Birla Institute of Technology & Science, Pilani
Second Semester 2022-2023

Online B.Sc. Computer Science Programme
Comprehensive Make-up Exam

<u>Question & Answers:</u>

Question 1:
Suppose $n=m^2$, so that n and m are integers and n is an even number. Prove that m is also an even number.

Answer:

For the sake of contradiction,

Let $n=m^2$, so that n and m are integers and n is an even number. Let us assume that m is odd.

Therefore, m can be expressed as

$m = 2k + 1$, where k is an integer.

Therefore,

$n = m^2$

$= (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2+2k) + 1$ ---(i)

We know,

Any number which can be expressed in the form of 2x+1 (where x is an integer) is odd.

Therefore, from (i)

$n = 2p + 1$ (where $p=2k^2+2k$)

= n is odd.

This contradicts our assumption that n is even.

Hence, m must be even.

Hence proved.

Question 2:

Provide a counterexample to the following statement:

If fog is one-to-one, then f is one-to-one.

Answer:

$f(x) = x^2$

x belongs to R $g(x) = x^{3/2}$

Therefore $f(g(x)) = x^3$

<u>Proving f(g(x)) is one-to-one:</u>

Let f(g(x)) be h(x)

To prove, let $h(x_1) = h(x_2)$, where x_1, x_2 belongs to the domain of h(x)

Therefore,

$h(x_1) = h(x_2)$

- ⇨ $x_1^3 = x_2^3$
- ⇨ $x_1^3 - x_2^3 = 0$
- ⇨ $(x_1 – x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$
- ⇨ $(x1 – x2) = 0$ (as $x_1^2 + x_1x_2 + x_2^2$ has no real roots)
- ⇨ $x_1 = x_2$
- ⇨ h(x) is one-to-one

<u>Proving f(x) is not one-to-one:</u>

To prove, let $f(x1) = f(x2)$

Therefore,

$f(x1) = f(x2)$

- ⇨ $x_1^2 = x_2^2$
- ⇨ $x_1^2 – x_2^2 = 0$
- ⇨ $(x_1 + x_2)(x_1 – x_2) = 0$
- ⇨ $(x_1 + x_2) = 0$ or $(x_1 – x_2) = 0$
- ⇨ $x_1 = x_2, - x_2$
- ⇨ f(x) is not one-to-one

Hence this is a perfect counter example.

Question 3:

Give an example of two uncountably infinite sets A and B, such that A- B is

(a) Finite

(b) Countably infinite

(c) Uncountably infinite

Answer:

(a)

Let A and B be the set of real numbers (R).

Here, A – B = {} (nullset), which is finite.

(b)

Let A be the set of real numbers (R) and B be the set of irrational numbers.

Here, A – B = set of all real numbers – set of all irrational numbers

= C, where C is the set of all rational numbers.

We know, rational numbers are countably infinite.

Hence, here A – B is countably infinite.

(c)

Let A be the set of real numbers (R) and B = {x | x belongs to [0,1]}

Here, A – B = R – [0,1]

= (-infinity, 0) U (1,infinity)

which is uncountably infinite.

Question 4:

Consider the vertex set V = {v_1, … , v_n}.

How many distinct directed graphs exist that have V as their vertex set?

Answer:

Let a,b belongs to V.

Each pair (a, b) is a possible edge.

There are n choices for a and (n-1) choices for b.

So, the total number of possible edges are n(n-1).

Now for each edge, it may be there in the graph or it may be not there.

So, for each edge, there are 2 possibilities.

Hence, for n(n-1) edges, the total number of possible distinct edges are $2^{n(n-1)}$

Question 5:

Given an undirected graph, its degree sequence is the monotonic nonincreasing sequence of the degrees of the vertices of the graph. For example, the cycle C4 has the degree sequence 2, 2, 2, 2 and the wheel W4 has the degree sequence 4, 3, 3, 3, 3.

(a) What is the degree sequence of the complete bipartite graph $K_{m,n}$, where m and n are positive integers. Briefly justify.

(b) Provide a counterexample to the following statement:

If two simple undirected graphs have the same degree sequence, then they are isomorphic.

Answer:

(a)

Let $K_{m,n}$ be a bipartite graph with two disjoint set of vertices v_1 and v_2, where $|v_1| = m$ and $|v_2| = n$

For each vertex v belongs to v_1,

Each can connect to n vertices of v_2.

For each vertex v belongs to v_2,

Each can connect to m vertices of v_1.

<u>Case 1: (when m<=n)</u>

The degree sequence of the complete bipartite graph $K_{m,n}$ is

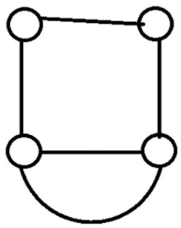
{ n, n, n, … , n, m, m, m, … , m }
(m times) (n times)

<u>Case 2: (when m>n)</u>

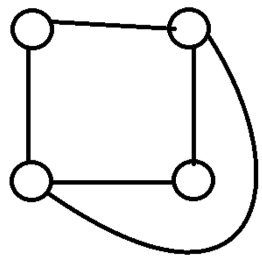
The degree sequence of the complete bipartite graph $K_{m,n}$ is

{ m, m, m, … , m, n, n, n, … , n }
(n times) (m times)

(b)



Graph G_1



Graph G_2

Degree sequence of G_1 = {2,2,3,3}

Degree sequence of G_2 = {2,2,3,3}

The degree sequences of G_1 and G_2 is same. But if we see both the graphs, they can't be isomorphic. In the graph G_1 the vertices of degree 3 are consecutive, while in graph G_2 they are not.

Hence, this is a proper counterexample of the statement.

Question 6:

Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in that strongly connected component.

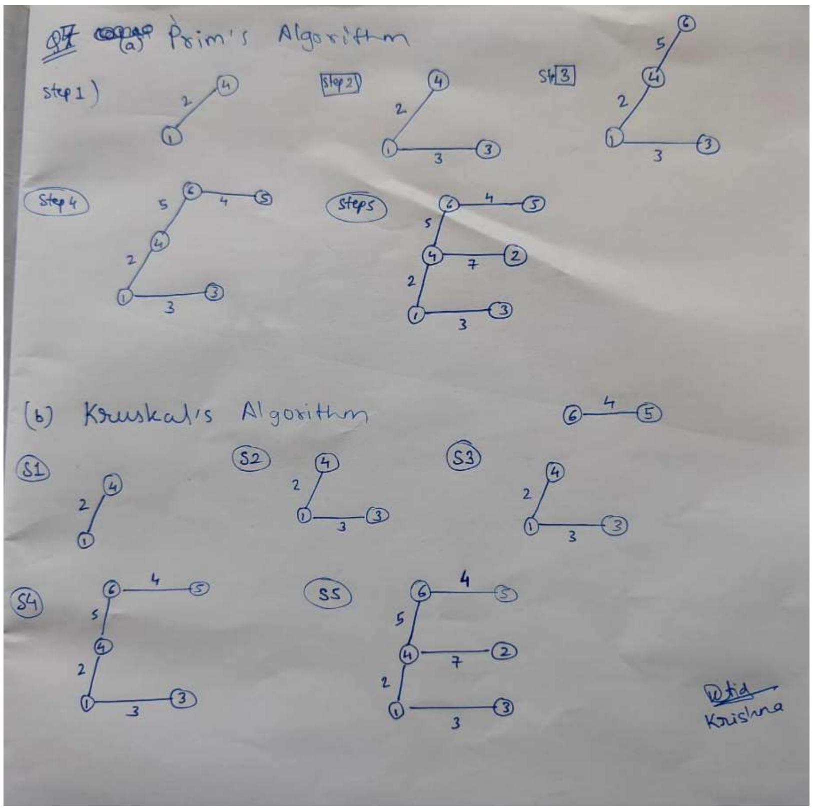
Answer:

Let a, b, c, ... , z be the directed path. Since z and a are in the same strongly connected component, there is a directed path from z to a. This path appended to the given path gives us a circuit. We can reach any vertex on the original path from any other vertex on that path by going around this circuit.

Question 7:

For the following weighted undirected graph, briefly trace the steps taken by the following algorithms in determining the minimum spanning tree: (a) Prim's Algorithm (b) Kruskal's Algorithm.

Answer:



Question 8:

Prove that the union of two subgroups of a group need not necessarily be a subgroup.

Answer:

Seeking a contradiction, let us assume that the union $H_1 \cup H_2$ is a subgroup of G.

Since $H_1 \not\subset H_2$, there exists an element $a \in H_1$ such that $a \notin H_2$.
Similarly, as $H_2 \not\subset H_1$, there exists an element $b \in H_2$ such that $b \notin H_1$.

As we are assuming $H_1 \cup H_2$ is a group, we have $ab \in H_1 \cup H_2$.
It follows that either $ab \in H_1$ or $ab \in H_2$.

If $ab \in H_1$, then we have

$b = a^{-1}(ab) \in H_1$

as both a^{-1} and ab are elements in the subgroup H_1.
This contradicts our choice of the element b.

If $ab \in H_2$, then we have

$a = (ab)\, b^{-1} \in H_2$

as both b^{-1} and ab are elements in the subgroup H_2.
This contradicts our choice of the element b.

In either case, we reached a contradiction.
Thus, we conclude that the union $H_1 \cup H_2$ is not a subgroup of G.

Birla Institute of Technology & Science, Pilani
Second Semester 2022-2023

Online B.Sc. Computer Science Programme
Comprehensive Regular Exam

Course No.	: BCS ZC219
Course Title	: Discrete Mathematics
Nature of Exam	: Open Book
Weightage	: 50%, 150 marks
Duration	: 2 Hours 30 Minutes
Date of Exam	: 18 Feb 23

No. of Pages = 2
No. of Questions = 9

Note to Students:
1. All parts of a question should be answered consecutively. Each answer should start from a fresh page.
2. Assumptions made if any, should be stated clearly at the beginning of your answer.

Q.1. Is the implication operation in Propositional Logic associative? That is, is it always the case that
$(p \rightarrow q) \rightarrow r = p \rightarrow (q \rightarrow r)$? Prove or disprove.

[15 Marks]

Q.2. It is convenient to have a new kind of existential quantifier, which we will call ∃1 quantifier ("there exists one" quantifier). Specifically ∃1 x P(x) is true if and only if there exists exactly one element x in the universe for which P(x) is true. Write down a predicate logic formula for ∃1 x P(x) -- i.e. a formula for ∃1 x P(x) that involves only the standard existential/universal quantifiers and logical connectives in Predicate Logic. (You do not need to explain why the formula is correct.)

[15 Marks]

Q.3. Prove, using mathematical induction, that 2n ≥ n + 12 for all natural numbers n ≥ 4.

[15 Marks]

Q.4.
Consider the set S={2, 6, 8, 16, 18, 32, 36}

(a) [10 marks] Draw the Hasse Diagram corresponding to the “divides” partial ordering relation on this set, i.e. (a,b) is in the relation iff a divides b.
(b) [5 marks] Write down the maximal and minimal element(s) of this poset.
(c) [5 marks] Is this poset a lattice? Briefly justify.

[20 Marks]

Q.5. I assign an individual C programming assignment to class wherein the character limit for the program is 100 characters. If C has a character set of size 128, and everyone submits a non-empty program of 100 characters or less (which may or may not be a valid C program), what is the minimum number of students my class must have, to guarantee that two submitted assignments are identical, character-by-character? Briefly justify your answer.

[15 Marks]

Q.6. Is every graph whose chromatic number is at most 4, planar? Prove or disprove.

[20 Marks]

Q.7.

Solve the recurrence relation $f(0) = 2$, $f(n) = f(n-1)+2$.

[15 Marks]

Q.8.

Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

[20 Marks]

Q.9. Let $(S, *)$ be a group, with $e \in S$ being the identity element. Suppose for $a,b \in S$, we have $a*b=a$. Prove that $b=e$.

[15 Marks]

****

1. $(p \to q) \to r = p \to (q \to r)$

p	q	r	$Q = p \to q$	$LHS = Q \to r$	$P = q \to r$	ans $= p \to P$
0	0	0	1	0	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Implication is not associative as $LHS \neq RHS$

OR

If all p, q and r are false

$\to$ $p \to (q \to r)$ is true as p is false

But $p \to q \to r$ is false

but $p \to q$ is true

$\Rightarrow$ if p and r are false but q is true

then $p \to (q \to r)$ is true because p is false

but $p \to q$ is true, r false so $(p \to q) \to r$ false

They have different values, these statements are not equivalent

$\therefore$ $\to$ is not associative

2. $\exists x (P(x) \wedge \forall y (P(y) \rightarrow (x=y)))$ $\rightarrow$ There exists x such that $P(x)$ is true, and for all y, if $P(y)$ is true, then $y=x$

This ensures that there is exactly one element x in the universe for which $P(x)$ is true.

3. First prove that the inequality holds for smallest value of $n=4$

For $n=4$

$2n = 2 * 4 = 8$

$n+12 = 4+12 = 16$

$2(n) >= n+12$

Assume inequality holds for k and $k+1$

$2(k+1) >= (k+1) + 12$ (Inductive Step)

$2k \geq k+12$

add 2 both sides

$2k+2 \geq k+12+2$

Simplify

$2(k+1) \geq k+14$

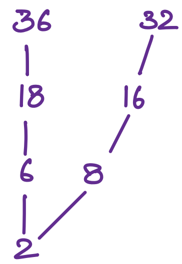
By mathematical induction

$2k \geq k+12$ holds for some integer k, then it holds for $k+1$

if base case $n=4$ holds true, it holds for all natural numbers $n \geq 4$

4. a divides b $\longrightarrow$ $\{2, 6, 8, 16, 18, 32, 36\}$

Hasse-diagram



(b) Maximal element

36, 32

Minimal elements

2

(c) Is this POSET a lattice?

meet semilattice $\forall x,y \in S,\ GLB(x,y) \neq \phi$

join semilattice $\forall x,y \in S,\ LUB(x,y) \neq \phi$

Consider the incomparable pairs

$GLB(18,16) = 2$

$LUB(18,16) = \phi$

Hence this is not a lattice, as its a meet semilattice but not join semilattice

5. Total number of distinct programs of 100 characters or less:

No.of possibilities for each char (128) ^ No.of char (100)

Total distinct programs $\Rightarrow$ 128 ^100

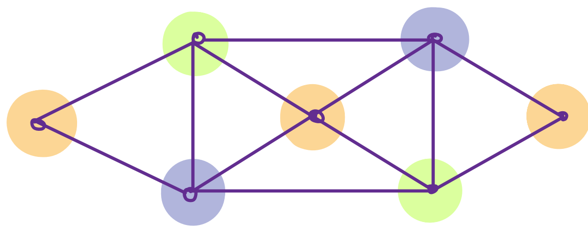
According to Pigeonhole Principle, atleast one pigeonhole must contain more than one pigeon.

$\Rightarrow$ (128 ^ 100 + 1) students

6. The 4 color theorem states that chromatic number of a planar graph is no greater than 4

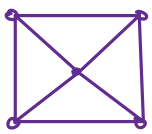
If the regions of a planar graph are colored so that adjacent regions have different colors, then no more than 4 colors are required

$$\chi(G) \leq 4$$



Euler's formula → $v - e + f = 2$

f → faces



1 face : 3 edges

$3 \geq 3(1)$

$\therefore e \geq 3f$

$e = 8$

$f = 5$

$2e \geq 3f$

$\therefore f = 2 - v + e$

$3f = 6 - 3v + 3e$

We know, $2e \geq 3f$

$\therefore 2e \geq 6 - 3v + 3e$

$\boxed{e \leq 3v - 6}$

7. $f(0) = 2, \; f(n) = f(n-1) + 2$

Using iterative method

$f(1) = f(0) + 2 = 2 + 2 = 4$

$f(2) = f(1) + 2 = 4 + 2 = 6$

$f(3) = f(2) + 2 = 6 + 2 = 8$

In General

$$\boxed{f(n) = 2n + n}$$

8. Given a simple undirected connected graph $G=(V,E)$

$e \in E$ is a cut edge if $G' = (V, E-\{e\})$ has atleast 2 non-empty connected components

Theorem: $e \in E$ is a cut edge $\Leftrightarrow$ e doesn't belong to a circuit in G

Suppose: $e \in E$ belongs to a circuit in G

circuit $\rightarrow e_1, \ldots, e_k$ with $e = e_i$ for $1 \leq i \leq k$

Removal of e_i still allows traversal

Hence G is still connected $\longrightarrow$ e can't be cut edge $\rightarrow$ Proof by contradiction

9. $(S, *)$ is a group

$e \in S \rightarrow$ identity element

Suppose $a, b \in S \Rightarrow a * b = a$

$b * e = e * b = b$ (property of identity element)

given $a * b = a$

Multiply both sides with a^{-1}

$a^{-1} * (a * b) = a * a^{-1}$ $(a * a^{-1} = e$ Inverse element)

$(a^{-1} * a) * b = e * a$ (Associative Property)

$e * b = a$ $(a * a^{-1} = e$ Inverse element)

$e =$ identity $(a * e = a$ Identity element)

$e * b = b$

$b = a$

Hence if $a * b = a$, b must be identity

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Second Semester 2022-2023

Online B.Sc. Computer Science Programme
Comprehensive Make-up Exam

Course No.	: BCS ZC219
Course Title	: Discrete Mathematics
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Weightage	: 50%, 150 marks
Duration	: 2 Hours 30 Minutes
Date of Exam	: 03 Mar 23

No. of Pages = 2
No. of Questions = 8

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Q.1. Suppose $n=m^2$, so that n and m are integers and n is an even number. Prove that m is also an even number.

[20 Marks]

Q.2. Provide a counterexample to the following statement:
If $f \circ g$ is one-to-one, then f is one-to-one.

[20 Marks]

Q.3. Give an example of two uncountably infinite sets A and B, such that A - B is
(a) Finite
(b) Countably infinite
(c) Uncountably infinite

[15 Marks]

Q.4. Consider the vertex set $V=\{v_1, \dots , v_n\}$.
How many distinct directed graphs exist that have V as their vertex set?

[15Marks]

Q.5. Given an undirected graph, its degree sequence is the monotonic nonincreasing sequence of the degrees of the vertices of the graph. For example, the cycle C_4 has the degree sequence 2, 2, 2, 2 and the wheel W_4 has the degree sequence 4, 3, 3, 3, 3.
(a) What is the degree sequence of the complete bipartite graph $K_{m,n}$, where m and n are positive integers. Briefly justify. [5 Marks]
(b) Provide a counterexample to the following statement:
If two simple undirected graphs have the same degree sequence, then they are isomorphic. [20 Marks]

Q.6. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in that strongly connected component

[15 Marks]

Q.7. For the following weighted undirected graph, briefly trace the steps taken by the following algorithms in determining the minimum spanning tree: (a) Prim's Algorithm (b) Kruskal's Algorithm.

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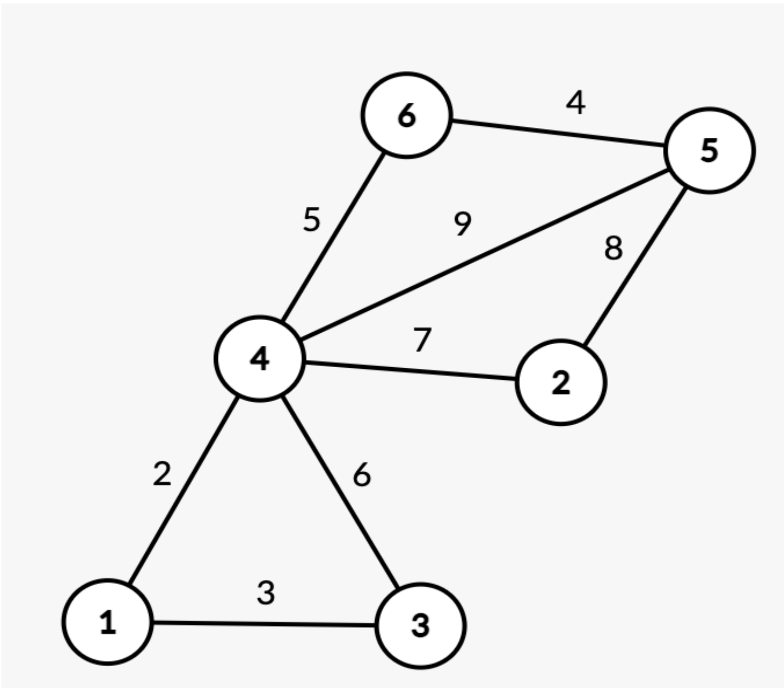

Q.8. Prove that the union of two subgroups of a group need not necessarily be a subgroup.

[20 Marks]

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1 Assume $n = m^2$, n is even

By definition $n = 2k$

$$\Rightarrow 2k = m^2 \quad \text{—} (1)$$

$$(1) / 2$$

$$k = \frac{m^2}{2}$$

m^2 and k are integers

(Parity of squares) Since m^2 is integer and $2k$ is even, m must be even

2. Let $f: R \to R$ $\quad$ $g: R \to R$

$f(x) = x^2$ $\quad$ $g(x) = x$

$$f \circ g \Rightarrow (f \circ g)(x) = f(g(x)) = f(x) = x^2$$

$$(f \circ g)(x) = x^2 \neq y^2 = (f \circ g)(y) \quad \text{(injective)}$$

However $f(x) = x^2$ is not one to one

because it maps to x & $-x$

$f(2) = f(-2) = 4$

Hence $f \circ g$ is one to one but f is not

3. (1) A-B is finite: $A \to N$ (natural), $B \to$ even natural

A-B all odd numbers $\to$ finite

(2) A-B is countably infinite: $A \to$ Integers, $B \to$ +ve integers

A-B $\to$ all -ve integers $\to$ infinite

③ A - B is uncountably infinite :

$A = R \rightarrow$ Real

$B = Q \rightarrow$ Rational

$A - B \rightarrow$ irrational numbers

$\downarrow$

uncountable set

4\. Each edge between two vertices can exist or not

No. of possible edges $= n(n-1)$

No. of distinct directed graph $= 2^{(n(n-1)/2)}$

5\. https://www.chegg.com/homework-help/questions-and-answers/subjective-question-hence-write-answer-text-field-given--given-undirected-graph-degree-seq-q109487972

8\. Let $G = (Z, +)$ be a group

$H_1 = 3Z = \{0, \pm 3, \pm 6, \pm 9 \ldots\}$

$H_2 = 2Z = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

$H_1 \cup H_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \ldots\}$

$(2, 3) \in H_1 \cup H_2$

$2 + 3 = 5$

$5 \notin H_1 \cup H_2$

Hence $H_1 \cup H_2$ does not hold closure property which is essential for subgroup

Hence H_1 and H_2 are not subgroups