Assignment-1

Krutik Mehta - EE18BTECH11027

Download all python codes from

https://github.com/Krutikmehta/IDP6/tree/main/Assignment_1/codes

and latex-tikz codes from

https://github.com/Krutikmehta/IDP6/tree/main/Assignment_1

1 Problem

1.1. Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{1.1.1}$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{1.1.2}$$

1.2. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{1.2.1}$$

and $H(k)$ using h(n).

1.3. Compute

$$Y(k) = X(k)H(k) \tag{1.3.1}$$

2 Solution

2.1. Let $W_N = e^{-j2\pi/N}$

We can express X as Matrix Multiplication of DFT Matrix and x.

$$X = \left[W_N^{ij}\right]_{N\times N} x, \quad i, j = 0, 1, \ldots, N-1 \tag{2.1.1}$$

2.2. Properties :

a) symmetry property :

$$W_N^{k+N/2} = -W_N^k$$

b) Periodicity property :

$$W_N^{k+N} = W_N^k$$

c)

$$W_N^2 = W_{N/2}$$

2.3. Using properties to derive FFT from DFT :

$$\mathcal{X}(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \tag{2.3.1}$$

$$= \sum_{n=even} x(n)W_N^{kn} + \sum_{n=odd} x(n)W_N^{kn} \tag{2.3.2}$$

$$= \sum_{m=0}^{2} x(2m)W_N^{2mk} + \sum_{m=0}^{2} x(2m+1)W_N^{(2m+1)k} \tag{2.3.3}$$

using property c, we get,

$$\mathcal{X}(k) = \sum_{m=0}^{2} x(2m)W_{N/2}^{mk} + W_N^k \sum_{m=0}^{2} x(2m+1)W_{N/2}^{mk} \tag{2.3.4}$$

$$= X_1(k) + W_N^k X_2(k) \tag{2.3.5}$$

- X_1(k) and X_2(k) are 3 point DFTs of x(2m) and x(2m+1) , m=0,1,2.
- X_1(k) and X_2(k) are periodic, Hence X_1(k+3) = X_1(k) and X_2(k+3) = X_1(k).

2.4. Calculating X1 and X2

$$X_1(k) = \sum_{m=0}^{2} x(2m)W_3^{mk} \tag{2.4.1}$$

$$X_2(k) = \sum_{m=0}^{2} x(2m+1)W_3^{mk} \tag{2.4.2}$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 & 0 & 0 & 0 \\ W_3^0 & W_3^1 & W_3^2 & 0 & 0 & 0 \\ W_3^0 & W_3^2 & W_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_3^0 & W_3^0 & W_3^0 \\ 0 & 0 & 0 & W_3^0 & W_3^1 & W_3^2 \\ 0 & 0 & 0 & W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.4.3}$$

$$\hat{X} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = B_1 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.4.4}$$

$$\hat{X} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = B_1 P \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} \tag{2.4.5}$$

where P = $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

P is an odd-even permutation matrix.

$$\implies \hat{X} = B_1 P x \tag{2.4.6}$$

2.5. Calculating X

$$X(k) = X_1(k) + W_N^k X_2(k) \tag{2.5.1}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^2 \\ 1 & 0 & 0 & W_6^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^5 \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.5.2}$$

$$X = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = B_2 \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.5.3}$$

$$\mathcal{X} = B_2 \hat{X} \tag{2.5.4}$$

$$X = B_2 B_1 P x \tag{2.5.5}$$

2.6. From the above exercise we have factorised the W matrix

$$W = B_2 B_1 P \tag{2.6.1}$$

2.7. Hence IFFT can be computed as;

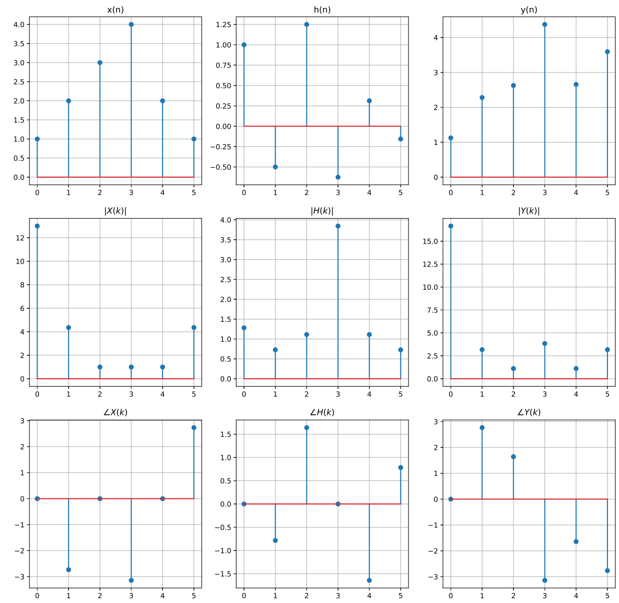
$$\begin{aligned} x &= \frac{1}{N} W^H X \\ x &= \frac{1}{6} P^H {B_1}^H {B_2}^H X \end{aligned} \tag{2.7.1}$$

2.8. The following code computes Y and generates magnitude and phase plots of X, H, Y

$$\begin{aligned} X &= B_2 B_1 P x \\ H &= B_2 B_1 P h \\ Y &= H X \\ y &= \frac{1}{6} P^H {B_1}^H {B_2}^H y \end{aligned} \tag{2.8.1}$$

https://github.com/Krutikmehta/IDP6/tree/main/Assignment_1/codes/EE18BTECH11027.py

2.9. The following plots are obtained



2.10. Benefits of FFT

- The DFT which would require n^2 operations for computation, but in the modified version, as most of the elements are zeros or ones, there are $(\frac{n}{2})^2$ operations.
- If we recursively perform this decomposition, it would result in $\frac{n}{2} \log n$ operations.