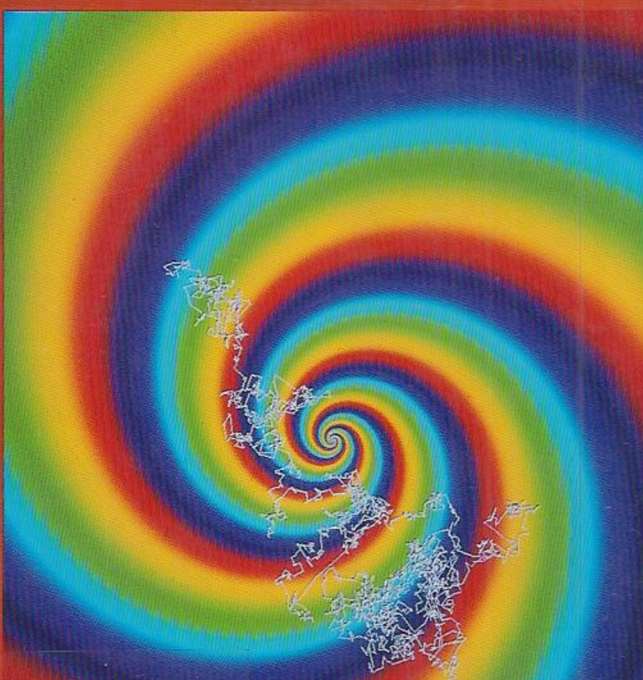
FRACTALS, CHAOS, POWER LAWS

Minutes from an Infinite Paradise



MANFRED SCHROEDER

SCIENCE

$19.95

FRACTALS, CHAOS, POWER LAWS

Minutes from an Infinite Paradise

MANFRED SCHROEDER

"As notable as the book's broad sweep is the author's good-natured, humorous presentation. The willing reader can sit back and enjoy an all-encompassing, irrepressibly enthusiastic tour, ranging from psychophysics to quasicrystals, from gambling strategies to Bach concertos, from the construction of Cantor sets to the design of concert halls."

— *Physics Today*

"Such a richness of topics and amazing splendor of illustrations!"

— *Mathematics Magazine*

"An inviting exposition for a literate but not highly scientific audience."

— *American Mathematical Monthly*

In *Fractals, Chaos, Power Laws*, Manfred Schroeder demonstrates how the concept of self-similarity illuminates our understanding of the universe. The connections between abstract mathematics and everyday experience become clear, as Schroeder explores exciting recent mathematical developments (intergalactic voids, non-linear dynamics, the Mandelbrot set, among others) and reveals the self-similar aspects of snowflakes, concert hall acoustics, pinball machines, computer graphics, the theory of relativity, and much more.

About the Author

Manfred Schroeder is a pioneer in the artistic potential of computer graphics, a world-renowned expert in concert hall acoustics, and holder of over 45 U.S. patents. He divides his time between Berkeley Heights, California and Goettingen, Germany. Dr. Schroeder is the author of *Number Theory in Science and Communication*.

ISBN 0-7167-2357-3
90000>
EAN
9 780716 723578

W. H. Freeman and Company
41 Madison Avenue, New York, NY 10010
20 Beaumont Street, Oxford, England OX12NQ

Ich sage euch:
man muss noch Chaos in sich haben,
um einen tanzenden Stern gebären zu können.
Ich sage euch:
ihr habt noch Chaos in euch.

Yea verily, I say unto you:
A man must have Chaos yet within him
To birth a dancing star.
I say unto you:
You have yet Chaos in you.

—FRIEDRICH NIETZSCHE,
Thus Spake Zarathustra

Fractals, Chaos, Power Laws

Minutes from an Infinite Paradise

Manfred Schroeder

W. H. Freeman and Company
New York

Library of Congress Cataloging-in-Publication Data

Schroeder, M. R. (Manfred Robert), 1926–
Fractals, chaos, power laws: minutes from an infinite paradise/
Manfred Schroeder.
p. cm.
Includes bibliographical references and index.
ISBN 0-7167-2136-8
1. Symmetry (Physics) 2. Self-similarity 3. Recursion 4. Scaling I. Title.
QC174.17.S9S38 1990
530.1—dc20 90-36763

Dedication page: Portrait of Georg Cantor. (Akademie der Wissenschaften zu Göttingen)

BIBLIOTECA DE LA UNIVERSIDAD NACIONAL DE MEX
MATE

BIBLIOTECA DE LA UNIVERSIDAD NACIONAL DE MEXICO
MATEMATICAS

MB-1224

QC
174
.17
S9
S345

Copyright © 1991 by W. H. Freeman and Company

No part of this book may be reproduced by any mechanical, photographic, or electronic process, or in the form of a phonographic recording, nor may it be stored in a retrieval system, transmitted, or otherwise copied for public or private use, without written permission from the publisher.

Printed in the United States of America

4 5 6 7 8 9 VB 9 9 8 7 6 5 4 3

To Georg Cantor

From his paradise no one shall ever evict us

—DAVID HILBERT, defending Cantor's set theory

Also by Manfred Schroeder

Number Theory in Science and Communication

Gm- 17064

CONTENTS

Color Plates

Plate 1 Three heavenly bodies meet. (A) As a double star (yellow and red, approaching from the left) and a single star (blue, closing in from the right) get close to each other, their orbits become wildly chaotic. After some time, they separate again, the double star orbiting off to the upper left and the single star receding to the lower right. (B) In this three-body rendezvous there is a switch of partners. The yellow member of the incoming double star exchanges its red partner for the single blue star, dancing away with its new mate to the lower left. The discarded red partner wanders off alone to the upper right. (Courtesy of Wolf Dieter Brandt.)

Plate 2 This fanciful self-similar leaf was generated by iterated affine transformations. (Courtesy of Holger Behme.)

Plate 3 Newton's iteration has three basins of attraction ("countries," shown in red, green, and blue). They meet at a fractal border with the following bizarre property: wherever two countries meet, the third is also present. Paradoxically there are no border *lines,* only three-sided border *posts.* Would international borders so designed promote peace? (Courtesy of Holger Behme.)

Plate 4 "Rainbow to Infinity," combines a large number of logarithmic spirals in different colors. (Courtesy of Holger Behme.)

Plate 5 "Clouds over Eastern Europe," a photograph taken by the author from his home near the crumbling "wall."

Plate 6 Real-world fractals. (A) Peeling paint at the Berkeley Swim Club, a multiply connected fractal. (B) Fractal growths in microorganisms: red algae on a rock in Point Reyes National Seashore, California.

Plate 7 Plants and flowers produced by turtle algorithms. (A) Rose campion, also called dusty miller (*Lychnis coronaria*), raised by Przemyslaw Prusinkiewicz and James Hanan. (B) Vervain (Verbena). (C) Lilac twig, grown by Prusinkiewicz and Hanan. (D) "The Garden of L," planted by Prusinkiewicz, Hanan, David Fracchia, and Deborah Fowler. (E) Plant with basipetal flowering sequences. (F) Flower field, fertilized with stochastic L-systems. (Plates 7A–F copyright © 1988 by P. Prusinkiewicz, University of Regina.)

Plate 8 (A) The Mandelbrot set (black) of complex numbers c for which the iteration $z_{n+1} = z_n^2 + c$ with $z_0 = 0$ stays bounded. Colored areas signify c-values for which the iteration is unbounded. For c-values from the main "cardioid" area of the **M** set, the iteration has a period length equal to 1. The circular disk to the left of the cardioid has a period length of 2, and so on. Each disk or "wart" comprises c-values for a given finite period length. The **M** set is a connected set, but it is not known whether it is locally connected. (B) A blow-up in the complex plane reveals a wart and filaments connecting it to smaller replicas of the **M** set. (C) Another enlargement in the complex plane shows an eleven-armed "whirlpool" in which smaller **M** sets swim. The arms themselves embrace smaller whorls whose parts, no doubt, contain still smaller whorls. (D) Self-similar descent: zeroing in on one of the "baby" **M** sets frolicking in the whirlpool reveals its patent parentage, the "mother" **M** set shown in A. (Courtesy of Holger Behme.)

Plate 9 Self-organized genetic drift between 16 different "species," $n = 1$ to 16 (shown in different rainbow colors). (A) Initially, the different species are randomly intermingled on a square lattice. At every click of the evolutionary clock, each lattice point occupied by species n will change any of its four nearest neighbors that belong to the species $n - 1$ to its own species number ($n = 0$ corresponds to $n = 16$). (B) At a later stage, different genes dominate larger and larger coherent areas, but many fine-grained neighborhoods persist. (C) Still later in the evolutionary process, a new kind of genetic pattern emerges: spirals with periodically repeating species. (D) Although the genetic interactions are strictly local, large spirals are the surviving dominant pattern. As in quasicrystals and numerous other natural phenomena, local rules engender long-range order and global designs. (Courtesy of Holger Behme.)

PREFACE

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

—HERMANN WEYL

The unifying concept underlying fractals, chaos, and power laws is self-similarity. Self-similarity, or invariance against changes in scale or size, is an attribute of many laws of nature and innumerable phenomena in the world around us. Self-similarity is, in fact, one of the decisive symmetries that shape our universe and our efforts to comprehend it.

Symmetry itself is one of the most fundamental and fruitful concepts of human thought [Wey 81]. By symmetry we mean an *invariance* against change: something stays the same, in spite of some potentially consequential alteration. Mirror symmetry, that is, invariance against "flipping sides," is perhaps the most widely noticed symmetry. Nature built many of her organisms in nearly symmetric ways, and most fundamental laws of physics, such as Newton's law of gravitation, have an *exact* mirror symmetry: there is no difference between left and right in the attraction of heavenly (and most earthbound) bodies. However, the nonconservation of parity in radioactive decay—that is, the *violation* of point symmetry in the "weak" interactions—has finally taught even the physicists to take the distinction between right and left seriously.

Another important symmetry is invariance with respect to geometric translation. Our trust in invariance under transpositions in space and time is, in fact, so unlimited that we believe that the laws of nature are the same all over the cosmos—and that they have been, and will remain so, for all time.

An equally momentous symmetry is invariance with respect to rotation. A circle is invariant under rotation around its center by any angle. A square can be rotated only through angles that are multiples of $360°/4 = 90°$; it is said to have a fourfold symmetry axis. A regular hexagon has a sixfold symmetry. While the rotational symmetry of a flower or a starfish may be imperfect, the exact isotropy found in the fundamental laws of nature is one of the most powerful principles in elucidating the structure of individual atoms, complicated molecules, and entire crystals. Transposition, rotation, and mirror symmetries, acting together, shape crystals from diamonds to snowflakes. And the same three symmetries govern much of what we find pleasing in ornamental designs.

An even more astounding symmetry is the exact identity of like elementary particles. There simply is no difference between an electron here and an electron there—on a distant star, for example. In fact, the perfect identity of photons, the particles of light, has disqualified them from being counted as so many identifiable individuals, resulting in a new kind of particle statistics, discovered by S. N. Bose and rendered palatable by Einstein—a way of counting not heretofore encountered in a world filled with tangible objects.

It was one of the greatest mathematicians of our century, Emmy Noether, who first pointed out the connection between the symmetries of the fundamental laws of physics with respect to displacements in space and time and rotations, on the one hand, and the conservation of linear momentum, energy, and angular momentum on the other. (Noether taught at Göttingen, where David Hilbert, overcoming obstinate prejudice, had finally secured a faculty position for her. In the dismantling of German science in 1933, she was forced to leave Göttingen. She died at Bryn Mawr in 1935.)

Other symmetries have had equally profound consequences in our understanding of the universe we inhabit. Invariance against uniform motion has given us special relativity, a fusion of space and time into space-time and, as its best-known consequence, the equation $E = mc^2$. The equivalence of acceleration and gravity postulated by Einstein is the basis of his general theory of relativity, which further revolutionized our appreciation of space, time, and matter.

Yet, among all these symmetries flowering in the Garden of Invariance, there sprouts one that, until recently, has not been sufficiently cherished: the ubiquitous invariance against changes in size, called *self-similarity* or, if more than one scale factor is involved, *self-affinity*. The enormously fruitful concepts of self-similarity and self-affinity pervade nature from the distribution of atoms in matter to that of the galaxies in the universe. And in mathematics, too, self-similarity is deeply entrenched. Some 300 years ago the German philosopher and polymath Gottfried Wilhelm Leibniz used the scaling invariance of the infinitely long straight line for its definition. Cantor sets and Weierstrass functions are other early examples—albeit less smooth—of self-similar structures in mathematics, later joined by Julia sets and other marvels of set theory.

It is perhaps symptomatic that with *set theory* still another abstract branch of mathematics has penetrated the real world. There simply seems to be no limit to Eugene Wigner's "unreasonable effectiveness" of mathematics. Indeed, who would have thought that such utterly mathematical constructions as *Cantor sets*, invented solely to reassure the skeptics that sets could both have zero measure and still be uncountable, would make a real difference in any practical realm, let alone become a pivotal concept? Yet this is precisely what happened for many natural phenomena from gelation, polymerization, and coagulation in colloidal physics and chemistry to nonlinear systems in innumerable branches of science. Percolation, dendritic growth, fracture surfaces, electrical discharges (lightnings and Lichtenberg figures), and the composition of quasicrystals are best described by set-theoretic constructs.

Or take the weird functions Karl Weierstrass invented a hundred years ago purely to prove that a function could be both everywhere continuous and yet nowhere differentiable. The fact that such an analytic pathology describes something in the real world—nay, is *elemental* to understanding the strange attractors of nonlinear dynamic systems (such as the double swing and the three-body problem)—gives one pause.

The word *symmetric* is of ancient Greek parentage and means well-proportioned, well-ordered—certainly nothing even remotely chaotic. Yet, paradoxically, self-similarity, the topic of this tome, alone among all the symmetries gives birth to its very antithesis: *chaos*, a state of utter confusion and disorder. As we shall endeavor to show, the genesis of chaos is, in fact, closely related to self-similarity and its inherent lack of "smoothness."

Perhaps not surprisingly, self-similarity entails numerous paradoxes in measurements of time, length, and even musical pitch. Think of Zeno's tardy turtle, pursued—but never overtaken—by swift Achilles. Why do certain lengths increase without bound when we measure them with ever smaller yardsticks? How would Euclid have explained plane geometric figures whose areas scale not as the squares of their apparent perimeters but as some lesser power, such as 1.77 and other fractional exponents? What should we think of musical sounds that, when scaled *up* in frequency, sound—incredibly—*lower* in pitch? How are such monstrosities possible? And how can we describe them in a consistent, meaningful manner?

Here a particularly felicitous thought by Felix Hausdorff comes to the rescue. His and Abram Besicovitch's new ways of looking at dimension dethrones it from its integer position and propels it into the realm of real numbers, giving us one of the sharpest tools—the *Hausdorff dimension* and its ramifications—with which to attack the strange sets that self-similarity breeds.

And while recalling some of the glorious names of the past, we should never forget our great contemporary, the inimitable Benoit B. Mandelbrot, who, single-handed, rescued set theory's most brittle functions and "dustiest" sets from near-oblivion and planted them right in the middle of our daily experience and consciousness. Yes, for all these years, we *have* been living with fractal arteries, not far from fractal river systems draining fractal mountainscapes under fractal clouds, toward fractal coastlines. But, kin to Molière's would-be gentleman, we lacked the proper prose—*fractal*, noun and adjective—that Benoit B. begot.

But our story also has a silent and immobile hero: the digital computer. There can be little doubt that computers have acted as the most forceful forceps in extracting fractals from the dark recesses of abstract mathematics and delivering their geometric intricacies into bright daylight. In fact, the impact of fractal images, often of unimagined beauty and appeal, has given computer graphics a surprising new dimension.

Synopsis

We open our treatise with one of the most charming uses that similarity was ever put too: the young Einstein's proof of Pythagoras's theorem. By adding just a single straight line, in the right place, to a right triangle and applying plenty of similarity, the popular theorem is proved without further prodding.

We then invade the unlimited domain of *self*-similarity as manifest in fractals, multifractals, and the scaling laws of physics, psychophysics, and boundless other fields.

In phase spaces, we encounter deterministic chaos and strange attractors. Percolation and other phase transitions lead us to critical exponents and a hierarchy of different dimensions. Following Poincaré, we immerse ourselves in the self-similarities of iterated mappings, from baker transformations and Bernoulli shifts to logistic parabolas and circle maps. Neither tori, *cantori*, nor Arnold tongues will faze us as we (sur)mount devil's staircases to unwind among the rational winding numbers festooning Farey trees.

And when we talk about nonlinear dynamics we must remember some of the great contributors of recent vintage: Siegel, Moser, Lorenz, Wilson, Feigenbaum, and—last but not least—the great Russian "school" exemplified by such names as Lyapunov, Arnol'd, Sinai, Chirikov, Alexeev, Anosov, Pesin, and the recently deceased master mathematician Kolmogorov.

Cayley trees, also known as Bethe lattices, will provide us with a fitting point of departure for many a practical fractal, such as our bronchial and vascular systems. Cellular automata concern us as models of both biological growth and chemical reactions.

We are also strangely attracted to symbolic dynamics, kneading (and needing) the Morse-Thue sequence and, especially, the Fibonacci rabbit sequence and their discrete self-similarities that, indiscreetly, tell us so much about period doubling, mode locking, frustrated Ising spins, and fivefold symmetric quasicrystals. Many of these subjects were shrouded in mystery and beset by paradoxes before the sharp scalpels, fashioned by scaling and renormalization theories, revealed the underlying tissue and made them tractable. In fact, it is no accident that viable fundamental field theories in physics are renormalizable, as they must be if they are to shun sham scales.

And, of course, we will not hesitate to run down random fractals, from Brownian motion to diffusion-limited aggregation and stock market hiccups (some hiccups of late!). The poor gambler's ruin and the St. Petersburg paradox will provide further food for fractal reflections.

These, then, are some of the exciting, and sobering, themes sounded in the present volume. The aim of this exposition is to enhance the reader's understanding of self-similarity, perhaps the most pregnant of all of nature's symmetries, and to illustrate the wide-ranging applications of scaling invariance in physics, chemistry, biology, music, and—particularly—the visual arts, as manifested in the recent renaissance of computer graphics through fractal images and their iterative beauty.

Acknowledgments

This book owes its existence to many sources. Apart from a brief encounter, in my dissertation, with chaos among the normal modes of concert halls, a "nonintegrable" system if there ever was one, my main stimulus came from the early demonstrations by Heinz-Otto Peitgen and Peter Richter of fractal Julia sets. Their beautiful images, and the intriguing mathematics which underlies them, as epitomized in their book *The Beauty of Fractals*, have made a lasting impression on me.

My first meeting with Mandelbrot's work was his analysis of word frequencies in natural and artificial languages, which touched upon my own interests in computer speech synthesis and recognition. Mandelbrot's monumental monograph *The Fractal Geometry of Nature* influenced me immeasurably, as it did so many other people.

I have also greatly benefited from Robert Devaney's books and lectures on chaos, Jens Feder's *Fractals*, Michael Barnsley's *Fractals Everywhere*, and Dietrich Stauffer's charming *Introduction to Percolation Theory*.

I learned a lot about new developments during the 1988 Gordon Conference on Fractals, organized by Richard Voss and Paul Meakin, which brought together many of the world's outstanding experts in the field.

At the University of Göttingen, it was mainly through the work of Werner Lauterborn that I saw chaos in action in nonlinear dynamic systems from cavitation bubbles to Toda chains. My own students at the Drittes Physikalisches Institut provided both stimulus and rectification during a series of lecture courses on self-similarity, fractals, and chaos. Heinrich Henze and Karl Lautscham made skillful additions to the demonstrations that accompanied these lectures. Holger Behme, Wolf Dieter Brandt, and Tino Gramss contributed many of the computer graphics in this book.

The late Walter Kaufmann-Bühler of Springer Verlag, New York, and Ronald Graham, a longtime colleague at Bell Laboratories, provided early encouragement.

Hildegard Franks and Esperanza Plata in Murray Hill, New Jersey, supported by Maija Sutter and Irena Schönke in Göttingen, converted nearly illegible

handwriting to readable typescript. Liane Liebe and Gisela Kirschmann-Schröder took loving care of the illustrations. Anny Schroeder, as during a previous itinerary, furnished much needed logistic support.

Jerry Lyons persuaded me to deliver myself and the manuscript into the hands of W. H. Freeman and Company, and I have not regretted heeding his appeal one bit. I greatly enjoyed working with him; with Georgia Lee Hadler, senior editor; with designer Nancy Singer; and with illustration coordinator Bill Page. The professional competence of my copy editor, Richard Mickey, proved beyond compare.

Last, but not least, I thank Hans-Werner Strube, Allan Hurd, Victor Klee, and Paul Meakin for their expert reviews of the manuscript and their manifold corrections and cogent comments.

Murray Hill and Göttingen
May 1990

Manfred Schroeder