Modulo Signal Recovery Algorithm

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Problem Statement

Input:

- Discrete-time signal $x_{\text{mod}}[n]$ of length N
- Time vector $t_s[n]$ corresponding to $x_{\text{mod}}[n]$
- Parameters: Energy E, Frequency W, Oversampling factor OF, Modulo parameter λ, Max signal value c

Output:

- Reconstructed signal $x_{\text{rec}}[n]$ and error metrics

Parameter Initialization

- Set Modulo threshold: $\Delta = 2\lambda$
- Set filter window K and predictor coefficients $h_{\text{coeffs}}[k]$ ($k = 0 \ldots 2K + 1$), designed for bandwidth W

Modulo Operation

For $n = 1$ to N:

$$x_{\text{mod}}[n] = x_{\text{orig}}[n] - \Delta \cdot \text{Round}\left(\frac{x_{\text{orig}}[n]}{\Delta}\right)$$

or equivalently,
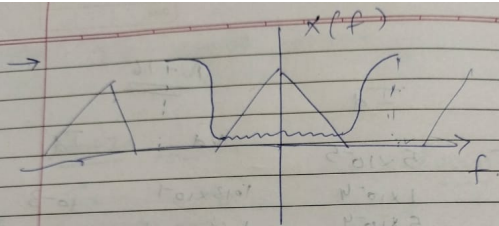
$$x_{\text{mod}}[n] = x_{\text{orig}}[n] + z_n,$$

where

$$z_n \in \{m\Delta : m \in \mathbb{Z}\}$$

is the folding term representing integer multiples of the modulo threshold Δ.

- $z_{\text{max}} = \left\lfloor \frac{c}{\Delta} + 0.5 \right\rfloor \cdot \Delta$
- $z_n \in \{-z_{\text{max}}, ..., z_{\text{max}}\}$
- $x_{\text{rec}}[1 : N_{\text{init}} - 1] = x_{\text{mod}}[1 : N_{\text{init}} - 1]$
- $z[n] = 0$ for all n

$$X(f) \cdot H(f) \longleftrightarrow x(t) * h(t)$$

$h \rightarrow$ chebyshev coefficient filter.

$$x(nT_s) = x_\lambda(nT_s) - Z(nT_s)$$

$$x(nT_s) * h(nT_s) = x_\lambda * h - Z * h$$

minimize the convolution

Main Recovery Loop (Detailed)

For $n = N_{\text{init}}$ to N:

- For each $m \in c_n$:
$$z^{\text{next}}[n] = m \cdot \Delta$$

- Build z^{full} by setting $z^{\text{full}}[1 : n-1] = z[1 : n-1]$, $z^{\text{full}}[n] = z^{\text{next}}[n]$

- Compute the error term as the absolute value of the difference between two convolutions:
$$e_m = \left| (x_{\text{mod}} * h_{\text{coeffs}})[n] - (z^{\text{full}} * h_{\text{coeffs}})[n] \right|,$$

  where $*$ denotes convolution.

- Select m^* minimizing e_m, assign $z[n] = m^* \cdot \Delta$

- Recover the unfolded sample:
$$x_{\text{rec}}[n] = x_{\text{mod}}[n] - z[n]$$

Signal Reconstruction and Metrics

- Use sinc interpolation to compute $x_{\text{rec,cont}}(t)$ from $\{x_{\text{rec}}[n]\}$
- Mean squared error:

$$\text{MSE} = \frac{\sum_n (x_{\text{orig}}[n] - x_{\text{rec}}[n])^2}{\sum_n (x_{\text{orig}}[n])^2}$$

Results

- With this new algorithm we are able to predict the samples with **ZERO MSE**
- compared to linear prediction approach where MSE in samples is of order **10e-30**
- Using Whittaker–Shannon to reconstruct the continous-analog version of signal , it is found MSE for both algorithms is almost same(MSE(dbs) differs after 21st decimal) for high oversampling factors.
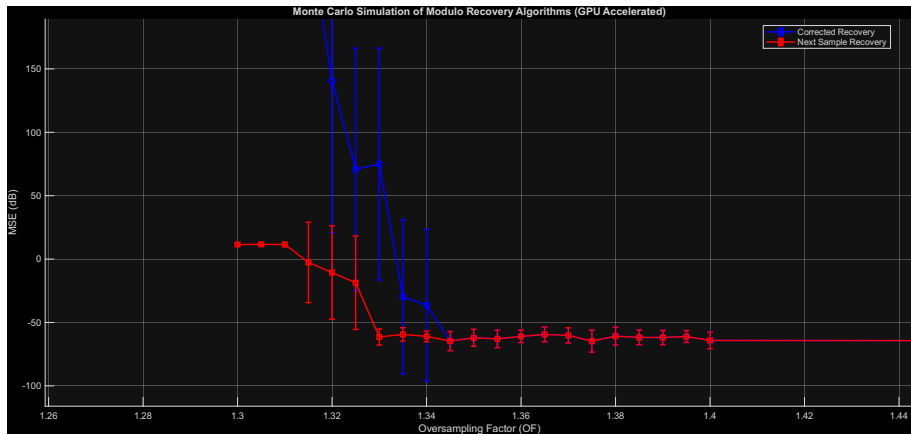
Figure: Blue:Linear prediction,Red:New Algo
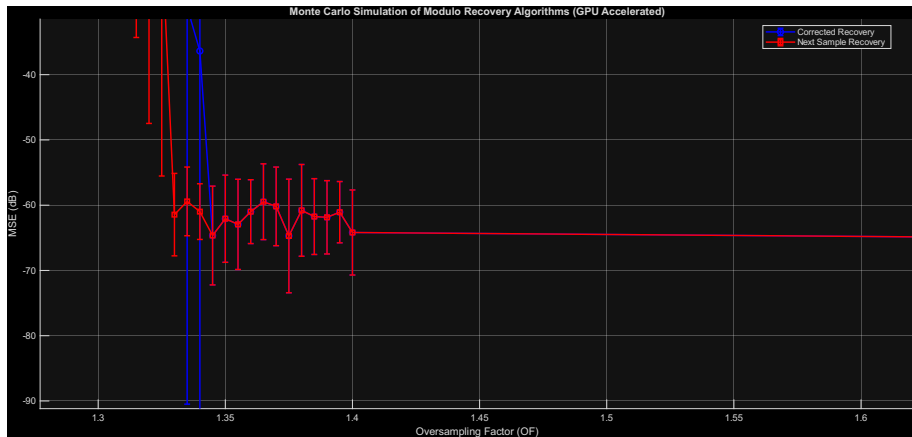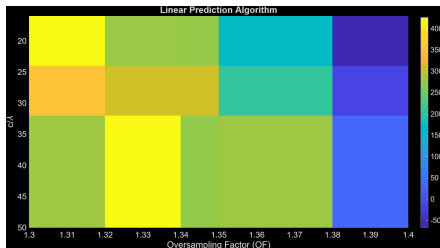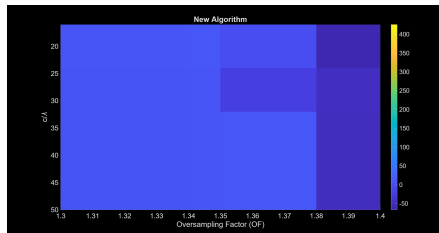
Results for low oversampling factors

Figure: Blue:Linear prediction,Red:New Algo

(a) Old Method

(b) New Method

Figure: Comparison between old and new methods

- Recover signal folding sequence $z(n)$ without prior knowledge of N_λ.
- what if initial samples of signal are folded ones??
- Use an **exhaustive search** combined with constraints derived from bandlimited signal theory.
- Key insight: Predict z by minimizing error between modulo-derived term and linear predictor output.

Exhaustive Search to Predict First Folding Sequence

- Define folding candidates over the set

$$\mathcal{C} = \{-p\Delta, \ldots, 0, \ldots, p\Delta\}^n$$

  where n is filter length, p is search range.

- Compute error function:

$$f(z) = \left| (x_{\mathrm{mod}} * h_{\mathrm{coeffs}})[n] - (z^{\mathrm{full}} * h_{\mathrm{coeffs}})[n] \right|,$$

  This time we don't know any element in z^{full}. Hence we perform exhaustive search to find combination of $z(n)$ that minimise the error function.

Bandlimited Signal Constraints(From Papoulis Paper)

Consecutive recovered signal differences must satisfy:

$$|x_{rec}(i+1) - x_{rec}(i)| < T,$$

with

$$T = BW \cdot T_s \sqrt{\frac{E \cdot BW}{3\pi}}.$$

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- Use this to **discard invalid candidates** from exhaustive search.
- Select $z_{\min}$ that both minimizes $f(z)$ **and** satisfies the bounded difference condition.
- If none satisfy, fallback to unconstrained minimum with warning.

- For each $z_{\text{candidate}}$:

$$x_{\text{rec}} = x_{\text{mod}}(1:n) - z_{\text{candidate}}$$

- Verify bounded consecutive differences:

$$\max_i |x_{\text{rec}}(i+1) - x_{\text{rec}}(i)| < T$$

- Only **valid** candidates are considered.

Estimation of N_λ from Modulo Signal

- Goal: Determine N_λ, the index before which samples can be considered unfolded (i.e., not modulo folded).
- Based on the key relation from the paper:

$$x_n * h_n = x_{\text{mod}} * h_n - z * h_n,$$

  where h_n is the Chebyshev predictor and x_{mod} is the folding sequence.
- Assuming the sample is unfolded ($z = 0$), test:

$$(x_n * h_n)^2 < \frac{\Delta^2}{4}.$$

- If true, sample n is likely unfolded.
- Algorithmically:
    - Start from $n = \text{length}(h_{\text{coeffs}}) + 1$.
    - For each n, compute the convolution $x_{\text{mod}} * h_{\text{coeffs}}$.
    - Increase N_λ while $(x_{\text{mod}} * h_{\text{coeffs}}(n))^2 < \frac{\Delta^2}{4}$.
    - Stop at the first n violating this condition.
- Benefit: Accurately leverage unfolded initial samples for better recovery performance.

$$e_n = (x_n)_{true} - \sum_{q=1}^{2*} x_{n-i} h_i$$

$$e_n = x_n * h_n$$

$$e_n^* = (x_n^* * h_n)^*$$

$* \longrightarrow$ modulo operation

if $|e_n| < \frac{\Delta}{2}$

$\therefore \quad x_n * h_n = x_n * h - 2 * h$

for unfolded samples $\underline{2=0}$

$\therefore \quad |x_n * h| < \frac{\Delta}{2}$

If inequality does not hold the sample in "folded one"!

$\Rightarrow N_\lambda$ estimated.