Dolph-Tschebyscheff Array

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Dolph-Tschebyscheff Array

- We can get a minimized/optimized beam width for a desired side lobe level and vice versa.

- d</=$\lambda/2$

- A compromise between uniform and binomial arrays.

- Its excitation coefficients are related to Tchebyscheff polynomials.

- A Dolph-Tschebyscheff array with no side lobes (or side lobes of $-\infty$ dB) reduces to the binomial design.

- While binomial arrays have very low level minor lobes, they exhibit larger beam widths (compared to uniform and Dolph-Tschebyscheff designs).

- The wide variations between the amplitudes of the different elements of an array, especially for an array with a large number of elements.

- This leads to very low efficiencies for the feed network.

Array Factor

$$(\mathrm{AF})_{2M}(\mathrm{even}) = \sum_{n=1}^{M} a_n \cos[(2n-1)u]$$

$$(\mathrm{AF})_{2M+1}(\mathrm{odd}) = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

where

$$u = \frac{\pi d}{\lambda}\cos\theta \quad \text{.....................(A)}$$

- **The array factor of an array of even or odd number of elements with symmetric amplitude excitation is nothing more than a summation of *M or M + 1 cosine terms*.**

- **The largest harmonic of the cosine terms is one less than the total number of elements of the array.**

The array factor reduces to

$$
\begin{aligned}
m &= 0 \quad \cos(mu) = 1 \\
m &= 1 \quad \cos(mu) = \cos u \\
m &= 2 \quad \cos(mu) = \cos(2u) = 2\cos^2 u - 1 \\
m &= 3 \quad \cos(mu) = \cos(3u) = 4\cos^3 u - 3\cos u \\
m &= 4 \quad \cos(mu) = \cos(4u) = 8\cos^4 u - 8\cos^2 u + 1 \\
m &= 5 \quad \cos(mu) = \cos(5u) = 16\cos^5 u - 20\cos^3 u + 5\cos u \\
m &= 6 \quad \cos(mu) = \cos(6u) = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1 \\
m &= 7 \quad \cos(mu) = \cos(7u) = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u \\
m &= 8 \quad \cos(mu) = \cos(8u) = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u \\
&\qquad\qquad - 32\cos^2 u + 1 \\
m &= 9 \quad \cos(mu) = \cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u \\
&\qquad\qquad - 120\cos^3 u + 9\cos u
\end{aligned}
$$

Tschebyscheff polynomial

- Is given as

$$T_m(z) = \cos[m \cos^{-1}(z)] \qquad -1 \le z \le +1$$

$$T_m(z) = \cosh[m \cosh^{-1}(z)]^{\dagger} \qquad z < -1, z > +1$$

- Let u = $\cos^{-1}(z)$
- $T_m(z)$ == $\cos(mu)$

❖ for different values of m, each cosine term, whose argument is an integer times a fundamental frequency, can be rewritten as a series of cosine functions

- *As z = cosu, the polynomial becomes:*

$$
\begin{array}{ll}
m=0 & \cos(mu)=1=T_0(z)\\
m=1 & \cos(mu)=z=T_1(z)\\
m=2 & \cos(mu)=2z^2-1=T_2(z)\\
m=3 & \cos(mu)=4z^3-3z=T_3(z)\\
m=4 & \cos(mu)=8z^4-8z^2+1=T_4(z)\\
m=5 & \cos(mu)=16z^5-20z^3+5z=T_5(z)\\
m=6 & \cos(mu)=32z^6-48z^4+18z^2-1=T_6(z)\\
m=7 & \cos(mu)=64z^7-112z^5+56z^3-7z=T_7(z)\\
m=8 & \cos(mu)=128z^8-256z^6+160z^4-32z^2+1=T_8(z)\\
m=9 & \cos(mu)=256z^9-576z^7+432z^5-120z^3+9z=T_9(z)
\end{array}
$$

And each is related to a Tchebyscheff (Chebyshev) polynomial *Tm(z)*.

•**The Tchebyscheff polynomials are valid only in the $-1 \le z \le +1$ *range.***
•***Because | cos(mu)| ≤ 1, each Tschebyscheff polynomial is***
|Tm(z)| ≤ 1 for −1 ≤ z ≤ +1.
•***For |z| > 1, the Tschebyscheff polynomials are related* to the hyperbolic cosine functions.**

- The recursion formula for Tschebyscheff polynomials is

$$\boxed{T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z)}$$

- Each polynomial can also be computed using

$$\boxed{\begin{array}{ll} T_m(z) = \cos[m\cos^{-1}(z)] & -1 \leq z \leq +1 \\ T_m(z) = \cosh[m\cosh^{-1}(z)]^{\dagger} & z < -1, z > +1 \end{array}}$$

•***Since the array factor of an even or odd number of elements is a summation of cosine terms whose form is the same as the Tschebyscheff polynomials***

• *The unknown coefficients of the array factor can be determined by equating the series representing the cosine terms of the array factor to the appropriate Tschebyscheff polynomial.*

• *The order of the polynomial should be one less than the total number of elements of the array.*

Properties of Tchebyscheff polynomials

- 1. All polynomials, of any order, pass through the point (-1, 1).
- 2. Within the range $-1 \le z \le 1$, *the polynomials have values within −1 to +1.*
- 3. All roots occur within $-1 \le z \le 1$, *and all maxima and minima have values of* +1 and −1, respectively.

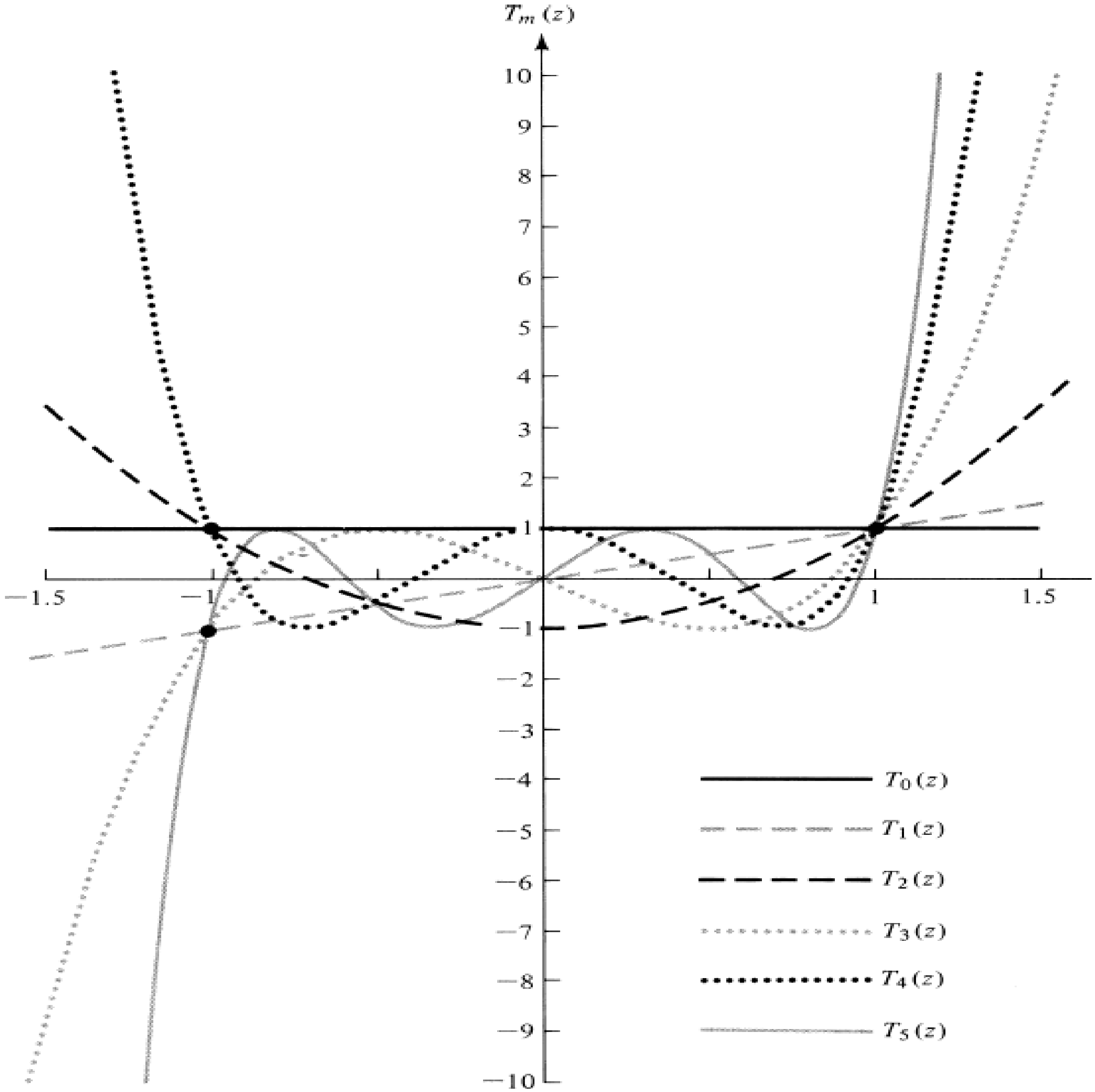

Figure 6.21 Tschebyscheff polynomials of orders zero through five.

Designing a Tschebyscheff array

- Assumptions:
- The number of elements, spacing between the elements, and ratio of major-to-minor lobe intensity ***(RO= main lobe max./side lobe level)*** *are known.*
- Side lobe level below main lobe max in dB = 20 $\log_{10}$RO
- i)Select the appropriate array factor as given by (A)
- ii) Expand the array factor. Replace each cos(*mu) function (m = 0, 1, 2, 3, . . .) by* its appropriate series expansion .

- iii) Determine the point z = *z0 such that Tm(z0) = R0 (voltage ratio). Where m = (no. of sources-1).*
- *The order m of the Tschebyscheff polynomial is always one less than the total number of elements.*
- Equate the array factor to a *Tm(z)* . The *Tm(z) chosen should be of order m*
- *where m is an integer equal to* one less than the total number of elements of the designed array.
- This will allow the determination of the excitation coefficients *an's*.
- Write the array factor of (A) using the coefficients found in above step.

Design a broadside Dolph-Tschebyscheff array of 10 elements with spacing *d between the* elements and with a major-to-minor lobe ratio of 26 dB. Find the excitation coefficients and form the array factor.

- The array factor is given by

$$(\mathrm{AF})_{2M} = \sum_{n=1}^{M=5} a_n \cos[(2n-1)u]$$

$$u = \frac{\pi d}{\lambda} \cos\theta$$

- When expanded, the array factor can be written as

$$\begin{aligned}(\mathrm{AF})_{10} = a_1 \cos(u) + a_2 \cos(3u) \\ + a_3 \cos(5u) + a_4 \cos(7u) + a_5 \cos(9u)\end{aligned}$$

- Replace cos*(u), cos(3u), cos(5u), cos(7u), and cos(9u) by their series expansions*
- *R0 (dB) = 26 = 20 log10(R0) or R0 (voltage ratio) = 20.*
- *Determine z0 by* equating *R0 to T9(z0). Thus*

  R0 = 20 = T9(z0) = $\cosh[9 \cosh^{-1}(z_0)]$ = 1.0851

- *Also*
- where *P is an integer equal to one less than the number of array elements (in this* case *P* = *9).*

$$\boxed{z_0 = \frac{1}{2}\left[\left(R_0 + \sqrt{R_0^2 - 1}\right)^{1/P} + \left(R_0 - \sqrt{R_0^2 - 1}\right)^{1/P}\right]}$$

In normalized form, the a_n coefficients can be written as

$$\begin{array}{lcl} a_5 = 1 & & a_5 = 0.357 \\ a_4 = 1.357 & & a_4 = 0.485 \\ a_3 = 1.974 & \text{or} & a_3 = 0.706 \\ a_2 = 2.496 & & a_2 = 0.890 \\ a_1 = 2.798 & & a_1 = 1 \end{array}$$

Using the first (left) set of normalized coefficients, the array factor can be written as:

$$(AF)_{10} = 2.798 \cos(u) + 2.496 \cos(3u) + 1.974 \cos(5u) + 1.357 \cos(7u) + \cos(9u)$$

where $u = [(\pi d/\lambda) \cos \theta]$.

Tschebyscheff Polynomial of $n = 9$

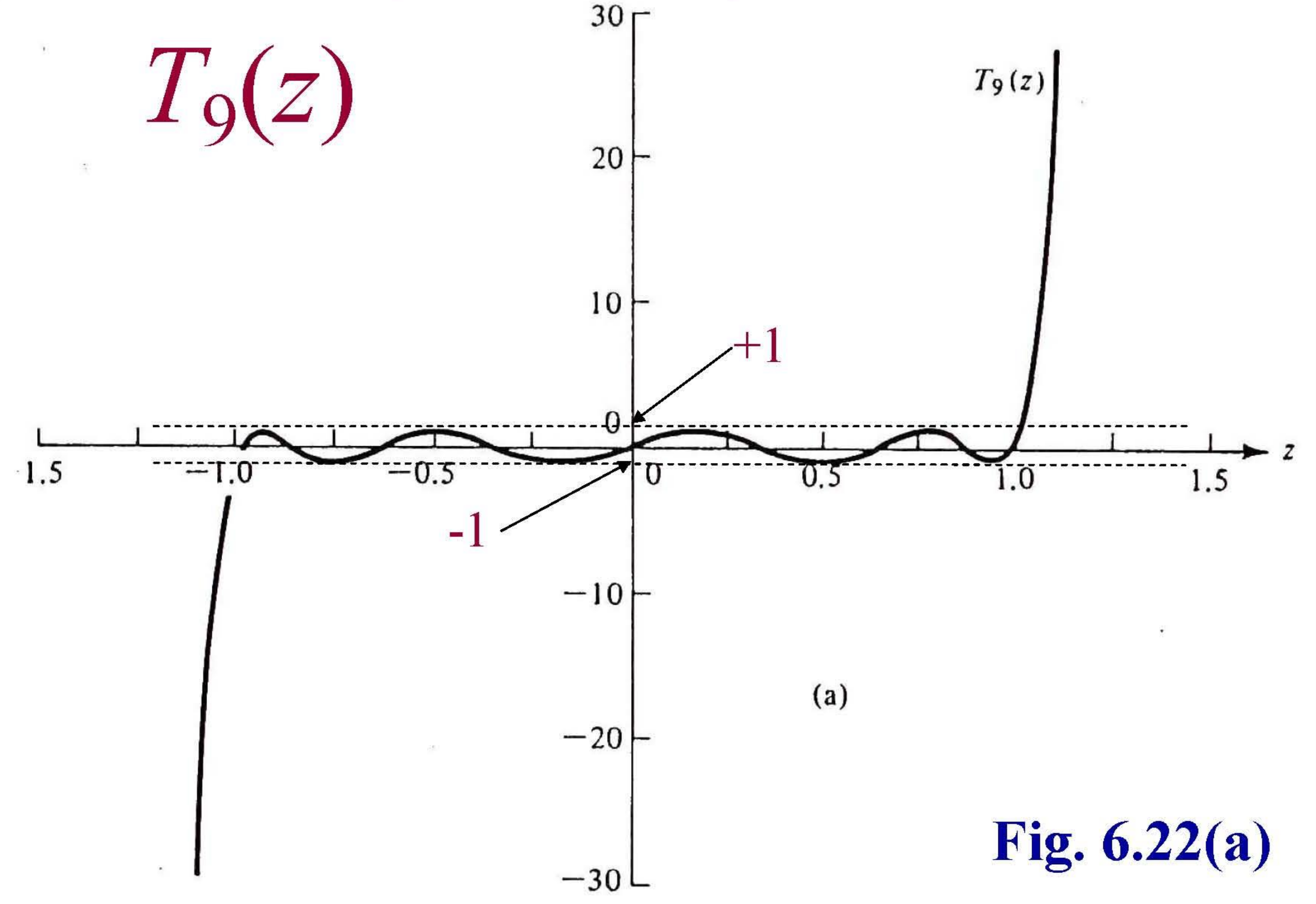

Fig. 6.22(a)

$$\cos(u) = \frac{z}{z_0} = \frac{z}{1.0851}$$

$$\begin{aligned}
(\mathrm{AF})_{10} = {} & z[(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5)/z_0] \\
& + z^3[(4a_2 - 20a_3 + 56a_4 - 120a_5)/z_0{}^3] \\
& + z^5[(16a_3 - 112a_4 + 432a_5)/z_0{}^5] \\
& + z^7[(64a_4 - 576a_5)/z_0{}^7] \\
& + z^9[(256a_5)/z_0{}^9] \\
& = 9z - 120z^3 + 432z^5 - 576z^7 + 256z^9
\end{aligned}$$

Matching similar terms allows the determination of the a_n's. That is,

$$\begin{aligned}
& 256a_5/z_0^9 = 256 && \Rightarrow a_5 = 2.0860 \\
& (64a_4 - 576a_5)/z_0{}^7 = -576 && \Rightarrow a_4 = 2.8308 \\
& (16a_3 - 112a_4 + 432a_5)/z_0{}^5 = 432 && \Rightarrow a_3 = 4.1184 \\
& (4a_2 - 20a_3 + 56a_4 - 120a_5)/z_0{}^3 = -120 && \Rightarrow a_2 = 5.2073 \\
& (a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5)/z_0 = 9 && \Rightarrow a_1 = 5.8377
\end{aligned}$$

$$\frac{z}{z_0} = \cos u = \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

$$z = z_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) \qquad \text{(6-74)}$$

1. $d = \lambda/4,\ N = 10,\ R_{OVR} = 20 \ \Rightarrow z_0 = 1.0851$

$$z = z_o \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) = 1.0851\cos\left(\frac{\pi}{4}\cos\theta\right)$$

$$\theta = 0^\circ: z = 1.0851\cos\left(\frac{\pi}{4}\right) = 1.0851(0.707) = 0.7673$$

$$\theta = 90^\circ: z = 1.0851\cos(0) = 1.0851$$

$$\theta = 180^\circ: z = 1.0851\cos\left(-\frac{\pi}{4}\right) = 1.0851(0.707) = 0.7673$$

2. $d = \lambda/2,\ N = 10,\ R_{OVR} = 20 \ \Rightarrow z_0 = 1.0851$

$$z = 1.0851\cos\left(\frac{\pi}{2}\cos\theta\right)$$

$$\theta = 0^\circ : z = 1.0851\cos\left(\frac{\pi}{2}\right) = 0$$

$$\theta = 90^\circ : z = 1.0851\cos(0) = 1.0851$$

$$\theta = 180^\circ : z = 1.0851\cos\left(-\frac{\pi}{2}\right) = 0$$

TABLE 6.9 Values of the Abscissa z as a Function of θ for a 10-Element Dolph-Tschebyscheff Array with $R_0 = 20$

	$d = \lambda/4$	$d = \lambda/2$	$d = 3\lambda/4$	$d = \lambda$
θ	z (Eq. 6-74)	z (Eq. 6-74)	z (Eq. 6-74)	z (Eq. 6-74)
0°	0.7673	0.0	−0.7673	−1.0851
10°	0.7764	0.0259	−0.7394	−1.0839
20°	0.8028	0.1026	−0.6509	−1.0657
30°	0.8436	0.2267	−0.4912	−0.9904
40°	0.8945	0.3899	−0.2518	−0.8049
50°	0.9497	0.5774	0.0610	−0.4706
60°	1.0025	0.7673	0.4153	0.0
70°	1.0462	0.9323	0.7514	0.5167
80°	1.0750	1.0450	0.9956	0.9276
90°	1.0851	1.0851	1.0851	1.0851
100°	1.0750	1.0450	0.9956	0.9276
110°	1.0462	0.9323	0.7514	0.5167
120°	1.0025	0.7673	0.4153	0.0
130°	0.9497	0.5774	0.0610	−0.4706
140°	0.8945	0.3899	−0.2518	−0.8049
150°	0.8436	0.2267	−0.4912	−0.9904
160°	0.8028	0.1026	−0.6509	−1.0657
170°	0.7764	0.0259	−0.7394	−1.0839
180°	0.7673	0.0	−0.7673	−1.0851

Array Factor Power Pattern of a 10-Element Broadside Dolph-Tschebyscheff Array

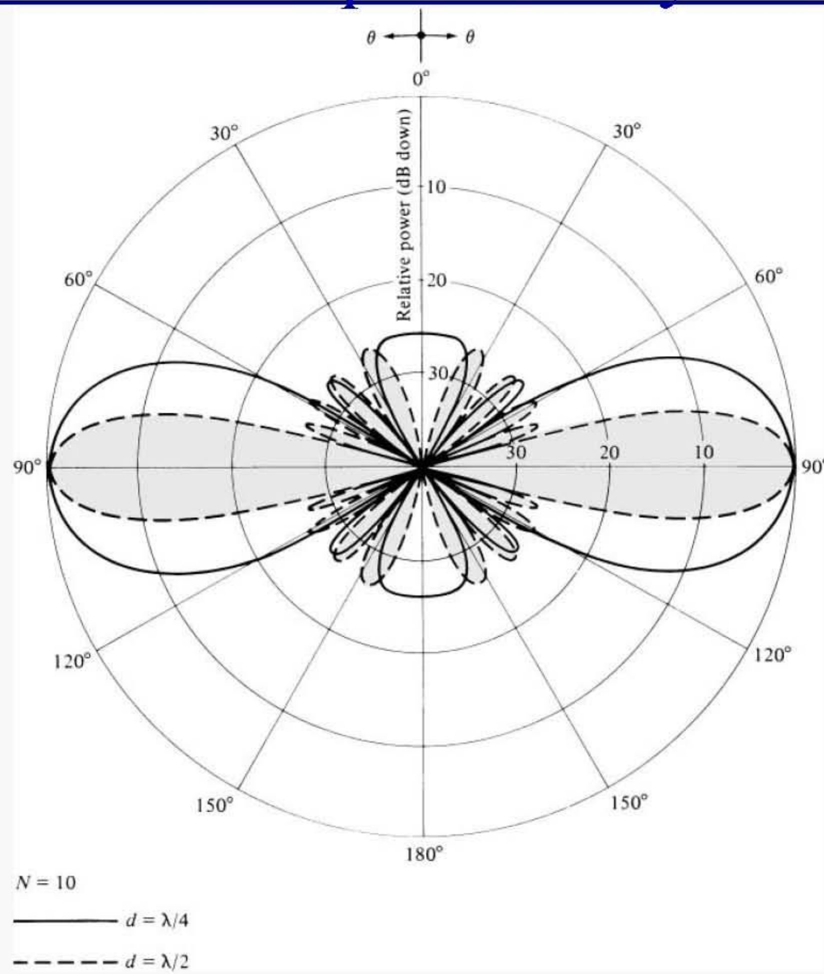

Fig. 6.23

- The directivity of a Dolph-Tschebyscheff array, with a given side lobe level,increases as the array size or number of elements increases.

- For a given array length, or a given number of elements in the array, the directivity does not necessarily increase as the side lobe level decreases.

- As a matter of fact, a −15 dB side lobe array has smaller directivity than a −20 dB side lobe