String Stability of Interconnected Systems

D. Swaroop and J. K. Hedrick

Abstract— **In this paper we introduce the notion of string stability of a countably infinite interconnection of a class of nonlinear systems. Intuitively, string stability implies uniform boundedness of all the states of the interconnected system for all time if the initial states of the interconnected system are uniformly bounded. It is well known that the input–output gain of all the subsystems less than unity guarantees that the interconnected system is input–output stable. We derive sufficient ("weak coupling") conditions which guarantee the asymptotic string stability of a class of interconnected systems. Under the same "weak coupling" conditions, string-stable interconnected systems remain string stable in the presence of small structural/singular perturbations. In the presence of parameter mismatch, these "weak coupling" conditions ensure that the states of all the subsystems are all uniformly bounded when a gradient-based parameter adaptation law is used and that the states of all the systems go to zero asymptotically.**

I. Introduction

Earlier research on interconnected systems focused on vehicle-following applications [17], [14], [8], [23], [11], control of distributed systems (e.g., regulation of seismic cables, vibration control in beams, etc.) [7], [19], signal processing [4], and power systems [5]. Loosely speaking, string stability of an interconnected system implies uniform boundedness of the state of all the systems. For example, in automated vehicle-following applications, tracking (spacing) errors should not amplify downstream from vehicle to vehicle for safety. Similarly, deflection at any point in a beam or a rod should remain bounded at all times. Spatial discretization and control of such distributed systems have a relevance to the problem of string stability for interconnected systems. Although a precise definition of string stability was not coined, Kuo and Melzer [17] and Levine and Athans [14] were seeking optimal control solutions to the automated vehicle-following problem. Chu defined string stability in the context of vehicle following [11]. In [4], Chang introduces a stronger version of stability for interconnected systems, namely, "γ-stability" for infinite interconnection of linear digital processors. Intuitively, "γ-stability" ensures that the state of all the systems decays to zero exponentially in time and system index. In this paper, we generalize the concept of string stability to a class of interconnected systems and seek sufficient conditions to guarantee their string stability. We also examine their robustness to structural and singular perturbations.

Manuscript received July 22, 1994; revised July 3, 1995. Recommended by Associate Editor, A. M. Bloch. This work was supported in part by the PATH program at the University of California, Berkeley.

The authors are with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720 USA.

Publisher Item Identifier S 0018-9286(96)02099-5.

This paper is organized as follows: In Section I, we define string stability and asymptotic string stability, we present "weak coupling" conditions that guarantee string stability for a class of interconnected systems, and we demonstrate that exponential string stability is preserved under small structural perturbations. In Section II, we prove that every exponentially string-stable interconnected system is string stable in the presence of small singular perturbations. In Section III, we discuss direct adaptive control of such interconnected systems. In Section IV, we provide an example of longitudinal controller design for vehicle-following systems.

II. String Stability

We use the following notations: $\|f_i(\cdot)\|_\infty$, or simply $\|f_i\|_\infty$ denotes $\sup_{t\geq 0}|f_i(t)|$, and $\|f_i(0)\|_\infty$ denotes $\sup_i |f_i(0)|$. For all $p < \infty$, $\|f_i(\cdot)\|_p$ or $\|f_i\|_p$ denotes $\left(\int_0^\infty |f_i(t)|^p\,dt\right)^{\frac{1}{p}}$ and $\|f_i(0)\|_p$ denotes $\left(\sum_1^\infty |f_i(0)|^p\right)^{\frac{1}{p}}$.

Consider the following interconnected system:

$$\dot{x}_i = f(x_i, x_{i-1}, \cdots, x_{i-r+1}) \tag{1}$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0\ \forall i \leq j$, $x \in \mathcal{R}^n$, $f\colon \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{r \text{ times}} \to \mathcal{R}^n$ and $f(0, \cdots, 0) = 0$.

Definition 1: The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is string stable, if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_i(0)\|_\infty < \delta \Rightarrow \sup_i \|x_i(\cdot)\|_\infty < \epsilon$.

Definition 2: The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is asymptotically (exponentially) string stable if it is string stable and $x_i(t) \to 0$ asymptotically (exponentially) for all $i \in \mathcal{N}$.

A more general definition of string stability is the following one.

Definition 3 (l_p String Stability): The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is l_p string stable if for all $\epsilon > 0$, there exists a δ such that

$$\|x_i(0)\|_p < \delta \Leftrightarrow \sup_t \left(\sum_1^\infty |x_i(t)|^p\right)^{\frac{1}{p}} < \epsilon.$$

Definition 1 of string stability can be restated as l_∞-string stability of Definition 3. Henceforth, we will deal with string stability according to Definition 1. The following theorem proves, under some "weak coupling" conditions, that any countably infinite interconnection of exponentially stable nonlinear systems is string stable. Clearly, a string of uncoupled exponentially stable systems is exponentially string stable. Intuitively, any interconnection of exponentially stable systems is string stable, if the interconnections are sufficiently weak. The following lemmas will be useful in proving the theorems in this paper.

Lemma 1: Let r be a constant positive integer. Define $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$, $\beta_j > 0$. If $\sum_1^r \beta_j < 1$, the rth degree polynomial $P_r(z)$ has all its roots inside the unit circle.

Proof: Let z_0 be such that $P_r(z_0) = 0$ and $|z_0| > 1$. Then

$$1 = \sum_1^r \beta_j z_0^{-j} \le \sum_1^r \beta_j < 1$$

which is a contradiction. This proves the lemma.

Lemma 2: Let $V_i(t) \ge 0 \; \forall t \ge 0$, $i \in \mathcal{N}$ and if

$$\dot{V}_i \le -\beta_0 V_i + \sum_1^\infty \beta_j V_{i-j}$$

with $\beta_0 > 0$ and $\beta_j \ge 0$, $j = 1, 2, \cdots$ and $\beta_0 > \sum_1^\infty \beta_j$. For all $j \le 0$, V_j should be read as zero. Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|V_i(0)\|_\infty < \delta \Rightarrow \sup_i \|V_i\|_\infty < \epsilon.$$

Proof: Let $M = \frac{\beta_0}{\beta_0 - \sum_1^\infty \beta_j} > 1$. It suffices to show that $\|V_i\|_\infty \le M\|V_i(0)\|_\infty$. We prove this by induction. From the inequality, it follows that:

$$\|V_i\|_\infty \le V_i(0) + \sum_1^\infty \frac{\beta_j}{\beta_0}\|V_{i-j}\|_\infty.$$

For $i = 1$, $\|V_i\|_\infty \le V_i(0)$ and the induction hypothesis is valid. Assuming that the hypothesis is valid through the integer i

$$\begin{aligned}\|V_{i+1}\|_\infty &\le V_{i+1}(0) + \sum_1^\infty \frac{\beta_j}{\beta_0} M\|V_i(0)\|_\infty \\ &\le \left(1 + M\sum_1^\infty \frac{\beta_j}{\beta_0}\right)\|V_i(0)\|_\infty \\ &= M\|V_i(0)\|_\infty.\end{aligned}$$

This proves that the induction hypothesis is valid for all i. Therefore, $\sup_i \|V_i\|_\infty \le M\|V_i(0)\|_\infty$.

Theorem 1 (Weak Coupling Theorem for String Stability): If the following conditions are satisfied:

- f is globally Lipschitz in its arguments, i.e.,

$$\begin{aligned}|f(y_1, \cdots, y_r) - f(z_1, \cdots, z_r)| \\ \le l_1|y_1 - z_1| + \cdots + l_r|y_r - z_r|.\end{aligned} \tag{2}$$

- The origin of $\dot{x} = f(x, 0, \cdots, 0)$ is globally exponentially stable.

Then for sufficiently small l_i, $i = 2, \cdots, r$, the interconnected system is globally exponentially string stable.

Proof: Since the origin of $\dot{x} = f(x, 0, \cdots, 0)$ is exponentially stable, by the converse Lyapunov theorem, there exists a Lyapunov function $V(x)$ and four positive constants α_l, α_h, α_1, α_3 such that

$$\alpha_l\|x\|^2 \le V(x) \le \alpha_h\|x\|^2 \tag{3}$$

$$\frac{\partial V}{\partial x} f(x, 0, \cdots, 0) \le -\alpha_1\|x\|^2 \tag{4}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_3\|x\|. \tag{5}$$

For the sake of convenience, we denote $V(x_i)$ by V_i. Then

$$\begin{aligned}\dot{V}_i &= \frac{\partial V_i}{\partial x_i} f(x_i, x_{i-1}, \cdots, x_{i-r+1}) \\ &= \frac{\partial V_i}{\partial x_i} f(x_i, 0, \cdots, 0) \\ &\quad + \frac{\partial V_i}{\partial x_i}[f(x_i, x_{i-1}, \cdots, x_{i-r+1}) - f(x_i, 0, \cdots, 0)] \\ &\le -\alpha_1\|x_i\|^2 + \alpha_3\|x_i\|\left(\sum_{j=2}^r l_j\|x_{i-j+1}\|\right).\end{aligned}$$

Using the inequality that $xy \le \frac{x^2+y^2}{2}$, the above equation results in

$$\dot{V}_i \le -\frac{\left(\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j\right)}{\alpha_h} V_i + \frac{\alpha_3}{2\alpha_l}\sum_{j=2}^r l_j V_{i-j+1}. \tag{6}$$

If $\sum_{j=2}^r l_j$ is sufficiently small such that $\sum_{j=2}^r l_j < \frac{2\alpha_l\alpha_1}{\alpha_3(\alpha_l + \alpha_h)}$, then $\frac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h} > \frac{\alpha_3}{2\alpha_l}\sum_{j=2}^r l_j > 0$. Consequently, string stability follows from Lemmas 1 and 2.

Let $d > 1$. Define $V(d^{-1}, t) = \sum_{j=1}^\infty V_i(t) d^{-i}$. Clearly, $V(d^{-1}, t)$ is defined whenever the weak coupling conditions are satisfied and whenever $\|x_i(0)\|_\infty$ exists

$$\dot{V} = \sum_{j=1}^\infty \dot{V}_i(t) d^{-i} \le -V d^{-(r-1)} P_r(d) \frac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h}.$$

Here $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$ where $\beta_j = \frac{\alpha_3}{2\alpha_l}\frac{\alpha_h}{\alpha_1 - \frac{\alpha_3}{2}\sum l_j} l_j$. Clearly, $P_r(d) > 0$ whenever $d > 1 > \rho(P_r(z))$, the spectral radius of the polynomial $P_r(z)$. $V \to 0$ exponentially and hence, $V_i(t)$, $x_i(t) \to 0$ exponentially.

The above theorem can easily be generalized to nonautonomous interconnected systems. Consider the following nonautonomous interconnection:

$$\dot{x}_i = f(x_i, x_{i-1}, \cdots, x_{i-r+1}, t)$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0 \; \forall i \le j$, $x \in \mathcal{R}^n$, $f: \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{r \text{ times}} \times \mathcal{R} \to \mathcal{R}^n$ and $f(0, \cdots, 0) = 0$.

Remark: If the following conditions are satisfied:

- f is globally Lipschitz in its arguments, i.e.,

$$\begin{aligned}|f(y_1, \cdots, y_r, t) - f(z_1, \cdots, z_r, t)| \\ \le l_1|y_1 - z_1| + \cdots + l_r|y_r - z_r|.\end{aligned}$$

- The origin of $\dot{x} = f(x, 0, \cdots, 0, t)$ is globally exponentially stable, i.e., there exists a Lyapunov function, $V(x)$ such that

$$\alpha_l\|x\|^2 \le V(x, t) \le \alpha_h\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, \cdots, 0, t) \le -\alpha_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_4\|x\|.$$

Then for sufficiently small l_i, $i = 2, \cdots, r$, the interconnected system is globally exponentially string stable.

Proof: Let $V(x_i, t) = V_i$. Clearly, $\dot{V}_i \le -\alpha_3\|x_i\|^2 + \alpha_4\|x_i\|\sum_1^r l_j\|x_{i-j}\|$. By the same arguments used in Theorem 1, the desired conclusion follows.

Another simple class of interconnected systems that arise in the context of vehicle-following systems is given by

$$\dot{x}_i = f(x_i, x_{i-1}, \dot{x}_{i-1}) \tag{7}$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0 \,\forall\, i \le j$, $x \in \mathcal{R}^n$, $f: \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ and $f(0, 0, 0) = 0$. If the following conditions are satisfied:

- f is globally Lipschitz in its arguments, i.e.,

$$\begin{aligned}&|f(y_1, y_2, y_3) - f(z_1, z_2, z_3)| \le \\ &l_1|y_1 - z_1| + \cdots + l_2|y_2 - z_2| + d_1|y_3 - z_3|\end{aligned}$$

.

- The origin of $\dot{x} = f(x, 0, 0)$ is globally exponentially stable, i.e., there exists a Lyapunov function, $V(x)$ such that

$$\begin{aligned}\alpha_l\|x\|^2 \le V(x) &\le \alpha_h\|x\|^2 \\ \frac{\partial V}{\partial x} f(x, 0, 0) &\le -\alpha_3\|x\|^2 \\ \left\|\frac{\partial V}{\partial x}\right\| &\le \alpha_4\|x\|.\end{aligned}$$

Then for sufficiently small $l_2 + d_1$, the interconnected system is globally exponentially string stable.

Proof: From the Lipschitz property

$$\begin{aligned}\|\dot{x}_i\| \le & l_1\|x_i\| + (l_2 + d_1 l_1) \\ & \cdot [\|x_{i-1}\| + d_1\|x_{i-2}\| + \cdots + d_1^{i-2}\|x_1\|.\end{aligned}$$

Let $V(x_i) = V_i$. Then

$$\begin{aligned}\dot{V}_i \le -\frac{\alpha_3}{2\alpha_h}V_i + & \frac{\alpha_4}{2\alpha_l}(l_2 + d_1 l_1)[(1 + d_1 + \cdots + d_1^{i-2})V_i \\ & + (V_{i-1} + d_1 V_{i-2} + \cdots + d_1^{i-2}V_1)].\end{aligned}$$

By the above lemma, string stability follows for sufficiently small $l_1 + l_2 + d_1$. Define $V(t) = \sum_1^\infty V_i d^{-1}$ for any $d > 1$. Then, $\dot{V} \le -KV$ where $K > 0$. Consequently, exponential stability is guaranteed.

From the definition of string stability, it is clear that the string stability of an interconnected system guarantees the stability of every subsystem. Under some stronger coupling condition, $\alpha_1 > \frac{\alpha_3}{2}\sum_{j=2}^r l_j$, any finite interconnections of one is asymptotically string stable. In the vehicle-following applications, although the number of vehicles in every platoon (electronically interconnected system of vehicles) will be finite, it is necessary that the stability of the platoon be independent of the size of the platoon to prevent the saturation of the input actuators. Another interesting feature about the string stability of an interconnection of exponentially stable systems is that it is preserved under small structural perturbations. Consider

$$\dot{x}_i = f(x_i, \cdots, x_{i-r+1}) + \epsilon f_p(x_i, \cdots, x_{i-r+1}).$$

Assume $f_p(0, \cdots, 0) = 0$ and $\|f_p(p_1, \cdots, p_r) - f_p(q_1, \cdots, q_r)\| \le \sum_{j=1}^r l_{fj}\|p_j - q_j\|$. From Theorem 1, the interconnection of the perturbed systems is string stable if

$$(\alpha_1 - \alpha_3\epsilon l_{f1}) - \sum_{j=2}^r \frac{\alpha_3(l_j + \epsilon l_{fj})}{2} > \frac{\alpha_h}{\alpha_l}\sum_{j=2}^r \frac{\alpha_3(l_j + \epsilon l_{fj})}{2}.$$

This condition is satisfied when

$$\epsilon < \frac{\alpha_1 - \frac{\alpha_3(\alpha_l + \alpha_h)}{2\alpha_l}\sum_{j=2}^r l_j}{\alpha_3 l_{f1} + \frac{\alpha_3(\alpha_l + \alpha_h)}{2\alpha_l}\sum_{j=2}^r l_{fj}}.$$

This concludes the proof that string stability is robust to small structural perturbations.

Remark (Weak Coupling for l_2 string stability): Consider the following interconnected system in which every subsystem is connected only to its neighboring subsystems:

$$\dot{x}_i = f(x_{i-1}, x_i, x_{i+1}) \qquad i \in \mathcal{N}.$$

As before, $x_i \equiv 0 \;\forall i \le 0$. If the following conditions are satisfied:

- f is globally Lipschitz in its arguments, i.e.,

$$\begin{aligned}&|f(x_1, x_2, x_3) - f(y_1, y_2, y_3)| \\ &\qquad \le l_1|x_1 - y_1| + l_2|x_2 - y_2| + l_3|x_3 - y_3|.\end{aligned}$$

- The origin of $\dot{x} = f(0, x, 0)$ is globally exponentially stable.

Then for sufficiently small $l_1 + l_3$, the interconnected system is globally exponentially l_2 string stable.

Proof: Since the origin of $\dot{x} = f(0, x, 0)$ is exponentially stable, by the converse Lyapunov theorem, there exists a Lyapunov function $V(x)$ such that

$$\begin{aligned}\alpha_l\|x\|^2 \le V(x) &\le \alpha_h\|x\|^2 \\ \frac{\partial V}{\partial x} f(0, x, 0) &\le -\alpha_1\|x\|^2 \\ \left\|\frac{\partial V}{\partial x}\right\| &\le \alpha_3\|x\|.\end{aligned}$$

For the sake of convenience we denote $V(x_i)$ by V_i. As in Theorem 1, we obtain

$$\dot{V}_i \le -\frac{\left(\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3)\right)}{\alpha_h}V_i + \frac{\alpha_3}{2\alpha_l}(l_1 V_{i-1} + l_3 V_{i+1}).$$

Define $V(d, t) = \sum_1^\infty d^{-i}V_i(t)$ for d sufficiently close to and greater than unity. Note that $V(d, 0)$ is defined if $\|x_i(0)\|_2$ is defined. Differentiating V, we obtain

$$\begin{aligned}\dot{V}(d, t) &= \sum_1^\infty d^{-i}\dot{V}_i \\ &\le -\left(\frac{\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3)}{\alpha_h} - \frac{\alpha_3}{2\alpha_l}(l_1 d^{-1} + l_3 d)\right)V.\end{aligned}$$

Define $P_2(d) = \frac{\left(\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3)\right)}{\alpha_h} - \frac{\alpha_3}{2\alpha_l}(l_1 d^{-1} + l_3 d)$. If $l_1 + l_3 < \frac{2\alpha_1\alpha_l}{\alpha_3(\alpha_l + \alpha_h)}$, then $P_2(1) > 0$ and $P_2(d_0) < 0$ for all $d_0 \ge \frac{2\alpha_1\alpha_l}{\alpha_h\alpha_3 l_3}$. By the intermediate value theorem, there exists a $d^* > 1$ such that $P_2(d^*) = 0$ and $P_2(d) > 0$ for all

$1 < d < d^*$. As a result

$$\dot{V}(d, t) \leq -P_2(d)V(d, t) \leq 0 \qquad 1 < d < d^*$$

which implies that

$$V(d, t) \leq V(d, 0)e^{-P_2(d)t} \leq V(d, 0).$$

This guarantees exponential l_2-string stability of the interconnected system. d^* is a performance measure associated with this interconnected system. For a general interconnected system, only l_2-string stability can be guaranteed.

It is desirable that the string-stability property be preserved in the presence of parasitic actuator dynamics. In the next section, we present the conditions which guarantee string stability of the origin of the interconnected system in the presence of such parasitic actuator dynamics. From here on, we consider "look-ahead or lower-triangular systems" only, and therefore the results would apply for l_∞-string stability. The results for lower-triangular systems that follow can easily be extended to l_2-string stability of general interconnected systems.

III. String Stability of Singularly Perturbed Interconnected Systems

Before proceeding to study the string stability of the interconnected system, we present a result on the stability of a singularly perturbed system from [13].

Theorem 2 (Robustness of Exponentially Stable Nonlinear Systems to Singular Perturbations): Consider the autonomous singularly perturbed system

$$\dot{x} = f_1(x, z) \tag{8}$$

$$\epsilon\dot{z} = g_1(x, z) \tag{9}$$

where $x \in \mathcal{R}^n$, $z \in \mathcal{R}^m$ and assume that the origin is an isolated equilibrium point and the functions f_1 and $_1$ are locally Lipschitz in an open connected set that contains the origin. Let $z = h_1(x)$ be an isolated root of $0 = g(x, z)$, such that $h_1(0) = 0$. Let $y = z - h_1(x)$. If the following conditions are satisfied:

- The reduced system is exponentially stable, i.e., there exists positive constants α_l, α_h, α_1, α_3 and a Lyapunov function $V(x)$ such that

$$\alpha_1\|x\|^2 \leq V(x) \leq \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial x} f_1(x, h_1(x)) \leq \alpha_1\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_3\|x\|.$$

- The boundary layer system is exponentially stable, uniformly for frozen x, i.e., there exists positive constants β_l, β_h, α_2, α_1, and a Lyapunov function $W(x, y)$ such that

$$\beta_l\|y\|^2 \leq W(x, y) \leq \beta_h\|y\|^2$$
$$\frac{\partial W}{\partial y} g(x, y + h_1(x)) \leq -\alpha_2\|y\|^2$$
$$\left\|\frac{\partial W}{\partial (x, y)}\right\| \leq \alpha_4\|x \quad y\|.$$

- There exist positive constants, β_2 and γ such that

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h_1}{\partial x}\right] f_1(x, y + h_1(x)) \leq \beta_2\|x\|\,\|y\| + \gamma\|y\|^2.$$

Let $\epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$. Then the origin of the singularly perturbed system is exponentially stable for $0 < \epsilon < \epsilon^*$.

Proof: See Theorem 2.1 and [13, Corollary 2.2].

Intuitively, the origin of the perturbed interconnected system will be string stable if the origin of every perturbed subsystem is stable and the origin of the "reduced" interconnected system is string stable. This observation leads us to the following theorem.

Consider the following perturbed interconnected system:

$$\dot{x}_i = f(x_i, z_i, x_{i-1}, \cdots, x_{i-r+1}) \tag{10}$$

$$\epsilon\dot{z}_i = g(x_i, z_i) \qquad i \in \mathcal{N} \tag{11}$$

where

$$f: \mathcal{R}^n \times \mathcal{R}^m \times \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{(r-1)\text{ times}} \rightarrow \mathcal{R}^n$$

$g: \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}^m$. Let $f(0, \cdots, 0) = 0$, $g(0, 0) = 0$, and let $z_i = h(x_i, \cdots, x_{i-r+1})$ be an isolated root of $0 = g(x_i, z_i)$. Let $y_i = z_i - h(x_i)$, $h(0) = 0$, and f, g, and h be sufficiently smooth Lipschitz functions.

Theorem 3 (Robustness of Exponentially Stable Interconnected Systems to Singular Perturbations): If the following conditions are satisfied:

1) Let there exist a Lyapunov function, $V(x_i)$, such that

$$\alpha_l\|x_i\|^2 \leq V(x_i) \leq \alpha_h\|x_i\|^2$$
$$\frac{\partial V}{\partial x_i} f(x_i, h(x_i), x_{i-1}, \cdots, x_{i-r+1}) \leq -\alpha_1\|x_i\|^2 + \sum_{j=2}^{r} \alpha_{1j}\|x_{i-j+1}\|^2$$

with $\alpha_{1j} > 0$ and $\alpha_1 > \frac{\alpha_h}{\alpha_l}\sum_{j=2}^{r}\alpha_{1j}$

$$\left\|\frac{\partial V}{\partial x_i}\right\| \leq \alpha_3\|x_i\|.$$

These conditions imply the string stability of the interconnected of reduced (unperturbed) systems.

2) There exists a Lyapunov function $W(x_i, y_i)$ such that

$$\beta_l\|y_i\|^2 \leq W(x_i, y_i) \leq \beta_h\|y_i\|^2$$
$$\frac{\partial W}{\partial y_i} g(x_i, y_i + h(x_i)) \leq -\alpha_2\|y_i\|^2$$

$$\left(\frac{\partial W}{\partial x_i} - \frac{\partial W}{\partial y_i}\frac{\partial h}{\partial x_i}\right) f(x_i, y_i + h(x_i), \cdots, x_{i-r+1}) \leq \beta_2\|x_i\|\,\|y_i\| + \gamma\|y_i\|^2 + \sum_{j=2}^{r}\gamma_j\|x_{i-j+1}\|^2$$

with $\gamma_j > 0$. This condition implies the exponential stability of the singularly perturbed individual systems.

Then the singularly perturbed interconnected system is string stable.

Proof: If $\sum_{j=2}^{r} \alpha_{1j} \neq 0$, let $k = \min\left\{\frac{\sum_{j=2}^{r}\alpha_{1j}}{\sum_{j=2}^{r}\gamma_j}, \frac{\alpha_h}{\beta_h}\right\}$. Otherwise, let $k = \min\left\{\frac{\alpha_1}{\sum_{j=2}^{r}}, \frac{\alpha_h}{\beta_h}\right\}$. Define $\nu(x_i, y_i) = \frac{1}{2}(V(x_i) + kW(x_i, y_i))$. Using shorthand notation ν_i for $\nu(x_i, y_i)$, V_i for $V(x_i)$, and W_i for $W(x_i, y_i)$, there exists a β_1 such that

$$\frac{\partial V_i}{\partial x_i}[f(x_i, y_i + h(x_i), \cdots, x_{i-r+1}) - f(x_i, h(x_i), \cdots, x_{i-r+1})] \leq \beta_1 \|x_i\| \, \|y_i\|$$

$$\frac{\alpha_l \|x_i\|^2 + k\beta_l \|y_i\|^2}{2} \leq \nu_i \leq \frac{\alpha_h \|x_i\|^2 + k\beta_h \|y_i\|^2}{2}$$

$$\begin{aligned}\dot{\nu}_i &= \frac{1}{2}[\dot{V}_i + k\dot{W}_i] \\ &= \frac{1}{2}\left[-\alpha_1\|x_i\|^2 + \beta_1\|x_i\| \, \|y_i\| + \sum_{j=2}^{r}\alpha_{1j}\|x_{i-j+1}\|^2\right] \\ &\quad + \frac{k}{2}\left[-\frac{\alpha_2}{\epsilon}\|y_i\|^2 + \beta_2\|x_i\|\|y_i\| + \gamma\|y_i\|^2 + \sum_{j=2}^{r}\gamma_j\|x_{i-j+1}\|^2\right] \\ &\leq -\lambda(\epsilon)(\|x_i\|^2 + \|y_i\|^2) + \sum_{j=2}^{r}\frac{\alpha_{1j} + k\gamma_j}{2}\|x_{i-j+1}\|^2\end{aligned}$$

where the equation is shown at the bottom of the page. Since $\lambda(\epsilon)$ is a continuous function of ϵ, define

$$F(\epsilon) = \frac{2\lambda(\epsilon)}{\alpha_h} - \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j).$$

Since $k\sum_{j=2}^{r}\gamma_j < \sum_{j=2}^{r}\alpha_{1j}$ from Assumption 1, it follows that $F(0) > 0$ and $F\left(\frac{4\alpha_1\alpha_2 k}{4\alpha_1\gamma k + (\beta_1 + \beta_2 k)^2}\right) < 0$. By the intermediate value theorem, there exists ϵ_d such that $0 < \epsilon_d < \frac{4\alpha_1\alpha_2 k}{4\alpha_1\gamma k + (\beta_1 + \beta_2 k)^2}$ and $\forall\, 0 < \epsilon < \epsilon_d$, $F(\epsilon) > 0$. Therefore

$$\begin{aligned}\dot{\nu}_i &\leq -\lambda(\epsilon)(\|x_i\|^2 + \|y_i\|^2) + \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j)\frac{\|x_{i-j+1}\|^2}{2} \\ &\leq -\frac{2\lambda(\epsilon)}{\alpha_h}\nu_i + \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j)\nu_{i-j+1}.\end{aligned}$$

By an argument similar to that in Theorem 1, there exists a constant $K > 0$, such that $\|\nu_i(\cdot)\|_\infty \leq K\|\nu_i(0)\|_\infty$. This proves that the interconnection of singularly perturbed systems is string stable $\forall\, 0 < \epsilon < \epsilon_d$.

It also follows, by an argument similar to that in Theorem 1, that $\nu_i \to 0$ exponentially.

The above theorem justifies the use of control based on the reduced (unperturbed) system model.

IV. Adaptive Control of Interconnected Systems

Consider the following open-loop interconnected system:

$$\dot{\xi}_i = f_o(\xi_i, \xi_{i-1}, \cdots, \xi_{i-r+1}) + g(\xi_i)u_i$$

where $\xi_i \in \mathcal{R}^{p+q+1}$ and where p and q are positive integers. As assumed earlier, $\xi_j \equiv 0$ for all $j \leq 0$. f_o, g are smooth vector fields, $u_i \in \mathcal{R}$ and $i \in \mathcal{N}$. The output of the ith subsystem is $h_i = h(\xi_i)$ with $h_i \in \mathcal{R}$. The objective is to find a control such that the states of the closed-loop interconnected system are always bounded and go to the origin, $\xi_i = 0$, $i \in \mathcal{N}$, asymptotically.

The following assumptions are used for obtaining the control effort and analyzing the closed-loop behavior of the interconnected system:

- There exists a global diffeomorphism $z_i = \phi_1(\xi_i)$, $y_i = \phi_2(\xi_i)$ with $z_i \in \mathcal{R}^{p+1}$, $y_i \in \mathcal{R}^q$, and

$$\begin{aligned}\dot{z}_i^{(1)} &= z_i^{(2)} \\ \dot{z}_i^{(2)} &= z_i^{(3)} \\ &\vdots \\ \dot{z}_i^{(p)} &= z_i^{(p+1)} \\ \dot{z}_i^{(p+1)} &= \theta_f^T W_f(\xi_i, \xi_{i-1}, \cdots, \xi_{i-r+1}) + \theta_g^T W_g(\xi_i)u_i \\ \dot{y}_i &= \eta(z_i, y_i)\end{aligned}$$

  where $z_i^{(j)}$ is the jth component of the vector z_i and $z_i^{(1)} = h_i$. The above condition implies the adaptive linearizability of the open-loop system with a strict relative degree equal to $p + 1$. For details on adaptive linearizability, see Sastry and Isidori [22]. The vector fields, f_o, g are implicitly assumed to be linearly parameterizable in the constant parameters and will, henceforth, be represented by $\hat{\theta}_f$ and $\hat{\theta}_g$, respectively. Similarly, the parameter estimation errors are given by θ_f, and θ_g. y_i represents the state that will be rendered unobservable by an input–output linearizing control. In other words, the dynamics of y_i represent the internal dynamics of the ith system.
- (The origin of) $\dot{y}_i = \eta(0, y_i)$ is globally exponentially stable. This assumption states that the zero dynamics of every system in the open-loop interconnected system is exponentially stable. This assumption is required to establish that y_i is bounded uniformly in i when z_i is uniformly bounded in i. A more general form of internal dynamics that arises in such interconnected systems is of the following form:

$$\dot{y}_i = \eta(z_i, y_i, z_{i-1}, y_{i-1}, \cdots, z_{i-r+1}, y_{i-r+1}).$$

To analyze the closed-loop interconnected system with such an internal dynamics, additional weak coupling conditions similar to those in Theorem 1 (on the magnitude

---

$$\lambda(\epsilon) = \frac{4\alpha_1 k(\alpha_2 - \epsilon\gamma) - \epsilon(\beta_1 + \beta_2 k)^2}{4(\epsilon\alpha_1 + k(\alpha_2 - \epsilon\gamma) + \sqrt{(\epsilon\alpha_1 - k(\alpha_2 - \epsilon\gamma))^2 + \epsilon^2(\beta_1 + \beta_2 k)^2})}$$

of the Lipschitz constants associated with y arguments of η) have to be imposed to conclude that y_i is uniformly bounded in i if z_i are uniformly bounded in i. To keep the analysis simple, we will, however, not use this general form of internal dynamics.

- Every system avails the information of its state and the information of the states of "r" systems preceding it. This assumption is necessary to generate a feedback linearizing law which guarantees that the states of all the systems are bounded uniformly in i. The above assumption enables us to define S_i such that $S_i = 0$ describes the desired closed-loop (string stable) dynamics. For example, one could define

$$S_i = z_i^{(p)} + \delta_1 z_i^{(p-1)} + \cdots + \delta_p z_i^{(1)} - \delta_{p+1} z_{i-1}^{(1)}$$

where $s^p + \delta_1 s^{p-1} + \cdots + \delta_p$ is a Hurwitz polynomial with real roots. On the surface, $S_i = 0$, $\|z_i\|_\infty < \|z_{i-1}\|_\infty$ if δ_{p+1} is sufficiently small. In matrix form, S_i can be defined compactly as

$$\dot{x}_i = f_d(x_i, \cdots, x_{i-r+1}) + b_\lambda S_i$$

where $b_\lambda = [0, \cdots, 0, 1]^T$ and $x_i = [z_i^{(1)}, \cdots, z_i^{(p)}]^T$. f_d is a smooth vector field and it satisfies the weak coupling conditions described in Theorem 1 so that the dynamics on the surface $S_i = 0$ is string stable. Algebraically, S_i should be understood as

$$S_i = z_i^{(p)} + \psi_d(x_i, x_{i-1}, \cdots, x_{i-r+1}).$$

Here ψ_d is a smooth scalar function.

The control input u_i should be chosen to drive S_i to the surface $S_i = 0$. To obtain the control effort, differentiate S_i

$$\begin{aligned}\dot{S}_i &= z_i^{(p+1)} + \dot{\psi}_d(x_i, \cdots, x_{i-r+1}) \\ &= \theta_f^T W_f(\xi_i, \cdots, \xi_{i-r+1}) + \theta_g^T W_g(\xi_i) u_i \\ &\quad + \dot{\psi}_d(x_i, \cdots, x_{i-r+1}).\end{aligned}$$

Choose u_i such that

$$\begin{aligned}&\hat{\theta}_f^T W_f(\xi_i, \cdots, \xi_{i-r+1}) + \hat{\theta}_g^T W_g(\xi_i) u_i \\ &+ \dot{\psi}_d(x_i, \cdots, x_{i-r+1}) = -\lambda S_i.\end{aligned}$$

Obtaining control effort requires inversion of $\hat{\theta}_g W_g$ which may be singular. If it is known that $|\theta_g W_g| > C$ where C is a generic positive constant, projection algorithms could be employed to counter this problem.

As seen earlier, the closed-loop dynamics of any adaptively linearizable nonlinear systems with a coupling (interconnecting) control law can be cast in the following form:

$$\begin{aligned}\dot{x}_i &= f_d(x_i, x_{i-1}, \cdots, x_{i-r+1}) + b_\lambda S_i \\ \dot{S}_i &= -\lambda S_i + \tilde{\theta}_i^T W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1}) \\ \dot{y}_i &= \phi(x_i, y_i, S_i)\end{aligned} \tag{12}$$

where $b_\lambda = [0 \cdots 0\ 1]^T$, $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$ where $\hat{\theta}_i$ is the estimate of the parameter, and θ_i is the actual (constant) value of the parameter. From the first equation, $S_i = 0$ describes the desired closed-loop dynamics. The second equation describes the dynamics of S_i, and the third equation indicates the behavior of the internal dynamics associated with this system. Any adaptively linearizable nonlinear system with a coupling (interconnecting) control law yields this form of equations. To analyze the effect of parameter adaptation, we assume the following:

1) There exists a Lyapunov function $V(x_i)$ (for convenience, V_i), such that

$$\begin{aligned}&\alpha_l \|x_i\|^2 \le V_i \le \alpha_i \|x_i\|^2 \\ &\frac{\partial V_i}{\partial x_i} f_d(x_i, x_{i-1}, \cdots, x_{i-r+1}) \\ &\le -l_1 \|x_i\|^2 + \sum_{j=2}^{r} l_j \|x_{i-j+1}\|^2 \\ &\left\| \frac{\partial V_i}{\partial x_i} \right\| \le \alpha_1 \|x_i\|\end{aligned}$$

with $l_1 > \frac{\alpha_h}{\alpha_l} \sum_{j=2}^{r} l_j$.

2) There exists a Lyapunov function $W_z(y_i)$ (for convenience, W_i), such that

$$\begin{aligned}&\beta_l \|y_i\|^2 \le W_i \le \beta_h \|y_i\|^2 \\ &\frac{\partial W_i}{\partial y_i} \phi(x_i, S_i, y_i) \le -\alpha_2 \|y_i\|^2 + \alpha_3 \|y_i\| \, |S_i| \\ &\qquad + \alpha_4 \|y_i\| \, \|x_i\| \\ &\left\| \frac{\partial W_i}{\partial y_i} \right\| \le \alpha_5 \|y_i\|.\end{aligned}$$

We assume the exponentially stable behavior of the zero dynamics. Assumptions 1 and 2 enable the string stability of the interconnected system in the absence of parameter mismatch.

3) $W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1})$ is bounded for all its bounded arguments.

Theorem 4 (Effectiveness of Parameter Adaptation for Interconnected Systems): Under the above mentioned conditions, the following parameter adaptation law:

$$\dot{\hat{\theta}}_i = -\Gamma W(x_i, \cdots, x_{i-r+1}) S_i, \qquad \Gamma > 0$$

guarantees that for all bounded $\|x_i(0)\|_\infty$, $\|S_i(0)\|_\infty$, $\|\tilde{\theta}_i(0)\|_\infty$

- $\sup_i \|x_i(\cdot)\|_\infty$, $\sup_i \|S_i(\cdot)\|_\infty$, $\sup_i \|\tilde{\theta}_i(\cdot)\|_\infty$ are bounded;
- $x_i(t)$, $S_i(t) \to 0$ asymptotically for all i.

Proof: Let $V_{ai} = S_i^2 + \tilde{\theta}_i^T \Gamma^{-1} \tilde{\theta}_i$. Using the adaptation law

$$\begin{aligned}\dot{V}_{ai} &= -2\lambda S_i^2 \\ &\Rightarrow V_{ai}(t) \le V_{ai}(0) \le S_i^2(0) + \frac{\|\tilde{\theta}_i(0)\|^2}{\lambda_{\min}(\Gamma)}\end{aligned}$$

$$\sup_i \|S_i(\cdot)\|_\infty \le \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}.$$

Similarly

$$\sup_i \|\tilde{\theta}_i(\cdot)\|_\infty \le \sqrt{\lambda_{\max}(\Gamma)} \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}$$

and

$$\sup_i \int_0^\infty S_i^2\, dt = \sup_i \|S_i(\cdot)\|_2^2 \le \frac{V_{ai}(0)}{2\lambda} \le \frac{S_i^2(0) + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}{2\lambda}.$$

Calculating $\dot{V}_i$ along the trajectories of x_i

$$\dot{V}_i = \frac{\partial V_i}{\partial x_i}[f_d(x_i, \cdots, x_{i-r+1}) + b_\lambda S_i] \le -l_1\|x_i\|^2 + \sum_{j=2}^{r} l_j\|x_{i-j+1}\|^2 + \alpha_1\|x_i\|\,|S_i|.$$

Since $\sup_i \|S_i\|_\infty \le K$ where $K := \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}$

$$\dot{V}_i \le -l_1\|x_i\|^2 + \sum_{j=2}^{r} l_j\|x_{i-j+1}\|^2 + \alpha_1 K\|x_i\|.$$

Define $e_i = \sqrt{V_i}$. Then

$$\dot{e}_i \le -\frac{l_1}{2\alpha_h}e_i + \sum_2^r \frac{l_j}{2\alpha_l}e_{i-j+1} + \frac{\alpha_1}{2\alpha_l}|S_i|$$

$$\|e_i\|p \le \frac{\alpha_h}{l_1}\sum_2^r \frac{l_j}{\alpha_l}\|e_{i-j+1}\|_p + \frac{\alpha_h\alpha_1}{l_1\sqrt{\alpha_l}}\|S_i\|_p$$

where $p = 2, \infty$. Since $l_1 > \frac{\alpha_h}{\alpha_l}\sum_2^r l_j\|e_i\|_p \le M\|S_i\|_p$ where $M > 0$ is a constant. Since $\sup_i \{\|S_i\|_\infty, \|S_i\|_2\} <$ max $\left\{K, \sqrt{\frac{V_{ai}(0)}{2\lambda}}\right\} < \infty$, it follows that $\sup_i \{\|e_i\|_\infty, \|e_i\|_2\} < K_1$ for some positive K_1. This implies that $\sup_i\{\|x_i\|_\infty, \|x_i\|_2\} < \frac{K_1}{\sqrt{\alpha_l}}$. By Assumption 2 (that the zero dynamics of every individual system is minimum phase), $\sup_i \|y_i(\cdot)\|_\infty$ exists. By Assumption 3, $W(x_i, S_i, y_i, x_{i-1}, \cdots, x_{i-r+1})$ is bounded. Therefore, $S_i \in L_\infty$. Consequently, by Barbalat's lemma, $S_i \to 0$.

Observe that $\sup_i\|\dot{e}_i\|_\infty$ is bounded, since

$$\dot{e}_i \le -\frac{l_1}{2\alpha_h}e_i + \sum_2^r \frac{l_j}{2\alpha_l}e_{i-j+1} + \frac{\alpha_1}{2\alpha_l}|S_i|.$$

Since $\sup_i \{\|e_i\|_\infty, \|e_i\|_2\}$ are bounded, by Barbalat's lemma, $e_i \to 0$. Therefore, $V_i, x_i \to 0$.

Remarks:

1) In Assumption 1, $S_i \equiv 0$ yields the desired "string stable" dynamics.
2) Designing decentralized adaptive controllers for interconnected systems can be done in two steps:
   a) Identify the desired closed-loop (string stable) dynamics. Design a controller to achieve the desired closed-loop dynamics in the absence of parametric uncertainty.
   b) Use a gradient adaptation law to update the parameters.
3) The dynamics of S_i are usually given by

$$\dot{S}_i = -\lambda \operatorname{sign}(S_i) + \tilde{\theta}_i W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1})$$

then the following adaptation law should be used:

$$\dot{\hat{\theta}}_i = -\Gamma W(x_i, \cdots, x_{i-r+1}) \operatorname{sign}(S_i), \quad \Gamma > 0$$

to conclude that $\sup_i \|x_i\|_\infty$, $\sup_i\|S_i\|_\infty$, $\sup_i\|\tilde{\theta}\|_\infty$ are bounded and that $x_i(t), S_i(t) \to 0$ asymptotically for all i.

The proof of the above remark is similar to the proof of Theorem 4.

V. Example: Vehicle-Following Systems

For a good overview on vehicle-following systems, the readers are referred to [2], [3], [11], [8], and [23]. The longitudinal constant spacing vehicle-following controller designed by Hedrick [8] and used for parameter adaptation by Swaroop [28] will be considered here. A simple longitudinal vehicle dynamic model for the ith vehicle in the platoon is given by

$$\ddot{x}_i = \frac{u_i - c_i\dot{x}_i^2 - F_i}{M_i}$$

where x_i is the position of the ith vehicle in the platoon with respect to an inertial frame, u_i represents the propulsive/braking effort, $c_i\dot{x}_i^2$ is the aerodynamic drag force, and F_i is the tire drag acting on the ith vehicle. The control objectives are:

- $\epsilon_i(t)$ is defined as $x_i(t) - x_{i-1}(t) = L_i$, where L_i is the desired constant intervehicular spacing. $\epsilon_i(t)$ should go to zero asymptotically (exponentially) for every lead vehicle maneuver.
- String stability of the platoon should be guaranteed, i.e., given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\epsilon_i(0)\|_\infty < \delta \Rightarrow \sup_i\|\epsilon_i\|_\infty < \epsilon$.

In designing the controller [8], [28], it is assumed that the lead vehicle velocity and acceleration and lead vehicle relative position information can be communicated to every controlled vehicle. Define

$$S_i = \dot{\epsilon}_i + q_1\epsilon_i + q_3(v_i - v_l) + q_4\left(x_i - x_l + \sum_1^i L_j\right)$$

and u_i is chosen such that $\dot{S}_i + \lambda S_i = 0$ for some $\lambda > 0$. u_i is given by

$$u_i = c_i\dot{x}_i^2 + F_i + \frac{1}{1+q_3} \cdot [\ddot{x}_{i-1} + q_3\ddot{x}_l - q_1\dot{\epsilon}_i - q_4(v_i - v_l) - \lambda S_i].$$

The spacing error dynamics are given by

$$\ddot{\epsilon}_i + \left(\frac{q_1+q_4}{1+q_3} + \lambda\right)\dot{\epsilon}_i + \frac{\lambda(q_1+q_4)}{q_1+q_3} = \frac{1}{1+q_3}[\ddot{\epsilon}_{i-1} + (q_1+\lambda)\dot{\epsilon}_{i-1} + q_1\lambda\epsilon_{i-1}].$$

Let $z_i = \dot{\epsilon}_i + \lambda \epsilon_i$. The dynamics of the "$z$" interconnected system is

$$\dot{z}_i = -\frac{q_1 + q_4}{1 + q_3} z_i + \frac{1}{1 + q_3} \dot{z}_{i-1} + \frac{q_1}{1 + q_3} z_{i-1}.$$

Platoon string stability is guaranteed if the above interconnected system is string stable. The above interconnected system is in the same form as in (7). String stability is guaranteed if q_1, q_3, and q_4 are chosen appropriately.

The theorems developed in this paper are sufficient conditions to guarantee string stability for nonlinear systems. For linear systems, one could use input–output stability and some results of this paper to conclude about string stability. For example, Laplace transformation of the above equation yields

$$\hat{Z}_i(s) = \frac{s + q_1}{s + \frac{q_1 + q_4}{1 + q_3}} \hat{Z}_{i-1}(s).$$

If q_1, q_3, q_4 are chosen such that $q_1 q_3 > q_4$

$$\|z_i\|_\infty \le \frac{q_1}{q_1 + q_4} \|z_{i-1}\|_\infty + K_1 |z_i(0)| + K_2 |z_{i-1}(0)|$$

for some positive K_1, K_2. String stability and, consequently, exponential string stability follow immediately.

VI. Conclusions

In this paper we defined string stability, asymptotic, and exponential string stability, for countably infinite interconnection of nonlinear systems. We derived sufficient conditions to guarantee string stability for a class of interconnected systems and demonstrated their robustness to small singular/structural perturbations. The interconnections considered "look ahead" lower-triangular interconnected systems) or are "banded" (finite "look ahead" and "look back"). We presented parameter adaptation law (gradient type) to regulate the states of all systems and to ensure uniform boundedness of all the states and parameter estimation errors. Finally, we have illustrated the theory developed in this paper with a controller design for vehicle-following systems.

References

[1] E. Barbieri, "Stability analysis of a class of interconnected systems," *J. Dynamic Syst., Measurements, Contr.*, vol. 115, no. 3, pp. 546–551, Sept. 1993.

[2] R. E. Fenton and J. G. Bender, "A study of automatic car following," *IEEE Trans. Veh. Technol.*, vol. VT-18, no. 3, Nov. 1969.

[3] R. J. Caudill and W. L. Garrard, "Vehicle follower longitudinal control for automated transit vehicles," *J. Dynamic Syst., Measurements, Contr.*, vol. 99, no. 4, pp. 241–248, Dec. 1977.

[4] S. S. L. Chang, "Temporal stability of n-dimensional linear processors and its applications," *IEEE Trans. Circuits Syst.*, vol. CAS-27, no. 8, pp. 716–719, Apr. 1980.

[5] E. J. Davison and N. Tripathi, "The optimal decentralized control of a large power system: Load and frequency control," *IEEE Trans. Automat. Contr.*, vol. AC-23, no. 2, pp. 312–325, 1978.

[6] C. A. Desoer and M. Vidyasagar, *Feedback System: Input–Output Properties.* New York: Academic, 1975.

[7] M. L. El-Sayed and P. S. Krishnaprasad, "Homogenous interconnected systems: An example," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 4, pp. 894–901, Aug. 1981.

[8] J. K. Hedrick, D. H. McMahon, V. K. Narendran, and D. Swaroop, "Longitudinal vehicle controller design for IVHS systems," in *Proc. 1990 Amer. Contr. Conf.*, San Diego, CA.

[9] L. R. Hung, R. Su, and J. Myer, "Global transformations of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 1, pp. 24–31, 1983.

[10] A. Isidori, *Nonlinear Control Systems* New York: Springer-Verlag, 1989.

[11] K. C. Chu, "Decentralized control of high speed vehicle strings," *Transportation Res.*, pp. 361–383, June 1974.

[12] ——, "Optimal decentralized regulation for a string of coupled systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, no. 6, pp. 243–246, June 1974.

[13] P. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design.* New York: Academic, 1986.

[14] J. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Trans. Automat. Contr.*, vol. AC-11, no. 11, pp. 355–361, Nov. 1966.

[15] L. E. Peppard, "String stability of relative motion PID vehicle control systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, no. 3, pp. 529–531, Oct. 1974.

[16] A. N. Michel, "Stability analysis of interconnected systems," *SIAM J. Contr.*, vol. 12, no. 3, pp. 554–579, Aug. 1974.

[17] S. M. Melzer and B. C. Kuo, "Optimal regulations of systems described by a countably infinite number of objects," *Automatica*, vol. 7, pp. 359–366, 1971.

[18] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[19] U. Ozguner and E. Barbieri, "Decentralized control of a class of distributed parameter systems," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1985, pp. 932–935.

[20] W. Rudin, *Principles of Mathematical Analysis.* New York: McGraw-Hill, 1964.

[21] S. S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[22] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 11, pp. 1123–1131, Nov. 1989.

[23] S. Sheikholeslam and C. A. Desoer, "Control of interconnected systems: The platoon problem," *IEEE Trans. Automat. Contr.*, Feb. 1992.

[24] ——, "Indirect adaptive control of a class of interconnected nonlinear dynamical systems," *Int. J. Contr.*, vol. 57, no. 3, pp. 743–765, 1993.

[25] D. D. Siljak, "Stability of large-scale systems under structural perturbations," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-2, pp. 657–663, 1972.

[26] ——, "On stability of large-scale systems under structural perturbations," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-3, pp. 415–417, 1973.

[27] J. J. E. Slotine and W. Li, *Applied Nonlinear Control.* Englewood Cliffs, NJ: Prentice-Hall, 1991.

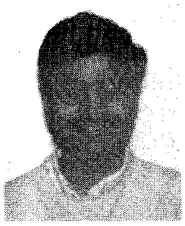
[28] D. Swaroop and J. K. Hedrick, "Direct adaptive longitudinal control of vehicle platoons," *Conf. Decision Contr.*, Orlando, FL, Dec. 1994.

[29] W. E. Thompson, "Exponential stability of interconnected systems," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 504–506, 1970.

[30] M. Vidyasagar, *Nonlinear Systems Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1978.

[31] ——, *"Input–output analysis of large-scale interconnected systems,"* vol. 29 of *Lecture Notes in Control and Information Sciences.* New York: Springer-Verlag, 1981.

[32] M. Vidyasagar and A. Vanelli, "New relationships between input–output and Lyapunov stability," *IEEE Trans. Automat. Contr.*, vol. 27, no. 2, pp. 481–483, April 1982.



D. Swaroop received the B.Tech degree from the Indian Institute of Technology, Madras, in 1989 and the M.S. and Ph.D. degrees from the University of California, Berkeley, in 1992 and 1994, respectively, all in mechanical engineering.

Since 1995 he has been a Visiting Postdoctoral Researcher for the University of California PATH program. His research interests include modeling, applied nonlinear and adaptive control theory, real-time control of mechanical systems, and vibrations.

Dr. Swaroop is a member of ASME.

J. K. Hedrick was a Professor of mechanical engineering at Massachusetts Institute of Technology from 1974–1988. He is currently the Director of the Vehicle Dynamics Laboratory in the Department of Mechanical Engineering at the University of California, Berkeley. His research interests include the development of advanced control theory and its application to a broad variety of transportation systems including high-speed ground vehicles, passenger and freight rail vehicles, automobiles, heavy trucks, and aircraft. He has consulted for many industries in the transportation area including problems in design, vibration, isolation, and electronic controller design. He has authored over 60 publications and edited several texts.

Dr. Hedrick has served on many national committees including the Transportation Research Board, the American National Standards Institute, ISO, and the NCHRP. He is currently a member of the Board of Directors of the International Association of Vehicle Systems Dynamics and Editor of the *Vehicle System Dynamics Journal*. He is a member of SAE and a Fellow of the ASME where he has served as Chairman of the Dynamic System and Controls Division.