CALCULUS 3

TD1-3

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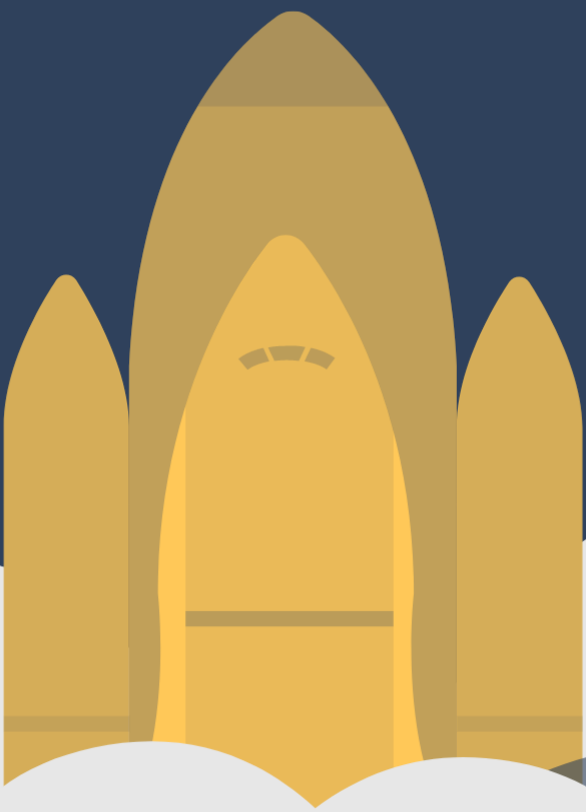

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TD1: Function of Several Variables

I2–TD1
(Function of Several Variables)

1. Sketch the level curves $f(x,y)=c$ and the level surfaces $f(x,y,z)=c$ of the functions for the indicated values of c.

   (a) $f(x,y)=y^2-x^2;\quad c=0,\pm1,\pm2,\pm3$ (b) $f(x,y,z)=4x^2+4y^2-z^2;\quad c=0,1$

2. Find the limit (if it exists) of the following functions.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{1-\cos(xy)}{xy^2}$

   (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{xy^2+yz^2}{x^2+2y^2+3z^2}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{\sinh x^2+\sinh y^2}$

   (e) $\displaystyle\lim_{(x,y)\to(0,1)}\frac{\ln\sqrt{1+\sqrt{x^2+(y-1)^4}}}{\sin\sqrt{x^2+(y-1)^4}}$

   (f) $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{\sin x}{x}(y+1),\frac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}\right)$

3. Let $\alpha\in\mathbb{R}$. Determine the value of α so that the function f has limit at $(0,0)$.

$$f(x,y)=\frac{x^\alpha y}{x^2+y^2}.$$

4. Let $\alpha_1,\alpha_2,\beta_1,\beta_2$ and γ be positive real numbers and $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=\begin{cases}\dfrac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^\gamma}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

   Show that f is continuous at $(0,0)$ if and only if $\dfrac{\alpha_1}{\beta_1}+\dfrac{\alpha_2}{\beta_2}>\gamma$.

5. Let $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\left(x^2+y^2\right)^x, & \text{if } (x,y)\neq(0,0)\\ 1, & \text{if } (0,0)\end{cases}$$

   (a) Is f continuous at $(0,0)$?

   (b) Find the partial derivatives of f at the point different from the origin.

   (c) Do the partial derivatives of f exist at the point $(0,0)$?

6. Show that the function $f:\mathbb{R}^2\to\mathbb{R}$ is in class C^1

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$

8. Suppose that $w = f(u,v)$ is a differentiable function of $u = \dfrac{y-x}{xy}$ and $v = \dfrac{z-x}{xz}$. Show then that:

$$x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} + z^2 \frac{\partial w}{\partial z} = 0.$$

9. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \arcsin\left(\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}\right) \quad \text{and} \quad g(x,y) = \arctan x - \arctan y.$$

(a) Calculate the partial derivative of f and g.

(b) Simplify f.

10. Find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ by using the appropriate Chain Rule.

(a) $f(x,y) = y^2 x + 2x^2$, $x(u,v) = u + v$ and $y(u,v) = 2u - v$.

(b) $f(x,y) = y^2 \ln(1 + y + x^2)$, $x(u,v) = uv$ and $y(u,v) = u^2 + v^2$.

11. Let $g(r,\theta) = f(x = r\cos\theta, y = r\sin\theta)$. Verify that for all $(r,\theta) \in \mathbb{R}_+^* \times \mathbb{R}$:

$$\Delta f(x,y) = \Delta f(r\cos\theta, r\sin\theta) = \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)$$

where $\Delta f(x,y) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$.

12. Let $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and $\alpha \in \mathbb{R}$. We say that f is an homogenous function of degree α if

$$\forall (x,y) \in \mathbb{R}^2,\ \forall \lambda > 0 :\ f(\lambda x, \lambda y) = \lambda^\alpha f(x,y).$$

(a) Show that if f is homogenous of degree α, then its partial derivatives are homogenous of degree $\alpha - 1$.

(b) Show that f is homogenous of degree α if and only if

$$\forall (x,y) \in \mathbb{R}^2 :\ x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$$

(c) Suppose that $f \in \mathcal{C}^2$ and homogenous of degree α. Show that

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y) + 2xy\frac{\partial^2 f}{\partial x \partial y}(x,y) + y^2\frac{\partial^2 f}{\partial y^2}(x,y) = \alpha(\alpha-1)f(x,y)$$

13. Differentiate implicitly to find the first partial derivatives of z

(a) $x^2 + 2yz + z^2 = 1$

(b) $\tan(x+y) + \tan(y+z) = 1$

(c) $z = e^x \sin(y+z)$

(d) $x \ln y + y^2 z + z^2 = 0.$

14. Determine the gradient of the following functions:

(a) $f(x,y) = \arctan \dfrac{x+y}{x-y}$

(b) $f(x,y) = (x+y)\ln(2x-y)$

(c) $f(x,y,z) = \sin(x+y)\cos(y-z)$

(d) $f(x,y,z) = (x+y)^z.$

15. Determine the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\frac{\partial f}{\partial x}(x,y) = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = x^2 + 2y$$

16. Let $f : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ be a function defined by

$$f(x,y) = \int_0^{\frac{\pi}{2}} \ln\left(x^2 \sin^2 t + y^2 \cos^2 t\right) dt.$$

(a) Show that for all $x, y > 0$: $\nabla f(x,y) = \left(\dfrac{\pi}{x+y}, \dfrac{\pi}{x+y}\right)$.

(b) Deduce that for all $x, y > 0$: $f(x,y) = \pi \ln\left(\dfrac{x+y}{2}\right)$.

17. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = \left(\int_0^x e^{-t^2}\, dt\right)^2 \quad \text{and} \quad g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\, dt$$

(a) Show that for all $x \in \mathbb{R}$:

$$f(x) + g(x) = \frac{\pi}{4}$$

(b) Deduce that

$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$$

18. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function in class $\mathcal{C}^1$ and let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$g(x,y,z) = f(x-y, y-z, z-x).$$

Show that

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0.$$

19. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be in class $\mathcal{C}^1$ such that $f(1,0,2) = 0$, $\dfrac{\partial f}{\partial x}(1,0,2) = 1$, $\dfrac{\partial f}{\partial y}(1,0,2) = -2$ and $\dfrac{\partial f}{\partial z}(1,0,2) = 1$. Compute

$$\lim_{t \to 0} \frac{f(e^t, \sin t, 2e^t)}{f(\cos t, t, 2-t)}.$$

20. Determine $f : \mathbb{R} \to \mathbb{R}$ twice differentiable such that for function φ defined by
$$\varphi(x, y) = f\left(\frac{x}{y}\right) \quad \text{satisfies} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

21. Compute the directional derivative along u, at the indicated points:
    (a) $f(x, y) = x\sqrt{y-3} \quad u = (-1, 6) \quad a = (2, 12)$.
    (b) $f(x, y, z) = \dfrac{1}{x+2y-3z}, \quad u = (12, -9, -4) \quad a = (1, 1, -1)$.

22. Then determine the Jacobbien matrix at a given point a.
    (a) $f(x, y, z) = \left(\frac{1}{2}(x^2 - z^2), \sin x \sin y\right), \ D = \mathbb{R}^3, \ a = (1, 1, 0)$.
    (b) $f(x, y) = \left(xy, \frac{1}{2}x^2 + y, \ln(1 + x^2)\right), \ D = \mathbb{R}^2, \ a = (1, 1)$.

23. Suppose a hill is described mathematically by using the model $z = f(x, y) = 300 - 0.01x^2 - 0.005y^2$, where x, y and z are measured in feet. If you are at the point $(50, 100, 225)$ on the hill, in what direction should you aim your toboggan if you want to achieve the quickest descent? What is the maximum rate of decrease of the hight of the hill at this point?

24. Compute the Hessian matrix of f at the given point.
    (a) $f(x, y) = xy^2 + x^2 + y^2 + 1$ at $(0, 0)$.
    (b) $f(x, y, z) = x^3 + x^2z + y^2z + z^3$ at $(1, 0, -1)$

25. Given the symmetric matrix $A = (a_{ij})_n$, a vector $b \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$. We define the map $f : \mathbb{R}^n \to \mathbb{R}$ by
$$f(x) = x.Ax + b.x + c = \sum_{p=1}^{n} x_p \left(\sum_{q=1}^{n} a_{pq}x_q\right) + \sum_{p=1}^{n} b_p x_p + c.$$
Determine the Hessian matrix of f.

(Application of Function of Several Variables)

26. Find the equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.
    (a) $f(x, y) = xy + \ln \sqrt[3]{1 + x^2 + 2y^4}$ at $(1, 0)$.
    (b) $F(x, y, z) = y \ln xz^2 - 2 = 0$ at $(e, 2, 1)$.

27. Show that the tangent plane to the Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) is given by $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1$.

28. Find the global extrema of the function over the region R.
    (a) $f(x, y) = x^2 + y^2 + x + 1, \quad R = \{(x, y) \in \mathbb{R}^2 \mid y \leq \sqrt{9 - x^2}; -3 \leq x \leq 3\}$.
    (b) $f(x, y) = x^2 - xy + y^2 + 3x - 2y + 1, \quad R = \{(x, y) \in [-2, 0] \times [0, 1]\}$.

29. Determine the local extrema and/or saddle points of the following functions.

(a) $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$
(b) $f(x,y) = 2x^3 + (x-y)^2 - 6y$
(c) $f(x,y) = \cos x + \cos y + \sin(x+y)$
(d) $f(x,y,z) = x^2 - xy + z^2 - 2xz + 6z$
(e) $f(x,y,z) = (x^2 + 2y^2 + 1)\cos z$
(f) $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xz + 1$.

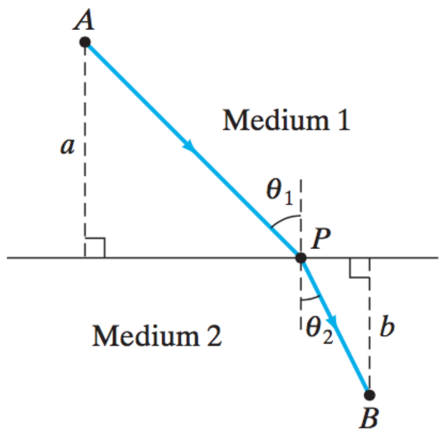
30. A company manufacturers two type of sneakers, running shoes and basketball shoes. The total revenue from x_1 units of running shoes and x_2 units of basketball shoes is $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$, where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximise the revenue.

31. For each of the following functions, use Lagrange Multipliers to find all extrema of f subject to the given constraints.

(a) $f(x,y) = x^2 + y$ constrain : $x^2 + 2y^2 = 1$
(b) $f(x,y,z) = xyz$ contrain: $2x + 3y + z = 6$
(c) $f(x,y,z) = x^2 + y^2 + z^2$, constraint $5x^2 + 9y^2 + 6z^2 + 4yz - 1 = 0$.
(d) $f(x,y,z) = x^2 + y^2 + z^2$, constraint $x^2 + y^2 + 2z^2 - 4 = 0$ and $xyz - 1 = 0$.
(e) $f(x,y,z) = x + y + z$, constrains: $y^2 - x^2 = 1$; and $x + 2z = 1$
(f) $f(x,y,z,w) = 3x + y + w$, constraints $3x^2 + y + 4z^3 = 1$ and $-x^3 + 3z^4 + w = 0$.

32. A cargo container (in the shape of a rectangular solid) must have a volume of $480m^3$. The bottom will cost 50$ per square meter to construct and sides and the top will cost 30$ per square meter to construct. Use Lagrange Multiplier to find the dimensions of the container of this size that has minimum cost.

(a) Figuer 1



(b) Figuer 2

33. Your company must design a storage tank for Super Suds liquid laundry detergent. The customer's specifications call for a cylindrical tank with hemispherical ends (see Figure 1), and the tank is to hold $8000\,m^3$ of detergent. Suppose that it costs twice as much (per square foot of sheet metal used) to machine the hemispherical ends of the tank as it does to make the cylindrical part. What is the minimum cost of manufacturing the tank?

34. A ray of light travels at a constant speed in a uniform medium, but in different media (such as air and water) light travels at different speeds. For example, if a ray of light passes from air to water, it is bent (or refracted) as shown in Figure 2. Suppose the speed of light in

medium 1 is v_1 and in medium 2 is v_2. Then, by Fermat's principle of least time, the light will strike the boundary between medium 1 and medium 2 at a point P so that the total time the light travels is minimized.

(a) Determine the total time the light travels in going from point A to point B via point P as shown in Figure 2.

(b) Use the method of Lagrange multipliers to establish **Snell's law of refraction:** that the total travel time is minimized when

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

(Hint: The horizontal and vertical separations of A and B are constant.)

35. Find the largest sphere centered at the origin that can be inscribed in the ellipsoid

$$3x^2 + 2y^2 + z^2 = 6.$$

36. Use Lagrange multiplier to show that the distance from a point (x_0, y_0) to the line $ax+by = c$ is defined by

$$D = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

37. Find the distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 - 225 = 0$.

38. Find the distance from between the ellipsoid (E) and plane (P), where

$$(E):\ 2x^2 + y^2 + 2z^2 - 8 = 0 \quad \text{and} \quad (P):\ x + y + z - 10 = 0.$$

39. Find the vertices of the ellipsoide $4x^2 + 9y^2 + 6z^2 + 4yz - 4 = 0$

40. Find the point closest to the point $(2, 5, -1)$ and on the line of intersection of the planes $x - 2y + 3z = 8$ and $2z - y = 3$.

41. Find the nearest and farthest distance from the origin to the conic section obtained by intersecting of the cone $z^2 = x^2 + y^2$ and the plane $x + y - z + 2 = 0$

42. (a) Find the maximum of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} x_k^2$$

subject to the constraint $\displaystyle\sum_{k=1}^{n} x_k^2 = 1$.

(b) Deduce that for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$: $\displaystyle\left|\prod_{k=1}^{n} x_k\right| \le \left(\frac{\|x\|}{\sqrt{n}}\right)^n$

TD1
(Function of Several Variables)

2. Find the limit (if it exists) of the following functions

(a). $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} f(x,y)$

+ choose path : x=0

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

+ choose path x=y

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(y,y) = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}$$

Thus $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exists.

(b). $\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy^2}$

$$\frac{1-\cos(xy)}{xy^2} \sim \frac{\frac{x^2y^2}{2}}{xy^2} = \frac{x}{2} \to 0$$

Thus $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$

(c). $\lim_{(x,y)\to(0,0)} \frac{xy^2+yz^2}{x^2+2y^2+3z^2} = \lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$

The limit exists and equals zero

$$|f(x,y,z) - 0| = \left|\frac{xy^2+yz^2}{x^2+2y^2+3z^2}\right| = \frac{\left|(x^2)^{\frac{1}{2}}y^2 + (y^2)^{\frac{1}{2}}z^2\right|}{x^2+2y^2+3z^2}$$

$$\le \frac{(x^2+2y^2+3z^2)^{\frac{1}{2}}y^2 + (x^2+2y^2+3z^2)^{\frac{1}{2}}z^2}{x^2+2y^2+3z^2}$$

$$\le \frac{(x^2+2y^2+3z^2)^{\frac{1}{2}}(x^2+2y^2+3z^2)}{x^2+2y^2+3z^2} = x^2+2y^2+3z^2 \to 0$$

Thus $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2+yz^2}{x^2+2y^2+3z^2} = 0$

(d). $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{sinhx^2+sinhy^2}$

$$\frac{xy}{sinhx^2+sinhy^2} \sim \frac{xy}{x^2+y^2} \quad => The\ limit\ does\ not\ exist\ (from\ (a))$$

Thus $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{sinhx^2+sinhy^2}$ is not exist .

(f). $\lim\limits_{(x,y)\to(0,0)} \left(\frac{sinx}{x}(y+1), \frac{sinx^2+siny^2}{\sqrt{x^2+y^2}}\right)$

$$\frac{sinx}{x}(y+1) \to 1$$

$$\frac{sinx^2+siny^2}{\sqrt{x^2+y^2}} \sim \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{sinx}{x}(y+1), \frac{sinx^2+siny^2}{\sqrt{x^2+y^2}}\right) = (1,0)$$

4. f is continuous at (0,0) <=> $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > r$ that $\alpha_1, \alpha_2, \beta_1, \beta_2\ and\ r > 0$

$$f_{(x,y)} = \begin{cases} \frac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r} & , if\ (x,y) \neq (0,0) \\ 0 & , if\ (x,y) = (0,0) \end{cases}$$

+ suppose $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > r$, we will show that f is continuous at (0,0)

That is , $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$$|f(x,y) - 0| = \left|\frac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}\right| = \frac{(|x|^{\beta_1})^{\frac{\alpha_1}{\beta_1}}(|y|^{\beta_2})^{\frac{\alpha_2}{\beta_2}}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}$$

$$\leq \frac{(|x|^{\beta_1}+|y|^{\beta_2})^{\frac{\alpha_1}{\beta_1}}(|x|^{\beta_1}+|y|^{\beta_2})^{\frac{\alpha_2}{\beta_2}}}{(|x|^{\beta_1}+|y|^{\beta_2})^r} = (|x|^{\beta_1}+|y|^{\beta_2})^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-r} \to 0$$

we get $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} - r > 0, or\ \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > r$

Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. So , f is continuous at (0,0)

+ conversely, suppose that f is continuous at (0,0) . That is $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

choose the path : $x = t^{\frac{1}{\beta_1}}$ and $y = t^{\frac{1}{\beta_2}}$
when (x,y) → $(0,0)$ $=> t \to 0$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 => \lim_{(x,y)\to(0,0)} f(t^{\frac{1}{\beta_1}}, t^{\frac{1}{\beta_2}}) = 0$$

$$<=> \lim_{(x,y)\to(0,0)} \frac{|t|^{\frac{1}{\beta_1}}|t|^{\frac{1}{\beta_2}}}{(|t|+|t|)^r} = 0$$

$$<=> \frac{1}{(2)^r} \lim_{(t)\to(0)} |t|^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-r} = 0$$

we get $\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} - r > 0, or\ \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > r$
Thus f is continuous at (0,0) if and only if

6. Show that the function f: $\mathbb{R}^2 \to \mathbb{R}$ is in class C^1

$$f_{(x,y)} = \begin{cases} \frac{x^2y^3}{x^2+y^2} & ,if\ (x,y) \neq (0,0) \\ 0 & ,if\ (x,y) = (0,0) \end{cases}$$

- For $(x,y) \neq (0,0)$, f is continuous
- For $(x,y) = (0,0)$

$$\left|f_{(x,y)} - f_{(0,0)}\right| = \left|\frac{x^2y^3}{x^2+y^2} - 0\right| \leq \frac{(x^2+y^2)|y^3|}{x^2+y^2} = |y^3| \to 0$$

$$So, \lim_{(x,y)\to(0,0)} f_{(x,y)} = f_{(0,0)} \Rightarrow f\ is\ continuous\ at\ (0,0)$$

- For $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy^3(x^2+y^2) - 2x(x^2y^3)}{(x^2+y^2)^2} = \frac{2xy^5}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3x^2y^2(x^2+y^2) - 2y(x^2y^3)}{(x^2+y^2)^2} = \frac{3x^4y^2+x^2y^4}{(x^2+y^2)^2}$$

- For $(x,y) = (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{x\to 0} \frac{f_{(h,0)} - f_{(0,0)}}{h} = \lim_{x\to 0} \frac{\frac{h^2(0)^2}{h^2-0^2} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{x\to 0} \frac{f_{(0,h)} - f_{(0,0)}}{h} = \lim_{x\to 0} \frac{\frac{0^2(h)^2}{0^2-h^2} - 0}{h} = 0$$

Thus $\frac{\partial f}{\partial x}\ and\ \frac{\partial f}{\partial y}$ exist

+ $\frac{\partial f}{\partial x}\ and\ \frac{\partial f}{\partial y}$ are continuous?

- For $(x,y) \neq (0,0) \Rightarrow \frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous*
- For $(x,y) = (0,0)$

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| = \left|\frac{2xy^5}{(x^2+\ y^2)^2} - 0\right| \leq\ 2\frac{|x|(x^2+y^2)^{\frac{5}{2}}}{(x^2+y^2)^2} = 2|x|(x^2+y^2)^{\frac{1}{2}} \to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(0,0) \Rightarrow \frac{\partial f}{\partial x} \text{ is continous at } (0,0)$$

Similarly ,$\frac{\partial f}{\partial x}$ *is also continous at* $(0,0)$

$$\Rightarrow \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ are continuous on } \mathbb{R}^2$$

Thus f is C^1

8. Suppose that w = f(u,v) is a differentiable function of $u = \frac{y-x}{xy}$ and $v = \frac{z-x}{xz}$. show then that:

$$x^2\frac{\partial \omega}{\partial x} + y^2\frac{\partial \omega}{\partial y} +\ z^2\frac{\partial \omega}{\partial z} = 0$$

By using Chain rule's theorem

- $\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial u} \times \frac{\partial u}{\partial x} +\ \frac{\partial \omega}{\partial v} \times \frac{\partial v}{\partial x}$

$$= \frac{\partial \omega}{\partial u} \times \frac{-xy(y-x)}{x^2y^2} + \frac{\partial \omega}{\partial v} \times \frac{-xz(z-x)}{x^2z^2}$$

$$= \frac{\partial \omega}{\partial u}\left(-\frac{1}{x^2}\right) + \frac{\partial \omega}{\partial v}\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow x^2\frac{\partial \omega}{\partial x} = -\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial v} \qquad (1)$$

- $\frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial u} \times \frac{\partial u}{\partial y} +\ \frac{\partial \omega}{\partial v} \times \frac{\partial v}{\partial y}$

$$= \frac{\partial \omega}{\partial u} \times \frac{xy - x(y-x)}{x^2y^2} + \frac{\partial \omega}{\partial v} \times 0$$

$$= \frac{\partial \omega}{\partial u}\left(\frac{1}{y^2}\right)$$

$$\Rightarrow y^2\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial u} \qquad (2)$$

• $\frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial u} \times \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial v} \times \frac{\partial v}{\partial z}$

$$= \frac{\partial \omega}{\partial u} \times 0 + \frac{\partial \omega}{\partial v} \times \frac{xz - x(z-x)}{x^2 z^2}$$
$$= \frac{\partial \omega}{\partial z}\left(\frac{1}{z^2}\right)$$

$$\Rightarrow z^2 \frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial v} \qquad (3)$$

From (1) , (2) , (3) : $x^2 \frac{\partial \omega}{\partial x} + y^2 \frac{\partial \omega}{\partial x} + z^2 \frac{\partial \omega}{\partial z} = -\frac{\partial \omega}{\partial u} - \frac{\partial \omega}{\partial v} + \frac{\partial \omega}{\partial u} + \frac{\partial \omega}{\partial v} = 0$

Therefore $x^2 \frac{\partial \omega}{\partial x} + y^2 \frac{\partial \omega}{\partial y} + z^2 \frac{\partial \omega}{\partial z} = 0$

10. Find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ by using the appropriate Chain Rule

(a) , $f_{(x,y)} = y^2 x + 2x^2$, $x_{(u,v)} = u + v$ and $y_{(u,v)} = 2u - v$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}$$
$$= (y^2 + 4x)1 + 2xy \times 2$$
$$= y^2 + 4xy + 4x$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial v}$$
$$= (y^2 + 4x)1 + 2xy \times (-1)$$
$$= y^2 - 2xy + 4x$$

Therefore $\frac{\partial f}{\partial v} = y^2 + 4xy + 4x$

$$\frac{\partial f}{\partial v} = y^2 - 2xy + 4x$$

(b), $f_{(x,y)} = y^2 \ln(1 + y + x^2)$, $x_{(u,v)} = uv$ and $y_{(u,v)} = u^2 - v^2$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}$$
$$= y^2 \frac{2x}{1 + y + x^2} \times v + \left[2y \ln(1 + y + x^2) + \frac{y^2}{1 + y + x^2}\right] 2u$$
$$\Rightarrow \frac{\partial f}{\partial u} = \frac{y^2}{1 + y + x^2}[2xv + 2u] + 4yu \ln(1 + y + x^2)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= y^2 \frac{2x}{1+y+x^2} \times u + \left[2y\ln(1+y+x^2) + \frac{y^2}{1+y+x^2}\right] 2v$$
$$= \frac{y^2}{1+y+x^2}[2xu + 2v] + 4yv\ln(1+y+x^2)$$

Thus $\frac{\partial f}{\partial u} = \frac{y^2}{1+y+x^2}[2xv + 2u] + 4yu\ln(1+y+x^2)$
$$\frac{\partial f}{\partial v} = \frac{y^2}{1+y+x^2}[2xu + 2v] + 4yv\ln(1+y+x^2)$$

12. f *is homogenous of degree* α*,if*

$\forall (x, y) \in \mathbb{R}^2, \forall \lambda > 0 : f(\lambda x, \lambda y) = \lambda^\alpha f(x, y)$
(a). Show that if f is homogenous of degree α, then its partial derivatives are homogenous
of degree $\alpha - 1$

f *is homogenous of degree* α*,if* $f(\lambda x,\lambda y)= \lambda^\alpha f(x,y)$
$\lambda\frac{\partial f}{\partial x}(\lambda x,\lambda y)= \lambda^\alpha \partial f\ \partial y(x,y)$
$\frac{\partial f}{\partial x}(\lambda x,\lambda y)= \lambda^{\alpha-1} \partial f\ \partial y(x,y)$

Then $\frac{\partial f}{\partial x}$ *is homogenous of degree* $\alpha-1$

Similarly,we have $\frac{\partial f}{\partial x}$ *is also an homogenous of degree* $\alpha-1$

Thus: f *is homogenous of degree* α*,then its partial derivatives are homogenous of degree* $\alpha-1$

(b) Show that f *is homogenous of degree* α *if and only if*

$\forall (x,y)\in \mathbb{R}^2 : x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)\neq\alpha f(x,y)$

Suppose that f *is homogenous of degree* $\forall (x,y)\in \mathbb{R}^2, \forall\lambda>0$
$f(\lambda x,\lambda y)= \lambda^\alpha f(x,y)$
Then differentiate both side with respect to λ*,we get*

$(*)\ x\frac{f(\lambda x,\lambda y)}{\partial x}+yf(\lambda x,\lambda y)\frac{f(\lambda x,\lambda y)}{\partial y}=\alpha\lambda^{\alpha-1}f(x,y)$

we have from (a) $\frac{\partial f}{\partial x}$& $\frac{\partial f}{\partial y}$ *are homogenous of degree* α

Since$(*)$ *we can written to*:

$x\lambda^{\alpha-1}\frac{f(\lambda x,\lambda y)}{\partial x}+y\lambda^{\alpha-1}f(\lambda x,\lambda y)\frac{f(\lambda x,\lambda y)}{\partial y}=\alpha\lambda^{\alpha-1}f(x,y)$

$\frac{xf(x,y)}{\partial x}+\frac{yf(x,y)}{\partial y}=\alpha f(x,y)$ (True)

+ *Comersly,suppose that*

$$\frac{xf(x,y)}{\partial x}+\frac{yf(x,y)}{\partial y}=\alpha f(x,y) \qquad (1)$$

concept:*we need to show that* f *is homogenous of degree* α

f *is homogenous of degree* α

change $x\to\lambda x$,$y\to\lambda y$ *we get*

$$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=\alpha f(\lambda x,\lambda y)$$

consider function $\varphi(\lambda)=f(x\lambda,y\lambda)$

$$=>\ \varphi'(\lambda)=\frac{x\partial f(\lambda x,\lambda y)}{\partial x}+\frac{y\partial f(\lambda x,\lambda y)}{\partial y} \qquad (2)$$

following (1) and (2)

$$\varphi'(\lambda)=\frac{a}{\lambda}\varphi(\lambda)\quad <=>\frac{\varphi'(\lambda)}{\varphi(\lambda)}=\frac{a}{\lambda}$$

$$=>\ \int\frac{\varphi'(\lambda)}{\varphi(\lambda)}d\lambda=\int\frac{a}{\lambda}d\lambda$$

$$=>\ \ln\varphi(\lambda)=\alpha\ln\lambda+k$$

$$=>\ \quad =\alpha\ln\lambda+\ln c$$

$$=>\ \quad =\ln C\lambda^{\alpha}$$

λ=1=> $\varphi(1)=C=f(x,y)$

we get $\varphi(\lambda)=f(x\lambda,y\lambda)=\lambda^{\alpha}f(x,y)$ (*true*)

Therefore f *is homogenous of degree* α *if and only if* $\forall(x,y)\in\mathbb{R}^2$:

$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=\alpha f(x,y)$

(c) *Suppose that* $f\in C^2$.

Show that we have by (b):$\forall(x,y)\in\mathbb{R}^2$:

$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=\alpha f(x,y)$

Differentiate both side with respect to x *and* y,*we get*:

$$\frac{\partial f}{\partial x}+x\,\frac{\partial^2 f}{\partial^2 x}+y\,\frac{\partial^2 f}{\partial x\partial y}=a\frac{\partial f}{\partial x}\ \times(x)\ (*)$$

$$\frac{\partial f}{\partial x}+y\,\frac{\partial^2 f}{\partial^2 y}+x\,\frac{\partial^2 f}{\partial x\partial y}=a\frac{\partial f}{\partial x}\ \times(y)\ (**)$$

take (*)+(**) we get

$$x^2\,\frac{\partial^2 f}{\partial^2 x}+2\,\frac{\partial^2 f}{\partial x\partial y}+y^2\,\frac{\partial^2 f}{\partial^2 x}=(a-1)\left[x\frac{\partial f}{\partial x}(\mathrm{x,y})+y\frac{\partial f}{\partial y}(\mathrm{x,y})\right]$$
$$=(a-1)af(\mathrm{x,y})$$

Thus $x^2\,\frac{\partial^2 f}{\partial^2 x}+2\,\frac{\partial^2 f}{\partial x\partial y}+y^2\,\frac{\partial^2 f}{\partial^2 x}=(a-1)af(\mathrm{x,y})$

14. Determine the gradient of the following functions :

(a). $f(\mathrm{x,y})=arctan\frac{x+y}{x-y}$

let $z=f(\mathrm{x,y})=arctan\frac{x+y}{x-y}$

$\tan z=\dfrac{x+y}{x-y}$

$(x-y)\tan z\ =x+y$

$<=>F(x,y,z)=(x-y)\tan z\ -x-y$

$$\frac{\partial f}{\partial x}=\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\ =-\frac{\tan z-1}{(x-y)(1+\tan^2 z)}\ =\frac{-2y}{(\mathrm{x-y})^2+(\mathrm{x+y})^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\ =-\frac{-\tan z-1}{(x-y)(1+\tan^2 z)}\ =\frac{2x}{(\mathrm{x-y})^2+(\mathrm{x+y})^2}$$

Thus , $\nabla f\ =\left(\frac{-2y}{(\mathrm{x-y})^2+(\mathrm{x+y})^2}\,,\frac{2x}{(\mathrm{x-y})^2+(\mathrm{x+y})^2}\right)$

(b). $f(\mathrm{x,y})=(x+y)\ln(2x-y)$

$$\frac{\partial f}{\partial x}=\ln(2x-y)+\frac{2(x+y)}{(2x-y)}\ =\frac{(2x-y)\ln(2x-y)+2(x+y)}{(2x-y)}$$

$$\frac{\partial f}{\partial y}=\ln(2x-y)-\frac{(x+y)}{(2x-y)}=\frac{(2x-y)\ln(2x-y)-(x+y)}{(2x-y)}$$

Thus $\nabla f\ =\left(\frac{(2x-y)\ln(2x-y)+2(x+y)}{(2x-y)}\,,\frac{(2x-y)\ln(2x-y)-(x+y)}{(2x-y)}\right)$

(c). $f(\mathrm{x,y,z}) = \sin(x+y)\cos(y-z)$
$$\frac{\partial f}{\partial x} = \cos(x+y)\cos(y-z)$$
$$\frac{\partial f}{\partial y} = \cos(x+y)\cos(y-z) + (-\sin(y-z)\sin(x+y))$$
$$= \cos(x+2y-z)$$
$$\frac{\partial f}{\partial z} = \sin(x+y)\sin(y-z)$$

Thus , $\nabla f = (\cos(x+y)\cos(y-z)\ ,\cos(x+2y-z)\,,\sin(x+y)\sin(y-z))$

(d). $f(\mathrm{x,y,z}) = (x+y)^z = e^{\mathrm{z}\ln(x+y)}$
$$\frac{\partial f}{\partial x} = z(x+y)^{z-1}$$
$$\frac{\partial f}{\partial y} = z(x+y)^{z-1}$$
$$\frac{\partial f}{\partial z} = \ln(x+y)\,e^{\mathrm{z}\ln(x+y)} = (x+y)^z\ln(x+y)$$

Thus , $\nabla f = (z(x+y)^{z-1}\,, z(x+y)^{z-1}, (x+y)^z\ln(x+y))$

16. $f(\mathrm{x,y}) = \int_0^{\frac{\pi}{2}} \ln(x^2\sin^2 t + y^2\cos^2 t)\,dt$

(a). show that for all x,y >0 , $\nabla f(x,y) = \left(\frac{\pi}{x+y}, \frac{\pi}{x+y}\right)$

Differentiate f with respect to x and y , we get
$$\frac{\partial f}{\partial x}(x,y) = \int_0^{\frac{\pi}{2}} \frac{2x\sin^2 t}{x^2\sin^2 t + y^2\cos^2 t}\,dt \quad (1)$$

$$\frac{\partial f}{\partial y}(x,y) = \int_0^{\frac{\pi}{2}} \frac{2y\cos^2 t}{x^2\sin^2 t + y^2\cos^2 t}\,dt \quad (2)$$

By multiplying (1) by x and (2) by y , then ass them side , we get
$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = 2\int_0^{\frac{\pi}{2}} 1\,dt = \pi \quad (3)$$

Now,

By multiplying (1) by y and (2) by x , then ass them side , we get
$$y\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) = \int_0^{\frac{\pi}{2}} \frac{2yx}{x^2\sin^2 t + y^2\cos^2 t}\,dt = I \ (4)$$

we compute I now

$$I = \int_0^{\frac{\pi}{2}} \frac{2yx}{x^2 \sin^2 t + y^2 \cos^2 t} dt$$

$$I = \int_0^{\frac{\pi}{2}} \frac{2yx}{x^2 \tan^2 t + y^2} \times \frac{1}{\cos^2 t} dt$$

let u = tan t = > du $=\frac{1}{\cos^2 t}$ dt

$$I = \int_0^{+\infty} \frac{2yx}{x^2u^2 + y^2} du$$

$$I = \int_0^{+\infty} \frac{2}{(\frac{x}{y}u)^2 + 1} \times \frac{x}{y} du$$

let b $=\frac{x}{y}u$ = > db $=\frac{x}{y}du$

$$I = \int_0^{+\infty} \frac{2}{(b)^2 + 1} db$$

$$= \pi$$

From (3) and (4)

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \pi \quad (5)$$

$$y\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) = \pi \quad (6)$$

from (5) and (6)

we get

$\frac{\partial f}{\partial x}(x,y) = \frac{\pi}{x+y}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{\pi}{x+y}$

Thus $\nabla f(x,y) = \left(\frac{\pi}{x+y}, \frac{\pi}{x+y}\right)$

(b). Deduce that for all x,y > 0 : f(x,y) = $\pi \ln \frac{x+y}{2}$

From (a), we have

$\frac{\partial f}{\partial x}(x,y) = \frac{\pi}{x+y}$ (7)

$\frac{\partial f}{\partial y}(x,y) = \frac{\pi}{x+y}$ (8)

Integral (7), with respect to x , we have

f(x,y) $=\pi \ln(x+y) + c(y)$ (9)

$$=> \frac{\partial f}{\partial y}(x,y) = \frac{\pi}{x+y} + c'(y) = \frac{\pi}{x+y}, (from\ eq\ 8)$$

$$=> c'(y) = 0 => c(y) = k$$

So, f(x,y) = $\pi \ln(x+y) + k$ (10)

From the original eq , we have f(1,1)=0

and eq (10), we also have f(1,1)= $\pi \ln 2 \ + k$
Then , $\pi \ln 2 \ + k = 0 => k = -\pi \ln 2$

Thus , $f(x,y) = \pi \ln(x+y) - \pi \ln 2 \ = \pi \ln\left(\frac{x+y}{2}\right)$

18. $g\ (x,y,z) = f(x-y, y-z, z-x)$
show that $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0$

Let u = x-y , v= y-z , w =z-x
$=> g\ (x,y,z) = f(u,v,w)$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$
$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$
$$\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial v}.\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

we add both sides , we obtain
$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0$$

20. Determine f : $\mathbb{R}^2 \rightarrow \mathbb{R}$ twice differentiable such that for function φ defined by $\varphi(x,y) = f(\frac{x}{y})$ satisfies $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$

We have
$$\frac{\partial\varphi}{\partial x}\ (x,y) = \frac{1}{y}f'(\frac{x}{y})$$
$$=> \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{y^2}f''(\frac{x}{y})$$
and

$$\frac{\partial\varphi}{\partial y}\ (x,y) = \frac{xy}{y^2}f'(\frac{x}{y})$$
$$=> \frac{\partial^2\varphi}{\partial y^2} = \frac{2x}{y^3}f'\left(\frac{x}{y}\right) + \frac{x^2}{y^3}f''\left(\frac{x}{y}\right)$$
$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$
$$\frac{1}{y^2}f''\left(\frac{x}{y}\right) + \frac{2x}{y^3}f'\left(\frac{x}{y}\right) + \frac{x^2}{y^3}f''\left(\frac{x}{y}\right) = 0$$

$$f''\left(\frac{x}{y}\right) + 2\frac{x}{y}f'\left(\frac{x}{y}\right) + \frac{x^2}{y^2}f''\left(\frac{x}{y}\right) = 0$$
$$2\frac{x}{y}f'\left(\frac{x}{y}\right) + \left(1 + \frac{x^2}{y^2}\right)f''\left(\frac{x}{y}\right) = 0$$
let t = $\frac{x}{y}$, we have
$$2tf'(t) + (1+t)f''(t) = 0$$
$$((1+t^2)f'(t))' = 0$$
$$(1+t^2)f'(t) = c$$
$$f'(t) = \frac{c}{1+t^2}$$
$$f(t) = c\arctan(t) + k$$

Thus $f(t) = c\arctan(t) + k$, $(c, k \in \mathbb{R})$

22. Determine the Jacobbien matrix at a given point a

(a). $f(x,y,z) = \left(\frac{1}{2}(x^2 - z^2), \sin x \sin y\right)$,$D = \mathbb{R}^3$, a = (1,1,0)

$\frac{\partial f}{\partial x}(x,y,z) = (x, \sin x \sin y)$ $\qquad$ $\frac{\partial f_1}{\partial x}(a) = 1$, $\frac{\partial f_2}{\partial x}(a) = \sin 1 \cos 1$

$\frac{\partial f}{\partial y}(x,y,z) = (0, \sin y \cos x)$ $\qquad$ => $\qquad$ $\frac{\partial f_1}{\partial y}(a) = 0$, $\frac{\partial f_2}{\partial y}(a) = \sin 1 \cos 1$

$\frac{\partial f}{\partial z}(x,y,z) = (-z, 0)$ $\qquad$ $\frac{\partial f_2}{\partial z}(a) = 0$, $\frac{\partial f_2}{\partial z}(a) = 0$

Then the Jacobbien matrix at a point a is given by

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a) & \frac{\partial f_1}{\partial y}(a) & \frac{\partial f_2}{\partial z}(a) \\ \frac{\partial f_2}{\partial x}(a) & \frac{\partial f_2}{\partial y}(a) & \frac{\partial f_2}{\partial z}(a) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sin 1 \cos 1 & \sin 1 \cos 1 & 0 \end{pmatrix}$$

(b). $f(x,y) = (xy, \frac{1}{2}x^2 + y, \ln(1+x^2))$ $D = \mathbb{R}^3$, a = (1,1)

$\frac{\partial f}{\partial x}(x,y) = (y, x, \frac{2x}{1+x^2})$ $\qquad$ $\frac{\partial f_1}{\partial x}(a) = 1$, $\frac{\partial f_2}{\partial x}(a) = 1$, $\frac{\partial f_3}{\partial x}(a) = 1$

$\frac{\partial f}{\partial y}(x,y) = (x, 1, 0)$ $\qquad$ $\frac{\partial f_1}{\partial y}(a) = 1$, $\frac{\partial f_2}{\partial y}(a) = 1$, $\frac{\partial f_3}{\partial y}(a) = 0$

Then the Jacobbien matrix at a point a is given by

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a) & \frac{\partial f_1}{\partial y}(a) \\ \frac{\partial f_2}{\partial x}(a) & \frac{\partial f_2}{\partial y}(a) \\ \frac{\partial f_3}{\partial x}(a) & \frac{\partial f_3}{\partial y}(a) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

24. Compute the Hessian matrix of f at the given point.

(a). $f(x,y) = xy^2 + x^2 + y^2 + 1$ at point (0,0)
we have

$$\frac{\partial f}{\partial x}(x,y) = y^2 + 2x$$

$$\frac{\partial f}{\partial y}(x,y) = 2xy + 2y$$

Then

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 => \frac{\partial^2 f}{\partial x^2}(0,0) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 2y => \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2y => \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2x + 2 => \frac{\partial^2 f}{\partial y^2}(0,0) = 2$$

Thus the Hessian matrix is

$$Hf(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(b). $f(x,y,z) = x^3 + x^2 z + yz^2 + z^3$ at (1,0,-1)
we have

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 + 2xz$$

$$\frac{\partial f}{\partial y}(x,y,z) = 2yz$$

$$\frac{\partial f}{\partial z}(x,y,z) = x^2 + y^2 + 3z^2$$

Thus the Hessian matrix is

$$Hf(a)=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(1,0,-1) & \frac{\partial^2 f}{\partial y\partial x}(1,0,-1) & \frac{\partial^2 f}{\partial z\partial x}(1,0,-1) \\ \frac{\partial^2 f}{\partial x\partial y}(1,0,-1) & \frac{\partial^2 f}{\partial y^2}(1,0,-1) & \frac{\partial^2 f}{\partial z\partial y}(1,0,-1) \\ \frac{\partial^2 f}{\partial x\partial z}(1,0,-1) & \frac{\partial^2 f}{\partial y\partial z}(1,0,-1) & \frac{\partial^2 f}{\partial z^2}(1,0,-1) \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -6 \end{pmatrix}$$

26. Find the equation of the tangent plane and _nd symmetric equations of the normal line to
the surface at the given point.

(a). $f(x,y)=xy+\ln\sqrt[3]{1+x^2+2y^4}$ at (1,0)
$f(x,y)=xy+\ln\sqrt[3]{1+x^2+2y^4}$
we have
$\frac{\partial f}{\partial x}(x,y)=y+\frac{1}{3}\frac{2x}{1+x^2+2y^4}=>\frac{\partial f}{\partial x}(1,0)=\frac{1}{3}$
$\frac{\partial f}{\partial y}(x,y)=x+\frac{1}{3}\frac{8y^3}{1+x^2+2y^4}=>\frac{\partial f}{\partial x}(1,0)=1$

Note that $z_0=f(x_0,y_0)=f(1,0)=\frac{1}{3}\ln 2$
The tangent plane to the surface at the point $\left(1,0,\frac{1}{3}\ln 2\right)$
$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)-(z-z_0)=0$
$\frac{1}{3}(x-1)+(y-0)-\left(z-\frac{1}{3}\ln 2\right)=0$
$x+3y-3z+\ln 2-1=0$

Thus tangent plane is
$x+3y-3z+\ln 2-1=0$

(b). $F(x,y,z)=y\ln xz^2-2=0$ at (e,2,1)
we have
$\frac{\partial F}{\partial x}(x,y,z)=\frac{y}{x}=>\frac{\partial F}{\partial x}(\mathrm{e},2,1)=\frac{2}{e}$
$\frac{\partial F}{\partial y}(x,y,z)=\ln xz^2=>\frac{\partial f}{\partial x}(\mathrm{e},2,1)=1$
$\frac{\partial F}{\partial z}(x,y,z)=y\frac{2xz}{xz^2}=>\frac{\partial f}{\partial x}(\mathrm{e},2,1)=4$
The tangent plane to the surface at the point (e,2,1)

$$\frac{\partial F}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial F}{\partial y}(x_0,y_0,z_0)(y-y_0)-\frac{\partial F}{\partial z}(x_0,y_0,z_0)(z-z_0)=0$$

$$\frac{2}{e}(x-e)+(y-2)+4(z-1)=0$$

$$\frac{2}{e}x+y+4z-8=0$$

Thus tangent plane is $\frac{2}{e}x+y+4z-8=0$

28. Find the global extrema of the function over the region R.

(a). $f(x,y)=\ x^2+y^2+x+1\ ,\ \ R=\{(x,y)\in\mathbb{R}^2|y\le\ \sqrt{9-x^2}\ ;\ -3\le x\le 3\ \}$

$0\le x^2+y^2\le 9$

+ For $(x,y)\in x^2+y^2=9$

we use Lagrange Multiplier Method

$$\begin{cases}\nabla \mathrm{f}=\gamma\nabla \mathrm{g}\\ \mathrm{g(x,y)}=0\end{cases}\quad \text{where}\begin{cases}\mathrm{f(x,y)}=x^2+y^2+x+1\\ \mathrm{g(x,y)}=x^2+y^2-9\end{cases}$$

⇨ $(2x+1\ ,2y)=\gamma(2\mathrm{x},2\mathrm{y})\ ,x^2+y^2=9$

$$⇨\begin{cases}2x+1=2\gamma \mathrm{x}\\ 2\mathrm{y}\quad =2\gamma\mathrm{y}\\ x^2+y^2=9\end{cases} => \begin{cases}2x(\gamma+1)=1 & (1)\\ 2y(1-\gamma)=0 & (2)\\ x^2+y^2=9 & (3)\end{cases}$$

from (2) : y =0 or $\gamma=1$

if y=0 , then substitute in to eq (3) => $x=\pm 3$

then eq (1) we get $\gamma=\frac{7}{6}\ and\ \gamma=\frac{5}{6}$

So, the critical point are

$a_1=\left(3,0,\frac{7}{6}\right),a_2=\left(-3,0,\frac{5}{6}\right)$

$$=>\begin{cases}f(3,0)=13\\ f(-3,0)=7\end{cases}$$

If $\gamma=1$, Then substitute into eq. (1) it is impossible

$$\nabla \mathrm{f}=0\ => \begin{cases}2\mathrm{x}+1=0\\ 2\mathrm{y}=0\end{cases} => \begin{cases}x=-\frac{1}{2}\\ y=0\end{cases}$$

So , the critical point off is $(-\frac{1}{2},0)$

⇨ $f\left(-\frac{1}{2},0\right)=\frac{3}{4}$

Thus max $\mathrm{f(x,y)}=13$ at the point (3,0)

$\min f(x, y) = \frac{3}{4}$ at the point $(-\frac{1}{2}, 0)$

30. Find x_1 and x_2 so as to maximise revenue
We have
$R\ (x_1 , x_2) = -5{x_1}^2 - 8{x_2}^2 - 2x_1\, x_2 + 42x_1 + 102\, x_2$

⇨ $\frac{\partial R}{\partial x_1} = -10x_1 - 2x_2 + 42$ (1)
⇨ $\frac{\partial R}{\partial x_2} = -16x_2 - 2x_1 + 102$ (2)
$\nabla R = 0$

$$\Rightarrow \begin{cases} -10x_1 - 2x_2 + 42 = 0 \ (3) \\ -16x_2 - 2x_1 + 102 = 0 \ (4) \end{cases}$$

from (3) and (4)
we got $x_1 = 3$, $x_2 = 6$

Thus $x_1 = 3$, $x_2 = 6$

32. Use lagrange Multiplier to find the dimesions of the continer of this size that has minimum cost.
let x,y,z be the dimension of the contine with x , y ,z > 0
then , The optimal function is the cost of the continer
$f(x, y, z) = 50yz + 30yz + 2(30xy) + 2(30xz)$
$= 80yz + 60xy + 60xz$
The volume of the continer
let $g(x, y, z) = x.y.z = 480$
+ use lagrange Multiplier
we have$L(\gamma, x, y, z) = f(x, y, z) - \gamma(g(x, y, z) - c)$
$= 80yz + 60xy + 60xz - \gamma(x.y.z - 480)$

$\nabla L = 0 \quad <=> x.y.z - 480 = 0 => x.y.z = 480$
$60y + 60z - \gamma yz = 0 \quad \times (x) \quad (1)$
$80z + 60x - \gamma xz = 0 \quad \times (y) \quad (2)$
$80y + 60x - \gamma xy = 0 \quad \times (z) \quad (3)$
+ (1) and (2)
⇨ $x = \frac{3}{4}y$
+ (2) and (3)
=> $z = y$

But we have
$x.y.z = 480$

$\frac{3}{4} y.y.y = 480$

$=> y = \sqrt[3]{360} => x = \frac{3\sqrt[3]{360}}{4}$ and $z = \sqrt[3]{360}$

Thus $x = \frac{3\sqrt[3]{360}}{4}, y = \sqrt[3]{360}, z = \sqrt[3]{360}$

34. (a). Determine the table time of light

$T = \frac{AP}{V_1} + \frac{PB}{V_2}$

that $AP = \frac{a}{\cos\theta_1}$ and $PB = \frac{b}{\cos\theta_2}$

⇨ $T = \frac{a}{\cos\theta_1} + \frac{b}{\cos\theta_2}$

(b). By using lagrange multiplier show that

$\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

$T(\theta_1, \theta_2) = \frac{a}{\cos\theta_1} + \frac{b}{\cos\theta_2} (\theta_1, \theta_2 \neq \frac{\pi}{2})$

constrains $L = a\tan\theta_1 + b\tan\theta_2$

$L(\gamma, x, y) = L(\gamma, \theta_1, \theta_2) - T(\theta_1, \theta_2) + \gamma(a\tan\theta_1 + b\tan\theta_2))$

$\nabla L(\gamma, x, y) = (a\tan\theta_1 + b\tan\theta_2, \frac{a\sin\theta_1}{V_1\cos^2\theta_1} + \frac{a}{\cos^2\theta_1}, \frac{b\gamma}{\cos\theta_2} + \frac{b\sin\theta_2}{V_2\cos^2\theta_2})$

if $\nabla L = 0$ $\begin{cases} a\tan\theta_1 + b\tan\theta_2 = 0 \\ a\sin\theta_1 + V_1 a\gamma = 0 \\ b\sin\theta_2 + V_2 b\gamma = 0 \end{cases}$

⇨ $\begin{cases} a\tan\theta_1 + b\tan\theta_2 = 0 \\ a\sin\theta_1 + V_1 a\gamma = 0 \Rightarrow \sin\theta_1 = -V_1\gamma \\ b\sin\theta_2 = -V_2\gamma \end{cases}$

we get

$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2} => \frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

Therefore $\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

36. Use Lagrange multiplier to show that the distance from a point $(x_0, y_0)$to the line $ax + by = c$ is defined by D $= \frac{ax_0 + by_0 - c}{\sqrt{a^2+b^2}}$

let (x, y) *be a point on the plance* (p): $ax + by = c$

The optimal function is distance from (x_0, y_0) and (x, y)

$$D(x,y) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Constrain $g(x,y) = ax + by = c$

Find the critical of $f(x,y) = (x-x_0)^2 + (y-y_0)^2$

Lagrange function

let $\mathrm{L}(\lambda, x, y) = f(x,y) - \lambda(g(x,y) - c)$

$$= (x-x_0)^2 + (y-y_0)^2 - \lambda(ax + by - c)$$

Substitute into (1):

$$(1) \Rightarrow -a\left(x_0 + \frac{\lambda a}{2}\right) - \mathrm{b}\left(y_0 + \frac{\lambda b}{2}\right) + c = 0$$

$$-ax_0 - \frac{\lambda a^2}{2} - by_0 - \frac{\lambda b^2}{2} + c = 0$$

$$-\frac{\lambda}{2}(a^2 + b^2) - (ax_0 + by_0) + c = 0$$

$$\lambda = \frac{2(-ax_0 - by_0 + c)}{a^2 + y^2} = \frac{2}{a}(x - x_0) = \frac{2}{b}(y - y_0)$$

$$\Rightarrow \begin{cases} \frac{2}{a}(x - x_0) = \frac{2(-ax_0 - by_0 + c)}{a^2 + y^2} \\ \frac{2}{b}(y - y_0) = \frac{2(-ax_0 - by_0 + c)}{a^2 + y^2} \end{cases} \Rightarrow \begin{cases} x - x_0 = \frac{a(-ax_0 - by_0 + c)}{a^2 + y^2} \\ y - y_0 = \frac{b(-ax_0 - by_0 + c)}{a^2 + y^2} \end{cases}$$

Then the distance from (x_0, y_0) $to\ (p) is$

$$D = \sqrt{\left(\frac{a(-ax_0 - by_0 + c)}{a^2 + b^2}\right)^2 + \left(\frac{b(-ax_0 - by_0 + c)}{a^2 + b^2}\right)^2}$$

$$D = \sqrt{\frac{a^2(-ax_0 - by_0 + c)^2}{(a^2 + b^2)^2} + \frac{b^2(-ax_0 - by_0 + c)^2}{(a^2 + b^2)^2}}$$

$$D = \sqrt{\frac{(-ax_0 - by_0 + c)^2(a^2 + b^2)}{(a^2 + b^2)^2}} \Rightarrow \frac{|ax_0 - by_0 + c|}{\sqrt{a^2 + b^2}}$$

$thus\ distance\ from\ (x_0, y_0)\ to\ plane\ (p)$

$$\frac{|ax_0 - by_0 + c|}{\sqrt{a^2 + b^2}}$$

38. Find the distance from between the ellipsoid (E) and place (P) , where (E) : $2x^2 + y^2 + 2z^2 - 8 = 0$ and (P): $x + y + z = -10 = 0$

we have (E) : $2x^2 + y^2 + 2z^2 - 8 = 0$

(P): $x + y + z - 10 = 0$

let $\mathrm{M(x, y, z)} \in$ (E)

The optimize function is the distance from M to place

$$\mathrm{d}(\mathrm{x},\mathrm{y},\mathrm{z}) = \mathrm{d}(\mathrm{M},(\mathrm{P})) = \frac{|x+y+z-10|}{\sqrt{1+1+1}} = \frac{\sqrt{3}}{3}|x+y+z-10|$$

the constrain $g(x) = 2x^2 + y^2 + 2z^2 - 8 = 0$

we need to prove that the minimize

$f(x) = x + y + z - 10$ subject to $g(x,y,z)$

let $L(\gamma,x,y) = f(x) - \gamma(g(x) - c)$

$$\nabla L(\gamma,x,y,z) = 0 \iff \begin{cases} 2x^2 + y^2 + 2z^2 = 8 & (1) \\ 1 - 4x\gamma = 0 \Rightarrow x = \frac{1}{4\gamma} \\ 1 - 2y\gamma = 0 \Rightarrow y = \frac{1}{2\gamma} \\ 1 - 4z\gamma = 0 \Rightarrow z = \frac{1}{4\gamma} \end{cases}$$

then (1): => $x = \pm 1$

then $y = \pm 2$, $z = x = \pm 1$

+ For x=1 , y=2 ,z=1

So , D = d(1,2,1) = $2\sqrt{3}$

+ For x=-1 , y=2 ,z=-1

So , D = d(-1,2,-1) = $\frac{14\sqrt{3}}{3}$

40. Find the point closet to the point (2,5, -1) and on the line of intersection of the planes x-2y+3

$let\ f(x,y,z) = (x-2)^2 + (y-3)^2 + (z+1)^2$

$$Find\ the\ minimum\ of\ f(x,y,z)\text{ ,two constrain}\begin{cases} x-2y+3z=8 \\ 2z-y=3 \end{cases} \Leftrightarrow \begin{cases} x-2y+3z-8=0 \\ 2z-y-3=0 \end{cases}$$

$$if\ \Delta(f+\lambda_1 g_{1.} + \lambda_2 g_2)(x,y,z) = 0$$

$$we\ got\ \begin{cases} z+2y+x-10=0 \\ x-2y+3z=8 \\ 2z-y-3=0 \end{cases} \qquad \Rightarrow y=2z-3\ ,$$

$$\Rightarrow z=\frac{7}{3}\ ,\ \Rightarrow y=\frac{5}{3}\ ,\ \Rightarrow x=\frac{13}{3}$$

$Thus\ (\frac{13}{3},\frac{5}{3},\frac{7}{3})\ is\ the\ closet\ point\ to\ (2,5,-1)$.

I2-Calculus 3-TD2
(Multiple Integrations)

1. Draw the region and evaluate the following integral

(a) $\displaystyle\int_0^1\int_1^2 (y+2x)\,dy\,dx$ (b) $\displaystyle\int_0^1\int_{x^2}^{x} xy\,dy\,dx$ (c) $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy$

2. Evaluate the following double integral

(a) $\displaystyle\iint_R (x+y^2)\,dA$, R is the region bounded by $x=0$, $y=x^2$ and $x+y=2$.

(b) $\displaystyle\iint_R (1+2y)\,dA$, R is the region bounded by $x=0$, $x=2$, $y=x$ and $y=2+x^2$

(c) $\displaystyle\iint_R x\,dA$, R is the region bounded by the lines $y=x$, $y=x-1$, $y=0$ and $y=2$.

(d) $\displaystyle\iint_R \sqrt{xy-y^2}\,dA$, where R is a triangle with vertices $(0,0),(10,1)$ and $(1,1)$.

(e) $\displaystyle\iint_R \left(x^2+y^2\right)\,dA$, R is the area enclosed by $y=4x$, $x+y=3, y=0$ and $y=2$

3. Evaluate the following double integral by using change variables

(a) $\displaystyle\iint_R (x+y)\,dA$, R is the region bounded by $y=x$, $y=x-2$, $y=-x$ and $y=2-x$

(b) $\displaystyle\iint_R \frac{y}{x}dA$, $R=\{(x,y)\in\mathbb{R}^2 : x<y<2x, x<y^2<2x\}$

(c) $\displaystyle\iint_R \frac{xy}{y^2-x^2}\,dA$, where R is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1, x^2-y^2=4$ and the ellipses $\dfrac{x^2}{4}+y^2=1, \dfrac{x^2}{16}+y^2/4=1$.

(d) $\displaystyle\iint_R \frac{(x-y)^2}{(x+y+1)^2}\,dA$, R is the region bounded by the lines $x=0; y=0$ and $x+y=1$.

4. Evaluate the following integrals by using Polar coordinates.

(a) $\displaystyle\iint_R \cos(x^2+y^2)\,dA$, where $R=\Big\{(x,y)\in\mathbb{R}^2 \,|\, x^2+y^2<4\Big\}$.

(b) $\displaystyle\iint_R \frac{y^2}{x^2+y^2}dA$, $R=\{(x,y)\in\mathbb{R}^2 : (x-2)^2+y^2<4, x>2\}$.

(c) $\displaystyle\iint_R x\,dA$, $R=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1, x+y>1\}$.

5. Use any method to compute the following integral on the given region R.

(a) $\displaystyle\iint_R \frac{x}{x^2+y^2}\,dA$, R is the region bounded by $y=x, y=0$ and $x=1$.

(b) $\displaystyle\iint_R \frac{y^2\cos(x^2+y^2)}{x^2}\,dA, R=\{(x,y)\in\mathbb{R}^2 : 1<x^2+y^2<4; 0<y<x\}$

(c) $\displaystyle\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA$, R is the region bounded by $y=x, y=0$ and $x=1$.

(d) $\displaystyle\iint_R (x^2-y^2)\,dA, R=\left\{(x,y)\in\mathbb{R}^2 : \frac{(x-1)^2}{4}+\frac{(y-1)^2}{9}<1\right\}$

(e) $\displaystyle\iint_R |x-y|dA,\ R=\{(x,y)\in\mathbb{R}^2 : 0<x<1, 0<y<1\}$

(f) $\displaystyle\iint_R (x^2+y^2)\,dA,\ R=\{(x,y)\in\mathbb{R}^2 : x^2+(y-2)^2<1\}$.

(g) $\displaystyle\iint_R \left(x^2+\frac{y^2}{9}\right)\sin\left(2\arctan\frac{y}{3x}\right)\,dA, R=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1, 0<y<x\}$.

(h) $\displaystyle\iint_R \frac{y}{x^3}\sin\left(\pi-\frac{\pi y^2}{x^2}\right)\,dA$, where $R=\{(x,y)\in\mathbb{R}^2 : 0<\sqrt{2}y<x,\ 1<x^2-y^2<4\}$.

(i) $\displaystyle\iint_R (x^3-3xy^2)\,dA,\ R=\{(x,y)\in\mathbb{R}^2 :\ (x+1)^2+y^2\le 9; (x-1)^2+y^2\ge 1\}$.

6. Calculate $I=\displaystyle\iint_R \frac{x}{(1+x^2)(1+xy)}\,dA$ where

$$R=\{(x,y)\in\mathbb{R}^2 \mid 0<x<1;\ 0<y<1\}$$

Then deduce the value of $J=\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$.

7. Find the region area R defined by:

(a) R is bouned by the $y=x^2$ and $x=y^2$.

(b) R is the region in the first quardrant and bouned by the $y=x^2, y=4x^2$, $x+y=1$ and $x+y=2$

8. Find the area of the surface S over the region R.

(a) S is the surface of $f(x,y)=1+x-2y$, R is a square with vertices $(0,0),(3,0),(0,3)$ and $(3,3)$.

(b) S is the paraboloid $y=9-x^2-z^2$ that lies between the planes $y=0$ and $y=5$.

(c) S is comprises the parts of the cylinder $x^2+z^2=1$ that lie within the cylinder $y^2+z^2=1$.

9. Find the mass and the center of mass of the lamina occupying the region R and having mass density ρ.

(a) R is the region bounded by $y=3-x, y=0$ and $x=1$; $\rho(x,y)=2xy$.

(b) R is the region bounded by the parabola $y=4-x^2$ and the x-axis; and $\rho(x,y)=y$.

10. Find the moments of inertia I_x, I_y, and I_o.

(a) R is the triangular region with vertices $(0,0),(2,1)$ and $(4,0)$; $\rho(x,y)=x$.

(b) R is the region in the first quadrant bounded by the circle $x^2+y^2=1$; $\rho(x,y)=x+y$.

11. Evaluate the following triple integral

(a) $\displaystyle\int_0^1 \int_0^{3x} \int_0^{y} (y + x^2)\, dz\, dy\, dx$

(b) $\displaystyle\int_0^{\ln 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} dx dy dz$

(c) $\displaystyle\int_1^e \int_1^x \int_0^{1/(xy)} 2 \ln y\, dz\, dy\, dx$

(d) $\displaystyle\int_0^{\pi/2} \int_0^{2\cos^2\theta} \int_0^{4-r^2} r \sin\theta\, dz\, dr\, d\theta$

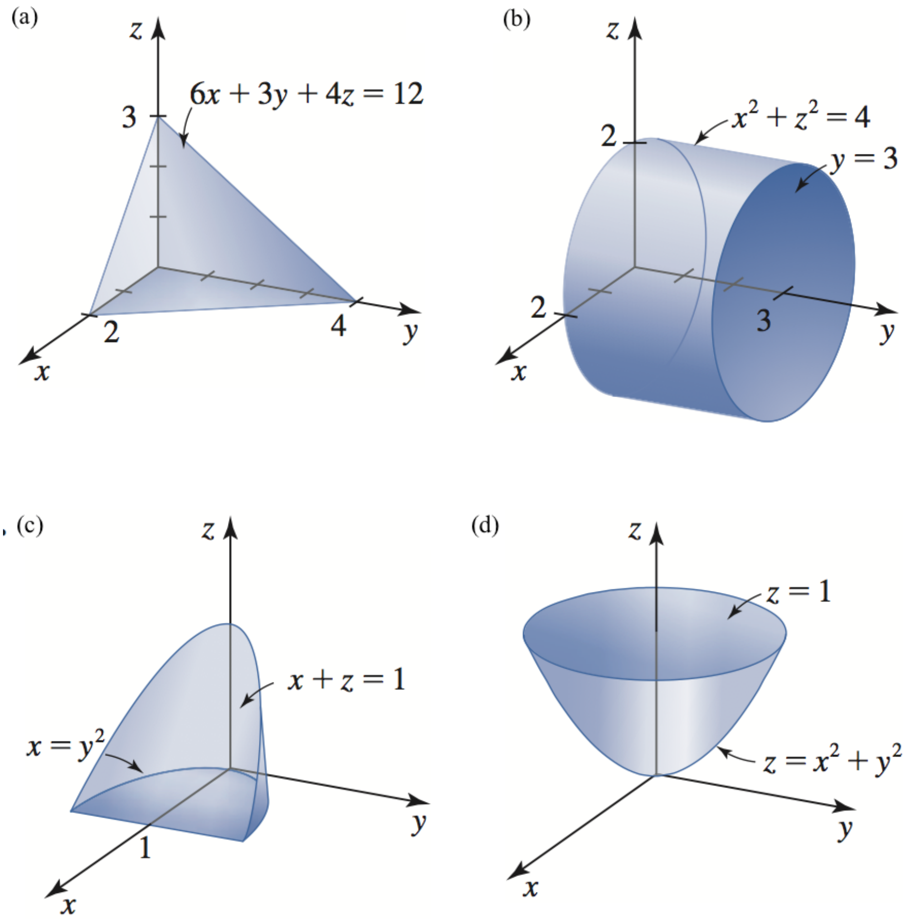
12. Compute $\displaystyle\iiint_Q \frac{z^3}{(x+y+z)}\, dV$

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1\}$$

Hint: Use change variables

$$x = x + y + z, \quad v = \frac{z}{y+z}, \quad w = \frac{y+z}{x+y+z}$$

13. The figure shows the region of integration for $\displaystyle\iiint_Q f(x, y, z) dV$. Express the triple integral as an iterated integral in six different ways using different orders of integration.



14. Evaluate the following integral by using Cylindrical coordinates

(a) $\displaystyle I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} \frac{e^z}{\sqrt{x^2+y^2}}\, dz\, dy\, dx.$

(b) $J = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy$.

(c) $K = \iiint_Q \sqrt{x^2+y^2} dV$, Q is the solid bounded by $z = x^2 + y^2$ and the plane $z = 4$.

(d) $L = \iiint_Q \frac{1}{x^2+y^2} dV$, Q is the solid bounded above by $z = 4 - x^2 + y^2$ and below by the sphere $x^2 + y^2 + z^2 = 9$.

(e) $\iiint_Q dV$, Q is the solid bounded by cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $y + z = 3$.

15. Evaluate the following integral by using Spherical coordinates

(a) $I = \iiint_Q (x^2+y^2+z^2)\, dV$, Q is the solid region bounded below by the cone $z^2 = x^2+y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

(b) $J = \iiint_Q z\, dV$, Q is the solid region bounded below by the cone $x^2+y^2+(z-2)^2 = 1$.

16. Find the volume of the solid Q by using triple integrals.

(a) Q is a thetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,3)$.

(b) $Q = \{(x,y,z) \in \mathbb{R}^3 : 2x + y + z - 3 = 0, x > 0, y > 0, z > 0\}$.

(c) Q is bounded by $z = 9 - x^2, z = 0, y = 0$ and $y = 2x$.

(d) Q is bounded by $z = x^2 + y^2$ and $z = 1$.

(e) Q is enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(f) Q is bounded by paraboloid $z = x^2 + y^2$, and inside the elliptic cylinder $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

(g) Q is bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and below by the paraboloid $8z = x^2 + y^2$.

(h) Q is volume inside the paraboloid $x^2 + y^2 = 2\mu z$ and outside the cone $x^2 + y^2 = \lambda^2 z^2$, where $\lambda, \mu > 0$.

(i) Q is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1$.

(j) Q is bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2 + y^2 = 4$, and the plane $z = 0$.

(k) $Q = \{(x,y,z) : x^{2/3} + y^{2/3} + z^{2/3} \leq 1\}$.

(l) Q is bounded by $x^2 + 4y^2 + z^2 = 1$

(m) Q is bounded by $x^2 + \frac{1}{2}y^2 + \frac{3}{4}z^2 + xz = 1$

17. Evaluate the integral $\iiint_Q f(x,y,z)\, dV$.

(a) $f(x,y,z) = \frac{1}{(x+y+z+1)^3}$, Q is the solid region bounded by the coordinates planes and the plane $x + y + z = 1$.

(b) $f(x,y,z) = xy$, $Q = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq x+y\}$.

(c) $f(x,y,z) = z$, Q is the region bounded by the cylinder $x^2 + z^2 = 1$ and the planes $y = x, y = 2x$, and $z = 0$.

(d) $f(x,y,z) = \sqrt{x^2+y^2}$, Q is the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = 3$.

(e) $f(x,y,z) = \sqrt{x^2+y^2}$, $Q = \{(x,y,z) \in \mathbb{R}^3 : x+y+z < 2, x^2+y^2 < 1, z > 0\}$.

(f) $f(x,y,z) = y$, Q is the part of the solid in the first octant lying inside the paraboloid $z = 4 - x^2 - y^2$.

(g) $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$, Q is the part of unit ball $x^2+y^2+z^2 \leq 1$ lying in the first octant.

(h) $f(x,y,z) = xz$, Q is the solid bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by the cone $z = \sqrt{x^2+y^2}$.

(i) $f(x,y,z) = \dfrac{1}{\sqrt{(x-2)^2+y^2+z^2}}\,dV$; $Q = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 < 1\}$

(j) $f(x,y,z) = \dfrac{1}{x^2+y^2}\,dV$; $Q = \{(x,y,z) \in \mathbb{R}^3 :\ x, y > 0; 2 < x^2+y^2 < 2\sqrt{2}(x+y); 0 < z < x+y\}$

18. Find the mass and centre of mass of the solid Q of given mass density.

(a) Q is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x+y+z = 1$. The density at a point P of Q is directly proportional to the distance between P and the yz-plane.

(b) Q is the solid bounded by the cylinder $x^2+y^2 = 1$ in the first octant and the plane $z+y = 1$; the density $\rho(x,y,z) = x^2+y^2+z^2$.

(c) Q is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$; the mass density at any point on the solid is directly proportional to the square of its distance form the origin.

19. Find the moment of inertia about the z-axis of a solid bounded by the cylinder $x^2+y^2 = 4$ and the planes $z = 0$ and $z = 3$ if the mass density at any point on the solid is directly proportional to its distance from the xy-plane.

20. Let Q be a uniform solid of mass m bounded by the spheres $\rho = a$ and $\rho = b$, where $0 < a < b$. Show that the moment of inertia of Q about a diameter of Q is

$$I = \frac{2m}{5}\left(\frac{b^5-a^5}{b^3-a^3}\right).$$

21. (a) Use the result of Exercise above to fine the moment of inertia of a uniform solid ball of mass m and radius b about a diameter of the ball.

(b) Use the result of Exercise above to find the moment of inertia of a hollow spherical shell of mass m and radius b about a diameter of the shell.

TD2

(Multiple Integrations)

2. *Evaluate the following double integral*

$(a)\iint_R \left(x^2+y^2\right)dA$, R *is the region bounded by* $x=0, y=x^2$ *and* $x+y=2$

$$=\int_0^1\int_{x^2}^{2-x}\left(x+y^2\right)^2 dydx$$

$$=\int_0^1\left[x(2-x)+\frac{(2-x)^3}{3}-x^3-\frac{x^6}{3}\right]dx$$

$$=\int_0^1\frac{-x^6-4x^4+3x^2-6x+8}{3}dx$$

$$=\frac{1}{3}\left(\frac{-1}{7}-1+1-3+8\right)=\frac{34}{21}$$

$(b)\iint_R (1+2y)dA$, R *is the region bounded by* $x=0, x=2, y=x$ *and* $y=2+x^2$

$$=\int_0^2\int_x^{2+x^2}(1+2y)dydx$$

$$=\int_0^2\left(6-x+4x^2+x^4\right)dx$$

$$=\left(12-2+\frac{32}{3}+\frac{32}{5}\right)=\frac{406}{15}$$

$(c)\iint_R xdA$, R *is the region bounded by the lines* $y=x, y=x-1, y=0$ *and* $y=2$

$$=\int_0^2\int_{x-1}^{x} xdydx$$

$$=\int_0^2\left(x^2-x^2+x\right)dx$$

$$=2$$

$(d)\iint_R \sqrt{xy-y^2}\,dA$, *where* R *is a triangle with vertices* $(0,0),(10,1)$, *and* $(1,1)$

$$=\int_0^1\int_y^{10y}\sqrt{xy-y^2}\,dxdy$$

$$=\int_0^1 18y^2dy=6$$

$$(e)\iint_R \left(x^2+y^2\right)dA \text{ , } R \text{ is the area enclosed by } y=4x, x+y=3, y=0 \text{ and } y=2$$

$$=\int_0^2\int_{\frac{y}{4}}^{3-y}\left(x^2+y^2\right)dxdy$$

$$=\int_0^2\left(\frac{27-27y+9y^2-y^3}{3}+3y^2-y^3-\frac{y^3}{192}-\frac{y^3}{4}\right)dy$$

$$=\frac{463}{48}$$

$$4.(a)\iint_R \cos\left(x^2+y^2\right)dA, \text{ where } R=\left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<4\right\}$$

$$\text{by using coordonate polar } r^2=x^2+y^2, J=r$$

$$=\iint_{R'}\cos\left(r^2\right)erdrd\theta$$

$$=\int_0^{2\pi}\int_0^2\frac{\left(r^2\right)'}{2}\cos\left(r^2\right)drd\theta=\int_0^{2\pi}\frac{1}{2}\sin(4)d\theta=\pi\sin(4)$$

$$(b)\iint_R\frac{y^2}{x^2+y^2}dA, R=\left\{(x,y)\in\mathbb{R}^2:(x-2)^2+y^2<4, x>2\right\}$$

$$=\frac{1}{2}\iint_{R'}r(1-\cos 2\theta)dA'=\int_0^{\frac{\pi}{4}}\int_{\frac{2}{\cos\theta}}^{4\cos\theta}r(1-\cos 2\theta)drd\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(4\frac{1-\cos 4\theta}{2}-4\frac{1-\cos\theta}{1+\cos\theta}\right)d\theta$$

$$=\frac{\pi}{2}-2+4\arctan\left(\frac{1}{2}\right)$$

$$(c)\iint_R xdA, R=\left\{(x,y)\in\mathbb{R}^2:x^2+y^2<1, x+y>1\right\}$$

$$=\int_0^1\int_{1-x}^{\sqrt{1-x^2}}xdydx=\int_0^{\frac{\pi}{2}}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1}r^2\cos\theta drd\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{3}\cos\left(1-\frac{1}{(\cos\theta+\sin\theta)^3}\right)d\theta=\frac{1}{6}$$

$8.$ *Find the area of the surface S over the redion* $\mathbb{R}$

b. S is the paraboloid $y = 9 - x^2 - z^2$ *that lies between the plans* $y = 0$ *and* $y = 5$

surface area $\iint_R \sqrt{1 + f_x{}^2 + f_z{}^2}\, dA$

$f_x = -2x,\ f_z = -2z$

$+y = 0 \Rightarrow x^2 + z^2 = 9$

$+y = 5 \Rightarrow x^2 + z^2 = 4$

$$S = \iint_R \sqrt{1 + (2x)^2 + (2x)^2}\, dA$$

$$= \int_0^{2\pi} \int_2^3 \sqrt{1 + 4r^2} \times r dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(\sqrt[3]{37} - \sqrt[3]{17}\right) d\theta$$

$$= \frac{\pi}{6}\left(\sqrt[3]{37} - \sqrt[3]{17}\right)$$

$10.(b)$ *R is the region in the first quadrant by the circle* $x^2 + y^2 = 1; \rho(x, y) = x + y$

$$\iint_R y^2 \rho(x, y)$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (x + y) y^2 dy dx$$

$$= \int_0^1 \frac{x(1 - x^2)\sqrt{1 - x^2}}{3} dx + \frac{1}{4}\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

let $v = (1 - x^2) \Rightarrow dV = -2x^2$

$$\Rightarrow I_x = \int_1^0 -\frac{v\sqrt{v}}{6} dv + \frac{2}{15}$$

$$= \frac{1}{18} + \frac{9}{15} = \frac{1}{5}$$

$$I_y = \iint_R x^2 \rho(x, y)\, dA$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x^2 (x + y) dy dx$$

$$= \int_0^1 (x^2\sqrt{1 - x^2} + \frac{1}{2} x^2 (1 - x^2) dx$$

$$=\frac{1}{15}+\int_0^1 x^2\sqrt{1-x^2}\,dx$$

$$let\ v=1-x^2 \Rightarrow x^2=1-v$$

$$dv=-2x$$

$$\Rightarrow I_y=\frac{1}{15}+\int_1^0-\frac{v^{1/2}(1-v)}{2}dv=\frac{1}{5}$$

$$I_0=I_x+I_y=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

$$12.We\ have\ u=x+y+z\Rightarrow x=u-(y+z),\left(y+z=\frac{z}{v}\right)$$

$$x=u-\frac{z}{v}(1)$$

$$but\ w=\frac{\frac{z}{v}}{u}=\frac{z}{uv}$$

$$\Rightarrow z=uvw$$

$$then: x=u-uw=u(1-w)$$

$$let\ z=v(y+z)=uvw\Leftrightarrow y+z=uw$$

$$\Rightarrow y=uw-uvw=uw(1-v$$

$$So\begin{cases} x=u(1-w) & (1)\\ y=uw(1-v) & (2)\\ z=uvw & (3)\end{cases}$$

$$(1):\begin{cases}\frac{\partial x}{\partial u}=1-w\\ \frac{\partial x}{\partial v}=0\\ \frac{\partial x}{\partial w}=-u\end{cases}\quad (1):\begin{cases}\frac{\partial y}{\partial u}=w-vw\\ \frac{\partial y}{\partial v}=-uw\\ \frac{\partial y}{\partial w}=u-uv\end{cases}\quad (3):\begin{cases}\frac{\partial z}{\partial u}=vw\\ \frac{\partial z}{\partial v}=uw\\ \frac{\partial z}{\partial w}=uv\end{cases}$$

$$\Rightarrow J = \begin{vmatrix} 1-w & 0 & -u \\ w-vw & -uw & u-uv \\ vw & uw & uv \end{vmatrix}$$

$$= -u^2w^2 - u^2w = -\left(u^2w^2 + u^2w\right)$$

$$A = \iiint_Q \frac{z^3 dv}{x+y+z}$$

$$= \int_{w_1}^{w_2}\int_{v_2}^{v_2}\int_{u_1}^{u_1} \frac{u^3w^3v^3}{u}\left(u^2w^2 + u^2w\right)dudvdw$$

$(1)\ x = u(1-w), (x = 0)$

$\Leftrightarrow 0 = u(1-w) \Rightarrow u = 0, w = 1$

$(2): y = uw(1-v), (y = 0)$

$0 = uw(1-v)$

$\Rightarrow u = 0, w = 0, v = 1$

$(3)\ z = 0 = uvw \Rightarrow u = v = w = 0$

$for\ \ x + y + z = 1 = u$

$u = 0 \Rightarrow u = 1\ or\ 0 \le u \le 1$

$v = 0 \Rightarrow v = 1\ or\ 0 \le v \le 1$

$w = 0 \Rightarrow w = 1\ or\ 0 \le w \le 1$

$$\int_{w_1}^{w_2}\int_{v_2}^{v_2}\int_{u_1}^{u_1} \frac{u^3w^3v^3}{u}\left(u^2w^2 + u^2w\right)dudvdw = \frac{11}{600}$$

$14.c.k = \iiint_Q \sqrt{x^2 + y^2}$, Q *is solid bounded by* $z = x^2 + y^2$ *and the plane* $z = 4$

by $u\sin g$ *cylindrical coordinates*

$$k = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r.rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2\left(4 - r^2\right)drd\theta$$

$$= \int_0^{2\pi}\left(\frac{64}{15}\right)d\theta = \frac{64}{15}(2\pi)$$

$(d)\iiint_Q \frac{1}{x^2+y^2}dV$, *Q is the solid bounded above by* $z=4-x^2+y^2$ *and below by the sphere* $x^2+y^2+z^2=9$

$$=\int_0^{2\pi}\int_0^a\int_{\sqrt{9-r^2}}^{4-r}\frac{1}{r}dzdrd\theta$$

$$=\int_0^{2\pi}\int_0^a\frac{1}{r}\left[(4-r)-\left(\sqrt{9-r^2}\right)\right]drd\theta$$

$$=2\pi\left[4\ln a-a-\sqrt{9-a^2}+3\ln\left(\frac{\sqrt{9-a^2}+3}{a}\right)\right]$$

16. *f Q is bounded by paraboloid* $z=x^2+y^2$, *and inside the elliptic cylinder* $\frac{x^2}{9}+\frac{y^2}{4}=1$

$$let\begin{cases}u=\frac{x}{3}\\ v=\frac{y}{2}\Rightarrow |J|=6\end{cases}$$

$v=\iiint 6dzdudv$ *by using cylindrical coordinate*

$$\begin{cases}u=r\cos\theta\Rightarrow z=9r^2\cos^2\theta+4r^2\sin^2\theta\\ v=r\sin\theta\end{cases}$$

$$v=6\int_0^{2\pi}\int_0^1\int_0^{9r^2\cos^2\theta+4r^2\sin^2\theta}rdzdrd\theta$$

$$=6\int_0^{2\pi}\int_0^1 r\left(4+5r^2\cos^2\theta\right)drd\theta$$

$$=6\int_0^{2\pi}\left(1+\frac{5}{4}\cos^2\theta\right)d\theta=\frac{39}{2}\pi$$

20. *Show that the moment of inertia of Q about a diameter of Q*

$I = \iiint_Q \left(x^2 + y^2\right) dm\left(P\right)$

$\sin ce\ Q\ is\ uniform\ \dfrac{dm\left(P\right)}{dV} = \dfrac{M}{V} \Rightarrow dm\left(P\right) = \dfrac{M}{V} dV$

$I = \dfrac{M}{V} \iiint_Q \left(x^2 + y^2\right) dv\ in\ sherical\ coordinates$

$\iiint_Q f\left(x, y, z\right) dv = \int_{Q_1}^{Q_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1}^{\rho_2} F\left(\rho, \varphi, \theta\right) \rho^2 \sin \varphi d\rho d\varphi d\theta$

$I = \dfrac{M}{V} \int_0^{2\pi} \int_0^{\pi} \int_a^b \rho^4 \sin^3 \varphi d\rho d\varphi d\theta$

$= \dfrac{M}{V}, \dfrac{b^5 - a^5}{5} \times 2\pi \times \dfrac{4}{3} = \dfrac{8\pi}{15} . \dfrac{m}{v} . \left(b^5 - a^5\right)$

$v = \dfrac{4}{3}\pi\left(b^3 - a^3\right)$

I2-TD3
(Vector Analysis)

1. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

   (a) $\mathbf{F}(x,y) = e^{x+y}\mathbf{i} + e^{xy}\mathbf{j}$

   (b) $\mathbf{F}(x,y) = \dfrac{xy^2}{(1+x^2)^2}\mathbf{i} + \dfrac{x^2y}{1+x^2}\mathbf{j}$

   (c) $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \dfrac{z^2}{y}\mathbf{j} + 2z\ln y\mathbf{k}$.

   (d) $\mathbf{F}(x,y,z) = (e^{-yz} - yze^{xyz})\mathbf{i} + xz(e^{-yz} + e^{xyz})\mathbf{j} + xy(e^{-yz} - e^{xyz})\mathbf{k}$

2. Determine the map $g \in C^\infty$ with $g(0) = 0$ that makes the vector field

$$\mathbf{F}(x,y) = (y\sin x + xy\cos x + e^y)\mathbf{i} + (g(x) + xe^y)\mathbf{j}$$

   conservative. Find a potential for the resulting $\mathbf{F}$.

3. Compute the scalar line integral $\displaystyle\int_C f\,ds$, where f and C are as indicated.

   (a) $f(x,y) = 1 + xy^2$; $C : \mathbf{r}(t) = (t, 3t^2),\ t \in [0,1]$.

   (b) $f(x,y) = x + y$; C is the perimeter of the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$.

   (c) $f(x,y) = x$; Let C be the union of the parabolic arc $y = 4 - x^2$ going from the point $A = (-2,0)$ to the point $B = (2,0)$, and the circle $x^2 + y^2 = 4$ from B to A.

   (d) $f(x,y,z) = xyz$; $C : \mathbf{r}(t) = (t, 2t, 3t),\ t \in [0,2]$

   (e) $f(x,y,z) = \dfrac{z}{x^2+y^2}$; $C : \mathbf{r}(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t, e^{2t}),\ t \in [0,5]$

   (f) $f(x,y,z) = 2x - y^{1/2} + 2z^2$; $C = C_1 + C_2$, where $C_1 : \mathbf{r}_1(t) = (t, t^2, 0),\ t \in [0,1]$ and $C_2 : \mathbf{r}_2(t) = (1, 1, t-1),\ t \in [1,3]$.

   (g) $f(x,y,z) = x + y + z$; C is the line segment from $(1,1,2)$ to $(3,-1,1)$.

   (h) $f(x,y,z) = \sqrt{1 + yz^2}$; C is the intersection of the surfaces $x^2 + z^2 = 4$ and $y = x^2$.

   (i) $f(x,y,z) = 2x - \sqrt{y} + 2z^2$; $C : \mathbf{r}(t) = \begin{cases} (t, t^2, 0), & 0 \le t \le 1 \\ (1, 1, t-1), & 1 \le t \le 3. \end{cases}$

4. Compute the vector line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ and C are as indicated.

   (a) $\mathbf{F}(x,y) = (\sin y, x)$; $C : \mathbf{r}(t) = (t^2 - 1, t),\ t \in [0,\pi]$.

   (b) $\mathbf{F}(x,y) = (x, y+1)$; $C : \mathbf{r}(t) = (1 - \sin t, 1 - \cos t),\ t \in [0, 2\pi]$.

   (c) $\mathbf{F}(x,y) = (xy, y - x)$; C is the curve $y = x^2$ from $(1,1)$ to $(3,9)$.

   (d) $\mathbf{F}(x,y,z) = (yz^2, xz^2, 2xyz)$; $C = C_1 + C_2$, where C_1 is the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$, $t \in (0, 2\pi)$, and C_2 is the line segment from $(1,0,2\pi)$ to $(2\pi, 2\pi, 2\pi)$.

(e) $\mathbf{F}(x,y,z) = (\sqrt{x^3+y^3+5}, z, x^2)$; C is the intersection of the elliptical cylinder $y^2 + 2z^2 = 1$ with the plane $x = -1$, oriented in the counterclockwise direction when viewed from far out the positive x-axis.

5. Evaluate the differential form of the vector field.

(a) $\int_C xydx + (x^2+y^2)dy$; C is the perimeter of the square with $(0,0), (1,0), (1,1)$ and $(0,1)$, oriented in the counterclockwise direction.

(b) $\int_C x^2ydx - xydy$; C is the curve with equation $y^2 = x^3$, from $(1,-1)$ to $(1,1)$.

(c) $\int_C (x-y)^2dx - (x+y)^2dy$; C is the portion of $y = |x|$, from $(-2,2)$ to $(1,1)$.

(d) $\int_C yzdx - xzdy + xydz$; C is the line segment form $(1,1,2)$ to $(5,3,1)$.

(e) $\int_C zdx + xdy + ydz$; C is the curve obtained by intersecting the surface $z = x^2$ and $x^2 + y^2 = 4$ and oriented counterclockwise around the z-axis.

(f) $\int_C dx + (x+y)dy + (x^2+xy+y^2)dz$; C is the intersection of $z = x^2 + y^2$ and $x^2+y^2+z^2 = 1$, oriented in the counterclockwise direction when viewed from high up the positive z-axis.

6. (6.1). Show that $\mathbf{F}$ is conservative and find the potential function of $\mathbf{F}$. (6.2). Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the given $\mathbf{F}$ and curve C from points A to B.

(a) $\mathbf{F}(x,y) = 2xy\mathbf{i} + (1+x^2-y^2)\mathbf{j}$; $A = (1,0)$ and $B = (2,3)$.

(b) $\mathbf{F}(x,y) = (x + \tan^{-1} y)\mathbf{i} + \frac{x+y}{1+y^2}\mathbf{j}$; $A = (0,0)$ and $B = (1,1)$.

(c) $\mathbf{F}(x,y,z) = 2xyz^2\mathbf{i} + x^2z^2\mathbf{j} + 2x^2zy\mathbf{k}$; $A = (1,0,1)$ and $B = (1,2,-1)$.

(d) $\mathbf{F}(x,y,z) = e^x\mathbf{i} + (xe^y + \ln z)\mathbf{j} + \frac{y}{z}\mathbf{k}$; $A = (0,1,1)$ and $B = (1,0,2)$.

7. Use Green's Theorem to evaluate the line integrals.

(a) $\int_C 2xy^2dx - x^2ydy$; C is the boundary of the region lying between the graph of $y = 0, y = \sqrt{x}$ and $x = 9$.

(b) $\int_C 2\arctan\frac{y}{x}dx + \ln(x^2+y^2)dy$; $C : x = 4 + 2\cos\theta,\ y = 4\sin\theta$.

(c) $\int_C (x-3y)dx + (x+y)dy$ C is the region lying between the graphs of $x^2+y^2 = 1$ and $x^2+y^2 = 9$.

8. Find the equation of the tangent plane to the parametric surface represented by $\mathbf{r}$ at the specified point.

(a) $\mathbf{r}(u,v) = (u-v)\mathbf{i} + (u-v)\mathbf{j} + v^2\mathbf{k}$; $(2,0,1)$.

(b) $\mathbf{r}(u,v) = u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^2\mathbf{k}$; $(u,v) = (1,\pi)$.

9. Find the area of the surface.

   (a) The part of the plane $\mathbf{r}(u,v) = (u+2v-1)\mathbf{i} + (2u+3v+1)\mathbf{j} + (u+v+2)\mathbf{k}$ $0 \le u \le 1$; $0 \le v \le 2$.

   (b) The part of the plane $z = 8 - 2x - 3y$ that lies in side the cylinder $x^2 + y^2 = 4$.

   (c) The surface $\mathbf{r}(u,v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + u\mathbf{k}$ $0 \le u \le \pi$; $0 \le v \le 2\pi$.

10. Find $\iiint_S f(x,y,z)dS$.

    (a) $f(x,y,z) = x+y$; S is the part of the plane $3x+2y+z=6$ in the first octant.

    (b) $f(x,y,z) = xz$; S is part of the plane $y+z=4$ inside the cylinder $x^2+y^2=4$.

    (c) $f(x,y,z) = x + \frac{y}{\sqrt{4z+5}}$; S is the surface with vector representation $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (v^2-1)\mathbf{k}$, $0 \le u \le 1$; $-1 \le v \le 1$.

    (d) $f(x,y,z) = z\sqrt{1+x^2+y^2}$; S is the helicoid with vector representation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$; $0 \le v \le 2\pi$.

11. Find the mass of the surface S having the given density mass function.

    (a) S is the part of the plane $x+2y+3z=6$ in the first octant; the density at any point of S is directly proportional to the square of distance between the point to yz-plane.

    (b) S is the hemisphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$; the density at any point on S is directly proportional to the distance between the point to xy-plane.

12. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$, that is, find the flux of $\mathbf{F}$ across $\mathbf{S}$. If S is closed, use the positive (outward) orientation.

    (a) $\mathbf{F}(x,y,z) = (2x, 2y, z)$; S is the part of the paraboloid $z = 4-x^2-y^2$ above the xy-plane; $\mathbf{N}$ point upward.

    (b) $\mathbf{F}(x,y,z) = (x^2, xy, xz)$; S is the surface of tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

13. **[Electric Charge]** Find the total charge on the part of the hemisphere $z=\sqrt{25-x^2-y^2}$ that lies directly above the plane region $R = \{(x,y) : x^2+y^2 \le 9\}$ if the charge density at any point on the surface is directly proportional to the distance between the point and xy-plane.

14. **[Flow of a Fluid]** The flow of a fluid is described by the vector field $\mathbf{F}(x,y,z) = (2x, 2y, 3z)$. Find the rate of flow of the fluid upward through the surface S that is the part of the plane $x+3y+z=6$ in the first octant.

15. **[Flow of a Liquid]** The flow of a liquid is described by the vector field $\mathbf{F}(x,y,z) = (x, y, 3z)$. If the mass density of the fluid is 1000 (in appropriate units), find the rate of flow (mass per unit time) upward of the liquid through the surface S that is part of the paraboloid $z = 9-x^2-y^2$ above the xy-plane.

16. Verify the Divergence Theorem by evaluating $\iint_S \mathbf{F}\cdot\mathbf{N}dS$.

(a) $\mathbf{F}(x,y,z) = (2xy, -y^2, 3yz)$; S is the cube bounded by the plane $x = 0, x = 2, y = 0, y = 2, z = 0$ and $z = 2$.

(b) $\mathbf{F}(x,y,z) = (xz, zy, 2z^2)$; S is the surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

17. Use the Divergence Theorem to find the flux of $\mathbf{F}$ across $\mathbf{S}$; that is, calculate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$.

(a) $\mathbf{F}(x,y,z) = (xy^2, 2yz, -3x^2y^3)$; S is the surface of the cube bounded by the planes $x = \pm 1, y = \pm 1$ and $z = \pm 1$.

(b) $\mathbf{F}(x,y,z) = (x+1, yz^2 + \cos xz, 2y^2z + e^{\tan x})$; S is the sphere $x^2 + y^2 + z^2 = 1$.

(c) $\mathbf{F}(x,y,z) = (xz, x^2y, y^2z + 1)$; S is the surface region that lies between the cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$.

18. (a) Use the Divergence Theorem to verify that the volume of the solid bounded by a surface S is

$$\iint_S x dy dz = \iint_S y dz dx = \iint_S z dx dy.$$

(b) Verify the result of part (a) for the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$.

19. Verify that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N} \, dS = 0$ for any closed surface S.

20. Given the vector field $\mathbf{F}(x,y,z) = (x,y,z)$, verify that $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = 3V$, where V is the volume of the solid bounded by the closed surface S.

21. Verify Stokes's Theorem by evaluating $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot d\mathbf{r}$ as a line integral and as a double integral.

(a) $\mathbf{F}(x,y,z) = (-y+z, x-z, x-y)$; S is part of the paraboloid $z = 9 - x^2 - y^2$ and $z \geq 0$.

(b) $\mathbf{F}(x,y,z) = (y, z, x)$; S is part of the plane $2x + 2y + z = 6$ lying in the first octant.

22. Use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(a) $\mathbf{F}(x,y,z) = (\arctan \frac{y}{x}, \ln x^2 + y^2, 1)$; C is a triangle with vertices $(0,0,0), (1,1,1)$ and $(0,0,2)$.

(b) $\mathbf{F}(x,y,z) = (3xz, e^{xz}, 2xy)$; C is a circle obtained by intersecting the cylinder $x^2+z^2 = 1$ with the plane $y = 3$ oriented in a counterclockwise direction when viewed from the right.

(c) $\mathbf{F}(x,y,z) = (-\ln\sqrt{x^2+y^2}, \arctan \frac{x}{y}, 1)$; S is the surface of $z = 9 - 2x - 3y$ over $r = 2\sin\theta$ in the first octant.

(d) $\mathbf{F}(x,y,z) = (xyz, y, z)$; S is the surface of $z = x^2$, $0 \leq x \leq a$, $0 \leq y \leq a$. $\mathbf{N}$ is the downward unit normal to the surface.

TD3

(Vector Analysis)

2. Determine the map $g \in c\infty\, with g(0) = 0\ that\ makes\ the\ vector\ field$

$F(x, y) = (ysinx + xycosx + e^x)i + (g(x) + xe^y)j\ conservatice.$

Let $M(x, y) = sinx + xcosx + e^y\ and\ N(x, y) = g(x) + xe^y$

$\frac{\partial M}{\partial y}(x, y) = sinx + xcosx + e^y\ and \frac{\partial N}{\partial x} = g'(x) + e^y$

$F(x, y) is\ conservartive\ iff \frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$

$Then, sinx + xcosx + e^y = g'(x) + e^y => g'(x) = sinx + xcosx$

We get , $g(x) = \int sinx + xcosx)dx = xsinx$

<u>Therefore:</u> $g(x) = xsinx\ that\ makes\ F(x, y) conservative.$

+ Find a potential for the resulting F.

By using the result above ,$\frac{\partial N}{\partial x}(x, y) = sinx + xcosx + e^y$

By above answer , we get

$$\begin{cases} f_x(x, y) = ysinx + xycosx + e^y\ (1) \\ f_y(x, y) = xsinx + xe^y\ (2) \end{cases}$$

Follow by (2) : $\int f_y(x, y)dy = \int(xsinx + xe^y)dy = xysinx + xe^y + c(x)\ (*)$

$=> f_x(x, y, z) = xycosx + ysinx + e^y + c'(x)$ (3)

Follow by (1) and (2) : $ysinx + xycosx + e^y = xycosx + ysinx + e^y + c'(x)$

$=> c'(x) = 0 => c(x) = k, k \in \mathbb{R}$

Follow by (*) :

<u>Therefore:</u> $f(x, y, z) = xysinx + xe^y + k, k \in \mathbb{R}\ is\ a\ potential$

4. Compute the vector line integral $\int Fdr$, where f and c are as indicated

a). $F(x, y) = (siny, x); C: r(t) = (t^2 - 1, t), t \in [0,1]$

Follow by $\int Fdr = \int_0^{\pi}(2tsint + t^2 - 1)dt$

<u>Therefore:</u> $\int Fdr = \frac{3\pi - \pi^3}{3}$

b). $F(x, y) = (x, y + 1); C: r(t) = (1 - sint, 1 - cost), t \in [0,2\pi]$

We have , $C: r(t) = (1 - sint, 1 - cost) => r'(t) = (-cost, sint)$

And $F\big(r(t)\big) = F(1 - sint, 1 - cost) = (1 - sint, 2 - cost)$

Follow by $\int Fdr = \int_0^{2\pi}(1 - sint, 2 - cost)(-cost, sint)dt$

$$= \int_0^{2\pi}(-cost + 2sint)dt = 0$$

c). $F(x, y) = (xy, y - x); C\ is\ the\ curve\ y = x^2\ from(1,1)to\ (0,2)$

We have, $C: r(t) = (t, t^2)\ and\ F\big(r(t)\big) = F(t, t^2) = (t^3, t^2 - t)$

$$r'(t) = (1,2t)and\ F\big(r(t)\big) = (t^3, t^2 - 1)$$

Follow by $\int Fdr = \int_1^3(t^3, t^2 - t)(t, t^2)dt = \frac{384}{5}$

d). $F(x, y, z) = (yz^2, xz^2, 2xyz); C = C_1 + C_2\ , where\ C_1\ is\ the\ helix$

$r(t) = (cost, sint, t), t \in (0,2\pi)and\ C_2 is\ the\ line\ segment$

$from\ (1,0,2\pi)\ to\ (2\pi, 2\pi, 2\pi)$

$+C_1: r(t) = (cost, sint, t), t \in [0,2\pi]$

$=> r_1'(t) = (-sint, cost, 1) => F\big(r_1(t)\big) \times r_1'(t) = 0$

$+C_2: r_2(t) = \big((2\pi - 1)t + 1, 2\pi t, 0\big)$

$r_2'(t) = (-sint, cost, 1) => F\big(r_1(t)\big) r_2'(t) = 0$

Therefore: $\int Fdr = 0$

e). $F(x, y, z) = \left(\sqrt{x^3 + y^3 + 5}, z, x^2\right)$, [….]

We have, $y^2 + 2z^2 = 1 <=> \begin{cases} y = cost \\ z = 1/\sqrt{2}sint \end{cases}, t \in [0,2\pi]$

We get, $C: r(t) = \left(-1, cost, \frac{1}{\sqrt{2}sint}\right) => r'(t) = \left(0, -sint, \frac{1}{\sqrt{2}cost}\right)$

And $F\big(r(t)\big) = \left(\sqrt{(-1 + \cos^3 t + 5)}\ , \frac{1}{\sqrt{2}sint}, 1\right)$

Follow by $\int Fdr = \int_0^{2\pi}(-\frac{1}{2}\sin^2 t + 1/\sqrt{2}cost)dt = -\sqrt{2}/2\pi$

6. show that F is conservative and find the potential function of F.

Compute $\int Fdr$ for the given F and curve C from points A to B.

a). $F(x, y) = 2xyi + (1 + x^2 - y^2)j\ ; A = (1,0)\ and\ B = (2,3)$

Let $M(x,y) = 2xy \; and \; N(x,y) = 1 + x^2 - y^2$

$\frac{\partial M}{\partial y} = 2x \; adn \frac{\partial N}{\partial x} = 2x$

Showing that , $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

That is , F is conservative

$=> \exists f \; such \; that \; F = \nabla f$

$=> \begin{cases} f_x = 2xy \; (1) \\ f_y = 1 + x^2 - y^2 \; (2) \end{cases}$

Follow by (1): $\int f_x dx = \int 2xydx = x^2y + c(y) \; => f(x,y) = x^2y + c(y)$

Then , $f_y(x,y) = x^2 + c'(y) = 1 + x^2 - y^2 => c'(y) = 1 - y^2$

We get, $c(y) = y - \frac{y^3}{3} + k \, , k \in \mathbb{R}$

And the potential function of F is $F(x,y) = x^2y + y - \frac{y^3}{3} + k \, , k \in \mathbb{R}$

We have $c: y = 3x - 3 \, , x \in [1,2]$

Or $r(t) = (t, 3t - 3) => r'(t) = (1,3) \, , \; t \in [1,2]$

And $F\big(r(t)\big) = (2t(3t-3), 1 + t^2 - (3t-3)^2)$

Follow by $\int Fdr = \int_1^2 (-18t^2 + 48t - 24)dt = 6$

b). $F(x,y) = (x + \tan^{-1} y) \, i + \frac{x+y}{1+y^2} j \;\; , A = (0,0) and \; B = (1,1)$

Let $F(x,y) = Mi + Nj$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \; and \frac{\partial N}{\partial x} = \frac{x+y}{1+y^2}$$

Showing that , $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus , F is conservative

$=> \exists f \; such \; that \; \nabla \mathrm{f} = \mathrm{F}$

$=> \begin{cases} f_x = x + \tan^{-1} y \;\; (1) \\ f_y = \frac{x+y}{1+y^2} \;\; (2) \end{cases}$

Follow by (1): $\int f_x dx = \int (x + \tan^{-1} y)dx = \frac{x^2}{2} + xtan^{-1}y + c(y)$

$f_y = \frac{x}{1+y^2} + c'(y) => c'(y) = \frac{y}{1+y^2} => c(y) = \frac{1}{2}\ln(1+y^2) + k, k \in \mathbb{R}$

The potential function of F is

$f(x,y) = \frac{x^2}{2} + x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) + k, k \in \mathbb{R}$

$\int Fdr = f(r(b)) - f(r(a)) = f(1,1) - f(0,0) = \frac{1+ln2}{2} + \frac{\pi}{4}$

c). $F(x,y,z) = (2xyz^2; x^2z^2; 2x^2zy) = (M,N,P)$

$A = (1,0,1)\ and\ B = (1,2,-1)$

$\frac{\partial P}{\partial y} = 2x^2z\ and\ \frac{\partial N}{\partial z} = 2x^2z \ => \ \frac{\partial P}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial P}{\partial x} = 4xzy\ and\ \frac{\partial M}{\partial z} = 4xzy \ => \ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial x}$

$\frac{\partial N}{\partial x} = 2xz^2\ and\ \frac{\partial M}{\partial y} = 2xz^2 \ => \ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Thus, F is conservative.

$=> \ \exists f\ such\ that\ \nabla f = F$

$=> \begin{cases} f_x = 2xyz^2 \ (1) \\ f_y = x^2z^2 \ (2) \\ f_z = 2x^2zy \ (3) \end{cases}$

Follow by (1): $=> f_y = x^2z^2 + c'(y,z) \ (4)$

Follow by (2) and (4): $=> c'(y,z) = 0$

Then , $c(y,z) = c(z)$

$f_x = 2xyz^2 + c(z) => f_z = 2x^2yz + c'(z) \ (5)$

By (3) and (5): $=> c(z) = k\ , k \in \mathbb{R}$

The potential function of F is $f(x,y,z) = x^2yz^2 + k\ , k \in \mathbb{R}$

$\int Fdr = f(r(b)) - f(r(a)) = 2$

d). $F(x,y,z) = \left(e^y, xe^y + lnz, \frac{y}{z}\right)\ , A = (0,1,1)\ and\ B = (1,0,2)$

$\frac{\partial P}{\partial y} = \frac{1}{z}\ and\ \frac{\partial N}{\partial z} = \frac{1}{z} \ => \ \frac{\partial P}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial P}{\partial x} = 0\ and\ \frac{\partial M}{\partial z} = 0 \ => \ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial x}$

$\frac{\partial N}{\partial x} = e^y \; and \, \frac{\partial M}{\partial y} = e^y \quad => \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Thus , F is conservative.

$=> \; \exists f \; such \; that \; \nabla \mathrm{f} = \mathrm{F}$

$$=> \begin{cases} f_x = e^y \;\; (1) \\ f_y = xe^y + lnz \;\; (2) \\ f_z = \frac{y}{z} \;\; (3) \end{cases}$$

Follow by (1): $=> f(x, y, z) = xe^y + c(y, z)$

Then $=> f_y = xe^y + c'(y, z) \;\; (4)$

By (2) and (4) : $we \; get \; c'(y, z) = lnz$

And $=> c(y, z) = ylnz + c(z) => \;\; f(x, y, z) = xe^y + ylnz + c(z)$

$f_z = \frac{y}{z} + c'(z) = \frac{y}{z} \;\; => c(z) = k, k \in \mathbb{R}$

Therefore , $f(x, y, z) = xe^y + ylnz + k \, , k \in \mathbb{R}$ is potential function of F

$\int Fdr = f\big(r(b)\big) - f\big(r(a)\big) = 1$

8. Find the equation of the tangent plane to the parametric surface represent by r at the specified point.

a). $r(u, v) = (u - v)i + (u + v)j + v^2 k$

$r_u(u, v) = (1,1,0) and \; r_v(u, v) = (-1, -1,2v)$

$r_v(u_0, v_0) = r_v(2,1) = (-1, -1,2)$

Then,$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = (2, -2,0)$

<u>Thus:</u> The equation of a plane in $2(x - 2) - 2y = 0$

b). $r(u, v) = ucosvi + 2usinvj + u^2 k, (u, v) = (1, \pi)$

$r_u(u, v) = (cosv, 2sinv, 2u) and \; r_v(u, v) = (-usinv, 2ucosv, 0)$

Then, $N = r_u(u_0, v_0) r_v(u_0, v_0) = \begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{vmatrix} = (4,0,2)$

<u>Therefore,</u> The equation of a plane is 4(x+1)+2(z-1)=0

10. Find $\iint f(x, y, z) ds$

a).

we get , $\iint f(x,y,z)ds = \iint f(x,y,z)\sqrt{1+g_x^2+g_y^2}dA$

Let $(s): z = g(x,y) = 6-3x-2y$

Since $g_x = -3$ and $g_y = -2$

$\sqrt{1+g_x^2+g_y^2} = \sqrt{14}$

Then, $\iint f(x,y,z)ds = \int_0^2 \int_0^{\frac{6-3x}{2}} \sqrt{14}(x+y)dydx = 5\sqrt{14}$

b). $f(x,y,z) = xz; S\ is\ part\ of\ plane\ y+z=4\ inside\ the\ cylinder\ x^2+y^2=4$

Let $(s): z = g(x,y) = 4-y$

Since $g_x = 0$ and $g_y = -1$

$\sqrt{1+g_x^2+g_y^2} = \sqrt{2}$

$\iint f(x,y,z) = \iint (4x-xy)\sqrt{2}dA$

By using Polar coordinate

Then, $\iint f(x,y,z)ds = \sqrt{2}\int_0^{2\pi}\int_0^2 (4rcos\theta - r^2cos\theta sin\theta)rdrd\theta = 0$

c). $f(x,y,z) = x + \frac{y}{\sqrt{4z+5}}$

we have , $r(u,v) = (4,v,v^2-1)$

And $r_u(u,v)\times r_v(u,v) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = (0,-2v,1)$

$\|r_u(u,v)\times r_v(u,v)\| = \sqrt{4v^2+1}$

We get, $\iint f(x,y,z)ds = \int_{-1}^1\int_0^1\left(4\sqrt{4v^2+1}+v\right)dv = \int_0^1\sqrt{4v^2+1}dv$

$\iint f(x,y,z)ds = \frac{1}{2}\left(\sqrt{5}+\frac{1}{2}\ln\left(2+\sqrt{5}\right)\right)$

d). $f(x,y,z) = z\sqrt{1+x^2+y^2}$, S is the helicoid with vector representation r(u,v)

$r_u = (cosv, sinv, 0) and\ r_v = (-usinv, ucosv, 1)$

$\|r_u \times r_v\| = \sqrt{15}$

We get, $\iint f(x,y,z)ds = \int_0^{2\pi}\int_0^1 \frac{4}{3}vdv = \frac{8}{3}\pi^2$

11. Find the mass of the surface S having the given density mass function.

a). S is the part of the plane $x + 2y + 3z = 6$ in the first octant the density at any point of S is clirectly proportional to the square of distance between the point to yz-plane.

We get, $\frac{f(x,y,z)}{x^2} = k\ , k \in \mathbb{R}$

$=> f(x, y, z) = kx^2\ .That\ is\ the\ density\ of\ (S).$

Then , $mass(S) = \iint kx^2\sqrt{g_x^2 + g_y^2 + 1}dA$

Let (s): $g(x, y) = z = \frac{6-x-2y}{3}$

Since , $g_x = -\frac{1}{3}\ and\ g_y = -\frac{2}{3}$

The region R is given by $R = \{(x, y) \in \mathbb{R}^2 : 0 < x < 6, 0 < y < \frac{6-x}{2}\}$

We get , $mass(S) = \int_0^6 \int_0^{\frac{6-x}{2}} \frac{kx^2}{3}\sqrt{14}dydx = 18\sqrt{14}k, k \in \mathbb{R}$

b). S is the hemisphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$ the density at any point on S is directly to the distance between the point to xy-plan.

By using hypothesis above , $\frac{f(x,y,z)}{z} = k, k \in \mathbb{R}$

$=> f(x, y, z) = kz$

Then, $mass(S) = \iint f(x, y, z)ds = \iint kz\sqrt{g_x^2 + g_y^2 + 1}dA$

Let (S): $g(x, y) = z = \sqrt{1 - x^2 - y^2}$

$=> g_x(x, y) = \frac{-x}{\sqrt{1-x^2-y^2}}\ and\ g_y(x, y) = \frac{-y}{\sqrt{1-x^2-y^2}}$

We have $\begin{cases} z = \sqrt{x^2 + y^2} \\ z^2 + x^2 + y^2 = 1 \end{cases} => x^2 + y^2 = 1/2$

So, the region R is given by $R = \{(x, y) \in \mathbb{R}: x^2 + y^2 \leq \left(\frac{1}{\sqrt{2}}\right)^2$

By using Polar coordinates , we have

$mass(S) = \int_0^{2\pi} \int_0^{1/\sqrt{2}} kdrd\theta = \frac{\pi k}{2}, k \in \mathbb{R}$

12. Find $\iint F.Nds, that\ is\ , find\ the\ flux\ of\ F\ across\ S$

a). S is the part of the plane x+2y+3z=6 in the first octant, the density at any point of S is directly proportional to the square of distance between the point to yz-plane.

We have , $F(x,y,z) = (2x, 2y, z)\ where\ M = 2x, N = 2y\ and\ p = z$

Let (S): $g(x,y) = z = 4 - x^2 - y^2$

Since, $g_x(x,y) = -2x\ and\ g_y(x,y) = -2y$

We get , $\iint F.Nds = \iint (4x^2 + 4y^2 + 4 - x^2 - y^2)dA$

$= \int_0^{2\pi} \int_0^2 (3x^2 + 3y^2 + 4)dA = 40\pi$

b). $F(x,y,z) = (x^2, xy, xz)$, S is the surface of the surface of tetrahedrion with vertices (0,0,0),(1,0,0),(0,2,0) and (0,0,3).

We get , $\iint F.Nds = \iint -M_{g_x} - N_{g_y} + P)dA$

Let $M(x,y,z) = x^2\ , N = xy\ and\ P = xz$

Let $g(x,y) = z = 1/2(6 - 6x - 3y)$

$$g_x(x,y) = -3\ and\ g_y(x,y) = -3/2$$

We get, $\iint F.Nds = \int_0^1 \int_0^{2-2x} 3xdydx$

$= \int_0^1 \frac{3}{2(4-8x+4x^2)dx} = 2$

13. By using the above hypothesis : $\frac{f(x,y,z)}{z} = k, k \in \mathbb{R}$

Then, $f(x,y,z) = kz\ \ (charge\ density)$

Let $Q = \iint f(x,y,z)ds = \iint kz\sqrt{1 + g_x^2 + g_y^2}dA$

Let $g(x,y) = z = \sqrt{25 - x^2 - y^2}$

Then, $g_x(x,y) = \frac{-x}{\sqrt{25-x^2-y^2}}\ and\ g_y(x,y) = \frac{-y}{\sqrt{25-x^2-y^2}}$

We get, $Q = \iint kz\frac{5}{g(x,y)} = 5k\int_0^{2\pi}\int_0^3 rdrd\theta = 45\pi k, k \in \mathbb{R}$

14. Let $R = \iint F.Nds = \iint \left(-M_g - N_{g_y} + p\right)dA$

We have , $F(x,y,z) = (2x, 2y, 3z)$

Let $M(x,y,z) = 2x\ , N = 2y\ and\ P = 3z$

Let $g(x,y) = z = 6 - x - 3y$

$$=> g_x(x,y) = -1\ and\ g_y(x,y) = -3$$

We get, $R = \iint \big(2x + 6y + 3(6 - x - 3y)\big)dA$

The region R is given by $R = \{(x, y) \in \mathbb{R} := 0 \leq x \leq 6\ , 0 \leq y \leq 2 - x/3$

$R = \int_0^6 \int_0^{2-x/3} (-x - 3y + 18)dydx = 84$

16. Verify the divergence Theorem by evaluating $\iint F.Nds$.

a). $F(x, y, z) = (2xy, -y^2, 2yz)$;S is the cube bounded by the plane x=0,x=2,y=0,y=2,z=0,and z=2

+Divergence Theorem $\iint F.Nds = \iiint divF\ dv$

$\iiint divFdv = \int_0^2 \int_0^2 \int_0^2 (2y - 2y + 3z)dxdydz = 24$

+Evaluate $\iint F.Nds$

We get, $\iint F.Nds$ =$(\iint F.Nds)_{s_1} + (\iint F.Nds)_{s_2} + (\iint F.Nds)_{s_3} + (\iint F.Nds)_{s_4} + (\iint F.Nds)_{s_5}$+$(\iint F.Nds)_{s_6}$

For $(\iint F.Nds)_{s_1}$

We have $(s_1): z = g_1(x, y) = 0, N_1 = (0,0,-1)$

$$(\iint F.Nds)_{s_1} = 0$$

For $(\iint F.Nds)_{s_2}$

We have $y = g_2(x, z) = 0, N_2 = (0,-1,0)$

$$(\iint F.Nds)_{s_2} = 0$$

For $(\iint F.Nds)_{s_3}$

We have $(s_3): z = g_3(x, y) = 1; N_3 = (0,0,1)$

$$(\iint F.Nds)_{s_3} = \int_0^2 \int_0^2 3ydxdy = 12$$

For $(\iint F.Nds)_{s_4}$

We have $(s_4): z = g(x, y) = 0; N_4 = (0,0,-1)$

$$(\iint F.Nds)_{s_4} = 0$$

For $(\iint F.Nds)_{s_5}$

We have $(s_5): z = g(x, y) = 0, N_4 = (0,0,-1)$

$$(\iint F.Nds)_{s_5} = 0$$

For $(\iint F.Nds)_{s_6}$

We have , $(s_6): z = g_6(x,y) = 1, N_6 = (0,0,1)$

$$\iint F.Nds)_{s_6} = \int_0^2 \int_0^2 3ydxdy = 12$$

Thus, $\iint F.Nds = 12 + 12 = 24$

b). $F(x,y,z) = (xz, zy, 2z^2); S\ is\ the\ surface\ bounded\ by\ z = 1 - x^2 - y^2, z = 0$

+Divergence Theorem $\iiint div\ FdV = \iint \int_0^{1-x^2-y^2} 6zdzdA$

By using Cylindrical coordinate , we have

$$\iiint div\ FdV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6zrdzdrd\theta = \pi$$

+Evaluate $\iint F.Nds$

Let $(s): z = g(x,y) = 1 - x^2 - y^2$

$g_x(x,y) = -2x\ and\ g_y(x,y) = -2y$

By using Polar coordinates

We get, $\iint F.Nds = \int_0^{2\pi} \int_0^1 (2 - 2r^2) rdrd\theta = \pi$

18. a). Use the Divergence Theorem to verify that the solid bounded by a surface S is

$$\iint xdydz = \iint ydzdx = \iint zdxdy$$

+Let $f(x,y,z) = (x,0,0)\ and\ N = (1,0,0)$

We get $\iint F.Nds = \iint xdydz = \iiint divFdv = V$ (1)

+Let $f(x,y,z) = (0,y,0)\ and\ N = (0,1,0)$

We get $\iint F.Nds = \iint ydxdz = \iiint divFdv = V$ (2)

+Let $f(x,y,z) = (0,0,z)\ and\ N = (0,0,1)$

We get $\iint F.Nds = \iint zdxdy = \iiint divFdv = V$ (1)

Follow by (1) , (2) and (3) :

$thus, \iint xdydz = \iint ydzdx = \iint zdxdy$

b). Verify that the result of part (a) for the cube bounded

by x=0, x=a, y=0, y=a, z=0 and z=a

By using the result of part (a) for the cube :

+Follow by (1): $V = \iint xdydz = \int_0^a \int_0^a adydz = a^3$

+Follow by (1): $V = \iint ydxdz = \int_0^a \int_0^a adxdz = a^3$

+Follow by (1): $V = \iint zdxdy = \int_0^a \int_0^a adxdy = a^3$

And also, $V_{cube} = a \times a \times a = a^3$

Therefore: The result of part (a) is verified.

20. Given the vector field F(x,y,z)=(x,y,z) , verify that $\iint F.Nds = 3V$, where V is the volume of the solid bounded by the closed surface S.

By using Divergence Theorem:

$$\iint F.Nds = \iiint divFdV = \iiint 3dV = 3V$$

Therefore: $\iint F.Nds = 3V$

22. Use Stokes's Theorem to evaluate $\int Fdr$

a). $F(x,y,z) = \left(\frac{arctany}{x}, lnx^2 + y^2, 1\right); C\ is\ a\ triangle\ with\ vertices\ (0,0,0), (1,1,1) and\ (0,0,2).$

Since $C = C_1 U C_2 U C_3$ is a simple closed curve , then by using Stokes's Theorem;

We have $\oint Fdr = \iint (curlF).Nds$

$$+ \quad curl\ F = \nabla F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ arctany/x & lnx^2 + y^2 & 1 \end{vmatrix}$$

$$=> curlF = \left(0,0, \frac{2x}{x^2} - \frac{x}{x^2+y^2}\right)$$

We know that the equation of plane is x-y=0 ⇔ y=x

Let $(s): y = g(x,z) = x , (x,z) \in \mathbb{R}^2$

$$=> N = (-g_x, 1, -g_z) = (-1,1,0)$$

We get , $\oint Fdr = \iint 0ds = 0$

C). $f(x,y,z) = \left(-ln\sqrt{x^2+y^2}, \frac{arctanx}{y}, 1\right); S\ is\ the\ surface\ of$

$z = 9 - 2x - 3y\ over\ r = 2sin\theta\ in\ the\ first\ octant.$

$+ \quad curl\ F = \nabla F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -ln\sqrt{x^2+y^2} & arctanx/y & 1 \end{vmatrix} = \left(0,0,\frac{2y}{x^2+y^2}\right)$

Let $(s): g(x,y) = z = 9 - 2x - 3y$

$N = (-g_x, -g_y, 1) = (2,3,1)$

By using Stokes's Theorem , we have

$$\oint Fdr = \iint (curF)Nds = \iint \frac{2y}{x^2+y^2} dA$$

By using Polar coordinate , we have

$$\oint Fdr = \int_0^{\pi/2} \int_0^{2sin\theta} 2sin\theta drd\theta = \int_0^{\pi/2} (2 - 2cos2\theta)d\theta = \pi - 1$$

d). $F(x,y,z) = (xyz, y, z); S\ is\ the\ surface\ of\ z = x^2, 0 \le x \le a, 0 \le y \le a\ .N\ is\ the\ downword\ unit\ normal\ to\ the\ surface.$

$+ \quad curl\ F = \nabla F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & y & z \end{vmatrix} = (0, xy, -xz)$

Let $(s): g(x,y) = z = x^2$

$$N = (-g_x, -g_y, -1) = (-2x, 0, -1)$$

By using Stokes's Theorem, we have

$$\oint Fdr = \iint (curlF).Nds = \int_0^a \int_0^a x^3 dydx = \frac{a^5}{4}$$