Continuous distributions.

1. *Leaks if a water pipe form a Poisson process with 4 leaks/year on avarage.*

(a) Compute the probability that during the first 6 months there will be no leaks, in the next 6 months there will be 3 leaks and during the second year there will be 3 leaks.

(b) Let T be the time of the first leak. Compute $P(T \leq t)$.

Solution. (a) Let $N(a, b)$ denote the number of leaks at the interval (a, b) (where a and b are measured in years). Then $N(a, b)$ has Poisson distribution with parameter $4(b - a)$. Thus

$$P(N(0, 1/2) = 0, N(1/2, 0) = 3, N(1, 2) = 3) = P(N(0, 1/2) = 0)P(N(1/2, 1) = 3)P(N(1, 2) = 3)$$

$$= e^{-2} \times \frac{2^3}{3!} \times e^{-4}\frac{4^3}{3!} = e^{-8}\frac{512}{36} \approx 0.0048$$

(b) $P(T > t) = P(N(0, t) = 0) = e^{-4t}$. Thus $P(T \leq T) = 1 - e^{-4t}$.

2. *Suppose that X has probability distribution function equal to cx^2, if $x \in [0, 1]$ and equal to 0 otherwise.*

(a) Find c. (b) Compute the cumulative distribution function. (c) Compute EX and VX. (d) Compute median, 25th and 75th percentiles.

Solution. (a) Since

$$1 = \int_0^1 p(x)dx = \int_0^1 cx^2dx = \frac{c}{3}$$

$c = 3$.

(b) For $x \in [0, 1]$

$$F_X(x) = \int_0^x (3u^2)du = x^3.$$

$$(c) \quad EX = \int_0^1 x \times 3x^2dx = \int_0^1 3x^3dx = \frac{3}{4}. \quad E(X^2) = \int_0^1 x^2 \times 3x^2dx = \int_0^1 3x^4dx = \frac{3}{5}.$$

$$\text{Thus} \quad V(X) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{48 - 45}{80} = \frac{3}{80}.$$

(d) The pth percentile is found from the equation $F_X(x_p) = 100p$. Thus $x_p^3 = 100p$ and $x_p = (100p)^{1/3}$. Hence

$$m = x_{50} = \left(\frac{1}{2}\right)^{1/3}, \quad x_{25} = \left(\frac{1}{4}\right)^{1/3}, \quad x_{75} = \left(\frac{3}{4}\right)^{1/3}.$$

3. *Let X have the cumulative distribution function $F(x) = \frac{x+x^3}{2}$ if $0 \leq x \leq 1$, 0 if $x \leq 0$ and 1 if $x \geq 1$. Compute*

(a) probability density function (b) EX (c) VX.

Solution. For $x \in [0, 1]$ we have $p(x) = F'(x) = \frac{1+3x^2}{2}$. Thus

$$EX = \int_0^1 x \times \frac{1 + 3x^2}{2}dx = \int_0^1 \frac{x}{2}dx + \int_0^1 \frac{3x^3}{2} = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}.$$

$$E(X^2) = \int_0^1 x^2 \times \frac{1 + 3x^2}{2}dx = \int_0^1 \frac{x^2}{2}dx + \int_0^1 \frac{3x^4}{2} = \frac{1}{6} + \frac{3}{10} = \frac{5 + 9}{30} = \frac{14}{30} = \frac{7}{15}.$$

$$V(X) = \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{7}{15} - \frac{25}{64} = \frac{7 \times 64 - 25 \times 15}{15 \times 64} = \frac{73}{960}.$$

4. *A passanger arrives at the bus stop. His waiting time T has uniform distribution on $[0, 10]$.*
(a) Compute the cumulative distribution function.
(b) Compute $P(2 \le T \le 5)$, $P(T > 3)$.
(c) Compute ET and VT.

Solution. The density if $\frac{1}{10}$ for $0 \le t \le 10$. Thus

$$F_T(t) = \int_0^t \frac{1}{10} ds = \frac{t}{10}.$$

Therefore

$$P(2 \le T \le 5) = \frac{5}{10} - \frac{2}{10} = \frac{3}{10}. \quad P(T > 3) = 1 - \frac{3}{10} = \frac{7}{10}.$$

Next

$$EX = \int_0^{10} t \times \frac{1}{10} = \frac{t^2}{20}\Big|_0^{10} = \frac{100}{20} = 5. \quad E(X^2) = \int_0^{10} t^2 \times \frac{1}{10} = \frac{t^3}{30}\Big|_0^{10} = \frac{1000}{30} = \frac{100}{3}.$$

Hence

$$V(X) = \frac{100}{3} - 5^2 = \frac{100}{3} - 25 = \frac{25}{3}.$$

5. *Suppose that a random variable X has proability density function $p(x) = e^{-x}$ if $x > 0$ and 0 otherwise. Find the median of X.*

Solution. If $X \sim$Exp(λ) then $F_X(x) = 1 - e^{-\lambda x}$. Hence the equation

$$F_X(m) = \frac{1}{2} \Leftrightarrow 1 - e^{-\lambda m} = \frac{1}{2} \Leftrightarrow e^{-\lambda m} = \frac{1}{2} \Leftrightarrow e^{\lambda m} = 2 \Leftrightarrow \lambda m = \ln 2 \Leftrightarrow m = \frac{\ln 2}{\lambda}.$$

In our case $\lambda = 1$ so $m = \ln 2$.

6. *Let Z be standard normal random variable. Find its median, 25th and 75th percentile.*

Solution. Using the normal table we find

$$z_{0.25} = -0.67, \quad z_{0.50} = 0, \quad z_{0.75} = 0.67.$$

7. *The weight of a box of appples has normal distribution with mean 30 lbs and standard deviation 2 lbs. Compute the probability that the box weights*
(a) exactly 31 pounds; (b) between 30 and 32 pounds (c) less than 29 pounds.

Solution. $W = 30+2Z$ where Z is the standard normal random variable. We have $P(W = 31) = 0$ since W has continuous distribution.

Next, using the normal table we find

$$P(30 < W < 32) = P(30 < 30 + 2Z < 32) = P(0 < Z < 1) = P(Z < 1) - P(Z < 0)$$
$$= 0.8413 - 0.5 = 0.3413 \text{ and}$$
$$P(W < 29) = P(30 + 2Z < 29) = P(Z < -1) = P(Z < 1) - P(Z < 0) = 0.8413 - 0.5 = 0.1587.$$

8. *A coin is tossed 100 times compute approximately the probability that*
(a) there are exactly 50 heads. (b) the number of heads is between 47 and 52 (inclusive).

Solution. Let X be the number of heads. Then $X \sim$Bin$(100, 0.5)$. Hence by normal approximation to binomial distribution $X \approx N(50, 25)$. That is $Z \approx 50 + 5Z$ where Z is the standard normal random variable. Hence using the normal table we conclude that

$$P(X = 50) = P(49.5 < X < 50.5) \approx P(49.5 < 50 + 5Z < 50.5)$$
$$= P(-0.1 < Z < 0.1) = P(Z < 0.1) - P(Z < -0.1) \approx 0.54 - 0.46 = 0.08,$$
$$P(47 \le X \le 52) = P(46.5 < X < 52.5) = P(46.5 < 50 + 5Z < 52.5)$$
$$= P(-0.7 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.7) \approx 0.76 - 0.31 = 0.45.$$

9. *Let $Y \sim Bin(25, p)$. Approximate $P(Y \le 10)$ without and with continuity correction and compare the results with the binomial table for $p = 0.3, 0.4, 0.5$.*

Solution. Let Φ be the normal cumulative distribution function. The approximation without correction is

$$P(Y \le 10) \approx \Phi\left(\frac{10 - 25p}{\sqrt{25p(1-p)}}\right).$$

The approximation with correction is

$$P(Y \le 10) \approx \Phi\left(\frac{10.5 - 25p}{\sqrt{25p(1-p)}}\right).$$

The results are summarized in the following table

p	without cor.	with cor.	actual
0.3	0.862	0.905	0.902
0.4	0.500	0.579	0.585
0.5	0.159	0.212	0.212

10. *Let $Y \sim Bin(100, 0.06)$. Approximate $P(Y = 3)$ using Poisson and normal approximation and compare the result with the direct computation using the binomial formula.*

Solution. Note that

$$P(Y = 3) = \binom{100}{3}(0.06)^3(0.94)^{97} = \frac{100 \times 99 \times 98}{3!}(0.06)^3(0.94)^{97} \approx 0.086.$$

Next, the Poisson approximation is $Y \approx$Pois$(100 \times 0.06) =$Pois(6). Hence

$$P(Y = 3) \approx e^{-6}\frac{6^3}{3!} \approx 0.089.$$

Finally the normal approximation says that $Y \sim N(100 \times 0.06, 100 \times 0.06 \times 0.94) = N(6, 5.64)$ $P(Y = 3) = P(Y \le 3) - P(Y \le 2)$ which according to the normal approximation is close to

$$\Phi\left(\frac{3.5 - 6}{\sqrt{5.64}}\right) - \Phi\left(\frac{2.5 - 6}{\sqrt{5.64}}\right) \approx \Phi(-1.05) - \Phi(-1.47) \approx 0.1469 - 0.0708 \approx 0.076.$$

11. *Lifetime T of a certain component has exponetial distribution with mean 10 years.*

(a) Compute the probability that $T > 5$, $T > 10$, $T > 15$.

(b) Compute the probability that componewnt will work for 15 years or more given that it worked for 5 years.

(c) Let S be the lifetime of the component measured in decades, thus $S = T/10$. Compute the distribution of S.

Solution. We have $P(T>t)=e^{-\lambda t}$ where $10=ET=\dfrac{1}{\lambda}$. Thus $\lambda=\frac{1}{10}$. (a) Accordingly

$$P(T>5)=e^{-5/10}=e^{-0.5},\quad P(T>10)=e^{-10/10}=e^{-1},\quad P(T>15)=e^{-15/10}=e^{-1.5}.$$

$$(b)\quad P(T>15|T>5)=\frac{P(T>15 \text{ and } T>5)}{P(T>5)}=\frac{P(T>15)}{P(T>5)}=\frac{e^{-1.5}}{e^{-0.5}}=e^{-1}=P(T>10).$$

(c) $P(S>s)=P(T/10>s)=P(T>10s)=e^{-10s/10}=e^{-s}$. Thus $S\sim$Exp(1).

12. *In problem 1 let T_2 and T_3 be the times of the second and third leak respectively. Find their probability density functions.*

Solution.

$$P(T_2>t)=P(N(0,t)<2)=P(N(0,t)=0)+P(N(0,t)=1)=e^{-4t}+(4t)e^{-4t}.$$

Thus $P(T_2<t)=1-(e^{-4t}+(4t)e^{-4t})$. Therefore

$$p_2(t)=-\left(e^{-4t}+(4t)e^{-4t}\right)'=4e^{-4t}+4(4t)e^{-4t}-4e^{-4t}=4^2te^{-4t}.$$

That is, $T_2\sim\Gamma(2,1/4)$. Likewise

$$P(T_3>t)=P(N(0,t)<3)=P(N(0,t)=0)+P(N(0,t)=1)+P(N(0,t)=2)$$

$$=e^{-4t}+(4t)e^{-4t}+\frac{(4t)^2}{2}e^{-4t}.$$

Thus $P(T_3<t)=1-\left(e^{-4t}+(4t)e^{-4t}+\frac{(4t)^2}{2}e^{-4t}\right)$. Therefore

$$p_3(t)=-\left(e^{-4t}+(4t)e^{-4t}+\frac{(4t)^2}{2}e^{-4t}\right)'=4e^{-4t}+4(4t)e^{-4t}-4e^{-4t}+4\frac{(4t)^2}{2}e^{-4t}-(4^2t)e^{-4t}=\frac{4^3t^2}{2}e^{-4t}.$$

That is, $T_3\sim\Gamma(3,1/4)$.

13. *Let Z be the standard normal random variable and $X=Z^2$. Find the distribution of X.*

Solution.

$$P(X\le x)=P(Z^2\le x)=P(-\sqrt{x}\le Z\le\sqrt{x})=\frac{2}{\sqrt{2\pi}}\int_0^{\sqrt{x}}e^{-s^2/2}ds.$$

By the Fundamental Theorem of Calculus

$$p_X(x)=\frac{2}{\sqrt{2\pi}}\left(\sqrt{x}\right)'e^{-(\sqrt{x})^2/2}=\frac{1}{\sqrt{2\pi x}}e^{-x/2}.$$

That is $X\sim\Gamma\left(\frac{1}{2},2\right)$.

14. *A malfunctioning of a certain engine could be caused by two components. Lifetimes of those components are independent and have exponential distributions with means 2 and 3 years respectively. Find the average lifetime of the engine.*

Solution. Let T_1 and T_2 be lifetime of the first and the second componet respectively. Thus T_1 ~Exp(1/2) and T_2 ~Exp(1/3). Then $T = \min(T_1, T_2)$. Hence

$$P(T > t) = P(T_1 > t, T_2 > t) = P(T_1 > t)P(T_2 > t)$$
$$= \exp\left(-\frac{t}{2}\right)\exp\left(-\frac{t}{3}\right) = \exp\left(-\left(\frac{t}{2} + \frac{t}{3}\right)\right) = \exp\left(-\frac{5t}{6}\right).$$

Hence T ~Exp(6/5).