UEC747: ANTENNA AND WAVE PROPAGATION

Jan-May 2021

Lecture 8: Vector Calculus

Dr Rajesh Khanna, Professor ECE

and

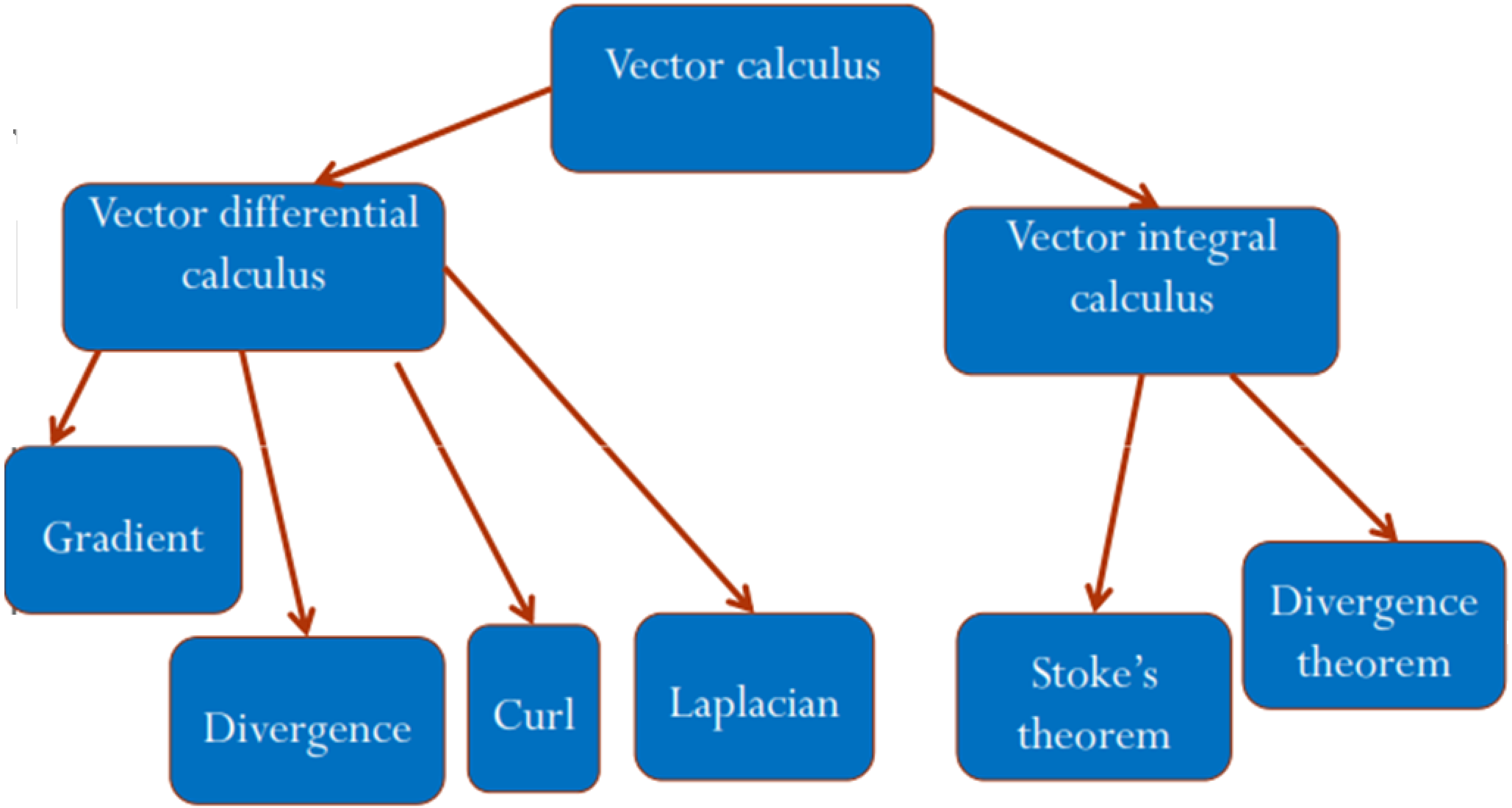
Dr Amanpreet Kaur, Assistant Professor, ECE

REVIEW OF VECTOR ANALYSIS

we will discuss basic mathematical operation like vector addition, subtraction, multiplication, vector integrals, vector differential operator and vector relation in other coordinates e.g. Cartesian, cylindrical and spherical. Knowledge of these will play a great role in the study of antenna theory etc. We will review following for operations

- Vector addition and subtraction
- Vector Multiplication (Dot and cross products)
- Vector Integral (Line and surface integrals)
- Vector Differentiation (Introduction to differential operators)

Vector Calculus



Vector differential calculus

- The vector differential operator ∇, called “del” or “nebla”, is defined in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

where $\mathbf{a}_x$.$\mathbf{a}_y$ and $\mathbf{a}_z$ are unit vectors in the directions of coordinates axes x,y, and z, respectively.

- Since a vector, in general, is a function of space and time both and ∇ operator is a vector space function operator.
- It is defined by means of partial derivates w.r.t. space which is a simply a method of keeping time fixed (if the vector is a function of time) and finding the behavior of a vector spatially in the region at that particular instant of time.

Using ∇ as multiplication Operator

Vector differential operator possesses properties similar to ordinary vector. Any Vector **A** can be differentiated by the same rules as is done in vector multiplication.

- 1. The gradient of a scalar V, written, as (∇V)
- 2. The divergence of a vector A, written as (∇ • **A**)
- 3. The curl of a vector A, written as (∇ X **A**)
- 4. The Laplacian of a scalar *V, written as* ($\nabla^2 V$)

∇ Operator in Cylindrical Coordinate

To obtain ∇ in terms of ρ, ϕ, and z, we recall from eq.

$$\rho = \sqrt{x^2 + y^2}, \qquad \tan \phi = \frac{y}{x}$$

$$\frac{\partial}{\partial x} = \cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin \phi \frac{\partial}{\partial \rho} + \frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\boxed{\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}}$$

∇ Operator in Spherical Coordinate

Similarly, to obtain ∇ in terms of r, θ, and ϕ, we use

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\phi = \frac{y}{x}$$

to obtain

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

So the solution for above equations is

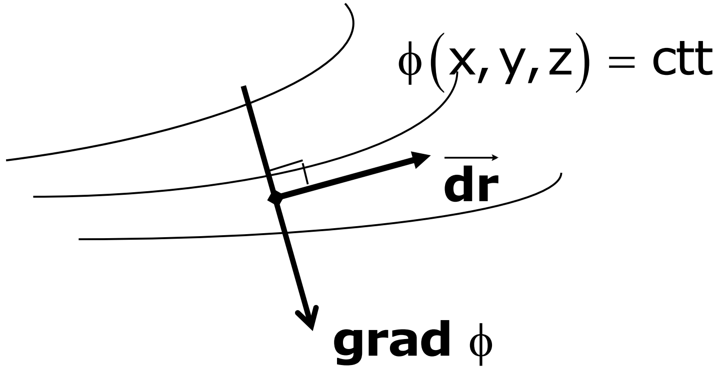
$$\boxed{\nabla = \mathbf{a}_r\frac{\partial}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}}$$

Gradient of a scalar V, written, as (∇V)

- The first type of multiplication is the multiplication of a vector by a scalar and the resultant is another vector. If the vector differential operator ∇ is operated with a scalar quantity say V (x, y , z), then this operation is known as Gradient and is abbreviated as Grad. Gradient of written as ∇V or grad V will be given by

$$\nabla \mathbf{V} = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Physical Interpretation of gradient



1- Definition. ϕ(x,y,z) is a differentiable scalar field

$$\nabla\phi = \frac{d\phi}{dx}\mathbf{a}_x + \frac{d\phi}{dy}\mathbf{a}_y + \frac{d\phi}{dz}\mathbf{a}_z$$

2 – Physical meaning: $\nabla\phi$ is a vector that represents both the magnitude and the direction of the maximum space rate of increase of Φ

- Let scalar function T represents temperature, and then ∇T or grad T will represent temperature gradient or space rate of change of temperature with distance. Thus although temperature is a scalar quantity having magnitude only and no direction, the temperature gradient ∇T is a vector quantity having magnitude and direction both in the direction of maximum rate of change of temperature .
- *The gradient may be defined as the vector that represents both the magnitude and the direction of the maximum spatial rate of change of a scalar function .It depends upon the position where the gradient is to be evaluated and it may have different magnitude and directions at different locations in space,*

Example 11

- Find the gradient of these scalars:

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = r^2 z \cos 2\phi$

(c) $W = 10R \sin^2 \theta \cos\phi$

Solution to Example 11

(a) Use gradient for Cartesian coordinate:

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$= 2e^{-z}\cos 2x \cosh y \mathbf{a}_x + e^{-z}\sin 2x \sinh y \mathbf{a}_y$$

$$- e^{-z}\sin 2x \cosh y \mathbf{a}_z$$

Solution to Example 11

(b) Use gradient for cylindrical coordinate:

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$

$$= 2rz\cos 2\phi\mathbf{a}_r - 2rz\sin 2\phi\mathbf{a}_\phi$$

$$+ r^2\cos 2\phi\mathbf{a}_z$$

Solution to Example 11

(c) Use gradient for Spherical coordinate:

$$\nabla W = \frac{\partial W}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\,\mathbf{a}_R + 10\sin 2\theta\cos\phi\,\mathbf{a}_\theta$$

$$-10\sin\theta\sin\phi\,\mathbf{a}_\phi$$

Divergence

The operator ∇ is of vector form, a scalar product can be obtained as :

$$\nabla \bullet A = \left(\frac{d}{dx}\mathbf{a}_x + \frac{d}{dy}\mathbf{a}_y + \frac{d}{dz}\mathbf{a}_z \right) \bullet (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

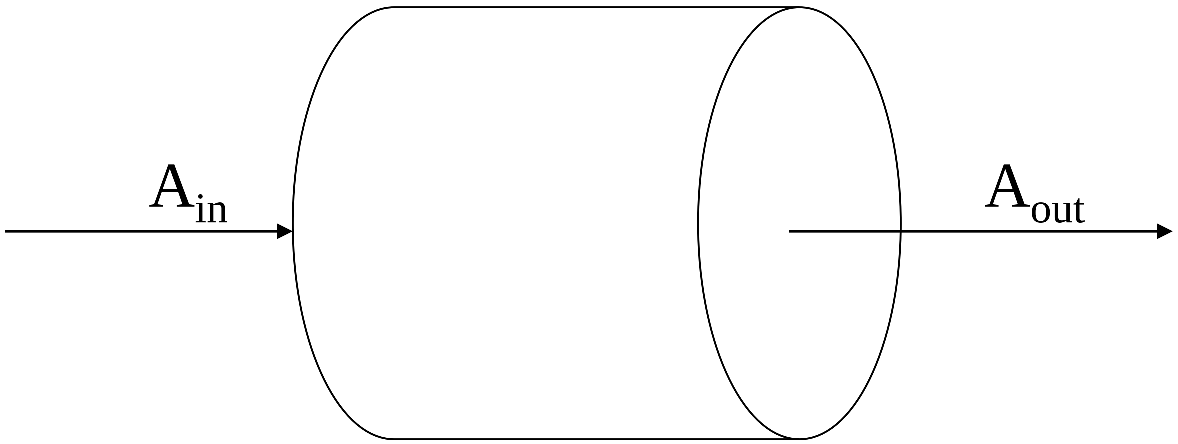
$\nabla \cdot$A is the net flux of A per unit volume at the point considered, counting vectors into the volume as negative, and vectors out of the volume as positive.

$$\boxed{A \cdot B = A_x B_x + A_y B_y + A_z B_z}$$

- Divergence of a vector function **A** at each point gives the rate per unit volume at which the physical entity is issuing from that point. If the vector **A** at a particular point (say x) represent flow of heat and if ∇.A is positive at x, then at point x either there must be source or heat must be leaving the point x so that temperature at point x is decreasing .Similarly if A represents velocity of a fluid and ∇.A is positive at a point x, then either there must be source of fluid at x or the density at point x is decreasing. Hence, for an incompressible fluid (like water) of non negative zero value of divergence at a point x represents the rate at which the fluid is being gained or removed at point x. Therefore, this gives the measure of strength of source or sink at point x. However, if there are no sources or sinks in some portion of the incompressible fluid then the divergence is zero at every point of that region. Symbolically, if
- ∇.**A** = Excess flow of outward flow over inward flow i.e. divergence of fluid.
- ∇.**A** = Excess flow of inward flow over outward flow i.e. convergence of fluid.
- Hence the name divergence is derived from this interpretation.

Output - input : the net rate of mass flow from unit volume

Divergence



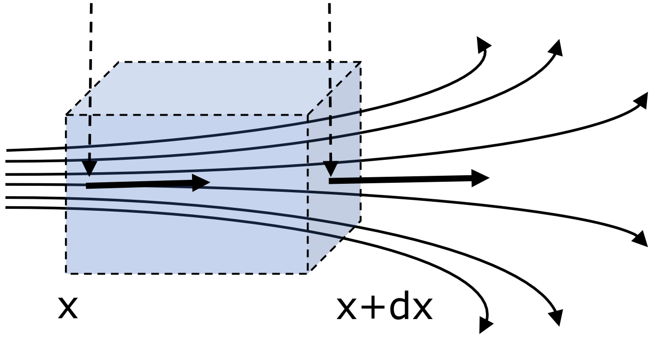
$$\nabla \cdot A > 0$$

The flux leaving the one end must exceed the flux entering at the other end.
The tubular element is "divergent" in the direction of flow.
Therefore, the operator $\nabla\cdot$ is frequently called the "divergence" :

Divergence of a vector $$\boxed{\nabla \cdot A = \text{div } A}$$

$\vec{\mathbf{v}}(x, y, z)$ is a differentiable vector field

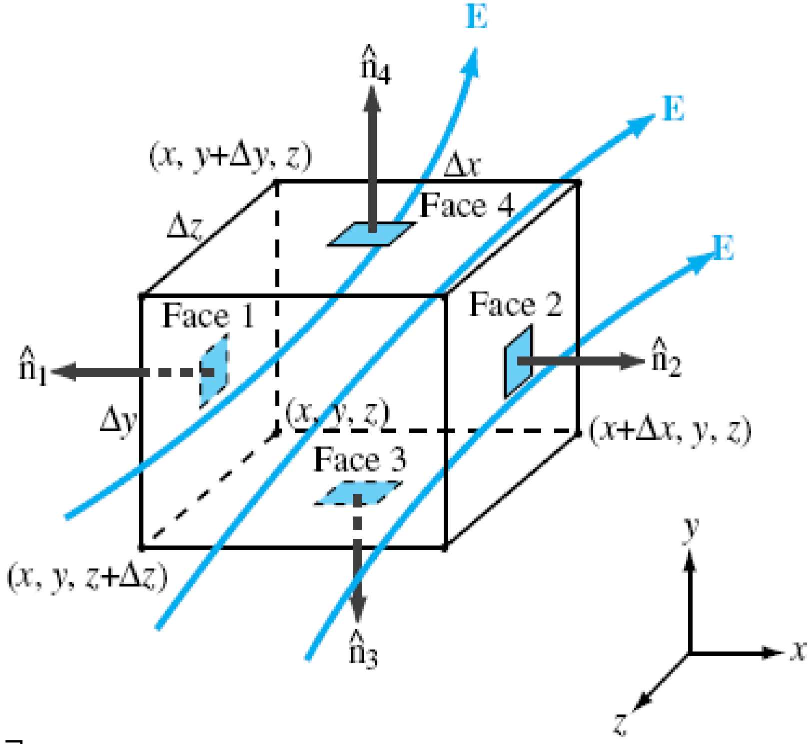
$$\text{div } \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \partial_u v^u$$



Physical Interpretation of Divergence

To further understand the divergence, consider water motion through a rectangular parallelepiped of infinitesimal volume $\Delta x\Delta y\Delta z$ as shown in figure. If ρ_m is the mass density of the fluid, the flow into the volume through the left hand face is $\rho_m v_x \Delta y\Delta z$, where v_x is the average velocity component of water velocity through the left-hand side. The corresponding velocity through the right-hand side is

$$\left[\rho_m v_x + \frac{\partial(\rho_m v_x)}{\partial x}\Delta x\right]\Delta y\Delta z$$



so that the net flow along x direction is $\frac{\partial(\rho_m v_x)}{\partial x}\Delta x\Delta y\Delta z$

The net outward flow in the y direction is therefore $\frac{\partial(\rho_m v_y)}{\partial y}\Delta x\Delta y\Delta z$

Similarly the net outward flow in the z direction is $\frac{\partial(\rho_m v_z)}{\partial z}\Delta x\Delta y\Delta z$

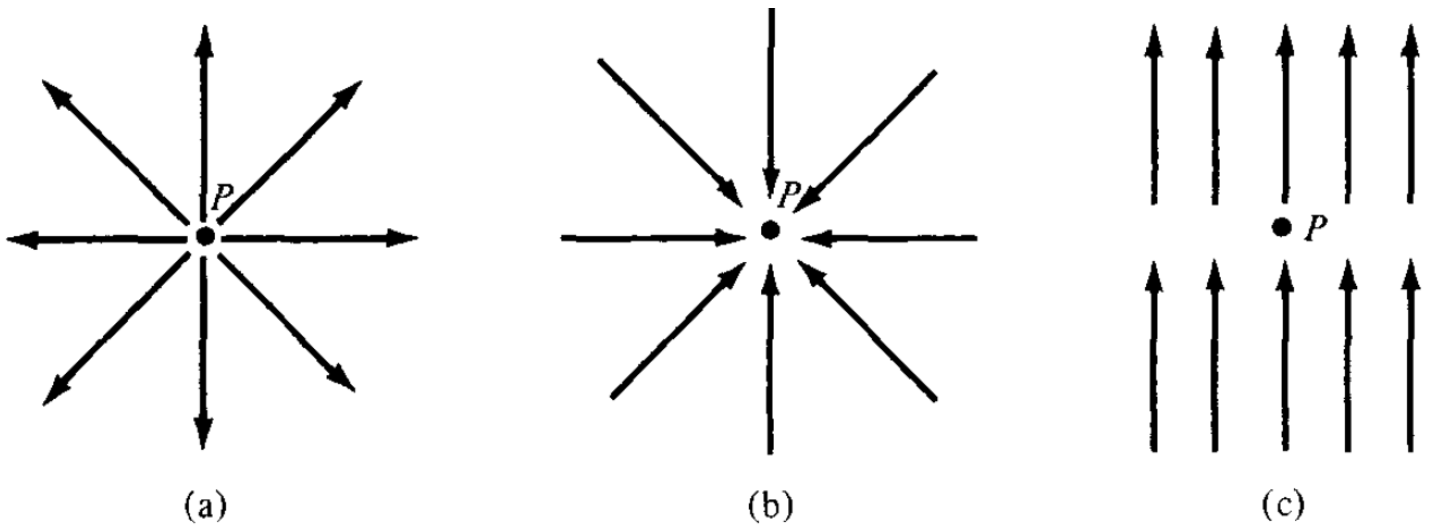
The net outward flow per unit volume is $\left[\frac{\partial(\rho_m v_x)}{\partial x} + \frac{\partial(\rho_m v_y)}{\partial y} + \frac{\partial(\rho_m v_z)}{\partial z}\right] = div(\rho_m \mathrm{v}) = \nabla.(\rho_m \mathrm{v})$

This is the divergence of the fluid at the point x, y, z. As water fluid cannot diverge from, nor converge toward, a point.

Hence, for water $\nabla.(\rho_m \mathbf{v})$ always equals zero.

Divergence

The case of a compressible fluid or gas such as stream is different. When the value on a steam boiler is opened, there is a positive value for the divergence at each point within the boiler. There is a net outward flow of stream for each element volume. On the other hand, when an evacuated light bulb is broken, there is momentarily a negative value for divergence in the space that was formerly the interior of the bulb.



(a) Positive divergence, (b) negative divergence, (c) zero divergence.

Example 12

Find divergence of these vectors:

(a) $P = x^2 yz\mathbf{a}_x + xz\mathbf{a}_z$

(b) $Q = r\sin\phi\,\mathbf{a}_r + r^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $W = \dfrac{1}{R^2}\cos\theta\,\mathbf{a}_R + R\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution to Example 12

(a) Use divergence for Cartesian coordinate:

$$\nabla \bullet \mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(x^2 yz\right) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$= 2xyz + x$$

Solution to Example 12

(b) Use divergence for cylindrical coordinate:

$$\nabla \bullet \mathrm{Q} = \frac{1}{r}\frac{\partial}{\partial r}\left(rQ_r\right) + \frac{1}{r}\frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \sin\phi\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(r^2 z\right) + \frac{\partial}{\partial z}\left(z\cos\phi\right)$$

$$= 2\sin\phi + \cos\phi$$

Solution to Example 12

(c) Use divergence for Spherical coordinate:

$$\nabla \bullet \mathrm{W} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 W_R\right) + \frac{1}{R\sin\theta}\frac{\partial\left(W_\theta \sin\theta\right)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial W_\phi}{\partial\phi}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(\cos\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(R\sin^2\theta\cos\phi\right)$$

$$+ \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(\cos\theta\right)$$

$$= 2\cos\theta\cos\phi$$

Curl of a vector field

- The third type of vector multiplication of a vector A by del operator is the cross product or vector product i.e. a vector quantity. The resultant operation is called curl.
- **Definition.** The **curl of a is an axial (or rotational) vector whose magnitude is the maximum circulation** of A per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.
- Thus, curl or rotation of a vector field A is defined as vector function of space obtained by taking the vector product of the vector ∇ and A.
- The curl of vector A is an axial (rotational) vector whose magnitude is the maximum circulation of **A** per unit area. The curl is used to find whether vector A is rotating or not.
- Curl direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

Curl of Vector A

$$\nabla x \mathrm{A} = \left(\frac{\partial}{\partial x}\mathrm{a}_x + \frac{\partial}{\partial y}\mathrm{a}_y + \frac{\partial}{\partial z}\mathrm{a}_z \right) X \left(A_x \mathrm{a}_x + A_y \mathrm{a}_y + A_z \mathrm{a}_z \right)$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Curl in cylindrical coordinates

- For cylindrical coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_r - \left[\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right]\mathbf{a}_\phi + \frac{1}{r}\left[\frac{\partial\left(rA_\phi\right)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\mathbf{a}_z$$

Curl in spherical coordinates

- For spherical coordinates:

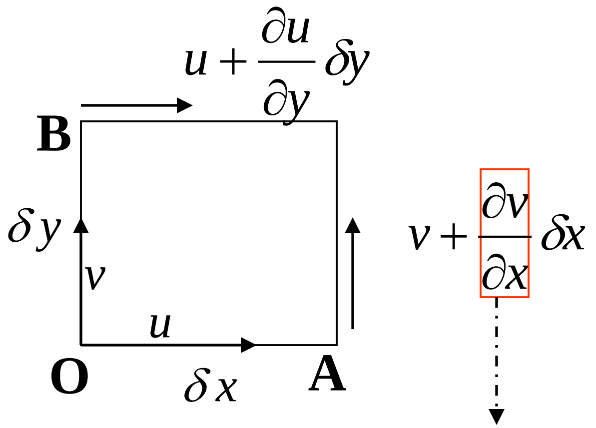
$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & R\mathbf{a}_\theta & R\sin\theta\,\mathbf{a}_\phi \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta$$

$$+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial(A_R)}{\partial\theta}\right)\mathbf{a}_\phi$$

Physical Interpretation of curl

Assume a two-dimensional fluid element

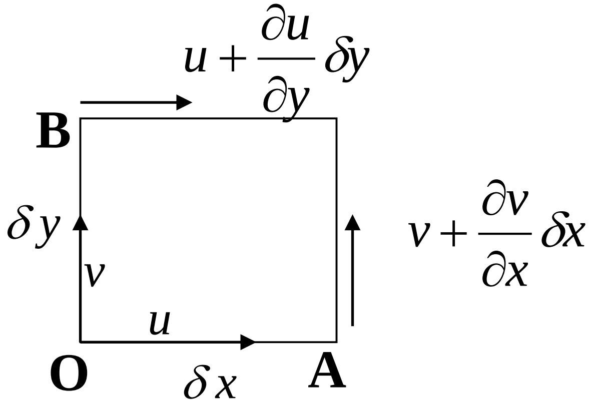


$$\nabla\times\mathbf{u}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix}=k\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Regarded as the angular velocity of OA, direction : **k**

Thus, the angular velocity of OA is $k\frac{\partial v}{\partial x}$; similarily, the angular velocity of OB is $-k\frac{\partial u}{\partial y}$

The angular velocity of the fluid element is the average of the two angular velocities :



$$\boxed{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}\mathbf{k}$$

$$\omega$$

$$\boxed{\nabla \times \mathbf{u} = 2\omega\,\mathbf{k}}$$

$$\boxed{\nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

This value is called the “vorticity” of the fluid element, which is twice the angular velocity of the fluid element. This is the reason why it is called the “curl” operator.

Examples on Curl Calculation

Given $\mathbf{A} = e^{xy}\,\mathbf{a_x} + \sin xy\,\mathbf{a_y} + \cos^2(xz)\,\mathbf{a_z}$

$$\nabla \times \mathbf{A} = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2(xz) \end{vmatrix}$$

$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a_y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_x}(0-0) + \mathbf{a_y}(0 + 2z\cos xz \sin xz) + \mathbf{a_z}\left(y\cos xy - xe^{xy}\right)$$

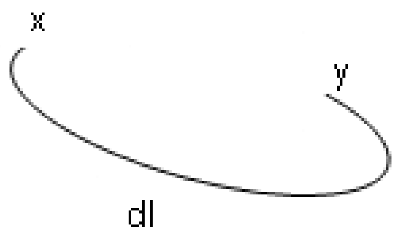
$$\nabla \mathbf{x} \mathbf{A} = \mathbf{a_y}(z\sin 2xz) + \mathbf{a_z}\left(y\cos xy - xe^{xy}\right)$$

Integration of Vector

- There are certain integrals involving vector quantities. These integrals are useful in deriving vector operations. The derivations are presented in Cartesian coordinates. There are three types of integrals for vector quantities
- Line Integral
- Surface Integral
- Volume Integral

Line Integral

- The line integral can be easily understood from the example of calculating the work W required to push a vehicle with a force F from point x to point y along a line path as shown in figure



Pushing vehicle along the path dl from x to point y

- The line integral is of the form

$$w = \int_x^y \mathrm{F.dl}$$

- The integral states that no work is will be expended in moving the cart if the direction of applied force is perpendicular to the path of motion. If we wish to push the vehicle around the path to its original position then it is called closed line integral

$$\mathrm{w} = \oint \mathbf{F.dl}$$

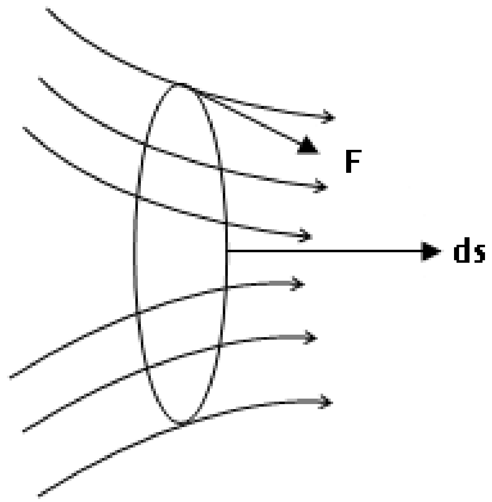
Surface Integral

The surface integral tells us the amount of field **F** that is passing through the surface **Δs** which has differential element **ds.** The surface integral is written as

$$\int_{\Delta s} \mathbf{F.ds}$$

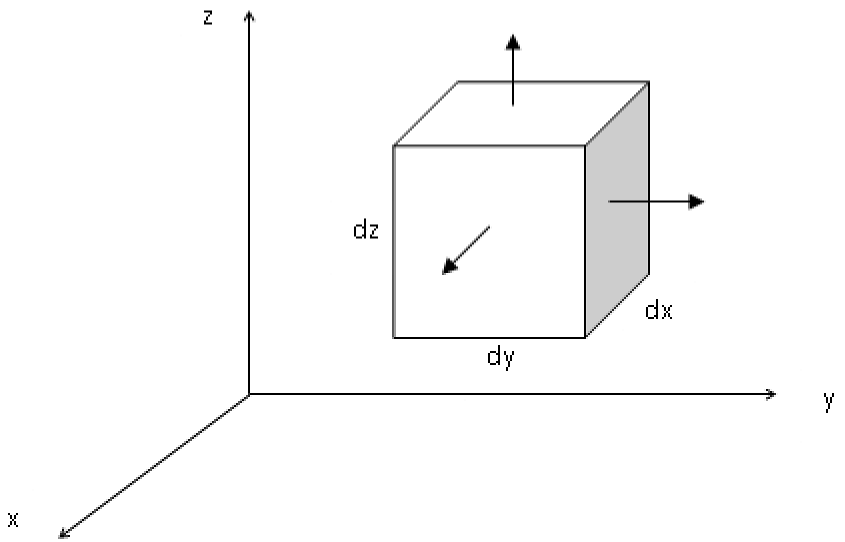
where **F** is field vector and **ds** is differential surface area. The differential surface area is a vector whose direction is specified by normal to the surface in outward direction.

For a non closed surface the direction has to be specified since there is no obvious outward direction. Using right hand thumb rule it is taken to be taken in the direction of the thumb if the fingers of the right hand follow the perimeter of the surface in counter clockwise direction.
The vector field F is confined to the surface then F.ds is zero. This means the vector F does not pass through the surface a shown in figure.



In case we integrate over the entire closed surface then surface integral is defined as $\oint \mathbf{F.ds}$

- For the cubical structure shown in figure
- Two differential surface vectors associated with surfaces dxdy perpendicular to z axis are
- $\mathbf{ds}$=dxdy$\mathbf{a}_z$ and $\mathbf{ds}$=dxdy(-$\mathbf{a}_z$)
- These two vectors are in opposite direction.
- Similarly the differential surface vectors for other
- four surfaces are defined.



Six differential surface vectors associated with cube

Surface Integral

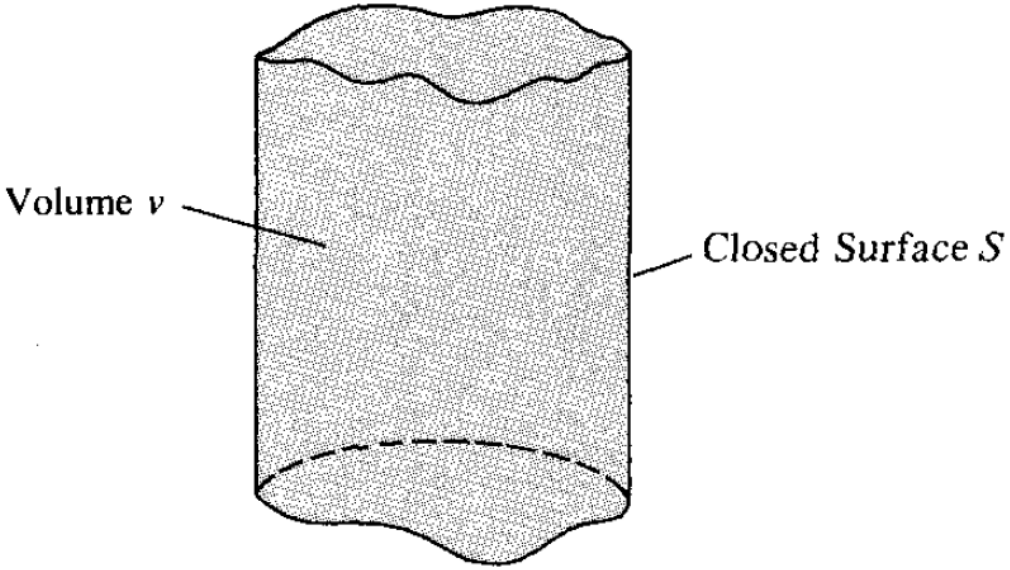
- In antenna theory we come across various volume integrals of vector quantities such as volume charge density ρ_v. The volume integral for above charge density for particular volume gives us total charge in the volume.

$$Q = \int_{\Delta s} \rho_{\mathrm{v}} dv$$

Gauss' Divergence Theorem

- The divergence theorem states that the total outward flux of a vector field A through the closed surface S is the same as the volume integral of the divergence of A

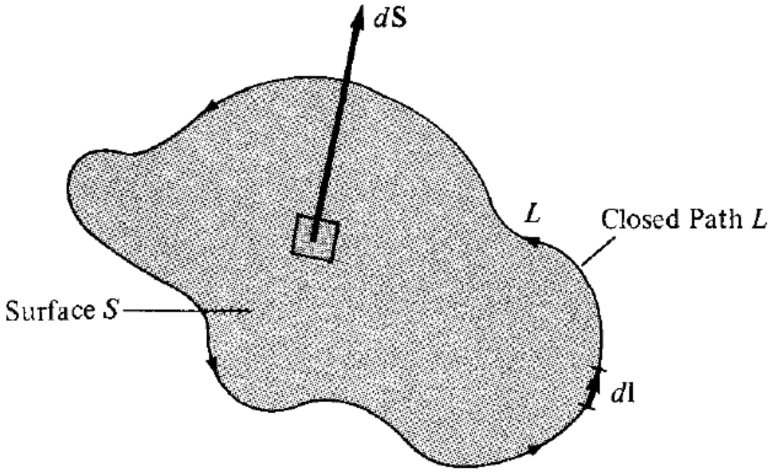
$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv$$



- The theorem applies to any volume v bounded by the closed surface *S*

Stokes’ Theorem

Considering a surface **S** having element **dS** and curve **C** denotes the curve :



If there is a vector field **A**, then the line integral of **A** taken round C is equal to the surface integral of $\nabla \times$ A taken over **S** :

$$\boxed{\int_C \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \int_S \nabla\times\mathbf{A}\,dS}$$

Two-dimensional system

$$\boxed{\int_C (A_x dx + A_y dy) = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy}$$

Stokes's Theorem

- Converts surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S

$$\int_S (\nabla \times \mathrm{B}) \cdot d\mathrm{s} = \oint_C \mathrm{B} \cdot d\mathrm{l} \quad \text{(Stoke's theorem)}$$

Gradient of a vector field

For Cartesian coordinate:

$$\nabla T = \frac{dT}{dL} = \frac{\partial T}{\partial x}\mathrm{a}_x + \frac{\partial T}{\partial y}\mathrm{a}_y + \frac{\partial T}{\partial z}\mathrm{a}_z$$

For Circular cylindrical coordinate:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \quad \text{(cylindrical)}$$

For Spherical coordinate:

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(spherical)}$$

Divergence of a vector field

For Cartesian coordinate:

$$\nabla \bullet \mathrm{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

For Circular cylindrical coordinate:

$$\nabla \bullet \mathrm{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

For Spherical coordinate:

$$\nabla \bullet \mathrm{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial\left(A_\theta \sin\theta\right)}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl in Cartesian coordinates

- For Cartesian coordinates:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- For cylindrical coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

- For spherical coordinates:

$$\nabla \times \mathrm{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathrm{a}_R & \mathrm{Ra}_\theta & \mathrm{Rsin}\,\theta\mathrm{a}_\phi \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

Laplacian of a Scalar

- **Laplacian** of a scalar *V* is denoted by $\nabla^2 V$.

$$
\begin{aligned}
\nabla^2 V &= \nabla \bullet \nabla V \\
&= \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \bullet \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right] \\
&= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}
$$

- The result is a scalar.

Laplacian

- In cartesian coordinates

$$\nabla^2 V = \left[\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right] \bullet \left[\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right]$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial^2 x}a_x + \frac{\partial^2 V}{\partial^2 y}a_y + \frac{\partial^2 V}{\partial^2 z}a_z$$

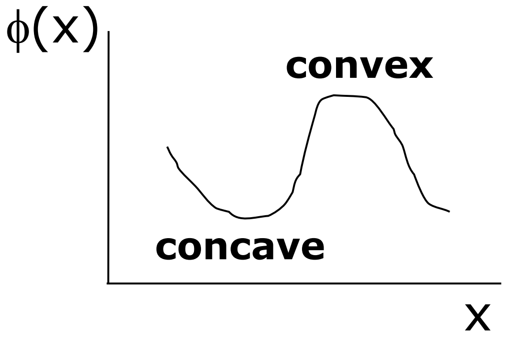
- In Cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

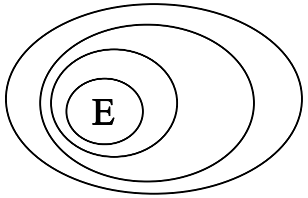
- In Spherical Coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

- **Laplacian: physical meaning**



As a second derivative, the one-dimensional Laplacian operator is related to minima and maxima: when the second derivative is positive (negative), the curvature is concave (convexe).



In most of situations, the 2-dimensional Laplacian operator is also related to local minima and maxima. If v_E is positive:

$\Delta\phi = -v_E$: maximum in E (ϕ(E) > average value in the surrounding)

$\Delta\phi = v_E$: minimum in E (ϕ(E) < average value in the surrounding)

Laplacian

- Laplacian of a vector: $\nabla^2 \mathbf{A}$ is defined as the gradient of the divergence of **A** minus the curl of the curl of A;

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

- **Only** for the cartesian coordinate system;

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

3. Differential operators

- **Summary**

Operator	**grad**	**div**	**curl**	**Laplacian**
is	a vector	a scalar	a vector	a scalar (*resp.* a vector)
concerns	a scalar field	a vector field	a vector field	a scalar field (*resp.* a vector field)
Definition	$\nabla\phi$	$\nabla \cdot \vec{\mathbf{v}}$	$\nabla \times \vec{\mathbf{v}}$	$\nabla^2\phi$ (*resp.* $\nabla^2\vec{v}$)

Classification of Vector Fields

- Solenoidal Vector Field: A vector field A is said to be solenoidal (or divergenceless) if

$$\nabla \bullet A = 0$$

- Such a field has neither source nor sink of flux, flux lines of A entering any closed surface must also leave it.

Classification of Vector Fields

- A vector field A is said to be irrotational (or potential) if

$$\nabla \times A = 0$$

- In an irrotational field A, the circulation of A around a closed path is identically zero.
- This implies that the line integral of A is independent of the chosen path
- An irrotational field is also known as a *conservative field*

THANKS