UEC747: ANTENNA AND WAVE PROPAGATION

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Lecture 5: Coordinate Transformation

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Coordinate Transformations

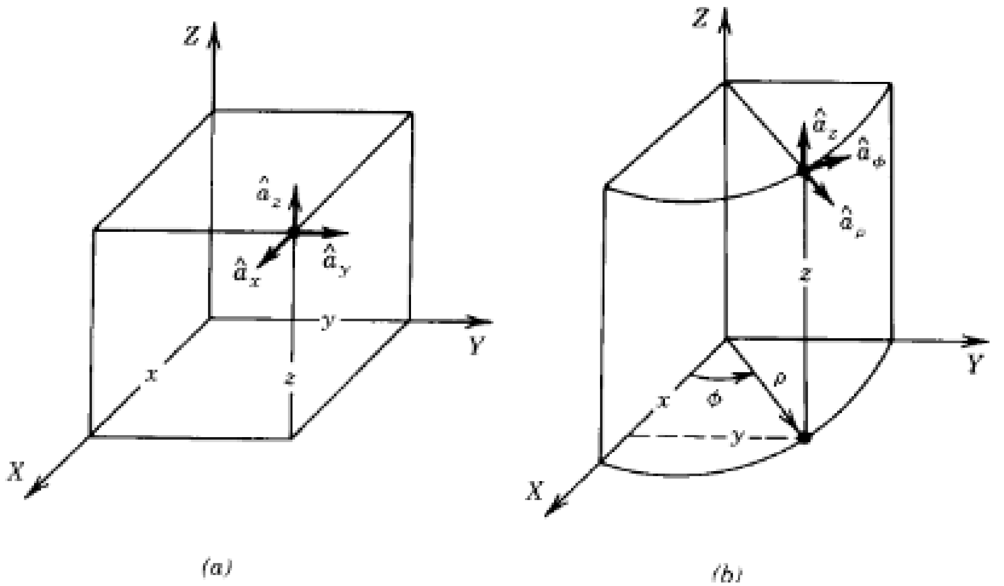
- A vector in cylindrical coordinate system can be transformed to a vector in Cartesian coordinates or vice versa.
- The transformation finds unknown coefficients associated with each of these vectors and the relationship between unit vectors.
- The transformation is found by dot product of vector **A** in cylindrical coordinate system with the unit vectors in Cartesian coordinate system.
- Given $\mathbf{A}= \mathbf{a_r} A_r + \mathbf{a_\phi} A_\phi + \mathbf{a_z} A_z$
- To convert it to Cartesian dot product of vector **A** in cylindrical coordinate system with the unit vectors in Cartesian coordinate system.

For example

$$A_x=\mathbf{A.a_x}= (\mathbf{a_r} A_r + \mathbf{a_\phi} A_\phi + \mathbf{a_z} A_z).\ \mathbf{A_x}$$

$$= A_r\ (\mathbf{a_r. a_x}) + A_\phi\ (\mathbf{a_\phi} . \mathbf{a_x})$$

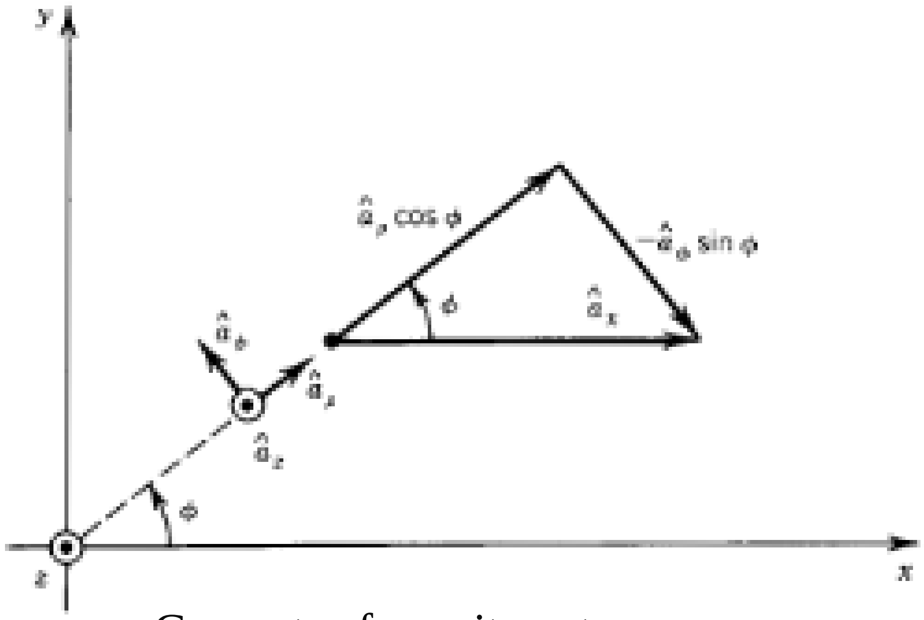
where $\mathbf{a_z. a_x}=0$



It can be seen that $\mathbf{a_r} \cdot \mathbf{a_x} = \cos\phi$

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \cos(\pi/2+\phi) = -\sin\phi$$

Therefore $A_x = A_r \cos\phi - A_\phi \sin\phi$



Geometry for unit vector a_x.

$$A_y = \mathbf{A}.\mathbf{a}_y$$

$$= A_r \, (\mathbf{a_r} \cdot \mathbf{a}_y) + A_\phi \, (\mathbf{a}_\phi \cdot \mathbf{a}_y)$$,where $\mathbf{a_z} \cdot \mathbf{a}_y = 0$

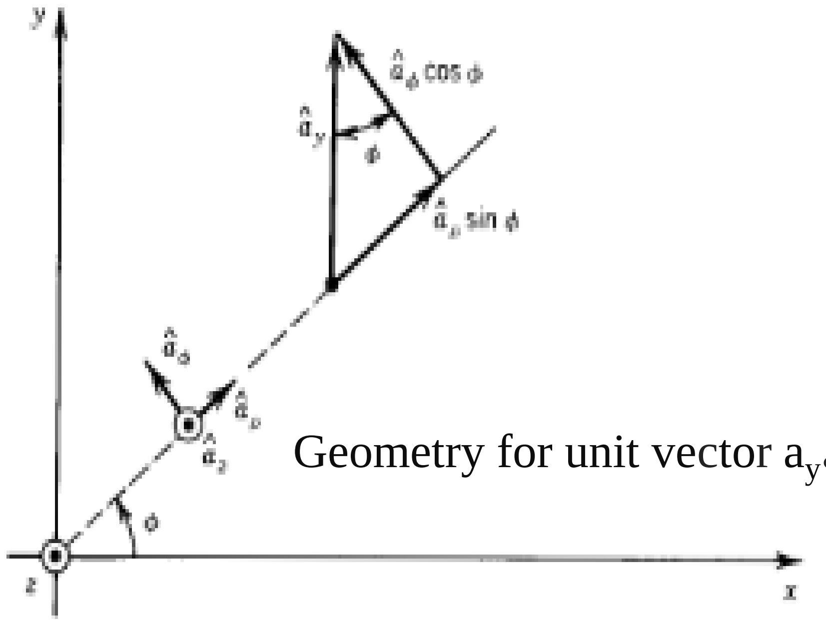
$$\mathbf{a_r} \cdot \mathbf{a}_y = \cos(\pi/2-\phi) = \sin\phi$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$$

Hence $A_y = A_r \sin\phi + A_\phi \cos\phi$

A_z terms are same in both coordinate system Hence A_z remains same. In above vector **A** was assumed to be constant .But it can also be function of independent variables. These variables must be transformed from cylindrical to Cartesian coordinate system by

$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z$$



Geometry for unit vector a_y.

The inverse transformation from Cartesian to cylindrical coordinate system is given by

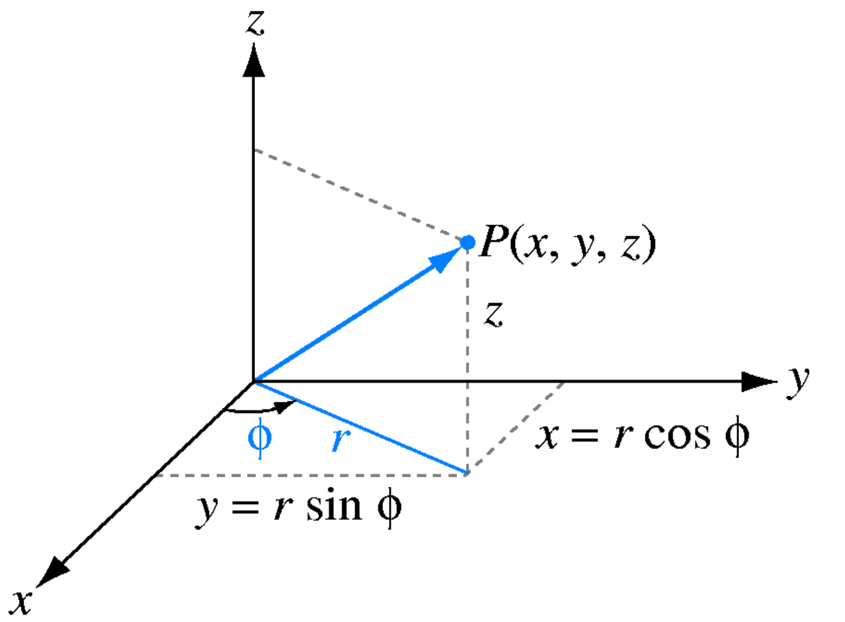
$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}\right\}$$

Cartesian to Cylindrical Transformation

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

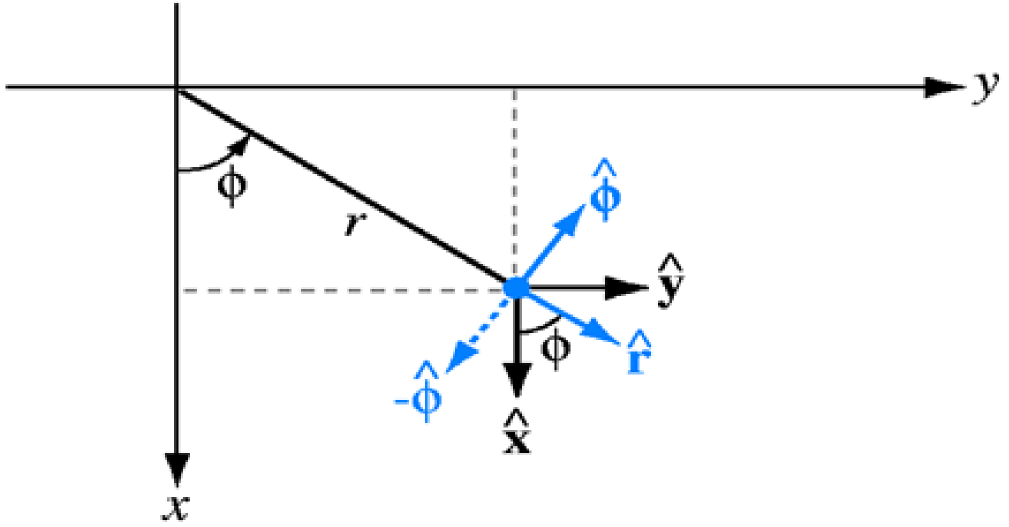


$$\boxed{\begin{aligned}\rho &= +\sqrt{x^2 + y^2}\\ \phi &= \tan^{-1}(y/x)\\ z &= z\end{aligned}}$$

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{z} = \hat{z}$$



Cylindrical Base Vectors (contd.)

- We can express cylindrical base vectors in terms of **Cartesian** base vectors. First, we find that:

$$a_\rho \cdot a_x = \cos\phi \qquad a_\phi \cdot a_x = -\sin\phi \qquad a_z \cdot a_x = 0$$

$$a_\rho \cdot a_y = \sin\phi \qquad a_\phi \cdot a_y = \cos\phi \qquad a_z \cdot a_y = 0$$

$$a_\rho \cdot a_z = 0 \qquad a_\phi \cdot a_z = 0 \qquad a_z \cdot a_z = 1$$

We can use these results to write **cylindrical** base vectors in terms of **Cartesian** base vectors, or vice versa!

$$\boxed{a_\rho = (a_\rho . a_x)A_x + (a_\rho . a_y)A_y + (a_\rho . a_z)A_z}$$

$$\boxed{a\rho = \cos\phi\, Ax + \sin\phi\, Ay}$$

$$\boxed{a_x = (a_x . a_\rho)A_\rho + (a_y . a_\rho)A_\rho + (a_z . a_\rho)A_\rho}$$

$$\boxed{a_x = \cos\phi\, A_\rho - \sin\phi\, A_\phi}$$

$$\mathbf{A} = (\hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi) A_x + (\hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi) A_y + \hat{a}_z A_z$$

$$\mathbf{A} = \hat{a}_\rho (A_x \cos\phi + A_y \sin\phi) + \hat{a}_\phi (-A_x \sin\phi + A_y \cos\phi) + \hat{a}_z A_z$$

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

In matrix form, the transformation matrix for rectangular-to-cylindrical components is given by

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Where

$$[A]_{rc} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $[A]_{rc}$ is an orthonormal matrix (its inverse is equal to its transpose), we can write the transformation matrix for cylindrical-to-rectangular components as

$$[A]_{cr} = [A]_{rc}^{-1} = [A]_{rc}^{t} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

or
$$\begin{aligned} A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\ A_z &= A_z \end{aligned}$$

Cylindrical-to-Spherical (and Vice Versa)

- we can write that the cylindrical and spherical coordinates are related by

$$\rho = r \sin\theta$$

$$z = r \cos\theta$$

- In a geometrical approach similar to the one employed in the previous section, we can show that the cylindrical-to-spherical transformation of vector components is given by

$$A_r = A_\rho \sin\theta + A_z \cos\theta$$

$$A_\theta = A_\rho \cos\theta - A_z \sin\theta$$

$$A_\phi = A_\phi$$

cylindrical-to-spherical transformation matrix

In matrix form by

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

$$[A]_{cs} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix}$$

Cylindrical Base Vectors to Spherical

- Finally, we can write **cylindrical** base vectors in terms of **spherical** base vectors, or vice versa, using the following relationships:

$$a_\rho \cdot a_r = \sin\theta \qquad a_\phi \cdot a_r = 0 \qquad a_z \cdot a_r = \cos\theta$$
$$a_\rho \cdot a_\theta = \cos\theta \qquad a_\phi \cdot a_\theta = 0 \qquad a_z \cdot a_\theta = -\sin\theta$$
$$a_\rho \cdot a_\phi = 0 \qquad a_\phi \cdot a_\phi = 1 \qquad a_z \cdot a_\phi = 0$$

- **For example:**

$$\boxed{a_\rho = (a_\rho . a_r)A_r + (a_\rho . a_\theta)A_\theta + (a_\rho . a_\phi)A_\phi}$$

$$\boxed{a_\rho = \sin\theta\, A_r + \cos\theta\, A_\theta}$$

- **or**

$$\boxed{a_\theta = (a_\theta . a_\rho)A_\rho + (a_\theta . a_\phi)A_\phi + (a_\theta . a_z)A_z}$$

$$\boxed{a_\theta = \cos\theta\, A_\rho - \sin\theta\, A_z}$$

$$[A]_{sc} = [A]_{cs}^{-1} = [A]_{cs}^{t} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

Spherical-to-cylindrical transformation is accomplished by

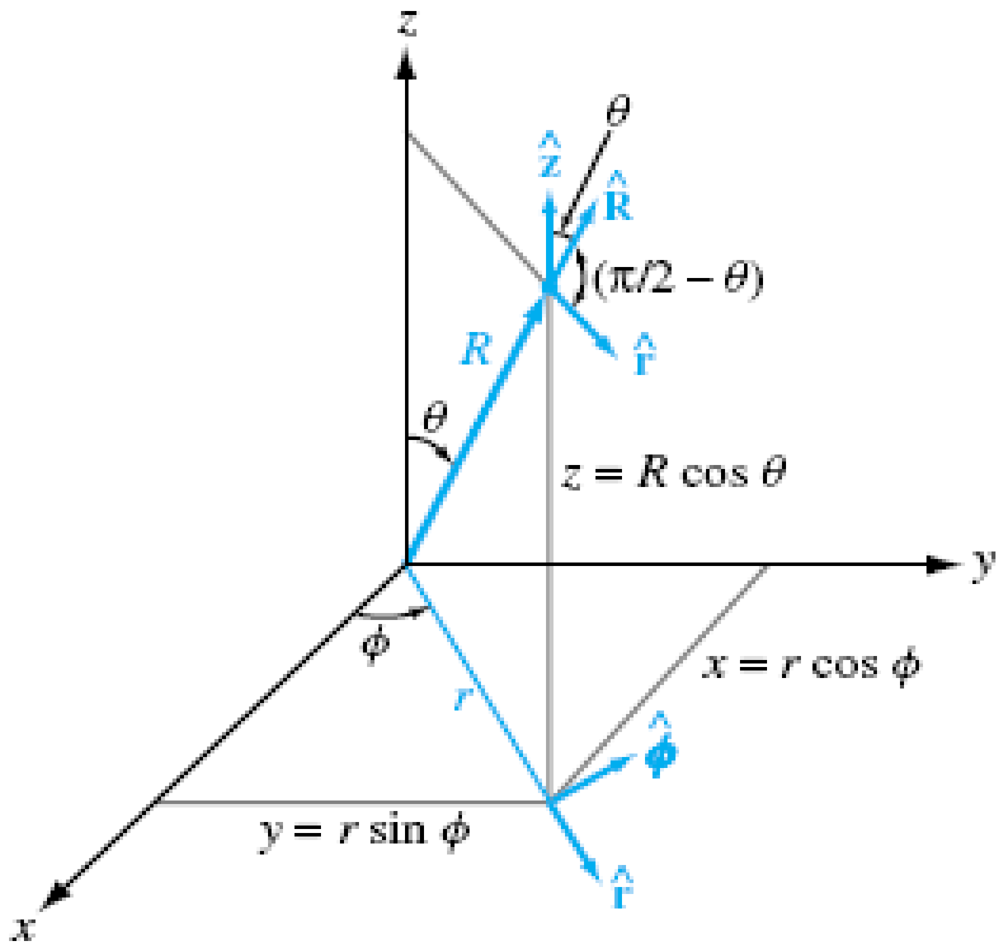
$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

or

$$\begin{aligned} A_\rho &= A_r \sin\theta + A_\theta \cos\theta \\ A_\phi &= A_\phi \\ A_z &= A_r \cos\theta - A_\theta \sin\theta \end{aligned}$$

Cartesian to Spherical Transformations

- Relationships between *(x, y, z)* and *(r, θ, Φ)* are



A vector in spherical coordinate system can be transformed to a vector in Cartesian coordinates or vice versa. The transformation finds unknown coefficients associated with each of these vectors and the relationship between unit vectors. The transformation is found by dot product of vector **A** in spherical coordinate system with the unit vectors in Cartesian coordinate system. For example

$$A_x = \mathbf{A}.\mathbf{a}_x$$

$$= (\mathbf{a}_r A_r + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi).\ \mathbf{a}_x$$

$$= A_r\ (\mathbf{a}_r.\ \mathbf{a}_x) + A_\theta\ (\mathbf{a}_\theta.\ \mathbf{a}_x) + A_\phi\ (\mathbf{a}_\phi\ .\ \mathbf{a}_x)$$

- It can be seen that

$$\mathbf{a_r} \cdot \mathbf{a}_x = \sin\theta\cos\phi$$

$$\mathbf{a}_\theta \cdot \mathbf{a}_x = \cos\theta\cos\phi$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi$$

Therefore $A_x = A_r \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$

- Similarly

$$A_y = \mathbf{A} \cdot \mathbf{a}_y$$

$$= A_r (\mathbf{a_r} \cdot \mathbf{a}_y) + A_\theta (\mathbf{a}_\theta \cdot \mathbf{a}_y) + A_\phi (\mathbf{a}_\phi \cdot \mathbf{a}_y)$$

$$\mathbf{a_r} \cdot \mathbf{a}_y = \sin\theta\sin\phi$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\theta\sin\phi$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$$

$A_z = \mathbf{A} . \mathbf{a}_z$

$= A_r \; (\mathbf{a}_r . \mathbf{a}_z) + A_\theta \; (\mathbf{a}_\theta . \mathbf{a}_z) + A_\phi \; (\mathbf{a}_\phi . \mathbf{a}_z)$

$\mathbf{a}_r . \mathbf{a}_z = \cos\theta$

$\mathbf{a}_\phi . \mathbf{a}_z = -\sin\theta$

$\mathbf{a}_\phi . \mathbf{a}_z = 0$

Spherical-to-rectangular components related by

$$A_x = A_r \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_y = A_r \sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

Vector Algebra Using Orthonormal Base Vectors

Q: Just why do we express a vector in terms of 3 orthonormal base vectors? Doesn't this just make things even more complicated ??

A: Actually, it makes things **much** simpler. The **evaluation** of vector operations such as addition, subtraction, multiplication, dot product, and cross product all become straightforward **if** all vectors are expressed using the **same** set of base vectors.

Vector Algebra Using Orthonormal Base Vectors (contd.)

Dot Product

Say we take the **dot product** of A and $\vec{B}$:

$$\boxed{\vec{A}.\vec{B} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)}$$

$$\begin{aligned}
&= A_x\hat{a}_x.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)\\
&+A_y\hat{a}_y.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)\\
&+A_z\hat{a}_z.\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)
\end{aligned}$$

$$\begin{aligned}
&= A_xB_x\left(\hat{a}_x.\hat{a}_x\right) + A_xB_y\left(\hat{a}_x.\hat{a}_y\right) + A_xB_z\left(\hat{a}_x.\hat{a}_z\right)\\
&+A_yB_x\left(\hat{a}_y.\hat{a}_x\right) + A_yB_y\left(\hat{a}_y.\hat{a}_y\right) + A_yB_z\left(\hat{a}_y.\hat{a}_z\right)\\
&+A_zB_x\left(\hat{a}_z.\hat{a}_x\right) + A_zB_y\left(\hat{a}_z.\hat{a}_y\right) + A_zB_z\left(\hat{a}_z.\hat{a}_z\right)
\end{aligned}$$

Q: I thought this was suppose to make things **easier** !?!

Vector Algebra Using Orthonormal Base Vectors (contd.) ECE230

A: Be patient! Recall that these are **orthonormal** base vectors, therefore:

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1 \qquad \hat{a}_x.\hat{a}_y = \hat{a}_y.\hat{a}_z = \hat{a}_z.\hat{a}_x = 0$$

- As a result, our **dot product** expression reduces to this simple expression:

$$\boxed{\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

We can apply this to the expression for determining the **magnitude** of a vector:

$$\boxed{\left|\vec{A}\right|^2 = \vec{A}.\vec{A} = A_x^2 + A_x^2 + A_z^2} \qquad \boxed{\left|\vec{A}\right| = \sqrt{\vec{A}.\vec{A}} = \sqrt{A_x^2 + A_x^2 + A_z^2}}$$

Vector Algebra Using Orthonormal Base Vectors (contd.)

- Let us revisit previous example, where we expressed a vector using two different sets of basis vectors:

$$\vec{A} = 2.0\hat{a}_x + 1.5\hat{a}_y \qquad \vec{A} = 2.5\hat{a}_y$$

- Therefore, the magnitude of A is determined to be:

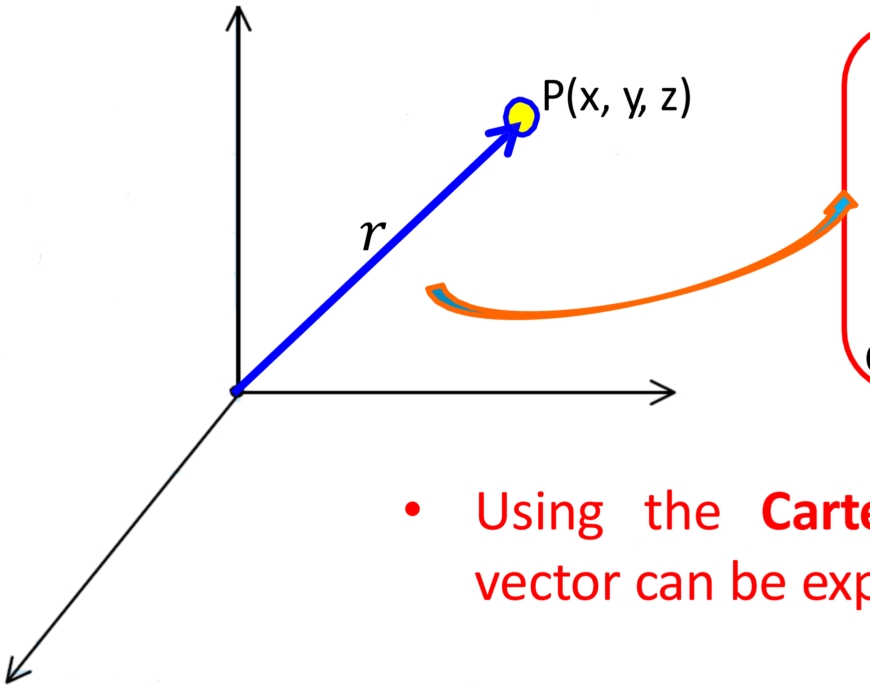
$$\left|\vec{A}\right| = \sqrt{2^2 + 1.5^2} = 2.5 \qquad \left|\vec{A}\right| = \sqrt{2.5^2} = 2.5$$

Q: Hey! We get the **same** answer from both expressions; is this a **coincidence**?

A: No! Remember, both expressions represent the **same** vector, only using different sets of base vectors. The magnitude of vector A is 2.5, **regardless** of how we choose to express A.

The Position Vector

- Consider a point whose location in space is specified with Cartesian coordinates (e.g., P(x, y, z)). Now consider the **directed distance** (a vector quantity!) extending from the origin to this point.



This **particular** directed distance—a vector beginning at the **origin** and extending outward to a point—is a **very important** and fundamental directed distance known as the **position vector** r

- Using the **Cartesian** coordinate system, the position vector can be explicitly written as:

$$\boxed{\overline{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}$$

The Position Vector (contd.)

- Note that given the **coordinates** of some point (e.g., x =1, y =2, z =-3), we can easily determine the **corresponding position vector** (e.g., $r = a_x + 2a_y - 3a_z$).
- Moreover, given some **specific position vector** (e.g., $r = 4a_y - 2a_z$), we can easily determine the **corresponding coordinates of that point** (e.g., x =0, y =4, z =-2).
- In other words, a position vector r is an alternative way to denote the location of a point in space! We can use **three coordinate values** to specify a point's location, **or** we can use a **single position vector** r .

I see! The position vector is essentially a **pointer.** Look at the end of the vector, and you will find the **point specified**!

The magnitude of r

- Note the **magnitude** of any and all position vectors is:

$$|r| = \sqrt{r.r} = \sqrt{x^2 + y^2 + z^2} = r$$

Q: Hey, this makes **perfect sense**! Doesn't the coordinate value r have a **physical** interpretation as the **distance** between the **point** and the **origin**?

A: That's right! The **magnitude** of a **directed distance** vector is equal to the **distance** between the two points—in this case the distance between the **specified point** and the **origin**!

Alternative forms of the position vector

- Be **careful**! Although the position vector **is correctly** expressed as:

$$\boxed{\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}$$

- It is **NOT CORRECT** to express the position vector as:

$$\boxed{\bar{r} \neq \rho\hat{a}_\rho + \phi\hat{a}_\phi + z\hat{a}_z}$$

$$\boxed{\bar{r} \neq r\hat{a}_r + \theta\hat{a}_\theta + \phi\hat{a}_\phi}$$

NEVER, EVER express the position vector in either of these two ways!

It should be **readily apparent** that the two expression above **cannot** represent a position vector—because **neither** is even a directed distance!

Alternative forms of the position vector (contd.)

Q: Why sure—it is **of course** readily apparent to me—but why don't you go ahead and explain it to those with **less i**nsight!

A: Recall that the **magnitude** of the position vector r has units of **distance**. Thus, the **scalar components** of the position vector must **also** have units of distance (e.g., meters). The coordinates x, y, z, ρ and r **do** have units of distance, but coordinates θ and ϕ do **not**.

Thus, the vectors θa_{θ} and ϕa_{ϕ} **cannot** be vector components of a position vector—or for that matter, any other **directed distance**!

Alternative forms of the position vector (contd.)

- Instead, we can use **coordinate transforms** to show that:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

$$= \rho\cos\phi\hat{a}_x + \rho\sin\phi\hat{a}_y + z\hat{a}_z$$

$$= r\sin\theta\cos\phi\hat{a}_x + r\sin\theta\sin\phi\hat{a}_y + r\cos\theta\hat{a}_z$$

ALWAYS use one of these three expressions of a position vector!!

Note that in **each** of the three expressions above, we use **Cartesian base vectors**. The **scalar components** can be expressed using Cartesian, cylindrical, or spherical **coordinates**, but we must always use **Cartesian base vectors**.

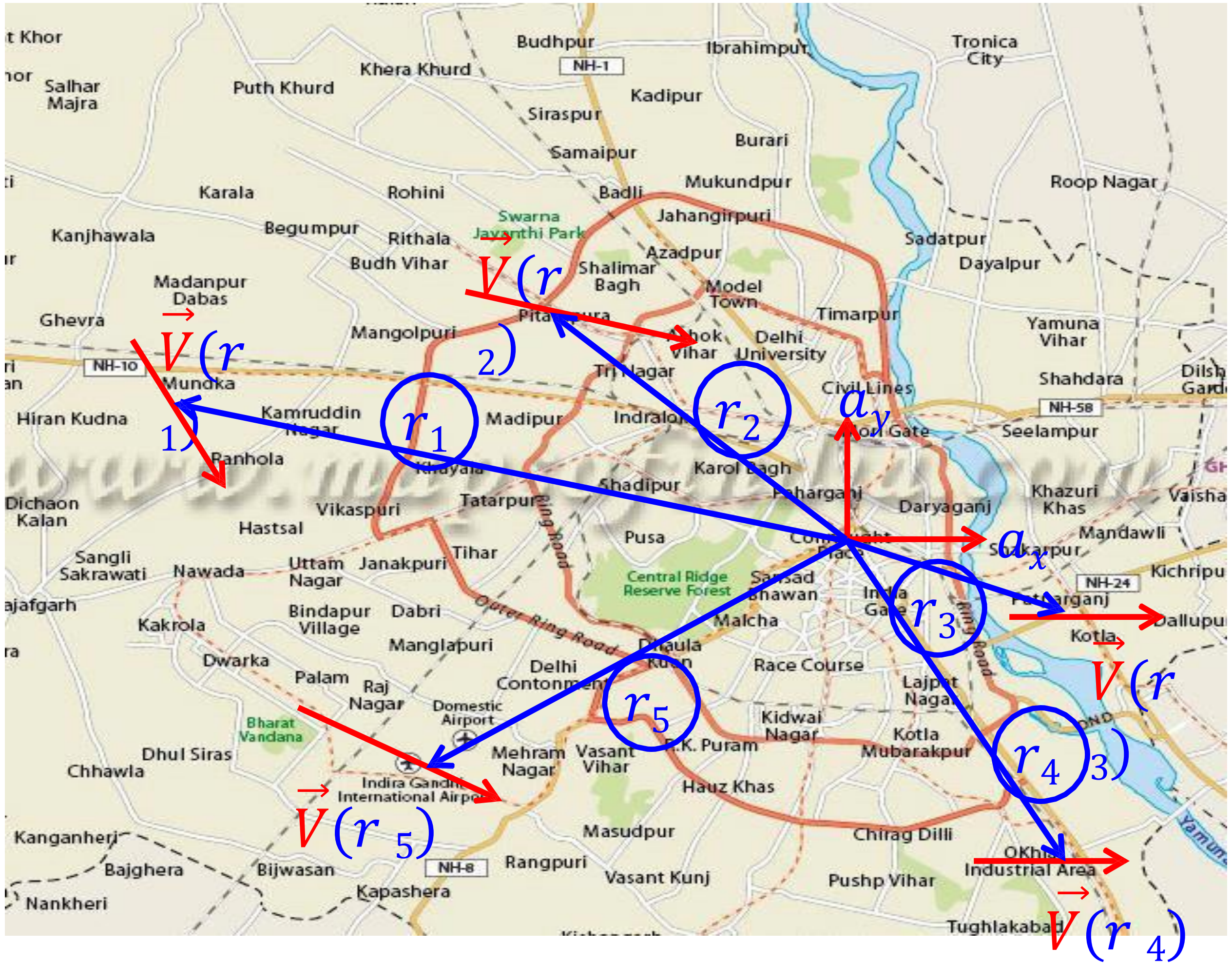
Alternative forms of the position vector (contd.)

Q: Why must we **always** use Cartesian base vectors? You said that we could express **any** vector using spherical or base vectors. Doesn't this **also** apply to position vectors?

A: The reason we **only** use Cartesian base vectors for constructing a position vector is that Cartesian base vectors are the only base vectors whose directions are **fixed**—independent of position in space!

Vector Field Notation

- **Consider the vector field $V(\vec{r})$**, which describes the **wind velocity** across the state of Delhi.



In this map, the **origin** has been placed at Connaught Place. The **locations** of Delhi locality can thus be identified using **position vectors** (units in kms)

Vector Field Notation (contd.)

$\bar{r}_1 = -400\hat{a}_x + 20\hat{a}_y$ → The **location** of Mundka

$\bar{r}_2 = -90\hat{a}_x + 70\hat{a}_y$ → The **location** of Pitampura

$\bar{r}_3 = 30\hat{a}_x - 5\hat{a}_y$ → **The location** of Patparganj

$\bar{r}_4 = 40\hat{a}_x - 90\hat{a}_y$ → The **location** of Okhla Industrial Estate

$\bar{r}_5 = -130\hat{a}_x - 70\hat{a}_y$ → The **location** of IGI Airport locality

- Evaluating the **vector field** $\vec{V}(r)$ at these locations provides the wind velocity **at** each Delhi locality (units of kmph).

$\vec{V}(\bar{r}_1) = 15\hat{a}_x - 17\hat{a}_y$ → The **wind velocity** in Mundka

$\vec{V}(\bar{r}_2) = 15\hat{a}_x - 9\hat{a}_y$ → The **wind velocity** in Pitampura

$\vec{V}(\bar{r}_3) = 11\hat{a}_x$ → **The wind velocity** in Patparganj

$\vec{V}(\bar{r}_4) = 7\hat{a}_x$ → The **wind velocity** in Okhla Industrial Estate

$\vec{V}(\bar{r}_5) = 9\hat{a}_x - 4\hat{a}_y$ → The **wind velocity** in IGI Airport locality

Vector Field Notation (contd.)

- From **vector field** $A(r)$, we can find the **magnitude** and **direction** of the discrete vector A that is **located** at the **point** defined by position vector r .
- This **discrete vector** A does **not** "extend" from the origin to the point described by position vector r . Rather, the discrete vector A describes a quantity **at that point**, and that point only. The magnitude of vector A does **not** have units of distance! The **length** of the arrow that represents vector A is merely symbolic—its length has **no** direct physical meaning.
- On the other hand, the position vector r , being a directed distance, **does** extend from the origin to a specific **point** in space. The magnitude of a position vector r **is** distance—the length of the **position vector** arrow **has** a direct physical meaning!
- Additionally, we should again note that a vector field need not be static. A **dynamic** vector field is likewise a function of **time**, and thus can be described with the notation:

$$\boxed{A(r , t)}$$

THANKS