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String Stability of Interconnected Systems

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Abstract

In this paper we introduce the notion of string stability of a countably infinite interconnection of a class of nonlinear systems. Intuitively, string stability implies uniform boundedness of all the states of the interconnected system for all time if the initial states of the interconnected system are uniformly bounded. It is well known that the I/O gain of all the subsystems less than unity guarantees that the interconnected system is I/O stable. We derive sufficient ("weak coupling") conditions which guarantee asymptotic string stability of a class of interconnected systems. Under the same "weak coupling" conditions, string stable interconnected systems remain string stable in the presence of small structural/singular perturbations. In the presence of parameter mismatch, these "weak coupling" conditions ensure that the states of all the subsystems are all uniformly bounded when gradient based parameter adaptation law is used and that they go to zero asymptotically.

1. Introduction

Earlier research on interconnected systems focussed on vehicle following applications [8], [7], [4], [11], [5], control of distributed systems, (e.g. regulation of seismic cables, vibration control in beams etc.,) [3], [9], signal processing [1], power systems [2]. Loosely speaking, string stability of an interconnected system implies uniform boundedness of the state of all the systems. For example, in automated vehicle following applications, tracking (spacing) errors should not amplify upstream from vehicle to vehicle for safety. Although string stability was not precisely defined, Kuo and Melzer [8], Levine and Athans [7] were seeking optimal control solutions to the automated vehicle following problem. Chu defined string stability in the context of vehicle following [5]. In [1], Chang introduces a stronger version of stability for interconnected systems, namely, "γ -stability" for infinite interconnection of linear digital processors. Intuitively, "γ-stability" ensures that the state of all the systems decays to zero exponentially in time and system index. In this paper, we generalize the concept of string stability to a class of interconnected systems and seek sufficient conditions to guarantee their string stability. We also examine their robustness to structural and singular perturbations.

This paper is organized as follows. In section 5.1, we define string stability and asymptotic string stability, we present "weak coupling" conditions that guarantee string stability for a class of interconnected systems and we demonstrate that exponential string stability is preserved under small structural perturbations. In section 5.2, we prove that every exponentially string stable interconnected system is string stable in the presence of small singular perturbations. In section 5.3, we discuss direct adaptive control of such interconnected systems.

2. String Stability

We use the following notations: $||f_i(\cdot)||_\infty$ or simply $||f_i||_\infty$ denotes $\sup_{t\geq 0} \ |f_i(t)|$ and $||f_i(0)||_\infty$ denotes $\sup_i \ |f_i(0)|$. For all $p < \infty$, $||f_i(\cdot)||_p$ or $||f_i||_p$ denotes $(\int_0^\infty |f_i(t)|^p dt)^{\frac{1}{p}}$ and $||f_i(0)||_p$ denotes $(\sum_1^\infty |f_i(0)|^p)^{\frac{1}{p}}$.

Consider the following interconnected system:

$$\dot{x}_i = f(x_i, x_{i-1}, \ldots, x_{i-r+1}) \qquad (1)$$

where $i \in \mathcal{N}, x_{i-j} \equiv 0 \ \forall \ i \leq j$, $x \in \mathcal{R}^n$, $f : \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{rtimes} \rightarrow \mathcal{R}^n$ and $f(0, \cdots, 0) = 0$.

Definition 1: The origin $x_i = 0, \quad i \in \mathcal{N}$ of the interconnected system 1 is string stable, if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $||x_i(0)||_\infty < \delta \Rightarrow \sup_i \ ||x_i(\cdot)||_\infty < \epsilon$.

Definition 2: The origin $x_i = 0, \quad i \in \mathcal{N}$ of the interconnected system 1 is asymptotically (exponentially) string stable, if it is string stable and $x_i(t) \longrightarrow 0$ asymptotically (exponentially) for all $i \in \mathcal{N}$, forall $||x_i(0)||_\infty$

Definition 3 (l_p string stability): The origin $x_i = 0, \quad i \in \mathcal{N}$ of the interconnected system 1 is l_p

string stable if for all $\epsilon > 0$, there exists a δ such that

$$||x_i(0)||_p < \delta \iff \sup_t \; ||x_i(t)||_p < \epsilon$$

where

$$||x_i(t)||_p = (\sum_1^{\infty} |x_i(t)|^p)^{\frac{1}{p}}$$

Henceforth, we will deal with string stability according to definition 1. The following theorem proves, under some "weak coupling" conditions, that any countably infinite interconnection of exponentially stable nonlinear systems is string stable. Clearly, a string of uncoupled exponentially stable systems is exponentially string stable. Intuitively, any interconnection of exponentially stable systems is string stable, if the interconnections are sufficiently weak.

Lemma 1: Let r be a constant positive integer. Define $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$, $\beta_j > 0$. If $\sum_1^r \beta_j < 1$, the $r-th$ degree polynomial $P_r(z)$ has all its roots inside the unit circle.

Lemma 2: Let $V_i(t) \geq 0 \;\; \forall t \geq 0, \;\; i \in \mathcal{N}$ and if

$$\dot{V}_i \leq -\beta_0 V_i + \sum_1^r \beta_j V_{i-j}$$

with $\beta_j > 0, \;\; j = 1, 2, \ldots, r$. For all $j \leq 0$, V_j should be read as 0. Then,

$$||V_i(\cdot)||_\infty \leq V_i(0) + \sum_1^r \frac{\beta_j}{\beta_0} ||V_{i-j}(\cdot)||_\infty$$

Lemma 3: Let $V_i(t) \geq 0, \;\; \forall t \geq 0, \;\; i \in \mathcal{N}$ and if $\beta_j > 0$ for all j and if

- $\dot{V}_i \leq -\beta_0 V_i + \sum_1^r \beta_j V_{i-j}$
- $\beta_0 > \sum_1^r \beta_j$

then, given any $\epsilon > 0$, $\exists \delta > 0$ such that

$$\sup_i V_i(0) < \delta \Rightarrow \sup_i ||V_i(\cdot)||_\infty < \epsilon$$

Proofs of Lemmas 1, 2 and 3 can be found in [13].

Theorem 1 (Weak Coupling Theorem for String Stability): If the following conditions are satisfied:

- f is globally Lipschitz in its arguments, i.e

$$|f(y_1, \ldots, y_r) - f(z_1, \ldots, z_r)| \leq l_1|y_1 - z_1| + \cdots + l_r|y_r - z_r|$$

- the origin of $\dot{x} = f(x, 0, \ldots, 0)$ is globally exponentially stable.

Then, for sufficiently small $l_i, i = 2, \ldots, r$, the interconnected system is globally exponentially string stable.

Proof: Since the origin of $\dot{x} = f(x, 0, \ldots, 0)$ is exponentially stable, by Converse Lyapunov Theorem, there exists a Lyapunov function $V(x)$ and four positive constants $\alpha_l, \alpha_h, \alpha_1, \alpha_3$ such that

$$\alpha_l ||x||^2 \leq V(x) \leq \alpha_h ||x||^2 \quad (2)$$

$$\frac{\partial V}{\partial x} f(x, 0, \ldots, 0) \leq -\alpha_1 ||x||^2 \quad (3)$$

$$||\frac{\partial V}{\partial x}|| \leq \alpha_3 ||x|| \quad (4)$$

For the sake of convenience, we denote $V(x_i)$ by V_i. Then,

$$\dot{V}_i = \frac{\partial V_i}{\partial x_i} f(x_i, x_{i-1}, \ldots, x_{i-r+1}) = \frac{\partial V_i}{\partial x_i} f(x_i, 0, \ldots, 0)$$
$$+ \frac{\partial V_i}{\partial x_i}[f(x_i, x_{i-1}, \ldots, x_{i-r+1}) - f(x_i, 0, \ldots, 0)]$$
$$\leq -\alpha_1 ||x_i||^2 + \alpha_3 ||x_i|| (\sum_{j=2}^r l_j ||x_{i-j+1}||)$$

Using the inequality that $xy \leq \frac{x^2+y^2}{2}$, the above equation results in

$$\dot{V}_i \leq -\frac{(\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j)}{\alpha_h} V_i + \frac{\alpha_3}{2\alpha_l} \sum_{j=2}^r l_j V_{i-j+1} \quad (5)$$

If $\sum_{j=2}^r l_j$ is sufficiently small such that $\sum_{j=2}^r l_j < \frac{2\alpha_l \alpha_1}{\alpha_3(\alpha_l + \alpha_h)}$, then $\frac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h} > \frac{\alpha_3}{2\alpha_l}\sum_{j=2}^r l_j > 0$. Consequently, string stability follows from Lemmas 1, 2 and 3.

Let $d > 1$. Define $V(d^{-1}, t) = \sum_{j=1}^{\infty} V_i(t) d^{-i}$. Clearly, $V(d^{-1}, t)$ is defined whenever the weak coupling conditions are satisfied and whenever $||x_i(0)||_\infty$ exists.

$$\dot{V} = \sum_{j=1}^{\infty} \dot{V}_i(t) d^{-i} \leq -V d^{-(r-1)} P_r(d) \frac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h}$$

Here $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$ where $\beta_j = \frac{\alpha_3}{2\alpha_l} \frac{\alpha_h}{\alpha_1 - \frac{\alpha_3}{2}\sum l_j} l_j$. Clearly, $P_r(d) > 0$ whenever $d > 1 > \rho(P_r(z))$, the spectral radius of the polynomial $P_r(z)$. $V \to 0$ exponentially and hence, $V_i(t), x_i(t) \to 0$ exponentially.

An interesting feature about the string stability of an interconnection of exponentially stable systems is that it is preserved under small structural perturbations. It is desirable that the string stability property be preserved in the presence of parasitic actuator dynamics. In the next section, we present the conditions which guarantee string stability of the origin of the interconnected system in the presence of such parasitic actuator dynamics.

3. String Stability Of Singularly Perturbed Interconnected Systems

Before proceeding to study the string stability of the interconnected system, we present a result on the stability of a singularly perturbed system from [6].

Theorem 2 (Robustness of Exponentially Stable Nonlinear Systems to Singular Perturbations): Consider the autonomous singularly perturbed system

$$\dot{x} = f_1(x, z) \quad (6)$$

$$\epsilon \dot{z} = g_1(x, z) \quad (7)$$

where $x \in \mathcal{R}^n, z \in \mathcal{R}^m$ and assume that the origin is an isolated equilibrium point and the functions f_1 and g_1 are locally Lipschitz in an open connected set that contains the origin. Let $z = h_1(x)$ be an isolated root of $0 = g(x, z)$, such that $h_1(0) = 0$. Let $y = z - h_1(x)$. If the following conditions are satisfied

- The reduced system is exponentially stable, i.e there exists positive constants $\alpha_l, \alpha_h, \alpha_1, \alpha_3$ and a Lyapunov function $V(x)$ such that

$$\alpha_1 ||x||^2 \leq V(x) \leq \alpha_h ||x||^2$$

$$\frac{\partial V}{\partial x} f_1(x, h_1(x)) \leq -\alpha_1 ||x||^2$$

$$||\frac{\partial V}{\partial x}|| \leq \alpha_3 ||x||$$

- The boundary layer system is exponentially stable, uniformly for frozen x, i.e there exists positive constants $\beta_l, \beta_h, \alpha_2, \alpha_4$ and a Lyapunov function $W(x, y)$ such that

$$\beta_l ||y||^2 \leq W(x, y) \leq \beta_h ||y||^2$$

$$\frac{\partial W}{\partial y} g(x, y + h_1(x)) \leq -\alpha_2 ||y||^2$$

$$||\frac{\partial W}{\partial (x, y)}|| \leq \alpha_4 ||x \quad y||$$

- There exist positive constants, β_2 and γ such that

$$[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} \frac{\partial h_1}{\partial x}] f_1(x, y + h_1(x)) \leq \beta_2 ||x|| ||y|| + \gamma ||y||^2$$

Let $\epsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2}$. Then the origin of the singularly perturbed system is exponentially stable for $0 < \epsilon < \epsilon^*$.

Proof: See Theorem 2.1 and Corollary 2.2 of [6]

Intuitively, origin of the perturbed interconnected system will be string stable if origin of every perturbed subsystem is stable and the origin of the "reduced" interconnected system is string stable. This observation leads us to the following theorem.

Consider the following perturbed interconnected system:

$$\dot{x}_i = f(x_i, z_i, x_{i-1}, \ldots, x_{i-r+1}) \quad (8)$$

$$\epsilon \dot{z}_i = g(x_i, z_i) \qquad i \in \mathcal{N} \quad (9)$$

where $f : \mathcal{R}^n \times \mathcal{R}^m \times \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{(r-1) times} \rightarrow \mathcal{R}^n$, $g : \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}^m$. Let $f(0, \ldots, 0) = 0; g(0, 0) = 0$ and let $z_i = h(x_i, \ldots, x_{i-r+1})$ be an isolated root of $0 = g(x_i, z_i)$. Let $y_i = z_i - h(x_i)$ and let $h(0) = 0$, and f, g, h be sufficiently smooth Lipschitz functions.

Theorem 3 (Robustness of Exponentially stable Interconnected Systems to Singular Perturbations): If the following conditions are satisfied:

1. Let there exist a Lyapunov function, $V(x_i)$, such that

$$\alpha_l ||x_i||^2 \leq V(x_i) \leq \alpha_h ||x_i||^2$$

$$\frac{\partial V}{\partial x_i} f(x_i, h(x_i), x_{i-1}, \ldots, x_{i-r+1}) \leq -\alpha_1 ||x_i||^2 + \sum_{j=2}^{r} \alpha_{1j} ||x_{i-j+1}||^2$$

   with $\alpha_{1j} > 0$ and $\alpha_1 > \frac{\alpha_h}{\alpha_l} \sum_{j=2}^{r} \alpha_{1j}$.

$$||\frac{\partial V}{\partial x_i}|| \leq \alpha_3 ||x_i||$$

   These conditions imply the string stability of the interconnected of reduced(unperturbed) systems.

2. There exists a Lyapunov function $W(x_i, y_i)$ such that

$$\beta_l ||y_i||^2 \leq W(x_i, y_i) \leq \beta_h ||y_i||^2$$

$$\frac{\partial W}{\partial y_i} g(x_i, y_i + h(x_i)) \leq -\alpha_2 ||y_i||^2$$

$$(\frac{\partial W}{\partial x_i} - \frac{\partial W}{\partial y_i} \frac{\partial h}{\partial x_i}) f(x_i, y_i + h(x_i), \ldots, x_{i-r+1}) \leq \beta_2 ||x_i + \sum_{j=2}^{r} \gamma_j ||x_{i-j+1}||^2$$

   with $\gamma_j > 0$. This condition implies the exponential stability of the singularly perturbed individual systems.

Then, the singularly perturbed interconnected system is string stable.

Proof: This theorem is proved using a composite Lyapunov function for each subsystem of the form $\frac{1}{2}(V(x_i) + kW(x_i, y_i))$ where k is a positive constant and the composite Lyapunov functions satisfy the conditions of Theorem 1 for sufficiently small singular perturbations. For details, see [13]. This theorem justifies the use of control based on reduced(unperturbed) system model.

4. Adaptive Control of Interconnected Systems

Under the assumption of adaptive linearizability of each subsystem, the closed loop dynamics of the interconnection of nonlinear subsystems with a coupling (interconnecting) control law can be cast in the following form: (For details, see [13]).

$$\dot{x}_i = f_d(x_i, x_{i-1}, \ldots, x_{i-r+1}) + b_\lambda S_i$$

$$\dot{S}_i = -\lambda S_i + \tilde{\theta}_i^T W(x_i, y_i, S_i, x_{i-1}, \ldots, x_{i-r+1})$$

$$\dot{y}_i = \phi(x_i, y_i, S_i) \qquad (10)$$

where $b_\lambda = [0 \cdots 0 \ \ 1]^T, \tilde{\theta}_i = \hat{\theta}_i - \theta_i$ where $\hat{\theta}_i$ is the estimate of the parameter and θ_i is the actual (constant) value of the parameter. From the first equation, $S_i = 0$ describes the desired closed loop dynamics. The second equation describes the dynamics of S_i and the third equation indicates the behavior of the internal dynamics associated with this system. Any adaptively linearizable nonlinear system with a coupling (interconnecting) control law yields this form of equations. In order to analyze the effect of parameter adaptation, we assume the following :

1. There exists a Lyapunov function $V(x_i)$ (for convenience, V_i), such that

$$\alpha_l ||x_i||^2 \leq V_i \leq \alpha_i ||x_i||^2$$

$$\frac{\partial V_i}{\partial x_i} f_d(x_i, x_{i-1}, \ldots, x_{i-r+1}) \leq -l_1 ||x_i||^2 + \sum_{j=2}^{r} l_j ||x_{i-j+1}||^2$$

$$||\frac{\partial V_i}{\partial x_i}|| \leq \alpha_1 ||x_i||$$

with $l_1 > \frac{\alpha_h}{\alpha_l} \sum_{j=2}^{r} l_j$.

2. There exists a Lyapunov function $W_z(y_i)$ (for convenience, W_i) , such that

$$\beta_l ||y_i||^2 \leq W_i \leq \beta_h ||y_i||^2$$

$$\frac{\partial W_i}{\partial y_i} \phi(x_i, S_i, y_i) \leq -\alpha_2 ||y_i||^2 + \alpha_3 ||y_i|| |S_i| + \alpha_4 ||y_i|| ||x_i||$$

$$||\frac{\partial W_i}{\partial y_i}|| \leq \alpha_5 ||y_i||$$

We assume the exponentially stable behavior of the zero dynamics. Assumptions 1 and 2 enable the string stability of the interconnected system in the absence of parameter mismatch.

3. $W(x_i, y_i, S_i, x_{i-1}, \ldots, x_{i-r+1})$ is bounded for all its bounded arguments.

Theorem 4 (Effectiveness of Parameter Adaptation for Interconnected systems): Under the above mentioned conditions, the following parameter adaptation law

$$\dot{\hat{\theta}}_i = -\Gamma W(x_i, \ldots, x_{i-r+1}) S_i, \qquad \Gamma > 0$$

guarantees that for all bounded $||x_i(0)||_\infty, ||S_i(0)||_\infty, ||\tilde{\theta}_i(0)||_\infty$,

- $\sup_i \ ||x_i(\cdot)||_\infty, \sup_i \ ||S_i(\cdot)||_\infty, \sup_i \ ||\tilde{\theta}_i(\cdot)||_\infty$ are bounded.
- $x_i(t), S_i(t) \to 0$ asymptotically for all i.

Proof: Let $V_{ai} = S_i^2 + \tilde{\theta}_i^T \Gamma^{-1} \tilde{\theta}_i$. Using the adaptation law,

$$\dot{V}_{ai} = -2\lambda S_i^2$$

$$\Rightarrow V_{ai}(t) \leq V_{ai}(0) \leq S_i^2(0) + \frac{||\tilde{\theta}_i(0)||^2}{\lambda_{min}(\Gamma)}$$

$$\sup_i \ ||S_i(\cdot)||_\infty \leq \sqrt{||S_i(0)||_\infty^2 + \frac{||\tilde{\theta}_i(0)||_\infty^2}{\lambda_{min}(\Gamma)}}$$

Similarly,

$$\sup_i \ ||\tilde{\theta}_i(\cdot)||_\infty \leq \sqrt{\lambda_{max}(\Gamma)} \sqrt{||S_i(0)||_\infty^2 + \frac{||\tilde{\theta}_i(0)||_\infty^2}{\lambda_{min}(\Gamma)}}$$

and

$$\sup_i \int_0^\infty S_i^2 dt = \sup_i \ ||S_i(\cdot)||_2^2 \leq \frac{V_{ai}(0)}{2\lambda}$$

$$\leq \frac{S_i^2(0) + \frac{||\tilde{\theta}_i(0)||_\infty^2}{\lambda_{min}(\Gamma)}}{2\lambda}$$

Calculating $\dot{V}_i$ along the trajectories of x_i,

$$\dot{V}_i = \frac{\partial V_i}{\partial x_i} [f_d(x_i, \ldots, x_{i-r+1}) + b_\lambda S_i]$$

$$\leq -l_1 ||x_i||^2 + \sum_{j=2}^{r} l_j ||x_{i-j+1}||^2 + \alpha_1 ||x_i|| |S_i|$$

Since $\sup_i \ ||S_i||_\infty \leq K$ where $K := \sqrt{||S_i(0)||_\infty^2 + \frac{||\tilde{\theta}_i(0)||_\infty^2}{\lambda_{min}(\Gamma)}}$.

$$\dot{V}_i \leq -l_1 ||x_i||^2 + \sum_{j=2}^{r} l_j ||x_{i-j+1}||^2 + \alpha_1 K ||x_i||$$

Define $e_i = \sqrt{V_i}$. Then

$$\dot{e}_i \leq -\frac{l_1}{2\alpha_h} e_i + \sum_{2}^{r} \frac{l_j}{2\alpha_l} e_{i-j+1} + \frac{\alpha_1}{2\alpha_l} |S_i|$$

$$||e_i||_p \leq \frac{\alpha_h}{l_1} \sum_{2}^{r} \frac{l_j}{\alpha_l} ||e_{i-j+1}||_p + \frac{\alpha_h \alpha_1}{l_1 \sqrt{\alpha_l}} ||S_i||_p$$

where $p = 2, \infty$. Since $l_1 > \frac{\alpha_h}{\alpha_l} \sum_2^r l_j$, $||e_i||_p \leq M||S_i||_p$ where $M > 0$ is a constant. Since $\sup_i \{||S_i||_\infty, ||S_i||_2\} < \max\{K, \sqrt{\frac{V_{ai}(0)}{2\lambda}}\} < \infty$, it follows that $\sup_i \{||e_i||_\infty, ||e_i||_2\} < K_1$ for some positive K_1. This implies that $\sup_i \{||x_i||_\infty, ||x_i||_2\} < \frac{K_1}{\sqrt{\alpha_l}}$. By Assumption 2 (that the zero dynamics of every individual system is minimum phase), $\sup_i ||y_i(\cdot)||_\infty$ exists. By Assumption 3, $W(x_i, S_i, y_i, x_{i-1}, \ldots, x_{i-r+1})$ is bounded. Therefore, $\dot{S}_i \in L_\infty$. Consequently, by Barbalat's Lemma, $S_i \to 0$.

Observe that $\sup_i ||\dot{e}_i||_\infty$ is bounded, since

$$\dot{e}_i \leq -\frac{l_1}{2\alpha_h} e_i + \sum_2^r \frac{l_j}{2\alpha_l} e_{i-j+1} + \frac{\alpha_1}{2\alpha_l}|S_i|$$

Since $\sup_i \quad sup_i\{||e_i||_\infty, ||e_i||_2\}$ are bounded, by Barbalat's Lemma, $e_i \to 0$. Therefore, $V_i, x_i \to 0$.

Remarks:

1. In Assumption1, $S_i \equiv 0$ yields the desired "string stable" dynamics.
2. Designing decentralized adaptive controllers for interconnected systems can be done in two steps :
   (a) Identify the desired closed loop (string stable) dynamics. Design a controller to achieve the desired closed loop dynamics in the absence of parametric uncertainty.
   (b) Use a gradient adaptation law to update the parameters.
3. The dynamics of the sliding surface is usually given by

   $$\dot{S}_i = -\lambda sign(S_i) + \tilde{\theta}_i W(x_i, y_i, S_i, x_{i-1}, \ldots, x_{i-r+1})$$

   then the following adaptation law should be used

   $$\dot{\tilde{\theta}}_i = -\Gamma W(x_i, \ldots, x_{i-r+1}) sign(S_i), \quad \Gamma > 0$$

   to conclude that $\sup_i ||x_i||_\infty, \sup_i ||S_i||_\infty, \sup_i ||\tilde{\theta}_i||_\infty$ are bounded and that $x_i(t), S_i(t) \to 0$ asymptotically for all i.
   The proof of the above remark is similar to the proof of Theorem 4.

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