Precoding and Signal Shaping for Digital Transmission. Robert F. H. Fischer

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Appendix C Introduction to Lattices

In practice, signal constellations form a regular grid. For analysis of the transmission properties of (large) signal sets, the boundary effects are often disregarded and an infinite set of points is assumed. A powerful tool for that is the theory of *lattices*, regular arrays of points in N-dimensional space. In this Appendix, the basics of lattices are discussed and the fundamental concepts are introduced. Much more detailed discussion can be found, e.g., in [CS88, For88a, For88b, FW89, For89]. Here we focus mainly on the application of lattices to digital transmission schemes.

C.1 DEFINITION OF LATTICES

Consider an infinite, discrete set of vectors (points) $\boldsymbol{\lambda}$ in Euclidean space $\mathbb{R}^N$, i.e.,

$$\Lambda = \{\, \boldsymbol{\lambda} \,\} \subset \mathbb{R}^N \quad \text{with} \quad \boldsymbol{\lambda} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{bmatrix} \in \mathbb{R}^N \,. \tag{C.1.1}$$

Geometrically, Λ forms an *N-dimensional lattice* if it constitutes a regular arrangement of an infinite number of points in N dimensions.[1] Because of the underlying Euclidean space, we can endow a lattice with attributes such as distances, volumes or shapes.

Regarding its algebraic properties, a lattice is a (*Abelian*) *group* under ordinary vector addition in $\mathbb{R}^N$. Mathematically, for Λ beeing a *lattice*, the following requirements have to be met:

G1:	Closure:	$\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j \in \Lambda, \ \forall \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j \in \Lambda$
G2:	Associativity:	$(\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j) + \boldsymbol{\lambda}_k = \boldsymbol{\lambda}_i + (\boldsymbol{\lambda}_j + \boldsymbol{\lambda}_k)$
G3:	Identity:	$\exists\, \mathbf{0} \in \Lambda$ with $\boldsymbol{\lambda}_i + \mathbf{0} = \mathbf{0} + \boldsymbol{\lambda}_i = \boldsymbol{\lambda}_i, \ \forall \boldsymbol{\lambda}_i \in \Lambda$
G4:	Inverse:	$\exists -\boldsymbol{\lambda}_i \in \Lambda$ with $\boldsymbol{\lambda}_i + (-\boldsymbol{\lambda}_i) = \mathbf{0}, \ \forall \boldsymbol{\lambda}_i \in \Lambda$
G5:	Commutativity:	$\boldsymbol{\lambda}_i + \boldsymbol{\lambda}_j = \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_i, \ \forall \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j \in \Lambda$

Given a set of N linear independent vectors $\boldsymbol{b}_i$, because of the group structure, each lattice point $\boldsymbol{\lambda} \in \Lambda$ can be expressed as a *linear combination* of integer multiples of these *basis vectors* $\boldsymbol{b}_i$

$$\boldsymbol{\lambda} = \sum_{i=1}^{N} k_i \boldsymbol{b}_i \ , \qquad k_i \in \mathbb{Z} \ . \tag{C.1.2}$$

The basis vectors have to be linearly independent and to span the N-space.[2] Note that the choice of the basis is not unique. It is common to combine the basis vectors $\boldsymbol{b}_i$ as columns into the *generator matrix*

$$\boldsymbol{B} = [\, \boldsymbol{b}_1, \, \boldsymbol{b}_2, \, \cdots, \, \boldsymbol{b}_N \,] \ . \tag{C.1.3}$$

With it, the lattice can be specified as

$$\Lambda = \left\{ \boldsymbol{B} \begin{bmatrix} k_1 \\ \vdots \\ k_N \end{bmatrix} \;\middle|\; k_i \in \mathbb{Z}, \, i = 1, 2, \ldots, N \right\} \triangleq \boldsymbol{B}\mathbb{Z}^N \ . \tag{C.1.4}$$

Note, the generator matrix is not unique; any integer linear combination of the basis vectors, which preserves the full rank of $\boldsymbol{B}$, can be used as well. The right hand side expression of (C.1.4)—its meaning should be intuitively clear—is defined for convenience. In addition, for the combined *scaling* and *translation* operation, we write briefly, in terms of sets

$$a\Lambda + \boldsymbol{b} \triangleq \left\{ a\boldsymbol{\lambda} + \boldsymbol{b} \;\middle|\; \boldsymbol{\lambda} \in \Lambda \right\} , \quad \text{with} \quad a \in \mathbb{R}, \, \boldsymbol{b} \in \mathbb{R}^N \ . \tag{C.1.5}$$

[1] Here, we only treat *real lattices*, i.e., points taken from $\mathbb{R}^N$. Generalizations, e.g., by using complex vectors instead of real vectors, are not considered, since we do not require them for our applications.

[2] We assume that an N-dimensional lattice spans N dimensions and does not degenerate to a smaller number of dimensions, i.e., a subspace.

A *fundamental region* $\mathcal{R}(\Lambda) \subseteq \mathbb{R}^N$ of a lattice Λ (i) includes one and only one point of Λ, and (ii) when shifting it to any lattice point, i.e., considering $\mathcal{R}(\Lambda) + \lambda$, and when λ ranges through all points in Λ, the whole real N-space $\mathbb{R}^N$ is tiled. Thus, a fundamental region is a building block for the entire N-space, and in the notation of sets, we may write

$$\mathbb{R}^N = \mathcal{R}(\Lambda) + \Lambda \,. \tag{C.1.6}$$

Here we define an "addition" of sets as follows

$$\mathcal{A} + \mathcal{B} \triangleq \left\{ \boldsymbol{a} + \boldsymbol{b} \mid \boldsymbol{a} \in \mathcal{A},\, \boldsymbol{b} \in \mathcal{B} \right\} . \tag{C.1.7}$$

Note that the basis vectors $\boldsymbol{b}_i$ span the *fundamental parallelepiped*

$$\mathcal{P} = \left\{ \boldsymbol{B} \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix} \mid r_i \in \mathbb{R},\, 0 \le r_i < 1,\, i = 1, 2, \ldots, N \right\} \triangleq \boldsymbol{B}\, [0,\, 1)^N \,, \tag{C.1.8}$$

which is a particular fundamental region of the lattice.

The concept of *arithmetic modulo* Λ is closely connected to the fundamental region. Two points λ_1 and λ_2 are called *equivalent modulo* Λ, if their difference is a lattice point, i.e., $\lambda_1 - \lambda_2 \in \Lambda$. We define a *modulo reduction* with respect to Λ as that point out of all equivalent points which lies in the fundamental region containing the origin

$$\boldsymbol{x} \bmod \Lambda \triangleq \boldsymbol{x} + \lambda \,, \quad \text{with} \quad \lambda \in \Lambda \quad \text{so that} \quad \boldsymbol{x} + \lambda \in \mathcal{R}(\Lambda) \,. \tag{C.1.9}$$

The *Voronoi region* $\mathcal{R}_V(\Lambda)$ of a lattice Λ is that fundamental region in which every point is closer to the origin than to any other lattice point. Because we deal with Euclidean N-space, distances are measured as the Euclidean norm of the difference vector. Hence, the Voronoi region is a special fundamental region and contains only the minimum weight (= distance from the origin) point of equivalent points (ties are resolved arbitrarily). In the literature, the Voronoi region is sometimes defined with respect to a particular point (e.g., [CS88]). Because of the regular arrangement of the lattice points, all these regions are congruent and we only look at that region—and call it Voronoi region—containing the origin.

Note that the set $\mathcal{R}_V(\Lambda) + \Lambda$ of translates of the Voronoi region constitute the decision regions for a minimum-distance decoder.

Example C.1: *Lattice, Basis, Fundamental Region, Voronoi Region*

Lattice $\mathbb{Z}$ In one dimension, the set $\mathbb{Z}$ of integers forms a lattice. The generator matrix is simply $\boldsymbol{B} = 1$, and the fundamental parallelepiped is the interval $[0,\, 1)$. Calculation modulo $\mathbb{Z}$ gives the "fractional part." The Voronoi region is the interval $\mathcal{R}_V(\mathbb{Z}) = (-0.5,\, 0.5]$, where we have arbitrarily chosen the point 0.5 to be included. Figure C.1 depicts a portion of the lattice. Note that in all subsequent figures, only the relevant part of the lattice is shown, as a lattice always contains an infinite number of points.

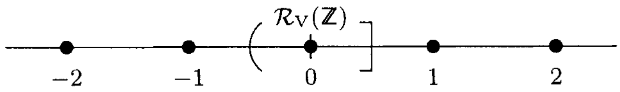

Fig. C.1 The lattice $\mathbb{Z}$ and its Voronoi region $\mathcal{R}_V(\mathbb{Z})$.

Lattice $\mathbb{Z}^N$ Taking the N-fold Cartesian product of the lattice $\mathbb{Z}$ results in the lattice $\mathbb{Z}^N$. The generator matrix is the N-dimensional identity matrix $\boldsymbol{I}$

$$\boldsymbol{B} = \boldsymbol{I} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

and the fundamental parallelepiped is the hypercube $[0,\ 1)^N$. There are many more choices for a fundamental region in N-dimensions compared to the one-dimensional case, where it is unique. Figure C.2 shows the lattice $\mathbb{Z}^2$, a possible fundamental region, and the Voronoi region $(-0.5,\ 0.5]^2$.

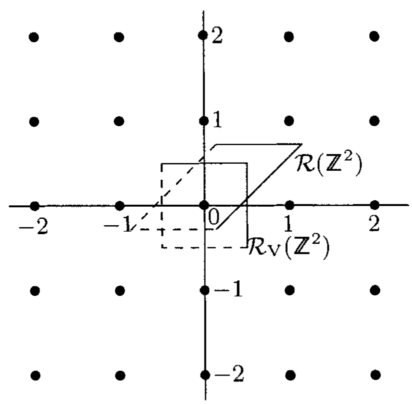

Fig. C.2 The lattice $\mathbb{Z}^2$ with an example of a fundamental region $\mathcal{R}(\mathbb{Z}^2)$ and the Voronoi region $\mathcal{R}_V(\mathbb{Z}^2)$. The points lying on the solid lines belong to the region, whereas the points on the dashed lines do not.

Lattice $\boldsymbol{A}_2$ In lattice theory, the two-dimensional *hexagonal lattice* is denoted as $\boldsymbol{A}_2$, because it is one representative of a broader class of lattices $\boldsymbol{A}_n$ (note, $\boldsymbol{A}_1 = \mathbb{Z}$) [CS88]. A possible generator matrix is

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}, \tag{C.1.10}$$

i.e., the basis vectors are chosen to be $\boldsymbol{b}_1 = [1, 0]^\mathsf{T}$ and $\boldsymbol{b}_2 = [1/2, \sqrt{3}/2]^\mathsf{T}$. The lattice $\boldsymbol{A}_2$ is shown in Figure C.3 and the relevant quantities are displayed. Note, the Voronoi region is a hexagon, which names the lattice.

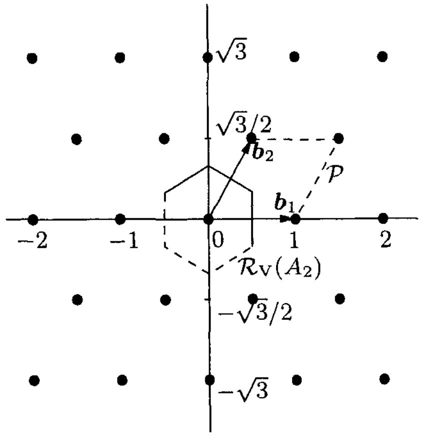

Fig. C.3 The lattice A_2 with its fundamental parallelepiped and Voronoi region $\mathcal{R}_V(A_2)$, and the basis vectors. The points lying on the solid lines belong to the region, whereas the points on the dashed lines do not.

C.2 SOME IMPORTANT PARAMETERS OF LATTICES

Now, we will discuss some important parameters for dealing with lattices.

One of the primary parameters of a lattice is the *minimum squared distance* $d^2_{\min}(\Lambda)$ between any two points from Λ. Mathematically, because of the group properties of the lattice, $d^2_{\min}(\Lambda)$ is identical to the minimum, nonzero weight of the lattice points ($|\lambda|$: Euclidean norm of λ)

$$\begin{aligned} d^2_{\min}(\Lambda) &\triangleq \min_{\substack{\lambda_i, \lambda_j \in \Lambda \\ \lambda_i \neq \lambda_j}} |\lambda_i - \lambda_j|^2 \\ &= \min_{\lambda_i \in \Lambda \setminus \{0\}} |\lambda_i|^2 . \end{aligned} \tag{C.2.1}$$

The ***channel coding problem*** [CS88] of lattice theory is to find lattices which, under certain additional constraints, maximize the minimum squared distance.

A *volume (fundamental volume)* $V(\Lambda)$, which is defined as the volume of a fundamental region, is associated with each lattice. Note that, in contrast to its shape, the volume of a fundamental region is unique, and hence is also equal to the volume of the Voronoi region. Using the generator matrix $\boldsymbol{B}$, from basic geometry, the volume $V(\Lambda)$ is given as

$$V(\Lambda) \triangleq |\det(\boldsymbol{B})| = \sqrt{\det(\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B})} . \tag{C.2.2}$$

Sometimes, an equivalent quantity, the *determinant of* Λ is used

$$\det\Lambda \triangleq \det(\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B}) = V^2(\Lambda)\,. \tag{C.2.3}$$

Using the minimum squared distance, the *packing radius* $\rho(\Lambda)$ of the lattice Λ is defined as

$$\rho(\Lambda) \triangleq \frac{1}{2}\sqrt{d^2_{\min}(\Lambda)}\,. \tag{C.2.4}$$

The interpretation of this quantity is as follows: spheres centered at the lattice points can have a maximum radius $\rho(\Lambda)$ without intersecting each other. Regarding the Voronoi region, $\rho(\Lambda)$ is the *inradius*, i.e., the radius of the largest sphere, which can be inscribed into the Voronoi region.

The *density* $\Delta(\Lambda)$ of a lattice Λ is the portion of N-space that is occupied by the above-mentioned spheres

$$\Delta(\Lambda) \triangleq \frac{\text{Volume of one } N\text{-dim. sphere with radius } \rho(\Lambda)}{V(\Lambda)}\,. \tag{C.2.5}$$

The ***packing problem*** [CS88] is to find that lattice which has maximum density.

Counting the number of lattice points which are at minimum distance from a fixed lattice point gives the *kissing number* $\tau(\Lambda)$ (the terminology is borrowed from the billiards). Because the lattice is well structured, the kissing number can be calculated as

$$\tau(\Lambda) \triangleq \left|\left\{\lambda \in \Lambda \mid |\lambda|^2 = d^2_{\min}(\Lambda)\right\}\right|\,, \tag{C.2.6}$$

and $|\cdot|$ denotes cardinality if sets are considered. In channel coding, the kissing number is usually known as *number of nearest neighbors*.

Closely related to the packing problem is the question known as ***kissing number problem*** [CS88], dating back (at least in low dimensions) some hundred years [CS88, page 21]: How many equal-sized spheres can be arranged around another sphere, so that they all touch it. Here, the spheres are assumed to be rigid bodies and may not intersect.

Contrary to the packing radius, the *covering radius* $\rho_c(\Lambda)$ is the largest distance a point in $\mathbb{R}^N$ can have to the closest lattice point. Again keeping the group properties of Λ in mind, we have the definition

$$\begin{aligned}\rho_c(\Lambda) &= \sup_{x\in\mathbb{R}^N}\inf_{\lambda\in\Lambda}|x-\lambda| \\ &= \sup_{x\in\mathcal{R}_{\mathrm{V}}(\Lambda)}|x|\,.\end{aligned} \tag{C.2.7}$$

Geometrically, the covering radius is the *circumradius*, i.e., $\rho_c(\Lambda)$ is the radius of the smallest sphere, which circumscribes the Voronoi region.

With this in mind, the ***covering problem*** [CS88] seeks for the least dense way of completely covering space with equal-sized and overlapping N-spheres.

The last parameter we discuss in this section is the *normalized second moment* $\mathsf{G}(\mathcal{R})$ (or dimensionless second moment) of a region $\mathcal{R}$. It is defined as the ratio of the variance per dimension (the Nth part of the moment of inertia) and the volume normalized to two-dimensions

$$\mathsf{G}(\mathcal{R}) \triangleq \frac{\frac{1}{N}\int_{\mathcal{R}} \frac{1}{V(\mathcal{R})}|\boldsymbol{r}|^2\,\mathrm{d}\boldsymbol{r}}{V(\mathcal{R})^{(2/N)}} = \frac{\int_{\mathcal{R}} |\boldsymbol{r}|^2\,\mathrm{d}\boldsymbol{r}}{N\,V(\mathcal{R})^{(1+2/N)}}\,. \tag{C.2.8}$$

For lattices, we define the normalized second moment to be the respective quantity of the Voronoi region, i.e.

$$\mathsf{G}(\Lambda) \triangleq \mathsf{G}(\mathcal{R}_{\mathrm{V}}(\Lambda))\,. \tag{C.2.9}$$

This parameter is useful, e.g., when designing an appropriate boundary for the signal constellation (signal shaping).

Moreover, the ***quantization problem*** [CS88] is to find an N-dimensional lattice, for which the normalized second moment is minimum, as (assuming an uniformly distributed input in N dimensions) $\mathsf{G}(\Lambda)$ is proportional to the mean-squared quantization error.

Example C.2: *Packing and Covering*

Lattice $\mathbb{Z}^2$ Figure C.4 depicts the packing and covering for the $\mathbb{Z}^2$ lattice. The packing radius is $\rho(\mathbb{Z}^2) = 1/2$, and the covering radius $\rho_c(\mathbb{Z}^2) = 1/\sqrt{2}$. Since the squared Euclidean distance is $d^2_{\min}(\mathbb{Z}^2) = 1$, and the volume equals $V(\mathbb{Z}^2) = 1$, the density of this lattice $\mathbb{Z}^2$ is $\Delta(\mathbb{Z}^2) = \pi/4$. By inspection, the kissing number is 4 and the normalized second moment calculates to $1/12 \approx 0.0833$.

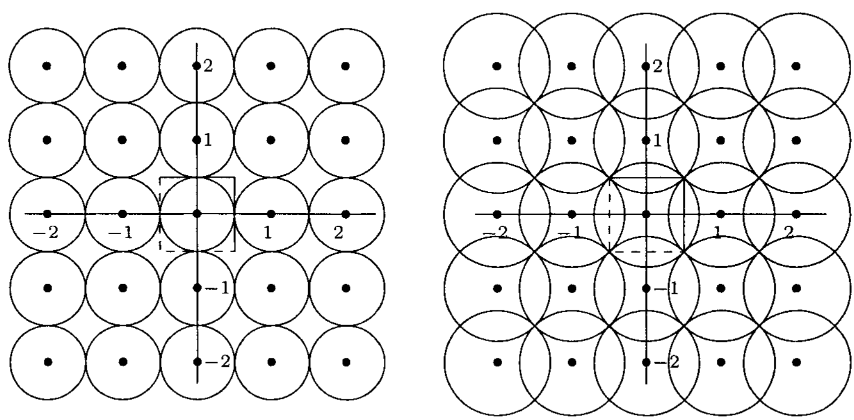

Fig. C.4 Packing and covering for the lattice $\mathbb{Z}^2$.

Lattice $\boldsymbol{A}_2$ Packing and covering are depicted for the hexagonal lattice $\boldsymbol{A}_2$ in Figure C.5. Here, for a squared Euclidean distance of $d^2_{\min}(\boldsymbol{A}_2) = 1$, the packing radius is $\rho(\boldsymbol{A}_2) = 1/2$ and the covering radius equals $\rho_c(\boldsymbol{A}_2) = 1/\sqrt{3}$. From basic geometry, the volume calculates to $V(\boldsymbol{A}_2) = \sqrt{3}/2$; the density of $\boldsymbol{A}_2$ is $\Delta(\boldsymbol{A}_2) = \pi/(2\sqrt{3})$, and the kissing number is 6. The normalized second moment reads $5/(36\sqrt{3}) \approx 0.0802$. As one can see, with respect to all parameters, the $\boldsymbol{A}_2$ lattice can be judged to be better than the $\mathbb{Z}^2$ lattice. It has been proved

[CS88] that the two-dimensional hexagonal lattice is the best with respect to all questions of lattice theory. Comparing Figures C.4 and C.5, we see that the lattice $\boldsymbol{A}_2$ has a denser packing, as well as a thinner covering—the areas of double covering are much smaller.

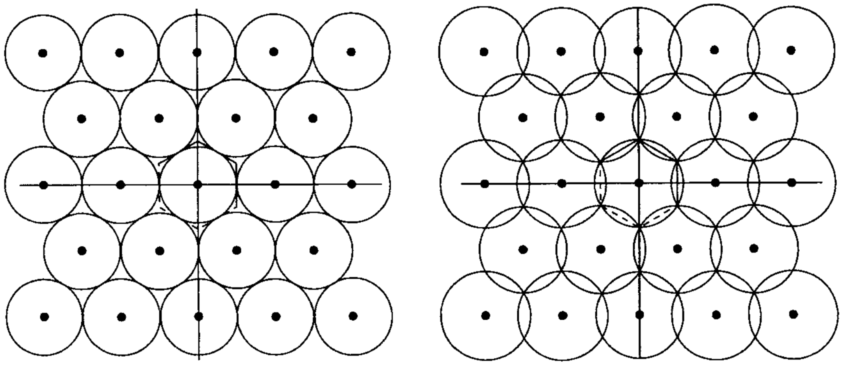

Fig. C.5 Packing and covering for the lattice $\boldsymbol{A}_2$.

C.3 MODIFICATIONS OF LATTICES

Given a lattice Λ, we now discuss some modifications, which preserve lattice property. The proofs are obvious and hence omitted here.

Scaling Let Λ be an N-dimensional lattice and $r \in \mathbb{R}$ a real scalar. Then, $r\Lambda \triangleq \{r\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in \Lambda\}$ is a lattice, too, consisting of all scaled vectors $r\boldsymbol{\lambda}$ of the initial lattice Λ. If $\boldsymbol{B}$ is the generator matrix of Λ, the respective generator matrix of $r\Lambda$ reads $r\boldsymbol{B}$. Scaling changes the parameters according to $d^2_{\text{min}}(r\Lambda) = r^2 d^2_{\text{min}}(\Lambda)$, $V(r\Lambda) = r^N V(\Lambda)$, $\rho(r\Lambda) = r\rho(\Lambda)$, $\rho_c(r\Lambda) = r\rho_c(\Lambda)$, and density, kissing number, and normalized second moment are scaling-invariant.

Orthonormal Transformation Let $\boldsymbol{R}$ be a real-valued orthonormal matrix, i.e., $\boldsymbol{R}^{\mathsf{T}}\boldsymbol{R} = \boldsymbol{I}$. The transformed set of points $\boldsymbol{R}\Lambda$ is again a lattice, which is generated by $\boldsymbol{R}\boldsymbol{B}$. Since $\boldsymbol{R}$ is orthonormal, we have

$$V(\boldsymbol{R}\Lambda) = \sqrt{\det(\boldsymbol{B}^{\mathsf{T}}\boldsymbol{R}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{B})} = \sqrt{\det(\boldsymbol{B}^{\mathsf{T}}\boldsymbol{B})} = V(\Lambda) \; ,$$

i.e., the volume and all other parameters are preserved. Geometrically, this transformation is a rotation in N dimensions.

Reflection The lattice property is also preserved, if Λ is reflected at an $N-1$ dimensional plane. Mathematically, this operation is described by a matrix $\boldsymbol{U}$ with integer entries and $\det(\boldsymbol{U}) = \pm 1$. Here, all parameters of the lattice are unaffected.

If one lattice can be transformed into another lattice by combined rotation, reflection, and scaling, the lattices are called *equivalent*.

Cartesian Product Let Λ be an N-dimensional lattice. The M-fold Cartesian product of Λ, denoted by Λ^M, is a lattice, too. It is defined as the set of MN-dimensional vectors of the form $\left[\boldsymbol{\lambda}_1^{\mathsf{T}}, \boldsymbol{\lambda}_2^{\mathsf{T}}, \ldots, \boldsymbol{\lambda}_M^{\mathsf{T}}\right]^{\mathsf{T}}$, with $\boldsymbol{\lambda}_i \in \Lambda, i = 1, 2, \ldots, M$.

Note that **translation** of a lattice, i.e., taking $\Lambda + \boldsymbol{t}$, $\boldsymbol{t} \in \mathbb{R}^N$, does not give a lattice in general! Only if the translation vector $\boldsymbol{t}$ is itself a lattice point are the lattice characteristics preserved. From the group properties we have $\Lambda + \boldsymbol{t} = \Lambda$, if and only if $\boldsymbol{t} \in \Lambda$.

With respect to digital transmission over the AWGN channel, translation of a lattice has an effect on the error probability. But note, translation of the boundary region of a constellation may cause an increase in average transmit power.

Example C.3: *Modification of Lattices*

Lattice $\mathbb{Z}$:

Scaling The lattice $\mathbb{Z}$ can be scaled by an arbitrary scalar r. For example, the set of even integers $2\mathbb{Z}$ is a lattice.

Cartesian Product By taking the 2-fold Cartesian product of the integer lattice $\mathbb{Z}$, we get the "square lattice" $\mathbb{Z}^2$

$$\mathbb{Z}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1, x_2 \in \mathbb{Z} \right\} .$$

This procedure can of course be generalized to N dimensions.

Lattice $\mathbb{Z}^N$:

Scaling The lattice $\mathbb{Z}^N$ scaled by the scalar r is again a lattice, which can also be interpreted as the N-fold Cartesian product of the one-dimensional lattice $r\mathbb{Z}$

$$r\mathbb{Z}^N = (r\mathbb{Z})^N .$$

Rotation Let the combined scaling/rotation matrix be

$$\boldsymbol{R} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} ,$$

which performs rotation by 45° counterclockwise and scaling by $\sqrt{2}$. $\boldsymbol{R}\mathbb{Z}^2$ is the rotated and scaled version of $\mathbb{Z}^2$, which can also be described by

$$\boldsymbol{R}\mathbb{Z}^2 = \boldsymbol{D}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1, x_2 \in \mathbb{Z}, \text{ and } x_1 + x_2 \in 2\mathbb{Z} \right\} .$$

$\boldsymbol{D}_2$ is also referred to as the two-dimensional checkerboard lattice.

Because $\boldsymbol{R}^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ performs a rotation by 90° and a scaling by 2, and because $\mathbb{Z}^2$ rotated by 90° coincides with itself, the following holds

$$\boldsymbol{R}\left(\boldsymbol{R}\mathbb{Z}^2\right) = 2\mathbb{Z}^2 = (2\mathbb{Z})^2 .$$

Figure C.6 depicts the lattices $\mathbb{Z}^2$, $\boldsymbol{R}\mathbb{Z}^2$, and $2\mathbb{Z}^2$ and the respective Voronoi regions.

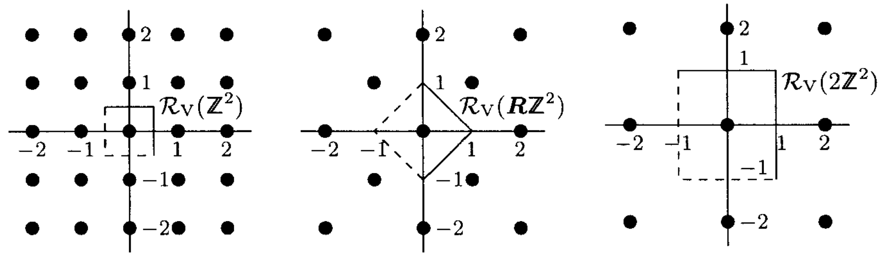

Fig. C.6 Lattices $\mathbb{Z}^2$, $\boldsymbol{R}\mathbb{Z}^2$, and $2\mathbb{Z}^2$.

C.4 SUBLATTICES, COSETS, AND PARTITIONS

From the last example, it can be seen that lattices $\boldsymbol{R}\mathbb{Z}^2$ and $2\mathbb{Z}^2$ are subsets of the lattice $\mathbb{Z}^2$. This leads us to subgroups: A subset $\mathcal{S}$ of a group $\mathcal{G}$, which by itself forms a group, i.e., meets the above group requirements **G1** through **G5**, is called a *subgroup* of the group $\mathcal{G}$. A subset of a lattice, which itself exhibits lattice properties, is called a *sublattice*.

Since real Euclidean space $\mathbb{R}^N$ forms a group (and thus can also be interpreted as a nondiscrete lattice), any real lattice ($\boldsymbol{\lambda}_i \in \mathbb{R}^N$) is a sublattice of $\mathbb{R}^N$.

Given a group $\mathcal{G}$, any subgroup $\mathcal{S}$ induces a *coset decomposition*. The group can be partitioned into $|\mathcal{G}|/|\mathcal{S}|$ disjoint *cosets*. One of them is equal to $\mathcal{S}$, the others are given as translates of the subgroup $\mathcal{S}$, i.e., to each element of $\mathcal{S}$ an element $g \in \mathcal{G}$, but $g \notin \mathcal{S}$ is "added."

Speaking in geometric terms, we have *lattice partitions*. Let Λ be a lattice and $\Lambda' \subset \Lambda$ a sublattice thereof. The lattice partition is the set of cosets, i.e., a set the elements of which are sets, and is usually denoted by Λ/Λ'. The (finite) cardinality of Λ/Λ', i.e., the number m of cosets including the sublattice itself, is called the *order* or *depth* of the partition and is written $m = |\Lambda/\Lambda'|$.

The cosets can be specified by a set $\mathcal{A} = [\Lambda/\Lambda'] \triangleq \{\boldsymbol{a}\}$, comprising m *coset leaders* or *coset representatives* $\boldsymbol{a}$. In the context of lattices, the denomination $[\cdot]$ symbolizes the extraction of the m coset leaders from the lattice partition; one particular representative is chosen from each element of Λ/Λ' (which are sets). The cosets can then be written as

$$\Lambda' + \boldsymbol{a}, \quad \text{with} \quad \boldsymbol{a} \in \mathcal{A} . \tag{C.4.1}$$

The sublattice itself is the *zero coset* with coset leader $\boldsymbol{a} = \mathbf{0}$. It follows from the definition of cosets that all points in one coset are equivalent modulo Λ', i.e., their differences lie in Λ' (cf. Section C.1). In order for $\mathcal{A}$ to specify m disjoint subsets, no pair of coset leaders must not be equivalent modulo Λ'. Finally, in order to unambiguously define $\mathcal{A}$, here we force the coset leaders to lie within the Voronoi region $\mathcal{R}_V(\Lambda')$ of the sublattice. Note, for the fundamental volumes $V(\Lambda') = m \cdot V(\Lambda)$ holds. Unfortunately, the other parameters of Λ', e.g., the minimum distance, cannot be calculated as easily as the volume.

Based on the cosets, the lattice Λ can be expressed as

$$\Lambda = \bigcup_{\boldsymbol{a} \in \mathcal{A}} (\Lambda' + \boldsymbol{a}) , \tag{C.4.2a}$$

or for short [For88a]

$$\Lambda = \Lambda' + [\Lambda/\Lambda'] . \tag{C.4.2b}$$

Note that algebraically the partition Λ/Λ' forms the *quotient group* or *factor group* [BB91], where cosets are the group elements. Defining the sum of two cosets as that coset which is given by the sum of the respective coset leaders (and a possible modulo reduction to $\mathcal{R}_V(\Lambda')$), the set Λ/Λ' has group properties (e.g., the identity element is the sublattice Λ').

In the design of coded modulation schemes, lattice partitions and the indexing of the cosets by their coset leaders, plays an important role. Since the partition Λ/Λ' is a group, codes based on it are linear codes [For88a]. Only when the constellation is restricted to a finite region, due to the boundary effects, the code becomes nonlinear, i.e., here the sum of any two codewords is no longer a valid codeword.

Returning to Euclidean space, any translate $\Lambda + \boldsymbol{t}$, $\boldsymbol{t} \in \mathbb{R}^N$, of a lattice Λ is simply a coset of Λ in $\mathbb{R}^N$. Here, the set of coset representatives is identical to the Voronoi region $\mathcal{R}_V(\Lambda)$.

Of course, the partitioning procedure can be applied more than once. Given lattices Λ, Λ', and Λ'', with $\Lambda'' \subset \Lambda' \subset \Lambda$, a *lattice partition chain* is induced. Considering the definition of a lattice partition, a lattice partition chain is the set of sets which again are sets of sets (i.e., nested sets). A lattice partition chain is concisely written as

$$\Lambda/\Lambda'/\Lambda'' . \tag{C.4.3}$$

Now, Λ can be uniquely expressed as

$$\Lambda = \bigcup_{\boldsymbol{a}' \in \mathcal{A}'} \bigcup_{\boldsymbol{a} \in \mathcal{A}} (\Lambda'' + \boldsymbol{a}' + \boldsymbol{a}) = \Lambda'' + [\Lambda'/\Lambda''] + [\Lambda/\Lambda'] , \tag{C.4.4}$$

where $\mathcal{A}$ and $\mathcal{A}'$ are the set of coset representatives of the partition steps Λ/Λ' and Λ'/Λ'', respectively. If the partitions Λ/Λ' and Λ'/Λ'' are of order $m = |\Lambda/\Lambda'|$

and $m' = |\Lambda'/\Lambda''|$, respectively, then the order of Λ/Λ'' is $m \cdot m'$. Repeating the partitioning, partition chains of arbitrary depth can be generated.

In practice, *binary lattices* [FW89] are the most common. Such a lattice Λ is a sublattice of $\mathbb{Z}^N$, and has $2^n\mathbb{Z}^N$ as a sublattice for some integer n. Hence, for binary lattices, $\mathbb{Z}^N/\Lambda/2^n\mathbb{Z}^N$ is a partition chain. This fact guarantees that the addressing of lattice points can be done by iterative binary partitioning, and thus the labels of the points can be written as binary numbers.

Example C.4: *Coset Decomposition and Partitions*

Lattice $\mathbb{Z}$ The integer lattice $\mathbb{Z}$ can be repeatedly two-way partitioned, which results in the partition chain

$$\mathbb{Z}/2\mathbb{Z}/4\mathbb{Z}/8\mathbb{Z}/16\mathbb{Z}/\cdots$$

The coset representatives for the ith partition step ($2^{(i-1)}\mathbb{Z}/2^i\mathbb{Z}$, $i \in \mathbb{N}$) are

$$\mathcal{A} = [\mathbb{Z}/2^i\mathbb{Z}] = \left\{0, 2^{(i-1)}\right\} .$$

Lattice $\mathbb{Z}^2$ The lattice $\boldsymbol{R}\mathbb{Z}^2$, with the rotation operator $\boldsymbol{R}$ given in the last example, is a sublattice of $\mathbb{Z}^2$. The order of $\mathbb{Z}^2/\boldsymbol{R}\mathbb{Z}^2$ is 2, and the set of coset representatives is

$$\mathcal{A} = [\mathbb{Z}^2/\boldsymbol{R}\mathbb{Z}^2] = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} .$$

Figure C.7 illustrates the lattices $\mathbb{Z}^2$, $\boldsymbol{R}\mathbb{Z}^2$, and its coset and shows the set $\mathcal{A}$ of coset leaders.

Partitioning the lattice $\boldsymbol{R}\mathbb{Z}^2$ once again results in the partition $\boldsymbol{R}\mathbb{Z}^2/2\mathbb{Z}^2$ or, starting with $\mathbb{Z}^2$, we have the four-way partition $\mathbb{Z}^2/2\mathbb{Z}^2$. Figure C.8 shows the four cosets and the four

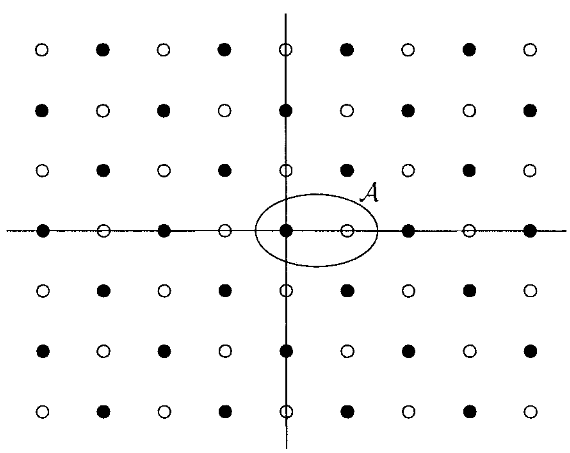

Fig. C.7 Two-way partitioning of the lattice $\mathbb{Z}^2$. The solid dots represent the sublattice $\boldsymbol{R}\mathbb{Z}^2$, the open dots the coset $\boldsymbol{R}\mathbb{Z}^2 + \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

coset representatives

$$\mathcal{A} = [\mathbb{Z}^2/2\mathbb{Z}^2] = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} .$$

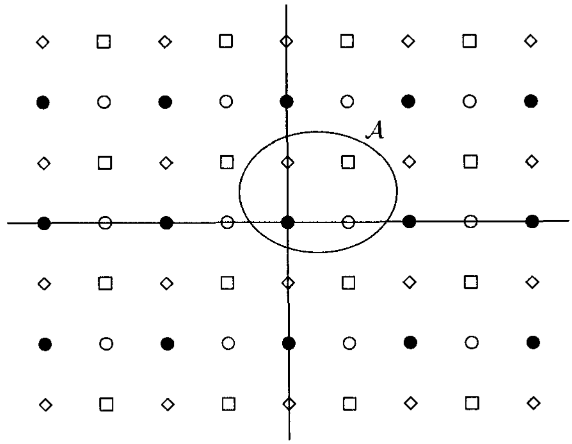

Fig. C.8 Four-way partitioning of the lattice $\mathbb{Z}^2$. The solid dots represent the sublattice $2\mathbb{Z}^2$, the others mark the three further cosets of $2\mathbb{Z}^2$.

Lattice A_2 Rotating the hexagonal lattice A_2 by 90° and scaling it by $\sqrt{3}$ gives a sublattice of A_2, which induces a three-way partitioning. Rotation and scaling can be described by the *ternary rotation operator* [For89]

$$R' = \begin{bmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix} .$$

A set of coset representatives is

$$\mathcal{A} = [A_2/R'A_2] = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \right\} .$$

Since $R'^2 = -3I$, we have the infinite three-way partitioning chain

$$A_2/R'A_2/3A_2/3R'A_2/9A_2/\cdots$$

On the other hand, scaling the A_2 lattice by a factor of 2 results in the sublattice $2A_2$, which leads to a four-way partition. Here, the coset representatives are

$$\mathcal{A} = [A_2/2A_2] = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1, \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix} \right\} .$$

Repeated partitioning gives the partition chain

$$A_2/2A_2/4A_2/8A_2/16A_2/\cdots$$

Last, we consider a seven-way partition. Here, performing a rotation by $\tan^{-1}(\sqrt{3}/2)$ degree and scaling by $\sqrt{7}$, i.e., applying

$$\boldsymbol{R}'' = \begin{bmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

gives a sublattice. The set of seven coset representatives is

$$\begin{aligned} \mathcal{A} &= [\boldsymbol{A}_2/\boldsymbol{R}''\boldsymbol{A}_2] \\ &= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} \right\}. \end{aligned}$$

The reader is encouraged to sketch the lattices, sublattices, and cosets as an exercise.

C.5 SOME IMPORTANT LATTICES AND THEIR PARAMETERS

Table C.1 collocates the parameters of some important examples for lattices [CS88, For88a, FW89]. These include the integer lattices $\mathbb{Z}^N$, the two-dimensional hexagonal lattice $\boldsymbol{A}_2$, the four-dimensional checkerboard lattice $\boldsymbol{D}_4$ (also called Schläfli lattice), the Gosset lattice $\boldsymbol{E}_8$. In this section, a short description of these lattices is given. For higher-dimensional lattices, such as the Barnes-Wall lattice Λ_{16} or the Leech lattice Λ_{24}, and for an in-depth discussion of important lattices and their properties, the reader is referred to the literature (e.g., [CS88, Chapter 4]).

Table C.1 Parameters of some important lattices.

	$d^2_{\min}$	V	ρ	Δ	τ	ρ_c	G
$\mathbb{Z}$	1	1	$\frac{1}{2}$	1	2	1	$\frac{1}{12}$
$\mathbb{Z}^2$	1	1	$\frac{1}{2}$	$\frac{\pi}{4}$	4	$\frac{1}{\sqrt{2}}$	$\frac{1}{12}$
$\mathbb{Z}^N$	1	1	$\frac{1}{2}$	$\frac{\pi^{N/2}}{2^N (N/2)!}$	$2N$	$\frac{\sqrt{N}}{2}$	$\frac{1}{12}$
$\boldsymbol{A}_2$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\pi}{2\sqrt{3}}$	6	$\frac{1}{\sqrt{3}}$	$\frac{5}{36\sqrt{3}}$
$\boldsymbol{D}_4$	2	2	$\frac{1}{\sqrt{2}}$	$\frac{\pi^2}{16}$	24	1	≈ 0.0766
$\boldsymbol{E}_8$	2	1	$\frac{1}{\sqrt{2}}$	$\frac{\pi^4}{348}$	240	1	≈ 0.0717

Lattice $\mathbb{Z}^N$ The N-dimensional integer (or cubic) lattice ("square lattice" in two dimensions) is defined as

$$\mathbb{Z}^N \triangleq \left\{ \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} \mid \lambda_i \in \mathbb{Z},\ i = 1, 2, \ldots, N \right\}. \tag{C.5.1}$$

The generator matrix reads

$$\boldsymbol{B}_{\mathbb{Z}^N} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

The lattice $\mathbb{Z}$ is the unique lattice in one dimension.

Lattice $\boldsymbol{A}_N$ The family of N-dimensional lattices $\boldsymbol{A}_N$ is favorably described in $N+1$ dimensions as

$$\boldsymbol{A}_N \triangleq \left\{ \sqrt{2}\,[\lambda_0, \ldots, \lambda_N]^{\mathsf{T}} \in \mathbb{Z}^{N+1} \mid \textstyle\sum_{i=0}^{N} \lambda_i = 0 \right\}, \tag{C.5.2}$$

which describes an N-dimensional hyperplane in $N+1$-dimensional space. The normalization ($\sqrt{2}$) is chosen to make $d_{\min}^2 = 1$. This family includes $\boldsymbol{A}_1 = \mathbb{Z}$ and the two-dimensional hexagonal lattice $\boldsymbol{A}_2$ (cf. the above examples), which is the optimum lattice in two dimensions. Here, a generator matrix is given by

$$\boldsymbol{B}_{\boldsymbol{A}_2} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}.$$

Lattice $\boldsymbol{D}_N$ The two-dimensional checkerboard lattice $\boldsymbol{D}_2$ and the Schläfli lattice $\boldsymbol{D}_4$ are representatives for lattices $\boldsymbol{D}_N$, defined as

$$\boldsymbol{D}_N \triangleq \left\{ [\lambda_1, \ldots, \lambda_N]^{\mathsf{T}} \in \mathbb{Z}^N \mid \textstyle\sum_{i=1}^{N} \lambda_i \text{ even} \right\}. \tag{C.5.3}$$

Here, $d_{\min}^2 = 2$ is valid. $\boldsymbol{D}_4$ is the densest lattice in 4 dimensions; it can be described by the generator matrix

$$\boldsymbol{B}_{\boldsymbol{D}_4} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lattice $\boldsymbol{E}_8$ The eight-dimensional lattice $\boldsymbol{E}_8$ is called Gosset lattice, and for $d_{\min}^2 = 2$ it can be defined as follows:

$$\boldsymbol{E}_8 \triangleq \left\{ [\lambda_1, \ldots, \lambda_8]^{\mathsf{T}} \mid \text{all } \lambda_i \in \mathbb{Z}, \text{ or all } \lambda_i \in \mathbb{Z} + 1/2 \text{ and } \textstyle\sum_{i=1}^{8} \lambda_i \text{ even} \right\}. \tag{C.5.4}$$

It can be shown that the Gosset lattice, which is the densest lattice in eight dimensions, can be composed as the union of the lattice $\boldsymbol{D}_8$ and a translated version thereof [CS88]

$$\boldsymbol{E}_8 = \boldsymbol{D}_8 \cup \left(\boldsymbol{D}_8 + [1/2, \ldots, 1/2]^{\mathsf{T}} \right). \tag{C.5.5}$$

Alternatively, the lattice can be defined by its generator matrix

$$B_{E_8} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

Because fast quantization algorithms exist for E_8, this lattice is widely used in lattice quantizers [CS82, CS83].

Some Useful Partition Chains We give some selected lattice partition chains and the corresponding squared minimum distances for reference and without proof.

In one dimension, we have the partition chain and its corresponding squared distances

$$\mathbb{Z}/2\mathbb{Z}/4\mathbb{Z}/8\mathbb{Z}/16\mathbb{Z}/\cdots \qquad 1/4/16/64/256/\cdots$$

For $N = 2$, and $R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, a two-way partition chain for $\mathbb{Z}^2$ exists, where at each step the squared distance increases by a factor of 2:

$$\mathbb{Z}^2/R\mathbb{Z}^2/2\mathbb{Z}^2/2R\mathbb{Z}^2/4\mathbb{Z}^2/\cdots \qquad 1/2/4/8/16/\cdots$$

The hexagonal lattice A_2 has a three-way partition chain based on the rotation operator $R' = \begin{bmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}$. Here, the distances increase by factors of 3:

$$A_2/R'A_2/3A_2/3R'A_2/9A_2/\cdots \qquad 1/3/9/27/81/\cdots$$

Another partition (four-way) is based on scaling A_2 by a factor of 2. Correspondingly, the distances increase by factors of 4

$$A_2/2A_2/4A_2/8A_2/16A_2/\cdots \qquad 1/4/16/64/256/\cdots$$

Last, we regard the seven-way partition induced by the rotation operator $R'' = \begin{bmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$, where the distances increase by factors of 7. Since no multiple of 2π is divisible by the angle of the rotation (which is an irrational number), a scaled version of the initial A_2 lattice never occurs and no partition step is a scaled version of any prior partition.

A standard binary partition chain for coded modulation in 4 dimensions is the following. Using the four-dimensional rotation operator $R_4 = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$, where R is the two-dimensional scaling/rotation matrix, a partition chain is obtained, where the minimum distance does not increase at every partition step [For88a]

$$\mathbb{Z}^4/D_4/R_4\mathbb{Z}^4/R_4D_4/2\mathbb{Z}^4/2D_4/2R_4\mathbb{Z}^4/\cdots \qquad 1/2/2/4/4/8/8/16/\cdots$$

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