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Leader Velocity Tracking and String Stability in Homogeneous Vehicle Formations With a Constant Spacing Policy.

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Abstract— **For vehicle platoons, the leader following control structure is well known for being capable of achieving string stability. In this paper, the linear string case is modified so that each follower tracks the instantaneous velocity of the leader in addition to the position of its predecessor. We show that it is possible to achieve string stability, under some basic assumptions, with this approach. We also discuss some of the benefits of the use of this method such as lowered coordination requirements and simplified communication needs.**

I. Introduction

In recent decades, formation control of autonomous vehicles has received a lot of attention from researchers [1], [2], [3], [4], [5]. Moreover, the simple case of a 1-D platoon of linear vehicles has been studied extensively (see [6] and the references therein), considering diverse alternatives to achieve coordinated movement of the string.

A simple solution for the linear case is to equip every member of the formation with a compensator that stabilizes its position in closed loop, using as a reference the position of its predecessor on the string and a desired constant inter-vehicle spacing. The internal stability of the whole system is assured by the design of the compensator and the assumption of linear dynamics for each vehicle.

Several works have shown that the architecture depicted above suffers from a drawback known as "string instability" namely, the amplification of disturbances along the string as a response to a disturbance in a single vehicle. This problem occurs in homogeneous linear control whenever the vehicle model has two integrators and regardless of the chosen compensator parameters [6].

The term "string stability" has been defined in many different ways but in this work we will consider the approach used in [7], where string stability refers to the case where a particular transfer function has a frequency magnitude peak that is bounded independently of the string length. Many solutions to this highly undesirable phenomena, due to the obvious implications in safety and performance, have been given. In [3], [4] the concept of "time headway" is used to guarantee that the system is string stable by allowing an inter-vehicle spacing that grows with the velocity. Heterogeneous control, namely equipping each vehicle with possible varying dynamics with an index dependent (in the position within the string sense) controller, has been proposed to overcome the difficulty [8], [9], [10], unfortunately this only helps if control "bandwidths" are allowed to diverge (either to 0 or ∞) as the string length grows [7]. Additionally, "leader following" approaches [6], [11] obtain string stability by providing every follower with the position of the leader, at the cost of increasing network requirements. The novel approach of this work is based on this latter technique.

The leader following structure in [6], [8], [11], [12], requires that each vehicle has communication of a "global" (leader derived) position set point. This in turn implies that every vehicle has access to their relative position to the leader in the string. Once this is known, the position of the leader is followed by keeping a distance that depends on this indexing. In this paper a modification where the leader only broadcasts its velocity to the followers is discussed. Under some simple assumptions on the vehicle dynamics and compensator design we derive conditions under which string stability is achieved with this approach.

The remainder of this paper is organized as follows. Section II gives some preliminary assumptions on the problem to be studied. Section III presents two control structures, the leader-predecessor following and the novel modification with leader velocity tracking. In Section IV we develop expressions for the inter-vehicle error dynamics which provide the key step to the main result of this work, which is presented in section V. Numerical examples and further discussion of the main result are shown in section VI. Final remarks and future lines of work are presented in section VII.

II. Preliminaries

We consider a platoon of N vehicles that travel in a straight line. Each vehicle is assumed to be described by the same linear scalar model given by

$$H(s) = \frac{1}{s^2}\tilde{H}(s), \tag{1}$$

where $\tilde{H}(s)$ is a proper, stable and minimum phase transfer function which satisfies $\tilde{H}(0) \neq 0$. For any stabilizing controller $K(s)$ for $H(s)$ we have that [7]

$$||T(s)||_\infty > 1, \tag{2}$$

where $T(s) = K(s)H(s)/(1 + K(s)H(s))$.

If we let $\underline{X}(s) = [X_1(s) \quad \cdots \quad X_N(s)]^T$ denote the vector of Laplace transforms for the vehicle positions we have (omitting initial conditions)

$$X_i(s) = H(s)(U_i(s) + D_i(s)), \quad i = 2, \dots \quad N, \tag{3}$$

where $D_i(s)$ denotes the input disturbance to the i-th vehicle. Additionally we assume that the leader of the platoon has

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dynamics described by $X_1(s) = H(s)D_1(s)$, i.e. its position is only governed by the input $D_1(s)$.

III. CONTROL STRUCTURES

The control objective is to keep a tight formation, that is, to minimize the errors, $E_i^{(p)}(s) = X_{i-1}(s) - X_i(s) - \delta/s$, where δ is a desired constant inter-vehicle spacing. We consider two control policies.

A. Leader-predecessor following

For simplicity we assume that the cars are initially positioned in the desired formation. The leader following structure is implemented by the use of the inputs

$$\begin{aligned} U_i(s) =& K(s)(\eta_i E_i^{(p)}(s) + (1-\eta_i)E_i^{(l)}(s)) \\ =& K(s)(\eta_i X_{i-1}(s) + (1-\eta_i)X_1(s) - X_i(s)), \end{aligned} \quad (4)$$

where $K(s)$ is a compensator (assumed to be identical for each car) that stabilizes the model $H(s)$ and $\eta_i \in [0,1]$. The latter implies that each car takes the weighted average spacing error of its predecessor and leader to regulate its position. Drawbacks of this scheme include (i) the requirement for each follower to be aware of its own position within the string since the leader spacing errors are given by $E_i^{(l)}(s) = X_1(s) - X_i(s) + (i-1)\delta/s$; (ii) the requirement for each vehicle to have a high accuracy absolute position reference to be able to compute $X_1(s) - X_i(s)$.

B. Leader velocity tracking and predecessor following

The control structure that this work proposes drops the leader position knowledge for each vehicle, exchanging it for measurements of the position of its predecessor and the velocity of the leader. The input for each car takes the form

$$\begin{aligned} U_i(s) =& K_p(s)(X_{i-1}(s) - X_i(s)) + K_v(s)s \times \\ & [\eta_i(X_{i-1}(s) - X_i(s)) + (1-\eta_i)(X_1(s) - X_i(s))] \\ =& \tilde{K}(s)(\tilde{\eta}_i(s)X_{i-1}(s) + (1-\tilde{\eta}_i(s))X_1(s) - X_i(s)), \end{aligned} \quad (5)$$

where $\tilde{K}(s) = K_p(s) + sK_v(s)$ is a compensator that stabilizes the model $H(s)$ and $\tilde{\eta}_i(s) = (K_p(s) + \eta_i sK_v(s))/\tilde{K}(s)$ with $\eta_i \in [0,1]$. This choice tries to keep a tight inter-vehicle spacing and at the same time tries to track the leader velocity in every follower. The latter has a connection with time headway policies from the fact that the control input for each automobile is computed with the use of its own velocity. The control structures described in above share a similar mathematical description, differing only in the compensator and the dynamic weights of the errors. This can be seen in the last lines of (4) and (5).

IV. VEHICLE STRING DYNAMICS

Figure 1 describes a block diagram of the interconnection which yields

$$\underline{\mathrm{X}}(s) = (I - H(s)K(s)G(s))^{-1}H(s)\underline{\mathrm{D}}(s), \quad (6)$$

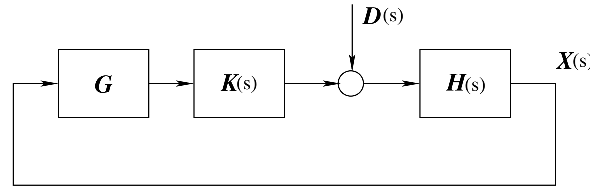

Fig. 1. Block diagram of the interconnection for the proposed control structures.

where I denotes the $N \times N$ identity matrix and $G(s)$ is given by

$$G(s) = \begin{bmatrix} 0 & & & & \\ 1 & -1 & & & \\ 1-\eta_3(s) & \eta_3(s) & -1 & & \\ \vdots & & \ddots & \ddots & \\ 1-\eta_N(s) & & \dots & \eta_N(s) & -1 \end{bmatrix}. \quad (7)$$

For the following derivations we will drop the argument of the transfer functions in order to simplify the exposition.

***Lemma** 1:* Let $T = KH/(1+KH)$ and $S = 1 - T$. Then the spacing errors defined by

$$\underline{\mathrm{E}} = \begin{bmatrix} E_2 \\ \vdots \\ E_N \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix} \underline{\mathrm{X}}, \quad (8)$$

are given by

$$\begin{aligned} \underline{\mathrm{E}} =& \begin{bmatrix} 1 \\ \mathcal{T}_2 \\ \vdots \\ \mathcal{T}_{N\text{-}1} \end{bmatrix} SHD_1 \\ &+ \begin{bmatrix} -1 & & & \\ \mathcal{S}_{2,1} & -1 & & \\ \vdots & \ddots & -1 & \\ \mathcal{S}_{N,1} & \dots & \mathcal{S}_{N\text{-}1,N\text{-}2} & -1 \end{bmatrix} SH \begin{bmatrix} D_2 \\ \vdots \\ D_N \end{bmatrix}. \end{aligned} \quad (9)$$

where $\mathcal{T}_k$ satisfies

$$\mathcal{T}_k = \eta_{k+1}T + (\eta_{k+1}T - 1)\sum_{i=3}^{k} T^{k-i+1} \prod_{j=i}^{k} \eta_j, \quad (10)$$

and $\mathcal{S}_{k,n}$ is given by

$$\mathcal{S}_{k,n} = (1 - \eta_{k+1}T)\left(\prod_{i=n+2}^{k} \eta_i\right) T^{k-n-1}, \quad (11)$$

for $n \le k-2$, and $\mathcal{S}_{k,n} = 1 - \eta_{k+1}T$ for $n = k-1$.

Proof: The result follows in a straightforward fashion after the computation of $(I - H(s)K(s)G(s))^{-1}$. ■

The previous lemma gives an expression for the inter-vehicle spacing dynamics when input disturbances occur on any element of the string. The following corollary gives the simplified expressions when the measurement weights η_i are equal.

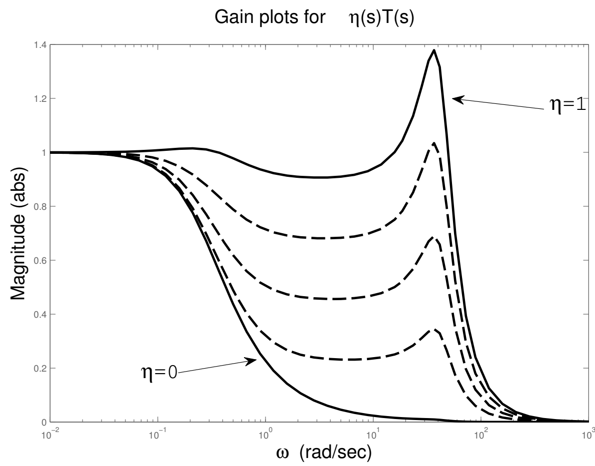

Fig. 2. Magnitude plots of $\tilde{\eta}(s)T(s)$ for $\alpha = 4$.

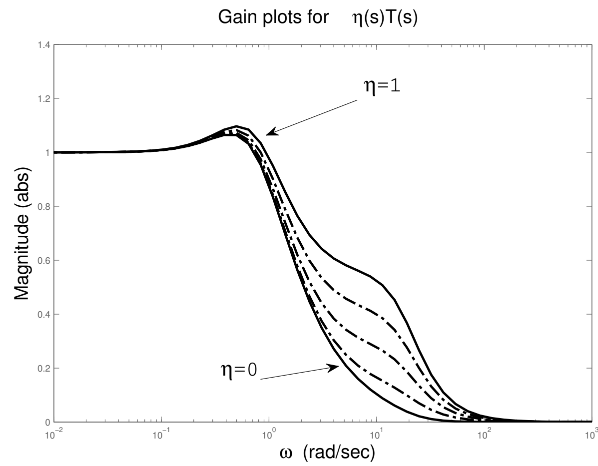

Fig. 3. Magnitude plots of $\tilde{\eta}(s)T(s)$ for $\alpha = 0.5$.

***Corollary** 1:* If $\eta_i = \eta$ for $i = 3 \dots N$, i.e. the string is homogeneous, then

$$\mathcal{T}_k = (\eta T)^{k-1}, \text{ and } \mathcal{S}_{k,n} = (1 - \eta T)(\eta T)^{k-n-1}. \quad (12)$$

V. String stability of the proposed structure

For the leader following structure we have that the string stability condition is given by [6]

$$||\eta T(s)||_\infty \leq 1 \Rightarrow \eta \leq ||T(s)||_\infty^{-1}. \quad (13)$$

This implies that the disturbance propagation along the string does not amplify independently of the number of vehicles on it. Since $1 < ||T(s)||_\infty < \infty$, there always exist $\eta \in [0, 1)$ such as the condition is satisfied.

Analogously, for the velocity tracking structure the string stability condition takes the form

$$||\tilde{\eta}(s)T(s)||_\infty \leq 1. \quad (14)$$

Moreover $T(0) = 1$ and from the definition $\tilde{\eta}(0) = 1$. Therefore the string stability condition becomes $||\tilde{\eta}(s)T(s)||_\infty = 1$. The interconnected system with velocity leader tracking will be string stable if

$$\left|\frac{K_p(j\omega) + j\omega\eta K_v(j\omega)}{\tilde{K}(j\omega)}\right| |T(j\omega)| \leq 1, \quad \forall\omega \quad (15)$$

and $\eta \in [0, 1)$. This condition is necessary for string stability. As the examples will show its not always possible to satisfy and it depends on the design of the compensator pair K_p and K_v, together with the plant dynamics.

VI. Numerical Examples and discussion

***Example** 1:* The most simple case arises when $\tilde{H}(s) = 1$, i.e. $H(s) = 1/s^2$. The string stability condition given in the previous section yields

$$\left|\frac{K_p(j\omega) + j\omega\eta K_v(j\omega)}{(j\omega)^2 + j\omega K_v(j\omega) + K_p(j\omega)}\right| < 1, \quad \forall\omega \quad (16)$$

Then, this condition will be satisfied if $K_p(s)$ and $K_v(s)$ are chosen appropriately. For constant controller gains $K_p(s) = k_p$ and $K_v(s) = k_v$ the condition becomes

$$\left|\frac{k_p + \eta k_v j\omega}{(j\omega)^2 + k_v j\omega + k_p}\right| < 1, \quad \forall\omega. \quad (17)$$

The magnitude peaks occur at $\omega = 0$ and $\omega_c = \sqrt{k_p - k_v^2/2}$. As noted before, $\tilde{\eta}(0)T(0) = 1$. If $\eta = 0$, which is the most favorable case, the magnitude peak at ω_c is

$$|\tilde{\eta}(j\omega_c)T(j\omega_c)| = 4\frac{u^2}{4u-1}, \quad (18)$$

with $u = k_p/k_v^2$. The condition for ω_c to be real and different from 0 is $u > 1/2$ which implies $4u^2/(4u - 1) > 1$. Therefore string stability can be achieved for $H(s) = s^{-2}$ with $K_p = k_p$ and $K_v = k_v$, $k_p, k_v \in \mathbb{R}$ when $k_p/k_v^2 \leq 1/2$ and $k_p, k_v > 0$ (for stability).

***Example** 2:* We consider the vehicle dynamics [6]

$$H(s) = \frac{1}{s^2(0.1s + 1)}, \quad (19)$$

and the compensator pair $K_p(s)$ and $K_v(s)$ given by

$$K_p(s) = \frac{2s + 1}{(0.05s + 1)}, \quad K_v(s) = \alpha K_p(s), \quad (20)$$

where $\alpha \in \mathbb{R}$. The stability of the system is guaranteed for any value of α. The string stability condition for this case becomes

$$|\tilde{\eta}(j\omega)T(j\omega)| = \left|\frac{1 + \eta\alpha j\omega}{1 + \alpha j\omega}T(j\omega)\right| \leq 1. \quad (21)$$

We are free to choose $\eta = 0$ (which is the case where each car tracks solely the velocity of the leader), which flattens the spectrum. Then we need the critical α to satisfy the condition. A simple computation yields that for $\alpha > 1.11$ we have $||T(s)/(1 + \alpha s)||_\infty = 1$. It is clear that for larger

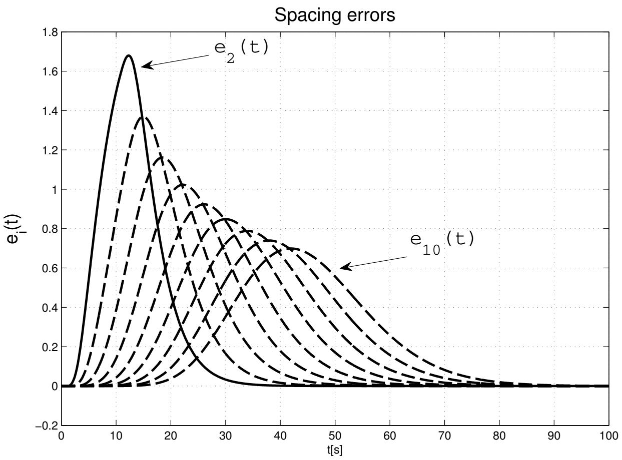

Fig. 4. Transient of the predecessor errors for leader movement when $\alpha = 4$ and $\eta = 0$.

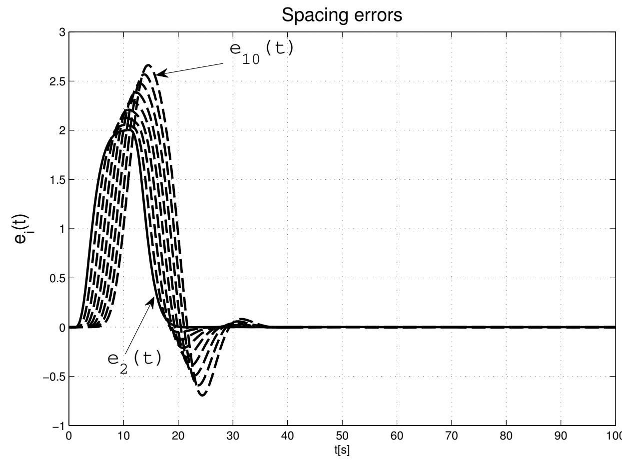

Fig. 5. Transient of the predecessor errors for leader movement when $\alpha = 0.5$ and $\eta = 0$.

values of α we are allowed to choose larger values of η. Moreover $\eta = 1$ always yields string instability due to the fact $||T(s)||_\infty > 1$.

Figures 2 and 3 show several magnitude plots of the product $\tilde{\eta}(s)T(s)$ for $\alpha = \{0.5, 4\}$. We see that for $\alpha = 4$ there are values of $\eta \in [0, \eta_c]$ where $||\tilde{\eta}(s)T(s)||_\infty = 1$ which is attained at $\omega_c = 0[rad/s]$. On the contrary, for $\alpha = 0.5$ we see on figure 3 that no value of η will achieve string stability. Thus, the design of the controllers $K_p(s)$ and $K_v(s)$ becomes paramount in the pursuit of string stability.

As noted above, this control structure presents some benefits when compared to the leader following structure. If a platoon of vehicles is moving at a constant speed, a new vehicle can join the formation by just following the last member of the string and tracking the velocity that is being broadcasted by the leader. No information about the number of cars in the string is needed. Another feature resides on the zero-frequency response of the string. An input disturbance to a follower of the string (i.e. any vehicle other than the leader) affects the errors of the vehicles behind it as

$$E_i = (1 - \tilde{\eta}(s)T(s))A_i(s)D_j(s), \quad j < i, \qquad (22)$$

where $A(0) \neq 0$. Hence, if $\tilde{\eta}(s)$ is static, a constant disturbance will have a non-zero effect in steady state. On the other hand, the dynamic selection of the weight defined in (5) satisfies $\tilde{\eta}(0) = 1$. Since the vehicle dynamics force $T(0) = 1$, the steady state error for constant disturbances in followers is 0.

Figures 4 and 5 depict the transient response of the string for $\alpha = 4$ and $\alpha = 0.5$ respectively, when $\eta = 0$, to the input

$$D_1(s) = \frac{1}{s^2}(e^{-s} - e^{-3s} - e^{-11s} + e^{-13s}), \qquad (23)$$

and $D_i(s) = 0$ for $i = 2, \ldots, N$.

As the string stability condition predicts, the first case exhibits string stability, whereas the second exhibits string instability even under the most favorable (from a string stability viewpoint) selection of η.

A final observation from these example is that in order to achieve string stability with this structure, a performance trade-off has to be done. The leader following scheme provides the possibility of scaling all the frequency content of the disturbances by a $0 < \eta < 1$ factor, independently of the chosen dynamics. The example suggest that we must have a slower transient response in order to achieve string stability. Additionally, when compared to the leader following structure [6], the leader velocity tracking structure delivers a worse performance as measured by the leader spacing error $E_i^{(l)}(s) = X_1(s) - X_i(s) + (i-1)\delta/s$. This is illustrated by figure 6, where the leader spacing error for the i-th vehicle is obtained as the sum of all the predecessor follower errors with less index $E_i^{(l)}(s) = \sum_{j \leq i} E_j^{(p)}(s)$.

VII. Conclusions and Future Works

In this paper we have analyzed a modification to the leader-following controller structure for the platooning problem of a linear string of vehicles. The main difference relies in the information communicated by the leader, namely its velocity, retaining the string stability capabilities of the original. This allows much simpler coordination requirements compared to the original method (each vehicle must know its position in the string). In addition, only constant disturbances in the leader yield a non-zero error in the string, whereas in the leader following structure, positional offsets have a ubiquitous impact in the string. Numerical examples have shown that some of the drawbacks of the structure are slower disturbance response and performance loss in the leader following sense.

Future works should be directed to more explicit string stability conditions and combinations of the current structure with other strategies that achieve string stability, such as time headway and heterogeneous controllers.

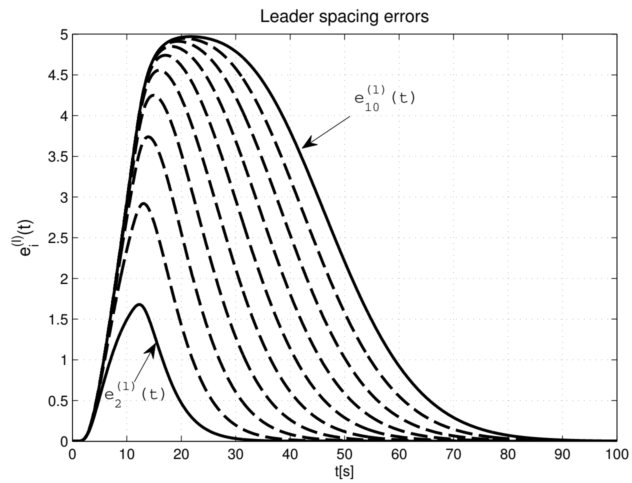

Fig. 6. Transient of the leader errors for $\alpha = 4$ and $\eta = 0$.

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