Propositions & Logic

Proposition is a declarative sentence (that is, one that declares a fact) that is either true or false, but not both.

- **Examples of propositions**
  - can be true
    - Britain is a part of Europe
    - $2^2 = 4$
  - can be false - still a valid proposition
    - Earth is the nearest planet to the sun
    - $2^2 = 10$
- **not-propositions**
  - $x + 2 = 10$ is not a proposition because it depends on x
  - *“Please give me some water”* is not a proposition because it is not a declarative sentence - it is a request
  - *“This statement is false”* is a **paradox** - it is neither true, nor false
  - *“I always tell lies”* is a **paradox** - it is neither true, nor false
  - *“What is the color of the sky”* is a question, not a declarative sentence

Atomic propositions are propositions that can not be further divided into different propositions.
example: $p: \text{ India is a country}$

Compound propositions are propositions that are formed by combining one or more atomic propositions using connectives.
example: $q: \text{ India is a country, and it is a part of Asia}$

Logical Operators / Connectives

Truth Table The truth table shows the output for each possible truth value of inputs.

For k variables, the truth table will have 2^k rows (since each variable has 2 possibilities - T / F)

Connectives are the operators that are used to combine one or more propositions.

Negation / Not $(\neg)$

Example

$$p: \text{ It is snowing today}$$
$$\neg p: \text{ It is not snowing today}$$

$q: \text{ x is less than } 10$ (that is, $x < 10$)
$\neg q: \text{ x is more than } 10\text{, or x is equal to } 10$ (that is, $x \geq 10$)

Truth Table

p	$\neg p$
F	**T**
T	**F**

Conjunction / And $(\land)$

Similar to multiplication $(\cdot)$

Example

$$p: \text{ It is snowing today}$$
$$q: \text{ John is carrying an umbrella}$$
$$p \land q: \text{ It is snowing today, and John is carrying an umbrella}$$

Truth Table

p	q	$p \land q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

Disjunction / Or $(\vee)$

Similar to addition $(+)$

Example

$p:$ I'm watching a movie
$q:$ I'm eating
$p \wedge q:$ I'm watching a movie or I'm eating (note - both together is possible for OR)

Truth Table

p	q	$p \vee q$
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

Logical Equivalences $(\equiv)$

$p \wedge T \equiv p$ $p \vee F \equiv p$	$p \cdot 1 = p$ $p + 0 = p$	Identity (no effect)
$p \vee T \equiv T$ $p \wedge F \equiv F$	$p + 1 = 1$ $p \cdot 0 = 0$	Domination
$p \vee p \equiv p$ $p \wedge p \equiv p$	$p + p = p$ $p \cdot p = p$	Idempotence (repetition doesn't matter)
$\neg(\neg p) \equiv p$	$\overline{\overline{p}} = p$	Double Negation
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	$p + q = q + p$ $p \cdot q = q \wedge p$	Commutative (can shuffle things if same operator)
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p + q) + r = p + (q + r)$ $(p \cdot q) \cdot r = p \cdot (q \cdot r)$	Associative
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p + qr = (p + q) \cdot (q + r)$ $p \cdot (q + r) = pq + pr$	Distributive
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\overline{p \cdot q} = \overline{p} + \overline{q}$ $\overline{p + q} = \overline{p} \cdot \overline{q}$	De-Morgan's (break the line, change the sign)
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	$p + pq = p$ $p \cdot (p + q) = p$	Absorption
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	$p + \overline{p} = 1$ $p \cdot \overline{p} = 0$	Negation

Conditionals ($\overset{\text{premise}}{p} \rightarrow \overset{\text{conclusion}}{q}$)

p implies q	q if p
if p, then q	q whenever p
p is sufficient for q	q unless $\neg p$
p only if q	q is necessary for p
	q follows from p
	q provided that p

Tautology	always true
Contradiction	always false
Contingency	sometimes true, sometimes false

$$
\begin{array}{c|c}
p \rightarrow q & \text{implication} \\
\neg q \rightarrow \neg p & \text{contrapositive} \\
(p \rightarrow q) \equiv (\neg q \rightarrow \neg p) & \text{implication} \equiv \text{contrapositive} \\
\hline
q \rightarrow p & \text{converse} \\
\neg p \rightarrow \neg q & \text{inverse} \\
(q \rightarrow p) \equiv (\neg p \rightarrow \neg q) & \text{converse} \equiv \text{inverse}
\end{array}
$$

Conditional Equivalences ($\equiv$)

$p \to q$	$\equiv$	$\neg p \lor q$	implication to connectives
$p \to q$	$\equiv$	$\neg q \to \neg p$	implication $\equiv$ contrapositive
$q \to p$	$\equiv$	$\neg p \to \neg q$	converse $\equiv$ inverse
$(p \to q) \land (p \to r)$	$\equiv$	$p \to (q \land r)$	
$(p \to q) \lor (p \to r)$	$\equiv$	$p \to (q \lor r)$	
$(p \to q) \land (q \to r)$	$\equiv$	$(p \lor q) \to r$	
$(p \to q) \lor (q \to r)$	$\equiv$	$(p \land q) \to r$	
$p \iff q$	$\equiv$	$(p \to q) \land (q \to p)$	bi-implication
$p \iff q$	$\equiv$	$(p \land q) \lor (\neg p \land \neg r)$	(if and only if)
$p \iff q$	$\equiv$	$pq + \bar{p}\bar{q}$	(equivalent)

Predicates & Quantifiers

- **Predicate:** a proposition about some object(s)
  - $P(x) : x < 10$
  - $P(a, b, c) : a^2 + b^2 = c^2$
  - $P(x) : x \text{ is a student}$
- **Domain/Universe of Discourse**: all possible values of the variable(s) under consideration
  - $U = \text{all integers}$
  - $U = \text{all students}$
  - $U = \text{everything}$
- **Quantifiers**: convert predicates into propositions about the universe

Universal Quantifier - ForAll $(\forall)$

$\forall x : P(x)$ means that $P(x)$ is true for all values of $x \in U$

Prove: must show that it is true for all values
Disprove: find 1 **counterexample**

Examples:

- All students are hard-working
  - $U =$ all students
    $\forall x : \text{hardworking}(x)$
  - $U =$ everthing
    $\forall x : \text{student}(x) \to \text{hardworking}(x)$
- All that glitters is not gold
  - $U =$ everything
    $\neg\Big(\forall x : \text{Glitter}(x) \to \text{Gold}(x)\Big)$

To combine multiple facts, we always use implication $(\to)$ with forall $(\forall)$

Existential Quantifier - Exists $(\exists)$

$\exists x : P(x)$ means that $P(x)$ is true for at least 1 value of $x \in U$ (there might be more for which it is true, but at least 1)

Prove: find 1 **evidence**
Disprove: must show that it is false for all values

Examples:

- Some students are hard-working
    - $U =$ all students
      $\exists x : \text{hardworking}(x)$
    - $U =$ everthing
      $\exists x : \text{student}(x) \wedge \text{hardworking}(x)$
- All that glitters is not gold
    - $U =$ everything
      $\exists x : \text{Glitter}(x) \wedge \neg\text{Gold}(x)$

Nested Quantifiers

- if all quantifiers are same, then order does not matter
- if quantifiers are different, then order matters

Examples

- every number except 0 has a multiplicative inverse
  $\forall x : \left[x \neq 0 \to \left(\exists y : x \cdot y = 1\right)\right]$
- every number except 0 has exactly 1 multiplicative inverse
  $\forall x : \left[x \neq 0 \to \exists y : \left(x \cdot y = 1 \land \left[\neg \exists z : (y \neq z \land x \cdot z = 1)\right]\right)\right]$
- definition of rational numbers
  $\forall x : \left[\text{Rational}(x) \iff \exists p \exists q : \left(x = p/q, p, q \in \mathbb{Z}, q \neq 0, gcd(p, q) = 1\right)\right]$
- definition of even numbers
  $\forall x : \left[\text{Even}(x) \iff \exists k : \left(x = 2 \cdot k, k \in \mathbb{Z}\right)\right]$

De-Morgan's Law

- move the negation inwards, and change the quantifier

$$\begin{array}{ccc} \neg \exists x : P(x) & \equiv & \forall x : \neg P(x) \\ \text{no student is short} & \equiv & \text{all students are tall} \end{array}$$

$$\begin{array}{ccc} \neg \forall x : P(x) & \equiv & \exists x : \neg P(x) \\ \text{not all students are rich} & \equiv & \text{there is some poor student} \end{array}$$

Rules of Inference

Rules of inference	Tautology	Name
$\begin{array}{r} p \\ p \to q \\ \hline \therefore q \end{array}$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\begin{array}{r} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$\begin{array}{r} p \to q \\ \hline \therefore p \to (p \wedge q) \end{array}$	$(p \to q) \to (p \to (p \wedge q))$	Absorption (logic)
$\begin{array}{r} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction introduction
$\begin{array}{r} p \wedge q \\ \hline \therefore p \end{array}$	$(p \wedge q) \to p$	Conjunction elimination
$\begin{array}{r} p \\ \hline \therefore p \vee q \end{array}$	$p \to (p \vee q)$	Disjunction introduction
$\begin{array}{r} p \to q \\ r \to q \\ p \vee r \\ \hline \therefore q \end{array}$	$((p \to q) \wedge (r \to q) \wedge (p \vee r)) \to q$	Disjunction elimination
$\begin{array}{r} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\begin{array}{r} p \vee p \\ \hline \therefore p \end{array}$	$(p \vee p) \to p$	Idempotency
$\begin{array}{r} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution (logic)
$\begin{array}{r} p \to q \\ q \to p \\ \hline \therefore p \leftrightarrow q \end{array}$	$((p \to q) \wedge (q \to p)) \to (p \leftrightarrow q)$	Biconditional introduction

Common Sets

Naturals $(\mathbb{N})$	$\{1, 2, 3, 4 \ldots\}$
Whole $(\mathbb{W})$	$\{0, 1, 2, 3, 4 \ldots\} = \mathbb{N} \cup \{0\}$
Integers $(\mathbb{Z})$	$\{\ldots - 3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$
Positive Integers $(\mathbb{Z}^+)$	$\{1, 2, 3, 4 \ldots\} = \mathbb{N}$
Negative Integers $(\mathbb{Z}^-)$	$\{-1, -2, -3 \ldots\}$
Non-Negative Integers	$\{0, 1, 2, 3 \ldots\} = \mathbb{Z}^+ \cup \{0\}$
Whole $(\mathbb{W})$	$\{0, 1, 2, 3, 4 \ldots\} = \mathbb{N} \cup \{0\}$
Rationals $(\mathbb{Q})$	x is rational iff it can be written as $x = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$ and $gcd(p, q) = 1$ (that is, it can be converted to its lowest terms)
Irrationals $(\mathbb{I})$	x is irrational iff it is not rational
Reals $(\mathbb{R})$	all numbers on the number line $= \mathbb{Q} \cup \mathbb{I}$
Even	n is even iff it can be written as $n = 2k$ where $k \in \mathbb{Z}$
Odd	n is odd iff it can be written as $n = 2k + 1$ where $k \in \mathbb{Z}$
Perfect Square	n is a perfect square iff it can be written as $n = k^2$ where $k \in \mathbb{Z}$

Proofs

Theorem	statement which can be shown to be true
Axiom/Postulate	statements we assume to be true
Lemma	less important theorem that is helpful in the proof of other results
Corollary	theorem that can be established directly from a theorem that has been proved
Conjecture	statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.
Vacuous Proof	Proof for $p \to q$ is vacuous if p is always false (nothing to prove)
Trivial Proof	Proof for $p \to q$ is trivial if q is always true (obvious to prove)

Direct Proofs

- to show $p \to q$
- assume that p is true
- use axioms, definitions, and previously proven theorems, together with rules of inference
- show that q must also be true

Example 1: Prove that "If n is an odd integer, then n^2 is odd.

Direct Proof

Without loss of generality, consider an odd number n.
By definition of odd, we can write n as $n = 2k + 1, k \in \mathbb{Z}$

Consider n^2. We can write it as

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 2 \cdot (2k^2 + 2k) + 1 \\ &= 2m + 1 && \{m = 2k^2 + 2k \end{aligned}$$

Therefore, n^2 is also odd
$\square$

Example 2: Prove that if m and n are both perfect squares, then nm is also a perfect square

Direct Proof

Consider two perfect squares m and n.

By the definition of a perfect square, we can write $m = s^2$ and $n = t^2$ where $s, t \in \mathbb{Z}$

Consider nm. We can write as

$$\begin{aligned} nm &= s^2 \times t^2 \\ &= (st)^2 \\ &= k^2 && \{k = st \end{aligned}$$

Therefore, nm is also a perfect square
$\square$

Proof by Contraposition

- to show $p \to q$
- use the contrapositive $\neg q \to \neg p$, since it is equivalent to the implication
- show that the contrapositive is true

Example 1: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof by Contraposition

Contrapositive: if n is even, then $3n + 2$ is even

Consider an even number n. By definition, we can write n as $n = 2k, k \in \mathbb{Z}$

Therefore,

$$\begin{aligned} 3n + 2 &= 3 \cdot (2k) + 2 \\ &= 6k + 2 \\ &= 2 \cdot (3k + 1) \\ &= 2m \qquad \begin{cases} m = 3k + 1 \end{cases} \end{aligned}$$

Therefore, $3n + 2$ is also even.
$\square$

Example 2: Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Proof by Contraposition

Contrapositive (using De-Morgan's law):
"if a and b are positive integers such that $\left(a > \sqrt{n}\right)$ **AND** $\left(b > \sqrt{n}\right)$, then $n \neq ab$"

Consider two positive integers number $a > \sqrt{n}$ and $b > \sqrt{n}$.
Then,

$$\begin{aligned} ab &= \left(\sqrt{n} + \delta_1\right) \cdot \left(\sqrt{n} + \delta_1\right) \quad \begin{cases} \delta_1 > 0, \delta_2 > 0 \end{cases} \\ &= n + \sqrt{n} \cdot (\delta_1 + \delta_2) \\ &> n \end{aligned}$$

Therefore, $ab \neq n$
$\square$

Proof by Contradiction

- to prove p
- assume (for the sake of contradiction) that p is false
- show that this assumption leads to a contradiction
- hence, p must be true

Example 1: Prove that $\sqrt{2}$ is irrational

Proof by Contradiction

Assume, for the sake of contradiction, that $\sqrt{2}$ is rational.

By definition of rational, we should be able to write it as

$$\sqrt{2} = p/q, \text{ where } \begin{cases} p, q \in \mathbb{Z} \\ q \neq 0 \\ gcd(p, q) = 1 \end{cases}$$

Now,

$$\begin{aligned} 2 &= \frac{p^2}{q^2} \quad \text{squaring both sides} \\ 2q^2 &= p^2 \end{aligned}$$

Therefore, we have that p^2 is even, which means that p is even (by Lemma 1)

> **Lemma 1:** if p^2 is even, then p must be even
>
> Proof by contraposition: $(p \text{ is odd}) \to (p^2 \text{ is odd})$
> Consider an odd number p
> By definition of off, we can write $p = 2k + 1, k \in \mathbb{Z}$
> Now,
>
> $$\begin{aligned} p^2 &= (2k+1)^2 \\ &= 2(2k^2+2)+1 \\ &= 2m+1 \qquad \left\{ m = 2k^2 + 2 \right. \end{aligned}$$
>
> Therefore, p^2 is odd
> $\square$

Therefore,

$$\begin{aligned} 2q^2 &= p^2 \\ 2q^2 &= (2m)^2 \quad \left\{ \because p \text{ is even} \right. \\ q^2 &= 2m^2 \end{aligned}$$

Therefore, q^2 is even, which means that q is also even (by Lemma 1)

Thus, $gcd(p, q) = 2$, which is a contradiction.

$\square$

Proof by Constructive Evidence

- to show $\exists x : P(x)$
- find evidence for x which satisfies $P(x)$

Example 1: Prove that $\exists a, b, c \in \mathbb{Z} : a^2 + b^2 = c^2$

Proof by Constructive Evidence

Let $a = 3, b = 4, c = 5$
Then,

$$\begin{aligned} a^2 + b^2 &= 3^2 + 4^2 \\ &= 25 \\ &= c^2 \end{aligned}$$

$\square$

Proof by Non-constructive Evidence

- to show $\exists x : P(x)$
- do some magic! (no other way to put it)

Example 1: Show that there exist irrational numbers x and y such that x^y is rational.

Proof by Non-constructive Evidence

We know that $\sqrt{2}$ is irrational

Consider $n = \sqrt{2}^{\sqrt{2}}$. Two cases arise.

- **Case 1:** $n = \sqrt{2}^{\sqrt{2}}$ is rational
  In this case, we've found two irrational numbers $x = \sqrt{2}$ and $y = \sqrt{2}$ such that x^y is rational.

- **Case 2:** $n = \sqrt{2}^{\sqrt{2}}$ is irrational
  Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$.
  Then, $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational

$\square$

Mathematical Induction

- Does infinite proofs in 1
- Similar to recursion
- 3 step process
    1. Induction hypothesis $H(n)$
    2. Base cases: $H(n_0), H(n_1), \ldots$
    3. Induction Step: $H(n) \to H(n+1)$

Example 1 - Prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

Proof by Induction

Induction Hypothesis: $H(n) : 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}, n \in \mathbb{N}$

Base Cases:

- $1 = \dfrac{1(1+1)}{2}$. Therefore, $H(1)\checkmark$
- $1 + 2 = 3 = \dfrac{2(2+1)}{2}$. Therefore, $H(2)\checkmark$
- $1 + 2 + 3 = 6 = \dfrac{3(3+1)}{2}$. Therefore, $H(3)\checkmark$

Inductive Step: To show $H(n) \to H(n+1)$

$$1 + 2 + 3 + \cdots + n + (n+1)$$

$$= \Big[1 + 2 + 3 + \cdots + n\Big] + (n+1)$$

$$= \left[\frac{n(n+1)}{2}\right] + (n+1) \qquad \{\because H(n)$$

$$= \frac{(n+1)\cdot(n+2)}{2}$$

$$\therefore H(n+1)$$

$\square$

Strong Induction

- similar to induction
- assume everything below in the Induction Step
  $$\Bigg(H(n_0) \wedge H(n_1) \wedge \cdots \wedge H(n)\Bigg) \rightarrow H(n+1)$$

Example 1 - Show that if n is an integer greater than 1, then n can be written as the product of primes

Proof by Strong Induction

Induction Hypothesis: $H(n) : n = p_1^{\alpha_1} \cdot p_2^{\alpha_1} \cdots p_k^{\alpha_1}$ where p_i are prime numbers, and $\alpha_i \in \mathbb{W}$

Base Cases:

- $2 = 2^1$. Therefore, $H(2)\checkmark$
- $3 = 3^1$. Therefore, $H(3)\checkmark$
- $4 = 2^2$. Therefore, $H(4)\checkmark$
- $5 = 5^1$. Therefore, $H(5)\checkmark$
- $6 = 2^1 \cdot 3^1$. Therefore, $H(6)\checkmark$

Inductive Step: To show $\Big(H(n_0) \wedge H(n_1) \wedge \cdots \wedge H(n)\Big) \rightarrow H(n+1)$

Consider the number $(n+1)$. Since $n+1 > 1$, two cases arise.

- **Case 1:** $(n+1)$ is prime
  Here, we already have $(n+1) = (n+1)^1$.
- **Case 2:** $(n+1)$ is composite
  By definition of compositie, we can write $(n+1) = ab$ where $a, b \in \mathbb{N}$ and $a < n+1$ and $b < n+1$
  Now,

$$\begin{aligned} n+1 &= ab \\ &= \underbrace{(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})}_{a} \cdot \underbrace{\left(q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_k^{\beta_k}\right)}_{b} \quad \Big\{ \because H(a), H(b) \end{aligned}$$

Therefore, $(n+1)$ can be written as a product of prime numbers.

$\square$

Sequences and Sums

- $\langle a_0, a_1, a_2 \ldots a_n \rangle$
- $a_0 =$ first term
- $a_n =$ n-th term
- $n = $ number of terms
- sequences are always ordered

Arithmetic Progression (AP)

$$a, \quad (a+d), \quad (a+2d) \quad \ldots \quad a+(n-1)d$$

- $d =$ common difference
- $\Big[a+(n-1)d\Big]$ is the n-th term of AP

Sum of first n terms of AP

$$\begin{aligned} \sum_{i=1}^{n} a+(i-1)d \quad &= \quad \sum_{i=0}^{n-1} a+id \\ &= \frac{n}{2}\Big[2a+(n-1)d\Big] \\ &= \frac{n}{2}\Big[\text{first term} + \text{last term}\Big] \end{aligned}$$

Geometric Progression (GP)

$$a, \quad ar, \quad ar^2 \quad \dots \quad ar^{n-1}$$

- $r =$ common ratio
- ar^{n-1} is the n-th term of GP

Sum of first n terms of GP

$$\sum_{i=1}^{n} ar^{i-1} \quad = \quad \sum_{i=0}^{n-1} ar^{i} \quad = \quad a\frac{r^n - 1}{r - 1}$$

Sum of ∞ terms of GP

Only when $|r| < 1$

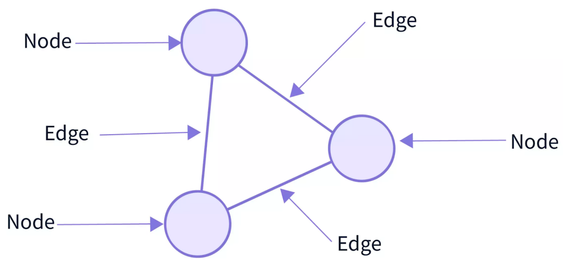
$$\sum_{i=0}^{\infty} ar^{i} \quad = \quad \frac{a}{1 - r}$$

Useful Summations

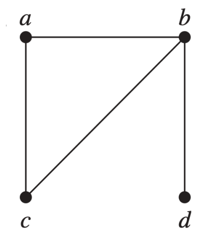
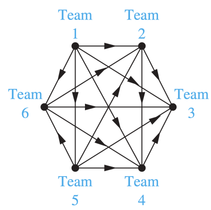
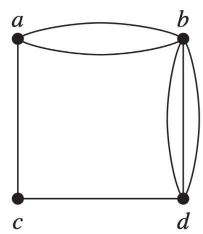
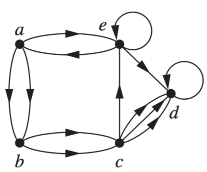
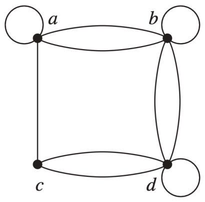
$$
\begin{aligned}
1 + 2 + 3 + \cdots + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + 3^2 + \cdots + n^2 &= n \cdot \frac{(n+1)}{2} \cdot \frac{(2n+1)}{3} \\
&= \frac{n(n+1)(2n+1)}{6} \\
1^3 + 2^3 + 3^3 + \cdots + n^3 &= \left[\frac{n(n+1)}{2}\right]^2 \\
1 + 2 + 4 + 8 + \cdots + 2^n &= 2^{n+1} - 1 \\
1 + 3 + 9 + 27 + \cdots + 3^n &= \frac{3^{n+1} - 1}{2} \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \infty \text{ terms} &= 2 \\
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} &\approx \log_e n
\end{aligned}
$$

Graphs

$G = (V, E)$, where V are vertices, E are edges, G is graph

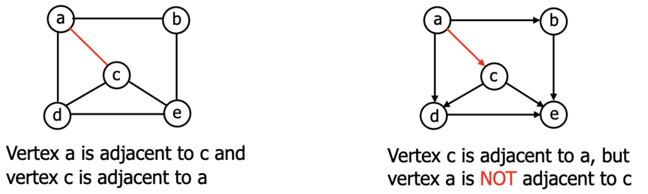
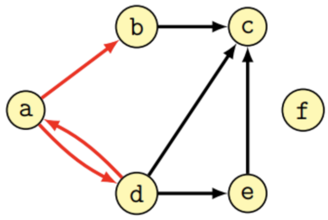
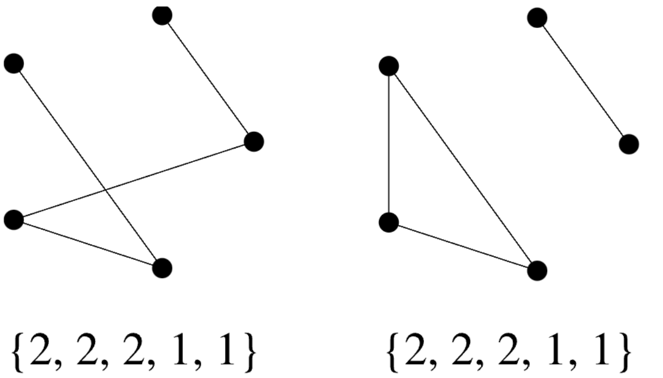


Types of graphs

Type	edges?	self-loops?	multi-edges?	V vs E	example
Simple	undirected	No	No	$0 \le e \le \frac{v(v-1)}{2}$	
Simple Directed	directed	No	No	$0 \le e \le v(v-1)$	
Multi	undirected	No	Yes	$0 \le e \le \infty$	
Multi Directred	directed	Yes	Yes	$0 \le e \le \infty$	
Pseudo	undirected	Yes	Yes	$0 \le e \le \frac{v(v+1)}{2}$	
Mixed	directed + undirected	Yes	Yes	$0 \le e \le \infty$	

- If the edges have weights associated with them, then we call it a weighted graph (weighted simple undirected, weighted simple directed, weighted mixed, ...)

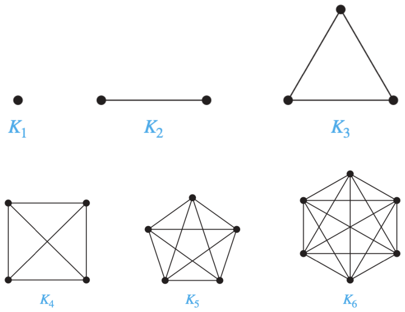
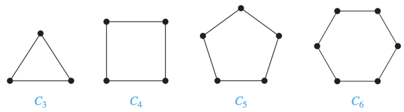
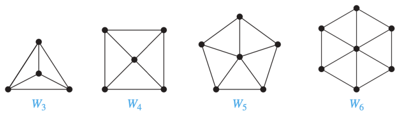
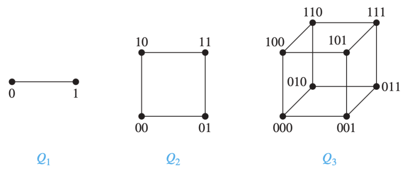
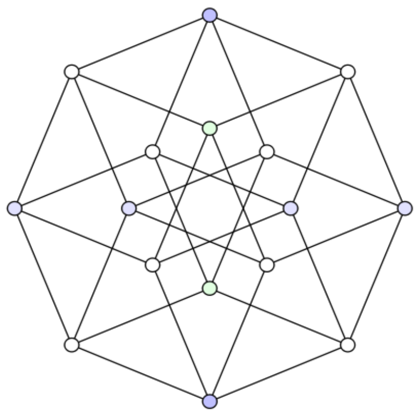
Basic Graph Definitions

Adjacent Vertices / Neighbors	in undirected graph, vertices a and b are adjacent if they're connected by an edge	
Incident	edge e is incident on vertices a and b if it connects them	 • (a, b) is incident from a to b • (a, d) is incident from a to d • (d, a) is incident from d to a • b is adjacent to a • d is adjacent to a • a is adjacent to d
Adjacency / Neighborhood	$\text{Neighborhood}(a)$ = the set of all vertices that are adjacent to a	
Degree $\deg(v_i)$	in undirected graphs, $\text{degree}(a)$ = number of vertices adjancent to it. self-loops count as 2 degree for undirected.	
In-Degree $\deg^-(v_i)$	in directed graphs, $\text{in-degree}(a)$ = number of incoming edges self-loops provide both 1 in-degree and 1 out-degree (total contribution of 2)	
Out-Degree $\deg^+(v_i)$	in directed graphs, $\text{out-degree}(a)$ = number of outgoing edges self-loops provide both 1 in-degree and 1 out-degree (total contribution of 2)	
Degree Sequence	sequence of degrees of the graph	
Isolated Vertex	vertex with degree 0	
Pendant Vertex	vertex with degree 1	
Initial / Start Vertex	For a directed edge (a, b) from a to b, the initial vertex is a	
Terminal / End Vertex	For a directed edge (a, b) from a to b, the initial vertex is b	

Basic Graph Theorems

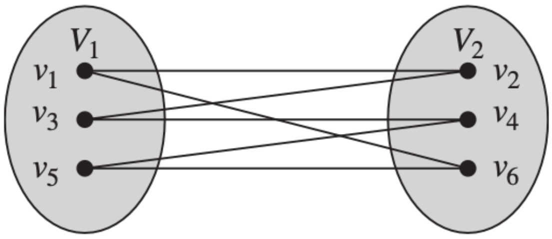
Handshaking Theorem	For any undirected graph $G = (V, E)$, with e edges, $$2e = \sum_{v_i \in V} \text{degree}(v_i)$$ This applies even for multi-edges and self loops. Self loops count as 2 degree
Collolary 1	An undirected graph has an even number of vertices of odd degree.
Directed Handshaking Theorem	For any directed graph $G = (V, E)$, with e edges, $$e = \sum_{v_i \in V} \text{in-degree}(v_i) = \sum_{v_i \in V} \text{out-degree}(v_i)$$ This applies even for multi-edges and self loops. Self loops count as 2 degree

Special Graphs

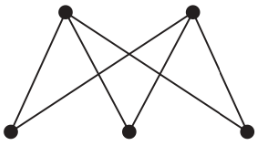
Type	Meaning	Examples	Property
Complete Graph K_n	Simple undirected graph with all possible edges		$v = n$ $e = \frac{n(n-1)}{2}$
Cycle C_n	a closed loop		$v = n$ $e = n$
Wheel W_n	add a center vertex to the cycle and add spokes		$v = n + 1$ $e = 2n$
Hypercube (n-cube) Q_n	n-dimensional hypercube	 	$v = 2^n$ $e = n \cdot 2^{n-1}$

Bi-partite Graphs

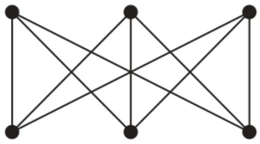
- vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that there are no internal edges inside V_1 or inside V_2



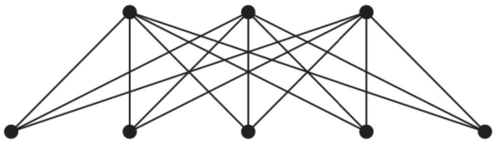
Complete Bi-Partitle Graph has all possible edges from V_1 to V_2



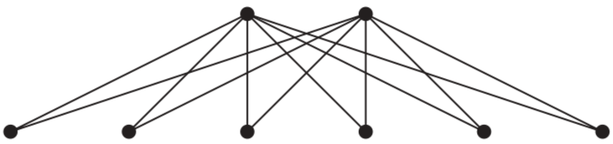
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

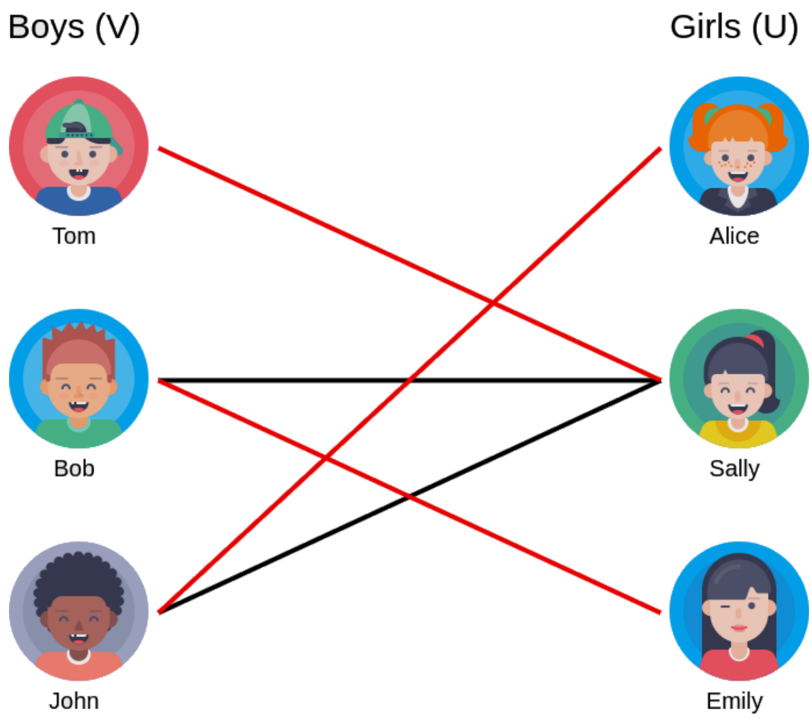
Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Matching

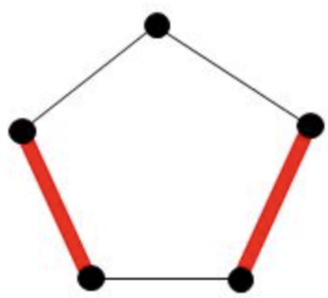
definition: Matching M in a simple graph $G = (V, E)$ is a subset of the set E of edges of the graph such that no two edges are incident with the same vertex. In other words, a matching is a subset of edges such that if s, t and u, v are distinct edges of the matching

Rules of Matching

- think of it like marriages b/w boys and girls
- 1 boy cannot marry multiple girls, and 1 girl cannot marry muktiple boys (no polygamy)
- Some boy / girl can remain un-married

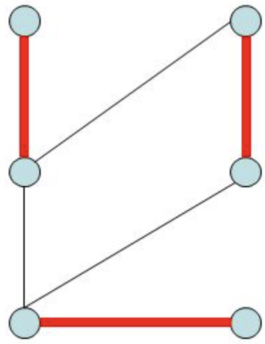
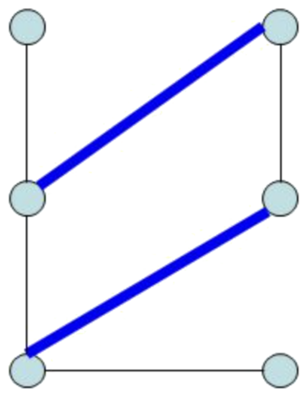


Matching can also be in a non-biartite graph



A matching in general graph

Matching definitions

Matched vertex	got married to someone	
Unmatched vertex	left unmarried	
Complete/Perfect Matching	every boy got married (some girls might remain unmarried)	
Maximum Matching	we have maximized the number of marriages	 Maximum Matching
Maximal Matching	we can't add more marriages	 Maximal Matching

Hall's Marriage Theorem

The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets $A \subset V_1$.

Subgraphs

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.

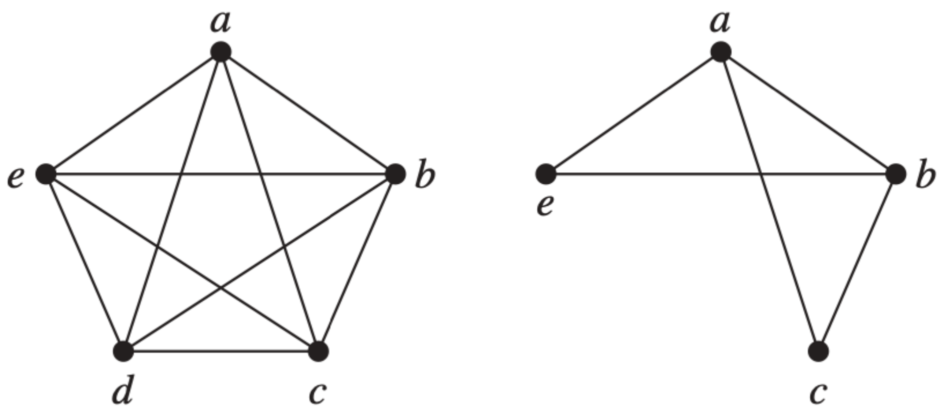


FIGURE 15 **A subgraph of K_5.**

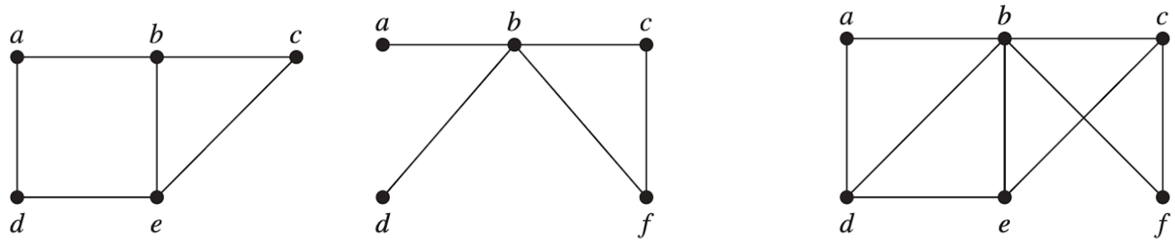
Proper Subgraph: A subgraph H of G is a proper subgraph of G if $H \neq G$

Induced Graph

The subgraph induced by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

Graph Union

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertexset $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$



G_1 G_2 $G_1 \cup G_2$

Adjacency List Representation

- Prefered for Sparse Graphs
- checking adjacency of 2 vertices is expensive: $\mathcal{O}(v)$
- is space efficient: $\mathcal{O}(v + e)$

Undirected

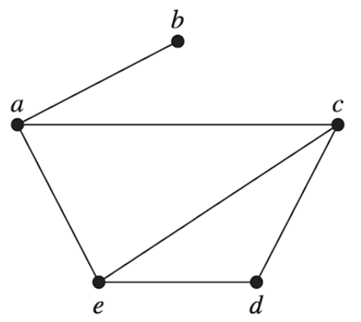


FIGURE 1 A simple graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Directed

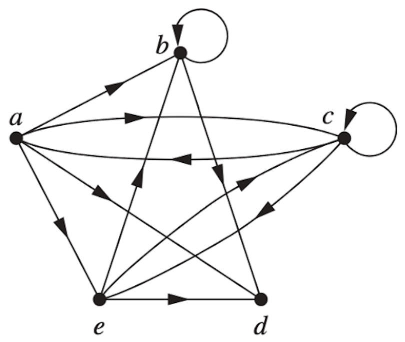


FIGURE 2 A directed graph.

TABLE 2 An Adjacency List for a Directed Graph.

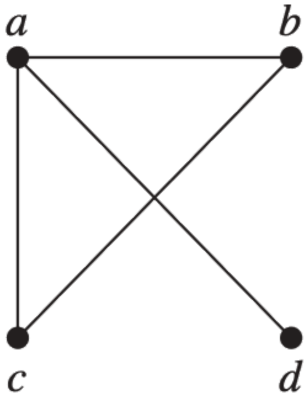
Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrix Representation

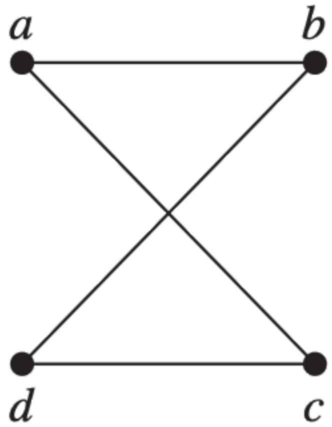
- Prefered for Dense Graphs
- checking adjacency of 2 vertices is fast: $\mathcal{O}(1)$
- takes a lot of space: $\mathcal{O}(v^2)$

Undirected

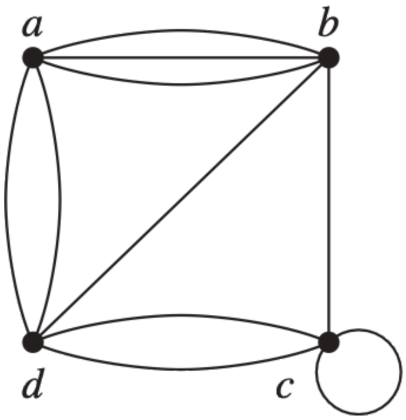
- Adjacency matrix is always symmetric for undirected graphs
- Diagonal is always zero for simple graphs (because no self loops)



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



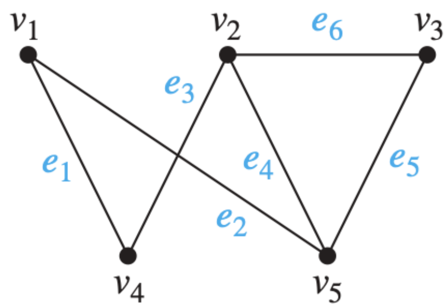
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



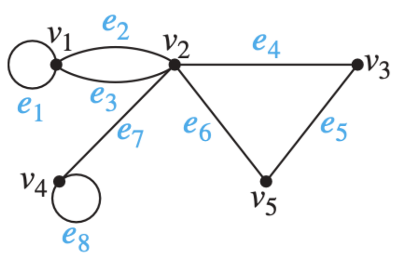
$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

Incidence Matrix Representation

- useful for multi-graphs
- represents which edges are inceident on which vertices



$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}.
\end{array}
$$

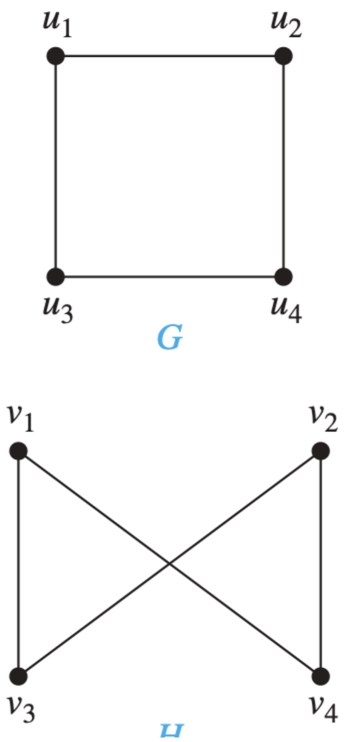


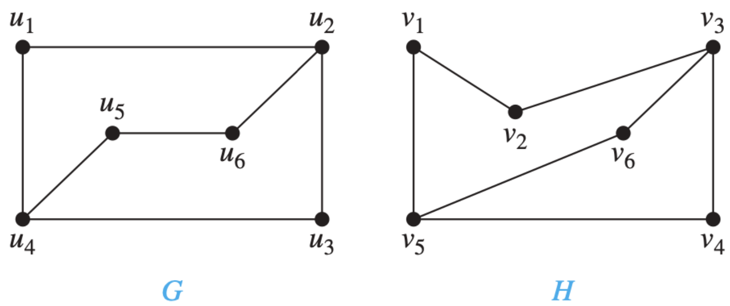
$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{array}
$$

Graph Isomorphism

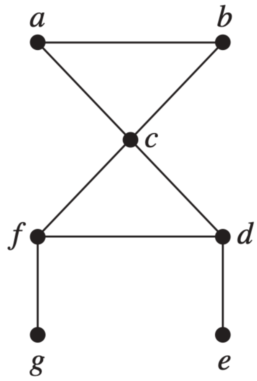
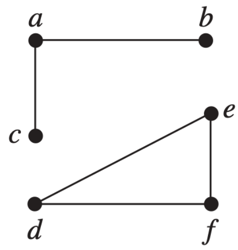
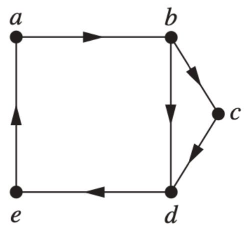
- two graphs are isomorphic if they can be re-drawn to look the same
  - The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function $f : V_1 \to V_2$ with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all $a, b \in V_1$.
  - Such a function f is called an **isomorphism**.
- Two simple graphs that are not isomorphic are called **nonisomorphic**.

Example



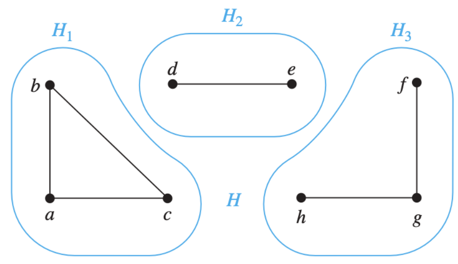
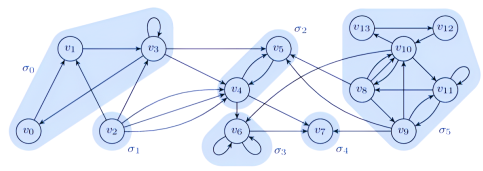
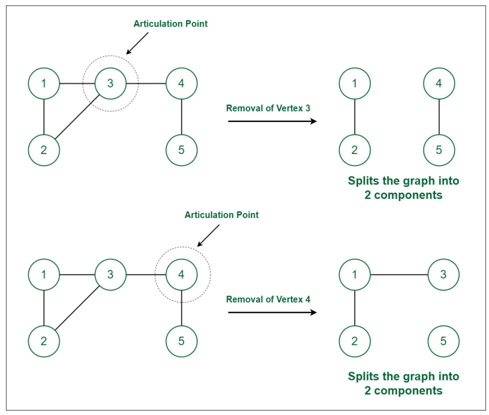
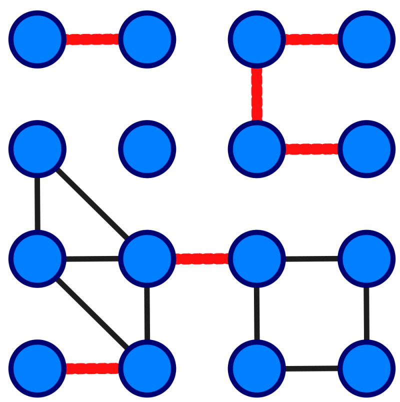


Graph Connectivity - basic definitions

Path	A path $p = v \xrightarrow{*} u$ of length n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ such that there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i.	
Simple Path	path which does not contain the same edge more than once.	
Circuit	The path is a circuit if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero	
Simple Circuit	circuit which does not contain the same edge more than once.	
Connected Graph	An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.	 G_1
Disconnected Graph	An undirected graph that is not connected is called disconnected.	 G_2
Strongly Connected Graph	there is a path $a \xrightarrow{*} b$ and a path $b \xrightarrow{*} a$ for all vertices $a, b \in V$	 G
Weakly Connected Graph	A directed graph is weakly connected if the underlying undirected graph is connected	

Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Graph Connectivity - measures

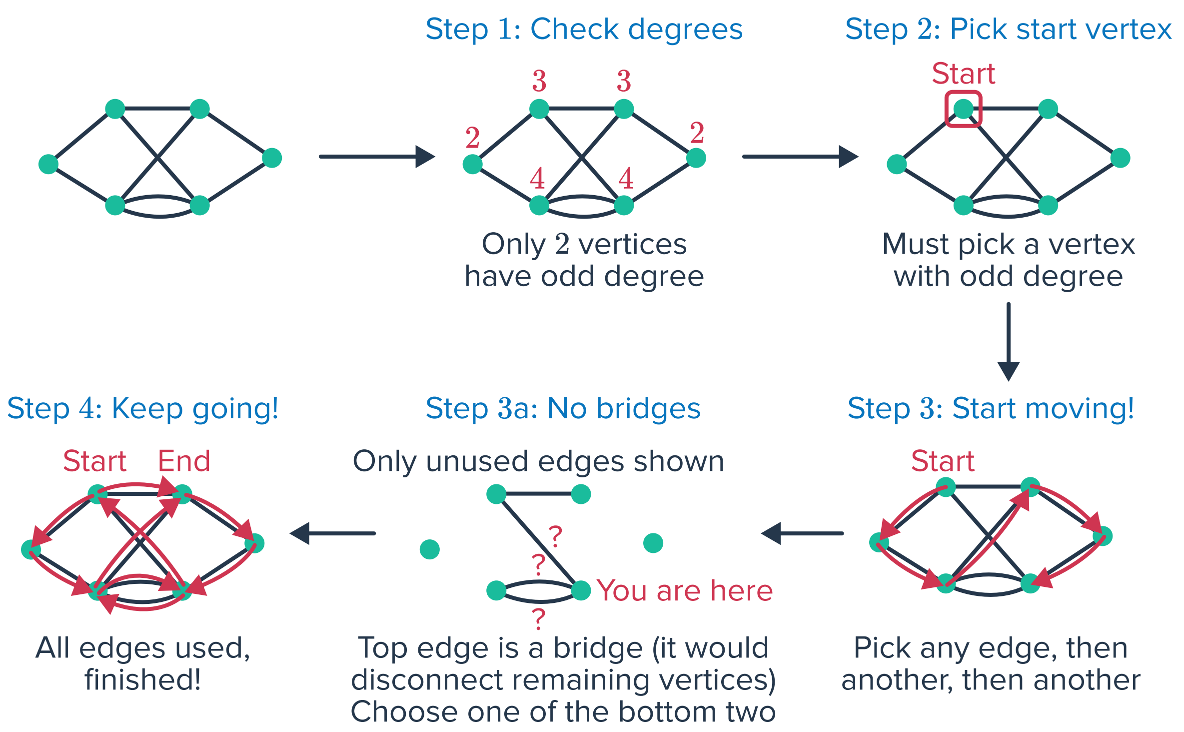
Connected Components	a subgraph which is connected	 **FIGURE 3** **The graph H and its connected components H_1, H_2, and H_3.**
Strongly Connected Components	there is a path $a \xrightarrow{*} b$ and a path $b \xrightarrow{*} a$ for all vertices $a, b \in V$	
Cut Vertex / Articulation Point	vertex which when removed makes a connected graph disconnected	
Vertex Cut	set of vertices, which on removal make the connected graph disconnected	
Cut Edge / Bridge	edge which when removed makes a connected graph disconnected	
Edge Cut	set of edges which on removal make a connected graph disconnected	
Non-Separable Graph	graph with no cut-vertex	
Vertex Connectivity: $\kappa(G)$	$\kappa(G)$ is the minimum number of vertices that you need to remove to make the graph disconnected or to make the get a graph with a single vertex	
Edge Connectivity: $\lambda(G)$	$\lambda(G)$ is the minimum number of edges that you need to remove to make the graph disconnected	

Relation b/w Edge and Vertex Connectivity

$$\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$$

Euler Paths & Circuits

Euler path	simple path containing every edge of G ● repeating vertices is allowed ● repeating edges is not allowed
Euler Circuit	Euler path which has same start and end vertex (closed loop)

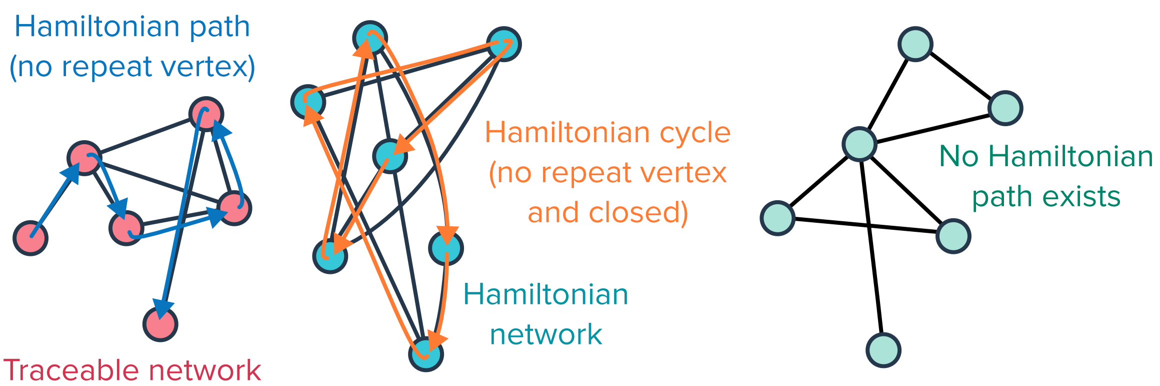


Necessary & Sufficient Conditions:

- **Euler Circuit**: all vertices must have even degree
- **Euler Path but not circuit**: exactly 2 vertices with odd degree

Hamiltonian Paths & Circuits

Hamiltonian path	simple path that passes through every vertex exactly once ● repeating vertices is not allowed ● repeating edges is not allowed



Hamiltonian Circuit | Hamiltonian path which has same start and end vertex

Note: only the start/end vertex is repeated

Necessary & Sufficient Conditions:

Not known

Sufficient Conditions

- if any condition matches, a Hamiltonian path exists. But it is not necessary that every graph with a Hamiltonian path must satisfy these conditions
  - **Dirac's Theorem:** If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.
  - **Ore's Theorem:** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Planar Graphs

- A graph is called planar if it can be drawn in the plane without any edges crossing
- otherwise, non-planar

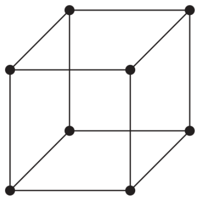


FIGURE 4 The graph Q_3.

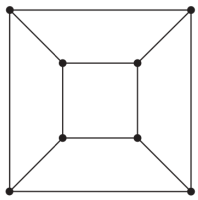


FIGURE 5 A planar representation of Q_3.

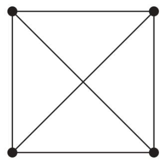


FIGURE 2 The graph K_4.

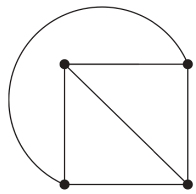
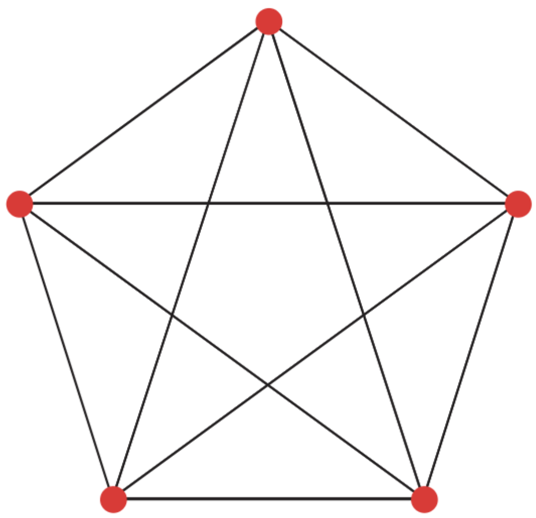
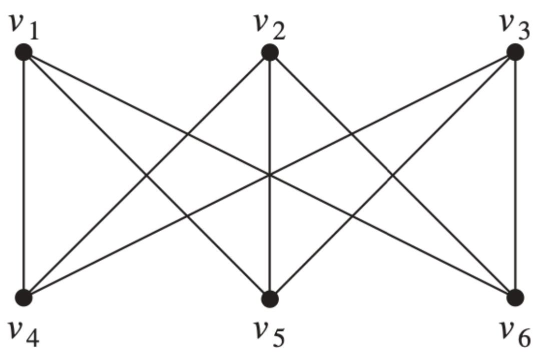


FIGURE 3 K_4 drawn with no crossings.

Minimal Non-Planar graphs

Min Vertices: K_5	Min Edges: $K_{3,3}$
 K_5	

Euler's Formula for Planar graphs

Connected

$$r = e - v + 2$$

- r is the number of regions / faces
- e is the number of edges
- v is the number of vertices

General

$$r = e - v + k + 1$$

- k is the number of connected components

Corrolary 1

For connected planar simple graph with more than 3 vertices, $e \leq 3v - 6$

Corrolary 2

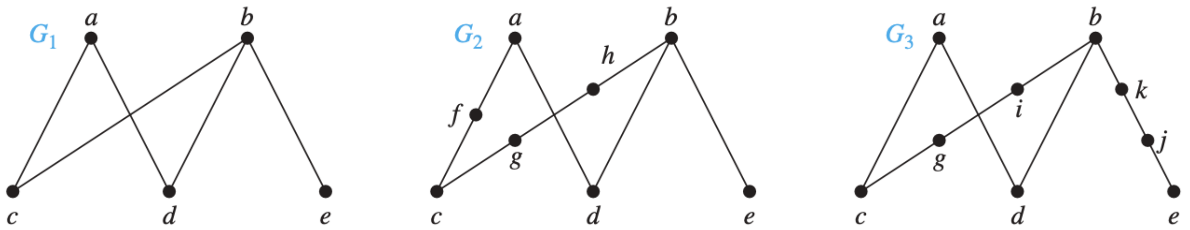
A connected planar simple graph has a vertex of degree not exceeding five.

Corrolary 3

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$

Kuratowski’s Theorem

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.



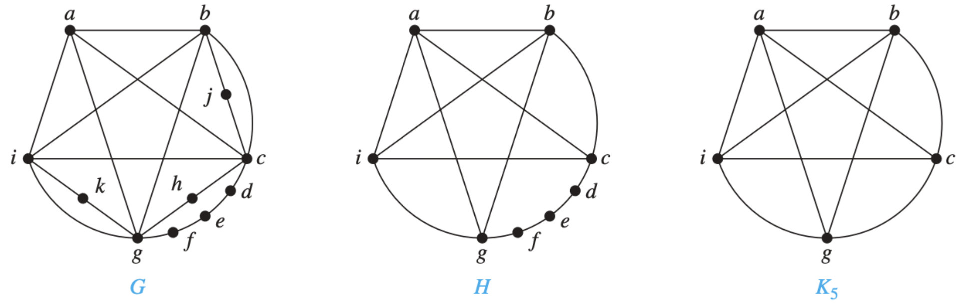


FIGURE 13 **The undirected graph *G*, a subgraph *H* homeomorphic to K_5, and K_5.**

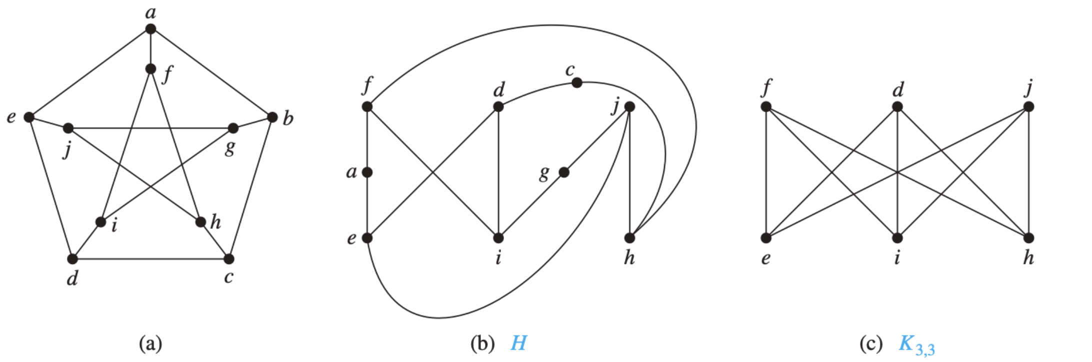


FIGURE 14 **(a) The Petersen graph, (b) a subgraph *H* homeomorphic to $K_{3,3}$, and (c) $K_{3,3}$.**

Graph Coloring

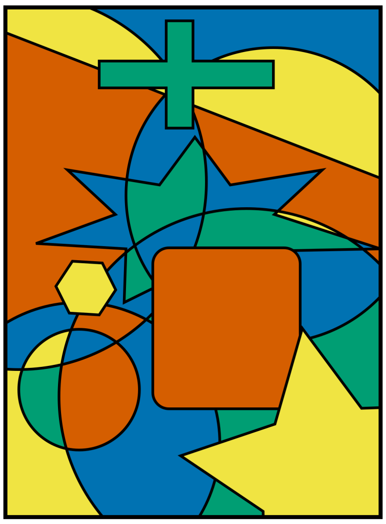
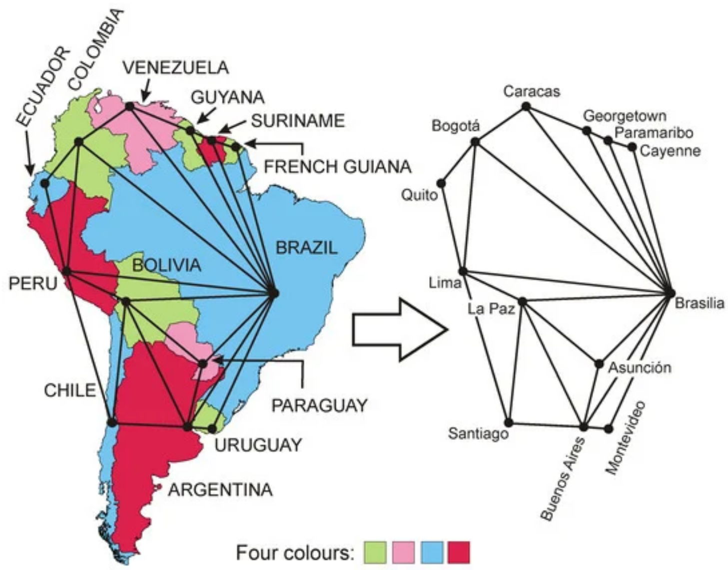
- **Coloring** of a simple graph: color each vertex so that no two adjacent vertices are assigned the same color
- **Chromatic number** $\chi(G)$: least number of colors needed for a coloring of this graph

4 color theorem

The chromatic number of a planar graph is no greater than four.

- all planar graphs have $\chi(G) \leq 4$
- non-planar graph can have $\chi(G) \leq 4$ - example, all bipartite graphs have $\chi(G) = 2$ irrespective of whether they're planar or not

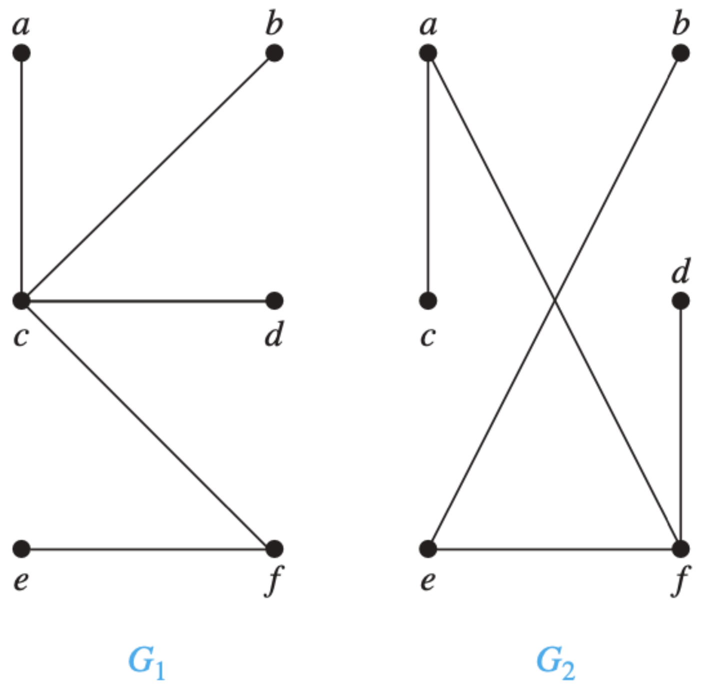
Note: all 2d maps are planar

Political Map of South America   Corresponding dual graph

Trees

- A tree is a connected undirected graph with no simple circuits (no cycles)
- A tree is a connected, acyclic graph

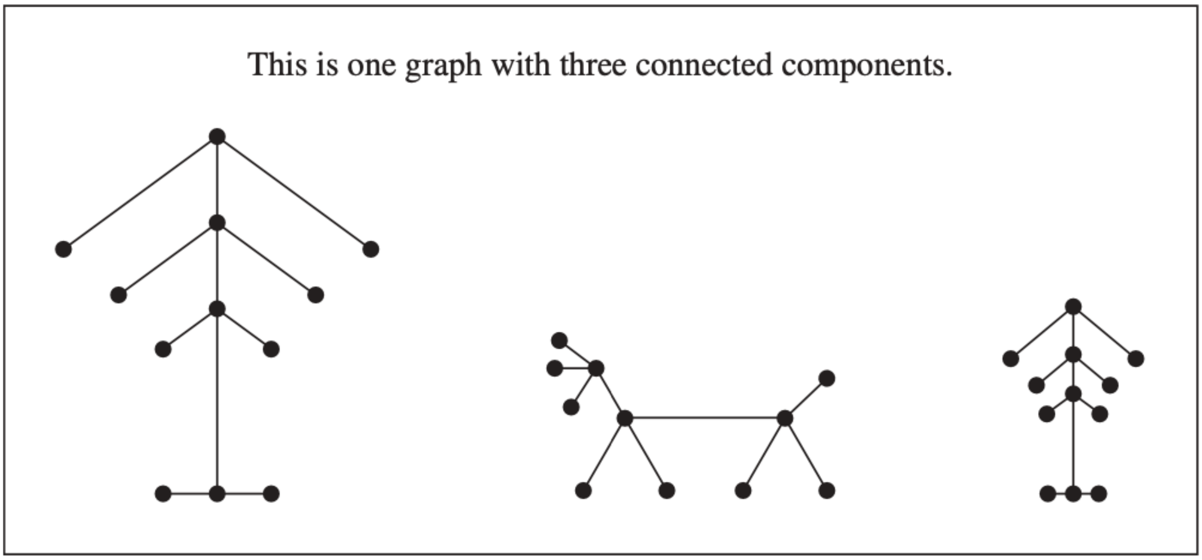


Theorem

An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Forest

a graph with many disconnected trees



Rooted Tree

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

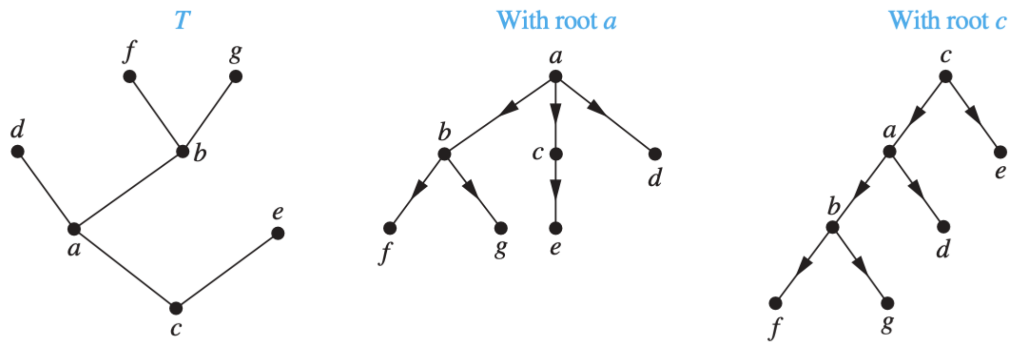
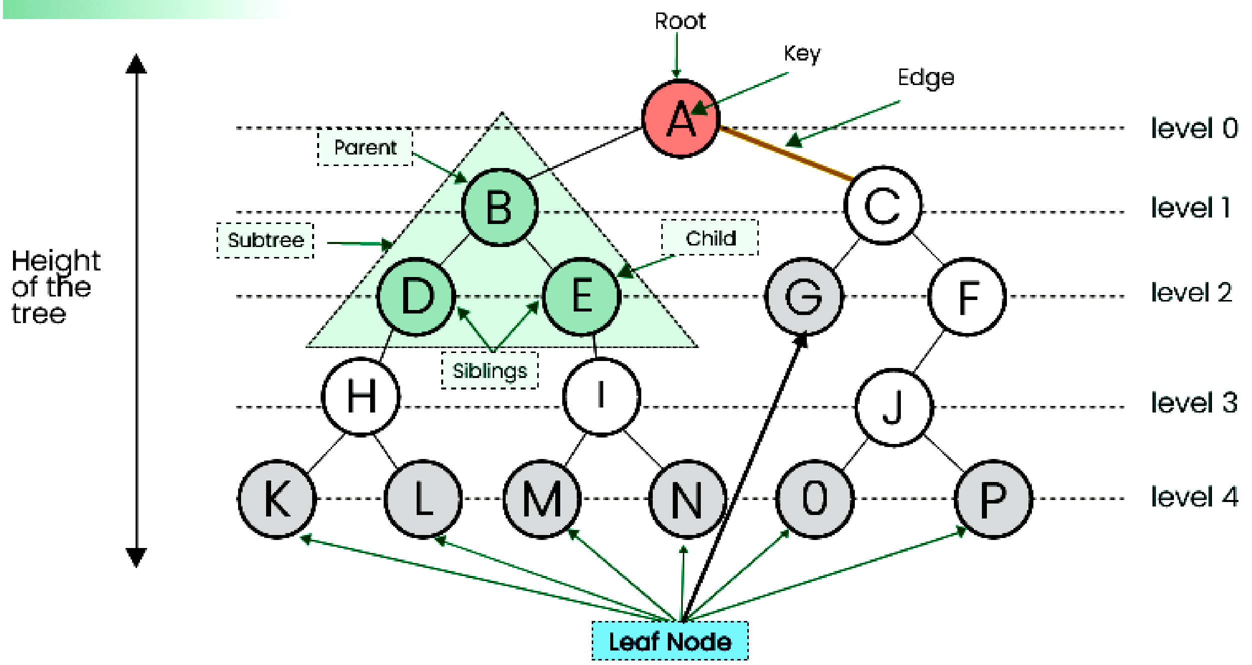
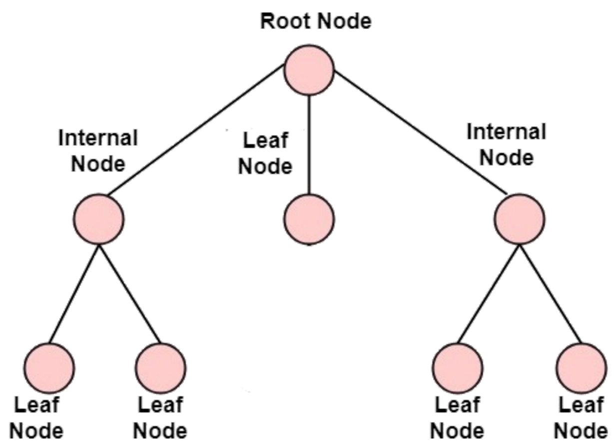


FIGURE 4 **A tree and rooted trees formed by designating two different roots.**

Root	Top vertex of a rooted tree
Internal Vertex	vertex which has children (includes root if the root has children)
Leaf Vertex	vertex with no children (doesn't have to be at the bottom - can be present at a higher level)

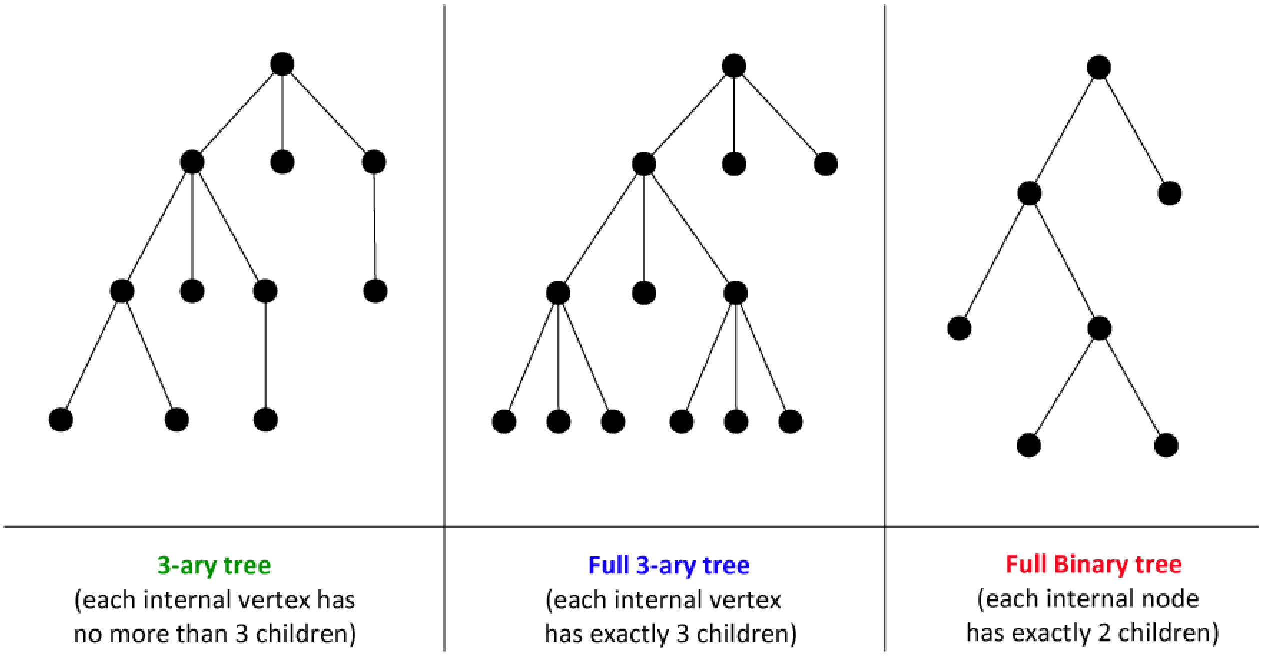
Tree Data Structure





$m-$ary tree & full m-ary tree

- $m-$**ary:** A rooted tree is called an m-ary tree if every internal vertex has no more than m children.
    - **Binary tree:** $m = 2$
    - **Ternary tree:** $m = 3$
- **full** $m-$**ary:** The tree is called a full m-ary tree if every internal vertex has exactly m children.



Tree properties

- A tree with n vertices has $n-1$ edges.
- A full $m-$ary tree with i internal vertices contains $n = mi + 1$ vertices.
- A full $m-$ary tree with
    - n vertices has $i = \frac{n-1}{m}$ internal vertices and $l = \frac{(m-1)\cdot n+1}{m}$ leaves

        > each vertex except the root has a parent, and each parent has exactly m children

    - i internal vertices has $n = mi + 1$ vertices and $l = (m-1) \cdot i + 1$ leaves

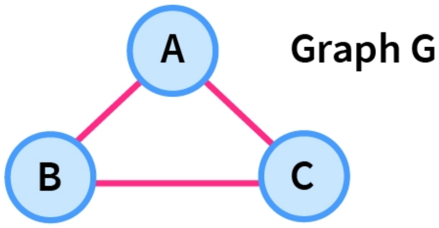
        > each internal vertex has m children, and there is also a root

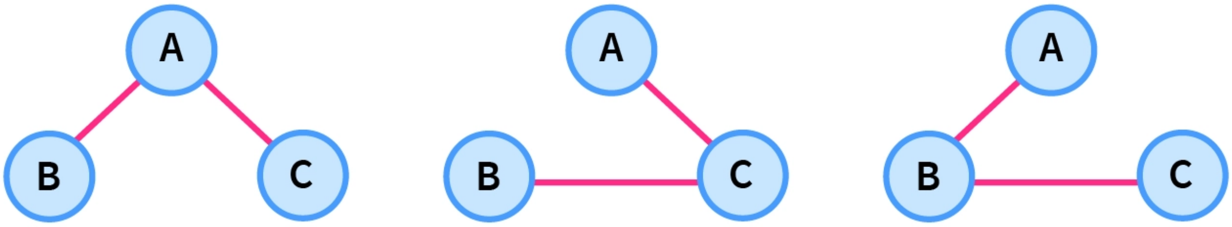
    - l leaves has $n = \frac{ml-1}{m-1}$ vertices and $i = \frac{l-1}{m-1}$ internal vertices

        > $l \text{ leaves } + \frac{l}{m} \text{ parents } + \frac{l}{m^2} \text{ grandparents} + \cdots + 1 \text{ root}$

Spanning Trees

- Tree containing all vertices of the graph





Spanning Trees, subgraph of G

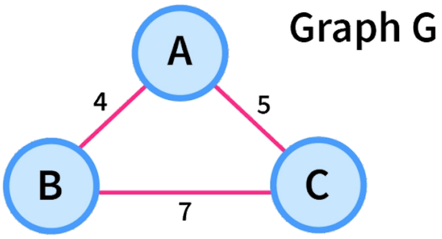
- All possible spanning trees for graph G have the same number of edges and vertices.
- Spanning trees do not have any cycles.
- A Spanning tree is a minimally connected sub-graph, which means if we remove any edge from the spanning tree then it becomes disconnected.
- A Spanning tree is a maximally acyclic sub-graph, which means if we add an edge to the spanning tree then it becomes cyclic.
- A connected graph G can have more than one spanning tree.
- A Spanning tree always contains $n-1$ edges, where n is the total number of vertices in the graph G.
- The total number of spanning trees that a complete graph of n vertices can have is n^{n-2}.
- We can construct a spanning tree by removing atmost $e-n+1$ edges from a complete graph G

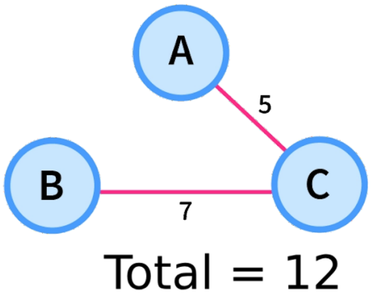
Theorem

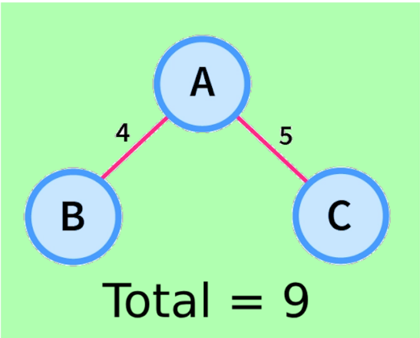
A simple graph is connected if and only if it has a spanning tree.

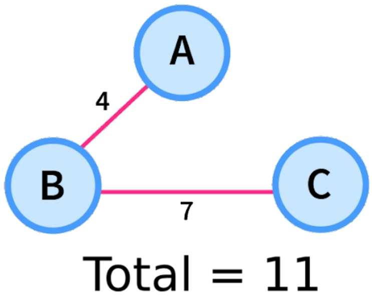
Minimum Spanning Tree

- A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.



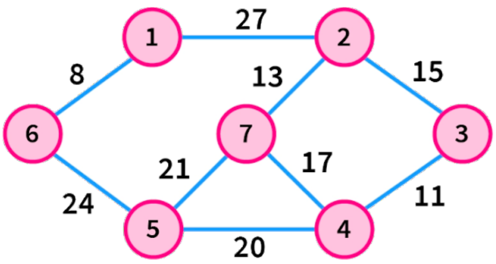


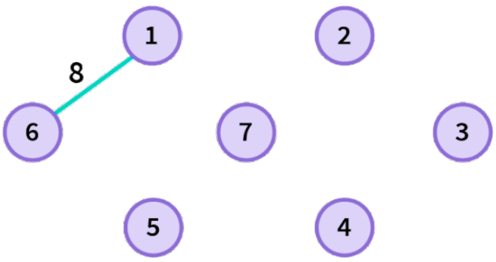


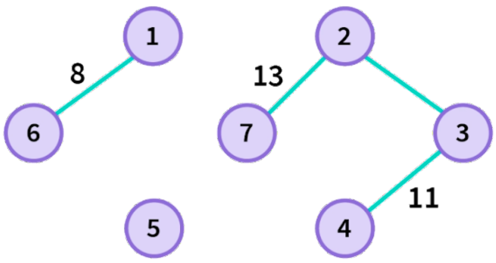


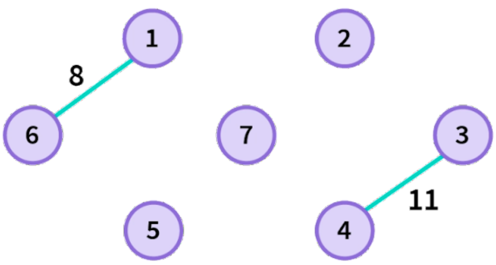
Kruskal's Algorithm for MST

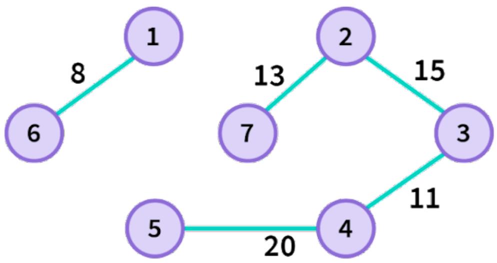
1. Sort all the edges of the graph in the increasing order of their weight.
2. Pick the edge with the smallest weight.
3. Check if it forms a cycle with the spanning tree formed so far.
4. Include the current edge if it does not form any cycle.
   Otherwise discard it.
5. Repeat step #3 until there $v - 1$ edges in the spanning tree

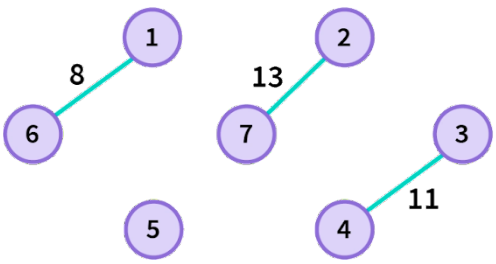


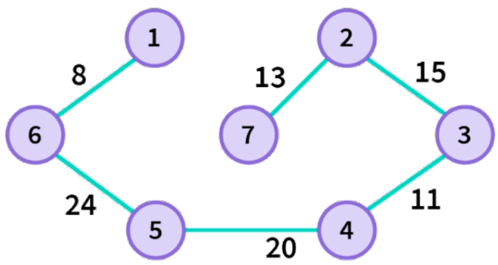












Prim’s Algorithm for MST

1. Select a starting vertex.
2. Select an edge e connecting the tree vertex and fringe vertex that has minimum weight. Fringe vertices are the vertices adjacent to visited vertices but not yet visited.
3. Add the selected edge and the vertex to the minimum spanning tree T
4. Repeat step #2 and step #3 until the vertices adjacent to the visited vertex are unvisited.