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Computational Physics: Problem Set 2

Discussion: 22/02/2023

$\boxed{1}$ Euler integration I – damped harmonic oscillator

Consider the undriven damped harmonic oscillator defined by the initial value problem

$$\partial_t^2 u(t) + \gamma \partial_t u(t) + u(t) = 0, \quad u(0) = 1, \partial_t u(0) = 0, \tag{1}$$

for $t > 0$.

a) Convince yourself that every harmonic oscillator can be brought to the form of Eq. (1) by rescaling the time variable $t \to \Omega t$.
Transform the Eq. (1) to a system of first order ODEs. Find the eigenvalues (complex eigenfrequencies) of the system and pick values for γ that lead to the four main regimes: undamped, underdamped, critically damped, overdamped.

b) Implement the Euler method for the 2×2 matrix problem

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & \gamma \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

c) Analytically find the two linearly independent analytical solutions to the ODE. From them, construct the analytical reference solutions $\tilde{y}(t)$ for the given initial conditions and the values of γ that you picked.

d) Run the program for the the time interval $0 \leq t \leq 100$.
Do this for each of the four values of γ picked in subproblem a) and using the time steps $\Delta t \in \{1, 0.1, 0.01, 0.001\}$ (16 calculations in total).
For each combination plot the error $E(t) = ||y(t) - \tilde{y}(t)||$ on appropriate semi-log and a double-log plot to see how the error evolves over time and it scales with the time step size. What is the convergence order of the method?

$\boxed{2}$ Euler integration II – Kepler problem

Consider the Kepler problem, i.e. the problem of a particle orbiting another particle to which it is attracted by a force (gravitation, electrostatic force). The particle is characterized by its position vector $\mathbf{r} = (x, y)^T$ and momentum vector $\mathbf{p} = (p_x, p_y)^T$. It obeys the system of ODEs

$$\begin{aligned} \partial_t \mathbf{r} &= \frac{1}{m}\mathbf{p}, \\ \partial_t \mathbf{p} &= -\frac{G\mathbf{r}}{|\mathbf{r}|^3}. \end{aligned}$$

a) Bring the problem to the form
$\partial_t \mathbf{r} = \mathbf{p}, \qquad \partial_t \mathbf{p} = -\mathbf{r}/|\mathbf{r}|^3$
by rescaling the time and the momentum variable.

b) Implement an Euler scheme for the rescaled problem and run it for the initial conditions $\mathbf{r}(0) = (1,0)^T$, $\mathbf{p}(0) = (0,p_0)^T$ with values $p_0 \in \{0, 0.3, 1, 2\}$ and various time steps $\Delta t \in \{1, 0.1, 0.01, 0.001\}$. Simulate for a time intervals that seem appropriate to you (may depend on the parameters). Summarize your observation and discuss if they are physically plausible.

Remark: Write the programs in a clean way and save them separately. You will need them again.

Students are encouraged to solve these problems in groups. They should be able to informally present their solution in the problem class on 22/02/2023 at the white board (no PowerPoint slides to be prepared).