Problem 1

(A) $N=4$

$$a_K^H a_L = \begin{cases} 1 & K=L \\ 0 & \text{otherwise} \end{cases}$$

$K=(0,1)$
$L=(0,1)$

we need to show that:

$a_0^H a_0 = a_1^H a_1 = 1$

and

$a_0^H a_1 = a_1^H a_0 = 0$

a_K^H = conjugate transpose of a (a_K^H calculated using ctranspose() on MATLAB)

$$a_0^* = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad a_1^* = \frac{1}{2}\begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

$$a_0^H a_0 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 1$$

$$a_1^H a_1 = \frac{1}{2}\begin{bmatrix} 1 & -j & -1 & j \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{j}{2} \\ -\frac{1}{2} \\ -\frac{j}{2} \end{bmatrix}$$

$$= \frac{1}{4} - \frac{j^2}{4} + \frac{1}{4} - \frac{j^2}{4}$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

$$a_0^H a_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{j}{2} \\ -\frac{1}{2} \\ -\frac{j}{2} \end{bmatrix} = \frac{1}{4} + \frac{j}{4} - \frac{1}{4} - \frac{j}{4} = 0$$

$$a_1^H a_0 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{4} - \frac{j}{4} - \frac{1}{4} + \frac{j}{4} = 0$$

$a_0^H a_0 = a_1^H a_1 = 1 \qquad a_0^H a_1 = a_1^H a_0 = 0$

the relationship holds.

(b)

$$U = \sum_{K=0}^{N-1} V(K,L)\, A^*(K,L)$$

$$A^*(K,L) = a_K^* a_L^{*T}$$

$$A^*(0,0) = a_0^* \cdot a_0^H = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^*(2,2) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$A^*(3,3) = \frac{1}{4} \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} \begin{bmatrix} 1 & -j & -1 & j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -j & -1 & j \\ -j & -1 & j & 1 \\ -1 & j & 1 & -j \\ j & 1 & -j & -1 \end{bmatrix}$$

(c)

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = A^H V A^* = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

[MATLAB]

$$= \frac{1}{4} \cdot \begin{bmatrix} 3 & -j & 1 & j \\ -j & 1 & j & 3 \\ 1 & j & 3 & -j \\ j & 3 & -j & 1 \end{bmatrix}$$

1d) $$A(0,0) + A(2,2) + A(3,3) = \frac{1}{4}\left[\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -j & -1 & j \\ -j & -1 & j & 1 \\ -1 & j & 1 & -j \\ j & 1 & -j & -1 \end{bmatrix}\right]$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -j & 1 & j \\ -j & 1 & j & 3 \\ 1 & j & 3 & -j \\ j & 3 & -j & 1 \end{bmatrix}$$

Problem 2

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$V = C U C^T$ where $C = \begin{cases} \frac{1}{\sqrt{N}} & \text{for } K=0,\ 0 \le n \le N-1 \\ \sqrt{\frac{2}{N}} \cos\left(\frac{\pi(2n+1)K}{2N}\right), & 1 \le K \le N-1,\ 0 \le n \le N-1 \end{cases}$

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0.65 & 0.27 & -0.27 & -0.65 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0.27 & -0.65 & 0.65 & -0.27 \end{bmatrix} \qquad C^T = \begin{bmatrix} \frac{1}{2} & 0.65 & \frac{1}{2} & 0.27 \\ \frac{1}{2} & 0.27 & -\frac{1}{2} & -0.65 \\ \frac{1}{2} & -0.27 & -\frac{1}{2} & 0.65 \\ \frac{1}{2} & -0.65 & \frac{1}{2} & -0.27 \end{bmatrix}$$

Using MATLAB

```
C =

    0.5000    0.5000    0.5000    0.5000
    0.6500    0.2700   -0.2700   -0.6500
    0.5000   -0.5000   -0.5000    0.5000
    0.2700   -0.6500    0.6500   -0.2700


CT =

    0.5000    0.6500    0.5000    0.2700
    0.5000    0.2700   -0.5000   -0.6500
    0.5000   -0.2700   -0.5000    0.6500
    0.5000   -0.6500    0.5000   -0.2700


U =

     1     1     0     0
     0     1     0     0
     0     0     1     0
     0     0     1     1

>> untitled2

V =

    1.5000   -0.0000   -0.5000         0
         0    1.3418         0   -0.8450
    0.5000         0    0.5000         0
         0    0.1458         0    0.6398
```

$$V = \begin{bmatrix} 1.5 & 0 & -0.5 & 0 \\ 0 & 1.34 & 0 & -0.85 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.146 & 0 & 0.64 \end{bmatrix}$$

Problem 3

(A) $Q(K,L) = 1 + ((1+K+L)q)$ where $q = 4$

```
[1] import numpy as np
```

```
q = 4

Q = np.zeros((8,8))

for k in range(Q.shape[0]):
  for l in range(Q.shape[1]):
    Q[k][l] = 1+((1+k+l)*q)
print(Q)
```

```
[[ 5.  9. 13. 17. 21. 25. 29. 33.]
 [ 9. 13. 17. 21. 25. 29. 33. 37.]
 [13. 17. 21. 25. 29. 33. 37. 41.]
 [17. 21. 25. 29. 33. 37. 41. 45.]
 [21. 25. 29. 33. 37. 41. 45. 49.]
 [25. 29. 33. 37. 41. 45. 49. 53.]
 [29. 33. 37. 41. 45. 49. 53. 57.]
 [33. 37. 41. 45. 49. 53. 57. 61.]]
```

(B) $Y(K,L) = \text{round}\left(\frac{V(K,L)}{Q(K,L)}\right)$

```
V = np.array([[186,-18,15,-9,23,-9,-14,19],
              [21,-34,26,-9,-11,11,14,7],
              [-10,-24,-2,6,-18,3,-20,-1],
              [-8,-5,14,-15,-8,-3,-3,8],
              [-3,10,8,1,-11,18,18,15],
              [4,-2,-18,8,8,-4,1,-7],
              [9,1,-3,4,-1,-7,-1,-2],
              [0,-8,-2,2,1,4,-6,0]])
Y = np.round(V/Q)
print(Y)
```

```
[[37. -2.  1. -1.  1. -0. -0.  1.]
 [ 2. -3.  2. -0. -0.  0.  0.  0.]
 [-1. -1. -0.  0. -1.  0. -1. -0.]
 [-0. -0.  1. -1. -0. -0. -0.  0.]
 [-0.  0.  0.  0. -0.  0.  0.  0.]
 [ 0. -0. -1.  0.  0. -0.  0. -0.]
 [ 0.  0. -0.  0. -0. -0. -0. -0.]
 [ 0. -0. -0.  0.  0.  0. -0.  0.]]
```

(C)

```
Y_reorder = np.array([37,-2,2,-1,-3,1,-1,2,-1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,-1,-1,0,0,1,0,0,0,0,-1,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0])
print(Y_reorder)
```

```
[37 -2  2 -1 -3  1 -1  2 -1  0  0  0  0  0  1  0  0  0  1  0  0  0  0  0
 -1 -1  0  0  1  0  0  0  0 -1  0  0  0  0  0  0  0 -1  0  0  0  0  0  0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0]
```

(D) number between -128 and 127 is 1 byte
Sequence of zeros = 2 bytes
Smaller than -128 and larger than 127 = 3 bytes

count $= 1+1+1+1+1+1+1+1+1+1+2+1+2+1+2+1+1+2+1+2+1+2+1+2$
$= 30$

Compression factor $= \frac{Y}{U} = \frac{30}{64} = 46.9\% \approx 47\%$