo II*, starting from the European base formed by the station points *Belle-Ile (France), Padrela (Portugal), Lagos (Portugal) and Agadir (Morocco)* and connecting these stations with the *Porto-Santo, Santa-Maria, Graciosa* and *Flores* Islands.

These independent stations, located in each of the island groups of the *Azores* archipelago, also constituted the object of a connexion between them by means of astro-geodetic triangulation, which allowed the results obtained to be compared with those provided by artificial-satellite work and, obviously, an appraisal of the results' validity to be made.

The results of the spatial-triangulation calculations were nearly as good as those obtained within the framework of the *France-Algeria* connexion *(Levallois* and *Kovalevsky 1971).*

21.3 The World Network of Triangulation by means of Artificial Satellites

During the period 1966—1971, the *U.S. Coast and Geodetic Survey* implemented a World triangulation project, realized by means of observations on the *Pageos* artificial satellite. This World network (Fig. 21.3; *Levallois* and



21.2. Working Scheme of the Connexion *Europe — Azores Islands*

Kovalevsky 1971, Ramsayer 1968 b) was formed by 45 station points, to a great extent uniformly distributed over the Earth's surface, with an average distance between them of about 4,000 km. As many as 14,500 spatial-direction observations were made by means of ballistic cameras of the *Wild BC-4* type [1].

The network scale was derived from measurements carried out at the station points, i.e. by measuring a total of 7 continental bases (in the *U.S.A.*, *Europe — the Tromsö-Catania traverse —* and *Australia*), being up to about 3,000 km in length, either from conventional astro-geodetic triangulations,

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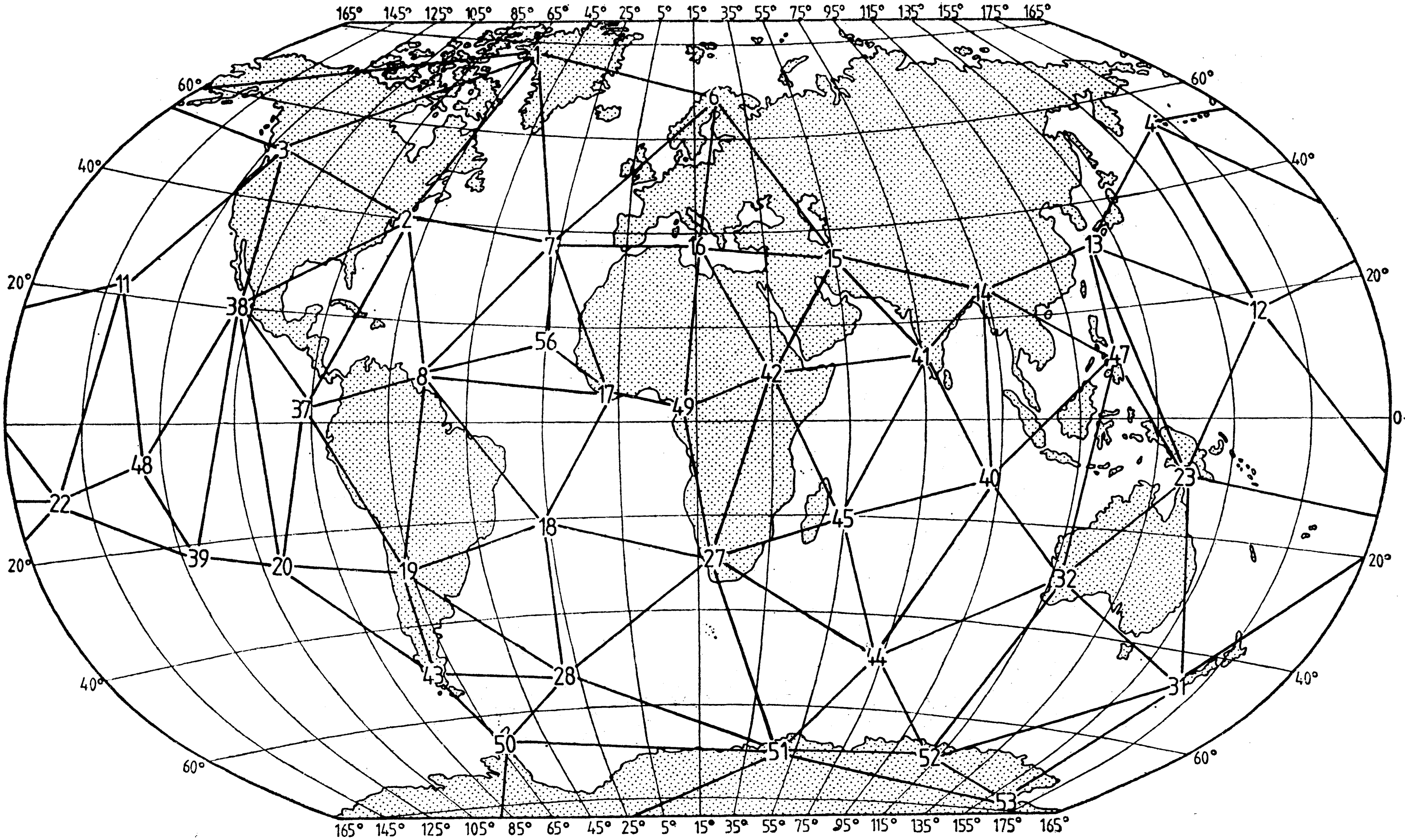
[1] These allow both active and passive artificial satellites to be photographed, achieving an accuracy in determining a spatial direction of about $\pm 2''$.



21.3. World Triangulation Network Using Artificial Satellites

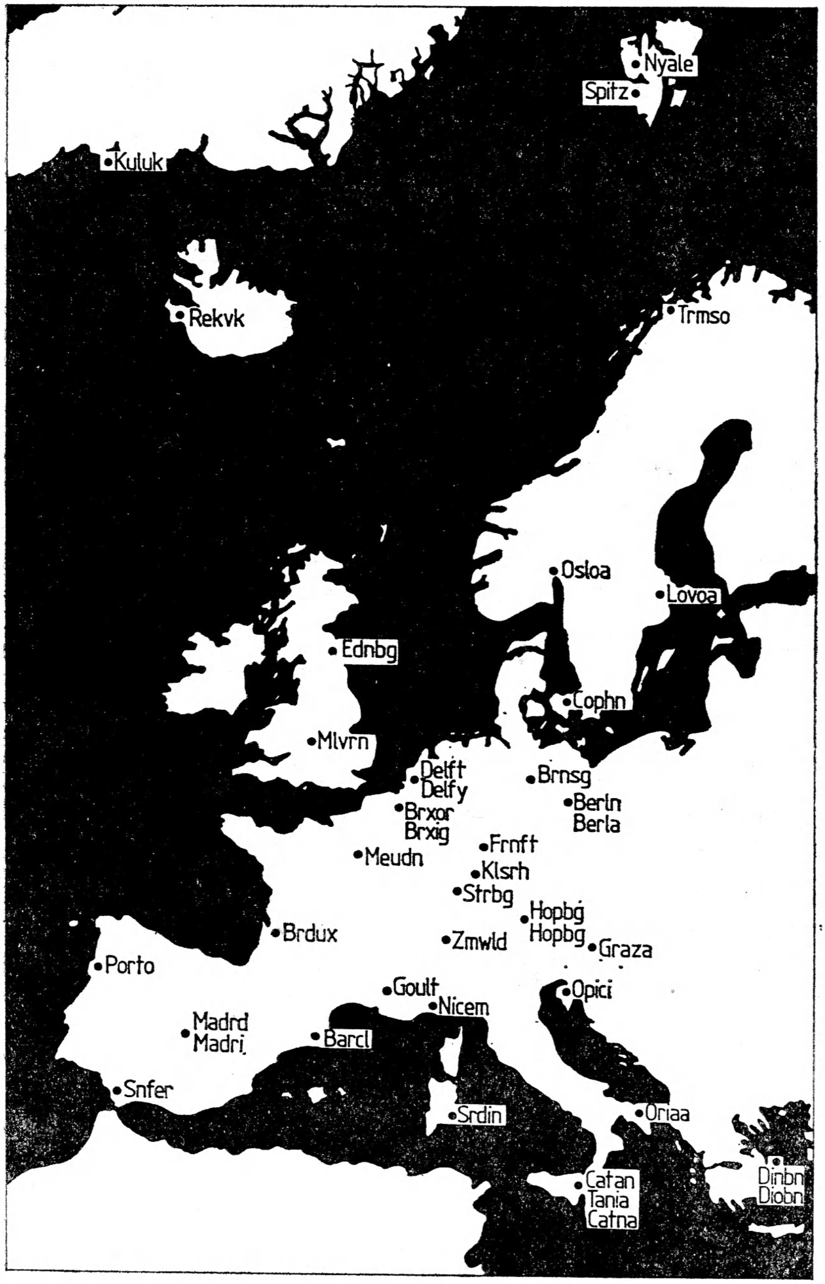

Nyale
Spitz
Kuluk
Rekvk
Trmso
Osloa
Lovoa
Ednbg
Cophn
Mlvrn
Delft
Delfy
Brnsg
Brxor
Brxig
Berln
Berla
Frnft
Meudn
Klsrh
Strbg
Hopbg
Hopbg
Brdux
Zmwld
Graza
Porto
Goult
Opici
Nicem
Madrd
Madri
Barcl
Snfer
Srdin
Oriaa
Catan
Tania
Catna
Dinbn
Diobn

Fig. 21.4.

or from intercontinental bases determined by means of traverses measured with high-precision geoidmeters and by geodetic-astronomy determinations at a great number of point along the corresponding traverses.

As was shown in the report of the 18th *COSPAR* Meeting *(Varna,* May 29th—June 7th) the three-dimensional positions of the 45 station points are determined with an average error of $\pm$ 4.5 m; except for four station points, the error generally did not exceed 6 m. The *Goddard Space Center* has published the results of this triangulation under the designation of *Gem 5* and *Gem 6.*

21.4 Western European Triangulation by means of Satellites (WEST)

The programme of the *Western European* satellite-triangulation has been based upon optical observations on 5 passive artificial satellites *(Pageos, Echo I, Echo II, Explorer 19* and *Explorer 39),* most of the observations (68%) being carried out on the *Pageos* satellite. The observations were made from August 1966 up to June 1972, a period in which over 3,650 simultaneous photographic plates were obtained; out of these, about 80% were calculated up to the year 1976 *(Ehrnsperger 1978).*

Remark. Additionally, 29 photographic plates obtained at 4 station points of the *densifying network of the North-American continent* were included in the final calculation, with a view to strengthening the *WEST* network in *Northern Europe.*

The purpose of the *WEST* programme has been to determine the spatial coordinates for 32 station points with 42 observing stations (Fig. 21.4) located on the territory of 17 European countries, with distances between them of up to some 1,000 km.

Fig. 21.4. The station points of the "WEST" satellite triangulation net: KULUK — Kulusuk airport (Groenland); NYALE — Ny — Ålesund; SPITZ — Spitzbergen (Norway); — REKVK — Reykjavik (Island); TRMSO — Tromsö (Norway); OSLO — Oslo (Norway); LOVOA — Stockholm (Sweden); EDNBG — Edinburg (Scotland or U.K.); COPHN — Copenhagen (Denmark); MLVRN — Malvern (England); DELFT — Delft (Holland); DELFY — Delft airport (Holland); BRNSG — Braunschweig, 2 observational points (W. Germany); BERLN — — Berin (E. Germany); BERLA — Berlin (E. Germany); BRXOR — Bruxelles (Belgium); BRXIG — Bruxelles (Belgium); FRNFT — Frankfurt am Mein (W. Germany); MEUDN — — Meudon observatory (France); KLSRH — Karlsruhe (W. Germany); STRBG — Strasbourg observatory (France); HOPBG — Hohen peissenberg, 2 observational points (W. Germany); BRDUX — Bordeaux observatory (France); ZMWLD — Zimmerwald (Switzerland); GRAZA — — Graz (Austria); PORTO — Porto observatory (Portugal); GOULT — Goult (France); NICEM — Nice (France); OPICI — Triest (Italy); MADRD — Madrid (Spain); MADRI — — Madrid (Spain); BARCL — Barcelona university (Spain); SNFER — San Fernando (Spain); SRDIN — Cagliari (Italy); ORIAA — Oria (Italy); DINBN — Dionysos (Greece); DIOBN — — Dionysos (Greece); CATAN — Catania (Italy); TANIA — Catania (Italy); CATNA — — Etna observatory (Italy).

Although this programme used altogether 14 types of various ballistic cameras *(BC-4, IGN, K-17, K-17 modified, Schmidt-Hewitt, TA-120, BMK Zeiss, FK Zeiss, K-37, Antares, Baker-Nunn, Refractor, Schmidt telescope* and *parallactic camera),* the adjusting computations, carried out by *Deutches Geodätisches Forschungsinstitut* in *München* (the *1975 solution* was presented at *the 16th General Assembly of the International Union of Geodesy and Geophysics* at *Grenoble—Ehrnsperger 1975),* showed that 20 station points were determined with an average error of between ± 2 m and $\pm$ **6 m.**

Remark. Within the framework of the operations carried out, two European traverses *Tromsö — Catania* and *Malvern—Graz* were also calculated; the results were as follows (*Ehrnsperger 1978*):

Hohenpeissenberg — Tromsö	2,467,733.52 $\pm$ 1.20 m
Hohenpeissenberg — Catania	1,194,842.47 $\pm$ 1.20 m
Hohenpeissenberg — Graz	346,945.38 $\pm$ 0.27 m
Hohenpeissenberg — Malvern	1,046,479.89 $\pm$ 0.65 m.

21.5 Brief Survey of Other Triangulation Operations by means of Artificial Satellites

In addition to the work mentioned in Sections 21.1—21.4, there were also set in motion a few programmes which have provided important results for determining or improving the local or the world geodetic networks, viz.:

(1) *The Smithsonian Astrophysical Observatory (SAO) — Massachusetts* has determined, from 15 stations for optically monitoring the artificial satellites with *Baker-Nunn* cameras, scattered over the entire terrestrial Globe, the satellites' geocentric positions with an accuracy of $\pm$ 15— $\pm$25 m, on the basis of combined geometrical and dynamic methods. At the same time, *SAO* has also given in 1966 the parameters of the so-called *Smithsonian Standard Earth 1 (SSE 1).* In the year 1969, *SAO* published *SSE 2,* determined with the aid of the coordinates of 46 stations, of which 19 were *Baker-Nunn* station and 12 — laser stations. The mean accuracy of this operation was of about $\pm$ 10 m.

Finally, in May 1973, at the *Athens* symposium, there appears *SSE 3* containing 105 stations, of which 23 were stations for the optical monitoring of the artificial satellites and 13 were laser stations (considered as base stations). The attained accuracy was $\pm$ 2— $\pm$3 m, for the base stations and $\pm$ 3 — $\pm$ 6 m for the optical stations.

(2) In the *U.S.A. the densifying network of the North-American continent* was successfully accomplished, within the framework of the World project of satellite triangulation.

The adjustment, based on 419 satellite events and totalling 966 photographic plates obtained with the ballistic camera *Wild BC-4,* covers most of the *North-American* continent together with some parts of *Greenland, Iceland* and *Norway.*

This network includes 23 station points located at distances of about 1,000 km between them, of which six are station points in the World network.

The adjustment was carried out by the *U.S. National Geodetic Survey* and relied upon the six station points of the World network.

(3) Two major programmes for measuring large geodetic traverses were initiated by *the Astronomical Council of the Academy of Sciences of the U.S.S.R.*, in the years 1970 and 1971. One programme concerns the *Arctica—Antarctica chord* and the second one a great arc *East-West* from *Sakhalin* and *Eastern Siberia* through *Africa* up to *North America*. A certain number of ballistic cameras *AFU-75* take part in the still continuing observations. From the year 1973, the laser-satellite system has also participated in this programme.

(4) In the year 1971 was started the international geodetic experiment *ISAGEX*, under the guidance of the *French National Centre for Space Research*, in which 50 stations have taken part, and over 1,500 photographic plates have been obtained. Several types of ballistic cameras *(Baker-Nunn, GSFC-MOTS, French ones* and *AFU-75* were used, by means of which several satellites were observed: *Geos 1* and *2, Be-B, BE-C, Péole, D 1 C* and *D 1 D*. As well as space geometrical geodesy, the aim of the measuring campaign was the study of the terrestrial gravitational field.

In addition to these major programmes, some other programmes in various regions *(Japanese Islands, Eastern Europe, Europe* short-arc programme*)* were initiated.

The *Europe-Africa* programme was terminated and reduced.

Light-reflection balloons are regularly utilized in spatial triangulation by the *Finnish Geodetic Institute*.

One must also mention a large number of studies and adjustments carried out at various institutes, representing remarkable contributions to the use of geometrical techniques in Space Geodesy. Probably the most important one is the general adjustment of about 160 station points distributed over the entire surface of the Earth, with the scale established on the basis of the *Secor* observations. This work was performed by the *Ohio State University*.

22
General Survey of Other Geometrical Methods of Spatial Geodesy by means of Artificial Satellites

Since the artificial satellite is monitored along a considerable orbital circle-arc, of the order of 20° up to 25° with respect to the Earth's centre, the telemetry lasers provide several utilization possibilities of a purely geometrical nature, as will be shown in what follows.

22.1 The 4-Laser Method[1]

It is assumed that three laser telemeters are installed at three known station points P_1, P_2, P_3 and that one sights from these points the same artificial satellite S^* (Fig. 22.1). The satellite's position at a given time moment is determined by its tripolar coordinates (D, α, β') derived from the three points of known coordinates.

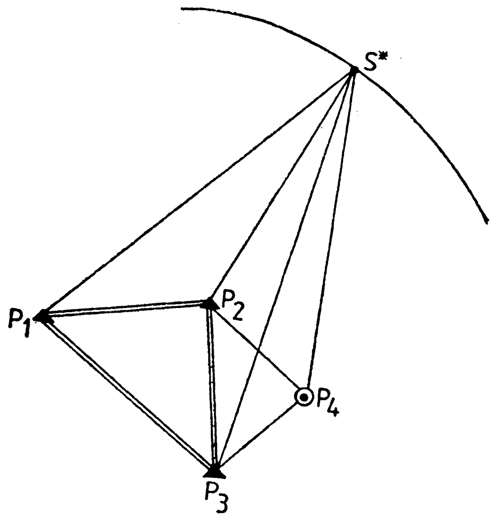


Fig. 22.1. Four-Laser Method: P_1, P_2, P_3 — station points of known coordinates; P_4 — station point of unknown coordinates

[1] In the technical literature it is also found under the designation of *trispheration method* (*Arnold 1970, Izotov et al. 1974*).

If at the same moment one measures, synchronously with the others, the distance to the same artificial satellite by means of a fourth laser station P_4, one defines a first locus of the unknown point P_4, which must be on the sphere centred on the artificial satellite, with the measured distance as radius.

Considering several positions S_1^*, S_2^*, S_3^*, ... of the artificial satellite, distance measurements to these positions will give as loci for the point P_4 a series of spheres of radii $P_4S_2^*$, $P_4S_3^*$, ... Consequently, the unknown station point may be determined as an intersection of the spheres whose centres are successively S_1^*, S_2^*, S_3^*, ... with the corresponding radii $P_4S_1^*$, $P_4S_2^*$, $P_4S_3^*$, ...

Remark. During 7 min one observes 8,400 distances from the Earth, which is enough data for removing the random errors from the calculation (*Ramsayer 1965*).

Consequently, this presupposes the knowledge of the coordinates of the three terrestrial station points, which can be provided by Classical Geodesy, e.g. by the triangulation method.

Remark. Classical Geodesy and particularly the triangulation method cannot provide the form of the P_1, P_2, P_3 triangle, especially the lengths of its sides, with an accuracy comparable with that of the laser method. A distance of the order of some thousands of kilometers may be determined in Classical Geodesy, according to the particular case (chain or surface network) with an accuracy of $\pm$ 4–5 m, whereas the laser telemetry gives a precision under $\pm$ 1.0 m.

From the geometrical point of view, the $P_1P_2P_3P_4$ tetrahedron, ignoring its spatial orientation, is determined by six conditions: taking the vertex P_1 as origin, the P_2 vertex is defined by the distance P_1P_2, which represents a first condition; the P_3 vertex is defined by the distances P_1P_3 and P_2P_3, i.e. two conditions, and the P_4 vertex is defined by the distances P_1P_4, P_2P_4 and P_3P_4, consequently another three conditions.

Thus, in order to define the corresponding tetrahedron it is necessary to make six independent observations. Inasmuch as the six edges of this tetrahedron, defining the four terrestrial station points, caunot be directly measured, for reasons of invisibility, between them, one must have recourse to their indirect measurement by means of the artificial satellite. If one sights simultaneously this satellite from the four station points, each observation provides four data, of which three are needed for establishing the satellite's position, which is defined by three parameters. Consequently, for determining the geometry of the figure one observation can still be used. Thus, it is necessary that from the four station points one should take, simultanously at least six times, the corresponding sights to the satellite, in order that the geometry of the figure be established and that this figure be accessible to calculation; 24 distance measurements follow therefrom. Assuming this geometrical figure to be known, every new station point will be determined starting from the three station points which are known through the measurement of 12 distances.

This requirement is fulfilled by sighting the artificial satellite in three different positions, simultaneously from the four station points.

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[1] A tetrahedron is well defined by its six edges.

All of these conditions assume that the measurements must be independent from the geometrical point of view.

Remark. If the three positions of the artificial satellite and the unknown station point P_4 are coplanar, then P_4 is not well determined in the direction perpendicular to the common plane. Therefore, it is necessary to vary the passages and to regard the measurements referring to the same passage, which are very close in space and time, as dependent on this passage.

22.2 The Optical-Telemetric Method

The principle of this method is quite plain, viz.: from the station point P_1 one observes an artificial satellite S^* equipped with a device for transmitting light flashes, e.g. an artificial satellite *Geos 2* which emits groups of 7 flashess every 4 s and one simultaneously measures the distance with a laser telemeter. If the light flashes and the distances are referred to the same time base, then one can know simultaneously the distances and the moment at which each flash is produced; one may thus determine the spatial coordinates of the station point P_1. If one proceeds in the same manner at another station point P_2, then one obtains a triangle $P_1P_2S^*$ in which one knows the distances $\overline{P_1S^*}$, $\overline{P_2S^*}$ and the angle $\widehat{P_1S^*P_2}$ given by the stellar directions. In order to determine the station point P_2, one may apply the basic geometry formula in the triangle $P_1S^*P_2$, thus obtaining the distance $\overline{P_1P_2}$, viz.:

$$\overline{P_1P_2}^2 = \overline{P_1S^*}^2 + \overline{P_2S^*}^2 - 2\overline{P_1S^*}\;\overline{P_2S^*}\cos\widehat{P_1S^*P_2}. \tag{22.1}$$

With a view to assessing the accuracy of this method, one can consider a particular case, viz. assume the triangle $P_1S^*P_2$ to be right-angled and isosceles and the distance P_1S^* to be of the order of some 1,400 km (Fig. 22.2). Taking into account that here:

$$\overline{P_1S^*} = \overline{P_2S^*} = \frac{\overline{P_1P_2}}{\sqrt{2}},$$

it follows $\overline{P_1P_2} \approx 2{,}000$ km.

From (22.1) one gets, applying the law of error propagation:

$$\mu^2_{\overline{P_1P_2}} = \frac{1}{2}\mu^2_{\overline{P_1S^*}} + \frac{1}{2}\mu^2_{\overline{P_2S^*}} + \frac{1}{2}\mu^2_{\widehat{P_1S^*P_2}}\overline{P_1S^*}^2,$$

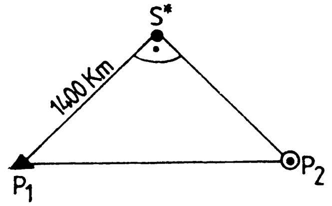


Fig. 22.2. The $P_1S^*P_2$ Right-Angled Isosceles Triangle

which, taking into account that the distances P_1S^* and P_2S^* are measured by means of the laser telemeter, i.e. at least $\mu_{\overline{P_1S^*}} = \mu_{\overline{P_2S^*}} = \pm$ $\pm$ 1.5 m, and

$$\mu_{\widehat{P_1S^*P_2}} = \pm 1'' \approx \frac{1}{200\ 000},$$

finally yields for an isolated determination:

$$\mu_{\overline{P_1P_2}} \approx 5\text{m}.$$

Remark. The optical-telemetric method could be successfully utilized for replacing the geodimeter continental-base measurement by bases resulting from spatial determinations.

One can also use a method more elaborate than that just described. Thus, carrying out more measurements, one may take into consideration the basic equations of the space observations (17.3.1) yielding the correction equations for directions (e.g. (17.40)) and for distances respectively, taking the form (17.42):

$$I_1(\mathrm{d}X_{S^*} - \mathrm{d}X_{P_1}) + J_1(\mathrm{d}Y_{S^*} - \mathrm{d}Y_{P_1}) + K_1(\mathrm{d}Z_{S^*} - \mathrm{d}Z_{P_1}) + \\ + D^{calc}_{P_1S^*} - L^{obs}_{P_1S^*} = v_D, \tag{22.2}$$

where I_1, J_1, K_1 are the direction cosines of the direction P_1S^*; X_{P_1}, Y_{P_1}, Z_{P_1} — the coordinates of the station point P_1, and X_{S^*}, Y_{S^*}, Z_{S^*} — the artificial satellite's coordinates at the moment concerned. Obviously, the measurement series may be incomplete, i.e., for example, one of the station points may have no telemetric measurements etc. As a rule the measurement does not give a continuous row of traces at distances in the sighting cadence, there being many empty places so that in order to determine the distance to the satellite at a given moment it is necessary to resort to an interpolating method. Hitherto, an orbital method has been utilized to this end, i.e. one has calculated the elements of an orbit which, for a position close to the corresponding station point, would best coincide with the observed distance elements. Moreover, this method has allowed one to define the internal accuracy of the operation of distance measuring, by means of a dispersion analysis, by the value of about ± 1.5 m. Such a method is indispensable when one studies large arcs. For sufficiently short arcs, such as would be the case when limiting the application of the method to a purely geometrical form near the flashes emitted by the artificial satellite or by the goniometric laser, one can use a plain interpolating method with respect to time. In this case, the distance measurements must be corrected for refraction, since the optical path travelled by pulses extends into the atmosphere. Thus, for calculating, in a first approximation this correction one may utilize the relation (*Levallois* and *Kovalevsky 1971*, p. 79 (3—06)):

$$\mathrm{d}D = \frac{1}{\cos\beta'}(n_0 - 1)\,H_e,$$

in which $H_e = 8$ km and n_0 — the refractive index of the atmosphere.

Remark. The dD correction is not negligible.

22.3 The Orbital Method

This method consists of utilizing the spatial positions of an artificial satellite, known on the basis of a dynamic theory, in order to determine by means of optical or distance observations the unknown coordinates of any station point.

The orbital method is used when, by force of geographical circumstances, the satellite can be simultaneously observed only from three station points of known coordinates and from the point to be determined — due to its remoteness — only up to or after the satellite's passage over the three station points. In this case, the orbital elements determined by means of the observations at the three base points are extrapolated onto the orbit sector observed from the station point to be determined.

Measuring the distance from this station point to the satellite, for two or several passages, allows one to determine the positions looked for.

This method will not be developed here, since it presents a series of difficulties in application and is still less accurate from the geodetic point of view.

Details in connexion with the orbital method are to be found in the technical literature (e.g.: *Arnold 1970, Levallois* and *Kovalevsky 1971, Boiko et al. 1972, Izotov et al. 1974*).

22.4 The Method of the Simultaneity Circle

As is shown by its designation, the method of the simultaneity circle, which was worked out already in 1958 by *C. Popovici*, utilizes the notion of *simultaneity circle* — a great circle on the celestial sphere passing through two simultaneous positions of the same artificial satellite observed from two terrestrial station points. If αS_1^*, δS_1^* and αS_2^*, δS_2^* respectively are the satellite's right ascensions and declinations, simultaneously observed from two terrestrial station points P_1 and P_2 of known coordinates, and A, D are the analogous values for the direction vector looked for between the station points P_1 and P_2, then the relation derived with the aid of the simultaneity circle is *(Popovici 1962, Arnold 1970)*:

$$\tan \delta_{S_1^*} \sin (A - \alpha S_2^*) + \tan \delta S_2^* \sin (\alpha_{S_1^*} - A) + \tan D \sin (\alpha S_2^* - \alpha S_1^*) = 0.$$

The method of the simultaneity circle was applied for the first time to determining the satellite *Sputnik II*, for the case in which one knows only the satellite's trace at a station point. Subsequently, the method has been developed and applied to numerous geodetic problems utilizing artificial satellites. Thus, with the help of this method there were determined geocentric positions of the artificial satellites from visual observations. During the period 1961—1965 the method was extended and utilized both for determining geocentric coordinates of the artificial satellites, of the observing-station points at which simultaneous observations are made (*Laplace* azimuths), and for orienting the large geodetic systems, for determining the Earth's mass centre, for checking and adjusting the spatial triangulation etc.

For instance, the utilization of the simultaneity-circle method for the calculation of the space absolute directions and of the *Laplace* azimuths presents particular advantages, inasmuch as it is not necessary to know the approximate coordinates of the observing-station points, the determination being carried out homogeneously for the entire spatial-triangulation network, within the same coordinate system, and the results can be readily utilized in the following stage — the network's adjustment. Such calculations have been performed for the directions of the triangle *Poznan-Riga-Bucharest*, making use of simultaneous photographic observations of the *Echo I* artificial satellite (June 1963).

Remark. These are *the first* directions to be determined in *Europe* with the aid of the Earth's artificial satellites.

22.5 Experiment of Using Laser Measurements

In 1968, *France* has laid the foundations of an experiment using two laser telemeters, one installed at the Astronomical Observatory of *Saint-Michel-de-Provence (France)* and the other at the Astronomical Observatory of *San Fernando (Spain)*. The aim of this experiment consisted of measuring the *Saint-Michel—San Fernando* base by observing the light flashes emitted by the *Geos 2* artificial satellite in the year 1968, while taking photographs against the stellar background. The corresponding experiment was combined with the first testing of the goniometric laser in *France* and was preceded by a geodetic development of spatial triangulation by means of the ballistic cameras of the *National Institute of Geodesy*, on the African coast *Dakar—Fort Lamy*, by simultaneous sightings at the *Pageos* passive artificial satellite, with a general extension to *South Africa*. The observing operations, having taken place within the period December 1968 — March 1969, were accomplished through an important act of international cooperation.

A few indications concerning the calculation method for the base *Saint-Michel — San Fernando* **are briefly presented here** *(Levallois* and *Kovalevsky 1971):*

(1) If the light beam originated at the time moment t, then the artificial satellite's trace was recorded at the moment $t + \Delta t$ and the return of the beam was registered by the laser-telemeter photocell at the time moment $t + \Delta t + \Delta t'$. The observed distance to the artificial satellite at the moment $t + \Delta t + \Delta t'$ is given by the relation:

$$D = c\,\frac{\Delta t + \Delta t'}{2},$$

in which c is the speed of light.

(2) By means of a first rough linearization for eliminating the mistaken distance measurements, one has carried out a check of the whole of the observed-distance measurements, with a view to avoiding their introduction into the subsequent calculations. The linearized values, regarded as correct, were obtained by calculating the satellite's simplified orbit, which allowed interpolating (or extrapolating), always in terms of satellite time, the distance values to any time moment so that the various measurements participating in the resulting observations (photographic measurements and laser measurements from other station points) may be synchronized. Consequently, all

these measurements were perfectly synchronized with respect to the satellite times.

(3) Afterwards, the various observation relations corresponding to simultaneous observations were worked out and all the results of several observation series were tackled by the least squares method. After some preliminary results, one has obtained for the distance *Saint-Michel — San Fernando* a value differing by about + 12 m from that determined by conventional geodesy; the corresponding distance is of the order of 1,300 km.

(4) The coordinate-error ellipsoid closely approximated a sphere of a 3 m radius; the *Saint-Michel—San Fernando* distance was obtained with a standard deviation of $\pm$ 5 m. These results show that when the laser method is completely elaborated and, in particular, when one is able to reduce the pulse durations to 2 or 3 ns (which will secure an excellent determination of the instantaneous distance), one will be able to measure terrestrial bases of the order of 1,000 or 1,500 km with an accuracy of approximately 3×10^{-6}.

Remark. Inasmuch as, until this experiment, the accuracy of the laser measurements had not been checked in *France* by direct observation, in the summer of the year 1969, purely terrestrial measurement operations for the geodetic connexion of the Island of *Corsica* with the European continent were carried out, by means of the quadrilateral (Fig. 22.3) *Coudon* (near *Toulon*), *Chauve* mountain (near *Nice*), *Stello* moutain (*Corsica*) and *Cinto* mountain (*Corsica*), whose sides (stretched out above the sea) reach values of 200—250 km.

This connexion was achieved by combining conventional-triangulation measurements (with theodolite) with laser measurements (Fig. 22.3). To this end, the two laser telemeters utilized for sighting at the *Geos 1* satellite from *San Fernando* and *Saint-Michel* were installed at *Coudon* and on the *Chauve* mountain respectively. The sides *Chauve* mountain — *Cinto* mountain and *Chauve* mountain — *Coudon* formed the object of numerous direct measurements, both by day and by night, at the same time.

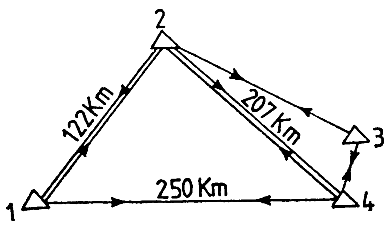


Fig. 22.3. The *Corsica — Europe* Connexion:
1 — *Coudon*; 2 — *Mt. Chauve*; 3 — — *Mt. Stello*; 4 — *Mt. Cinto* laser measurement; optical sighting

Since the sight *Coudon—Chauve* mountain passed over land and its height above the ground hardly exceeded 200— — 300 m, the measurement result was not quite certain, because of the uncertainty concerning the value of the refractive index of the air in the lower layers. On the other hand, the sight *Chauve* mountain — *Cinto* mountain, which passed at least 1,000— — 1,200m above sea-level, was freed from the influence of the air's refractive index, the measurement result being accepted with a high degree of confidence.

Table 22.1 gives the comparative results, taken from *Levallois* and *Kovalevsky* (*1971*, p. 83), of the laser measurements and of those obtained by conventional triangulation in the *Europe 1950 system*.

Table 22.1. *Comparative Results*

Sight	Laser measurement	Europe 1950 system
Mt. Chauve-Coudon	121 525.45 m	121 525.86 m
Mt. Chauve-Mt.Cinto	206 983.52 m	206 982.94 m

The weights of the direction and length observations were chosen identical, a very close agreement being noticed between the European geodetic network and the laser measuremenss.

In conclusion, it can be said that the *French* experiment has proved that it is necessary to trust equally both the conventional triangulation and the laser telemetry, as well as the present value admitted for the speed of light.

Fifth part

Methods for Determining the Terrestrial Ellipsoid and the Geoid

The Earth's form and size can be determined if the coordinates of the points of its physical surface are known in a well-defined system.

In order to determine the coordinates of the Earth's physical surface, by using the measurements of the geodetic elements on this surface, one needs to know the following main reference surfaces: the reference ellipsoid, the general terrestrial ellipsoid and the geoid. For their determination a series of methods have hitherto been used which can be classified into: (1) *astro-geodetic methods*; (2) *gravity methods*; (3) *satellite methods*.

Some of these methods are now no longer used in determining the Earth's form and dimensions. However, because of their historical and educational importance, we have decided to deal with them, albeit somewhat briefly in the present treatise.

In subsequent chapters we will analyse the main methods for determining the form and the size of the Earth, and within the framework of each method we present both the conventional and the modern procedures.

23

The Determination of the Reference Ellipsoid by Using Astro-Geodetic Methods

The astro-geodetic triangulation networks, which cover wide territories, allow the determination of the local reference ellipsoid by means of the methods to be presented in what follows. If there existed an astro-geodetic triangulation network covering the Earth's entire surface, one could determine the general terrestrial ellipsoid by the same methods. However, such a network cannot be brought into being directly; it can only be achieved by joining together the continental triangulations, which, however, calls for knowledge of the general terrestrial ellipsoid itself. This is why astro-geodetic methods are not langer utilized for determining the general terrestrial ellipsoid.

For the same reasons, it is not possible to determine the geoid on the whole terrestrial surface making use of the astro-geodetic networks. On the basis of such networks one can only carry out local determinations of the geoid. Since the methodology of such determinations has been described in Chapter. 6, we will here present only the methods for determining the local reference ellipsoid.

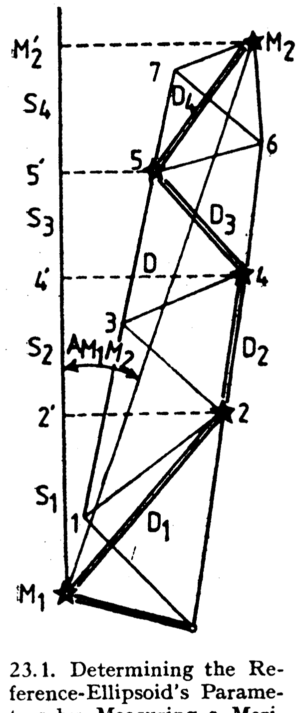
Determining the reference ellipsoid is equivalent to establishing its size and orientation in the Earth's body. The dimensions of the ellipsoid are given by two parameters: the semi-major axis a and the flattening f, while for its orientation one needs knowledge of the geodetic coordinates B_0, L_0, H_0^E of the fundamental point of the astro-geodetic triangulation, as well as an (initial) azimuth A_0 towards another point of the network. According to (10.4), (10.5), (10.10) and (3.49), these elements are:

$$
\begin{gathered}
B_0 = \Phi_0 - \xi_0 - 0{,}171\, H_0 \sin 2B_0;\ L_0 = \Lambda_0 - \eta_0 \sec \Phi; \\
A_0 = \alpha_0 - \eta_0 \tan B_0;\ H_0^E = N_0 + H_0^0.
\end{gathered}
\tag{23.1}
$$

Thus, the basic quantities which allows determination of the reference parameters are: a, f, Φ_0, Λ_0, α_0, H_0^0, ξ_0, η_0, N_0. The astronomical coordinates Φ_0, Λ_0, α_0, are determined by the methods of the position astronomy and the orthometric altitude is determined by the methods described in section 3.5.2. Therefore, in what follows we will only tackle the methods for determining the quantities a, α, ξ_0, η_0, N_0. When *Molodenski*'s theory is used, the geoid undulation N_0 is replaced by the height anomaly ζ_0 and the orthometric altitude by the normal height.

23.1 Determining the Reference Ellipsoid from Measurements of Meridian and Parallel Arcs

Let us consider a triangulation chain connecting the point M_1 to the point M_2. As a result of its adjustment one gets the length D and the azimuth $A_{M_1M_2}$ of the geodetic line which joins the chain's end-points (Fig. 23.1).



23.1. Determining the Reference-Ellipsoid's Parameters by Measuring a Meridian Arc

The length and the azimuth of the geodetic line are obtained by solving the inverse geodetic problem on the arc $\overset{\frown}{M_1M_2}$. The geodetic line D is projected onto the meridian passing through the point M_1, the distance:

$$S = D \cos A^{m}_{M_1M_2}\, \Delta S, \tag{23.2}$$

being obtained, in which ΔS is a corrective coefficient, and the mean azimuth $A^{m}_{M_1M_2}$ is calculated by means of the relation:

$$A^{m}_{M_1M_2} = \frac{A_{M_1M_2} + A_{M_2M_1} \pm 180^\circ}{2}. \tag{23.3}$$

If in the triangulation chain in Fig. 23.1 there exist astronomical determinations at the points M_1, 2, 4, 5, M_2, the arc $\overset{\frown}{M_1M_2}$ may be regarded as being formed by the partial arcs $\overset{\frown}{M_12}$, $\overset{\frown}{24}$, $\overset{\frown}{45}$, $\overset{\frown}{5M_2}$ whose projections onto the meridian passing through M_1 are: S_1, S_2, S_3, S_4.

The length of a meridian arc (9.8) may be presented in a simplified form, integrating the relation (9.9) termwise:

$$S_{1\text{-}2} = a(1-e)^2 \left\{ A'(B_2 - B_1) - \frac{B'}{2}(\sin 2B_2 - \sin 2B_1) + \right.$$

$$\left. + \frac{C'}{4}(\sin 4B_2 - \sin 4B_1) - \ldots \right\}.$$

Replacing the sine difference by a product of sines and cosines of the same angle and substituting the values of the coefficients A', B', C', one gets:

$$s_{1-2} = a(1 - e^2)\left\{\left(1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \ldots\right)(B_2 - B_1) - \right.$$

$$-\left(\frac{3}{4}e^2 + \frac{15}{16}e^4 + \ldots\right)\sin(B_2 - B_1)\cos(B_2 + B_1) +$$

$$\left. + \left(\frac{15}{128}e^4 + \ldots\right)\sin 2(B_2 - B_1)\cos 2(B_2 + B_1)\right\}.$$

Considering:

$$\sin(B_2 - B_1) \approx (B_2 - B_1) - \frac{(B_2 - B_1)^3}{6};$$

$$\sin 2(B_2 - B_1) \approx 2(B_2 - B_1),$$

one obtains:

$$s_{1-2} = a(1 - e^2)(B_2 - B_1)\left\{1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 - \left(\frac{3}{4}e^2 + \frac{15}{16}e^4\right)\cdot\cos 2B_m + \right.$$

$$+ \frac{1}{8}(B_2 - B_1)\cos 2B_m + \frac{15}{96}e^4(B_2 - B_1)^2\cos 2B_m +$$

$$\left. + \frac{15}{64}e^4\cos 4B_m.\right\}.$$

Taking into consideration only the terms in e^2, one arrives at the approximate relation:

$$s = \frac{a(B_2 - B_1)''}{\rho''}\left\{1 - \left(\frac{1}{4} + \frac{3}{4}\cos 2B_m\right)e^2\right\}, \tag{23.4}$$

in which:

B_1, B_2 are the geodetic latitudes of the ends of the meridian arc referred to the reference ellipsoid to be determined;

B_m is the arithmetical average of the latitudes B_1 and B_2;

a, e are the the parameters of the sought ellipsoid.

Inasmuch as the actual parameters a and e of the reference ellipsoid are not known, one introduces their approximate values a_0 and e_0. In this case one obtains:

$$a = a_0 + \Delta a;\ e = e_0 = e_0 + \Delta e, \tag{23.5}$$

the problem's unknowns now becoming Δa and Δe. The eccentricity e being determined, on the basis of (8.3) one may derive the flattening f, so that the reference ellipsoid is defined. Introducing (23.5) into (23.4) yields:

$$\begin{aligned} S = a_0 \frac{(B_2 - B_1)''}{\rho''} \left[1 - \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m\right) e_0^2\right] + \\ + \Delta a \frac{(B_2 - B_1)''}{\rho''} \left[1 - \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m\right) e_0^2\right] - \\ - a_0 \frac{(B_2 - B_1)''}{\rho''} \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m\right) \Delta e^2. \end{aligned} \tag{23.6}$$

The expression of the radius of curvature of the meridian of the ellipsoid a_0, e_0, at a mean latitude B_m is, according to (9.5):

$$M_0^m = \frac{a_0(1 - e_0^2)}{(1 - e \sin^2 B_m)^{3/2}} = a_0 (1 - e_0^2) \left(1 + \frac{3}{2} e_0^2 \sin^2 B_m\right).$$

Expressing sin B_m in terms of cos $2B_m$:

$$\sin^2 B_m = \frac{1}{2} - \frac{1}{2} \cos 2B_m,$$

one gets the following form for the meridian's radius of curvature:

$$M_m^0 = a_0 \left[1 - \left(\frac{1}{4} + \frac{3}{4} \cos 2\, B_m\right) e_0^2\right]. \tag{23.7}$$

From (23.6) and (23.7) one obtains:

$$\begin{aligned} \frac{S}{M_m^0} \rho'' = (B_2 - B_1)'' + (B_2 - B_1)'' \frac{\Delta a}{a_0} - \\ - (B_2 - B_1)'' \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m\right) \Delta e^2. \end{aligned} \tag{23.8}$$

The left-hand side of (23.8) represents the latitude difference of two points located on the a_0, e_0 ellipsoid, corresponding to the distance S and to the mean latitude B_m:

$$\frac{S}{M_m^0} \rho'' = (B_2^0 - B_1^0). \tag{23.9}$$

Introducing (23.9) into (23.8) and expressing in this relation the latitudes B_2 and B_1 by means of (10.5), one gets:

$$\begin{aligned} (B_2^0 - B_1^0)'' = [(\Phi_2 - \xi_2) - (\Phi_1 - \xi_1)] - 0'',171\, (H_2 \sin 2B_2 - H_1 \sin 2B_1) + \\ + (B_2 - B_1)'' \frac{\Delta a}{a_0} - (B_2 - B_1)'' \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m\right) \Delta e^2, \end{aligned}$$

whence one may derive:

$$\xi_2 = \xi_1 + (B_2^0 - B_1^0)'' \frac{\Delta a}{a_0} - (B_2^0 - B_1^0)'' \left(\frac{1}{4} + \frac{3}{4} \cos 2B_m^0\right) \Delta e^2 +$$

$$+ (\Phi_2 - \Phi_1)'' - 0'',171 \, (H_2 \sin 2B_2^0 - H_1 \sin 2B_1^0) - (B_2^0 - B_1^0)''. \qquad (23.10)$$

In (23.10) the differences $(B_2 - B_1)$ were replaced by $(B_2^0 - B_1^0)$ and B_m by B_m^0.

Introducing the notations:

$$(\Phi_2 - \Phi_1)'' - 0'',171(\sin 2B_2^0 \cdot H_2 - \sin 2B_1^0 \cdot H_1) - (B_2^0 - B_1^0)'' = l;$$

$$(B_2^0 - B_1^0) = p; \; -(B_2^0 - B_1^0)\left(\frac{1}{4} + \frac{3}{4} \cos 2B_m^0\right) = q,$$

(23.10) may be written in the form:

$$\xi_2 = \xi_1 + p \frac{\Delta a}{a_0} + q \, \Delta e^2 + l. \qquad (23.11)$$

Equation (23.11) can be regarded as an error equation. For each of the n segments constituting the projection of the geodetic line onto the meridian which passes through the initial point of the triangulation chain one may write an error equation of the form (23.11), which leads to:

$$\begin{aligned} \xi_2 &= \xi_1 + p_1 \frac{\Delta a}{a_0} + q_1 \Delta e^2 + l_1; \\ \xi_3 &= \xi_2 + p_2 \frac{\Delta a}{a_0} + q_2 \Delta e^2 + l_2; \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \xi_n &= \xi_{n-1} + p_{n-1} \frac{\Delta a}{a_0} + q_{n-1} \Delta e^2 + l_{n-1}. \end{aligned} \qquad (23.12)$$

The quantities ξ_i will be considered as random errors in the error equations. The equations (23.12) are not independent and, consequently, cannot be solved by the least squares method. In order to obtain the independence of these equations, one replaces the quantity ξ_{i+1} in the right-hand side of the i equation by its value given by the $i - 1$ equation. One gets:

$$\xi_2 = \xi_1 + p_1 \frac{\Delta a}{a_0} + q_1 \Delta e^2 + l_1;$$

$$\xi_3 = \xi_1 + (p_1 + p_2) \frac{\Delta a}{a_0} + (q_1 + q_2) \Delta e^2 + (l_1 + l_2);$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\xi_n = \xi_1 + (p_1 + p_2 + \dots + p_{n-1}) \frac{\Delta a}{a_0} + (q_1 + q_2 + \dots + q_{n-1}) \Delta e^2 +$$

$$+ (l_1 + l_2 + \dots + l_{n-1}).$$

Denoting by P_i, Q_i, L_i the unknowns' coefficients in the previous equations, one arrives at the following system of error equations:

$$\begin{aligned}
\xi_1 &= \xi_1; \\
\xi_2 &= \xi_1 + P_1 \frac{\Delta a}{a_0} + Q_1 \Delta e^2 + L_1; \\
\xi_3 &= \xi_1 + P_2 \frac{\Delta a}{a_0} + Q_2 \Delta e^2 + L_2; \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
\xi_n &= \xi_1 + P_{n-1} \frac{\Delta a}{a_0} + Q_{n-1} \Delta e^2 + L_{n-1}.
\end{aligned} \tag{23.13}$$

The relations (23.13) represent a system of error equations in which the unknowns are $\xi_1, \frac{\Delta a}{a_0}$ and Δe^2, the quantities Δe^2 playing the role of random errors. The system (23.13) is, consequently, solved under the condition of minimum:

$$\Sigma \xi^2 = \min. \tag{23.14}$$

The condition of minimum (23.14) expresses the fact that the ellipsoid is determined in such a way that it should best approximate the geoid profile along the given meridian arc.

Solving the system of error equations (13.12) yields the quantities $\Delta a/a_0$, Δe^2 and then, with the aid of (23.5), one gets the parameters a and e of the reference ellipsoid. The component ξ_1 whose value is determined along with $\frac{\Delta a}{a_0}$ and Δe^2 is necessary for orienting the ellipsoid in the Earth's body. This is achieved by determining with the help of (10.5a) the latitude of the first point of the triangulation chain:

$$B_1 = \Phi_1 - \xi_1 - 0'',171 \cdot H_1 . \sin 2B_1$$

On the basis of this relation one can orientate the triangulation chain in such a manner that the meridian sections of the ellipsoid and of the geoid are superimposed, the reference ellipsoid's orientation thus being achieved.

In the case in which one measures arcs of parallel, the problem is solved analogously and one gets error equations of the form:

$$\begin{aligned}
\eta_1 &= \eta_1; \\
\eta_2 &= \eta_1 + P'_1 \frac{\Delta a}{a_0} + Q'_1 \Delta e^2 + L'_1; \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
\eta_n &= \eta_1 + P'_{n-1} \frac{\Delta a}{a_0} + Q'_{n-1} \Delta e^2 + L'_{n-1},
\end{aligned} \tag{23.15}$$

in which:

$$P'_i = \sum_{k=1}^{i} p'_k; \quad Q'_i = \sum_{k=1}^{i} q'_k; \quad L'_i = \sum_{k=1}^{i} q'_k; \quad i = 1, 2, \ldots, n;$$

$$p'_k = (L^0_{k+1} - L^0_k) \cos B_0; \quad q'_k = \frac{1}{2} (L^0_{k+1} - L^0_k) \cos B_0 \sin^2 B_0;$$

$$l'_k = [(\Lambda_{k+1} - \Lambda_k) - (L^0_{k+1} - L^0_k)] \cos B_0.$$

The equation system (23.15) is solved under the condition $\Sigma\eta^2 = \min$. For determining the quantities a, e^2 and η_1 it is necessary to apply the method to two arcs of parallel at latitudes palpably different, since measuring the arcs of parallel as arcs of circle yields the radius of the latter, from which, knowing the latitude, one can determine the values a and η_1.

In practice, the determination of the parameters and of the orientation of the reference ellipsoid is carried out on the basis of processing in common the measurements of arcs of parallel and of arcs of meridian, under the condition $\Sigma(\xi^2 + \eta^2) = \min$. Along with getting the reference ellipsoid's parameters, one also obtains the quantities ξ_1, and $\eta_1 \sec B_1$, on whose basis one can determine the initial elements of the triangulation:

$$B_0 = \Phi_1 - \xi_1; \quad L_0 = \Lambda_1 - \eta_1 \sec B_0; \quad A_0 = \alpha_1 - \eta_1 \tan B_0. \tag{23.16}$$

The method of determining the parameters and the orientation of the reference ellipsoid, presented in this section, is based on some assumptions which are not wholly fulfilled in practice. One of these assumptions consists of regarding the ξ and η components of the deflection of the vertical as random errors which should obey the condition $\Sigma\xi = 0$, (respectively $\Sigma\eta = 0$). One knows, however, that the geoid, as well as its small undulations of a local character, also has large undulations of a continental character. Consequently, the conditions $\Sigma\xi = 0$ (respectively $\Sigma\eta = 0$) are fulfilled only if the meridian-arc measurements (respectively those of arcs of parallel) are extended to the Earth's entire surface. In the case of separate arc measurements, the components of the deflection of the vertical contain, in addition to an accidental part, an important systematic part, which leads to the conclusion that the application to this case of the least squares method is of an arbitrary character. On the other hand, this arbitrary character is strengthened by solving the problem under the condition $\Sigma\xi^2 + \eta^2 = \min.$, since the quantities ξ_i and η_i are dependent on one another.

The measured meridian arc is reduced to the geoid's level. In deducing the method theoretically, the S arc is defined by means of (23.4) as a quantity on the ellipsoid. This corresponds to developing the S arc, whose magnitude is known on the geoid's surface, on the surface of the a_0, e_0 ellipsoid. This is the reason why solving the problems of large triangulations on the basis of this principle is designated as the method of unfolding. The application of the method of unfolding to calculating the reference ellipsoid's parameters making use exclusively of geodetic measurements does not take account of the geoid undulation along the chosen profile, which leads to considerable errors. A way of improving the accuracy of this method consists of solving the problem in two approximations. The complete elimination of these errors is obtained only by fully replacing the method of unfolding with the projection method, which was analysed in Chapter. 10.

23.2 Determining the Reference Ellipsoid by Using the Method of Surfaces

The method of surfaces is applied when a geodetic network covering a large area is available. In order to facilitate the presentation, this network is regarded, in the following, as formed by a polygon of triangulation chains (Fig. 23.2); (the problem is solved in the same manner for a compact network too).

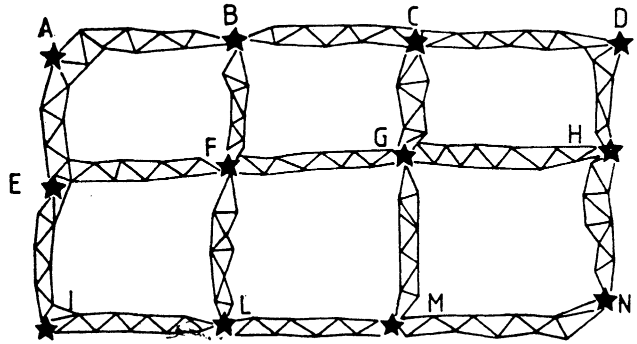


Fig. 23.2. Determining the Reference-Ellipsoid's Parameters by the Method of Surfaces

At the intersection of the triangulation chains being considered, astronomical determinations of latitude, longitude and azimuth are carried out. By adjusting these triangulation chains on any reference ellipsoid a_0, f_0, one determines the geodetic coordinates of all the triangulation points denoted by B_A^0, L_A^0, A_A^0, B_B^0, L_B^0, A_B^0, ..., B_N^0, L_N^0, A_N^0.

As initial point of the triangulation the point A is taken. Utilizing the inverse geodetic problem, one also determines the lengths and the azimuths of the geodetic lines joining the intersection points of the triangulation chains. The coordinates and the azimuths determined at the points $A, B, \ldots, N$, calculated on the reference ellipsoid, which is to be determined, are denoted by $B_A L_A A_A$, $B_B L_B A_B$, ..., $B_N L_N A_N$ and the components of the deflection of the vertical with respect to the same ellipsoid by ξ_A, η_A, ..., ξ_N, η_N.

Setting:

$$a = a_0 + \Delta a;\ f = f_0 + \Delta f, \tag{23.17}$$

the problem may be stated thus: determining the corrections Δa and Δf on the basis of the condition:

$$\Sigma(\xi^2 + \eta^2) = \text{min.}, \tag{23.18}$$

The condition (23.18) embodies the fact that the reference ellipsoid is determined in such a manner that it best approximates the geoid on the surface over which the triangulation network is spread.

On the basis of (10.4), (10.5a), (10.10) one can write the equalities:

$$\begin{aligned}
B_A &= B_A^0 + \mathrm{d}B_A = \Phi_A - \xi_A;\\
L_A &= L_A^0 + \mathrm{d}L_A = \Lambda_A - \eta_A \sec \Phi_A;\\
A_A &= A_A^0 + \mathrm{d}A_A = \alpha_A - \eta_A \tan \Phi_A;\\
&\dots\dots\dots\dots\dots\dots\\
B_N &= B_N^0 + \mathrm{d}B_N = \Phi_N - \xi_N;\\
L_N &= L_N^0 + \mathrm{d}L_N = \Lambda_N - \eta_N \sec \Phi_A;\\
A_N &= A_N^0 + \mathrm{d}A_N = \alpha_N - \eta_N \tan \Phi_A.
\end{aligned}$$

The last two equalities of each previous equation group are mutually dependent, for which reason one will give up, in subsequent developments, the second equality of each equation group.

In this way one gets:

$$\begin{aligned}
\xi_A &= \Phi_A - B_A^0 - \mathrm{d}B_A;\\
\eta_A \tan \Phi_A &= \alpha_A - A_A^0 - \mathrm{d}A_A;\\
&\dots\dots\dots\dots\dots\dots\\
\xi_N &= \Phi_N - B_N^0 - \mathrm{d}B_N;\\
\eta_N \tan \Phi_N &= \alpha_N - A_N^0 - \mathrm{d}A_N.
\end{aligned} \tag{23.19}$$

In (23.19) $\mathrm{d}B_A$, $\mathrm{d}L_A$, $\mathrm{d}A_A \dots$ etc. denote the corrections of the geodetic coordinates and azimuths when passing from the initial ellipsoid a_0, f_0 to the reference ellipsoid a, f, looked for. Consequently, the problem can be solved making use of the differential relations analysed in Section 12.4. The total derivatives of the geodetic latitude and longitude may thus be written:

$$\begin{aligned}
\mathrm{d}B_k &= \left(\frac{\partial B_k^0}{\partial B_A^0}\right)\mathrm{d}B_A + \left(\frac{\partial B_k^0}{\partial A_A^0}\right)\mathrm{d}A_A + \left(\frac{\partial B_k^0}{\partial a}\right)\Delta a + \left(\frac{\partial B_k^0}{\partial f}\right)\Delta f;\\
\mathrm{d}L_k &= \mathrm{d}L_A + \left(\frac{\partial l_{Ak}^0}{\partial B_A^0}\right)\mathrm{d}B_A + \left(\frac{\partial l_{Ak}^0}{\partial A_A^0}\right)\mathrm{d}A_A + \left(\frac{\partial l_{Ak}^0}{\partial a}\right)\Delta a + \left(\frac{\partial l_{Ak}^0}{\partial f}\right)\Delta f.
\end{aligned} \tag{23.20}$$

Putting:

$$\left(\frac{\partial B_k^0}{\partial B_A^0}\right) = p_1^{Ak};\ \left(\frac{\partial B_k^0}{\partial A_A^0}\right) = p_2^{Ak};\ \left(\frac{\partial B_k^0}{\partial a}\right) = p_3^{Ak};\ \left(\frac{\partial B_k^0}{\partial f}\right) = p_4^{Ak};$$

$$\left(\frac{\partial l_{Ak}^0}{\partial B_A^0}\right) = q_1^{Ak};\ \left(\frac{\partial l_k^0}{\partial A_A^0}\right) = q_2^{Ak};\ \left(\frac{\partial l_k^0}{\partial a}\right) = q_2^{Ak};\ \left(\frac{\partial l_k^0}{\partial f}\right) = q_4^{Ak}$$

and in view of the equalities (23.20), the relations (23.19) become, for any point K (Fig. 23.3):

$$\xi_k = \Phi_k - B_k^0 - p_1^{Ak}\, \mathrm{d}B_A - p_2^{Ak}\, \mathrm{d}A_A - p_3^{Ak}\, \Delta a - p_4^{Ak}\, \Delta f;$$

$$\eta_k \sec \Phi_k = \Lambda_k - L_k^0 - \mathrm{d}L_A - q_1^{Ak}\, \mathrm{d}B_A - q_2^{Ak}\, \mathrm{d}A_A - q_3^{Ak}\, \Delta a - q_4^{Ak}\, \Delta f.$$

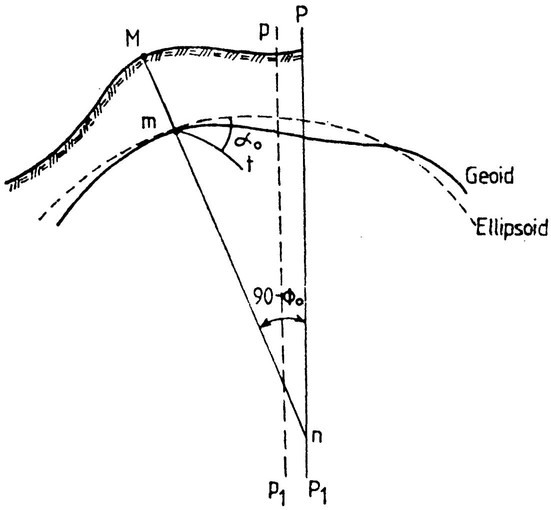


Fig. 23.3. Orienting the Reference Ellipsoid

Replacing in the previous relations the values $\mathrm{d}B_A$, $\mathrm{d}L_A$, $\mathrm{d}A_A$ from (23.18) yields:

$$\xi_k = \Phi_k - B_k^0 - p_1^{Ak}\,(\Phi_B - B_A^0) + p_1^{Ak}\,\xi_A - p_2^{Ak}\,(\alpha_A - A_A^0) +$$
$$+ p_2^{Ak}\,\eta_A \tan \Phi_A - p_3^{Ak}\,\Delta a - p_4^{Ak}\,\Delta f; \tag{23.21}$$

$$\eta_k \sec \Phi_k = \Lambda_k - L_k^0 - (\Lambda_A - L_A^0) + \eta_A \sec \Phi_A - q_1^{Ak}(\Phi_A - B_k^0) +$$
$$+ q_1^{Ak}\,\xi_A - q_2^{Ak}\,(\alpha_A - A_A^0) + q_2^{Ak}\,\eta_A \tan \Phi_A - q_3^{Ak}\,\Delta a - q_4^{Ak}\,\Delta f.$$

If one considers the reference ellipsoid oriented at the initial point A according to astronomical measurements, then $\Phi_A = B_A^0$, $\alpha_A = A_A^0$, $\Lambda_A = L_0^A$ and the relations (23.20) become much simpler:

$$\xi_k = (\Phi_k - B_k^0) + p_1^{Ak}\,\xi_A + p_2^{Ak}\,\eta_A \tan \Phi_A - p_3^{Ak}\,\Delta a - p_4^{Ak}\,\Delta f;$$

$$\eta_k \sec \Phi_k = (\Lambda_k - L_k^0) + q_1^{Bk}\,\xi_A + [1 + q_2^{Ak} \sin \Phi_A]\,\eta_A \sec \Phi_A -$$
$$- q_3^{Ak}\,\Delta a - q_4^{Ak}\,\Delta f. \tag{23.22}$$

Equations of the form (23.22) are formed for all the intersection points of the triangulation chains, leading to a system of error equations. This system is solved by the least squares method, on the basis of the condition of minimum (23.18). The solution yields the quantities Δa and Δf, and on the basis of (23.17) one gets the parameters of the reference ellipsoid. Solving the system (23.22) equally yields the quantities ξ_A, η_A, by means of which one orientates the reference ellipsoid by establishing the coordinates of the initial point:

$$B_A = \Phi_A - \xi_A;\ L_A = \Lambda_A - \eta_A \sec \Phi_A;\ A_A = \alpha_A - \eta_A \tan \Phi_A. \qquad (23.23)$$

This method underlay the determination of *Hayford's* ellipsoid in the year 1909 which was used in geodetic work in *Romania* in the period 1950—1951.

23.3 Determining the Reference Ellipsoid by Using Differential Relations

Let us consider an astro-geodetic network for whose processing one has adopted an arbitrary reference ellipsoid a_0, f_0, the orientation of which is given by the arbitrary coordinates B_0 and L_0 of the triangulation's fundamental point as well as by the quantities ζ_0 and A_0 (the geodetic azimuth towards a point B of the network). Here one has adopted the notation ζ for the geoid's undulation, in order to avoid confusion arising from using the same letter N to denote both the geoid's undulation and the radius of curvature of the prime vertical. One assumes that the processing of the network is carried out by means of the projection method (Chapter 10). The problem of the determination of the actual reference ellipsoid may be stated as that of determining the corrections of the parameters of the initial ellipsoid: da_0, df_0 (or de_0), dB_0, dL_0, dA_0 and $d\xi_0$.

The coordinates of the network's fundamental point, on the ellipsoid looked for, are:

$$B_0 + \delta B_0 = \Phi_0 - \xi_0 - 0'', 171 \cdot H_0 \sin 2B_0;\ L_0 + \delta L_0 = \Lambda_0 - \eta_0 \sec \Phi_0;$$
$$A_0 + \delta A_0 = \alpha_0 - \eta_0 \tan \Phi_0;\ \zeta_0' + \delta\zeta_0 = \zeta_0. \qquad (23.24)$$

Let us consider a triaxial coordinate system with its origin at the centre of the initial ellipsoid a_0, f_0. The centre of the ellipsoid being sought will have in this system the coordinates δx_0, δy_0, δz_0 and any point on the geoid's surface will have the coordinates:

$$x = (N + \zeta') \cos B \cos L;\ y = (N + \zeta') \cos B \sin L;$$
$$z = N(1 - e^2) \sin B + \zeta' \sin B, \qquad (23.25)$$

where B and L denote the coordinates of the corresponding point situated on the geoid, referred to the initial ellipsoid.

For the same point, the rectangular coordinates referred to the centre of the reference ellipsoid looked for are:

$$x + \delta x; \; y + \delta y; \; z + \delta z.$$

The corrections δx, δy, δz are functions of the variations of the geodetic coordinates, of the major semi-axis and of the flattening. Therefore they can be written:

$$\begin{aligned}
\delta x &= \frac{\partial x}{\partial B}\,\delta B + \frac{\partial x}{\partial L}\,\delta L + \frac{\partial x}{\partial \zeta}\,\delta\zeta + \frac{\partial x}{\partial a}\,\delta a + \frac{\partial x}{\partial f}\,\delta f;\\
\delta y &= \frac{\partial y}{\partial B}\,\delta B + \frac{\partial y}{\partial L}\,\delta L + \frac{\partial y}{\partial \zeta}\,\delta\zeta + \frac{\partial y}{\partial a}\,\delta a + \frac{\partial y}{\partial f}\,\delta f;\\
\delta z &= \frac{\partial z}{\partial B}\,\delta B + \frac{\partial z}{\partial L}\,\delta L + \frac{\partial z}{\partial \zeta}\,\delta\zeta + \frac{\partial z}{\partial a}\,\delta a + \frac{\partial z}{\partial f}\,\delta f.
\end{aligned} \tag{23.26}$$

Inasmuch as A stands for the *Laplace* azimuth, equations two and three in (23.24) are dependent. One therefore gives up in this demonstration the third equation. This is why in (23.26) the termes in δA are absent. Working out the partial derivatives of the relations (23.25), and also considering (9.5) and (9.50) and introducing these values into the relation (23.26), one gets:

$$\begin{aligned}
\delta x &= -(M+\zeta)\sin B\cos L\,\delta B - (N+\zeta)\cos B\sin L\,\delta L +\\
&\quad + \cos B\cos L\,\delta\zeta + N\cos B\cos L\frac{\delta a}{a} + M\cos B\cos L\sin^2 B\frac{\delta f}{1-f};\\
\delta y &= -(M+\zeta)\sin B\sin L\,\delta B + (N+\zeta)\cos B\cos L\,\delta L +\\
&\quad + \cos B\sin L\,\delta\zeta + N\cos B\sin L\frac{\delta a}{a} + M\cos B\sin L\sin^2 B\frac{\delta f}{1-f};\\
\delta z &= (M+\zeta)\cos B\,\delta B + \sin B\,\delta\zeta + N(1-e^2)\sin B\frac{\delta a}{a} -\\
&\quad - M(1+\cos^2 B - e^2\sin^2 B)\sin B\frac{\delta f}{1-f}.
\end{aligned} \tag{23.27}$$

From these relations one can extract the values of the corrections δB, δL, $\delta\zeta$ in the form:

$$\begin{aligned}
\delta B &= -\frac{1}{M}\sin B\cos L\,\delta x - \frac{1}{M}\sin B\sin L\,\delta y + \frac{1}{M}\cos B\,\delta z +\\
&\quad + \frac{N}{M}e^2\sin B\cos B\frac{\delta a}{a} + (2 - e^2\sin^2 B)\sin B\cos B\frac{\delta f}{1-f};
\end{aligned} \tag{23.28}$$

$$\delta L = -\frac{1}{N}\sec B \sin L\,\delta x + \frac{1}{N}\sec B \cos L\,\delta y;$$

$$\delta\zeta = \cos B \cos L\,\delta x + \cos B \sin L\,\delta y + \sin B\,\delta z - N(1 - e^2\sin^2 B)\frac{\delta a}{a} +$$

$$+ M(1 - e^2\sin^2 B)\sin^2 B\frac{\delta f}{1-f}. \qquad (23.28)$$

If one writes the relations (23.24) for any point, one gets:

$$\xi = (\Phi - B) - \delta B - 0'', 171 \cdot H \sin 2B;$$

$$\eta = (\Lambda - L)\cos B - \delta L;\ \zeta = \zeta' + \delta\zeta. \qquad (23.29)$$

Replacing (23.28) into (23.29) yields:

$$\xi'' = \frac{\rho''}{M}\sin B \cos L\,\delta x_0 + \frac{\rho''}{M}\sin B \sin L\,\delta y_0 - \frac{\rho''}{M}\cos B\,\delta z_0 -$$

$$- \rho'' e^2 \sin B \cos B\frac{\partial a}{a} - \rho''(2 - e^2\sin^2 B)\sin B \cos B\frac{\delta f}{1-f} +$$

$$+ (\Phi - B) - 0'',171 H \cdot \sin 2B; \qquad (23.30)$$

$$\eta'' = \frac{\rho''}{N}\sin L\,\delta x_0 - \frac{\rho''}{N}\cos L\,\delta y_0 + (\Lambda - L)\cos\Phi;$$

$$\zeta = \cos B \cos L\,\delta x_0 + \cos B \sin L\,\delta y_0 + \sin B\,\delta z_0 -$$

$$- N(1 - e^2\sin^2 B)\frac{\delta a}{a} + M(1 - e^2\sin^2 B)\sin^2 B\frac{\delta f}{1-f} + \zeta'.$$

Regarding the quantities ξ, η, ζ as random errors, the relations (23.30) may be considered error equations. Writing the three equalities in (23.30) for each point of the given astro-geodetic triangulation network yields a system of error equations which can be solved under the condition of minimum $\Sigma(\xi^2 + \eta^2 + \zeta^2) = \min$.

From solving this system one obtains the looked for parameters of the reference ellipsoid.

By means of this procedure one can determine very accurately the linear parameter δa of the ellipsoid. The flattening f is determined, however, with a much smaller accuracy. For determining this parameter precisely it is advisable to utilize observations on the Earth's artificial satellites.

24

The Determination of the Earth's Form from Astro-Geodetic and Gravity Measurements

The astro-geodetic measurements can serve as a basis for determining the dimensions of the terrestrial ellipsoid, but the parameters of the ellipsoid obtained in this manner do not entirely meet the accuracy requirements imposed by the present stage of development of Geodesy. The method presented in Section 23.3 permits one to determine the ellipsoid's semi-major axis with a high degree of accuracy. However, the same accuracy is not reached in determining the flattening f; in order to determine this parameter, one needs to use gravity data and the results of observations on the Earth's artificial satellites.

If the Earth's mass M, the gravity potential W, the difference $C - (A + B)/2$ of the moments of inertia and the rotational angular velocity ω of the Earth were known, Physical Geodesy could satisfactorily settle the problem of determining the Earth's form and size. Since these quantities are known only approximately, Physical Geodesy cannot be used as a self-sufficient method for the determination of the form and dimensions of the Earth. Therefore, to this end, one utilizes together the results of astronomical, geodetic and gravity measurements. As a result of these measurements one obtains either the terrestrial ellipsoid's parameters, as was shown in Section 23.3, or the geoid's global form, the physical constants of the Earth or the parameters of the general terrestrial ellipsoid, as we shall show in this chapter.

24.1 Global Determination of the Geoid

Stokes' formula (5.6) allows, (when one knows the gravity anomaly on the Earth's entire surface), one to determine the geoid's undulations with respect to an ellipsoid having the same mass as the Earth's total mass M, the same potential U_0 as the geoid's actual potential W_0 and its centre coincident with the Earth's mass centre. If such an ellipsoid is available, *Stokes'* formula offers the possibility of determining the geoid as "mathematical surface of the Earth", as it was termed by *F. Gauss*.

If, however, the available ellipsoid has a mass $M' \neq M$ and a potential $U'_0 \neq W_0$, then for the global determination of the geoid one must use *Stokes'* generalized formula (5.11):

$$N = \frac{G\,\Delta M}{R\gamma} - \frac{\Delta W}{\gamma} + \frac{R}{4\pi\gamma}\iint\limits_{\sigma} \Delta g\, S(\psi)\, d\sigma. \tag{24.1}$$

In (24.1), writing:

$$N_0 = \frac{G\,\Delta M}{R\gamma} - \frac{\Delta W}{\gamma}, \tag{24.2}$$

the formula for determining the geoid's undulations is obtained in the form:

$$N = N_0 + \frac{R}{4\pi\gamma}\iint\limits_{\sigma} \Delta g\, S(\psi)\, d\sigma = N_0 + N'. \tag{24.3}$$

If in determining the geoid's surface one utilizes only the second term of (24.3):

$$N' = \frac{R}{4\pi}\iint\limits_{\sigma} \Delta g\, S(\psi) d\sigma, \tag{24.4}$$

then instead of the geoid's surface S one determines the surface S', parallel to the surface S at a distance N_0 from it (Fig. 24.1).

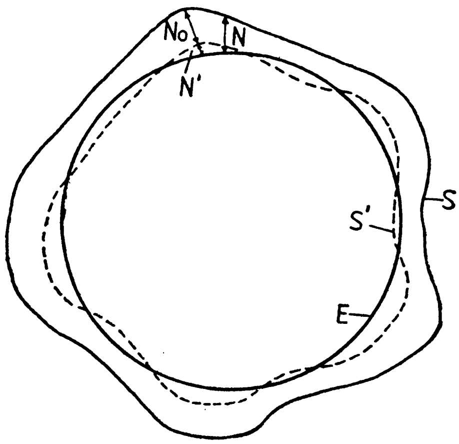


Fig 24.1. Geoid's Determination by Using Stokes' Formula

In Fig. 24.1, E represents the surface of the ellipsoid $U'_0 \neq W_0$, $M' \neq M$, S' — the geoid's surface as determined from (24.4) and S — the actual surface of the geoid, whose determination can be carried out by means of (24.3).

For the global determination of the geoid one consequently needs to know the constant N_0 and the distribution of the gravity anomaly over the Earth's entire surface.

24.1.1 Determining the Constant N_0

If one knows precisely the differences ΔM and ΔW between the actual mass and potential of the Earth and the mass and normal potential of the ellipsoid being used, then the constant N_0 may easily be determined by means of (24.2). In this situation, the geoid can be determined on the actual scale, without carrying out any kind of distance measurement, consequently without having recourse to geodetic and astronomical measurements.

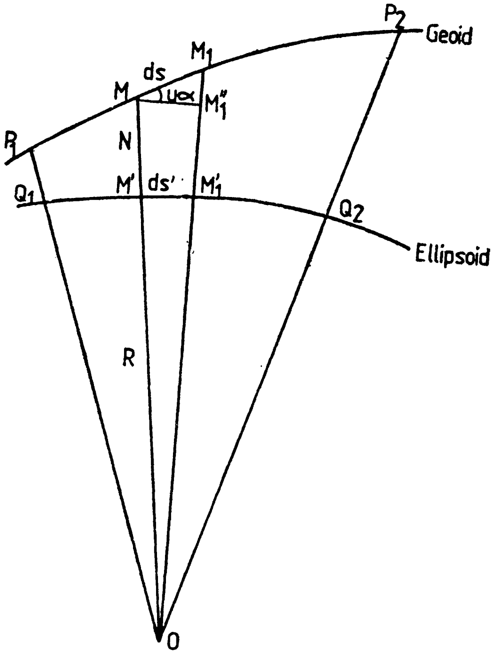


Fig. 24.2. Determination of the Constant N_0 by Using Astronomical and Geodetic Measurements

In practice, however, the actual mass and potential of the Earth are not accurately known. In this case, one needs geodetic and astronomical measurements in order to determine the constant N_0.

Let P_1 and P_2 be two points situated on the geoid, and Q_1 and Q_2 their projections onto the ellipsoid along the normals to the latter (Fig. 24.2). The distances between the points P_1 and P_2 (respectively Q_1 and Q_2) are denoted by s and s'.

Considering on the geoid two points M and M_1, infinitely close, and their projections M' and M_1' onto the ellipsoid, and considering also the deflection of the vertical u_α at M, from the triangles $MM_1''O$ and $M'M_1'O$ one deduces the equality:

$$\frac{ds \cos u_\alpha}{R+N} = \frac{ds'}{R},$$

whence, taking $\cos u_\alpha \approx 1$, one gets:

$$ds = ds' \frac{R+N}{R} = ds' + \frac{N}{R} ds',$$

Inasmuch as the coefficient of the quantity ds' is very small, in the last term of the previous relation, one can take $ds' \approx ds$, without introducing perceptible errors. In this case:

$$ds = ds' + \frac{N}{R} ds.$$

The distance P_1P_2 can now be obtained as:

$$s = s' + \frac{1}{R}\int_{Q_1}^{Q_2} N \, ds.$$

In view of (24.3), this equality may also be presented in the form:

$$s - s' = \frac{1}{R}\int_{Q_1}^{Q_2} (N' + N_0) \, ds = N_0 \frac{s}{R} + \frac{1}{R}\int_{Q_2}^{Q_1} N' \, ds,$$

whence one derives the value of the constant N_0:

$$N_0 = \frac{R}{s}(s - s') - \frac{1}{s}\int_{Q_1}^{Q_2} N' \, ds. \tag{24.5}$$

In order to determine the constant N_0 by means of (24.5), one needs gravity and geodetic measurements. Indeed, the quantity N' is determined using *Stokes'* formula (24.4) making use of gravity measurements on the Earth's entire surface. The distance s is obtained from measuring the distance between two points of the Earth's physical surface and reducing it on the geoid. The distance s' between the points Q_1 and Q_2 on the ellipsoid may be determined if one knows the geodetic coordinates B and L of these points.

Once the constant N_0 is determined by means of (24.5), one can determine, on the basis of (24.3), the geoid on its actual scale, utilizing, to this end, a single distance measurement. In the case in which a large number of measured distances is available, as actually happens in practice, the constant N_0 is determined by an adjustment applying the least squares method.

24.1.2 Examples of Global Determinations of the Geoid

The application of *Stokes'* formula for determining the geoid's undulations needs knowledge of the gravity anomalies over the Earth's entire surface.

In order to realize this desideratum it is necessary:

(1) *To know the value of gravity at every point of the terrestrial surface.*

(2) *To have the gravity values determined in a well-defined system.*

(3) *To have the same ellipsoid adopted for calculating the gravity anomaly.*

(4) *To adopt the same type of anomaly on the whole surface of the Earth.*

In practice, however, none of these conditions is completely fulfilled.

A large part of the Earth's surface, consisting of the surface of the sea and even of continental areas, is still unexplored by gravity techniques. In order to obtain a complete gravity map of the entire surface of the Earth one has attempted to fill in these gaps making use, to this end, of various extrapolation methods. Thus, *A. Uotila* has calculated the free-air anomalies as effects of the topographic masses, corresponding to the zero isostatic anomaly. A similar contribution was made by *L. Kivioja* who, unlike *A. Uotila*, has also used, in extrapolating the anomalies, actual, measured, values of gravity. An attempt with good results was made by *K. Arnold*, who determined the gravity anomalies utilizing observations on successive passages of the Earth's artificial satellites.

In order to obtain the gravity field in the same system, all the gravity measurements are carried out in the unitary *Potsdam* system. On the basis of national maps, local gravity-anomaly maps have been compiled, using, however, different ellipsoids: *Clarke, Hayford, Cassinis* etc. Obtaining a global gravity map by putting together the various national maps is, therefore, impossible. In passing to a unitary anomaly system it is necessary to adopt a unique value for the normal gravity, which should serve as a basis for transforming the local gravity maps into a unified World map.

Hitherto, the global form of the geoid has been calculated by several authors, making use of various gravity methods and materials. The first geoid calculation was carried out by *A. Hirvonen*, who compiled the global map of the geoid on the basis of the determination of the undulations of the latter at 62 points. The calculations were carried out making use of *Stokes'* formula and as basic material were utilized the anomaly mean values within areas of $5° \times 5°$, where gravity determinations were available, whereas in the other zones the free-air anomalies corresponding to the zero isostatic anomalies were used.

Equally on the basis of *Stokes'* formula, geoid determinations were carried out by *Tanni* (who in 1948 determined the global form of the geoid and in 1949 its form on the territory of *Central Europe*) and by *Heiskanen* who determined the *Columbus* geoid in 1957.

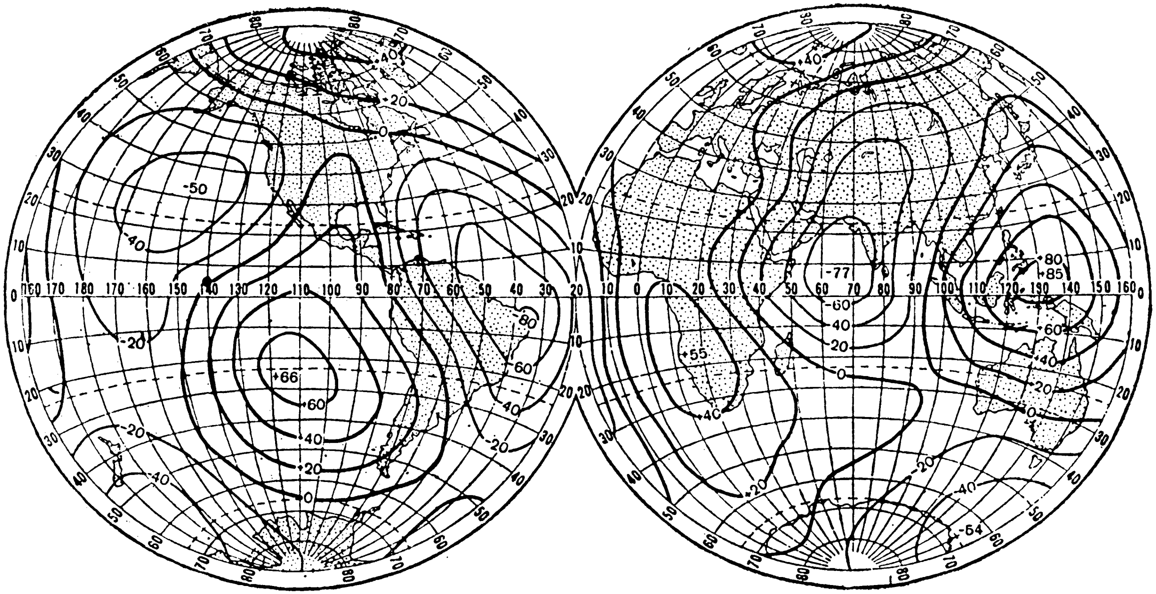
In 1961, *Irene Fischer*, making use of the totality of 1st-order astro-geodetic measurements existing at that date, with their connexions without solutions of continuity (joining the meridian arc *Cape of Good Hope—Cairo* and that of the *Andes Mountains*), determined the astro-geodetic global geoid with respect to the spheroid $f = 1/298.3$; $a = 6{,}378{,}166$ m.

Global determinations of the geoid utilizing the expansions of the potential and of the gravity anomaly in spherical harmonics were carried out by *I. D. Zhongolovich* in the year 1952 (Fig. 24.3), *M. Kaula* in 1962, *A. Uotila* in 1962 and *M. Bursa* in 1969 (Fig. 24.4).

24.2 Determining the Earth's Physical Constants

The Earth's main physical constants are: the mass M and the gravity potential W. Since the universal constant of gravitation G is not precisely known, instead of the mass M the product GM will be considered as a physical constant of the Earth, and designated as *the Earth's gravitational geocentric constant.*

In connexion with the determination of the Earth's physical constants, it should be remembered that for the case of any reference ellipsoid ($U_0 \approx W_0$;



Fid. 24.3. The Geoid as Determined by I. Zhongolovich; Triaxial Reference Ellipsoid: f_1 1: 296.6; f_2 32,000; $\Lambda = 6°$

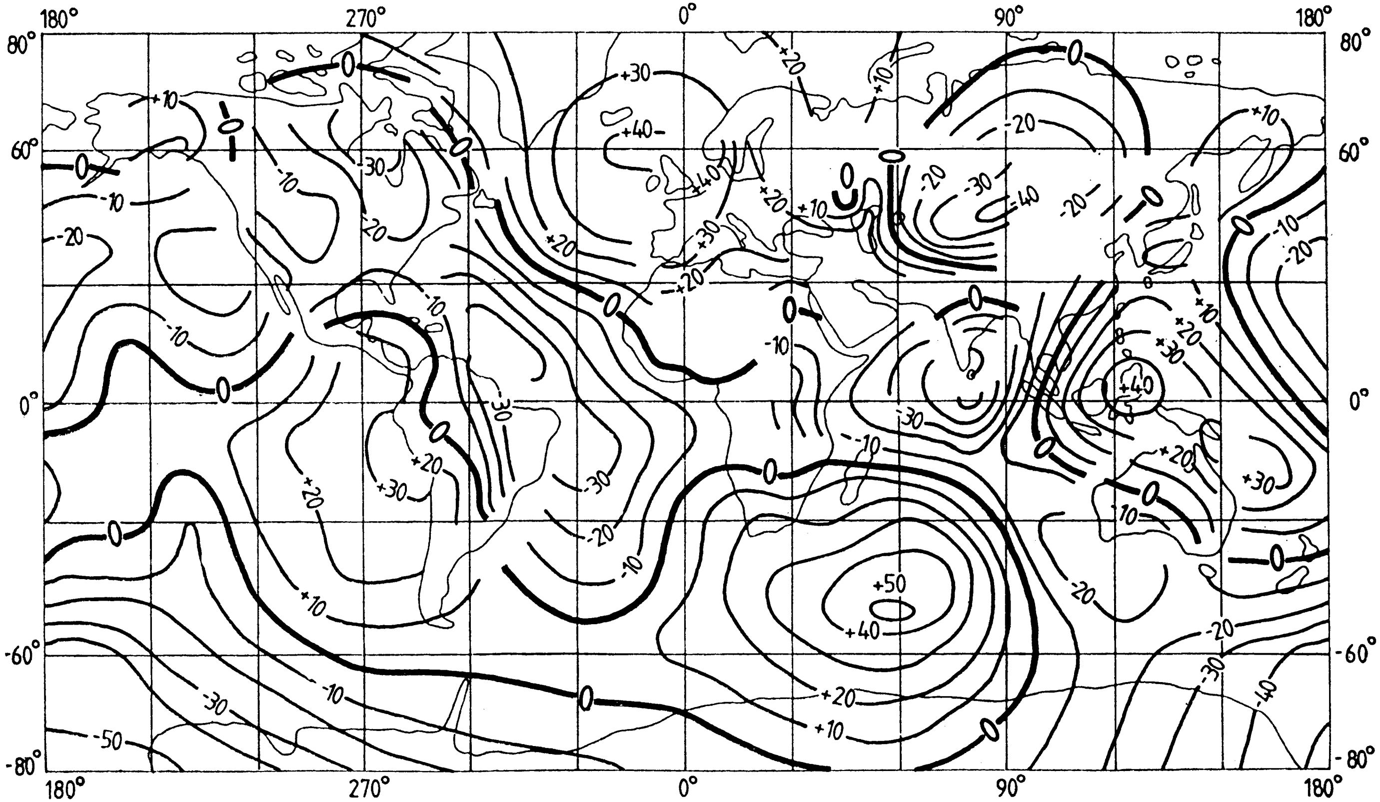


Fig. 24.4. The Geoid as Determined by M. Bursa; Triaxial Reference Ellipsoid: $a = 6{,}378{,}173$ m; $e^2 = 0.006\,704\,953$; $e_1^2 = 0.000\,021\,275$; $\Lambda = 14°.8$

$\Delta M \approx 0$), the boundary condition in the spherical approximation (4.25) is written as:

$$\Delta g = -\frac{\partial T}{\partial \rho} - \frac{2g_m}{R} N.$$

Considering $g_m = \gamma$ and according to (4.18):

$$N = \frac{T - \Delta W}{\gamma},$$

the previous boundary condition may be written as:

$$\Delta g = -\frac{\partial T}{\partial \rho} - \frac{2\gamma}{R}\left(\frac{T - \Delta W}{\gamma}\right) = -\frac{\partial T}{\partial \rho} - \frac{2T}{R} + \frac{2\Delta W}{R}. \tag{24.6}$$

Introducing into this relation the values of the perturbing potential T and of its derivative from (4.30) and (4.32) yields:

$$\begin{aligned} \Delta g &= \frac{1}{R}\sum_{n=0}^{\infty}(n+1)\,T_n(\theta, \lambda) - \frac{2}{R}\sum_{n=0}^{\infty} T_n(\theta, \lambda) + \frac{2\Delta W}{R} = \\ &= \frac{1}{R}\sum_{n=0}^{\infty}(n-1)\,T_n(\theta, \lambda) + \frac{2\Delta W}{R}. \end{aligned} \tag{24.7}$$

On the other hand, the gravity anomaly can be expressed by a harmonic series of the form:

$$\Delta g = \sum_{n=0}^{\infty} \Delta g_n. \tag{24.8}$$

From (24.7) and (24.8) one gets:

$$\sum_{n=0}^{\infty} \Delta g_n = \frac{1}{R}\sum_{n=0}^{\infty}(n-1)T_n(\theta, \lambda) + \frac{2\Delta W}{R}. \tag{24.9}$$

For $n = 0$, the equality (24.9) becomes:

$$\Delta g_0 = \frac{1}{R}\,T_0(\theta, \lambda) + \frac{2\,\Delta W}{R}. \tag{24.10}$$

Expressing the zero-degree harmonic of the perturbing potential, utilizing (5.7), yields:

$$\Delta g_0 = \frac{G\Delta M}{R^2} + \frac{2\Delta W}{R}. \tag{24.11}$$

The zero-degree harmonic of the gravity anomaly can be expressed as a surface harmonic by means of (2.41). Indeed, for $n = 0$ this relation becomes:

$$\Delta g_0 = \frac{1}{4\pi}\iint_{\tau} \Delta g \, d\sigma. \tag{24.12}$$

The relations (24.11) and (24.2) form an equation system which, when solved with respect to GM and W, leads to:

$$G\Delta M = R^2 \cdot \Delta g_0 + 2\, R g_m N_0;\ \Delta W = R\, \Delta g_0 + g_m N_0, \qquad (24.13)$$

in which γ was replaced with an average value g_m of the gravity on the Earth's entire surface.

From the relations (24.13) one may determine the constants GM and ΔW, from which one can afterwards calculate the quantities GM and W_0 as physical constants of the terrestrial ellipsoid which meets *Stokes'* conditions (has the same potential as the geoid and its mass equals the Earth's mass):

$$M = M' + \Delta M;\ W_0 = U_0' + \Delta W. \qquad (24.14)$$

The quantities M and W_0 can be calculated only under the condition that the gravity anomaly over the entire surface of the Earth is known. This condition is imposed by the relation (24.12).

24.3 Determining the Gravity Field and the Parameters of the General Terrestrial Ellipsoid

The general terrestrial ellipsoid is that ellipsoid which: (1) *contains the same volume as the geoid;* (2) *has its centre as the Earth's mass centre;* (3) *has its dimensions chosen in such a way that the sum of the geoid's undulations is zero.*

For such an ellipsoid, the gravity field is completely defined if one knows: the geocentric gravitational constant GM, the geoid's potential W_0, the difference of the moments of inertia $C - (A + B)/2$ and the rotational angular velocity ω.

If these parameters are known, one can determine both the semi-major axis a and the flattening f of the general terrestrial ellipsoid.

For solving the problem, one assumes an ellipsoid of semi-major axis a_0 and flattening f_0, whose potential U_0' differs from the potential W_0 of the geoid. One also assumes that the centre of this ellipsoid does not coincide with the Earth's mass centre and, as a consequence, the principal moments of inertia of this ellipsoid will differ from the Earth's moments of inertia. Considering as known the mass M' of this ellipsoid, one may determine the actual mass of the Earth, either by means of (24.14), as has been shown in the previous section, or utilizing the density φ of the single layer which, according to (7.12) generates the perturbing potential T. In this latter case, the geocentric gravitational constant GM is calculated by means of the relation:

$$GM = GM' + \int_S \varphi\, dS, \qquad (24.15)$$

in which the density φ is obtained by solving the integral equation (7.16), for which the integration is extended over the entire surface of the Earth.

The mass of the ellipsoid a_0, f_0 is chosen as equal to the mass of the Earth, as has been previously shown. In this case, as the difference $M - M'$ vanishes, the zero-order harmonic in the series expansion of the perturbing potential will be, according to (5.7):

$$T_0 = \frac{G\Delta M}{R} = 0. \tag{24.16}$$

If one calculates the value ζ of the height anomaly, with respect to the initial ellipsoid a_0, f_0, one gets, according to *Bruns'* formula (4.18):

$$\zeta = \frac{T + \Delta W}{\gamma}$$

or, on the basis of (4.30):

$$\zeta = \frac{T_0}{\gamma} + \frac{T_1}{\gamma} + \frac{1}{\gamma}\sum_{n=2}^{\infty} T_n + \frac{\Delta W}{\gamma}. \tag{24.17}$$

The 1st-order harmonic of the potential's series expansion is, according to (5.9):

$$T_1 = \frac{GM}{R^2}[x_0 \cos L \cos B + y_0 \sin L \cos B + z_0 \sin B],$$

in which x_0, y_0, z_0 denote the coordinates of the centre of the ellipsoid a_0, f_0, referred to the Earth's mass centre. Considering:

$$\frac{GM}{R^2} \approx \gamma,$$

the previous relation becomes:

$$\frac{T_1}{\gamma} = x_0 \cos L \cos B + y_0 \sin L \cos B + z_0 \sin B. \tag{24.18}$$

Regarding the sum:

$$T' = \sum_{n=2}^{\infty} T_n,$$

as a first approximation of the expansion in series of the perturbing potential, the term T'/γ represents, according to *Bruns'* equation (4.5), a first approximation of the value of the height anomaly:

$$\frac{T'}{\gamma} = \frac{1}{\gamma}\sum_{n=2}^{\infty} T_n = \zeta_0. \tag{24.19}$$

Taking into consideration the relations (24.16), (24.18) and (24.19), the relation (24.17) becomes:

$$\zeta - \zeta_0 = x_0 \cos L \cos B + y_0 \sin L \cos B + z_0 \sin B + \frac{\Delta W}{\gamma}. \tag{24.20}$$

If at several astronomical points one determines the height anomaly both by astro-geodetic levelling and gravimetrically, the difference between the values obtained is due to the difference between the centre of the ellipsoid a_0, f_0 and the Earth's mass centre and also to the difference between the potential U'_0 of the ellipsoid a_0, f_0 and the potential W_0 of the geoid. The right-hand side of (24.20) itself expresses the sum of these differences, and one may consequently consider that the left-hand side of this relation represents the difference between the height anomalies determined astro-geodetically and those determined gravimetrically.

Differentiating (24.20) with respect to the geodetic coordinates B and L yields:

$$
\begin{aligned}
&- x_0 \cos L \sin B - y_0 \sin L \sin B + z_0 \cos B - \frac{\Delta W}{\gamma^2}\frac{\partial \gamma}{\partial B} = \frac{\partial \zeta}{\partial B} - \frac{\partial \zeta_0}{\partial B};\\
&- x_0 \sin L \cos B + y_0 \cos L \cos B = \frac{\partial \zeta}{\partial L} - \frac{\partial \zeta_0}{\partial L}.
\end{aligned}
\tag{24.21}
$$

But, according to (5.15), the left-hand side of (24.21) may be presented in the form:

$$
\begin{aligned}
&\frac{\partial \zeta}{\partial B} = - M\xi; \qquad &&\frac{\partial \zeta_0}{\partial B} = - M\xi_0;\\
&\frac{\partial \zeta}{\partial L} = - \eta N \cos B; \qquad &&\frac{\partial \zeta_0}{\partial L} = - \eta_0 N \cos B,
\end{aligned}
\tag{24.22}
$$

in which ξ and η represent the components of the astro-geodetic deflections of the vertical, and ξ_0 and η_0 the components of the gravimetric deflection of the vertical.

On the basis of (24.22), the relations (24.21) can be written in the form:

$$
\begin{aligned}
&x_0 \cos L \sin B + y_0 \sin L \sin B - z_0 \cos B + \frac{\Delta W}{\gamma^2}\frac{\partial \gamma}{\partial B} = M(\xi - \xi_0);\\
&x_0 \sin L - y_0 \cos L = N(\eta - \eta_0).
\end{aligned}
\tag{24.23}
$$

Equations of the type (24.20) and (24.23) may be written for a large number of astronomical points. The only unknowns appearing in these equations, x_0, y_0, z_0 and ΔW can be determined utilizing the least squares procedure.

On the basis of the quantity ΔW determined in this way, one can deduce the geoid's potential W_0 using (24.14). Similarly, knowing the coordinates x_0, y_0, z_0, one may determine the principal moments of inertia A, B and C, making use of the relations derived in section 2.3.5. On the basis of these values, one determines from (3.14) the factor J_2 of the dynamic form of the Earth.

Once the constants W_0, GM, J_2 and ω have been determined, one can determine the differences between the semi-major axis a_0 and the flattening f_0 of the initial ellipsoid and the semi-major axis a and the flattening f corresponding to the general terrestrial ellipsoid.

To this end, considering $b = a(1-f)$ and $e'^2 = 2f + 3f^2 + \dots$, the relations (3.41) and (3.42) can be presented in the form (*Heiskanen* and *Moritz* 1967):

$$GM = a_0^2\gamma_{0E}\left(1 - f_0 + \frac{3}{2}m\right);\ W_0 = a_0\,\gamma_{0E}\left(1 - \frac{2}{3}f_0 + \frac{11}{6}m\right).$$

Solving these equations for a and γ_{0E} yields:

$$a_0 = \frac{GM}{W_0}\left(1 + \frac{1}{3}f_0 + \frac{1}{3}m\right);\ \gamma_{0E} = \frac{W_0^2}{GM}\left(1 + \frac{1}{3}f_0 - \frac{13}{6}m\right). \tag{24.24}$$

In the spherical approximation these relations may be presented in the form:

$$a_0 = \frac{GM}{W_0};\ \gamma_{0E} = \frac{W_0^2}{GM}. \tag{24.25}$$

Differentiating the relations (24.24) with respect to M, W_0 and f one gets:

$$\mathrm{d}a = \frac{G\mathrm{d}M}{W_0}\left(1 + \frac{1}{3}f_0 + \frac{1}{3}m\right) - \frac{GM}{W_0^2}\mathrm{d}W_0\left(1 + \frac{1}{3}f_0 + \frac{1}{3}m\right) + $$
$$+ \frac{GM}{W_0}\frac{1}{3}\mathrm{d}f;$$

$$d\gamma_{E0} = -\frac{W_0^2}{G^2M^2}\mathrm{d}M\left(1 + \frac{1}{3}f_0 - \frac{13}{6}m\right)\frac{2W_0\,\mathrm{d}W}{GM}\left(1 + \frac{1}{3}f_0 - \frac{13}{6}m\right) +$$
$$+ \frac{W_0^2}{GM}\frac{1}{3}\mathrm{d}f.$$

Disregarding the terms containing f and m leads to:

$$\begin{aligned} \mathrm{d}a &= \frac{G\,\mathrm{d}M}{W_0} - \frac{GM}{W_0^2}\mathrm{d}W_0 + \frac{GM}{W_0}\frac{1}{3}\mathrm{d}f; \\ \mathrm{d}\gamma_{0E} &= -\frac{W_0^2}{GM^2}\mathrm{d}M + \frac{2W_0}{GM}\mathrm{d}W + \frac{W_0^2}{GM}\frac{1}{3}\mathrm{d}f. \end{aligned} \tag{24.26}$$

Taking (24.25) into consideration, the previous relations become:

$$\begin{aligned} \mathrm{d}a &= \frac{1}{a_0\,\gamma_{0E}}G\,\mathrm{d}M - \frac{\mathrm{d}W_0}{\gamma_{0E}} + \frac{1}{3}a_0\,\mathrm{d}f; \\ \mathrm{d}\gamma_{0E} &= -\frac{1}{a^2}G\,\mathrm{d}M + \frac{2}{a}\mathrm{d}W + \frac{1}{3}\gamma_{0E}\,\mathrm{d}f. \end{aligned} \tag{24.27}$$

On the other hand differentiating (3.19) yields:

$$\mathrm{d}f = \frac{3}{2}\,\mathrm{d}J_2, \tag{24.28}$$

in which $\mathrm{d}J_2$ represents the difference between the moments of inertia of the ellipsoid a_0, f_0 and those calculated for the ellipsoid a, f by means of the coordinates x_0, y_0, z_0 (or obtained with the aid of the Earth's artificial satellites). With the help of the quantity $\mathrm{d}f$ thus calculated one determines by means of (24.27) the increases $\mathrm{d}a$ and $\mathrm{d}\gamma_E$.

One can now determine the quantities:

$$\begin{gathered} a = a_0 + \mathrm{d}a:\ f = f_0 + \mathrm{d}f;\ \gamma_E = \gamma_{0E} + \mathrm{d}\gamma_E; \\ U_0 = U_0' + \Delta W;\ M = M' + \int_S \varphi\,\mathrm{d}S, \end{gathered} \tag{24.29}$$

which represent the parameters of the general terrestrial ellipsoid and of its gravitational field.

25
Dynamic Geodesy

The principle of Dynamic Geodesy may be summarized as follows: one observes the geodetic artificial satellites with sufficient accuracy from stations of coordinates and at time moments t precisely known. One also assumes known a few physical parameters of the Earth such as: the gravitational field, the atmosphere's density at the satellite's level, the luni-solar attraction etc. On the basis of the theory of the artificial satellites' movement, the observational data and the above-mentioned parameters being available as primary information, one interpolates the future positions of the satellite at other moments t. At the t moments one also makes new observations of the satellites. The positions interpolated at the moment t and those observed at the same moment will differ especially due to the fact that the physical parameters are not well known. From comparing the observed elements with those interpolated one may calculate corrections of the physical parameters, by means of which one can afterwards improve the orbit calculations.

Inasmuch as the artificial satellites have trajectories relatively close to the Earth's surface (compared with the other celestial bodies), these will be especially influenced by the asymmetry of the gravitational field and by the resistance of the atmosphere; therefore, the most important data obtained in Dynamic Geodesy refer to these two parameters. There are also other phenomena influencing the satellites' movement and on which Dynamic Geodesy provides important data. These are: the radiation pressure, the attraction of other celestial bodies, the magnetic effect etc. The physical parameters which modify the trajectory of a satellite are termed perturbations. For a better understanding of the problem, in what follows, we will analyse both the satellites' non-perturbed orbits and the perturbing functions and the movement of the satellites under the influence of these functions.

25.1 Non-Perturbed Orbits

One considers the satellite S (Fig. 25.1) of negligible mass, rotating around the body of mass M. The satellite's position, assumed to move in vacuum, in a completely symmetrical gravitational field, may be given either by the

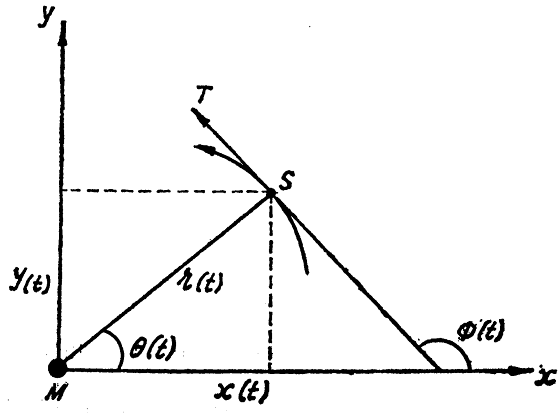


Fig. 25.1. Satellite's Plane Coordinates

rectangular coordinates x, y, or the polar ones r, θ. They are connected by the well-known formulae:

$$x = r \cos \theta ;$$
$$y = r \sin \theta ;$$
$$r = \sqrt{x^2 + y^2} ; \qquad (25.1)$$
$$\theta = \tan^{-1} \frac{y}{x} .$$

The satellite's coordinates are given for the moment t. The variation of these coordinates with respect to the time is obtained using:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta ;$$
$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta ; \qquad (25.2)$$
$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{x\dot{x} + y\dot{y}}{r} ;$$
$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{x\dot{y} - y\dot{x}}{r^2} .$$

The tangential velocity in the direction of the satellite's movement (T) is given by the derivative $\mathrm{d}T/\mathrm{d}t$ and the relationship between the tangential velocity and the variation of the rectangular coordinates is *(Mueller 1964)*:

$$\dot{x} = \dot{T} \cos \Phi ;$$
$$\dot{y} = \dot{T} \sin \Phi ;$$
$$\dot{T} = \sqrt{\dot{x}^2 + \dot{y}^2} ; \qquad (25.3)$$
$$\Phi = \tan^{-1} \frac{\dot{y}}{\dot{x}} .$$

The areal velocity $\dot{A}$ is defined by the well-known relation:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = x\dot{y} - y\dot{x} = h,$$

and the tangential and areal accelerations are given by:

$$\ddot{T} = \frac{\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y}}{\dot{T}}; \tag{25.4}$$

$$\dot{h} = x\,\ddot{y} - y\,\ddot{x} = 2\,r\,\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}.$$

The attraction force between the masses M and m (m being the unit mass) is:

$$F = -G\frac{M}{r^2}$$

which, resolving, may be expressed by its components as:

$$F_x = F\cos\theta = F\frac{x}{r} = \ddot{x}; \tag{25.5}$$

$$F_y = F\sin\theta = F\frac{y}{r} = \ddot{y}.$$

From these two equations one can derive *Kepler*'s second law:

$$x\,\ddot{y} - y\,\ddot{x} = \dot{h} = 2\ddot{A} = 0$$

or $\dot{A}$ = const.

Taking (25.2) and (25.4) into consideration, one can deduce from the equations (25.5):

$$F\dot{r} = \dot{T}\,\ddot{T},$$

and by differentiating the relations (25.2) with respect to time and taking account of (25.4), one gets:

$$\dot{T}\,\ddot{T} = \dot{r}\,(\ddot{r} - r\,\dot{\theta}^2) = F\,\dot{r}.$$

From this relation one can deduce the differential equation of the motion in the form:

$$F = \ddot{r} - r\,\dot{\theta}^2$$

The solution of this equation is:

$$r = \frac{h^2}{G^2\,[1 + e\cos(\theta - \theta_0)]}, \tag{25.6}$$

where G is the universal constant of gravitation, and e and θ_0 are integration constants. Depending on the eccentricity e, the orbit can be a circle, an ellipse, a parabola or a hyperbola. The geodetic satellites generally have elliptical orbits with one of the foci at the Earth's centre. The elliptical orbits are particularly important for Dynamic Geodesy, for which reason the geometry of the elliptical orbits will be analysed in what follows.

The elements of an artificial satellite's orbit have been defined in Subchapt. 19.2 as being: semi-major axis a, eccentricity e, latitude argument of the perigee ω, moment of passage at the perigee τ, right ascension of the ascending node Ω and inclination i. The first three elements a, e, ω define the form and the position of the orbit in its own plane, the fourth one defines the satellite's position on the orbit and the last two, Ω and i, define the orbit in space.

In order to define the eccentric anomaly E, one introduces the coordinate system x, y (Fig. 19.2), using which the coordinates of the satellite S may be defined:

$$x = r \cos v;$$

$$y = r \sin v.$$

From the figure also follows:

$$\begin{aligned} x &= a \cos E - ae = r \cos v; \\ y &= b \sin E = a(1 - e^2)^{1/2} \sin E = r \sin v. \end{aligned} \tag{25.7}$$

From (25.7) one obtains:

$$r = a(1 - e \cos E), \tag{25.8}$$

and further from (25.7) and (25.8):

$$\begin{aligned} \sin v &= (1 - e^2)^{1/2} \frac{\sin E}{1 - e \cos E}; \\ \cos v &= \frac{\cos E - e}{1 - e \cos E}; \\ \tan v/2 &= \left(\frac{1 + e}{1 - e}\right)^{1/2} \tan \frac{E}{2}. \end{aligned} \tag{25.9}$$

According to *Kepler*'s third law, the average movement $\bar{n}$ is given by the relation:

$$\bar{n} = \sqrt{GM}\, a^{-3/2}. \tag{25.10}$$

If at the moment t the satellite finds itself at S, then the mean anomaly $\bar{M}$ may be defined by the relation:

$$\bar{M} = \bar{n}(r - \tau) = a^{-3/2} (GM)^{1/2} (t - \tau), \tag{25.11}$$

being the true anomaly corresponding to the movement of a fictitious satellite having a constant velocity. Consequently, the mean anomaly was denoted by $\bar{M}$ so as not to be confused with the Earth's mass M.

The connexion relation between the eccentric anomaly and the mean one is obtained making use of *Kepler*'s equation (*Caputo* 1967), in the form:

$$E - e \sin E = \bar{n}(t - \tau) = \bar{M}. \tag{25.12}$$

Expanding (25.12) in Bessel functions up to the 7th power of e yields *(Mueller* 1964*)*:

$$E = \bar{M} + \left(e - \frac{e^3}{8} + \frac{e^5}{192} - \frac{e^7}{9216}\right) \sin \bar{M} + \tag{25.13}$$
$$+ \left(\frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48}\right) \sin 2\, \bar{M} + \left(\frac{3e^3}{8} - \frac{27\, e^5}{128} + \frac{243\, e^7}{5120}\right) \sin 3\, \bar{M} + \ldots$$

A first approximation of the equation (25.13) is:

$$E_1 = \bar{M} + e \sin \bar{M} + \frac{1}{2} e^2 \sin 2\, \bar{M}. \tag{25.14}$$

In order to solve *Kelper's* equation one may use an iterative process. Expanding the relation (25.12) in a *Taylor* series and disregarding the 2nd-order terms, one gets:

$$\bar{M} = (1 - e \cos E)\, \Delta E$$

whence:

$$\Delta E = \frac{\Delta \bar{M}}{1 - e \cos E}\,. \tag{25.15}$$

As a first approximation to E, one makes use of the value E_1 given by (25.14), whose introduction into (25.12) yields a first approximation $\bar{M}_1$ of $\bar{M}$. One calculates the difference $\Delta \bar{M} = \bar{M}_1 - \bar{M}$ and from (25.15) a first correction ΔE_1. The second approximation to E is obtained as $E_2 = E_1 + \Delta E_1$. The process continues until the difference between the actual value $\bar{M}$ and the nth-order approximation of this value doesn't exceed an imposed tolerance.

One can establish a few relations between the elements of the elliptical orbit *(Caputo* 1967*)*, frequently utilized in Dynamic Geodesy. To this end, one introduces the notations:

$$p = \exp jv;\ s = \exp jE;\ t = \exp j\bar{M};\ j = \sqrt{-1}. \tag{25.16}$$

Between the parameters s and t there exists, on the basis of *Kepler*'s theorem (25.12), the following relation:

$$t = s \exp \frac{e}{2}\left(\frac{1}{s} - s\right).$$

From (25.9) and (25.16) one derives:

$$\tan \frac{E}{2} = -j \frac{s - 1}{s + 1};$$
$$\tan v = -j \frac{p - 1}{p + 1}. \tag{25.17}$$

Introducing the notation:

$$\frac{(1-e)^{1/2}}{(1+e)^{1/2}}=\frac{1-\beta}{1+\beta}\,; \quad \beta=\frac{1-(1-e^2)^{1/2}}{e},$$

and also taking account of (25.17), from (25.9) one gets:

$$s=\frac{p+\beta}{1+\beta p}\,; \quad p=\frac{s-\beta}{1-\beta s}\,.$$

The relation between the parameters p or s and the radius vector may also be obtained *(Caputo 1967):*

$$\frac{r}{a}=1-\frac{e}{2}\left(s+\frac{1}{s}\right)=\frac{1}{1+\beta^2}(1-\beta s)\left(1-\frac{\beta}{s}\right); \tag{25.18}$$

$$\frac{r}{a}=\frac{(1-\beta^2)^2}{1+\beta^2}\,\frac{1}{(1+\beta p)(1+\beta p^{-1})}\,.$$

25.2 Relationships between the Elements of the Satellite Orbit and the Geocentric, Topocentric and Rectangular Coordinate Systems

Inasmuch as, in the complex process of observing the artificial satellites and of processing the resulting data, one utilizes the geocentric, topocentric and rectangular coordinate systems, it is useful to establish relationships between these systems and the orbital elements of the satellites.

To this end, one takes into consideration the Cartesian coordinate systems X, Y, Z (Fig. 25.2; the Z axis directed towards the North Pole, the X axis towards the first point of Aries γ and the Y axis perpendicular to the XZ plane) and x, y defined as in Fig. 19.1.

From Fig. 25.2 one can immediately deduce the relationship between the rectangular coordinates X, Y, Z and the coordinates α and δ:

$$\begin{aligned} X &= r\cos\delta\cos\alpha \\ Y &= r\cos\delta\sin\alpha \\ Z &= r\sin\delta \end{aligned} \tag{25.19}$$

In order to pass from the x, y system (Fig. 19.1) to the X, Y, Z system one must perform three successive rotations: a rotation by ω round the OZ axis (perpendicularly to the orbit's plane), a second one by i round the axis passing through the ascending node (Fig. 19.1) and the third rotation by Ω round the OZ axis. The relationship between the two systems is achieved by

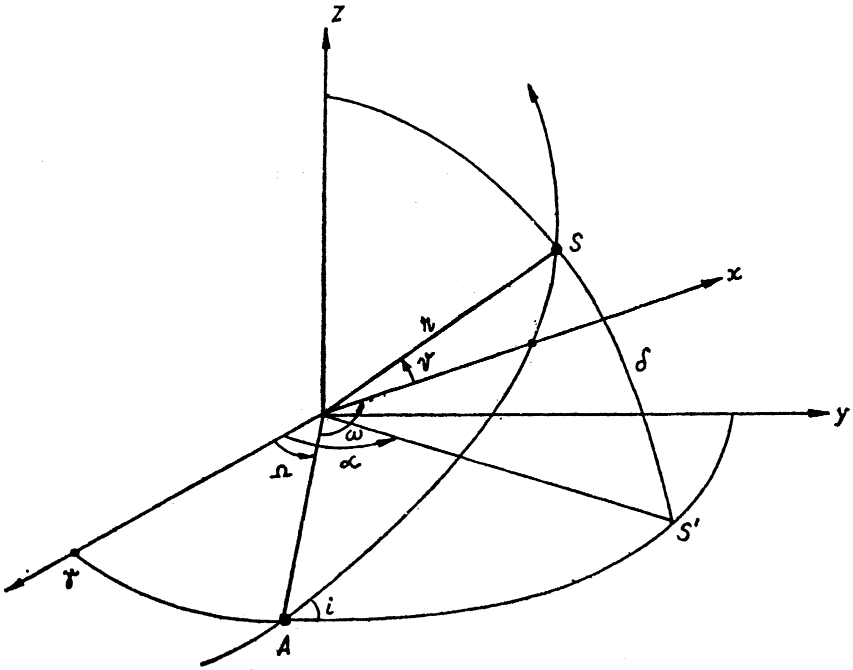


Fig. 25.2. The Connexion between the Orbit's Elements and the Rectangular Coordinate System

means of the matricial equation presented in subchapter 18.2 and reintroduced here with the notations currently used in Dynamic Geodesy:

$$\begin{Vmatrix} X \\ Y \\ Z \end{Vmatrix} = \mathbf{R} \begin{Vmatrix} x \\ y \\ z \end{Vmatrix}$$

R representing the rotation matrix which, according to what has been shown above, has the form:

$$\mathbf{R} = \begin{Vmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & -\cos i \end{Vmatrix} \begin{Vmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{Vmatrix} =$$

$$= \begin{Vmatrix} \cos\Omega\cos\omega - \cos i\sin\Omega\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos i\cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\omega\sin\Omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{Vmatrix} =$$

$$= \begin{Vmatrix} R(xX) & R(yX) & R(zX) \\ R(xY) & R(yY) & R(zY) \\ R(xZ) & R(yZ) & R(zZ) \end{Vmatrix} \tag{25.20}$$

In (25.20) the quantities $R(x\,X)$, $R(y\,Y)$,... represent the rotation cosines *(Mueller 1964)*. Since the Z component is zero, the relations for passing from the $x, y, z = 0$ in X, Y, Z systems take the form:

$$\begin{aligned} X &= xR(xX) + YR(y\,X); \\ Y &= xR(xY) + YR(yY); \\ Z &= xR(xZ) + YR(yZ). \end{aligned} \tag{25.21}$$

25.3 Perturbed Orbits

In Section 25.1, there were analysed the equations of the motion of an artificial satellite for the ideal case of a spherical Earth situated at one of the ellipse's foci, with a uniform density, without any hypothesis referring to other forces which would influence the satellite's movement. Actually, however, the satellite's movement is continuously perturbed by a series of perturbing factors which may be classified into: *gravitational* and *non-gravitational* ones. The first group includes firstly the effect of the sphericity of the terrestrial gravitational field, producing the main perturbation in an artificial satellite's movement. This category also includes the effect of the luni-solar attraction and the relativity effect. The latter have, however, a smaller influence on the artificial satellite's movement. To the group of non-gravitational perturbations belong: the atmosphere's resistance, the electromagnetic effect and the radiation pressure.

One can make another classification of the perturbations, according to the manner in which the orbital elements vary with time, viz.: *secular perturbations* (in which the variations of the orbital elements are considered as linear in time), *long-period perturbations* and *short-period ones*.

Due to the perturbing factors, the satellite's orbit is continuously changing, which constitutes a serious handicap for practical work. This is why one introduces the notion of *osculating movement*, which assumes that the satellite is moving along an orbit whose elements are continuously varying. If at the moment t_0 all the perturbing forces disappear, the satellite will move along a non-perturbed orbit, tangent to the actual orbit at the moment t_0 but having as initial elements for the position and velocity vectors those of the actual orbit at the moment t_0. This ellipse is called the *osculating ellipse*. The perturbations of the satellite's orbit can be obtained by integrating with respect to time the variation of the orbit's elements with respect to the initial Keplerian orbit.

25.3.1 Equations of Variation of the Orbit Elements in Terms of Perturbing Force

An artificial satellite's motion is influenced by a series of factors which, as was shown above, may be of a gravitational or non-gravitational nature. If all of these factors combine into a single perturbing force R, then one can determine the variations of the satellite's orbital elements, due to this perturbing function, by means of the well-known equations of *Lagrange:*

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{\bar{n}a}\frac{\partial R}{\partial \bar{M}};\\
\frac{de}{dt} &= \frac{1-e^2}{\bar{n}a^2e}\frac{\partial R}{\partial \bar{M}} - \frac{\sqrt{1-e^2}}{\bar{n}a^2e}\frac{\partial R}{\partial \omega};\\
\frac{di}{dt} &= \frac{1}{\bar{n}a^2\sqrt{1-e^2}\sin i}\left(\frac{\partial R}{\partial \omega}\cos i - \frac{\partial R}{\partial \Omega}\right);\\
\frac{d\Omega}{dt} &= \frac{1}{\bar{n}a^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i};\\
\frac{d\omega}{dt} &= \frac{-\cos i}{\bar{n}a^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{\bar{n}a^2e}\frac{\partial R}{\partial e};\\
\frac{dT}{dt} &= \frac{1-e^2}{\bar{n}^2a^2e}\frac{\partial R}{\partial e} + \frac{2}{\bar{n}^2a}\frac{\partial R}{\partial a}.
\end{aligned}
\tag{25.22}
$$

Instead of the last relation in (25.22) one can utilize the relation:

$$
\frac{d\bar{M}}{dt} = \bar{n} - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{\bar{n}a^2e}\frac{\partial R}{\partial e}.
$$

The variations of the orbit's elements may also be expressed as functions of the rectangular components S, T and W of the perturbing force, by means of *Gauss*' equations:

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2a^2}{b}\sqrt{\frac{a}{GM}}\left(eS\sin v + \frac{p}{r}T\right);\\
\frac{de}{dt} &= \frac{b}{a}\sqrt{\frac{a}{GM}}\left[S\sin v + \left(\frac{r+p}{p}\cos v + \frac{er}{p}\right)T\right];\\
\frac{di}{dt} &= \frac{r}{b}\sqrt{\frac{a}{GM}}W\cos(\omega + v);
\end{aligned}
\tag{25.23}
$$

$$\frac{d\Omega}{dt}=\frac{r}{b}\sqrt{\frac{a}{GM}}\,W\frac{\sin(\omega+v)}{\sin i}; \tag{25.23}$$

$$\frac{d\omega}{dt}=\frac{b}{a}\sqrt{\frac{a}{GM}}\left[-\left[\frac{1}{e}S\cos v+\frac{r+p}{ep}T\sin v-\frac{r}{p}W\sin(\omega+v)\cot i\right]\right.$$

in which one has used the notations:

$$p=\frac{b^2}{a}=a(1-e^2); \tag{25.24}$$

$$r=\frac{p}{1+e\cos v},$$

and S, T, W denoted the coordinate system oriented as follows: the S axis along the radius vector, W perpendicular to the orbit's plane and T perpendicular to S and W.

The components S, T and W can be expressed in terms of the perturbing force in the form *(Heiskanen and Moritz 1967):*

$$S=\frac{\partial R}{\partial r};$$

$$T=-\frac{\cos(\omega+v)\sin i}{r\sin\theta}\frac{\partial R}{\partial\theta}+\frac{\cos i}{r\sin^2\theta}\frac{\partial R}{\partial\lambda}; \tag{25.25}$$

$$W=-\frac{\cos i}{r\sin\theta}\frac{\partial R}{\partial\theta}-\frac{\cos(\omega+v)\sin i}{r\sin^2\theta}\frac{\partial R}{\partial\lambda}.$$

The elements used in (25.24) are defined in Fig. 25.3.

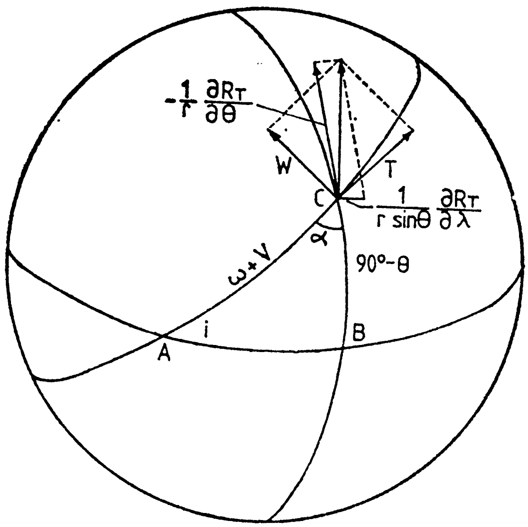


Fig. 25.3. Components of the Perturbing Force

If the perturbing function and the constant GM are well known, then the elements of the satellite's orbit may be calculated for a given epoch by adding to the elements corresponding to an initial epoch the variation of these elements obtained by integrating the equations (25.22).

In order to obtain the variations of the orbital elements for a certain time period, other motion equations are also utilized, in which instead of the Keplerian elements other types of parameters are used. An example of such parameters is given by the *Delaunay* variables, replacing the elements $\bar{M}$, W and Ω:

$$L = a\sqrt{GM};\ l = \bar{M};$$

$$G = L\sqrt{1 - e^2};\ g = \omega;$$

$$H = G\cos i;\ h = \Omega.$$

25.3.2 Perturbation of the Satellite Orbit Due to the Asphericity of the Earth's Gravitational Field

In deriving the motion equations of an artificial satellite along a Keplerian orbit, it was assumed that it is moving in a spherical gravitational field of the form:

$$V_S = GM/R. \tag{25.26}$$

In fact, the satellite's motion takes place in the actual gravitational field, which — according to what has been shown in Section 2.3.5 — has the form:

$$V = \frac{GM}{R}\left[1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{R}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\, P_{nm}(u)\right]\cdot \tag{25.27}$$

One can, consequently, state that the Keplerian orbit is affected by a perturbing function arising from the difference between the actual gravitational field defined by (25.27) and a spherical field given by (25.26):

$$R^t = \frac{GM}{R}\left\{\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{R}\right)^n P_{nm}(u)[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right\}\cdot \tag{25.28}$$

If the perturbing function R^t is expressed as a function of the elements of the Keplerian orbit, one can obtain the derivatives $\partial R^t/\partial S_i$ (in which S_i are the orbital elements $S_1 = a$, $S_2 = e, \ldots$) and one then derives from *Lagrange*'s equations (25.22) the variations $\mathrm{d}a/\mathrm{d}t$, $\mathrm{d}e/\mathrm{d}t$ etc. By integrating these variations, one can eventually find the perturbations of the motion for the integration interval.

Expressing the Perturbing Force in terms of the Orbit Elements. The main perturbation of a gravitational nature is that arising from the lack of symmetry of the actual potential of the Earth. This function is expressed by the relation (25.28), which, however, cannot be used in this form in Dynamic Geodesy. In what follows this relation will be expressed as a function of the elements of the Keplerian orbit *(Kaula 1966)*.

The Legendre polynomials entering into the terms of the harmonic series of the potential expansion may be written in the form:

$$P_{lm}(\sin\varphi) = \cos^m \varphi \sum_{t=0}^{K} \frac{(-1)^t [2(l-t)]!\,\sin^{l-m-2t}\varphi}{2^l - t!(l-t)!\,(l-m-2t)!} =$$

$$= \cos^m \varphi \sum_{t=0}^{K} T_{lmt} \sin^{l-m-2t}\varphi\,; \qquad (25.29)$$

$$K = \frac{l-m}{2},\ \text{if } l-m \text{ even;}$$

$$K = \frac{l-m-1}{2} \text{ if } l-m \text{ odd.}$$

In the relations (25.29) the unknown angle φ will subsequently be expressed as a function of the elements of the Keplerian orbit.

In order to be able to utilize (25.28), one will also have to express $\cos m\lambda$ and $\sin m\lambda$ as functions of the same Keplerian elements.

For this, we recall that:

$$\cos m\lambda = R_e(\cos\lambda + j\sin\lambda)^m = R_e \sum_{s=0}^{m} \binom{m}{s} j^s \cos^{m-s}\lambda\,\sin^s\lambda, \qquad (25.30)$$

in which R_e represents the real part of the series expansion, $j = \sqrt{-1}$ and:

$$\binom{m}{s} = \frac{m!}{s!\,(m-s)!}.$$

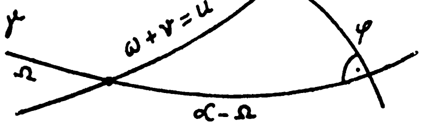
Analogously one gets:

$$\sin m\lambda = R_e \sum_{s=0}^{m} \binom{m}{s} j^{s-1} \cos^{m-s}\lambda\,\sin^s\lambda. \qquad (25.31)$$

Inasmuch as the angle λ will have also to be expressed as a function of the orbital elements, one resorts to the spherical triangle in Fig. 25.4, from which it follows that:

$$\lambda = (\alpha - \Omega) + (\Omega - \theta)$$

where θ is the Greenwich sidereal time.



25.4. The Connexion between the Satellite Coordinates and the Elements of the Keplerian Orbit

Replacing λ by the values obtained above, one deduces:

$$\cos m\lambda = \cos[m(\alpha - \Omega) + m(\Omega - \theta)] = \cos m(\alpha - \Omega) \cos m(\Omega - \theta) - \\ - \sin m(\alpha - \Omega) \sin m(\Omega - \theta). \quad (25.32)$$

From Fig. 25.3 it also follows that:

$$\cos(\omega + v) = \cos(\alpha - \Omega) \cos \varphi + \sin(\alpha - \Omega) \sin \varphi \cos \pi/2 = \\ = \cos(\alpha - \Omega) \cos \varphi; \quad (25.33)$$

$$\cos \varphi = \cos(\omega + v)\cos(\alpha - \Omega) + \sin(\omega + v) \sin(\alpha - \Omega) \cos i.$$

Introducing the value of $\cos \varphi$ obtained from the first relation into the second relation (25.33) yields:

$$\sin(\alpha - \Omega) = \frac{\sin(\omega + v) \cos i}{\cos \varphi}. \quad (25.34)$$

From (25.32) and (25.33) one gets:

$$\sin \varphi = \sin(\omega + v)\sin i. \quad (25.35)$$

If in (25.32) the trigonometrical functions $\sin m(\alpha - \Omega)$ and $\cos m(\alpha - \Omega)$ are expressed by series expansions of the form (25.30) and (25.31) and afterwards the trigonometrical functions of the angle $(\alpha - \Omega)$ are substituted according to (25.33) and (25.34), one gets:

$$\cos m\lambda = R_e \sum_{s=0}^{m} \binom{m}{s} j^s \frac{\cos^{m-s}(\omega + v) \sin^s(\omega + v) \cos^s i}{\cos^m \varphi} [\cos m(\Omega - \\ - \theta) + j \sin m(\Omega - \theta)]; \quad (25.36)$$

$$\sin m\lambda = R_e \sum_{s=0}^{m} \binom{m}{s} j^s \frac{\cos^{m-s}(m + v) \sin^2(\omega + v) \cos^2 i}{\cos^m \varphi},$$

which represents the expression of the trigonometrical functions $\cos m\lambda$ and $\sin m\lambda$ in terms of the elements of the Keplerian orbit. Into the relations (25.36) there also enters as an unknown element the angle φ.

In order to eliminate this unknown as well, one considers any term of the expansion of the perturbing function:

$$R_{lm}^{t} = GM \frac{a_e^l}{R^{l+1}} P_{lm}(\sin \varphi)(C_{lm} \cos m\lambda + S_{lm} \sin m\lambda), \quad (25.37)$$

in which, in view of (25.35), the Legendre polynomials are:

$$P_{lm}(\sin \varphi) = \cos^m \varphi \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t} i \sin^{l-m-2t}(\omega + v). \quad (25.38)$$

Now substituting (25.36) (in which $\cos^m \varphi$ is taken outside the sum) and (25.38) into (25.37), $\cos^m \varphi$ disappears and the expression obtained has the form:

$$R_{lm}^t = GM \frac{a_e^l}{R^{l+1}} \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t} i \, R_e [(C_{lm} - jS_{lm}) \cos m(\Omega - \theta) +$$

$$+ (S_{lm} + jC_{lm}) \sin m(\Omega - \theta)] \times \tag{25.39}$$

$$\times \sum_{s=0}^{m} \binom{m}{s} j^s \sin^{l-m-2t+s} (\omega + v) \cos^{m-s} (\omega + v) \cos^s i.$$

For the further transformation of (25.39), one recalls the trigonometric relation:

$$\sin^{l-m-2t+s} (\omega + v) \cos^{m-s} (\omega + v) =$$

$$= \frac{(-j)^{l-m-2t+s}}{2^{l-2t}} \sum_{c=0}^{l-m-2t+s} \sum_{d=0}^{m-s} \binom{l-m-2t+s}{c} \binom{m-s}{d} \times$$

$$\times (-1)^c [\cos (l - 2t - 2c - 2d) (\omega + v) + j \sin (l - 2t - 2c - 2d)(\omega + v)].$$

Introducing this expression into (25.39), after some transformations, one gets:

$$R_{lm}^t = G \frac{a_e^l}{R^{l+1}} \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t} i \, (-1)^{k+t} \sum_{s=0}^{m} \binom{m}{s} \frac{\cos^s i}{2^{l-2t}} \times$$

$$\times \sum_{c=0}^{l-m-2t+s} \sum_{d=0}^{m-s} \binom{l-m-2t+s}{c} \binom{m-s}{d} (-1)^c \times \tag{25.40}$$

$$\times \left\{ \begin{bmatrix} C_{lm} \\ S_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos [(l - 2t - 2c - 2d)(\omega + v) + m(\Omega - \theta)] + \right.$$

$$\left. + \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin [(l - 2t - 2c - 2d)(\omega + v) + m(\Omega - \theta)] \right\}.$$

The relation (25.40) may still be formulated by introducing the quantity $p = t - c - d$ and replacing the quantity d by $d = p - c$. In this manner one removes an index from the product. The expression in R_{lm}^t will have the form:

$$R_{lm}^t = GM \frac{a_e^l}{R^{l+1}} \sum_{p=0}^{l} F_{lmp} (i) \left\{ \begin{bmatrix} C_{lm} \\ -S_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos[(l - 2p)(\omega + v) + \right.$$

$$\left. + m(\Omega - \theta)] + \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin [(l - 2p)(\omega + v) + m(\Omega - \theta)] \right\}, \tag{25.41}$$

in which:

$$F_{lmp}(i) = \Sigma_t \frac{(2l-2t)!}{t!(l-t)!\,2^{2l-2t}} \sin^{l-m-2t} i \times$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \frac{\cos^s i \; \Sigma_c \binom{l-m-2t+s}{c}}{(l-m-2t)!} \binom{m-s}{p-t-c} (-1)^{c-k}, \tag{25.42}$$

where k is the integral part of $(1-m)/2$, t varies from 0 to p or k (depending on which is the smaller), and the summation is carried out over the values of c for which the binomial coefficients are different from zero.

In (25.41) there is still one single transformation remaining to be applied, viz. the values R and v must be replaced as functions of a, $\overline{M}$ and e. To this end one utilizes the relations *(Caputo 1967)*:

$$\frac{1}{R^{l+1}} \begin{bmatrix} \cos \\ \sin \end{bmatrix} [(l-2p)(w+v) + m(\Omega - \theta)] =$$

$$= \frac{1}{a^{l+1}} \sum_{-\infty}^{+\infty} G_{lpq} \begin{pmatrix} \cos \\ \sin \end{pmatrix} [(l-2p)\omega + (l-2p+q)\overline{M} + m(\Omega - \theta)],$$

where G_{lpq} is the eccentricity function which will be discussed later on.

The final result of expressing the perturbing function R^t_{lm} in terms of the elements of the Keplerian orbit is:

$$R^t_{lm} = GM \frac{a_e^l}{a^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{+\infty} G_{lpq}(e)\, S_{lmpq}(\omega, \overline{M}, \Omega, \theta), \tag{25.43}$$

in which $F_{lmp}(i)$ is given by (25.42) and S_{lmpq} is:

$$S_{lmpq} = \begin{bmatrix} C_{lm} \\ -S_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos[(l-2p)\omega + (l-2p+q)\overline{M} + m(\Omega - \theta)] +$$

$$+ \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin[(l-2p)\omega + (l-2p+q)\overline{M} + m(\Omega - \theta)]. \tag{25.44}$$

From (25.44) one remarks that for $q = 2p-1$ the term with $\overline{M}$ disappears, i.e. there will exist long period perturbations. In this case the eccentricity function will have the form:

$$G_{lpq}(e) = (1-e^2)^{1/2-l} \sum_{d=0}^{p'-1} \binom{l-1}{2d+l-2p'} \binom{2d+l-2p'}{d} \left(\frac{e}{2}\right)^{2d+l-2p'}, \tag{25.45}$$

where:

$$p' = \begin{cases} p \text{ if } p \leqslant \dfrac{n}{2} \\[2ex] l-p \text{ if } p > \dfrac{n}{2} \end{cases}$$

If $q \neq 2p - 1$ (case of the short-period terms), the eccentricity function will be:

$$G_{lpq}(e) = (-1)^q (1 + \beta^2)^l \beta^{|q|} \sum_{k=0}^{\infty} P_{lpqk} Q_{lpqk} \beta^{2k}, \qquad (25.46)$$

in which:

$$\beta = \frac{e}{1 + \sqrt{1 - e^2}};$$

$$P_{lpqk} = \sum_{r=0}^{k} \binom{2p'-2l}{h-r} \frac{(-1)^r}{r!} \left[\frac{(l - 2p' + q')e}{2\beta} \right]^r;$$

$$Q_{lpqk} = \sum_{r=0}^{k} \binom{-2p'}{h-r} \left[\frac{(l - 2p' + q')e}{2\beta} \right]^r \frac{1}{r!}$$

where:

$$h = \begin{cases} k - q' \text{ for } q' < 0 \\ k \text{ for } q' > 0 \end{cases}$$

$$q' = \begin{cases} q \text{ for } p \leqslant \dfrac{l}{2} \\ -q \text{ for } p > \dfrac{l}{2} \end{cases}$$

The relation (25.43) offers the possibility of calculating the variation of the orbital elements due to the perturbing function of the terrestrial potential. Taking account of the relation derived in section 2.3.5 between the coefficients C_{lm}, J_{lm}, S_{lm} and K_{lm}, the relation (25.44) may be written in the form *(Mueller 1964)*:

$$S_{lmpq}(\omega, \bar{M}, \Omega, \theta_G) = \begin{bmatrix} -J_{lm} \\ K_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos[(l - 2p)\omega + (l - 2p + q)\bar{M} +$$

$$+ m(\Omega - \theta_G)] - \begin{bmatrix} K_{lm} \\ J_{lm} \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin[(l - 2p)\omega + (l - 2p + q)\bar{M} + m(\Omega - \theta_G)]. \qquad (25.47)$$

Determining the Perturbations of the Elements of the Satellite Orbit. The relations derived in the foregoing paragraphs offer the possibility of obtaining the variation of the elements of the satellite's orbit due to the asymmetry of the terrestrial gravitational field. To this end, one calculates the values F_{lmp}, S_{lmpq}, G_{lpq} (for given values of the indices l, m), with the aid of (25.42), (25.44) and (24.46) and one introduces them into (25.43). In this way one gets the perturbing function expressed in terms of the orbital elements. Then differentiating the perturbing function with respect to the orbital elements yields the quantities $\partial R_{lm}/\partial a$, $\partial R_{lm}/\partial_i$, etc., which on being introduced into the Lagrange equations (25.22) lead to obtaining the variations of the orbital elements within a given time interval. It must be mentioned that these variations are due only to the terms R^{t}_{lm} for a given l and a given m. The result will have the form:

$$\frac{\mathrm{d}a_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\frac{2F_{lmp}\,G_{lpq}\,S'_{lmpq}}{\bar{n}\,a^{l+2}}(l-2p+q);$$

$$\frac{\mathrm{d}e_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\frac{F_{lmp}G_{lpq}S'_{lmpq}}{\bar{n}\,a^{l+3}e}\sqrt{1-e^2}\,[\sqrt{1-e^2}\,(l-2p+q)-(l-2p)];$$

$$\frac{di_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\frac{F_{lmp}G_{lpq}S'_{lmpq}}{\bar{n}\,a^{l+3}\sqrt{1-e^2}\sin i}([l-2p)\cos i-m];$$

$$\frac{\mathrm{d}\Omega_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\frac{F'_{lmp}G_{lpq}S_{lmpq}}{\bar{n}\,a^{l+3}\sqrt{1-e^2}\sin i}; \tag{25.48}$$

$$\frac{\mathrm{d}\omega_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\frac{S_{lmpq}}{\bar{n}\,a^{l+3}}\left[\frac{\sqrt{1-e^2}}{e}F_{lmp}G'_{lpq}-\frac{\cot i}{\sqrt{1-e^2}}F'_{lmp}G_{lpq}\right];$$

$$\frac{\mathrm{d}\bar{M}^{*}_{lm}}{\mathrm{d}t}=\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\frac{F_{lmp}S_{lmpq}}{\bar{n}\,a^{l+3}}\left[-\frac{1-e^2}{e}G'_{lpq}+2(n+1)G_{lpq}\right].$$

In the relations (25.48) the following notations were used:

$$F'_{lmp}=\frac{\mathrm{d}\,F_{lmp}}{\mathrm{d}t};\quad G'_{lpq}=\frac{\mathrm{d}G_{lpq}}{\mathrm{d}t}.$$

and the perturbation of the average anomaly has been defined as:

$$\bar{M}*=\int_0^t \bar{n}\,\mathrm{d}t-\bar{n}\,(t-T).$$

In order to obtain the perturbation of the orbital elements within a given time interval, one integrates the relations (25.48) and one gets the perturbations Δa_{lm}, Δe_{lm} etc. again for a given set of l, m values. The total perturbation, due to the entire perturbing function, is obtained by summing from $m=0$ to $m=l$ and for $l=2$ to $l=\infty$:

$$\Delta a=\sum_{l=2}^{\infty}\sum_{m=0}^{l}\Delta a_{lm};$$

$$\Delta e=\sum_{l=2}^{\infty}\sum_{m=0}^{l}\Delta e_{lm}.$$

25.4 The Luni—Solar Perturbation

One regards the Earth's mass M as concentrated at the mass centre T, also considered as the origin of the geocentric coordinate system X, Y, Z. Let P be a perturbing body (e.g. the *Moon*) whose mass m' is concentrated

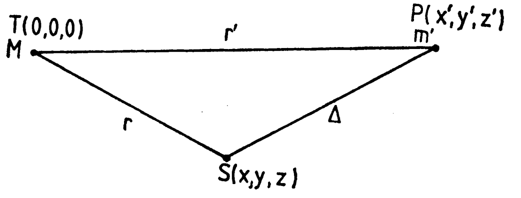


Fig. 25.5. The Perturbation Due to the Moon's Attraction

at its mass centre, of coordinates x', y', z'. The satellite, of mass unity, is located, at the point S of coordinates x, y, z (Fig. 25.5). The distance between the Earth and the satellite is denoted by r, that between the Earth and the perturbing body by r' and the distance satellite — perturbing body by Δ.

On the satellite S acts the attractive force $\vec{\mathbf{F}}_1$ exercised by the Earth (from which should be subtracted the force $\vec{\mathbf{F}}_2$ with which the Moon attracts the Earth) and the force $\vec{\mathbf{F}}_3$ of attraction of the satellite by the Moon:

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 - \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 = \frac{GM}{r^2} - \frac{GM'}{r'^2} + \frac{GM}{\Delta^2}. \tag{25.49}$$

According to Newton's third law:

$$F = m\,a$$

and, since the satellite's mass is regarded as equal to unity, the previous relation may be written in the form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F_x; \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F_y; \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = F_z.$$

In view of (25.49), the above relations become:

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= = \frac{GM \cdot x}{r^3} + GM'\left(\frac{x' - x}{\Delta^3} - \frac{x'}{r'^3}\right); \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\frac{GMy}{r^3} + GM'\left(\frac{y' - y}{\Delta^3} - \frac{y'}{r'^3}\right); \\ \frac{\mathrm{d}^2z}{\mathrm{d}t^2} &= -\frac{GMz}{r^3} + GM'\left(\frac{z' - z}{\Delta^3} - \frac{z'}{r'^3}\right). \end{aligned} \tag{25.50}$$

The second term on the right-hand side of the above relations represents the partial derivatives $\partial R_L/\partial x$, $\partial R_L/\partial y$, $\partial R_L/\partial z$ of the function:

$$R_L = Gm'\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3}\right), \tag{25.51}$$

being the perturbing function due to the attraction of the Moon. In the case in which one considers the simultaneous attraction of the Moon (of mass m' situated at the distance Δ' from the satellite) and of the Sun (of mass m'' situated at the distance Δ'' from the satellite), the luni-solar perturbing function becomes:

$$R_{LS} = Gm'\left(\frac{1}{\Delta'} - \frac{xx' + yy' + zz'}{r'^3}\right) + Gm''\left(\frac{1}{\Delta''} - \frac{xx'' + yy'' + zz''}{r''^3}\right). \tag{25.52}$$

In (25.52), r' and r'' denote the distances from the Earth to the Moon, and to the Sun respectively.

25.5 Perturbations of the Artificial Satellites' Orbits Caused by the Resistance of the Atmosphere and by Radiation Pressure

As well as the main perturbing factor of the artificial satellites' orbits — the terrestrial gravitational field — and the luni-solar attraction, noticeable perturbing influences are also due to the atmospheric resistance and to the radiation pressure. These latter ones will be briefly analysed next.

25.5.1 Orbit Perturbations Due to the Atmosphere's Resistance

In order to evaluate the resistance effect generated as a consequence of the satellite's friction against the air layers of the atmosphere, it is necessary to know the density of the latter and the velocity of the air currents along the orbit.

The density of the atmosphere depends on the altitude with respect to the Earth's physical surface, the solar activity, the variations of the geomagnetic field and the transition from day to night. The resistance effect of the atmosphere is smaller for very high satellites and for those having a reduced profile and a large mass.

A method for determining the perturbations in the satellites' orbits due to the resistance effect of the atmosphere is as follows: one supposes that this effect on the average anomaly $\overline{M}$ may be presented in the form of a power series:

$$\overline{M} = \overline{M}_0 - \overline{M}_1 t - \overline{M}_2 t^2 - \overline{M}_3 t^3 - \ldots,$$

in which t represents the observation moment and $\overline{M}_1, \overline{M}_2, \ldots$ are coefficients which can be determined by processing the observational data on satellites by the least squares method. Assuming that the influence of the atmosphere's resistance decreases rapidly with time, in the expansion of the average anomaly one only considers the terms:

$$\bar{M} = \bar{M}_0 - \bar{M}_1 t - \bar{M}_2 t^2.$$

If from processing the observations one obtains the coefficients $\bar{M}_0$, $\bar{M}_1$, $\bar{M}_2$, differentiating the expression for the average anomaly then yields the mean angular velocity:

$$\bar{n} = \frac{\mathrm{d}\bar{M}}{\mathrm{d}t} = \bar{M}_1 + 2\bar{M}_2 t.$$

With respect to an initial epoch t, this velocity is modified by:

$$\Delta\bar{n} = 2\,\bar{M}_2 t.$$

Once the value of $\bar{M}_2$ is known, one can determine $\Delta\bar{n}$ and $\bar{n}$, on the basis of which one can determine, according to Kepler's third law the variation of the semi-major axis, in the form:

$$\Delta a = \frac{4}{3}\,\frac{a}{\bar{n}}\,\bar{M}_2 t.$$

With the aid of the quantities $\Delta\bar{n}$ and Δa one determines the resistance effect of the atmosphere on the eccentricity e in the form:

$$\Delta e = (1 - e)\frac{\Delta a}{a} = -\frac{2}{3}(1 - e)\frac{\Delta\bar{n}}{\bar{n}}.$$

In the end, the elements Δa and Δe, obtained in this way, underlie the determination of the resistance effect of the atmosphere on the rotation motion of the nodes' line and on the apsides' line, which may be determined in the form:

$$\Delta\omega = \frac{1}{3\bar{n}}\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)\left[\frac{7 - e}{1 + e}\right]\bar{M}_2 t^2;$$

$$\Delta\Omega = \frac{1}{3\bar{n}}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)\left[\frac{7 - e}{1 + e}\right]\bar{M}_2 t^2.$$

This somewhat approximate method cannot be utilized for operations needing a greater accuracy, such as those for determining the tesseral and sectorial spherical functions of the terrestrial gravitational field.

A more accurate method consists of splitting up the vector of the satellite's velocity into its components S and T and expressing them in terms of the orbital elements. In this case the influence of the atmosphere's resistance on the two components can be expressed as a function of the density of the atmosphere, of the satellite's section and of the orbital elements. Afterwards, with the help of (25.22), one derives by integration the perturbations of a satellite generated by the resistance of the atmosphere.

25.5.2 Orbit Perturbations Due to the Solar Radiation Pressure

The radiation pressure exercised by the light of the Sun on the satellites has as an effect a perturbation of their orbits. The variation of the satellite's velocity produced as a consequence of this effect is:

$$t = k\frac{A}{m}\,\frac{1}{c}\,I_0\left(\frac{a_\varepsilon}{r_\varepsilon}\right)^2\left[\left(1-\frac{v_n}{c}\right)-\frac{1}{c}\,v_s\right],$$

in which k is a numerical constant depending on the reflective properties of the satellite, A is the satellite's average section, m its mass, c is the speed of light, a_ε the semi-major axis of the orbital ellipse along which the Earth travels round the Sun, v_n is the component of the satellite's velocity in the Sun-Earth direction and v_s is the satellite's relative velocity with respect to the Sun.

For calculating the effect of the radiation presure, the perturbation t is broken down along the directions S, T, W which are expressed as functions of the orbital elements of the Sun's apparent motion with respect to the Earth. With the aid of the components S, T, W, by integrating the relations (25.22), one gets the perturbations of the satellite's orbit. In this case a numerical integration is to be preferred, inasmuch as the integration only extends over the time period during which the satellite is subjected to the solar radiations and not over the period during which it is in the Earth's shadow.

The perturbations due to the radiation effect are greater than those due to the atmosphere's resistance for the satellites of great height (over 1,000 km). At heights of 800—900 km the effects are equal for the minimum and the maximum respectively, of solar activity.

The perturbations due to the radiation effect are felt in particular by the balloon satellites *(Echo I, Echo II, Pageos)*, which have a small weight and a larger diameter.

25.6 Determining the Parameters of the Earth's Gravitational Field by Using Artificial Satellites

For determining the orbit of an artificial satellite it is necessary to know both the geocentric coordinates of the observing station and the coefficients of the expansion of the normal potential in spherical functions. But in order to determine the geocentric coordinates of the position of the observing station one needs to know the coefficients of the normal potential's expansion and, conversely, for the determination of the latter knowledge of the geocentric coordinates of the observing stations is necessary. These two categories of unknowns can only be determined simultaneously by a process of successive approximations.

In order to illustrate such a process we now present the stages of a complete calculation of dynamic geodesy *(Levallois* and *Kovalevsky 1971):*

(1) *The determination of the constant GM* is carried out by means of observations on space probes. These must be sufficiently far from the Earth that the influence of the perturbing force R_T is small enough. In this case one can use an approximate expansion of the normal potential in spherical functions.

(2) *The determination of the amplitude of the secular and long-period terms* is performed by repeated observations on a certain number of satellites over a period of a few months. Within the framework of this stage one determines the zonal harmonics, considering the tesseral harmonics and the short-period terms as random values.

(3) *The calculation of the tesseral harmonic* is achieved by studying short arcs of the orbits of a large number of satellites. It is advisable that for this determination a large number of station distributed as uniformly as possible on the entire surface of the Earth should be utilized. It is within the framework of this stage that the initial geocentric coordinates of the observing stations are also corrected.

In what follows the principle guiding the practical realization of each stage will be shown separately.

25.6.1 Determining the Constant GM

Up to now the most accurate determination of the GM value has been based on the observation of space probes very far from the Earth.

Taking into consideration only the attraction of the Earth and of the Moon, the differential equations of the motion of such a space probe are of the type (25.50):

$$\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\frac{GM}{r^3}x + GM\frac{m'}{M}\left(\frac{x'-x}{\Delta^3} - \frac{x'}{r'^3}\right) + \frac{\partial R_T}{\partial x};\\
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\frac{GM}{r^3}y + GM\frac{m'}{M}\left(\frac{y'-y}{\Delta^3} - \frac{y'}{r'^3}\right) + \frac{\partial R_T}{\partial y};\\
\frac{\mathrm{d}^2z}{\mathrm{d}t^2} &= -\frac{GM}{r^3}z + GM\frac{m'}{M}\left(\frac{z'-z}{\Delta^3} - \frac{z'}{r'^3}\right) + \frac{\partial R_T}{\partial z}.
\end{aligned} \tag{25.53}$$

The partial derivatives of the perturbing function R_T decrease rapidly with the distance r between the space probe and the Earth. Since from the beginning the condition that the probe should be very far from the Earth was imposed, it follows that it suffices that in the perturbing function R_T only the first zonal harmonics be taken into consideration. These harmonics are already known with an accuracy acceptable for the purpose required in this case.

For the determination of the value of GM from the relations (25.53), it is also necessary to know the ratio m'/M of the Moon's mass to the Earth's mass. This ratio is determined by measuring by means of the Doppler method the radial velocity of a planetary probe. The best determinations of this ratio give the value (*Levallois* and *Kovalevsky 1971*):

$$M/m' = 81.3015 \pm 0.001. \tag{25.54}$$

This ratio being given and knowing the partial derivatives of the perturbing function to the necessary approximation, the relations (25.53) may be considered as a system of differential equations which can be integrated. In the end one gets a system of equations having as unknowns the coordinates x_t, y_t, z_t of the probe at the moment t, its velocity $\mathrm{d}x_t/\mathrm{d}t$, $\mathrm{d}y_t/\mathrm{d}t$, $\mathrm{d}z_t/\mathrm{d}t$ at the same moment, as well as the parameter GM. The study of the lunar probes *Ranger* have allowed the determination of the most probable value of this parameter as being:

$$GM = 398\,601.3\ \mathrm{km}^3/\mathrm{s}^2.$$

The parameter GM may also be determined from Moon observations. Starting from *Kepler*'s third law:

$$\bar{n}^2 a_e^2 = G(M + m'), \tag{25.55}$$

where $\bar{n}$ is the Moon's angular velocity of revolution and a_l is the semi-major axis of the Moon's orbit, one can deduce the parameter as:

$$GM = \frac{\bar{n}^2}{1 + \dfrac{m'}{M}}\, a_l^3 \tag{25.56}$$

For the ratio m'/M, one takes the value given by (25.54) and the orbital radius a_l may be determined with the help of radar measurements of the distance from the observing station to the Moon. Knowing these elements, one can determine the parameter GM with sufficient accuracy on the basis of (25.29) without carrying out any other distance measurement on the Earth's surface.

For the 1967 geodetic reference ellipsoid the value:

$$GM = 398\,601\ \mathrm{km}^3/\mathrm{s}^2,$$

has been adopted, which is deduced from observing some space probes, as well as from Moon observations, by utilizing the relation (25.29).

25.6.2 Determining the Zonal Harmonics

One considers the perturbing function due to the asymmetry of the gravitational field to be expressed only by means of the zonal coefficients J_n:

$$R_n^T = -\frac{GM}{a_e}\sum_{n=2}^{\infty}\left(\frac{a_e}{R}\right)^{n+1} J_n P_n\,(\cos\theta). \tag{25.57}$$

Differentiating this function with respect to r and θ and introducing these values into (25.25) one gets the components S, T and W of the perturbing force.

Substituting these values into (25.23) yields the variations of the orbital elements as functions of the coefficients $J_2, J_3, \ldots$ The perturbation of the orbit's elements for a given period is obtained by integrating these elements in the form:

$$\Delta a = 0$$

$$\Delta e = -\frac{1-e^2}{e} \tan i \cdot \Delta i;$$

$$\Delta i = 3\pi e\left(\frac{a_e}{p}\right)\left(1 - \frac{5}{4}\sin^2 i\right)\cos i \cos \omega\, J_3 +$$

$$+\frac{45}{16}\pi e\left(\frac{a_e}{p}\right)^4\left(1 - \frac{7}{6}\sin^2 i\right)\sin 2i \sin 2\omega\, e J_4 \ldots;$$

$$\Delta\Omega = -3\pi\left(\frac{a_e}{p}\right)^3 \cos i J_2 + 3\pi\left(\frac{a_e}{p}\right)^3\left(1 - \frac{15}{4}\sin^2 i\right)\cot i \sin \omega e J_3 +$$

$$+\frac{15}{2}\pi\left(\frac{a_e}{p}\right)^4\left(1 - \frac{7}{4}\sin^2 i\right)\cos i\, J_4 \ldots; \tag{25.58}$$

$$\Delta\omega = 6\pi\left(\frac{a_e}{p}\right)^2\left(1 - \frac{5}{4}\sin^2 i\right)J_2 + 3\pi\left(\frac{a_e}{p}\right)^3\left(1 - \frac{5}{4}\sin^2 i\right)\sin i \sin \omega\, eJ_3 -$$

$$-15\pi\left(\frac{a_e}{p}\right)^4\left[\left(1 + \frac{31}{8}\sin^2 i + \frac{49}{16}\sin^4 i\right) + \left(\frac{3}{8} - \frac{7}{16}\sin^2 i\right)\sin^2 i \cos 2\omega\right]J_4 \ldots$$

The 2nd-order terms were disregarded in these equations but for the practical determination of the zonal harmonics they must be taken into consideration.

Analysing the relations (25.58) shows that: (a) the semi-major axis of the orbit does not have secular or long-duration variations; (b) the eccentricity and the inclination have long-period variations and (c) the quantities Ω and ω have both long-period and secular variations.

From observations on a large number of satellites with a wide variation range of the inclination one determines the perturbations of the orbits. For each observed position one forms an equation system of the type (25.58), from which one can, then, determine the coefficients of the zonal harmonics.

The even zonal harmonics $J_2, J_4, \ldots$ may be better determined from the perturbations of the ascending node $\Delta\bar{\Omega}$ and from those of the perigee rotation

$\Delta\bar{\omega}$ which can be expressed as functions of the even zonal harmonics in the form *(Levallois* and *Kovalevsky 1971):*

$$\frac{d\Omega}{dt} = -\frac{3}{2} J_2 \bar{n} \frac{R^2}{p^2} \cos i +$$

$$+ J_2^2 \frac{\bar{n}R^4}{p^4} \cos i \left[\left(-\frac{45}{8} + \frac{3e^2}{4} + \frac{9e^4}{32}\right) + \left(\frac{57}{8} - \frac{69}{32} e^2 - \frac{27}{64} e^4\right) \sin^2 i\right] +$$

$$+ J_4 \frac{\bar{n}R^4}{p^4} \cos i \left(\frac{15}{4} - \frac{105}{16} \sin^2 i\right) \left(1 + \frac{3e^2}{2}\right) +$$

$$+ J_6 \frac{\bar{n}R^6}{p^6} \cos i \left(-\frac{105}{16} + \frac{945}{32} \sin^2 i - \frac{3465}{128} \sin^4 i\right)\left(1 + 5e^2 + \frac{15}{8} e^4\right) \ldots$$

$$\frac{d\bar{\omega}}{dt} = J_2 \frac{\bar{n}R^2}{p^2}\left(3 - \frac{15}{4} \sin^2 i\right) +$$

$$+ J_2^2 \frac{\bar{n}R^4}{p^4}\left[\left(\frac{27}{2} - \frac{15e^2}{6} - \frac{9e^2}{16}\right) + \left(-\frac{507}{16} + \frac{171e^2}{32} + \frac{99e^4}{64}\right) \sin^2 i +\right.$$

$$\left. + \left(\frac{1185}{64} - \frac{675}{128} e^2 - \frac{135}{128} e^4\right) \sin^4 i\right] +$$

$$+ J_4 \frac{\bar{n}R^4}{p^4}\left[\left(-\frac{3}{8} + \frac{15}{8} \sin^2 i - \frac{105}{64} \sin^4 i\right)\left(10 + \frac{15}{2} e^2\right) +\right.$$

$$\left. + \left(-\frac{15}{4} + \frac{165}{16} \sin^2 i - \frac{105}{16} \sin^4 i\right)\left(1 + \frac{3}{2} e^2\right)\right] +$$

$$+ J_6 \frac{\bar{n}R^6}{p^6}\left[\left(\frac{5}{16} - \frac{105}{32} \sin^2 i + \frac{945}{128} \sin^4 i - \frac{1155}{256} \sin^6 i\right) \times\right.$$

$$\times \left(21 + \frac{105}{2} e^2 + \frac{105}{8} e^4\right) +$$

$$+ \left(\frac{105}{16} - \frac{1155}{32} \sin^2 i + \frac{7245}{128} \sin^4 i - \frac{3465}{128} \sin^6 i\right)\left(1 + 5e^2 + \frac{15}{8} e^4\right) \cdots$$

In a general form, these relations may be written, after integration, as:

$$\bar{\Omega} = \left[\sum_{n=1}^{\infty} A_n (a, e, i) J_{2n} + B_n(a, e, i) J_2^2\right] t + \Omega_0;$$

$$\bar{\omega} = \left[\sum_{n=1}^{\infty} A'_n (a, e, i) J_{2n} + B'_n(a, e, i) J_2^2\right] t + \omega_0.$$

Considering the elements $\overline{\Omega}$ and $\overline{\omega}$ as being known from observing a large number of satellites and assuming that the value of J_2^2 is fairly accurately known, one can write, for the satellite j, an equation system of the form:

$$\Omega'_j = \sum_{n=1}^{P} A_{nj} \, J_{2n}; \qquad \omega'_j = \sum_{n=1}^{P} A'_{nj} \, J_{2n}. \tag{25.59}$$

From the complete equation system written for the j satellites one can determine the even zonal harmonics, taking care that the number of equations is greater than the number of zonal harmonics which one wishes to determine.

The odd zonal harmonics are determined with the help of the perturbations of the inclination, of the eccentricity, of the ascending node and of the perigee's argument, which can be expressed as functions of the zonal harmonics in the form:

$$\begin{aligned}
\Delta i_k &= \sum_{p=1}^{P} B_{pk} \sin (2p - 1) \, \overline{\omega}_k ; \\
\Delta e_k &= \sum_{p=1}^{P} B'_{pk} \sin (2p - 1) \overline{\omega}_k ; \\
\Delta \Omega_k &= \sum_{p=1}^{P} A_{pk} \cos (2p - 1) \, \overline{\omega}_k ; \\
\Delta \omega_k &= \sum_{p=1}^{P} A'_{pk} \cos (2p - 1) \, \overline{\omega}_k .
\end{aligned} \tag{25.60}$$

The coefficients A_{pk}, A'_{pk}, B_{pk} and B'_{pk} are functions of the elements a_k, e_k, i_k of the satellite k and of the odd zonal harmonics $J_3, J_5, \ldots, J_{2p-1}$. The form of these coefficients *(Levallois* and *Kovalevsky 1971)* has not been given here, as it is more complicated.

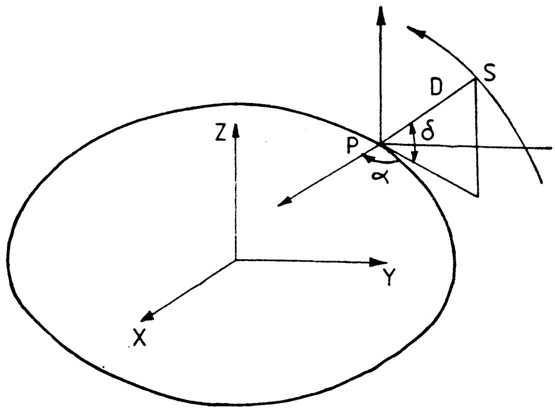
Observing k satellites yields $4Pk$ equations with P unknowns J_{2P-1}. The observed satellites must have their elements as different as possible from one another, in order to achieve a more accurate determination of the P unknowns. The fact must still be emphasized that in determining the even and the odd zonal harmonics one utilizes a series of common elements, which implies that the two types of unknowns are correlated. In order to improve the results it is necessary to apply a process of successive approximations.

25.6.3 Determining the Tesseral Harmonics and the Positions of the Observation Stations

The tesseral harmonics have a much shorter period than the zonal ones, so that the application of the procedures utilized for determining the latter does not work for the former.

Let the X, Y, Z coordinate system have origin at the Earth's mass centre, with the X-axis situated in the equatorial plane and passing through the first point of Aries γ and the Y-axis in the same plane and perpendicular to the X-axis. This coordinate system does not rotate with the Earth, as it is an astronomical system, fixed with respect to the stars. Let also P be a point situated on the ellipsoid (Fig. 25.6), of coordinates X_P, Y_P, Z_P determin-

Fig. 25.6. Satellite's Polar Coordinates



ed in the previously described coordinate system (actually the point P is not on the ellipsoid but by applying the reduction methods which were analysed in the second part of the book it may be brought onto the ellipsoid). In the same coordinate system, the satellite at S would have the coordinates X_S, Y_S, Z_S. By observing a satellite one determines its spherical coordinates D, α, δ; D being the distance to the satellite, α the right ascension and δ the declination. The last two elements are determined by photographing the satellite against a stellar background. From Fig. 25.6, one obtains the following relations between the rectangular and the spherical coordinates of the satellite:

$$X_S - X_P = D \cos \delta \cos \alpha; \; Y_S - Y_P = D \cos \delta \sin \alpha;$$

$$Z_S - Z_P = D \sin \delta. \tag{25.61}$$

whence it follows that:

$$\alpha = \tan^{-1} \frac{Y_S - Y_P}{X_S - X_P};$$

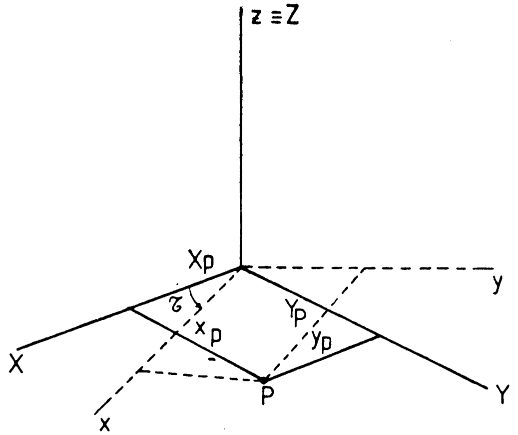
$$\delta = \tan^{-1} \frac{(Z_S - Z_P)}{\sqrt{(X_S - X_P)^2 + (Y_S - Y_P)^2}} \tag{25.62}$$

$$D = \sqrt{(X_S - X_P)^2 + (Y_S - Y_P)^2 + (Z_S - Z_P)^2}$$

The satellite's rectangular coordinates in the X, Y, Z system may be expressed in terms of time, of the orbit's elements and of the tesseral harmonics:

$$
\begin{aligned}
X &= X(t;\, a_0, e_0, i_0, \Omega_0, \omega_0, T_0;\, J_{nm}, K_{nm});\\
Y &= Y(t;\, a_0, e_0, i_0, \Omega_0, \omega_0, T_0;\, J_{nm}, K_{nm});\\
Z &= Z(t;\, a_0, e_0, i_0, \Omega_0, \omega_0\, T_0;\, J_{nm}, K_{nm}).
\end{aligned}
\tag{25.63}
$$

The coordinates of the point P in the same system X, Y, Z are still unknown. In order to determine them, one can express them in terms of



25.7. The Connexion between the X, Y, Z and the x, y, z Coordinate Systems

geocentric coordinates x_p, y_p, z_p. This coordinate system has the z axis parallel to the Z axis, the x axis in the equatorial plane passing through the zero Greenwich meridian and the y axis in the equatorial plane perpendicular to x (Fig. 25.7). The angle τ between the X and x axes has the value $\tau = \omega t$ in which ω represents the Earth's rotational angular velocity.

From Fig. 25.7 follows the connexion between the coordinates of the point P in the two systems:

$$
\begin{aligned}
X_P &= x_P \cos\tau - y_P \sin\tau;\\
Y_P &= x_P \sin\tau + y_P \cos\tau;\\
Z_P &= z_P.
\end{aligned}
\tag{25.64}
$$

Now substituting into (25.62) the values of X_P, Y_P, Z_P given by (25.64) and the values of X, Y, Z given by (25.63) one gets for the quantities α, δ and D relations whose general form is:

$$
\begin{aligned}
\alpha &= \alpha(x_P, y_P, z_P;\, t;\, a_0, e_0, i_0, \Omega_0, \omega_0\, T_0;\, J_{nm}, K_{nm});\\
\delta &= \delta(x_P, y_P, z_P;\, t;\, a_0, e_0, i_0, \Omega_0, \omega_0\, T_0;\, J_{nm}, K_{nm});\\
D &= D(x_P, y_P, z_P;\, t;\, a_0, e_0, i_0, \Omega_0, \omega_0\, T_0;\, J_{nm}, K_{nm}).
\end{aligned}
\tag{25.65}
$$

For each observation carried out on a satellite one obtains an equation of the form (25.65). Obtaining a sufficiently large number of such equations and solving them by the least squares method one can determine: the geocentric coordinates of the station, the orbital elements and the parameters of the gravitational field. This is the principle of the orbital method.

25.7 The Geoid Determination

As was shown in Chapter 5, the astro-geodetic or gravimetric determination of the geoid presupposes a great volume of astronomical, geodetic and gravity data uniformly distributed over the Earth's surface. The lack of these data has led to compromise solutions: in the gravimetrically explored zones one has utilized the available data and for the other zones one has interpolated by some procedure the gravity anomalies. The accuracy of the results is conditional upon the limited possibilities of predicting the gravity anomalies.

These shortcomings may be got round to some extent by using the methods of Dynamic Geodesy, offering two major possibilities of solving the problem.

The first way starts by assuming the existence of a known reference ellipsoid, from whose parameters the coefficients of the expansion of its normal potential U_0 into spherical functions according to (3.30) can be determined.

From observations on artificial satellites one can determine the coefficients of the expansion in spherical harmonics of the actual potential V of the Earth.

According to (4.1), from the difference of these potentials one gets the perturbing potential T, from which one can determine, on the basis of *Bruns'* formula (4.5), the geoid's undulation with respect to the reference ellipsoid.

The second way is based on the determination of the free-air anomalies according to (4.26) from the perturbing potential, deduced as before. The free-air anomalies determined in this manner may be used directly in *Stokes'* formula (5.6), the geoid's undulation thus being determined. In order to improve the accuracy of this method, one may combine the determinations of the free-air anomalies obtained by measurements with those resulting from artificial satellites' observation, with the understanding that in the explored regions one utilizes the values obtained from measurements and in the other areas one utilizes the anomalies calculated on the basis of the relation (4.26) from the perturbing potential obtained by the methods of Dynamic Geodesy.

Similarly with this second method, one may obtain the gravimetric deflections of the vertical using *Vening Meinesz*'s formulae (5.22).

The modern laser-techniques have allowed one to use a direct method for determining the geoid's profile over the ocean's surface by satellite altimetry. This method will be briefly described in § 25.7.4.

25.7.1 Some Results Obtained in Determining the Earth's Gravitational Field and the Geoid

At the beginnings of the utilization of artificial satellites for geodetic purposes, it was possible to obtain the geocentric coordinates of the monitoring stations and the parameters of the gravitational field with an accuracy which today seems completely approximate. Thus, one could determine, at that time, only three or four coefficients of the expansion of the gravitational potential in spherical functions and the geocentric coordinates could only be known with an accuracy of about 500 m. The idea of the hydrostatic equilibrium of the Earth was also accepted. The advances achieved since that time in the knowledge of the parameters of the gravitational field, of the geocentric coordinates of a world network of satellite monitoring stations, as well as in the field of determining the geoid were rapid: one has arrived at accuracies of $\pm$ 1—2 m in determining the geocentric coordinates and at determining a greater number of spherical harmonics, at present the order being reached of $l = 23$ for the zonal harmonics and $l = m = 30$ for the tesseral harmonics.

The investigations in the domain of determining the Earth's gravitational field and the global form of the geoid are very numerous, being grouped along three main directions: gravity techniques, purely satellite methods and combined satellite and gravity procedures.

Some of the most significant results of these investigations will be presented in the sequel. A comparative analysis of all the results obtained so far will not be made here as it constitutes the object of a vast piece of research work which is being performed at the *Smithsonian Astrophysical Observatory* (*S.A.O.*).

25.7.2 Some Results Obtained in Determining the Parameters of the Earth's Gravitational Field

The gravimetric method of representing the gravitational field assumes the existence of a world gravity survey. Since three quartes of the Earth's surface is covered by water, the existence of such a survey is not yet available and, consequently, the representation of the gravitational field in this way is not convincing.

The first attempt at getting the representation of the gravitational field from gravity data as measured on the Earth's physical surface was made in the year 1928 by *W. Heiskanen* who had at his disposal the isostatic gravity anomalies within 865 standard surfaces of the size $1^\circ \times 1^\circ$. Afterwards, *U. Uotila* in the year 1962 gave a representation based on 11,294 average free-air anomalies within surfaces of $1^\circ \times 1^\circ$, of which only 2,535 were located in the southern hemisphere. The number of $1^\circ \times 1^\circ$-sized areas in which measured gravity values are available is, however, small in comparison with the total number of 64,800 such areas which exist on the entire surface of the Earth.

There are, however, theoretical possibilities of determining by extrapolation the mean values of the anomalies in zones which are not surveyed gravimetrically. This was done by *L. Kivoja* in the year 1963, on the basis of the mean values used by *W. Heiskanen* and *U. Uotila*. Based on these data he made a harmonic analysis of the Earth's gravitational field, giving the values of the coefficients A_{nk} and B_{nk} up to the order $n = 8$. Contributions in this direction were also made by *H. Jeffreys* (1961), *D. Zhongolovich* (1957) and *M. Kaula* in the year 1959 but the most complete analysis of the gravitational field carried out on the basis of gravity data obtained from direct measurements remains that of *L. Kivoja*.

Dynamic Geodesy allows the determination of the Earth's gravitational field from observing the artificial satellites, according to the methodology put forward in principle in Section 25.5. This determination presents the advantage of not depending on the existence of a world gravity survey, the method's principle consisting of expressing the actual Earth's potential in an infinite series of spherical harmonics, whose coefficients are determined from observations on artificial satellites.

Inasmuch as the processing of the data provided by the satellite observation yields a finite number of coefficients of the infinite series defining the actual potential of the Earth, one may consider that the error in knowing this potential exactly, the truncation error, is given by the disregarded geopotential harmonics. The truncation error will be the smaller the more coefficients in the potential's expansion in spherical functions are known.

The zonal harmonics are deduced (25.6.2) by analysing the secular displacement of the node line and of the *apsis* line for the even zonal coefficients, and from the analysis of the long-period variation of the terms e, i, Ω and ω, for the odd zonal coefficient respectively.

On the basis of results of satellite observations, with the aid of the *Baker Nunn* camera, *Y. Kozai* determined in 1964 the values of the even and of the odd zonal coefficients up to the 14th, and 13th order respectively, and in the years 1967, 1969 and 1971 the values of these coefficients up to the 20th and 21st order respectively, whose values are given in Table 25.1.

Table 25.1. *Coefficients of the Even and of the Odd Zonal Harmonics x* 10^6 *as Derived by Y. Kozai*

n	1967	1969	1971	n	1967	1969	1971
2	1082.639	1082.628	1082.637	3	−2.565	−2.538	−2.539
4	−1.609	−1.593	−1.617	5	−0.174	−0.230	−0.234
6	0.542	0.592	0.555	7	−0.419	−0.361	−0.348
8	−0.128	−0.118	−0.209	9	−0.022	−0.100	−0.159
10	−0.338	−0.354	−0.240	11	0.176	0.202	0.323
12	0.053	−0.042	−0.190	13	−0.146	−0.123	−0.333
14	−0.174	−0.073	0.105	15	−0.065	−0.174	0.108
16	0.449	0.187	0.024	17	−0.052	0.085	−0.218
18	−0.324	−0.231	−0.103	19	−0.075	−0.216	0.084
20	0.334	−0.005	−0.126	21	—	0.145	−0.086

Also in the year 1971 the French research workers *A. Cazenave, F. Forestier, F. Nouel* and *J. Pieplu*, using the observations of three satellites in the neighbourhood of the equator *(Peole, Dial* and *SAS)*, determined the system of zonal coefficients $FR = 71$ (Table 25.2).

Table 25.2. *System of the Coefficients of the Zonal Harmonics* FR 71

n	$Jn \cdot 10^6$	n	$Jn \cdot 10^6$
2	1082.637	3	−2.543
4	−1.619	5	−0.226
6	0.558	7	−0.365
8	−0.209	9	−0.118
10	−0.233	11	0.236
12	−0.188	13	−0.202
14	0.085	15	−0.081
16	0.048	17	−0.027
18	−0.137	19	−0.112
20	−0.087	21	0.106

More recent results in determining the zonal coefficients are those of *A. Wagner* (1972) who, processing data obtained from observing 21 satellites (of which 3 were in the neighbourhood of the equator), obtained the values of the coefficients of the even and of the odd zonal harmonics up to the 20th, and 21st orders respectively (Table 25.3).

Table 25.3. *Coefficients of the Zonal Harmonics as Calculated by A. Wagner as well as those Calculated within the Standard Earth Programme*

n	$Jn \cdot 10^6$			n	$Jn \cdot 10^6$		
	Wagner	SE I	SE III		Wagner	SE I	SE III
2	1082.635	1082.645	1082.636	3	−2.541	−2.546	−2.540
4	−1.600	−1.649	−1.619	5	−0.230	−0.210	−0.230
6	0.530	0.646	0.552	7	−0.364	−0.333	−0.345
8	−0.200	−0.270	−0.204	9	−0.081	−0.053	−0.162
10	−0.224	−0.054	−0.232	11	0.137	0.302	0.317
12	−0.208	−0.357	−0.196	13	−0.101	−0.114	−0.336
14	0.166	0.179	0.101	15	−0.072	—	0.104
16	0.003	—	0.043	17	−0.204	—	−0.227
18	−0.086	—	−0.077	19	0.047	—	0.083
20	−0.085	—	−0.108	21	0.015	—	−0.070
22	—	—	0.075	23	—	—	0.111

Having plenty of material at his disposal, *A. Wagner* was able to perform a very thorough checking: a few orbital arcs were not introduced into the processing operation and, after having determined the set of zonal coefficients, were used for checking the system. The results were more than satisfactory.

The tesseral harmonics are determined simultaneously with the determinations of the stations' positions through a process of succesive approximations.

The first determinations of the coefficients of the non-zonal harmonics were carried out in the *U.S.A.* in the years 1965—1966 by *R. Anderle,* who used the Doppler observations performed in the navy's laboratories, and by *M. Gaposchkin,* who utilized the optical observations made with S.A.O. stations equipped with *Baker Nunn* photographic cameras.

Important results were obtained by *R. Anderle* concerning the influence of the terms which were not taken into consideration in expanding the gravity potential in spherical harmonics. He has shown that the truncation error due to the limitation of the harmonic series to $N < n$ $(n \to \infty)$ leads to errors affecting even the coefficients of the Nth order which are taken into consideration in expanding the geopotential in spherical harmonics. This is explained by the fact that the error arising from neglecting the terms $n > N$ is re-distributed in the Nth-order terms. From *R. Anderle*'s studies it followed that the limitation of the series to the terms (7,7) leads to errors in determining these terms which may reach values of 0.2×10^{-6} and the limitation to the term (12,12) reduces the errors to 0.05×10^{-6}.

Determinations of the tesseral harmonics were also carried out by *H. Guier* and *R. Newton,* their results being in good agreement with those of *M. Gaposchkin.*

Particularly important results were obtained within the framework of the "*Standard Earth*" programme, initiated and performed within the *S.A.O.* framework in three stages: *Standard Earth I* in the year 1966, *Standard Earth II* in 1969 and *Standard Earth III* in 1973.

The parameters defining a "*Standard Earth*" consist of: a set of Cartesian coordinates of the stations for satellite monitoring and a set of coefficients of the expansion of the terrestrial potential in spherical harmonics. Both types of data are expressed in an orthogonal geocentric coordinate system defined by the *Conventional International Origin* (*C.I.O.*) and by the *Mean Greenwich Observatory,* which determines the zero meridian.

For calculating the parameters of the Standard Earth the following types of data are utilized:

(1) *Individual satellite observations,*

(2) *Simultaneous satellite observations,*

(3) *Gravity observations carried out on the Earth's surface,*

(4) *Data obtained from conventional triangulation.*

The individual observations combined with terrestrial gravity data serve to determin the gravitational field, while the triangulation data, the individual observations and the simultaneous ones serve to determine the geocentric coordinates.

The geocentric coordinates and the coefficients of the spherical harmonics were obtained for "*Standard Earth I and II*" using the same processing operation of the initial data, whereas for "*Standard Earth III*" these quantities were calculated separately. Thus, the orbits affected by large errors in the gravity field, such as e.g. the resonances which are included in the determination of the potential's coefficients, are not used in determining the geocentric coordinates.

The values of the coefficients of the zonal harmonics in the expansion of the gravity potential as calculated within the framework of the "*Standard Earth*" programme are given in Table 25.3 and the values of the non-zonal coefficients are given in Table 25.4.

Table 25.4. *Coefficients of the Non-Zonal Harmonics as Determined by H. Guier and R. Newton as well as by M. Gaposchkin in the Standard Earth I — 1966 and Standard Earth II — 1969 Systems*

n	m	$C_{nm} \cdot 10^6$			$C_{nm} \cdot 10^6$		
		Guier Newton	Standard Earth I	Standard Earth II	Guier Newton	Standard Earth I	Standard Earth II
2	2	2.38	2.38	2.43	−1.20	−1.35	−1.39
3	1	1.84	1.94	1.89	0.21	0.27	0.24
3	2	1.22	0.73	0.77	−0.68	−0.54	−0.69
3	3	0.66	0.56	0.69	0.98	1.62	1.42
4	1	−0.56	−0.57	−0.62	−0.44	−0.47	−0.46
4	2	0.42	0.33	0.33	0.44	0.66	0.66
4	2	0.84	0.85	0.89	0.00	−0.19	−0.16
4	4	−0.21	−0.05	−0.13	0.19	0.23	0.37
5	1	0.14	−0.08	−0.07	−0.17	−0.10	−0.06
5	2	0.27	0.63	0.58	−0.34	−0.23	−0.33
5	3	0.09	−0.52	−0.41	0.10	0.01	0.07
5	4	−0.49	0.26	−0.29	−0.26	0.06	0.21
5	5	−0.03	0.16	0.10	−0.67	0.59	−0.62
6	1	0.00	−0.05	−0.06	0.10	−0.03	0.02
6	2	−0.16	0.07	0.01	−0.16	−0.37	−0.40
6	3	0.53	−0.05	−0.05	0.05	0.03	0.01
6	4	−0.31	−0.04	0.01	−0.51	−0.52	−0.45
6	5	−0.18	−0.31	−0.30	−0.50	−0.46	−0.46
6	6	0.01	−0.04	−0.11	−0.23	−0.16	−0.21
7	1	0.13	0.20	0.15	0.09	0.16	0.08
7	2	0.46	0.36	0.26	0.06	0.16	0.11
7	3	0.39	0.25	0.29	−0.21	0.02	−0.17
7	4	−0.14	−0.15	−0.24	0.00	−0.10	−0.06
7	5	−0.06	0.08	−0.07	−0.19	0.05	0.02
7	6	−0.45	−0.21	−0.10	0.75	0.06	−0.05
7	7	0.09	0.06	0.01	−0.14	0.10	0.02
8	1	−0.15	−0.08	−0.09	−0.05	0.06	0.10
8	2	0.03	0.03	0.02	−0.04	0.04	0.00
8	3	−0.05	−0.04	−0.04	0.22	0.00	0.00
8	4	−0.07	−0.21	−0.17	−0.04	−0.01	−0.03
8	5	0.08	−0.05	−0.21	0.00	0.12	0.00
8	6	−0.02	−0.02	−0.29	0.67	0.32	0.19
8	7	0.17	−0.01	0.05	−0.07	0.03	−0.05
8	8	0.15	−0.25	−0.19	0.09	0.10	0.26

Within the framework of *Standard Earth III*, the completely normalized non-zonal coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ of the non-zonal harmonics up to $n = 25$ and $m = 14$ were calculated.

From merely scanning the table values one cannot assess the accuracy in determining these coefficients but one can appreciate as a particularly significant fact in determining the parameters of the Earth's gravitational field the similarity between the values in the systems $FR = 71$, *Wagner* and *Standard Earth III*, which endows the results with a high degree of confidence.

25.7.3 The Determination of the Geoid

The advantages offered by the combination of the gravity methods with the dynamic ones have led several authors to approach the problem of the determination, along with the coefficients of the expansion of the potential in spherical functions, of the geoid's global form.

In the year 1966, *M. Gaposchkin* determined, by photographic observations from 12 *Baker Nunn* stations, the gravity potential by means of an expansion in spherical functions up to the 8th order and in addition the global form of the geoid. These determinations were made within the framework of the *"Standard Earth I 1966"* programme. The precision of the geoid which was obtained (Fig. 25.8) is difficult to estimate; it should fit into the general precision in obtaining the geoid's undulations by dynamic methods, i.e. $\pm$ 7 m.

From the potential obtained by *M. Gaposchkin* one has calculated the average gravity anomalies within surfaces of $20° \times 20°$ and these values were compared with those obtained from measurements. The differences are of $\pm$ 6.6 mgal, which constitutes an encouraging result.

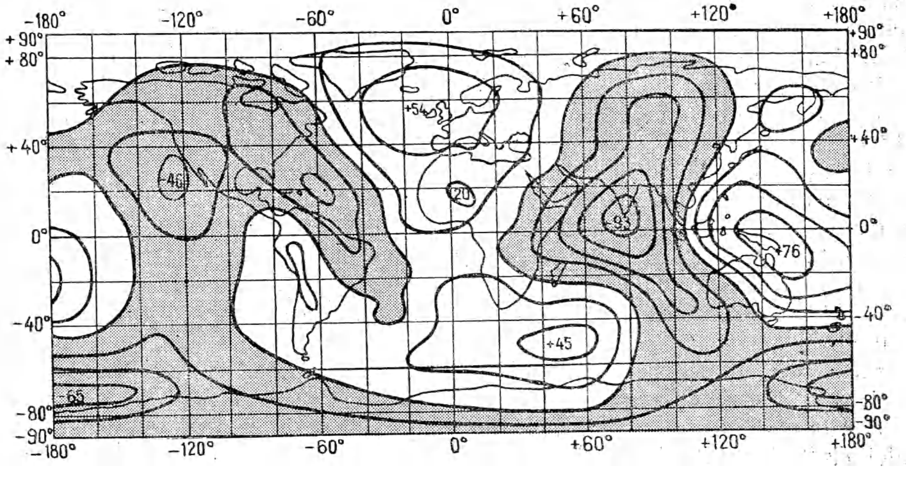


Fig. 25.8. The Geoid as Determined by M. Gaposchkin — 1966

Geoid determinations were also carried out by *M. Kaula*, using the combined dynamic and gravimetric method, *R. Anderle* (1966), utilizing Doppler observations, *Izsak* (1966), using photographic observations, and others.

In the year 1969, within the framework of the *"Standard Earth II"* programme, a new geoid, with respect to the reference ellipsoid 1: 298.255 was determined. The geoid's undulations correspond to the coefficients of the spherical harmonics given in Table 25.4. The errors in the geoid's undulations are estimated by the authors at $\pm$ 3—4 m.

Within the *"Standard Earth III"* programme, a new geoid (Fig. 25.9) was determined, which agrees very well with that determined in the year 1969 within the framework of the same programme. It should be emphasized that no differences appeared between the two geoids, either in the regional configuration or in the detailed one. The precision of the geoid determined in the year 1973 with respect to an average ellipsoid $f = 1: 298.256$ is estimated at $\pm$ 2.5 m.

One must also remember the determination in the year 1975 of the *GRIM-1* geoid (Fig. 25.10). The determination of this geoid is underlain by 140.000 direction and distance measurements for 10 satellites, the operations being carried out in cooperation by *Sonderforschungsbereich 78 Satellitengeodäsie der T.U. München* and *Groupe de Recherches Spatiales Toulouse.*

The undulations of this geoid were calculated with respect to the reference ellipsoid having as parameters: $GM = 398{,}601.3$ km^3 s^2; $a = 6{,}378, 155$ m; $f = 1: 298.255$ and their representation was made by level curves with a spacing of 10 m.

In 1976, through the cooperation of the same two institutes, there appeared *"The GRIM 2 Earth Gravity Field Model"*. The following types of data underlay the determination of this model *(Balmino et al. 1976):*

(a) satellite data consisting of laser, optical and interferometric measurements distributed into 60 arcs of 10 to 24 days. All the data used for determining the *GRIM 1* model were re-evaluated and additionally observational data concerning the satellites *D1C* and *BE-B*, those on the satellites *Péole*, *SAS* and *DIAL* (*Cazenave et al.* 1972) as well as all the data of laser measurements on the *French* satellite *Starlette* were utilized.

(b) the gravity data consisted of the 34,400 average values of the free-air anomaly calculated for areas of 1° × 1° by the *DEFENSE MAPPING AGENCY*. The gravity data were referred to the international geodetic system 1967.

Processing the available data yielded the completely normalized coefficients $\bar{C}_{lm}$ and $\bar{S}_{lm}$ up to $l = m = 30$. It is to be stressed that the coefficients of the zonal harmonics (up to $l = 23$) show a particularly good agreement with those determined within the framework of the *"Standard Earth III"* experiment. The coordinates of 38 terrestrial stations were also obtained.

As a continuation of the work carried out for determining the *GRIM 2* model, the *"Detailed Gravimetric Geoid for the North Atlantic"* was calculated, which was considered as the preliminary solution of the *GRIM 3* (*P.G.*3) model *(Albuisson et al. 1979)*. For the calculation of the undulations of a regional character one has utilized the spherical harmonics of the reference

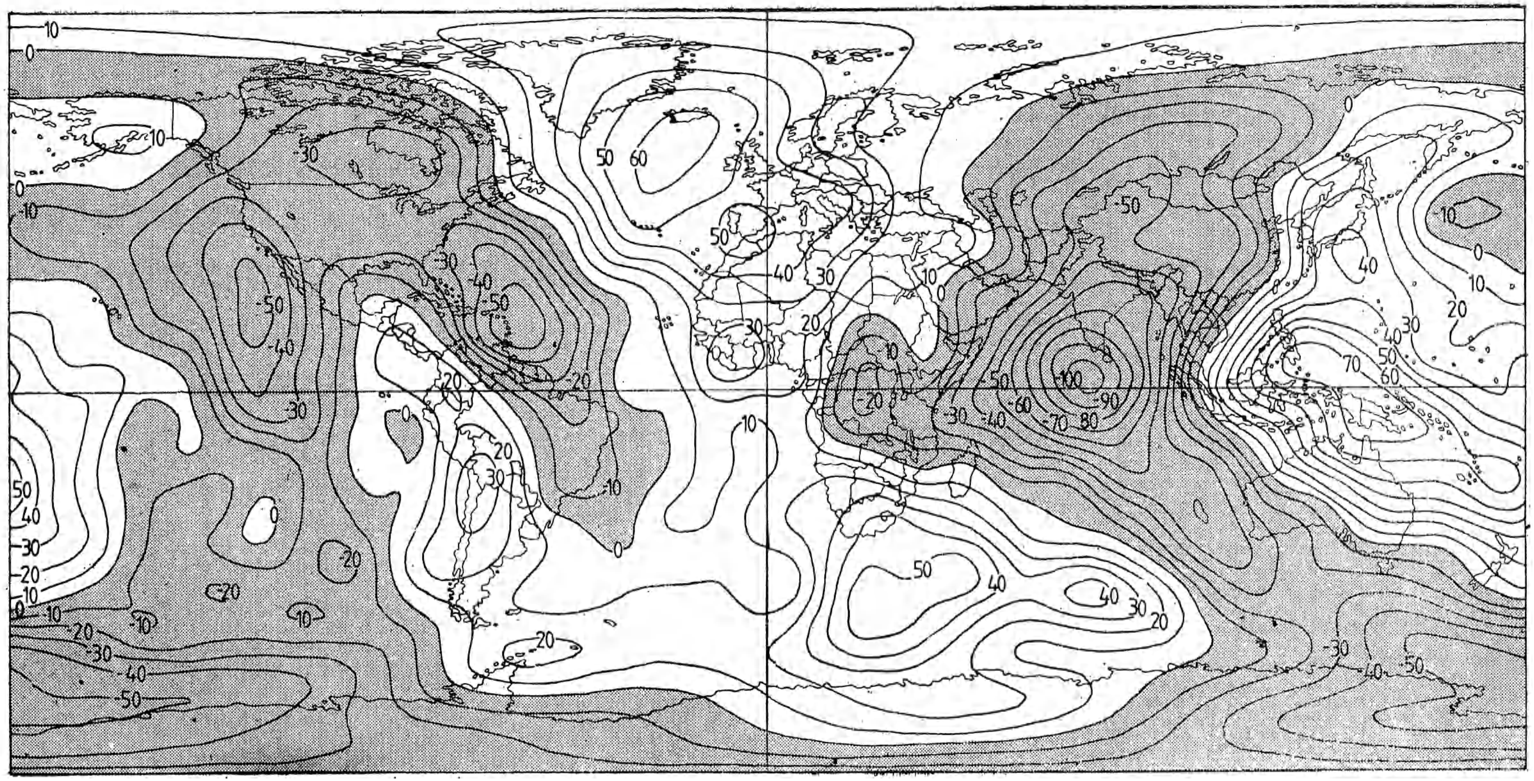


Fig. 25.9. The Geoid as Determined within the "Standard Earth III" — 1975 Programme

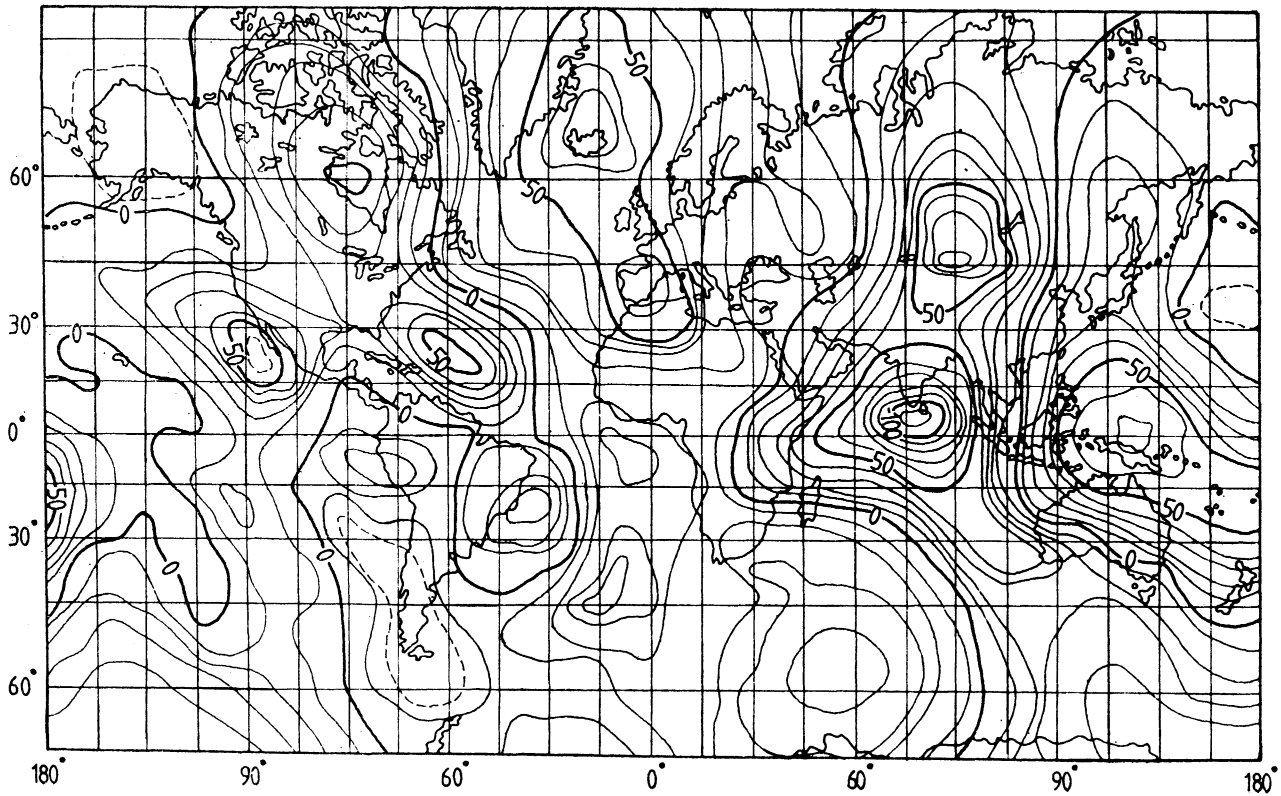


Fig. 25.10. The GRIM 1 — 1975 Geoid

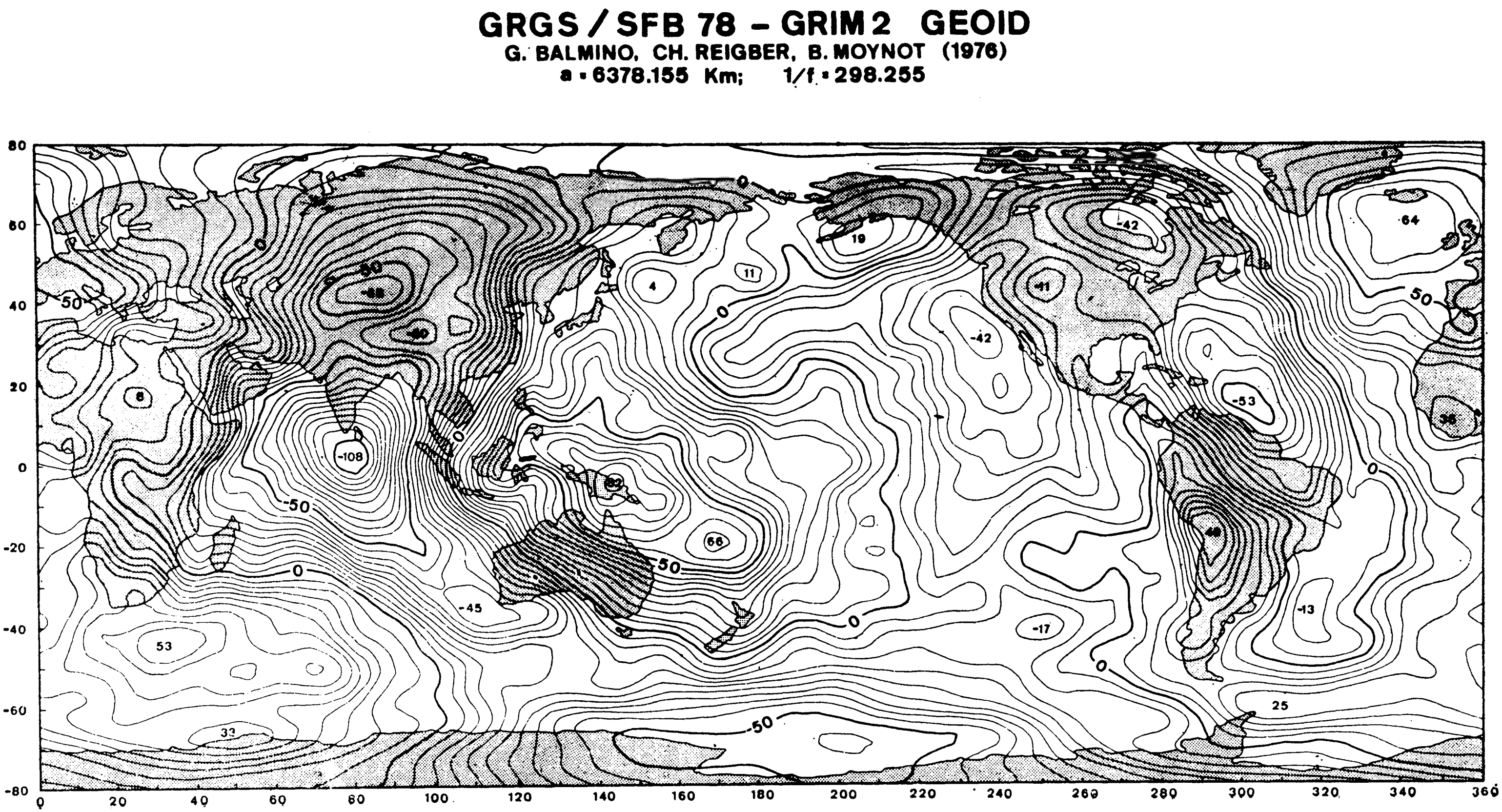


Fig. 25.11. The GRIM 2 Geoid

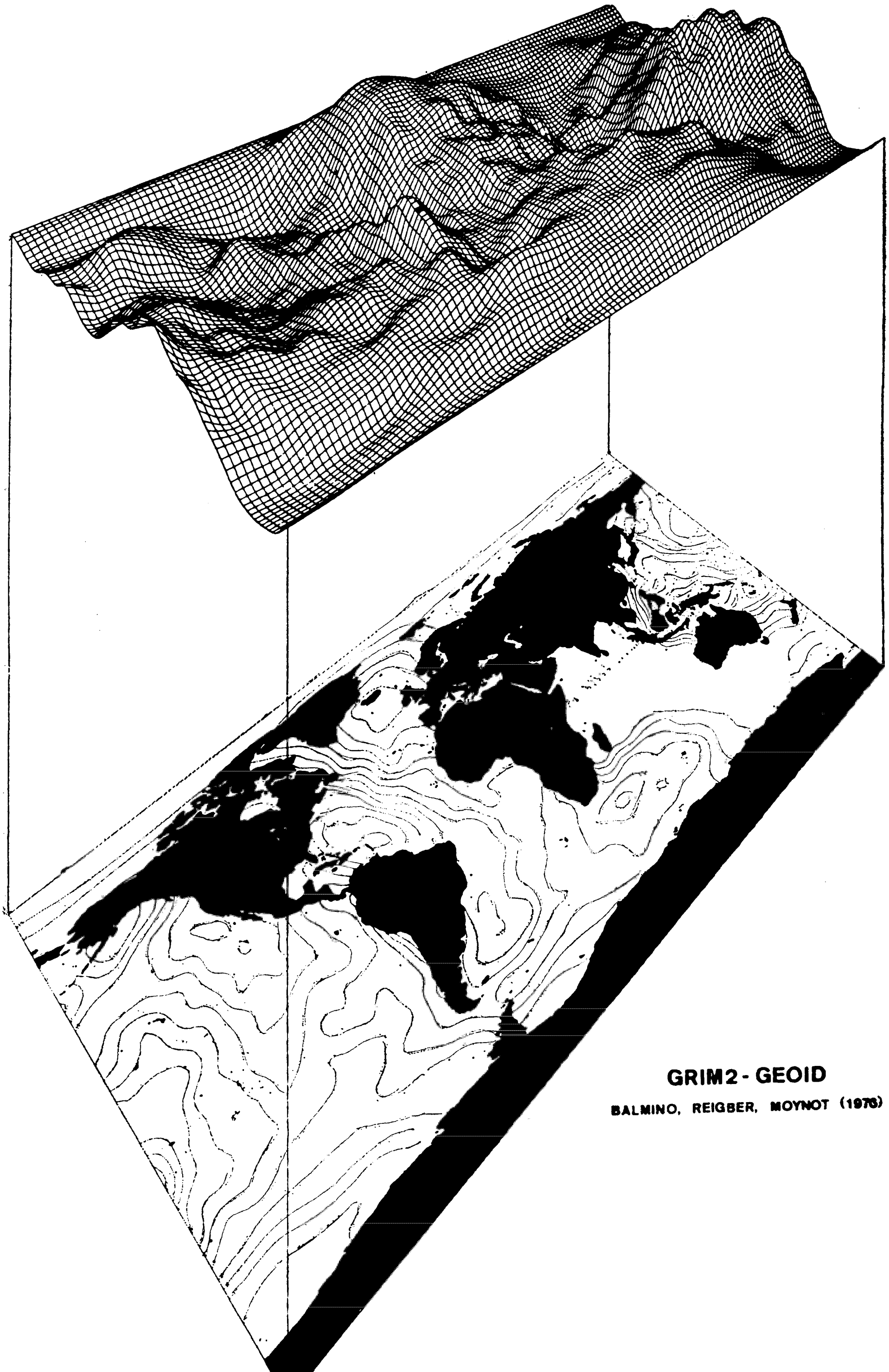


Fig. 25.12. The GRIM 2 Geoid

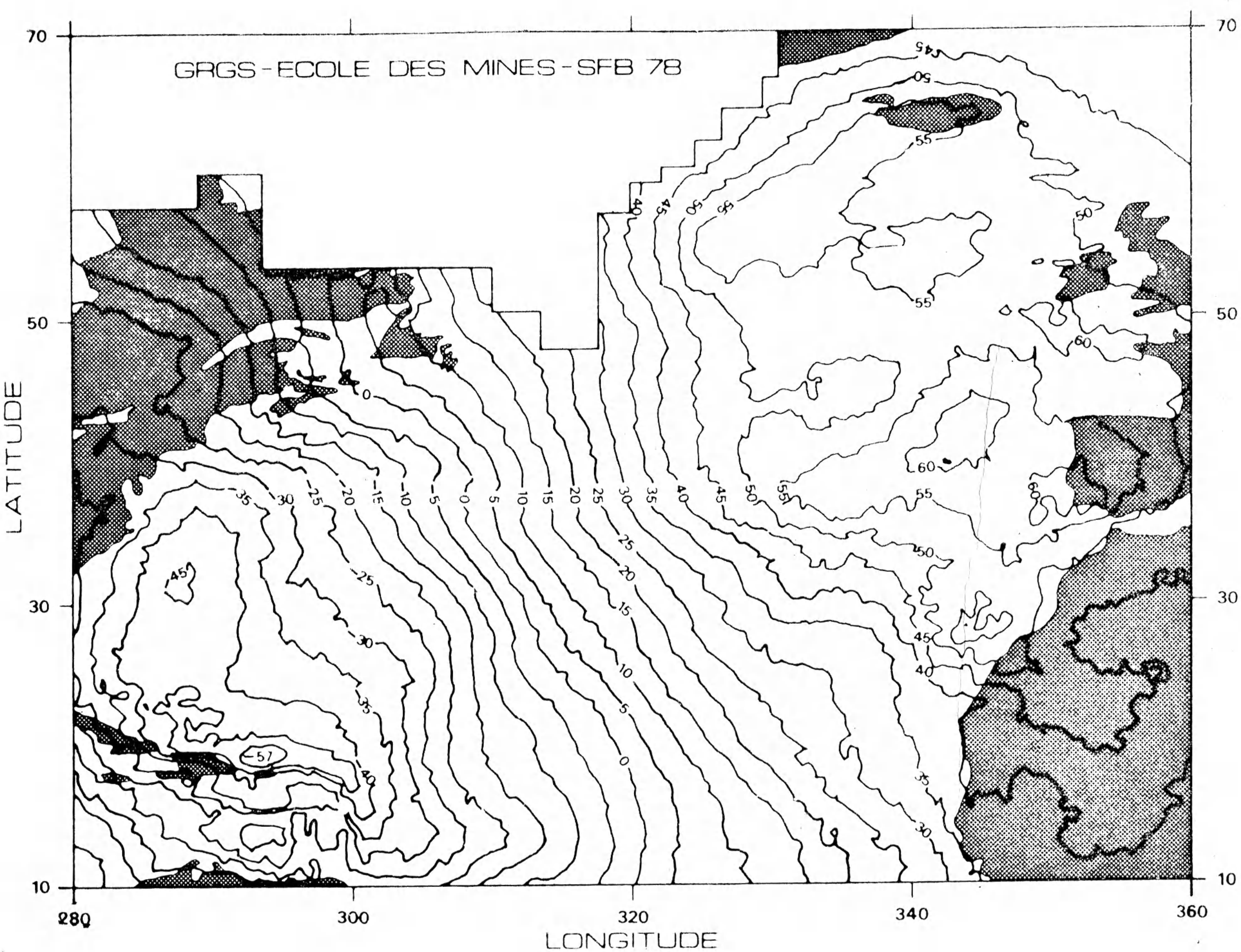


Fig 25.13. The North Atlantic's Geoid

model (up to $l = m = 30$), while those of a local character were obtained by integrating with the help of *Stokes* formula the residual anomalies as follows: for $0° \leqslant \psi \leqslant 5°$ one has utilized the average anomalies on areas of $4' \times 4'$ (245,000 such areas) and for $5° \leqslant \psi \leqslant 20°$ the average anomalies on areas of $1° \times 1°$ (35,180 areas).

For each point of the terrestrial surface the geoid's undulation was calculated by means of the relation:

$$\Delta N(\varphi, \lambda) = \Delta N_0 + \Delta N_1 + \Delta N_2,$$

in which:

$$\Delta N_0 = R \sum_{n \geqslant 2}^{N_{max}} \sum_{m=0}^{n} (\bar{C}^*_{nm} \cos m\lambda + \bar{S}^*_{nm} \sin m\lambda) P_{nm} (\sin \varphi);$$

$$\Delta N_1 = \frac{R}{4\pi\gamma} \iint_{0 \leqslant \psi \leqslant 5°} (\Delta g_1 - \Delta g_R)\, S(\psi)\, d\sigma;$$

$$\Delta N_2 = \frac{R}{4\pi\gamma} \iint_{5° \leqslant \psi \leqslant 20°} (\Delta g_2 - \Delta g_R)\, S(\psi)\, d\sigma.$$

Here the following notations were used:

$\bar{C}^*_{nm}, \bar{S}^*_{nm}$ difference between the harmoinic coefficients *P.G.3* and those of the chosen reference ellipsoid;
Δg_R field of the reference gravity anomalies *P.G.3*;
Δg_1 anomalies on the $4' \times 4'$ areas;
Δg_2 anomalies on the $1° \times 1°$ areas.

The preliminary constants utilized were $GM = 398{,}601.3$ kg^{3}s^2, $a_e = 6{,}378.155$ km, $f = 1:298.255$.

It is difficult to appreciate the accuracy of the geoid (Fig. 25.13) for the area of *The North Atlantic*. Nevertheless, a few parameters were estimated which may indicate some precision characteristics, viz.: the truncation error, which may reach 0.7 to 3.1 m, the error due to the choice of the reference field ΔN_0, which, by comparison with other regional values within the same area, shows a difference of about 5 m on the average etc.

On the whole, however, the geoid determined may be considered to have a fair accuracy, as is also confirmed by the comparisons which were made with satellite-altimetry work.

25.8 Satellite Altimetry

Satellite altimetry consists of the direct measurement, with the aid of an altimeter installed on the satellite and having a vertical orientation, of the distance down to the area under the satellite.

If one knows the positions of the satellite S (Fig. 25.14) and of the points under the satellite, one can determine, with the help of the radar-altimetry measurements of the altitude h, the geoid's undulation by means of the relation *(Groten* 1979*):*

$$\vec{\mathbf{N}} = \vec{\mathbf{r}} - \vec{\mathbf{R}} + \vec{\mathbf{h}}.$$

If one parametrizes the satellite's ellipsoidal altitude $h(a_0; b_{nk})$, the topography of the ocean surface $\tau(b_j)$ and the geoid's undulation $\zeta(b_i)$ one can

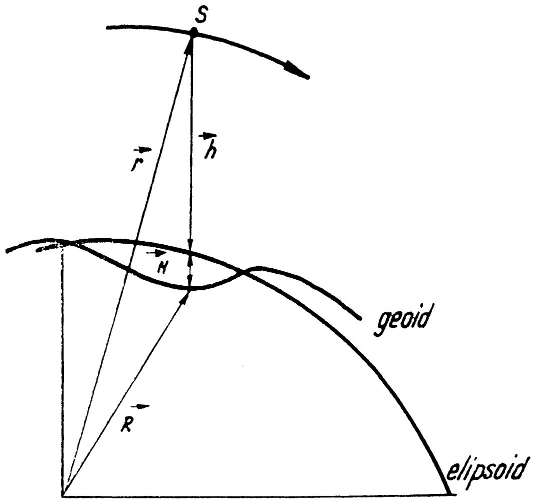


Fig. 25.14. Principle of Satellite Altimetry

obtain the equation of an altitude measurement in the form *(Lelgemann* 1979*):*

$$A + v = h(a_0; b_{nk}) - \tau(b_j) - \zeta(b_i) - \delta A - \delta h(\mathrm{d}\bar{x}),$$

in which:

A — altimetric observation;

δA — altimeter error;

$\mathrm{d}a$ — diference between the semi-major axes of the reference ellipsoid and of the ellipsoid corresponding to the geoid W.

Radar altimetry is mainly affected by three types of error: errors in the accurate determination of the satellite's coordinates on the trajectories covering the area being studied, errors due to the position of the terrestrial base points and errors due to the ocean topography.

The determination of the satellite's coordinates may be performed by purely geometrical procedures (§ 19.5), by dynamic procedures (Chapter. 25) or — the way used most — by combined procedures.

If one knows the perturbing forces affecting the satellite's motion (asymmetry of the gravitational field, luni-solar attraction, atmospheric drag), one can calculate the probable elements of the satellite's orbit at the moment t_i. The determination of the satellite's coordinates by using only the calculation of long arcs does not secure sufficient accuracy for satellite altimetry, for which reason one also considers the calculation of short arcs, securing a higher accuracy in the interpolation process within the framework of the geometrical method.

Since the Keplerian elements of the orbit are not suitable for calculations, one introduces the *Hill* orbital elements *(Izsak 1963)*, viz.: $\dot{r}$; $G = r^2 u$; $H = G \cos i = GI$, with r (radius vector); u (satellite's argument); Ω (longitude of the ascending node), and the equations of the motion are:

$$\frac{d\dot{r}}{dt} = \left(-\frac{\mu}{r^2} + \frac{G^2}{r^3}\right) + \frac{\partial R}{\partial r};$$

$$\frac{d\dot{r}}{dt} = \dot{r} - \frac{\partial R}{\partial r};$$

$$\frac{dG}{dt} = \frac{\partial R}{\partial u};$$

$$\frac{du}{dt} = \frac{G}{r^2} - \frac{\partial R}{\partial G} + \frac{I}{G}\frac{\partial R}{\partial I};$$

$$\frac{dI}{dt} = \frac{1}{G}\frac{\partial R}{\partial \Omega} - \frac{I}{G}\frac{\partial R}{\partial u};$$

$$\frac{d\Omega}{dt} = -\frac{1}{G}\frac{\partial R}{\partial I}.$$

The above-defined *Hill* elements have been considered as the most suitable in the studies carried out for the *SURGE* [1] experiment for determining the geoid in the *North Sea (Lelgemann 1979)*.

The coordinates of the base points must be known in the coordinate system being utilized in determining the satellite's position and using the same reference ellipsoid.

A particular complication in satellite altimetry is due to the topography of the planetary ocean. The separation of the temporal irregularities generated by tides and by meteorological factors is a problem which cannot be solved by geodesists alone. In addition, the different density of the oceans, due to large-depth currents changing the salinity, calls in question the hypothesis of the equipotential surface of the oceans (these being assumed to coincide with the geoid).

The satellite-altimetry studies are still in their infancy. From among the investigations contemplated for the future, great importance attaches to the project of performing and analysing the accuracy of the satellite altimetry worked out by the *"Seasat User Group of Europe"* *(SURGE)*.

The analysis methods for validating the satellite-altimetry experiment which were worked out by *SURGE* are *(Lelgemann 1979)*:

— Accurate calculation of the coordinates for all the observed orbits and of the base points located in the *North Sea* area.

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[1] *Seasat User Group of Europe*

— Analysis and contingent completion of the levelling in the analysed area.

— Calculation of the satellite's ellipsoidal altitude at the moment of carrying out the satellite altimetry, from accurate calculation of short arcs.

— Calculation of the geoid's undulations and separation of the radar-altimeter errors, as well as of the residual orbital errors.

— Calculation of the deflections of the vertical on the basis of the altimetric geoid, calculated as above, and the comparison of their values with the measurement data obtained at points in the *North Sea* area.

— Calculation of the astro-gravimetric geoid and, on its basis, of the deflections of the vertical and comparison with the altimetric data.

— Determination of the equipotential surfaces within the water masses.

25.9 Interpretation of the Results of Satellite Geodesy

Obtaining the coordinates of the observation satellites as well as the spherical functions of the potential by means of the orbital method is of particular importance in the study of the Earth's shape. The contribution of Dynamic Geodesy to specifying the Earth's shape will be briefly analysed in what follows.

The coordinates of a world network of observing stations allow one to correct and to connect the networks of local triangulation, calculated on different ellipsoids, into a unified coordinate system — the geocentric system. This has multiple practical and theoretical applications.

The geopotential's spherical functions allow the determination of the Earth's potential in the external space and therefrom of the gravity anomalies and of the geoid. Also, knowing the gravity anomalies, one can determine, with the aid of *Veining Meinesz*'s formula, the gravimetric deflections of the vertical.

The analysis of the main spherical functions may lead to significant conclusions concerning the Earth's figure.

The first spherical function J_2 is essential within the framework of the average terrestrial ellipsoid's flattening. The connexion between the spherical function J_2 and the flattening is obtained by means of the relation (3.60), in which, if one expresses the second eccentricity e' in terms of the flattening, one gets:

$$J_2 = \frac{C - \dfrac{A+B}{2}}{Ma^2} = \frac{2}{3}f - \frac{1}{3}m - \frac{1}{3}f^2 + \frac{2}{21}fm,$$

the quantities f and m being given by the formula (3.18).

From observations on the artificial satellites one can determine the value of the spherical function J_2 from which one may then determine the flattening of the mean terrestrial ellipsoid. The most exact value obtained in this way for the flattening is *(Arnold 1970)*:

$$f = 1{:}\ 298.255 \pm 0.005.$$

which corresponds to an error in the difference between the semi-major and the semi-minor axes of the ellipsoid of $\pm$ 0.3 m.

This manner of determining the flattening f has the advantage that it requires no hypothesis regarding the masses' disposition inside the Earth.

The sectorial spherical function J_{22} expresses the connexion between the two main moments of inertia A and B, whose axes lie in the equator's plane:

$$J_{22} = \frac{A - B}{4Ma^2} = -1.816 \cdot 10^{-6}$$

The analysis of the spherical function J_{22} has confirmed the fact that the terrestrial equator has not a circular form but an elliptical one. The flattening of the equator's ellipse f_e, obtained from analysing the sectorial spherical function J_{22}, is:

$$f_e = 1{:}\ 91{,}827,$$

facing the West longitude 15°.4. This means that the difference between the semi-major axis of the equator's ellipse (located approximately on the western shore of Africa) and the semi-minor axis is 69.5 $\pm$ 0.8 m.

The zonal spherical function J_3 is also very important for the study of the Earth's figure. It makes itself conspicuous by its magnitude:

$$J_3 = 2.565 \times 10^{-6},$$

compared with the subsequent zonal functions.

By analysing the global geoid's shape as determined by satellite methods, one has established a 15-meter rise at the *North Pole*, a depression in the zone of the northern middle latitudes of —5 m and again a 5-meter rise in the zone of the southern middle latitudes and a depression of —15 m at the *South Pole*. This is the so-called *pear shape* of the Earth, due to which the zonal spherical function J_3 has a palpably greater value than the subsequent zonal functions.

The geoid's asymmetry with respect to the equator (the pear shape) was a surprise for specialists, since thereby the hypothesis according to which Earth's normal figure should be that of a fluid mass in rotation is contradicted. New hypotheses were put forward concerning the causes of the geoid's asymmetrical shape, connected with the asymmetry of the topography and of the density of the terrestrial crust in the two hemispheres of the Earth.

26
The Geodetic Reference Systems

The processing of the geodetic measurements implies the definition of a geodetic-coordinate system on the surface of an ellipsoid, as well as knowledge of the gravity field for the chosen ellipsoid. The parameters defining the dimensions, the orientation and the gravity field for the chosen ellipsoid constitute the geodetic reference system.

The parameters of the reference ellipsoid are determined by means of one of the procedures described in the preceding chapters; starting from an arbitrary ellipsoid a_0, f_0, one determines, by processing the geodetic measurements, the corrections Δa and Δf for the passage to the ellipsoid which best approximates the Earth's form. The accuracy in obtaining the parameters a, f depends on the accuracy, the quantity and the distribution of the geodetic data over the Earth's surface. If for deducing these parameters one utilizes the geodetic measurements on a certain surface, a local reference ellipsoid will be obtained. The connexion of the triangulation networks processed on different reference ellipsoids may be achieved by referring these networks to a unique geodetic reference system.

The first geodetic reference system was adopted by the *International Association of Geodesy* in *Madrid* in the year 1924 and in *Stockholm* in 1930. This system was characterized by 3 defining constants:

$$a = 6{,}378{,}388 \text{ m}; \; f = 1:297.0;$$

$$\gamma = \gamma_E (1 + \beta \sin^2 \Phi + \beta_2 \sin^2 2\Phi) = \tag{26.1}$$

$$= 978.0490(1 + 0.0052884 \sin^2 \Phi - 0.0000059 \sin^2 2\Phi).$$

The first two constants were adopted by the *General Assembly of the International Association of Geodesy* in *Madrid* (1924) and the value of the normal gravity at the equator was adopted in *Stockholm* (1930).

Since the year 1930 the volume of geodetic measurements and their accuracy have increased considerably, so that the geodetic reference system 1924—1930 no longer fully satisfied the new requirements of Geodesy. Thus, by analysing the displacement of the ascending nodes and of apsis lines of the Earth's artificial satellites it was possible to establish that the value of the flattening f was unsatisfactory. It was, consequently, necessary to adopt a new reference system whose defining parameters should be determined with the corresponding accuracy and which should at the same time satisfy the needs of astronomy too. At the XIVth *General Assembly of the International*

Association of Geodesy (Lucerne, 1967), the following were adopted as **parameters of the Earth's figure and of its gravitational field:**

(1) *The Earth's equatorial radius a;*

(2) *The geocentric gravitational constant GM of the Earth, including the atmosphere;*

(3) *The factor of the Earth's dynamic form* J_2,

which correspond to the parameters adopted by the XIIIth General Assembly of the *International Astronomical Union.* **The numerical values of the adopted parameters are:**

$$a = 6{,}378{,}160 \text{ m}; \; GM = 398{,}603 \times 10^9 \text{ m}^3 \text{ s}^{-2}; \; J_2 = 10{,}827 \times 10^{-7}. \tag{26.2}$$

These constants underlie the derivation of the parameters of the general terrestrial ellipsoid as equipotential surface $U =$ const. and of its gravitation field. The normal potential U may be determined according to *Stokes'* theorem if one knows the semi-major axis a, the semi-minor axis b, the mass M and the rotational angular velocity ω, without making hypotheses concerning the distribution of the Earth's internal density. Another way of determining the normal potential is that utilizing the expansion of the normal potential into a series of zonal ellipsoidal harmonics in terms of the linear eccentricity E. The coefficients of this expansion are derived from the condition that the general terrestrial ellipsoid should be an equipotential surface. In this case, the normal potential is uniquely determined if one knows the semi-major axis a, the geocentric gravitational constant GM, the factor of the Earth's dynamic form J_2 and the rotational angular velocity ω.

The equipotential ellipsoid determined in this way serves for all geodetic purposes: as surface for processing the geodetic measurements, as reference surface for Satellite Geodesy and for Physical Geodesy, and as a base for calculating the normal gravity on the physical surface and in the external space.

On the basis of the constants a, GM, J_2 and ω one can determine all the geometrical and dynamic parameters of the reference ellipsoid. Thus, from the relation (3.56) one can obtain the eccentricity e^2 in the form:

$$e^2 = 3J_2 + \frac{2m\, e' e^2}{15 q_0}, \tag{26.3}$$

in which m and q_0 have the meanings given by (3.18) and (3.26) respectively. Considering in (3.18) $be' = ae$, one gets:

$$m = \frac{\omega^2 a^3 e}{GMe'}. \tag{26.4}$$

Substituting (26.4) into (26.3) yields:

$$e^2 = 3J_2 + \frac{2}{15} \frac{\omega^2 a^3}{GM} \frac{e^3}{g_0}. \tag{26.5}$$

Solving the equation (26.5) iteratively, one gets the eccentricity e on whose basis one determines the geometrical elements of the ellipsoid:

$$b = a\sqrt{1-e^2};\quad f = \frac{a-b}{a};\quad E = \sqrt{a^2-b^2};\ c = a^2/b. \tag{26.6}$$

The physical constants: normal potential U_0, gravitational potential V and normal gravity are determined on the basis of the quantities determined from the relations (26.6), with the help of the relations (3.18), (3.28), (3.36) and (3.37):

$$\begin{aligned}
U_0 &= \frac{GM}{b}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{e'^{2n}}{2n+1} + \frac{1}{3}m\right];\\
V &= \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty} J_{2n}\left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta)\right];\\
\gamma_E &= \frac{GM}{ab}\left(1 - m - \frac{m}{6}\frac{e'q_0'}{q_0}\right);\\
\gamma_P &= \frac{GM}{a^2}\left(1 + \frac{m}{3}\frac{e'q_0'}{q_0}\right).
\end{aligned} \tag{26.7}$$

For calculating the normal gravity one may utilize *Somigliana*'s formula (3.47):

$$\gamma = \frac{a\gamma_E\cos^2\Phi + b\gamma_P\sin^2\Phi}{\sqrt{a^2\cos^2\Phi + b^2\sin^2\Phi}}.$$

The XIVth *General Assembly of the International Union of Geodesy and Geophysics* has recommended the following formulae for the practical calculation of the normal gravity:

(1) Somigliana's formula expressed as a function of the eccentricity e:

$$\gamma = \gamma_E\frac{1 + k\sin^2\Phi}{\sqrt{1 - e^2\sin^2\Phi}}, \tag{26.8}$$

in which

$$K = \frac{b\gamma_P}{a\gamma_P} - 1. \tag{26.9}$$

(2) Pizzetti's formula:

$$\gamma = G_1\sqrt{1 + e'^2\cos^2\Phi} + \frac{G_2}{\sqrt{1 + e'^2\cos^2\Phi}}, \tag{26.10}$$

in which:

$$G_1 = \frac{a\gamma_E - b\gamma_P}{e'^2 b};\quad G_2 = \frac{b\gamma_P - a\gamma_E}{e'^2 b} \tag{26.11}$$

(3) The formula of the normal gravity expanded in series:

$$\gamma = 978.03185(1 + 0.005278895 \sin^2\Phi + 0.000023462 \sin^4\Phi), \quad (26.12)$$

whose error is of $\pm$ 0.004 mgal.

One can also utilize the formula (3.50), by adopting for f^* and f_4 the corresponding numerical values:

$$\gamma = 978.0318(1 + 0.0053024 \sin^2 \Phi - 0.000\,0059 \sin^2 2\Phi), \quad (26.13)$$

which secures, however, an accuracy of ± 0.1 mgal.

On the basis of (26.5) and (26.12) the main geometrical and physical constants of the general terrestrial ellipsoid were calculated, whose values are given in Table 26.1.

Table 26.1. *Geometrical and Physical Constants of the General Terrestrial Ellipsoid*

Constant's name	Notation	Formula	Value
Semi-major axis	a	—	6,378,160 m (exact)
Semi-minor axis	b	$a\sqrt{1-e^2}$	6,356,774.5161 m
Linear eccentricity	E	$\sqrt{a^2-b^2}$	521,864.6732 m
Polar curvature radius	c	$\dfrac{a^2}{b}$	6,399,617,4290 m
First eccentricity	e	—	$e^2 = 0.006\,694\,605\,32856$
Second eccentricity	e'	$\dfrac{e}{\sqrt{1-e^2}}$	$e'^2 = 0.006\,739\,725\,12832$
Flattening	f	$\dfrac{a-b}{a}$	0.003 352 923 71299
Flattening inverse	f^{-1}	$\dfrac{a}{a-b}$	298.247 167 427
Quarter of meridian	Q	$c\int_0^{\pi/2} \dfrac{d\Phi}{(1+e'^2\cos^2\Phi)^{3/2}}$	10,002,001.2313 m
Mean radius	R_1	$\dfrac{2a+b}{3}$	6,371,031.5054 m
Radius of the sphere with the same surface	R_2	$c\left[\int_0^{\pi/2} \dfrac{\cos\Phi}{(1+e'^2\cos^2\Phi)^2}\, d\Phi\right]^{1/2}$	6,371,029.9148 m
Radius of the sphere with the same volume	R_3	$3\sqrt{a^2b}$	6,371,023.5234 m
Geocentric gravitational constant	GM	—	$398\,603\cdot 10^9 \text{m}^3\text{s}^{-2}$(exact)

Table 26.1 (continued)

1	2	3	4
Angular rotational velocity	ω	—	7.292 115 1467·10^{-5} (rad/s (exact)
Normal potential on the ellipsoid	U_0	$\frac{GM}{b}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{e'^{2n}}{2n+1}+\frac{1}{3}\right]$	6,263,703.0523 kgal.m
Spherical armonics coefficients	J_2	$J_{2n}=(-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\times\left(1-n+5n\frac{c-A}{ME^2}\right)$	0.001 0827 (exact)
	J_4		−0.000 002 371 264 40
	J_6		0.000 000 006 08516
	J_8		−0.000 000 000 01428
—	m	$\frac{\omega^2a^2b}{GM}$	0.003 449 801 43430
Normal gravity at the equator	γ_E	$\frac{GM}{ab}\left(1-m-\frac{m}{6}\frac{e'q_0'}{q_0}\right)$	978.031 84558 gal
Normal gravity at the pole	γ_P	$\frac{GM}{a^2}\left(1+\frac{m}{3}\frac{e'q_0'}{q_0}\right)$	983.217 72792 gal
Gravimetric flattening	f^*	$\frac{\gamma_P-\gamma_E}{\gamma_E}$	0.005 302 365 52330

In defining the geocentric gravitational constant, the mass M has been regarded as the Earth's total mass, the atmosphere included. The derivation of the relations (26.7), as indeed the entire theory of Physical Geodesy, is underlain by the hypothesis according to which the normal potential is a harmonic function outside the ellipsoid, which implies the absence of any mass in the exterior of the latter. If one took into consideration the effect of the atmosphere's mass as well, then one would have to modify in a very complicated manner the formulae of the potential and of the normal gravity.

The influence of the atmosphere's mass being very small, a modification of the formulae of the normal gravity and potential has no practical justification. It has been agreed that the potential theory should be applied to the case of an ellipsoid without atmosphere but whose mass comprises the Earth's entire mass, the atmosphere included. If one wishes to take into consideration the influence of the atmosphere, then corrections can be applied directly to the gravity's measured values. In this way, the atmosphere's influence is taken out of the geodetic reference system and placed in the domain of measurement. The following relation is recommended for calculating the influence of the atmosphere on gravity:

$$\delta g = \frac{GM_{(atm)}}{\sqrt{(b^2 + E^2)\left(b^2 + \frac{1}{2}E^2\right)}} . \tag{26.14}$$

On the basis of this relation, one calculates the value δg as a function of the altitude H. This value varies between 0.87 mgal for $H = 0$ and 0.01 mgal for $H = 32$ km.

The gravity anomalies calculated in the 1967 reference system will differ from those calculated in the 1930 system for two reasons:

(a) *The passage from Cassinis's ellipsoid to the reference ellipsoid 1967 introduces a difference:*

$$\gamma_{1967} - \gamma_{1930} = -(17.2 - 13.6 \sin^2 \Phi) \text{ mgal}; \tag{26.15}$$

(b) *The modification of the absolute value of gravity at Potsdam from 981.274 gal to 981.260 gal.*

In this manner, the correction which must be added to the anomaly calculated with *Cassinis*'s formula in order to pass to the anomaly in the new system is:

$$\Delta g = (3.2 - 13.6 \sin^2 \Phi) \text{ mgal}.$$

These corrections are tabled as functions of the latitude of the measured point.

Recognizing that *the Geodetic Reference System 1967* no longer represents the size, shape and the Earth's gravity field with sufficient accuracy, the *International Union of Geodesy and Geophysics* has recommended at its *XVIIth General Assembly*, which took place at Canberra in 1979, the introduction of the **Geodetic Reference System 1980.**

The Geodetic Reference System 1980 is equally underlain by the theory of the geocentric equipotential ellipsoid. The calculation formulae for this reference system are also the same as those used for calculating the 1967 System.

The parameters of *the 1980 System* recommended by *the International Union of Geodesy and Geophysics are*:

— *the Earth's equatorial radius*:

$$a = 6{,}378{,}137 \text{ m};$$

— *the geocentric gravitational constant* (including the atmosphere):

$$GM = 3{,}986{,}005 \times 10^8 \text{ m}^3\text{s}^{-2};$$

— *the dynamical form factor of the Earth* (excluding the deformations due to the earth tides);

$$J_2 = 108{,}263 \times 10^{-8}$$

— *the Earth's angular rotational velocity*:

$$\omega = 7{,}292{,}115 \times 10^{-11} \text{ rad s}^{-1}.$$

The semi-minor axis of the reference ellipsoid will be parallel to the direction defined by the *Conventional International Origin* and the first meridian will be parallel to the zero meridian adopted by the *B.I.H.*

Sixth part

Determination of the Recent Movements of the Earth's Crust

Geodesy, whose object is the study of the Earth's shape and size as well as of its gravity field, aims at determining not only the parameters defining these important features of our planet but also *the temporal modifications affecting these quantities.* The phenomena lying within the scope of this complex of problems, which most frequently turn out to be mutually interdependent in a significant way, may be summed up (*Stange 1976*) as follows:

– secular modifications of the Earth's figure (form and dimensions) as well as of the gravity field;

– modifications in the Earth's rotational motion and velocity;

– the Pole motion;

– the Earth tides;

– movements of the Earth's crust.

The above-mentioned phenomena constitute the object of investigation of **Geodynamics,** defined (*Melchior 1971*) as the dynamics of the Earth – Moon system (each of these bodies regarded as undergoing deformation), as well as of the artificial satellites, in their capacity as indicators of the acting forces. Along with other disciplines, such as e.g. astronomy, geology, geophysics, geomorphology, oceanography among others, geodesy makes an important contribution to studying the Earth's dynamic properties by carrying out repeatedly and at certain time intervals geodetic measurements of high accuracy and wide range. Therefore, the studies performed in this field are sometimes also called studies of **four-dimensional geodesy,** thereby being understood the association of a fourth coordinate, namely the time, with the three coordinates with which one usually operates in geodesy.

A few contributions of Geodesy within the framework of geodynamics have already been mentioned in previous chapters of the present treatise. Taking into consideration the great volume of geodetic work now being carried out in the field of the determination of recent crustal movements, as well as their importance, both scientific and of a practical nature, the last part of this book addresses itself particularly to these problems, at the same time also discussing other aspects of the participation of Geodesy within the framework of geodynamics programmes.

As was shown in Chapter 1 (Figs. 1.1, 1.2 and 1.3), *the Earth's crust* is represented by a layer of variable thickness (on the average 30 km), deeper in the mountainous areas, where it may reach values up to 60 km and much thinner under the water-covered zones. Together with the upper part of the Earth's mantle, the crust constitutes the *lithosphere* (totalling approximately 100 km), a complex zone from the physical-chemical point of view but accessible to some important investigation methods: seismic, electrometric, magnetic etc.

Under the influence of a multitude of factors, both internal and external to the lithosphere, with an older or more recent action, the Earth's crust undergoes spatial displacements varying as to extent and intensity. The determination of the recent movements of the Earth's crust represents one of the important research objects for several branches of science and technology, which has led, particularly during the last two decades, to the initiation of comprehensive programmes of interdisciplinary observations, studies, processings and interpretations.

Within the framework of this complex of preoccupations, each of the involved disciplines pursues not only the general common objectives but also particular objectives of their own, utilizing special working methods to this end. The results which are obtained are used for scientific purposes as well as for technical studies for locating, designing and exploiting public works, mines etc.

A principal feature of the determinations of the Earth's crust movements by means of geodetic methods is represented by the high accuracy, in comparison with that which can be obtained by the methods utilized by other disciplines. At the same time one must not forget the fact that Geodesy can determine only the recent movements of the Earths' crust, viz. those displacements which have occurred during the time interval between two or more epochs of geodetic measurements, whereas other disciplines can estimate approximately, on the basis of hypotheses and specific methods of investigation, crustal movements which have occurred within much greater time intervals, of the order of millennia and even of millions of years.

One must also mention the fact that Geodesy most frequently emphasizes the displacements manifesting themselves at the upper part of the Earth's surface, these not always being identical with the crustal movements proper, which take place at some depth inside the Earth.

Geodesy is directly interested in studying the transformations occuring in the interior of the Earth's crust, since these lead to modifications in the gravity field which in their turn generate changes in the geoid's form. Using the means at its disposal, which will be examined in the following chapters, Geodesy is able to determine not only the crustal movements but also their consequences, realized by terrain displacements, both vertical and horizontal, beginning with some areas of limited extent and ending with continents in their entirety.

The modifications induced in the geoid's form as well as the modifications in the spatial position of some parts of the physical surface constitute the main arguments for the inclusion, in recent times, of the investigations of the crustal movements in the category of geodetic preoccupations in the domain of the study of the Earth's figure.

27

Geodetic Methods for Determining the Recent Movements of the Earth's Crust

27.1 Introductory Remarks

In order to determine the recent movements of the Earth's crust, Geodesy utilizes several specific observing, processing and interpreting methods. Some of these (gravity methods, triangulation, trilateration, trigonometric levelling) were tackled in the other parts of the book, so that in this chapter we will give a more thorough treatment of high-precision geometrical levelling as the main method for determining the vertical displacements. For reasons of unity of presentation alone, brief references will be made to other geodetic determination methods too, and in the last two chapters both the processing and the presentation of the results will include nearly all the geodetic methods presently being used.

27.1.1 Factors Contributing to the Movements of the Earth's Crust

A displacement of a portion of the Earth's crust may be caused, most frequently, by a multitude of factors looked in a complex interdependence, so that their separation is difficult to achieve. Just in order to facilitate the presentation, a few main causes leading to displacements of the Earth's crust will be distinctly tackled, in what follows.

(1) *As a consequence of the modifications which occurred after the last glacial stage large areas of the Earth's surface were totally and suddenly freed from the pressure due to the weight of ice, which led to modifications of the equilibrium state in which the Earth's crust was with respect to the Earth's mantle.* The process of restoration of the equilibrium state is proceeding slowly, with a speed depending on the viscosity of the magma, which determines vertical displacements of the terrestrial crust over large areas, and appears to have a continuous and relatively uniform character.

(2) *One of the relatively recent hypotheses, known under the name of plate tectonics or global tectonics theory justifies plausibly enough the macro-movements of the Earth's crust.* According to this theory, the lithosphere, bearing on it the continents as well as the oceans, consists of a number of mobile plates

floating on the asthenosphere. Unlike the lithosphere, which consists of rigid layers (sedimentary — at the upper part, basaltic — in all areas and granitic in the continental areas), the asthenosphere is formed by a material of a viscous nature.

Being a hypothesis, confirmed so far by geophysical methods, the number of lithospheric plates varies from one author to another but in something between 6 and 11. In Fig. 27.1 there is presented one of the variants accepted in the geodetic literature *(Koch 1973)*, in which the probable displacement direction of the lithosphere's tectonic plates is emphasized. Both the direction and the magnitude of these displacements are different. The hypothesis of global tectonics does not rule out the possibility of the notion of *tectonic subplate (micro-plate)*, by means of which one can explain some phenomena of the Earth's crust of a dynamic nature over territories of a smaller extent. Due to the displacements of the lithosphere, the following phenomena may take pace:

— *In the oceanic areas*, where the tectonic plates have divergent movements, the magma forces itself from the asthenosphere towards the surface forming by solidification new portions of terrestrial crust, viz. the ridges on the ocean bottom (chains of submarine mountains which, with a total length of $\approx$80,000 km, greatly exceed the total length and area of the chains of continental mountains).

— *In the case in which two tectonic plates have reciprocally convergent movements*, one of the plates slides beneath the other, partially melting in the asthenosphere. As a consequence, the phenomenon produces the dissappearance of a few portions of terrestrial crust, being accompanied by earthquakes, either intermediate ones (at depths of 60—300 km) or deep ones (at depths of 300—700 km).

— Finally, *the reciprocal plate slippings in the form of frictions*, as a result of which parts of the Earth's crust either don't appear or disappear are accompanied by the occurrence of transcurrent faults (e.g. the *San Andreas* fault in *California*) and of shallow earthquakes.

The lithospheric tectonic plates are not plane but have the form of spherical caps. Hence their displacements over the asthenosphere form a rotational motion around a pole called a *motion pole*, which is not to be confused with the geographical or rotational pole of the Earth.

The movements of the tectonics plates also generate in this way the movements of the continents (known as *continental drift*) as well as the macro-modifications in the structure of the gravity field and, consequently, in the geoid's form.

(3) Another category of factors producing displacements of the Earth's crust is represented by *the action of the volcanoes*, displacements which are, as a rule, of a local character.

(4) *The modifications occurring in the structure of the upper part of the terrestrial crust, such as those due to certain natural factors* (changes in the level of the water-table, solutions of the component chemical substances etc., as well as transformations produced by such human activities as mining exploitations, great hydro-energetic complexes etc.) may generate such local displacements of the Earth's crust.

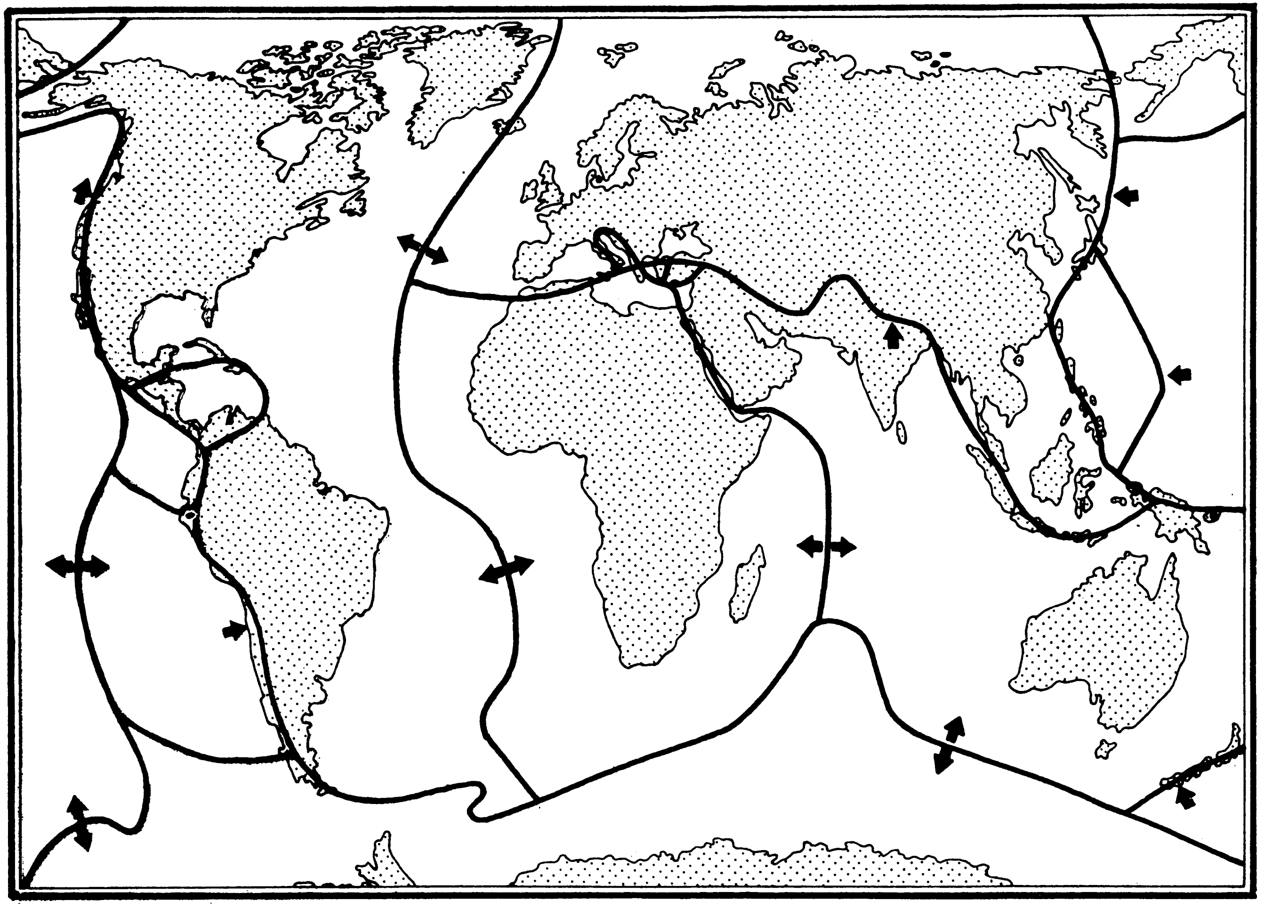


Fig. 27.1. Tectonic Plates of the Lithosphere and Their *Probable* Displacement Direction (*Koch 1973, Kolaczek and Wilson 1973*)

(5) *The sea and ocean tides* determine displacements of the terrestrial crust, particularly along the coasts.

(6) Although having a more restricted sphere of action, *the influences of the atmosphere* (modifications in the air pressure, the systematic action of the wind etc.) may also be mentioned as second-order factors generating displacements of the Earth's crust.

27.1.2 Classification of the Geodetic Determinations of the Movements of the Earth's Crust

In the technical literature one at present finds several categories of geodetic methods by means of which one may determine the movements of the Earth's crust and which can be classified from several points of view.

Classification with respect to the reference system. Depending on the reference system to which the geodetic determinations of the Earth's crust are referred, the following types of classifications are used:

(1) *Absolute* and, *relative determination* respectively.

(2) *Determinations of horizontal* and, *vertical movements* respectively.

(3) *Four-dimensional determinations.*

Although the displacement of a point situated on the Earth's surface is a *spatial motion*, geodetic determination methods cannot accomplish in all situations a complete description of this motion, i.e. they don't always have a *four-dimensional character*. It is necessary to mention, from this point of view, that up to now the determinations of the vertical movements have prevailled compared with the determinations of the horizontal movements.

The **absolute displacements** may be defined as modifications of the position of the geodetic points recorded within the framework of the astronomical coordinate system Φ, Λ, H (Section 3.4). In such a system there is the possibility of completely describing the spatial temporal variations of the positions of the geodetic points, i.e. the determinations have a four-dimensional character. But from practical considerations, derermined not only by the possible accuracy of the geodetic operations but also by their cost, there is a consensus to distinguish and also to determine practically the following two categories of displacements:

— *Absolute vertical displacements*, defined as temporal modifications of the position of a network of geodetic points along the lines of force passing through these points;

— *Absolute horizontal displacements*, defined as temporal modifications of the positions of the geodetic points with respect to the meridians and parallels passing through these points, on the assumption that they remain in a fixed position.

The definition of the absolute displacements is based on the hypothesis that the position of the coordinate system is maintained constant in time. Actually, however, due to multiple causes (such as e.g. the polar movement) the specified hypothesis only holds within certain limits. Consequently, in order to maintain the necessary rigour in the given definitions, one needs to make a distinction between the displacements proper of the geodetic points and the variations which intervene in the position of the coordinate system itself. These aspects complicate very much the practical programmes of determinations so that, one is forcibly led to the following two solutions:

— one introduces *total separations* as regards the determinations of the temporal variations occurring in the reference system, on the one hand, and of the modifications in the position of the geodetic points, on the other hand, in the sense that each category of displacements is determined separately, ignoring the other category;

— one resorts to *iterative solutions*, in which one takes into consideration successively the reciprocal influence of the two categories of displacements, at every new stage of determination.

The mere illustration of the accuracy necessary for the geodetic determinations of absolute displacements, can be seen e.g. from the following reasoning, in which we can also accept some approximations which do not effect the final conclusions. For the present purpose, the lines of force could also be sufficiently well approximated by the normals to the ellipsoid which pass through the geodetic points. Taking into consideration the magnitude of the normal to the ellipsoid (Table 9.3) it follows that absolute displacements of the order of one decimeter need a relative determination accuracy of the order of 10^{-8}. Conversely, in the situation when the repeated geodetic measurements, carried out in point networks which would cover the Earth's surface as completely as possible, achieved a relative accuracy of determination of the order of 10^{-8} there would exist the possibility of determining variations of the order of a decimeter in the magnitude of the parameters describing the form and the dimensions of the Earth (e.g. variations of the semi-axes a and b of the reference ellipsoid). This performance has not yet been achieved at present for two main reasons:

— the geodetic, astro-geodetic and gravimetric networks cover the Earth's surface only partially. Unfortunately, there is not a unanimous consensus as regards the exchange of information and of numerical data, without which the enunciated problem remains practically insoluble;

— the position of the geodetic points in such *global networks* is not yet determined with the accuracy stated as necessary (Chapters. 19, 20 and 21), although the possibilities are more and more promising from this point of view.

From considerations of a practical nature, one also faces situations in which by absolute vertical displacement one understands the modification of the altitudes of the points situated on the terrestrial surface with respect to mean level of any sea. This last definition is, however, an approximate one since, due to extremely complex causes, the mean level of every sea adjust itself with respect to the Earth's mass centre.

The relative displacements are, in most of situations, a local character, being currently defined as follows:

— *The relative vertical displacements* at points situated on the terrestrial surface are represented by modifications of their altitudes with respect to the level surface passing through one of these points, whose position is regarded as non-modified;

— *The relative horizontal displacements* are represented by modifications of the coordinates of the geodetic points with respect to a point considered as fixed. Currently it is from this point that one begins to carry out the geodetic observations (in several series), keeping fixed the orientation of a direction and the length of a side (the so-called *"free network of the conventional type"*).

It must be mentioned that during recent decades more and more geodetic determinations of displacements of points situated on the Earth's surface have been derived by applying the theories of processing "free networks" in which none of the network's points, nor any other element of the network, were considered as fixed (Section 15.5). In such situations the relative character of the determinations lies in the manner of obtaining the provisional coordinates with respect to which one afterwards gets, by the optimization process implied by such a processing, the final coordinates of the point network.

Classification with respect to the volume of the determinations of the recent crustal movements. From this point of view one distinguishes:

(1) *Global determinations* (when one envisages the entire terrestrial surface) realized by world geodetic networks.

(2) *Continental determinations*, realized by international or national networks for large territories.

(3) *Local determinations*, on smaller territories, being of particular interest from the scientific or technical-economic point of view.

Classification with respect to the accuracy of the geodetic determinations. The accuracy in determining the recent movements of the Earth's crust may also constitute a classification criterion. Thus, if the position of any point P situated on the terrestrial surface can be determined by geodetic methods with the mean "a posteriori" error: $\pm m_P$, one can speak of a modification of the position of this point by a quantity ΔD_P, between two determination series, only when:

$$| \Delta D_P | > t_k \cdot | m_P | \tag{27.1}$$

where the coefficient $t_k = t(S, f)$ depends on the statistical reliability S with which one operates (usually $S = 95\%$) and on the number of additional observations f existing in the geodetic network in the two series of determinations. Depending on S and f, one can extract from the tables of the *Student distribution* the coefficient t_k (for the most frequent cases met with in practice, t_k lies between 2 and 3).

In this manner, the Student statistical test, which has found an application almost unanimously recognized in the determinations of the movements of the Earth's crust by means of geodetic methods, (*Wolf 1968, Böhm 1965, Ghiţău 1970*) distinguishes these determinations, qualitatively, setting them in order according to both the accuracy which can be obtained by a certain working method and the magnitude of the displacement being determined.

27.2 Repeated Geometrical Levelling

The repeated geometrical levelling so far represents the main geodetic method for determining the recent movements of the Earth's crust, as regards both the volume of determinations and the accuracy of the results obtained. The utilization of the repeated levelling leads to the determination of the vertical (absolute and relative) movements of the Earth's crust within the global, continental and local networks.

27.2.1 Constructing the Geometrical Levelling Networks

The layout of the networks of higher-order repeated geometrical levelling is carried out in cooperation with specialist geologists-geophysicists, particularly as regards the delimination of the zone for locating the bench marks, so that their vertical displacements should correspond to the actual movements of the Earth's crust, associated harmful influences (unstable areas, undergoing landslides or soil erosion, possibilities of bench mark deterioration etc.) being eliminated as much as possible.

The 1st-order levelling network of the *Socialist Republic of Romania*, carried out by the "*Direcția Topografică Militară*" (*DTM*) in the period 1960—1971 was presented in Fig. 3.7 and technical details concerning the bench marks and the levelling marks used for its realization are given in Fig. 27.2.

The lines of geometrical levelling are laid out along railways and main roads. In *Romania*, the realization of the 1st-order levelling network was made by means of reinforced-concrete levelling bench marks of the types I, II and III and by cast-iron levelling marks of the types *A* and *B* (Fig. 27.2).

The *Socialist Republic of Romania* has stuck to the comprehensive project of carrying out a levelling connexion between the *Baltic Sea* and the *Black Sea*, in which the interested *European* socialist countries are taking part. The responsible body is the "*Institutul de geodezie, fotogrammetrie, cartografie şi organizarea teritoriului*" (*IGFCOT*), which has worked out the layout of this work (Fig. 27.3). The main levelling line crosses the *Romanian* territory from *Constanța* to *Oradea*.

After carrying out studies on the possibility of improving the realization of the levelling points, particularly in areas covered by Quaternary formations made up of silt, loess, contractile clay, the *IGFCOT* specialists have reached the conclusion that there is a need to manufacture a depth bench mark, which is represented in Fig. 27.4. The planting depths vary and may reach up to 25—30 m, depending on the actual geological characteristics of the soil.

Every higher-order levelling network is connected to a so-called fundamental (origin) zero point whose position is generally considered as non-modifiable in time. The higher-order repeated levelling in *Romania* is

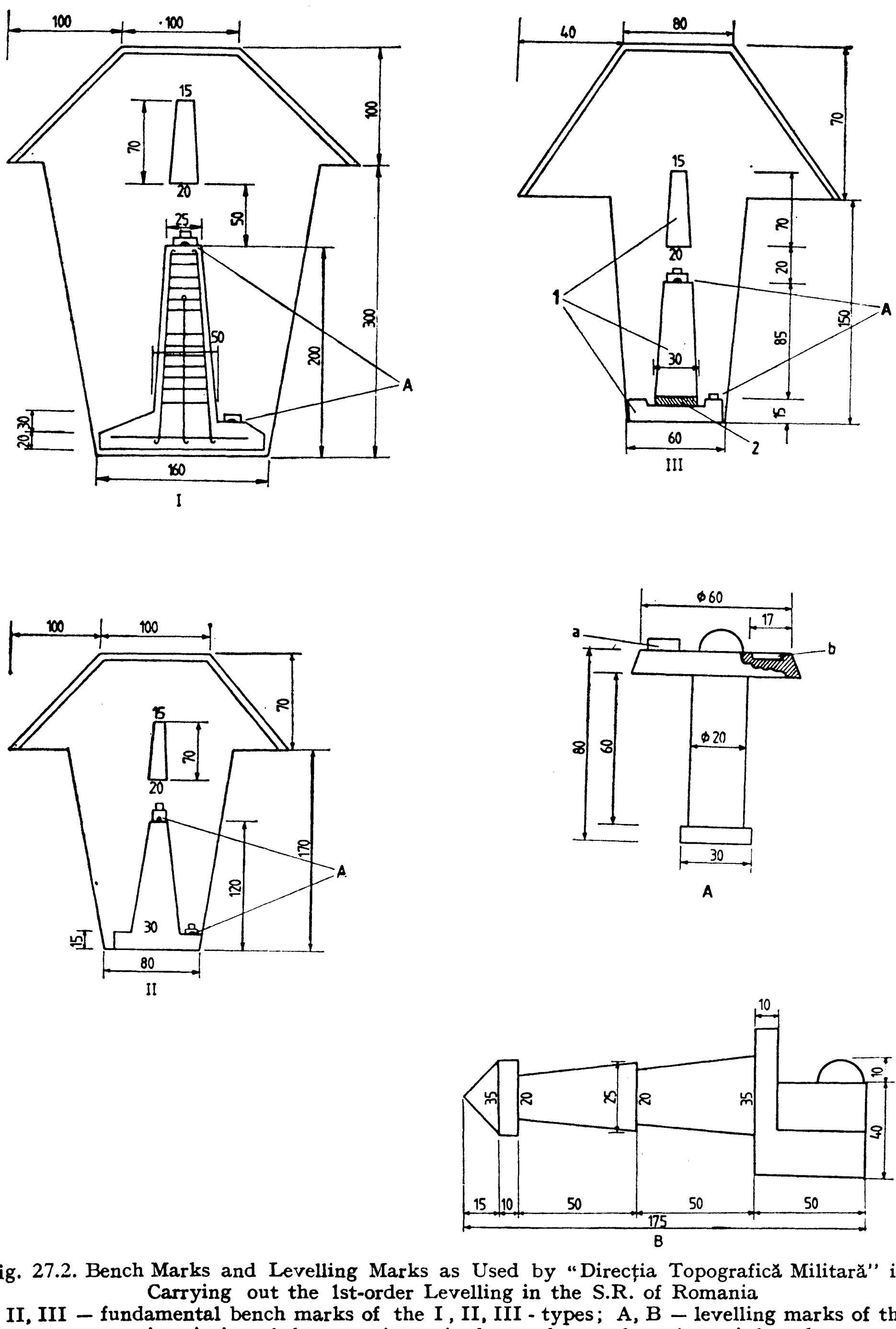


Fig. 27.2. Bench Marks and Levelling Marks as Used by "Direcția Topografică Militară" in Carrying out the 1st-order Levelling in the S.R. of Romania
I, II, III — fundamental bench marks of the I, II, III - types; A, B — levelling marks of the A, B-types; a — inscription of the operating unit; b — order number; 1 — reinforced concrete; 2 — concrete.

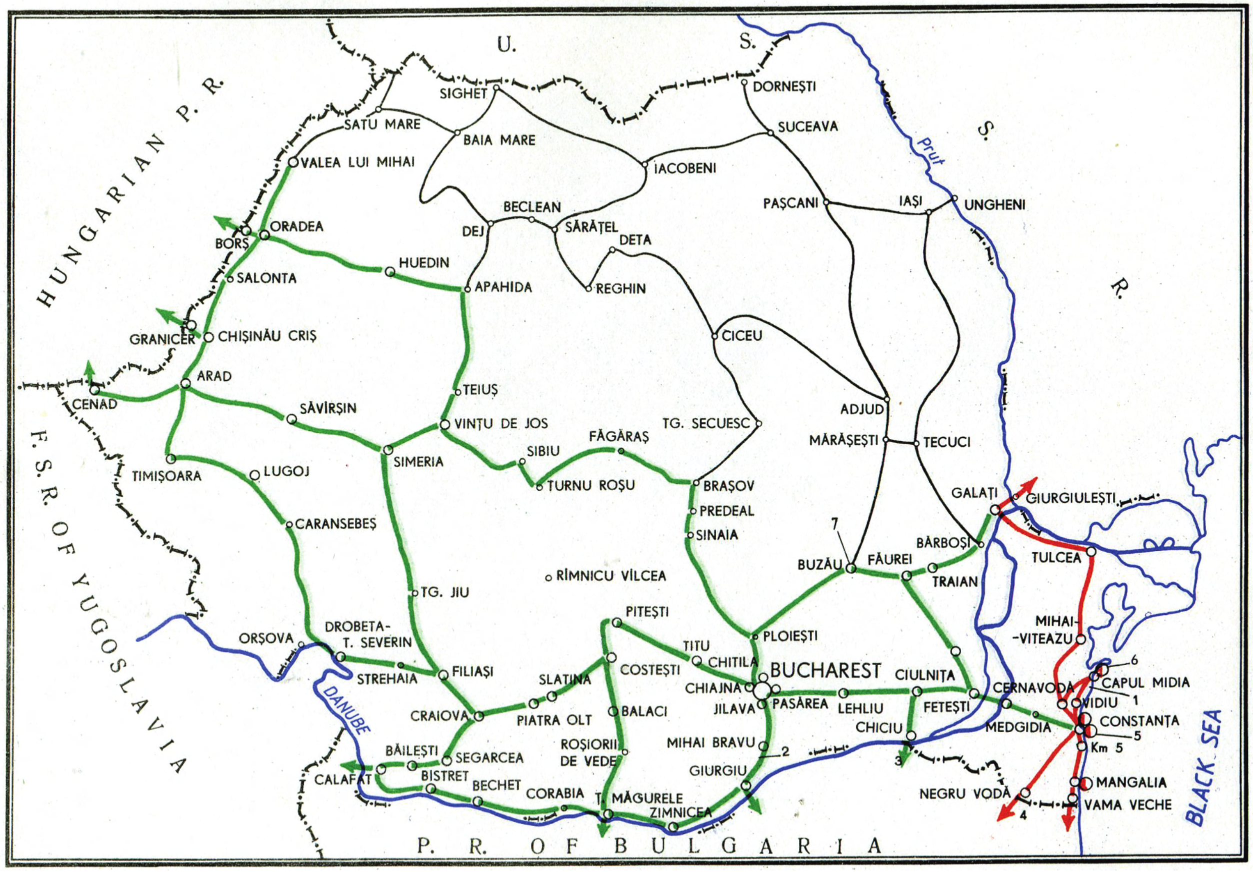


Fig. 27.3. Sketch of the High-Precision Repeat Levelling Lines on the Territory of the S.R. of Romania Being Performed by "Institutul de Geodezie, Fotogrammetrie, Cartografie şi Organizarea Teritoriului":

1 — levelling line of connexion of the tide gauges on the Black Sea Shore; 2 — high-precision repeat levelling lines on the territory of the S.R. of Romania; 3, 4 — connexions with the neighbouring countries; 5 — installed tide gauges; 6 — designed tide gauges; 7 — depth bench marks.

referred to the "*Black Sea zero*" (the period prior to the year 1954) and to the "Baltic Sea zero" (the period 1954—1971) respectively. There are several determinations of the difference between the "*Black Sea zero*" and the "*Baltic Sea zero*", carried out at various times, which may be studied in Fig. 27.5.

As can be seen, the results differ somewhat amongst themselves due to the different methods by which the necessary connexions in the field were carried out as well as to the processing methods used. As a consequence, the practical difficulties in connecting levelling results referred to different zero points are still great and the precision which is obtained is well below that needed in higher-order levelling work. It is expected that by means of the above-mentioned international levelling network, connecting several national fundamental zero points, the level differences between these points should be obtained more correctly. The difficulties lie not only in the non-homogeneity of the measurements carried out during different time periods (apparatuses, technology etc.) but also in the vertical displacements of the levelling bench marks and even of the fundamental zero points.

In *Romania* other fundamental "*zero points*" were also utilized for a few operations:

(1) *The Sulina zero point,* established on the basis of observations at the target in the harbour, in the year 1857.

(2) *The Adriatic Sea zero point* is 0.153 m higher (*Drăghici* and *Toader* 1974) with respect to the Black Sea zero (a value established in the year 1923).

The instruments used in the higher-order levelling are designated as high-precision instruments, with the help of which one can obtain, using a suitable working method, mean errors smaller than $\pm$ 0.5 mm/km: *Zeiss Ni-004, Wild N-3, Zeiss Ni-002* etc.

The targets used have an invar band with simple or double graduation, being little influenced by temperature modifications, humidity and other atmospheric factors.

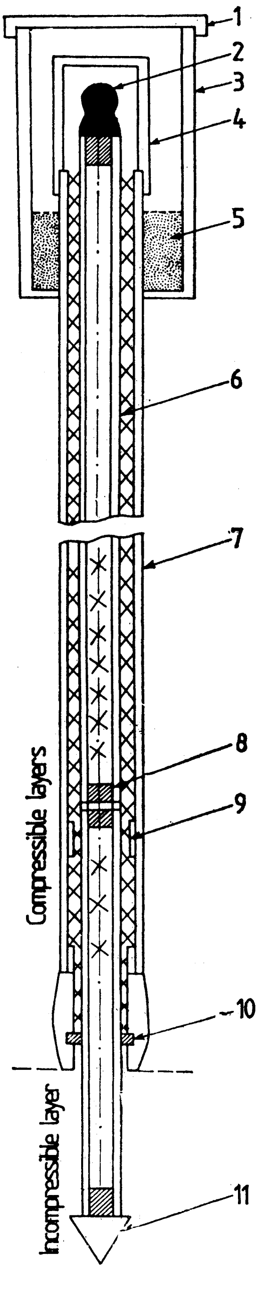


Fig. 27.4. Depth Bench Mark Type of "Institutul de Geodezie, Fotogrammetrie, Cartografie și Organizarea Teritoriului": 1 — bench mark's lid; 2 — bench mark proper; 3 — control room; 4 — protection lid; 5 — gravel and sand; 6 — bench mark-holding steel tube; 7 — protecting steel tube; 8 — component-parts connecting piece; 9 — coupling and guiding muff; 10 — protecting joint; 11 — water-tight joint; 12 — steel drill.

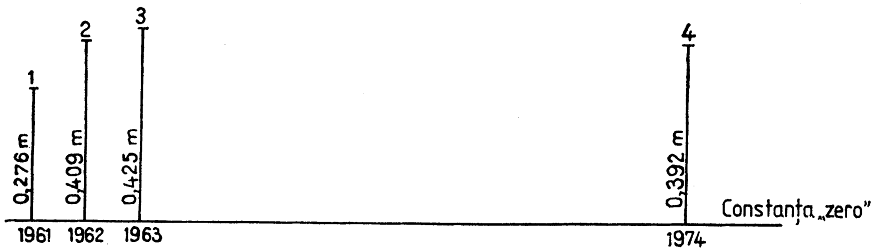


Fig. 27.5. Level Differences between "Black Sea Zero" and "Baltic Sea Zero": 1 — determination of "Direcția Topografică Militară"; 2, 3 — determinations of "Comisia de Stat a Apelor"; 4 — determination of "Institutul de Geodezie, Fotogrammetrie, Cartografie și Organizarea Teritoriului".

Generally, the graduation errors of the invar targets are less than $\pm$ 0.02 mm; when calibrating them the length between two consecutive divisions on the target is determined with an error of $\pm$ 0.01 mm.

In order to reduce the total error in determining the level difference between the extremities of a levelling line, one utilizes a special working method which is described in highly specialized contributions *(Lallemand 1912, Jordan 1932, Botez 1969* and others).

27.2.2 Systematic Residual Errors and Accidental Errors. Preliminary Corrections

Carrying out the measuring operations using suitable working methods cannot completely eliminate the following error sources:

Systematic errors:
— errors due to atmospheric refraction;
— errors of poles' verticality;
— errors of poles' calibration;
— temperature influences both on instrument and particularly on the utilized poles;
— influences of Moon's and Sun's attraction, particularly for the levelling lines oriented along the North-South direction.

Accidental errors:
— errors of pole reading;
— errors of bringing to coincidence the precision torus-level and of automatic horizontalization of the sighting axis respectively, in the case of instruments with a pendulum;
— other random errors arising from carrying out the field operations.

Before being submitted to rigorous analyses and processings, the measured level differences are corrected for the errors due to the action of physical factors whose influence could not be eliminated by the working method employed.

The correction due to atmospheric refraction. The theory of atmospheric refraction in geometric levelling work has been developed by *T. J. Kukkamäki* and subsequently taken over for analysis and thorough examination in numerous contributions, the working formula being:

$$c_r = -10^{-5}\,\gamma\,\Sigma\left(\frac{D}{50}\right)^2 \Delta t\,\Delta H, \tag{27.2}$$

in which: γ is a coefficient which may be extracted from *Kukkamäki's* tables as a function of the month, day, hour and temperature at which the levelling operations were carried out, as well as of the latitude of the station point;

D — the sighting's length, in m;

$\Delta t = t_2 - t_1$ — the temperature difference at the heights of 50 cm and 250 cm above the ground respectively, in °C;

ΔH — the level difference measured in pole units (0.5 m).

A consensus has not yet been reached as regards the application of the correction (27.2) in routine work.

The correction due to the Moon's and Sun's attraction. The Moon's and Sun's attraction is realized along any levelling line by the quantity c_a by which the readings a and b on the two poles (Fig. 27.6) due to the change of the vertical's direction, of the horizontal plane respectively, are modified depending on the position of the two heavenly bodies:

$$\Delta H = a - b = (a' + c_a) - (b' - c_a) = a' - b' + 2c_a. \tag{27.3}$$

where a', b' are the actual readings on the poles and a, b the corresponding correct values. These influences are very small (maximum values of the order of $\pm$ 0.04 mm km) but are of a systematic character.

The formula for calculating the correction due to the Moon's and Sun's attraction for a levelling line is *(Helmert 1924, Simonsen 1965, Thurm 1971* and others.):

$$c_a = c_\odot + c_☾ = s[k_\odot \sin 2z_\odot \cos(A_\odot - A) + k_☾ \sin^2 2z_☾ \cos(A_☾ - A)], \tag{27.4}$$

in which: $c_\odot$, $c_☾$ are the components of the correction for the Sun, and for the Moon respectively;

s is the length of the levelling line;

$z_\odot, z_☾$ are the zenithal distances of the Sun and of the Moon respectively;

$A_\odot, A_☾, A$ — the azimuths of the Sun, of the Moon and of the levelling line respectively;

$k_\odot$, $k_☾$ — coefficients whose values, for 1 km of levelling line are:

$$k_\odot \approx 0.0265; \quad k_☾ \approx 0.0571. \tag{27.5}$$

The correction due to the poles' temperature during the measurement. One deduces this correction from the well-known relation existing between the invar band's length at the calibration temperature t_c and the corresponding length at the working temperature t:

$$c_t = \alpha(t - t_c)\,\Delta H, \tag{27.6}$$

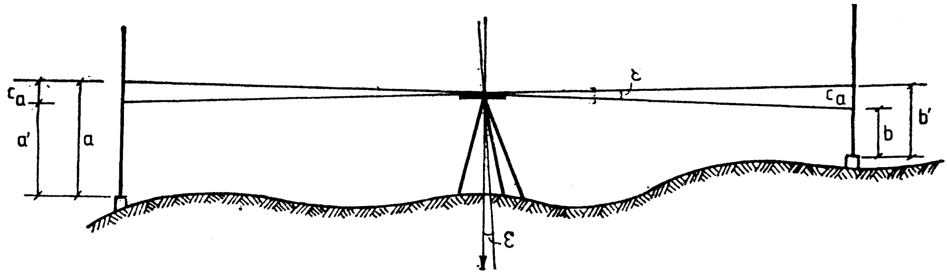


Fig. 27.6. Influence of the Sun's and Moon's Attraction in the High-Precision Geometrical Levelling

in which α represents the invar's expansion coefficient ($\alpha \approx 4 \times 10^{-7}/1°$ C)

The correction of the poles' calibration. If one denotes by l_d the length of the pole up to the division d as determined when calibrating it and by l'_d the actual length, considered when carrying out the observations in the field, the correctlon c_c^d is determined by the relation:

$$c_e^d = l_d - l'_d. \tag{27.7}$$

This correction must be applied to all the readings a and b incorporating the division d (as a direct reading on the pole).

The correction of reduction to the normal-altitude system. The necessity and the calculating formula of this correction were presented in § 3.5.3.

27.2.3 Evaluating the Accuracy of the Levelling Operations

Calculating the accuracy prior to carrying out the adjustment. The field observations being considered as twofold sets of measurements, one may calculate prior to adjustment several categories of random and systematic mean errors, which can be found in highly specialized contributions *(Lallemand 1912, Vignal 1936, Jordan 1932, Krasovski 1953)*.

To this end one utilizes the existing differences between the values obtained for the measuring directions to and from respectively, for sections as well as for the entire levelling lines, and also the misclosures occuring in the network's polygons.

Calculating the accuracy after the adjustment.
After the rigorous adjustment of the observations one may calculate:
(1) *The mean error of the weight unit:*

$$m_0 = \pm \sqrt{\frac{[pvv]}{r}}, \tag{27.8}$$

where: v — represents the corrections calculated by adjustment;
r — the number of condition equations for the network considered.

(2) *The individual errors m_i of the altitudes of all the points*:

$$m_i = m_0 \sqrt{Q_{ii}}, \tag{27.9}$$

where Q_{ii} is the corresponding quadratic weight coefficient.

(3) *The mean errors for any of the level differences ΔH intervening in the network, regarded as functions F of the adjusted elements*:

$$m_{\Delta H} = m_0 \sqrt{Q_{FF}}, \tag{27.10}$$

where Q_{FF} is the corresponding weight coefficient.

(4) *In the case of repeated levelling one can also determine the errors of the displacements δH resulting from the rigorous processing, also regarded as functions f of the adjusted elements*:

$$m_{\delta H} = m_0 \sqrt{Q_{ff}}. \tag{27.11}$$

27.2.4 On the Correlation of the Levelling Observations

The existing correlations between the original observations may be revealed with the help of the rectangular terms Q_{ij} (of covariance) in the $\mathbf{Q}$ matrix of the weighting coefficients.

The correlations which can appear in the levelling operations are of a physical and algebraic nature respectively. The ways of determining the correlations may be followed in more detailed contributions *(Lucht 1971)*, which deal with such problems. As an illustration of the manner of determining the algebraic correlations, one can examine the situation arising from the utilization of a coefficient of pole calibration (denoted by g), assumed, in the interests of simplifying the presentation, to be the same for a pair of utilized poles. The original level differences, as determined with this pair of poles (denoted by ΔH_1^0, ΔH_2^0, ..., ΔH_n^0) are to be multiplied by the calibration coefficient:

$$\Delta H = g \Delta H^0.$$

In such situations, applying the formulae (13, 28) and (426, 3) from the work of *Wolf* (*1968*) yields the following correlation matrix $\mathbf{Q}(\Delta H)$ between the observations being considered *(Ghițău* 1970*)*:

$$\mathbf{Q}_{(\Delta H)} = \left\| \begin{array}{lll} \mu_0^2 D_1 + \mu_g^2 \Delta H_1^2; & \mu_g^2 \Delta H_1 \, \Delta H_2; & \mu_g^2 \Delta H_1 \, \Delta H_3 \ldots \\ \mu_g^2 \Delta H_1 \, \Delta H_2; & \mu_0^2 \cdot D_2 + \mu_g^2 \Delta H_2^2; & \mu_g^2 \Delta H_2 \, \Delta H_3 \ldots \\ \mu_g^2 \Delta H_1 \, \Delta H_3; & \mu_g^2 \Delta H_2 \, \Delta H_3; & \mu_0^2 D_3 + \mu_g^2 \Delta H_3^2 \ldots \\ \ldots & \ldots & \ldots. \end{array} \right. \tag{27.12}$$

where μ_0 is the mean error of one kilometer of twofold levelling, multiplied by g;

μ_g is the error in determining the calibration factor g; D_1, D_2, D_3 ... are the lengths of the levelling lines in km.

27.3 Triangulation-Trilateration Network for Determining the Horizontal Movements of the Earth's Crust

By repeating the observations in the triangulation-trilateration networks, with the corresponding processing of the results, one can determine the horizontal displacements of the Earth's crust in the area covered by the considered network (these two geodetic methods were presented at length in part three of the treatise as regards both the observing technologies and the calculating procedures).

Up to now the determination of the horizontal displacements of the terrestrial crust have been carried out over relatively small areas, of a local character, of the size of a province or of a country; carrying out determinations of a global character as in the case of the repeated levelling is not yet contemplated. This situation is due to the attainable accuracy, relatively small in comparison with the magnitude of the displacements proper (relation 27.1), as well as to the complexity of the operations needed, which require large fund allocations and a much longer performing time than for the geometric-levelling networks.

In local networks, located in areas where the horizontal movements have a pronounced character, triangulation, trilateration and, sometimes, precision polygonometry have given good results. It must be emphasized that, for the determination of the horizontal displacements, combinations of angle and distance measurements with high-precision instruments are extremely advisable. In order to illustrate the previous statements let us examine the example provided by the experimental polygon *Karlsruhe* (Fig. 27.7) where the horizontal movements are of approximately 2 mm year *(Lichte 1972, Kunz 1973)*.

Thus, in a period of 100 years, the points A and B in the triangulation network may move by 0.2 m. If these points are situated at the ends of a

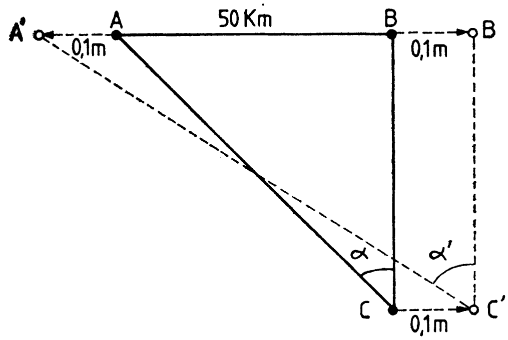


Fig. 27.7. Example of Interpreting Horizontal Displacements in the Triangulation-Trilateration Networks *(after Lichte 1972, Kunz 1973)*

length of 50 km, a modification of the opposite angle α by only $1''.15$ results, a value which barely fits into the interval established by the relation 27.1. It follows that the triangulation method could not provide a convincing answer in connexion with the horizontal movements likely to occur in the area, even after this long time interval.

On the other hand, the distance measuring instruments which exist at present may provide relatively certain answers after a smaller number of years. For such purposes one recommends the utilization of laser geodimeters since the results are influenced to a smaller extent by meteorological conditions in comparison with those obtained when utilizing micro-wave apparatuses.

If the triangulation network has shorter sides, e.g. of the order of 1—2 km, the relative determination accuracy increases, so that for territories of limited extent one could get acceptable results even in the case of small displacements, after a relatively shorter time interval (10—20 years).

Therefore, in local networks the sides have lengths under 1 km and in networks for larger areas the side lengths seldom exceed 10—15 km *(Kasahara 1965, Kurochkin 1973, Kolmogorov 1973, Meade 1965, Holdahl 1977, Savage 1975, 1976* among others*)*. Exceptions are represented by tectonic or intensively studied areas — for more general scientific purposes — where either the horizontal movements are much larger or the geodetic observations are carried out experimentally with an extremely large number of instruments, which offers multiple possibilities of preliminary analyses and qualitative improvements of the final results. A few of these results will be presented in 29.4.1.

In conclusion, one must also remember that in order to determine the altitudes of the points in the triangulation-trilateration networks one utilizes, in the general case, the trigonometric levelling method, presented in the third part of the book. There, where the conditions will allow it, one will of course make use, to this end, of the geometric levelling method.

The trigonometric levelling, characterized by a substantial output in the field operation, is accompanied by important errors of vertical refraction, which must be taken into account in processing the repeated observations. It is this element which represents, in fact, the main obstacle to constructing the three-dimensional base networks as was shown in the fourth part of this treatise.

27.4 Astro-Geodetic Methods for Determining the Movements of the Continents

A first possibility for determining the continental movements consists of comparing the astronomical coordinates Φ ,Λ of astronomical observatories located on different continents, a comparison which is carried out at certain time intervals, gradually as one determines new mean values of the astronomical coordinates of these observatories.

The necessity for determining the displacement of the continents (also known as continental drift) was stressed for the first time within the framework of the *General Assembly of the International Astronomical Union,* held in *Rome* in 1922. During the last 50 years, an important number of astronomical observatories have been under constant watch, a great number of determination being carried out annually.

Another possibility for determining the continental movements is represented by the use of the geodetic networks performed by utilizing the Earth's artificial or natural satellites. In recent years, a particular development has been achieved by the determinations based on laser observations (Section 19.4.3), as well as by the *Doppler* determinations (Section 19.4.2), offering a higher accuracy as compared with the other methods previously used.

Since the principles of determining the astronomical coordinates Φ and Λ, and the coordinates of the terrestrial points respectively by satellite observation have already been presented in Section 14.5 and in Chapts. 19—22, in this section, we will only present the practical ways of using these categories of astro-geodetic methods for determining the movements of a continental character of the Earth's crust, the particular difficulties arising, as well as the accuracy which is achieved. References to actual results obtained so far will be made in § 29.4.2.

27.4.1 Astronomical Observations for Determining the Movements of the Continents

Although astronomical observations are corrected for instrumental and refraction errors, for aberration and annual parallax influences, as well as for those of nutation and precession, nevertheless, for one and the same point, one notices annual variations of the final coordinates Φ, Λ, as determined from these observations (Table 29.1). **These variations are generated by several factors, of which the most significant are:**

(1) *Inherent errors in establishing the corrections to be applied to original observations, due to ignoring — except within certain limits of precision — the phenomena which have triggered them.*

(2) *The stars' own movements used in determining the astronomical latitude.*

(3) *Secular pole movements.*

(4) *Horizontal displacements of the station.*

(5) *Local variations in the position of the point's vertical.*

Things become still more complicated when one compares differences between determinations carried out at several astronomical observatories, at certain time intervals, with a view to calculating the metric modifications which have occurred in their relative position, inasmuch as the above-mentioned factors act completely differently in time and in space.

Therefore, the determinations of the movements of the Earth's crust of a continental character can be significant only when they arise from comparing results obtained at great time intervals, of the order of 1—5 decades. Equally, the greater the geodetic distances between the astronomical observatories taken into consideration —of the order of hundreds and thousands of kilometers —, the smaller are the relative errors in determining the continental movements.

The comparison of sets of astronomical coordinates determined at great time intervals also introduces, however, a series of specific errors such as:

(1) *The different precision of the star catalogues used at the considered measurement epochs*, which would call for the recalculation of the old determinations with the new catalogues, characterized by a much greater accuracy than those utilized 50—60 years ago. Of course, corresponding adaptations of the elements provided by catalogues are necessary, as well as the application of a unified calculation scheme for all determinations, both old and new. Details on star catalogues were given in Section 20.1.

(2) *The errors in determining and even in defining the time*, the essential element in the determinations of astronomical longitudes.

Thus, up to the year 1967 one has utilized the ephemeris time, deduced from observing the bodies of the solar system, in which the second was defined as "the fraction 1/31,556,925.9747 of the tropical year, for the year 1900, January 0, at 12 hours of ephemeris time". The ephemeris time is uniform, its determination accuracy being of the order of 10^{-9}, equivalent to that provided by quartz clocks. In the year 1970 the definition of the atomic time, as fundamental reference time was adopted, in which the second is defined as being equal to "9,192,631,770 periods of the radiation corresponding to the transition between the levels $F = 4$, $mF = 0$ and $F = 3$, $mF = 0$ of the fundamental state 2 $S\frac{1}{2}$ of the Cesium 133 atom". The uniformity of the atomic time is of the order of 10^{-12}, which represents remarkable progress compared with that of the ephemeris time, having led to the increase of the determination accuracy of the astronomical longitudes in the modern period, as compared with that obtained a few decades ago.

In order to illustrate the above considerations, one presents in Table 27.1 the mean corrections $\Delta\lambda_\lambda$ for 6 astronomical observatories in Europe, for intervals of 10 years each, at two different measurement epochs, as well as the errors $\mu_{\Delta\tau}$ of these corrections as calculated on the basis of comparing the mean results with the annual ones. The data contained in Table 27.1 reflect the progress achieved in the determination precision of the longitudes

Table 27.1. *Exemplifying the Accuracy in Determining the Astronomical Longitude* (after Ţeglov 1973)

Observatory	Epoch 1936.5		Epoch 1962.5		Differences $\Delta\lambda_2-\Delta\lambda_1$ ms
	Corrections $\Delta\lambda_1$ ms	Errors $\mu_{\Delta\lambda_1}$ ms	Corrections $\Delta\lambda_2$ ms	Errors $\mu_{\Delta\lambda_2}$ ms	
Greenwich	− 8.2	±2.7	+1.4	±0.6	+ 9.6
Hamburg	+ 4.8	2.6	−5.0	2.9	− 9.8
Leningrad	− 2.4	3.5	+5.1	4.7	+ 7.5
Paris	−17.4	2.9	+2.3	0.8	+19.7
Potsdam	+18.3	3.8	+4.8	2.6	−13.5
Pulkowo	− 0.2	3.8	−5.9	1.6	− 5.7
Mean values	− 0.8	±3.2	+0.4	±2.2	+ 1.3

and, at the same time, the existence of systematic errors in the longitudes of these observatories.

Remark. In order to provide a comparison criterion, we recall that at the mean latitude of 45° the arc of terrestrial parallel of 1″ has a magnitude of about 21.9 m, so that ± 1 ms has as equivalent the length of ±0.33 m.

27.4.2 The Pole Movement

One of the factors generating variations in the determinations of astronomical latitudes at stationary points is constituted by the pole movement (polar wandering). In this theory concerning the dynamics of rigid bodies, *L. Euler* reaches the conclusion that the rotational pole (the point at which the rotational axis pierces the Earth's surface) performs a movement of circular rotation, with a period of 305 days. Actually, as the Earth is an elastic body, the rotational axis doesn't coincide with the main axis of inertia (axis of figure). Due to the Earth's elastic deformations, to the mass displacements on its surface and inside it, to the perturbing action exerted by the Moon and the Sun on the Earth's rotation movement (including the rotational velocity), the rotational axis does not occupy a fixed position.

Therefore, one usually utilizes the notions of instantaneous rotational axis and pole respectively. As a result of the elastic properties of the planet, the instantaneous pole has a much more irregular motion than that described by *Euler* in the case of the rigid Earth. The oscillatory motion of the terrestrial pole in the case of the elastic Earth was described for the first time by the American astronomer *S. C. Chandler* in the year 1891 (whence also the terms *Chandler motion, Chandler period*) being characterized by a variable period (430 ... 437 days) and an amplitude of 0″.1 ... 0″.2 (the metric equivalent: 3 m ... 6 m).

Using its determinations, astronomy may identify a relative motion of the instantaneous rotational pole with respect to the *Conventional International Origin (CIO)*. This origin is defined by the following initial latitudes:

Mizusawa (Japan)	39° 08′ 03″.602
Kitab (U.R.S.S.)	1″.850
Carloforte (Italy)	8″.941
Gaithersburg (U.S.A.)	13″.202
Ukiah (U.S.A.)	12″.086

The 5 latitudes were obtained as arithmetical means of the determinations carried out between the years 1900 and 1905, with the exception of the *Kitab* observatory, for which the period considered was 1935—1940. Thus, the average pole defined in this manner by the *International Astronomical Union* and adopted by the *International Association of Geodesy* in the year 1960 merely has a conventional meaning.

One of the basic problems of modern astronomy consists of determining the coordinates x, y of the instantaneous rotational pole with respect to the conventional origin, a problem dealt with by the *International Polar Motion Service* (based at *Mizusawa*), by the coordination of the determinations carried out at the 5 above-mentioned observatories. The equations from which one derives these quantities, at a given moment, as functions of the latitude variations $\Delta\Phi$ recorded at these observatories are *(Melchior* 1971*):*

$$x = -0.4359\,\Delta\Phi_M + 0.1227\,\Delta\Phi_K + 0.4483\,\Delta\Phi_C + 0.1232\,\Delta\Phi_G - 0.2583\,\Delta\Phi_U;$$
$$y = -0.2636\,\Delta\Phi_M - 0.3133\,\Delta\Phi_K + 0.0172\,\Delta\Phi_C + 0.3382\,\Delta\Phi_G + 0.2559\,\Delta\Phi_U. \tag{27.13}$$

The present interest in the polar motion has been considerably reinforced by the established fact that changes in this motion due to large earthquakes may contain information concerning pre-seismic and post-seismic activities. Such motions will be still more clearly observable by means of long baseline interferometry (*LBI*) and by laser techniques in distance measurements. For instance, the *Chile* earthquake in the year 1960 (magnitude 8.5) has led, according to the performed calculations, to a change in the polar motion corresponding to a displacement by 65 cm of the rotational pole.

Authorized institutions periodically publish diagrams of the form given in Fig. 27.8, describing the polar motions as determined within certain time intervals.

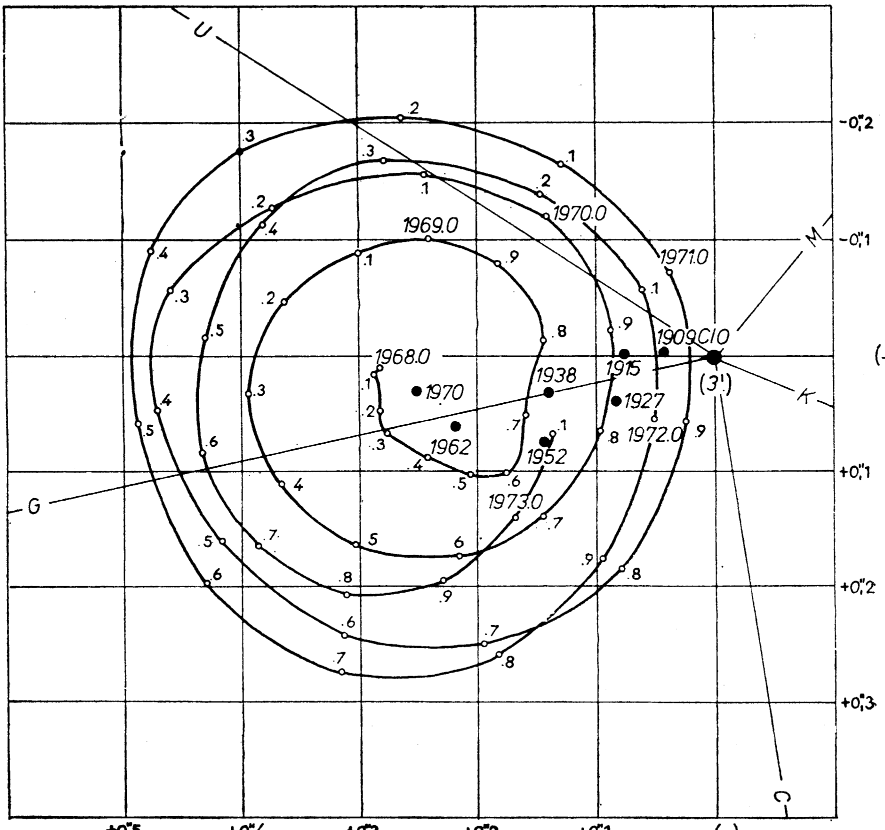


Fig. 27.8. Polar Motion for the Period 1968.0 — 1973.1

The latitudes of all astronomical or astro-geodetic stations are referred to *CIO* by correcting the determinations carried out taking into consideration the polar motion. Only after having applied this correction can the astronomical determinations be utilized for calculating the continents' displacements.

27.4.3 Utilization of Laser and Radio-Technical Observations for Determining the Continental Movements

The determination of the continental movements may be carried out, in principle, by analysing the time modifications which affect the magnitude of distances between points located on different continents. This is provided still more faithfully by analysing the modifications of this type which occur in the three coordinates of the points concerned within a certain time interval. As was mentioned in the introduction to this part of the book, an essential element in such studies is the accuracy with which the determinations are carried out.

At present, the following categories of methods, which have been dealt with more fully in other chapters of this treatise, are in an advanced stage of experimentation:

— *laser observations* (goniometric and telemetric ones), dealt with in § 19.4.3;

— *Doppler observations* — tackled in § 19.4.2;

— *very long base interferometry* — presented in § 19.4.2;

— *satellite altimetry.*

The observations proper are carried out either from on geodetic satellites, specially equipped for such purposes, or on apparatuses installed on the Moon, and in the future may be even on other cosmic bodies, as well as on high targets launched by means of balloons. One notes the introduction, into this complex of problems, of measuring instruments and techniques which until recently were being utilized in other fields (e.g. in navigation or in radioastronomy) but with an accuracy insufficient for the geodetic purposes we are examining, an accuracy which is continuously improving. It is rather difficult to foresee the limits of this since the studies carried out in this direction are of a quite remarkable intensity, which will lead to spectacular results in the coming decades.

One must also mention the interdisciplinary nature of these studies and primarily the relations which are being established with the meteorological services in order to bring the necessary corrections to the post-calculated elements as well as to the results obtained.

Of course, such determinations, described only as potential ideals presuppose extremely complex and expensive endowments, which for the time being constitutes an important hindrance to their general application and leads to their utilization only in countries with a well-developed economy.

The first distance measurements utilizing laser pulses towards an Earth's artificial satellite were made in the year 1964 and had an accuracy of about 10 m. At present, the technical literature *Nottarp and Seeger 1971, Campbell et al. 1973, Kolaczek and Wilson 1973, Schneider 1975, Stange 1975* among others) indicates possibilities of determining distances of the order of thousands of kilometers with an accuracy of the order of a decimeter or even better (the principles of space trilateration were presented in Section19.6).

It is true that up to now what can be estimated relatively correctly, in nearly all cases, is the internal accuracy (obtained by one and the same instrument in a long sequence of measurements). The international geodetic programmes contemplate the achievement, in the not too remote future, space networks of very great dimensions in which a large number of such instruments will be included, located on different continents. Undoubtedly, the accuracy which will be obtained in processing the observations carried out in the whole network will be somewhat smaller at the beginning as compared with the internal accuracy obtained at present, but the development foreseen in this field is particularly encouraging.

Thanks to this extremely high accuracy achieved in measuring very large distances, the determination of the lithospheric plates' displacements — which are estimated at a maximum of 15 cm/year — as well as of other horizontal displacements, of a general or local character of the same order of magnitude, has become possible.

By utilizing the *Doppler* method (Section 19.4), the motion of the instantaneous rotational pole may be determined with a precision of $\pm 0''.01$ or even better, higher than, or at least comparable with, the precision obtained by the International *Polar Motion Service*, evaluated at $\pm 0''.015$. The determinations carried out at *Dahlgren (U.S.A.)* for a period of 4 months on 3 of the Earth's artificial satellites have provided an accuracy of $\pm 0''.007$ (22 cm) for the average position of the pole within a 5 day interval. The *DPMS (Dahlgren Polar Motion Service)* results are now included in the data of the *International Polar Motion Service*. In this manner one completes the information pieces obtained in the conventional way as described in § 27.4.2, which are needed for the astro-geodetic processing of the observations devoted to determining the continental movements.

Within the same context, one must mention the investigations of very long baseline interferometry *(VLBI)*, which offer particular possibilities concerning the accuracy in determining very large distances (about 5 ... 10 cm for 5,000 ... 6,000 km).

The above-examined investigations are used not only for the purpose of determining the continents' movements but also for geodynamics studies, such as e.g. the time modifications of the Earth's rotation parameters.

27.5 Other Methods Used for Determining the Crustal Movements

In addition to the methods previously mentioned, which have a pronounced geodetic character, there is another category of methods utilized by Geodesy in close connexion with other disciplines, by means of which one performs the processing and the interpretation of their data with a view to deducing the movements of the Earth's crust. The methods to be presented in the following are seldom used separately, independently. These are mostly utilized along with the other methods examined so far, so that the informational sources are particularly complex and diverse.

27.5.1 The Gravity Method for Determining the Crustal Movements

In order to determine the geoid's undulations, the deflections of the vertical, the observations' reductions on the ellipsoid as well as for other geodetic operations, Geodesy needs gravity observations. More and more, geodetic institutes are studying, in some countries, by means of continuous recordings of the gravity's variations, the time modifications of the Earth's mantle parameters due to the Moon's and Sun's atractive influence.

The variations established in the gravity's magnitude for certain time intervals, at a given point, may be induced by several factors, from among which the vertical movements of the Earth's crust and the physical-chemical modifications in the sub-crustal layer, leading to the density's modification, are the most significant ones.

Of course, these are accompanied by other influences too, of which those of the Moon's and Sun's attraction are also considerable.

The relation (3.2) makes clear the interdependence existing between the variation $\mathrm{d}H$ of the altitude of a point and the variation $\mathrm{d}W$ of the potential at the same point:

$$g = -\frac{\mathrm{d}W}{\mathrm{d}H}$$

If one takes account of the fact that the level surface itself, with respect to which the altitudes are defined, manifests certain time variations δN, it follows that the absolute vertical displacement δr of the point concerned is obtained from the relation:

$$\delta r = \delta N + \delta H$$

More detailed studies *(Biro 1973, Torge 1976, Strang van Hees 1977)* propose solutions for solving this complex of problems, in which hypotheses of a geophysical nature are also sometimes involved.

Thus, *Strang van Hees* derives the relations connecting the elements mentioned within the framework of the elastic *Earth*:

$$\begin{aligned} \delta W &= -\frac{1}{2} R \frac{\gamma}{p} \delta g, \\ \delta r &= -\frac{1}{2} \frac{R}{g} \frac{h}{p} \delta g, \\ \delta N &= -\frac{R}{2g} \frac{1+k}{p} \delta g. \end{aligned} \tag{27.14}$$

where: R = Gauss average radius (§ 9.2.2) and h, k, γ and p are constants which can be determined by astronomical and geophysical observations and measurements: $h = 0.59$; $k = 0.28$; $\gamma = 0.70$; $p = 1.19$.

With these values, which can be taken as average quantities, the equations (27.14) become:

$$\delta W(\mathrm{m^2s^{-2}}) = -\ 18.7\ \delta g\ (\mathrm{mgal}),$$

$$\delta r\ (\mathrm{m}) = -\ 1.61\ \delta g\ (\mathrm{mgal}),$$

$$\delta N\ (\mathrm{m}) = -\ 3.49\ \delta g\ (\mathrm{mgal}).$$

Using the above relations the quantities δW, δr and δN become determinable by high-precision δg measurements. It must also be emphasized that the method is meaningful only when gravimetric instruments of the highest precision are used, since otherwise the reduced accuracy of a series of gravity determinations has the erroneous effect of affirming the existence of movements of the Earth's crust within the investigated area.

It follows, for what is under examination here, that an error of ± 0.001 mgal in determining the gravity's variation δg would lead to an error of approximately ± 0.002 m in the determination of the absolute vertical displacement δr of the point concerned. The gravimetric field instruments utilized at present unfortunately furnish measurements affected by errors greater by at least one order of magnitude, which recommends that the method be used only in zones with large vertical movements or in rugged areas, of difficult access, where repeated levelling could not provide better results either,

At present, intensive efforts are under way to produce new types of small-sized gravimetric instruments which allow, under field conditions, absolute gravity determinations with errors of the order of ± 0.001 mgal, which would represent a remarkable step in the possibility of applying the gravity methods to the study of the vertical movements of the Earth's crust.

Usually, the repeated gravity determinations refer to limited areas, of a local character, investigated by various methods, frequently utilizing gravity profiles (and, to a lesser extent, gravity networks) along certain directions established within the framework of the interdisciplinary cooperation existing in the case of such studies (*Boulanger 1962*, *Schleusener* and *Torge 1971*, *Boedecker 1978*, *Schneider 1975* among others). The density of points, the careful marking of the gravity stations as well as taking the necessary steps in carrying out the gravity observations (great speed of execution, adoption of suitable sequence of operations — in order to avoid certain instrumental errors etc.) ought to ensure the necessary accuracy for determining the movements of the Earth's crust.

Analogously, there should exist the theoretical possibility of proceeding to the determination of the horizontal movements of the Earth's crust too, by utilizing the horizontal gradients of gravity. Since the accuracy achieved as well as the existing volume of such gravity determinations are both somewhat less, this possibility is not at present used for the stated purpose.

The gravity measurements carried out continuously at stationary points serve first of all for controlling the gravity field but may be extremely useful also for controlling the stability, along the vertical, of the levelling bench marks and other points in their immediate neighbourhood. Hence there is a proposal *(Biro 1973)* to locate such "supercontrol" points near the fundamental zero points or near the tide gauge stations.

27.5.2 The Method of Continuous Recording of the Sea Level

The continuous recording of the time variations of the sea level, performed either by means of complex installations called tide gauges or mareographs, or at tide-poles, offer in principle the possibility of determining the vertical movements of the Earth's crust in the corresponding area. The causes of these variations are extremely complex (Fig. 27.9), which generate great difficulties in the numerical filtering of the results for the purpose of distinguishing one of the main component parameters.

At present the world 1st-order tide-gauge network numbers some 260 installations in whose results various disciplines are interested: oceanography, geodesy, geophysics, meteorology, seismology etc. The field of *Coastal Geodesy*, the name ascribed to the complex programme of determinations and measurements carried out for the above-mentioned purpose, continues to attract ever more followers.

Every tide gauge is connected to a group of 3—5 levelling bench marks (one of these being designated as the "main bench mark"), located in its neighbourhood, and, by means of "coastal levelling", to the state levelling network. The utilization of the tide-gauge recordings for determining the recent vertical movements of the Earth's crust is possible only if the connexion to the general-levelling network was made within several time periods. Additionally, information of a continuous character concerning the weather conditions, the temperature and the density of the sea water, is essential in order to be able to carry out the subsequent mathematical processings.

In *Romania*, in the year 1895, the "*Institutul Geografic Militar*" *(IGM)* installed at *Constanţa* (near the *Genoese beacon*) a mareometer which was in operation up to the year 1903 *(Drăghici* and *Toader* 1974*)*. In the year 1910, *IGM* re-installed a mareometer which was destroyed in the year 1916, the determinations being limited to readings at a tide-pole up to the year 1932. The "*Direcţia Porturilor Maritime*" installed a tide gauge in the year 1932, and this is still in operation.

Recently, new tide gauges were installed on the *Romanian* littoral, viz. in August 1974 in the *Tomis* harbour and in the year 1975 in the *Mangalia* harbour.

H. Montag (1970) makes an attempt at classifying the errors which occur in exploiting and analysing the tide-gauge results (Fig. 27.10). The total errors with which the annual mean level is determined are of the order of ± 3 — ± 5 mm. Inasmuch as the tide gauges are installed along the coast, at intervals of 100—300 km, there is the possibility of utilizing them as "absolute-control points" for the levelling determinations connected with these points as well as of processing in common their data in order to determin the recent vertical movements of the Earth's crust.

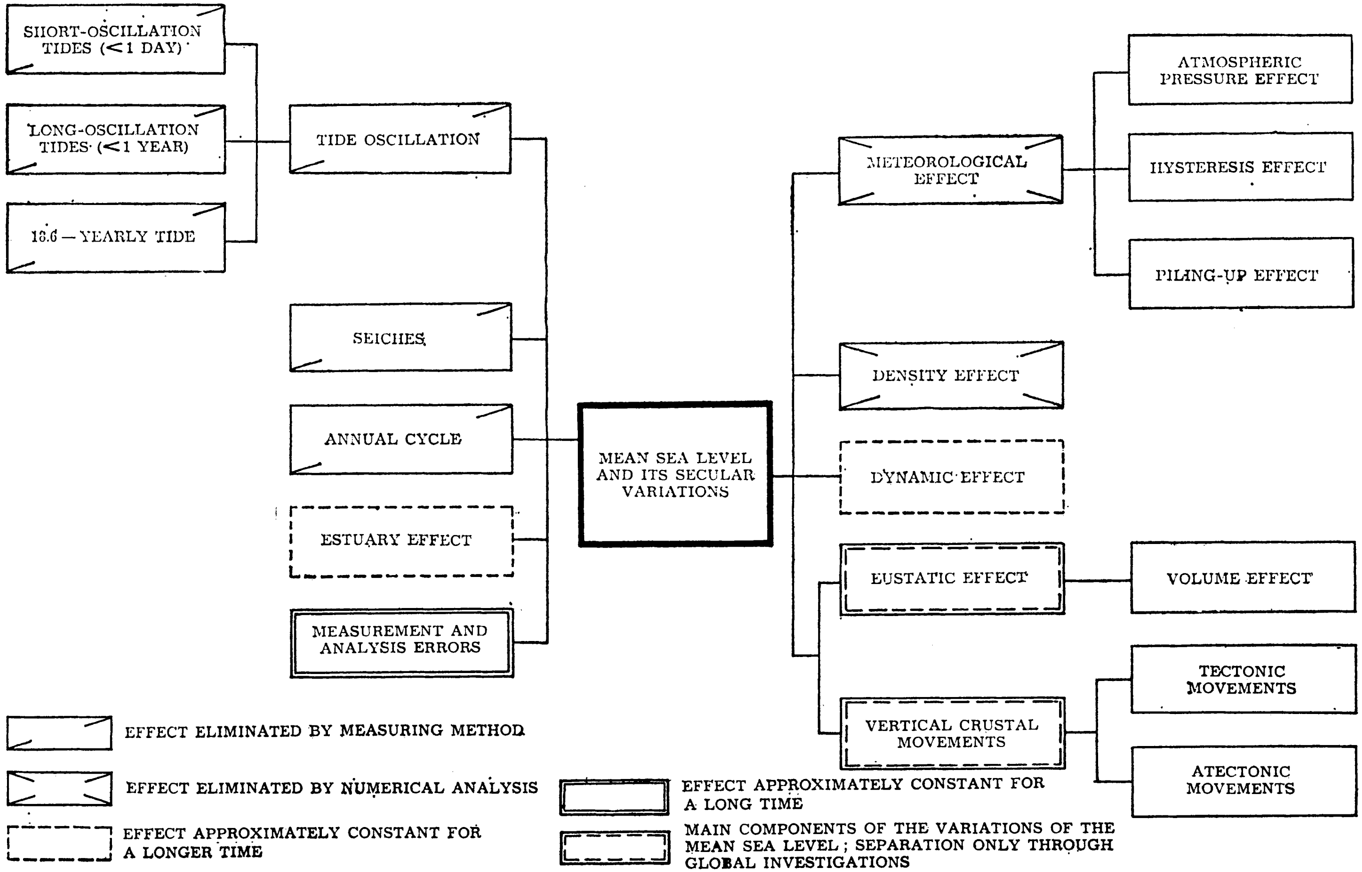


Fig. 27.9. Causes Generating the Variation of the Mean Sea Level *(after Montag 1970)*.

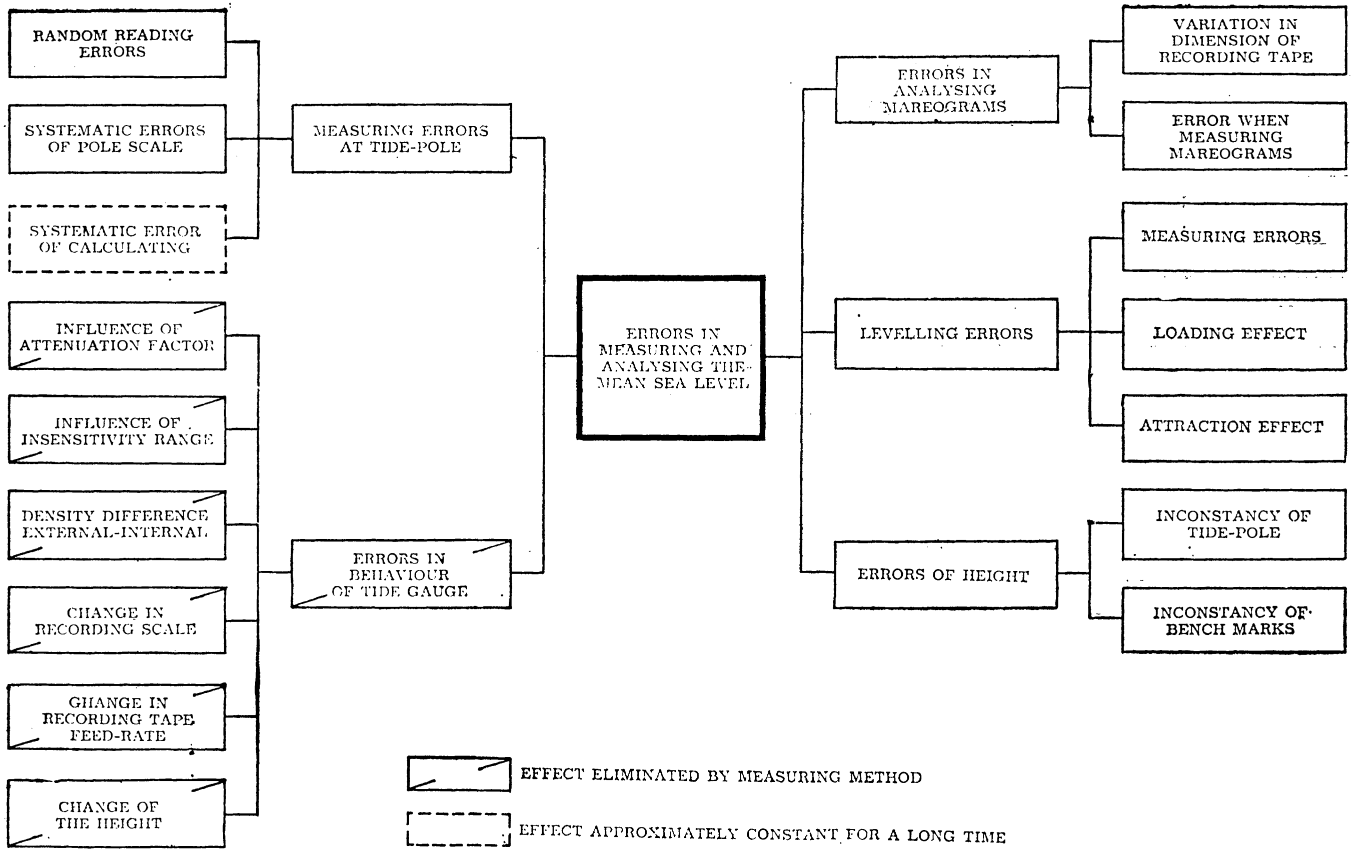


Fig. 27.10. Errors Intervening in Exploiting and Analysing the Recordings of the Tide Gauges *(Montag 1970)*

From the recordings carried out at a certain tide gauge, denoted by z, one can form various mean values which will be utilized in the subsequent processing calculations, to be presented in § 28.2.4. Usually one operates with daily means, denoted by z_d, monthly means, denoted by z_m, annual means, denoted by z_a, and with means for an interval of 18.6 years, denoted by $z_{18.6}$:

$$
\begin{aligned}
z_d &= \frac{1}{n_d}\,\Sigma z; \quad z_m = \frac{1}{n_m}\,\Sigma z_d; \\
z_a &= \frac{1}{n_a}\,\Sigma z_m; \quad z_{18.6} = \frac{1}{12 \cdot 18.6}\,\Sigma_{18.6} z_m
\end{aligned}
\tag{27.15}
$$

where n_d, n_m, n_a denote the number of recordings utilized for the daily, monthly, and annual means respectively. By taking these means one can eliminate, in a good approximation, the influences exerted by the Sun and especially by the Moon within these time intervals. It is necessary that $n_d > 4$ and that the observations should be distributed as uniformly as possible during a 24-hour period.

For areas of more restricted dimensions, of a local character or even of the size of a small-area country, it is possible to apply repeatedly, and with a much greater frequency, the procedures described in Sections 27.2, 27.3, 27.5 and even some methods presented in Section 27.4. As well as this, the observations are usually formed into surface networks, which offers the possibility of obtaining an additional number of observations compared with that strictly needed, so that the results which are obtained by the processing enjoy a higher degree of probability.

For the processing of the observations repeated within such networks, we will present in the sequel a functional-stochastic model conceived in such a way that it complies with these desiderata, containing within its structure, in principle, nearly all the categories of determinations obtained by means of the geodetic methods presented in Chapter. 27.

The theoretical base for constructing this model derives from the principles of Three-Dimensional Geodesy, presented in Chapters. 17 and 18. However, to achieve a model which also has a practical applicability, adaptations and modifications were made, so that the volume of calculations be somewhat more reduced as compared with what would result from the exact application of the principles of Three-Dimensional Geodesy.

By specializing this general model, one can either obtain solutions for processing a certain category of observations, or limitations or possibly extensions of the area concerned etc. Incorrectly shaped calculation models may falsify reality, providing erroneous results which *F. R. Helmert* called **"theory errors"**. Therefore, the correctness of the final results is achieved by corresponding steps taken not only within the framework of the technological process of obtaining the results but also during the processing stage, i.e. when producing the functional-stochastic model used in the calculations of rigorous adjustment.

28

Processing the Geodetic Observations Carried out for Determining the Recent Crustal Movements

The present conception in processing the geodetic observations in general and the repeated ones in particular includes the idea of putting together in a single block all the informational sources, whose analysis may lead to conclusions which are better founded compared with the case of the separate processing of each category of geodetic observations.

The multitude of phenomena combining in the generation of the movements of the Earth's crust as well as their complex forms of manifestation also impose, however, a certain separation of the possibilities examined, so that the investigation takes on an iterative character, in the analysis of the constituent parts, on the basis of accepting simplifying hypotheses. This solving method is conditioned by particular practical difficulties in determining, e.g., simultaneously both the local movements (in a certain area) and the global displacements, while taking into account the stars own movements, the variations of the gravity field (those at the zero points of the geodetic networks included), the modifications occurring in the Earth's rotation etc.

To these difficulties also should be added those due to the fact that the parameters entering the domain of such determinations differ widely as to their order of magnitude and time variability, the present achievable accuracy in their determination also being extremely varied.

This is why the solutions which have been found are usually of a relative and temporary character, being subject to continuous improvement.

For areas of larger dimensions, of a continental or even global character, the processing techniques derive from the nature of the procedures being used and the corresponding displacements are obtained by comparing coordinates or even lengths determined repeatedly, at certain time intervals (Section 27.4). As has already been mentioned, what is essential for determinations of this type is the separation of the movements proper of the Earth's crust from the other phenomena of a geodynamic nature (polar motion, Earth tides, modifications in the parameters of the Earth's movement and so on) among several others.

28.1 A Functional-Stochastic Model for the Three-Dimensional Determination of the Recent Movements of the Earth's Crust within Areas of Limited Size

The functional-stochastic model to be presented here may be applied within the framework of areas of limited size, of the magnitude of a province or of a single, small-area country.

28.1.1 Initial (Known) Elements

The coordinate system. The calculations will be developed in the ellipsoidal global coordinate system, described in § 17.1.2.2 by using the geodetic coordinates of the points concerned (B = latitude, L = longitude) and the ellipsoidal altitude H^E. In order to simplify the notations which will follow, we shall henceforth give up the superscript E, so that the ellipsoidal altitude of any point P will be simply denoted by H_p.

The network's fixed point. In what follows, we shall envisage free networks in the conventional conception or, as may happen, constrained networks.

Therefore, we will assume that there is the possibility of choosing a fixed point (denoted by F), located within an area which is relatively stable from the point of view of the movements of the Earth's crust. Its choice must be made in cooperation with specialist geologists-geophysicists, and on the basis of comparing geodetic measurements repeated at certain time intervals (subjected to a preliminary processing).

The ellipsoidal coordinates (B_F, L_F, H_F), as well as the astronomical coordinates (Φ_F, Λ_F) of the fixed point will be regarded as constant in time.

One also assumes as known at the fixed point: an initial astronomical azimuth towards a point f, α_{Ff}, an initial distance D^0_{Ff} reduced to the normal to the ellipsoid (§ 10.4.2), the deflections of the vertical ξ_F, η_F and the height's anomaly ζ_F.

Remark. In order to avoid notation ambiguities over the letter N (for the normal and for the geoid's undulation rspectively, the notion of "height's anomaly", denoted by ζ, will always be utilized in what follows although in principle one may also use the geoid's undulations for the same purposes.

In the case of narrow networks, developing more along a certain direction, it is useful to choose two fixed points at the two ends, it thus being easier to control the stability within the central zone. In this case, knowledge of the initial azimuth and side is no longer absolutely necessary.

Note. In processing the repeated geodetic observations, particularly within the framework of local networks of late, the principles presented in Section 15.5 have been applied, i.e. one does away with the necessity of choosing fixed elements within the network. Processings of this type, which present advantages but also particular disadvantages, can be deduced by specializing the model which will be exhibited in what follows and by applying a solution of the kind examined in Section 15.5.

The deflections of the vertical and the height's anomalies

For solving the problem considered as correctly as possible it is necessary to determine, perhaps locally, the quasi-geoid within the investigated area, so that at all the points of the network one knows the height's anomalies ζ, as well as the deflections of the vertical ξ and η. These elements may be assumed to be constant in time, which corresponds to the hypothesis that between two series of observations the gravity potential remains constant. In the opposite case, one should perform a new determination of the quasi-geoid for each series.

The vertical gradient of gravity. For the points at which one carries out repeated gravity determinations of a high accuracy, one needs knowledge of the vertical gradient of gravity $\partial g/\partial H$ which will also be assumed unvarying in time.

Provisional coordinates. From a preliminary processing one knows the provisional coordinates of all the points in the network concerned, which for any point P will be denoted by B'_P, L'_P, H'_P.

Measurements (determinations). The measurement of the movements of the Earth's crust, carried out for geodetic determinations, between the points P and R, at the moment t, will be symbolically denoted by M^{Qi}_{PR}, where M **may have the following meanings**:

- $M = d$ — azimuthal directions;
- $M = D$ — distances;
- $M = \alpha$ — determinations of astronomical azimuths (in limited numbers):
- $M = \beta$ — zenithal angles (for small distances only, under 6 km);
- $M = g$ — gravity determinations of the highest accuracy;
- $M = \Delta H$ — differences of ellipsoidal altitudes (as determined by geometrical levelling);
- $M = \Phi$ — determinations of astronomical latitudes (in extremely limited numbers);
- $M = \Lambda$ — determinations of astronomical longitudes (in extremely limited numbers).

Remarks:

(1) For the calculation model which is presented in this chapter, the astronomical determinations are limited to what is strictly necessary, since for areas of reduced dimensions the achieved accuracy does not satisfy the inequality (27.1), for the reasons shown in § 27.4.1. We will thus assume that the determinations of astronomical latitudes and longitudes are carried out in an extremely limited number, viz. at certain fixed points in the network (where they will be regarded as constant), as well as at other points in the network, situated at 100 – 150 km. The astronomical azimuths may be utilized to a somewhat greater extent, due to the higher accuracy in determination, especially in areas with large displacements, in order to diminish contingent transverse errors which may occur in the network.

(2) Giving up the necessity of determining Φ and Λ at all points of the network involves important structural modifications in the formulae borrowed from Three-Dimensional Geodesy, imposing simultaneously the obligation of the preliminary reduction of all geodetic observations to the normal to the reference ellipsoid. This is how the studies which are presented in this chapter differ from the known theories of Three-Dimensional Geodesy, put forward in Chapters. 17 and 18 of the present treatise.

(3) Apart from gravity determinations of the highest accuracy, like those specified in § 27.5.1, one needs gravity measurements carried out in order to reduce the geodetic observations onto the ellipsoid, measurements whose accuracy may be somewhat less.

(4) For the azimuthal directions — d, the distances — D and the zenithal angles — β, we have adopted the same notations as in Chapter. 17.

The variance-covariance matrix of the measurements. Strictly rigorously, for processing the geodetic observations the variance-covariance matrix Φ of the measurements under consideration would be necessary. One mostly operates, however, just with the weights' matrix, obtained from the matrix $\mathbf{Q}$ by disregarding the covariance terms.

The necessity of introducing the variance-covariance matrix into calculations is justified in the case of processing the repeated geodetic measurements for the following two main reasons:

(1) *The group of determinations under consideration involved in a certain measurement epoch is characterized by a heterogeneous accuracy* (Chapter. 27).

(2) *Depending on the measuring instruments and technology utilized, the observations carried out have a certain degree of accuracy, which differs from one measurement epoch to another.*

28.1.2 Description of then Fuctional-Stochastic Model

One can apply the following two methods of forming the functional-stochastic model:

(1) *Through a strict application of the ideas of Three-Dimensional Geodesy* (Chapters. 17 and 18), *i.e. by the direct processing of the measurements $M°$, with a view to deducing the probable values M in the sense of the least squares method.*

The previous remarks (1) and (2) stressed the main difficulties implied by adopting this calculation method for the present purpose.

(2) *The measurements $M°$ are preliminarily reduced to the normal to the ellipsoid, leading to the quantities M', which are then rigorously processed, in order to obtain the probable values M.*

In principle, the results obtained by means of the two methods must be identical. This hypothesis is then only realizable when sufficiently accurate determinations of the deflections of the vertical ξ, η and of the height's anomalies ζ are available, so that the errors of reduction are of the same order of magnitude as the measurement errors. From the point of view of the volume of calculations, the second way is much more advantageous since the functional model involves fewer unknowns.

Inasmuch as the quantities M' refer specifically to the ellipsoid's surface, one will make here too, as in pure Three-Dimensional Geodesy, a division into two groups of the equations with which one subsequently works, viz.:

(1) *In the first group* (also called the position group) are included: azimuthal directions, distances measured by means of electromagnetic instruments, Laplace azimuths and contingent astronomical longitudes, which corresponds to the usual calculations of ellipsoidal coordinates B, L.

(2) *In the second group* (also designated as the height group) are included: geometric levelling (along with gravity measurements needed for calculating the ellipsoidal altitudes), zenithal angles, highest-precision gravity determinations and contingent determinations of astronomical latitudes, which correspond to the usual calculations of altitudes H.

This separation, which takes place effectively in Conventional Geodesy, here has only a *formal* role, since the two groups of equations will contain common linking unknowns.

The actual details of this separation, as well as the solutions provided by *Dufour, Wolf* etc. were examined in § 18.7.3.

In order to determine the reductions which are applied to observations carried out in the field and occur in the first group, one needs elements of the second group, so that one may apply iterative calculating procedures with good results. The convergence is less rapid in areas with large level-differences, due particularly to the influence these exert on reducing inclined distances.

The functional model. Inasmuch as carrying out all the geodetic operations within a measurement epoch covers a greater time interval, one will assume that the positions of the networks's points are affected, even within this interval, by the Earth's crust movements. Therefore, the functional model will contain supplementary unknowns, denoted by $\dot{B}_P$, $\dot{L}_P$, $\dot{H}_P$ which represent the projections of the displacement speed of the point P onto the corresponding directions:

$$\dot{B}_P = \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_P ; \; \dot{L}_P = \left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)_P ; \; \dot{H}_P = \left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_P . \tag{28.1}$$

The time needed for carrying out a single geodetic observation being much smaller compared with the time necessary for carrying out all the observations in the network, one can ascribe a certain measurement moment t_i to every determination taken into account in forming the model.

Consequently, the coordinates $\widetilde{B}_P^i$, $\widetilde{L}_P^i$, $\widetilde{H}_P^i$ of the point P at a certain measurement moment t_i, affected only by its displacement speed (a systematic component), may be written in the form:

$$\begin{aligned} \widetilde{B}_P^i &= B_P'^{\mathrm{I}} + \mathrm{d}B_P + \Delta t_{\mathrm{I}i}\dot{B}_P ; \\ \widetilde{L}_P^i &= L_P'^{\mathrm{I}} + \mathrm{d}L_P + \Delta t_{\mathrm{I}i}\dot{L}_P ; \\ \widetilde{H}_P^i &= H_P'^{\mathrm{I}} + \mathrm{d}H_P + \Delta t_{\mathrm{I}i}\dot{H}_P , \end{aligned} \tag{28.2}$$

where: $B_P'^{\mathrm{I}}$, $L_P'^{\mathrm{I}}$, $H_P'^{\mathrm{I}}$ are the provisional coordinates of the point P at an initial moment t_{I}, so that (Fig. 28.1):

$$B_P^{\mathrm{I}} = B_P'^{\mathrm{I}} + \mathrm{d}B_P ; \; L_P^{\mathrm{I}} = L_P'^{\mathrm{I}} + \mathrm{d}L_P ; \; H_P^{\mathrm{I}} = H_P'^{\mathrm{I}} + \mathrm{d}H_P , \tag{28.3}$$

and:

$$\Delta t_{\mathrm{I}i} = t_i - t_{\mathrm{I}} . \tag{28.4}$$

The equations (28.2) describe the actual situation, of the displacement of the point P, for small time intervals only. For the purpose of describing such displacements during larger time intervals, when the hypothesis of the movement's linearity can no longer be accepted, it will be necessary to introduce, for one and the same point, several unknowns of the type (28.1), $\dot{B}_P^{I-II}$, $\dot{B}_P^{II-III}$ etc., between the measurement epochs I, II, III *(Wolf 1965, 1968, Ghiţău 1970, 1973* and others*)*. In order to avoid formulae which are too complicated such a situation will not be tackled here.

A clear distinction must be made between the actual time variation of the coordinate of a point and the variation which will be determined by calculation (Fig. 28.1), between which there will exist differences denoted by $\tilde{v}$. The differences $\tilde{v}$ have a random character when there exists a correctly shaped functional-stochastic model and the measurements have a high degree of precision.

The functional-stochastic model will comprise the following categories of unknowns:

(1) *Orientation unknowns.* At a point P, for the measurement moment t_i of the azimuthal observations by the series' method (which will be taken into consideration, just as in Chapter 17), it is necessary to consider the orientation unknown do_P^i of the corresponding series:

$$o_P^i = o_P'^i + do_P^i,$$

where $o_P'^i$ denotes the provisional value of the mean orientation angle.

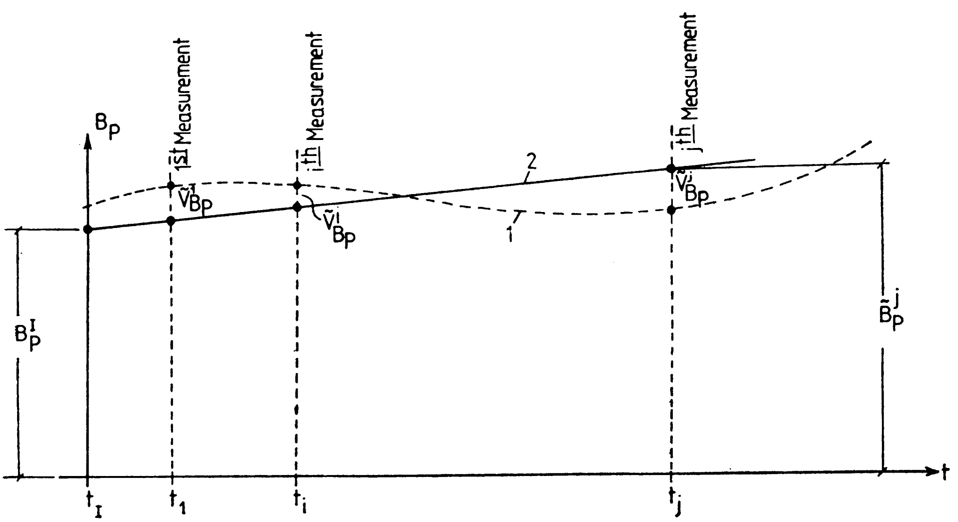


Fig. 28.1. Time Variation of the Latitude of Point P:
1 — actual variation; 2 — probable linearized variation (in the sense of the least squares method); $\tilde{v}$ — variations of a random nature.

For all the observations carried out within several measurement epochs at all the network's points, the following orientation unknowns:

$$\mathrm{d}o^T = \| \mathrm{d}o_A^1, \ldots, \mathrm{d}o_A^i, \ldots, \mathrm{d}o_B^1, \ldots, \mathrm{d}o_B^i, \ldots, \mathrm{d}o_P^1, \ldots, \mathrm{d}o_P^i, \ldots \| . \tag{28.5}$$

will be considered.

(2) *Refractive indices.* As was mentioned in Sections 17.2 and 17.3, in the most general case, for every station and for every measurement of zenithal angle a refractive index k should be introduced:

$$\mathbf{k}^T = \| k_A^1 \ldots, k_A^i, \ldots, k_B^1, \ldots, k_B^i, \ldots, k_P^1, \ldots, k_P^i, \ldots \| . \tag{28.6}$$

Solutions of this kind would very much increase the number of unknowns contained in the functional model and one could therefore introduce a single refractive index for an entire measurement epoch, so that:

$$\mathbf{k}^T = \| k_{\mathrm{I}}, k_{\mathrm{II}}, \ldots \| . \tag{28.7}$$

The pronounced reduction in the number of unknowns which is achieved by adopting such solutions may, however, be harmful for the correctness of the formation of the functional model. A more rational solution could be obtained by dividing the area covered by the geodetic network taken into consideration into several zones denoted by $a, b, \ldots$, in which the refractive index may be regarded as (characteristically) constant, so that:

$$\mathbf{k}^T = \| k_{\mathrm{I}a}, k_{\mathrm{I}b}, \ldots, k_{\mathrm{II}a}, k_{\mathrm{II}b}, \ldots \| . \tag{28.8}$$

The cost necessities as well as those of modelling the actual situation as correctly as possible are better served by this latter solution.

(3) *Scale unknowns.* For every electronic instrument for directly measuring the distances, one may introduce a supplementary scale unknown, which will be determined by adjustment:

$$\mathbf{m}^T = \| m_1, m_2, \ldots \| . \tag{28.9}$$

(4) *Additional calibration unknowns.* On completion of the scale unknowns (28.9), in the case of utilizing electronic instruments for directly measuring distances, one can also introduce, as supplementary unknowns, additional calibration coefficients:

$$\mathbf{a}^T = \| a_1, a_2, \ldots \| . \tag{28.10}$$

(5) *Coordinates and altitudes, with their displacements rates.* These unknowns are represented by the increases $\mathrm{d}B$, $\mathrm{d}L$, $\mathrm{d}H$ and the displacement rates $\dot{B}, \dot{L}, \dot{H}$ occurring in relations of the (28.2) type. In order to express all the unknowns involved in the functional model in metric units, it is advisable to operate with the quantities $\mathrm{d}\bar{B}$ and $\mathrm{d}\bar{L}$, defined as in (17.26), which will thus replace the geodetic-coordinate increases $\mathrm{d}B$ and $\mathrm{d}L$:

$$(M + H)\frac{\mathrm{d}B''}{\rho''} = \mathrm{d}\bar{B}\ ; \ (N + H)\ \frac{\cos B\ \mathrm{d}L''}{\rho''} = \mathrm{d}\bar{L} \cdot \tag{17.26}$$

where M and N are the main radii of curvature. The increases $\mathrm{d}\bar{B}$ and $\mathrm{d}\bar{L}$ have the meaning of elementary meridian and parallel arcs, calculated at the ellipsoidal altitude H of the point concerned.

One similarly defines the rates of linear displacement:

$$\frac{M+H}{\rho''}\left(\frac{\mathrm{d}B''}{\mathrm{d}t}\right)_P = \left(\frac{\mathrm{d}\bar{B}}{\mathrm{d}t}\right)_P = \dot{\bar{B}}_P\,;\ \frac{(N+H)\cos B}{\rho''}\left(\frac{\mathrm{d}L''}{\mathrm{d}t}\right)_P =$$

$$= \left(\frac{\mathrm{d}l}{\mathrm{d}t}\right)_P = \dot{\bar{L}}_P \tag{28.11}$$

vhich replace the rates of angular displacement $\dot{B}_p$, $\dot{L}_p$ defined by (28.1). Over the whole of the geodetic network under consideration, one will consequently operate with the following vectors:

$$\begin{aligned} \mathbf{dB}^T &= \|\ \mathrm{d}\bar{B}_A, \dot{\bar{B}}_A, \ldots, \mathrm{d}\bar{B}_P, \dot{\bar{B}}_P, \mathrm{d}\bar{B}_R, \dot{\bar{B}}_R, \ldots \|\,; \\ \mathbf{dL}^T &= \|\ \mathrm{d}\bar{L}_A, \dot{\bar{L}}_A, \ldots, \mathrm{d}\bar{L}_P, \dot{\bar{L}}_P, \mathrm{d}\bar{L}_R, \dot{\bar{L}}_R, \ldots \|\,; \\ \mathbf{dH}^T &= \|\ \mathrm{d}H_A, \dot{H}_A, \ldots, \mathrm{d}H_P, \dot{H}_P, \mathrm{d}H_R, \dot{H}_R, \ldots \|. \end{aligned} \tag{28.12}$$

Therewith, the total vector of unknowns appearing in the functional model denoted by $\mathbf{x}$ has the form:

$$\mathbf{x}^T = \|\ \underbrace{\mathbf{dB}^T, \mathbf{dL}^T, \mathbf{do}^T, \mathbf{m}^T, \mathbf{a}^T,}_{\text{position } \mathbf{x}_1^T}\ \underbrace{\mathbf{dH}^T, \mathbf{k}^T}_{\text{height } \mathbf{x}_2^T} \|. \tag{28.13}$$

The stochastic model consists of the following main components:

(1) *In order to find an optimum solution, in the sense of the least squares method, one introduces variations $\tilde{v}$ of a random nature of the displacement rates of the geodetic points* (Fig. 28.1):

$$B_P^{i} = \bar{B}_P^{i} + \tilde{v}_{BP}^{i}\,;\ L_P^{i} = \bar{L}_P^{i} + \tilde{v}_{LP}^{i}\,;\ H_P^{i} = H_P^{i} + \tilde{v}_{HP}^{i}. \tag{28.14}$$

Equally, for the practical reasons previously mentioned, it is useful to operate with other quantities denoted by $\tilde{v}_{\bar{B}}$, $\tilde{v}_{\bar{L}}$ expressed in terms of to $\tilde{v}_B$ and $\tilde{v}_L$ by means of relations analogous to (17.26). In this manner, considering all the points lying in the geodetic network, will result the general vector v which may also be divided into two components, in the position—height conception mentioned in § 18.7.3 and 28.1.2:

$$\tilde{v}^T = \|\ \underbrace{\tilde{v}^1_{B_A}, \tilde{v}^1_{L_A}, \ldots, \tilde{v}^i_{B_A}, \tilde{v}^i_{L_A}, \ldots, \tilde{v}^1_{B_A}, \tilde{v}^i_{L_A}, \ldots, \tilde{v}^i_{B_A}, \tilde{v}^i_{L_A}, \ldots}_{\text{position } \tilde{\mathbf{v}}_1^T} \tag{28.15}$$

$$\underbrace{\ldots, v^2_{H_A}, \ldots, v^i_{H_A}, \ldots, v^i_{H_P}, \ldots, v^i_{H_P}, \ldots \|,}_{\text{height } \tilde{\mathbf{v}}_2^T}$$

The components of the vector $\tilde{v}^T$ have the meaning of *corrections* representing the differences (of a random nature) which exist between the actual displacements of the geodetic points and the displacements described by (28.2). The corrections $\tilde{v}$ are correlated quantities, having the variance-covariance matrix denoted by $\tilde{\mathbf{Q}}$. Some practical methods for determining the matrix $\tilde{\mathbf{Q}}$ were given by *Ghițău* (*1970*).

(2) *For every measurement, carried out at the moment t_i, between the geodetic points P and R, preliminarily reduced to the normal to the ellipsoid — denoted by M'^{i}_{PR} — one will determine a correction denoted by v^{i}_{MPR} so that the probable value M^{i}_{PR} follows from the well-known equality:*

$$M^{i}_{PR} = M'^{i}_{PR} + v^{i}_{M_{PR}}. \tag{28.16}$$

Associated with the measurements M^i is the variance-covariance matrix $\mathfrak{Q}$ (s. § 28.1.1.7).

(3) *The adjustment will be developed under the condition of minimum:*

$$\Omega = \mathbf{v}^T\, \mathfrak{Q}^{-1}\mathbf{v} + \tilde{\mathbf{v}}^T\, \tilde{\mathfrak{Q}}^{-1}\tilde{\mathbf{v}} = \text{minimum}. \tag{28.17}$$

28.1.3 Forms of the Equations of the Corrections

To every measurement carried out corresponds an equation of the most general form, in which both corrections v and $\tilde{v}$, as well as unknowns x, specific to Three-Dimensional Geodesy appear. As the basic principles have already been presented in Section 17.3 of the treatise, we will here give the results corresponding to the problem under examination, using the specializations already mentioned.

Azimuthal direction reduced to the normal to the ellipsoid — d'. The direction measured at the moment t_i, from the point P towards the point R, denoted by d^{0i}_{PR}, may be reduced to the normal to the ellipsoid by applying the correction c_3 (formula (10.14) in Part Two of the treatise), which expresses the influence of the deflection of the vertical:

$$\mathrm{d}'^{i}_{PR} = \mathrm{d}^{0i}_{PR} + c_3. \tag{28.18}$$

By introducing the relations (28.2), (28.11) and (28.14) into formula (17.46), one gets for the azimuthal direction reduced to the normal to the ellipsoid the following equation:

$$\begin{aligned} &- v^{i}_{d_{PR}} + a_1\tilde{v}^{i}_{\bar{B}_P} + a_2\tilde{v}^{i}_{\bar{L}_P} + a_3\tilde{v}^{i}_{H_P} + a_4\tilde{v}^{i}_{\bar{B}_R} + a_5\tilde{v}^{i}_{\bar{L}_R} + a_6\tilde{v}^{i}_{H_R} + \\ &+ a_1\mathrm{d}\bar{B}_P + a_2\mathrm{d}\bar{L}_P + a_3\mathrm{d}H_P + a_4\mathrm{d}\bar{B}_R + a_5\mathrm{d}\bar{L}_R + a_6\mathrm{d}H_R + a_{10}\dot{\bar{B}}_P + \\ &+ a_{11}\dot{\bar{L}}_P + a_{12}\dot{H}_P + a_{13}\dot{\bar{B}}_R + a_{14}\dot{\bar{L}}_R + a_{15}\dot{H}_R - do^{i}_P + [A'_{PR} - \\ &- (d'^{i}_{PR} + o'^{i}_P)] = 0, \end{aligned} \tag{28.19}$$

where the coefficients $a_1, \ldots, a_6$ were defined by means of the relations (17.29) and the other coefficients are:

$$\begin{aligned} &a_{10} = a_1\Delta t_{1i}\,; \quad a_{11} = a_2\Delta t_{1i}\,; \quad a_{12} = a_3\Delta t_{1i}\,; \\ &a_{13} = a_4\Delta t_{1i}\,; \; a_{14} = a_5\Delta t_{1i}\,; \; a_{15} = a_6\Delta t_{1i}. \end{aligned} \tag{28.20}$$

The calculation of the coefficients a, of the auxiliary constants $\bar{Q}, \bar{R}, \bar{P}$, of the provisional azimuth A'_{PR} (relations (17.11) and (17.15) and of the provisional orientation angle o'^{i}_{PR} is performed using the provisional coordinates B'^{I}_{P}, L'^{I}_{P}, B'^{I}_{R}, L'_{R}.

Distance reduced to the normal to the ellipsoid — D'. The distance directly measured by means of an electromagnetic instrument, at the moment t_i, between the points P and R is modified by the physical corrections special to the instrument used, the distance D^{0j}_{PR} at the level of the ellipsoidal altitudes H^E_P and H^E_R being obtained (Fig. 10.5). After adjustment, by considering the scale and additional factors (m and a respectively), the condition:

$$(D^{0j}_{PR} + v^{j}_{DPR})(1 - m) + a = D'_{PR} + \mathrm{d}D^{j}_{PR} \tag{28.21}$$

must be artisfied, where D'_{PR} represents the provisional distance calculated with the provisional geodetic coordinates B'^{I}_{P}, L'^{I}_{P}, B'_{R}, L'^{I}_{R} (formulae (17.12) and (17.15)). If one takes into consideration the formula (17.31), expressing the correction $\mathrm{d}D_{PR}$, together with the supplementary relations (28.2), (28.11) and (28.14) it follows from (28.21) that:

$$\begin{aligned} &- v^{j}_{DPR} + e_1\tilde{v}^{j}_{\bar{B}P} + e_2\tilde{v}^{j}_{\bar{L}P} + e_3\tilde{v}^{j}_{HR} + e_4\tilde{v}^{j}_{\bar{B}R} + e_5\tilde{v}^{j}_{\bar{L}R} + e_6\tilde{v}^{j}_{HR} + \\ &+ e_1\mathrm{d}\bar{B}_P + e_2\mathrm{d}\bar{L}_P + e_3\mathrm{d}H_P + e_4\mathrm{d}\bar{B}_R + e_5\mathrm{d}\bar{L}_R + e_6\mathrm{d}H_R + e_9\dot{\bar{B}}_P + \\ &e_{10}\dot{\bar{L}}_P + e_{11}\dot{H}_P + e_{12}\dot{\bar{B}}_R + e_{13}\dot{\bar{L}}_R + e_{14}\dot{H}_R + D^{0j}_{PR}\cdot m - a + (D'_{PR} - \\ &- D^{0j}_{PR}) = 0, \end{aligned} \tag{28.22}$$

where the coefficients $e_1, \ldots, e_6$ have been defined by means of (17.32) and the other coefficients are:

$$\begin{aligned} &e_6 = e_1\Delta t_{Ij};\ e_{10} = e_2\Delta t_{Ij};\ e_{11} = e_3\Delta t_{Ij}; \\ &e_{12} = e_4\Delta t_{Ij};\ e_{13} = e_5\Delta t_{Ij};\ e_{14} = e_6\Delta t_{Ij}. \end{aligned} \tag{28.23}$$

For calculating the coefficients e one utilizes the provisional geodetic coordinates B'^{I}, $L^{I'}$.

Laplace azimuth — A'. If one takes into consideration the Laplace azimuth A^{i}_{Pf}, determined at the moment t_i (formulae (10.9) and (10.10)) from the point P towards the point f, one may use as basic equation:

$$A^{i}_{Pf} + v^{i}_{APf} = A^{i}_{Pf} + \mathrm{d}A^{i}_{Pf}, \tag{28.24}$$

in which A'_{Pf} represents the provisional value of the azimuth, determinable by means of (17.11) and (17.15), by utilizing the provisional geodetic coordinates of these points and the correction $\mathrm{d}A$ is to be expressed with respect to the relations (17.27), (28.2), (28.11) and (28.14). In this manner one gets:

$$\begin{aligned} &- v^{i}_{APf} + a_1\tilde{v}^{i}_{\bar{B}P} + a_2\tilde{v}^{i}_{\bar{L}P} + a_3\tilde{v}^{i}_{HP} + a_4\tilde{v}^{i}_{\bar{B}f} + a_5\tilde{v}^{i}_{\bar{L}f} + a_6\tilde{v}^{i}_{Hf} + \\ &+ a_1\mathrm{d}\bar{B}_P + a_2\mathrm{d}\bar{L}_P + a_3\mathrm{d}H_P + a_4\mathrm{d}\bar{B}_f + a_5\mathrm{d}\bar{L}_f + a_6\mathrm{d}H_f + a_{10}\dot{\bar{B}}_P + \\ &a_{11}\dot{\bar{L}}_P + a_{12}\dot{H}_P + a_{13}\dot{\bar{B}}_R + a_{14}\dot{\bar{L}}_R + a_{15}\dot{H}_R + (A'_{Pf} - A'_{Pf}) = 0, \end{aligned} \tag{28.25}$$

where the coefficients $a_1, \ldots, a_6$ are calculated by means of the relations (17.29) and the coefficients $a_{10}, \ldots, a_{15}$ from (28.20).

Remark. If the point P is a fixed point (denoted by F), then:

$$\tilde{v}_{\bar{B}F} = \tilde{v}_{\bar{L}F} = \tilde{v}_{H_F} = \mathrm{d}\bar{B}_F = \mathrm{d}\bar{L}_F = \mathrm{d}\bar{H}_F = \dot{\bar{B}}_F = \dot{\bar{L}}_F = \dot{H}_F = 0. \tag{28.26}$$

Such situations may also occur in the case of the other types of equations previously examined.

Zenithal angle reduced to the normal to the ellipsoid — β'. If one considers the zenithal angle β_{PR}^{oi} measured at the moment t_i, from the point P towards the point R, reduced afterwards to the normal to the ellipsoid (the relations (10.17)), denoted by $\beta_{PR}'^{i}$, the necessary equation is obtained from the relations (17.47), (28.2), (28.11) and (28.15):

$$\begin{aligned} &- \tilde{v}^i_{\beta PR} + f_1\tilde{v}^i_{\bar{B}P} + f_2\tilde{v}^i_{\bar{L}P} + f_3\tilde{v}^i_{HP} + f_4\tilde{v}^i_{\bar{B}R} + f_5\tilde{v}^i_{\bar{L}R} + f_6\tilde{v}_{HR} + \\ &+ f_1\mathrm{d}\bar{B}_P + f_2\mathrm{d}\bar{L}_P + f_3\mathrm{d}H_P + f_4\mathrm{d}\bar{B}_R + f_5\mathrm{d}\bar{L}_R + f_6\mathrm{d}H_R + f_6\dot{\bar{B}}_P + \\ &+ f_{10}\dot{\bar{L}}_P + f_{11}\dot{H}_P + f_{12}\dot{\bar{B}}_R + f_{13}\dot{\bar{L}}_R + f_{14}\dot{H}_R - D'_{PR}\cdot \mathrm{d}k^i_P + [\beta'_{PR} - \\ &\qquad - (\beta'^i_{PR} + D'_{PR}\cdot k')] = 0. \end{aligned} \tag{28.27}$$

in which the coefficients $f_1, \ldots, f_6$ have been defined by means of the relations (17.34)—(17.37) and:

$$\begin{aligned} &f_9 = f_1\Delta t_{1i}\,; f_{10} = f_2\Delta t_{1i}\,; f_{11} = f_3\Delta t_{1i}\,; \\ &f_{12} = f_4\Delta t_{1i}\,; f_{13} = f_5\Delta t_{1i}\,; f_{14} = f_6\Delta t_{1i}\,; \end{aligned} \tag{28.28}$$

k' is the provisional value for the influence of the vertical refraction, such as e.g.:

$$k' = 0.13/2\ R, \tag{28.29}$$

where R is the mean radius of curvature so that:

$$k = k' + \mathrm{d}k^i_P; \tag{28.30}$$

β'_{PR} is the provisional value for the zenithal angle, calculable by means of the relations (17.13) and (17.15), by utilizing the provisional coordinates B'^{I}, L'^{I}, H'^{I} of the points P and R.

Gravity determinations of the highest precision — g^0. Let us assume that at the points P and R were carried out gravity determinations of the highest precision at the moments t_1 and t_i: g_P^{01}, g_P^{0i} and g_R^{01}, g_R^{0i} respectively. Inasmuch as $g = g(B, L, H)$ and the coordinates of the points P and R have time variations, one may write:

$$\begin{aligned} g_P^{0i} + v^i_{gP} = g_P^{01} + v^1_{gP} + \left(\frac{\partial g}{\partial B}\right)_P \left(\frac{\mathrm{d}\bar{B}}{\mathrm{d}t}\right)_P \Delta t_{1i} + \left(\frac{\partial g}{\partial L}\right)_P \left(\frac{\mathrm{d}\bar{L}}{\mathrm{d}t}\right)_P \Delta t_{1i} + \\ + \left(\frac{\partial g}{\partial H}\right)_P \left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_P \Delta t_{1i} + \ldots \end{aligned} \tag{28.31}$$

and analogously for the point R. In this manner, the spatial displacements of any point P could be determined, in principle, by means of gravity measurements of the highest precision, repeated at certain time intervals. With the notations already adopted the relation (28.31) can be written in the form:

$$v^1_{g_P} - v^i_{g_P} + c^P_1\dot{\bar{B}}_P\Delta t_{1i} + c^P_2\dot{\bar{L}}\Delta t_{1i} + c^P_3\dot{H}_P\Delta t_{1i} + (g^{01}_P - g^{0i}_P) = 0, \tag{28.32}$$

where:

$$c^P_1 = \left(\frac{\partial g}{\partial \bar{B}}\right)_P;\ c^P_2 = \left(\frac{\partial g}{\partial \bar{L}}\right);\ c^P_3 = \left(\frac{\partial g}{\partial H}\right), \tag{28.33}$$

represent the horizontal gradients and the vertical gradient of gravity respectively.

An equation similar to (28.32) may be written for the point R too. Since one usually determines gravity differences between the points P and R, denoted for the measurement moments t_1 and t_i by δg^{01}_{RP} and δg^{0i}_{PR} respectively, the equation which is to be used in the model under examination is obtained by subtracting the relations of the type (28.32) written for the points P and R:

$$v^1_{g_R} - v^i_{g_R} - v^1_{g_R} + v^i_{g_R} + c^1_R\dot{\bar{B}}_R\Delta t_{1i} + c^R_2\dot{\bar{L}}_R\Delta t_{1i} + c^R_3\dot{H}_R\Delta t_{1i} -$$

$$- c^P_1\dot{\bar{B}}\Delta t_{1i} - c^P_2\dot{\bar{L}}\ \Delta t_{1i} - c^P_3\dot{H}\ \Delta t_{1i} + (\delta g^{01}_{PR} - \delta g^{0i}_{PR}) = 0. \tag{28.34}$$

Remarks:

(1) In the free term of (28.34), the gravity differences dg^0 remain gravimetrically non-reduced, i.e. they are subjected only to instrumental reductions.

(2) The coefficients c are determined experimentally, as they have different values from one point to an other. New methods for the practical determination of the gravity gradients are presented by *H. Wolf* (1972).

(3) In most cases, the coefficients c_1 and c_2 are much smaller in size compared with c_3, so that one can decide to ignore their contribution, the equation (28.34) thus taking a much simpler form. In such a case, one can, of course, only bring out the vertical displacement of the points P and R taken into consideration.

(4) As was mentioned both in Part One of the treatise and in 27.5.1, the gravity variations may be provoked not only by the spatial displacements of the gravimetric points but also by other factors wich as e.g.: Earth tides, physical-chemical modifications within the subcrustal ayer etc. We will, not, however, go into these aspects in any detail in the functional model examined here.

Difference of ellipsoidal altitudes (as determined by geometrical levelling) —H. After having applied to the level differences determined by geometrical levelling the necessary corrections (section 3.5 and formulae (10.1), (10.2)), one can obtain the difference of ellipsoidal altitudes between the points R and S, at the moment t_k, denoted by ΔH^k_{RS}. Inasmuch as:

$$\Delta H^k_{RS} + v^k_{\Delta H_{RS}} = H^k_S - H^k_R,$$

one can obtain, by using the relations (28.2) and (28.14):

$$v^k_{\Delta H_{RS}} + v^k_{H_R} - \tilde{v}^k_{H_S} + dH_R - dH_S + \dot{H}_R\Delta t_{1k} -$$

$$- \dot{H}_S\Delta t_{1k} + [\Delta H^k_{RS} - (H'^I_S - H'^I_R)] = 0. \tag{28.35}$$

Astronomical latitude reduced to the normal to the ellipsoid — Φ'. For networks of large dimensions, one can also utilize determinations of astronomical latitudes Φ in a limited number, so that their utilization within the functional model does not harm the general precision in calculating the movements of the Earth's crust. For the reduction to the normal to the ellipsoid the relation (10.5) will be applied, the quantity Φ' being obtained. After adjustment, for any point P at which such determinations were not made, the condition equation

$$\Phi'^{i}_{P} + v^{i}_{\Phi_P} = B'^{I}_{P} + \mathrm{d}B_P + \Delta t_{1i}\dot{B}, \tag{28.36}$$

must be fulfilled, whence immediately results the relation needed in calculations.

Astronomical longitude reduced to the normal to the ellipsoid — Λ'. In a manner analogous to that in the case of the astronomical latitudes, the astronomical longitude ψ may be reduced to the normal to the ellipsoid by applying the formula (10.4), ψ' being obtained after which one can write:

$$\Lambda'^{i}_{P} = v^{i}_{\Lambda_P} = L'^{I}_{P} + \mathrm{d}L_P + \Delta t_{1i}\dot{L}. \tag{28.37}$$

To conclude the description of the equations which may occur in the functional-stochastic model examined, it is useful to embody the corrections v in a vector which is also divided into two groups in the position-height conception, specific to the calculation model examined:

$$\underbrace{\mathbf{v}^T = \| \, v^1_{dPR} \ldots v^i_{dPR} \ldots v^1_{DPR} \ldots v^i_{DPR} \ldots v^1_{APf} \ldots v^i_{APf} \ldots v^1_{\Lambda P} \ldots {}^{i}_{\Lambda P} v \ldots}_{\text{position } \mathbf{v}_1^T}$$

$$\underbrace{\ldots v^1_{\beta_{PR}} \ldots v^i_{\beta_{PR}} \ldots v^1_{g_P} \ldots v^i_{g_P} \ldots v^1_{\Delta H_{PR}} \ldots v^i_{\Delta H_{PR}} \ldots v^1_{\Phi_P} \ldots v^i_{\Phi_P} \ldots \|.}_{\text{height } \mathbf{v}_2^T} \tag{28.38}$$

28.2 Particular Functional-Stochastic Models for Determining the Recent Crustal Movements

Depending on the nature and magnitude of the displacements of the Earth's crust, as well as regarding the volume of the available geodetic observations, one may utilize particular functional-stochastic models, obtained either by using the model described in Section 28.1 or by other developments based on the considerations presented in Chapter 27, of which some will be examined in the sequel.

28.2.1 Functional-Stochastic Models for Determining the Vertical Movements

The most numerous geodetic determinations of the movements of the Earth's crust hitherto carried out were performed by repeated geometric levelling. This situation has come about due to the particularly high accuracy which has characterized such operations for several decades, as well as the somewhat smaller cost in comparison with other geodetic working methods (triangulation, astro-geodetic levelling etc.). Numerous studies have now been made in which functional-stochastic models are proposed for processing the observations of repeated geometrical levelling. These will non be examined both in general and in particular.

A general functional-stochastic model (Ghiţău model). One may assume that there exists some arbitrary number (greater than 2) of measurement epochs of a geometric-levelling network. For general networks, such operations began in the XIXth century and, as a consequence, one can no longer accept the linearity of the variation of the altitude of a point R during such a time interval but only between two measurement epochs. Consequently, instead of the last relation in the group (28.2), it will be necessary to write for a determination at the moment t_i situated, e.g., within the IVth measurement epoch, the following relation *(Wolf 1963, 1968):*

$$\widetilde{H}_R^i = H'^i_R + \mathrm{d}H_R + \Delta t_{\mathrm{I-II}}\dot{H}_R^{\mathrm{I-II}} + \Delta t_{\mathrm{II-III}}\dot{H}_R^{\mathrm{II-III}} + \Delta t_{\mathrm{III}-i}\dot{H}_R^{\mathrm{III-IV}}, \qquad (28.39)$$

in which $\dot{H}_R^{\mathrm{I-II}}$, $\dot{H}_R^{\mathrm{II-III}}$, $\dot{H}_R^{\mathrm{III-IV}}$ are the movement speeds along the vertical of the point R between the measurement epochs I, II, III and IV, which are to be detemined (as unknowns) by the rigorous processing of the geodetic observations.

The greater number of unknowns which the general model will contain is justifiable in the case of the repeated geometrical levelling by the existence of a relatively large number of measurement epochs.

Taking into consideration the relations (28.39) and (28.14) describing the variation of a systematic and random nature respectively of the altitudes of the points R and S, then equation (28.36) becomes more complex *(Ghiţău 1970)*, leading to a model of a general character *(the Ghiţău model):*

$$\begin{aligned} v^i_{\Delta H_{RS}} + \widetilde{v}^i_{H_{RS}} - \widetilde{v}^i_{H_S} + \mathrm{d}H_R - \mathrm{d}H_S + \Delta t_{\mathrm{I-II}}\dot{H}_R^{\mathrm{I-II}} + \Delta t_{\mathrm{II-III}}\dot{H}_R^{\mathrm{II-III}} + \\ + \Delta t_{\mathrm{III}-i}\dot{H}_R^{\mathrm{III-IV}} - \Delta t_{\mathrm{I-II}}\dot{H}_S^{\mathrm{I-II}} - \Delta t_{\mathrm{II-III}}\dot{H}_S^{\mathrm{II-III}} - \Delta t_{\mathrm{III}-i}\dot{H}_S^{\mathrm{III-IV}} + \\ + [\Delta H^i_{RS} - (H'^{\mathrm{I}}_S - H'^{\mathrm{I}}_R)] = 0. \end{aligned} \qquad (28.40)$$

Usually, in processing the repeated geometrical levelling one operates with the level differences ΔH *un-reduced to a certain altitude system.*

The adjustment is subsequently developed on the basis of a condition of minimum of the form (28.17).

The Wolf-Jacobs functional-stochastic model. In his doctoral thesis, *E. Jacobs* develops a functional-stochastic model, in which he accepts the constancy (partial and sometimes even total constancy) of the displacement speeds during the entire time interval and disregards the existence of the corrections v, which corresponds to the proposal of *H. Wolf* from the year 1963:

$$\dot{H}_R^{\text{I-II}} = \dot{H}_R^{\text{II-III}} = \dot{H}_R^{\text{III-IV}} = \dot{H}_R; \; \tilde{v}_{HR} = \tilde{v}_{Hs} = 0. \tag{28.41}$$

In this manner the equation (28.40) becomes much simpler:

$$v^i_{\Delta H_{RS}} = \mathrm{d}H_S - \mathrm{d}H_R + \Delta t_{1i}\dot{H}_S - \Delta t_{1i}\dot{H}_R + [(H'^1_S - H'^1_R) - \Delta H^i_{RS}], \tag{28.42}$$

as specific to indirect observations.

This particular model contains fewer unknowns and may be applied with good results for relatively restricted areas, characterized by systematic movements of the Earth's crust, as in the zone of the carboniferous basin of the *Ruhr (F. R. of Germany)*, investigated by *Jacobs*, where the subsidences have also reached maximum values of —0.7 m/year (at some points).

The adjustment is developed only on the basis of the well-known relation:

$$\mathbf{v}^T\mathbf{Q}^{-1}\mathbf{v} = \text{minimum},$$

the matrin $\mathbf{Q}^{-1}$ sometimes being replaced by the diagonal matrix of the weights $\mathbf{P}$.

The Hermanovski functional-stochastic model. In several of his contributions, *A. Hermanovski* lays the foundations of a model in which only the components of a random nature v from the general equation (28.40) ower and the systematic components are disregarded:

$$\dot{H}_R^{\text{I-II}} = \dot{H}_R^{\text{II-III}} = \dot{H}_R^{\text{III-IV}} = 0. \tag{28.43}$$

The particular merit of the *Hermanovski* model lies in introducing, for the first time, the necessity of approaching the components v in such a way that the adjustment should develop on the basis of the condition of minimum (28.17). The actual proposals contain, however, a few inexactitudes and non-sequiturs *(Ghiţău 1970)*.

The geokinetic functional-stochastic model. In several contributions, *L. Bendefi* and *L. Mălzer* introduce the notion of *geokinetic model*, obtained by subtracting the equations of the form (28.40) or of a simpler form (e.g. (28.42)), written for two successive measurement epochs. In this manner, the unknowns dH are eliminated before the processing proper, not being contained in the functional model, which, generally, forms a simpler structure.

The previously specified papers contain, however, some structural inexactitudes *(Ghiţău 1970)*. *Replacing two groups of equations by a single group, obtained by subtracting the corresponding equations from the two groups, one can provide the same results as in the block processing of the initial observations only if:*

(1) *One takes into consideration the mathematical correlations introduced by the above-mentioned linear transformation.*

(2) *To the equations obtained by transformation one adds one of the initial groups.*

These aspects are neglected in the papers mentioned. Additionally, in the case of *H. Mälzer* the proposal is made that the processing of the observations should be made on the basis of the condition of minimum of the modifications established in the level differences:

$$[(\Delta H^{\mathrm{I}}_{RS} - \Delta H^{\mathrm{II}}_{RS})^2) = \text{minimum} \qquad (28.44)$$

This construction of the calculation model has as consequence an *optimum distribution* (in the sense of the least squares method) of the vertical movements of the Earth's crust, in the whole network, which obviously contravenes reality, inasmuch as the network's points may have displacements differing greatly as to intensity, and obeying no law of normal distribution. Applying the models proposed by *Jacobs* and *Mälzer* to one and the same volume of repeated observations, completely different results were obtained *(Ghiţău 1970)*, which shows the impossibility of utilizing the geokinetic model (with the condition of minimum (28.44)) in areas with pronounced movements of the Earth's crust.

The Finnish functional-stochastic model. The Finnish geodetic school has the distinction of giving first the extremely exact and comprehensive determinations of the vertical movements of the Earth's crust, presented in several papers of *E. Kääriainen* in the years 1953, 1966, 1975 etc. The model used in adjustment is based on the determination of rates of time modifications of the level differences. In the notation utilized in this chapter, one should operate with quantities of the form ΔH obtained from:

$$\Delta \dot{H}_{AB} = \dot{H}_B - \dot{H}_A. \qquad (28.45)$$

Inasmuch as such calculations may be made for all level differences existing in the network, the necessity arises of introducing the condition equations of the form:

$$\Sigma_P \Delta \dot{H} = 0, \qquad (28.46)$$

written for every polygon P of the levelling network concerned (actual or fictitious polygon — between adjacenk old points). The modification rate of the level difference between the points A and B, measured at the moments t_i and t_j, is calculated from the formula:

$$\Delta \dot{H}^{ij}_{AB} = \frac{\Delta H^j_{AB} - \Delta H^i_{AB}}{\Delta t_{ij}}. \qquad (28.47)$$

As a consequence, the processing of the repeated levelling would be carried out in the following order:

(1) *One calculates the modification rates of every level difference by means of* (28.47).

(2) *One carries out a preliminary adjustment of these rates on the basis of equations of the form* (28.46), *from which result the most probable values.*

(3) *The level differences measured at different moments within a measurement epoch are reduced to one and the same moment, depending on the specific*

modification rates. Thus, e.g., the level difference measured between the bench marks A and B at the moment t_i, situated within the measurement epoch I, is reduced to the moment t_1 using the relation

$$\Delta H^{\mathrm{I}}_{AB} = \Delta H^{i}_{AB} - \Delta \dot{H}^{\mathrm{I-II}}_{AB} \cdot \Delta t_{\mathrm{I}i}. \qquad (28.48)$$

(4) *The level differences reduced by means of* (28.48) *are separately adjusted, for each measurement epoch, in the conventional manner, after which one calculates the altitudes of the levelling bench marks in each measurement epoch.* The difference between the values obtained for the altitude of a bench mark then provides the movement along the vertical sought.

The Finnish model offers remarkable practical advantages, which has led to its utilization in several countries. The final results differ with respect to those obtained from the strictly rigorous processing, inasmuch as in the order given, one inevitably accepts a series of approximations. The estimation of the accuracy in determining the displacements is also clumsier than in the case of models 28.2.1, inascmuch as the unknowns $\dot{H}_A$, $\dot{H}_B$, ... and their errors are calculated after the processing has been carried out.

Functional-stochastic model for describing the trend of vertical movement of an area. In the studies presented in this chapter, every levelling bench mark has been regarded as an independent element. One may, however, also investigate the movement which would be made by an area where there are several levelling bench marks, as a unitary self-contained whole. Such an investigation may clarify the *movement trend* in the future of this area *(Ghiţău 1970)*.

The displacement made by the area is displayed differently at each of the bench marks it contains, as a function of their horizontal coordinates (X, Y). Thus, in the hypothesis of a linear displacement of the area, one may write for any of the bench marks P:

$$\dot{H}_P = R_{00} + R_{10}X_P + R_{01}Y_P, \qquad (28.49)$$

where the parameters R_{00}, R_{10}, R_{01} represent unknowns to be determined by the adjustment of the geodetic observations.

If the investigated area has a more complex movement, the equation (28.49) is insufficient and the utilization of a more complex functional model of the form:

$$\dot{H}_P = R_{00} + R_{10}X_P + R_{01}Y_P + R_{11}X_PY_P + R_{20}X_P^2 + R_{02}Y_P^2 + \ldots \qquad (28.50)$$

will be necessary.

The expressions of the displacement rates, (28.49) or (28.50), according to the case, are to be introduced into equations of the form (28.40), (28.42) etc., the necessary calculation model being obtained.

The coefficients R_{00}, R_{10}, R_{01}, ... may then be utilized for estimating the trend of displacement of the investigated area in the future.

28.2.2 Functional-Stochastic Models for Determining the Horizontal Movements

The researches carried out until now for determining the horizontal movements of the Earth's crust have had a local character or have covered a few somewhat extended areas but with large horizontal displacements. The motivation for this state of affairs, as well as the present technical possibilities were analysed in Section 27.3.

The processing of the repeated geodetic observations carried out for this purpose was performed in the conventional manner, separately for every measurement epoch, after which one has determined, just by taking differences, the movements along the directions X, Y or B, L, according to the case, depending on the coordinate system being used.

28.2.3 The Functional-Stochastic Model Used for Determining the Continental Movements of the Earth's Crust

The astro-geodetic measurements carried out following the methods described in Section 27.4 are processed separately, independently, with the known conventional calculation methods. One takes into consideration, as was specified in Section 27.4, all possible sources of errors, by applying specific corrections, so that the results obtained are characterized by a high accuracy.

The movements of a continental character of the Earth's crust are then calculated by taking the differences between the determinations performed at certain time intervals.

28.2.4 Functional-Stochastic Models for Processing the Tide Gauge Recordings with a view to Determining the Crustal Movements

There exist numerous theories for processing the tide gauge recordings (*Lennon 1965, Montag 1970, Jakubovski 1965, Puruckherr 1973* among others), by means of which one determines, on the basis of hypotheses of a statistical nature, the time variations of the mean annual sea level in the area where the tide gauge is installed.

The calculation method which is presented in what follows is based on the principle of successive approximations and proposes to itself as main objective the simultaneous processing of the repeated levelling and of the tide gauge recordings within the framework of a geodetic network of large dimensions, of the type presented in Fig. 29.1, devoted to determinations of recent crustal movements. The results which will be obtained at one stage of the calculation will submitted to an analysis of a statistical significance and, in the case of an affirmative result, will subsequently be used as known elements or as initial data for the next calculation stages.

It is advisable that the recordings obtained at a certain tide gauge be submitted to an independent analysis so that by numerical filtering there should progressively be eliminated the influences which don't constitute the main objective of the calculations. To this end, one will utilize, at various stages of the processing, according to necessities, the averages determined by means of (27.15).

The calculations may begin by determining, in a first approximation, the modification rate of the sea level, denoted by $\dot{z} = \mathrm{d}z/\mathrm{d}t$. To this end, one may utilize the average $z_{18.6}$, referring to a long time period, so that they offer good estimating possibilities at this calculation stage. For every mean value $z_{18.6}$ one can write the following correction equation:

$$v_{18.6} = z^{\mathrm{I}}_{18.6} + (t_{18.6} - t_{\mathrm{I}})\dot{z}_{18.6} - z_{18.6},$$

which is obtained from relations of the form (28.2), (28.3), (28.14), under the assumption $z_{18.6}$ = constant during the interval $\Delta t_{\mathrm{I}i} = t_{\mathrm{I}} - t_{\mathrm{I}}\cdot$, where, $z^{\mathrm{I}}_{18.6}$ denoted a mean value of the sea level for the initial moment t_{I} (arbitrarily chosen).

The next iteration in the calculation process is represented by the determination of the meteorological influences on the variation of the sea level. Inasmuch as the atmospheric pressure, denoted by B, and the wind's action are reciprocally independent, their common influence on the sea level may be estimated by utilizing the monthly means z_m (27.15) within the framework of every calendar year on the basis of the following regression *(Montag* 1967*)*:

$$v_m = z^{\mathrm{I}}_m + K^1_m(B^1_m - B_{\mathrm{I}}) + K^2_m(B^2_m - B_{\mathrm{I}}) + \ldots + [z_m - (t_i - t_{\mathrm{I}})\dot{z}_{18.6}].$$

$K^1_m, K^2_m, \ldots$ denoted the coefficients of influence of the atmospheric pressure characteristic for the meteorological stations 1, 2, ..., which have provided the mean monthly values B^1_m, B^2_m, ..., and B_{I} represents a mean value of the atmospheric pressure for the moment t_{I}. In an analogous manner one can also determine other meteorological influences, as well as the influence of the density and temperature of the water *(Montag 1967, 1970, Lennon* 1965, among others). The coefficients in the corresponding regressions will be symbolically denoted by D and the corresponding factors by F.

The final stage in the processing carried out for each tide gauge separately may be represented by the determination of the influence of a periodic nature exerted by the Moon, by utilizing the following type of equations:

$$v_a = z^{\mathrm{I}}_a + (t_a - t_{\mathrm{I}})\dot{z}_a + k_1 \sin \Omega\Delta T_{\mathrm{I}a} + k_2 \cos \Omega\Delta T_{\mathrm{I}a} - \\ -\left[z_a - \sum_i K_i(B^i_a - B_{\mathrm{I}}) - \sum D_a F_a\right],$$

where:

$$k_1/k_2 = \mathrm{cotg}\ \Phi,$$

Ω and Φ being the angular velocity and the phase respectively of the oscillatory movement describing the above-mentioned influence during periods larger than 18.6 years, which is considered to be the nodal tide. Including the parameter z_a at this stage of the calculation ensures a better determination of the influence coefficients k_1 and k_2.

After the preliminary stage described above, one can calculate corrected values of the mean annual sea-level, which are free from the influence of the meteorological factors, of the density and temperature of the water, as well as from the influences exerted by the Moon and the Sun. The preliminarly corrected values, denoted by (Z), may be calculated from the following relation:

$$(Z) = z_a - \sum_i K^i (B_a^i - B_\mathrm{I}) - \sum D_a F_a - k_1 \sin \Omega \Delta T_{\mathrm{I}a} - k_2 \cos \Omega \Delta T_{\mathrm{I}a}$$

inasmuch as the influence coefficients K^i, D_a, k_1, k_2 are known.

At the stage of common processing of the tide-gauge recordings and of the repeated levelling, the values (Z) will represent initial elements, which will receive new corrections, depending on the interdependence existing between the categories of observations contributing to this processing. If one utilizes the relations (28.2), (28.3) and (28.14), one may write, in an analogous manner, for a tide gauge m, at the moment t :

$$[(Z_m^J)] = (Z_m^J) + v_{Zm}^J = Z_m^\mathrm{I} + \mathrm{d}Z_m + \Delta t_{\mathrm{I}J} \dot{Z}_m.$$

The displacement rate $\dot{Z}_m$ of the sea level contains as its main element, at this stage, the displacement rate along the vertical of the Earth's crust denoted by $\dot{H}_m$ but also other elements, of a eustatic or local nature, depending on the variation of the water volume, denoted by $\dot{E}_m$:

$$\dot{Z}_m = \dot{H}_m + \dot{E}_m.$$

If we accept that the rates of displacement of the Earth's crust in the area of the tide gauge m and of the corresponding main levelling bench mark, designated by M, are equal:

$$\dot{H}_m = \dot{H}_M,$$

then it follows that

$$v_{Zm}^J = Z_m^\mathrm{I} + \mathrm{d}Z_m + \Delta t_{\mathrm{I}J} \dot{H}_M + \Delta T_{\mathrm{I}J} \dot{E}_M - (Z_m^J). \tag{28.51}$$

The relation (28.51) represents the connexion equation between the tide-gauge recordings and the repeated-levelling observations, inasmuch as the expression of the correction v_{ZM}^J which is to be applied to the mean annual value, preliminarily corrected, obtained from the recordings at the tide gauge m includes in its structure the element $\dot{H}_M$ which appears in all the equations of the type (28.35) or (28.40), specific to the operations of repeated geometrical levelling. For instance, if between the main reference bench marks M and N of the tide gauges m and n one measures, at the moment t_J, the level difference ΔH_{MN}^J, then, under the assumption $\dot{H} =$ constant, one gets from (28.40):

$$v_{\Delta H_{MN}}^J + \tilde{v}_{H_M}^J - \tilde{v}_{H_N}^J + \mathrm{d}H_M - \mathrm{d}H_N + \Delta t_{\mathrm{I}J} \dot{H}_M - \Delta t_{\mathrm{I}J} \dot{H}_N +$$
$$+ [H_{MN}^J - (H_N'^\mathrm{I} - H_M'^\mathrm{I})] = 0. \tag{28.52}$$

The foundation is thereby laid for the functional-stochastic model necessary for the simultaneous processing of the two categories of observations as reflected by the common elements $\dot{H}_M$ and $\ddot{H}_M$ respectively, which intervene both in expressions of the kind of (28.51), arising from processing the tide-gauge recordings corrected at preliminary calculation stages and in expressions of the type of (28.52), specific to the repeated levelling.

Consequently, for the countries having in operation any number of tide gauges, connected to the general levelling network and for which recordings extending over a large number of years are available, it will be possible to carry out the simultaneous processing of the repeated levelling observations and of the tide-gauge recordings by the corresponding synthesis of the models described in 28.2.1 and 28.2.4.

28.3 Possibilities of Adjusting the Repeated Geodetic Observations

28.3.1 Grouping in the "Position"-"Height" Mode

The functional-stochastic model described in Section 28.1 may be processed either in toto or, as was mentioned in § 28.1.2, separately for the two groups: the "position" and the "height" ones. Corresponding to this division, the method of calculation described in § 18.7.3 is subsequently followed. The connexion between the notations and the formulae utilized may be readily made.

The vector of the unknowns $\mathbf{x}$ which intervene in the the model has been divided in formula (28.13) into the components $\mathbf{x}_1$ and $\mathbf{x}_2$ and, analogously, the vector of the corrections $\tilde{\mathbf{v}}$ into the components $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ in formula (28.15). The manner of dividing the vector $\mathbf{v}$ into its components $\mathbf{v}_1$ and $\mathbf{v}_2$ was explained in § 18.7.3 and is realized in the functional-stochastic model, utilized in the relation (28.38).

The family of equations described in § 28.1.3 **may be written in the form:**

$$\begin{aligned} &\textit{1st Group}\quad \mathbf{E}\mathbf{v}_1 + \mathbf{B}_1\tilde{\mathbf{v}}_1 + \mathbf{B}_2\tilde{\mathbf{v}}_2 + \mathbf{C}_1\mathbf{x}_1 + \mathbf{C}_2\mathbf{x}_2 + \mathbf{w}_1 = \mathbf{o};\\ &\textit{2nd Group}\quad \mathbf{E}\mathbf{v}_2 + \mathbf{B}_3\tilde{\mathbf{v}}_1 + \mathbf{B}_4\tilde{\mathbf{v}}_2 + \mathbf{C}_3\mathbf{x}_1 + \mathbf{C}_4\mathbf{x}_2 + \mathbf{w}_2 = \mathbf{o}, \end{aligned} \tag{28.53}$$

where $\mathbf{E}$ represents the unit matrix (of various dimensions).

The method of calculation specific to Three-Dimensional Geodesy is represented by an iterative process described in § 18.7.3 from which follow the corrections v and $\tilde{v}$, as well as the unknowns x_1 and x_2 appearing in the system (28.53).

The probable coordinates of the points of the geodetic network at any moment t may then be calculated by means of (28.2) and (28.14).

In connexion with solving the system (28.53), one can still make the specification that, since the corrections v have the coefficient ± 1, one may make use of the *Helmert* transformation:

$$\tilde{\mathbf{v}} \equiv \mathbf{z} \tag{28.54}$$

so that the corrections $\tilde{v}$ also play the role of unknowns. In this manner one can pass from the system (28.53), specific to the general form of adjustment (in which appear corrections as well as unknowns), to a system of equations specific to indirect observations, which may more easily be programmed for electronic computers. Thus, for the first group we would obtain:

$$\begin{aligned} \tilde{\mathbf{v}}_1 &= \mathbf{E}\mathbf{z}_1; \\ \tilde{\mathbf{v}}_2 &= \mathbf{E}\mathbf{z}_2; \\ \mathbf{v}_1 &= -\mathbf{B}_1\mathbf{z}_1 - \mathbf{B}_2\mathbf{z}_2 - \mathbf{C}_1\mathbf{x}_1 - \mathbf{C}_2\mathbf{x}_2 - \mathbf{w}_1. \end{aligned} \tag{28.55}$$

It must be said that on undergoing this transformation the functional-stochastic model did not suffer structural modifications but only formal ones, so that the final results are the same as in the case of solving the first group in the system (28.53).

In the case of very large networks, the block solving of the equations in one group of (28.55) may be cumbersome, even with large-capacity electronic computers. Therefore one can also recommend other divisions into groups, from other points of view, presented in detail by *Ghiţău (1970)*, which will be mentioned briefly in what follows.

28.3.2 Grouping in the Static-Kinetic Sense

In the first group one carries out an adjustment in the conventional static sense, i.e. one does not include the unknowns describing the displacement rates of the geodetic points. In the second group one also includes the unknowns $\dot{B}, \dot{L}, \dot{H}$, which describe the kinetic aspect of the given problem. Between the groups one must ensure such a connexion that the final results are identical with those which would be obtained by a block adjustment.

28.3.3 Territorial Grouping

One may make a division into groups corresponding to a territorial division of the area under investigation. In such cases, one can apply with very good results the *Helmert-Pranis Pranievich* method of group adjustment.

28.3.4 Iterative Procedures

The groups may also be formed with respect to the measurement epochs. In this manner, the totality of the measurements carried out within a certain epoch will constitute an independent group, which is adjusted conventionally, separately.

After this first step, the speed of measurements changing may be calculated, using relations of (28.47) type, after which the observations from each measurement epoch will be "centred", that is, they will be brought to the central moments placed in the, I, II, III ... epoches, denoted by t_I, t_{II}, t_{III} ... . In such a way one eliminates in a first aproximation, the influence of geodetic points deplacements on the results obtained in every measurement epoch.

The described cycle is repeated until stabilization of the results is reached. This method of solving has numerous practical advantages. When the calculations are worked out carefully particularly when the area under investigation is small, the procedure converges rapidly and the results come close to the results obtained with by a rigorous solution.

28.4 Accuracy Estimation and Statistical Analysis of the Determinations of Recent Crustal Movements

By solving the system of equations (28.53) transformed by means of the relations (28.54) — (28.55) — block or group solving —, one obtains not only the values of the unknowns x and of the corrections v and $\tilde{v} = z$ but also their determination errors, denoted by m_x, m_z. For a complete estimation of the accuracy in determining the movements of the Earth's crust it is necessary to retain, in its entirety the inverse matrix of the system of normal equations, from which one may extract, as necessary, the quadratic or rectangular weight coefficients which are useful in the corresponding calculations.

After completing the calculation operations, the results obtained are submitted to a statistical analysis, which may lead to structural improvements of the functional-stochastic model and, as a result one may carry out a new processing, with an improved model. As was stated in § 27.1.2. the *Student* statistical test is used, at present to this end almost without exception.

For improvements in the structure of the model one examines the ratio:

$$t_k = x/m_x \tag{28.56}$$

for each of the unknowns x and, similarly, for the unknowns $z = \tilde{v}$. In the case when the value t_k determined is maller than the value t_s extracted from the *Student* tables one can neglect its contribution to the future functional-stochastic model, which will be utilized at a new stage of the processing. This elimination must be made carefully and, to begin with, it is perhaps advisable to discard only the unknowns x or z, for which the coefficient t_k is far too small compared with t_s. This precaution is suggested by the fact that every structural modification of the model used leads to final results noticeably different from the initial ones, new analyses being needed, after which the corresponding decisions are taken.

Thus arises the necessity of processing by steps, by utilizing functional-stochastic models increasing perfect in structure, the initial informational material, represented by the amount of measurement remaining, however, the same.

At every new step one also investigates the ratio of the weights of the different groups of available measurements, both after the various categories of geodetic observations (azimuthal directions, distances, level differences etc.) and after the existing measurement epochs I, II, III, ... Direct rigorous procedures, as well as iterative procedures for solving this problem are known, by means of which one can determine new weights for the next adjustment step. From among these, the iterative procedure of *Helmert* has enjoyed a wide application in such complex processes of calculations, due especially to the simplicity of its realization.

Thus, for example, if one examines the weight ratio between the measurement epochs I, II, III, ... one will calculate the following *global weight coefficients* P_{I}^{*}, P_{II}^{*}, P_{III}^{*} by means of the relation:

$$P_{I}^{*}: P_{II}^{*}: P_{III}^{*}: \ldots = \frac{n_{I}}{\Omega_{I}} : \frac{n_{II}}{\Omega_{II}} : \frac{n_{III}}{\Omega_{III}}, \tag{28.57}$$

in which: n_{I}, n_{II}, ..., represent the number of existing geodetic observations within the measurement epochs I, II, ... and Ω_{I}, Ω_{II} ... are calculated using the relation (28.17) for each epoch separately. The smaller the ratio n/Ω, the less accurate are the geodetic observations in the corresponding epoch. For the smallest ratio obtained one can then choose any value for the corresponding coefficient P^{*} (e.g. the value 1) and the other coefficients $P_{\frac{1}{2}}^{*}$ are then determined by utilizing the relation (28.57)) One divides the variance-covariance matrices $\mathbf{Q}_{I}$, $\mathbf{Q}_{II}$ ... corresponding to the measurement epochs by the coefficients P^{*} thus (or one multiplies the matrices of the weights $\mathbf{P}_{I}$, $\mathbf{P}_{II}$, ... when the covariance existing between observations is disregarded).

In an analogous manner one can also proceed with the study of the ratio between the categories of available geodetic observations, within the framework of each measurement epoch.

Consequently, the next adjustment step will develop on the basis of an improved functional-stochastic model from which the statistically insignificant unknowns have been eliminated, as well as with an improved variance-covariance matrix (respectively with better estimated weights).

After the final adjustment step, when the results have become stable, one calculates the spatial displacements of each geodetic point, for the components mentioned in this chapter: B, L, H. Thus, e.g., the movement of the point P along the meridian, between the moments i and j will be calculated from the relation:

$$\Delta B_{P}^{ij} = B_{P}^{j} - B_{P}^{i}, \tag{28.58}$$

the coordinates B_{P}^{j} and B_{P}^{i} being calculated, along with their errors, within the framework of the corresponding measurement epochs. It is also easy to determine the error $m_{\Delta B_{P}}$ of this displacement, so that one can establish if this displacement is significant or not from the statistical point of view:

$$t_{\Delta B_{P}} = \Delta B_{P}^{ij}/m_{\Delta B_{P}}. \tag{28.59}$$

If $t_{\Delta B_P}$ is greater than the coefficient t_S extracted from the *Student* tables, then one can state that there exists a displacement of the point P in the meridian direction between the measurement moments i and j. If the opposite is the case, the established displacement can be ascribed to measurement of errors.

One can also proceed analogously for the displacements ΔL in the direction of the parallel as well as for displacements along the direction of the normal to the ellipsoid ΔH. One may also calculate a total displacement ΔD_P^{ij} within this time interval:

$$\Delta D_P^{ij} = \sqrt{(\Delta B_P^{ij})^2 + (\Delta L_P^{ij})^2 + (\Delta H_P^{ij})^2}, \tag{28.60}$$

which is submitted to a similar analysis, by means of (27.1), a complete answer thus being given to the problem under examination.

29

Examples of Determinations of Recent Movements of the Earth's Crust

As was mentioned in Section 27.2 and § 28.2.1, most of the determinations of the recent movements of the Earth's crust have been performed by the method of repeated geometrical levelling. These determinations have frequently been embodied in vast programmes of investigations of an interdisciplinary character, which (particularly in the case of very large networks) evolved on the basis of international cooperation.

29.1 Map of the Recent Crustal Movements in Eastern Europe

On the basis of an international cooperation between the national geodetic, geophysical and geological services of the *People's Republic of Bulgaria, Czechoslovak Socialist Republic, German Democratic Republic, Hungarian People's Republic, Polish People's Republic, Socialist Republic of Romania, Soviet Union* and *Federative Socialist Republic of Yugoslavia,* as far back as 1963, as a consequence of a proposal made within the framework of theXIIIth General Assembly of the *International Union of Geodesy and Geophysics,* large-scale operations concerning the compilation of the map of recent crustal movements in Eastern Europe, representing a unique performance in the field of topical maps, were initiated.

A first draft of this map was presented in the year 1971 at the XVth *General Assembly of the International Union of Geodesy and Geophysics* but the work was finally completed in 1973, resulting in a map to the scale 1: 2,500,000 (made up of 6 sheets), and drafted in Russian and English (Fig. 29.1).

The working schedule of this work foresees that in the period 1973—1980, there should be published in each of the participating countries national monographs presenting the work carried out, with more detailed interpretations for every area which is interesting from the point of view of the movements of the Earth's crust.

The map covers an area of about 6,500,000 km^2, having been submitted to a thorough scientific investigation in which specialists from various interested fields took part.

J. D. Boulanger, the coordinator of the work, classifies the basic methods which were utilized in constructing this map as follows:

(1) *Continuous recordings provided by oceanography*. More than 150 tide gauges were used, of which 30 were considered as constraint points in the calculations of adjusting the national levelling networks. The average accuracy of determination of the movements of the Earth's crust by means of the tide gauges used is estimated at $\pm$ 0.3 mm/year. It shauld be mentioned that at the meeting points of the isolines (curves of equal displacements) no errors greater than $\pm$0.2 mm/year were obtained.

(2) *Repeated geometrical levelling*. It was appreciated that within the framework of of this complex of determinations, repeated geometrical levelling represents the main method. The levelling observations evolve over a time period varying between 50 and 75 years. The total lenth of the levelling lines is about 135,000 km. The contribution of the *U.S.S.R.*'s 1st-order network to this network is dominant: 80,000 km of levelling lines, 59 levelling polygons, 9,600 levelling bench marks. The perimeter of the levelling polygons lying within the network varies between 300 and 3,700 km.

(3) *Geological-geomorphological methods*. Since the oceanographic and geodetic methods have a discrete character, numerous geological-geomorphological interpretations along the levelling lines were necessary, in order to complete the isoline tracing.

The methods of processing the oceanographic and geodetic observations have been a major object of study for all participating countries. From this point of view, attention should be drawn, in the final publications intended to present the map of recent vertical movements of the Earth's crust in *Eastern Europe*, to the contributions of *H. Montag* and *I. I. Entin*.

Special attention was paid to the selection of the levelling lines in order to ensure an overall accuracy in the final network, by applying in each participating country the formulae contained in the present book in § 27.2.4.

One has taken into consideration level differences not reduced to a certain altitude system. Consequently, the hypothesis of a constant gravitational field within the considered time interval was accepted.

I. I. Entin describes the sequence of the processing operations of the geodetic observations in the following way:

1. *Calculating the modification rates of the level differences for every levelling line and adjusting them separately for each participating country.*

2. *Comparing the results obtained at the borders of the participating countries*. From this point of view, large differences have also emerged, viz. at the borders of the *Soviet Union* with the *Polish People's Republic*, the *Hungarian People's Republic* and the *Czechoslovak Socialist Republic*, of up to 10 mm/year. At the other state borders the differences were small, of the order of 1—2 mm/year or less.

3. *After reconciling the operations previously mentioned, each country carried out internally the adjustment of the repeated levelling according to the method it considered suitable*. Generally, the calculation model described in § 28.2.1.5 was accepted.

The accuracy of the displacement rates determined by means of the operations previously mentioned, calculated at 226 points in the levelling network, lies between $\pm$0.3 and $\pm$ 2.6 mm/year.

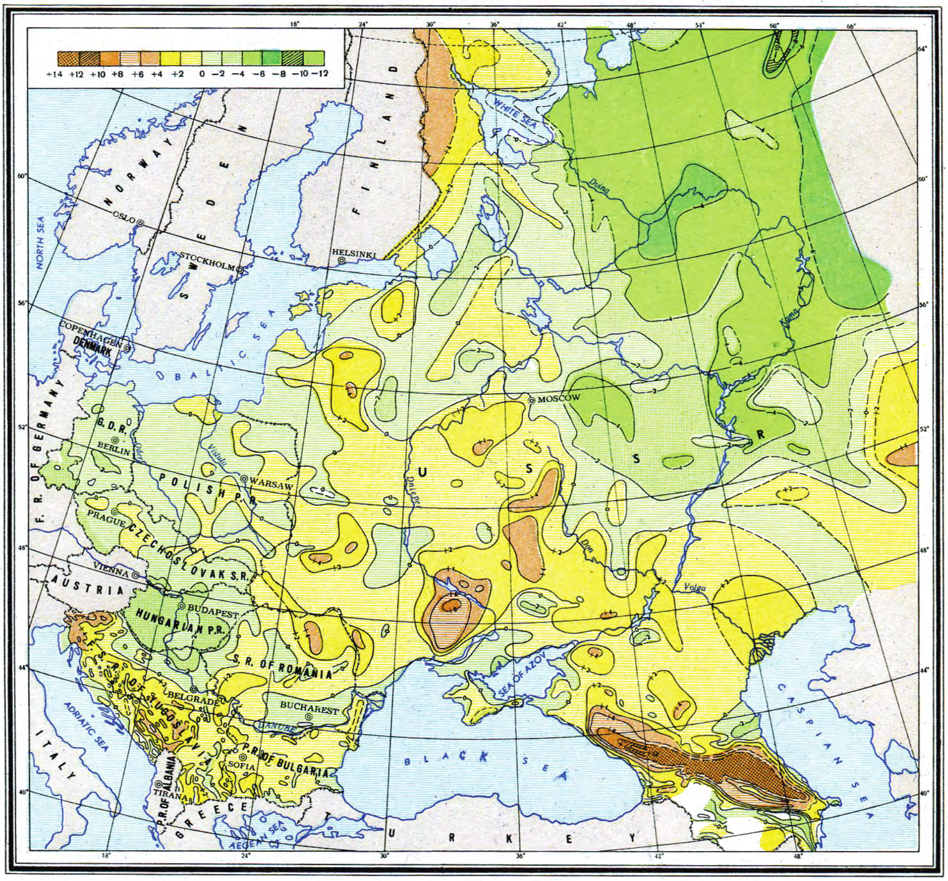


Fig. 29.1. Map of the Recent Vertical Movements of the Earth's Crust in Eastern Europe, Edited by the International Union of Geodesy and Geophysics — with the Participation of the People's Republic of Bulgaria, Czechoslovak Socialist Republic, German Democratic Republic, Hungarian People's Republic, Socialist Republic of Romania, Soviet Union, Federative Socialist Republic of Yugoslavia.

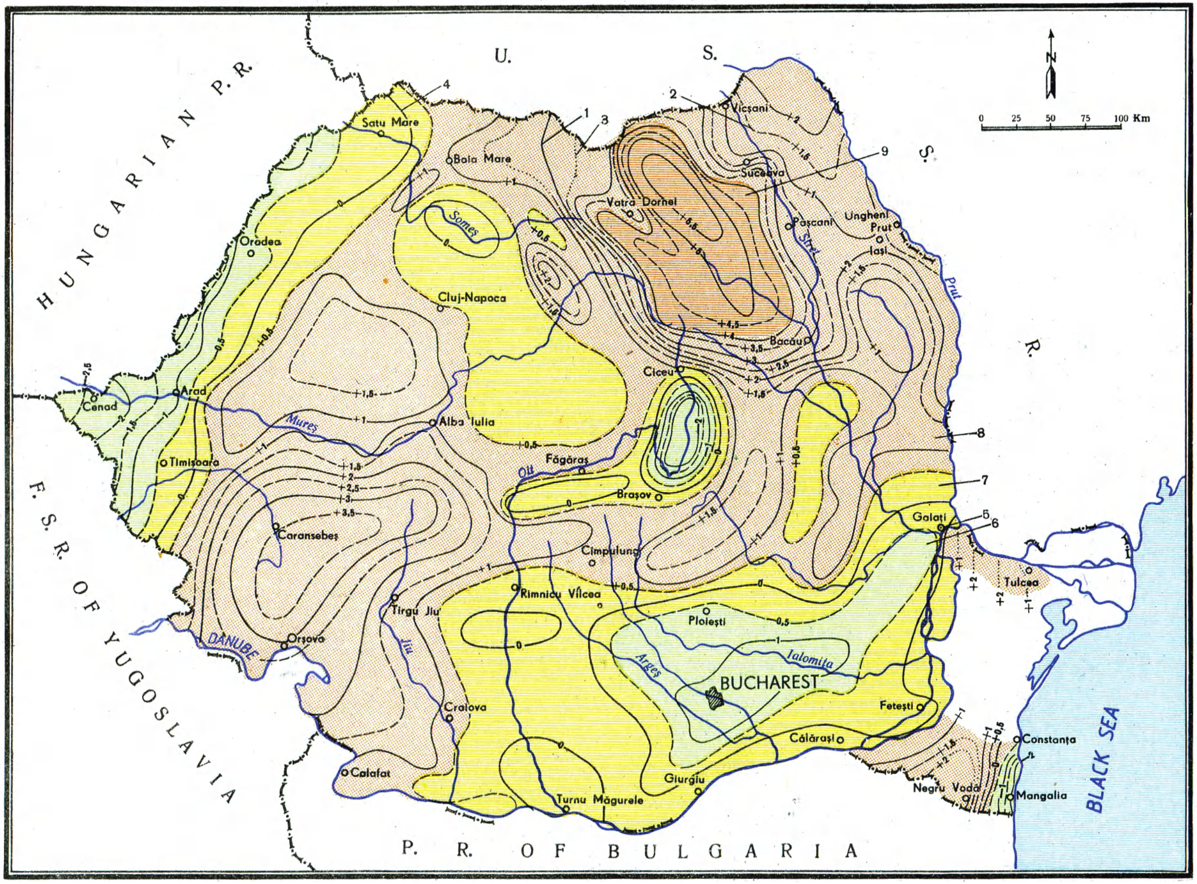


Fig. 29.2. Map of the Recent Vertical Crustal Movements on the Territory of the Socialist Republic of Romania (1975) Stage :

1 — main isolines, in mm/year ; 2 — secondary isolines, in 0.5 mm/year ; 3 — hypothetical isolines ; 4 — zero-rate isolines ; 5 — origin point (rate — 0.2 mm/year) ; 6 — subsidence areas (from — 0.5 mm/year to — 0.2 mm/year) ; 7 — relatively stable areas (from — 0.5 mm/year to+0.5 mm/year) ; 8 — upheaval areas (from 1 mm/year to 5 mm/year) ; 9 — intensive upheaval areas (over 4 mm/year).

The results provided by the map represented in Fig. 29.1 show vertical movements of the Earth's crust using between the maximum limits of — 12 and +13.5 mm/year respectively. It is estimated that the differences revealed by the map are due mainly to the differences existing within the geomorphological structure of the Earth's crust in *Eastern Europe*. The north-western area is characterized by crustal up-heavals of between 8 and 10 mm/year. The central area, of large dimensions, is characterized by relatively small displacements of 1 to 2 mm/year. The southern and eastern areas, in which the *Carpathian Mountains*, the *Balkan Mountains*, the *Yugoslav Alps*, the *Caucasus Mountains* are located, are also generally characterized by rises of the Earth's crust of between 2 and 13 mm/year (in a few local areas the rises even exceed this latter limit).

29.2 Map of the Vertical Crustal Movements on the Territory of the Socialist Republic of Romania

In order to work out the Map of the recent vertical movements on the territory of the *Socialist Republic of Romania*, data either published or available in the archieves of *"Direcția Topografică Militară"*, *"Institutul de Geologie și Geofizică"*, *"Institutul de Geodezie, Fotogrammetrie, Cartografie și Organizarea Teritoriului"*, and *"Centrul de Fizica Pămîntului și Seismologie"* *(Cornea et al. 1978)* were used. One of the issues, viz. that represented in Fig. 29.2, was integrated into the Map of the recent vertical movements of the Earth's crust in *Eastern Europe* (Fig. 29.1). It is to be remarked that in the year 1977 the above-mentioned institutions published a new edition of the map in which were represented:

— elements of vertical movement of the Earth's crust;
— tectonic and neo-tectonic elements;
— seismological elements,

an overall picture thus being achieved of phenomena in a state of close interdependence.

The system for working out the map was described in Section 29.1.

As origin the bench mark *Giurgiulești*, located near the state border with the *U.S.S.R.* was taken, for which one has accepted the displacement rate along the vertical of — 0.2 mm/year extracted from the preliminary map (stage 1971) of the vertical movements in *Eastern Europe*. In this manner it was possible to achieve a better connexion with the results obtained up to 1971 by the other socialist countries which took part in working out the map presented in Fig. 29.1.

The main levelling networks used are the following:

(1) *From the network constructed during the period 1881—1896 by the specialists of the Geographical Institute in Vienna* (on the whole about 2,900 km), *in Banat and Transylvania, one has utilized only two levelling lines, of total length of about 200 km, strictly necessary for closing the polygons of the "old levelling network"*. This work is referred to the *Adriatic Sea* as zero point. In the other areas of the country the old levelling was carried out in three

stages: 1895—1913, 1916—1941 and 1949—1960, with respect to the *Black Sea* as zero point.

(2) *The "new" network taken into consideration in working out the map is the network determined by "Direcția Topografică Militară" during the period 1960—1971* (Fig. 3.7), *with respect to the Baltic Sea* as *zero point.*

The mean time interval between the old and the new observations of repeated geometrical levelling is 36 years, the extrems being 49—67 years (33.6% of the network) and 6—11 years (17.7% of the network) respectively.

The technology and the instruments used differ very much from one period to the another, so that the old and new measurements respectively are characterized by systematic and random errors differing widely. The average misclosures in the polygons of the old network are 57 mm and in the polygons of the new network, 50 mm. One has taken into consideration level differences not reduced to a certain altitude system.

The levelling network utilized in working out the map comprises 72 levelling lines, of a total length of about 6,000 km, forming 14 levelling polygons (the perimeters of the polygons vary between 318 and 1,105 km). For the final interpretation and for tracing the isolines one has utilized altogether 1,499 levelling bench marks of the 1st and 2nd orders. The connexion with the neighbouring countries has been good, most of the proved departures being of $\pm$ 0.5 mm/year.

The processing of the repeated observations of geometrical levelling was made according to the calculation models presented in § 28.2.1.5 and Section 29.1.

The map of the recent vertical movement of the Earth's crust on the territory of the *Socialist Republic of Romania* shows up the pronounced trend of upheaval of the terrestrial crust in the area of the *Eastern Carpathians* (maximum values of over + 5 mm/year). This trend is somewhat smaller in the areas of the *Transylvanian Alps* (maximum values of + 3.5 mm/year) and of the Apuseni Mountains (maximum values of + 1.5 mm/year).

Transylvania's Basin appears as a quasi-stationary area (the displacements of + 0.5 mm/year are near to the limit of the determination errors).

Subsidences of about 2 mm/year occur in the area of the *Upper Valley* of the *Olt* (the *Brașov-Miercurea Ciuc* sectors), the *Pannonian Basin* (the *Sînnicolau Mare-Nădlac* sectors) and in the littoral zone *(Constanța-Mangalia)*. The eastern sector of the *Wallachian Depression* is also characterized by subsidences, albeit somewhat smaller (maximum —1 mm/year).

The stage reached by the year 1975 in working out the map of the recent vertical movements of the Earth's crust on the territory of the *Socialist Republic of Romania* shows a good correspondence with the information obtained by geological, geomorphological and geophysical methods, thus representing a success for Romanian Geodesy. **The map admits future improvements, which may be grouped as follows:**

(1) *Considering the new high-precision levelling measurements initiated in the year 1973 by "Institutul de Geodezie, Fotogrammetrie, Cartografie și Organizarea Teritoriului"*, as well as other similar work carried out by specialized institutions in the country during the last few years.

(2) *Examining the possibilities of utilizing the tide-gauge recordings carried out on the Romanian littoral.*

(3) *Applying a more improved functional-stochastic model in processing the repeated measurements of geometrical levelling,* presented in the present treatise in Sections 28.4 and 29.3, in which there should appear as unknowns the displacement rates of the bench marks and not the rates of modification of the levelling lines (numerous calculation correlations are thus eliminated).

It is necessary to emphysize the fact that the calculation models presented in Sections 28.3 and 28.4 and in § 29.3.1 do not start out from the absolute necessity of repeating the observations in the whole of the network or of compulsorily framing the levelling lines in polygons, since one does not finally consider the adjustment principles specific to indirect observations. In other words, one could eliminate in this way some of the levelling lines which were introduced into the actual processing only in order to be able to form the levelling polygons needed for applying the method of conditioned observations and which are not characterized by an accuracy similar to that obtained on the other levelling lines taken into consideration. In the same context the proposed model allows one also to include into the future adjustment the most recent results, obtained in the actual operations of repeated geometrical levelling, even if at that moment they do not yet form closed polygons in their entirety.

29.3 Utilization of Some Improved Functional-Stochastic Models for Determining the Vertical Crustal Movements

In order to illustrate the calculation procedures mentioned in § 28.2.1, as well as for deducing a few conclusions of a practical nature, one has adjusted a local network of repeated geometrical levelling in the area of the carboniferous basin of the *Ruhr (F.R. of Germany)*, presented in Fig. 29.3, by utilizing several functional-stochastic models.

In connexion with the configuration of the levelling network, it is to be noted that generally this maintains its main configuration from one measurement epoch to another. Nevertheless, due to necessities of a practical nature as well as to objective causes, there may appear structural modifications either through the disappearance of certain bench marks or levelling lines or through the design of new levelling lines with a view to improving the

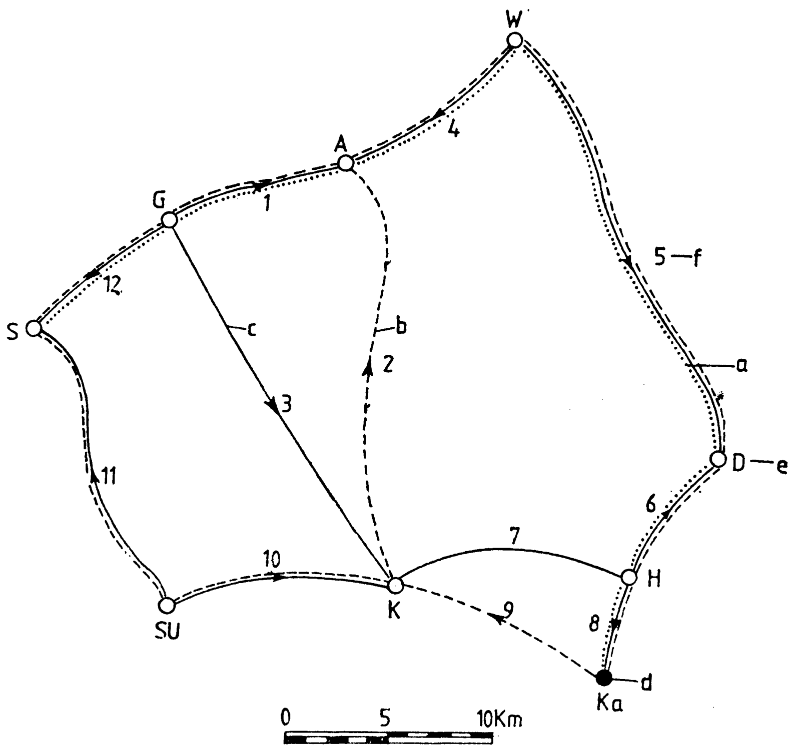


Fig. 29.3. Local Network of Repeated Geometrical Levelling in the Area of the Carboniferous Basin of the Ruhr (F.R. of Germany):
a, b, c — levelling observations carried out in the measurement epochs of the years 1920, 1935 and 1960; d — initial (fixed) point; e — shortened designation of the levelling bench mark; f — number of the levelling line.

network's structure and the conclusions likely to emerge after the processing. Consequently, the rigorous adjustment of the repeated levelling must also envisage modifications of a limited character which may occur in the configuration of the network under consideration. This is an argument for which the processing techniques based on the method of indirect observations or on procedures reducible by means of simple transformations to this method are given priority in practical applications.

The level differences measured in the measurement epochs of the years 1920, 1935 and 1960 are presented in Table 29.1. The approximate values for the altitudes $H_P'^{\mathrm{I}}$ at the central moment $t_{\mathrm{I}} = 1920$, and $H_P'^{\mathrm{II}}$ at the central moment $t_{\mathrm{II}} = 1935$ respectively (for the points SU and K) are given in Table 29.2. As initial point one has chosen the bench mark Ka, located in an area with vertical movements negligible compared with those of other bench marks: $H_{Ka} = \text{const.} = 36.2630$.

Table 29.1. *Measured Level Differences H*

Line No.	Length km	Network 1920		Network 1935		Network 1960	
		Observation year	Differences ΔH m	Observation year	Differences ΔH m	Observation year	Differences ΔH m
1	15	1921	24.7820	1930	24.7770	1962	24.7480
2	34	—	—	1934	11.4374	—	—
3	30	—	—	—	—	1956	13.3313
4	14	1921	21.3266	1935	21.3223	1962	21.3129
5	38	1921	1.1042	1935	1.1011	1956	1.1009
6	10	1921	0.4450	1935	0.4360	1956	0.4277
7	17	—	—	—	—	1956	9.2157
8	7	1921	5.4434	1935	5.4430	1956	5.4414
9	18	—	—	1934	3.7776	—	—
10	15	—	—	1934	0.3613	1952	0.3663
11	21	—	—	1931	3.8942	1952	3.8834
12	11	1922	17.5888	1930	17.5898	1950	17.5863

Table 29.2. *Provisional Altitudes of the Levelling Bench Marks*

Bench mark	Altitude H′ m
H	30.810
D	31.250
W	30.130
A	51.460
G	26.700
S	44.290
SU	40.390
K	40.030

29.3.1 Example of Utilizing the General Model (Ghiţău Model)

As was mentioned in § 28.2.1.1, for every observation an equation of the form (28.40) will be written in which the unknowns dH and H appears, as well as the corrections v and $\tilde{v}$ (the general case of the rigorous adjustment — condition equation with connexion unknowns). Consequently, for the three measurement epochs 26 such equations (Table 29.3) will result.

The provisional coordinates H' contained in Table 29.2 were determined by the separate processing of the observations carried out in the first measurement epoch. In order to determine the structure of the functional-stochastic model one can utilize the graphs of variation of the altitude of each bench mark obtained from the separate adjustments (Fig. 29.4) of the observations carried out in the other measurement epochs. These graphs may

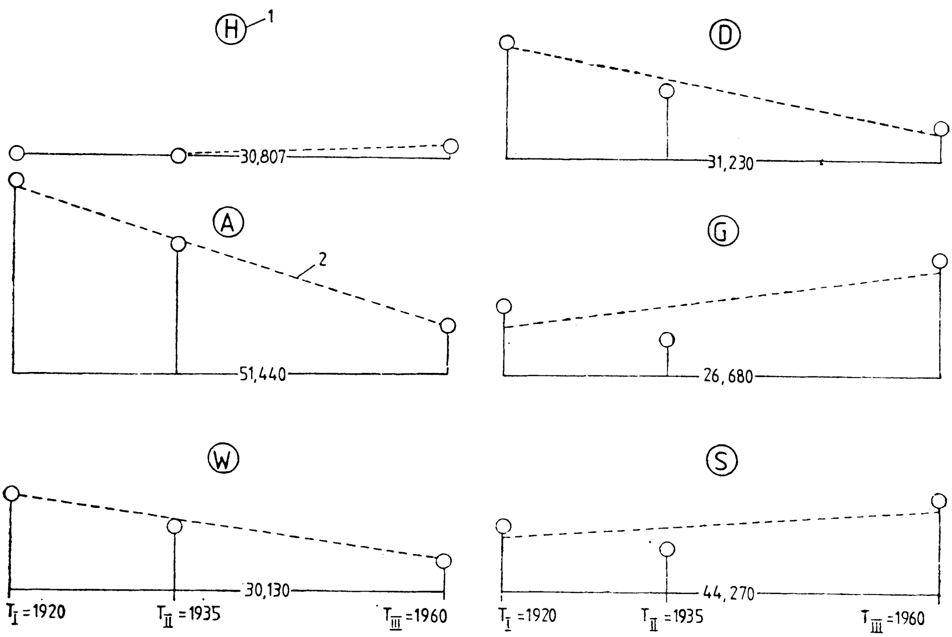


Fig. 29.4. Variations of the Altitudes of the Levelling Bench Marks as Determined by Separate Processings, on Measurement Epochs:
1 — shortened designation of the bench mark; 2 — probable variation, as graphically determined, of the altitude of the levelling bench mark.

also serve for determining, prior to the block adjustment, the ratio between the weights of the corrections v and $\tilde{v}$.

For the first iteration the functional model was given a very complex structure, including 8 unknowns dH, 14 unknowns H and 32 corrections v. The weights p, needed for the first iteration, of the levelling observations were calculated from the known relation: $p = 100\ L_{(\mathrm{km})}$. It must be mentioned that dH, v and the free terms in the equations (28.40) were expressed in mm and the displacement rates $\dot{H}$ in mm/10 years.

In order to determine the weights accompanying the corrections $\tilde{v}$ one has utilized the graphs in Fig. 29.4, by applying the known formula: $p_{\tilde{v}}^2 = 100/m_{\tilde{v}}^2$. The error $m_{\tilde{v}}^2$ was determined using a relation also known: $m_{\tilde{v}}^2 = [\delta\delta]/n$, where δ denotes the departures with respect to the probable variation as determined graphically and n the number of these departures. It turned out that $m_{\tilde{v}}^2 = 120 : 18 = 6.6$, so that $p_{\tilde{v}} = 15$.

The global equation-system was transformed according to *Helmert* (relation (28.54)), yielding a system of 58 correction equations with 54 unknowns. The results of the adjustment after this initial iteration may be followed in Table 29.3, in column 55 (the corrections v) and in row 27 (the

corrections $\tilde{v}$, the unknowns dH and H along with the determination errors).
From among the results one mentions:

(1) *The standard deviation of the weight unit is* $m_0 = \pm 0.17$ mm/km.

(2) *The maximum value of the corrections v* (in absolute value) is $|v^9_{wD}| = 0.61$ mm and the minimum value $|v^1_{HKa}| = 0.01$ mm.

(3) *Analogously, for the corrections* $\tilde{v}$: $|\tilde{v}^1_H| = 0.69$ mm, $|\tilde{v}^1_w| = 4 \times 10^{-16}$ mm.

(4) The unknowns dH have the extreme values: $|dH^I_w| = 20.88$ mm and $|dH^I_H| = 0.41$ mm respectively while their errrors lie between the limits $\pm$ 3.00 and $\pm$ 1.30 mm.

(5) The unknowns $\dot{H}$ have the extreme values: $|\dot{H}^{II-III}_A| = 7.60$ mm/10 years and $|\dot{H}^{I-II}_H| = 0.29$ mm/10 years while their errors lie between the limits $\pm$ 3.19 and $\pm$ 0.87 mm/10 years.

These first results offer the possibility of improving the chosen model, viz. from the following points of view:

1. *From the total of 33 corrections $\tilde{v}$ initially chosen, 19 cannot be called significant* (their values equal about 1×10^{-14} mm) *and may therefore be eliminated from the next iteration.* A few corrections $\tilde{v}$ have values approximately constant for the same point but of opposite signs: $\tilde{v}^9_w \approx -\tilde{v}^{10}_w; \tilde{v}^8_G \approx -\tilde{v}^7_G$; $\tilde{v}^4_S \approx -\tilde{v}^7_S$.

2. *Some of the unknowns H are very small compared with their errors, such as e.g.:* $\dot{H}^{I-II}_H \approx, \dot{H}^{II-III}_H, \dot{H}^{I-II}_G, \dot{H}^{I-II}_G, \dot{H}^{I-II}_S$ *and therefore* the *Student* test *can be excluded from the next iteration.* To the same extent the differences between certain values of the unknowns $\dot{H}$ for the same point, such as e.g.: $\dot{H}^{I-II}_W \approx \dot{H}^{II-III}_W$; $\dot{H}^{I-II}_A \approx \dot{H}^{II-III}_A$ are also statistically insignificant.

Therefore, these may henceforth be regarded as equal to one another, which means that one will be able to operate with a single unknown $\dot{H}^{I-III}_W, \ldots$ instead of two unknowns.

After examining all the cases of this kind, there remain for the next iteration only 13 corrections $\tilde{v}$, 18 unknowns dH and 8 unknowns $\dot{H}$, so that the number of the supplementary observations increases from 4 to 10.

3. *In order to verify the initial hypotheses admitted when establishing the weights, the ratios gf the observation groups which represent each measurement epoch separately and then the ratios between the weights corresponding to the corrections v and $\tilde{v}$ respectively were investigated.* To this end one has worked with the global weighting coefficients P^* (relation (28.57)) for each of the groups previously mentioned, which are presented in Table 29.4 for each iteration. After the 4th iteration results very similar to those of the 3rd iteration were obtained, so that one has accepted as definitive the results of the 4th iteration. The final results after the 4th iteration are presented in Table 29.3: column 56 (for the corrections v), row 28 (for the corrections $\tilde{v}$ and the unknowns dH and H, as well as their mean errors).

In Table 29.4, the last row contains the standard deviations m_0 of the weight unit in mm/km, calculated for all the measurement epochs.

Table 29.4. *Global Weight-Coefficients P*

	Iteration 1		Iteration 2		Iteration 3		Iteration 4	
	Before	After	Before	After	Before	After	Before	After
$P_{\mathrm{I}}^{\bullet}$	1	6.1	6.1	2.4	2.5	1.0	1.2	0.9
$P_{\mathrm{II}}^{\bullet}$	1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$P_{\mathrm{III}}^{\bullet}$	1	0.1	0.1	0.3	0.2	0.2	0.2	0.2
$P_{\tilde{v}}^{\bullet}$	1	0.5	0.1	2.0	2.0	2.6	2.0	2.8
m_0		0.16		0.17		0.17		0.16

29.3.2 Example of Processing with the Wolf-Jacobs Model

The results of the last iteration from the processing previously examined show that the quantities $\tilde{v}$ are rather small compared with their errors. Therefore, for the case investigated one may also apply a functional-stochastic model containing no quantities $\tilde{v}$. One has carried out such an adjustment too, the results also being presented in Table 29.3: column 57 (for the corrections v) and row 29 (for the unknowns dH and $\dot{H}$, as well as their average errors). The calculations were carried out using as basic equation the expression (28.42).

The isolines which describe the vertical movements of the Earth's crust in the area under examination, obtained on the basis of the *Wolf-Jacobs* model, are presented in Fig. 29.5.

If one compares the results obtained by utilizing this model with the results of the 4th iteration of the general model utilized in § 29.3.1, **one finds the following mean and maximum differences, respectively** (in absolute value):

1) *For the corrections* v: 0.2 mm, and 0.6 mm respectively;

2) *For the unknowns* dH: 0.1 mm, and 0.1 mm respectively;

3) *For the unknowns* $\dot{H}$: 0.1 mm/10 years, and 0.1 mm/10 years respectively.

The differences between the corresponding individual errors turn out to be less than 0.1 mm. The mean error of the weight unit m_0 is of $\pm$ 0.18 mm/km.

In Table 29.5 are presented the adjusted values of the heights of all the bench marks in the network, for all the measurement periods, as well as the variations of these heights, along with their mean errors. A comparison of the values of the *Student* coefficient t_k calculated with $t_s = \pm 2.2$ (deduced for $f = 10$ and $S = 95\%$) shows that about 70% of the variations of the

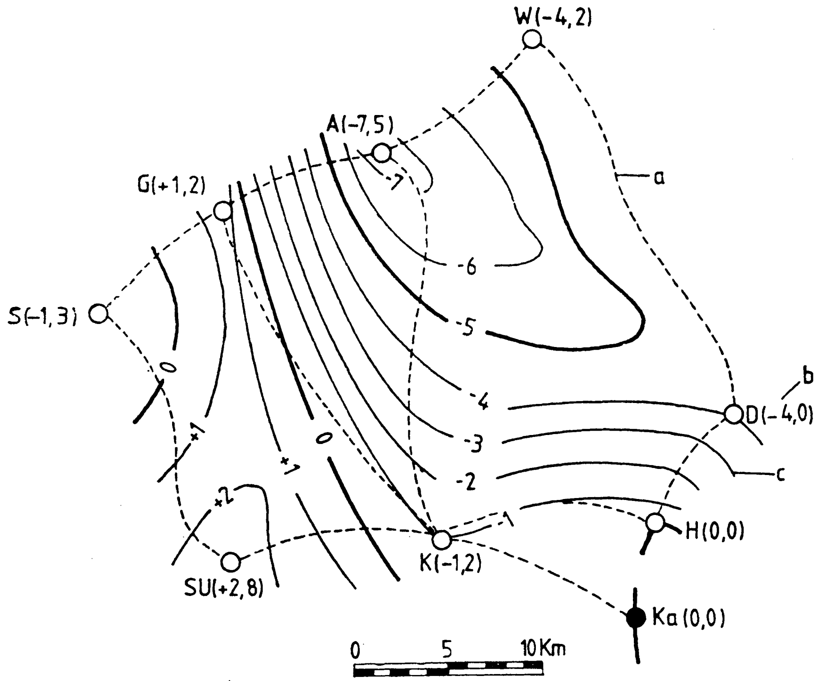


Fig. 29.5. Isolines for the Interval t_{II} and t_{III}:
a — levelling lines; b — displacements along the vertical, in mm/10 years; c — isolines.

Table 29.5. *Time Variations of the Altitudes of the Bench Marks*

Point's Shortened designation	t_I = 1920		t_{II} = 1935		t_{III} = 1960		Altitudes' variations					
							1920 – 1935			1935 – 1960		
	H_P^{I} m	m_H mm	H_P^{II} m	m_H mm	H_P^{III} m	m_H mm	ΔH_P mm	$m_{\Delta H}$ mm	t_k	ΔH_P mm	$m_{\Delta H}$ mm	t_k
H	30.8100	0.3	30.8100	0.3	30.8100	0.3						
D	31.2556	0.6	31.2460	0.6	31.2360	1.1	– 9.6	±0.8	12.8	– 10.0	+ 1.5	6.5
W	30.1510	0.9	30.1448	0.7	30.1344	1.6	– 6.2	±0.8	7.7	– 10.4	± 1.3	8.0
A	51.4780	0.9	51.4668	0.7	51.4482	1.5	– 11.2	±0.7	16.0	– 18.6	± 1.2	15.5
G	26.6948	0.8	26.6948	0.8	26.6977	1.6				+ 2.9	± 1.5	1.9
S	44.2835	0.8	44.2835	0.8	44.2803	2.5				– 3.2	±2.1	1.5
SU			40.3916	0.8	40.3985	2.3				+ 6.9	±2.3	3.0
K			40.0303	0.6	40.0274	1.6				– 2.9	± 1.7	1.7

heights of the network's bench marks may be regarded as statistically significant. Although some of the variations are apparently large ($\Delta H_S^{II-III} = -3.2$ mm, $\Delta H_K^{II-III} = -2.9$ mm), they cannot be considered as significant in the sense of the *Student* test, inasmuch as their mean errors are also large ($\pm$ 2.1 mm, respectively $\pm$ 1.7 mm) and thus the confidence interval is exceeded.

29.3.3 Example of Processing with the Model not Containing Systematic $\dot{H}$ Components

In order to demonstrate how much reality can be falsified by utilizing an incorrect functional-stochastic model, one has carried out an adjustment in which a model containing only stochastic quantities for describing the process of movement of the Earth's crust was used, quantities represented by the corrections v and $\tilde{v}$. The results of this adjustment are given in Table 29.3: column 58 (for the corrections v) and row 30 (for the corrections v and the unknowns dH, along with their mean errors). These results differ considerably from those obtained by utilizing the other models. **The mean and the maximum values of these differences are as follows:**

1) *For the corrections* v: 2.0 mm, and 8.2 mm respectively;
2) *For the corrections* $\tilde{v}$: 2.2 mm, and 5.1 mm respectively
3) *For the unknowns* dH: 3.4 mm, and 8.6 mm respectively.

The average error of the weight unit m_0 being :1.1 mm/km.

Also for the variations of the heights of the levelling bench marks, very different results were obtained compared with the case of the other models. Thus, e.g., for the bench mark W one has obtained the following values with the model from § 29.3.2 and with the model from § 29.3.3 respectively:

$\Delta H_W^{\text{II}-\text{III}} = -$ 17.m 4m, and $-$ 2.6 mm respectively; $m_{\Delta H_w} = \pm 1.2$ mm, and $\pm$ 4.6 mm respectively; $t_k = 14.5$, and 0.6 respectively.

It clearly follows that for such an area, where systematic vertical movements of the Earth's crust obviously occur, the utilization of a functional-stochastic model in which only the corrections v and $\tilde{v}$ intervene leads to erroneous results, quite remote from reality.

29.4 Other Examples of Determining Recent Movements of the Earth's Crust

29.4.1 Determinations of Horizontal Movements

The determinations of the horizontal movements of the Earth's crust, hitherto carried out, were of a local character and seldom of a general one, being carried out particularly in areas with large displacements. It was not possible to organize international cooperations of the kind mentioned in Section 29.1.

As an example one presents in Fig. 29.6 the results obtained in *Japan* by comparing the coordinates of the 1st-order geodetic points (a total of 156 points) calculated on the basis of the observations carried out in two measurement epochs: 1883—1909 and 1948—1958 respectively *(Kasahara* and *Sugimura* 1965*)*.

Examining the results obtained reveals an expansion of the territory in the south-western part with values of up to 4 m for the time interval

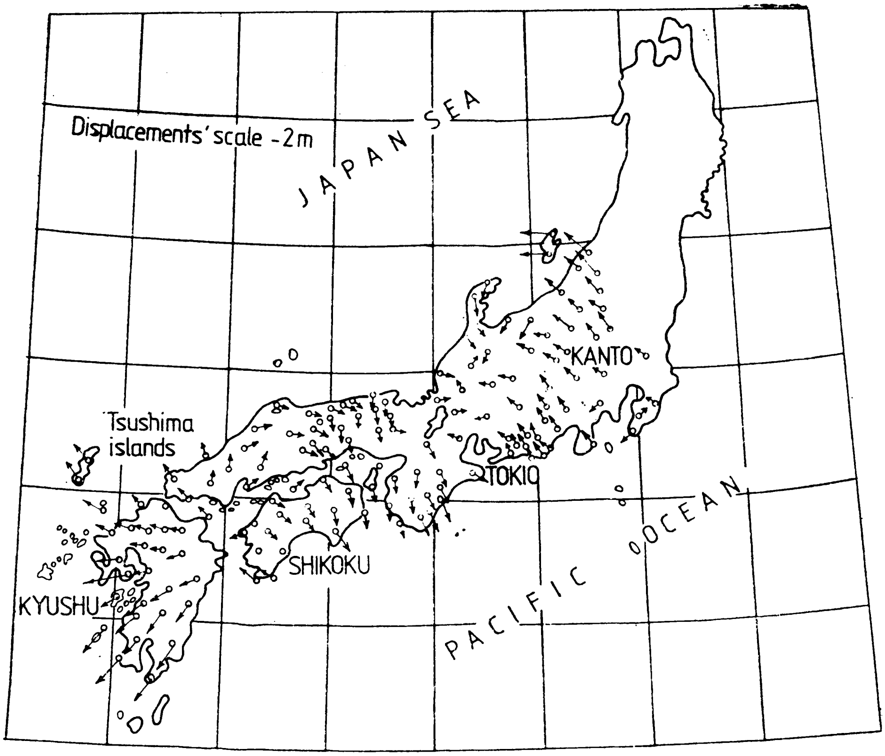


Fig. 29.6. Horizontal Displacements of the Points of Japan's 1st-Order Triangulation Network (*after Kasahara and Sugimura 1965 B*)

mentioned. The central area, with a pronounced topography, experiences a contraction, a result which has also been verified by means of repeated geometrical levelling, as a result of which rises of the terrestrial crust were obtained.

As was specified in Section 27.3, during the last few decades the determinations of horizontal movements of the Earth's crust developed rapidly due to the utilization of trilateration networks, since the measurement accuracy is greater. Such a network is presented by *Lichte* (1972) and by *Kunz* (1973) and was performed with a view to emphasizing the movements in the *Karlsruhe* area *(F.R. of Germany)*, preliminarily estimated at approximately 0.5 mm/year (Fig. 29.7). Although measured with the most accurate instruments existing at present, by applying improved technologies and working programmes, the coordinators of this work emphasize the fact that for the moment, due to the small displacements, one cannot draw definite geodetic conclusions (the observations begun in the year 1967). Future generations will have, however, a good comparison base which may be put to good use by carrying out new determinations, with, of course, still further improved instruments and working methods.

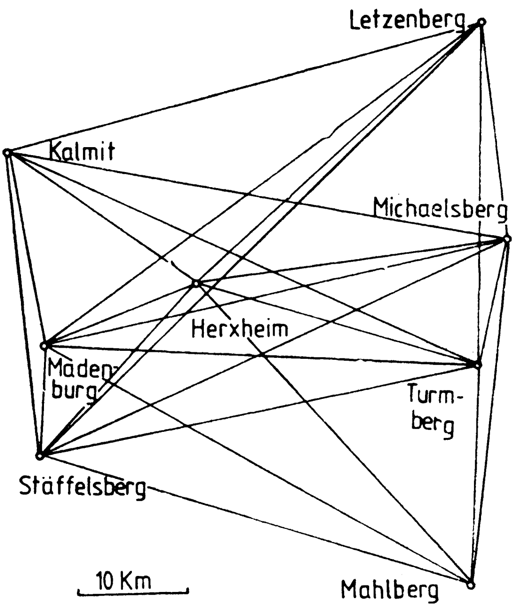


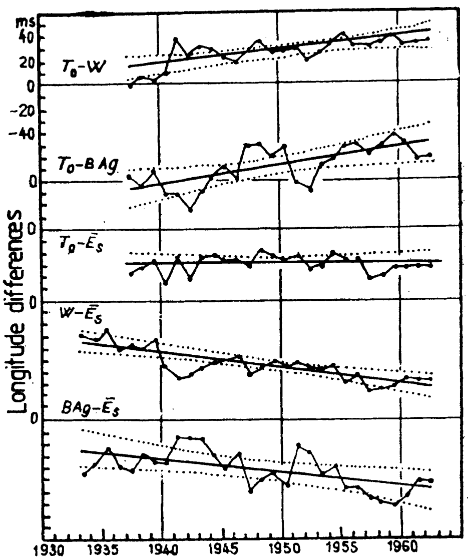
Fig. 29.7. Trilateration Network in the Karlsruhe Area (F.R. of Germany) for Determining the Horizontal Movements of the Earth's Crust (*after Lichte* 1972, *Kunz* 1973)

29.4.2 Examples of Determining the Movements of the Continents

As was shown in Section 27.4, processing the astronomical observations carried out in several observatories installed on different continents and repeated at certain time intervals, may reveal modifications in the relative position of the continents. But the errrors inherent in the determination of the astronomical coordinates, as well as the other factors previously mentioned (polar motion, errors in determining the observation time, catalogue errors etc.) lead, with rare exceptions, to the obtaining of different results from one author to another as regards the magnitude of the displacements of the continents. For this reason, the values of the determinations of this kind should always be accompanied by the author's name and the year (the stage) of the determination, since the answers given by various specialists differ in the case of time.

In Fig. 29.8 is presented the result of the studies carried out by the Japanese astronomers *M. Torao, S. Okazaki* and *S. Fujii* concerning the variation of the differences of the longitudes of 5 European observatories (their average is denoted by E_5), *Washington* (*W*), *Buenos Aires* (*B Ag*) and *Tokyo* (*To*) within the time interval 1933—1962 (except for the point *Tokyo*, for which the interval considered is 1937—1962).

Fig. 29.8. Determinations of the Time Variations of Longitude Differences (*after* *Tarao et al. 1965*).



The modification rates of the differences of longitudes between the stations mentioned, along with their determination errors (in mm/year) are presented in Table 29.6.

A. Stoyko and *N. Stoyko*, utilizing in their investigations observations carried out during a longer time period (1924—1964), deduce for the modification rates of the longitude differences between the observatories *Ottawa* and *Paris*, respectively *Washington-Paris*, the following values: —1.22 ms/year and —1.43 ms/year respectively. In these investigations other results are also presented, somewhat smaller, for the determinations contained in Table 29.6, viz.: + 0.46 ms/year for $To-W$ and +0.35 ms/year for $E_5 - W$.

Table 29.6. *Modification Rates of the Longitude Differences in ms/year*

T_0-W	T_0-BAg	T_0-E_5	E_5-W	E_5-BAg	$BAg-W$
$+1.05\pm0.19$	$+1.51\pm0.35$	$+0.04\pm0.19$	$+1.26\pm0.17$	$+1.11\pm0.29$	$+0.15\pm0.33$

All the studies undertaken make clear the anticlockwise-rotation trend of the *Pacific Ocean*'s coast and, as a consequence, one records a decrease of the latitude of the *Tokyo* observatory along with an increase of the latitudes of the observatories in *America*. But, as was specified in Section 27.4, the secular variations established in the latitudes of the astronomical observatories are much affected by the errors of the polar motion, so that it is difficult to make clear, with certainty, the influences due only to the horizontal movements of the Earth's crust.

The determinations of the horizontal movements of the Earth's crust by utilizing the laser and the *Doppler* equipment respectively in the observations of the Earth's artificial satellites and high targets present a quite remarkable prospect. For example, one presents in Fig. 29.9 a programme for determining the displacements of the European continental plate (also to be compared with Fig. 27.1). The southern limit of this plate crosses the area of the *Mediterranean Sea,* South of *Spain* through the *North of Africa,* continuing then through *Italy* and *Greece* towards *Asia Minor.* It is almost certain that the laser station *Dyonisos* (near *Athens*) is located on another tectonic plate, so that the time variations of the lengths indicated on Fig. 29.9 will be apt to serve in the future for determining the reciprocal displacements of the two plates.

In the same context, one must mention the *SAFE (San Andreas Fault Experiment)* programme, initiated in the year 1973, of laser determinations

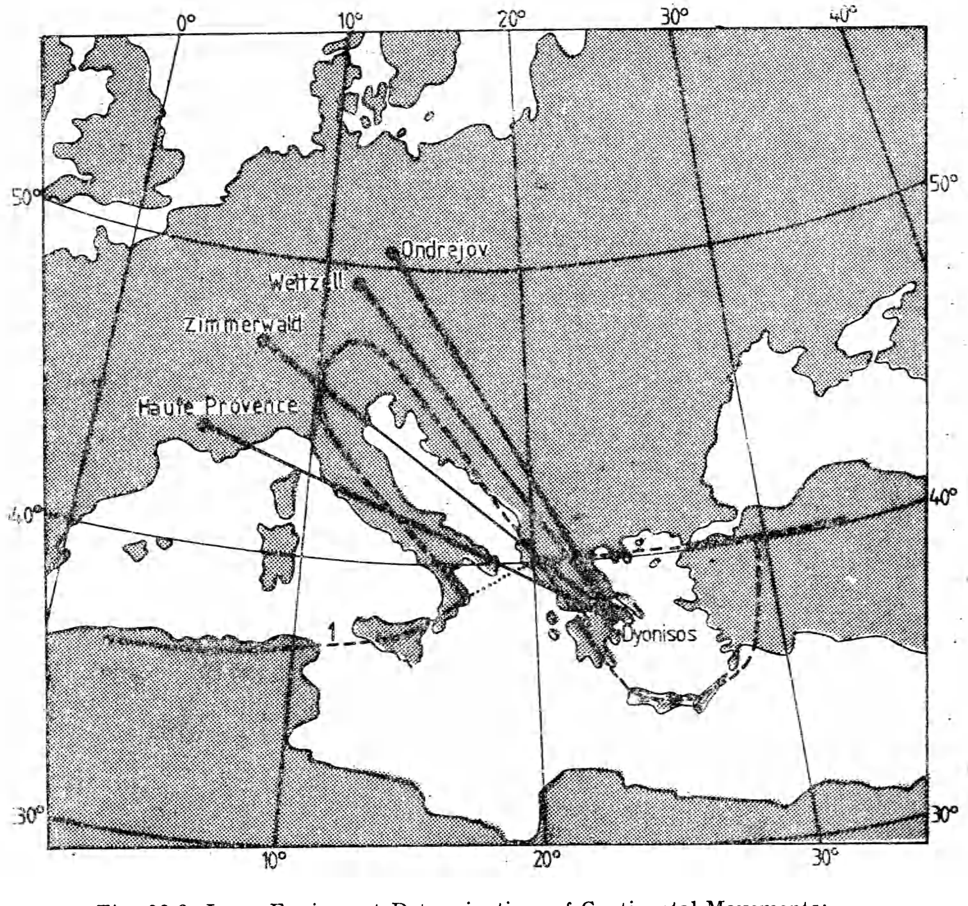


Fig. 29.9. Laser-Equipment Determinations of Continental Movements:
1 — limit of lithosphere's tectonic plates (*after Kolaczek and Wilson 1973*)

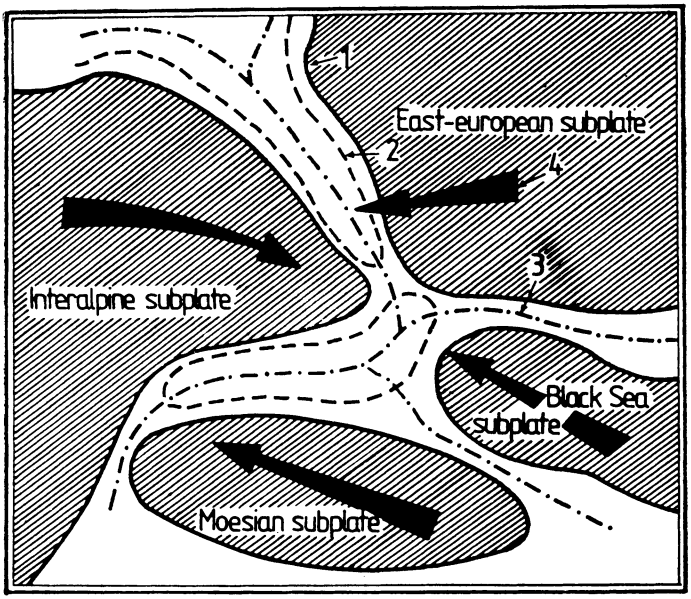


Fig. 29.10. Movement Trends of the Tectonic Subplates in the Vrancea Region, Socialist Republic of Romania *(after Airinei 1977 b)*:
1 — limits of regional gravity-high anomalies; limit of regional gravity-low anomalies; 3 — gravity-low axis; a — probable directions of movement of the tectonic subplates.

by utilizing the Earth's artificial satellites for detecting the movements of the *American* continental plate with respect to the *Pacific* plate along the *San Andreas* fault, near *Los Angeles*. This programme is combined with determinations of vertical displacements in the zone of *Palmdale* city, located on the same fault, near Los Angeles.

A similar programme of geodetic measurements may also be suggested for pursuing the reciprocal position in time of the lithosperic subplates in the *Vrancea* area (Fig. 29.10), in which occurred the March 4th, 1977, earthquake.

29.4.3 Vertical Movements of the Earth's Crust as Determined by Tide Gauge Recordings

Processing the continuous tide-gauge recordings has revealed a rise of the sea level of an average amount of about + 0.8 mm/year *(Jakubovski 1965, Montag 1970)*. As was specified in § 27.5.2, one may utilize to this end, with positive results, only observations covering a period longer than 18.6 years, which is the estimated period of the nodal tide. The results obtained in various countries differ more or less from the average value previously mentioned: + 2 mm/year at *Wilhelmshaven (F.R. of Germany)*, + 1.5 mm/year for the whole of tide gauges installed on the southern coast of the *United Kingdom*, + 1.3 mm/year for the *Dutch* coast etc. On the basis of the data recorded by the tide gauge at *Constanța* over a period of 42 years, the rate of increase of the *Black Sea*'s level was determined as being about + 2.7 mm/year *(Cornea et al. 1968)*.

Future Prospects in the Light of Present Geodetic Achievements

In the era of electronics and space research, the technological advances achieved have exerted a great influence on geodetic work. The following examples are significant in this direction:

— The introduction of the laser as a substitute for the conventional light sources in the distance-measuring instruments has increased their possibilities of measurement up to 80—100 km and has allowed the measurements to be carried out both by day and by night.

— The Earth's artificial satellite (e.g., Table 91.1) have become the main means of obtaining the positions of the terrestrial geodetic points.

The orbits of these satellites are now tracked and calculated with fair accuracy, so that by means of a corresponding Doppler receiver one can get positions of terrestrial geodetic points with a precision of about $\pm$ 1 m, which precision will undoubtedly be improved in the next decade.

— The large electronic computers have allowed not only the processing of the observations and the calculation of the artificial satellite's orbits but also the complete and rigorous utilization at present of geodetic data already obtained.

The basic aims of Geodesy concerning the solid Earth, but which extend also to oceans, the Moon and other planets, are now and will remain for the future also the following:

(1) Establishing and maintaining national and global three-dimensional geodetic networks on land, taking into consideration their time varation aspect.

(2) Measuring and representing the geodynamic phenomena (polar motion, Earth tides, crustal movements and others).

(3) Determining the Earth's gravitational field, its time variations included.

As a consequence of these, there results the task of determining the form and size of the Earth, as well as of securing a geodetic base for supporting the topo-photogrammetric work aimed at mapping the territories.

The dynamic behaviour of the Earth introduces an additional new dimension into the geodetic measurements — time. The significant tectonic movements are of the order of mm/year; studying them constitutes a true "challenge" for Geodesy. Otherwise, establishing and maintaining a reference framework, within which the time variable phenomena might be represented, form one of the geodetic purposes which are important for the not too remote future.

The task of keeping the geodetic networks "up to date" is a difficult one, inasmuch as it is complicated by changes taking place in the Earth's crust, natural changes or changes produced by human activities.

The methods based on optical telescopes for measuring the position of geodetic points on the Earth's surface, the polar motions and the Earth's rotation present fundamental limitations due to the influence of the phenomenon of atmospheric refraction, which renders almost impossible any significant improvement whatever beyond the present level of the achievements obtained so far. Certain improvements in measuring the polar motions and the Earth's rotation may, however, be obtained by taking weighted averages of measurements carried out at a large number of stations, making use of *Doppler* satellite tehniques (e.g., the *DPMS* operational system — § 19.4.2); the precision which can be attained is, however, only of $\pm$ 0.5—$\pm$ 1 m.

As regards the techniques of land measurements, the length of a measurement is generally less than 50 km, because of the Earth's curvature, of the terrain's shape and of the atmospheric limitations. As a consequence, the measurement of a long base is performed in fact by measuring shorter distances ($<$ 50 km) from which it is formed, an operation in which, obviously, measurement errors accumulate.

The accuracy of the base measurements by means of precision and levelling laser-instruments is also limited, since the entire measurement path passes through the atmosphere. A few atempts have been made, e.g., to obtain the polar motion from satellite laser-observations but up to now the accuracy achieved by isolated station points does not compare with that provided by *DPMS*. The *EPSOC* [1] campaign, initially intended for the same purpose, doesn't seem to furnish satisfactory results.

The space programmes have developed several new techniques which go beyond certain limitations of the conventional measurement systems and achieve an accuracy of about $\pm$ 1 cm in the long-base measurements necessary in determining the polar motion, the Earth's rotation, the movement of the tectonic plates and other smaller dynamic movements, such as, e.g., the Earth tides and the local crustal movements. From among the new techniques lately accomplished, two are most significant, viz. monitoring with precision laser instruments the Earth's artificial satellites and the reflectors installed on the Moon, and monitoring radio-interferometrically the natural astronomical sources (*quasars*[2]), and the radio signals from satellites respectively. Both types of base measurement-techniques presuppose the propagation of the signal from the space object to the terrestrial stations at each end of the base.

[1] *Earth Physics Satellite Observation Campaign.*

[2] Quasi-stellar objects, having an angular diameter generally smaller than 1″, which intensively emit radio-waves and waves belonging to the ultraviolet and infrared domains and, sometimes (e.g., the *quasar 3C 273*), nebulous diffuse jets, presenting spectral lines strongly removed towards the red limit of the visible spectrum.

It is, however, to be noted that for smaller distances (< 50 km) the conventional geodetic techniques become, generally, competitive with the extraterrestrial methods (which utilize balloons, aircrafts, Earth's artificial satellites, the Moon or strong radio signals emitted by quasars).

One knows that the Earth is in a state of continuous dynamic motion; the movements due to the Earth's crust tides, to the oceanic tides, to the polar motion, to the variations of the Earth's rotation and to other phenomena of this kind are of an amplitude of about a meter. It is self-evident that in analysing modern geodetic data one must also include models of these dynamic effects for determining the position of the geodetic points with an accuracy of the order of a centimeter. This represents both an advantage and a disadvantage. The datagathering and the ways of analysing them allow one to determine both the significant dynamic parameters and the position vector, but an incorrect modelling of the dynamic effect may generate errors in the determination of other parameters, such as, e.g., the components of the position vector. In addition, each system has certain other parameters (e.g. those connected with the orbit of the space object or the Moon — the *Moon libration* [1], or with the quasar's position or structure), which must be determined by means of the same measurements.

Although these operations represent additional complications, their results are of great value for Geodesy, as well as for other sciences, such as e.g., astronomy. Practice has shown that as soon as the accuracy of the measuring system has been improved, the better data have led to a better understanding and a better modelling of all the quantities involved, as well as to reducing the model errors in the utilized solutions. This process will continue in the future up to the moment when the final precision becomes comparable with the measurement accuracy.

The geodetic aspects highlighted up to now refer to *the geometrical ones* (or "coordinates" — determinations of geodetic points and bases) which represent, in the end, "the Earth's size".

In order to determine "the Earth's shape", connected with its gravitational field, it is necessary to have taken into consideration a group of requirements dealing with the physical aspects of Geodesy. If one considers the determinations deduced from satellite data, one may estimate that in broad terms the whole gravitational field has been mapped, but in insufficient detail for coping with the needs of oceanography and geophysics. A representation of this field is the geoid (Section 24.1. and § 25.7.3), and equipotential surface of the Earth's gravity field, almost coinciding with the non-disturbed surface of the oceans. But, due to the tides of the Sun or the Moon, to the waves, to atmospheric disturbances, to variations in the water's salinity and to the manner of circulation of the water in oceans, the surface of the oceans doesn't remain undisturbed.

The geoid's undulations seldom exceed 100 m and for most of the Earth they are less than 25 m.

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[1] Apparent movement of slight balancing of the lunar globe with respect to its mean position, which allows the observation from the Earth, although the Moon always presents the same side to the Earth.

Nevertheless, these irregularities represent significant indicators for the internal stresses and are essential for improving the geodetic results.

A direct observation (such as, e.g., *the satellite altimetry* — Section 25.8) of the height of the ocean's surface topography with respect to the surface of the reference ellipsoid allows one to find out the significant characteristics and, although the accuracy is below the value of $\pm$ 1 m, this surface departs from the geoid because of the oceanic currents.

The accuracy of $\pm$ 10 cm, e.g. that achieved by the *Seasat A* satellite (Table 19.1), will allow observations of the variations of the oceans' surface due to strong currents, although the effect of the currents with time mean-values will be superimposed on the effect of the geoid's variations. The two kinds of effects can be separated either by geodesists, if they can determine the geoid's variations with the necesary accuracy or by oceanographers, if they can evaluate the currents by geostrophic calculations. The success of the data analysis in improving the knowledge of the gravitational field/geoid with the data coming from the *Geos 3* satellite or from the experiments with the *Seasat* altimeter depends to a large extent on the oceanographic data accompanying them (e.g.: ocean agitation, ocean state), on the calibration of the instruments, as well as on other non-geodetic factors.

The insufficient knowledge of the gravitational field contributes to the greatest extent to the determination with a lower accuracy of the satellites' orbits, the precision of the global orbit being for a ususal geodetic satellite of order of 10—15 m ($\sim$ 10 m along the path, $\sim$7 m across the path and $\sim$5 m radially). Despite all these relative global uncertainties, the positions of the terrestrial stations have been determined with errors of less than $\pm$ 1 m by means of processing the data, so that the errors of the orbit have been adjusted over the area of interest (by utilizing multiple arcs or passages with different geometry and by means of filtering techniques) or using satellites flying at great heights, such as, e.g., *Lageos*, which are less affected by the gravitational field.

As regards lunar and planetary geodesy, this has the same goals as the geodesy oriented towards practical necessities but the measurement techniques are not identical. The instruments for the geodetic utilization of the bodies of the solar system are the telescopes and the artificial satellites provided with radar instruments for distance measurements, multi-spectral sensors, *Doppler* monitoring equipments and photographic cameras. Gravity meters and accelerators installed on the surface of these bodies have also been used.

The primary data which are analysed for the determination of the mass, shape and gravitational field of the bodies of the solar system are obtained by determining the orbital movement of the paths of the space-craft or of the asteroids. The measurement of the bodies' topography is made by means of radar, spectroscopy, occultations [1], television stereogrammetry, photogrammetric and *Doppler* techniques, as well as by radar astronomy for determining the rotation.

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[1] Temporary eclipses, for a certain observatory on the Earth, of one celestial body by another, due to their relative movements, beginning with immersion and terminating with emersion.

The techniques of primary analysis include, e.g., expansions in spherical harmonics of the observational data, the examination of the derived harmronic coefficients, the interpretation of the data referring to the distance up to the surface, the pressure at the surface and the application of classical figure-theory.

The *NASA* programme of planetary exploration, along with the large radars from *Arecibo (Porto Rico)* have led, during the last years, to a significant increase in geodetic knowledge of the Moon and a few planets *(Mars, Mercury, Venus, Jupiter, Uranus)*. It seems that the scientific interest for such measurements will be maintained in the future too, the following goals being foremost *(Committee on geodesy: Trends and perspectives)*, 1978.

— a noticeable improvement as regards the knowledge of the form and gravitational field of the Moon, by undertaking again, in the not too remote future, the project of an unmanned lunar orbitor, proposed by *NASA* which was delayed for financial reasons in the years 1977, 1978 and 1979. In this project a relay satellite for measuring the gravity variation on the far side of the Moon was also planned.

— an improvement of the gravimetric and altimetric data for the planet Mars, with the aid of a low-altitude orbiter; a more precise knowledge of the moment of inertia will be possible by utilizing an automatic astronomical observatory, installed on a space-craft which should land on Mars;

— obtaining gravimetric and altimetric information concerning the planet *Mercury*, by means of a polar orbitor. The determination of the moment of inertia from the rotation variations seems possible through the utilization of a radio-monitoring system implying the landing of an instrument on this planet;

— an investigation containing gravimetric and altimetric data on the planet *Venus* with the help of the *VOIR* [1] satellite, planned to be launched in the year 1983, first of all in order to obtain the radar picture of the planet;

— a more accurate determination of the zonal variations of the gravity on the planet *Jupiter*, utilizing to this end the *Galileo* satellite, planned to be launched in the year 1981;

— obtaining new knowledge about *Saturn* and *Titan* with the aid of the *Voyager* space-craft, launched in the year 1977 and of the *Galileo* satellite (1981);

— geodetic measurements on bodies located beyond *Mars*, in order to determine the masses of the large asteroids, for determining more correctly their densities and thereby their composition. It is hoped that the data concerning the masses of some asteroids will be improved by analysing the perturbations of the *Viking* space-craft, but significant improvements would require a special mission for asteroids.

The rapid evolution of geodetic techniques during the last two decades and the development of the new applications of Geodesy, in particular in the field of the dynamics of oceans and of the geophysics of the solid Earth offer

[1] *Venus* **O***rbiting* **I***maging* **R***adar*.

the possibilities of sketching some *directions of development of Geodesy*[1] in the more or less distant future, viz.:

(1) The high accuracy of the measurement instruments and equipments in the era of cosmic flights will surely transform three-dimensional Geodesy into a *four-dimensional one*. One will be able to measure the time variations of the seas' surface (the sea as topographic surface), of the physical surface of the continents and of the gravity field. National geodetic information will have to be framed within a continental and, in the end, a world-wide context, put in relationship with a cosmical inertial system, astronomically defined.

(2) The integration into continental systems of the fields of geodetic points performed at a national level. By including the additional astronomical measurements and the space distances one corrects the orientation and the scale of the position network, which is simultaneously completed by the introduction of vertical-angle measurements (the problem of the atmospheric refraction) and of the geometrical and astronomical levellings, so as to become genuine three-dimensional systems. Finally, these systems must be connected to the global system *(Wolf 1970)*.

(3) Continuing with the conditions, using geodetic methods with artificial satellites, of the global fields of geodetic points (minimum distance between points about 500 m). An increase of the accuracy by one order of magnitude is to be expected by the utilization of laser and *Doppler* equipment.

(4) As regards the determination of the external gravity field and of the geoid, important advances are to be expected due to technological developments such as, e.g.:

— the "*drag-free*" system by which the satellite is isolated from the atmospheric and from the solar radiation; the French satellite *Castor*, launched on May 17th, 1975 has already been equipped with such a system.

— continuing to improve the "Satellite-to-Satellite-Tracking" system for determining the higher-order harmonics; this system was realized for the first time in a so-called "high low" version in the satellite *Geos 3* in connexion with the geostationary satellite *ATS-F*. It is interesting to mention here the *Spacelab* mission, planned for the begining of the year 1980;

— satellite altimetry for directly determining the oceans' surfaces (e.g. the *Seasat* altimeter);

— airborne gravity meters and gradiometers for knowledge of the detailed structures of the gravity field;

— transportable automatic instruments for the astro-geodetic determination of the deflection of the vertical;

[1] For correctly establishing these directions valid for various time periods (from 3—5 years to 15—20 years), it is necessary to carry out *prognosis studies*. The main purpose of these studies must be the elimination of the probability that present actions provoke in a more or less distant future the situation in which one wouldn't be able to act freely, rapidly, effectively and efficiently. Foreseeing doensn't eliminate the unforeseen but may eliminate or reduce the negative consequences of the unforeseen or of the lack of preparation for facing it. This presupposes the actual application of the prognosis methods (e.g. *the Delphi method, the morphological method, the method of pertinence trees, creativity methods*). Details in connexion with these methods are to be found in the specialist literature (e.g., *Botez et al. 1971, Jantsch 1972, Rotaru 1978*).

— transportable gravity meters for absolute gravity determinations, which should ensure an accuracy of $\pm$ 0.003 mgal.

The coverage of the Earth with gravity values determined on the ground could improve the present situation, so that the gravity method might be completely utilized.

(5) The increase of accuracy, already obtained or to be expected, as regards the results of geodetic operations and the improved analysis procedures, allows one to suppose that, in the future, Geodesy can make an important contribution to the investigations of geodynamics.

If one starts at present, from the static problem of a global and regional understanding of the gravity field and from the possible conclusions following therefrom for geodynamics, then the main task of Geodesy in this field lies in confirming with certainty the recent movements of the Earth's crust and the gravity field. The order of magnitude of the time-evolving variations is, on the average, up to 1 cm/year in position (continental drift: 1—5 cm/year), 1 mm/year in altitude and 0.001—0.01 mgal/year in acceleration of gravity. The global and regional fields of existing points are not enough to confirm these modifications. An increase in accuracy by 1—2 orders of magnitude is needed. By utilizing laser distance measurements to the Earth's artificial satellites and to the Moon, as well as the *VLBI* techniques, it is expected to obtain for the global networks, in the near future, an accuracy of 1 dm. The relative accuracy of the continental fields of base geodetic points will also be improved to $\pm 1 \times 10^{-6}$ by the inclusion of the space distance measurements.

At last, it will be possible to increase the accuracy of the global network of gravity determinations by one order of magnitude (i.e. to $\pm$ 0.01 mgal) by utilizing the transportable instruments for measuring gravity and the instrumental capabilities of some well-known gravity meters.

As soon as the global network has such an accuracy, one will have to establish the global dimensions (continental drift) and the regional dimensions of recent variations, by measurements repeated after about 10—20 years.

A perpetual geodetic "surveillance" of the Earth's physical surface thus becomes possible. By means of high-precision absolute measurements of gravity and by simultaneous hydrometric observations and precision levelling one could possibly separate the vertical crustal movements from the variations of the sea level. To this end, one also utilizes today, particularly in the tectonically active areas, special geodynamic testing networks for discovering the recent modifications, e.g. in *Japan, U.S.A. (San Andreas Fault), U.S.S.R.* and *Northern Ireland*. It is expected that in the future special attention should be devoted to such systems.

(6) An active participation in measuring and analysing the tides of the Earth's crust. These variations of the gravity vector will be measured, as hitherto, at fixed observation stations (e.g. in *Europe*) but it is necessary to foresee additionally, in the future, mobile (field) stations, so that local and regional phenomena may also be included. It must be mentioned that the existing observation stations are not sufficient for solving the problems of geodynamics. These require an increased cooperation of Geodesy with

Geophysics, Geology and Astronomy. A promising beginning has already been made within the framework of the investigations in the *F.R. of Germany* with the installation for distance measuring with the 3rd-generation laser, of the type *S.F.B.-78*.

(7) New results are expected, in particular from the *VLBI* projects. As in *Eratosthenes*'s procedure, the radio-stars are intercepted by various telescopes at the Earth's surface. The radio signals, e.g. those emitted by quasars, don't simultaneously reach the recording installations of the terrestrial stations. The difference between the recorded time moments, as established by high-precision analysers (correlators), is proportional to the difference between the coordinates of the measuring stations, which, however, must be calculated.

In the near future, the first results of a world geodetic *VLBI* network are to be presented (prospective accuracy $\pm$ 1—5 m). The results of the first measurements on the American continent have been very encouraging.

(8) As regards the processing of the geodetic observations, the most important scientific objectives in future research could be summed up as:

— clarifying the mathematical and probabilistic foundation of the linear standard estimation techniques utilized in Geodesy and their correlations; this objective will lead to increasing the role of linear algebra (matricial algebra) in the theory of adjustments (particularly the theory of the generalized inverse — a special geodetic preoccupation), of mathematical statistics and of functional and numerical analysis;

— extending the utilization of the collocation method (generally regarded as a prediction technique), as a technique for unifying and extending the adjustment, in particular in connexion with the determination of the gravitational field from incomplete data;

— planning the geodetic measurement systems, within whose framework an important role will have to be assigned to the optimization process, i.e. to identifying the economic aspect (the "cost" factor must be minimum) with the geometrical one (the accuracy and reliability of results must be optimum). These aspects, which are frequently contradictory, e.g. "the more reliable the more expensive", must be submitted to decision theory which should provide criteria to be used as a quantitative base for decision-making.

Such criteria will be based on carefully assessing the social reasons and the interpretation of these cost analyses in terms of scientific achievements, as well as the non-geodetic requirements (e.g. geophysical ones) and their translation into geodetic requirements;

— finding a greater diversity of adjusting techniques (e.g. other than the adjustment by means of the least squares method). One must investigate various optimization criteria in the adjustment theory and their applicability to geodetic problems. These methods must be compared and therefrom one must select specific criteria for choosing the best method for all major problems in geodesy and geodynamics. One must also identify the assesable quantities which may be obtained from specific observation types;

— the unique, valid and practical solution of the generalization problem (random and mixed effects), analysing the variation and the comparison of the numerical results obtained from geodetic applications. It is necessary to identify the parameters of every geodetic problem for which the variance analysis leads to significant results. One must also determine the capability of each method to detect systematic errors in the model;

— working out new methods for establishing the weights in the various geodetic solutions using heterogeneous data.

(9) In the year 1801, the geodesist *Soldner* was the first to calculate "the gravitative deflection of the light" passing through the Sun. With the aid of his model of comets, he obtained about a century earlier the same formula as *A. Einstein* for checking the *M. Born - A. Einstein* correspondence. The discovery constitutes the starting point of a central feature of the present, of the cosmic era, viz. of the problem of the existence of the so-called "*black holes*". On the basis of the gravitational interaction one may imagine the existence of a stellar formation (massive celestial bodies) which shrinks to become a "singularity" (i.e. invisible). Round such bodies, irradiates a gravitational field of such an intensity that light can't cross it. The ray of light is bent so strongly on the basis of *Soldner*'s formula that it "collapses". A recovery is excluded.

The problem of the existence of such "black holes" has in recent times become a problem much discussed by cosmonauts, astronomers, astrophysicists, forecasters and last but by no means the least important of those interested in it, by geodesists [1].

It follows, in conclusion, that as regards our planet there are still problems awaiting solution, that however accurate and definite the results of measurements may be at a given moment, the evolution of the terrestrial phenomena always calls for a forward-looking view on the conclusions and the final results obtained. Perhaps during the coming decades, by using improved technical and computational methods it will be possible to discern and reveal more easily the highly intricate phenomena which constitue the dynamics of the form and size of the Earth.

[1] On September 13, 1978, scientific satellite weighing about 4,500 kg, equipped with instruments for studying X-ray sources in the Cosmos, such as quasars, pulsars and black holes was launched at *Cape Canaveral*.

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Abbreviations

Acta techn. Acad. scient. hung.	Acta technica Academiae scientiarum hungaricae
Allgem. Vermessungs-Nachr.	Allgemeine Vermessungs-Nachrichten
Ann. Astron. Obs. Tokyo	Annals of the Tokyo Astronomical Observatory
Arb. geod. Inst. Potsdam	Arbeiten aus dem geodätischen Institut Potsdam
Astron. J.	The Astronomical Journal
Astron. Nachr.	Astronomische Nachrichten
Beiter. z. angew. Geophys.	Beiträge zur angewandten Geophysik
Ber. Tätigk. Inst. f. angew. Geod.	Berichte über die Tätigkeit des Instituts für angewandte Geodäsie
Biull. nabliud. i.s.Z.	Biulleten' nabliudeniya iskusstvennyh sputnikov Zemli
Boll. geod. e sci. affini	Bollettino di geodesia e scienze affini
Bull. géod.	Bulletin géodésique (IAG)
Canad. J. Earth sci.	Canadian Journal of Earth sciences
Gerlands Beitr. Geophys.	Gerlands Beiträge zur Geophysik
Géod. és kartogr.	Géodezia és kartográfia
Geod. i aerofotos'emka	Geodeziya i aerofotos'emka
Geod. i kartogr.	Geodezja i kartografia
Geod. i kartografiya	Geodeziya i kartografiya
Geod. u. Geophys. Veröff.	Geodätische und Geophysikalische Veröffentlichungen
Geofys. sb.	Geofysikální sborník
J. Geod. Soc. Japan	Journal of the Geodetic Society of Japan
J. Geophys. Res.	Journal of Geophysical Research
J. Res. Nat. Bur. Standards	Journal of Research of the National Bureau of Standards
Mitt. Chef. Griegs-Kart.u.V.	Mitteilungen aus dem Chefkriegskarten- und Vermessungswesen
Mitt. Inst. theoret. Geod.	Mitteilungen aus dem Institut für theoretische Geodäsie
Nachr. Kart. -u.V.	Nachrichten aus dem Karten- und Vermessungswesen
Nachr. Reichsvermess. Dienst	Nachrichten aus dem Reichsvermessungsdienst
Nauch. inform.	Nauchnye informatsii
Proc. Roy. Acad.	Proceedings Royal Academy
Pubbl. Comm. geod. ital.	Pubblicazione Commissione geodetica italiana
Publ. Isost. Inst. IAG	Publications of the Isostatic Institute of the International Association of Geodesy
Publ. I. G. N.	Publications de l'Institut géographique national
Publ. Finn. Geod. Inst.	Publications of the Finnish Geodetic Institute
Publ. Isost. Inst.	Publications of the Isostatic Institute
Publ. sp. Bull. géod.	Publication spéciale du Bulletin géodésique

Raumf. f. Erde	Raumfahrt für die Erde
Rev. geod. cad. organiz. terit.	Revista de geodezie, cadastru și organizarea teritoriului
Rev. geod. organiz. terit.	Revista de geodezie și organizarea teritoriului
Rev. Roum. géol. géophys. géogr.	Revue roumaine de géologie, géophysique et géographie, (série de) géophysique
Sovrem. dvizh. zem. kory	Sovremennye dvizheniya zemnoi kory
Ster. u. Weltr.	Sterne und Weltraum
Studia geophys. et geod.	Studia geophysica et geodaetica
Tech. Bull. U.S. Dept. Commerce	Technical Bulletin U.S. Department of Commerce. Coast and Geodetic Survey
Trans. Cambridge Philosoph. Soc.	Transactions of Cambridge Philosophical Society
Trans. Roy. Inst. Technol.	Transactions of the Royal Institute of Technology
Tr. Tsentr. n. -i. in-ta geod. aeros'emky i kartogr.	Trudy Tsentral'nogo nauchnogo-issledovatel' skogo Instituta geodezii, aeros'emki i kartografii
Veröff. balt. geod. Komm.	Veröffentlichungen der baltischen geodätischen Kommission
Veröff. Dtsch. geod. Komm. Bayer. Akad. Wiss.	Veröffentlichungen der Deutschen geodätischen Kommission bei der Bayerischen Akademie der Wissenschaften
Veröff. Finn. Geod. Inst.	Veröffentlichungen des Finnischen Geodätischen Institutes
Veröff. Geod. Inst. Potsdam	Veröffentlichungen des Geodätischen Institutes in Potsdam
Veröff. Inst. f. angew. Geod.	Veröffentlichungen des Institutes für angewandte Geodäsie
Veröff. Inst. f. Erdm.	Veröffentlichungen des Institutes für Erdmessung
Vermesstechn.	Vermessungstechnik
Vermesstechn. Rundsch.	Vermessungstechnische Rundaschau
Wiss. Arb. Lehrst. f. Geod. Photogramm. u. Kartograph. Techn. Univ.	Wissenschaftliche Arbeiten der Lehrstühle für Geodäsie, Photogrammetrie und Kartographie an der Technischen Universität
Wiss. Z. Techn. Univ.	Wissenschaftliche Zeitschrift der technischen Universität
Z. f. angew. Math. u. Phys.	Zeitschrift für angewandte Mathematik und Physik
Z.f. Geophys.	Zeitschrift für Geophysik
Z. f. V.	Zeitschrift für Vermessungswesen

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