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1 Improving guarantees of randomized algorithms

a)

The modification of the randomized algorithm A would be running it iteratively until it outputs the correct answer. Based on the fact that the initial algorithm fails with probability 99%, the idea of iterative runs could be modeled as a geometrical distribution with probability of success $p = 0.01$. As we know, the expected value of a geometric distribution is calculated as $\frac{1}{p} = \frac{1}{0.01} = 100$. Therefore, running 100 times the initial algorithm A gives a new expected running time of $100 \cdot O(n^2) \approx O(n^2)$ for our approach.

b)

We define X the actual running time of randomized algorithm B. Initially, the algorithm B runs in expected time $T(n)$. We stop the algorithm at $a \cdot T(n)$ running time with $a > 0$.Guaranteeing that this algorithm is finished within $a \cdot T(n)$ with probability larger than or equal to 0.95, it implies that: $Pr(X \geq a \cdot T(n)) \leq 0.05$. Based on the Markov inequality and the fact that $E[X] = T(n)$ we get:

$$
\begin{aligned}
Pr(X \geq a \cdot T(n)) &\leq \frac{E[X]}{a \cdot T(n)} \implies \\
Pr(X \geq a \cdot T(n)) &\leq \frac{T(n)}{a \cdot T(n)} \implies \\
Pr(X \geq a \cdot T(n)) &\leq \frac{1}{a}
\end{aligned}
$$

If we assign a = 20 then the inequality will be $Pr(X \geq a \cdot T(n)) \leq 0.05$, so that implies if we run our initial algorithm B and stopped it after $20 \cdot T(n)$ we can guarantee the success of it with bounded probability of 0.95. Given the variance of algorithm B equals $\sqrt{n}$ then through Chebyshev inequality we get:

$$
\begin{aligned}
Pr(X \geq a \cdot T(n)) = Pr(X - T(n) \geq (a-1) \cdot T(n)) = Pr(X - E[X] \geq (a-1) \cdot T(n)) &\geq \frac{Var(X)}{(a-1)^2 \cdot T(n)^2} \implies \\
&\geq \frac{\sqrt{n}}{(a-1)^2 \cdot T(n)^2}
\end{aligned}
$$

Again, we need:

$$
\begin{aligned}
Pr(X \geq a \cdot T(n)) \geq 0.05 &\implies \\
\frac{\sqrt{n}}{(a-1)^2 \cdot T(n)^2} \geq 0.05 &\implies \\
\frac{n^{\frac{1}{4}}}{(a-1) \cdot T(n)} \geq \frac{\sqrt{5}}{10} &\implies \\
a \geq \frac{2 \cdot \sqrt{5} n^{\frac{1}{4}}}{T(n)} + 1 &
\end{aligned}
$$

c)

We define X the random variable that describes the number of wrong answers in k tries. If function f returns correct answer with probability 0.7 then it is easy to assume that X random variable is a Binomial distribution over k tries $X \sim Binomial(k, 0.3)$.Since we care about the event of algorithm making a mistake, which happens only if the most frequent answer is a wrong one and by frequent we mean equal or more than the half of k tries. So the event we need to investigate is $X \geq \frac{k}{2}$. Using Chernoff-Hoeffding bound and the expected value of X, $E[X] = 0.3 \cdot k$, we have:

$$Pr(X \geq \frac{k}{2}) = Pr(X \geq (1 + \frac{2}{3}) \cdot 0.3 \cdot k) \leq (\frac{e^{\frac{2}{3}}}{(1+\frac{2}{3})^{1+\frac{2}{3}}})^{0.3 \cdot k} \approx 0.9461^k$$

Given the fact that we need the above inequality to be upper bounded with 2^{-t}, we compute k as:

$$\begin{aligned}
0.9461^k \leq 2^{-t} \implies& \\
k \geq log_{(0.9461)}(\frac{1}{2}) \cdot t \implies& \\
k \geq 12.51 \cdot t&
\end{aligned}$$

Since k represent the number of trials it has to be integer, so we compute the upper floor and the final answer is: $k \geq \lceil 12.51 \rceil \cdot t$

2 Exercise 4.10 from MU

a)

We define Y_i as the expected winning money a player receives in i-th game. Based on exercise's description the expected value of $Y_i, E[Y_i]$ is the following:

$$E[Y_i] = 2 \cdot \frac{4}{25} + 99 \cdot \frac{1}{200} - 1 \cdot \frac{167}{200} = -0.02$$

Also, we define as Y the total turns in one million games and its expected value $E(Y)$ is, based on that each winning money of each player Y_i is independent, the following:

$$Y = \sum_{i=1}^{10^6} Y_i \implies$$

$$E[Y] = \sum_{i=1}^{10^6} E[Y_i] = -0.02 \cdot 10^6$$

We know that each play winning money can receive values from -1 (no win at all) to 99 (win 100 with 1 dollar input), so $Pr(-1 \leq Y_i \leq 99) = 1$. Based on that, we can apply the Hoeffding Bound as follows to answer the question about the probability of giving 10000 dollars over the first one million games:

$$Pr(Y \geq 10^4) = Pr(Y \geq 10^6 \cdot 0.01) = Pr(Y \geq 10^6 \cdot (-0.02 + 0.03)) \leq e^{\frac{-2 \cdot 10^6 \cdot (0.03)^2}{(99-(-1))^2}} = e^{-0.18} \approx 0.8353$$

b)

Using the random variable of sub question (a) we define X the same random variable as Y ($X = Y$) since Y expresses the total returns throughout a million games, which is equal to the total net loss of the casino.

Hence:

$$
\begin{aligned}
E[e^{t\cdot X}] &= E[e^{t\cdot \sum_{i=1}^{10^6} X_i}] \Longrightarrow \\
&= E[\prod_{i=1}^{10^6} e^{t\cdot X_i}] \Longrightarrow \\
&= \prod_{i=1}^{10^6} E[e^{t\cdot X_i}]
\end{aligned}
$$

Also based on subquestion (a) we know that:

$$
E[e^{t\cdot X_i}] = e^{2\cdot t}\cdot \frac{4}{25} + e^{99\cdot t}\cdot \frac{1}{200} + e^{-1\cdot t}\cdot \frac{167}{200}
$$

Therefore, previous equation takes its final form as:

$$
\begin{aligned}
E[e^{t\cdot X}] &= \prod_{i=1}^{10^6} E[e^{t\cdot X_i}] \Longrightarrow \\
&= \prod_{i=1}^{10^6} (e^{2\cdot t}\cdot \frac{4}{25} + e^{99\cdot t}\cdot \frac{1}{200} + e^{-1\cdot t}\cdot \frac{167}{200})
\end{aligned}
$$

c)

For $Pr(X \geq 100)$, we apply the Markov's inequality:

$$
\begin{aligned}
Pr(X \geq 100) = Pr(e^X \geq e^{100}) = Pr(e^{t\cdot X} \geq e^{100\cdot t}) &\leq \frac{E[e^{t\cdot X}]}{e^{100\cdot t}} \Longrightarrow \\
&\leq \frac{\prod_{i=1}^{10^6}(e^{2\cdot t}\cdot \frac{4}{25} + e^{99\cdot t}\cdot \frac{1}{200} + e^{-1\cdot t}\cdot \frac{167}{200})}{e^{100\cdot t}} \Longrightarrow \\
&\leq \prod_{i=1}^{10^6}(e^{-98\cdot t}\cdot \frac{4}{25} + e^{-t}\cdot \frac{1}{200} + e^{-101\cdot t}\cdot \frac{167}{200})
\end{aligned}
$$

Asigning the value $t = 0.0006$ to the above inequality gives us:

$$
\begin{aligned}
Pr(X \geq 100) &\leq \prod_{i=1}^{10^6}(e^{-98\cdot t}\cdot \frac{4}{25} + e^{-t}\cdot \frac{1}{200} + e^{-101\cdot t}\cdot \frac{167}{200}) \xrightarrow{t=6\cdot 10^{-4}} \\
&\leq \prod_{i=1}^{10^6}(e^{-98\cdot 6\cdot 10^{-4}}\cdot \frac{4}{25} + e^{-6\cdot 10^{-4}}\cdot \frac{1}{200} + e^{-101\cdot 6\cdot 10^{-4}}\cdot \frac{167}{200}) \approx 16\cdot 10^{-4}
\end{aligned}
$$

3 Permutation Routing on the Hypercube

a)

The address of a node of the Hypercube is $(x_1, ..., x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, ..., x_n)$. Based on the hint we consider the packets with source in the form $(x_1, ..., x_{\frac{n}{2}}, 0, 0, ..., 0)$. The total number of packets are 2^n, where n is the size of our cube. Fixing latter half of address's bits being zero, then the number of nodes having the form $(x_1, ..., x_{\frac{n}{2}}, 0, 0, ..., 0)$ are $2^{\frac{n}{2}}$. So there are $2^{\frac{n}{2}}$ packets in a transpose permutation with a source address in form $(x_1, ..., x_{\frac{n}{2}}, 0, ..., 0)$ and a destination in form $(0, ..., 0, x_1, ..., x_{\frac{n}{2}})$.

Running the bit-fixing routing algorithm, after $\frac{n}{2}$ first bits each of those $2^{\frac{n}{2}}$ packets will have as a source address $(0, ..., 0, 0, ..., 0)$. From those $2^{\frac{n}{2}}$ packets,exactly half of them, in particular those who have $x_1 = 1$ in their destination, moving to address $(0, ..., 0, 1, ..., 0)$ will fix their next bit, since the destinatition is in form $(0, ..., 0, x_1, ..., x_{\frac{n}{2}}) = (0, ..., 0, 1, ..., x_{\frac{n}{2}})$. The number of those buckets is $\dfrac{2^{\frac{n}{2}}}{2} = 2^{\frac{n}{2}-1}$. Hence, if we suppose the best case in which bits in destination $(0, ..., 0, 1, x_2, ..., x_{\frac{n}{2}})$ starting from x_2 to x_n are zero as our source then at least $2^{\frac{n}{2}-1} = \frac{2^{\frac{n}{2}}}{2} = \frac{\sqrt{2^n}}{2} = \frac{\sqrt{N}}{2}$ packets traverse the edge between nodes $(0, ..., 0, 0, ..., 0)$ and $(0, ..., 0, 1, ..., 0)$, which takes $\Omega(\sqrt{N})$.

b)

b).1

Based on exercise, the packets we do care, have source's address in form $(x_1, ...x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, ..., x_n)$ =$(1, ..., 1, 0..., 0)$ with the number of "ones" is equal to k and the number of "zeros" is equal to k, and have destination's address in form $(x_{\frac{n}{2}+1}, ..., x_n, x_1, ...x_{\frac{n}{2}})$ =$(0, ..., 0, 1..., 1)$ with the number of "ones" is equal to k and the number of "zeros" is equal to k. The question to be answered is what is the expected number of packets that going through the node 0^n, or in other words what is the expected probability μ to randomly pick the "ones" from the source address and turn them into "zeros" to turn the source's address to $(0, ..., 0, 0..., 0)$. Note that we pass through the node 0^n, if we have corrected all the "ones" to "zeros" before correcting the "zeros" to "ones".

The total number of "ones" in the source address is the permutation of choosing those k "ones" out of the first $\frac{n}{2}$ bits, where k will be chosen later. So, the total number of "ones" is $\dbinom{\frac{n}{2}}{k}$.Moreover, we know that the sum of "ones" and "zeros" are $2 \cdot k$. The probability of choosing randomly a "one" with each draw being independent from the others is $\dfrac{1}{\binom{2\cdot k}{k}}$. So, the expected probability to randomly pick the "ones" from the source address and turn them into "zeros" to pass from the node 0^n is $\mu = \dfrac{\binom{\frac{n}{2}}{k}}{\frac{1}{\binom{2\cdot k}{k}}}$.

b).2

Let $k = \frac{n}{2}$, then the initial inequality takes the following form:

$$(\frac{2 \cdot k}{k})^k \leq \binom{2 \cdot k}{k} \leq (\frac{2 \cdot e \cdot k}{k})^k \Longrightarrow$$

$$\frac{1}{(\frac{2\cdot e\cdot k}{k})^k} \geq \frac{1}{\binom{2\cdot k}{k}} \geq \frac{1}{(\frac{2\cdot e\cdot k}{k})^k} \Longrightarrow$$
$$\frac{\binom{\frac{n}{2}}{k}}{\binom{2\cdot k}{k}} \geq \frac{\binom{\frac{n}{2}}{k}}{(\frac{2\cdot e\cdot k}{k})^k} \Longrightarrow$$
$$\frac{\binom{\frac{n}{2}}{k}}{\binom{2\cdot k}{k}} \geq \frac{\binom{\frac{n}{2}}{k}}{(\frac{2\cdot e\cdot k}{k})^k} \geq \frac{(\frac{n}{2\cdot k})^k}{(\frac{2\cdot e\cdot k}{k})^k} \Longrightarrow$$
$$\mu \geq \frac{(\frac{n}{2\cdot k})^k}{(\frac{2\cdot e\cdot k}{k})^k} \Longrightarrow$$
$$\mu \geq (\frac{n}{4\cdot e\cdot k})^k \xrightarrow{k=\frac{n}{8\cdot e}}$$
$$\mu \geq 2^{\frac{n}{8\cdot e}} = 2^{\Omega(n)}$$

b).3

We define as X_i as the indicator random variable of i-th packet that goes through node 0^n. So:

$$X_i = \begin{cases} 1 & \text{if packet i goes through node } 0^n \\ 0 & \text{if packet i does not go through node } 0^n \end{cases}$$

Hence,the sum of all possible X_i packets is defined as a random variable X, as follows $X = \sum_{i=1}^{2^n} X_i$. We want to calculate the probability that at least $\frac{B}{2}$ packets go through node 0^n, where $B = \mu$ based on the subquestion b(ii). Having known the lower bound of μ from previous subquestion, we will use the lower tail of Chernoff bounds in order to calculate the probability of the sum of X_i packets that go through 0^n is at most $\frac{B}{2}$ and then calculate the complimentary probability, which is the one asked. So:

$$Pr(X \leq (1-\delta)\cdot\mu) \leq e^{-\frac{\mu\cdot\delta^2}{2}} \xrightarrow{\delta=\frac{1}{2}}$$
$$Pr(X \leq \frac{\mu}{2}) \leq e^{-\frac{\mu}{8}} \Longrightarrow$$
$$Pr(X \leq \frac{\mu}{2}) \leq e^{-\frac{2^{\frac{n}{8\cdot e}}}{8}} \Longrightarrow$$
$$Pr(X \leq \frac{\mu}{2}) \leq e^{-2^{\frac{n}{8\cdot e}-3}} \Longrightarrow$$
$$Pr(X \leq \frac{B}{2}) \leq e^{-2^{\frac{n}{8\cdot e}-3}}$$

So the asked probability of the sum of X_i packets that go through 0^n is at least $\frac{B}{2}$, is calculated as $Pr(X \geq \frac{B}{2}) = 1 - Pr(X \leq \frac{B}{2}) = 1 - e^{-2^{\frac{n}{8\cdot e}-3}}$, which is as expected a high probability.

b).4

Based on subquestion b(ii), we have calculated the lower bound of expected number of packets that go through 0^n node as $2^{\frac{n}{8\cdot e}}$. Passing through a particular node, a packet has n possible edges to traverse. Hence, lowering the bound of required number of steps, $2^{\frac{n}{8\cdot e}}$ packets have to pass from the intermediate node 0^n by using at least $\frac{2^{\frac{n}{8\cdot e}}}{n}$ steps.