1 Detecting defects

a)

We define E the event of find at least one defective cookie. The complimentary probability of event E, $Pr(\overline{E})$ is not to find defective cookies.Since we pick each cookie independently and uniformly at random, not finding defective cookies after k tries is equal to:

$$Pr(\overline{E}) = (1-p)^k \implies Pr(E) = 1 - (1-p)^k \tag{1}$$

Given $p \geq a$ and $k \geq \frac{ln(100)}{a}$ it is true that:

$$\begin{aligned}
k &\geq \frac{ln(100)}{p} \implies \\
k \cdot p &\geq ln(100) \implies \\
-k \cdot p &\leq ln(0.01) \implies \\
e^{-k \cdot} &\leq 0.01 \xrightarrow{(1-x)^k \leq e^{-k}} \\
(1-p)^k &\leq 0.01 \implies \\
1-(1-p)^k &\geq 0.99 \xrightarrow{(1)} \\
Pr(E) &= 0.99
\end{aligned}$$

b)

We define W the event all the unreliable workers have been detected and fired and W_i the event that the worker i has been detected as reliable. Therefore :

$$Pr(W) = 1 - Pr[\bigcup_{i \in n} W_i]$$

Assuming that the $Pr(W) \geq 0.99$ it implies that:

$$Pr[\bigcup_{i \in n} W_i] \leq 0.01$$

Based on union bound inequality, we know that:

$$Pr[\bigcup_{i \in n} W_i] \leq n \cdot Pr(W_i)$$

With regards to subquestion (a) for a reliable worker W_i it is true that $Pr(W_i) \leq (1-a)^k$. So:

$$\begin{aligned}
n \cdot (1-a)^k &\leq 0.01 \xrightarrow{(1-x) \leq e^{-x}} \\
n \cdot e^{-a \cdot k} &\leq 0.01 \implies \\
e^{-a \cdot k} &\leq \frac{0.01}{n} \implies
\end{aligned}$$

$$
\begin{aligned}
-a \cdot k &\leq ln(\frac{0.01}{n}) \implies \\
-a \cdot k &\leq ln(\frac{1}{100 \cdot n}) \implies \\
k &\geq \frac{ln(100 \cdot n)}{a} \implies \\
k_{min} &= \frac{ln(100 \cdot n)}{a}
\end{aligned}
$$

2 Jensen's Inequality

a)

Given that f is concave function we can assume that f is a twice differentiable function and $f''(x) \leq 0$. We define $\mu = E[X]$ where X is a random value. In order to proof the Jensen's inequality change for concave functions we will use the Taylor expansion of f function based on the fact that f is twice differentiable. So:

$$
\begin{aligned}
f(x) &= f(\mu) + f'(\mu) \cdot (x - \mu) + \frac{f''(c) \cdot (x-\mu)^2}{2} \xrightarrow{f''(x) \leq 0} \\
&\leq f(\mu) + f'(\mu) \cdot (x - \mu)
\end{aligned}
$$

Using linearity of expectations to the previous equation and based on the identity of $E[cX] = c[X]$ where c is a constant the previous inequatily turn into:

$$
\begin{aligned}
E[f(x)] &\leq E[f(\mu) + f'(\mu) \cdot (x - \mu)] \implies \\
&\leq E[f(\mu)] + f'(\mu) \cdot E[(x - \mu)] \implies \\
&\leq f(\mu) + f'(\mu)(\cdot E[x] - E[\mu)] \implies \\
&\leq f(E[X])
\end{aligned}
$$

So Jensen's Inequality for a concave f function turns into $E[f(x)] \leq f(E[X])$

b)

We define arithmetic mean of n numbers as $A(x_1, ..., x_n) = \frac{x_1 + ... + x_n}{n} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$

Also, we define geometric mean of n numbers as $G(x_1, ..., x_n) = (x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}}$

We want to prove that $\frac{1}{n} \cdot \sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} (x_i)^{\frac{1}{n}}$.
Based on subquestion (a) we define f(x) as a concave function, so it is true that:

$$
\sum_{i=1}^{n} p_i \cdot f(x_i) \leq f(\sum_{i=1}^{n} p_i \cdot x_i) \tag{2}
$$

where $\sum_{i=1}^{n} p_i = 1$.We choose $f(x) = ln(x)$ since $ln(x)$ is a concave function since it is a twice differentiable function with $f''(x) = -\frac{1}{x^2} \leq 0$. So equation (1) will turn into:

$$\sum_{i=1}^{n} p_i \cdot ln(x_i) \leq ln(\sum_{i=1}^{n} p_i \cdot x_i) \Longrightarrow$$
$$\sum_{i=1}^{n} ln(x_i^{p_i}) \leq ln(\sum_{i=1}^{n} p_i \cdot x_i) \Longrightarrow$$
$$ln(\prod_{i=1}^{n} x_i^{p_i}) \leq ln(\sum_{i=1}^{n} p_i \cdot x_i) \xrightarrow{ln(x)\nearrow}$$
$$\prod_{i=1}^{n} (x_i)^{p_i} \leq \sum_{i=1}^{n} p_i \cdot x_i \xrightarrow{p_i=\frac{1}{n}, \forall i \in [1,..,n]}$$
$$\prod_{i=1}^{n} (x_i)^{\frac{1}{n}} \leq \sum_{i=1}^{n} \frac{1}{n} \cdot x_i \Longrightarrow$$
$$\prod_{i=1}^{n} (x_i)^{\frac{1}{n}} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

c)

We choose $f(x) = sin(x)$ which is a twice differentiable function and $f''(x) = -sin(x) \leq 0$, so f is a concave function.It is true that: $sin(60) = sin(\frac{A+B+C}{3}) = \frac{\sqrt{3}}{2}$. So based on the fact that f is concave we know that:

$$E[f(x)] \leq f(E[X]) \Longrightarrow$$
$$E[sin(x)] \leq sin(E[X]) \xrightarrow{x=\frac{A+B+C}{3}}$$
$$E[sin(\frac{A+B+C}{3})] \leq \frac{\sqrt{3}}{2} \Longrightarrow$$
$$\frac{sin(A) + sin(B) + sin(C)}{3} \leq \frac{\sqrt{3}}{2} \Longrightarrow$$
$$sin(A) + sin(B) + sin(C) \leq 3 \cdot \frac{\sqrt{3}}{2}$$

3 Random children

a)

We define X the random variable of the event that a girl was born after a number of succeeding boy's births. Assuming that (1-p) is the probability that a boy was born and p the probability that a girl was born then the probability of the birth of a girl after k boy's birth is the following:

$$Pr(X = k) = (1 - p)^{k-1} \cdot p$$

The excepted value E[X] is given as:

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} k \cdot Pr(X=k) \implies \\
&= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p \implies \\
&= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k} \xrightarrow{c=1-p} \\
&= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} k \cdot c^{k} \implies \\
&= \frac{p}{1-p} \cdot c \cdot \sum_{k=1}^{\infty} k \cdot c^{k-1} \implies \\
&= \frac{p}{1-p} \cdot c \cdot \sum_{k=1}^{\infty} \frac{d}{dc}(c^{k}) \implies \\
&= \frac{p}{1-p} \cdot c \cdot \frac{d}{dc} \sum_{k=1}^{\infty} c^{k} \implies \\
&= \frac{p}{1-p} \cdot c \cdot \frac{d}{dc}(\frac{c}{1-c}) \implies \\
&= \frac{p}{1-p} \cdot c \cdot \frac{1}{(1-c)^2} \implies \\
&= \frac{1}{p}
\end{aligned}
$$

So the total number of births since the first arrival of a daughter if $p = \frac{1}{2}$ are equal to $E[X] = \frac{1}{p} = 2$. So the expected value of a girl is 1 and the expected value of a boy is also 1.

b)

Following the same formula as in subquestion (a), the total number of births since the first arrival of a daughter is equal to: $E[X] = \frac{1}{p} = \frac{10}{4} = 2.5$ So the expected value of a girl is 1 and the expected value of a boy is 1.5.

c)

Assuming that after k births the number of girls are going to be born are either 1 or 0, since either a girl will be born and stop the process or k boys will be born and stop the process, the distribution of having a girl is a Bernoulli.So if we define X the number of girls who are probable to be born then:

$$
\begin{aligned}
&E[X] = 1 \cdot Pr(X=1) + 0 \cdot Pr(X=0) \implies \\
&E[X] = Pr(X=1) = \sum_{i=1}^{k} (1-p)^{i-1} \cdot p = 1 - (1-p)^k = 1 - (\frac{1}{2})^k
\end{aligned}
$$

Based on the assumption that we define Y the number of boys who are probable to be born then the expected value is:

$$E[Y] = \sum_{i=1}^{k} i \cdot Pr(Y = i) \Longrightarrow$$
$$E[Y] = \sum_{i=1}^{k-1}[i \cdot (1-p)^i \cdot p] + k \cdot (1-p)^k \Longrightarrow$$
$$E[Y] = p \cdot (1-p) \cdot \sum_{i=1}^{k-1}[i \cdot (1-p)^{i-1}] + k \cdot (1-p)^k \Longrightarrow$$
$$E[Y] = p \cdot (1-p) \cdot \sum_{i=1}^{k-1} \frac{d}{di} \cdot (1-p)^i + k \cdot (1-p)^k \Longrightarrow$$
$$E[Y] = p \cdot (1-p) \cdot \frac{d}{di} \cdot \sum_{i=1}^{k-1}(1-p)^i + k \cdot (1-p)^k \Longrightarrow$$
$$E[Y] = 1 - (\frac{1}{2})^k$$

d)

Using the same formulas of subquestion (c) the expected value of girls $E[X]$ is:

$$E[X] = Pr(X = 1) = \sum_{i=1}^{k}(1-p)^{i-1} \cdot p = 1 - (1-p)^k = 1 - (0.6)^k$$

While the expected value of boys $E[Y]$ is:

$$E[Y] = \sum_{i=1}^{k-1}[i \cdot (1-p)^i \cdot p] + k \cdot (1-p)^k \Longrightarrow$$
$$E[Y] = \sum_{i=1}^{k-1}[i \cdot (0.6)^i \cdot 0.4] + k \cdot 0.6^k \Longrightarrow$$
$$E[Y] = \frac{3}{2} \cdot 5^{-k} \cdot (5^k - 3^k)$$

4 Random counter

a)

We define n the number of the counter and k the number of bits we need to represent them. Assuming that the counter works deterministically then:

$$k = \lfloor log_2(n) \rfloor + 1$$

For example if the number of counter is n = 5 then $k = \lfloor log_2(5) \rfloor = k = \lfloor 2.32 \rfloor + 1 = 3$. We will need 3 bits.

b)

With regards to total expectation we need to compute $E[2^{X_i}]$. We are going to condition the expected value based on the previous value of X, X_{i-1}.So:

$$E[2^{X_i}] = E[E[2^{X_i}|X_{i-1} = x_{i-1}]] \tag{3}$$

For the inner expected value of the previous equation $E[2^{X_i}|X_{i-1} = x_{i-1}]$, we are going to compute it thinking intuitively. In the previous step the counter had a value $X_{i-1} = x_{i-1}$. The counter either changed to the next number of the counter $x_{i-1}+1$ with a probability of $p = 2^{-x_{i-1}}$ or stayed in the same value x_{i-1} with a probability of $1-p = 1-2^{-x_{i-1}}$. So, the inner expected value of $E[2^{X_i}|X_{i-1} = x_{i-1}]$ will be:

$$E[2^{X_i}|X_{i-1} = x_{i-1}] = 2^{x_{i-1}+1} \cdot 2^{-(x_{i-1})} + 2^{x_{i-1}} \cdot (1-2^{-x_{i-1}}) \Longrightarrow$$
$$E[2^{X_i}|X_{i-1} = x_{i-1}] = 1 + 2^{x_{i-1}}$$

Using equation (3):

$$E[2^{X_i}] = E[E[2^{X_i}|X_{i-1} = x_{i-1}]] \Longrightarrow$$
$$E[2^{X_i}] = E[1 + 2^{x_{i-1}}] = 1 + E[2^{x_{i-1}}]$$

Working iteratively in the previous equation over the $E[2^{x_{i-1}}]$ terms, it is true that:

$$E[2^{X_i}] = E[1+2^{x_{i-1}}] = 1 + E[2^{x_{i-1}}] = 1 + 1 + E[2^{x_{i-2}}] = ... = i+1$$

c)

Assuming that:

$$2_i^X \leq 10 \cdot E[2_i^X] \xrightarrow{E[2_i^X]=m+1, i=m}$$
$$2^{X_i} \leq 10 \cdot (m+1) \Longrightarrow$$
$$X_i \leq log_2(10 \cdot (m+1)) \Longrightarrow$$
$$log_2(X_i) \leq log_2(log_2(10 \cdot (m+1)))$$

So, with asymptotic notation the number of bits we will need is $O(log_2(log_2(10 \cdot (m+1))))$

d)

Following the same idea as in subquestion (b), with regards to total expectation we need to compute $E[2^{2 \cdot X_i}]$. We are going to condition the expected value based on the previous value of X, X_{i-1}.So:

$$E[2^{2 \cdot X_i}] = E[E[2^{2 \cdot X_i}|X_{i-1} = x_{i-1}]] \tag{4}$$

The inner expected value of $E[2^{2 \cdot X_i}|X_{i-1} = x_{i-1}]$ will be:

$$E[2^{2 \cdot X_i}|X_{i-1} = x_{i-1}] = 2^{2 \cdot x_{i-1}+1} \cdot 2^{-(x_{i-1})} + 2^{2 \cdot x_{i-1}} \cdot (1-2^{-x_{i-1}}) \Longrightarrow$$
$$E[2^{X_i}|X_{i-1} = x_{i-1}] = 3 \cdot 2^{x_{i-1}} + 2^{2 \cdot (x_i - 1)}$$

Using equation (4):

$$E[2^{2\cdot X_i}] = E[E[2^{2\cdot X_i}|X_{i-1} = x_{i-1}]] \Longrightarrow$$
$$E[2^{2\cdot X_i}] = E[3 \cdot 2^{x_{i-1}} + 2^{2\cdot(x_i-1)}] = 3 \cdot E[2^{x_{i-1}}] + E[2^{2\cdot x_{i-1}}] = 3i + E[2^{2\cdot x_{i-1}}]$$

Working iteratively in the previous equation over the $E[2^{2\cdot x_{i-1}}]$ terms, it is true that:

$$E[2^{2\cdot X_i}] = E[3 \cdot 2^{x_{i-1}} + 2^{2\cdot(x_i-1)}] = 3i + E[2^{2\cdot x_{i-1}}] = 3i + 3(i-1) + E[2^{2\cdot x_{i-2}}] \Longrightarrow$$
$$E[2^{2\cdot X_i}] = 3i + 3(i-1) + 3(i-2) + ... + 3(i-i) \Longrightarrow$$
$$E[2^{2\cdot X_i}] = 3 \cdot (i(i+1) - \frac{(i+1)i}{2}) = 3 \cdot \frac{i(i+1)}{2}$$

As long as the variance we know that $Var(2^X) = E[2^{2\cdot X}] - (E[2^X])^2 = 3 \cdot \frac{i(i+1)}{2} - (i+1)^2$