Standard Dissipative Materials with Gradient Variables

Theory and Computation

by

Kristoffer Carlsson
Magnus Ekh
Fredrik Larsson
Kenneth Runesson

PUBLISHED BY XXXXXXXXXXXXXX

XXXXXXXXXXXXXX.COM

First printing, xxxxxxxxxxxxxxx

Contents

1. Introduction

In this chapter we present

1.1 ???

This text presents

2. Thermodynamics with Gradient Variables

In this chapter we present the basic relations of continuum thermodynamics for solid material behavior with "nonlocal", i.e. gradient, effects. Constitutive relations are established for the most general situation of non-isothermal behavior.

2.1 Literature overview

TRUESDELL [**Truesdell1968**], NOLL [**Noll1958**], TRUESDELL & NOLL [**Truesdell1965**], and COLEMAN [**Coleman1964**], text-books by LEMAITRE & CHABOCHE [**Lemaitre1990a**] and MAUGIN [**Maugin1992**].

2.1.1 Modeling school A?

sdffsdfdsfdsfsd fsd fsd fsd f sdf sd

fsdfds

2.1.2 Modeling school B?

fsdfsd fsd fs dfs

sfsdfs

sfsdfsd

2.2 Thermodynamic concepts in the presence of gradient variables

gfdgfdg dfg dg df

gdffgd

2.2.1 Thermodynamic system

Definition 2.2.1 — **Thermodynamic system.** The standard definition of a *thermodynamic system* is a given amount of matter that is closed in the sense that it can not exchange matter with the surroundings; however, it can exchange energy in various forms.[a]

[a] This definition excludes "growth" whereby the mass balance involves a source term.

When establishing balance equations, it is convenient to consider an arbitrary part of an existing finite body whose (actual) boundary is subjected to given boundary conditions. In other words, we consider the "cut-out" amount of matter that currently occupies the spatial region $\mathscr{B}$ with boundary $\partial\mathscr{B}$ (with outward unit normal $\boldsymbol{n}$), as shown in Figure **??**. Here, we consider only small deformations, whereby the mass density ρ is taken as a material parameter (rather than a field variable) that is unaffected by the deformation.

This "body" is acted upon by two types of loads (actions) as follows:

Macroforces

Macroforces (standard, representing "local" continuum models):

A volume-specific force $\boldsymbol{b}$ (force per unit volume) is acting in the interior of $\mathscr{B}$, and surface tractions $\boldsymbol{t}_{\mathrm{n}}$ are acting on $\partial\mathscr{B}$ from the surrounding matter (from which the considered body part is cut-out). A volume-specific heat source r (power per unit volume) acts in the interior of $\mathscr{B}$, and the thermal power q_n is supplied via $\partial\mathscr{B}$.

Moreover, $\boldsymbol{u}$ is the displacement vector, that is energy-conjugated to $\boldsymbol{b}$.

Microforces

Microforces (non-standard, representing "non-local" or "gradient" continuum models):

A volume-specific "micro-force" b^μ is acting in the interior of $\mathscr{B}$, and a surface "micro-traction" t_n^μ is acting on $\partial\mathscr{B}$ from the surrounding matter [1] It is assumed that there is no micro-source of heat present.

Energy transfers

In standard fashion, energy is supplied to the system in the form of *mechanical power* $\mathscr{P}_\mathrm{ext}$ (from the mechanical loads $\boldsymbol{b}$ and $\boldsymbol{t}$) and in the form of *thermal power* $\mathscr{Q}_\mathrm{ext}$ (from the thermal "loads" r and q). However, energy is also supplied from "micro-power" $\mathscr{P}_\mathrm{ext}^\mu$ (generated by b^μ and t_n^μ). One part of the supplied energy may be converted into *kinetic energy* $\mathscr{K}$, which is manifested by macroscopic motion of the body ($\dot{\boldsymbol{u}} \neq 0$); hence, it is assumed that the micro-motion $\dot{k}$ does not constitute any part of $\mathscr{K}$ since it is not associated with any "micro-inertia". The remaining part of the energy is stored as *internal energy* $\mathscr{E}$, which has to be parameterized suitably in terms of thermodynamically independent variables. How the total energy can change is governed by the 1st law of thermodynamics.

In order to complete the characterization of a thermodynamic system, we shall also introduce the *entropy* $\mathscr{S}$. How the entropy can change due to the supply of heat power is governed by the 2nd law of thermodynamics, which formally takes the same form as in the standard situation without microforces present.

2.3 Mechanical balance laws

2.3.1 Momentum balance law – Equilibrium equation

The total momentum $\mathscr{P}$ of the body $\mathscr{B}$ in Figure ?? is given as

$$\mathscr{P} = \int_{\mathscr{B}} \rho \dot{\boldsymbol{u}} \,\mathrm{d}V \tag{2.1}$$

whereas the resultant $\mathscr{F}_\mathrm{ext}$ of the externally applied mechanical loads on $\mathscr{B}$ is

$$\mathscr{F}_\mathrm{ext} = \int_{\mathscr{B}} \boldsymbol{b} \,\mathrm{d}V + \int_{\partial\mathscr{B}} \boldsymbol{t}_\mathrm{n} \,\mathrm{d}S \tag{2.2}$$

The global format of the momentum balance reads

$$\dot{\mathscr{P}} = \mathscr{F}_\mathrm{ext} \tag{2.3}$$

The result in (2.3) can be localized as follows: We may use the relation $\boldsymbol{t}_\mathrm{n} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ between the (symmetric) stress tensor $\boldsymbol{\sigma}$ and the total traction $\boldsymbol{t}_\mathrm{n}$ along the boundary $\partial\mathscr{B}$ together with Green's theorem to transform the boundary integral in (2.2) into a volume integral, which gives

$$\mathscr{F}_\mathrm{ext} = \int_{\mathscr{B}} [\boldsymbol{b} + \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}] \,\mathrm{d}V \tag{2.4}$$

Upon inserting (2.1) and (2.4) into (2.3), we obtain the equations of motion

$$\rho \ddot{\boldsymbol{u}} - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} = \boldsymbol{b} \tag{2.5}$$

[1] To simplify notation, we assume in this chapter that b^μ and t_n^μ are scalar quantities corresponding to a scalar "micro-displacement" $u^\mu = k$, which is identical to the internal variable.

When inertia forces are ignored ($\rho\ddot{\boldsymbol{u}} = \boldsymbol{0}$), then (2.5) represents the quasistatic equilibrium equation.

In a completely analogous fashion, we establish the resultant $\mathscr{F}^{\mu}_{\text{ext}}$ of the externally applied micro-forces on $\mathscr{B}$ as

$$\mathscr{F}^{\mu}_{\text{ext}} = \int_{\mathscr{B}} b^{\mu}\,\mathrm{d}V + \int_{\partial\mathscr{B}} t^{\mu}_{\mathrm{n}}\,\mathrm{d}S \tag{2.6}$$

Clearly, the global format of micro-force equilibrium is

$$\mathscr{F}^{\mu}_{\text{ext}} = 0 \tag{2.7}$$

This result can be localized as follows: We may use the relation $t^{\mu}_{\mathrm{n}} = \boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{n}$ between the micro-stress vector[2] $\boldsymbol{\sigma}^{\mu}$ and the micro-traction t^{μ}_{n} along the boundary $\partial\mathscr{B}$ together with Green's theorem to transform the boundary integral in (2.6) into a volume integral, whereby the balance condition (2.7) gives

$$\mathscr{F}^{\mu}_{\text{ext}} = \int_{\mathscr{B}} [b^{\mu} + \boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{\nabla}]\,\mathrm{d}V = 0 \tag{2.8}$$

Localizing this result, we obtain the equation of micro-equilibrium as

$$-\boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{\nabla} = b^{\mu} \tag{2.9}$$

in complete analogy with the standard quasistatic equilibrium equation.

2.3.2 Energy balance – The first law of thermodynamics

The kinetic energy $\mathscr{K}$ is defined as

$$\mathscr{K} \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathscr{B}} \rho|\dot{\boldsymbol{u}}|^2\,\mathrm{d}V \tag{2.10}$$

whereas the internal energy is represented as

$$\mathscr{E} \stackrel{\text{def}}{=} \int_{\mathscr{B}} \rho e\,\mathrm{d}V \tag{2.11}$$

The precise way in which the total energy $\mathscr{E} + \mathscr{K}$ may change in a given thermodynamic process is discussed next.

We define the standard forms of mechanical and thermal power supply as

$$\mathscr{P}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\,\mathrm{d}V + \int_{\partial\mathscr{B}} \boldsymbol{t}_{\mathrm{n}} \cdot \dot{\boldsymbol{u}}\,\mathrm{d}S \tag{2.12}$$

$$\mathscr{Q}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} r\,\mathrm{d}V + \int_{\partial\mathscr{B}} q_{\mathrm{n}}\,\mathrm{d}S \tag{2.13}$$

whereas the power supply from micro-forces is

$$\mathscr{P}^{\mu}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} b^{\mu}\dot{k}\,\mathrm{d}V + \int_{\partial\mathscr{B}} t^{\mu}_{\mathrm{n}}\dot{k}\,\mathrm{d}S \tag{2.14}$$

The global format of the 1st law (axiom) of thermodynamics reads

$$\dot{\mathscr{E}} + \dot{\mathscr{K}} = \mathscr{P}_{\text{ext}} + \mathscr{P}^{\mu}_{\text{ext}} + \mathscr{Q}_{\text{ext}} \tag{2.15}$$

[2] $\boldsymbol{\sigma}^{\mu}$ is a vector (1st order tensor) in the present case due to the assumption that t^{μ}_{n} is a scalar quantity.

We may use the relations $\boldsymbol{t}_\mathrm{n} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$ and $t_\mathrm{n}^\mu = \boldsymbol{\sigma}^\mu\cdot\boldsymbol{n}$ together with Green's theorem to transform the boundary integrals in (2.12) and (2.14) into volume integrals, which gives

$$\mathscr{P}_\mathrm{ext} = \int_\mathscr{B} \boldsymbol{b}\cdot\dot{\boldsymbol{u}}\,\mathrm{d}V + \int_\mathscr{B} [\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}}]\cdot\boldsymbol{\nabla}\,\mathrm{d}V = \int_\mathscr{B} [\boldsymbol{b}+\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}]\cdot\dot{\boldsymbol{u}}\,\mathrm{d}V + \int_\mathscr{B} \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V \tag{2.16}$$

$$\mathscr{P}_\mathrm{ext}^\mu = \int_\mathscr{B} b^\mu\dot{k}\,\mathrm{d}V + \int_\mathscr{B} [\boldsymbol{\sigma}^\mu\dot{k}]\cdot\boldsymbol{\nabla}\,\mathrm{d}V = \int_\mathscr{B} [b^\mu+\boldsymbol{\sigma}^\mu\cdot\boldsymbol{\nabla}]\dot{k}\,\mathrm{d}V + \int_\mathscr{B} \boldsymbol{\sigma}^\mu\cdot\dot{\boldsymbol{g}}\,\mathrm{d}V \tag{2.17}$$

Here, we introduced the strain and gradient operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{g}$

$$\boldsymbol{\varepsilon}[\boldsymbol{u}] \overset{\mathrm{def}}{=} (\boldsymbol{u}\otimes\boldsymbol{\nabla})^\mathrm{sym} \quad\Rightarrow\quad \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}] \tag{2.18a}$$

$$\boldsymbol{g}[k] \overset{\mathrm{def}}{=} k\otimes\boldsymbol{\nabla} = \boldsymbol{\nabla}k \quad\Rightarrow\quad \dot{\boldsymbol{g}} = \boldsymbol{g}[\dot{k}] \tag{2.18b}$$

In order to obtain the last expression in (2.16), we used that $\boldsymbol{\sigma}$ is symmetrical, which implies that $\boldsymbol{\sigma}:[\dot{\boldsymbol{u}}\otimes\boldsymbol{\nabla}] = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}$.

Now, inserting the equations of equilibrium in (2.5) and micro-equilibrium in (2.9) into (2.16) and (2.17), we obtain

$$\mathscr{P}_\mathrm{ext} = \int_\mathscr{B} \ddot{\boldsymbol{u}}\cdot\rho\dot{\boldsymbol{u}} + \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V = \int_\mathscr{B} \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\rho|\dot{\boldsymbol{u}}|^2\right] + \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V \tag{2.19}$$

$$\mathscr{P}_\mathrm{ext}^\mu = \int_\mathscr{B} \boldsymbol{\sigma}^\mu\cdot\dot{\boldsymbol{g}}\,\mathrm{d}V \tag{2.20}$$

Likewise, we may introduce the heat flux vector $\boldsymbol{h}$ and use the relation $q_\mathrm{n} = -\boldsymbol{h}\cdot\boldsymbol{n}$ to transform the boundary integral in (2.13) into a volume integral. We then obtain

$$\mathscr{Q}_\mathrm{ext} = \int_\mathscr{B} \mathscr{Q}\,\mathrm{d}V \quad \text{with} \quad \mathscr{Q} \overset{\mathrm{def}}{=} r - \boldsymbol{\nabla}\cdot\boldsymbol{h} \tag{2.21}$$

where $\mathscr{Q}$ is the specific external thermal power supply.

Inserting (2.19), (2.20) and (2.21) into (2.15), we may express the energy balance equation as

$$\int_\mathscr{B} [\rho\dot{e} - \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^\mu\cdot\dot{\boldsymbol{g}} - \mathscr{Q}]\,\mathrm{d}V = 0 \tag{2.22}$$

Localizing this result, i.e. the integrand must vanish identically, we obtain

$$\rho\dot{e} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^\mu\cdot\dot{\boldsymbol{g}} + \mathscr{Q} = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^\mu\cdot\dot{\boldsymbol{g}} + r - \boldsymbol{h}\cdot\boldsymbol{\nabla} \tag{2.23}$$

This is the *local format* of the energy equation.

R In a locally *isometric* process ($\mathscr{W} = 0$), the net heat input is converted entirely into internal energy, whereas in a locally *adiabatic* process ($\mathscr{Q} = 0$), the internal energy gained is the work done by the stresses. **REINTERPRET!!** □

2.3.3 Entropy inequality – Dissipation inequalities

We introduce the *entropy function* $\mathscr{S}$

$$\mathscr{S} = \int_{\mathscr{B}} \rho s \, \mathrm{d}V \tag{2.24}$$

where s is the entropy density (entropy per unit mass). Associated with the existence of $\mathscr{S}$, we define $\mathscr{R}_{\text{ext}}$ as the rate of *input of entropy* from the exterior into $\mathscr{B}$. The standard expression is

$$\mathscr{R}_{\text{ext}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} \frac{r}{\theta} \, \mathrm{d}V + \int_{\partial\mathscr{B}} \frac{q_{\text{n}}}{\theta} \, \mathrm{d}S \tag{2.25}$$

The global format of the 2nd law (=axiom) of thermodynamics is *defined* as the inequality

$$\dot{S} - \mathscr{R}_{\text{ext}} \geq 0 \tag{2.26}$$

Upon transforming the surface integral in (2.25) into a volume integral, we first obtain the representation

$$\mathscr{R}_{\text{ext}} = \int_{\mathscr{B}} \mathscr{R} \mathrm{d}V \quad \text{with} \quad \mathscr{R} \stackrel{\text{def}}{=} \frac{\mathscr{Q} + \boldsymbol{h} \cdot \boldsymbol{\nabla}[\ln\theta]}{\theta} \tag{2.27}$$

Inserting (2.24) and (2.27) into (2.26), we may express the entropy inequality as

$$\int_{\mathscr{B}} [\rho \dot{s} - \mathscr{R}] \, \mathrm{d}V \geq 0 \quad \text{or} \quad \int_{\mathscr{B}} [\theta \rho \dot{s} - \mathscr{Q} - \boldsymbol{h} \cdot \boldsymbol{\nabla}[\ln\theta]] \, \mathrm{d}V \geq 0 \tag{2.28}$$

which is commonly known as the global version of the *Clausius-Duhem-Inequality (CDI)*.

Remark: The result in (2.28) may be localized (in standard fashion) in the sense that the integrand is non-negative; however, this is never exploited henceforth. □

Alternatively, $(2.28)_2$ may be rewritten as the *dissipation inequality*

$$\mathscr{D} = \mathscr{D}_{\text{mech}} + \mathscr{D}_{\text{therm}} \geq 0 \tag{2.29}$$

where

$$\mathscr{D}_{\text{mech}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} [\rho \theta \dot{s} - \mathscr{Q}] \, \mathrm{d}V, \tag{2.30a}$$

$$\mathscr{D}_{\text{therm}} \stackrel{\text{def}}{=} \int_{\mathscr{B}} [-\boldsymbol{h} \cdot \boldsymbol{\nabla}[\ln\theta]] \, \mathrm{d}V = \int_{\mathscr{B}} [-\frac{\boldsymbol{h} \cdot \boldsymbol{\nabla}\theta}{\theta}] \, \mathrm{d}V \tag{2.30b}$$

are the *mechanical* and *thermal* part, respectively, of the total dissipation. Upon introducing the local format of the energy equation into (2.30a), we obtain the alternative expression

$$\mathscr{D}_{\text{mech}} = \int_{\mathscr{B}} [-\rho \dot{e} + \rho \theta \dot{s} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{\mu} \cdot \dot{\boldsymbol{g}}] \, \mathrm{d}V \tag{2.31}$$

It is common to impose separately the (sufficient but not always necessary) conditions

$$\mathscr{D}_{\text{mech}} \geq 0, \tag{2.32a}$$

$$\mathscr{D}_{\text{therm}} \geq 0 \tag{2.32b}$$

which are known as the *Clausius-Planck-Inequality (CPI)* and the *Fourier-Inequality (FI)*, respectively. This approach is taken subsequently.

2.4 Constitutive framework - Isothermal conditions

2.4.1 Canonical format - 2-field format

We first consider isothermal conditions, whereby the (absolute) temperature $\theta = \theta_0$ serves only as a parameter in the constitutive model. We may the directly introduce the mass-specific free energy density $\psi \stackrel{\text{def}}{=} e - \theta_0 s$ with the parameterization $\psi(\boldsymbol{\varepsilon}, k, \boldsymbol{g})$. From (2.31), we then obtain

$$\begin{aligned}
\mathscr{D}_{\text{mech}}(\dot{\boldsymbol{u}}, \dot{k}) &= \int_{\mathscr{B}} [-\rho\dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{\mu} \cdot \dot{\boldsymbol{g}}] \, \mathrm{d}V \\
&= \int_{\mathscr{B}} \left[\left[\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}] + \left[-\rho \frac{\partial \psi}{\partial k} \right] \dot{k} + \left[\boldsymbol{\sigma}^{\mu} - \rho \frac{\partial \psi}{\partial \boldsymbol{g}} \right] \cdot \boldsymbol{g}[\dot{k}] \right] \mathrm{d}V \\
&\geq 0 \quad \forall \dot{\boldsymbol{u}}, \dot{k}
\end{aligned} \tag{2.33}$$

that must hold for any given thermodynamic process defined by given fields $\dot{\boldsymbol{u}}, \dot{k}$. Note that these fields (in space-time) are treated as independent in this context.

The constitutive relations for the Standard Dissipative material with gradients under isothermal conditions are established as follows:

- Introduce the mass-specific dissipation potential function $\phi(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}})$, which (i) is convex[3] and (ii) satisfies the condition $\phi(\mathbf{0}, 0, \mathbf{0}) = 0$. The corresponding global dissipation functional is

$$\Phi(\dot{\boldsymbol{u}}, \dot{k}) \stackrel{\text{def}}{=} \int_{\mathscr{B}} \rho \phi(\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}], \dot{k}, \boldsymbol{g}[\dot{k}]) \, \mathrm{d}V$$

- Introduce the global constitutive potential $\chi(\dot{\boldsymbol{u}}, \dot{k}) \stackrel{\text{def}}{=} \mathscr{D}_{\text{mech}}(\dot{\boldsymbol{u}}, \dot{k}) - \Phi(\dot{\boldsymbol{u}}, \dot{k})$, and the constitutive assumption that χ has a saddle point in the sense that

$$(\dot{\boldsymbol{u}}, \dot{k}) = \arg \left[\min_{\dot{\boldsymbol{u}}'} \max_{\dot{k}'} \chi(\dot{\boldsymbol{u}}', \dot{k}') \right]$$

  That the potential χ is stationary in the space of $(\dot{\boldsymbol{u}}, \dot{k})$ for an actual thermodynamic process introduces a constraint on the relation between these fields in space-time.

The directional (partial) derivatives of $\chi(\dot{\boldsymbol{u}}, \dot{k})$ for variations $\delta\dot{\boldsymbol{u}}$ of $\dot{\boldsymbol{u}}$ and $\delta\dot{k}$ of $\dot{k}$ are denoted $\chi'_{\dot{u}}(\dot{\boldsymbol{u}}, \dot{k}; \delta\dot{\boldsymbol{u}})$ and $\chi'_{\dot{k}}(\dot{\boldsymbol{u}}, \dot{k}; \delta\dot{k})$, respectively. The stationarity condition corresponding to the saddle-

[3] A function $F(\underline{x}) : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ is convex iff, for any pair $\underline{x}_1, \underline{x}_2$, the following inequality holds:

$$F(\alpha \underline{x}_1 + [1-\alpha]\underline{x}_2) \leq \alpha F(\underline{x}_1) + [1-\alpha] F(\underline{x}_2)$$

. For a smooth convex function $F(\underline{x}) : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ the following result holds:

$$F(\underline{x}_2) - F(\underline{x}_1) \geq \left[\frac{\partial F}{\partial \underline{x}}(\underline{x}_1) \right]^{\mathrm{T}} [\underline{x}_2 - \underline{x}_1] \quad \forall \underline{x}_1, \underline{x}_2 \in \mathbb{R}^{\mathrm{n}}$$

point property then becomes

$$
\begin{aligned}
\chi'_{\dot{u}}(\dot{\boldsymbol{u}},\dot{k};\delta\dot{\boldsymbol{u}}) &= \int_{\mathscr{B}} \left[\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}]\,\mathrm{d}V - \int_{\mathscr{B}} \left[\rho\frac{\partial\phi}{\partial\dot{\boldsymbol{\varepsilon}}}\right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}]\,\mathrm{d}V \\
&= \int_{\mathscr{B}} \left[\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} - \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{\varepsilon}}}\right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}] = 0, \quad \forall\delta\dot{\boldsymbol{u}} \in ??? \qquad (2.34a) \\
\chi'_{\dot{k}}(\dot{\boldsymbol{u}},\dot{k};\delta\dot{k}) &= \int_{\mathscr{B}} \left[-\rho\frac{\partial\psi}{\partial k}\delta\dot{k} + \left[\boldsymbol{\sigma}^{\mu} - \rho\frac{\partial\psi}{\partial\boldsymbol{g}}\right]\cdot\boldsymbol{g}[\delta\dot{k}]\right]\mathrm{d}V - \int_{\mathscr{B}} \left[\rho\frac{\partial\phi}{\partial\dot{k}}\delta\dot{k} + \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\cdot\boldsymbol{g}[\delta\dot{k}]\right]\mathrm{d}V \\
&= -\int_{\mathscr{B}} \left[\rho\frac{\partial\psi}{\partial k} + \rho\frac{\partial\phi}{\partial\dot{k}} + \left[\boldsymbol{\sigma}^{\mu} - \rho\frac{\partial\psi}{\partial\boldsymbol{g}} - \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\right]\cdot\boldsymbol{\nabla}\right]\delta\dot{k}\,\mathrm{d}V \\
&\quad + \int_{\partial\mathscr{B}} \left[\boldsymbol{\sigma}^{\mu} - \rho\frac{\partial\psi}{\partial\boldsymbol{g}} - \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\right]\cdot\boldsymbol{n}\,\delta\dot{k}\,\mathrm{d}S = 0 \quad \forall\delta\dot{k} \in ??? \qquad (2.34b)
\end{aligned}
$$

Upon localizing the result in (2.34), we obtain the constitutive identities

$$\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} - \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{\varepsilon}}} = \boldsymbol{0} \quad \text{in } \mathscr{B} \qquad (2.35a)$$

$$\rho\frac{\partial\psi}{\partial k} + \rho\frac{\partial\phi}{\partial\dot{k}} + \boldsymbol{\sigma}^{\mu}\cdot\boldsymbol{\nabla} - \left[\rho\frac{\partial\psi}{\partial\boldsymbol{g}} + \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\right]\cdot\boldsymbol{\nabla} = 0 \quad \text{in } \mathscr{B} \qquad (2.35b)$$

$$\boldsymbol{\sigma}^{\mu}\cdot\boldsymbol{n} - \left[\rho\frac{\partial\psi}{\partial\boldsymbol{g}} + \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\right]\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathscr{B} \qquad (2.35c)$$

Upon introducing the micro-stress variables

$$\kappa \overset{\text{def}}{=} \rho\frac{\partial\psi}{\partial k} + \rho\frac{\partial\phi}{\partial\dot{k}} \qquad (2.36a)$$

$$\boldsymbol{\xi} \overset{\text{def}}{=} \rho\frac{\partial\psi}{\partial\boldsymbol{g}} + \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}} \qquad (2.36b)$$

we may abbreviate the system (2.51b, 2.51c) as

$$\kappa + \boldsymbol{\sigma}^{\mu}\cdot\boldsymbol{\nabla} - \boldsymbol{\xi}\cdot\boldsymbol{\nabla} = 0 \quad \text{in } \mathscr{B} \qquad (2.37a)$$

$$\boldsymbol{\sigma}^{\mu}\cdot\boldsymbol{n} - \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathscr{B} \qquad (2.37b)$$

It is possible to eliminate the micro-stress $\boldsymbol{\sigma}^{\mu}$ from the constitutive equations by combining (2.37a) with the equation for micro-equilibrium in (2.9). As a result, (2.37a) is replaced by the equation

$$\kappa - \boldsymbol{\xi}\cdot\boldsymbol{\nabla} = b^{\mu} \quad \text{in } \mathscr{B} \qquad (2.38)$$

Moreover, from (2.37b), we note that the micro-traction on $\mathscr{B}$ can be expressed in terms of $\boldsymbol{\xi}$ as $t_{\mathrm{n}}^{\mu}(\overset{\text{def}}{=} \boldsymbol{\sigma}^{\mu}\cdot\boldsymbol{n}) = \boldsymbol{\xi}\cdot\boldsymbol{n}$; hence, we conclude that the variable $\boldsymbol{\xi}$ plays the role of a "shifted" micro-stress.

In the literature, e.g. BIOT [**Biot1965**], NGUYEN [**Nguyen2000**], the identity in (2.51b), or (2.37a), is known as Biot's equation. This is a constitutive evolution equation, which can be interpreted as a "derived" micro-equilibrium equation for $\boldsymbol{\xi}$ of the Helmholtz type, whereby the constitutive relation for $\boldsymbol{\xi}$ was given in (2.36b). It plays a role that is similar to the standard equilibrium equation, whereby the constitutive relation for $\boldsymbol{\sigma}$ is given directly in (2.51a). In conclusion, upon combining these equations, we are in the position to solve for the fields $\boldsymbol{u}(\boldsymbol{x},t)$ and $k(\boldsymbol{x},t)$ in space-time for any given body subjected to the appropriate loading and boundary conditions.

In order to comply with notation used in the literature, see e.g. GURTIN [**Gurtin2000**], we introduce the decomposition into "energetic" (superscript en) and "dissipative" (superscript di) parts as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{en}} + \boldsymbol{\sigma}^{\mathrm{di}} \tag{2.39a}$$

$$\kappa = \kappa^{\mathrm{en}} + \kappa^{\mathrm{di}} \tag{2.39b}$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\mathrm{en}} + \boldsymbol{\xi}^{\mathrm{di}} \tag{2.39c}$$

where

$$\boldsymbol{\sigma}^{\mathrm{en}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}), \quad \boldsymbol{\sigma}^{\mathrm{di}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{\varepsilon}}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \tag{2.40a}$$

$$\kappa^{\mathrm{en}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\psi}{\partial k}(\boldsymbol{\varepsilon},k,\boldsymbol{g}), \quad \kappa^{\mathrm{di}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\phi}{\partial\dot{k}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \tag{2.40b}$$

$$\boldsymbol{\xi}^{\mathrm{en}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\psi}{\partial\boldsymbol{g}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}), \quad \boldsymbol{\xi}^{\mathrm{di}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \stackrel{\mathrm{def}}{=} \rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) \tag{2.40c}$$

In conclusion, we may express (2.38) as

$$\kappa^{\mathrm{en}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}) + \kappa^{\mathrm{di}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}}) - \left[\boldsymbol{\xi}^{\mathrm{en}}(\boldsymbol{\varepsilon},k,\boldsymbol{g}) + \boldsymbol{\xi}^{\mathrm{di}}(\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}})\right]\cdot\boldsymbol{\nabla} = b^{\mu} \tag{2.41}$$

It remains to check that the CDI is satisfied for the *actual* thermodynamic process defined by $(\dot{\boldsymbol{u}},\dot{k})$ that satisfy the constitutive constraint equations (2.37). From (2.33) we obtain

$$\begin{aligned}
\mathscr{D}_{\mathrm{mech}}(\dot{\boldsymbol{u}},\dot{k}) &= \int_{\mathscr{B}}\left[\left[\boldsymbol{\sigma}-\rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\right]:\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}]+\left[-\rho\frac{\partial\psi}{\partial k}\right]\dot{k}+\left[\boldsymbol{\sigma}^{\mu}-\rho\frac{\partial\psi}{\partial\boldsymbol{g}}\right]\cdot\boldsymbol{g}[\dot{k}]\right]\mathrm{d}V\\
&= (2.40) = \int_{\mathscr{B}}\left[[\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{en}}]:\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}]+[-\kappa^{\mathrm{en}}]\dot{k}+\left[\boldsymbol{\sigma}^{\mu}-\boldsymbol{\xi}^{\mathrm{en}}\right]\cdot\boldsymbol{g}[\dot{k}]\right]\mathrm{d}V\\
&= \int_{\mathscr{B}}\left[\boldsymbol{\sigma}^{\mathrm{di}}:\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}]+\kappa^{\mathrm{di}}\dot{k}-\kappa\dot{k}+\boldsymbol{\xi}^{\mathrm{di}}\cdot\boldsymbol{g}[\dot{k}]+[\boldsymbol{\sigma}^{\mu}-\boldsymbol{\xi}]\cdot\boldsymbol{g}[\dot{k}]\right]\mathrm{d}V\\
&= \int_{\mathscr{B}}\left[\boldsymbol{\sigma}^{\mathrm{di}}:\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}]+\kappa^{\mathrm{di}}\dot{k}+\boldsymbol{\xi}^{\mathrm{di}}\cdot\boldsymbol{g}[\dot{k}]\right]\mathrm{d}V\\
&\quad -\int_{\mathscr{B}}\underbrace{[\kappa+[\boldsymbol{\sigma}^{\mu}-\boldsymbol{\xi}]\cdot\boldsymbol{\nabla}]}_{=0 \text{ from } (2.37\text{a})}\dot{k}\,\mathrm{d}V+\int_{\partial\mathscr{B}}\underbrace{[\boldsymbol{\sigma}^{\mu}-\boldsymbol{\xi}]\cdot\boldsymbol{n}}_{=0 \text{ from } (2.37\text{b})}\dot{k}\,\mathrm{d}S\\
&= \int_{\mathscr{B}}\underbrace{\left[\rho\frac{\partial\phi}{\partial\dot{\boldsymbol{\varepsilon}}}:\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}]+\rho\frac{\partial\phi}{\partial\dot{k}}\dot{k}+\rho\frac{\partial\phi}{\partial\dot{\boldsymbol{g}}}\cdot\boldsymbol{g}[\dot{k}]\right]}_{\geq 0}\mathrm{d}V \geq 0 \quad \forall\dot{\boldsymbol{u}},\dot{k}
\end{aligned} \tag{2.42}$$

That, indeed, $\mathscr{D}_{\mathrm{mech}} \geq 0$ follows directly from the properties of ϕ as given above.

Proof: Consider $X(\underline{x})$ convex and $X(\underline{0}) = 0$. From convexity follows that, for any given $\underline{x}$,

$$0 \leq X(\underline{x}) - \underbrace{X(\underline{0})}_{=0} \leq X'(\underline{x})^{\mathrm{T}}[\underline{x}-\underline{0}] \quad \Rightarrow \quad X'(\underline{x})^{\mathrm{T}}\underline{x} \geq 0 \tag{2.43}$$

Now, setting $X = \phi$ and $\underline{x} = (\dot{\boldsymbol{\varepsilon}},\dot{k},\dot{\boldsymbol{g}})$, we directly obtain the inequality in (2.42). □

2.5 Constitutive framework - Non-isothermal conditions

2.5.1 Canonical format - 3-field format

Next, we consider the general situation of non-isothermal conditions, whereby the (absolute) temperature θ (or the entropy s) is included as a thermodynamic variable in the constitutive model. The basic parameterization of the mass-specific internal energy density is $e(\boldsymbol{\varepsilon}, s, k, \boldsymbol{g})$. From (2.31), we now obtain

$$\begin{aligned}
\mathscr{D}_{\text{mech}}(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}) &= \int_{\mathscr{B}} [-\rho \dot{e} + \rho\theta\dot{s} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{\mu} \cdot \dot{\boldsymbol{g}}] \, \mathrm{d}V \\
&= \int_{\mathscr{B}} \left[\left[\boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}] + \left[\rho\theta - \rho \frac{\partial e}{\partial s} \right] \dot{s} + \left[-\rho \frac{\partial e}{\partial k} \right] \dot{k} + \left[\boldsymbol{\sigma}^{\mu} - \rho \frac{\partial e}{\partial \boldsymbol{g}} \right] \cdot \boldsymbol{g}[\dot{k}] \right] \mathrm{d}V \\
&\geq 0 \quad \forall \dot{\boldsymbol{u}}, \dot{s}, \dot{k} \qquad (2.44)
\end{aligned}$$

that must hold for any given thermodynamic process defined by given fields $\dot{\boldsymbol{u}}, \dot{s}, \dot{k}$. Note that these fields (in space-time) treated as independent in this context.

The constitutive relations for the Standard Dissipative material with gradients under non-isothermal conditions are established as follows:

- Introduce the mass-specific dissipation potential function $\phi(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}})$, which (i) is convex and (ii) satisfies the condition $\phi(\mathbf{0}, 0, \mathbf{0}) = 0$. The corresponding global dissipation functional is

$$\Phi(\dot{\boldsymbol{u}}, \dot{k}) \stackrel{\text{def}}{=} \int_{\mathscr{B}} \rho \phi(\boldsymbol{\varepsilon}[\dot{\boldsymbol{u}}], \dot{k}, \boldsymbol{g}[\dot{k}]) \, \mathrm{d}V$$

- Introduce the global constitutive potential $\chi(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}) \stackrel{\text{def}}{=} \mathscr{D}_{\text{mech}}(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}) - \Phi(\dot{\boldsymbol{u}}, \dot{k})$, and the constitutive assumption that χ has a saddle point in the sense that

$$(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}) = \arg \left[\min_{\dot{\boldsymbol{u}}'} \min_{\dot{s}'} \max_{\dot{k}'} \chi(\dot{\boldsymbol{u}}', \dot{s}', \dot{k}') \right]$$

  That the potential χ is stationary in the space of $(\dot{\boldsymbol{u}}, \dot{s}, \dot{k})$ for an actual thermodynamic process introduces a constraint on the relation between these fields in space-time.

The stationarity condition corresponding to the saddle-point property then becomes

$$\begin{aligned}
\chi'_{\dot{u}}(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}; \delta\dot{\boldsymbol{u}}) &= \int_{\mathscr{B}} \left[\boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}] \, \mathrm{d}V - \int_{\mathscr{B}} \left[\rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}] \, \mathrm{d}V \\
&= \int_{\mathscr{B}} \left[\boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} \right] : \boldsymbol{\varepsilon}[\delta\dot{\boldsymbol{u}}] = 0, \quad \forall \delta\dot{\boldsymbol{u}} \in ??? \qquad (2.45a) \\
\chi'_{\dot{u}}(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}; \delta\dot{s}) &= \int_{\mathscr{B}} \left[\rho\theta - \rho \frac{\partial e}{\partial s} \right] \delta\dot{s} \, \mathrm{d}V \\
&= 0, \quad \forall \delta\dot{s} \in ??? \qquad (2.45b) \\
\chi'_{\dot{k}}(\dot{\boldsymbol{u}}, \dot{s}, \dot{k}; \delta\dot{k}) &= \int_{\mathscr{B}} \left[-\rho \frac{\partial e}{\partial k} \delta\dot{k} + \left[\boldsymbol{\sigma}^{\mu} - \rho \frac{\partial e}{\partial \boldsymbol{g}} \right] \cdot \boldsymbol{g}[\delta\dot{k}] \right] \mathrm{d}V - \int_{\mathscr{B}} \left[\rho \frac{\partial \phi}{\partial \dot{k}} \delta\dot{k} + \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \cdot \boldsymbol{g}[\delta\dot{k}] \right] \mathrm{d}V \\
&= - \int_{\mathscr{B}} \left[\rho \frac{\partial e}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \left[\boldsymbol{\sigma}^{\mu} - \rho \frac{\partial e}{\partial \boldsymbol{g}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{\nabla} \right] \delta\dot{k} \, \mathrm{d}V \\
&\quad + \int_{\partial\mathscr{B}} \left[\boldsymbol{\sigma}^{\mu} - \rho \frac{\partial e}{\partial \boldsymbol{g}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{n} \, \delta\dot{k} \, \mathrm{d}S = 0 \quad \forall \delta\dot{k} \in ??? \qquad (2.45c)
\end{aligned}$$

Upon localizing the result in (2.45), we obtain the constitutive identities

$$\boldsymbol{\sigma} - \rho \frac{\partial e}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0} \quad \text{in } \mathscr{B} \tag{2.46a}$$

$$\theta - \frac{\partial e}{\partial s} = \mathbf{0} \quad \text{in } \mathscr{B} \tag{2.46b}$$

$$\rho \frac{\partial e}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{\nabla} - \left[\rho \frac{\partial e}{\partial \boldsymbol{g}} + \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{\nabla} = 0 \quad \text{in } \mathscr{B} \tag{2.46c}$$

$$\boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{n} - \left[\rho \frac{\partial e}{\partial \boldsymbol{g}} + \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\mathscr{B} \tag{2.46d}$$

From (2.46b) appears that the tempertaure θ is purely energetic. It is then convenient to introduce the mass-specific free energy density ψ via the Legendre transformation

$$\psi(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) = \inf_{\hat{s}} [e(\boldsymbol{\varepsilon}, \hat{s}, k, \boldsymbol{g}) - \theta \hat{s}] \tag{2.47}$$

Upon evaluating the inf, we establish the (stationarity) condition

$$\frac{\partial e}{\partial s}(\boldsymbol{\varepsilon}, s, k, \boldsymbol{g}) = \theta \tag{2.48}$$

which is precisely the constitutive relation for θ already obtained in (2.46b). Assuming that θ is monotonic in s (for fixed $\boldsymbol{\varepsilon}, k$ and $\boldsymbol{g}$, we may solve for $s = \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g})$ from (2.48) and insert into the expression for ψ in (2.47) to obtain

$$\psi(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) = e(\boldsymbol{\varepsilon}, \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}), k, \boldsymbol{g}) - \theta \bar{s}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \tag{2.49}$$

Hence, the conditions are

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial e}{\partial \boldsymbol{\varepsilon}}|_s + \left[\underbrace{\frac{\partial e}{\partial \bar{s}} - \theta}_{=0} \right] \frac{\partial \bar{s}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial e}{\partial \boldsymbol{\varepsilon}} \tag{2.50a}$$

$$\frac{\partial \psi}{\partial \theta} = \left[\underbrace{\frac{\partial e}{\partial \bar{s}} - \theta}_{=0} \right] \frac{\partial \bar{s}}{\partial \theta} - s = -s \tag{2.50b}$$

$$\frac{\partial \psi}{\partial k} = \frac{\partial e}{\partial k}|_s + \left[\underbrace{\frac{\partial e}{\partial \bar{s}} - \theta}_{=0} \right] \frac{\partial \bar{s}}{\partial k} = \frac{\partial e}{\partial k} \tag{2.50c}$$

$$\frac{\partial \psi}{\partial \boldsymbol{g}} = \frac{\partial e}{\partial \boldsymbol{g}}|_s + \left[\underbrace{\frac{\partial e}{\partial \bar{s}} - \theta}_{=0} \right] \frac{\partial \bar{s}}{\partial \boldsymbol{g}} = \frac{\partial e}{\partial \boldsymbol{g}} \tag{2.50d}$$

As a result, the constitutive identities (2.46) are replaced by

$$\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0} \quad \text{in } \mathscr{B} \tag{2.51a}$$

$$s + \frac{\partial \psi}{\partial \theta} = \mathbf{0} \quad \text{in } \mathscr{B} \tag{2.51b}$$

$$\rho \frac{\partial \psi}{\partial k} + \rho \frac{\partial \phi}{\partial \dot{k}} + \boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{\nabla} - \left[\rho \frac{\partial \psi}{\partial \boldsymbol{g}} + \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{\nabla} = 0 \quad \text{in } \mathscr{B} \tag{2.51c}$$

$$\boldsymbol{\sigma}^{\mu} \cdot \boldsymbol{n} - \left[\rho \frac{\partial \psi}{\partial \boldsymbol{g}} + \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}} \right] \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\mathscr{B} \tag{2.51d}$$

We may now put forward the same arguments as for the isothermal situation to conclude that the relevant Biot equation is the same as in (2.38), i.e.

$$\kappa - \boldsymbol{\xi} \cdot \boldsymbol{\nabla} = b^{\mu} \quad \text{in } \mathscr{B} \tag{2.52}$$

whereby the variable $\boldsymbol{\xi}$ plays the role of a "shifted" micro-stress.

As to the decomposition into "energetic" (superscript en) and "dissipative" (superscript di) parts, we note that s is purely energetic. In summary,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{en}} + \boldsymbol{\sigma}^{\text{di}} \tag{2.53a}$$

$$s = s^{\text{en}} \tag{2.53b}$$

$$\kappa = \kappa^{\text{en}} + \kappa^{\text{di}} \tag{2.53c}$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\text{en}} + \boldsymbol{\xi}^{\text{di}} \tag{2.53d}$$

where

$$\boldsymbol{\sigma}^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}), \quad \boldsymbol{\sigma}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \tag{2.54a}$$

$$s^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \stackrel{\text{def}}{=} -\frac{\partial \psi}{\partial \theta}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \tag{2.54b}$$

$$\kappa^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \underline{k}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}), \quad \kappa^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \underline{\dot{k}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \tag{2.54c}$$

$$\boldsymbol{\xi}^{\text{en}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}) \stackrel{\text{def}}{=} \rho \frac{\partial \psi}{\partial \boldsymbol{g}}(\boldsymbol{\varepsilon}, \theta, k, \boldsymbol{g}), \quad \boldsymbol{\xi}^{\text{di}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \stackrel{\text{def}}{=} \rho \frac{\partial \phi}{\partial \dot{\boldsymbol{g}}}(\dot{\boldsymbol{\varepsilon}}, \dot{k}, \dot{\boldsymbol{g}}) \tag{2.54d}$$

Finally, that $\mathscr{D}_{\text{mech}} \geq 0$ can be shown as for the isothermal situation.

3. Boundary value problems with gradient variables

In this chapter we present

3.1 Primary (canonical) problem formulation

3.1.1 Strong and weak format

We consider a given body occupying the domain Ω, and we restrict to quasistatic conditions (for simplicity). The strong format of the coupled problem of finding $\boldsymbol{u}(\boldsymbol{x},t), k(\boldsymbol{x},t)$ is space-time is given as

$$-\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} = \boldsymbol{b} \quad \text{in } \Omega \tag{3.1a}$$

$$\kappa - \boldsymbol{\xi}\cdot\boldsymbol{\nabla} = b^{\mu} \quad \text{in } \Omega \tag{3.1b}$$

Obvious choice of boundary conditions on the body with boundary Γ are of the Dirichlet and Neumann type as follows:

$$\boldsymbol{u} = \bar{u} \quad \text{on } \Gamma_{\mathrm{D}}, \quad \boldsymbol{t}_{\mathrm{n}} \stackrel{\text{def}}{=} \boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{t} \quad \text{on } \Gamma_{\mathrm{N}} \tag{3.2a}$$

$$k = \bar{k} \quad \text{on } \Gamma_{\mathrm{D}}^{\mu}, \quad t_{\mathrm{n}}^{\mu} \stackrel{\text{def}}{=} \boldsymbol{\xi}\cdot\boldsymbol{n} = \bar{t}^{\mu} \quad \text{on } \Gamma_{\mathrm{N}}^{\mu} \tag{3.2b}$$

where $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \Gamma_{\mathrm{D}}^{\mu} \cup \Gamma_{\mathrm{N}}^{\mu}$.

3.1.2 Time-discrete variational formulation

Function spaces $\mathbb{U}$, $\mathbb{U}^0$, $\mathbb{K}$, $\mathbb{K}^0$,

3.1.3 FEM

3.2 Dual problem formulation

3.2.1 Strong and weak format

3.2.2 Time-discrete variational formulation

3.2.3 FEM

4. Book template examples

4.1 Citation

This statement requires citation [Smi12]; this one is more specific [Smi13, page 122].

4.2 Lists

Lists are useful to present information in a concise and/or ordered way[1].

4.2.1 Numbered List

1. The first item
2. The second item
3. The third item

4.2.2 Bullet Points

- The first item
- The second item
- The third item

4.2.3 Descriptions and Definitions

Name Description
Word Definition
Comment Elaboration

4.3 Theorems

This is an example of theorems.

4.3.1 Several equations

This is a theorem consisting of several equations.

Theorem 4.3.1 — **Name of the theorem.** In $E = \mathbb{R}^n$ all norms are equivalent. It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{4.1}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer :} \tag{4.2}$$

4.3.2 Single Line

This is a theorem consisting of just one line.

[1]Footnote example...

Theorem 4.3.2 A set $\mathscr{D}(G)$ in dense in $L^2(G)$, $|\cdot|_0$.

4.4 Definitions

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 4.4.1 — **Definition name.** Given a vector space E, a norm on E is an application, denoted $||\cdot||$, E in $\mathbb{R}^+ = [0, +\infty[$ such that:

$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \tag{4.3}$$

$$||\lambda\mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{4.4}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \tag{4.5}$$

4.5 Notations

Notation 4.1. *Given an open subset G of* $\mathbb{R}^n$*, the set of functions* φ *are:*

1. *Bounded support G;*
2. *Infinitely differentiable;*

a vector space is denoted by $\mathscr{D}(G)$.

4.6 Remarks

This is an example of a remark.

R The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

4.7 Corollaries

This is an example of a corollary.

Corollary 4.7.1 — **Corollary name.** The concepts presented here are now in conventional employment in mathematics. Vector spaces are taken over the field $\mathbb{K} = \mathbb{R}$, however, established properties are easily extended to $\mathbb{K} = \mathbb{C}$.

4.8 Propositions

This is an example of propositions.

4.8.1 Several equations

Proposition 4.8.1 — **Proposition name.** It has the properties:

$$\big| ||\mathbf{x}|| - ||\mathbf{y}|| \big| \leq ||\mathbf{x} - \mathbf{y}|| \tag{4.6}$$

$$||\sum_{i=1}^{n} \mathbf{x}_i|| \leq \sum_{i=1}^{n} ||\mathbf{x}_i|| \quad \text{where } n \text{ is a finite integer} \tag{4.7}$$

4.8.2 Single Line

Proposition 4.8.2 Let $f,g \in L^2(G)$; if $\forall \varphi \in \mathscr{D}(G)$, $(f,\varphi)_0 = (g,\varphi)_0$ then $f = g$.

4.9 Examples

This is an example of examples.

4.9.1 Equation and Text

■ Example 4.1 Let $G = \{x \in \mathbb{R}^2 : |x| < 3\}$ and denoted by: $x^0 = (1,1)$; consider the function:

$$f(x) = \begin{cases} e^{|x|} & \text{si } |x - x^0| \leq 1/2 \\ 0 & \text{si } |x - x^0| > 1/2 \end{cases} \tag{4.8}$$

The function f has bounded support, we can take $A = \{x \in \mathbb{R}^2 : |x - x^0| \leq 1/2 + \varepsilon\}$ for all $\varepsilon \in]0; 5/2 - \sqrt{2}[$. ■

4.9.2 Paragraph of Text

■ Example 4.2 — **Example name.** Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

■

4.10 Exercises

This is an example of an exercise.

Exercise 4.1 This is a good place to ask a question to test learning progress or further cement ideas into students' minds. ■

4.11 Problems

Problem 4.1 What is the average airspeed velocity of an unladen swallow?

4.12 Vocabulary

Define a word to improve a students' vocabulary.
Vocabulary 4.1 — **Word.** Definition of word.

4.13 Table

4.14 Figure

Treatments	**Response 1**	**Response 2**
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table 4.1: Table caption

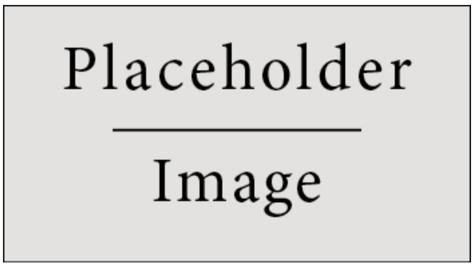

Figure 4.1: Figure caption

Bibliography

Books

[Smi12] John Smith. *Book title*. 1st edition. Volume 3. 2. City: Publisher, Jan. 2012, pages 123–200 (cited on page 21).

Articles

[Smi13] James Smith. “Article title”. In: 14.6 (Mar. 2013), pages 1–8 (cited on page 21).

Index