# UEC747: ANTENNA AND WAVE PROPAGATION

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## Lecture 4: Review of Coordinate System

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# Coordinate Systems

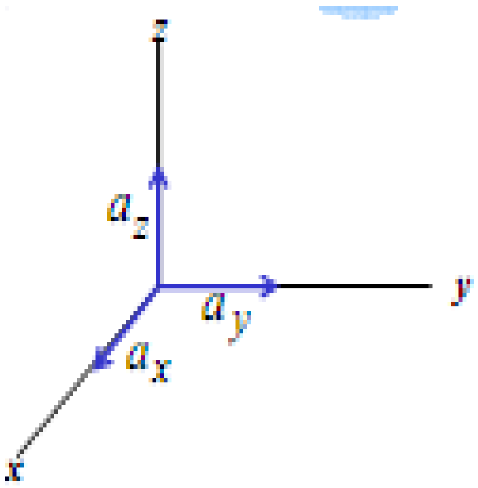
- The vector quantity can be of one dimensional or n-dimensional. Hence it is necessary to understand the coordinate systems in which these vector quantities are represented.
- In this book we will frequently encounter problems where antenna is considered as source radiating in space. To be able to specify the fields at a point in space caused by an antenna, we have to make reference to a coordinate system. As antenna radiates in free space we have to consider three dimensional coordinate systems. In antenna theory the antenna is considered as a point source radiating in free space the spherical coordinate system is generally applicable for antenna analysis.

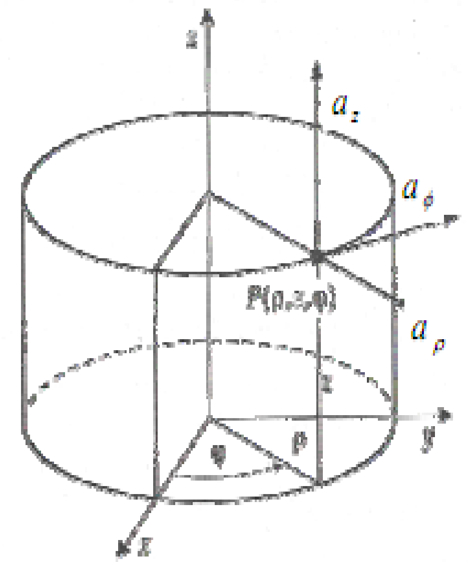
# Coordinate Systems

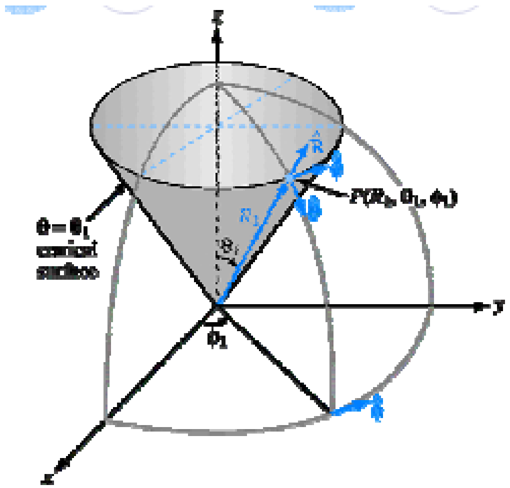
A 3-Dimensional(3-D) coordinate system is specified by intersection of three surfaces.

Each surface is described by $\xi_1$=constant, $\xi_2$=constant and $\xi_3$=constant, where $\xi_i$ is the i$^{th}$ axis of the orthogonal coordinate system.

An orthogonal coordinate system is defined when these three surfaces are mutually orthogonal at a point.



*Choice is based on symmetry of problem*

*<u>Examples:</u>*

*Sheets - RECTANGULAR*

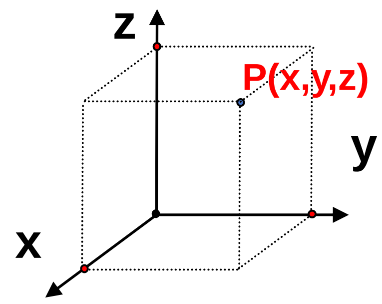
*Wires/Cables - CYLINDRICAL*

*Spheres - SPHERICAL*

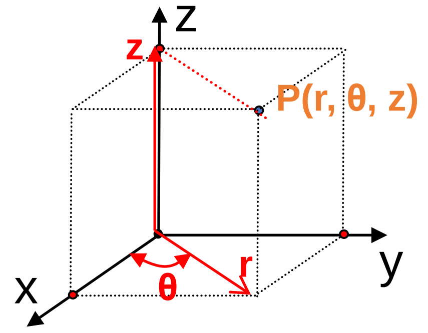
## **Orthogonal Coordinate Systems:** (coordinates mutually perpendicular)

# 3 PRIMARY COORDINATE SYSTEMS:

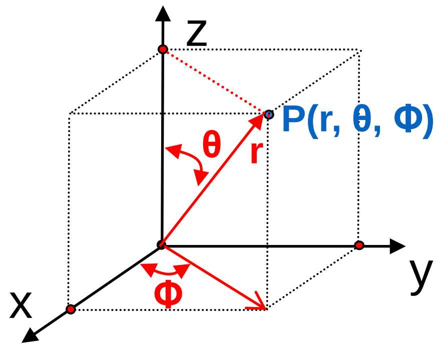
- RECTANGULAR
- CYLINDRICAL
- SPHERICAL




**Rectangular Coordinates**




**Cylindrical Coordinates**

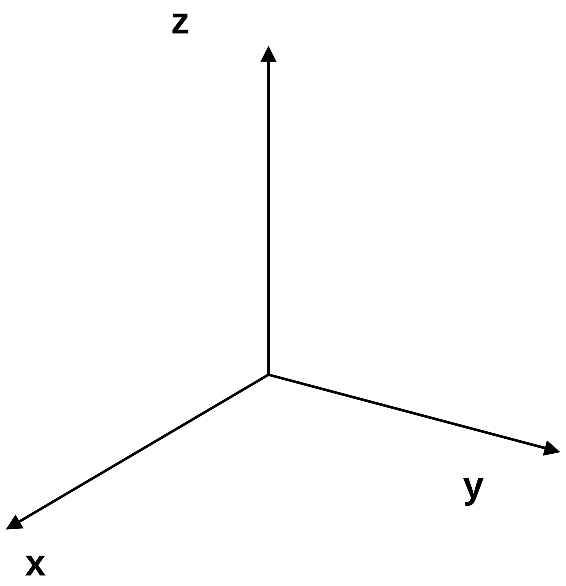


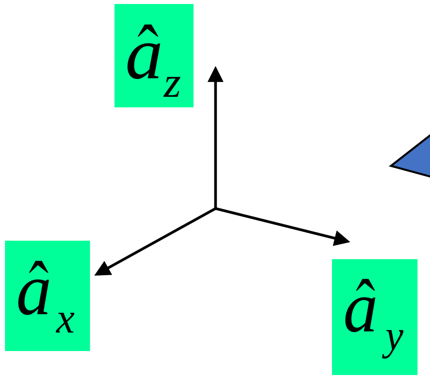

**Spherical Coordinates**

- In Cartesian coordinate system all of these surfaces are planes and they are specified by independent variables x, y and z separately being constant.
- In cylindrical coordinate system two surfaces are planes and one cylinder.
- In spherical coordinate system surfaces are a sphere, a plane and a cone.

# VECTOR REPRESENTATION: UNIT VECTORS

## Rectangular Coordinate System




Unit Vector Representation for Rectangular Coordinate System

The Unit Vectors imply :

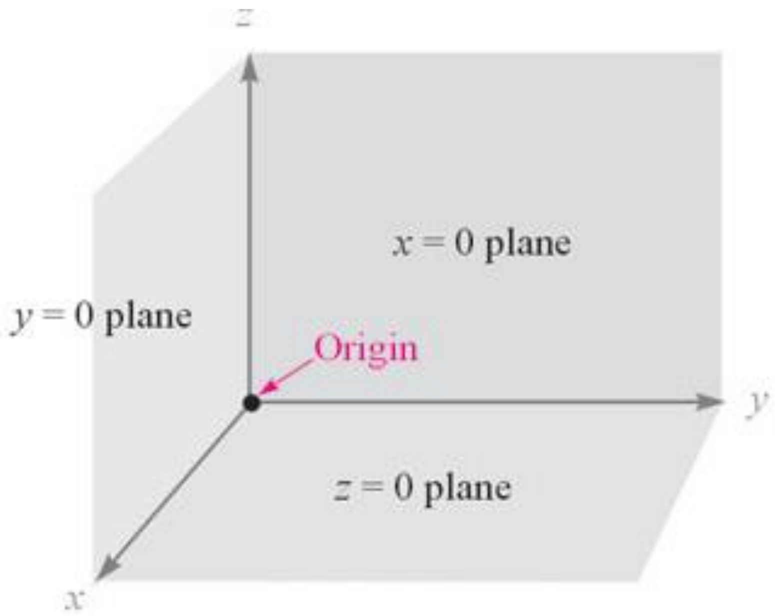
$\hat{a}_x$ ⇒ Points in the direction of increasing **x**

$\hat{a}_y$ ⇒ Points in the direction of increasing **y**

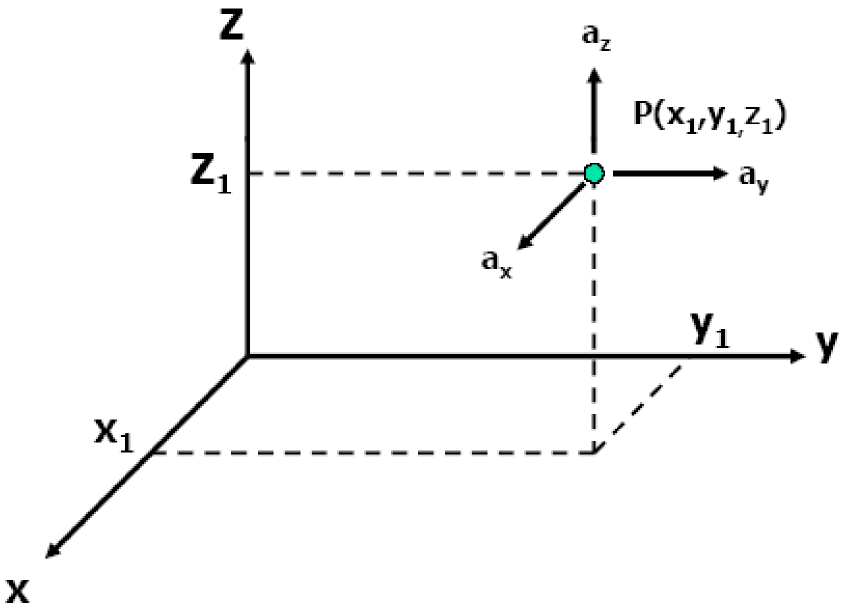
$\hat{a}_z$ ⇒ Points in the direction of increasing **z**

# Cartesian coordinates

- Consists of three mutually orthogonal axes $(x, y, z)$ and a point in space is denoted as $P(x, y, z)$

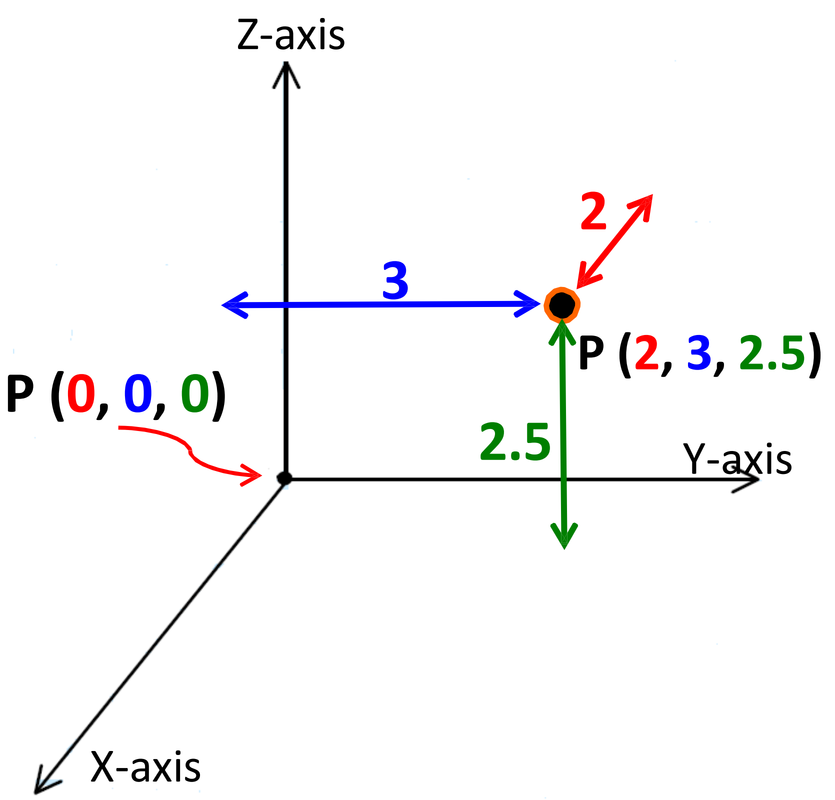



- Unit vector of $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ in the direction of increasing coordinate value.



# Cartesian Coordinates

- Note the coordinate values in the Cartesian system effectively represent the **distance** from a **plane** intersecting the origin.
- For **example**, x =3 means that the point is **3 units** from the y-z **plane** (i.e., the x = 0 plane).
- Likewise, the y coordinate provides the **distance** from the x-z (y=0) plane, and the z coordinate provides the **distance** from the x-y (z =0) plane.
- Once **all three** distances are specified, the **position** of a point is **uniquely** identified.



# Cartesian Coordinates

**Differential quantities:**

Differential distance:

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

Differential surface:

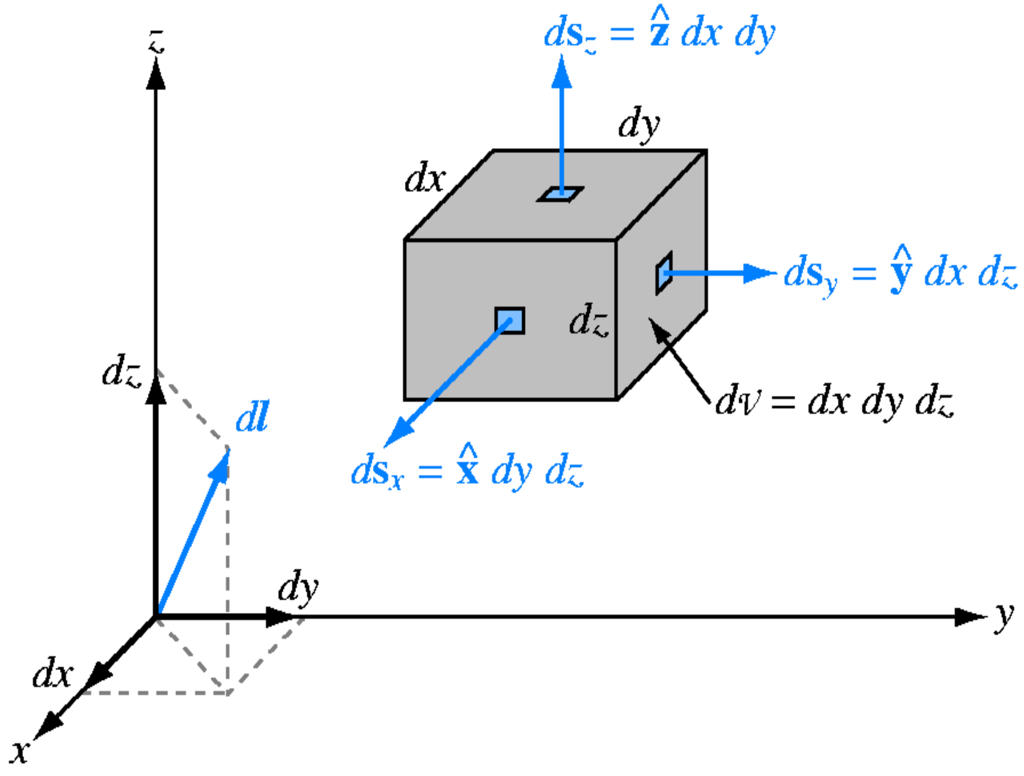
$$d\vec{s}_x = \hat{x}dydz$$

$$d\vec{s}_y = \hat{y}dxdz$$

$$d\vec{s}_z = \hat{z}dxdy$$

Differential Volume:

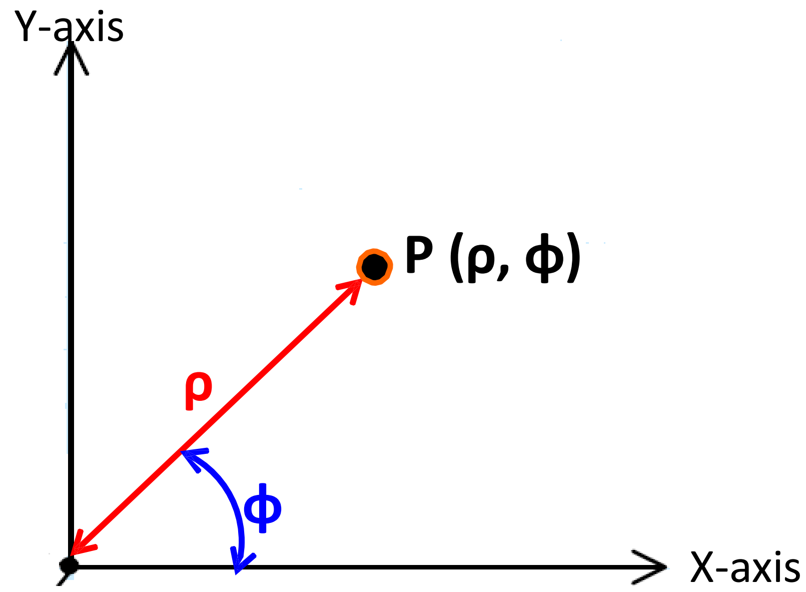
$$dv = dxdydz$$

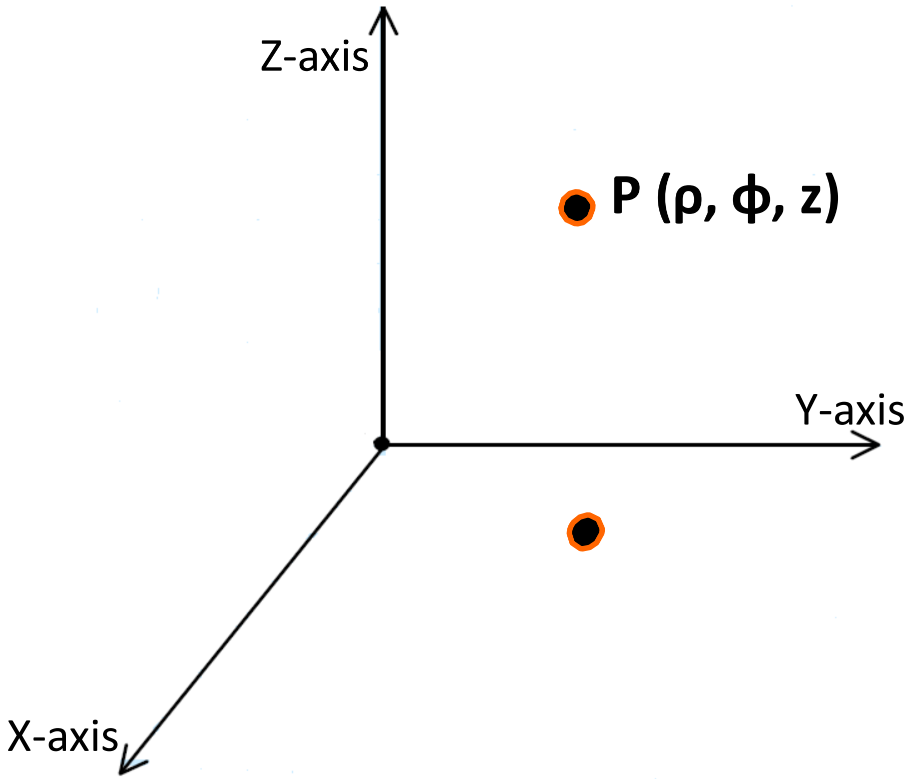


# Cylindrical Coordinates

- You're also familiar with **polar coordinates**. In **two** dimensions, we specify a point with **two** scalar values, generally called ρ and ϕ.

We can extend this to **3**-dimensions, by adding a **third** scalar value z. This method for identifying the position of a point is referred to as **cylindrical coordinates**.

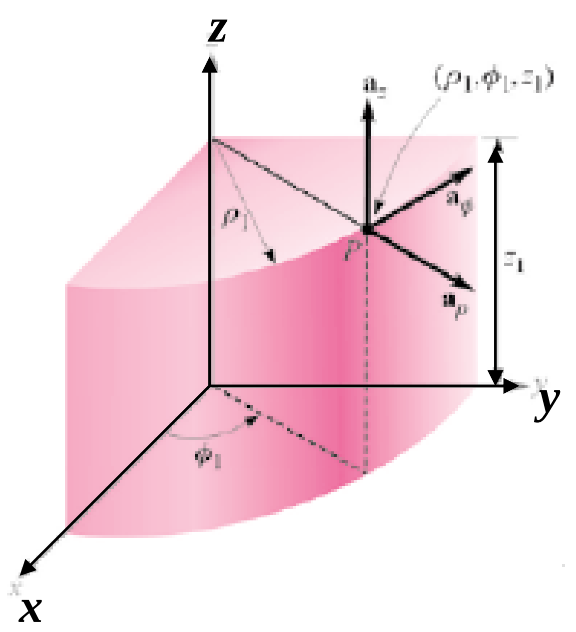


# Cylindrical Coordinates

**Note the physical significance of each parameter of cylindrical coordinates:**

1. The value $\rho$ indicates the **distance** of the point from the **z-axis** ($0 \le \rho < \infty$).
2. The value $\phi$ indicates the **rotation angle** around the **z-axis** ($0 \le \phi < 2\pi$), **precisely** the same as the angle $\phi$ used in **spherical** coordinates.
3. The value $z$ indicates the **distance** of the point from the x-y ($z = 0$) plane ($-\infty < z < \infty$), **precisely** the same as the coordinate **z** used in **Cartesian** coordinates.
4. Once **all three** values are specified, the **position** of a point is **uniquely** identified.
5. Unit vector of $\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z$ in the direction of increasing coordinate value.




**Form by three surfaces or planes:**

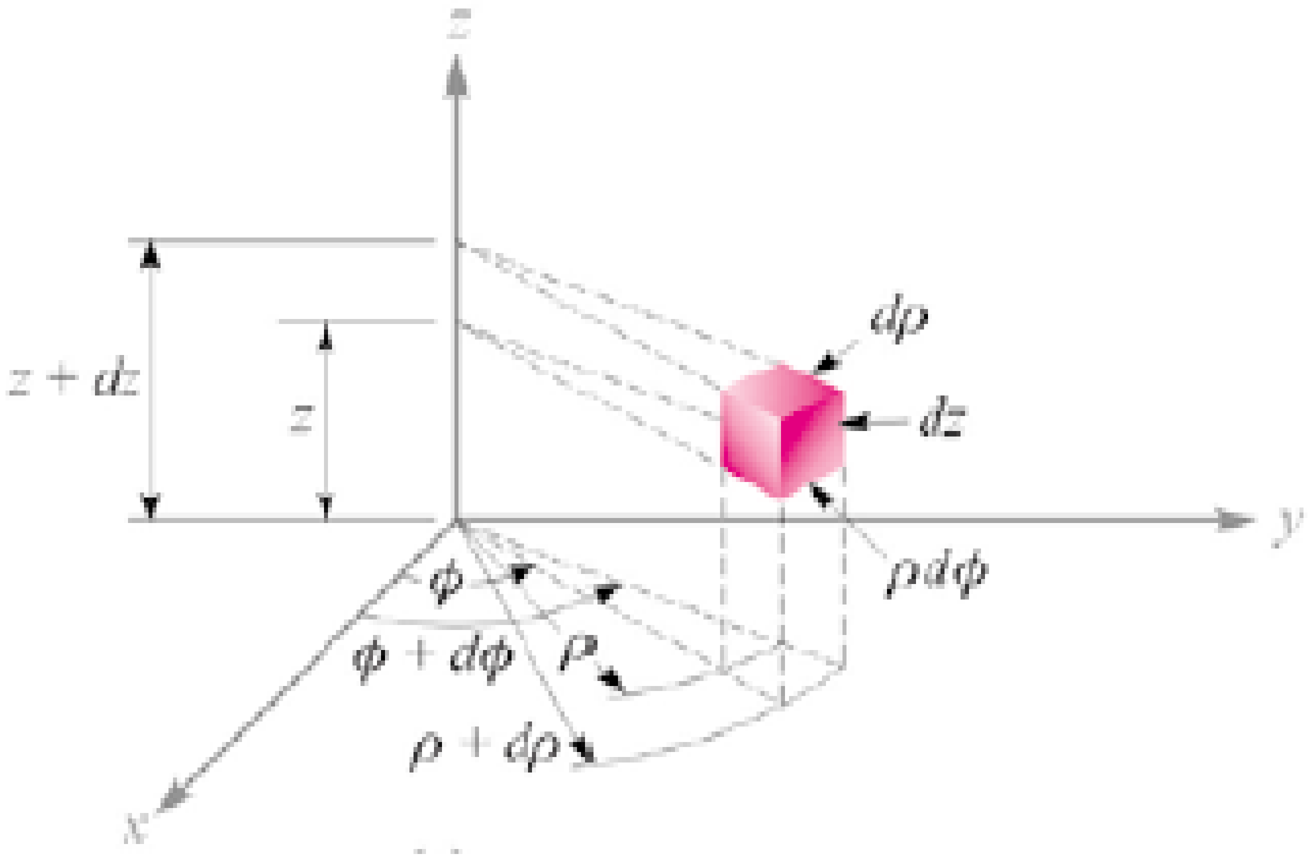
- Plane of z (constant value of z)
- Cylinder centered on the z axis with a radius of $r$. Some books use the notation $\rho$.
- Plane perpendicular to x-y plane and rotate about the z axis by angle of $\phi$

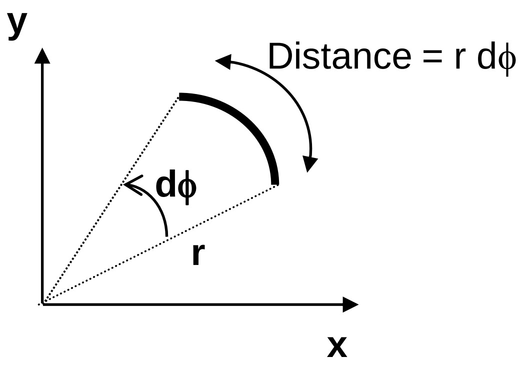
# Cylindrical Coordinates:

- Increment in **length** for direction is: $\phi$

$$\boxed{\rho d\phi}$$

- $d\phi$ is not increment in length!




Differential Distances: **( dr, rdϕ, dz )**

# Cylindrical Coordinates:

Differential Distances: **( dρ, rdϕ, dz )**

$$d\vec{l} = d\rho \bullet \hat{a}\rho + \rho \bullet d\phi \bullet \hat{a}_\phi + dz \bullet \hat{a}_z$$

Differential Surfaces:

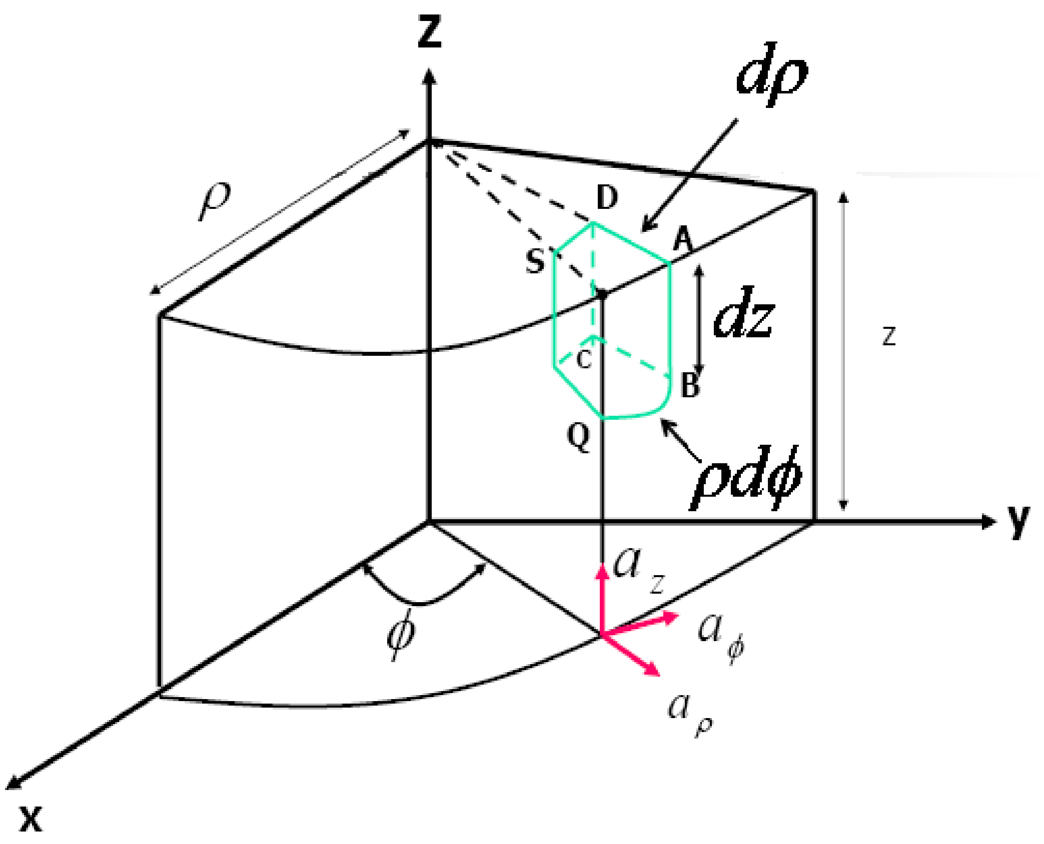
$$d\vec{s}_\rho = \rho d\phi \bullet dz \bullet \hat{a}_\rho$$

$$d\vec{s}_\phi = d\rho \bullet dz \bullet \hat{a}_\phi$$

$$d\vec{s}_z = \rho d\phi \bullet d\rho \bullet \hat{a}_z$$

Differential Volume:

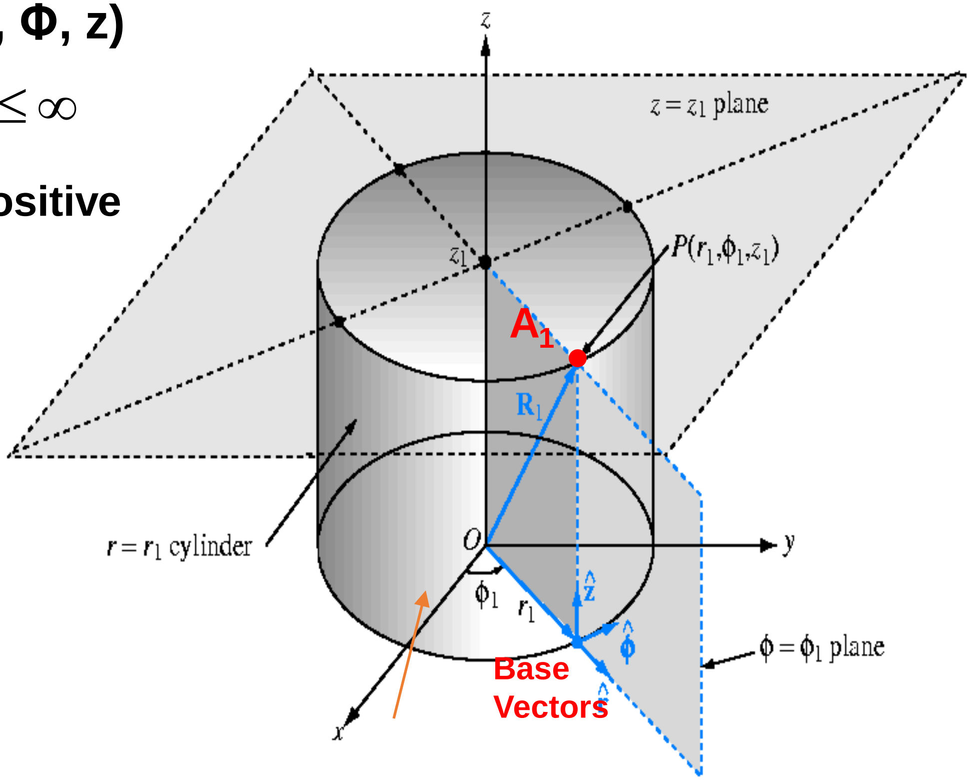
$$dv = \rho \, d\rho \, d\phi \, dz$$



# Cylindrical Coordinates

**( ρ, Φ, z)**

- **ρ radial distance in x-y plane** $0 \le r \le \infty$
- **Φ azimuth angle measured from the positive x-axis** $0 \le \Phi < 2\pi$
- **Z** $-\infty < z < \infty$




## Vector representation

$$\vec{A} = \hat{a}\left|\vec{A}\right| = \hat{\rho}A_\rho + \hat{\Phi}A_\Phi + \hat{z}A_z$$

## Magnitude of A

$$\left|\vec{A}\right| = \sqrt[+]{\vec{A}\cdot\vec{A}} = \sqrt[+]{A_\rho^2 + A_\Phi^2 + A_z^2}$$

## Position vector A

$$\hat{\rho}\rho_1 + \hat{z}z_1$$

## Base vector properties

$$\hat{\rho}\times\hat{\Phi} = \hat{z},$$

$$\hat{\Phi}\times\hat{z} = \hat{\rho},$$

$$\hat{z}\times\hat{\rho} = \hat{\Phi}$$

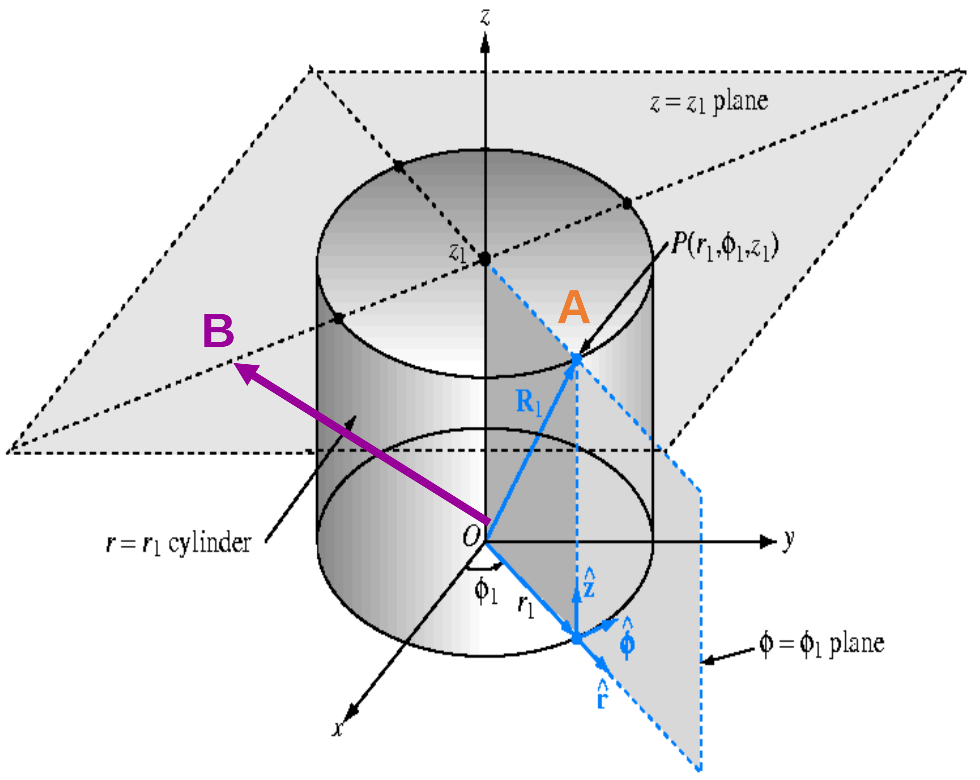
# Cylindrical Coordinates

## Dot product:

$$\vec{A}\cdot\vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

## Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$



# Example 6

A cylinder with radius of $\rho$ and length of $L$

Determine:

(i) The volume enclosed.

(ii) The surface area of that volume.

# Solution to Example 6

- (i) For volume enclosed, we integrate;

$$V = \int_v dV$$

$$= \int_{\rho=0}^{\rho} \int_{\phi=0}^{\phi=2\pi} \int_{z=0}^{z=L} \rho d\phi d\rho dz$$

$$= \left[\frac{\rho^2}{2}\right]_0^{\rho} [\phi]_0^{2\pi} [z]_0^{z=L}$$

$$= \left(\frac{\rho^2}{2}\right)(2\pi)(L)$$

$$= \pi\rho^2 L$$

# Solution to Example 6

- (ii) For surface area, we add the area of each surfaces;

$$S = \underbrace{\int_{\phi=0}^{2\pi}\int_{z=0}^{L} \rho d\phi dz}_{sides} + \underbrace{\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \rho d\phi d\rho}_{bottom} + \underbrace{\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \rho d\phi d\rho}_{top}$$

$$= (\rho)\ {}_{\phi=}^{2\pi}[\phi]\ {}_{z=0}^{z=L}[z] + {}^{\rho}_{0}\left[\frac{\rho^2}{2}\right]\ {}_{0}^{2\pi}[\phi] + {}^{\rho}_{0}\left[\frac{\rho^2}{2}\right]\ {}_{0}^{2\pi}[\phi]$$

$$= 2\pi\rho L + \pi\rho^2 + \pi\rho^2$$

$$= 2\pi\rho L + 2\pi\rho^2$$

# Example 7

The surfaces $\rho = 3, \rho = 5, \phi = 100^0, \phi = 130^0, z = 3, z = 4.5$

define a closed surface. Find:

(a) The enclosed volume.

(b) The total area of the enclosing surface.

# Solution to Example 7

(a) The enclosed volume;

$$V = \int_{\rho=3}^{5} \int_{\phi=1.745}^{2.269} \int_{z=3}^{4.5} \rho d\rho d\phi dz$$

$$= \left[\frac{\rho^2}{2}\right]_3^5 \left[\phi\right]_{1.745}^{2.269} \left[z\right]_3^{4.5}$$

$$= (8)(0.524)(1.5)$$

$$= 6.288$$

Must convert $\phi$ into radians

# Solution to Example 7

(b) The total area of the enclosed surface:

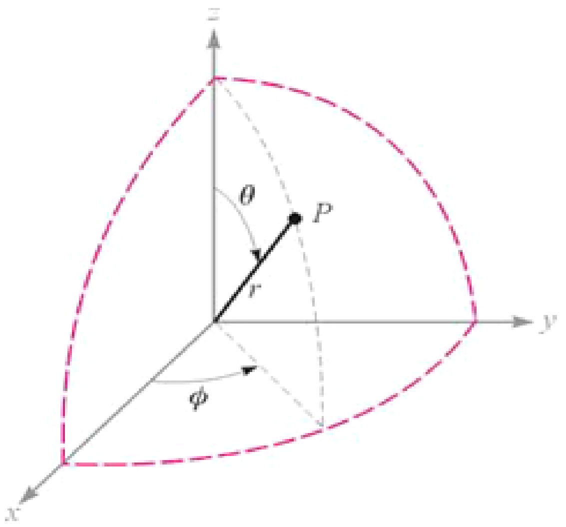
$$Area = 2\int_{\phi=1.745}^{2.269}\int_{\rho=3}^{5}\rho\, d\rho\, d\phi + \int_{z=3}^{4.5}\int_{\phi=1.745}^{2.269} 3\, d\phi\, dz$$

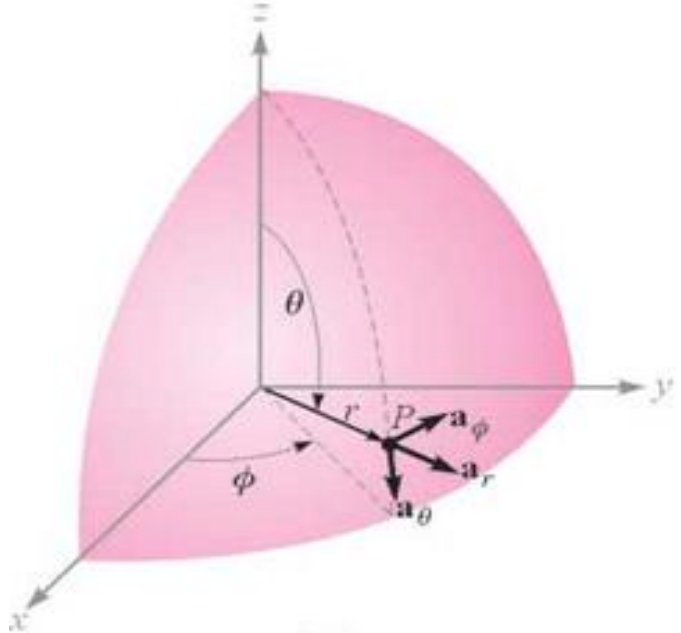
$$+\int_{z=3}^{4.5}\int_{\phi=1.745}^{2.269} 5\, d\phi\, dz + 2\int_{z=3}^{4.5}\int_{\rho=3}^{5} d\rho\, dz$$

$$= 20.7$$

# VECTOR REPRESENTATION: UNIT VECTORS

## Spherical Coordinate System



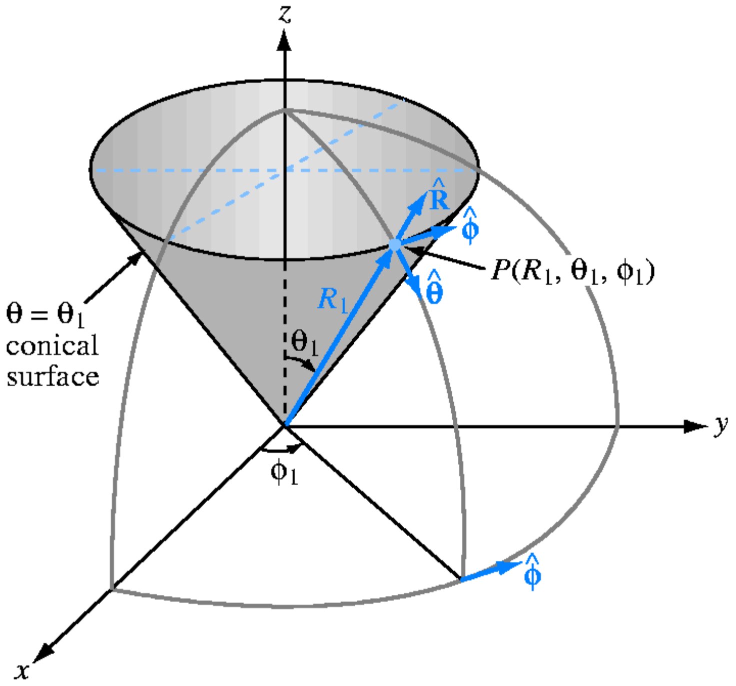




Figure 3-13

The Unit Vectors imply :

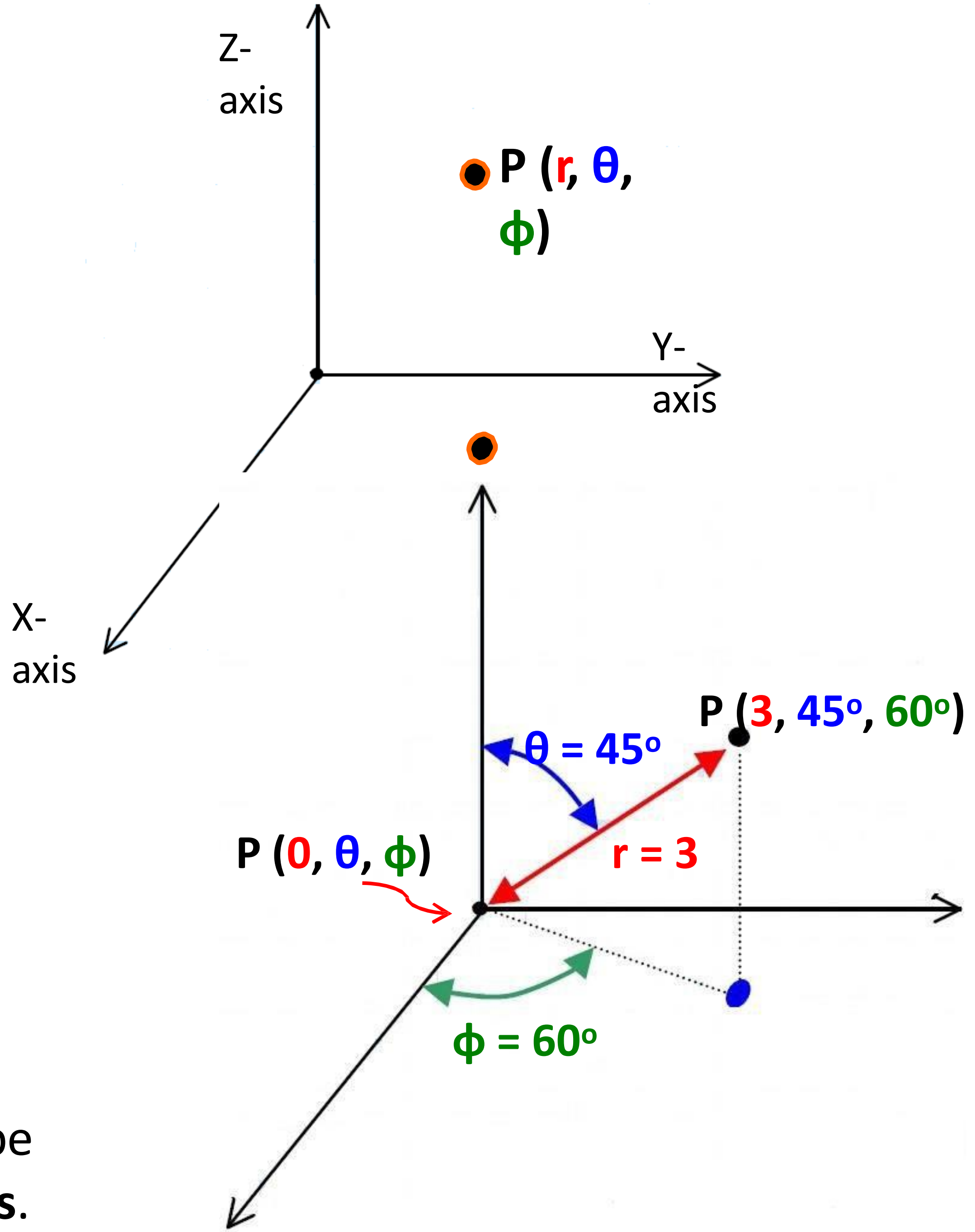
$\hat{a}_r$ ➡ Points in the direction of increasing **r**

$\hat{a}_\theta$ ➡ Points in the direction of increasing θ

$\hat{a}_\phi$ ➡ Points in the direction of increasing φ

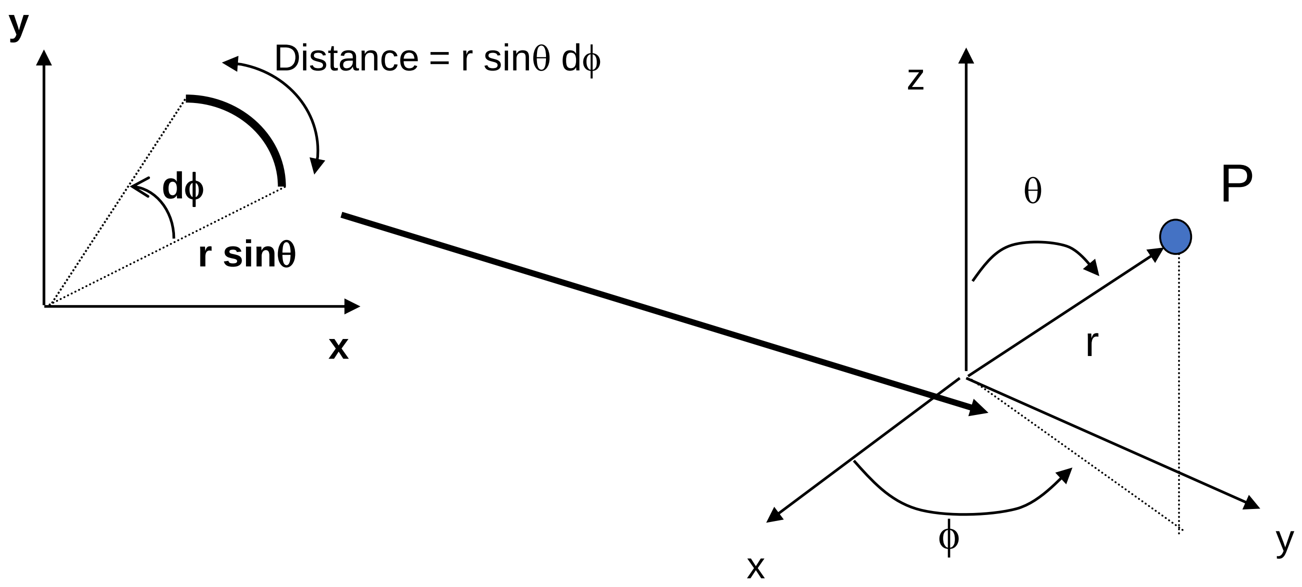
# Spherical Coordinates

- For spherical coordinates, r ($0 \le r < \infty$) expresses the **distance** of the point from the **origin** (i.e., similar to **altitude**).
- Angle **θ** ($0 \le \theta \le \pi$) represents the angle formed **with the z-axis** (i.e., similar to **latitude**).
- Angle **ϕ** ($0 \le \phi < 2\pi$) represents the rotation angle around the z-axis, **precisely** the same as the **cylindrical** coordinate ϕ (i.e., similar to **longitude**).

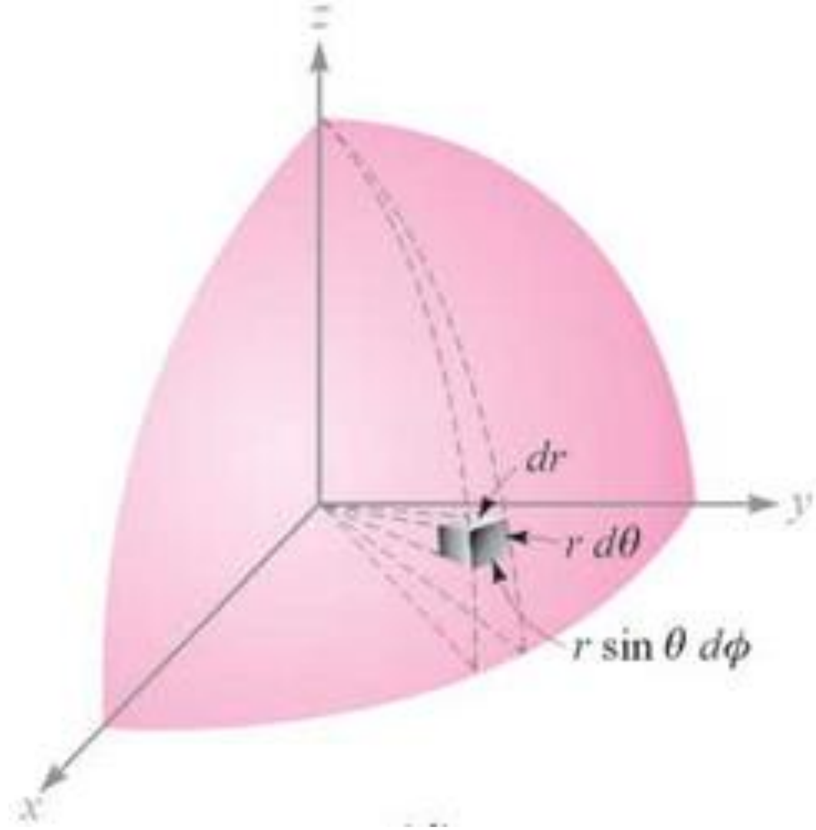



Thus, using **spherical** coordinates, a point in space can be unambiguously defined by **one distance** and **two angles**.

# Spherical Coordinates:




Differential Distances: **( dr, rdθ, r sinθ dϕ )**



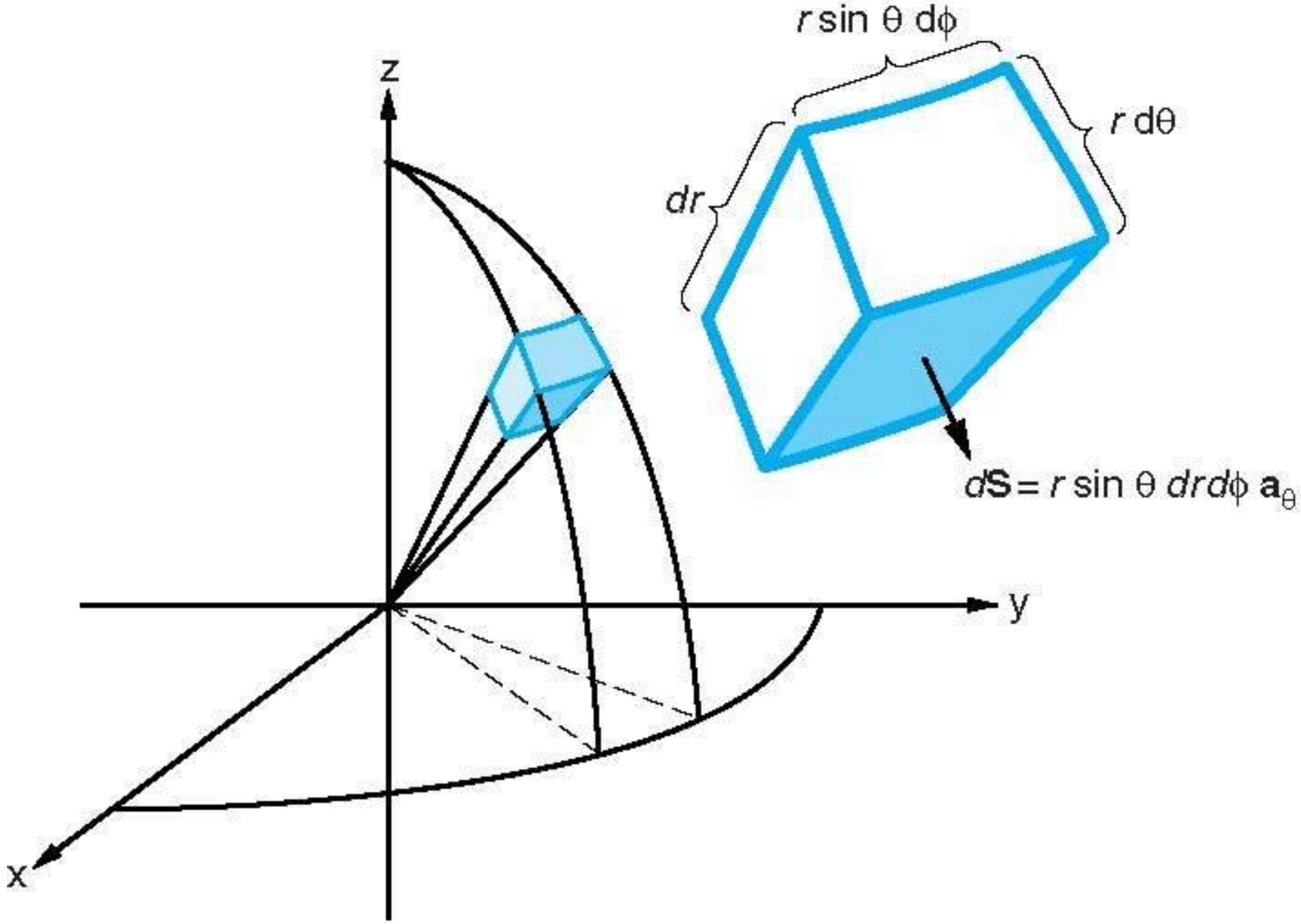
# Spherical Coordinates

- Differential Surface

$$dS_r = r^2 \sin\theta d\theta d\phi \mathbf{a}_r$$

$$dS_\theta = r \sin\theta dr d\phi \mathbf{a}_\theta$$

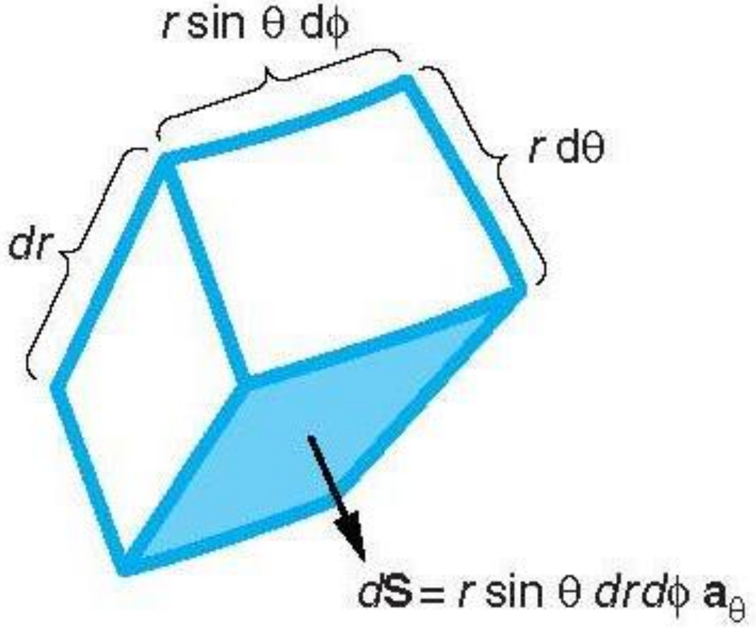
$$dS_\phi = r dr d\theta \mathbf{a}_\phi$$

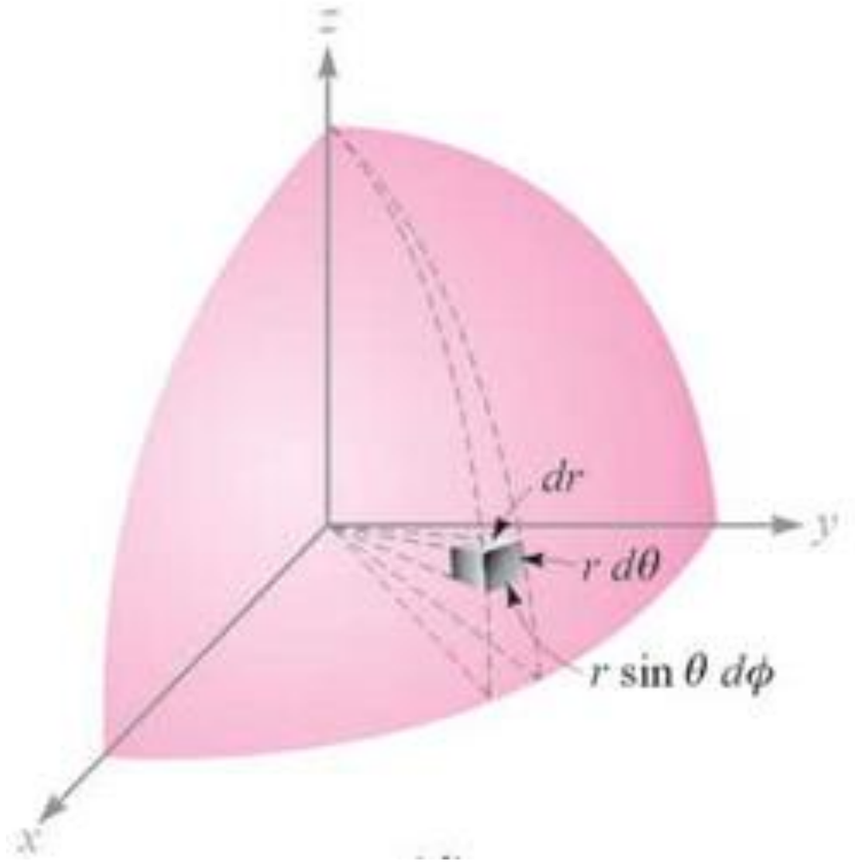


# Spherical Coordinates

- Differential Volume

$$dV = r^2 \sin\theta dr d\theta d\phi$$



# Spherical Coordinates

**Differential quantities:**

**Length:**

$$d\vec{l} = \hat{R}dl_R + \hat{\Theta}dl_\Theta + \hat{\Phi}dl_\Phi$$
$$= \hat{R}dR + \hat{\Theta}Rd\Theta + \hat{\Phi}R\sin\Theta d\Phi$$

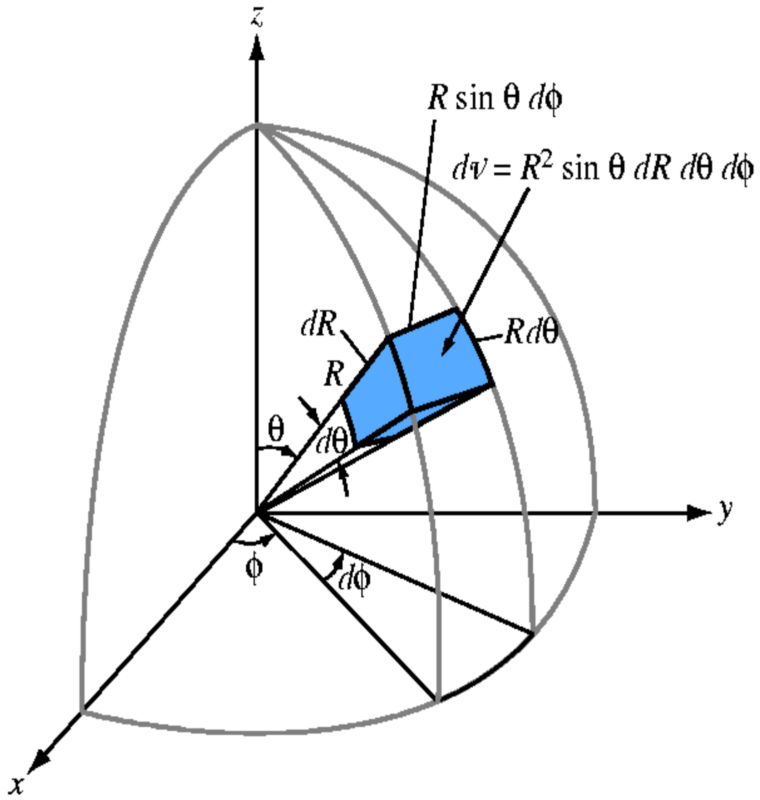
$$dl_R = dR$$
$$dl_\Theta = Rd\Theta$$
$$dl_\Phi = R\sin\Theta d\Phi$$

**Area:**

$$d\vec{s}_R = \hat{R}dl_\Theta dl_\Phi = \hat{R}R^2\sin\Theta d\Theta d\Phi$$
$$d\vec{s}_\Theta = \hat{\Theta}dl_R dl_\Phi = \hat{\Theta}R\sin\Theta dRd\Phi$$
$$d\vec{s}_\Phi = \hat{\Phi}dl_R dl_\Theta = \hat{\Phi}RdRd\Theta$$

**Volume:**

$$dv = R^2\sin\Theta dRd\Theta d\Phi$$




Back

# Example 8

A sphere of radius 2 cm contains a volume charge density $\rho_v$ given by

$$\rho_v = 4\cos^2\theta \quad \left(\mathrm{C/m^3}\right)$$

Find the total charge $Q$ contained in the sphere.

Solution

$$Q = \int_v \rho_v dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2\times10^{-2}} \left(4\cos^2\theta\right) R^2 \sin\theta dR d\theta d\phi$$

$$= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\Bigg|_0^{2\times10^{-2}} \sin\theta \cos^2\theta d\theta d\phi$$

$$= \frac{32}{3}\times10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi} d\phi = 44.68 \quad (\mu\mathrm{C})$$

# Spherical Coordinates

**(R, θ, Φ)**

**Vector representation**

$$\vec{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$

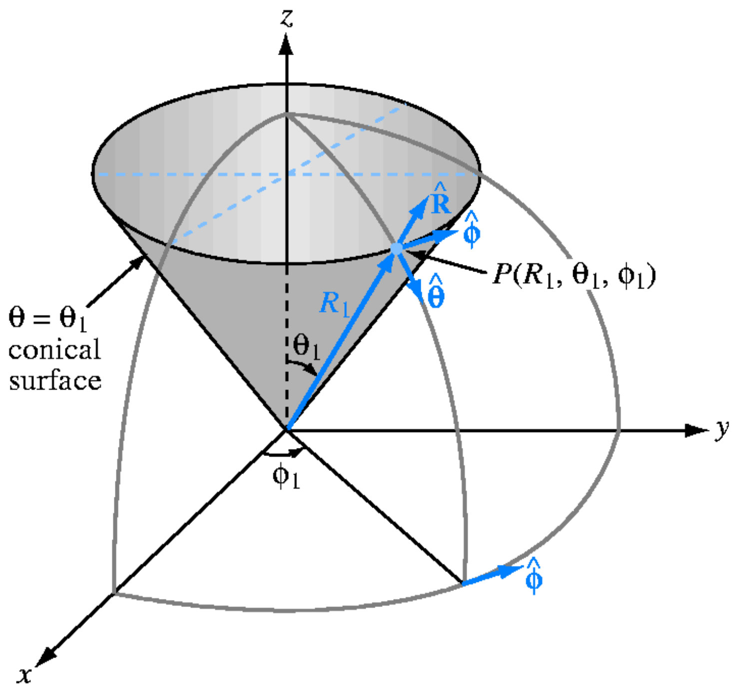



Figure 3-13

**Magnitude of A**

$$\left|\vec{A}\right| = \sqrt[+]{\vec{A}\cdot\vec{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$$

**Position vector A**

$$\hat{R}R_1$$

**Base vector properties**

$$\hat{R}\times\hat{\Theta} = \hat{\Phi}\,, \qquad \hat{\Theta}\times\hat{\Phi} = \hat{R}\,, \qquad \hat{\Phi}\times\hat{R} = \hat{\Theta}$$

# Spherical Coordinates

## Dot product:

$$\vec{A}\cdot\vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

## Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$$

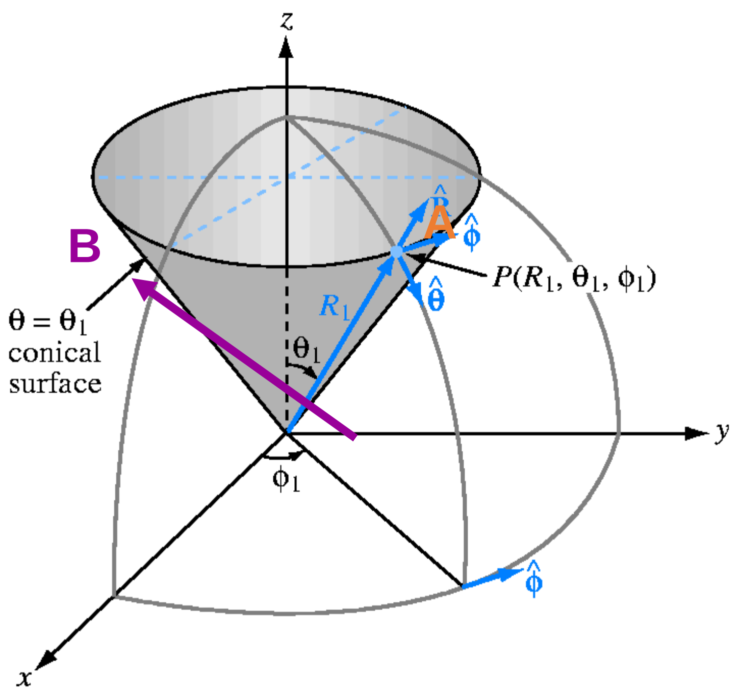



Figure 3-13

Back

# VECTOR REPRESENTATION: UNIT VECTORS

Summary

| RECTANGULAR Coordinate Systems | CYLINDRICAL Coordinate Systems | SPHERICAL Coordinate Systems |
|---|---|---|
| $\left(\hat{a}_x \quad \hat{a}_y \quad \hat{a}_z\right)$ | $\left(\hat{a}_\rho \quad \hat{a}_\phi \quad \hat{a}_z\right)$ | $\left(\hat{a}_r \quad \hat{a}_\theta \quad \hat{a}_\phi\right)$ |

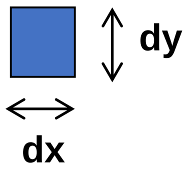
**NOTE THE ORDER!**

$\rho, \phi, z$ $\qquad$ $r, \theta, \phi$

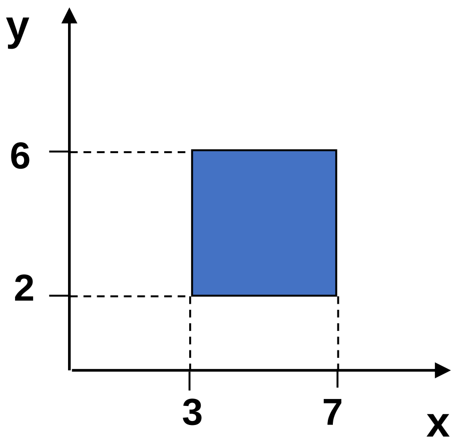
Note: We do not emphasize transformations between coordinate systems

# AREA INTEGRALS

• integration over 2 "delta" distances ⟶




Example:




$$\text{AREA} = \boxed{\int_3^7 \int_2^6 dy \bullet dx} = 16$$

Note that: ***z* = *constant***

In this course, area & surface integrals will be on similar types of surfaces e.g. **r =constant** or **$\phi$ = constant** or **$\theta$ = constant** et c….

# SURFACE NORMAL

Representation of differential surface element:

***Vector is NORMAL to surface***

$$d\vec{s} = dx \bullet dy \bullet \hat{a}_z$$

# DIFFERENTIALS FOR INTEGRALS

Example of Line differentials

$$d\vec{l} = dx \bullet \hat{a}_x$$ or $$d\vec{l} = dr \bullet \hat{a}_r$$ or $$d\vec{l} = rd\phi \bullet \hat{a}_\phi$$

Example of Surface differentials

$$d\vec{s} = dx \bullet dy \bullet \hat{a}_z$$ or $$d\vec{s} = rd\phi \bullet dz \bullet \hat{a}_r$$

Example of Volume differentials ⇨ $$dv = dx \bullet dy \bullet dz$$

# Line Integral

- The line integral is the integral of the tangential component of **A** along Curve L

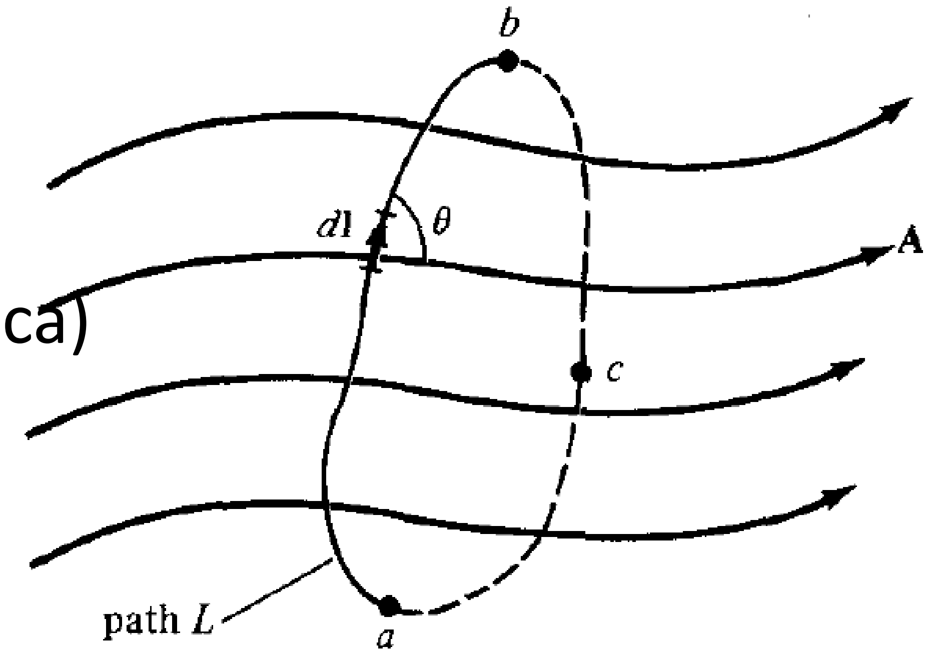
$$\int_L \vec{A}d\vec{l} = \int_a^b \left|\vec{A}\right| \cos\theta d\vec{l}$$

- Closed contour integral (abca)

Circulation of **A** around L

$$\oint_L \vec{A}d\vec{l}$$

**A** is a vector field

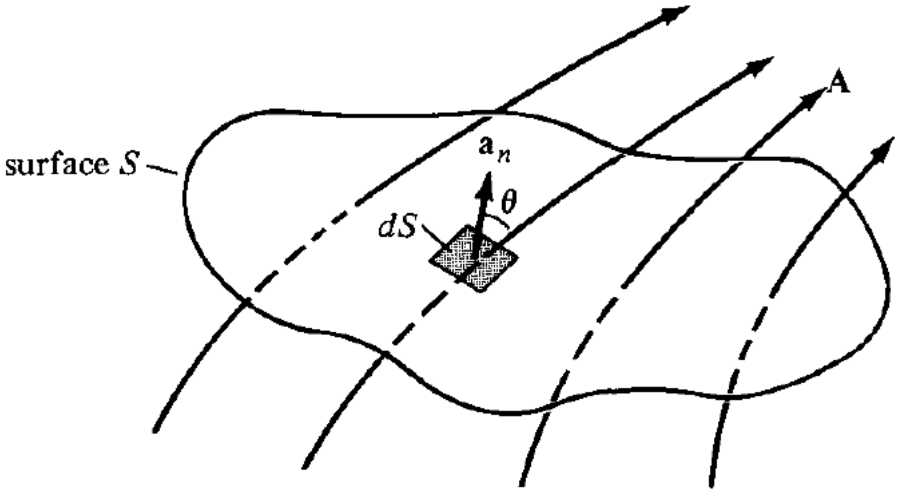


# Surface Integral (flux)

- Vector field A containing the smooth surface S
- Also called; Flux of A through S

**A** is a vector field

$$\psi = \int_S |\vec{A}| \cos\theta dS = \int_S \vec{A}\vec{a}_n dS = \int_S \vec{A} d\vec{S}$$




- Closed Surface Integral

  Net outward flux of A from S

$$\psi = \oint_S \vec{A} d\vec{S}$$

# Volume Integral

- Integral of scalar $\rho_V$ over the volume V

$$\int_V f_V dV = \iiint_V f_V dxdydz$$
$$= \iiint_V f_V \rho d\rho d\phi dz$$
$$= \iiint_V f_V r^2 \sin\theta dr d\theta d\phi$$

THANKS