Error Analysis of Modulo Unfolding Algorithms

November 18, 2025

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0.1 Methodology for Plot Construction

To systematically evaluate the reconstruction error, we consider two common axes of analysis.

For each value along the horizontal axis (either OF or c/λ), we performed 20 independent trials. In each trial:

1. A random bandlimited signal is generated with predefined spectral characteristics.

2. Modulo folding is applied using the specified threshold λ.

3. The signal is reconstructed using the algorithm under test.

4. The reconstructed signal is compared against the original, and the mean squared error (MSE) is computed.

To express the error on a logarithmic scale, the MSE is converted to decibels (dB) using:

$$\text{MSE (dB)} = 10 \log_{10} (\text{mean square error})$$

For each point along the x-axis, we compute the mean and standard deviation of the error (in dB) across the 20 trials. The resulting plots consist of:

- A dark solid line representing the **mean MSE** for each parameter value.

- A light shaded region around the line that captures the **uncertainty** or variability in the error, represented by $\pm$ one standard deviation.

This visualization allows us to assess not only the expected reconstruction performance of each algorithm, but also its robustness across different random signal realizations.

Comparison: Error vs Oversampling Factor

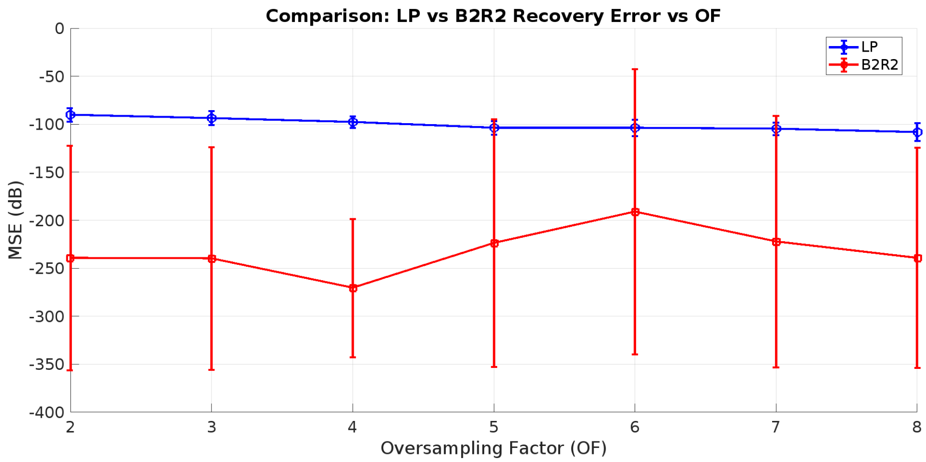

Figure 1: error vs OF

magnified view of linear prediction curves

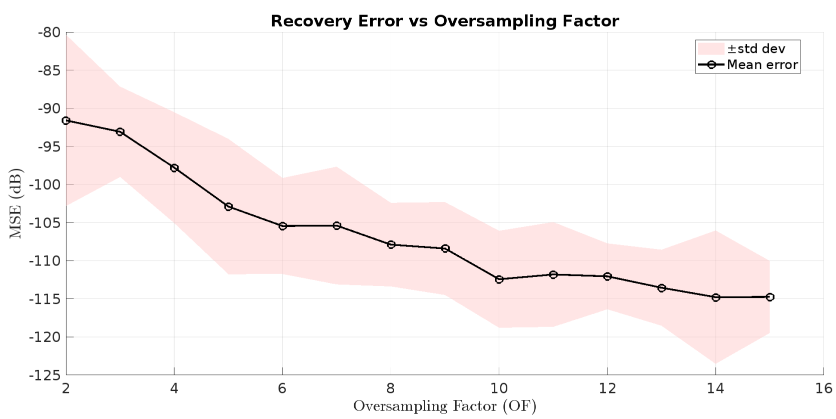

Figure 2: error vs OF

chebyshev filter based approach follows the principle that error must reduce as OF increases and with low uncertainity.
while B2R2 algo seems to go off with the intiution of error decreasing with OF increasing.Here are few causes:

- **20 samples taken**:for accurate variance and mean calculations to gain some insights we need to take atleast 50 samples to see the "gaussian" effect(central limit theorem).

- **Hyperparameters used:** In the code,we decay both mu and momentum as: mu = decay * mu;
mom = decay * mom;
But:

  Some realizations require longer steps to escape shallow regions .

  Others benefit from slow descent

  If you decay too fast or too slow:

  Recovery for that trial can diverge or converge poorly

  This is not **signal-dependent**, but depends on gradient shape — which varies heavily with the random signal's coefficient rather than spectral characteristics.

- N_λ **window length:** we sort of shrink "window" in the algorithm . Window size deppends on N_λ,which further depends on offset of randomly generated sinc functions.So a same bandwith signal can have different window length and so more PGD updates are required for some signal and so different errors can be obtained.

- **ok then why this** N_λ thing does not interfere much in linear prediction algorithm as there we just use samples till N_λ once to predict next set of samples.while in b2r2 algo we again and again use it to predict the samples.

Recovery Error vs $\frac{c}{\lambda}$

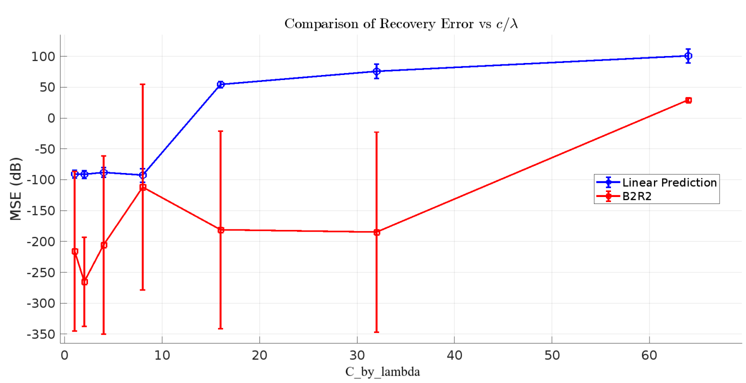

Figure 3: error vs $\frac{c}{\lambda}$

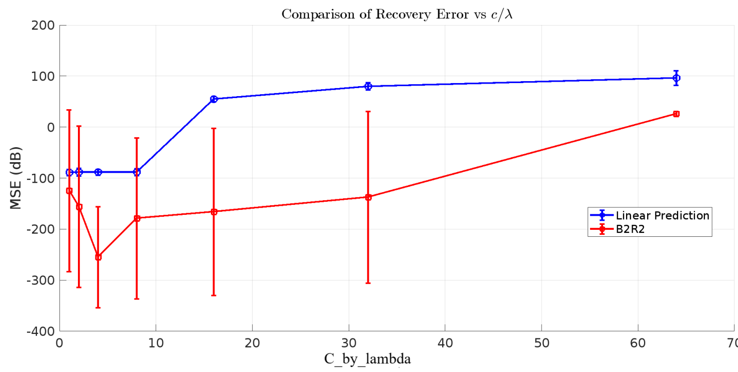

Figure 4: error vs $\frac{c}{\lambda}$

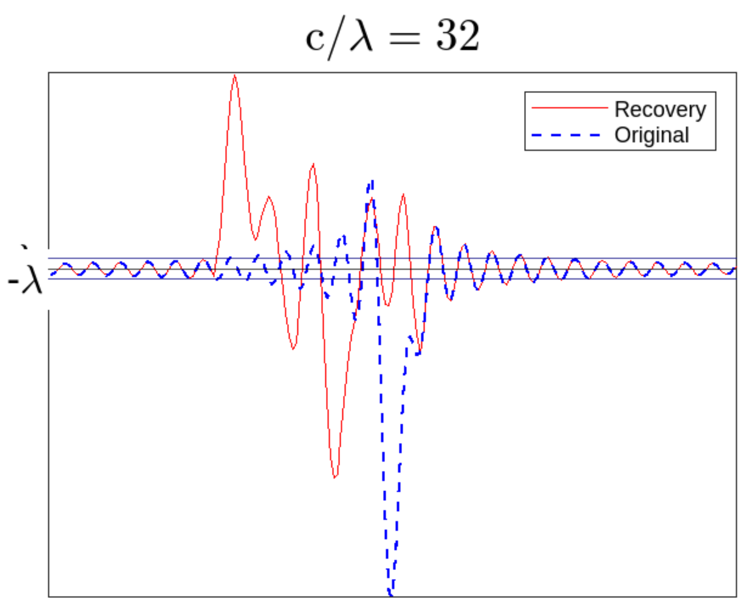

Figure 5: not working with c/lambda=32

$c/\lambda = 32$

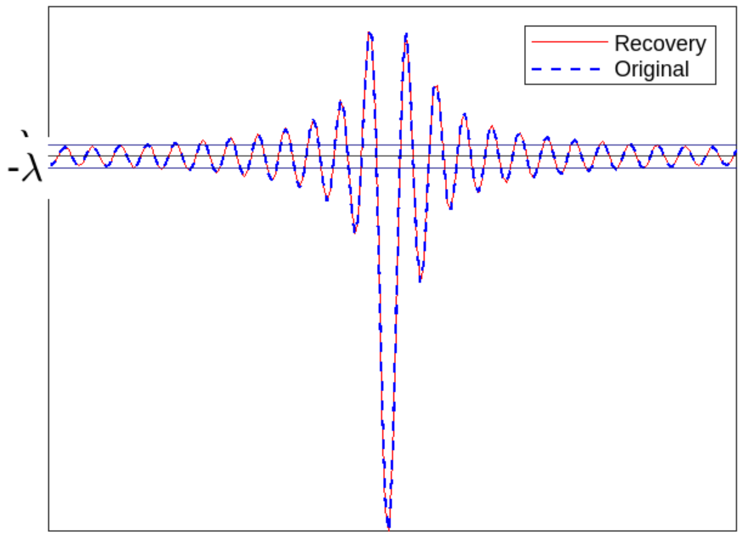

Figure 6: working with c/lambda

Algorithm 1 $\mathrm{B}^2\mathrm{R}^2$: Recovery from Modulo Samples

Require: Modulo samples x_λ, threshold λ, oversampling factor OF, window size $M = 2N + 1$

Ensure: Reconstructed signal $\hat{x}$

Initialize $\hat{x} \leftarrow x_\lambda$, $N \leftarrow \lfloor M/2 \rfloor$
Compute $\epsilon \leftarrow \min(10^{-3}, \max(10^{-6}, 10^{-4} \cdot \lambda \cdot \log(M+1) \cdot \sqrt{\mathrm{OF}}))$
Construct Band-pass Partial DFT matrix F_ρ and compute $G = F_\rho^\top F_\rho$
Initialize $z^0 \leftarrow \mathcal{BP}(-x_\lambda)$ $\triangleright$ Band-pass init in $\omega \in [\omega_m, \omega_s - \omega_m]$
repeat
    for $k = 1, 2, \ldots$ **do**
        $\nabla_k \leftarrow G(\hat{x} + z^{k-1}) + \mathrm{reg} \cdot z^{k-1}$
        $z^k \leftarrow z^{k-1} - \mu_k \nabla_k + m \cdot (z^{k-1} - z^{k-2})$
        Zero out entries of z^k outside $[-N, N]$
        if convergence $\left(\max |z^k - z^{k-1}| < \epsilon\right)$ **then**
            $z^k \leftarrow \mathrm{Quantize}(z^k, 2\lambda)$
            if $N < 1$ **then**
                Update $\hat{x}[0] \leftarrow \hat{x}[0] + z^k[0]$
                return $\hat{x}$
            else
                Update $\hat{x}[\pm N] \leftarrow \hat{x}[\pm N] + z^k[\pm N]$
                $N \leftarrow N - 1$
                Reset step size μ, momentum m
            end if
            Reinitialize: Zero out entries of z^k outside $[-N, N]$
            break
        end if
    end for
until converged
return $\hat{x}$