Joint distribution.

1. *John plays chess with Bill and Bob. For each game he gets 1 point for a win, 1/2 point for a draw and 0 points for a loss. Let X be outcome of his game with Bill and Y be outcome of his game with Bob. Suppose that the joint distribution of X and Y is given in the following table.*

$Y\backslash X$	0	1/2	1
0	.10	.07	.11
1/2	.07	.05	.10
1	.05	.10	.35

(a) Compute the marginal distributions of X and Y;
(b) Compute the probability that both games have the same result;
(c) Find the distribution of $X+Y$.
(d) Find the distribution of Y given that $X=\frac{1}{2}$.

Solution.

(a)

$Y\backslash X$	0	1/2	1	Y
0	.10	.07	.11	0.28
1/2	.07	.05	.10	0.22
1	.05	.10	.35	0.50
X	0.22	0.22	0.56	1

(b) $P(X=Y)=0.10+0.05+0.35=0.50$.

(c) $P(X+Y=0)=0.10$; $P(X+Y=0.5)=0.07+0.07=0.14$; $P(X+Y=1)=0.11+0.05+0.05=0.21$; $P(X+Y=1.5)=0.10+0.10=0.20$; $P(X+Y)=2=0.35$. So the answer is

X	0	0.5	1	1.5	2
P	0.10	0.14	0.21	0.20	0.35

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(d) Since $P(X=0.5)=0.22$ the answer is

$$P(Y=0|X=0.5)=\frac{0.07}{0.22}=\frac{7}{22}; \quad P(Y=0.5|X=0.5)=\frac{0.05}{0.22}=\frac{5}{22};$$

$$P(Y=1|X=0.5)=\frac{0.10}{0.22}=\frac{10}{22}.$$

2. *Suppose 50 % of all drivers have American cars, 40 % have Japanese cars and 10 % have European cars. Consider 15 consecutive cars crossing certain intersection.*

(a) What is the probability that 8 are American, 5 are Japanese and 2 are European; 9 American and 6 Japanese ?

(b) Find the marginal distribution of the number of American cars.

(c) Find the conditional distribution of the number of American cars given that there are two European cars.

Solution. (a) For each string of length 15 consisting of 8 As, 5 Js and 2 Es the probability that the observed cars will fit to this string is $(0.5)^8(0.4)^5(0.1)^2$. The number of such strings is $\binom{15}{8,5,2}=\frac{15!}{8!5!2!}$. Hence the answer is $\frac{15!}{8!5!2!}(0.5)^8(0.4)^5(0.1)^2$.

(b) Each car is American with probability 0.5 and non-American with probability 0.5. Hence, the number of American cars has Binomial distribution with parameters $(15,0.5)$.

(c) Since the number of European cars has Binomial distribution with parameters $(15, 0.1)$ the probability that there are two European cars is

$$\frac{15!}{13!2!}(0.9)^{13}(0.1)^2.$$

Next, similar to part (a) the probability that there are k American cars, $(13-k)$ Japanese cars and two European cars is

$$\frac{15!}{k!(13-k)!2!}(0.5)^k(0.4)^{13-k}(0.1)^2.$$

Hence the conditional probability to have k American cars given that there are 2 European cars is

$$\frac{\frac{15!}{k!(13-k)!2!}(0.5)^k(0.4)^{13-k}(0.1)^2}{\frac{15!}{13!2!}(0.9)^{13}(0.1)^2}$$

$$= \frac{13!}{k!(13-k)!}\frac{0.5^k(0.4)^{13-k}}{0.9^{13}} = \frac{13!}{k!(13-k)!}\left(\frac{5}{9}\right)^k\left(\frac{4}{9}\right)^{13-k}.$$

Thus the conditional distribution of the number of American cars given that there are 2 European cars is binomial with parameters $\left(13, \frac{5}{9}\right)$. This is reasonable. Indeed we know that among non-European cars $\frac{5}{9}$ is American and $\frac{4}{9}$ is Japanese. Hence if we know that there are 13 non-European cars, then the number of American cars is binomial with parameters $\left(13, \frac{5}{9}\right)$.

3. *Let (X, Y) have density $p(x, y) = k(2x + y)$ if $0 \le x \le 1$, $0 \le y \le 1$ and 0 otherwise.*
(a) Find the constant k.
(b) Compute $P(X > Y)$.
(c) Find the marginal distributions of X and Y.
(d) Find the distribution of Y given that $X = \frac{1}{4}$.
(e) Compute $E(X^2)$ and $E(Y^2)$.
(f) Compute VX, VY and $Cov(X, Y)$.

Solution. (a) Since

$$1 = \int_0^1\int_0^1 k(2x+y)dxdy = 2k\int_0^1 xdx + k\int_0^1 ydy = k + \frac{k}{2} = \frac{3k}{2}$$

$k = \frac{2}{3}$.

$$(b)\quad P(X > Y) = \int_0^1\left(\int_0^x \frac{2}{3}(2x+y)dy\right)dx = \frac{2}{3}\int_0^1\left(2x^2 + \frac{x^2}{2}\right)dx = \frac{5}{3}\int_0^1 x^2dx = \frac{5}{9}.$$

$$(c)\quad p_X(x) = \frac{2}{3}\int_0^1(2x+y)dy = \frac{2}{3}\left(2x+\frac{1}{2}\right) = \frac{4x}{3} + \frac{1}{3},$$

$$p_Y(y) = \frac{2}{3}\int_0^1(2x+y)dx = \frac{2}{3}(1+y) = \frac{2+2y}{3}.$$

$$(d) \quad p_Y\left(y|X=\frac{1}{4}\right) = \frac{\frac{2}{3}(2\times\frac{1}{4}+y)}{\int_0^1 \frac{2}{3}(2\times\frac{1}{4}+y)dy} = \frac{\frac{1}{3}+\frac{2y}{3}}{\int_0^1\left(\frac{1}{3}+\frac{2y}{3}\right)dy} = \frac{\frac{1}{3}+\frac{2y}{3}}{\frac{2}{3}} = \frac{1+2y}{2}.$$

(e) By part (c)

$$E(X^2) = \int_0^1 x^2\left(\frac{4x}{3}+\frac{1}{3}\right)dx = \int_0^1\left(\frac{4x^3}{3}+\frac{x^2}{3}\right)dx = \frac{1}{3}+\frac{1}{9} = \frac{4}{9},$$

$$E(Y^2) = \int_0^1 y^2\frac{2+2y}{3}dy = \int_0^1 \frac{2y^2+2y^3}{3}dy = \frac{2}{9}+\frac{1}{6} = \frac{7}{18}.$$

$$(f) \quad E(X) = \int_0^1 x\left(\frac{4x}{3}+\frac{1}{3}\right)dx = \int_0^1\left(\frac{4x^2}{3}+\frac{x}{3}\right)dx = \frac{4}{9}+\frac{1}{6} = \frac{11}{18}.$$

Hence $V(X) = E(X^2) - (EX)^2 = \frac{11}{18} - \frac{16}{81} = \frac{99-32}{162} = \frac{67}{162}$.

$$E(Y) = \int_0^1 y\frac{2+2y}{3}dy = \int_0^1 \frac{2y+2y^2}{3}dy = \frac{1}{3}+\frac{2}{9} = \frac{5}{9}.$$

Hence $V(Y) = E(Y^2) - (EY)^2 = \frac{5}{9} - \frac{49}{324} = \frac{5\times 36-49}{324} = \frac{131}{324}$. Next

$$E(XY) = \int_0^1\int_0^1 xy\times\frac{2}{3}(2x+y)\,dxdy = \int_0^1\int_0^1\frac{4x^2y}{3}dxdy + \int_0^1\int_0^1\frac{2xy^2}{3}dxdy$$

$$= \frac{4}{3}\times\frac{1}{3}\times\frac{1}{2}+\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3} = \frac{2}{9}+\frac{1}{9} = \frac{1}{3}.$$

Hence

$$E(XY) = \frac{1}{3} - \frac{11}{18}\times\frac{5}{9} = \frac{54-55}{162} = -\frac{1}{162}.$$

4. *Suppose that John's arrival time to a bus stop is uniform on the segment 1:00 to 1:10 and bus arrival time is uniform on the segment 1:00 to 1:20 and is independent of John's. What is the probability that John misses the bus; that he has to wait more than 5 min?*

Solution. Let J and B denote the arrival times of John and bus respectively. The joint density of (J, B) is

$$p_{J,B}(j,b) = p_J(j)p_B(b) = \frac{1}{10}\times\frac{1}{20} = \frac{1}{200}.$$

Hence (J, B) has a uniform distribution on the rectangle $[0,10]\times[0,20]$.

(a) The set where $B < J$ is a right triangle with both sides equal to 10. It has area $\dfrac{10\times 10}{2} = 50$. Thus $P(B < J) = \frac{50}{200} = \frac{1}{4}$.

(b) The set $B > J + 5$ is the trapezoid with bases 5 and 15 and height 10. The area of this trapezoid is $\frac{5+15}{2}\times 10 = 100$. Thus $P(J > B + 5) = \frac{100}{200} = \frac{1}{2}$.

5. *Let X and Y be independent, $X \sim Exp(1)$, $Y \sim Exp(2)$. Compute $P(X < Y)$.*

Solution. Let $X\sim\text{Exp}(a)$, $Y\sim\text{Exp}(b)$. Then the joint density of (X, Y) is $abe^{-ax}e^{-by}$. Hence

$$P(X > Y) = \int_0^\infty ae^{-ax}\left(\int_x^\infty be^{-by}dy\right)dx = \int_0^\infty ae^{-ax}e^{-ay}dx = \int_0^\infty ae^{-(a+b)x}dx = \frac{a}{a+b}.$$

In our case $a = 1$, $b = 2$ so the answer is $\frac{1}{3}$.

6. *X and Y are independent. Find the distribution of $Z = X + Y$ if*
(a) $X \sim Pois(2)$, $Y \sim Pois(3)$;
(b) $X \sim Uni(0,1)$, $Y \sim Uni(0,1)$;

Solution. (*a*) $\quad p_Z(n) = \sum_{k=0}^{n} P(X=k)P(Y=n-k) = \sum_{k=0}^{n} \frac{2^k}{k!}e^{-2} \times \frac{3^{n-k}}{k!}e^{-3}$
$= e^{-5}\frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} 2^k \times 3^{n-k} = \frac{5^n}{n!}e^{-5}$ where the last step uses the binomial theorem. Thus Z~Pois(5).

(b) The plausable values of Z are $0 \leq z \leq 2$. There are two cases.

(I) $0 \leq z \leq 1$. Then the set $x + y < z$ is triangle with vertices $(0,0)$, $(0,z)$ and $(z,0)$. The area of this triangle is $z^2/2$. So $F_Z(z) = P(Z \leq z) = \frac{z^2}{2}$ and $f_Z(z) = (z^2/2)' = z$.

(II) $0 \leq z \leq 1$. $x + y \leq z$ everywhere on the unit square except for the triangle with vertices $(1, z-1)$, $(z-1, 1)$ and $(1,1)$. This is the right triangle with legs $2 - z$. It follows that the area of the triangle is $(2-z)^2/2$ and so

$$F_Z(z) = P(Z \leq z) = 1 - \frac{(2-z)^2}{2}. \text{ Thus } f_Z(z) = \left(1 - \frac{(2-z)^2}{2}\right)' = 2 - z.$$

7. *Let (X,Y) be uniformly distributed in a triangle $x \geq 0$, $y \geq 0$, $2x + 3y \leq 6$.*
(a) Find marginal distributions of X and Y. (b) Compute $P(X > Y)$. (c) Compute VX, VY and $Cov(X,Y)$.

Solution. The triangle has sides 3 and 2 so its area is $\frac{3\times 2}{2} = 3$. So the denisity is $\frac{1}{3}$. (a) Accordingly

$$p_X(x) = \int_0^{\frac{6-2x}{3}} \frac{dy}{3} = \frac{6-2x}{9}, \quad p_Y(y) = \int_0^{\frac{6-3y}{2}} \frac{dx}{3} = \frac{6-3y}{6} = 1 - \frac{y}{2}.$$

(b) The set $X > Y$ is a traingle with side 3 and the vertex satisfying

$$2x + 3y = 6 \text{ and } x = y.$$

Thus $5y = 6$ and $y = \frac{6}{5}$. So the area of the triangle $X > Y$ is $\frac{3\times\frac{6}{5}}{2} = \frac{9}{5}$. Since the original rectangle has area 3 the answer is $\frac{9}{5}/3 = \frac{3}{5}$.

(c) By part (a)

$$EX = \int_0^3 x \times \frac{6-2x}{9} dx = \int_0^3 \frac{6x - 2x^2}{9} dx = 1,$$
$$E(X^2) = \int_0^3 x^2 \times \frac{6-2x}{9} dx = \int_0^3 \frac{6x^2 - 2x^3}{9} dx = \frac{3}{2}.$$

Hence $V(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$.

$$EY = \int_0^2 y \times \left(1 - \frac{y}{2}\right) dy = \int_0^2 \left(y - \frac{y^2}{2}\right) dy = \frac{2}{3}.$$
$$E(Y^2) = \int_0^2 y^2 \times \left(1 - \frac{y}{2}\right) dy = \int_0^2 \left(y^2 - \frac{y^3}{2}\right) dy = \frac{2}{3}.$$

Hence $V(Y) = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$. Next

$$E(XY) = \int_0^3 \left(\int_0^{\frac{6-2x}{3}} \frac{y}{3} dy \right) x dx = \int_0^3 x \times \frac{(6-2x)^2}{54} = \int_0^3 \frac{36x - 24x^2 + 4x^3}{54} dx = 3 - 4 + \frac{3}{2} = \frac{1}{2}.$$

Thus

$$\mathrm{Cov}(X, Y) = \frac{1}{2} - 1 \times \frac{2}{3} = -\frac{1}{6}.$$