

# **UEC747: ANTENNA AND WAVE PROPAGATION**

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**Lecture 7: Review of Vector Analysis**

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**and**

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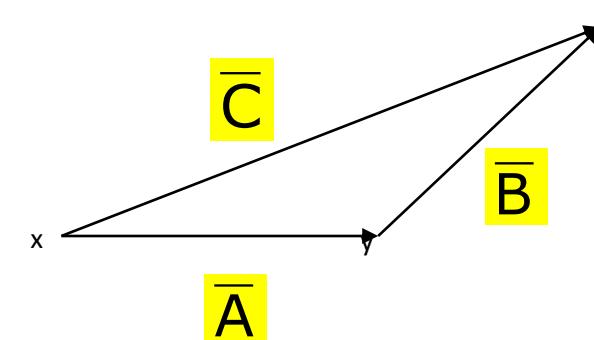
# REVIEW OF VECTOR ANALYSIS

we will discuss basic mathematical operation like vector addition, subtraction, multiplication, vector integrals, vector differential operator and vector relation in other coordinates e.g. Cartesian, cylindrical and spherical. Knowledge of these will play a great role in the study of antenna theory etc. We will review following for operations

- Vector addition and subtraction
- Vector Multiplication (Dot and cross products)
- Vector Integral (Line and surface integrals)
- Vector Differentiation (Introduction to differential operators)

# Vector>Addition

Consider the displacement of a point from location x to y and from y to z as shown below. The resultant linear displacement C from point x to z can be found by adding vectors  $\bar{A}$  and  $\bar{B}$ .



Two vectors  $\bar{A}$  and  $\bar{B}$  can be added together to give another vector  $\bar{C}$ ; that is,

$$\bar{C} = \bar{A} + \bar{B}$$

Vectors are added by adding their individual components. Thus, if

$$A_x a_x + A_y a_y + A_z a_z \quad \text{and}$$

$$\bar{B} = B_x a_x + B_y a_y + B_z a_z$$

$$C = (A_x + B_x) a_x + (A_y + B_y) a_y + (A_z + B_z) a_z$$

# Vector Addition

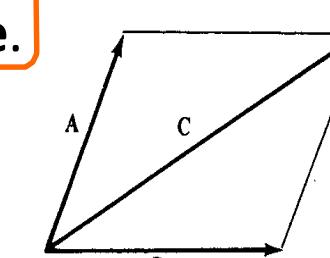
Q: Say we **add** two vectors  $\vec{A}$  and  $\vec{B}$  together; what is the **result**?

A: The addition of two vectors results in **another vector**, which we will denote as  $C$ . Therefore, we can say:

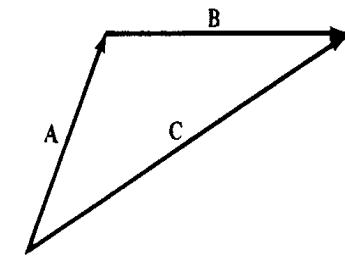
$$\vec{A} + \vec{B} = \vec{C}$$

The magnitude and direction of  $C$  is determined by the **head-to-tail rule**.

This is not a **provable** result, rather the head-to-tail rule is the **definition** of vector addition. This definition is used because it has many **applications** in physics.



**Parallelogram rule**

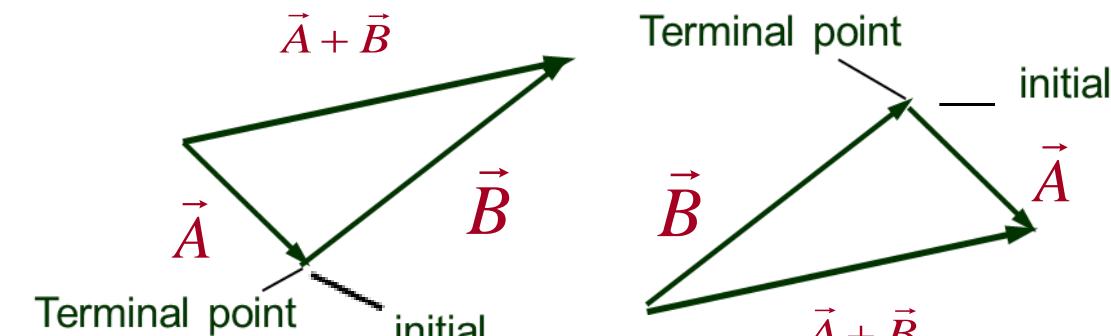
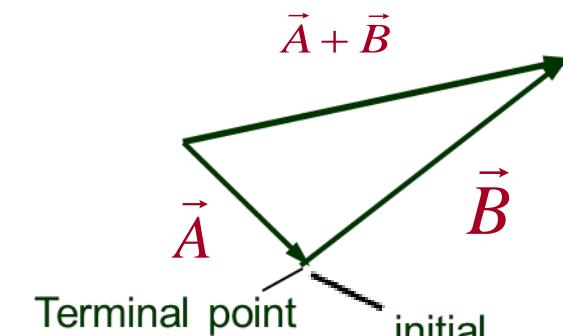


**Head-to-tail rule**

Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

$$\vec{D} = (A_x - B_x)\hat{a}_x + (A_y - B_y)\hat{a}_y + (A_z - B_z)\hat{a}_z$$



# Vector Subtraction

- Vector subtraction is similarly carried out as  
 $\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

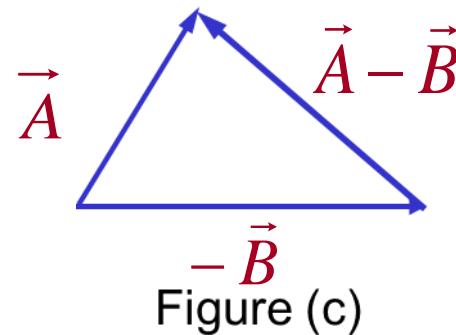
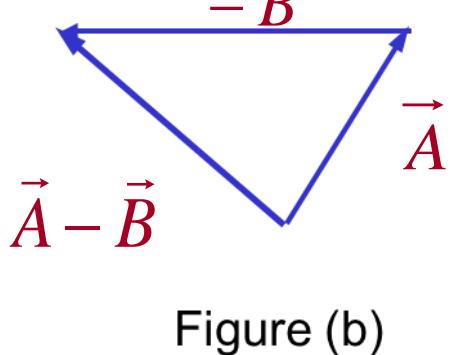
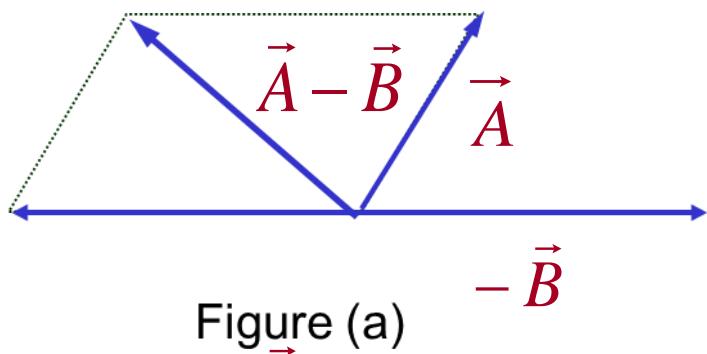


Figure ( C ) shows that vector  $\mathbf{D}$  is a vector that must be added to  $\mathbf{B}$  to give Vector  $\mathbf{A}$ .  
So if vector  $\mathbf{A}$  and  $\mathbf{B}$  are placed tail to tail then vector  $\mathbf{D}$  is a vector that runs from the tip of  $\mathbf{B}$  to  $\mathbf{A}$ .

## Vector Subtraction

- We consider the addition of a negative vector as a **subtraction**.

**Q:** Is  $\vec{A} + \vec{B} = \vec{B} - \vec{A}$ ?

**A:** What do **you** think ?



The three basic laws of algebra obeyed by any given vector **A**, **B**, and **C**, are summarized as follows:

<u>Law</u>	<u>Addition</u>	<u>Multiplication</u>
<b>Commutative</b>	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$k\bar{A} = \bar{A}k$
<b>Associative</b>	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$k(l\bar{A}) = (kl)\bar{A}$

From these two properties, we can conclude that the addition of **several** vectors can be executed in **any order**

**Distributive**     $k(\bar{A} + \bar{B}) = k\bar{A} + k\bar{B}$

where  $k$  and  $l$  are scalars

## Example 2: Vector Algebra

If

$$\vec{A} = 10\hat{x} - 4\hat{y} + 6\hat{z}$$

$$\vec{B} = 2\hat{x} + \hat{y}$$

Find:

- (a) The component of  $\vec{A}$  along  $\hat{y}$
- (b) The magnitude of  $3\vec{A} - \vec{B}$
- (c) A unit vector  $\vec{C}$  along  $\vec{A} + 2\vec{B}$

## Solution to Example 2

(a) The component of  $\vec{A}$  along  $\hat{y}$  is

$$A_y = -4$$

(b)

$$\begin{aligned}3\vec{A} - \vec{B} &= 3(10, -4, 6) - (2, 1, 0) \\&= (30, -12, 18) - (2, 1, 0) \\&= (28, -13, 18) \\&= 28\hat{x} - 13\hat{y} + 18\hat{z}\end{aligned}$$

## Solution to Example 2

Hence, the magnitude of  $\vec{3A} - \vec{B}$  is:

$$|3\vec{A} - \vec{B}| = \sqrt{(28)^2 + (-13)^2 + (18)^2} = 35.74$$

(c) Let  $\vec{C} = \vec{A} + 2\vec{B}$

$$\begin{aligned} &= (10, -4, 6) + (4, 2, 0) \\ &= (14, -2, 6) \\ &= 14\hat{x} - 2\hat{y} + 6\hat{z} \end{aligned}$$

## Solution to Example 2

So, the unit vector along  $\vec{\mathbf{C}}$  is:

$$\begin{aligned}\hat{\mathbf{c}} &= \frac{\vec{\mathbf{C}}}{|\mathbf{C}|} = \frac{(14, -2, 6)}{\sqrt{(14)^2 + (-2)^2 + (6)^2}} \\ &= \frac{14}{15.36} \hat{x} - \frac{2}{15.36} \hat{y} + \frac{6}{15.36} \hat{z} \\ &= 0.911 \hat{x} - 0.130 \hat{y} + 0.391 \hat{z}\end{aligned}$$

# Vector Multiplication

- Multiplication by scalar
- Scalar (**dot** ) product ( $\mathbf{A} \bullet \mathbf{B}$ )
- Vector (**cross**) product ( $\mathbf{A} \times \mathbf{B}$ )
- Scalar triple product  $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$
- Vector triple product  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

# Multiplication by a Scalar

- Consider a scalar quantity  $a$  and a vector quantity  $\vec{B}$ . We express the multiplication of these two values as:

$$a\vec{B} = \vec{C}$$

In other words, the product of a scalar and a vector is a **vector!**

**Q:** OK, but what **is** vector  $C$ ? What is the **meaning** of  $C$ ?

**A:** The resulting vector  $C$  has a **magnitude** that is equal to  $a$  times the **magnitude** of  $B$ . In other words: The **direction** of vector  $C$  is **exactly** that of  $B$ :

$$|\vec{C}| = a|\vec{B}|$$

→ Just to reiterate, multiplying a vector by a scalar changes the **magnitude** of the vector, but **not** its direction.

Example:: Multiplication of a scalar  $k$  to a vector  $A$  gives a vector that points in the same direction as  $A$  and magnitude equal to  $|kA|$

$$\overrightarrow{A}$$

$$\overrightarrow{kA}$$

$$|k| < 1$$

$$\overrightarrow{kA} \quad |k| > 1$$

The division of a vector by a scalar quantity is a multiplication of the vector by the reciprocal of the scalar quantity.

## Multiplication (contd.)

### Some important properties of vector multiplication:

1. The scalar-vector multiplication is **distributive**:

$$\vec{aB} + \vec{bB} = (\vec{a} + \vec{b})\vec{B}$$

2. also **distributive** as:

$$\vec{aB} + \vec{aC} = \vec{a}(\vec{B} + \vec{C})$$

3. Scalar-Vector multiplication is also **commutative**:

$$\vec{aB} = \vec{B}\vec{a}$$

4. Multiplication of a vector by a **negative** scalar is interpreted as:

$$-\vec{aB} = \vec{a}(-\vec{B})$$

5. **Division** of a vector by a scalar is the same as multiplying the vector by the **inverse** of the scalar:

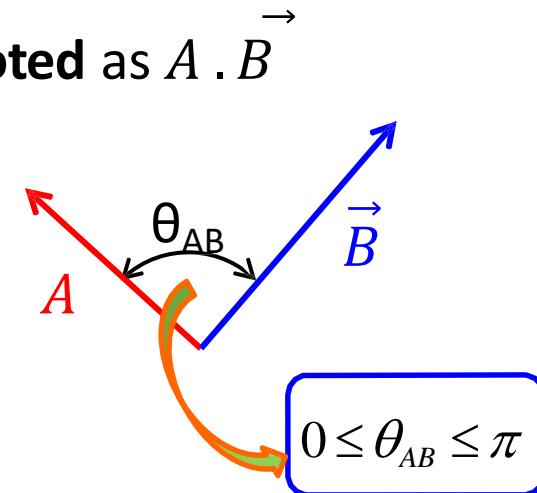
$$\frac{\vec{B}}{a} = \left(\frac{1}{a}\right)\vec{B}$$

## The Dot Product

- The **dot product** of two vectors,  $\vec{A}$  and  $\vec{B}$ , is denoted as  $\vec{A} \cdot \vec{B}$

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta_{AB}$$

angle  $\theta_{AB}$  is the angle formed **between** the vectors  $\vec{A}$  and  $\vec{B}$ .



**IMPORTANT NOTE:** The dot product is an operation involving **two vectors**, but the result is a **scalar** !! e.g.,:

$$\vec{A} \cdot \vec{B} = c$$



The dot product is also called the **scalar product** of two vectors.

- Note also that the dot product is **commutative**, i.e.,:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

## The Dot Product (contd.)

1. The dot product of a vector **with itself** is equal to the **magnitude** of the vector **squared**.

$$\vec{A} \cdot \vec{A} = |\vec{A}| \cdot |\vec{A}| \cos 0^\circ = |\vec{A}|^2 \rightarrow |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$$

2. If  $\vec{A} \cdot \vec{B} = 0$  (and  $\vec{A} \neq 0, \vec{B} \neq 0$ ), then it must be true that:

$$\theta_{AB} = 90^\circ$$

3. If  $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}|$ , then it must be true that:

$$\theta_{AB} = 0$$

4. If  $\vec{A} \cdot \vec{B} = -|\vec{A}| |\vec{B}|$ , then it must be true that:

$$\theta_{AB} = 180^\circ$$

5. The dot product is **distributive** with addition, such that:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

# Dot Product in Cartesian

- The dot product of two vectors of Cartesian coordinate below yields the sum of **nine scalar terms**, each involving the dot product of two unit vectors.

$$\overset{\rightarrow}{\mathbf{A}} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$$

$$\overset{\rightarrow}{\mathbf{B}} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

# Dot Product in Cartesian

- Since the angle between two unit vectors of the Cartesian coordinate system is  $90^0$ , we then have:

$$\mathbf{a}_x \bullet \mathbf{a}_y = \mathbf{a}_y \bullet \mathbf{a}_x = \mathbf{a}_x \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_x = \mathbf{a}_y \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_y = 0$$

- And thus, only three terms remain, giving finally:

$$\overrightarrow{\mathbf{A}} \bullet \overrightarrow{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

# Dot Product in Cartesian

- The two vectors,  $\vec{A}$  and  $\vec{B}$  are said to be perpendicular or **orthogonal** ( $90^\circ$ ) with each other if;

$$\vec{A} \bullet \vec{B} = 0$$

# Laws of Dot Product

- Dot product obeys the following:

- Commutative Law

$$\vec{\mathbf{A}} \bullet \vec{\mathbf{B}} = \vec{\mathbf{B}} \bullet \vec{\mathbf{A}}$$

- Distributive Law

$$\vec{\mathbf{A}} \bullet \vec{\mathbf{A}} = A^2 = |\mathbf{A}|^2$$

$$\vec{\mathbf{A}} \bullet \left( \vec{\mathbf{B}} + \vec{\mathbf{C}} \right) = \vec{\mathbf{A}} \bullet \vec{\mathbf{B}} + \vec{\mathbf{A}} \bullet \vec{\mathbf{C}}$$

# Properties of dot product

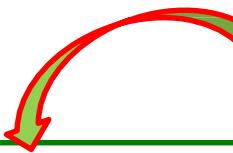
- Properties of dot product of unit vectors:

$$a_x \bullet a_x = a_y \bullet a_y = a_z \bullet a_z = 1$$

$$a_x \bullet a_y = a_y \bullet a_z = a_z \bullet a_x = 0$$

# The Cross Product

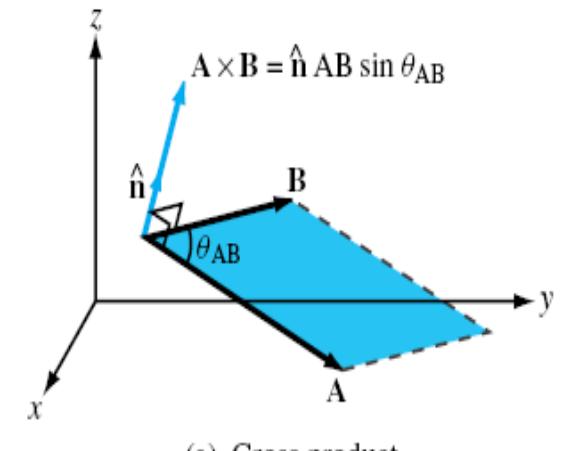
- The **cross product** of two vectors,  $\vec{A}$  and  $\vec{B}$ , is denoted as  $\vec{A} \times \vec{B}$ .



$$\vec{A} \times \vec{B} = \hat{a}_n |\vec{A}| |\vec{B}| \sin \theta_{AB}$$

$$0 \leq \theta_{AB} \leq \pi$$

Just as with the dot product, the angle  $\theta_{AB}$  is the angle between the vectors  $\vec{A}$  and  $\vec{B}$ . The unit vector  $\hat{a}_n$  is **orthogonal** to both  $\vec{A}$  and  $\vec{B}$  (i.e.,  $\hat{a}_n \cdot \vec{A} = 0$  and  $\hat{a}_n \cdot \vec{B} = 0$ .)



(a) Cross product

**IMPORTANT NOTE:** The cross product is an operation involving **two vectors**, and the result is also a **vector**. e.g.,:

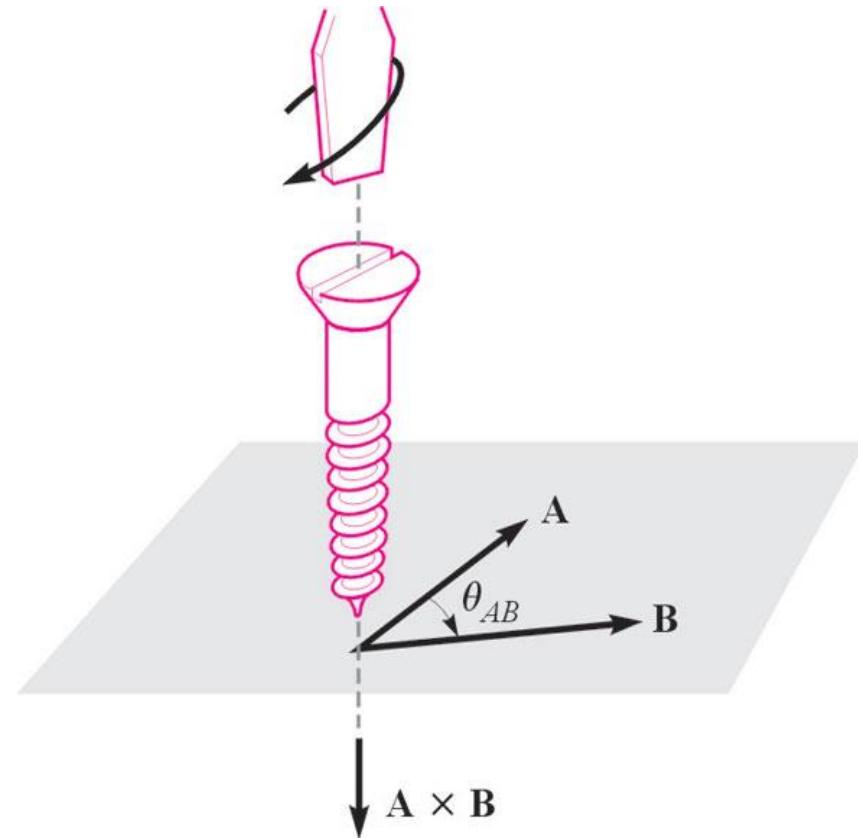
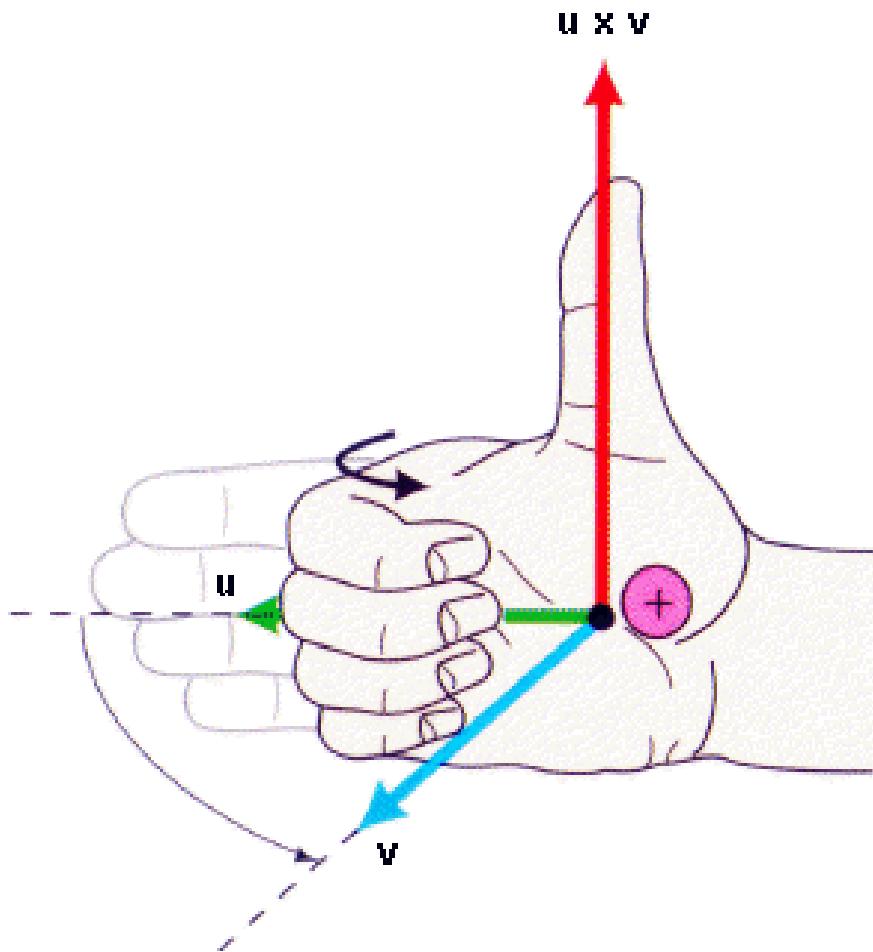
$$\vec{A} \times \vec{B} = \vec{C}$$

- The **magnitude** of vector  $\vec{A} \times \vec{B}$  is therefore:

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta_{AB}$$

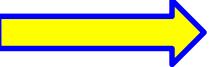
While the **direction** of vector  $\vec{A} \times \vec{B}$  is described by unit vector  $\hat{a}_n$ .

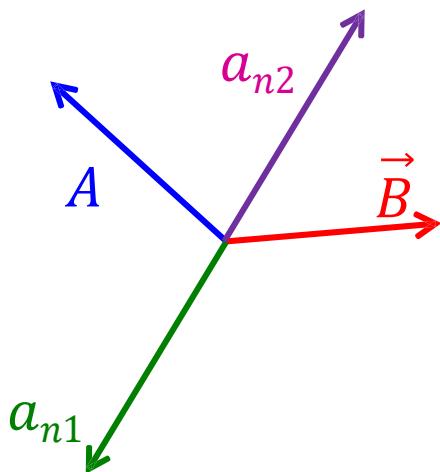
# Right-hand Rule



- It is also along one of the two possible perpendiculars which is in direction of **advance of right hand screw.**

## The Cross Product (contd.)

Problem!  There are two unit vectors that satisfy the equations  $\vec{a}_n \cdot \vec{A} = 0$  and  $\vec{a}_n \cdot \vec{B} = 0$ !! These two vectors are antiparallel.



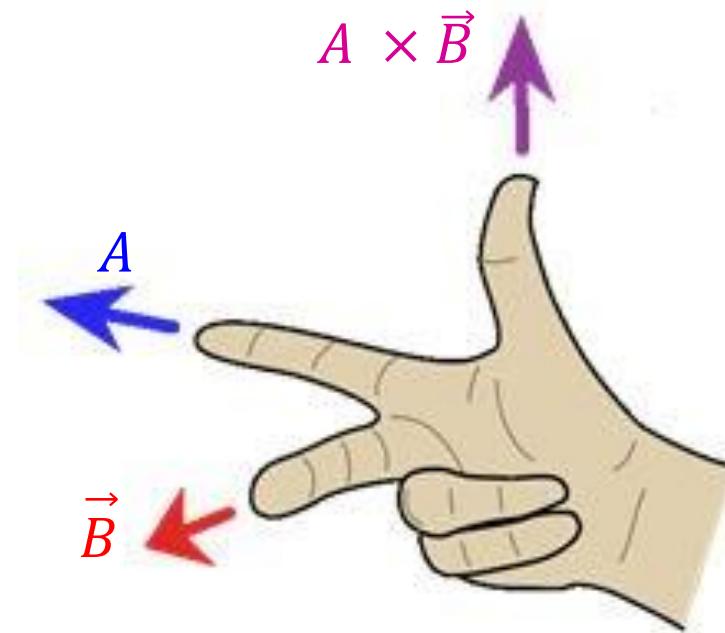
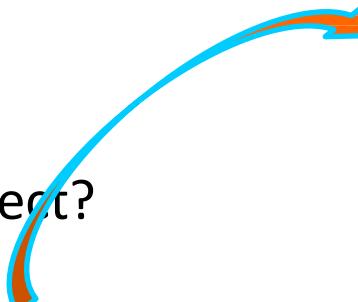
$$\vec{A} \cdot \hat{a}_{n1} = \vec{A} \cdot \hat{a}_{n2} = 0$$

$$\vec{B} \cdot \hat{a}_{n1} = \vec{B} \cdot \hat{a}_{n2} = 0$$

$$\hat{a}_{n1} = -\hat{a}_{n2}$$

Q: Which unit vector is correct?

A: Use the **right-hand rule**



## The Cross Product (contd.)

1. If  $\theta_{AB} = 90^\circ$  (i.e., **orthogonal**), then:

$$\vec{A} \times \vec{B} = \hat{a}_n |\vec{A}| |\vec{B}| \sin 90^\circ = \hat{a}_n |\vec{A}| |\vec{B}|$$

2. If  $\theta_{AB} = 0^\circ$  (i.e., **parallel**), then:

$$\vec{A} \times \vec{B} = \hat{a}_n |\vec{A}| |\vec{B}| \sin 0^\circ = 0$$

Note that  $\vec{A} \times \vec{B} = \mathbf{0}$  also if  $\theta_{AB} = 180^\circ$ .

3. The cross product is **not** commutative! In other words,  $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ .

While evaluating the cross product of two vectors,  
the **order** is important !

$$\vec{A} \times \vec{B} \neq -(\vec{B} \times \vec{A})$$

4. The **negative** of the cross product is:

$$-(\vec{A} \times \vec{B}) = \vec{A} \times (-\vec{B})$$

5. The cross product is also **not** associative:

$$\vec{A} \times \vec{B} \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$$

6. But, the cross product is **distributive**, in that:

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$$

# Cross product in Cartesian

- The cross product of two vectors of Cartesian coordinate:

$$\vec{\mathbf{A}} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$$

$$\vec{\mathbf{B}} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

yields the sum of nine simpler cross products, each involving two unit vectors.

# Cross product in Cartesian

- By using the properties of cross product, it gives

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y) \mathbf{a}_x + (A_z B_x - A_x B_z) \mathbf{a}_y + (A_x B_y - A_y B_x) \mathbf{a}_z$$

and be written in more easily remembered form:

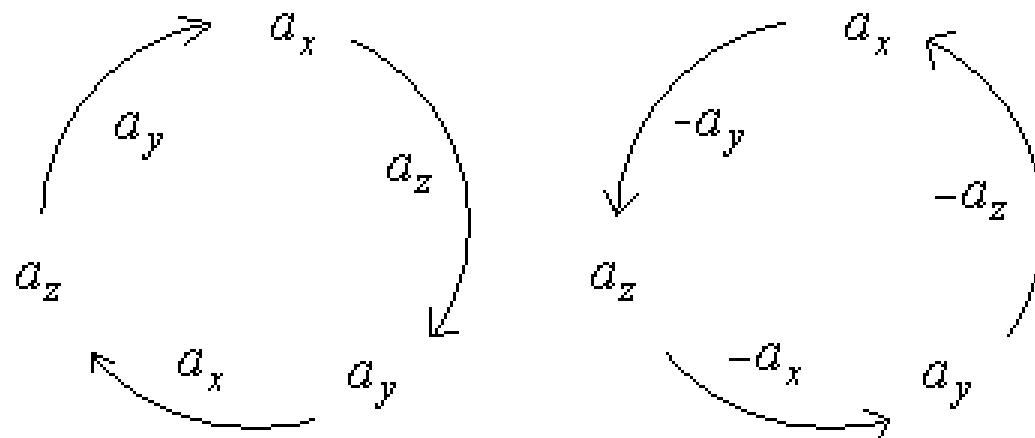
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

# Properties of Vector Product

➤ Properties of cross product of unit vectors:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Or by using cyclic permutation:



## Example 4:Dot & Cross Product

Determine the dot product and cross product of the following vectors:

$$\vec{\mathbf{A}} = 2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z$$

$$\vec{\mathbf{B}} = -\mathbf{a}_x - 5\mathbf{a}_y + 6\mathbf{a}_z$$

# Solution to Example 4

The dot product is:

$$\begin{aligned}\vec{\mathbf{A}} \bullet \vec{\mathbf{B}} &= A_x B_x + A_y B_y + A_z B_z \\ &= ((2)(-1)) + ((3)(-5)) + ((-4)(6)) \\ &= -41\end{aligned}$$

# Solution to Example 4

The cross product is:

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 3 & -4 \\ -1 & -5 & 6 \end{vmatrix} \\ &= [(3)(6) - (-4)(-5)]\mathbf{a}_x \\ &\quad - [(2)(6) - (-4)(-1)]\mathbf{a}_y \\ &\quad + [(2)(-5) - (3)(-1)]\mathbf{a}_z \\ &= -2\mathbf{a}_x - 8\mathbf{a}_y - 7\mathbf{a}_z\end{aligned}$$

## The Triple Product

- The **triple product** is not a “new” operation, as it is simply a combination of the **dot** and **cross** products.
- For example, the triple product of vectors  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  is denoted as:

$$\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$$

Q: Yikes! Does this mean:

$$(\overrightarrow{A} \cdot \overrightarrow{B}) \times \overrightarrow{C}$$

OR

$$\overrightarrow{A} \cdot (\overrightarrow{B} \times \overrightarrow{C})$$

A: The answer is **easy!** Only one of these two interpretations makes sense:

$$(\overrightarrow{A} \cdot \overrightarrow{B}) \times \overrightarrow{C} = \text{Scalar } \times \text{Vector} \quad \text{makes no sense}$$

$$\overrightarrow{A} \cdot (\overrightarrow{B} \times \overrightarrow{C}) = \text{Vector} \cdot \text{Vector} \quad \text{dot product}$$

# Scalar & Vector Triple Product

- A scalar triple product is

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

- A vector triple product is

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

known as the “bac-cab” rule.

# Example 1

- Given three vectors  $\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_z$   
 $\mathbf{Q} = 2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$   
 $\mathbf{R} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$

Determine

- $(\mathbf{P}+\mathbf{Q}) \times (\mathbf{P}-\mathbf{Q})$
- $\mathbf{Q} \bullet (\mathbf{R} \times \mathbf{P})$
- $\mathbf{P} \bullet (\mathbf{Q} \times \mathbf{R})$
- $\sin \theta_{QR}$
- $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$
- A unit vector perpendicular to both  $\mathbf{Q}$  and  $\mathbf{R}$

# Solution

$$\begin{aligned}(a) \quad (\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q}) &= \mathbf{P} \times (\mathbf{P} - \mathbf{Q}) + \mathbf{Q} \times (\mathbf{P} - \mathbf{Q}) \\&= \mathbf{P} \times \mathbf{P} - \mathbf{P} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P} - \mathbf{Q} \times \mathbf{Q} \\&= 0 + \mathbf{Q} \times \mathbf{P} + \mathbf{Q} \times \mathbf{P} - 0 \\&= 2\mathbf{Q} \times \mathbf{P} \\&= 2 \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} \\&= 2(1 - 0) \mathbf{a}_x + 2(4 + 2) \mathbf{a}_y + 2(0 + 2) \mathbf{a}_z \\&= 2\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z\end{aligned}$$

## Solution (cont')

(b) The only way  $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$  makes sense is

$$\begin{aligned}\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) &= (2, -1, 2) \cdot \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} \\ &= (2, -1, 2) \cdot (3, 4, 6) \\ &= 6 - 4 + 12 = 14.\end{aligned}$$

Alternatively:

$$\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

# Solution (cont')

To find the determinant of a 3 X 3 matrix, we repeat the first two rows and cross multiply; when the cross multiplication is from right to left, the result should be negated as shown below. This technique of finding a determinant applies only to a 3 X 3 matrix. Hence

$$\begin{aligned}\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) &= \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} \\ &= +6 +0 -2 +12 -0 -2 \\ &= 14\end{aligned}$$

# Solution (cont')

(c) From eq. (1.28)

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = 14$$

or

$$\begin{aligned}\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \cdot (5, 2, -4) \\ &= 10 + 0 + 4 \\ &= 14\end{aligned}$$

(d)

$$\begin{aligned}\sin \theta_{QR} &= \frac{|\mathbf{Q} \times \mathbf{R}|}{|\mathbf{Q}| |\mathbf{R}|} = \frac{|(5, 2, -4)|}{|(2, -1, 2)| |(2, -3, 1)|} \\ &= \frac{\sqrt{45}}{3\sqrt{14}} = \frac{\sqrt{5}}{\sqrt{14}} = 0.5976\end{aligned}$$

# Solution (cont')

(e) 
$$\begin{aligned}\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \times (5, 2, -4) \\ &= (2, 3, 4)\end{aligned}$$

Alternatively, using the bac-cab rule,

$$\begin{aligned}\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= \mathbf{Q}(\mathbf{P} \cdot \mathbf{R}) - \mathbf{R}(\mathbf{P} \cdot \mathbf{Q}) \\ &= (2, -1, 2)(4 + 0 - 1) - (2, -3, 1)(4 + 0 - 2) \\ &= (2, 3, 4)\end{aligned}$$

(f) A unit vector perpendicular to both  $\mathbf{Q}$  and  $\mathbf{R}$  is given by

$$\begin{aligned}\mathbf{a} &= \frac{\pm \mathbf{Q} \times \mathbf{R}}{|\mathbf{Q} \times \mathbf{R}|} = \frac{\pm (5, 2, -4)}{\sqrt{45}} \\ &= \pm (0.745, 0.298, -0.596)\end{aligned}$$

Note that  $|\mathbf{a}| = 1$ ,  $\mathbf{a} \cdot \mathbf{Q} = 0 = \mathbf{a} \cdot \mathbf{R}$ . Any of these can be used to check  $\mathbf{a}$ .

## Solution (cont')

(g) The component of  $\mathbf{P}$  along  $\mathbf{Q}$  is

$$\begin{aligned}\mathbf{P}_Q &= |\mathbf{P}| \cos \theta_{PQ} \mathbf{a}_Q \\&= (\mathbf{P} \cdot \mathbf{a}_Q) \mathbf{a}_Q = \frac{(\mathbf{P} \cdot \mathbf{Q}) \mathbf{Q}}{|\mathbf{Q}|^2} \\&= \frac{(4 + 0 - 2)(2, -1, 2)}{(4 + 1 + 4)} = \frac{2}{9}(2, -1, 2) \\&= 0.4444\mathbf{a}_x - 0.2222\mathbf{a}_y + 0.4444\mathbf{a}_z.\end{aligned}$$

THANKS