desired invariant factors and the controllability indices of (F_0, G_0) satisfy (9) with equality for every p. The set of controllability indices of the given pair (F, G) also satisfy (9) and hence are less than or equal to those of (F_0, G_0) in the associated partial ordering. Since the partial order is a finite one, there is a sequence of partitions $\{P_i\}$ so that P_0 is the set of controllability indices of (F_0, G_0) and P_r is the set of controllability indices of (F, G) and so that P_i covers P_{i+1} for $1 \leq i \leq r$. By applying Lemma 2 r times, we can construct a matrix $\hat{G}$ such that $(F_0, \hat{G})$ has the same controllability indices as (F, G). Then by Lemma 1, there is a matrix K such that $F - GK$ has the same invariant factors as F_0, which is the theorem. Notice that P_{i+1} need only be obtained from P_i by an operation of the type given in Lemma 2, which is less restrictive than insisting on the covering condition.

Since choosing $\psi_1(z)$ to have degree n always leads to the satisfaction of the hypothesis, a reachable system may always be made cyclic with arbitrary characteristic polynomial. In general, the controllability indices provide bounds on the sizes of the cyclic components of the closed-loop matrix. In the generic case, no two controllability indices differ by more than one, and the bounds of the theorem are weakest.

III. Conclusions

The proof given is a transcription of Rosenbrock's system matrix operations into state space language, although Lemma 2 seems to be of independent interest. It should not be surprising that the results of Brunovsky and Popov play an important role. The complexity of some of the calculations provides an illustration of the power of modern transfer function analysis in linear systems theory. Another constructive proof starting from the Brunovsky canonical form, as outlined by Kalman [2], is an alternative which has not yet been completely clarified.

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Technical Notes and Correspondence

String Stability of Relative-Motion PID Vehicle Control Systems

LLOYD E. PEPPARD

***Abstract*—An important performance criterion for moving vehicle longitudinal control systems is string stability in the face of perturbations in the motion of individual vehicles. Systems employing a moving-cell reference for each vehicle always exhibit string stability since there is no vehicle interaction within the string. This correspondence investigates the string stability for a class of relative-motion systems where a moving-cell position reference is not available. It is shown that string stability can be achieved only by using both forward and rearward intervehicle separation measurements.**

Introduction

Application of control system theory to the problem of controlling the longitudinal motion of vehicle strings has led to the development of systems employing both moving-cell [1] references and relative-motion measurements within the vehicle string. Of the latter type, the system reported by Levine and Athans [2] uses a control based on measurements from the entire string while those of Peppard and Gourishankar [3] and Fenton and Bender [4] use nearest-neighbor measurements. For these systems, string stability, or the property of the vehicle string to attenuate disturbances as they propagate down the string, is an important performance criterion. Since moving-cell systems employ on-board references for each vehicle, interaction between vehicles is precluded and hence, string stability is assured. As indicated in [5] this is a strong point in favor of moving-cell systems.

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For some applications, such as the regulation of aircraft separation [6], [7], it may be impractical to obtain measurements of each vehicle's position with respect to a moving-cell reference. Further, an important performance criterion for such systems is the maintenance of a specified inter-vehicle separation, even if one or more vehicles has for some reason deviated from its scheduled position in space. For such applications it may be possible to measure the deviation in velocity of each vehicle from a specified value and the separation between each vehicle and its nearest neighbors. The purpose of this correspondence is to show that the problem of string instability attributed to a relative-motion system using such measurements can be avoided by proper choice of the feedback control.

Relative-Motion Control Systems

It has previously been shown [9] that proportional-derivative (PD) type relative-motion systems using forward or forward plus rearward separation measurements can be designed to exhibit string stability. However, in order to provide for vehicle control in the face of constant (or slowly varying) force disturbances, a proportional-integral-derivative (PID) type feedback control must be considered. Such a control has been shown to be effective in regulating the motion of terrestrial vehicles over undulating terrain [8], [10] and is equally useful in compensating for slowly varying wind forces on terrestrial vehicles or aircraft.

The linearized state equations for the motion of the ith vehicle in a string can be written as

$$\begin{aligned}\dot{x}_i(t) &= y_i(t)\\ \dot{y}_i(t) &= -\mu y_i(t) + u_i(t) + w_i(t)\end{aligned} \tag{1}$$

where x_i is the position error, y_i is the velocity error, u_i is the control force per unit mass, μ is the linearized drag coefficient, and w_i is the disturbance force per unit mass.

As previously stated, we will assume that the available measurements for control use are the velocity error of each vehicle (y_i) and the distance to the vehicle ahead $(x_{i-1} - x_i)$ and to the vehicle behind $(x_i - x_{i+1})$. In addition, the system must exhibit zero separation error when $w_i(t) =$ constant. A PID control scheme which satisfies these conditions and uses only forward separation measurements is given by

$$u_i(t) = -h_1(x_i - x_{i-1}) - h_2 y_1 - h_3 \int_0^t (x_i - x_{i-1})\, d\tau. \quad (2)$$

For the "lead" vehicle ($i = 1$), $x_0 = 0$, and $x_1 = x_1(0) + \int_0^t y_1(\tau)\, d\tau$. The gains $h_1 - h_3$ can be determined using linear regulator theory [3] or pole placement techniques [11] to obtain a desirable vehicle response to constant (step) and initial condition (impulse) disturbances.

It can be shown [12], [7] that a necessary condition for string stability is $|X_i(\omega)/X_{i-1}(\omega)| \leq 1$ for all frequency ω. The system (1) using control (2) yields a transfer junction

$$X_i(s)/X_{i-1}(s) = \frac{h_3 + h_1 s}{s^3 + (\mu + h_2)s^2 + h_1 s + h_3} \quad (3)$$

from which it can be concluded (see [7]) that the necessary condition cannot be met for $h_3 \neq 0$ (i.e., for a PID control using forward separation measurements). Fig. 1(a) shows the position error response for the first 4 vehicles in a string to a step disturbance on the lead vehicle of $w_1(0) = 3.22$ ft/s^{-2} when (3) has one real pole and one complex-conjugate pair ($h_1 = 0.25$, $h_2 = 0.8$, $h_3 = 0.025$, $\mu = 0.1$). Fig. 1(b) shows the response when (3) has three real poles ($h_1 = 0.31$, $h_2 = 2.1$, $h_3 = 0.01$, $\mu = 0.1$). In both cases, individual vehicles are asymptotically stable but the disturbance buildup (string instability) is evident.

If rearward separation measurements are included in the control scheme (2), a PID control is given by

$$u_i(t) = -h_1'(x_i - x_{i-1}) - h_2' y_i - h_3' \int_0^t (x_i - x_{i-1})\, d\tau$$
$$-h_4'(x_i - x_{i+1}) - h_5' \int_0^t (x_i - x_{i+1})\, d\tau. \quad (4)$$

If we let $h_1' = \beta h_1$, $h_2' = h_2$, $h_3' = \beta h_3$, $h_4' = (1 - \beta)h_1$, and $h_5' = (1 - \beta)h_3$, $0 \leq \beta \leq 1$, then we can write

$$\left.\frac{X_i(s)}{X_{i+1}(s)}\right|_{X_{i-1}=0} = \frac{\beta(h_3 + h_1 s)}{s^3 + (\mu + h_2)s^2 + h_1 s + h_3} \quad (5)$$

and

$$\left.\frac{X_i(s)}{X_{i+1}(s)}\right|_{X_{i-1}=0} = \frac{(1 - \beta)(h_3 + h_1 s)}{s^3 + (\mu + h_2)s^2 + h_1 s + h_3}. \quad (6)$$

These transfer functions describe the response of the ith vehicle to that of the $(i + 1)$th when $X_{i-1} = 0$ and the $(i - 1)$th when $X_{i+1} = 0$. The parameter β can be thought of as the relative weighting of forward and rearward separation errors in the control scheme. To achieve equal attenuation of disturbances traveling in both directions, β is set equal to $\frac{1}{2}$. Both (5) and (6) can then be shown to satisfy the stability condition for any pole configuration of (5) and (6). Fig. 2 shows the response of the same system as Fig. 1(b) using the control (4) with $\beta = \frac{1}{2}$. The gains $h_1 - h_3$ are as before. It can be seen that the purturbation in the position of the lead vehicle is now attenuated as it propagates down the string. This is, of course, at the expense of the additional measurement requirement.

It might be noted that the failure of either a vehicle's forward or rearward separation sensor would only result in a performance for that vehicle similar to that of Fig. 1 (the gains h_1 and h_2 and h_3 would be halved if $\beta = \frac{1}{2}$). Also, depending on the actual desired intervehicle separation, it may not be necessary to use rearward measurements for each vehicle. Rather, these might be used for, say, every fourth vehicle to reduce the perturbation buildup over the entire string.

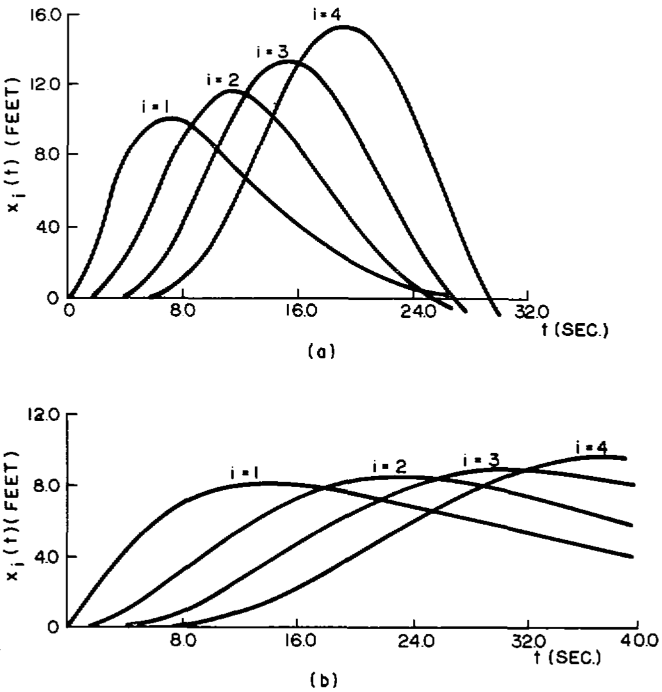

Fig. 1. Position error response of the first four vehicles in a string to a step disturbance on the lead-vehicle using forward separation measurements for (a) a real plus complex-conjugate pair and (b) an all-real pole configuration.

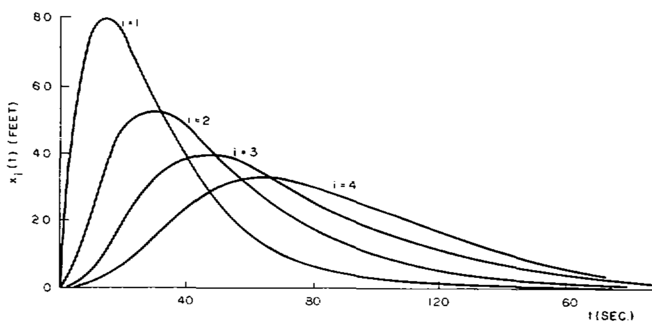

Fig. 2. Response corresponding to Fig. 1(b) using forward and rearward separation measurements.

Conclusions

It has been stated [8] that relative-motion systems of the type developed in this correspondence can exhibit undesirable shockwave effects due to perturbation buildup along the vehicle string. It has been shown here that a PID feedback control using only velocity error and separation measurements can be designed to exhibit string stability if both forward and rearward separation measurements are employed. The additional complexity of such a system compared to a moving-cell system may be justified in applications where it is impractical to determine the position of individual vehicles in the string but where separation measurements can be obtained.

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On Reentry Vehicle Tracking in Various Coordinate Systems

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Abstract—**Some simulation results are presented to demonstrate difficulties which may arise in choosing initialization procedures and prediction algorithms for reentry vehicle (RV) tracking filters operating in various coordinate systems.**

I. Introduction

Some recent publications [1], [2] on reentry vehicle tracking by phased-array radars have presented data indicating inferior performance of Cartesian (xyz) filters in comparison with range-direction-cosine (ruv) filters. The main purpose of this note is to indicate a possible explanation for the observed differences, and a simple solution involving a trivial modification of the initialization procedure. Also included are some comments on the choice of a prediction algorithm for the tracking filter.

II. Initialization Procedures

In some of the writer's experience with Monte Carlo tracking simulations, it has been noticed that when the random *a priori* position errors happen to be quite large, estimation accuracy may be badly degraded over the entire trajectory. In the simulations in which this effect was observed, it was found that the difficulty could be remedied by a trivial alteration of the procedure used to initialize the filter. It is suspected that a similar remedy would be effective in the cases of [1] and [2], where apparently the initial errors were large and were not varied randomly from one run to the next.

The two tracking filters simulated were essentially the same as the xyz and ruv extended Kalman filters described in [1], except that, in the xyz filter, the measurements were transformed from ruv to xyz coordinates before computation of the residuals (innovations) for insertion into the filter. The states estimated in each case are three position variables (x,y,z or r,u,v), three velocity variables ($\dot{x},\dot{y},\dot{z}$, or $\dot{r},\dot{u},\dot{v}$), and a function of the ballistic coefficient β (in this case ρ/β was used). The reentry vehicle initial conditions were

Altitude = 135,000 ft
Velocity = 24,000 ft/s
Flight-path angle = 35°

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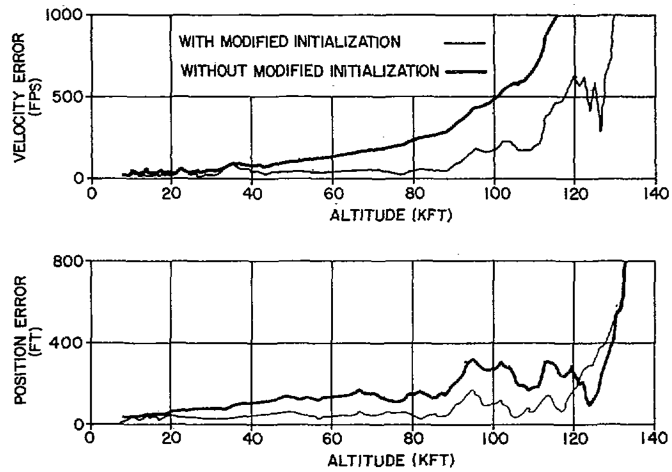

Fig. 1. Effect of initialization procedure on tracking accuracy of xyz Filter.

and the RV flew past the radar before impacting about 50 kft away. Ballistic coefficient β was 1500 psf, plus an exponentially correlated random process; this β model was assumed to represent random density variations as well as actual ballistic-coefficient variations.

Measurements were taken ten times per second, and rms measurement errors were approximately 8 ft in range and 1 mrad in angle (or, more correctly, 1 msine, since the simulated radar was of the phased array type).

A possible source of difficulty in the xyz filter arises because of the nonlinear nature of the measurements. When the measurements are transformed from ruv to xyz coordinates, the measurement-error covariance matrix (R) must be correspondingly transformed. The matrix used for this transformation is normally based on the position estimates available before each measurement. Before the first measurement, however, these estimates are greatly in error, and such a transformation results in a misorientation of the error ellipsoid corresponding to R. Since this ellipsoid is very flat due to the high accuracy of the range measurement, this misorientation causes the filter to believe that the first measurement is highly accurate in a direction in which it is not. The result is a severe bias in the estimates, which persists for a surprisingly long period of time (almost throughout the trajectory). This is illustrated in Fig. 1, for a particular case in which the initial (*a priori*) position error is approximately 32000 ft.

The remedy for this difficulty is quite simple, and costs nothing in computation time. Since the target position indicated by the first measurement is much more accurate than the *a priori* estimate, we simply wait until the first measurement becomes available, and use the indicated position in deriving the transformation matrix for R. This trivial modification results in vastly improved performance, as is apparent from Fig. 1. With this modification, the xyz filter has performance virtually identical to that of the seven-state ruv filter, which does not suffer from the above-mentioned difficulty. It should be noted that this difficulty will be present even if the measurement residuals for the xyz filter are calculated in ruv coordinates, as in [1]. In such a filter the R matrix need not be transformed, but the elements of the transformation matrix appear instead in the H matrix of the Kalman filter (the matrix of partial derivatives of the measurements with respect to the states). The same effect will therefore be observed, and a similar modification should be effective. The remedy suggested by Mehra [1] is to make the xyz filter an "iterative-sequential" filter by using the technique of Denham and Pines [3], which involves an iterative procedure after each measurement. Our results indicate that the difficulty can be circumvented without this added complexity (and the attendant increase in computational cost). With a trivial (and cost-free) modification at the initial measurement time, the performance of the xyz filter can be made identical to that of the ruv filter. This procecure is similar to the use of a single iteration on the first measurement *only*.