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Modelling and vibration control for a flexible string system in three-dimensional space

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Abstract: In this study, the control design and stability analysis are presented for a three-dimensional (3D) string system with the payload dynamics. A set of partial differential equations and ordinary differential equations (PDEs–ODEs) are developed to describe the motion of the 3D string system. The dynamic model considers the comprehensive effects of environmental loads, which are critical for the design of a string system. Based on the Lyapunov's direct method and the properties of the string system dynamics, three boundary control inputs are applied at the boundary to suppress the vibrations of the system under the external loads. Uniform boundedness of the 3D dynamics with the proposed control is achieved. Exponential stability is proved via the Lyapunov's direct method when there is no distributed load. Simulation examples are provided by using the finite difference method, and some useful conclusions are drawn.

1 Introduction

Vibration suppression problems of flexible systems in the industry, such as oil drilling and gas exploration in offshore engineering have received increasing attention. Due to the effects of external loads induced by winds, currents, waves and so on, flexible systems may produce large vibrations in a three-dimensional (3D) space. Large vibrations would degrade the performance of the systems and produce premature fatigue problems. Therefore, accurate analysis and reliable improvement should be performed for flexible systems to predict extreme responses at different points under environmental loads.

A flexible string system can be used to model various mechanical structures in the industry, such as the crane cable for transferring the payload [1] and thruster assisted mooring systems for positioning marine vessels [2]. For dynamic analysis, the flexible string system is regarded as a distributed parameter system (DPS) which is mathematically represented by partial differential equations (PDEs) coupled with ordinary differential equations (ODEs). Many remarkable results [3–6] have been presented for the different kinds of DPSs.

Approaches to control infinite-dimensional systems including the finite element method, the Galerkin's method, the Laplace transform, the assumed modes method and so on are all based on the truncated finite-dimensional models of the system. The truncated models are obtained via the model analysis or spatial discretisation, in which the system is represented by a finite number of modes by neglecting the higher frequency modes [7–12]. The problems arising from the truncation procedure in the modelling need to be carefully treated in practical applications. The truncated models used in these methods can cause the actual system to become unstable due to excitation of the unmodelled, high-frequency vibration modes (i.e. spillover effects) [13]. For purpose of suppressing vibrations of the flexible systems, in the literature, boundary control schemes which the control inputs are implemented at the boundaries to control all the modes, have been proposed on the basis of original PDE model [14–17]. A crane model is stabilised by using the angle feedback in [18]. The authors in [19] propose a boundary control law of an axially moving string based on the Lyapunov redesign method. Boundary control is designed for a cable with a gantry crane modelled by a string structure in [18], and the experiment is implemented to verify the control performance. In [20], a novel active disturbance refection control approach is used to stabilise a multi-dimensional wave equation with boundary control matched disturbance. The authors in [21] propose a new boundary output feedback control scheme for the Euler–Bernoulli beam equation described by PDE. Based on the PDE model, the 2D marine risers in [22] is ensured global stabilisation by a boundary controller. In [23], the non-linear PDEs of motion of the marine riser is derived and boundary control is employed for global stabilisation. In [24], a novel control method is designed for non-linear ODE systems with actuator dynamics modelled by a wave PDE, where the boundary is moving. In [25], stabilisation of discrete-time non-linear systems that are actuated via PDE systems is investigated. In [26], adaptive boundary output-feedback control is proposed for the hyperbolic partial integro-differential equations (PIDE). In [27–29], the boundary control is proposed for vibration suppression of the flexible string/riser/beam with constraints, where the barrier Lyapunov function is employed for stability analysis. The vibration of a non-uniform wind turbine tower (modelled as PDE system) in [30] is regulated with boundary control via disturbance observer.

In all the works mentioned above, the control design of flexible structures are restricted to one vertical plane, and only transverse deformation is taken into account. However, the string may move in a deflectable direction. The control performance will be affected if the coupling effects between motions in three directions are ignored. In addition, mathematical works in [31] show that even slight pace curvature introduces significant changes in the beam natural frequencies and especially on mode shapes, i.e. the coupling of the out-of-plane wave types, and extensional and flexural waves exhibits in the flexible beams [32]. To improve accuracy and reliability analysis, modelling and control of the string system in a 3D space is necessary. In [33], modelling and non-linear behaviour are discussed for a 3D system with lumped masses and elastic springs, where the stability of the motion is studied using the Lyapunov method. Recently, the boundary control is proposed for vibration suppression of a flexible marine riser in the 3D space in [23]. The riser is modelled as an Euler–Bernoulli beam with three coupled PDEs and Lyapunov's direct method is used to design the proposed

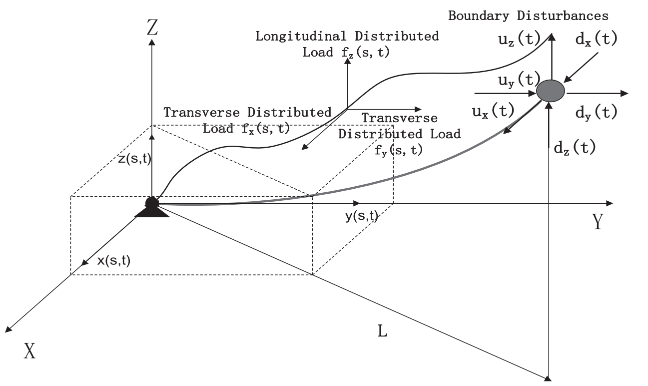

Fig. 1 *Non-linear 3D string system*

control and analyse the stability of the closed-loop system. Due to the non-linear coupling in the 3D space, the control design for the linear model of the string system [19] cannot be straightforwardly used. To the best of our knowledge, the result is the first complete solution of boundary control to a non-linear string system in 3D space for vibrations reduction under the distributed loads.

In this paper, the control problem of a string system in 3D space is addressed by using boundary control method. In 3D space, strong system couplings along the X, Y and Z axes lead to the higher non-linearities and result in a more complex mathematical model, shown in Section 2.1. Furthermore, the payload dynamics and the external loads along the string are also under consideration, making the control design more difficult.

2 Problem formulation

A typical string system is shown in Fig. 1. $x(s, t)$ and $y(s, t)$ are the transverse displacements in the X and Y directions at the position s for time t, $z(s, t)$ is the longitudinal displacement of the string at the position s for time t. $d_x(t), d_y(t), d_z(t)$ are the unknown time-varying boundary loads on the tip payload in the X, Y, Z directions, $f_x(s, t), f_y(s, t), f_z(s, t)$ are the unknown spatiotemporally varying distributed loads along the string in the X, Y, Z directions and $u_x(t), u_y(t), u_z(t)$ are the boundary control forces applied on the tip payload in the X, Y, Z directions. The other side of the string is fixed.

Remark 1: For clarity, notations $(\cdot)' = \partial(\cdot)/\partial s$, $(\cdot)'' = \partial^2(\cdot)/\partial s^2$, $\dot{(\cdot)} = \partial(\cdot)/\partial t$ and $\ddot{(\cdot)} = \partial^2(\cdot)/\partial t^2$ are utilised throughout this paper.

2.1 Dynamics of the 3D string system

In this section, mathematical model of the 3D string is derived by using the Hamilton's principle. The kinetic energy of the system $E_k(t)$ is given

$$E_k(t) = \frac{1}{2}M\left\{[\dot{x}(L,t)]^2 + [\dot{y}(L,t)]^2 + [\dot{z}(L,t)]^2\right\} + \frac{1}{2}\rho\int_0^L \left\{[\dot{x}(s,t)]^2 + [\dot{y}(s,t)]^2 + [\dot{z}(s,t)]^2\right\} \mathrm{d}s \tag{1}$$

where s and t represent the spatial and time variables, respectively, L denotes the length of the string, M is the mass of the payload and ρ is the mass per unit length of the string.

The potential energy $E_{\mathrm{p}}(t)$ due to the tension T and the axial stiffness EA is obtained from

$$E_{\mathrm{p}}(t) = \frac{1}{2}T\int_0^L \left\{\left[x'(s,t)\right]^2 + \left[y'(s,t)\right]^2\right\} \mathrm{d}s + \frac{1}{2}EA\int_0^L \left\{z'(s,t) + \frac{1}{2}\left[x'(s,t)\right]^2 + \frac{1}{2}\left[y'(s,t)\right]^2\right\}^2 \mathrm{d}s \tag{2}$$

The virtual work done by unknown distributed external loads $f_x(s, t)$, $f_y(s, t)$, $f_z(s, t)$ on the string and time-varying loads $d_x(t)$, $d_y(t)$, $d_z(t)$ on the tip payload is given by

$$\delta W_f(t) = d_x(t)\delta x(L,t) + d_y(t)\delta y(L,t) + d_z(t)\delta z(L,t) + \int_0^L [f_x(s,t)\delta x(s,t) + f_y(s,t)\delta y(s,t) + f_z(s,t)\delta z(s,t)]\,\mathrm{d}s \tag{3}$$

To reduce the vibrations, boundary control inputs $u(t) = [u_x(t),\ u_y(t),\ u_z(t)]$ is applied at the boundary of the string. The virtual work done by the control inputs is expressed as

$$\delta W_m(t) = u_x(t)\delta x(L,t) + u_y(t)\delta y(L,t) + u_z(t)\delta z(L,t) \tag{4}$$

Thus, the total virtual work done on the system is written as

$$\delta W(t) = \delta W_f(t) + \delta W_m(t) \tag{5}$$

By using the extended Hamilton's principle $\int_{t_1}^{t_2}[\delta E_k(t) - \delta E_{\mathrm{p}}(t) + \delta W(t)]\,\mathrm{d}t = 0$, we obtain the governing equations of the system as

$$f_x = \rho\ddot{x} - Tx'' - \frac{3EA}{2}x'^2x'' - EA\left[x''z' + x'z''\right] - \frac{EA}{2}\left[x''[y']^2 + 2x'y'y''\right] \tag{6}$$

$$f_y = \rho\ddot{y} - Ty'' - \frac{3EA}{2}y'^2y'' - EA\left[y''z' + y'z''\right] - \frac{EA}{2}\left\{[x']^2y'' + 2x'x''y'\right\} \tag{7}$$

$$f_z = \rho\ddot{z} - EAz'' - EAx'x'' - EAy'y'' \tag{8}$$

$\forall(s, t) \in (0, L) \times [0, \infty)$, and the boundary conditions as

$$x(0,t) = y(0,t) = z(0,t) = 0 \tag{9}$$

$$u_x(t) + d_x(t) = M\ddot{x}(L,t) + Tx'(L,t) + \frac{EA}{2}[x'(L,t)]^3 + EAx'(L,t)z'(L,t) + \frac{EA}{2}x'(L,t)[y'(L,t)]^2 \tag{10}$$

$$u_y(t) + d_y(t) = M\ddot{y}(L,t) + Ty'(L,t) + \frac{EA}{2}[y'(L,t)]^3 + EAy'(L,t)z'(L,t) + \frac{EA}{2}[x'(L,t)]^2y'(L,t) \tag{11}$$

$$u_z(t) + d_z(t) = M\ddot{z}(L,t) + EAz'(L,t) + \frac{EA}{2}[x'(L,t)]^2 + \frac{EA}{2}[y'(L,t)]^2 \tag{12}$$

Remark 2: With consideration of the non-linear string system in 3D space, the string system is represented as three non-linear non-homogeneous PDEs (6)–(8). Due to the existence of the non-linear non-homogeneous PDEs in our system, the model in our paper differs from the system governed by the homogeneous PDEs or the linear PDEs in [5, 18]. As a consequence, the control schemes in these papers are not suitable for our system.

The control objective is to suppress the vibrations of the string in the X, Y and Z directions by employing the boundary control laws $u_x(t), u_y(t)$ and $u_z(t)$.

2.2 Preliminaries

Lemma 1 [34]: Let $\phi(s,t) \in R$ be a function defined on $s \in [0, L]$ and $t \in [0, \infty)$ that satisfies the boundary condition

$$\phi(0,t) = 0, \quad \forall t \in [0, \infty), \tag{13}$$

then the following inequality holds

$$\phi^2 \le L \int_0^L [\phi']^2 \,\mathrm{d}s, \quad \forall x \in [0, L]. \tag{14}$$

2.3 Assumptions

Assumption 1: For the unknown distributed loads $f_x(s,t)$, $f_y(s,t)$, $f_z(s,t)$ and unknown boundary loads $d_x(t)$, $d_y(t)$, $d_z(t)$, we assume that there exists constants $\bar{f}_x, \bar{f}_y, \bar{f}_z, \bar{d}_x, \bar{d}_y, \bar{d}_z \in R^+$, such that $|f_x(s,t)| \le \bar{f}_x$, $|f_y(s,t)| \le \bar{f}_y$, $|f_z(s,t)| \le \bar{f}_z, \forall (s,t) \in [0, L] \times [0, \infty)$ and $|d_x(t)| \le \bar{d}_x$, $|d_y(t)| \le \bar{d}_y$, $|d_z(t)| \le \bar{d}_z, \forall t \in [0, \infty)$.

3 Boundary control design

In [35], we have presented the model-based boundary control for a flexible string system in the 3D space. In this paper, we further consider two cases for the flexible string system: (i) boundary control under external loads, i.e. f_x, f_y and f_z exist; and (ii) boundary control without external load, i.e. $f_x = f_y = f_z = 0$. For the first case, boundary control is introduced and uniformly boundedness of the closed-loop system is guaranteed. For the second case, without changing the control design and Lyapunov candidate function, we will prove the exponential stability of the closed-loop system.

Under the external loads, the control laws $u_x(t)$, $u_y(t)$ and $u_z(t)$ at the boundary of the string are proposed to suppress the vibrations in all the three directions. The control laws are given as

$$u_x(t) = -M\dot{x}'(L,t) - 2k_1\dot{x}(L,t) - \mathrm{sgn}[x'(L,t) + \dot{x}(L,t)]\bar{d}_x, \tag{15}$$

$$u_y(t) = -M\dot{y}'(L,t) - 2k_2\dot{y}(L,t) - \mathrm{sgn}[y'(L,t) + \dot{y}(L,t)]\bar{d}_y, \tag{16}$$

$$u_z(t) = -M\dot{z}'(L,t) - 2k_3\dot{z}(L,t) - \mathrm{sgn}[z'(L,t) + \dot{z}(L,t)]\bar{d}_z, \tag{17}$$

where k_1, k_2 and k_3 are the control gains.

Remark 3: All the signals in the proposed boundary control law can be measured directly by sensors or be obtained by a backward difference algorithm. In the proposed controller (15)–(17), $x(L,t)$, $y(L,t)$ and $z(L,t)$ can be measured through position sensors at the top boundary of the string. $\dot{x}(L,t)$, $\dot{y}(L,t)$, $\dot{z}(L,t)$ can be obtained by a 3D accelerometer attached at the top boundary of the string. $x'(L,t)$, $y'(L,t)$ and $z'(L,t)$ can be measured with an in inclinometer. In addition, $\dot{x}(L,t)$, $\dot{y}(L,t)$, $\dot{z}(L,t)$, $\dot{x}'(L,t)$, $\dot{y}'(L,t)$ and $\dot{z}'(L,t)$ can be counted by a backward difference algorithm based on the measured values.

3.1 Boundary control under external loads

Consider the Lyapunov candidate function as

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{18}$$

where

$$V_1(t) = \frac{\alpha}{2}EA\int_0^L \left(z' + \frac{x'^2}{2} + \frac{y'^2}{2}\right)^2 \mathrm{d}s + \frac{\alpha}{2}\rho\int_0^L \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\mathrm{d}s + \frac{\alpha}{2}T\int_0^L (x'^2 + y'^2)\,\mathrm{d}s, \tag{19}$$

$$V_2(t) = \frac{\alpha}{2}M[z'(L,t) + \dot{z}(L,t)]^2 + \frac{\alpha}{2}M[x'(L,t) + \dot{x}(L,t)]^2 + \frac{\alpha}{2}M[y'(L,t) + \dot{y}(L,t)]^2, \tag{20}$$

$$V_3(t) = \beta\rho\int_0^L s\left(\dot{x}x' + \dot{y}y' + \dot{z}z'\right)\mathrm{d}s, \tag{21}$$

where α and β are two positive constants.

Lemma 2: The Lyapunov candidate function given by (18), is upper and lower bounded as

$$0 \le \lambda_1(\theta(t) + V_2(t)) \le V(t) \le \lambda_2(\theta(t) + V_2(t)), \tag{22}$$

where λ_1 and λ_2 are positive constants, and

$$\theta(t) = \int_0^L [(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 + (x')^2 + (y')^2 + (z')^2 + (x')^4 + (y')^4 + (x'y')^2]\,\mathrm{d}s. \tag{23}$$

Proof: See Appendix 1. □

From the results of Lemma 2, we can further have the following lemma.

Lemma 3: The time derivative of the Lyapunov function in (18) is upper bounded with

$$\dot{V}(t) \le -\lambda V(t) + \varepsilon, \tag{24}$$

provided that k_1, k_2 and k_3 are selected for satisfying the following inequalities

$$\alpha k_1 - \frac{\beta\rho L}{2} \ge 0 \tag{25}$$

$$\alpha k_2 - \frac{\beta\rho L}{2} \ge 0 \tag{26}$$

$$\alpha k_3 - \frac{\beta\rho L}{2} \ge 0 \tag{27}$$

$$\alpha T - \alpha k_1 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_7} - \frac{\beta LT}{2} \ge 0 \tag{28}$$

$$\alpha T - \alpha k_2 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_8} - \frac{\beta LT}{2} \ge 0 \tag{29}$$

$$\alpha EA - \alpha k_3 - \frac{\beta LEA}{2} \ge 0 \tag{30}$$

where λ and ε are two positive constants.

Proof: See Appendix 2. □

Theorem 1: For the string system dynamics represented by (6)–(8) and boundary conditions (9)–(12), under Assumption 1, and the control laws (15)–(17), given that the initial conditions are bounded, and that the required state information $x(L,t)$, $y(L,t)$, $z(L,t)$, $\dot{x}(L,t)$, $\dot{y}(L,t)$ and $\dot{z}(L,t)$ are available, the closed-loop system is uniformly bounded as $|x(s,t)| \le D, |y(s,t)| \le D$ and $|z(s,t)| \le D$, $\forall s \in [0, L]$, where

$$D = \sqrt{\frac{L}{\lambda_1}\left(V(0)\,e^{-\lambda t} + \frac{\varepsilon}{\lambda}\right)}, \quad \forall s \in [0, L], \tag{31}$$

λ and ε are two positive constants.

Proof: From (24), We have

$$\frac{\partial}{\partial t}(Ve^{\lambda t}) \leq \varepsilon e^{\lambda t}. \tag{32}$$

Integration of the above inequalities, we obtain

$$V(t) \leq \left(V(0) - \frac{\varepsilon}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon}{\lambda} \leq V(0)\, e^{-\lambda t} + \frac{\varepsilon}{\lambda} \in \mathcal{L}_\infty, \tag{33}$$

which implies $V(t)$ is bounded. Utilising inequality (14) and (22), we have

$$\frac{1}{L}x^2(s,t) \leq \int_0^L [x'(s,t)]^2\,\mathrm{d}s \leq \theta(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty, \tag{34}$$

$$\frac{1}{L}y^2(s,t) \leq \int_0^L [y'(s,t)]^2\,\mathrm{d}s \leq \theta(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty, \tag{35}$$

$$\frac{1}{L}z^2(s,t) \leq \int_0^L [z'(s,t)]^2\,\mathrm{d}s \leq \theta(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{36}$$

From the above three inequalities and inequality (33), we can state the $\theta(t)$ is bounded $\forall t \in [0, \infty)$. Since $\theta(t)$ is bounded, $\dot{x}(s,t)$, $x'(s,t)$, $x''(s,t)$, $\dot{y}(s,t)$, $y'(s,t)$, $y''(s,t)$, $\dot{z}(s,t)$ and $z'(s,t)$ are bounded $\forall(s,t) \in [0,L] \times [0,\infty)$. From (1), the kinetic energy of the system is bounded and using property in [34], $\dot{x}'(s,t)$ and $\dot{y}'(s,t)$ are bounded $\forall(s,t) \in [0,L] \times [0,\infty)$. From the boundedness of the potential energy (2), we can use property in [34] to conclude that $z''(s,t)$ is bounded $\forall(s,t) \in [0,L] \times [0,\infty)$. Finally, using Assumption 1, (6)–(8) through (9)–(12) and the above statements, we can conclude that $\ddot{x}(s,t)$, $\ddot{y}(s,t)$ and $\ddot{z}(s,t)$ are also bounded $\forall(s,t) \in [0,L] \times [0,\infty)$. From Lemma 3 and the above proof, it is shown the deflection $x(s,t)$, $y(s,t)$ and $z(s,t)$ are uniformly bounded $\forall(s,t) \in [0,L] \times [0,\infty)$. □

Remark 4: From the above stability analysis, $\dot{x}(s,t)$, $\dot{y}(s,t)$, $\dot{z}(s,t)$, $x(s,t)$, $y(s,t)$ and $z(s,t)$ are all bounded $\forall(s,t) \in [0,L] \times [0,\infty)$, and we can conclude the control inputs of u_x, u_y and u_z are bounded $\forall t \in [0,\infty)$.

3.2 Boundary control without the distributed load

In this section, when the distributed loads $f_x(s,t) = f_y(s,t) = f_z(s,t) = 0$, we analyse the exponential stability of the system by using the same Lyapunov candidate function (18) and control laws (15)–(17) of Section 3.1.

Theorem 2: For the system dynamics described by (6)–(8) and boundary conditions (9)–(12), using the proposed control law (15)–(17), then the exponential stability under the condition $f_x(s,t) = f_y(s,t) = f_z(s,t) = 0$ can be achieved as $|x(s,t)| \leq D_0$, $|y(s,t)| \leq D_0$ and $|z(s,t)| \leq D_0$, $\forall s \in [0,L]$, where

$$D_0 = \sqrt{\frac{L}{\lambda_1}V(0)\, e^{-\lambda t}} \tag{37}$$

Furthermore, we have

$$\lim_{t\to\infty} |x(s,t)| = \lim_{t\to\infty} |y(s,t)| = \lim_{t\to\infty} |z(s,t)| = 0. \tag{38}$$

Proof: If $f_x(s,t) = f_y(s,t) = f_z(s,t) = 0$, we have $\varepsilon = 0$. From inequality (24), we obtain the time derivation of the Lyapunov candidate function (18) as

$$\dot{V}(t) \leq -\lambda V(t), \tag{39}$$

where $\lambda = \lambda_3/\lambda_2$. Multiplying (39) by $e^{\lambda t}$ yields

$$\frac{\partial}{\partial t}(Ve^{\lambda t}) \leq 0. \tag{40}$$

Integration of the above inequality yields

$$V(t) \leq V(0)\, e^{-\lambda t} \in \mathcal{L}_\infty, \tag{41}$$

which indicates that $V(t)$ is bounded. Using inequalities (34)–(36), we obtain

$$|x(s,t)| \leq \sqrt{\frac{L}{\lambda_1}V(0)\, e^{-\lambda t}}, \quad \forall s \in [0,L], \tag{42}$$

$$|y(s,t)| \leq \sqrt{\frac{L}{\lambda_1}V(0)\, e^{-\lambda t}}, \quad \forall s \in [0,L], \tag{43}$$

$$|z(s,t)| \leq \sqrt{\frac{L}{\lambda_1}V(0)\, e^{-\lambda t}}, \quad \forall s \in [0,L]. \tag{44}$$

Furthermore, we have

$$\lim_{t\to\infty} |x(s,t)| = \lim_{t\to\infty} |y(s,t)| = \lim_{t\to\infty} |z(s,t)| = 0. \tag{45}$$

From the above proof, we have the conclusion that the displacements $x(s,t)$, $y(s,t)$ and $z(s,t)$ exponentially converge to zero at the rate of convergence λ as $t \to \infty$ by using the proposed control laws, when external loads $f_x(s,t) = f_y(s,t) = f_z(s,t) = 0$. □

4 Numerical simulations

In this section, by using the finite difference method, simulations for a 3D string system with a payload under the external loads are carried out to verify the effectiveness of the proposed control laws (15)–(17). The string system, initially at rest, is excited by the boundary loads and the distributed loads. The external loads are given as

$$f_x(s,t) = f_y(s,t) = [3 + \sin(\pi st) + \sin(2\pi st) + \sin(3\pi st)]s/50 \tag{46}$$

$$f_z(s,t) = 3 + \sin(\pi st) + \sin(2\pi st) + \sin(3\pi st)/55s \tag{47}$$

$$d_x(t) = d_y(t) = 1 + 0.2\sin(0.2\pi t) + 0.3\sin(0.3\pi t) + 0.5\sin(0.5\pi t)/5 \tag{48}$$

$$d_z(t) = 0.5 + 0.2\sin(0.2\pi t) + 0.3\sin(0.3\pi t) + 0.5\sin(0.5\pi t)/5 \tag{49}$$

The corresponding initial conditions of the system are described as $x(s,0) = y(s,0) = z(s,0) = 0.2s$, $\dot{x}(s,0) = \dot{y}(s,0) = \dot{z}(s,0) = 0$. Parameters of the non-linear string system are given in Table 1.

Table 1 Parameters of the string

Parameter	Description	Value
M	mass of the tip payload	5 kg
L	length of string	1 m
T	tension	15 N
EA	axial stiffness	0.3 N
ρ	mass per unit length	0.1 kg/m

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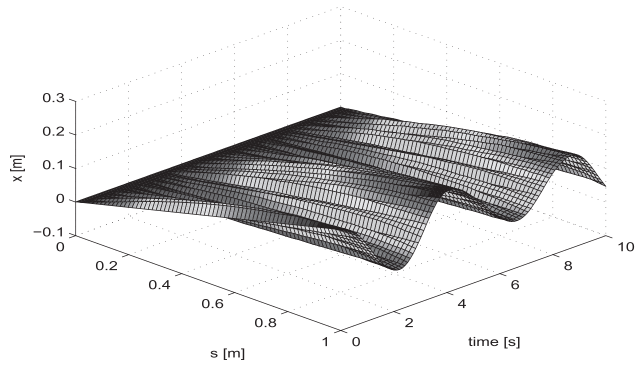

Fig. 2 *Displacement of the string at the X direction without control*

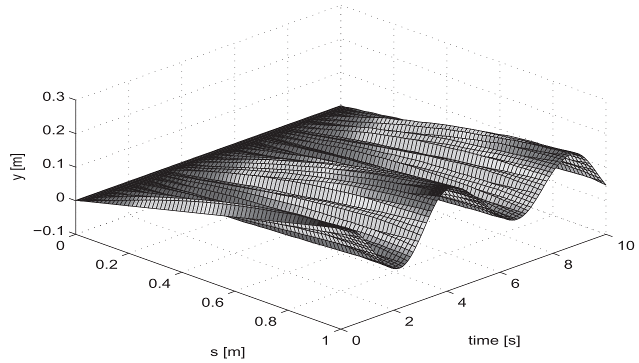

Fig. 3 *Displacement of the string at the Y direction without control*

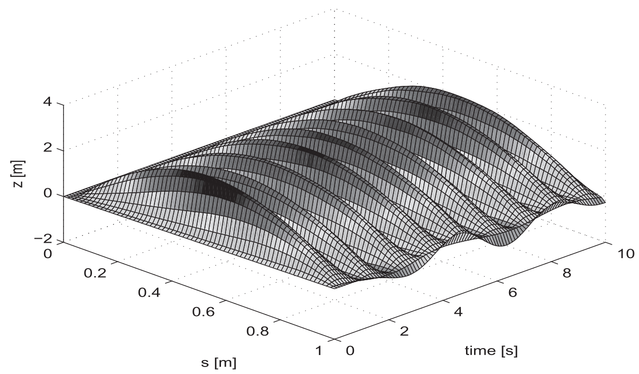

Fig. 4 *Displacement of the string at the Z direction without control*

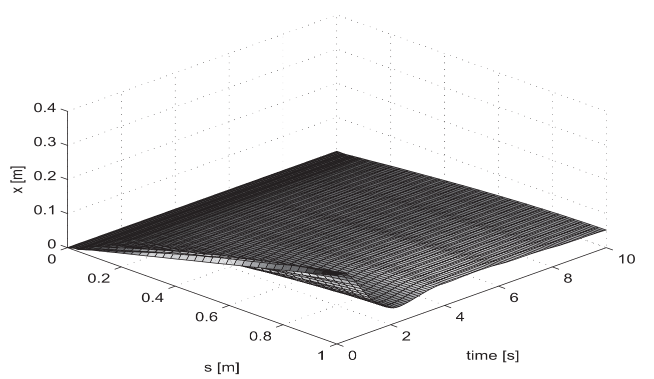

Fig. 5 *Displacement of the string at the X direction with the proposed control*

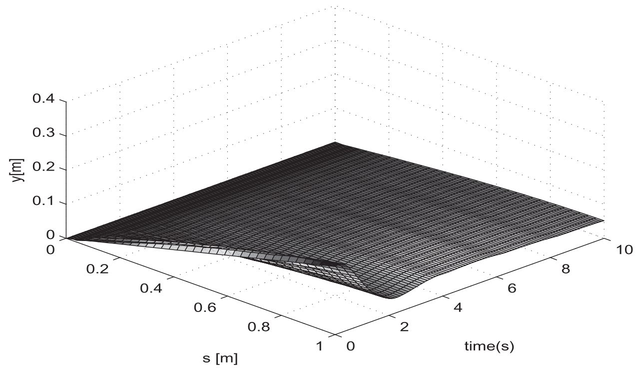

Fig. 6 *Displacement of the string at the Y direction with the proposed control*

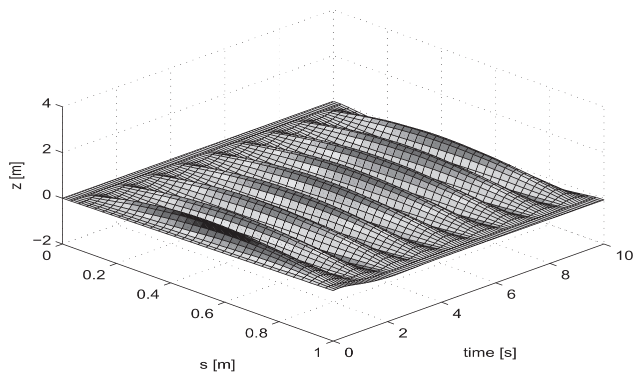

Fig. 7 *Displacement of the string at the Z direction with the proposed control*

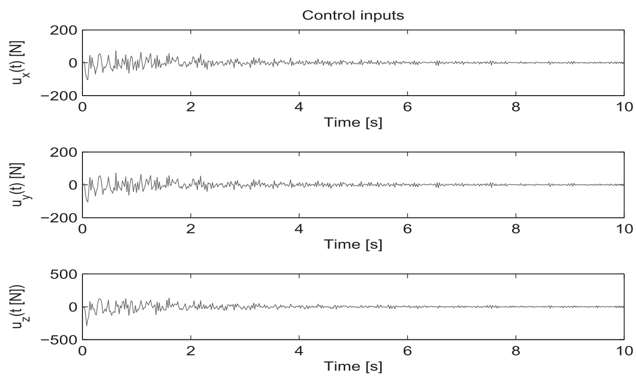

Fig. 8 *Control inputs*

The case without control inputs, i.e. $u_x(t) = u_y(t) = u_z(t) = 0$, is firstly considered. The results are plotted in Figs. 2–4, respectively. From Figs. 2–4, it can be seen that the external loads can cause large vibrations in X, Y and Z directions. When the proposed control laws (15)–(17) are applied, displacements of the string in the three directions are shown in Figs. 5–7. We choose control gains as $k_1 = k_2 = 3$, $k_3 = 0.2$. It can be observed that when the system is under the proposed control, displacements in the X, Y and Z directions are significantly suppressed. We can conclude that the proposed control can stabilise the string at the small neighbourhood of its equilibrium position. The corresponding control inputs are given in Fig. 8.

5 Conclusion

In this paper, the dynamic equations of motion for a 3D string system have been derived, and then used for the design of the boundary control. The models are reliable since the comprehensive effects of the external loads have been taken into account, which guarantee an accurate analysis of the string system. Under the proposed control scheme, all the signals of the closed-loop system are uniformly bounded. Numerical simulations have been provided to illustrate the effectiveness of the proposed control. In the future research, we will consider vibration control problem for the 3D flexible string with input and output constraints.

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8 Appendix

8.1 Appendix 1: Proof of Lemma 2

Proof: $V_1(t)$ can be rewritten as

$$\begin{aligned} V_1(t) = {} & \frac{\alpha}{2}\rho\int_0^L[\dot{x}^2+\dot{y}^2+\dot{z}^2]\,\mathrm{d}s + \frac{\alpha}{4}EA\int_0^L(x'y')^2\,\mathrm{d}s \\ & + \frac{\alpha}{2}T\int_0^L[x'^2+y'^2]\,\mathrm{d}s + \frac{\alpha}{2}EA\int_0^L z'^2\,\mathrm{d}s + \frac{\alpha}{8}EA\int_0^L x'^4\,\mathrm{d}s \\ & + \frac{\alpha}{8}EA\int_0^L y'^4\,\mathrm{d}s + \frac{\alpha}{2}EA\int_0^L z'x'^2\,\mathrm{d}s + \frac{\alpha}{2}EA\int_0^L z'y'^2\,\mathrm{d}s \end{aligned} \tag{50}$$

Using inequalities $|\phi_1\phi_2| \le (1/\delta)\phi_1^2 + \delta\phi_2^2$, $2[z'(s,t)]^2 \le [x'(s,t)]^2$ and $2[z'(s,t)]^2 \le [y'(s,t)]^2$ [36], we have

$$\begin{aligned} & -\frac{1}{2\delta}\int_0^L x'^2\,\mathrm{d}s - \delta\int_0^L x'^4\,\mathrm{d}s \\ & \le \int_0^L z'x'^2\,\mathrm{d}s \le \frac{1}{2\delta}\int_0^L x'^2\,\mathrm{d}s + \delta\int_0^L x'^4\,\mathrm{d}s \end{aligned} \tag{51}$$

$$\begin{aligned} & -\frac{1}{2\delta}\int_0^L y'^2\,\mathrm{d}s - \delta\int_0^L y'^4\,\mathrm{d}s \\ & \le \int_0^L z'y'^2\,\mathrm{d}s \le \frac{1}{2\delta}\int_0^L y'^2\,\mathrm{d}s + \delta\int_0^L y'^4\,\mathrm{d}s \end{aligned} \tag{52}$$

where δ is a positive constant.

Let the positive δ satisfying $T - (EA/2\delta) \geq 0$ and $(1/4) - \delta \geq 0$, we have

$$0 \leq \gamma_1\theta(t) \leq V_1(t) \leq \gamma_2\theta(t) \tag{53}$$

where γ_1 and γ_2 are defined as

$$\gamma_1 = \frac{\alpha}{2}\min\left[\rho, T - \frac{EA}{2\delta}, \quad \frac{1}{2}EA, \quad EA\left(\frac{1}{4} - \delta\right)\right] \tag{54}$$

$$\gamma_2 = \frac{\alpha}{2}\max\left[\rho, T + \frac{EA}{2\delta}, \quad EA, \quad EA\left(\frac{1}{4} + \delta\right)\right] \tag{55}$$

We can also obtain that

$$|V_3(t)| \leq \beta\rho L\int_0^L [\dot{x}^2 + x'^2 + \dot{y}^2 + y'^2 + \dot{z}^2 + z'^2]\,\mathrm{d}s$$
$$\leq \beta_1\theta(t), \tag{56}$$

where $\beta_1 = \beta\rho L$. Then, we have $-\beta_1\theta(t) \leq V_3(t) \leq \beta_1\theta(t)$. Considering β is a positive weighting constant satisfying $0 < \beta < (\gamma_1/\rho L)$, we have $0 < \beta_1 < \gamma_1$. Let $\beta_2 = \gamma_1 - \beta_1, \beta_3 = \gamma_2 + \beta_1$, we further have

$$0 \leq \beta_2\theta(t) \leq V_1(t) + V_3(t) \leq \beta_3\theta(t). \tag{57}$$

Given the Lyapunov candidate function in (18), we obtain

$$0 \leq \lambda_1(\theta(t) + V_2(t)) \leq V(t) \leq \lambda_2(\theta(t) + V_2(t)), \tag{58}$$

where $\lambda_1 = \min(\beta_2, 1)$ and $\lambda_2 = \max(\beta_3, 1)$ are positive constants. □

8.2 Appendix 2: Proof of Lemma 3

Proof: We differentiate (18) with respect to time to obtain

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3, \tag{59}$$

The first term of (59) is written as

$$\dot{V}_1 = \alpha\rho\int_0^L \dot{x}\ddot{x}\,\mathrm{d}s + \alpha\rho\int_0^L \dot{y}\ddot{y}\,\mathrm{d}s + \alpha\rho\int_0^L \dot{z}\ddot{z}\,\mathrm{d}s$$
$$+ \alpha EA\int_0^L\left(z' + \frac{1}{2}x'^2 + \frac{1}{2}y'^2\right)(\dot{z}' + x'\dot{x}' + y'\dot{y}')\,\mathrm{d}s$$
$$+ T\alpha\int_0^L x'\dot{x}'ds + T\alpha\int_0^L y'\dot{y}'\,\mathrm{d}s \tag{60}$$

Applying the boundary conditions, and using inequality $|\phi_1\phi_2| \leq (1/\delta)\phi_1^2 + \delta\phi_2^2, \delta > 0$, we have (see (61))

where $\delta_1, \delta_2, \delta_3$ are positive constants.

Differentiating $V_2(t)$ and substituting the boundary conditions and boundary control laws (15)–(17), and using Assumption 1, we have (see (62))

Differentiating $V_3(t)$, we have

$$\dot{V}_3(t) = \beta\rho\int_0^L s\left[\ddot{x}x' + \dot{x}\dot{x}' + \ddot{y}y' + \dot{y}\dot{y}' + \ddot{z}z' + \dot{z}\dot{z}'\right]\mathrm{d}s \tag{63}$$

Applying inequality $|\phi_1\phi_2| \leq (1/\delta)\phi_1^2 + \delta\phi_2^2, \delta > 0$ to $B_6(t)$ and $B_8(t)$, and using integration by parts, we have (see (64))

$$\dot{V}_1 \leq \alpha(Tx'(L,t) + EAx'(L,t)z'(L,t) + \frac{EA}{2}[x'(L,t)]^3 + \frac{EA}{2}x'(L,t)[y'(L,t)]^2)\dot{x}(L,t) + \alpha(Ty'(L,t) + EAy'(L,t)z'(L,t) + \frac{EA}{2}[x'(L,t)]^2y'(L,t)$$
$$+ \frac{EA}{2}[y'(L,t)]^3)\dot{y}(L,t) + \alpha\left(\frac{1}{2}EA[x'(L,t)]^2 + \frac{1}{2}EA[y'(L,t)]^2 + EAz'(L,t)\right) \times \dot{z}(L,t) + \alpha\delta_2\int_0^L \dot{y}^2\,\mathrm{d}s$$
$$+ \alpha\delta_1\int_0^L \dot{x}^2\,\mathrm{d}s + \alpha\delta_3\int_0^L \dot{z}^2\,\mathrm{d}s + \int_0^L \frac{\alpha}{\delta_1}f_x^2\,\mathrm{d}s + \int_0^L \frac{\alpha}{\delta_2}f_y^2\,\mathrm{d}s + \int_0^L \frac{\alpha}{\delta_3}f_z^2\,\mathrm{d}s. \tag{61}$$

$$\dot{V}_2(t) \leq -\alpha k_1[x'(L,t) + \dot{x}(L,t)]^2 + \alpha k_1[x'(L,t)]^2 - \alpha k_1[\dot{x}(L,t)]^2 - \alpha\left\{Tx'(L,t) + EAx'(L,t)z'(L,t) + \frac{EA}{2}[x'(L,t)]^3 + \frac{EA}{2}x'(L,t)[y'(L,t)]^2\right\}$$
$$\times [x'(L,t) + \dot{x}(L,t)] - \alpha k_2[y'(L,t) + \dot{y}(L,t)]^2 + \alpha k_2\{[y'(L,t)]^2 - [\dot{y}(L,t)]^2\} - \alpha\{Ty'(L,t) + \frac{EA}{2}[y'(L,t)]^3 + EAy'(L,t)z'(L,t)$$
$$+ \frac{EA}{2}[x'(L,t)]^2y'(L,t)\} \times [y'(L,t) + \dot{y}(L,t)] - \alpha k_3[z'(L,t) + \dot{z}(L,t)]^2 + \alpha k_3\{[z'(L,t)]^2 - [\dot{z}(L,t)]^2\}$$
$$- \alpha\left\{EAz'(L,t) + \frac{EA}{2}[x'(L,t)]^2 + \frac{EA}{2}[y'(L,t)]^2\right\}[z'(L,t) + \dot{z}(L,t)] \tag{62}$$

$$\dot{V}_3(t) \leq \left(\frac{3\beta L}{8}EA + \beta EAL\delta_7\right)[x'(L,t)]^4 + \left(\frac{\beta L}{2} + \frac{\beta EAL}{2\delta_7}\right)T[x'(L,t)]^2 + \frac{\beta L}{2}\rho[\dot{x}(L,t)]^2 - \left(\frac{\beta T}{2} - \beta L\delta_4 - \frac{\beta EA}{2\delta_9}\right)\int_0^L x'^2\,\mathrm{d}s$$
$$- \left(\frac{3\beta L}{8}EA - \delta_9\beta EA\right)\int_0^L x'^4\,\mathrm{d}s + \frac{\beta L}{2}\rho[\dot{y}(L,t)]^2 - \frac{\beta\rho}{2}\int_0^L \dot{x}^2\,\mathrm{d}s + \left(\frac{\beta L}{2} + \frac{\beta EAL}{2\delta_8}\right)T[y'(L,t)]^2 + \left(\frac{3\beta L}{8}EA + \beta EAL\delta_8\right)[y'(L,t)]^4$$
$$- \left(\frac{\beta T}{2} - \beta L\delta_5 - \frac{\beta EA}{2\delta_{10}}\right)\int_0^L y'^2\,\mathrm{d}s - \left(\frac{3\beta L}{8}EA - \delta_{10}\beta EA\right)\int_0^L y'^4\,\mathrm{d}s - \frac{\beta\rho}{2}\int_0^L \dot{y}^2\,\mathrm{d}s + \frac{\beta L}{2}EA[z'(L,t)]^2 - \frac{\beta\rho}{2}\int_0^L \dot{z}^2\,\mathrm{d}s$$
$$- \left(\frac{\beta EA}{2} - \beta L\delta_6\right)\int_0^L z'^2\,\mathrm{d}s + \frac{\beta L}{2}\rho[\dot{z}(L,t)]^2 + \frac{\beta L}{\delta_4}\int_0^L f_x^2\,\mathrm{d}s + \frac{\beta L}{\delta_5}\int_0^L f_y^2\,\mathrm{d}s + \frac{\beta L}{\delta_6}\int_0^L f_z^2\,\mathrm{d}s + \frac{3\beta}{4}EAL(x'y')^2 - \frac{3\beta}{4}EA\int_0^L (x'y')^2\,\mathrm{d}s, \tag{64}$$

where δ_4, δ_5, δ_6, δ_7, δ_8, δ_9 and δ_{10} are positive constants. Substituting (61), (62) and (64) into (59), and applying $|\phi_1\phi_2| \le (1/\delta)\phi_1^2 + \delta\phi_2^2$, $2[z'(s,t)]^2 \le [x'(s,t)]^2$ and $2[z'(s,t)]^2 \le [y'(s,t)]^2$ [36], we further obtain

$$
\begin{aligned}
\dot{V}(t) \le & -\left(\frac{\beta\rho}{2} - \alpha\delta_1\right)\int_0^L \dot{x}^2\,\mathrm{d}s - \left(\frac{\beta\rho}{2} - \alpha\delta_2\right)\int_0^L \dot{y}^2\,\mathrm{d}s \\
& - \left(\frac{\beta\rho}{2} - \alpha\delta_3\right)\int_0^L \dot{z}^2\,\mathrm{d}s - \left(\frac{\beta T}{2} - \beta L\delta_4 - \frac{\beta EA}{2\delta_9}\right)\int_0^L x'^2\,\mathrm{d}s \\
& - \left(\frac{\beta T}{2} - \beta L\delta_5 - \frac{\beta EA}{2\delta_{10}}\right)\int_0^L y'^2\,\mathrm{d}s - \left(\frac{\beta EA}{2} - \beta L\delta_6\right) \\
& \times \int_0^L z'^2\,\mathrm{d}s - \left(\frac{3\beta EAL}{8} - \delta_9\beta EA\right)\int_0^L x'^4\,\mathrm{d}s \\
& - \left(\frac{3\beta EAL}{8} - \delta_{10}\beta EA\right)\int_0^L y'^4\,\mathrm{d}s - \alpha k_2(y' + \dot{y})^2 \\
& - \frac{3\beta EA}{4}\int_0^L (x'y')^2\,\mathrm{d}s - \alpha k_1(x' + \dot{x})^2 - \alpha k_3(z' + \dot{z})^2 \\
& + \left(\frac{\alpha}{\delta_1} + \frac{\beta L}{\delta_4}\right)\int_0^L f_x^2\,\mathrm{d}s + \left(\frac{\alpha}{\delta_2} + \frac{\beta L}{\delta_5}\right)\int_0^L f_y^2\,\mathrm{d}s \\
& + \left(\frac{\alpha}{\delta_3} + \frac{\beta L}{\delta_6}\right)\int_0^L f_z^2\,\mathrm{d}s - \left(\alpha k_1 - \frac{\beta\rho L}{2}\right)\dot{x}^2 \\
& - \left(\alpha k_2 - \frac{\beta\rho L}{2}\right)\dot{y}^2 - \left(\alpha k_3 - \frac{\beta\rho L}{2}\right)\dot{z}^2 \\
& - \left(\alpha T - \alpha k_1 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_7} - \frac{\beta LT}{2}\right)x'^2 \\
& - \left(\alpha T - \alpha k_2 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_8} - \frac{\beta LT}{2}\right)y'^2 \\
& - \left(\alpha EA - \alpha k_3 - \frac{\beta LEA}{2}\right)z'^2 \\
& - \left(\frac{\alpha EA}{2} - \frac{3\beta LEA}{8} - \delta_7\left|\frac{3\alpha}{2} - \beta L\right|EA\right)x'^4 \\
& - \left(\frac{\alpha EA}{2} - \frac{3\beta LEA}{8} - \delta_8\left|\frac{3\alpha}{2} - \beta L\right|EA\right)y'^4 \\
& - \left(\alpha EA - \frac{3\beta EAL}{4}\right)(x'y')^2 \le -\lambda_3[\theta(t) + V_2(t)] + \varepsilon
\end{aligned}
\tag{65}
$$

where

$$
\begin{aligned}
\lambda_3 = \min\Bigg\{ & \frac{\beta\rho}{2} - \alpha\delta_1, \frac{\beta\rho}{2} - \alpha\delta_2, \frac{\beta\rho}{2} - \alpha\delta_3, \frac{\beta L}{2} - \beta L\delta_4 \\
& - \frac{\beta EA}{2\delta_9}, \frac{\beta L}{2} - \beta L\delta_5 - \frac{\beta EA}{2\delta_{10}}, \frac{\beta EA}{2} - \beta L\delta_6, \frac{3\beta EA}{4}, \\
& \times \frac{3\beta EAL}{8} - \delta_9\beta EA, \frac{3\beta EAL}{8} - \delta_{10}EA, \frac{2k_1}{M}, \frac{2k_2}{M}, \frac{2k_3}{M}\Bigg\} > 0
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\varepsilon = & \left(\frac{\alpha}{\delta_1} + \frac{\beta L}{\delta_4}\right)\int_0^L f_x^2\,\mathrm{d}s + \left(\frac{\alpha}{\delta_2} + \frac{\beta L}{\delta_5}\right)\int_0^L f_y^2\,\mathrm{d}s \\
& + \left(\frac{\alpha}{\delta_3} + \frac{\beta L}{\delta_6}\right)\int_0^L f_z^2\,\mathrm{d}s \le \left(\frac{\alpha L}{\delta_1} + \frac{\beta L^2}{\delta_4}\right)\overline{f}_x^2 \\
& + \left(\frac{\alpha L}{\delta_2} + \frac{\beta L^2}{\delta_5}\right)\overline{f}_x^2 + \left(\frac{\alpha L}{\delta_3} + \frac{\beta L^2}{\delta_6}\right)\overline{f}_x^2 \in \mathcal{L}_\infty
\end{aligned}
\tag{67}
$$

The design constants $k_1, k_2, k_3, \alpha, \beta, \delta_7, \delta_8$ are selected to satisfy the following conditions

$$\alpha k_1 - \frac{\beta\rho L}{2} \ge 0 \tag{68}$$

$$\alpha k_2 - \frac{\beta\rho L}{2} \ge 0 \tag{69}$$

$$\alpha k_3 - \frac{\beta\rho L}{2} \ge 0 \tag{70}$$

$$\alpha T - \alpha k_1 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_7} - \frac{\beta LT}{2} \ge 0 \tag{71}$$

$$\alpha T - \alpha k_2 - \frac{|(3\alpha/2) - \beta L|EA}{2\delta_8} - \frac{\beta LT}{2} \ge 0 \tag{72}$$

$$\alpha EA - \alpha k_3 - \frac{\beta LEA}{2} \ge 0 \tag{73}$$

$$\frac{\alpha EA}{2} - \frac{3\beta LEA}{8} - \delta_7\left|\frac{3\alpha}{2} - \beta L\right|EA \ge 0 \tag{74}$$

$$\frac{\alpha EA}{2} - \frac{3\beta LEA}{8} - \delta_8\left|\frac{3\alpha}{2} - \beta L\right|EA \ge 0 \tag{75}$$

$$\alpha EA - \frac{3\beta EAL}{4} \ge 0 \tag{76}$$

From inequality (58), we have

$$\dot{V}(t) \le -\lambda V(t) + \varepsilon, \tag{77}$$

where $\lambda = \lambda_3/\lambda_2 > 0$ and $\varepsilon > 0$. □