

# Standard Dissipative Materials with Gradient Variables

Theory and Computation

by

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# Contents

<b>1</b>	<b>Introduction</b>	<b>5</b>
1.1	???	5
<b>2</b>	<b>Thermodynamics with Gradient Variables</b>	<b>7</b>
2.1	<b>Literature overview</b>	<b>7</b>
2.1.1	Modeling school A?	7
2.1.2	Modeling school B?	7
2.2	<b>Thermodynamic concepts in the presence of gradient variables</b>	<b>7</b>
2.2.1	Thermodynamic system	7
2.3	<b>Balance laws</b>	<b>8</b>
2.3.1	Momentum balance	8
2.3.2	Energy balance	9
2.3.3	Entropy inequality – Dissipation inequalities	11
2.4	<b>Constitutive framework - Isothermal conditions</b>	<b>12</b>
2.4.1	Canonical format - 2-field format	12
2.5	<b>Constitutive framework - Non-isothermal conditions</b>	<b>14</b>
2.5.1	Canonical format - 3-field format	14
<b>3</b>	<b>Boundary value problems with gradient variables</b>	<b>19</b>
3.1	<b>Primary (canonical) problem formulation</b>	<b>19</b>
3.1.1	Strong and weak format	19
3.1.2	Time-discrete variational formulation	19
3.1.3	FEM	19
3.2	<b>Dual problem formulation</b>	<b>19</b>
3.2.1	Strong and weak format	19
3.2.2	Time-discrete variational formulation	19
3.2.3	FEM	19
<b>4</b>	<b>Book template examples</b>	<b>21</b>
4.1	<b>Citation</b>	<b>21</b>
4.2	<b>Lists</b>	<b>21</b>
4.2.1	Numbered List	21
4.2.2	Bullet Points	21
4.2.3	Descriptions and Definitions	21
4.3	<b>Theorems</b>	<b>21</b>
4.3.1	Several equations	21
4.3.2	Single Line	21

<b>4.4</b>	<b>Definitions</b>	<b>22</b>
<b>4.5</b>	<b>Notations</b>	<b>22</b>
<b>4.6</b>	<b>Remarks</b>	<b>22</b>
<b>4.7</b>	<b>Corollaries</b>	<b>22</b>
<b>4.8</b>	<b>Propositions</b>	<b>22</b>
4.8.1	Several equations .....	22
4.8.2	Single Line .....	23
<b>4.9</b>	<b>Examples</b>	<b>23</b>
4.9.1	Equation and Text .....	23
4.9.2	Paragraph of Text .....	23
<b>4.10</b>	<b>Exercises</b>	<b>23</b>
<b>4.11</b>	<b>Problems</b>	<b>23</b>
<b>4.12</b>	<b>Vocabulary</b>	<b>23</b>
<b>4.13</b>	<b>Table</b>	<b>23</b>
<b>4.14</b>	<b>Figure</b>	<b>23</b>
	<b>Bibliography</b> .....	<b>25</b>
	Books	25
	Articles	25
	<b>Index</b> .....	<b>27</b>

# 1. Introduction

In this chapter we present

## 1.1 ???

This text presents



## 2. Thermodynamics with Gradient Variables

In this chapter we present the basic relations of continuum thermodynamics for solid material behavior with “nonlocal”, i.e. gradient, effects. Constitutive relations are established for the most general situation of non-isothermal behavior.

### 2.1 Literature overview

TRUESDELL [Truesdell1968 ], NOLL [Noll1958 ], TRUESDELL & NOLL [Truesdell1965 ], and COLEMAN [Coleman1964 ], text-books by LEMAITRE & CHABOCHE [Lemaitre1990a ] and MAUGIN [Maugin1992 ].

---

#### 2.1.1 Modeling school A?

sdffsdfdsfdfsfd fsd fsd fsd f sdf sd  
fsdfds

#### 2.1.2 Modeling school B?

fsdfsd fsd fs dfs  
sfsdfs  
fsdfsd

### 2.2 Thermodynamic concepts in the presence of gradient variables

gfdgfdg dfg dg df  
gdffgd

#### 2.2.1 Thermodynamic system

Some introduction to thermodynamic system. How we use it.

**Definition 2.2.1 — Closed thermodynamic system.** The standard definition of a *closed thermodynamic system* is a macroscopic volume in space that is closed in the sense that it can not exchange matter with the surroundings. It can, however, exchange energy in various forms with the surroundings through the boundary of the volume.<sup>a</sup>

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<sup>a</sup>This definition excludes “growth” whereby the mass balance law involves a source term.

When establishing balance equations, it is convenient to consider an arbitrary part of an existing finite body whose boundary is subjected to given boundary conditions. In other words, we consider the “cut-out” amount of matter that occupies the spatial region  $\mathcal{B}$  with boundary  $\partial\mathcal{B}$  (with outward unit normal  $\mathbf{n}$ , as shown in Figure ??). Here, we consider only small deformations, whereby the mass density  $\rho$  is taken as a material parameter (rather than a field variable) that is unaffected by the deformation. This “body” is acted upon by two types of forces, macroforces and microforces.

**Definition 2.2.2 — Macroforces.** Macroforces represents “local” or “mechanical” forces in continuum models that are conjugated with the macro-displacement  $\mathbf{u}$  of the body. The

macroforces are made up from two types of forces. A body force  $\mathbf{b}$  (force per unit volume) acting in the interior of  $\mathcal{B}$ , and surface tractions  $\mathbf{t}_n$  acting on  $\partial\mathcal{B}$  from the surrounding matter (from which the considered body is cut-out from).

**Definition 2.2.3 — Microforces.** Microforces represents “non-local” or “gradient” forces in continuum models that are conjugated to the “micro-displacement”  $u^\mu = k$ , which is identical to the internal variable. A volume-specific “micro-force”  $b^\mu$  is acting in the interior of  $\mathcal{B}$ , and a surface “micro-traction”  $t_n^\mu$  is acting on  $\partial\mathcal{B}$  from the surrounding matter <sup>a</sup>

<sup>a</sup>To simplify notation, we assume in this chapter that  $b^\mu$  and  $t_n^\mu$  are scalar quantities

### Heat sources

A volume-specific heat source  $r$  (power per unit volume) acts in the interior of  $\mathcal{B}$ , and the thermal power  $q_n$  is supplied via  $\partial\mathcal{B}$ . It is assumed that there is no micro-source of heat present.

## 2.3 Balance laws

### 2.3.1 Momentum balance

$\mathcal{P}$  has no bold in this font

The total momentum  $\mathcal{P}$  of the body  $\mathcal{B}$  in Figure ?? is given as

$$\mathcal{P} = \int_{\mathcal{B}} \rho \dot{\mathbf{u}} dV, \quad (2.1)$$

whereas the resultant  $\mathcal{F}_{\text{ext}}$  of the externally applied mechanical loads on  $\mathcal{B}$  is

$$\mathcal{F}_{\text{ext}} = \int_{\mathcal{B}} \mathbf{b} dV + \int_{\partial\mathcal{B}} \mathbf{t}_n dS. \quad (2.2)$$

The global format of the momentum balance reads

$$\dot{\mathcal{P}} = \mathcal{F}_{\text{ext}}. \quad (2.3)$$

**Theorem 2.3.1** The equations of motion can be written as  $\rho \ddot{\mathbf{u}} - \boldsymbol{\sigma} \cdot \nabla = \mathbf{b}$ .

*Proof.* We may use the relation  $\mathbf{t}_n = \boldsymbol{\sigma} \cdot \mathbf{n}$  between the (symmetric) stress tensor  $\boldsymbol{\sigma}$  and the total traction  $\mathbf{t}_n$  along the boundary  $\partial\mathcal{B}$  together with Green’s theorem to transform the boundary integral in (2.2) into a volume integral, which gives

$$\mathcal{F}_{\text{ext}} = \int_{\mathcal{B}} \mathbf{b} + \boldsymbol{\sigma} \cdot \nabla dV \quad (2.4)$$

Upon inserting (2.1) and (2.4) into (2.3), we obtain

$$\int_{\mathcal{B}} \rho \ddot{\mathbf{u}} dV = \int_{\mathcal{B}} \mathbf{b} + \boldsymbol{\sigma} \cdot \nabla dV \quad (2.5)$$

Since the volume  $\mathcal{B}$  is arbitrary the integrands must be equal pointwise which is exactly 2.3.1. ■

**R** When inertia forces are ignored ( $\rho \ddot{\mathbf{u}} = \mathbf{0}$ ), then 2.3.1 represents the quasistatic equilibrium equation.



In a completely analogous fashion, we establish the resultant  $\mathcal{F}_{\text{ext}}^\mu$  of the externally applied micro-forces on  $\mathcal{B}$  as

$$\mathcal{F}_{\text{ext}}^\mu = \int_{\mathcal{B}} b^\mu dV + \int_{\partial\mathcal{B}} t_n^\mu dS. \quad (2.6)$$

Clearly, the global format of micro-force equilibrium is

$$\mathcal{F}_{\text{ext}}^\mu = 0. \quad (2.7)$$

**Theorem 2.3.2** The equation of micro-equilibrium can be written as  $-\boldsymbol{\sigma}^\mu \cdot \boldsymbol{\nabla} = b^\mu$

*Proof.* The proof is done in the same way as theorem 2.3.1. ■

### 2.3.2 Energy balance

We first define the standard forms of *mechanical* and *thermal* power supply.

**Definition 2.3.1 — Mechanical power.**

$$\mathcal{P}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} \mathbf{b} \cdot \dot{\mathbf{u}} dV + \int_{\partial\mathcal{B}} \mathbf{t}_n \cdot \dot{\mathbf{u}} dS, \quad (2.8)$$

**Definition 2.3.2 — Thermal power.**

$$\mathcal{Q}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} r dV + \int_{\partial\mathcal{B}} q_n dS, \quad (2.9)$$

We also introduce the concept of micropower.

**Definition 2.3.3 — Micropower.** The micropower is the power supply from the microforces

$$\mathcal{P}_{\text{ext}}^\mu \stackrel{\text{def}}{=} \int_{\mathcal{B}} b^\mu \dot{k} dV + \int_{\partial\mathcal{B}} t_n^\mu \dot{k} dS. \quad (2.10)$$

We first introduce the concepts of kinetic and total internal energy.

**Definition 2.3.4 — Kinetic energy.** The kinetic energy  $\mathcal{K}$  is defined as

$$\mathcal{K} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathcal{B}} \rho |\dot{\mathbf{u}}|^2 dV \quad (2.11)$$

**Definition 2.3.5 — Total internal energy.** The total internal energy  $\mathcal{E}$  is defined as

$$\mathcal{E} \stackrel{\text{def}}{=} \int_{\mathcal{B}} \rho e dV \quad (2.12)$$

where  $e$  is the internal energy per unit mass.

In standard fashion, energy is supplied to the system in the form of mechanical power  $\mathcal{P}_{\text{ext}}$  (from the mechanical loads  $\mathbf{b}$  and  $\mathbf{t}$ ) and in the form of *thermal power*  $\mathcal{Q}_{\text{ext}}$  (from the thermal “loads”  $r$  and  $q$ ). However, energy is also supplied from “micro-power”  $\mathcal{P}_{\text{ext}}^\mu$  (generated by  $b^\mu$  and  $t_n^\mu$ ). Part of the supplied energy may be converted into kinetic energy  $\mathcal{K}$ , which is manifested by macroscopic motion of the body ( $\dot{\mathbf{u}} \neq \mathbf{0}$ ); hence, it is assumed that the micro-motion  $\dot{k}$  does not constitute any part of  $\mathcal{K}$  since it is not associated with any “micro-inertia”. The remaining part of the energy is stored as internal energy  $\mathcal{E}$ , which has to be parameterized suitably in terms of thermodynamically independent variables. How the total energy can change is governed by the first law of thermodynamics.

**Physical law 2.3.3 — First law of thermodynamics.** This law concerns the conservation of energy in a thermodynamic system. It says that the change in the total energy of a thermodynamic system is equal to the external power flowing into the system... **FIX**

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{P}_{\text{ext}} + \mathcal{P}_{\text{ext}}^{\mu} + \mathcal{Q}_{\text{ext}} \quad (2.13)$$

The precise way in which the total energy  $\mathcal{E} + \mathcal{K}$  may change in a given thermodynamic process is discussed next.

**Theorem 2.3.4 — The local energy equation.** The energy in the system satisfies

$$\rho \dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \boldsymbol{\sigma}^{\mu} \cdot \dot{\mathbf{g}} + r - \mathbf{h} \cdot \nabla. \quad (2.14)$$

*Proof.* We may use the relations  $\mathbf{t}_n = \boldsymbol{\sigma} \cdot \mathbf{n}$  and  $t_n^{\mu} = \boldsymbol{\sigma}^{\mu} \cdot \mathbf{n}$  together with Green's theorem to transform the boundary integrals in (2.8) and (2.10) into volume integrals, which gives

$$\mathcal{P}_{\text{ext}} = \int_{\mathcal{B}} \mathbf{b} \cdot \dot{\mathbf{u}} \, dV + \int_{\mathcal{B}} [\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}] \cdot \nabla \, dV = \int_{\mathcal{B}} [\mathbf{b} + \boldsymbol{\sigma} \cdot \nabla] \cdot \dot{\mathbf{u}} \, dV + \int_{\mathcal{B}} \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, dV \quad (2.15)$$

$$\mathcal{P}_{\text{ext}}^{\mu} = \int_{\mathcal{B}} b^{\mu} \dot{k} \, dV + \int_{\mathcal{B}} [\boldsymbol{\sigma}^{\mu} \dot{k}] \cdot \nabla \, dV = \int_{\mathcal{B}} [b^{\mu} + \boldsymbol{\sigma}^{\mu} \cdot \nabla] \dot{k} \, dV + \int_{\mathcal{B}} \boldsymbol{\sigma}^{\mu} \cdot \dot{\mathbf{g}} \, dV \quad (2.16)$$

Here, we introduced the strain and gradient operators  $\boldsymbol{\epsilon}$  and  $\mathbf{g}$

**Maybe not new operators introduced in a proof...**

$$\boldsymbol{\epsilon}[\mathbf{u}] \stackrel{\text{def}}{=} (\mathbf{u} \otimes \nabla)^{\text{sym}} \Rightarrow \dot{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}[\dot{\mathbf{u}}], \quad (2.17a)$$

$$\mathbf{g}[k] \stackrel{\text{def}}{=} k \otimes \nabla = \nabla k \Rightarrow \dot{\mathbf{g}} = \mathbf{g}[\dot{k}]. \quad (2.17b)$$

In order to obtain the last expression in (2.15), we used that  $\boldsymbol{\sigma}$  is symmetrical, which implies that  $\boldsymbol{\sigma} : [\mathbf{u} \otimes \nabla] = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}$ .

Now, inserting the equations of equilibrium in (2.5) and micro-equilibrium in (??) into (2.15) and (2.16), we obtain

$$\mathcal{P}_{\text{ext}} = \int_{\mathcal{B}} \dot{\mathbf{u}} \cdot \rho \dot{\mathbf{u}} + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, dV = \int_{\mathcal{B}} \frac{d}{dt} \left[ \frac{1}{2} \rho |\dot{\mathbf{u}}|^2 \right] + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \, dV \quad (2.18)$$

$$\mathcal{P}_{\text{ext}}^{\mu} = \int_{\mathcal{B}} \boldsymbol{\sigma}^{\mu} \cdot \dot{\mathbf{g}} \, dV \quad (2.19)$$

Likewise, we may introduce the heat flux vector  $\mathbf{h}$  and use the relation  $q_n = -\mathbf{h} \cdot \mathbf{n}$  to transform the boundary integral in (2.9) into a volume integral. We then obtain

$$\mathcal{Q}_{\text{ext}} = \int_{\mathcal{B}} \mathcal{Q} \, dV \quad \text{with} \quad \mathcal{Q} \stackrel{\text{def}}{=} -\nabla \cdot \mathbf{h} \quad (2.20)$$

where  $\mathcal{Q}$  is the specific external thermal power supply.

Inserting (2.18), (2.19) and (2.20) into (2.13), we may express the energy balance equation as

$$\int_{\mathcal{B}} [\rho \dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \boldsymbol{\sigma}^{\mu} \cdot \dot{\mathbf{g}} - \mathcal{Q}] \, dV = 0 \quad (2.21)$$

Localizing this result, i.e. the integrand must vanish identically, we obtain theorem 2.3.4. ■

**R** In a locally *isometric* process ( $\mathcal{W} = 0$ ), the net heat input is converted entirely into internal energy, whereas in a locally *adiabatic* process ( $\mathcal{Q} = 0$ ), the internal energy gained is the work done by the stresses. **REINTERPRET!!**

### 2.3.3 Entropy inequality – Dissipation inequalities

In order to complete the characterization of a thermodynamic system, we shall also introduce the *entropy*  $\mathcal{S}$ . How the entropy can change due to the supply of heat power is governed by the second law of thermodynamics, which formally takes the same form as in the standard situation without microforces present.

We introduce the *entropy function*  $\mathcal{S}$

$$\mathcal{S} = \int_{\mathcal{B}} \rho s dV \quad (2.22)$$

where  $s$  is the entropy density (entropy per unit mass). Associated with the existence of  $\mathcal{S}$ , we define  $\mathcal{R}_{\text{ext}}$  as the rate of *input of entropy* from the exterior into  $\mathcal{B}$ . The standard expression is

$$\mathcal{R}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} \frac{r}{\theta} dV + \int_{\partial\mathcal{B}} \frac{q_n}{\theta} dS \quad (2.23)$$

The global format of the 2nd law (=axiom) of thermodynamics is *defined* as the inequality

$$\dot{\mathcal{S}} - \mathcal{R}_{\text{ext}} \geq 0 \quad (2.24)$$

Upon transforming the surface integral in (2.23) into a volume integral, we first obtain the representation

$$\mathcal{R}_{\text{ext}} = v \int_{\mathcal{B}} \mathcal{R} \quad \text{with} \quad \mathcal{R} \stackrel{\text{def}}{=} \frac{\mathcal{Q} + \mathbf{h} \cdot \nabla [\ln \theta]}{\theta} \quad (2.25)$$

Inserting (2.22) and (2.25) into (2.24), we may express the entropy inequality as

$$\int_{\mathcal{B}} [\rho \dot{s} - \mathcal{R}] dV \geq 0 \quad \text{or} \quad \int_{\mathcal{B}} [\theta \rho \dot{s} - \mathcal{Q} - \mathbf{h} \cdot \nabla [\ln \theta]] dV \geq 0 \quad (2.26)$$

which is commonly known as the global version of the *Clausius-Duhem-Inequality (CDI)*.

**R** The result in (2.26) may be localized (in standard fashion) in the sense that the integrand is non-negative; however, this is never exploited henceforth.

Alternatively, (2.26)<sub>2</sub> may be rewritten as the *dissipation inequality*

$$\mathcal{D} = \mathcal{D}_{\text{mech}} + \mathcal{D}_{\text{therm}} \geq 0 \quad (2.27)$$

where

$$\mathcal{D}_{\text{mech}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} [\rho \theta \dot{s} - \mathcal{Q}] dV, \quad (2.28a)$$

$$\mathcal{D}_{\text{therm}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} [-\mathbf{h} \cdot \nabla [\ln \theta]] dV = \int_{\mathcal{B}} \left[ -\frac{\mathbf{h} \cdot \nabla \theta}{\theta} \right] dV \quad (2.28b)$$

are the *mechanical* and *thermal* part, respectively, of the total dissipation. Upon introducing the local format of the energy equation into (2.28a), we obtain the alternative expression

$$\mathcal{D}_{\text{mech}} = \int_{\mathcal{B}} [-\rho \dot{e} + \rho \theta \dot{s} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^\mu \cdot \dot{\mathbf{g}}] dV \quad (2.29)$$

It is common to impose separately the (sufficient but not always necessary) conditions

$$\mathcal{D}_{\text{mech}} \geq 0, \quad (2.30a)$$

$$\mathcal{D}_{\text{therm}} \geq 0 \quad (2.30b)$$

which are known as the *Clausius-Planck-Inequality (CPI)* and the *Fourier-Inequality (FI)*, respectively. This approach is taken subsequently.

## 2.4 Constitutive framework - Isothermal conditions

### 2.4.1 Canonical format - 2-field format

We first consider isothermal conditions, whereby the (absolute) temperature  $\theta = \theta_0$  serves only as a parameter in the constitutive model. We may directly introduce the mass-specific free energy density  $\psi \stackrel{\text{def}}{=} e - \theta_0 s$  with the parameterization  $\psi(\boldsymbol{\varepsilon}, k, \mathbf{g})$ . From (2.29), we then obtain

$$\begin{aligned} \mathcal{D}_{\text{therm}}(\dot{\mathbf{u}}, \dot{k}) &= \int_{\mathcal{B}} [-\rho \dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^\mu \cdot \dot{\mathbf{g}}] dV \\ &= \int_{\mathcal{B}} \left[ \left[ \boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \left[ -\rho \frac{\partial \psi}{\partial k} \right] \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial \psi}{\partial \mathbf{g}} \right] \cdot \mathbf{g}[\dot{k}] \right] dV \\ &\geq 0 \quad \forall (\dot{\mathbf{u}}, \dot{k}) \end{aligned} \quad (2.31)$$

that must hold for any given thermodynamic process defined by given fields  $\dot{\mathbf{u}}, \dot{k}$ . Note that these fields (in space-time) are treated as independent in this context.

The constitutive relations for the Standard Dissipative material with gradients under isothermal conditions are established as follows:

- Introduce the mass-specific dissipation potential function  $\phi(\boldsymbol{\varepsilon}, \dot{k}, \mathbf{g})$ , which (i) is convex<sup>1</sup> and (ii) satisfies the condition  $\phi(\mathbf{0}, 0, \mathbf{0}) = 0$ . The corresponding global dissipation functional is

$$\Phi(\dot{\mathbf{u}}, \dot{k}) \stackrel{\text{def}}{=} \int_{\mathcal{B}} \rho \phi(\boldsymbol{\varepsilon}[\dot{\mathbf{u}}], \dot{k}, \mathbf{g}[\dot{k}]) dV$$

- Introduce the global constitutive potential  $\chi(\dot{\mathbf{u}}, \dot{k}) \stackrel{\text{def}}{=} \mathcal{D}_{\text{mech}}(\dot{\mathbf{u}}, \dot{k}) - \Phi(\dot{\mathbf{u}}, \dot{k})$ , and the constitutive assumption that  $\chi$  has a saddle point in the sense that

$$(\dot{\mathbf{u}}, \dot{k}) = \arg \left[ \min_{\dot{\mathbf{u}}'} \max_{\dot{k}'} \chi(\dot{\mathbf{u}}', \dot{k}') \right]$$

That the potential  $\chi$  is stationary in the space of  $(\dot{\mathbf{u}}, \dot{k})$  for an actual thermodynamic process introduces a constraint on the relation between these fields in space-time.

The directional (partial) derivatives of  $\chi(\dot{\mathbf{u}}, \dot{k})$  for variations  $\delta \dot{\mathbf{u}}$  of  $\dot{\mathbf{u}}$  and  $\delta \dot{k}$  of  $\dot{k}$  are denoted  $\chi'_{\dot{\mathbf{u}}}(\dot{\mathbf{u}}, \dot{k}; \delta \dot{\mathbf{u}})$  and  $\chi'_{\dot{k}}(\dot{\mathbf{u}}, \dot{k}; \delta \dot{k})$ , respectively. The stationarity condition corresponding to the saddle-

<sup>1</sup>A function  $F(\underline{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex iff, for any pair  $\underline{x}_1, \underline{x}_2$ , the following inequality holds:

$$F(\alpha \underline{x}_1 + [1 - \alpha] \underline{x}_2) \leq \alpha F(\underline{x}_1) + [1 - \alpha] F(\underline{x}_2)$$

. For a smooth convex function  $F(\underline{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  the following result holds:

$$F(\underline{x}_2) - F(\underline{x}_1) \geq \left[ \frac{\partial F}{\partial \underline{x}}(\underline{x}_1) \right]^T [\underline{x}_2 - \underline{x}_1] \quad \forall \underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$$

point property then becomes

$$\begin{aligned}\chi'_u(\dot{\mathbf{u}}, k; \delta \dot{\mathbf{u}}) &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] dV - \int_{\mathcal{B}} \left[ \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] dV \\ &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] = 0, \quad \forall \delta \dot{\mathbf{u}} \in ???\end{aligned}\quad (2.32a)$$

$$\begin{aligned}\chi'_k(\dot{\mathbf{u}}, k; \delta \dot{k}) &= \int_{\mathcal{B}} \left[ -\rho \frac{\partial \psi}{\partial k} \delta \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial \psi}{\partial \mathbf{g}} \right] \cdot \mathbf{g}[\delta \dot{k}] \right] dV - \int_{\mathcal{B}} \left[ \rho \frac{\partial \phi}{\partial \dot{k}} \delta \dot{k} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \cdot \mathbf{g}[\delta \dot{k}] \right] dV \\ &= - \int_{\mathcal{B}} \left[ \rho \frac{\partial \psi}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial \psi}{\partial \mathbf{g}} - \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \nabla \right] \delta \dot{k} dV \\ &\quad + \int_{\partial \mathcal{B}} \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial \psi}{\partial \mathbf{g}} - \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \mathbf{n} \delta \dot{k} dS = 0 \quad \forall \delta \dot{k} \in ???\end{aligned}\quad (2.32b)$$

Upon localizing the result in (2.32), we obtain the constitutive identities

$$\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0} \quad \text{in } \mathcal{B} \quad (2.33a)$$

$$\rho \frac{\partial \psi}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \boldsymbol{\sigma}^\mu \cdot \nabla - \left[ \rho \frac{\partial \psi}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \nabla = 0 \quad \text{in } \mathcal{B} \quad (2.33b)$$

$$\boldsymbol{\sigma}^\mu \cdot \mathbf{n} - \left[ \rho \frac{\partial \psi}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \mathbf{n} = 0 \quad \text{on } \partial \mathcal{B} \quad (2.33c)$$

Upon introducing the micro-stress variables

$$\boldsymbol{\kappa} \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} \quad (2.34a)$$

$$\boldsymbol{\xi} \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \quad (2.34b)$$

we may abbreviate the system (2.49b, 2.49c) as

$$\boldsymbol{\kappa} + \boldsymbol{\sigma}^\mu \cdot \nabla - \boldsymbol{\xi} \cdot \nabla = 0 \quad \text{in } \mathcal{B} \quad (2.35a)$$

$$\boldsymbol{\sigma}^\mu \cdot \mathbf{n} - \boldsymbol{\xi} \cdot \mathbf{n} = 0 \quad \text{on } \partial \mathcal{B} \quad (2.35b)$$

It is possible to eliminate the micro-stress  $\boldsymbol{\sigma}^\mu$  from the constitutive equations by combining (2.35a) with the equation for micro-equilibrium in (?). As a result, (2.35a) is replaced by the equation

$$\boldsymbol{\kappa} - \boldsymbol{\xi} \cdot \nabla = b^\mu \quad \text{in } \mathcal{B} \quad (2.36)$$

Moreover, from (2.35b), we note that the micro-traction on  $\mathcal{B}$  can be expressed in terms of  $\boldsymbol{\xi}$  as  $t_n^\mu \stackrel{\text{def}}{=} \boldsymbol{\sigma}^\mu \cdot \mathbf{n} = \boldsymbol{\xi} \cdot \mathbf{n}$ ; hence, we conclude that the variable  $\boldsymbol{\xi}$  plays the role of a “shifted” micro-stress.

In the literature, e.g. BIOT [Biot1965], NGUYEN [Nguyen2000], the identity in (2.49b), or (2.35a), is known as Biot’s equation. This is a constitutive evolution equation, which can be interpreted as a “derived” micro-equilibrium equation for  $\boldsymbol{\xi}$  of the Helmholtz type, whereby the constitutive relation for  $\boldsymbol{\xi}$  was given in (2.34b). It plays a role that is similar to the standard equilibrium equation, whereby the constitutive relation for  $\boldsymbol{\sigma}$  is given directly in (2.49a). In conclusion, upon combining these equations, we are in the position to solve for the fields  $\mathbf{u}(\mathbf{x}, t)$  and  $k(\mathbf{x}, t)$  in space-time for any given body subjected to the appropriate loading and boundary conditions.

In order to comply with notation used in the literature, see e.g. GURTIN [Gurtin2000], we introduce the decomposition into “energetic” (superscript en) and “dissipative” (superscript di) parts as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{en}} + \boldsymbol{\sigma}^{\text{di}} \quad (2.37a)$$

$$\kappa = \kappa^{\text{en}} + \kappa^{\text{di}} \quad (2.37b)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\text{en}} + \boldsymbol{\xi}^{\text{di}} \quad (2.37c)$$

where

$$\boldsymbol{\sigma}^{\text{en}}(\boldsymbol{\varepsilon}, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, k, \mathbf{g}), \quad \boldsymbol{\sigma}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.38a)$$

$$\kappa^{\text{en}}(\boldsymbol{\varepsilon}, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial k}(\boldsymbol{\varepsilon}, k, \mathbf{g}), \quad \kappa^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{k}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.38b)$$

$$\boldsymbol{\xi}^{\text{en}}(\boldsymbol{\varepsilon}, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \mathbf{g}}(\boldsymbol{\varepsilon}, k, \mathbf{g}), \quad \boldsymbol{\xi}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.38c)$$

In conclusion, we may express (2.36) as

$$\kappa^{\text{en}}(\boldsymbol{\varepsilon}, k, \mathbf{g}) + \kappa^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) - \left[ \boldsymbol{\xi}^{\text{en}}(\boldsymbol{\varepsilon}, k, \mathbf{g}) + \boldsymbol{\xi}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \right] \cdot \nabla = b^\mu \quad (2.39)$$

It remains to check that the CDI is satisfied for the *actual* thermodynamic process defined by  $(\dot{\mathbf{u}}, \dot{k})$  that satisfy the constitutive constraint equations (2.35). From (2.31) we obtain

$$\begin{aligned} \mathcal{D}_{\text{mech}}(\dot{\mathbf{u}}, \dot{k}) &= \int_{\mathcal{B}} \left[ \left[ \boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \left[ -\rho \frac{\partial \psi}{\partial k} \right] \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial \psi}{\partial \mathbf{g}} \right] \cdot \mathbf{g}[\dot{k}] \right] dV \\ &\stackrel{(2.38)}{=} \int_{\mathcal{B}} \left[ \left[ \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{en}} \right] : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \left[ -\kappa^{\text{en}} \right] \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \boldsymbol{\xi}^{\text{en}} \right] \cdot \mathbf{g}[\dot{k}] \right] dV \\ &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma}^{\text{di}} : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \kappa^{\text{di}} \dot{k} - \kappa^{\text{en}} \dot{k} + \boldsymbol{\xi}^{\text{di}} \cdot \mathbf{g}[\dot{k}] + \left[ \boldsymbol{\sigma}^\mu - \boldsymbol{\xi} \right] \cdot \mathbf{g}[\dot{k}] \right] dV \\ &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma}^{\text{di}} : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \kappa^{\text{di}} \dot{k} + \boldsymbol{\xi}^{\text{di}} \cdot \mathbf{g}[\dot{k}] \right] dV \\ &\quad - \int_{\mathcal{B}} \underbrace{\left[ \kappa + \left[ \boldsymbol{\sigma}^\mu - \boldsymbol{\xi} \right] \cdot \nabla \right] \dot{k}}_{=0 \text{ from (2.35a)}} dV + \int_{\partial \mathcal{B}} \underbrace{\left[ \boldsymbol{\sigma}^\mu - \boldsymbol{\xi} \right] \cdot \mathbf{n}}_{=0 \text{ from (2.35b)}} \dot{k} dS \\ &\stackrel{\geq 0}{=} \int_{\mathcal{B}} \underbrace{\left[ \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \rho \frac{\partial \phi}{\partial \dot{k}} \dot{k} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \cdot \mathbf{g}[\dot{k}] \right]}_{\geq 0} dV \geq 0 \quad \forall \dot{\mathbf{u}}, \dot{k} \end{aligned} \quad (2.40)$$

That, indeed,  $\mathcal{D}_{\text{mech}} \geq 0$  follows directly from the properties of  $\phi$  as given above.

**Proof:** Consider  $X(\underline{x})$  convex and  $X(\underline{0}) = 0$ . From convexity follows that, for any given  $\underline{x}$ ,

$$0 \leq X(\underline{x}) - \underbrace{X(\underline{0})}_{=0} \leq X'(\underline{x})^T [\underline{x} - \underline{0}] \quad \Rightarrow \quad X'(\underline{x})^T \underline{x} \geq 0 \quad (2.41)$$

Now, setting  $X = \phi$  and  $\underline{x} = (\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}})$ , we directly obtain the inequality in (2.40).  $\square$

## 2.5 Constitutive framework - Non-isothermal conditions

### 2.5.1 Canonical format - 3-field format

Next, we consider the general situation of non-isothermal conditions, whereby the (absolute) temperature  $\theta$  (or the entropy  $s$ ) is included as a thermodynamic variable in the constitutive

model. The basic parameterization of the mass-specific internal energy density is  $e(\boldsymbol{\varepsilon}, s, k, \mathbf{g})$ . From (2.29), we now obtain

$$\begin{aligned}\mathcal{D}_{\text{mech}}(\dot{\mathbf{u}}, \dot{s}, \dot{k}) &= \int_{\mathcal{B}} [-\rho \dot{e} + \rho \theta \dot{s} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^\mu \cdot \dot{\mathbf{g}}] dV \\ &= \int_{\mathcal{B}} \left[ \left[ \boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\dot{\mathbf{u}}] + \left[ \rho \theta - \rho \frac{\partial e}{\partial s} \right] \dot{s} + \left[ -\rho \frac{\partial e}{\partial k} \right] \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial e}{\partial \mathbf{g}} \right] \cdot \mathbf{g}[\dot{k}] \right] dV \\ &\geq 0 \quad \forall \dot{\mathbf{u}}, \dot{s}, \dot{k}\end{aligned}\tag{2.42}$$

that must hold for any given thermodynamic process defined by given fields  $\dot{\mathbf{u}}, \dot{s}, \dot{k}$ . Note that these fields (in space-time) treated as independent in this context.

The constitutive relations for the Standard Dissipative material with gradients under non-isothermal conditions are established as follows:

- Introduce the mass-specific dissipation potential function  $\phi(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}})$ , which (i) is convex and (ii) satisfies the condition  $\phi(\mathbf{0}, 0, \mathbf{0}) = 0$ . The corresponding global dissipation functional is

$$\Phi(\dot{\mathbf{u}}, \dot{k}) \stackrel{\text{def}}{=} \int_{\mathcal{B}} \rho \phi(\boldsymbol{\varepsilon}[\dot{\mathbf{u}}], \dot{k}, \mathbf{g}[\dot{k}]) dV$$

- Introduce the global constitutive potential  $\chi(\dot{\mathbf{u}}, \dot{s}, \dot{k}) \stackrel{\text{def}}{=} \mathcal{D}_{\text{mech}}(\dot{\mathbf{u}}, \dot{s}, \dot{k}) - \Phi(\dot{\mathbf{u}}, \dot{k})$ , and the constitutive assumption that  $\chi$  has a saddle point in the sense that

$$(\dot{\mathbf{u}}, \dot{s}, \dot{k}) = \arg \left[ \min_{\dot{\mathbf{u}}'} \min_{\dot{s}'} \max_{\dot{k}'} \chi(\dot{\mathbf{u}}', \dot{s}', \dot{k}') \right]$$

That the potential  $\chi$  is stationary in the space of  $(\dot{\mathbf{u}}, \dot{s}, \dot{k})$  for an actual thermodynamic process introduces a constraint on the relation between these fields in space-time.

The stationarity condition corresponding to the saddle-point property then becomes

$$\begin{aligned}\chi'_{\dot{\mathbf{u}}}(\dot{\mathbf{u}}, \dot{s}, \dot{k}; \delta \dot{\mathbf{u}}) &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] dV - \int_{\mathcal{B}} \left[ \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] dV \\ &= \int_{\mathcal{B}} \left[ \boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta \dot{\mathbf{u}}] = 0, \quad \forall \delta \dot{\mathbf{u}} \in ???\end{aligned}\tag{2.43a}$$

$$\begin{aligned}\chi'_{\dot{s}}(\dot{\mathbf{u}}, \dot{s}, \dot{k}; \delta \dot{s}) &= \int_{\mathcal{B}} \left[ \rho \theta - \rho \frac{\partial e}{\partial s} \right] \delta \dot{s} dV \\ &= 0, \quad \forall \delta \dot{s} \in ???\end{aligned}\tag{2.43b}$$

$$\begin{aligned}\chi'_{\dot{k}}(\dot{\mathbf{u}}, \dot{s}, \dot{k}; \delta \dot{k}) &= \int_{\mathcal{B}} \left[ -\rho \frac{\partial e}{\partial k} \delta \dot{k} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial e}{\partial \mathbf{g}} \right] \cdot \mathbf{g}[\delta \dot{k}] \right] dV - \int_{\mathcal{B}} \left[ \rho \frac{\partial \phi}{\partial \dot{k}} \delta \dot{k} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \cdot \mathbf{g}[\delta \dot{k}] \right] dV \\ &= - \int_{\mathcal{B}} \left[ \rho \frac{\partial e}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial e}{\partial \mathbf{g}} - \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \nabla \right] \delta \dot{k} dV \\ &\quad + \int_{\partial \mathcal{B}} \left[ \boldsymbol{\sigma}^\mu - \rho \frac{\partial e}{\partial \mathbf{g}} - \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \mathbf{n} \delta \dot{k} dS = 0 \quad \forall \delta \dot{k} \in ???\end{aligned}\tag{2.43c}$$

Upon localizing the result in (2.43), we obtain the constitutive identities

$$\boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0} \quad \text{in } \mathcal{B} \quad (2.44a)$$

$$\theta - \frac{\partial e}{\partial s} = 0 \quad \text{in } \mathcal{B} \quad (2.44b)$$

$$\rho \frac{\partial e}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \boldsymbol{\sigma}^\mu \cdot \nabla - \left[ \rho \frac{\partial e}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \nabla = 0 \quad \text{in } \mathcal{B} \quad (2.44c)$$

$$\boldsymbol{\sigma}^\mu \cdot \mathbf{n} - \left[ \rho \frac{\partial e}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \mathbf{n} = 0 \quad \text{on } \partial \mathcal{B} \quad (2.44d)$$

From (2.44b) appears that the temperature  $\theta$  is purely energetic. It is then convenient to introduce the mass-specific free energy density  $\psi$  via the Legendre transformation

$$\psi(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) = \inf_{\hat{s}} [e(\boldsymbol{\varepsilon}, \hat{s}, k, \mathbf{g}) - \theta \hat{s}] \quad (2.45)$$

Upon evaluating the inf, we establish the (stationarity) condition

$$\frac{\partial e}{\partial s}(\boldsymbol{\varepsilon}, s, k, \mathbf{g}) = \theta \quad (2.46)$$

which is precisely the constitutive relation for  $\theta$  already obtained in (2.44b). Assuming that  $\theta$  is monotonic in  $s$  (for fixed  $\boldsymbol{\varepsilon}, k$  and  $\mathbf{g}$ , we may solve for  $s = \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g})$  from (2.46) and insert into the expression for  $\psi$  in (2.45) to obtain

$$\psi(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) = e(\boldsymbol{\varepsilon}, \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}), k, \mathbf{g}) - \theta \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \quad (2.47)$$

Hence, the conditions are

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \Big|_s + \underbrace{\left[ \frac{\partial e}{\partial \bar{s}} - \theta \right]}_{=0} \frac{\partial \bar{s}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \quad (2.48a)$$

$$\frac{\partial \psi}{\partial \theta} = \underbrace{\left[ \frac{\partial e}{\partial \bar{s}} - \theta \right]}_{=0} \frac{\partial \bar{s}}{\partial \theta} - s = -s \quad (2.48b)$$

$$\frac{\partial \psi}{\partial k} = \frac{\partial e}{\partial k} \Big|_s + \underbrace{\left[ \frac{\partial e}{\partial \bar{s}} - \theta \right]}_{=0} \frac{\partial \bar{s}}{\partial k} = \frac{\partial e}{\partial k} \quad (2.48c)$$

$$\frac{\partial \psi}{\partial \mathbf{g}} = \frac{\partial e}{\partial \mathbf{g}} \Big|_s + \underbrace{\left[ \frac{\partial e}{\partial \bar{s}} - \theta \right]}_{=0} \frac{\partial \bar{s}}{\partial \mathbf{g}} = \frac{\partial e}{\partial \mathbf{g}} \quad (2.48d)$$

As a result, the constitutive identities (2.44) are replaced by

$$\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0} \quad \text{in } \mathcal{B} \quad (2.49a)$$

$$s + \frac{\partial \psi}{\partial \theta} = 0 \quad \text{in } \mathcal{B} \quad (2.49b)$$

$$\rho \frac{\partial \psi}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \boldsymbol{\sigma}^\mu \cdot \nabla - \left[ \rho \frac{\partial \psi}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \nabla = 0 \quad \text{in } \mathcal{B} \quad (2.49c)$$

$$\boldsymbol{\sigma}^\mu \cdot \mathbf{n} - \left[ \rho \frac{\partial \psi}{\partial \mathbf{g}} + \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}} \right] \cdot \mathbf{n} = 0 \quad \text{on } \partial \mathcal{B} \quad (2.49d)$$



We may now put forward the same arguments as for the isothermal situation to conclude that the relevant Biot equation is the same as in (2.36), i.e.

$$\boldsymbol{\kappa} - \boldsymbol{\xi} \cdot \nabla = b^\mu \quad \text{in } \mathcal{B} \quad (2.50)$$

whereby the variable  $\boldsymbol{\xi}$  plays the role of a “shifted” micro-stress.

As to the decomposition into “energetic” (superscript en) and “dissipative” (superscript di) parts, we note that  $s$  is purely energetic. In summary,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{en}} + \boldsymbol{\sigma}^{\text{di}} \quad (2.51a)$$

$$s = s^{\text{en}} \quad (2.51b)$$

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}^{\text{en}} + \boldsymbol{\kappa}^{\text{di}} \quad (2.51c)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\text{en}} + \boldsymbol{\xi}^{\text{di}} \quad (2.51d)$$

where

$$\boldsymbol{\sigma}^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}), \quad \boldsymbol{\sigma}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.52a)$$

$$s^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \stackrel{\text{def}}{=} -\frac{\partial \psi}{\partial \theta}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \quad (2.52b)$$

$$\boldsymbol{\kappa}^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \underline{k}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}), \quad \boldsymbol{\kappa}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\underline{k}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.52c)$$

$$\boldsymbol{\xi}^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \mathbf{g}}(\boldsymbol{\varepsilon}, \theta, k, \mathbf{g}), \quad \boldsymbol{\xi}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\mathbf{g}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\mathbf{g}}) \quad (2.52d)$$

Finally, that  $\mathcal{D}_{\text{mech}} \geq 0$  can be shown as for the isothermal situation.



## 3. Boundary value problems with gradient variables

In this chapter we present

### 3.1 Primary (canonical) problem formulation

#### 3.1.1 Strong and weak format

We consider a given body occupying the domain  $\Omega$ , and we restrict to quasistatic conditions (for simplicity). The strong format of the coupled problem of finding  $\mathbf{u}(\mathbf{x}, t), k(\mathbf{x}, t)$  is space-time is given as

$$-\boldsymbol{\sigma} \cdot \nabla = \mathbf{b} \quad \text{in } \Omega \quad (3.1a)$$

$$\kappa - \boldsymbol{\xi} \cdot \nabla = b^\mu \quad \text{in } \Omega \quad (3.1b)$$

Obvious choice of boundary conditions on the body with boundary  $\Gamma$  are of the Dirichlet and Neumann type as follows:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_D, \quad \mathbf{t}_n \stackrel{\text{def}}{=} \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_N \quad (3.2a)$$

$$k = \bar{k} \quad \text{on } \Gamma_D^\mu, \quad t_n^\mu \stackrel{\text{def}}{=} \boldsymbol{\xi} \cdot \mathbf{n} = \bar{t}^\mu \quad \text{on } \Gamma_N^\mu \quad (3.2b)$$

where  $\Gamma = \Gamma_D \cup \Gamma_N = \Gamma_D^\mu \cup \Gamma_N^\mu$ .

#### 3.1.2 Time-discrete variational formulation

Function spaces  $\mathbb{U}, \mathbb{U}^0, \mathbb{K}, \mathbb{K}^0$ ,

#### 3.1.3 FEM

### 3.2 Dual problem formulation

#### 3.2.1 Strong and weak format

#### 3.2.2 Time-discrete variational formulation

#### 3.2.3 FEM



## 4. Book template examples

### 4.1 Citation

This statement requires citation [Smi12]; this one is more specific [Smi13, page 122].

### 4.2 Lists

Lists are useful to present information in a concise and/or ordered way<sup>1</sup>.

#### 4.2.1 Numbered List

1. The first item
2. The second item
3. The third item

#### 4.2.2 Bullet Points

- The first item
- The second item
- The third item

#### 4.2.3 Descriptions and Definitions

**Name** Description

**Word** Definition

**Comment** Elaboration

### 4.3 Theorems

This is an example of theorems.

#### 4.3.1 Several equations

This is a theorem consisting of several equations.

**Theorem 4.3.1 — Name of the theorem.** In  $E = \mathbb{R}^n$  all norms are equivalent. It has the properties:

$$||\mathbf{x}|| - ||\mathbf{y}|| \leq ||\mathbf{x} - \mathbf{y}|| \quad (4.1)$$

$$||\sum_{i=1}^n \mathbf{x}_i|| \leq \sum_{i=1}^n ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer :} \quad (4.2)$$

#### 4.3.2 Single Line

This is a theorem consisting of just one line.

---

<sup>1</sup>Footnote example...

**Theorem 4.3.2** A set  $\mathcal{D}(G)$  is dense in  $L^2(G)$ ,  $|\cdot|_0$ .

#### 4.4 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

**Definition 4.4.1 — Definition name.** Given a vector space  $E$ , a norm on  $E$  is an application, denoted  $\|\cdot\|$ ,  $E$  in  $\mathbb{R}^+ = [0, +\infty[$  such that:

$$\|\mathbf{x}\| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \quad (4.3)$$

$$\|\lambda \mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\| \quad (4.4)$$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad (4.5)$$

#### 4.5 Notations

**Notation 4.1.** Given an open subset  $G$  of  $\mathbb{R}^n$ , the set of functions  $\phi$  are:

1. Bounded support  $G$ ;
2. Infinitely differentiable;

a vector space is denoted by  $\mathcal{D}(G)$ .

#### 4.6 Remarks

This is an example of a remark.

**R** The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field  $\mathbb{K} = \mathbb{R}$ , however, established properties are easily extended to  $\mathbb{K} = \mathbb{C}$ .

#### 4.7 Corollaries

This is an example of a corollary.

**Corollary 4.7.1 — Corollary name.** The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field  $\mathbb{K} = \mathbb{R}$ , however, established properties are easily extended to  $\mathbb{K} = \mathbb{C}$ .

#### 4.8 Propositions

This is an example of propositions.

##### 4.8.1 Several equations

**Proposition 4.8.1 — Proposition name.** It has the properties:

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\| \quad (4.6)$$

$$\|\sum_{i=1}^n \mathbf{x}_i\| \leq \sum_{i=1}^n \|\mathbf{x}_i\| \quad \text{where } n \text{ is a finite integer} \quad (4.7)$$

### 4.8.2 Single Line

**Proposition 4.8.2** Let  $f, g \in L^2(G)$ ; if  $\forall \varphi \in \mathcal{D}(G)$ ,  $(f, \varphi)_0 = (g, \varphi)_0$  then  $f = g$ .

## 4.9 Examples

This is an example of examples.

### 4.9.1 Equation and Text

■ **Example 4.1** Let  $G = \{x \in \mathbb{R}^2 : |x| < 3\}$  and denoted by:  $x^0 = (1, 1)$ ; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \quad (4.8)$$

The function  $f$  has bounded support, we can take  $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$  for all  $\varepsilon \in ]0; 5/2 - \sqrt{2}[$ . ■

### 4.9.2 Paragraph of Text

■ **Example 4.2 — Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris. ■

## 4.10 Exercises

This is an example of an exercise.

**Exercise 4.1** This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

## 4.11 Problems

**Problem 4.1** What is the average airspeed velocity of an unladen swallow?

## 4.12 Vocabulary

Define a word to improve a students' vocabulary.

**Vocabulary 4.1 — Word.** Definition of word.

## 4.13 Table

## 4.14 Figure

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 4.1: Table caption

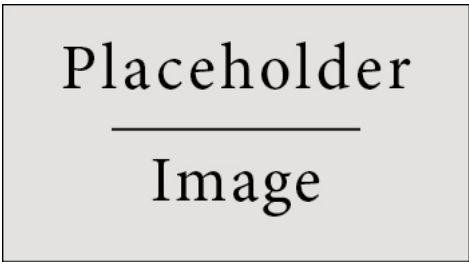


Figure 4.1: Figure caption



# Bibliography

## Books

[Smi12] John Smith. *Book title*. 1st edition. Volume 3. 2. City: Publisher, Jan. 2012, pages 123–200 (cited on page 21).

## Articles

[Smi13] James Smith. “Article title”. In: 14.6 (Mar. 2013), pages 1–8 (cited on page 21).



# Index

- boundary conditions, 19
- Citation, 21
- Corollaries, 22
- Definitions
  - Kinetic energy, 9
  - Macroforces, 7
  - Mechanical power, 9
  - Microforces, 8
  - Micropower, 9
  - Thermal power, 9
  - Total internal energy, 9
- Examples, 23
  - Equation and Text, 23
  - Paragraph of Text, 23
- Exercises, 23
- Figure, 23
- Lists, 21
  - Bullet Points, 21
  - Descriptions and Definitions, 21
  - Numbered List, 21
- Notations, 22
- Physical laws
  - First law of thermodynamics, 10
- Problems, 23
- Propositions, 22
  - Several Equations, 22
  - Single Line, 23
- Remarks, 22
- Table, 23
- Theorems, 21
  - Equilibrium, 8
  - Micro-equilibrium, 9
  - Several Equations, 21
  - Single Line, 21
- Thorem
  - The local energy equation, 10
- Vocabulary, 23