Assignment-1

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Download all python codes from

https://github.com/Krupateja/EE3025/blob/main/Assignment_1/ee18btech11015_1.py

and latex-tikz codes from

https://github.com/Krupateja/EE3025/tree/main/Assignment_1

1 Problem

1.1. Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{1.1.1}$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{1.1.2}$$

1.2. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{1.2.1}$$

and $H(k)$ using h(n).

1.3. Compute

$$Y(k) = X(k)H(k) \tag{1.3.1}$$

2 Solution

2.1. It is the output of LTI system when we give unit impulse as input.It can be found from the below dierence equation.

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \tag{2.1.1}$$

2.2. DFT of x(n) is

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{2.2.1}$$

We know that $W_N = e^{-j2\pi/N}$

We can express X as Matrix Multiplication of DFT Matrix and x.

$$X = \left[W_N^{ij}\right]_{N\times N} x, \quad i, j = 0, 1, \ldots, N-1 \tag{2.2.2}$$

2.3. Here N = 6

$$\implies W_6 = e^{-j2\pi/6} = \frac{1}{2} - \frac{\sqrt{3}}{2}j \tag{2.3.1}$$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5) \end{bmatrix} = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0\\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5\\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10}\\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15}\\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20}\\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix} \begin{bmatrix} 1\\ 2\\ 3\\ 4\\ 2\\ 1 \end{bmatrix} \tag{2.3.2}$$

$$\implies \begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5) \end{bmatrix} = \begin{bmatrix} 13 + 0j\\ -4 - \sqrt{3}j\\ 1 + 0j\\ -1 + 0j\\ 1 + 0j\\ -4 + \sqrt{3}j \end{bmatrix} \tag{2.3.3}$$

2.4. Similarly,

$$\begin{bmatrix} H(0)\\ H(1)\\ H(2)\\ H(3)\\ H(4)\\ H(5) \end{bmatrix} = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0\\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5\\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10}\\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15}\\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20}\\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix} \begin{bmatrix} 1\\ -0.5\\ 1.25\\ -0.625\\ 0.3125\\ -0.15625 \end{bmatrix} \tag{2.4.1}$$

$$\implies \begin{bmatrix} H(0)\\ H(1)\\ H(2)\\ H(3)\\ H(4)\\ H(5) \end{bmatrix} = \begin{bmatrix} 1.28125 + 0j\\ 0.51625 - 0.5142j\\ -0.07813 + 1.1096j\\ 3.84375 + 0j\\ -0.07183 - 1.1096j\\ 0.51625 + 0.5142j \end{bmatrix} \tag{2.4.2}$$

2.5. We can find Y using,

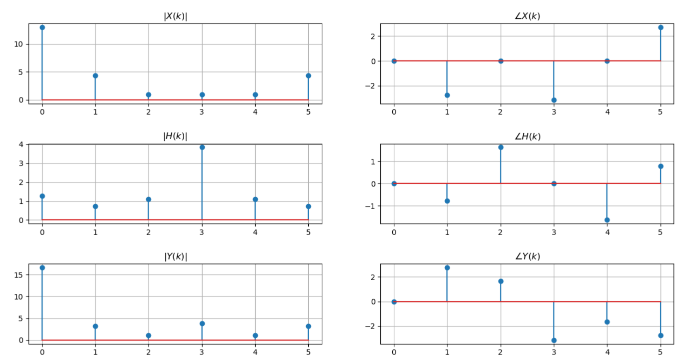
$$\begin{bmatrix} Y(0)\\ Y(1)\\ Y(2)\\ Y(3)\\ Y(4)\\ Y(5) \end{bmatrix} = \begin{bmatrix} X(0).H(0)\\ X(1).H(1)\\ X(2).H(2)\\ X(3).H(3)\\ X(4).H(4)\\ X(5).H(5) \end{bmatrix} \tag{2.5.1}$$

$$\implies \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} 16.6562 + 0j \\ -2.95312 + 1.16372j \\ -0.07813 + 1.1096j \\ -3.84375 + 0j \\ -0.07813 - 1.1096j \\ -2.95312 - 1.16372j \end{bmatrix} \tag{2.5.2}$$

2.6. The following code computes Y and generates magnitude and phase plots of X, H, Y

https://github.com/Krupateja/EE3025/blob/main/Assignment_1/ee18btech11015_1.py

2.7. The following plots are obtained



2.8. Consider the square property of Complex Exponentials

$$W_N^2 = W_{N/2} \tag{2.8.1}$$

2.9. For any N-point DFT we can write,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \ldots, N-1 \tag{2.9.1}$$

Dividing the inputs into even and odd indices and using Property(c),

$$X(k) = \sum_{n=even} x(n) W_N^{kn} + \sum_{n=odd} x(n) W_N^{kn} \tag{2.9.2}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{(2m+1)k} \tag{2.9.3}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_{N/2}^{mk} + W_N^k \sum_{m=0}^{N/2-1} x(2m+1) W_{N/2}^{mk} \tag{2.9.4}$$

Finally,

$$X(k) = X_1(k) + W_N^k X_2(k) \tag{2.9.5}$$

Taking N = 6 and expressing the even odd DFT's $X_1(k)$, $X_2(k)$ interms of matrices we get,

2.10. Let F_N be the N-point DFT Matrix. Using the property of Complex Exponentials we can express F_N in terms of $F_{N/2}$

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix} P_N \tag{2.10.1}$$

For N = 6

$$\implies F_6 = \begin{bmatrix} I_3 & D_3 \\ I_3 & -D_3 \end{bmatrix} \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix} P_6 \tag{2.10.2}$$

where I_3 is the 3x3 identity matrix

$$D_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^1 & 0 \\ 0 & 0 & W_3^2 \end{bmatrix} \tag{2.10.3}$$

$$P_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.10.4}$$

$$\implies P_6 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.10.5}$$

Let

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \end{bmatrix} = F_{N/2} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \end{bmatrix} \tag{2.10.6}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = F_{N/2} \begin{bmatrix} x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.10.7}$$

be the N/2 point DFTs.

2.11. By replacing the above results in the equation

$X = F_N x$, we get

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0\\ 0 & 1 & 0 & 0 & W_6^1 & 0\\ 0 & 0 & 1 & 0 & 0 & W_6^2\\ 1 & 0 & 0 & -W_6^0 & 0 & 0\\ 0 & 1 & 0 & 0 & -W_6^1 & 0\\ 0 & 0 & 1 & 0 & 0 & -W_6^2 \end{bmatrix} \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2)\\ X_2(0)\\ X_2(1)\\ X_2(2) \end{bmatrix} \tag{2.11.1}$$

Broke N-point DFT into 2 N/2-point DFTs using above method

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2) \end{bmatrix} = \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2) \end{bmatrix} + \begin{bmatrix} W_6^0 & 0 & 0\\ 0 & W_6^1 & 0\\ 0 & 0 & W_6^2 \end{bmatrix} \begin{bmatrix} X_2(0)\\ X_2(1)\\ X_2(2) \end{bmatrix} \tag{2.11.2}$$

$$\begin{bmatrix} X(3)\\ X(4)\\ X(5) \end{bmatrix} = \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2) \end{bmatrix} - \begin{bmatrix} W_6^0 & 0 & 0\\ 0 & W_6^1 & 0\\ 0 & 0 & W_6^2 \end{bmatrix} \begin{bmatrix} X_2(0)\\ X_2(1)\\ X_2(2) \end{bmatrix} \tag{2.11.3}$$

By doing this we can reduce our time complexity from O(N^2) to O(NlogN)

2.12. Let x(n) = $\left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1, 4, 3 \right\}$

$N = 8 = 2^3$ Breaking down it recursively,

$$F_8 = \begin{bmatrix} I_4 & D_4\\ I_4 & -D_4 \end{bmatrix} \begin{bmatrix} F_4 & 0\\ 0 & F_4 \end{bmatrix} P_8 \tag{2.12.1}$$

$$F_4 = \begin{bmatrix} I_2 & D_2\\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0\\ 0 & F_2 \end{bmatrix} P_4 \tag{2.12.2}$$

2-point FFT is a base case

$$F_2 \begin{bmatrix} x1\\ x2 \end{bmatrix} = \begin{bmatrix} x1 + x2\\ x1 - x2 \end{bmatrix} \tag{2.12.3}$$

2.13. The following code computes Y and generates magnitude and phase plots of X, H, Y

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https://github.com/Krupateja/EE3025/blob/
    main/Assignment_1/codes/
    ee18btech11015_2.py
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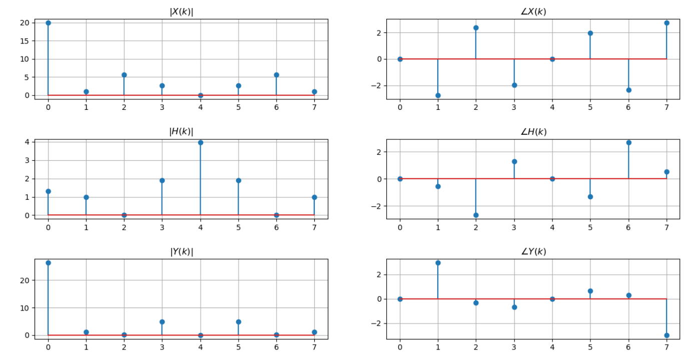
The following plots are obtained



2.14. The following code compares computation times for N-point DFTs and N-point FFTs of the form N = 2^n for n=0 to 11

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https://github.com/Krupateja/EE3025/blob/
    main/Assignment_1/codes/
    ee18btech11015_3.py
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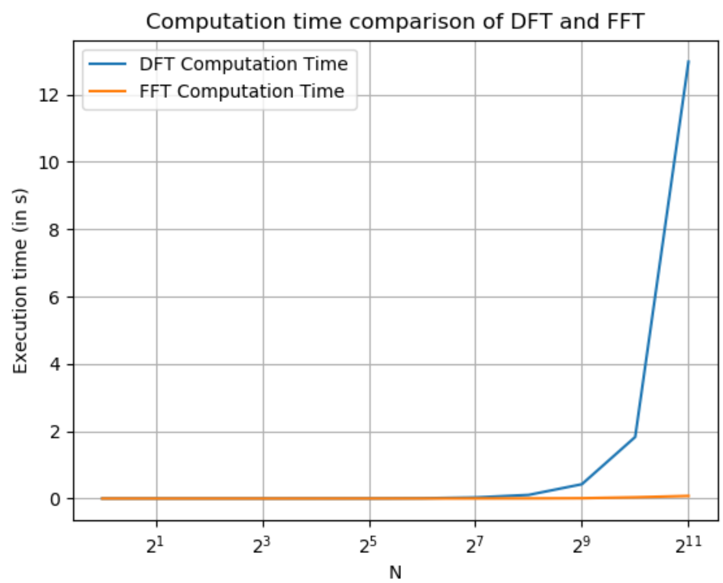
The following plots are obtained We observe that the computation time for DFT computation rises exponentially But FFT is much faster. In



DFT - Matrix multiplication of NxN matrix with Nx1 vector. which is $O(N^2)$ time complexity. In FFT - N-point FFT is broken down recursively into 2 N/2-point FFTs recursively. Additionally O(N) operation of Vector multiplication is performed on the N/2 point FFTs.

$$T(n) = 2T(n/2) + O(n) \tag{2.14.1}$$

Solving this recurrence gives $O(NlogN)$ time complexity.

2.15. Step-by-step visualisation 8-point FFTs into 4-point FFTs

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0)\\ X_1(1)\\ X_1(2)\\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0)\\ X_2(1)\\ X_2(2)\\ X_2(3) \end{bmatrix} \tag{2.15.1}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{2.15.2}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{2.15.3}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{2.15.4}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{2.15.5}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{2.15.6}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{2.15.7}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{2.15.8}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{2.15.9}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{2.15.10}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{2.15.11}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{2.15.12}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{2.15.13}$$

2.16. Below is the 8 point FFT

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$