An Omnibus Test of Normality for Moderate and Large Size Samples

Author(s): Ralph B. D'Agostino
Source: *Biometrika*, Vol. 58, No. 2 (Aug., 1971), pp. 341-348
Published by: Oxford University Press on behalf of Biometrika Trust
Stable URL: http://www.jstor.org/stable/2334522
Accessed: 23-03-2018 19:19 UTC

Biometrika (1971), **58**, 2, *p.* 341
Printed in Great Britain

An omnibus test of normality for moderate and large size samples

BY RALPH B. D'AGOSTINO
Boston University

SUMMARY

We present a test of normality based on a statistic D which is up to a constant the ratio of Downton's linear unbiased estimator of the population standard deviation to the sample standard deviation. For the usual levels of significance Monte Carlo simulations indicate that Cornish–Fisher expansions adequately approximate the null distribution of D if the sample size is 50 or more. The test is an omnibus test, being appropriate to detect deviations from normality due either to skewness or kurtosis. Simulation results of powers for various alternatives when the sample size is 50 indicate that the test compares favourably with the Shapiro–Wilk W test, $\sqrt{b_1}$, b_2 and the ratio of range to standard deviation.

1. INTRODUCTION

Shapiro & Wilk (1965) presented a test of normality based on a statistic W consisting essentially of the ratio of the square of the best, or approximately best, linear unbiased estimator of the population standard deviation to the sample variance. They supplied weights for the ordered sample observations needed in computing the numerator of W and also percentile points of the null distribution of W for samples of size 3 to 50. Subsequent investigation (Shapiro, Wilk & Chen, 1968) revealed that this test has surprisingly good power properties. It is an omnibus test, that is, it is appropriate for detecting deviations from normality due either to skewness or kurtosis, which appears to be superior to 'distance' tests, e.g. the chi-squared and Kolmogorov–Smirnov tests. It also usually dominates such standard tests as $\sqrt{b_1}$, third standardized sample moment; b_2, fourth standardized sample moment; and u, ratio of the sample range to the sample standard deviation.

Shapiro and Wilk did not extend their test beyond samples of size 50. A number of reasons indicate that it is best not to make such an extension. First, there is the problem of the appropriate weights for the ordered observations for the numerator of W. Each sample size requires a new set. The proliferation of tables is obvious and undesirable. However, even if the appropriate weights were computed from the expected values of the ordered observations from the standardized normal distribution (Harter, 1961), there would still be the uninviting problem of finding the appropriate null distribution of W. Because W's moments beyond the first are unknown, Cornish–Fisher expansions or similar techniques are not applicable. Further, the extension of the normal approximation for W based on Johnson's bounded curves (Shapiro & Wilk, 1968) when the sample is greater than 50 would require an extrapolation and the procedure for implementing it is not available. Simulation runs seem to be the only available way to obtain the null distribution.

We present a new test of normality applicable for samples of size 50 or larger which possesses the desirable omnibus property. It requires no tables of weights and for samples of size 50 or more its null distribution can be approximated by Cornish–Fisher expansions, using moments up to the fourth. The test is based on a statistic D which is up to a constant the ratio of Downton's (1966) linear unbiased estimator of the normal distribution standard deviation to the sample standard deviation.

The test statistic D, the mechanics of performing the test and probability points generated by Cornish–Fisher expansions are in § 2. In § 3 there are results of a simulation check on the Cornish–Fisher expansions for the null distribution of D. For the usual levels of significance these simulations indicate that the expansions adequately approximate the null distribution for samples of size 50 or more. Simulation results of powers for various alternative distributions, at the 10% level of significance with samples of size 50, appear in § 4. These show that the test based on D is omnibus and that its power properties compare favourably with other tests of normality such as W, $\sqrt{b_1}$, b_2 and u. In § 5 the computation of the moments of D and the details involved with the Cornish–Fisher expansions for the null distribution of D are discussed.

2. DESCRIPTION OF THE TEST AND TABLES

Let $X_1, \ldots, X_n$ represent a random sample of size n and let $X_{1,n} < \ldots < X_{n,n}$ represent the ordered observations derived from it. The statistic D is

$$D = \frac{T}{n^2 S}, \tag{2·1}$$

where

$$T = \sum_{i=1}^{n} \{i - \tfrac{1}{2}(n+1)\} X_{i,n}, \tag{2·2}$$

$$S^2 = \frac{\Sigma(X_i - \bar{X})^2}{n}, \tag{2·3}$$

$\bar{X}$ being the sample mean. Downton's (1966) original unbiased estimator of the normal distribution standard deviation is $2\sqrt{\pi}\, T/\{n(n-1)\}$. Thus, up to a constant, D is equal to the ratio of Downton's estimator to the sample standard deviation.

If the sample is drawn from a normal distribution the expected value of D and its asymptotic standard deviation are, respectively,

$$E(D) = \frac{(n-1)}{2\sqrt{(2n\pi)}} \frac{\Gamma(\frac{1}{2}n - \frac{1}{2})}{\Gamma(\frac{1}{2}n)}, \tag{2·4}$$

or approximately $(2\sqrt{\pi})^{-1} = 0{\cdot}28209479$, and

$$\operatorname{asd}(D) = \left\{\frac{12\sqrt{3} - 37 + 2\pi}{24\pi n}\right\}^{\frac{1}{2}} = \frac{0{\cdot}02998598}{\sqrt{n}}. \tag{2·5}$$

Details are given in § 5. An approximate standardized variable, possessing asymptotically mean zero and variance unity, is

$$Y = \frac{D - (2\sqrt{\pi})^{-1}}{\operatorname{asd}(D)}. \tag{2·6}$$

If the sample is drawn from a nonnormal distribution the expected value of Y tends to differ from zero. The direction of the difference depends on the alternative distribution. Simulation suggests that if the alternative distribution has greater than normal kurtosis, then Y tends to be on the average less than zero, while if the kurtosis is less than normality then Y tends to be greater than zero. The important point is that deviations from normality are reflected both in values of Y greater than or less than its normal distribution mean. To guard against all possibilities a two-sided test needs to be employed. Table 1 contains a collection of percentile points of Y for a number of different sample sizes. More extensive tables of both D and Y are available from the author upon request. These percentile points were computed using Cornish–Fisher expansions for the null distribution of D.

Table 1. *Percentile points of Y*

	Probability level									
n	0·5	1	2·5	5	10	90	95	97·5	99	99·5
50	−3·949	−3·442	−2·757	−2·220	−1·661	0·759	0·923	1·038	1·140	1·192
60	−3·846	−3·360	−2·699	−2·179	−1·634	0·807	0·986	1·115	1·236	1·301
70	−3·762	−3·293	−2·652	−2·146	−1·612	0·844	1·036	1·176	1·312	1·388
80	−3·693	−3·237	−2·613	−2·118	−1·594	0·874	1·076	1·226	1·374	1·459
90	−3·635	−3·100	−2·580	−2·095	−1·579	0·899	1·109	1·268	1·426	1·518
100	−3·584	−3·150	−2·552	−2·075	−1·566	0·920	1·137	1·303	1·470	1·569
150	−3·409	−3·009	−2·452	−2·004	−1·520	0·990	1·233	1·423	1·623	1·746
200	−3·302	−2·922	−2·391	−1·960	−1·491	1·032	1·290	1·496	1·715	1·853
250	−3·227	−2·861	−2·348	−1·926	−1·471	1·060	1·328	1·545	1·779	1·927
300	−3·172	−2·816	−2·316	−1·906	−1·456	1·080	1·357	1·528	1·826	1·983
350	−3·129	−2·781	−2·291	−1·888	−1·444	1·096	1·379	1·610	1·863	2·026
400	−3·094	−2·753	−2·270	−1·873	−1·434	1·108	1·396	1·633	1·893	2·061
450	−3·064	−2·729	−2·253	−1·861	−1·426	1·119	1·411	1·652	1·918	2·090
500	−3·040	−2·709	−2·239	−1·850	−1·419	1·127	1·423	1·668	1·938	2·114
550	−3·019	−2·691	−2·226	−1·841	−1·413	1·135	1·434	1·682	1·957	2·136
600	−3·000	−2·676	−2·215	−1·833	−1·408	1·141	1·443	1·694	1·972	2·154
650	−2·984	−2·663	−2·206	−1·826	−1·403	1·147	1·451	1·704	1·986	2·171
700	−2·969	−2·651	−2·197	−1·820	−1·399	1·152	1·458	1·714	1·999	2·185
750	−2·956	−2·640	−2·189	−1·814	−1·395	1·157	1·465	1·722	2·010	2·199
800	−2·944	−2·630	−2·182	−1·809	−1·392	1·161	1·471	1·730	2·020	2·211
850	−2·933	−2·621	−2·176	−1·804	−1·389	1·165	1·476	1·737	2·029	2·221
900	−2·923	−2·613	−2·170	−1·800	−1·386	1·168	1·481	1·743	2·037	2·231
950	−2·914	−2·605	−2·164	−1·796	−1·383	1·171	1·485	1·749	2·045	2·241
1000	−2·906	−2·599	−2·159	−1·792	−1·381	1·174	1·489	1·754	2·052	2·249

The mechanics of performing the test are simple. Draw a random sample of size n ($n \geqslant 50$). Order the observations and compute D either by (2·1) or by the equivalent formula

$$D = \frac{T}{n^{\frac{3}{2}}\{\Sigma(X_i - \bar{X})^2\}^{\frac{1}{2}}}, \tag{2·7}$$

where T is given by (2·2). As can be seen from (2·5) the range of D is small. Numerical accuracy of about 5 decimal places should be aimed at in calculating it. Next transform D to its standardized Y value by use of (2·6). Table 1 gives the critical values for a range of significance levels; e.g. if $n = 100$ and the test level is 0·10 the critical values are $-2{\cdot}075$ and 1·137.

3. Evaluating the Cornish–Fisher expansions

In § 5 we show that the null distribution of D and hence of Y is asymptotically normal and that Cornish–Fisher expansions using moments up to the fourth are available. Here we desire to demonstrate, using simulation results, that for most practical purposes these expansions adequately approximate the null distribution of D for samples as small as 50.

14,000 normal random samples of both sizes 50 and 60 were generated and the resulting

Y's were classified according to the Cornish–Fisher expansions' cumulative distributions of Table 1. The results are in Table 2, in percentages, where an observed percentage is equal to the percentage of the total number of samples for a given sample size yielding values of Y less than or equal to the corresponding Cornish–Fisher percentile, desired percentages. The variation is within normal sampling variation as measured by the chi-squared test and for two-sided tests at levels 0·01, 0·02, 0·05, 0·10 and 0·20 the observed levels and the desired levels agree to at least two decimal places. For most practical purposes this appears adequate. Because D is asymptotically normal it is reasonable to assume that if the Cornish–Fisher expansions work well for samples as small as 50 and 60 they will be adequate for larger size samples.

Table 2. *Simulation checks on Cornish–Fisher expansions for D*

	Desired percentages									
	0·50	1·00	2·50	5·00	10·00	90·00	95·00	97·50	99·00	99·50
Sample size (number of samples)	Observed percentages									
50 (14,000)	0·42	0·90	2·35	4·83	9·94	89·91	94·88	97·30	98·81	99·31
60 (14,000)	0·54	1·00	2·34	4·99	9·76	90·06	94·88	97·34	98·86	99·34
100 (1000)	0·90	1·60	3·10	5·80	10·80	90·40	95·20	98·00	99·20	99·60
40 (5000)	0·50	1·18	2·56	5·12	10·42	90·06	94·90	97·14	98·28	98·72
30 (1000)	0·50	0·80	2·00	4·50	10·20	88·00	93·90	96·00	97·30	97·70

Table 2 also contains simulation results for other sample sizes. For $n = 100$ (1000 samples), as is to be expected, there is good agreement. For $n = 30$ (1000 samples) and $n = 40$ (5000 samples) the Cornish–Fisher expansions and the observed match less well.

4. Empirical power study

To investigate the power of D we generated from 200 to 400 random samples, of size 50 for each of several alternative distributions, 42 in all, and performed a two-sided 10 percent level significance test on them. The alternatives cover a wide variety of possibilities, representing a good selection of third and fourth standardized moments, $\sqrt{\beta_1}$ and β_2 respectively. Most of them were also considered by Shapiro *et al.* (1968) in their comparative study of tests of normality. The main difference is our inclusion of Johnson's (1949) unbounded curves to represent more alternatives with $\beta_2 > 3{\cdot}0$. They used the double chi-squared distributions here. Table 3 contains the empirical powers for some of these alternatives. A complete table of all the alternatives and their powers is available from the author. We include for comparison with D the results of Shapiro *et al.* (1968) for W, $\sqrt{b_1}$, b_2 and u. We also include new power calculations for $\sqrt{b_1}$ and b_2 using Johnson's unbounded curves as alternatives.

In judging the comparative power of D a few points should be kept in mind. First, while Table 3 contains empirical powers for $n = 50$ we stress that this is the smallest sample size we suggest for the use of D. There is still available W. The main use of D comes when W is unavailable, for example when $n \geqslant 100$. However, even for $n = 50$, D is as powerful or more powerful than W for about one half of the alternative distributions considered.

Table 3. 10 % *level power in per cent* ($n = 50$)

Alternative	Distributions	Moments $\sqrt{\beta_1}$	Moments β_2	Test D	Test W	Test $\sqrt{b_1}$	Test b_2	Test u
Beta (p, q)	$p = q = 1$ (uniform)	0	1·80	70	96	1	93	99
	$p = q = 2$ (parabolic)	0	2·14	41	37	1	51	96
	$p = 2, q = 1$ (triang.)	−0·57	2·40	16	96	59	29	64
	$p = 3, q = 2$	−0·29	2·36	25	42	37	29	35
Chi-squared (ν)	$\nu = 1$	2·83	15·00	*	*	*	96	29
	$\nu = 2$ (exponential)	2·00	9·00	90	*	*	81	21
	$\nu = 4$	1·41	6·00	72	97	95	51	17
	$\nu = 10$	0·89	4·20	30	74	71	37	19
Johnson's bounded (γ, δ),	$\gamma = 0, \quad \delta = 0{\cdot}7071$	0	1·87	71	85	1	89	94
$\gamma + \delta \log\{x/(1-x)\}$ is a standard	$\gamma = 0{\cdot}533, \delta = 0{\cdot}5$	0·65	2·13	23	*	67	51	99
normal variable $0 \leqslant x \leqslant 1$	$\gamma = 0, \quad \delta = 2{\cdot}0$	0	2·63	14	7	4	10	7
	$\gamma = 1, \quad \delta = 1{\cdot}0$	0·73	2·91	17	8	9	8	10
	$\gamma = 1, \quad \delta = 2{\cdot}0$	0·28	2·77	7	19	17	11	13
	$\gamma = 0, \quad \delta = 0{\cdot}5$	0	1·63	59	†	0	90	†
Laplace (double exponential)	—	0	6·00	68	49	44	60	51
Logistic	—	0	4·2	30	20	29	34	29
Log normal (μ, σ^2)	$\mu = 0, \sigma^2 = 1$	6·18	113·94	99	*	*	90	43
Student t (ν)	$\nu = 2$	0	—	93	84	75	92	87
	$\nu = 4$	0	—	56	42	49	60	51
	$\nu = 10$	0	4·00	28	21	28	28	24
Tukey (λ), $R^\lambda - (1-R)^\lambda$,	$\lambda = 0{\cdot}1$	0	3·21	12	8	14	13	12
where R is uniform on	$\lambda = 0{\cdot}2$	0	2·71	10	12	5	9	17
unit interval	$\lambda = 0{\cdot}7$	0	1·92	72	78	2	81	87
	$\lambda = 1{\cdot}5$	0	1·75	70	97	1	98	99
	$\lambda = 3{\cdot}0$	0	2·06	54	55	1	60	75
	$\lambda = 5{\cdot}0$	0	2·90	15	24	3	2	5
	$\lambda = 10{\cdot}0$	0	5·38	*	*	43	90	58
Weibull (k), $kx^{k-1} e^{-x^k}$	$k = 0{\cdot}5$	6·62	87·72	*	*	*	98	44
	$k = 2{\cdot}0$	0·63	3·25	14	24	17	10	14
Johnson's unbounded (γ, δ),	$\gamma = 0, \delta = 0{\cdot}9$	0	83·08	91	†	74	89	†
$\gamma + \delta \sinh^{-1}(x)$ is a standard	$\gamma = 0, \delta = 1$	0	36·19	88	†	66	80	†
normal variable	$\gamma = 0, \delta = 2$	0	4·51	30	†	34	26	†
$-\infty \leqslant x \leqslant \infty$	$\gamma = 0, \delta = 3$	0	3·53	16	†	18	14	†
	$\gamma = 0, \delta = 4$	0	3·28	14	†	14	10	†
	$\gamma = 1, \delta = 2$	0·87	5·59	38	†	58	42	†

* 100 %. † Not considered by Shapiro *et al.* (1968).

Secondly, if the type of deviation from normality is known a test other than D may be *a priori* appropriate. Geary (1947) has shown that $\sqrt{b_1}$ and b_2 have optimal large sample properties if the deviation is due solely to skewness or kurtosis, respectively. Also the empirical results of Shapiro *et al.* (1968) indicate u has very good sensitivity for symmetric alternatives with short tails, for example the uniform distribution. The statistic D is most useful when the type of deviation from normality is unknown. It maintains good power over a wide spectrum of alternatives. It is as powerful as or more powerful than $\sqrt{b_1}$ for about half of the skewed distributions considered and always as powerful as or more powerful than

23-2

$\sqrt{b_1}$ for symmetric alternatives. For about three-quarters of both the symmetric and skewed alternatives D is as powerful as or more powerful than b_2. Also D is as powerful or more powerful than u for about two-thirds of the symmetric alternatives while almost always so for skewed alternatives.

Thirdly, even if the type of deviation from normality is known a serious practical consideration, namely the lack of adequate tables of critical values for an appropriate test, may lead to D's use. Except for $\sqrt{b_1}$ where a simple normal approximation of the null distribution exists (D'Agostino, 1970*b*) elaborate tables of critical values do not exist, nor are any approximations yet known for the other tests of normality, i.e. b_2 and u when $n > 50$.

5. ASYMPTOTIC NULL DISTRIBUTION OF D

If the distribution from which the sample is drawn is normal then D is asymptotically normal with variance of order $1/n$. This follows because both T/n^2 and S are asymptotically normal with variance of order $1/n$ and the ratio of two such variables, i.e. D, also is asymptotically normal with variance of order $1/n$. The result for T/n^2 follows by applying the technique of Chernoff, Gastwirth & Johns (1967) to it. For S and D we have standard large sample theory (Rao, 1965, pp. 319, 366 and 321).

The above result implies that the technique of Cornish–Fisher expansions (Fisher & Cornish, 1960) is appropriate for approximating the null distribution of D. These expansions require the cumulants of D. The expansion using the first four cumulants is as follows. If D_P and Z_P are the $100P$ percentile points $(0 < P < 1)$ of D and the standard normal distribution respectively, then the Cornish–Fisher expansion for D_P in terms of Z_P is

$$D_P = E(D) + V_P \sqrt{\{\mu_2(D)\}}, \tag{5·1}$$

where

$$V_P = Z_P + \frac{\gamma_1(Z_P^2-1)}{6} + \frac{\gamma_2(Z_P^3-3Z_P)}{24} - \frac{\gamma_1^2(2Z_P^3-5Z_P)}{36}. \tag{5·2}$$

Here $E(D), \mu_2(D), \gamma_1$ and γ_2 are respectively the mean, variance, third and fourth cumulant values of D.

It has been possible to compute the first four cumulants of D under the null hypothesis of normality. Barnett, Mullen & Saw (1967) give the first four noncentral moments of T, the numerator of D, and Kendall & Stuart (1969, p. 372) give the first four noncentral moments of $n^{\frac{1}{2}}S$. From these the noncentral moments of D up to the fourth can be computed. This follows because D and S are independent by sufficiency (Hogg & Craig, 1956) and thus

$$E(D^k) = \left(\frac{1}{n^{\frac{3}{2}}}\right)^k E\left(\frac{T}{n^{\frac{1}{2}}S}\right)^k \tag{5·3}$$

$$= \frac{1}{n^{3k/2}} \frac{E(T^k)}{E(n^{\frac{1}{2}}S)^k} \tag{5·4}$$

for $k > 0$. The first two noncentral moments of D are thus

$$E(D) = \frac{n-1}{2\sqrt{2n\pi}} \frac{\Gamma(\frac{1}{2}n-\frac{1}{2})}{\Gamma(\frac{1}{2}n)}, \tag{5·5}$$

or approximately $1/(2\sqrt{\pi})$, and

$$E(D^2) = \frac{1}{n^2}\left\{\frac{1}{4} + \left(\frac{1}{12}+\frac{3}{2\pi}\right)(n-2) + \frac{(n-2)(n-3)}{4\pi}\right\}. \tag{5·6}$$

The third and fourth noncentral moments are more complicated and will not be given. However, they are readily available using (5·4) and the above two references. Given the noncentral moments the needed cumulants can be computed (Kendall & Stuart, 1969, p. 70). These in turn can be used in the expansions.

Percentile points of Y are obtained from those of D by use of (2·6).

For $n \geqslant 200$ approximations can be used in the Cornish–Fisher expansions which produce percentile points for Y differing by at most four units in the third decimal place from those given in Table 1. The approximations are

$$\sqrt{\{\mu_2(D)\}} \simeq \frac{0\cdot029986}{\sqrt{n}}, \quad \gamma_1 \simeq -\frac{8\cdot463}{\sqrt{n}}, \quad \gamma_2 \simeq \frac{107\cdot9}{n}. \tag{5·7}$$

These were found by fitting functions by least squares to the actual values of $\sqrt{\{\mu_2(D)\}}$, γ_1 and γ_2. With these approximations we suggest use of the exact value of $E(D)$ given in (5·5). If exact values are not possible the approximation

$$E(D) \simeq \frac{1}{2\sqrt{\pi}}\sqrt{\left(\frac{n-1}{n}\right)}\left\{1+\frac{1}{4(n-1)}+\frac{1}{32(n-1)^2}-\frac{5}{128(n-1)^3}\right\} \tag{5·8}$$

should be adequate.

Finally, we mention that the normal approximation to D, i.e. taking Y of (2·6) as normal with mean zero and variance unity, does not appear to be appropriate except for extremely large n, well over 1000. Rather than its use, we suggest the use of either Table 1 or the Cornish–Fisher expansions in conjunction with the approximations (5·7) and (5·8).

6. Conclusions

We close by reviewing a bit of statistical history. Downton (1966) suggested the statistic

$$\sigma^* = \frac{2\sqrt{\pi}\,T}{n(n-1)}$$

as a quite efficient alternative for unbiasedly estimating the normal distribution standard deviation. Barnett *et al.* (1967) suggested the use of σ^* in the statistic

$$Y = \frac{\sqrt{n}\,(\bar{X}-\mu)}{\sigma^*}$$

for testing about the mean of a normal distribution. David (1968) showed σ^* was Gini's mean difference, a venerable statistical tool. D'Agostino (1970*a*) showed σ^* could be used as the basis for a very efficient estimator of the normal distribution standard deviation having a small mean squared error. Now we attempt to use it in D to give us a criterion for judging normality. In all these applications it does well.

I am grateful to Professor D. R. Cox and a referee for numerous suggestions especially with respect to the presentation of Table 1.

REFERENCES

BARNETT, F. C., MULLEN, K. & SAW, J. G. (1967). Linear estimation of a population scale parameter. *Biometrika* **54**, 551–4.

CHERNOFF, H., GASTWIRTH, J. L. & JOHNS, M. V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Ann. Math. Statist.* **38**, 52–72.

D'AGOSTINO, R. B. (1970*a*). Linear estimation of the normal distribution standard deviation. *Am. Statist.* **24**, 14–5.

D'AGOSTINO, R. B. (1970*b*). Transformation to normality of the null distribution of g_1. *Biometrika* **57**, 679–81.

DAVID, H. A. (1968). Gini's mean difference rediscovered. *Biometrika* **55**, 573–5.

DOWNTON, F. (1966). Linear estimates with polynomial coefficients. *Biometrika* **53**, 129–41.

FISHER, R. A. & CORNISH, E. A. (1960). The percentile points of distributions having known cumulants. *Technometrics* **2**, 209–25.

GEARY, R. C. (1947). Testing for normality. *Biometrika* **34**, 209–42.

HARTER, H. L. (1961). Expected values of normal order statistics. *Biometrika* **48**, 151–65.

HOGG, R. V. & CRAIG, A. J. (1956). Sufficient statistics in elementary distribution theory. *Sankhyā* **17**, 209–16.

JOHNSON, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika* **36**, 149–76.

KENDALL, M. G. & STUART, A. (1969). *The Advanced Theory of Statistics, I*, 3rd edition. London: Griffin.

RAO, C. R. (1965). *Linear Statistical Inference and its Applications*. New York: Wiley.

SHAPIRO, S. S. & WILK, M. B. (1965). An analysis of variance test for normality (complete samples). *Biometrika* **52**, 591–611.

SHAPIRO, S. S. & WILK, M. B. (1968). Approximations for the null distribution of the *W* statistic. *Technometrics* **10**, 861–6.

SHAPIRO, S. S., WILK, M. B. & CHEN, H. J. (1968). A comparative study of various tests for normality. *J. Am. Statist. Ass.* **63**, 1343–72.

[*Received June* 1970. *Revised November* 1970]

Some key words: Tests of normality; Application of Downton's estimator; Application of Gini's mean difference; Cornish–Fisher expansions; Omnibus test of normality; Shapiro–Wilk test; Skewness and kurtosis.