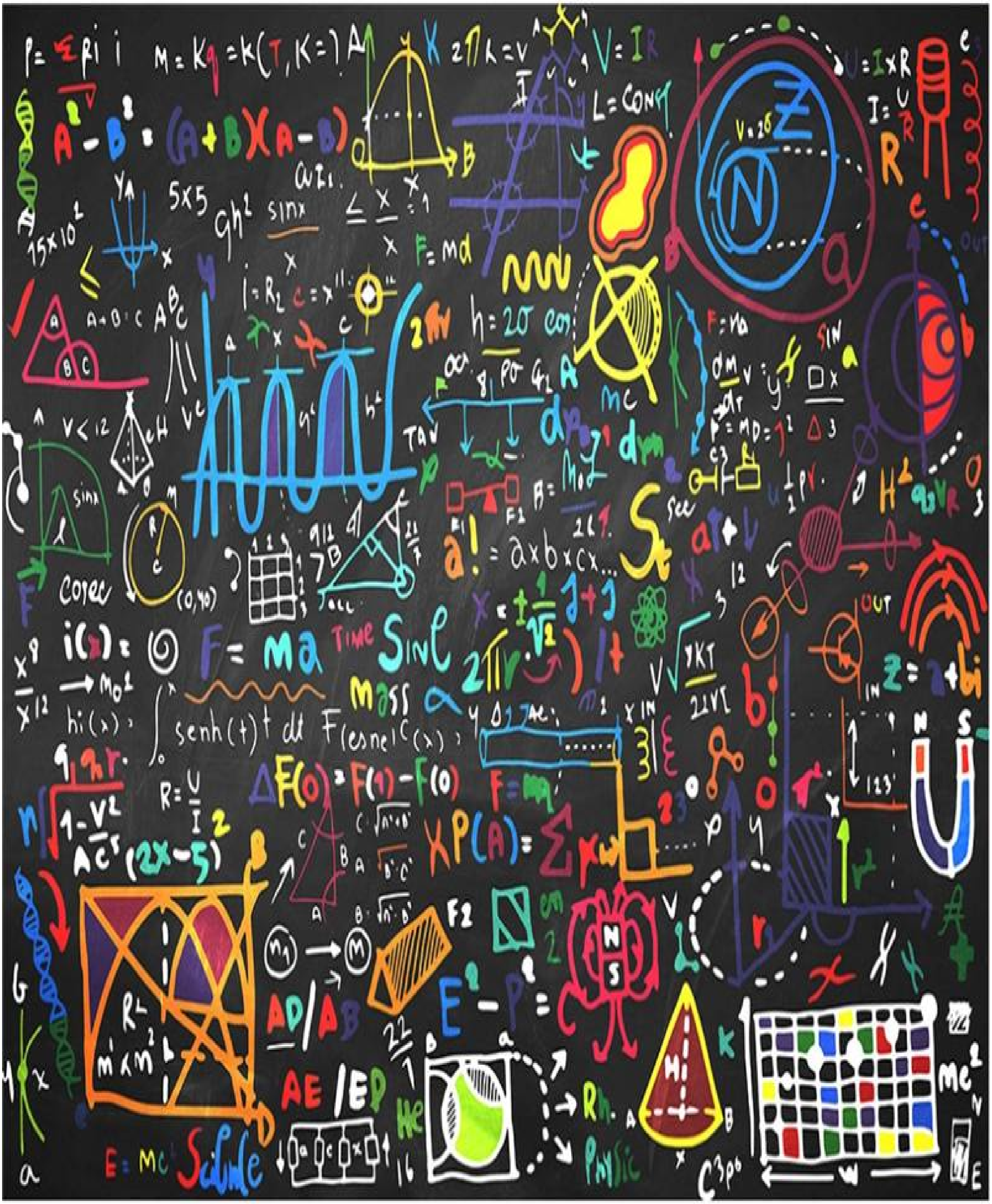


Taing Kimmeng

CalculusIII & Linear Algebra

TD5&6 (Double&Multiple Intergration)

I2-TD5
(Double Integrations)

1. Sketch the region and evaluate the iterated integral

(a) $\int_0^{\pi} \int_0^{\sin x} \cos x \, dy \, dx$

(b) $\int_0^1 \int_y^1 x^2 \sin(xy) \, dx \, dy$

(c) $\int_0^2 \int_{y/2}^1 e^{-x^2} \, dx \, dy$

(d) $\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x} \, dx \, dy$

2. Given $I = \int_0^1 \int_{y/2}^{(y/2)+2} (2x - y) \, dx \, dy$.

(a) Evaluate this integral and sketch the region R of integration in the xy-plane.

(b) Let $u = 2x - y$ and $v = y$. Find the region D in the uv-plane that corresponds to R.

(c) Use the change variables $u = 2x - y$ and $v = y$ to evaluate I.

3. Evaluate $J = \iint_R \frac{xy}{y^2 - x^2} \, dA$, where R is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1, x^2 - y^2 = 4$ and the ellipses $x^2/4 + y^2 = 1, x^2/16 + y^2/4 = 1$. (Hint: Sketch the region R, and use it to make an appropriate change of variables.)

4. $I = \iint_R \frac{(x-y)^2}{(x+y+1)^2} \, dA$, R is the region bounded by the lines $x = 0; y = 0$ and $x + y = 1$.

✓ 5. Compute $I = \iint_R \frac{y}{x} dA$, $R = \{(x, y) \in \mathbb{R}^2 : x < y < 2x, x < y^2 < 2x\}$

(Hint: Use the change variables $u = \frac{y}{x}$ and $v = \frac{y^2}{x}$.)

6. Compute $I = \iint_R \frac{y^2(2 + 2y^2 - x)}{(2-x)^4} \, dA$; $R = \left\{(x, y) \in \mathbb{R}^2 \mid x - 1 < y^2 < x, \ 0 < y < 2 - x\right\}$

(Hint: Using change variables $u = \frac{y}{2-x}$ and $v = y^2 - x$)

✓ 7. Compute $J = \iint_R (x + y) \, dA$, where R is the region bounded by the ellipse

$$13x^2 + 14xy + 10y^2 = 9.$$

(Hint: Use the change variables $u = 2x - y$ and $v = x + y$.)

8. Evaluate $K = \iint_R e^{x^2+xy+y^2} \, dA$, where is the region bounded by the ellipse

$$x^2 + xy + y^2 = 1$$

(Hint: Use change variables $u = x + \frac{y}{2}$ and $v = \frac{\sqrt{3}}{2} y$.)

✓ 9. Evaluate the following integrals by using Polar coordinates.

(a) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx\,dy$ (b) $\int_0^1 \int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$

(c) $\iint_R \sin(x^2+y^2)\,dA$, R is the region of unit circle centered at the origin.

(d) $\iint_R \frac{1}{\sqrt{4-x^2-y^2}}\,dA$, where $R = \left\{(x,y) \in \mathbb{R}^2 : y \le x; x \le \sqrt{1-y^2}\right\}$.

(e) $\iint_R \frac{y^2}{x^2+y^2} dA$, $R = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4, x > 2\}$.

(f) $\iint_R x\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : 1 < (x-2)^2 + y^2 < 4\}$.

(g) $\iint_R (x^2+y^2)\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-2)^2 < 1\}$.

(h) $\iint_R (x^2+y^2)\,dA$, $R = \{(x,y) \in \mathbb{R} \mid x^2+y^2 \le 2ax; x^2+y^2 \le 2ay\}$

(i) $\iint_R \frac{1+y^3}{(1+x^2+y^2)}\,dA$, $R = \{(x,y) \in \mathbb{R} \mid x < x^2+y^2 < 1\}$

10. Use any method to compute the following integral on the given region R.

(a) $\iint_R \frac{xy}{1+x^2+y^2}\,dA$; $R = \{(x,y) \in [0,1]^2 : x^2+y^2 > 1\}$.

(b) $\iint_R (x^3+y^3)\,dA$; $R = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{1}{y} < x < \frac{2}{y};\ x^2 < y < 2x^2\right\}$

(c) $\iint_R |x-y|\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$

(d) $\iint_R \frac{y^2\cos(x^2+y^2)}{x^2}\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4; 0 < y < x\}$

(e) $\iint_R (x^2-y^2)\,dA$, $R = \left\{(x,y) \in \mathbb{R}^2 : \frac{(x-1)^2}{4} + \frac{(y-1)^2}{9} < 1\right\}$

(f) $\iint_R xy\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : 4(x-1)^2 + 9(y-1)^2 < 36, y > 1, x-y > 0.\}$

(g) $\iint_R \frac{1}{\sqrt[4]{(x^2+y^2)^3}}\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 2(\sqrt{x^2+y^2}+x)\}$

(h) $\iint_R x(y-1)^2\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : 3x^2+y^2-2y < 2, 0 < y-1 < x\}$

(i) $\iint_R \left(x^2 + \frac{y^2}{9}\right) \sin\left(2\arctan\frac{y}{3x}\right)\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1, 0 < y < x\}$.

(j) $\iint_R \frac{y^2(2+2y^2-x)}{(2-x)^4}\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : x-1 < y^2 < x,\ 0 < y < 2-x\}$.

(k) $\iint_R \frac{y}{x^3} \sin\left(\pi - \frac{\pi y^2}{x^2}\right)\,dA$, where $R = \{(x,y) \in \mathbb{R}^2 : 0 < \sqrt{2}y < x,\ 1 < x^2-y^2 < 4\}$.

(l) $\iint_R \frac{(2x+y-3)^2}{(2y-x+6)^2}\,dA$, R is the square with vertices $(0,0), (2,1), (3,-1)$ and $(1,-2)$.

(m) $\iint_R (x^3 - 3xy^2)\, dA$, $R = \{(x, y) \in \mathbb{R}^2 :\ (x+1)^2 + y^2 \leq 9; (x-1)^2 + y^2 \geq 1\}$.

11. Find the region area R.

(a) $R = \{(x, y) \in \mathbb{R}^2 : y^2 - 6x < 0, x - y < 12, x^2 + y^2 > 16\}$

(b) $R = \left\{(x, y) \in \mathbb{R}^2 \mid 2a_1 x < y^2 < 2a_2 x;\ 2b_1 y < x^2 < 2b_2 y\right\}$; $a_1, a_2, b_1, b_2 \in (0, +\infty)$ and $a_1 < a_2; b_1 < b_2$.

(c) $R = \{(x, y) \in \mathbb{R}^2 : x < 0, y^2 < x^4(x+4)\}$

(d) Find the area of the region $R = \left\{(x, y) \in \mathbb{R}^2 \mid x^{2/3} + y^{2/3} = 1\right\}$.

(e) Find the area of the region $R = \left\{(x, y) \in \mathbb{R}^2 \mid (y - x)^2 = 1 - x^2\right\}$.

(f) R is bounded by the ellipse $5x^2 + 6xy + 5y^2 = 4$.

(g) R is bounded by the circle $r = 3\cos\theta$.

(h) R is bounded by one loop of the four-leaved rose $r = \cos 2\theta$.

(i) R is bounded by the cardioid $r = 3 - 3\sin\theta$.

(j) R is bounded by the lemniscate $r^2 = 4\cos 2\theta$.

(k) R is the region lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

(l) R is the region lies inside $r = 1 - \cos\theta$ and outside $r = \cos\theta$.

12. Let $f : \mathbb{R} \to \mathbb{R}_+^*$ and $R = \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 < 4\}$. Compute the integral

$$I = \iint_R \frac{2f(x) + 5f(y)}{f(x) + f(y)}\, dA$$

13. Calculate $I = \iint_R \frac{x}{(1 + x^2)(1 + xy)}\, dA$ where

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1;\ 0 < y < 1\}$$

Then deduce the value of $J = \int_0^1 \frac{\ln(1 + x)}{1 + x^2} dx$.

14. (a) Show the existence of the $I = \int_0^{\pi/2} \frac{\ln(1 + \cos x)}{\cos x}\, dx$

(b) Show that $I = \iint_R \frac{\sin y}{1 + \cos x \cos y}\, dA$ where

$$R = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < \frac{\pi}{2};\ 0 < y < \frac{\pi}{2}\right\}$$

(c) Deduce the value of I.

15. The objective of this exercise is to compute the value of $\int_0^1 \frac{\ln t}{1 - t^2}\, dt$.

(a) Compute $I = \iint_{0 \leq y \leq x \leq 1} \frac{1}{(1 + x^2)(1 + y^2)} dA$;

(b) Show that the following integrals are convergent

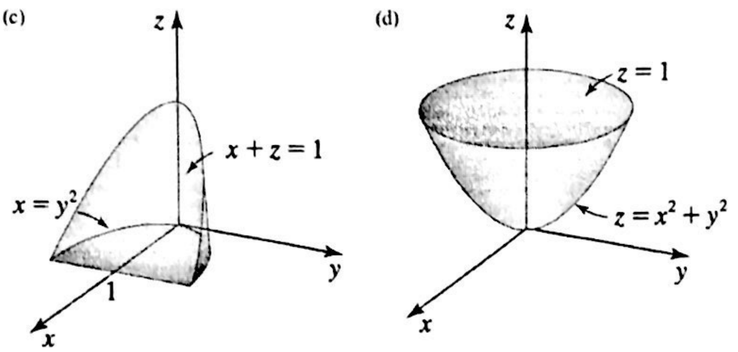
$$J = \int_0^{\pi/4} \frac{\ln(2\cos^2 t)}{2\cos 2t}\,dt \quad K = \int_0^{\pi/4} \frac{\ln(2\sin^2 t)}{2\cos 2t}\,dt \quad L = \int_0^{1} \frac{\ln t}{1-t^2}\,dt$$

(c) Show that $I = J$.

(d) Calculate $J + K$ and $J - K$ in terms of L.

(e) Deduce the values of K and L.

16. Use double integral to find the volume of the solid shown in the figures below:



17. Find the area of the surface S over the region R.

(a) S is the surface of $f(x, y) = 1 + x - 2y$, R is a square with vertices $(0, 0), (3, 0), (0, 3)$ and $(3, 3)$.

(b) S is the paraboloid $y = 9 - x^2 - z^2$ that lies between the planes $y = 0$ and $y = 5$.

(c) S is the part of the paraboloid $z = y^2 - x^2$ that lies above the annular region $R = \{(x, y) : 1 \le x^2 + y^2 \le 4\}$.

(d) S is the part of the sphere $x^2+y^2+z^2 = a^2$ that lies inside the cylinder $x^2-ax+y^2 = 0$.

(e) S is comprises the parts of the cylinder $x^2 + z^2 = 1$ that lie within the cylinder $y^2 + z^2 = 1$.

18. Find the mass and the center of mass of the lamina occupying the region R and having mass density ρ.

(a) R is the region bounded by $y = 3 - x, y = 0$ and $x = 1$; $\rho(x, y) = 2xy$.

(b) R is the region bounded by the parabola $y = 4 - x^2$ and the x-axis; and $\rho(x, y) = y$.

(c) R is the region bounded by the circle $r = 2\cos\theta$; $\rho(r, \theta) = r$.

(d) R is the region bounded by the cardioid $r = 1 + \cos\theta$; $\rho(r, \theta) = 3$

19. Find the moments of inertia I_x, I_y, and I_o.

(a) R is the triangular region with vertices $(0, 0), (2, 1)$ and $(4, 0)$; $\rho(x, y) = x$.

(b) R is the region bounded by $y = 3 - x, y = 0$ and $x = 1$; $\rho(x, y) = 2xy$.

(c) R is the region bounded by $y = \sqrt{x}, y = 0$ and $x = 4$; $\rho(x, y) = xy$.

(d) R is the region in the first quadrant bounded by the circle $x^2+y^2 = 1$; $\rho(x, y) = x+y$.

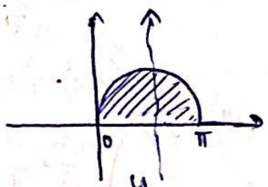
Double Integrations

1). Sketch the region and evaluate the interated integral

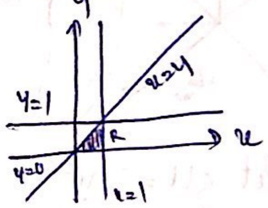
(a) $R = \{x, y \in \mathbb{R}^2,\ 0 \le x \le \pi,\ 0 \le y \le \sin x\}$

$$\int_0^{\pi}\int_0^{\sin x} \cos x \, dy\, dx = \frac{1}{2}\int_0^{\pi} \sin 2x \, dx = -\frac{1}{4}\cos 2x \Big|_0^{\pi}$$

$$= +\frac{1}{4}(1-1) = 0$$



b) $\displaystyle\int_0^1\int_y^1 x^2 \sin(xy)\, dx\, dy$

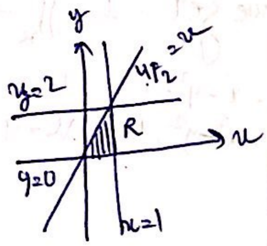


$$= \int_0^1\int_0^x x^2 \sin(xy)\, dy\, dx$$

$$= \int_0^1 x^2\left[-\frac{1}{x}\cos(xy)\right]_0^x dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{2}\sin x^2\right]_0^1 = \frac{1}{2} - \frac{1}{2}\sin 1$$

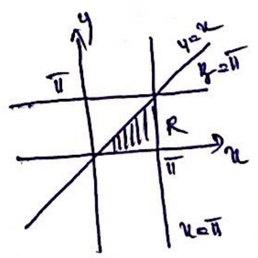
c). $\displaystyle C = \int_0^2\int_{y/2}^1 e^{-x^2}\, dx\, dy$



$$C = \int_0^1\int_0^{2x} e^{-x^2}\, dy\, dx$$

$$= \int_0^1 2x e^{-x^2}\, dx = \left[-e^{-x^2}\right]_0^1 = 1 - \frac{1}{e}$$

d). $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$

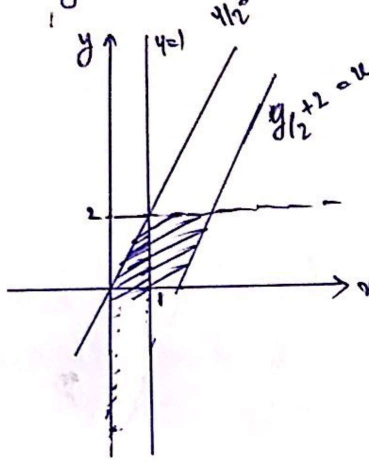


$$= \int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^{\pi} \sin x\, dx = -\cos x\Big|_0^{\pi} = -(-1-1) = 2$$

2. Given $I = \int_0^1 \int_{y/2}^{y/2+2} (2x-y)\,dx\,dy$

a. Evaluate this integral and sketch the region R of integration in the xy-plane



$$I = \int_0^1 \int_{y/2}^{y/2+2} (2x-y)\,dx\,dy = \int_0^1 \left[(y/2+2)^2 - (y/2)^2 - y(y/2+2-y/2)\right]dy$$

$$= \int_0^1 (2y+4-2y)\,dy = \int_0^1 4\,dy = \underline{4}$$

b. Let $u = 2x-y$ and $v = y$. Find the region D in the uv plane that corresponds to R.

Let $u = 2x-y$, $v = y$

Since $0 \le y \le 1$, $y/2 \le x \le \frac{y}{2}+2$

$\Rightarrow 0 \le v \le 1$, $0 \le u \le 4$

Thus $D = (u,v) \in \mathbb{R}^2 : 0 \le u \le 1,\ 0 \le v \le 1$

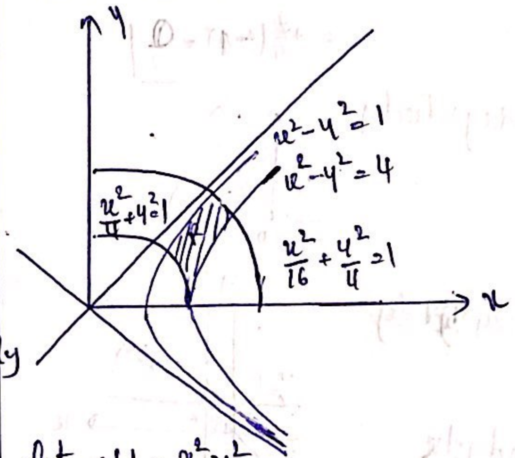
c. Use the change variables $u = 2x-y$ and $v = y$ to evaluated I

$$\begin{cases} u = 2x-y \\ v = y \end{cases}$$

$$\frac{1}{|J|} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2 \Rightarrow |J| = 1/2$$

$$I = \frac{1}{2}\int_0^1 \int_0^4 u\,du\,dv = \frac{1}{2} \times 1 \times \frac{4^2}{2} = \underline{4}$$

3. Evaluate $I = \iint_R \frac{xy}{y^2-x^2}\,dA$ where R is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1$, $x^2-y^2=4$ and ellipses $\frac{x^2}{4}+y^2=1$, $\frac{x^2}{16}+\frac{y^2}{4}=1$ (Hint: sketch the region R, and use it to make an appropriate change of variables).



let $\begin{cases} u = x^2-y^2 \\ v = \frac{x^2}{4}+y^2 \end{cases}$

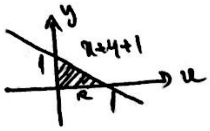
$$\frac{1}{|J|} = \begin{vmatrix} 2x & -2y \\ \frac{x}{2} & 2y \end{vmatrix} = 5xy \Rightarrow |J| = \frac{1}{5xy}$$

we get $I = \iint_R \frac{xy}{y^2-x^2}\,dA = -\frac{1}{5}\iint_D \frac{1}{u}\,dA'$

- $x^2-y^2=1 \Rightarrow u=1$
- $x^2+y^2=4 \Rightarrow u=4$
- $\frac{x^2}{4}+y^2=1 \Rightarrow v=1$
- $\frac{x^2}{16}+\frac{y^2}{4}=1 \Leftrightarrow \frac{x^2}{4}+y^2=4 \Rightarrow v=4$

$$\Rightarrow I = -\frac{1}{5}\int_1^4\int_1^4 \frac{1}{u}\,du\,dv = -\frac{1}{5}\times 3 \times \ln 4 = \underline{-\frac{6}{5}\ln 2}$$

④ $I=\iint_R \frac{(x-y)^2}{(x+y+1)^2}\,dA$, R is the region bounded by the lines $x=0$, $y=0$ and $x+y=1$



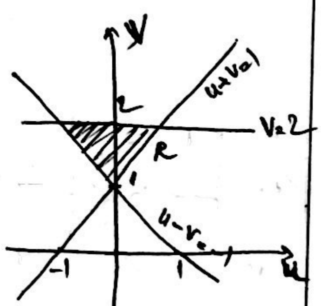
let $\begin{cases} U=x-y \\ V=x+y+1 \end{cases}$

$$\frac{1}{|J|}=\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}=2 \Rightarrow |J|=\frac{1}{2}$$

$\begin{cases} U+V=2x+1 \\ U-V=-2y-1 \end{cases}$

. if $x=0$ $\Rightarrow$ $\begin{cases} U+V=1 \\ U-V=-1 \end{cases}$
. if $y=0$
. if $x+y=1$ $\Rightarrow$ $V=2$

then $I=\iint_D \frac{u^2}{v^2}\,dv$



$$=\frac{1}{2}\int_1^2\int_{1-v}^{v-1}\frac{u^2}{v^2}\,du\,dv$$

$$=\frac{1}{6}\int_1^2 \frac{1}{v^2}\left((v^3-3v^2+3v-1)+(v^3-3v^2+3v-1)\right)dv$$

$$=\frac{1}{3}\int_1^2\left(v-3+\frac{3}{v}-\frac{1}{v^2}\right)dv=\frac{1}{3}\left(\frac{3}{2}-\cancel{3}+3\ln 2+\cancel{\frac{1}{2}}\ldots\right)$$

$$=\frac{1}{3}(-2+3\ln 2)$$

$$=-\frac{2}{3}+\ln 2$$

⑤ Compute $I=\iint_R \frac{y}{x}\,dA$, $R=\{(x,y)\in\mathbb{R}^2 : x<y<2x,\ x<y^2<2x\}$

(Hint: Use the change variable $u=\frac{y}{x}$ and $v=\frac{y^2}{x}$)

let $\begin{cases} u=y/x \\ v=y^2/x \end{cases}$

. $x<y<2x \Leftrightarrow 1<y/x<2 \Leftrightarrow 1<u<2$

. $x<y^2<2x \Leftrightarrow 1<\frac{y^2}{x}<2 \Leftrightarrow 1<v<2$

$\Rightarrow \begin{cases} x=u^{-2}v \\ y=u^{-1}v \end{cases}$

$$J=\begin{vmatrix} -2u^{-3}v & u^{-2} \\ -u^{-2}v & u^{-1} \end{vmatrix}=-2u^{-4}v+u^{-4}v=-u^{-4}v$$

$\Rightarrow |J|=u^{-4}v$

we get $I=\int_1^2\int_1^2 (u^{-4}v)\frac{u^{-1}v}{u^{-2}v}\,du\,dv$

$$=\int_1^2\int_1^2 u^{-3}v\,du\,dv=\frac{4-1}{2}\times\left(-\frac{1}{8}+\frac{1}{2}\right)=\frac{3}{2}\times\frac{3}{8}=\frac{9}{16}$$

⑥ Compute $I=\iint_R \frac{y^2(2+2y^2-x)}{(2-x)^4}\,dA$

$R=\{(x,y)\in\mathbb{R}^2 \mid x-1<y^2<x,\ 0<y<2-x\}$

(Hint Using change variable $u=\frac{y}{2-x}$ and $v=y^2-x$)

let $\begin{cases} u=\frac{y}{2-x} \\ v=y^2-x \end{cases}$

. $x-1<y^2<x \Leftrightarrow -1<y^2-x<0$
$\Leftrightarrow -1<v<0$

. $0<y<2-x \Leftrightarrow 0<\frac{y}{2-x}<1$
$\Leftrightarrow 0<u<1$

$$\frac{1}{J}=\begin{vmatrix} \frac{y}{(2-x)^2} & \frac{1}{2-x} \\ -1 & 2y \end{vmatrix}=\frac{2y^2}{(2-x)^2}+\frac{1}{(2-x)}$$

$$=\frac{2y^2+2-x}{(2-x)^2}$$

Since $\frac{y^2(2+2y^2-x)}{(2-x)^4}=\frac{y^2}{(2-x)^2}\cdot\frac{(2y^2+2-x)}{(2-x)^2}$
$=u^2\cdot\frac{1}{|J|}$

$$I=\iint_D u^2\cdot\frac{1}{|J|}\cdot|J|\,dA'=\int_{-1}^{0}\int_0^1 u^2\,du\,dv$$

$$=\frac{1}{3}(1-0)(0+1)=\frac{1}{3}$$

⑦ Compute $J = \iint_R (x+y)\,dA$, where R is the region bounded by the ellipse

$$13x^2+14xy+10y^2=9$$

(Hint: use the change varrable $u=2x-y$ and $v=x+y$)

let $\begin{cases} u=2x-y\\ v=x+y\end{cases}$

$\Rightarrow \begin{cases} x=\frac13(u+v)\\ y=\frac13(-u+2v)\end{cases}$

$\frac{1}{J'} = \begin{vmatrix}2&-1\\1&1\end{vmatrix} = 2+1=3 \Rightarrow |J'| = 1/3$

Then $D=\{(u,v)\in\mathbb{R}^2 \,/\, \frac{u^2}{3^2}+v^2\le 1\}$

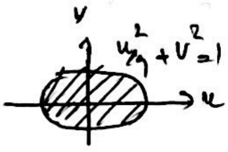
because $13x^2+14xy+10y^2=9$

$\frac{13}{9}(u^2+2uv+v^2)+\frac{14}{9}(v^2-u^2)+\frac{10}{9}(v^2-4vu+u^2)=9$

$13u^2+\cancel{26uv}+13v^2+28v^2-14u^2+10v^2-\cancel{40uv}+10u^2+\cancel{14uv}=81$

$9u^2+81v^2=81$

$\frac{u^2}{9}+v^2=1$



We get $J=\iint_R (x+y)\,dA = \frac13\iint_D v\,dA'$

let $s=\frac{u}{3}$ & $t=v \Rightarrow J''^{-1}=\begin{vmatrix}1/3&0\\0&1\end{vmatrix}=1/3$

$\Rightarrow s^2+t^2=1 \qquad \Rightarrow J''=3$

$\Rightarrow D'=\{(s,t)\in\mathbb{R}^2,\ s^2+t^2=1\}$

by using polar coordinates

$J=\frac13\cdot 3\int_0^1\int_0^{2\pi} r^2\sin\theta\,dr\,d\theta$

$=\frac13\cdot 4\int_0^{\pi/2}\sin\theta\,d\theta = \boxed{\frac43}$

⑧ Compute $K=\iint_R e^{x^2+xy+y^2}\,dA$ where R is the region bounded by the ellipse $x^2+xy+y^2=1$ (*)

Hint Use the change varrble $u=x+\frac{y}{2}$ and $v=\frac{\sqrt3}{2}y$)

let $\begin{cases}U=x+\frac{y}{2}\\ V=\frac{\sqrt3}{2}y\end{cases} \Rightarrow \begin{cases}x=U-\frac{\sqrt3}{3}V\\ y=\frac{2\sqrt3}{3}V\end{cases}$

$J=\begin{vmatrix}1&-\frac{\sqrt3}{3}\\0&\frac{2\sqrt3}{2}\end{vmatrix} = \frac{2\sqrt3}{2}$

Then take $\begin{cases}x=U-\frac{\sqrt3}{3}V\\ y=\frac{2\sqrt3}{3}V\end{cases}$ into (*)

$U^2-\cancel{\frac{2\sqrt3}{3}uv}+\frac19 V+\cancel{\frac{2\sqrt3}{3}uv}-\frac23V^2+\frac43V^2=1$

$u^2+v^2=1$

$D=\{(u,v)\in\mathbb{R}^2,\ u^2+v^2=1\}$

by using polar coordinates

$K=\frac{2\sqrt3}{3}\int_0^{2\pi}\int_0^1 re^{r^2}\,dr\,d\theta = \frac{2\sqrt3}{3}\times\frac12\times 2\pi\times(e-1)$

$=\boxed{\frac{2\pi\sqrt3}{3}(e-1)}$

③ Evaluate the following integrals by using Polar coordinates.

(a) $A = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx\,dy = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx\,dy$

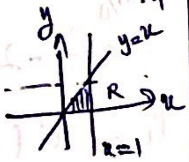
$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 2, -\sqrt{4-y^2} \le x \le \sqrt{4-y^2}\}$

by using polar coordinates

$A = \int_0^2 \int_0^{2\pi} r^2 \sin\theta\, dr\, d\theta$

$= \frac{1}{3} \cdot 8 \cdot 2 \int_0^{\pi/2} \sin\theta\, d\theta = \boxed{\frac{16}{3}}$

(b). $\int_0^1 \int_0^x \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$



by using polar coordinates

$r\cos\theta = 1$
$\Rightarrow r = \frac{1}{\cos\theta}$

$B = \int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta}} \frac{r}{|r|}\, dr\, d\theta$

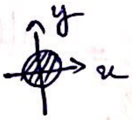
$= \int_0^{\pi/4} \frac{1}{\cos\theta}\, d\theta = \int_0^{\pi/4} \frac{\cos\theta}{1-\sin^2\theta}\, d\theta$

$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} \right) d\theta$

$= \frac{1}{2} \left[\ln|1+\sin\theta| - \ln|1-\sin\theta| \right]_0^{\pi/4}$

$= \frac{1}{2} \ln\left(\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2} \right) = \frac{1}{2} \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}} \right)$

c). $\iint_R \sin(x^2+y^2)\, dA$ R is the region of the unit circle centered at origin.



$C = \int_0^{2\pi} \int_0^1 r \sin(r^2)\, dr\, d\theta$

$= -\frac{1}{2} \int_0^{2\pi} \left[\cos(r^2) \right]_0^1 d\theta$

$= -\frac{1}{2} \int_0^{2\pi} \left[\cos 1 - \cos 0 \right] d\theta = -\frac{1}{2} \times 2\pi (\cos 1 - \cos 0)$

$= \pi - \pi \cos 1$

d). $\iint_R \frac{1}{\sqrt{4-x^2-y^2}}\, dA$ where $R = \{(x,y) \in \mathbb{R}^2 : y \le x,\ x^2 \le \sqrt{1-y^2}\}$

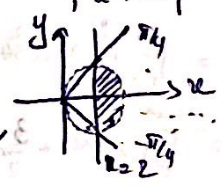
$d = \int_{-3\pi/4}^{\pi/4} \int_0^1 \frac{r}{\sqrt{4-r^2}}\, dr\, d\theta$

$= \left(\frac{\pi}{4} + \frac{3\pi}{4} \right) \left(\frac{1}{2} \right) \int_0^1 \frac{-2r\, dr}{\sqrt{4-r^2}}$

$= \frac{1}{2}\pi \left[\frac{(4-r^2)^{3/2}}{3/2} \right]_0^1 = \pi(2-\sqrt{3})$

e). $\iint_R \frac{y^2}{x^2+y^2}\, dA$, $R = \{(x,y) \in \mathbb{R}^2 : (x-2)^2+y^2 < 4 ; x > 2\}$

Analyze



$(x-2)^2 + y^2 < 4$

$r^2\cos^2\theta - 4r\cos\theta + \cancel{4} + r^2\sin^2\theta < \cancel{4}$

$r(r - 4\cos\theta) < 0$

$\Rightarrow r < 4\cos\theta$

• $x > 2 \Leftrightarrow r > \frac{2}{\cos\theta}$

$\Leftrightarrow \frac{2}{\cos\theta} < r < 4\cos\theta$

and $-\frac{\pi}{4} \le \theta \le \pi/4$

$D = \{(r,\theta) \in \mathbb{R}^2,\ -\pi/4 \le \theta \le \pi/4,\ \frac{2}{\cos\theta} \le r \le 4\cos\theta\}$

$E = \int_{-\pi/4}^{\pi/4} \int_{\frac{2}{\cos\theta}}^{4\cos\theta} \frac{r^3 \sin^2\theta}{r^2}\, dr\, d\theta$

$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \sin^2\theta \left(16\cos^2\theta - \frac{4}{\cos^2\theta} \right) d\theta$

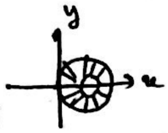
$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (4\sin^2 2\theta - 4\tan^2\theta)\, d\theta$

$= \int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta - 2\sec^2 + 2)\, d\theta$

$= \frac{3\pi}{2} - 4$

f). $\iint_R x\,dA \quad R=\{(x,y)\in\mathbb{R}^2 : 1<y^2+(x-2)^2<4\}$

Analysis



let $\begin{cases} x=r\cos\theta+2 \\ y=r\sin\theta \end{cases}$

• $1<x^2+(y-2)^2<4$

$\Leftrightarrow 1<r^2<4$

$\Leftrightarrow 1<r<2$ and $J=\begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}=r$

$D=\{(r,\theta)\in\mathbb{R}^2,\ 1<r<2,\ 0<\theta<2\pi\}$

$f=\int_0^{2\pi}\int_0^1 r(r\cos\theta+2)\,dr\,d\theta$

$=\int_0^{2\pi}\int_1^2 (r^2\cos^2\theta+2r)\,dr\,d\theta$

$=\int_0^{2\pi}\left(\frac{7}{3}\cos^2\theta+\frac{3}{2}\times2\right)d\theta$

$=\int_0^{2\pi}\left(\frac{7}{6}(1-\overset{0}{\cos2\theta})+\frac{3\times2}{2}\right)d\theta$

$=6\pi+\frac{7}{3}\pi=\frac{25\pi}{3}$

g). $\iint_R (x^2+y^2)\,dA, \quad R=\{(x,y)\in\mathbb{R}^2 : x^2+(y-2)^2<1\}$

let $\begin{cases} x=r\cos\theta \\ y=r\sin\theta+2 \end{cases}$

$J=r$

$D=\{(r,\theta)\in\mathbb{R}^2,\ 0<r<1$ & $0<\theta<2\pi\}$

$g=\int_0^{2\pi}\int_0^1 r(r^2\cos^2\theta+r^2\sin^2\theta+4r\sin\theta+4)\,dr\,d\theta$

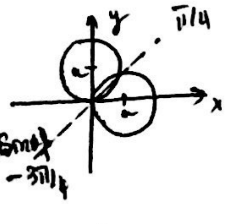
$=\int_0^{2\pi}\int_0^1 (r^3+4r+\overset{0}{4r^2\sin\theta})\,dr\,d\theta$

$=\frac{1}{4}\times2\pi+\frac{4}{2}\times2\pi=\frac{\pi}{2}+4\pi=\frac{9\pi}{2}$

h. $\iint_R (x^2+y^2)\,dA, \quad R=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2ax,\ x^2+y^2\le 2ay\}$

• $x^2+y^2=2ax \Leftrightarrow (x-a)^2+y^2=a^2$

• $x^2+y^2=2ay \Leftrightarrow x^2+(y-a)^2=a^2$



$R=R_1+R_2$

$R_1=\{(r,\theta)\in\mathbb{R}^2 \mid 0\le\theta\le\pi/4,\ 0<r<2a\sin\theta\}$

$R_2=\{(r,\theta)\in\mathbb{R}^2 \mid \pi/4\le\theta\le\pi/2,\ 0<r<2a\cos\theta\}$

h. $\iint_R f(x,y)\,dA=\iint_{R_1} f(x,y)\,dA+\iint_{R_2} f(x,y)\,dA$

$\therefore h=\int_0^{\pi/4}\int_0^{2a\sin\theta} r^3\,dr\,d\theta+\int_{\pi/4}^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$

$=\int_0^{\pi/4}4a^4\sin^4\theta\,d\theta+\int_{\pi/4}^{\pi/2}4a^4\cos^4\theta\,d\theta$

• $\int_0^{\pi/4}4\sin^4\theta\,d\theta=\int_0^{\pi/4}4\cdot\frac{(1-\cos2\theta)^2}{4}d\theta=\int_0^{\pi/4}(1-2\cos2\theta+\cos^2 2\theta)\,d\theta$

$=\int_0^{\pi/4}\left(1-2\cos2\theta+\frac{1}{2}+\frac{\cos4\theta}{2}\right)d\theta$

$=\frac{3\pi}{8}-1$

• $\int_{\pi/4}^{\pi/2}4\cos^4\theta\,d\theta=\int_0^{\pi/4}\left(1+2\cos2\theta+\frac{1}{2}+\frac{\cos4\theta}{2}\right)d\theta$

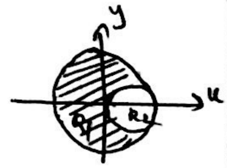
$=\frac{3\pi}{8}-1$

$h=a^4\left(\frac{3\pi}{8}-1+\frac{3\pi}{8}-1\right)=a^4\left(\frac{3\pi}{4}-2\right)$

i. $\iint_R \frac{1+y^3}{(1+x^2+y^2)}\,dA, \quad R=\{(x,y)\in\mathbb{R}^2 \mid x<x^2+y^2<1\}$

• $x=x^2+y^2 \Leftrightarrow (x-\frac{1}{2})^2+y^2=1/4$

• $x^2+y^2=1$



$\iint_R \frac{1+y^3}{1+x^2+y^2}\,dA=\int_0^{2\pi}\int_0^1 \frac{1+r^3\sin^3\theta}{1+r^2}\,r\,dr\,d\theta-\int_{-\pi/2}^{\pi/2}\int_0^{1/2} r\frac{1+r^3\sin^3\theta}{1+r^2}\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^1 \frac{r}{1+r^2}\,dr\,d\theta-\int_{-\pi/2}^{\pi/2}\int_0^{1/2}\frac{r}{1+r^2}\,dr\,d\theta$

$=2\pi\times\frac{1}{2}\ln2-\pi\times\frac{1}{2}\ln\frac{5}{4}$

$=\pi(\ln2-\ln\frac{5}{8})=\pi\ln\frac{16}{5}$

because $\int_0^{2\pi}\sin^3\theta\,d\theta=0$

$\int_{-\pi/2}^{\pi/2}\sin^3\theta\,d\theta=0$

(10) Use any method to compute the following on the given region R.

(a). $\iint_R \frac{xy}{1+x^2+y^2}dA$, $R=\{(x,y)\in[0,1]^2 : x^2+y^2>1\}$

$\iint_R f(x,y)\,dxdy = \iint_{R_1} f(x,y)\,dxdy - \iint_{R_2} f(x,y)\,dxdy$

$= \int_0^1\int_0^1 \frac{xy}{1+x^2+y^2}dxdy - \int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\cos\theta}{1+r^2}\, r\,drd\theta$

• $\int_0^1\int_0^1 \frac{xy}{x^2+y^2+1}dxdy = \int_0^1 \frac12 y\ln(1+x^2+y^2)\Big|_0^1 dy$
$= \int_0^1 \frac12 y\ln\left(\frac{2+y^2}{1+y^2}\right)dy$

$= \frac14\int_0^1 2y\ln(2+y^2)dy - \frac14\int_0^1 2y\ln(1+y^2)dy$

$\boxed{* \int \ln x\,dx = x\ln|x| - x + cte}$

$= \frac14\left[(2+y^2)\ln(2+y^2) - (2+\cancel{y^2}) - (1+y^2)\ln(1+y^2) + (1+\cancel{y^2})\right]_0^1$

$= \frac14\left[3\ln3 - 2\ln2 - 3 - 2\ln2\right] = \frac34\ln3 - \ln2 - 3$

• $\int_0^{\pi/2}\int_0^1 \frac{r^3}{1+r^2}\,drd\theta = \int_0^{\pi/4}\frac12\sin2\theta\,d\theta\int_0^1\left(r-\frac{r}{1+r^2}\right)dr$

$= -\frac14(-1-1)\left[\frac{r^2}{2}-\frac12\ln(1+r^2)\right]_0^1$

$= \frac12\left(\frac12-\frac12\ln2\right)$

$\iint_R f(x,y)\,dxdy = \frac34\ln3 - \ln2 - 3 - \frac14 + \frac14\ln2$

$= \frac34\ln3 - \frac34\ln2 - \frac{13}{4}$

$= \frac34\ln\left(\frac32\right) - \frac{13}{4}$

b) $\iint_R (x^3+y^3)dA$; $R=\{(x,y)\in\mathbb{R}^2 \mid \frac1y < x < \frac2y ;\ x^2<y<2x^2\}$

$\begin{cases} U = xy \\ V = \frac{y}{x^2}\end{cases}$

Then $\frac1y < x < \frac2y \Leftrightarrow 1 < xy < 2 \Leftrightarrow 1<U<2$

$x^2 < y < 2x^2 \Leftrightarrow 1 < \frac{y}{x^2} < 2 \Leftrightarrow 1<V<2$

$\frac1J = \begin{vmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2}\end{vmatrix} = \frac{3y}{x^2} = 3V$

$\Rightarrow |J| = \frac{1}{3V}$

and $x^3+y^3 = \frac{xy}{\frac{y}{x^2}} + (xy)^2\cdot\frac{y}{x^2} = \frac UV + U^2V$

Then

$b = \int_1^2\int_1^2 \frac{1}{3V}\left(\frac UV + U^2V\right)dudv$

$= \frac13\int_1^2\int_1^2\left(\frac{u}{v^2}+u^2\right)dudv = \frac13\int_1^2\left(\frac32\cdot\frac{1}{v^2}+\frac73\right)dv$

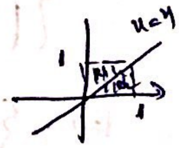
$= \frac13\left(-\frac32\left(\frac12-1\right)+\frac73\right) = \frac13\left(\frac34+\frac73\right)$

$= \frac14 + \frac79 = \frac{37}{36}$

c) $\iint_R |x-y|\,dA$, $R=\{(x,y)\in\mathbb{R}^2 : 0<x<1, 0<y<1\}$

for $R_1: x\ge y \Rightarrow |x-y| = x-y$

$R_2: x\le y \Rightarrow |x-y| = y-x$



$\iint_R |x-y|\,dA = \int_0^1\int_0^x (x-y)\,dydx + \int_0^1\int_x^1 (y-x)\,dydx$

$= \int_0^1\left(\int_0^x (x-y)\,dy + \int_x^1 (y-x)\,dy\right)dx$

$= \int_0^1\left(x^2 - \frac{x^2}{2} + \frac12 - \frac{x^2}{2} + x^2 - x\right)dx$

$= \int_0^1\left(x^2+\frac12-x\right)dx = \frac13+\frac12-\frac12$

$= \frac13$

d)- $\iint_R \frac{y^2\cos(x^2+y^2)}{x^2}\,dA$, $R=\{1<x^2+y^2<4,\ 0<y<x\}$

let $\begin{cases} U=x^2+y^2 \\ V=\frac{y}{x}\end{cases}$ then $\begin{cases}1<x^2+y^2<4 \\ 0<y<x\end{cases}$

$\Leftrightarrow \begin{cases}1<U<4\\ 0<V<1\end{cases}$

$\frac{1}{J}=\begin{vmatrix}2x & 2y\\ \frac{1}{x} & -\frac{y}{x^2}\end{vmatrix}=-\frac{2y}{x}-\frac{2y}{x}=-\frac{4y}{x}$

$\Rightarrow |J|=\frac{x}{4y}=\frac{1}{4V}$

then $\frac{y^2\cos(x^2+y^2)}{x^2}=V^2\cos U$

we get $d=\int_1^4\int_0^1 \frac{1}{4V}\cdot V^2\cos u\,du\,dv$

$=\frac{1}{4}\int_1^4 v\,dv\int_0^1\cos u\,du$

$=\frac{1}{8}(16-1)\sin 1=\frac{15}{8}\sin 1$

e)- $\iint_R (x^2+y^2)\,dA$, $R=\{(x,y)\in\mathbb{R}^2\ \frac{(x-1)^2}{4}+\frac{(y-1)^2}{9}<1\}$

let $\begin{cases}\left(\frac{x-1}{2}\right)=r\cos\theta\\ \left(\frac{y-1}{3}\right)=r\sin\theta\end{cases}$

$\Leftrightarrow \begin{cases}x=\frac{r\cos\theta}{2}-1\\ y=\frac{r\sin\theta}{3}-1\end{cases}$

$J=\begin{vmatrix}\frac{\cos\theta}{4} & -\frac{r\sin\theta}{4}\\ \frac{r\sin\theta}{9} & \frac{r\cos\theta}{9}\end{vmatrix}=\frac{1}{6}r$

$e=\int_0^{2\pi}\int_0^1\left(\frac{r^2\cos^2\theta}{4}-r\cos\theta-\frac{r^3\sin^2\theta}{9}+2r^2\frac{\sin\theta}{3}\right)\frac{1}{6}\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^1\frac{1}{6}\left(\frac{r^3}{8}(1+\cos 2\theta)-\frac{r^3}{18}(1-\cos 2\theta)\right)dr\,d\theta$

$=\frac{1}{6}\times 2\pi\left(\frac{1}{18}\times\frac{1}{4}-\frac{1}{18}\times\frac{1}{4}\right)=\frac{\pi}{24}\left(\frac{1}{4}-\frac{1}{9}\right)$

$=\frac{5\pi}{864}$

f- $\iint_R xy\,dA$, $R=\{(x,y)\in\mathbb{R}^2: 4(x-1)^2+9(y-1)^2<36,\ y>1,\ x-y>0\}$

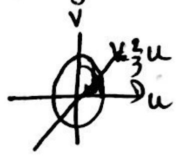
let $\begin{cases}U=\frac{x-1}{3}\\ V=\frac{y-1}{2}\end{cases}\Rightarrow\begin{cases}x=3U+1\\ y=2V+1\end{cases}$

$J=\begin{vmatrix}3&0\\0&2\end{vmatrix}=6$

$xy=(3U+1)(2V+1)=6UV+3U+2V+1$

$D'=\{(U,V)\in\mathbb{R}^2\ /\ U^2+V^2<1,\ V>0,\ V<\frac{2}{3}U\}$

$f=6\iint_{D'}(6UV+3U+2V+1)\,du\,dv$



$\begin{cases}U^2+V^2=1\\ V=\frac{2}{3}u\end{cases}$

$\Rightarrow U^2+\frac{4}{9}U^2=1\Rightarrow U^2=\frac{4}{13}\Rightarrow U=\frac{2}{\sqrt{13}}\Rightarrow V=\frac{3}{\sqrt{13}}$

$\tan\theta=\frac{V}{U}=\frac{3}{2}\Rightarrow\theta=\arctan\left(\frac{3}{2}\right)$

then $\begin{cases}0<r<1\\ 0<\theta<\arctan\left(\frac{3}{2}\right)\end{cases}$

$f=\int_0^{\arctan\frac{3}{2}}\int_0^1 r(6r^2\sin\theta\cos\theta+3r\cos\theta+2r\sin\theta+1)\,dr\,d\theta$

$=\int_0^{\arctan\frac{3}{2}}\left(\frac{3}{4}\sin 2\theta+\frac{3}{2}\cos\theta+\sin\theta+1\right)d\theta$

$=-\frac{3}{8}\left(\sin(2\arctan\tfrac{3}{2})-1\right)+\frac{3}{2}\sin(\arctan\tfrac{3}{2})-\cos(\arctan\tfrac{3}{2})-\arctan\frac{3}{2}$

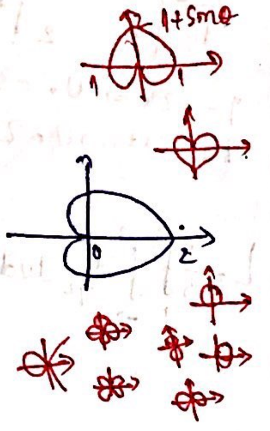
- $\sin(2\arctan\frac{3}{2})=2\cdot\frac{3/2}{\sqrt{1+(\frac{3}{2})^2}}\cdot\frac{1}{\sqrt{1+(\frac{3}{2})^2}}=\frac{3}{\frac{13}{4}}=\frac{12}{13}$
- $\sin(\arctan\frac{3}{2})=\frac{3\sqrt{13}}{13}$
- $\cos(\arctan\frac{3}{2})=\frac{2\sqrt{13}}{13}$

$f=-\frac{3}{8}\left(\frac{12}{13}-1\right)+\frac{3}{2}\cdot\frac{3\sqrt{13}}{13}-\frac{2\sqrt{13}}{13}-\arctan\frac{3}{2}$

$=\frac{3}{104}+\frac{5\sqrt{13}}{13}-\arctan\frac{3}{2}$

g)- $\iint_R \frac{1}{\sqrt[4]{(x^2+y^2)^3}}\,dA$

$R=\{(x,y)\in\mathbb{R}^2 \quad x^2+y^2<2(\sqrt{x^2+y^2}+x)\}$



$x^2+y^2 = 2\sqrt{x^2+y^2}\ 2x$

$r^2 = 2r+2r\cos\theta$

$r = 2+2\cos\theta$

$0<r<2+2\cos\theta$

$0<r<2\pi$

$I=\int_0^{2\pi}\int_0^{2+2\cos\theta}\frac{r}{r^{3/2}}\,drd\theta = \int_0^{2\pi}\frac{r^{1/2}}{1/2}\Big|_0^{2+2\cos\theta}d\theta$

$=\int_0^{2\pi}2\sqrt2\sqrt{1+\cos\theta}\,d\theta = 4\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta = 0$

h. $\iint_R x(y-1)^2\,dA$, $R=\{(x,y)\in\mathbb{R}^2 : 3x^2+y^2-2y<2,\ 0<y-1<x\}$

• $3x^2+y^2-2y<2 \iff x^2+\frac{(y-1)^2}{3}<1$

• $0<y-1<x$

let $u=x$

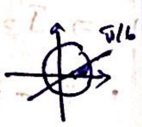
$v=\frac{y-1}{\sqrt3} \Rightarrow y=\sqrt3 v+1$ | $0<y-1<x \iff 0<v<\frac{u}{\sqrt3}$

$J=\begin{vmatrix}1&0\\0&\sqrt3\end{vmatrix}=\sqrt3$

$D=\{(u,v)\in\mathbb{R}^2 \mid u^2+v^2<1,\ 0<v<\frac{u}{\sqrt3}\}$

and $x(y-1)^2 = \sqrt3^2 u\left(\frac{y-1}{\sqrt3}\right)^2 = 3uv^2$

$h=3\sqrt3\iint_D uv^2\,dudv$



by using polar coordinates

$r\sin\theta = \frac{r\cos\theta}{\sqrt3} \Rightarrow \tan\theta = \frac{\sqrt3}{3} \Rightarrow \theta=\pi/6$

$\begin{cases}0<r<1\\0<\theta<\pi/6\end{cases}$

$h = 3\sqrt3\int_0^{\pi/6}\int_0^1 r^4\cos\theta\sin^2\theta\,drd\theta$

$=\frac{3\sqrt3}{5}\int_0^{\pi/6}\sin^2\theta\,d(\sin\theta)$

$=\frac{3\sqrt3}{5}\,\frac{\sin^3\theta}{3} = \frac{\sqrt3}{5}\left(\frac18-0\right)=\boxed{\frac{\sqrt3}{40}}$

$\frac{\frac{y}{3x}}{\sqrt{1+(\frac{y}{x})^2}}\cdot\frac{1}{\sqrt{1+(\frac{y}{3x})^2}}$

$2\sin\theta\cos\theta$

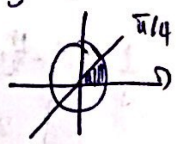
i. $\iint_R \left(x^2+\frac{y^2}{9}\right)\sin\left(2\arctan\frac{y}{3x}\right)$

$\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1,\ 0<y<x\}$

$\cdot\left(x^2+\frac{y^2}{9}\right)\sin\left(2\arctan\frac{y}{3x}\right)$

$=\left(x^2+\frac{y^2}{9}\right)\frac{1\times\frac{y}{3x}}{1+\left(\frac{y}{3x}\right)^2}\cdot 2$

$=\frac23\frac{y}{x}\,x^2\cancel{\left(1+\frac{y^2}{9x^2}\right)}\cdot\frac{1}{\cancel{1+\frac{y^2}{9x^2}}} = \frac23 xy$



$I = \int_0^{\pi/4}\int_0^1\frac23 r^3\sin\theta\cos\theta\,drd\theta$

$=\frac1{12}\int_0^{\pi/4}\sin 2\theta\,d\theta = -\frac1{24}(0-1)=\boxed{\frac1{24}}$

j. $\iint_R\frac{y^2(2+2y^2-x)}{(2-x)^4}\,dA$

$R=\{(x,y)\in\mathbb{R}^2 : x-1<y^2<x,\ 0<y<2-x\}$

let $V=y^2-x$

$U=\frac{y}{2-x}$

then • $x-1<y^2<x \iff -1<y^2-x<0$

$\iff -1<V<0$

• $0<y<2-x \iff 0<\frac{y}{2-x}<1$

$\iff 0<U<1$

$\frac1J=\begin{vmatrix}\frac{y}{(2-x)^2}&\frac{1}{2-x}\\-1&2y\end{vmatrix}=\frac{2y^2}{(2-x)^2}+\frac{1}{2-x}=\frac{2y^2+2-x}{(2-x)^2}$

$|J|=\frac{(2-x)^2}{2y^2+2-x}$

$$J = \int_{-1}^{0}\int_{0}^{1} \frac{y^2}{(2-x)^2}\,du\,dv = \int_{-1}^{0}\int_{0}^{1} u^2\,du\,dv$$

$$= \boxed{\frac{1}{3}}$$

K $\iint_R \frac{y}{x^3}\sin\left(\pi - \frac{\pi y^2}{x^2}\right)dA$, where

$R = \{(x,y) \in \mathbb{R}^2\ 0 < \sqrt{2}y < x,\ 1 < x^2 - y^2 < 4\}$

let $\begin{cases} u = y/x \\ v = x^2 - y^2 \end{cases}$

then . $0 < \sqrt{2}y < x \Leftrightarrow 0 < \frac{y}{x} < \frac{\sqrt{2}}{2}$

$\Leftrightarrow 0 < u < \frac{\sqrt{2}}{2}$

. $0 < x^2 - y^2 < 4 \Leftrightarrow 0 < v < 4$

$$\frac{1}{J} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 2x & -2y \end{vmatrix} = 2 - 2\frac{y^2}{x^2} = \frac{2(x^2-y^2)}{x^2}$$

$$|J| = \frac{x^2}{2(x^2-y^2)} = \frac{x^2}{2v}$$

$$K = \int_0^4\int_0^{\sqrt{2}/2} \frac{u}{2v}\sin(\pi - \pi u^2)\,du\,dv$$

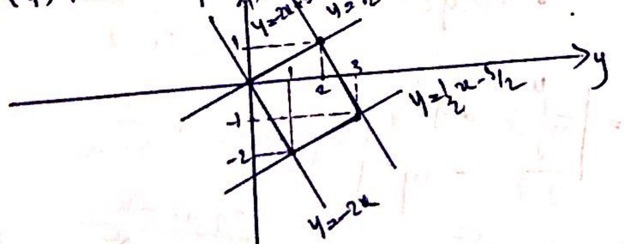
$$= \frac{\ln 2}{-2\pi}\int_0^{\sqrt{2}/2}(-2\pi u)\sin(\pi - \pi u^2)\,du$$

$$= \frac{\ln 2}{2\pi}\cos(\pi - \pi u^2)\Big|_0^{1/\sqrt{2}}$$

$$= \frac{\ln 2}{2\pi}(0-(-1)) = \boxed{\frac{\ln 2}{2\pi}}$$

L. $\iint_R \frac{(2x+y-3)^2}{(2y-x+6)^2}\,dA$

R is the square with vertices $(0,0), (2,1), (3,-1)$ & $(1,-2)$



let $\begin{cases} u = 2x+y-3 \\ v = 2y-x+6 \end{cases}$

$$\frac{1}{J} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5 \Rightarrow |J| = \frac{1}{5}$$

$y = -2x \Rightarrow u = -3$ $\quad$ $y = \frac{1}{2}x \Leftrightarrow v = 6$

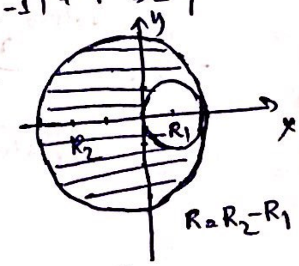
$y = -2x+5 \Rightarrow u = 2$ $\quad$ $y = \frac{1}{2}x - \frac{5}{2} \Rightarrow v = 1$

$$L = \frac{1}{5}\int_1^6\int_{-3}^{2} \frac{u^2}{v^2}\,du\,dv = \frac{1}{5}\int_1^6 \frac{1}{v^2}dv\int_{-3}^{2} u^2\,du$$

$$= -\frac{1}{5}\left(\frac{1}{6} - 1\right)\left(\frac{8+27}{3}\right)$$

$$= \frac{1}{\cancel{5}} \times \frac{\cancel{5}}{6} \times \frac{35}{3} = \boxed{\frac{35}{18}}$$

m. $\iint_R (x^3 - 3xy^2)\,dA$, $R = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \le 9,\ (x-1)^2 + y^2 \ge 1\}$



. $R_1 = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \ge 9\}$

let $\begin{matrix} u = x+1 \\ v = y \end{matrix} \Rightarrow \begin{cases} x = u+1 \\ y = v \end{cases}$

$$J = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

. $x^3 - 3xy^2 = (u+1)^3 - 3(u+1)v^2$

$= u^3 + 3u^2 + 3u + 1 - 3uv^2 - 3v^2$

by using polar coordination

$$M_1 = \int_0^{2\pi}\int_0^1 r(r^3\cos^3\theta + 3r^2\cos^2\theta + 3r\cos\theta + 1 - 3r^3\cos\theta\sin^2\theta - 3r^2\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{3}{2}r^3 + r - \frac{3}{2}r^3\right)dr\,d\theta$$

$$= \frac{1}{2} \times 2\pi = \pi$$

. $R_2 = \{(x+1)^2 + y^2 \le 9,\ (x,y) \in \mathbb{R}^2\}$

$\begin{matrix} u = x+1 \\ v = y \end{matrix} \Rightarrow \begin{cases} x = u-1 \\ y = v \end{cases}$

$$\Rightarrow J = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

. $x^3 - 3xy^2 = (u-1)^3 - 3(u-1)v^2$

$= u^3 - 3u^2 + 3u - 1 - 3uv^2 + 3v^2$

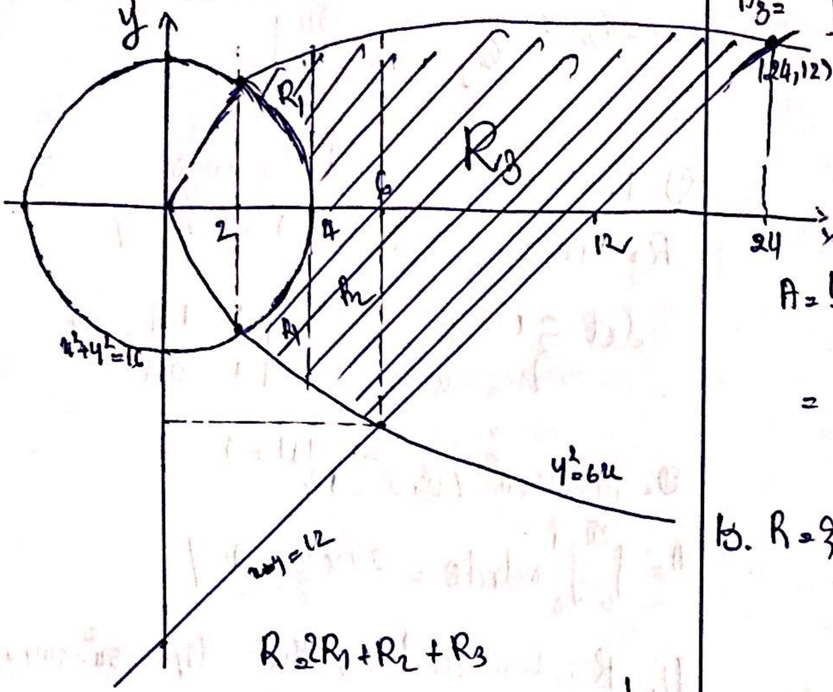
$$M_2 = \int_0^{2\pi}\int_0^3 r(r^3\cos^3\theta - 3r^2\cos^2\theta + 3r\cos\theta - 1 - 3r^3\cos\theta\sin^2\theta + 3r^2\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 -\frac{3}{2}\cancel{r^3} - r + \frac{3}{2}\cancel{r^3}\,dr\,d\theta = -\frac{9}{2}\pi \times 2 = -9\pi$$

$$M = -9\pi - \pi = \boxed{-10\pi}$$

1)- Find the region R

(a)- $R=\{(x,y)\in\mathbb{R}^2 : y^2-6x<0,\ x-y<12,\ x^2+y^2>16\}$



$R = 2R_1 + R_2 + R_3$

$R_1=\{(x,y)\in\mathbb{R}^2,\ 2<x<4,\ \sqrt{6x}<y\le-\sqrt{16-x^2}\}$

$R_2=\{(x,y)\in\mathbb{R}^2,\ 4<x<6,\ -\sqrt{6x}<y<\sqrt{6x}\}$

$R_3=\{(x,y)\in\mathbb{R}^2,\ 6<x<24,\ x-12<y<\sqrt{6x}\}$

$$A_1=2\int_2^4\int_{-\sqrt{6x}}^{\sqrt{16-x^2}}dy\,dx=2\int_2^4\left(\sqrt{16-x^2}+\sqrt{6x}\right)dx$$

$$K=2\int_2^4\sqrt{16-x^2}\,dx \quad \left|\begin{array}{l} \text{let } x=4\cos\theta \Rightarrow dx=-4\sin\theta\,d\theta \\ \text{if } x=2\Rightarrow\theta=\pi/3 \\ x=4\Rightarrow\theta=0\end{array}\right.$$

$$=2\int_0^{\pi/3}\sqrt{16(1-\cos^2\theta)}\,(-4\sin\theta)\,d\theta$$

$$=-32\int_0^{\pi/3}\sin^2\theta\,d\theta$$

$$=-\frac{32}{2}\int_0^{\pi/3}(1-\cos2\theta)\,d\theta$$

$$=-16\left(\frac{\pi}{3}-\left(\frac{\frac{\sqrt3}{2}-0}{2}\right)\right)=16\left(\frac{\sqrt3}{4}-\frac{\pi}{3}\right)$$

$$L=\int_2^4\sqrt6\,x^{1/2}\,dx=\frac{16\sqrt6-8\sqrt3}{3}$$

$$A_1=\frac{16\sqrt3}{4}-\frac{16\pi}{3}-\frac{16\sqrt6}{3}-\frac{8\sqrt3}{3}$$

$$A_2=\int_4^6\int_{-\sqrt{6x}}^{\sqrt{6x}}dy\,dx=2\int_4^6\sqrt{6x}\,dx=48-\frac{32\sqrt6}{3}$$

$$A_3=\int_6^{24}\int_{x-12}^{\sqrt{6x}}dy\,dx=\int_6^{24}\left(\sqrt{6x}-x+12\right)dx$$

$$=144$$

$$A=\frac{16\sqrt3}{4}-\frac{16\pi}{3}-\frac{16\sqrt6}{3}-\frac{8\sqrt3}{3}+48-\frac{32\sqrt6}{3}+144$$

$$=\frac{4}{3}\sqrt3-16\sqrt6+192-\frac{16\pi}{3}$$

b. $R=\{(x,y)\in\mathbb{R}^2 \mid 2a_1x<y^2<2a_2x\ ;\ 2b_1y<x^2<2b_2y\}$

$a_1,a_2,b_1,b_2\in(0,+\infty)$ and $a_1<a_2,\ b_1<b_2$

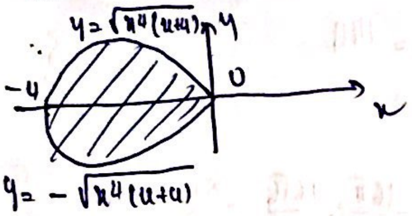
$$\text{let}\begin{cases}u=\dfrac{y^2}{2x}\\ v=\dfrac{x^2}{2y}\end{cases}\Rightarrow\frac{1}{J}=\begin{vmatrix}-\dfrac{y^2}{2x^2} & \dfrac{y}{x}\\ \dfrac{x}{y} & -\dfrac{x^2}{2y^2}\end{vmatrix}=\frac14-1=\frac{-3}{4}$$

$$\Rightarrow|J|=\frac43$$

- $2a_1x<y^2<2a_2x \Leftrightarrow a_1<\dfrac{y^2}{2x}<a_2$
  $\Leftrightarrow a_1<u<a_2$
- $2b_1y<x^2<2b_2y \Leftrightarrow b_1<\dfrac{x^2}{2y}<b_2$
  $\Leftrightarrow b_1<v<b_2$

$$B=\int_{a_1}^{a_2}\int_{b_1}^{b_2}\frac43\,dv\,du=\frac43(b_2-b_1)(a_2-a_1)$$

c)- $R = \{(x,y) \in \mathbb{R}^2 : x<0,\ y^2 < x^4(x+4)\}$



$$A = \int_{-4}^{0}\int_{-\sqrt{x^4(x+4)}}^{\sqrt{x^4(x+4)}} dy\,dx$$

$$= 2\int_{-4}^{0} x^2\sqrt{x+4}\,dx$$

let $u = x+4 \Rightarrow x = u-4$
$\Rightarrow dx = du$

- $x=0 \Rightarrow u=4$
- $x=-4 \Rightarrow u=0$

$$A = 2\int_0^4 (u-4)^2\sqrt{u}\,du$$

$$= 2\int_0^4 \left(u^{5/2} - 8u^{3/2} + 16u^{1/2}\right)du$$

$$= 2\left(\frac{4^{7/2}}{7/2} - \frac{8\cdot 4^{5/2}}{5/2} + \frac{16\cdot 4^{3/2}}{3/2}\right)$$

$$= 2\times 2^8\left(\frac{1}{7} - \frac{1}{5} + \frac{1}{3}\right)$$

$$= 2^9\left(\frac{15-21+35}{105}\right) = \frac{29\times 2^9}{105} = \frac{14848}{105}$$

d)- Find the area of the region
$R = \{(x,y) \in \mathbb{R}^2 \mid x^{2/3} + y^{2/3} = 1\}$

let $u = (r\cos\theta)^3$
$v = (r\sin\theta)^3$

$$\Rightarrow J = \begin{vmatrix} 3r^2\cos^3\theta & -3r^3\sin\theta\cos^2\theta \\ 3r^2\sin^3\theta & 3r^3\cos\theta\sin^2\theta \end{vmatrix}$$

$$= 9r^5\sin^2\theta\cos^4\theta + 9r^5\cos^2\theta\sin^4\theta$$

$$= 9r^5\sin^2\theta\cos^2\theta = \frac{9}{4}r^5\sin^2 2\theta = \frac{9}{8}r^5(1-\cos 4\theta)$$

$$A = \int_0^{2\pi}\int_0^1 \frac{9}{8}r^5(1-\cos 4\theta)\,dr\,d\theta.$$

$$= 2\pi\times\frac{\cancel{9}^{3}}{8}\times\frac{1}{\cancel{6}_{2}} = \frac{3\pi}{8}$$

e)- Find the area of the region
$R = \{(x,y) \in \mathbb{R}^2 \mid (y-x)^2 = 1-x^2\}$

let $\begin{cases} u = y-x \\ v = x \end{cases} \Rightarrow \frac{1}{J} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$
$\Rightarrow |J| = 1$

$D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 = 1\}$

$$A = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = 2\pi\times\frac{1}{2} = \pi$$

f)- R is bounded by the ellipse $5x^2+6xy+5y^2=4$

$5x^2+6xy+5y^2 = 4 \Leftrightarrow (x-y)^2 + (2x+2y)^2 = 4$

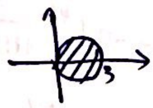
let $\begin{cases} u = x-y \\ v = 2x+2y \end{cases}$

$\frac{1}{J} = \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 2+2 = 4 \Rightarrow |J| = \frac{1}{4}$

$D = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 = 4\}$

$$A = \frac{1}{4}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = \frac{1}{4}\times 2\pi\times\frac{4}{2} = \pi$$

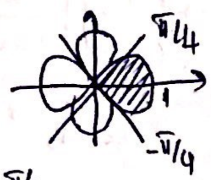
g)- R is bounded by circle $r = 3\cos\theta$.



$$A = \int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{9}{2}\cos^2\theta\,d\theta$$

$$= \frac{9}{4}\int_{-\pi/2}^{\pi/2}(1-\cos 2\theta)\,d\theta = \frac{9}{4}\times\pi = \frac{9}{4}\pi$$

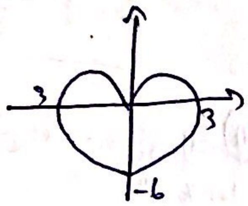
h_- R is bounded by one loop of the four-leaved rose $r = \cos 2\theta$



$$A = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1-\cos 4\theta)\,dr\,d\theta$$

$$= \frac{1}{4}\times\frac{\pi}{2} = \boxed{\frac{\pi}{8}}$$

I R is bounded by the cardioid $r = 3-3\sin\theta$
$= 3(1-\sin\theta)$

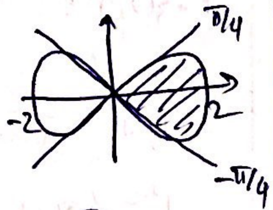


$$A = \int_0^{2\pi}\int_0^{3(1-\sin\theta)} r\,dr\,d\theta$$

$$= \frac{9}{2}\int_0^{2\pi}\Big(1-2\overset{0}{\sin\theta} + \frac{1}{2}(1-\overset{0}{\cos 2\theta})\Big)d\theta$$

$$= \frac{9}{2}\times\frac{3}{2}\times 2\pi = \boxed{\frac{27\pi}{2}}$$

J R is bounded by the lemniscate $r^2 = 4\cos 2\theta$
$r = \pm 2\sqrt{\cos 2\theta}$

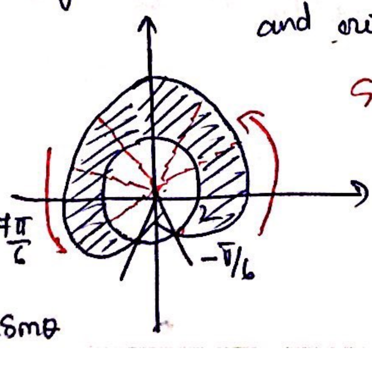


$$A = 2\int_{-\pi/4}^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 2\times 2\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta = 2\sin 2\theta\Big|_{-\pi/4}^{\pi/4} = \boxed{4}$$

K R is the region lies inside $r = 3+2\sin\theta$ and outside $r=2$

$r = 3+2\sin\theta$
$r = 2$
$3+2\sin\theta = 2$
$\theta = -\pi/6$
and $\theta = \pi - (-\pi/6) = \frac{7\pi}{6}$
$-\pi/6 < \theta < 7\pi/6$
$2 < r < 3+2\sin\theta$



$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}, \frac{7\pi}{6}$

$$A = \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\,dr\,d\theta$$

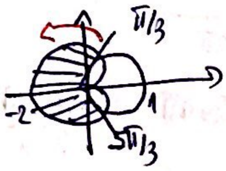
$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}(9+12\sin\theta + 2-2\cos 2\theta - 4)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6}(7+12\sin\theta - 2\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\Big(\frac{7\times 8\pi}{6} - 12\Big(-\frac{\sqrt3}{2}-\frac{\sqrt3}{2}\Big) - \Big(\frac{\sqrt3}{2}+\frac{\sqrt3}{2}\Big)\Big)$$

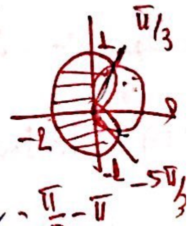
$$= \frac{1}{2}\Big(\frac{4\pi}{3} + 12\sqrt3 - \sqrt3\Big) = \boxed{\frac{7\pi}{3} + \frac{11\sqrt3}{3}}$$

L R is the region lies inside $r = 1-\cos\theta$ and outside $r = \cos\theta$



$\pi/3 \le \theta \le 5\pi/3$

$\cos\theta \le r \le 1-\cos\theta$



$$A = \int_{\pi/3}^{5\pi/3}\int_{\cos\theta}^{1-\cos\theta} r\,dr\,d\theta$$

$\frac{\pi}{3} - \pi$

$\pi - \pi/3$

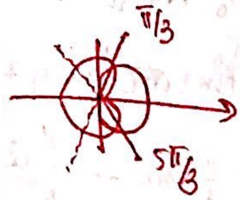
$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3}\Big(1-2\cos\theta + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}\cos 2\theta} - \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}\cos 2\theta}\Big)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3}(1-2\cos\theta)\,d\theta$$

$$= \frac{1}{2}\Big(\frac{4\pi}{3} - 2\Big(\frac{-\sqrt3}{2} - \frac{\sqrt3}{2}\Big)\Big) = \boxed{\frac{\pi}{3} + \sqrt3}$$

$\pi - \frac{3\pi}{4} = \frac{\pi}{4}$

$\pi + \frac{3\pi}{4} = \frac{7\pi}{4} = -\frac{3\pi}{4}$



12) Let $f: \mathbb{R} \to \mathbb{R}_+^*$ and $\{(x,y)\in\mathbb{R}^2 , x^2+y^2 \le 4\}$

Compute the integral

$$I = \iint_R \frac{2f(x)+5f(y)}{f(x)+f(y)} dA$$

$$= 2\iint_R dA + 3\iint_R \frac{f(y)}{f(x)+f(y)} dA$$

let $M = \iint_R \frac{f(y)}{f(x)+f(y)} dA$ & $N = \iint_R \frac{f(x)}{f(x)+f(y)} dA$

$$M+N = \iint_R dA = \int_0^{2\pi}\int_0^{2} r\,dr\,d\theta = 4\pi$$

$M-N = 0$

$$+\begin{cases} M+N = 4\pi \\ M-N = 0 \end{cases}$$

$M = 2\pi$

Then $I = 2\times 4\pi + 2\pi\times 3 = \underline{14\pi}$

13) Calculate $I = \iint_R \frac{x}{(1+x^2)(1+xy)} dA$ where

$R = \{(x,y)\in\mathbb{R}^2 \mid 0<x<1 ; 0<y<1\}$

Since $\frac{x}{(1+x^2)(1+xy)} = \frac{x+y}{(1+x^2)(1+y^2)} - \frac{y}{(1+y^2)(1+xy)}$

$$= \frac{1}{1+y^2}\left(\frac{x}{1+x^2} + \frac{y}{1+x^2} - \frac{y}{1+xy}\right)$$

$$I = \int_0^1\int_0^1 \frac{1}{1+y^2}\left(\frac{x}{1+x^2} + \frac{y}{1+x^2} - \frac{y}{1+xy}\right) dx\,dy$$

$$= \int_0^1 \frac{1}{1+y^2}\left[\frac{1}{2}\ln(1+x^2) + y\arctan x - \ln(1+xy)\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{1+y^2}\left(\frac{1}{2}\ln 2 + \frac{y\pi}{4} - \ln(1+y)\right) dy$$

$$= \left[\frac{1}{2}\ln 2 \arctan y + \frac{\pi}{8}\ln(1+y^2)\right]_0^1 - \int_0^1 \frac{\ln(1+y)}{1+y^2} dy$$

$$= \frac{\pi}{4}\ln 2 - \int_0^1 \frac{\ln(1+y)}{1+y^2} dy$$

and $I = \int_0^1\int_0^1 \frac{x}{(1+x^2)(1+xy)} dy\,dx$

$$= \int_0^1 \frac{\ln(1+x)}{(1+x^2)} dx$$

$$** \quad \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \int_0^1 \frac{\ln(1+y)}{1+y^2} dy$$

Then $I = \frac{\pi}{4}\ln 2 - I$

$$\Rightarrow I = \frac{\pi}{8}\ln 2$$

Then deduce the value of $J = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

$$\boxed{I = J = \frac{\pi}{8}\ln 2}$$

14) a) Show the existence of $I = \int_0^{\pi/2} \frac{\ln(1+\cos x)}{\cos x} dx$

for $x = \pi/2$

$$\lim_{x\to\pi/2} \frac{\ln(1+\cos x)}{\cos x}$$

let $t = \cos x$; if $x\to\pi/2 \Rightarrow t\to 0$

$$\Rightarrow \lim_{t\to 0}\frac{\ln(1+t)}{t} = \lim_{t\to 0}\ln(1+t)^{1/t} = \ln e = 1$$

for $x=0 \Rightarrow \lim_{x\to 0}\frac{\ln(1+\cos x)}{\cos x} = 0$

$f(x) = \frac{\ln(1+\cos x)}{\cos x}$ continuous for $x\in(0,\pi/2)$

$\Rightarrow \underline{I \text{ exist}}$

b) Show that $I = \iint_R \frac{\sin y}{1+\cos x\cos y} dA$ where

$R = \{(x,y)\in\mathbb{R}^2 \mid 0<x<\pi/2 , 0<y<\pi/2\}$

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin y}{1+\cos x\cos y} dy\,dx$$

$$= -\int_0^{\pi/2} \frac{\ln(1+\cos x\cos y)}{\cos x}\Big|_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \frac{\ln(1+\cos x)}{\cos x} dx$$

c) Deduce the value of I

we have $I = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin y}{1+\cos x\cos y}\,dx\,dy$

let $v = \tan\frac{x}{2} \Rightarrow x = 2\arctan v$

Then $dx = \frac{1}{1+v^2}\,dv$

$\cdot\ \cos x = \frac{1-v^2}{1+v^2}$ because $\tan x = \tan\left(2\cdot\frac{x}{2}\right)$

$\Rightarrow \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2v}{1-v^2}$

$\Rightarrow \cos x = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+\frac{4v^2}{(1-v)^2}}}$

$\cos x = \frac{1-v^2}{1+v^2}$

let $H = \int_0^{\pi/2} \frac{\sin y}{1+\cos x\cos y}\,dx$

$= \int_0^1 \frac{\sin y}{1+\cos x\cos y}\,dx = \int_0^1 \frac{\sin y}{1+\left(\frac{1-v^2}{1+v^2}\right)\cos y} \times \frac{2}{1+v^2}\,dv$

$= \sin y \int_0^1 \frac{2}{(1+\cos y)+(1-\cos y)v^2}\,dv$

let $a = 1+\cos y$ and $b = 1-\cos y$

$\Rightarrow H = 2\sin y \int_0^1 \frac{1}{a+bv^2}\,dv$

$= \frac{2}{a}\sin y \int_0^1 \frac{1}{\left(1+\sqrt{\frac{b}{a}}\,v\right)^2}\,dv$

$H = \frac{2}{a}\sin y\,\frac{\sqrt{b}}{\sqrt{a}} \arctan\sqrt{\frac{b}{a}}$

$= \frac{2\sin y}{\sqrt{ab}} \arctan\sqrt{b/a}$

$\Rightarrow H = \frac{2\sin y}{\sqrt{1-\cos^2 y}} \arctan\sqrt{\frac{1-\cos y}{1+\cos y}}$

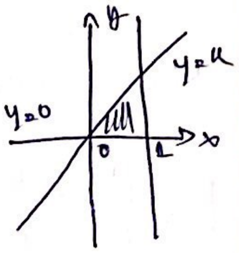
$\Rightarrow I = \int_0^{\pi/2} \frac{2\sin y}{\sqrt{1-\cos^2 y}} \arctan\sqrt{\frac{1-\cos y}{1+\cos y}}\,dy$

$= 2\int_0^{\pi/2} \arctan\left(\tan^2 y/2\right)dy$

$= 2\int_0^{\pi/2} \left(\frac{y}{2}\right)^2 dy = \boxed{\frac{\pi^2}{8}}$

(15) The objective of this exercise is to compute the value of $\int_0^1 \frac{\ln t}{1-t^2}\,dt$

(a) Compute $I = \iint_{0\le y\le x\le 1} \frac{1}{(1+x^2)(1+y^2)}\,dA$



$I = \int_0^1\int_0^x \frac{1}{(1+x^2)(1+y^2)}\,dy\,dx$

$= \int_0^1 \frac{1}{1+x^2}\arctan x\,dx$

$= \frac{1}{2}(\arctan x)^2\Big|_0^1 = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \boxed{\frac{\pi^2}{32}}$

(b) Show that the following integrals are convergent.

$J = \int_0^{\pi/4} \frac{\ln(2\cos^2 t)}{(2\cos 2t)}\,dt$

let $\begin{cases} f(t) = \frac{\ln(2\cos^2 t)}{2\cos 2t} \\ g(t) = 1 \end{cases}$

$\lim_{t\to\pi/4} \frac{f(t)}{g(t)} = \frac{\ln(2\cos^2 t)}{2\cos 2t} = \frac{1}{2}$

but $\int_0^{\pi/4} g(t)\,dt = \pi/4 < \infty$

Thus J convergent by limit comparison test

$K = \int_0^{\pi/4} \frac{\ln(2\sin^2 t)}{2\cos 2t}\,dt$

$\cdot = \pi/4$

$\lim_{t\to\pi/4} \frac{\ln(2\sin^2 t)}{2\cos 2t} = \lim_{t\to\pi/4} \frac{\frac{4\cos\theta\sin\theta}{2\sin^2 t}}{-2\sin\theta\cos\theta}$

$= \lim_{t\to\pi/4} -\frac{1}{2}\left(\frac{1}{\sin^2\theta}\right) = -\frac{1}{2}$

$\Rightarrow K$ convergent

$\cdot = 0$

$K = \int_0^{\pi/4} \frac{\ln(2\sin^2 t)}{2\cos 2t}\,dt = \int_0^{\pi/4} \frac{\ln 2}{2\cos 2t}\,dt + \int_0^{\pi/4} \frac{\ln\sin t}{\cos t}\,dt$

Then $K \sim \int_0^{\pi/4} \frac{\ln(\sin t)}{\cos 2t} dt \sim \int_0^{\pi/4} \ln(\sin t) dt \sim \int_0^{\pi/4} \ln t\, dt$

$K \sim \int_0^{\sqrt{2}/2} \ln t\, dt = \frac{\sqrt{2}}{2}\ln\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} < \infty$

K convergent

Thus K convergent

$L = \int_0^1 \frac{\ln t}{1-t^2} dt$

. $t \to 0$

$L = \int_0^1 \frac{\ln t}{1-t^2} dt \sim \int_0^1 \ln t\, dt = -1 < \infty$

L convergent

. $t \to 1$

$\lim_{t \to 1} \frac{\ln t}{1-t^2} = \lim_{t \to 1} \frac{1/t}{-2t} = -\frac{1}{2} < \infty$

L convergent

Thus L convergent

(c) Show that $I = J$

$I = \iint_{0 \le y \le x \le 1} \frac{1}{(1+x^2)(1+y^2)} d\theta$

by using polar coordonation

$I = \int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta}} \frac{1}{(1+r^2\cos^2\theta)(1+r^2\sin^2\theta)} r\, dr\, d\theta$

$= \int_0^{\pi/4} \int_0^{1/\cos\theta} \left(\frac{r\sin^2\theta}{1+r^2\sin^2\theta} - \frac{r\cos^2\theta}{1+r^2\cos^2\theta}\right) \frac{1}{\cos^2\theta + \sin^2\theta} dr\, d\theta$

$= \int_0^{\pi/4} \int_0^{1/\cos\theta} \frac{1}{\cos 2\theta} \left(\frac{r\sin^2\theta}{1+r^2\sin^2\theta} - \frac{r\cos^2\theta}{1+r^2\cos^2\theta}\right) dr\, d\theta$

$= \int_0^{\pi/4} \frac{1}{\cos 2\theta} \left[\frac{1}{2}\ln(1+r^2\sin^2\theta) - \frac{1}{2}\ln(1+r^2\cos^2\theta)\right]_0^{1/\cos\theta} d\theta$

$= \int_0^{\pi/4} \frac{1}{2\cos 2\theta} (\ln\cos^2\theta + \ln 2)\, d\theta$

$= \int_0^{\pi/4} \left(\frac{\ln 2\cos^2\theta}{2\cos 2\theta}\right) d\theta = J$

Thus $I = J$

d)- Calculate $J+K$ and $J-K$ in term of L

$J+K = \int_0^{\pi/4} \frac{\ln(4\cos^2 t \sin^2 t)}{2\cos 2t} dt$

$= \int_0^{\pi/4} \frac{\ln(\sin^2 2t)}{2\cos 2t} dt$

$= \int_0^{\pi/4} \frac{\ln(\sin 2t)}{\cos 2t} \cdot \frac{\cos 2t}{\cos 2t} dt$

let $U = \sin 2t \Rightarrow du = 2\cos 2t\, dt$

. $t = \pi/4 \Rightarrow U = 1$

. $t = 0 \Rightarrow U = 0$

. $\cos^2 2t = 1 - U^2$

$J+K \int_0^1 \left(\frac{\ln U}{1-U^2} \times \frac{1}{2}\right) du = \frac{1}{2}L$

$J-K = \int_0^{\pi/4} \frac{\ln(\cot^2 t)}{\cot^2 t - 1} \times \frac{1}{\sin^2 t} dt$

let $U = \cot(t) \Rightarrow du = -\frac{1}{\sin^2 t} dt$

$J-K = \int_0^1 \frac{\ln U}{1-U^2} du = L$

Thus $J+K = \frac{1}{2}L$
$J-K = \frac{1}{2}L$

e)- Deduce the values of K and L

we have $\begin{cases} J+K = \frac{1}{2}L \\ J-K = L \end{cases}$

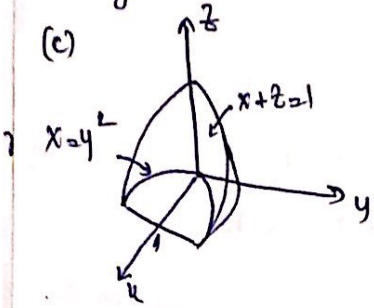
$2J = \frac{3}{2}L \Rightarrow L = \frac{4}{3}J = \frac{4}{3} \times \frac{\pi^2}{32} = \frac{\pi^2}{24}$

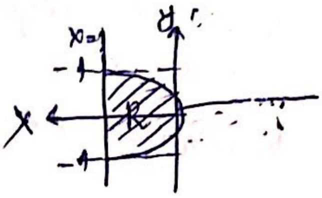
$K = \frac{1}{2}L - J = \frac{1}{2} \times \frac{\pi^2}{24} - \frac{\pi^2}{32} = \frac{-\pi^2}{96}$

Thus $K = \frac{-\pi^2}{96}$, $L = \frac{\pi^2}{24}$

(16) Use double integral to find the volume of the solide shown in the figures below:

(c)





$V=\iint_R f(x,y)\,dA$

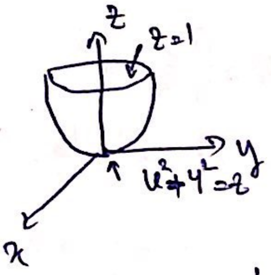
$f(x,y)=1-x$

$R=\{(x,y)\in\mathbb{R}^2 \,/\, x\geq y^2,\ x\leq 1\}$

$V=\int_{-1}^{1}\int_{y^2}^{1}(1-x)\,dx\,dy=\int_{-1}^{1}\left[x-\frac{x^2}{2}\right]_{y^2}^{1}dy$

$=\int_{-1}^{1}\left(\frac{1}{2}-y^2+\frac{y^4}{2}\right)dy=2\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)=\frac{8}{15}$

(d)



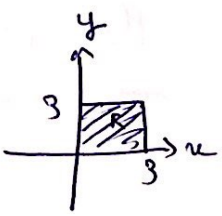
$V=\iint_R f(x,y)\,dA \quad / \ z=f(x,y)=1-x^2-y^2$

$V=\int_0^{2\pi}\int_0^1 r(1-r^2)\,dr\,d\theta=2\pi\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{\pi}{2}$

(17) Find the area of the surface S over the region R.

(a) S is the surface of $f(x,y)=1+x-2y$, R is a square with vertrces $(0,0)$, $(3,0)$, $(0,3)$ and $(3,3)$

$(S): f(x,y)=1+x-2y$



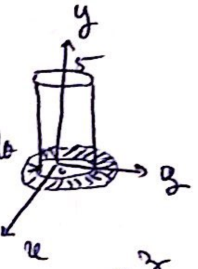
$R:\{(x,y)\in\mathbb{R}^2 \,/\, 0\leq x\leq 3,\ 0\leq y\leq 3\}$

$I=\int_0^3\int_0^3\sqrt{1+1+4}\,dx\,dy=\sqrt{6}\times 3\times 3=9\sqrt{6}$

(b) S is the paraboloid $y=9-x^2-z^2$ that lies between the planes $y=0$ and $y=5$

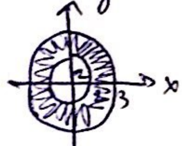
$(S): f(x,z)=9-x^2-z^2$

by using polar coordonation



$I=\int_0^{2\pi}\int_2^3 r\sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\,dr\,d\theta$

$=\int_0^{2\pi}\int_2^3 r\sqrt{1+4r^2}\,dr\,d\theta$



$=2\pi\times\frac{1}{8}\int_2^3\sqrt{1+4r^2}\,d(1+4r^2)$

$=\frac{2\pi}{8}\times\frac{2}{3}\left[1+4r^2\right]^{3/2}\Big|_2^3$

$=\frac{\pi}{6}\left(37^{3/2}-17^{3/2}\right)$

- $y=0 \Rightarrow x^2+z^2=9$
- $y=5 \Rightarrow x^2+z^2=4$
- $f_x=-2x$
- $f_z=-2z$

(c) S is the part of the paraboloid $z=y^2-x^2$ that lies above the annuler R region $R=\{(x,y): 1\leq x^2+y^2\leq 4\}$

$(S): f(x,y)=y^2-x^2$

$f_x=-2x \quad f_y=2y$

by using polar coordonate

$I=\int_0^{2\pi}\int_1^2 r\sqrt{1+4r^2}\,dr\,d\theta$

$=\frac{2\pi}{8}\times\frac{1}{6}\times\left[1+4r^2\right]^{3/2}\Big|_1^2$

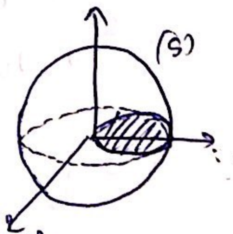
$=\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$

(d) S is the part of the sphere $x^2+y^2+z^2=a^2$ that lies inside the cylinder $x^2-ax+y^2=0$

- $x^2-ax+y^2=0 \Leftrightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$

$R=\{(x,y)\in\mathbb{R}^2 \,/\, \left(x-\frac{a}{2}\right)^2+y^2\leq\frac{a^2}{4}\}$

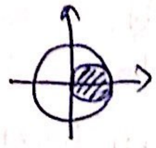
$z=f(x,y)=\sqrt{a^2-x^2-y^2}$



$f_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}, \quad f_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

$1+f_x^2+f_y^2=1+\frac{x^2+y^2}{a^2-(x^2+y^2)}=\frac{a^2}{a^2-(x^2+y^2)}$

by using polar coordinate



• $x^2-2ax+y^2\le 0$

$r^2-2ar\cos\theta\le 0$ / $r\ge 0$

$0\le r\le a\cos\theta$

$-\pi/2\le\theta\le\pi/2$

$$I=2\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$=2a\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)\frac{-2r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

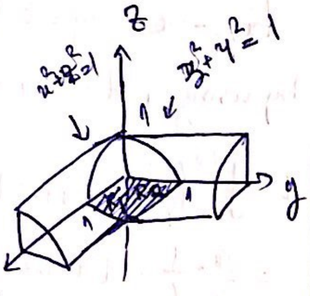
$$=-\frac{2a}{2}\int_{-\pi/2}^{\pi/2}\left[\frac{(a^2-r^2)^{1/2}}{1/2}\right]_0^{a\cos\theta}d\theta$$

$$=-2a\int_0^{\pi/2}a(\sin\theta-1)\,d\theta=-2a^2\int_0^{\pi/2}(\sin\theta-1)\,d\theta$$

$$=-2a^2\left(-\cos\theta\Big|_0^{\pi/2}-\frac{\pi}{2}\right)$$

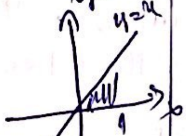
$$=\pi a^2-2a^2$$

e)- S is comprises the parts of the cylinder $x^2+z^2=1$ that lie within the cylinder $y^2+z^2=1$



let (S): $z=f(x,y)=\sqrt{1-x^2}$

$R:\{(x,y)\in\mathbb{R}^2,\ y\le x,\ y\ge 0,\ x\le 1\}$



$$S=2\int_0^1\int_0^x\sqrt{1+\frac{x^2}{1-x^2}}\,dy\,dx$$

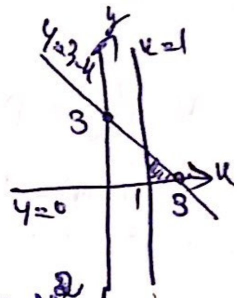
$$=2\int_0^1\int_0^x\frac{1}{\sqrt{1-x^2}}\,dy\,dx$$

$$=-\int_0^1\frac{-2x}{\sqrt{1-x^2}}\,dx=\left[-2(1-x^2)^{1/2}\right]_0^1=2$$

(18) Find the mass and the center of mass of the lamina occupying the region R and having mass density ρ.

a) R is the region bounded by $y=3-x$, $y=0$ and $x=1$ $\rho(x,y)=2xy$

$R=\{(x,y)\in\mathbb{R}^2 / 0\le y\le 3-x,\ 1\le x\le 3\}$



• mass of the lamina

$$m=\int_1^3\int_0^{3-x}2xy\,dy\,dx=\int_1^3 x(x-3)^2\,dx$$

$$=\int_1^3(x^3-6x^2+9x)\,dx=\frac{80}{4}-2\times 26+36$$

$$=4$$

$$M_x=\int_1^3\int_0^{3-x}2y^2x\,dy\,dx=\int_1^3\frac{2}{3}(3-x)^3x\,dx$$

$$=\frac{2}{3}\int_1^3(27x-27x^2+9x^3-x^4)\,dx$$

$$=9\times 8-6\times 26+30\times 26-\frac{2}{15}\times 242$$

$$=72-156+780-\frac{484}{15}=\frac{56}{15}$$

$$M_y=\int_1^3\int_0^{3-x}2yx^2\,dy\,dx=\int_1^3 9x^2-6x^3+x^4\,dx$$

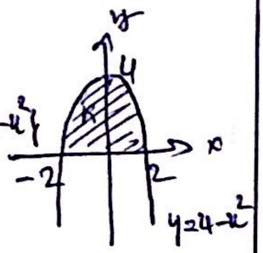
$$=26\times 3-6\times 20+\frac{242}{5}=\frac{32}{5}$$

$$\bar{x}=\frac{32/5}{4}=\frac{8}{5}$$

$$\bar{y}=\frac{56/15}{4}=\frac{14}{15}$$

Thus $m=4$; $(\bar{x},\bar{y})=\left(\frac{8}{5},\frac{14}{15}\right)$

b). R is the region bounded by the parabola $y=4-x^2$ and x-axis and $f(x,y)=y$



$R=\{(x,y)\in\mathbb{R}^2 / -2\le x\le 2,\ 0\le y\le 4-x^2\}$

$$m=\int_{-2}^{2}\int_0^{4-x^2} y\,dy\,dx$$

$$=\frac{1}{2}\int_{-2}^{2} 4-8x+x^2\,dx=\frac{1}{2}\left(16\times 4-\frac{128}{3}+\frac{64}{3}\right)$$

$$=\boxed{\frac{256}{15}}$$

$$M_x=\int_{-2}^{2}\int_0^{4-x^2} xy\,dy\,dx=\frac{1}{2}\int_{-2}^{2}16x-8x^3+x^5\,dx$$

$$=\frac{64}{12}-\frac{64}{12}=0$$

$$M_y=\int_{-2}^{2}\int_0^{4-x^2} y^2\,dy\,dx=\int_{-2}^{2}\frac{1}{3}(4-x^2)^3\,dx$$

$$=\frac{4096}{108}$$

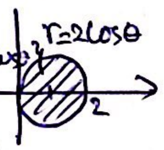
$$\bar{x}=\frac{4096/108}{256/15}=\frac{20}{9}$$

$$\bar{y}=\frac{0}{256/15}=0$$

Thus $m=256/15$, $(\bar{x},\bar{y})=\left(\frac{20}{9},0\right)$

c). R is the region by the circle $r=2\cos\theta$, $f(r,\theta)=r$

$R=\{(r,\theta)\in\mathbb{R}^2, -\pi/2\le\theta\le\pi/2,\ 0\le r\le 2\cos\theta\}$



$$m=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{8}{3}\cos^3\theta\,dr\,d\theta$$

$$=\frac{8}{3}\int_{-\pi/2}^{\pi/2}(\cos\theta-\cos\theta\sin^2\theta)\,d\theta$$

$$=\frac{8}{3}\left(1+1-\left(\frac{1}{3}+\frac{1}{3}\right)\right)=\frac{32}{9}$$

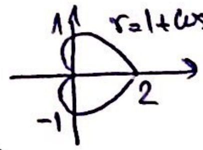
$$M_x=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\,dr\,d\theta=4\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta$$

$$=4\int_{-\pi/2}^{\pi/2}(1-\sin^2\theta)^2\,d(1-\sin\theta)=3\left[\frac{(1-\sin^2\theta)^3}{3}\right]_{-\pi/2}^{\pi/2}=\frac{32}{15}$$

$$M_y=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\sin\theta\,dr\,d\theta=0$$

Thus $m=\frac{32}{9}$, $(\bar{x},\bar{y})=(0,5/3)$

d). R is the region bounded by the cardioid $r=1+\cos\theta$, $f(r,\theta)=3$



$$M=\int_0^{2\pi}\int_0^{1+\cos\theta}3r\,dr\,d\theta=\int_0^{2\pi}\frac{3}{2}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=\frac{3}{2}\int_0^{2\pi}\left(1+\frac{1}{2}\right)d\theta=\frac{3}{2}\times\frac{3}{2}\times 2\pi=\frac{9}{2}\pi$$

$$M_x=\int_0^{2\pi}\int_0^{1+\cos\theta}3r^2\cos\theta\,dr\,d\theta$$

$$=\int_0^{2\pi}\cos\theta(1+3\cos^2\theta+3\cos\theta+\cos^3\theta)\,d\theta$$

$$=\int_0^{2\pi}\cos\theta+3\overset{0}{\cos^3\theta}+3\cos^2\theta+\cos^4\theta\,d\theta$$

$$=\int_0^{2\pi}\overset{0}{\cos\theta}+3\left(\frac{1+\overset{0}{\cos 2\theta}}{2}\right)+\frac{1}{4}\left(1+2\overset{0}{\cos 2\theta}+\cos^2 2\theta\right)d\theta$$

$$=\int_0^{2\pi}\left(\frac{3}{2}+\frac{3}{8}\right)d\theta=2\pi\times\frac{15}{8}=\frac{15\pi}{4}$$

$$M_y=\int_0^{2\pi}\int_0^{1+\cos\theta}3r^2\sin\theta\,dr\,d\theta=\int_0^{2\pi}(1+\cos\theta)^3\sin\theta\,d\theta$$

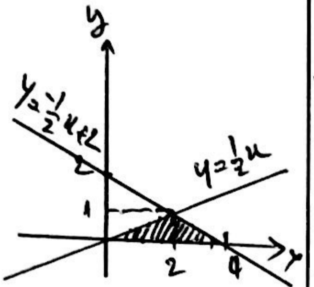
$$=-\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{2\pi}=0$$

Thus $m=\frac{9\pi}{2}$, $(\bar{x},\bar{y})=(0,5/6)$

(19) Find the moments of inertia I_x, I_y, I_0

a). R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(4,0)$

$\delta(x,y) = x$.



$R = \{(x,y)\in\mathbb{R}^2 / 0\le y\le 1,$
$2y \le x \le -2y+4\}$

$$I_x = \int_0^1\int_{2y}^{-2y+4} xy^2\,dx\,dy$$
$$= \frac{1}{2}\int_0^1 y^2(16-16y)\,dy = \frac{1}{2}\left[\frac{16}{3}-\frac{16}{4}\right] = \frac{2}{3}$$

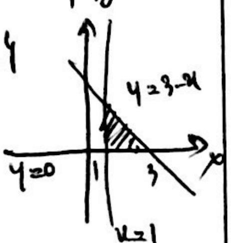
$$I_y = \int_0^1\int_{2y}^{-2y+4} x^3\,dx\,dy = \frac{1}{4}\int_0^1 (4-2y)^4-(2y)^4\,dy$$

$$= 4\left[-\frac{(2-y)^5+y^5}{5}\right]_0^1 = 24$$

$$I_0 = I_x+I_y = \frac{2}{3}+24 = \frac{74}{3}$$

b). R is the region bounded by $y=3-x$, $y=0$ and $x=1$, $\delta(x,y)=2xy$

$R = \{(x,y)\in\mathbb{R}^2 / 1\le x\le 3,\ 0\le y\le 3-x\}$



$$I_x = \int_1^3\int_0^{3-x} 2xy^3\,dy\,dx$$
$$= \frac{1}{2}\int_1^3 x(3-x)^4\,dx$$

let $t = 3-x \Rightarrow dt = -dx$

. $x=1 \Rightarrow t=2$

. $x=3 \Rightarrow t=0$

$$I_x = \frac{1}{2}\int_0^2 (3-t)t^4\,dt = \frac{1}{2}\left(\frac{2^6}{6}-\frac{3.2^5}{5}\right)$$

$$= \frac{64}{5}$$

$$I_y = \int_1^3\int_0^{3-x} yx^3\,dy\,dx = \int_1^3 x^3(9-6x+x^2)\,dx$$
$$= \left[\frac{9}{4}x^4-\frac{6}{5}x^5+\frac{1}{6}x^6\right]_1^3 = \frac{9}{4}\times 8-\frac{6}{4}\times 242+\frac{1}{6}\times 728$$
$$= 164/15$$

$$I_0 = I_x+I_y = \frac{164}{15}+\frac{64}{5} = \frac{356}{15}$$

c). R is the region bounded by $y=\sqrt{x}$, $y=0$ and $x=4$, $\delta(x,y)=xy$

$R = \{(x,y)\in\mathbb{R}^2 / 0\le x\le 4,\ 0\le y\le \sqrt{x}\}$

$$I_x = \int_0^4\int_0^{\sqrt{x}} xy^3\,dy\,dx$$
$$= \frac{1}{4}\int_0^4 x^3\,dx = \frac{1}{4^2}\cdot 4^4 = 16$$

$$I_y = \int_0^4\int_0^{\sqrt{x}} x^3y\,dy\,dx = \int_0^4 \frac{1}{2}x^4\,dx = \frac{1}{2}\times\frac{4^5}{5} = \frac{512}{5}$$

$$I_0 = I_x+I_y = 16+\frac{512}{5} = \frac{592}{5}$$

d). R is the region in the first quadrant bounded by circle $x^2+y^2=1$, $\delta(x,y)=x+y$

$R = \{(r,\theta)\in\mathbb{R}^2 / 0\le\theta\le\frac{\pi}{2},\ 0\le r\le 1\}$

$$I_x = \int_0^{\pi/2}\int_0^1 r^4\sin^2\theta\cos\theta + r^4\sin^3\theta\,dr\,d\theta$$
$$= \frac{1}{5}\int_0^{\pi/2}(\sin^2\theta\cos\theta+\sin^3\theta)\,d\theta = \frac{7}{60}$$

$$I_y = \int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\sin\theta + r^4\cos^3\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\frac{1}{5}(\cos^2\theta\sin\theta+\cos^3\theta)\,d\theta$$

$$= \frac{7}{60}$$

$$I_0 = I_x+I_y = \frac{7}{60}+\frac{7}{60} = \frac{7}{30}$$

I2-TD6
(Multiple Integrations)

1. Evaluate the following integrals.

(a) $\displaystyle\int_0^1\int_0^{3x}\int_0^{y}(y+x^2)\,dz\,dy\,dx$

(b) $\displaystyle\int_0^{\pi/2}\int_{x/2}^{\pi/2}\int_0^{1/y}\sin y\,dz\,dy\,dx$

(c) $\displaystyle\int_1^{e}\int_1^{x}\int_0^{1/(xy)}2\ln y\,dz\,dy\,dx$

(d) $\displaystyle\int_0^{\pi/2}\int_0^{2\cos^2\theta}\int_0^{4-r^2}r\sin\theta\,dz\,dr\,d\theta$

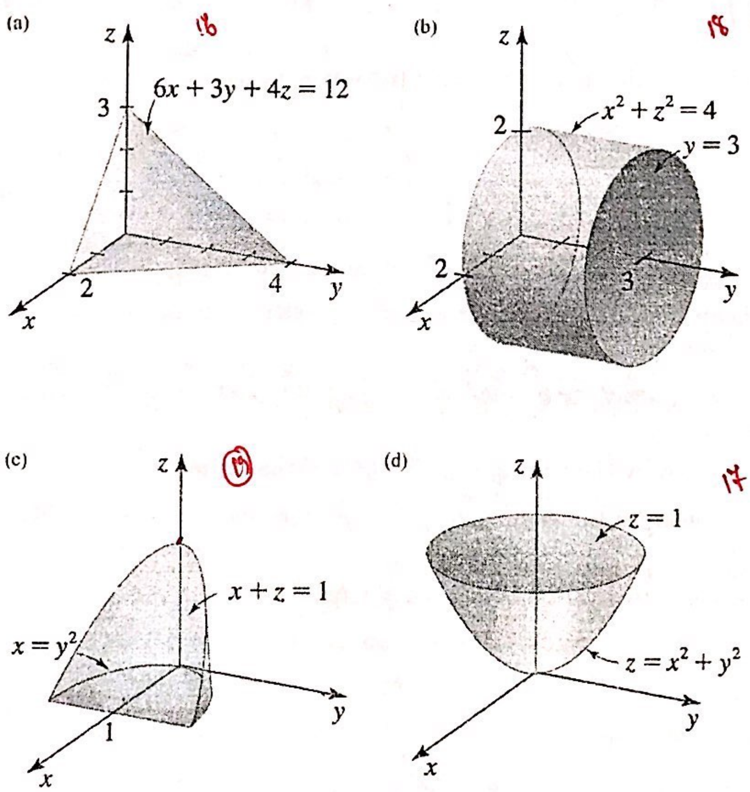
2. Compute $\displaystyle\iiint_Q \frac{z^3}{(x+y+z)}\,dV$

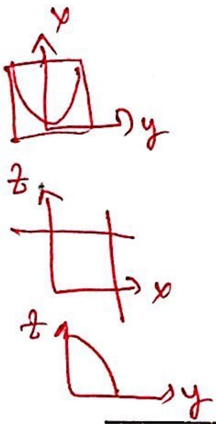
$$Q=\{(x,y,z)\in\mathbb{R}^3 \mid x\geq 0,\ y\geq 0,\ z\geq 0,\ x+y+z\leq 1\}$$

Hint: Use change variables

$$u=x+y+z,\quad v=\frac{z}{y+z},\quad w=\frac{y+z}{x+y+z}$$

3. The figure shows the region of integration for $\displaystyle\iiint_Q f(x,y,z)dV$. Express the triple integral as an iterated integral in six different ways using different orders of integration.





4. Evaluate the following integral by using Cylindrical coordinates

(a) $I = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3} \frac{e^z}{\sqrt{x^2+y^2}}\, dz\, dy\, dx.$

(b) $J = \int_{-1}^{1}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy.$

(c) $K = \iiint_Q \sqrt{x^2+y^2}$, Q is the solid bounded by $z = x^2+y^2$ and the plane $z = 4$.

(d) $L = \iiint_Q \frac{1}{x^2+y^2}$, Q is the solid bounded above by $z = 4-(x^2+y^2)$ and below by the sphere $x^2+y^2+z^2 = 9$.

5. Evaluate the following integral by using Spherical coordinates

(a) $I = \iiint_Q (x^2+y^2+z^2)\, dV$, Q is the solid region bounded below by the cone $z^2 = x^2+y^2$ and above by the sphere $x^2+y^2+z^2 = 9$.

(b) $J = \iiint_Q z\, dV$, Q is the solid region bounded below by the cone $x^2+y^2+(z-2)^2 = 1$.

6. Find the volume of the solid Q by using triple integrals.

(a) $Q = \{(x,y,z) \in \mathbb{R}^3 : 2x+y+z-3 = 0, x > 0, y > 0, z > 0\}$.

(b) Q is bounded by $z = 9-x^2, z = 0, y = 0$ and $y = 2x$.

(c) Q is enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(d) Q is bounded by paraboloid $z = x^2+y^2$, and inside the elliptic cylinder $x^2/9+y^2/4 = 1$.

(e) Q is bounded above by the sphere $x^2+y^2+z^2 = 9$ and below by the paraboloid $8z = x^2+y^2$.

(f) Q is volume inside the paraboloid $x^2+y^2 = 2\mu z$ and outside the cone $x^2+y^2 = \lambda^2 z^2$, where $\lambda, \mu > 0$.

(g) Q is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1$.

(h) Q is bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2+y^2 = 4$, and the plane $z = 0$.

(i) Q is bounded by the graph $r = 2\cos\theta$ and $r^2+z^2 = 4$.

(j) Q is bounded by the graph $r^2+z = 16, z = 0$, and $r = 2\sin\theta$.

(k) $Q = \{(x,y,z) : x^{2/3}+y^{2/3}+z^{2/3} \le 1\}$.

(l) Q is bounded by $x^2+4y^2+z^2 = 1$

(m) Q is bounded by $x^2+\frac{1}{2}y^2+\frac{3}{4}z^2+xz = 1$

7. Evaluate the integral $\iiint_Q f(x,y,z)\, dV$.

(a) $f(x,y,z) = xy$, $Q = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le x+y\}$.

(b) $f(x,y,z) = z$, Q is the region bounded by the cylinder $x^2 + z^2 = 1$ and the planes $y = x, y = 2x$, and $z = 0$.

(c) $f(x,y,z) = \sqrt{x^2+y^2}$, Q is the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = 3$.

(d) $f(x,y,z) = \sqrt{x^2+y^2}$, $Q = \{(x,y,z) \in \mathbb{R}^3 : x + y + z < 2, x^2 + y^2 < 1, z > 0\}$.

(e) $f(x,y,z) = y$, Q is the part of the solid in the first octant lying inside the paraboloid $z = 4 - x^2 - y^2$.

(f) $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$, Q is the part of unit ball $x^2 + y^2 + z^2 \leq 1$ lying in the first octant.

(g) $f(x,y,z) = xz$, Q is the solid bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by the cone $z = \sqrt{x^2+y^2}$.

(h) $f(x,y,z) = \dfrac{1}{\sqrt{(x-2)^2+y^2+z^2}}\,dV$; $Q = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 < 1\}$

(i) $f(x,y,z) = \dfrac{1}{x^2+y^2}\,dV$; $Q = \{(x,y,z) \in \mathbb{R}^3 :\ x, y > 0; 2 < x^2 + y^2 < 2\sqrt{2}(x + y); 0 < z < x + y\}$

8. Find the mass and centre of mass of the solid Q of given mass density.

(a) Q is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. The density at a point P of Q is directly proportional to the distance between P and the yz-plane.

(b) Q is the solid bounded by the cylinder $x^2 + y^2 = 1$ in the first octant and the plane $z + y = 1$; the density $\rho(x,y,z) = x^2 + y^2 + z^2$.

(c) Q is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$; the mass density at any point on the solid is directly proportional to the square of its distance form the origin.

9. Find the moment of inertia about the z-axis of a solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$ if the mass density at any point on the solid is directly proportional to its distance from the xy-plane.

10. Let Q be a uniform solid of mass m bounded by the spheres $\rho = a$ and $\rho = b$, where $0 < a < b$. Show that the moment of inertia of Q about a diameter of Q is

$$I = \frac{2m}{5}\left(\frac{b^5 - a^5}{b^3 - a^3}\right).$$

11. (a) Use the result of Exercise above to fine the moment of inertia of a uniform solid ball of mass m and radius b about a diameter of the ball.

(b) Use the result of Exercise above to find the moment of inertia of a hollow spherical shell of mass m and radius b about a diameter of the shell.

12. Let $f(x,y)$ be a continuous in the region $R = \{(x,y) \in \mathbb{R}^2 :\ x \leq b, y \geq a, y \leq x\}$. Show that

$$\int_a^b \int_a^x f(x,y)\,dy\,dx = \int_a^b \int_b^y f(x,y)\,dx\,dy.$$

Use the result above to show that

$$I = \int_a^x \int_a^y \int_a^z f(t)\,dt\,dz\,dy = \frac{1}{2}\int_a^x (x-t)^2 f(t)\,dt.$$

13. Find the region Q where the integral

$$\iiint_Q (1 - 2x^2 - 3y^2 - z^2)^{1/3} dV$$

has maximum value.

14. Compute the integral

(a) $I = \iiint_{0\leq x+y+z\leq 1} xyz\ dV$

(b) $I = \iiint_{\sqrt{x}+\sqrt{y}+\sqrt{z}\leq 1} xyz\ dV$

15. Compute the integral

$$\iiiint_V e^{x^2+y^2-u^2-v^2} dx\,dy\,du\,dv$$

where $V = \{(x,y,u,v) \in \mathbb{R}^4 : x^2+y^2+u^2+v^2 \leq 1\}$.

16. Calculate $I = \iiint_Q \dfrac{1}{(1+x^2z^2)(1+y^2z^2)} dV$ where

$$Q = \{(x,y,z) \in \mathbb{R}^3 \mid 0 < x < 1\,; 0 < y < 1\,; z > 0\}$$

Then deduce the value of $J = \displaystyle\int_0^\infty \left(\frac{\arctan t}{t}\right)^2 dt$

17. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left[\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right] dx_1\,dx_2\ldots dx_n.$$

18. Let $r > 0$ and $V_n(r)$ be the volume of the boll $B_n(0,r)$ in $\mathbb{R}^n$. Show by induction that for all integer $n \geq 2:\ V_n(r) = \alpha_n r^n$, where

$$\alpha_n = \begin{cases} \dfrac{2\pi}{n}\cdot\dfrac{2\pi}{n-2}\cdots\dfrac{2\pi}{2}, & \text{if } n \text{ is even} \\ \dfrac{2\pi}{n}\cdot\dfrac{2\pi}{n-2}\cdots\dfrac{2\pi}{3}.2, & \text{if } n \text{ is odd.} \end{cases}$$

Calculate $V_4(2)$ and $V_5(3)$.

I2 TD6 (Mutiple Integration)

① Evaluate the following integrals

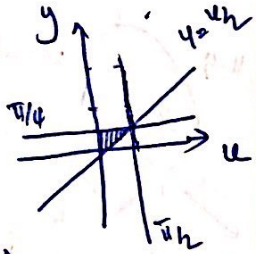
ⓐ $\int_0^1\int_0^{3x}\int_0^{y}(y+x^2)\,dz\,dy\,dx$

$= \int_0^1\int_0^{3x} y^2+x^2y\,dy\,dx = \int_0^1 9x^3+\frac{3x^4}{2}\,dx$

$= \frac{9}{4}+\frac{3}{10} = \frac{63}{20}$

ⓑ $\int_0^{\pi/2}\int_{x/2}^{\pi/2}\int_0^{1/y}\sin y\,dz\,dy\,dx$

$= \int_0^{\pi/2}\int_{x/2}^{\pi/2}\frac{\sin y}{y}\,dy\,dx$



$= \int_0^{\pi/4}\int_0^{2y}\frac{\sin y}{y}\,dx\,dy$

$= 2\int_0^{\pi/4}\sin y\,dy = -2\left(\frac{\sqrt{2}}{2}-1\right)$

$= 2-\sqrt{2}$

ⓒ $\int_1^e\int_1^x\int_0^{1/xy} 2\ln y\,dz\,dy\,dx$

$= \int_1^e\int_1^x \frac{2\ln y}{xy}\,dy\,dx$

$= \int_1^e \frac{2}{x}\left[\frac{(\ln y)^2}{2}\right]_1^x dx$

$= \int_1^e \frac{1}{2}\cdot\frac{2}{x}(\ln x)^2\,dx = \frac{1}{3}\left[(\ln x)^3\right]_1^e = \frac{1}{3}(1\times1\times1) = \frac{1}{3}$

ⓓ $\int_0^{\pi/2}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\,dz\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^{2\cos^2\theta}(4r-r^3)\sin\theta\,dr\,d\theta$

$= \int_0^{\pi/2}(8\cos^4\theta-4\cos^8\theta)\sin\theta\,d\theta$

$= \int_0^{\pi/2}(4\cos^8\theta-8\cos^4\theta)\,d(\cos\theta)$

$= \left[\frac{4}{9}\cos^9\theta-\frac{8}{5}\cos^5\theta\right]_0^{\pi/2} = -\left(\frac{4}{9}-\frac{8}{5}\right) = \frac{52}{45}$

② Compute $\iiint_Q \frac{z^3}{(x+y+z)}\,dv$

$Q=\{(x,y,z)\in\mathbb{R}^3 \mid x\geq0, y\geq0, z\geq0, x+y+z\leq1\}$

Hint: Use change varrbles

$u=x+y+z$, $v=\frac{z}{y+z}$, $w=\frac{y+z}{x+y+z}$

Let
$$\begin{cases} u=x+y+z & (1)\\ v=\frac{z}{y+z} & (2)\\ w=\frac{y+z}{x+y+z} & (3)\end{cases}$$

following (1)&(2)&(3)

$$\begin{cases} y+z=z/v\\ y+z=wu\end{cases} \Rightarrow \frac{z}{v}=wu \Rightarrow z=uvw$$

$\Rightarrow y = uw-z = uw-uvw = uw(1-v)$

$\Rightarrow x = u-wuv-uw(1-v)$

$= u-uw = u(1-w)$

$$J = \begin{vmatrix} 1-w & 0 & -u\\ w(1-v) & -uw & u(1-v)\\ vw & uw & uv\end{vmatrix} = \begin{vmatrix} 1-w & 0 & -u\\ w & 0 & u\\ vw & uw & uv\end{vmatrix}$$

$$= \begin{vmatrix} 1-w & 0 & -u\\ w & 0 & u\\ vw & uw & uv\end{vmatrix}\begin{matrix} 1-w & 0\\ w & 0\\ vw & uw\end{matrix} = uw^2$$

$\Rightarrow |J| = uw^2$

- $x=0 \Leftrightarrow u(1-w)=0 \Rightarrow u=0, w=1$
- $y=0 \Leftrightarrow uw(1-v)=0 \Rightarrow w=0, u=0, v=1$
- $z=0 \Leftrightarrow uvw=0 \Rightarrow u=0, v=0, w=0$
- $x+y+z=1 \Rightarrow u=1$

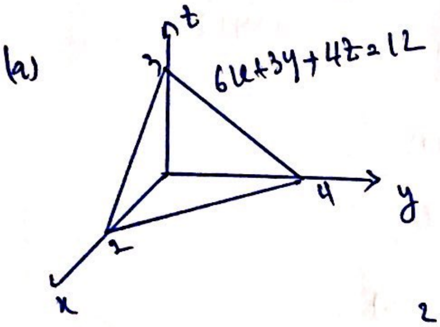
then $\frac{z^3}{(x+y+z)} = v^3w^3u^2$

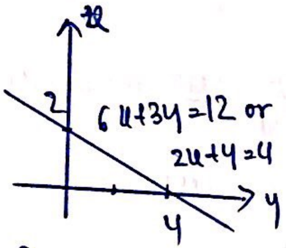
$I = \int_0^1\int_0^1\int_0^1 v^3u^3w^5\,du\,dv\,dw$

$= \frac{1}{4}\times\frac{1}{4}\times\frac{1}{6} = \frac{1}{96}$

⑧ The figure shows the region of integration for $\iiint_Q f(x,y,z)\,dV$. Express the triple integral as an iterated integral in six different ways using different orders of integration.

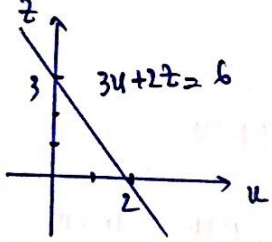
(a)





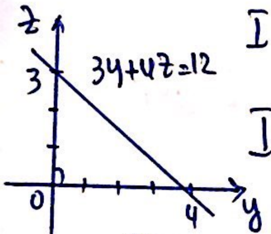
$$I_1=\int_0^2\int_0^{4-2x}\int_0^{\frac{1}{4}(12-6x-3y)} f(x,y,z)\,dz\,dy\,dx$$

$$I_2=\int_0^4\int_0^{\frac{1}{2}(4-y)}\int_0^{\frac{1}{4}(12-6x-3y)} f(x,y,z)\,dz\,dx\,dy$$



$$I_3=\int_0^3\int_0^{\frac{1}{3}(6-2z)}\int_0^{\frac{1}{3}(12-6x-4z)} f(x,y,z)\,dy\,dx\,dz$$

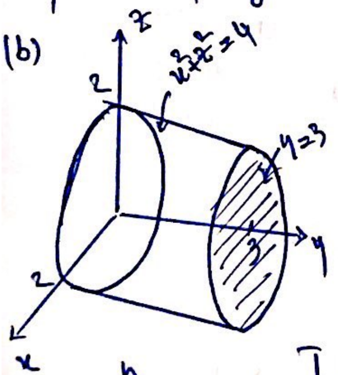
$$I_4=\int_0^2\int_0^{\frac{1}{2}(6-3x)}\int_0^{\frac{1}{3}(12-6x-4z)} f(x,y,z)\,dy\,dz\,dx$$

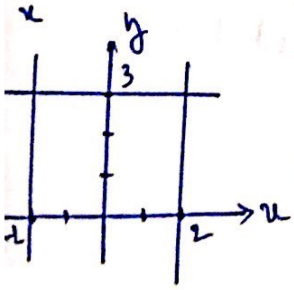


$$I_5=\int_0^3\int_0^{\frac{1}{3}(12-4z)}\int_0^{\frac{1}{6}(12-3y-4z)} f(x,y,z)\,dx\,dy\,dz$$

$$I_6=\int_0^4\int_0^{\frac{1}{4}(12-3y)}\int_0^{\frac{1}{6}(12-3y-4z)} f(x,y,z)\,dx\,dz\,dy$$

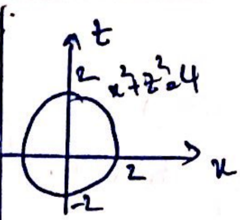
(b)





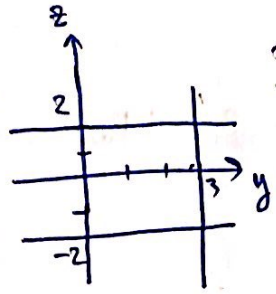
$$I_1=\int_0^3\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dx\,dy$$

$$I_2=\int_{-2}^{2}\int_0^{3}\int_0^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dy\,dx$$



$$I_3=\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_0^{3} f(x,y,z)\,dy\,dx\,dz$$

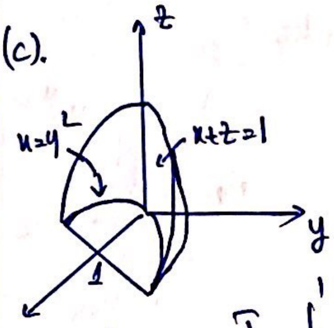
$$I_4=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{3} f(x,y,z)\,dy\,dz\,dx$$



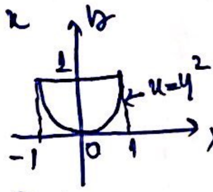
$$I_5=\int_{-2}^{2}\int_0^{3}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dy\,dz$$

$$I_6=\int_0^{3}\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy$$

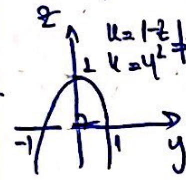
(c).



$$I_1=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x} f(x,y,z)\,dz\,dy\,dx$$

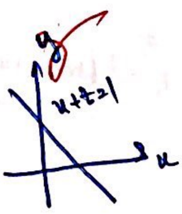


$$I_2=\int_{-1}^{1}\int_{y^2}^{1}\int_0^{1-x} f(x,y,z)\,dz\,dx\,dy$$



$$I_3=\int_0^1\int_{-1}^{1}\int_{y^2}^{1} f(x,y,z)\,dx\,dy\,dz$$

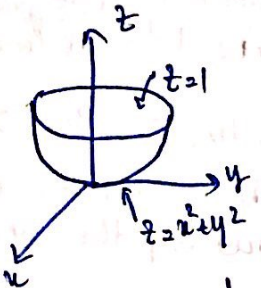
$$I_4=\int_{-1}^{1}\int_0^{1}\int_{y^2}^{1} f(x,y,z)\,dx\,dz\,dy$$

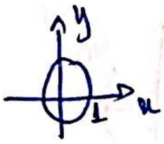


$$I_5=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x} f(x,y,z)\,dz\,dy\,dx$$

$$I_6=\int_{-1}^{1}\int_{y^2}^{1}\int_0^{1-x} f(x,y,z)\,dz\,dx\,dy$$

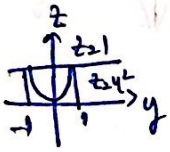
(d)



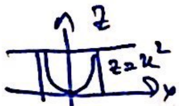


$$I_1 = \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\,dx\,dy\,dz$$

$$I_2 = \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\,dy\,dx\,dz$$



$$I_3 = \int_{-1}^1 \int_{-1}^1 \int_{x^2+y^2}^1 f(x,y,z)\,dz\,dx\,dy$$



$$I_4 = \int_{-1}^1 \int_{-1}^1 \int_{x^2+y^2}^1 f(x,y,z)\,dz\,dy\,dx$$

$$I_5 = \int_{-1}^1 \int_{y^2}^1 \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\,dx\,dz\,dy$$

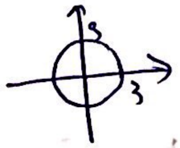
$$I_6 = \int_{-1}^1 \int_{x^2}^1 \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\,dy\,dz\,dx$$

④ Evaluate the following integral by using cylindrical coordinates

(a) $I = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 \frac{e^z}{\sqrt{x^2+y^2}}\,dz\,dy\,dx$

by using cylindrical coordinate

$Q = \{(r,\theta,z) \in \mathbb{R}^3 : 0 \le \theta \le 2\pi,\ 0 < r < 3,\ r \le z \le 3\}$



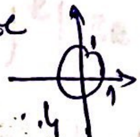
$$I = \int_0^{2\pi}\int_0^3\int_r^3 r\frac{e^z}{r}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (e^3 - e^r)\,dr\,d\theta$$

$$= 2\pi(3e^3 - e^3 + 1) = \underline{2\pi(2e^3+1)}$$

(b) $I = \int_{-1}^1 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{4-x^2-y^2} e^{x^2+y^2+z}\,dz\,dx\,dy$

by using cylindrical coordinate



$Q = \{(r,\theta,z) \in \mathbb{R}^2 : 0 \le \theta \le 2\pi,\ 0 < r < 2,\ 0 < z < 4-r^2\}$

$$I = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r e^{r^2+z}\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2\int_0^{4-r^2} re^{r^2}(e^z)\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2 r(e^4 - e^{r^2})\,dr$$

$$= 2\pi\left(2e^4 - \frac{e^4}{2} + \frac{1}{2}\right) = 2\pi\left(\frac{3e^4}{2} + \frac{1}{2}\right)$$

$$= \underline{\pi(3e^4+1)}$$

(c). $K = \iiint_Q \sqrt{x^2+y^2}\,dQ$, Q is the solid bounded by $z = x^2+y^2$ and plane $z=4$

by using cylindrical coordinate

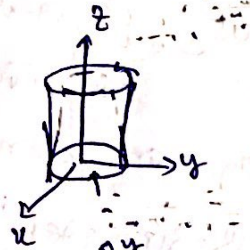
$$I = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r.r\,dz\,dr\,d\theta = 2\pi\int_0^2 (4r^2 - r^4)\,dr$$

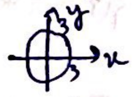
$$= 2\pi\left(\frac{4}{3}.8 - \frac{32}{5}\right) = 64\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \underline{\frac{128\pi}{15}}$$

(d). $I=\iiint_Q \frac{1}{x^2+y^2}$, Q is the solid bounded above by $z=4-(x^2+y^2)$ and below by the sphere $x^2+y^2+z^2=9$

Anlise

$$\begin{cases} z=4-(x^2+y^2)\\ z^2+x^2+y^2=9\end{cases}$$





by using cylindrical coordinate

$$I=\int_0^{2\pi}\int_3^2\int_{\sqrt{9-r^2}}^{4-r^2}\frac{r}{r^2}\,dz\,dr\,d\theta$$

$$=2\pi\int_3^2 \frac{1}{r}(4-r^2-\sqrt{9-r^2})\,dr$$

$$=2\pi\int_3^2\left(\frac{4}{r}-r-\frac{\sqrt{9-r^2}}{r}\right)dr$$

note $\int \frac{\sqrt{9-r^2}}{r}dr$

let $r=3\sin\theta \Rightarrow dr=3\cos\theta\, dr$

$$\Rightarrow \int \frac{\sqrt{9-9\sin^2\theta}}{3\sin\theta}3\cos\theta\, d\theta$$

$$=\int\frac{3\cos^2\theta}{\sin\theta}d\theta=3\int\frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$=3\int\frac{1}{\sin\theta}d\theta-3\int\sin\theta\, d\theta$$

$$=\frac{3}{2}\ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)+3\cos\theta$$

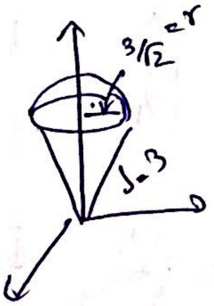
$$\Rightarrow 2\pi\int_3^2\frac{\sqrt{9-r^2}}{r}dr=\frac{3}{2}\ln\left(\frac{1-\cos 2}{1+\cos 2}\right)-\frac{3}{2}\ln\left(\frac{1-\cos 3}{1+\cos 3}\right)+3(\cos 2-\cos 3)$$

$$=\left[\frac{3}{2}\ln\left(\frac{\tan^2 1}{\tan^2 3/2}\right)+3(\cos(2)-\cos(3))\right]2\pi$$

$$I=2\pi\left(\ln\frac{2}{3}+\frac{5}{2}-\frac{3}{2}\ln\left(\frac{\tan^2 1}{\tan^2 3/2}\right)+3(\cos 2-\cos 3)\right)$$

5 Evaluate the following integral by using spherical coordinate.

(a) $I=\iiint_Q(x^2+y^2+z^2)\,dv$, Q is the solid region bounded below by the cone $z^2=x^2+y^2$ and above by the sphere $x^2+y^2+z^2=9$



anlise

$\tan\phi=\frac{r}{\rho}=\frac{3/\sqrt{2}}{3}=\frac{\sqrt{2}}{2}\Rightarrow\phi=\pi/4$

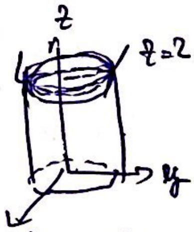
by using sphere coordinate

$$I=\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2.\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\frac{3^5}{5}\times 2\pi\times\left(1-\frac{\sqrt{2}}{2}\right)=\frac{243}{5}(2-\sqrt{2})\pi$$

(b). $J=\iiint_Q z\,dv$, Q is the solid region bounded below by the cone

$x^2+y^2+(z-2)^2=1$



let $u=x$, $v=y$, $w=z-2$ $\Rightarrow J=1$

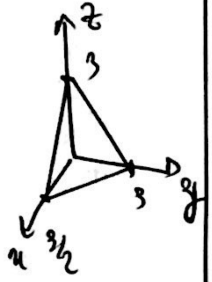
we get $Q'=\{(u,v,w)\in\mathbb{R}^3 / u^2+v^2+w^2\le 1\}$

$$J=\int_0^{2\pi}\int_0^{\pi}\int_0^1(\overset{0}{\rho^3\sin\varphi\cos\varphi}+2\rho^2\sin\varphi)\,d\rho\, d\varphi\, d\theta$$

$$=2\pi\times 4\times\frac{1}{3}=\frac{8\pi}{3}$$

⑥ Find the volume of the solide Q by using triple integrals.

(a) $Q = \{(x,y,z)\in\mathbb{R}^3 : 2x+y+z-3=0, x\geq 0, y\geq 0, z\geq 0\}$



$$V = \int_0^{3/2}\int_0^{3-2x}\int_0^{3-2x-y} dz\,dy\,dx$$

$$= \int_0^{3/2}\int_0^{3-2x} 3-2x-y \, dy\,dx$$

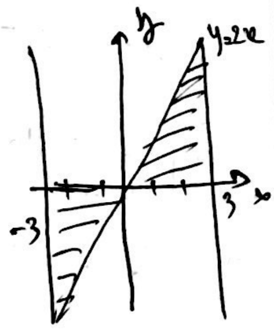
$$= \int_0^{3/2} (3-2x)^2 - \frac{1}{2}(3-2x)^2 \, dx$$

$$= \frac{1}{2}\int_0^{3/2} (9-12x+4x^2)dx$$

$$= \frac{1}{2}\left(9\times\frac{3}{2} - \frac{12}{2}\times\left(\frac{3}{2}\right)^2 + \frac{4}{3}\times\left(\frac{3}{2}\right)^3\right)$$

$$= \frac{1}{2}\left(\frac{27}{2} - \frac{27}{2} + \frac{9}{2}\right) = \frac{9}{4}$$

b)- Q is bounded by $z = 9-x^2$, $z=0$, $y=0$ and $y=2x$



$$V = \int_0^3\int_0^{2x}\int_0^{9-x^2} dz\,dy\,dx$$

$$= \int_0^3 2x(9-x^2)dx$$

$$= 2\cdot\frac{9}{2}\cdot(3)^2 - \frac{2}{3}\cdot 27$$

$$= 9(9-2) = 63$$

c)- Q is enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

let $u=\frac{x}{a}$, $v=\frac{y}{b}$, $w=\frac{z}{c}$ $\Rightarrow J = abc$

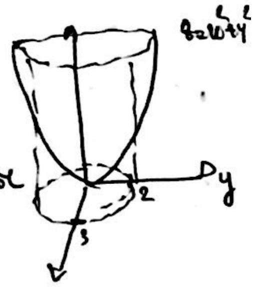
by using spherical coordinate

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\,\rho^2\cos\varphi\, d\rho\, d\varphi\, d\theta = abc\times 2\pi\times 2\times\frac{1}{3}$$

$$= \frac{4\pi}{3}abc$$

d)- Q is bounded by paraboloid $z=x^2+y^2$ and inside the elliptic cylinder $\frac{x^2}{9}+\frac{y^2}{4}=1$

let $u=\frac{x}{3}$, $v=y/2$, $w=z$ $\Rightarrow |J| = 6$



by using cylindrical coordinate

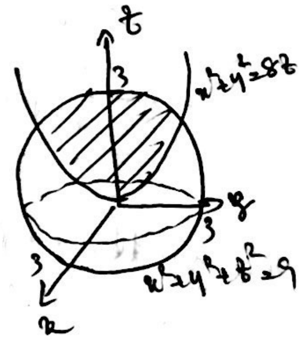
$$V = 6\int_0^{2\pi}\int_0^1\int_0^{9r^2\cos^2\theta+4r^2\sin^2\theta} r\,dz\,dr\,d\theta$$

$$= 6\int_0^{2\pi}\int_0^1 r(4+5r^2\cos^2\theta)dr\,d\theta$$

$$= 6\int_0^{2\pi}\int_0^1 (4r + \frac{5}{2}r^3 + \frac{5}{2}r^3\cos 2\theta)dr\,d\theta$$

$$= 12\pi\left(\frac{4}{2}+\frac{5}{8}\right) = \frac{39}{2}\pi$$

e)- Q is bounded above by the sphere $x^2+y^2+z^2=9$ and below by the paraboloid $8z = x^2+y^2$



$$\begin{cases} x^2+y^2+z^2=9 \\ 8z = x^2+y^2 \end{cases}$$

$z^2+8z-9=0$ ⟨ $z=1$, $z=-9$

$\Rightarrow x^2+y^2 = 8 = (2\sqrt{2})^2$

$x^2+y^2=0$

$\Rightarrow 0<r<2\sqrt{2}$

by using cylindrical coordinate

$$V = \int_0^{2\pi}\int_0^{2\sqrt{2}}\int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^{2\sqrt{2}}\left(r\sqrt{9-r^2} - \frac{r^3}{8}\right)dr$$

$$= 2\pi\left[-\frac{1}{2}\frac{(9-r^2)^{3/2}}{3/2} - \frac{r^4}{8\times 4}\right]_0^{\sqrt{8}}$$

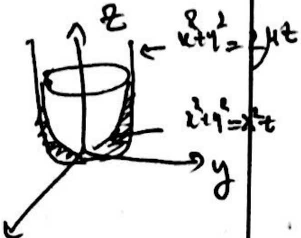
$$= -2\pi\left(\frac{1}{3}+2-9\right) = \frac{60\pi}{3} = 20\pi$$

f). Q is volume inside the paraboloid $x^2+y^2=2\mu z$ and outside the cone $x^2+y^2=\lambda^2 z^2$, where $\lambda, \mu > 0$

$$\begin{cases} x^2+y^2=2\mu z \\ x^2+y^2=\lambda^2 z^2 \end{cases}$$

$$\lambda^2 z^2 = 2\mu z$$

$$z(\lambda^2 z - 2\mu) = 0 \Rightarrow \begin{cases} z=0 \\ z=\frac{2\mu}{\lambda^2} \end{cases}$$



$$\Rightarrow x^2+y^2=\left(\frac{2\mu}{\lambda}\right)^2$$

by using cylindrical coordinate

$$V=\int_0^{2\pi}\int_0^{\frac{2\mu}{\lambda}}\int_{\frac{r^2}{2\mu}}^{\frac{r}{\lambda}} r\,dz\,dr\,d\theta$$

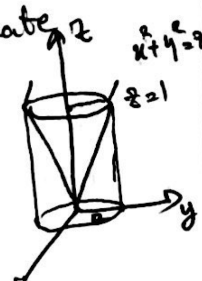
$$=2\pi\int_0^{\frac{2\mu}{\lambda}} r\left(\frac{r}{\lambda}-\frac{r^2}{2\mu}\right)dr$$

$$=2\pi\left(\frac{1}{3\lambda}\cdot\frac{8\mu^3}{\lambda^3}-\frac{1}{8\mu}\cdot\frac{16\mu^4}{\lambda^4}\right)$$

$$=2\pi\left(\frac{8\mu^2}{3\lambda^3}-\frac{\mu^3}{2\lambda^4}\right)$$

g). Q is bounded by the $z=\sqrt{x^2+y^2}$ and the plane $z=1$

by using cylindrical coordinate



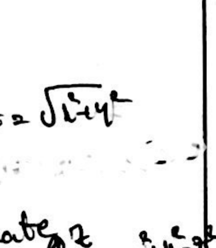
$$V=\int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^1 r-r^2\,dr$$

$$=2\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{3}$$

h). Q is bounded by the cone $z=\sqrt{x^2+y^2}$, and the plane and the cylinder $x^2+y^2=4$, and the plane

by using cylindrical coordinate



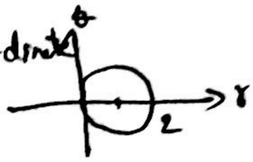
$$V=\int_0^{2\pi}\int_0^1\int_r^2 r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^2 2r-r^2\,dr$$

$$=2\pi\left(4-\frac{8}{3}\right)=\frac{8\pi}{3}$$

i). Q is bounded by the graph $r=2\cos\theta$ and $r^2+z^2=4$

by using cylindrical coordinate



$$V=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$=-\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}-2r\sqrt{4-r^2}\,dr\,d\theta$$

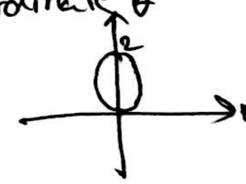
$$=-\int_{-\pi/2}^{\pi/2}\left(\frac{(4-4\cos^2\theta)^{3/2}}{3/2}-\frac{4^{3/2}}{3/2}\right)d\theta$$

$$=-\frac{2}{3}\int_{-\pi/2}^{\pi/2}\left(\sin\theta(1-\cos^2\theta)-8\right)d\theta$$

$$=-\frac{2}{3}(-8)\times\pi=\frac{16\pi}{3}$$

j). Q is bounded by the graph $r^2+z=16$, $z=0$ and $r=2\sin\theta$

by using cylindrical coordinate



$$V=\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{16-r^2} r\,dz\,dr\,d\theta$$

$$=\int_0^{\pi}\int_0^{2\sin\theta} 16r-r^3\,dr\,d\theta$$

$$=\int_0^{\pi} 16\cdot(1-\cos 2\theta)-\frac{8}{8}\left(\frac{3}{2}-2\cos 2\theta+\cos 4\theta\right)d\theta$$

$$=\frac{29\pi}{3}$$

k). $Q=\{(x,y,z): x^{2/3}+y^{2/3}+z^{2/3}\le 1\}$

let $u=x^{1/3}$, $v=y^{1/3}$, $w=z^{1/3}$ $\Rightarrow \begin{cases} x=u^3 \\ y=v^3 \\ z=w^3 \end{cases} \rightarrow |J|=27(uvw)^2$

By using spherical coordinate

$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^1 27\rho^6\sin^4\varphi\cos^2\theta\sin^2\theta\cos^2\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=3\int_0^{\pi}\sin^5\varphi\cos^2\varphi\,d\varphi\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta$$

$\therefore \int_0^{\pi} \sin\varphi\,(1-\cos^2\varphi)^2\cos^2\varphi\,d\varphi$

let $t=\cos\varphi \Rightarrow dt=-\sin\varphi\,d\varphi$

• if $\varphi=\pi \Rightarrow t=-1$

$\varphi=0 \Rightarrow t=1$

$\Rightarrow \int_{-1}^{1} t^2(1-2t^2+t^4)\,dt = \int_{-1}^{1} t^2-2t^4+t^6\,dt$

$= \frac{1}{3}-\frac{2}{5}+\frac{1}{7}-\left(-\frac{1}{3}+\frac{2}{5}-\frac{1}{7}\right) = \frac{16}{105}$

• $\int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{8}\int_0^{2\pi}(1-\cos(4\theta))\,d\theta = \frac{\pi}{4}$

$\Rightarrow V = \cancel{3}\left(\frac{\cancel{16}^{4}}{\cancel{105}_{35}}\times\frac{\pi}{\cancel{4}}\right) = \frac{4\pi}{35}$

l). Q is bounded by $x^2+4y^2+z^2=1$

let $\left.\begin{array}{l} u=x \\ v=2y \\ w=z \end{array}\right| \Rightarrow J=\frac{1}{2}$

$Q'=\{(u,v,w)\in\mathbb{R}^3,\ u^2+v^2+z^2=1\}$

by using sphere coordinate

$V=\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$=\frac{1}{6}\times 2\pi\times 2 = \frac{2\pi}{3}$

m. Q is bounded by $x^2+\frac{1}{2}y^2+\frac{3}{4}z^2+zx=1$

Since $x^2+\frac{1}{2}y^2+\frac{3}{4}z^2+xz=1$

$\left(x+\frac{1}{2}z\right)^2+\frac{1}{2}y^2+\frac{1}{2}z^2=1$

let $\left\{\begin{array}{l} u=x+\frac{1}{2}z \\ v=\frac{1}{\sqrt{2}}y \\ w=\frac{1}{\sqrt{2}}z \end{array}\right| \Rightarrow \frac{1}{J}=\begin{vmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}\begin{matrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{matrix}=\frac{1}{2}$

$\Rightarrow |J|=2$

$Q'=\{(u,v,w)\in\mathbb{R}^3,\ u^2+v^2+w^2=1\}$

by using sphere coordinate

$V=2\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{4\pi}{3}\times 2 = \frac{8\pi}{3}$

(7) Evaluate the integral $\iiint_Q f(x,y,z)\,dv$

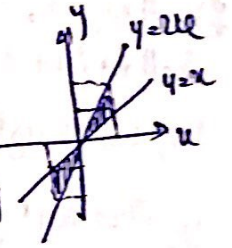
(a) $f(x,y,z)=xy$, $Q=\{(x,y,z)\in\mathbb{R}^3, 0\le x\le 1, 0\le y\le x^2, 0\le z\le x+y\}$

$V=\int_0^1\int_0^{x^2}\int_0^{x+y} xy\,dz\,dy\,dx$

$=\int_0^1\int_0^{x^2} x^2y+xy^2\,dy\,dx$

$=\int_0^1 \frac{x^6}{2}+\frac{x^7}{3}\,dx = \frac{1}{14}+\frac{1}{24} = \frac{19}{168}$

(b) $f(x,y,z)=z$, Q is the region bounded by the cylinder $z^2+x^2=1$ and the planes $y=x, y=2x$ and $z=0$



$R=R_1+R_2$

$R_1=\{(x,y)\in\mathbb{R}^2: 0\le x\le 1, x\le y\le 2x\}$

$R_2=\{(x,y)\in\mathbb{R}^2: -1\le x\le 0, 2x\le y\le x\}$

$V=2\int_0^1\int_0^{2x}\int_0^{\sqrt{1-x^2}} z\,dz\,dy\,dx = 2\int_0^1 2x\cdot\frac{1-x^2}{2}\,dx$

$=2\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}$

(c). $f(x,y,z)=\sqrt{x^2+y^2}$, Q is the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=1$ and $z=3$

by using cylindrical coordinate

$V=\int_0^{2\pi}\int_0^1\int_1^3 r^2\,dz\,dr\,d\theta = 2\pi\times\frac{1}{3}\times 2 = \frac{4\pi}{3}$

(d). $f(x,y,z)=\sqrt{x^2+y^2}$, $Q=\{(x,y,z)\in\mathbb{R}^3: x+y+z\le 2, x^2+y^2\le 1, z\ge 0\}$

by using cylindrical coordinate

$V=\int_0^{2\pi}\int_0^1\int_0^{2-r\cos\theta-r\sin\theta} r^2\,dz\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^1 2r^2-r^3\overset{\to 0}{\cos\theta}-r^3\overset{\to 0}{\sin\theta}\,dr\,d\theta$

$=2\pi\times 2\times\frac{1}{3} = \frac{4\pi}{3}$

~~e). $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$, Q is the part of unit ball $x^2+y^2+z^2 \le 1$ lying in the first octant.~~

e)- $f(x,y,z) = y$, Q is the part of the solid in the first octant lying inside the paraboloid $z = 4 - x^2 - y^2$

by using cylindrical coordinate

$$V = \int_0^{\pi/2}\int_0^2\int_0^{4-r^2} r^2\sin\theta\, dz\,dr\,d\theta = \int_0^2 4r^2 - r^4\, dr$$

$$= \left(\frac{4}{3}2^3 - \frac{2^5}{5}\right) = 2^5\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{64}{15}$$

f)- $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$, Q is the part of unit ball $x^2+y^2+z^2 \le 1$ lying in the first octant.

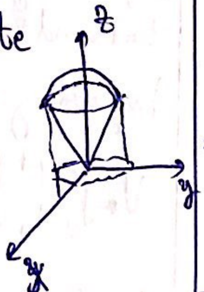
by using spherical coordinate

$$V = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 e^{\rho^3}\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{\pi}{6}(e-1)$$

g)- $f(x,y,z) = xz$, Q is the solid bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by the cone $z = \sqrt{x^2+y^2}$

by using cylindrical coordinate



$$V = \int_0^{2\pi}\int_0^2\int_r^{\sqrt{4-r^2}} r\cos\theta\, dz\,dr\,d\theta$$

$= 0$

because $\int_0^{2\pi}\cos\theta\, d\theta = 0$

h- $f(x,y,z) = \frac{1}{\sqrt{(x-2)^2+y^2+z^2}}\, dv$, $Q = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$

$$V = \int_{-1}^{1}\int_0^{2\pi}\int_0^{\sqrt{1-x^2}} \frac{r}{\sqrt{(x-2)^2+r^2}}\, dr\,d\theta\,dx$$

$$= 2\pi\int_{-1}^{1}\left[\left((x-2)^2+r^2\right)^{1/2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= 2\pi\int_{-1}^{1}(5-4x)^{1/2} - (x-2))\, dx = 2\pi\left[-\frac{(5-4x)^{3/2}}{6} - \frac{x^2}{2} + 2x\right]_{-1}^{1}$$

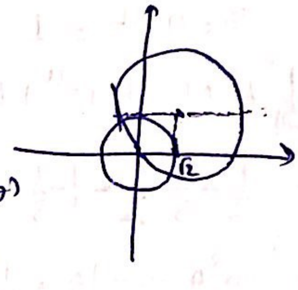
$$= 2\pi \times \frac{25}{3} = \frac{50\pi}{3}$$

i). $f(x,y,z) = \frac{1}{x^2+y^2}\, dv$ $Q = \{(x,y,z) \in \mathbb{R}^3 :$ $x, y \ge 0$, $0 < x^2+y^2 < 2\sqrt{2}(x+y)$, $0 < z < x+y\}$

• $x^2+y^2 = 2\sqrt{2}(x+y)$

$(x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 2^2$

• $x^2+y^2 = \sqrt{2}^2$



(+) $r^2 = r2\sqrt{2}(\cos\theta+\sin\theta)$

$r = 2\sqrt{2}(\sin\theta+\cos\theta)$

(−) $r^2 = \sqrt{2}^2 \Rightarrow r = \sqrt{2}$

Then $\sqrt{2} = 2\sqrt{2}(\sin\theta+\cos\theta)$

$\sin\theta + \cos\theta = 1/2$

let $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$\Rightarrow \frac{2t+1-t^2}{1+t^2} = 1/2 \Rightarrow \frac{4t+1-3t^2}{2(t^2+1)} = 0$ / $t^2+1 \ne 0$

$\Rightarrow 4t+1-3t^2 = 0 \Rightarrow \begin{cases} t = \frac{2+\sqrt{7}}{3} \\ t = \frac{2-\sqrt{7}}{3} \end{cases}$

$\Rightarrow 2\arctan\left(\frac{2-\sqrt{7}}{3}\right) \le \theta \le 2\arctan\left(\frac{2-\sqrt{7}}{3}\right)$

by using cylindrical coordinate

$$V = \int_{2\arctan\frac{2-\sqrt{7}}{3}}^{2\arctan\frac{2+\sqrt{7}}{3}}\int_{\sqrt{2}}^{2\sqrt{2}(\cos\theta+\sin\theta)}\int_0^{r(\cos\theta+\sin\theta)} dz\,dr\,d\theta$$

$$= \int_{2\arctan\frac{2-\sqrt{7}}{3}}^{2\arctan\frac{2+\sqrt{7}}{3}} (\cos\theta+\sin\theta)(4\sin\theta+4\cos\theta-2)\, d\theta$$

$$= \int_{2\arctan\frac{2-\sqrt{7}}{3}}^{2\arctan\frac{2+\sqrt{7}}{3}} (4 + 2\sin 2\theta - 2\cos\theta - 2\sin\theta)\, d\theta$$

$$= \left[4 - \cos 2\theta - 2(\sin\theta - \cos\theta)\right]_{2\arctan\left(\frac{2-\sqrt{7}}{3}\right)}^{2\arctan\frac{2+\sqrt{7}}{3}}$$

• $\int_{2\arctan\frac{2-\sqrt{7}}{3}}^{2\arctan\frac{2+\sqrt{7}}{3}} \cos\theta = \frac{\sqrt{7}}{2}$, $\int_{2\arctan\frac{2-\sqrt{7}}{3}}^{2\arctan\frac{2+\sqrt{7}}{3}} \sin\theta = \frac{\sqrt{7}}{2}$

$V = 11.01 - 0.66 = 11.35$

$\ln x$

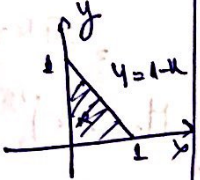
$1 - \ln x$

$\frac{1}{x} - x$

$$\int_0^1 \ln x\, dx = -\left[e^x\right]_{-\infty}^{0} = -1$$

8: Find the mass and centre of mass of the solid Q of given mass density:

ⓐ Q is the tetrahedron bounded by the plane $x=0$, $y=0$, $z=0$ and $x+y+z=1$ by the density at a point P of Q is directly proportional to the distance between P and the yz-plane.



let $\delta(x,y,z)=Kx$, $K\in\mathbb{R}$

$m=\int_0^1\int_0^{1-x}\int_0^{1-x-y} Kx\,dz\,dy\,dx$

$=\int_0^1\int_0^{1-x} Kx-Kx^2-Kxy\,dy\,dx$

$=\int_0^1 Kx-Kx^2-Kx^2+Kx^3-\frac{Kx}{2}(1-2x+x^2)\,dx$

$=\left(\frac12-\frac13-\cancel{\frac13}+\cancel{\frac14}-\cancel{\frac14}+\cancel{\frac13}-\frac18\right)K=\frac{12-8-3}{24}K=\boxed{\frac{K}{24}}$

$M_{yz}=\int_0^1\int_0^{1-x}\int_0^{1-x-y}x^2K\,dz\,dy\,dx$

$=\int_0^1\int_0^{1-x}K(x^2-x^3-x^2y)\,dy\,dx$

$=K\int_0^1(x^2-x^3-x^3+x^4-\frac{x^2}{2}(1-2x+x^2))\,dx$

$=K\left(\frac13-\frac14-\cancel{\frac14}+\frac15-\frac16+\cancel{\frac24}-\frac1{10}\right)=K\left(\frac16-\frac14+\frac1{10}\right)=\frac{K}{60}$

$M_{xz}=\int_0^1\int_0^{1-x}\int_0^{1-x-y}xyK\,dz\,dy\,dx$

$=K\int_0^1\int_0^{1-x}xy-x^2y-xy^2\,dy\,dx$

$=K\int_0^1\frac12x(1-x)^2-\frac12x^2(1-x)^2-\frac13x(1-x)^3\,dx$

$=K\int_0^1\frac12x-x^2+\frac12x^3-\frac12x^2+x^3-\frac12x^4-\frac13x+x^2-x^3+\frac13x^4\,dx$

$=K\int_0^1\frac16x-\frac12x^2+\frac12x^3-\frac16x^4\,dx=K\left(\frac1{12}-\frac16+\frac18-\frac1{30}\right)=\frac{K}{120}$

$M_{xy}=\int_0^1\int_0^{1-x}\int_0^{1-x-y}Kxz\,dz\,dy\,dx$

$=\int_0^1\int_0^{1-x}\frac{Kx}{2}(1-x-y)^2\,dy\,dx$

$=\frac K2\int_0^1\int_0^{1-x}x+x^3+y^2x-2x^2-2yx+2x^2y\,dy\,dx$

$=\frac K2\int_0^1 x-x^2+x^3-x^4+\frac x3-x^2+x^3-\frac{x^3}{3}-2x^2+2x^3$
$-x^3+2x^2-x+x^4-2x^3+x^2\,dx$

$=\frac K6\int_0^1-x^4+3x^3-3x^2+x\,dx$

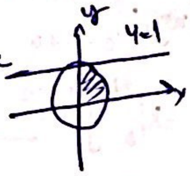
$=\frac K6\left(-\frac15+\frac34-1+\frac12\right)=\frac{K}{120}$

$(\bar x,\bar y,\bar z)=\left(\frac{\frac{K}{60}}{\frac{K}{24}},\frac{\frac{K}{120}}{\frac{K}{24}},\frac{K/120}{K/24}\right)=\left(\frac25,\frac15,\frac15\right)$

Thus $m=\frac{K}{24}$ and $(\bar x,\bar y,\bar z)=\left(\frac25,\frac15,\frac15\right)$

b). Q is the solid bounded by the cylinder $x^2+y^2=1$ in the first octant and the plane $z+y=1$ the density $\delta(x,y,z)=x^2+y^2+z^2$

By using cylinderical coordinate



$m=K\int_0^{\pi/2}\int_0^1\int_0^{1-r\sin\theta}(r^2+z^2)r\,dz\,dr\,d\theta$

$=K\int_0^{\pi/2}\int_0^1 r^3-r^4\sin\theta+\frac r3(1-3r\sin\theta+3r^2\sin^2\theta-r^3\sin^3\theta)\,dr\,d\theta$

$=K\int_0^{\pi/2}\frac14-\frac15\sin\theta+\frac16-\frac13\sin\theta+\frac18(1-\cos2\theta)-\frac1{15}\sin\theta(1-\cos^2\theta)\,d\theta$

$\frac34\times\frac14\times\frac12\times\pi$

$=K\int_0^{\pi/2}\frac{23}{24}-\frac35\sin\theta-\frac1{15}\sin\theta\cos^2\theta\,d\theta$

$=\left(\frac{23}{24}\times\frac{\pi}{2}+\frac35(0-1)-\frac1{15}\times\frac13(0-1)\right)K$

$=K\left(\frac{23\pi}{48}-\frac{26}{45}\right)\approx0.96K$

$M_{yz}=K\int_0^{\pi/2}\int_0^1\int_0^{1-r\sin\theta}r\cos\theta(r^2+z^2)\,dz\,dr\,d\theta$

$=\int_0^{\pi/2}\int_0^1Kr^3\cos\theta(1-r\sin\theta)+\frac{r\cos\theta}{3}(1-r\sin\theta)^3\,dr\,d\theta$

• $\int_0^1 r\cos\theta(1-r\sin\theta)^3=\frac{\cos\theta}{2}-\cos\theta\sin\theta+\frac{3\cos\theta\sin^2\theta}{4}-\frac{\cos\theta\sin^3\theta}{5}$

• $\int_0^1 r^3\cos\theta(1-r\sin\theta)=\frac{\cos\theta}{4}-\frac{\cos\theta\sin\theta}{5}$

$M_{yz}=K\int_0^{\pi/2}\frac34\cos\theta-\frac65\cos\theta\sin\theta+\frac34\cos\theta\sin^4\theta-\frac15\cos\theta\sin^3\theta\,d\theta$

$=\left(\frac34(1-0)+\frac3{10}(0-0)+\frac3{20}(1-0)-\frac1{20}(1-0)\right)$

$=K\frac{17}{20}=0.85K$

$$M_{xz} = k\int_0^{\pi/2}\int_0^1\int_0^{1-r\sin\theta} r\sin\theta(r^2+z^2)\,dz\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\int_0^1 r^2\sin\theta - r^4\sin^2\theta + \frac{r\sin\theta - 3r^2\sin^2\theta + 3r^3\sin^3\theta - r^4\sin^4\theta}{3}\,dr\,d\theta$$

$$= k\int_0^{\pi/2} \frac{1}{4}\sin\theta - \frac{1}{5}\sin^2\theta + \frac{1}{6}\sin\theta - \frac{1}{3}\sin^2\theta + \frac{1}{4}\sin^3\theta - \frac{1}{15}\sin^4\theta\,d\theta$$

$$= k\int_0^{\pi/2} \frac{5}{12}\sin\theta - \frac{8}{30}(1-\cos 2\theta) + \frac{1}{4}\sin^3\theta - \frac{1}{15}\sin^4\theta\,d\theta$$

$$= k\left(\frac{5}{12} + \frac{2\pi}{15} + \frac{1}{6} - \frac{\pi}{80}\right) = \frac{7}{48}(4-\pi)k = 0.12k$$

$$M_{xy} = k\int_0^{\pi/2}\int_0^1\int_0^{1-r\sin\theta} z(r^2+z^2)\,dz\,dr\,d\theta$$

$$= k\int_0^{\pi/2}\int_0^1 \frac{r^2 - 2r^3\sin\theta + r^4\sin^2\theta}{2} + \frac{(1-r\sin\theta)^4}{4}\,dr\,d\theta$$

$$= k\int_0^{\pi/2} \underbrace{\left(\frac{1}{6} - \frac{\sin\theta}{4} + \frac{\sin^2\theta}{6}\right.}_{\pi/8 - 1/4} + \frac{-(1-\sin\theta)^5+1}{20\sin\theta}\,d\theta$$

$$= k\left(\frac{\pi}{8} - \frac{1}{4} + x\right)$$

$$\int_0^{\pi/2} \frac{1-(1-\sin\theta)^5}{20\sin\theta}\,d\theta \quad / \quad \sin\theta \approx 1$$

$$\approx \frac{\pi}{40}$$

$$M_{xy} \approx k\left(\frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{40}\right) = \frac{1}{20}(3\pi - 5)k \approx 0.22k$$

Thus $m = 0.85k$, $(\bar{x},\bar{y},\bar{z}) \simeq (0.88, 0.12, 0.22)$

(c). Q is bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z=2$, the mass density at any point on the solid is directly proportional to the square of its distance from the origin.

by using cylindrical coordinate

$$m = k\int_0^{2\pi}\int_0^2\int_r^2 (z^2+r^2)\,r\,dz\,dr\,d\theta$$

$$= 2\pi k\int_0^2 \frac{r(2^3-r^3)}{3} + r^3(2-r)\,dr$$

$$= 2\pi k\left(\frac{2^5}{6} - \frac{2^5}{15} + \frac{2^5}{4} - \frac{2^5}{5}\right) = \frac{48\pi}{5}k$$

$$M_{yz} = k\int_0^{2\pi}\int_0^2\int_r^2 r\cos\theta(z^2+r^2)\,dz\,dr\,d\theta = \dots 0$$

$$M_{xy} = k\int_0^{2\pi}\int_0^2\int_r^2 z^3 + r^2 z\,dz\,dr\,d\theta$$

$$= 2\pi k\int_0^2 \frac{2^4}{4} - \frac{r^4}{4} + \frac{2^2r^2}{2} - \frac{r^4}{2}\,dr$$

$$= 2\pi k\left(\frac{2^5}{4} - \frac{2^5}{5} + \frac{2^5}{6} - \frac{2^5}{10}\right)$$

$$= 2\pi k \times \frac{7}{30} = \frac{7k\pi}{30}$$

$$M_{xz} = k\int_0^{2\pi}\int_0^2\int_r^2 r\sin\theta(z^2+r^2)\,dz\,dr\,d\theta = 0$$

Thus $m = \frac{48\pi k}{5}$, $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{7}{288}\right)$

(g) Find the moment of inertia

let $\delta(x,y,z) = k(x^2+y^2)$

$Q = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 4, 0 \le z \le 3\}$

by using cylindrical coordinate

$$I_z = \int_0^{2\pi}\int_0^2\int_0^3 kr^3\,dz\,dr\,d\theta = 2\pi \times \frac{k\,2^4}{4} \times \frac{3^2}{2}$$

$$= 36k\pi$$

Thus $I_z = 36k\pi$, $k \in \mathbb{R}$

(10) Show that the moment of inertia of Q about a diameter of Q is

$$I = \frac{2m}{5}\left(\frac{b^5-a^5}{b^3-a^3}\right)$$

choose (oz) be diameter of Q

Then $I_z = \iiint_Q (x^2+y^2)\,\delta(x,y,z)\,dV$

Since, mass of Q is m and it is uniform

$\delta(x,y,z) = \frac{m}{V}$, $V = \frac{4\pi}{3}(b^3-a^3)$

$\Rightarrow \delta = \frac{3m}{4\pi}(b^3-a^3)^{-1}$

by using spherical coordinate

$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_a^b \delta(\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= 2\pi \times \frac{3m\pi}{4(b^3-a^3)} \times \frac{(b^5-a^5)}{5} \times \underbrace{\int_0^{\pi}(-\cos^2\varphi+1)\sin\varphi\,d\varphi}_{4/3}$$

$$= 2\pi \times \frac{3m}{4(b^3-a^3)\pi} \times \frac{(b^5-a^5)}{5} \times \frac{4}{3} = \frac{2m}{5}\left(\frac{b^5-a^5}{b^3-a^3}\right)$$

Thus $\boxed{I = \frac{2m}{5}\left(\frac{b^5-a^5}{b^3-a^3}\right)}$

(11) (a) Use the result of Exercise above to find the moment of inertia of a uniform solid ball of mass m and radius b about a diameter of the ball.

by using the result above for $a=0$

we have $\boxed{I = \frac{2m}{5}b^2}$

(b) Use the result of Exercise above to find the moment of inertia of a hollow spherical shell of mass m and radius b about a diameter of the shell

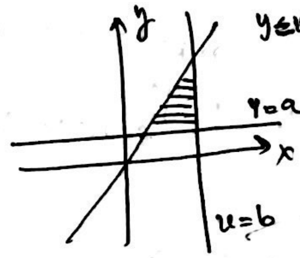
* A hollow spherical shell mean that $a \to b$, we have

$$I = \lim_{a\to b}\frac{2m(b^5-a^5)}{5(b^3-a^3)} = \left(\frac{2m}{\cancel{5}}\right)\frac{-\cancel{5}a^4}{-3a^2}$$

$\boxed{I = \frac{2}{3}mb^2}$

(12) Let $f(x,y)$ be a continuous on the region $R = \{(x,y)\in\mathbb{R}^2 : x\le b,\ y\ge a,\ y\le x\}$

Show that $\int_a^b\int_a^x f(x,y)\,dy\,dx = \int_a^b\int_b^y f(x,y)\,dy\,dx$



by Fubini's theorem

$\boxed{\int_a^b\int_a^x f(x,y)\,dy\,dx = \int_a^b\int_b^y f(x,y)\,dx\,dy}$

use the result above to show that

$$I = \int_a^x\int_a^y\int_a^z f(t)\,dt\,dz\,dy = \frac{1}{2}\int_a^x (x-t)^2 f(t)\,dt$$

$$I = \int_a^x\int_a^y\int_a^z f(t)\,dt\,dz\,dy$$

$$= \int_a^x\int_t^x\int_t^y f(t)\,dz\,dy\,dt$$

$$= \int_a^x f(t)\left[\frac{y^2}{2}-ty\right]_t^x dt = \int_a^x f(t)\left(\frac{x^2}{2}-tx+\frac{t^2}{2}\right)dt$$

$$\boxed{= \frac{1}{2}\int_a^x f(t)(x-t)^2\,dt}$$

(13) Find the region Q where the integral $\iiint_Q (1-2x^2-3y^2-z^2)^{1/3}\,dV$ has maximum value

let $I = \iiint_Q (1-2x^2-3y^2-z^2)^{1/3}\,dV$

I has maximum value if

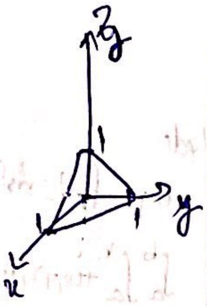
$0 < 1-2x^2-3y^2-z^2$

or. $2x^2+3y^2+z^2 < 1$

Thus $Q = \{(x,y,z)\in\mathbb{R}^3 \,/\, 2x^2+3y^2+z^2<1\}$

(14) Compute the integral

(a). $I = \iiint_{0\le x+y+z\le 1} xyz\,dV$



$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \frac{xy(1-x-y)^2}{2}\,dy\,dx$$

$$= \frac12\int_0^1\int_0^{1-x} (xy - x^3y + xy^3 - 2x^2y - 2y^2x + 2x^2y^2)\,dy\,dx$$

$$= \frac12\int_0^1 \frac{x(x+1)^2}{2} - \frac{x^3(1-x)^2}{2} + \frac{x(1-x)^4}{4} - \frac{2x^2(1-x)^2}{2} - \frac{2(1-x)^3x}{3} + \frac{2x^2(1-x)^3}{3}\,dx$$

$$= \frac12\left(\frac1{24} - \cancel{\frac1{120}} + \cancel{\frac1{120}} - \frac1{30} - \frac1{30} + \frac1{90}\right) = \frac12\left(\frac1{24} - \frac5{90}\right) = \boxed{-\frac1{144}}$$

(b). $I = \iiint_{0\le\sqrt{x}+\sqrt{y}+\sqrt{z}\le 1} xyz\,dV$

let $\begin{cases} x = u^4 \\ y = v^4 \\ z = w^4 \end{cases} \to J = 64u^3v^3w^3$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 64\rho^{27}\sin^{12}\varphi\cos^7\varphi\cos^7\theta\sin^7\theta\,d\rho\,d\varphi\,d\theta$$

$$= \frac{64}{28}\cdot\frac{1}{27}\cdot 2\pi\underbrace{\int_0^{\pi}\sin^{12}\varphi\cos^7\varphi\,d\varphi}_{0}$$

$= 0$

(15) Compute the integral

$\iiiint_V e^{x^2+y^2+u^2-v^2}\,dx\,dy\,du\,dv$

where $\{(x,y,u,v)\in\mathbb{R}^4 : x^2+y^2+u^2+v^2\le 1\}$

by using polar coordinate

$$I = \iint_{R'}\int_0^{2\pi}\int_0^{\sqrt{1-u^2-v^2}} re^{r^2}e^{-(u^2+v^2)}\,dr\,d\theta\,dA'$$

$$= \iint_{R'}\pi(e^{1-u^2-v^2}-1)e^{-u^2-v^2}\,dA'$$

$$= \iint_{R'}\pi(e^{1-2(u^2+v^2)} - e^{-(u^2+v^2)})\,dA'$$

by using polar coordinate

$$I = \int_0^{2\pi}\int_0^1 \pi r(e^{1-2r^2} - e^{-r^2})\,dr\,d\theta$$

$$= \frac12\pi\left[-\frac14 e^{1-2r^2} + e^{-r^2}\right]_0^1$$

$$= \frac12\pi\left(-\frac1{4e} + \frac1e + \frac14 e - 1\right)$$

$$= \frac12\pi\left(\frac{3}{4e} + \frac14 e - 1\right)$$

16) Calculate $I=\iiint_Q \frac{1}{(1+x^2z^2)(1+y^2z^2)}dv$

where $Q=\{(x,y,z)\in\mathbb{R}^3 \mid 0<x<1, 0<y<1, z>0\}$

Then deduce the value of

$$J=\int_0^{\infty}\left(\frac{\arctan t}{t}\right)^2 dt$$

we have $I=\int_0^{+\infty}\int_0^1\int_0^1 \frac{1}{(1+x^2z^2)(1+y^2z^2)}dx\,dy\,dz$

$$=\int_0^{+\infty}\int_0^1 \frac{1}{1+y^2z^2}\cdot\frac{1}{z}\arctan z\,dy\,dz$$

$$=\int_0^{+\infty}\left(\frac{\arctan z}{z}\right)^2 dz = J$$

but $I=\iint_R\int_0^{+\infty}\frac{1}{(1+x^2z^2)(1+y^2z^2)}dz\,dA$

$$=\iint_R \frac{1}{x^2-y^2}\left(x\arctan(xz)-y\arctan(yz)\right)\Big]_0^{+\infty} dA$$

$$=\iint_R \frac{1}{x^2-y^2}\left(\frac{\pi}{2}x-\frac{\pi}{2}y\right)dA$$

$$=\frac{\pi}{2}\iint_R \frac{1}{x+y}dA=\frac{\pi}{2}\int_0^1\int_0^1\frac{1}{x+y}dy\,dx$$

$$=\frac{\pi}{2}\int_0^1(\ln(1+y)-\ln y)dy$$

$$=\frac{\pi}{2}(2\ln 2-2+1)=\frac{\pi}{2}(2\ln 2-1)$$

Thus $I=J=\frac{\pi}{2}(2\ln 2-1)$

17) Evaluate

$$N=\lim_{n\to+\infty}\int_0^1\int_0^1\cdots\int_0^1 \cos^2\left[\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right]dx_1dx_2\ldots dx_n$$

$$N=\lim_{n\to+\infty}\int_0^1\int_0^1\cdots\int_0^1 \cos^2\left[\frac{\pi}{2n}\sum_{i=1}^n x_i\right]dx_1\ldots dx_n$$

let $U=-x_i+1$ if $x_i=0 \Rightarrow U_i=1$

$x_i=1 \Rightarrow U=0$

$J=(-1)^n$

~~$I=\frac{1}{2}\int_0^1\cdots\int_0^1\left(1+\cos\left(\frac{\pi}{n}(x_1+x_2+\cdots+x_n)\right)\right)dx_1dx_2\ldots dx_n$~~

Then $N=\lim_{n\to+\infty}\int_0^1\int_0^1\cdots\int_0^1\cos^2\left[\frac{\pi}{2n}(u_1+\cdots+u_n)\right]du_1\ldots du_n$

$$=\lim_{n\to+\infty}\int_0^1\cdots\int_0^1\cos^2\frac{\pi}{2n}[n-(x_1+\cdots+x_n)](-1)^n dx_1\ldots dx_n$$

$$=\lim_{n\to+\infty}\int_0^1\cdots\int_0^1\sin^2\frac{\pi}{2n}(x_1+\cdots+x_n)dx_1\ldots dx_n$$

$$N=1-\lim_{n\to+\infty}\int_0^1\cdots\int_0^1\cos^2\left(\frac{\pi}{2n}\sum_{i=1}^n x_i\right)dx_1\ldots dx_n$$

$\Leftrightarrow N=1-N \Rightarrow N=\frac{1}{2}$

(18) Let $r>0$ and $V_n(r)$ be the volume of the ball $B_n(0,r)$ in $\mathbb{R}^n$. Show by induction that for all integer $n\geq 2$:

$V_n(r)=\alpha_n r^n$, where

$$\alpha_n=\begin{cases}\frac{2\pi}{n}\cdot\frac{2\pi}{n-2}\cdots\frac{2\pi}{2} & \text{if } n \text{ is even}\\ \frac{2\pi}{n}\cdot\frac{2\pi}{n-2}\cdots\frac{2\pi}{3}\cdot 2 & \text{if } n \text{ is odd.}\end{cases}$$

Calculate $V_4(2)$ and $V_5(3)$

- For $n=3 \Rightarrow V_3(r)=\alpha_3 r^3=\frac{4\pi}{3}r^3$ true
- Suppose that $n=k$ is true, then $V_k(r)=\alpha_k r^k$
- We will show that it true until $n=k+1$

we have

$B_{k+1}(0,r)$ in $\mathbb{R}^{k+1}$

$$V_{k+1}=\underset{B_{k+1}(0,r)}{\iint\cdots\int} du_1\cdots du_{k+1}$$

$$=\int_{-r}^{r}du_{k+1}\underset{B_k(0,\sqrt{r^2-u_{k+1}^2})}{\iint\cdots\int}du_1\cdots du_k$$

$$=\alpha_k\int_{-r}^{r}(r^2-u_{k+1}^2)^{k/2}du_{k+1}$$

$$=2\alpha_k\int_0^r(r^2-u_{k+1}^2)^{k/2}du_{k+1}$$

Let $u_{k+1}=r\sin\theta,\ \theta\in[0,\pi/2]$

then $V_{k+1}(r)=2\alpha_k\int_0^{\pi/2}r^{k+1}\cos^{k+1}\theta d\theta$

$$=2\int_0^{\pi/2}\alpha_k\cos^{k+1}\theta d\theta)r^{k+1}$$

Let $\varphi_m=2\int_0^{\pi/2}\cos^m\theta d\theta$

$$=2\int_0^{\pi/2}\cos^{m-1}\theta\cos\theta d\theta$$

Let $u=\cos^{m-1}\theta$, $dv\ \cos\theta d\theta \Rightarrow v=\sin\theta d\theta$

we get $\varphi_m=2\left[[\sin\theta\cos^{m-1}\theta]_0^{\pi/2}+\int_0^{\pi/2}(m-1)\sin^2\theta\cos^{m-2}\theta d\theta\right]$

$$=2(m-1)\int_0^{\pi/2}\cos^{m-2}d\theta-(m-2)\varphi_m$$

$\Rightarrow \varphi_m+(m-1)\varphi_m=(m-1)\varphi_{m-2}$

$\Rightarrow m\varphi_m=(m-1)\varphi_{m-2}$

$\Rightarrow \frac{\varphi_m}{\varphi_{m-2}}=\frac{m-1}{m}\Rightarrow\varphi_m=\frac{m-1}{m}\varphi_{m-2}$

Then $\prod_{m=2}^{k}\frac{\varphi_m}{\varphi_{m-2}}=\prod_{m=2}^{k}\frac{m-1}{m}$

$\Rightarrow \frac{\varphi_k\varphi_{k-1}}{\varphi_0\varphi_1}=\frac{1}{k}$ where $\varphi_0=\pi,\ \varphi_1=2$

we get $\varphi_k\varphi_{k-1}=\frac{2\pi}{k}$

Since $V_{k+1}(r)=\alpha_{k+1}r^{k+1}$

It means $\alpha_{k+1}=\alpha_k\varphi_{k+1}=\varphi_k\varphi_{k+1}\alpha_{k-1}$

$\Rightarrow \alpha_k=\varphi_k\varphi_{k-1}\alpha_{k-2}$

we get $\alpha_k=\frac{2\pi}{k}\alpha_{k-2}$

$$=\frac{2\pi}{k}\cdot\frac{2\pi}{k-2}\cdot\frac{2\pi}{k-4}\cdots\alpha_4$$

$$=\frac{2\pi}{k}\cdot\frac{2\pi}{k-2}\cdots\frac{2\pi}{2}\text{ if } k \text{ is even}$$

and also $\alpha_k=\frac{2\pi}{k}\cdot\alpha_{k-2}$

$$=\frac{2\pi}{k}\cdot\frac{2\pi}{k-2}\cdots\alpha_k$$

$$=\frac{2\pi}{k}\cdot\frac{2\pi}{k-2}\cdots\frac{4\pi}{3}\cdot 2\text{ if } k \text{ is odd}$$

That is true for $n=k+1$

Consequently $V_4(2)=\alpha_4 2^4=\frac{2\pi}{4}\cdot\frac{2\pi}{2}\cdot 16=8\pi^2$

$V_5(2)=\alpha_5 3^5=2\times\frac{4\pi}{3}\times\frac{2\pi}{5}\times 3^5$

$$=\frac{648}{5}\pi^2$$

Thus $V_4(2)=8\pi^2$ & $V_5(3)=\frac{648}{5}\pi^2$

c). Repeat part (b), this time using the variable ordering (z, y, x). What does the second derivative that tell you new?

$$H_L(\lambda, z, y, x) = H_L(0,0,0,c) = \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

So $\left|\frac{\partial g}{\partial z}(0,0,c)\right| = 1 \neq 0$

and (8): $-d_3$, $-d_4$ by $n=3$, $k=1$

Then $-d_3 = -\begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2$

$-d_4 = -(-4) = 4$

Thus $a(0,0,c)$ is the minimum of f subject to constrain $g(x,y,z)$

d). Without marking any detail calculations, discuss why f must attain its minimum value at the point $(0,0,c)$

Since $f(x,y,z) = x^2+y^2$ and constrain, $g(x,y,z) = z-c$ are not dependent and $f(x,y,z) = x^2+y^2$ must be attain to minimum because $f(x,y,z)$ is not bounded above.

So f has no maximum value

Since $(0,0,c)$ is a critical point of f and f has no maximum value of f

Then $(0,0,c)$ is the minimum

Thus f must attain to minimum value at the point $(0,0,c)$

21) (a) Find the maximum

consider equation $\nabla(f+\lambda g)(x) = 0$

$$(f+\lambda g)(x_1, \dots, x_n) = \prod_{k=1}^{n} x_k^2 - \lambda \sum_{k=1}^{n} x_k^2$$

Then $\nabla(f+\lambda g) = \left(2x_1 \prod_{k=2}^{n} x_k^2 - \lambda 2x_1, \dots, 2x_n \prod_{k=1}^{n-1} x_k^2 - 2\lambda x_n\right)$

or
$$\begin{cases} 2x_1 x_2^2 \cdots x_n^2 - 2\lambda x_1 = 0 \\ 2x_2 x_1^2 x_3^2 \cdots x_n^2 - 2\lambda x_2 = 0 \\ \vdots \\ 2x_n x_1^2 \cdots x_{n-1}^2 - 2\lambda x_n = 0 \end{cases}$$

$$\begin{cases} 2x_1 (x_2^2 \cdots x_n^2 - \lambda) = 0 \\ 2x_2 (x_1^2 \cdots x_n^2 - \lambda) = 0 \\ \vdots \\ 2x_n (x_1^2 \cdots x_{n-1}^2 - \lambda) = 0 \end{cases}$$

for short $2x_i\left(\prod_{k=1}^{n} x_k - \lambda\right) = 0$ for $k \neq i$

Since $\sum_{k=1}^{n} x_k^2 = 1 \Rightarrow x_k \neq 0$

Then $\prod_{\substack{k=1 \\ k \neq j}}^{n} x_k - \lambda = 0$

we get
$$\begin{cases} x_2^2 x_3^2 \cdots x_n^2 = \lambda & (1) \\ x_1^2 x_2^2 \cdots x_n^2 = \lambda & (2) \\ \vdots \\ x_1^2 \cdots x_{n-1}^2 = \lambda & (n) \end{cases}$$

(1) divides (2) $\Rightarrow \frac{x_2^2}{x_1^2} = 1$ for $\lambda \neq 0$

$\left(\frac{x_2}{x_1}\right)^2 = 1 \Rightarrow \begin{cases} x_2 = -x_1 \\ x_2 = x_1 \end{cases}$, $x_2^2 = x_1^2$

Then $x_1^2 + x_1^2 + \dots + x_n^2 = 1 \Rightarrow x_1^2 = \frac{1}{n}$

Then for $1 \leq k \leq n$ $x_k^2 = 1/n$

Therefore $f(x_1, \dots, x_n) = 1/n^n$, for $\lambda \neq 0$

if $\lambda = 0 \Rightarrow x_1^2 x_2^2 \cdots x_n^2 = 0 \Leftrightarrow f(x_1, \dots, x_n) = 0$

Thus maximum of f equal $1/n^n$

b). Deduce for $x \in \mathbb{R}^n$:

$$\left|\prod_{k=1}^{n} x_k\right| \leq \left|\frac{\|x\|}{\sqrt{n}}\right|^n$$

we have $\prod_{k=1}^{n}\left(\frac{x_k}{\|x\|}\right)^2 \leq \prod_{k=1}^{n}(x_k^2) \leq \frac{1}{n^n}$

$$\Rightarrow \left\|\prod_{k=1}^{n} x_k\right\| \leq \left(\frac{\|n\|}{\sqrt{n}}\right)^n$$