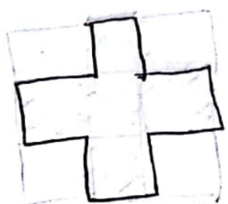
Q3. To prove: Median filter is a non-linear filter.

Assumption: The median filter is a cross-filter, i.e., it takes only these pixels into account as neighbors, given a pixel (i,j)

$$Nbhd(i,j) = \{(i-1,j),\ (i+1,j),\ (i,j-1),\ (i,j+1)\}.$$



Proof:

We aim to prove this by finding a counter-example to the statement "Median filter is a linear filter". This method holds as any filter which is not linear is a non-linear filter.

Let there be 2 images A, B of size 3×3.

In matrix representation,

$$A = \begin{bmatrix} 6 & 5 & 2 \\ 1 & 4 & 3 \\ 2 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 6 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

We pad A and B along all borders by the border values themselves.

$$A_{padded} = \begin{array}{c|ccc|c} & 6 & 5 & 2 & \\ \hline 6 & 6 & 5 & 2 & 2 \\ 1 & 1 & 4 & 3 & 3 \\ 2 & 2 & 5 & 1 & 1 \\ \hline & 2 & 5 & 1 & \end{array}, \quad B_{padded} = \begin{array}{c|ccc|c} & 2 & 4 & 2 & \\ \hline 2 & 2 & 4 & 2 & 2 \\ 3 & 3 & 6 & 1 & 1 \\ 1 & 1 & 3 & 5 & 5 \\ \hline & 1 & 3 & 5 & \end{array}$$

Let F_m be the cross median filter as explained before.

By applying F_m on A and B, we get

$$F_m(A) = \begin{bmatrix} 6 & 5 & 2 \\ 2 & 4 & 3 \\ 2 & 4 & 1 \end{bmatrix}, \quad F_m(B) = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 3 & 2 \\ 1 & 3 & 5 \end{bmatrix}$$

Now we calculate $F_m(A+B)$

$$A+B = \begin{bmatrix} 6 & 5 & 2 \\ 1 & 4 & 3 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 2 \\ 3 & 6 & 1 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 4 \\ 4 & 10 & 4 \\ 3 & 8 & 6 \end{bmatrix} \quad \text{s.t.}$$

$$(A+B)_{padded} = \begin{array}{c|ccc|c} & 8 & 9 & 4 & \\ \hline 8 & 8 & 9 & 4 & 4 \\ 4 & 4 & 10 & 4 & 4 \\ 3 & 3 & 8 & 6 & 6 \\ \hline & 3 & 8 & 6 & \end{array}$$

Thus, $Fm(A+B) = \begin{bmatrix} 8 & 9 & 4 \\ 4 & 8 & 4 \\ 3 & 8 & 6 \end{bmatrix}$

$$Fm(A) + Fm(B) = \begin{bmatrix} 6 & 5 & 2 \\ 2 & 4 & 3 \\ 2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 2 \\ 3 & 3 & 2 \\ 1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 9 & 4 \\ 5 & 7 & 5 \\ 3 & 7 & 6 \end{bmatrix}$$

$$\neq Fm(A+B)$$

$\therefore Fm(A+B) \neq Fm(A) + Fm(B)$

Thus Fm is a non-linear filter, as:

a function is linear if it is

a) homogenous $\rightarrow f(\alpha x) = \alpha f(x) \quad \forall \alpha$

b) additive $\rightarrow f(x+y) = f(x) + f(y)$

Since, $Fm(A+B) \neq Fm(A) + Fm(B)$,

Fm is not additive.

$\therefore$ The median filter is non-linear.

Q5. a) No.

The histogram distribution of both images will change upon application of the 3x3 averaging mask, and they will be unequal.

Consider Image 1. The white portion will remain white and the black portion will remain black - only the edge pixels on both sides will face a change in their intensity values (upon averaging mask).

eg. (intensity value written in each pixel)

255	255	255	0	0	0
255	255	255	0	0	0
255	255	255	0	0	0
255	255	255	0	0	0
255	255	255	0	0	0
255	255	255	0	0	0

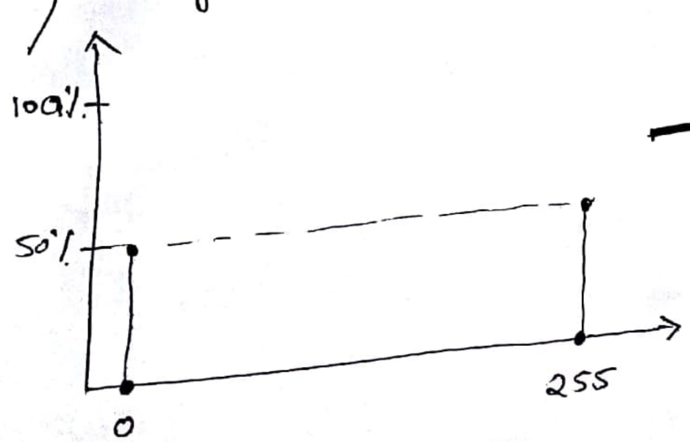
$\xrightarrow[3\times3]{\text{averaging}}$

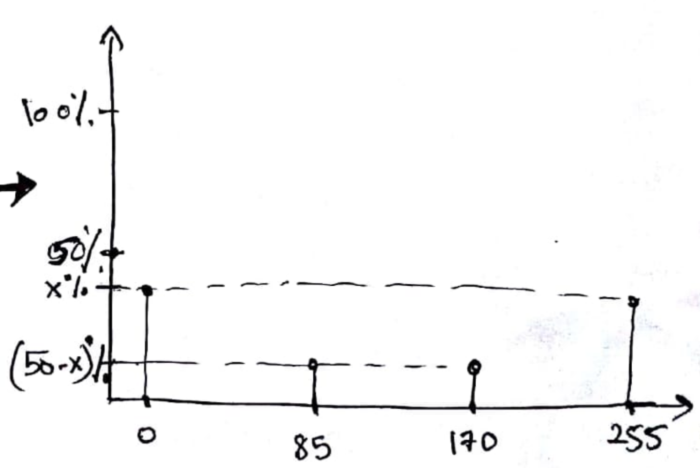
255	255	170	85	0	0
255	255	170	85	0	0
255	255	170	85	0	0
255	255	170	85	0	0
255	255	170	85	0	0
255	255	170	85	0	0

So the histogram of Image 1 changes from
{0 : 50%, 255 : 50%}
to {0 : 33%, 85 : 16.5%, 170 : 16.5%, 255 : 33%}
for the given Image 1.
~~Those~~ The histogram pixel count would depend on the image but the intensities get transformed from {0, 255} to {0, 85, 170, 255}.

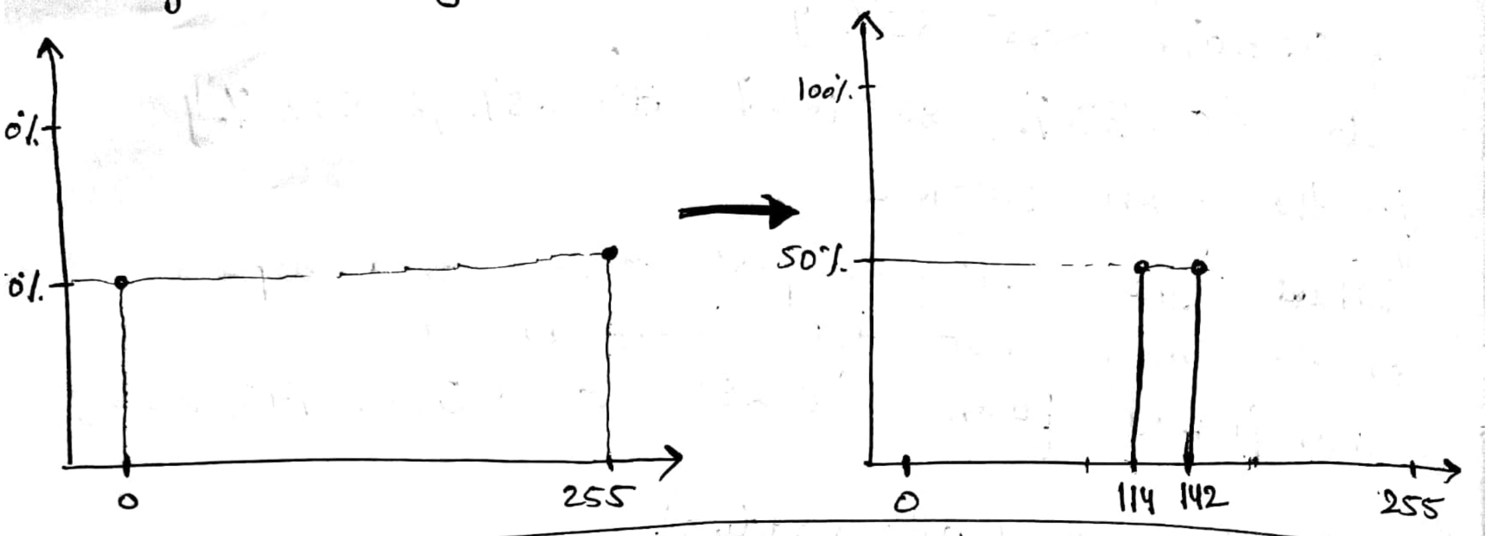
Consider Image 2. (with padding).
Each white square is surrounded by 4 white and 4 black squares.
Thus intensity 255 squares will transform to intensity $\frac{255 \times 5 + 0 \times 4}{9} = 142$ value,
and all intensity 0 squares will transform to intensity $\frac{255 \times 4 + 0 \times 5}{9} = 114$ value.
So the histogram changes from {0 : 50%, 255 : 50%} to {114 : 50%, 142 : 50%} for Image 2.
Clearly, the histograms for the transformed images are going to be different.

b) Histogram – Image 1





Histogram → Image 2



Q6 Object is enclosed by a square of size $q \times q$ pixels. Let the size of the smallest averaging mask that reduces the average intensity to one-tenth of original intensity in one pass be $p \times p$.

$(p \geq q)$.

We assume that the intensities of the object are very different from the intensities surrounding the object in the $p \times p$ filter.

~~Therefore, within the pxp filter,~~

the intensity of a pixel (i,j) upon averaging:

$$I'(i,j) = \frac{1}{p^2} \sum_{k=-\frac{(p-1)}{2}}^{\left(\frac{p-1}{2}\right)} \sum_{l=-\frac{(p-1)}{2}}^{\left(\frac{p-1}{2}\right)} I(k,l) \leq \frac{1}{10} I(i,j)$$

$$\Rightarrow \frac{1}{p^2}\left(q^2 \times I_{bright} + (p^2-q^2) \times I_{dark}\right) \leq \frac{1}{10} I_{bright}$$

Let $I_{bright} = 1$ and $I_{dark} = 0$

(extreme case).

$$\Rightarrow \frac{1}{p^2}\left(q^2(1) + (p^2-q^2)(0)\right) \leq \frac{1}{10}(1)$$

$$\Rightarrow \frac{q^2}{p^2} \leq \frac{1}{10}$$

$$\Rightarrow p^2 \geq 10 q^2$$

$$\Rightarrow p \geq \sqrt{10} \cdot q$$

$$\Rightarrow p \geq 3.162\, q$$

and p needs to be odd. (q is odd)

$$\Rightarrow p \geq 4q+1$$

$\Rightarrow$ The smallest averaging mask to achieve the $1/10^{th}$ intensity reduction in one pass is atleast 10 times the size of $q \times q$, and p being an integer needs to be atleast $4q+1$

Q7. 5x5 Gaussian filter :

$$\bar{f}(x,y) = \frac{1}{265}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 6 & 26 & 43 & 26 & 6 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

Original Image $\rightarrow f(x,y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Unsharp mask $= f(x,y) - \bar{f}(x,y)$
(5,5)

$$= \frac{-1}{265}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 6 & 26 & -222 & 26 & 6 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} //$$

such that upon applying the unsharp mask by a real valued amount 'k' to the image $f(x,y)$, we get the resultant image

$$f'(x,y) = f(x,y) + k.(\text{Unsharp mask})$$

$$= f(x,y) - \frac{k}{265}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 6 & 26 & -222 & 26 & 6 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

$$= \frac{-k}{265} \times \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 26 & 16 & 4 \\ 6 & 26 & \left(-222 - \frac{265}{k}\right) & 26 & 6 \\ 4 & 16 & 26 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} //$$

Q8. Given: $\frac{\partial f}{\partial x} = f(x+1) - f(x)$ —— (1)

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x) \quad \text{—— (2)}$$

To prove: Subtracting Laplacian from image is proportional to unsharp masking.

Proof:

Let the image pixel be $f(x,y)$.

We know that

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \cancel{f(x+1)}$$

$$= f(x+1,y) + f(x-1,y) - 2f(x,y) + f(x,y+1) + f(x,y-1) - 2f(x,y)$$

$$= f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \quad \text{—— (3)}$$

$\therefore f(x,y) - \nabla^2 f(x,y)$

$$= f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)]$$

$$= 6 \cdot f(x,y) - [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)] \quad \text{—— (4)}$$

Let the average filter be $\bar{f}(x,y) =$

$$\frac{1}{5}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) + f(x,y)] \quad \text{—— (5)}$$

From (4) & (5),

$$f(x,y) - \nabla^2 f(x,y) = 6f(x,y) - 5\bar{f}(x,y)$$

$$= f(x,y) + 5 \cdot [f(x,y) - \bar{f}(x,y)]$$

= Original image + k × (unsharp mask)

$\therefore$ Subtracting Laplacian from an image is proportional to the resultant image of unsharp masking.