Senoir
CalCulusIII & linear Algebra
TD7 (Vector Analysis)

I2-TD7
(Vector Analysis)

1. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

   (a) $\mathbf{F}(x,y) = e^{x+y}\mathbf{i} + e^{xy}\mathbf{j}$

   (b) $\mathbf{F}(x,y) = \dfrac{xy^2}{(1+x^2)^2}\mathbf{i} + \dfrac{x^2y}{1+x^2}\mathbf{j}$

   (c) $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \dfrac{z^2}{y}\mathbf{j} + 2z\ln y\mathbf{k}$.

   (d) $\mathbf{F}(x,y,z) = (e^{-yz} - yze^{xyz})\mathbf{i} + xz(e^{-yz} + e^{xyz})\mathbf{j} + xy(e^{-yz} - e^{xyz})\mathbf{k}$

2. Determine the map $g \in C^\infty$ with $g(0) = 0$ that makes the vector field

$$\mathbf{F}(x,y) = (y\sin x + xy\cos x + e^y)\mathbf{i} + (g(x) + xe^y)\mathbf{j}$$

   conservative. Find a potential for the resulting $\mathbf{F}$.

3. Compute the scalar line integral $\displaystyle\int_C f\,ds$, where f and C are as indicated.

   (a) $f(x,y) = e^x + xy^2$; $C : \mathbf{r}(t) = (t, 3t^2)$, $t \in [0,1]$.

   (b) $f(x,y) = x + y$; C is the perimeter of the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$.

   (c) $f(x,y) = x$; Let C be the union of the parabolic arc $y = 4 - x^2$ going from the point $A = (-2,0)$ to the point $B = (2,0)$, and the circle $x^2 + y^2 = 4$ from B to A.

   (d) $f(x,y,z) = xyz$; $C : \mathbf{r}(t) = (t, 2t, 3t)$, $t \in [0,2]$

   (e) $f(x,y,z) = \dfrac{z}{x^2+y^2}$; $C : \mathbf{r}(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t, e^{2t})$, $t \in [0,5]$

   (f) $f(x,y,z) = 2x - y^{1/2} + 2z^2$; $C = C_1 + C_2$, where $C_1 : \mathbf{r}_1(t) = (t, t^2, 0)$, $t \in [0,1]$ and $C_2 : \mathbf{r}_2(t) = (1, 1, t-1)$, $t \in [1,3]$.

   (g) $f(x,y,z) = x + y + z$; C is the line segment from $(1,1,2)$ to $(3,-1,1)$.

   (h) $f(x,y,z) = \sqrt{1 + yz^2}$; C is the intersection of the surfaces $x^2 + z^2 = 4$ and $y = x^2$.

   (i) $f(x,y,z) = 2x - \sqrt{y} + 2z^2$; $C : \mathbf{r}(t) = \begin{cases} (t, t^2, 0), & 0 \le t \le 1 \\ (1, 1, t-1), & 1 \le t \le 3. \end{cases}$

4. Compute the vector line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ and C are as indicated.

   (a) $\mathbf{F}(x,y) = (\sin y, x)$; $C : \mathbf{r}(t) = (t^2 - 1, t)$, $t \in [0,\pi]$.

   (b) $\mathbf{F}(x,y) = (x, y+1)$; $C : \mathbf{r}(t) = (1 - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$.

   (c) $\mathbf{F}(x,y) = (xy, y - x)$; C is the curve $y = x^2$ from $(1,1)$ to $(3,9)$.

   (d) $\mathbf{F}(x,y,z) = (yz^2, xz^2, 2xyz)$; $C = C_1 + C_2$, where C_1 is the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$, $t \in (0, 2\pi)$, and C_2 is the line segment from $(1,0,2\pi)$ to $(2\pi, 2\pi, 2\pi)$.

(e) $\mathbf{F}(x,y,z) = (\sqrt{x^3+y^3+5}, z, x^2)$; C is the intersection of the elliptical cylinder $y^2 + 2z^2 = 1$ with the plane $x = -1$, oriented in the counterclockwise direction when viewed from far out the positive x-axis.

5. Evaluate the differential form of the vector field.

(a) $\int_C xy\,dx + (x^2+y^2)dy$; C is the perimeter of the square with $(0,0), (1,0), (1,1)$ and $(0,1)$, oriented in the counterclockwise direction.

(b) $\int_C x^2y\,dx - xy\,dy$; C is the curve with equation $y^2 = x^3$, from $(1,-1)$ to $(1,1)$.

(c) $\int_C (x-y)^2dx - (x+y)^2dy$; C is the portion of $y = |x|$, from $(-2,2)$ to $(1,1)$.

(d) $\int_C yz\,dx - xz\,dy + xy\,dz$; C is the line segment form $(1,1,2)$ to $(5,3,1)$.

(e) $\int_C z\,dx + x\,dy + y\,dz$; C is the curve obtained by intersecting the surface $z = x^2$ and $x^2 + y^2 = 4$ and oriented counterclockwise around the z-axis.

(f) $\int_C dx + (x+y)dy + (x^2+xy+y^2)dz$; C is the intersection of $z = x^2+y^2$ and $x^2+y^2+z^2 = 1$, oriented in the counterclockwise direction when viewed from high up the positive z-axis.

6. (6.1). Show that $\mathbf{F}$ is conservative and find the potential function of $\mathbf{F}$. (6.2). Compute $\int_C \mathbf{F}\cdot dr$ for the given $\mathbf{F}$ and curve C from points A to B.

(a) $\mathbf{F}(x,y) = 2xy\mathbf{i} + (1+x^2-y^2)\mathbf{j}$; $A = (1,0)$ and $B = (2,3)$.

(b) $\mathbf{F}(x,y) = (x+\tan^{-1}y)\mathbf{i} + \frac{x+y}{1+y^2}\mathbf{j}$; $A = (0,0)$ and $B = (1,1)$.

(c) $\mathbf{F}(x,y,z) = 2xyz^2\mathbf{i} + x^2z^2\mathbf{j} + 2x^2zy\mathbf{k}$; $A = (1,0,1)$ and $B = (1,2,-1)$.

(d) $\mathbf{F}(x,y,z) = e^x\mathbf{i} + (xe^y + \ln z)\mathbf{j} + \frac{y}{z}\mathbf{k}$; $A = (0,1,1)$ and $B = (1,0,2)$.

7. Use Green's Theorem to evaluate the line integrals.

(a) $\int_C 2xy^2dx - x^2y\,dy$; C is the boundary of the region lying between the graph of $y = 0, y = \sqrt{x}$ and $x = 9$.

(b) $\int_C 2\arctan\frac{y}{x}dx + \ln(x^2+y^2)dy$; $C : x = 4 + 2\cos\theta,\ y = 4\sin\theta$.

(c) $\int_C (x-3y)dx + (x+y)dy$ C is the region lying between the graphs of $x^2+y^2 = 1$ and $x^2+y^2 = 9$.

8. Find the equation of the tangent plane to the parametric surface represented by $\mathbf{r}$ at the specified point.

(a) $\mathbf{r}(u,v) = (u-v)\mathbf{i} + (u-v)\mathbf{j} + v^2\mathbf{k}$; $(2,0,1)$.

(b) $\mathbf{r}(u,v) = u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^2\mathbf{k}$; $(u,v) = (1,\pi)$.

9. Find the area of the surface.

   (a) The part of the plane $\mathbf{r}(u,v) = (u+2v-1)\mathbf{i} + (2u+3v+1)\mathbf{j} + (u+v+2)\mathbf{k}$ $0 \le u \le 1$; $0 \le v \le 2$.

   (b) The part of the plane $z = 8 - 2x - 3y$ that lies in side the cylinder $x^2 + y^2 = 4$.

   (c) The surface $\mathbf{r}(u,v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + u\mathbf{k}$ $0 \le u \le \pi$; $0 \le v \le 2\pi$.

10. Find $\iiint_S f(x,y,z)dS$.

    (a) $f(x,y,z) = x + y$; S is the part of the plane $3x + 2y + z = 6$ in the first octant.

    (b) $f(x,y,z) = xz$; S is part of the plane $y + z = 4$ inside the cylinder $x^2 + y^2 = 4$.

    (c) $f(x,y,z) = x + \frac{y}{\sqrt{4z+5}}$; S is the surface with vector representation $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (v^2-1)\mathbf{k}$, $0 \le u \le 1$; $-1 \le v \le 1$.

    (d) $f(x,y,z) = z\sqrt{1+x^2+y^2}$; S is the helicoid with vector representation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$; $0 \le v \le 2\pi$.

11. Find the mass of the surface S having the given density mass function.

    (a) S is the part of the plane $x + 2y + 3z = 6$ in the first octant; the density at any point of S is directly proportional to the square of distance between the point to yz-plane.

    (b) S is the hemisphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2+y^2}$; the density at any point on S is directly proportional to the distance between the point to xy-plane.

12. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, that is, find the flux of $\mathbf{F}$ across $\mathbf{S}$. If S is closed, use the positive (outward) orientation.

    (a) $\mathbf{F}(x,y,z) = (2x, 2y, z)$; S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane; $\mathbf{N}$ point upward.

    (b) $\mathbf{F}(x,y,z) = (x^2, xy, xz)$; S is the surface of tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

13. **[Electric Charge]** Find the total charge on the part of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies directly above the plane region $R = \{(x,y) : x^2 + y^2 \le 9\}$ if the charge density at any point on the surface is directly proportional to the distance between the point and xy-plane.

14. **[Flow of a Fluid]** The flow of a fluid is described by the vector field $\mathbf{F}(x,y,z) = (2x, 2y, 3z)$. Find the rate of flow of the fluid upward through the surface S that is the part of the plane $x + 3y + z = 6$ in the first octant.

15. **[Flow of a Liquid]** The flow of a liquid is described by the vector field $\mathbf{F}(x,y,z) = (x, y, 3z)$. If the mass density of the fluid is 1000 (in appropriate units), find the rate of flow (mass per unit time) upward of the liquid through the surface S that is part of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.

16. Verify the Divergence Theorem by evaluating $\iint_S \mathbf{F} \cdot \mathbf{N}dS$.

(a) $\mathbf{F}(x,y,z) = (2xy, -y^2, 3yz)$; S is the cube bounded by the plane $x = 0, x = 2, y = 0, y = 2, z = 0$ and $z = 2$.

(b) $\mathbf{F}(x,y,z) = (xz, zy, 2z^2)$; S is the surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

17. Use the Divergence Theorem to find the flux of $\mathbf{F}$ across $\mathbf{S}$; that is, calculate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$.

(a) $\mathbf{F}(x,y,z) = (xy^2, 2yz, -3x^2y^3)$; S is the surface of the cube bounded by the planes $x = \pm 1, y = \pm 1$ and $z = \pm 1$.

(b) $\mathbf{F}(x,y,z) = (x+1, yz^2 + \cos xz, 2y^2z + e^{\tan x})$; S is the sphere $x^2 + y^2 + z^2 = 1$.

(c) $\mathbf{F}(x,y,z) = (xz, x^2y, y^2z + 1)$; S is the surface region that lies between the cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$.

18. (a) Use the Divergence Theorem to verify that the volume of the solid bounded by a surface S is

$$\iint_S x dy dz = \iint_S y dz dx = \iint_S z dx dy.$$

(b) Verify the result of part (a) for the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$.

19. Verify that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N}\, dS = 0$ for any closed surface S.

20. Given the vector field $\mathbf{F}(x,y,z) = (x,y,z)$, verify that $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = 3V$, where V is the volume of the solid bounded by the closed surface S.

21. Verify Stokes's Theorem by evaluating $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot dr$ as a line integral and as a double integral.

(a) $\mathbf{F}(x,y,z) = (-y+z, x-z, x-y)$; S is part of the paraboloid $z = 9 - x^2 - y^2$ and $z \geq 0$.

(b) $\mathbf{F}(x,y,z) = (y, z, x)$; S is part of the plane $2x + 2y + z = 6$ lying in the first octant.

22. Use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(a) $\mathbf{F}(x,y,z) = (\arctan \frac{y}{x}, \ln x^2 + y^2, 1)$; C is a triangle with vertices $(0,0,0), (1,1,1)$ and $(0,0,2)$.

(b) $\mathbf{F}(x,y,z) = (3xz, e^{xz}, 2xy)$; C is a circle obtained by intersecting the cylinder $x^2+z^2 = 1$ with the plane $y = 3$ oriented in a counterclockwise direction when viewed from the right.

(c) $\mathbf{F}(x,y,z) = (-\ln \sqrt{x^2+y^2}, \arctan \frac{x}{y}, 1)$; S is the surface of $z = 9 - 2x - 3y$ over $r = 2\sin\theta$ in the first octant.

(d) $\mathbf{F}(x,y,z) = (xyz, y, z)$; S is the surface of $z = x^2$, $0 \leq x \leq a$, $0 \leq y \leq a$. $\mathbf{N}$ is the downward unit normal to the surface.

I2. TD7

(Vector Analysis)

1. Determine whether the vector field is conservative.
If it is, find a potential function for the vector field.

a). $F(x,y) = e^{x+y}\,i + e^{xy}\,j$

We can be written as, $F(x,y) = \left(e^{x+y}, e^{xy}\right)$

Let $M(x,y) = e^{x+y}$ and $N(x,y) = e^{xy}$

$\Rightarrow \dfrac{\partial M}{\partial y}(x,y) = e^{x+y}$ and $\dfrac{\partial N}{\partial x}(x,y) = ye^{xy}$

Showing that, $\dfrac{\partial M}{\partial y}(x,y) \neq \dfrac{\partial N}{\partial x}(x,y)$.

Therefore, $\boxed{F(x,y) \text{ is not conservative.}}$

b). $F(x,y) = \dfrac{xy^2}{(1+x^2)^2}\,i + \dfrac{x^2y}{1+x^2}\,j$

We can be written as, $F(x,y) = \left(\dfrac{xy^2}{(1+x^2)^2}, \dfrac{x^2y}{1+x^2}\right)$

Let $M(x,y) = \dfrac{xy^2}{(1+x^2)^2}$ and $N(x,y) = \dfrac{x^2y}{1+x^2}$

$\Rightarrow \dfrac{\partial M}{\partial y}(x,y) = \dfrac{2xy}{(1+x^2)^2}$ and $\dfrac{\partial N}{\partial x}(x,y) = \dfrac{2xy}{(1+x^2)^2}$

Showing that, $\dfrac{\partial M}{\partial y}(x,y) = \dfrac{\partial N}{\partial x}(x,y)$

That is, $F(x,y)$ is conservative.

$\Rightarrow \exists f$ such that $F = \nabla f \Leftrightarrow \left(\frac{xy^2}{(1+x^2)^2}, \frac{x^2 y}{1+x^2} \right) = \left(f_x(x,y), f_y(x,y) \right)$

$$\Rightarrow \begin{cases} f_x(x,y) = \dfrac{xy^2}{(1+x^2)^2} & (1) \\ f_y(x,y) = \dfrac{x^2 y}{1+x^2} & (2) \end{cases}$$

Follow by (1): $\int f_x(x,y)\,dx = \int \frac{xy^2}{(1+x^2)^2}\,dx = -\frac{y^2}{2(1+x^2)} + C(y)$ (3)

$\Rightarrow f_y(x,y) = -\frac{y}{1+x^2} + C'(y)$ (4)

Follow by (2) and (4): $\frac{x^2 y}{1+x^2} = -\frac{y}{1+x^2} + C'(y)$

$\Rightarrow C'(y) = y \Rightarrow C(y) = \frac{y^2}{2} + K, K \in \mathbb{R}$ (5)

Follow by (3) and (5): $f(x,y) = -\frac{y^2}{2(1+x^2)} + \frac{y^2}{2} + K$

Therefore, $\boxed{f(x,y) = -\frac{y^2}{2(1+x^2)} + \frac{y^2}{2} + K}$ is a potential function of $F(x,y)$.

c). $F(x,y,z) = 3x^2 i + \frac{z^2}{y} j + 2z\ln y\, k$

We can be written as, $F(x,y,z) = \left(3x^2, \frac{z^2}{y}, 2z\ln y\right)$

Let $M(x,y,z) = 3x^2$ and $N(x,y,z) = \frac{z^2}{y}$ and $P(x,y,z) = 2z\ln y$.

$\Rightarrow \frac{\partial P}{\partial y}(x,y,z) = \frac{2z}{y}$ and $\frac{\partial N}{\partial z}(x,y,z) = \frac{2z}{y} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ (1)

- $\frac{\partial P}{\partial x}(x,y,z) = 0$ and $\frac{\partial M}{\partial z}(x,y,z) = 0 \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$ (2)
- $\frac{\partial N}{\partial x}(x,y,z) = 0$ and $\frac{\partial M}{\partial y}(x,y,z) = 0 \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ (3).

Follow by (1), (2) and (3): $\operatorname{curl} F(x,y,z) = 0$

Showing that, F is conservative.

$\Rightarrow \exists f$ such that $F = \nabla f \Leftrightarrow \left(3x^2, \frac{z^2}{y}, 2z\ln y\right) = \left(f_x(x,y,z), f_y, f_z\right)$.

$$\Rightarrow \begin{cases} f_x(x,y,z) = 3x^2 & (4) \\ f_y(x,y,z) = \frac{z^2}{y} & (5) \\ f_z(x,y,z) = 2z\ln y & (6) \end{cases}$$

Follow by (4): $\int f_x(x,y,z)\,dx = x^3 + C(y,z) \Rightarrow f(x,y,z) = x^3 + C(y,z)$. (*)

Then; $f_y(x,y,z) = C'(y,z)$ (7)

Follow by (5) and (7): $C'(y,z) = \frac{z^2}{y} \Rightarrow C(y,z) = \int \frac{z^2}{y}\,dy = z^2\ln y + C(z)$

Follow by (*): $f(x,y,z) = x^3 + z^2\ln y + C(z) \Rightarrow f_z(x,y,z) = +2z\ln y + C'(z)$ (8)

Follow by (6) and (8): $2z\ln y = 2z\ln y + C'(z) \Rightarrow C'(z) = 0$

$\Rightarrow C(z) = \int 0\,dz = k, \ k \in \mathbb{R}$

We get, $f(x,y,z) = x^3 + z^2 \ln y + k,\ k \in \mathbb{R}$.

<u>Therefore</u>, $\boxed{f(x,y,z) = x^3 + z^2 \ln y + k,\ k\in\mathbb{R} \text{ is a potential function of } F(x,y,z).}$

d). $F(x,y,z) = \left(e^{-yz} - yze^{xyz}\right) i + xz\left(e^{-yz} + e^{xyz}\right) j + xy\left(e^{-yz} - e^{xyz}\right) k$.

We can be written as, $F(x,y,z) = \left(e^{-yz} - yze^{xyz},\ xz\left(e^{-yz} - e^{xyz}\right),\ xy\left(e^{-yz} - e^{xyz}\right)\right)$.

Let $M(x,y,z) = e^{-yz} - yze^{xyz}$, $N(x,y,z) = xz\left(e^{-yz} - e^{xyz}\right)$
and $P(x,y,z) = xy\left(e^{-yz} - e^{xyz}\right)$.

- $\frac{\partial P}{\partial y}(x,y,z) = x\left(e^{-yz} - e^{xyz}\right) + xy\left(-ze^{-yz} - xze^{xyz}\right)$
- $\frac{\partial N}{\partial z}(x,y,z) = x\left(e^{-yz} + e^{xyz}\right) + xz\left(-ye^{-yz} + xze^{xyz}\right)$.

$\Rightarrow \frac{\partial N}{\partial z}(x,y,z) \neq \frac{\partial P}{\partial y}(x,y,z)$.

Showing that, $\operatorname{curl} F \neq 0$.

<u>Therefore</u>, $\boxed{F(x,y,z) \text{ is not conservative.}}$

2. Determine the map $g \in C^{\infty}$ with $g(0) = 0$ that makes the vector field $F(x,y) = (y\sin x + xy\cos x + e^y)i + (g(x) + xe^y)j$ conservative. Find a potential for the resulting F.

Let $M(x,y) = y\sin x + xy\cos x + e^y$ and $N(x,y) = g(x) + xe^y$

$\Rightarrow \frac{\partial M}{\partial y}(x,y) = \sin x + x\cos x + e^y$ and $\frac{\partial N}{\partial x}(x,y) = g'(x) + e^y$.

$F(x,y)$ is conservative iff $\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$

Then, $\sin x + x\cos x + \cancel{e^y} = g'(x) + \cancel{e^y} \Rightarrow g'(x) = \sin x + x\cos x$

We get, $g(x) = \int(\sin x + x\cos x)dx$

$$= -\cancel{\cos x} + x\sin x + \cancel{\cos x}$$

$$= x\sin x.$$

Therefore, $\boxed{g(x) = x\sin x \text{ that makes } F(x,y) \text{ conservative.}}$

* Find a potential for the resulting F.

By using the result above, $\frac{\partial N}{\partial x}(x,y) = \sin x + x\cos x + e^y$

$\Rightarrow \frac{\partial N}{\partial y}(x,y) = \frac{\partial M}{\partial x}(x,y)$. Showing that, $F(x,y)$ is conservative.

$\Rightarrow \exists f$ such that $F = \nabla f \Leftrightarrow (y\sin x + xy\cos x + e^y, x\sin x + xe^y) =$

$$(f_x(x,y), f_y(x,y)).$$

$$\Rightarrow \begin{cases} f_x(x,y) = y\sin x + xy\cos x + e^y & (1) \\ f_y(x,y) = x\sin x + xe^y & (2) \end{cases}$$

Follow by (2): $\int f_y(x,y)\,dy = \int (x\sin x + xe^y)\,dy = xy\sin x + xe^y + C(x)$ (*)

$\Rightarrow f_x(x,y,z) = xy\cos x + y\sin x + e^y + C'(x)$ (3).

Follow by (1) and (3): $\cancel{y\sin x} + \cancel{xy\cos x} + \cancel{e^y} = \cancel{xy\cos x} + \cancel{y\sin x} + \cancel{e^y} + C'(x)$

$\Rightarrow C'(x) = 0 \Rightarrow C(x) = K,\ K \in \mathbb{R}$.

Follow by (*): $f(x,y,z) = xy\sin x + xe^y + K$, $K \in \mathbb{R}$.

<u>Therefore</u>, $\boxed{f(x,y,z) = xy\sin x + xe^y + K,\ K \in \mathbb{R}}$ is a potential for the resulting $F(x,y)$.

3. Compute the scalar line integral $\int_C f\,ds$, where f and C are as indicated.

a). $f(x,y) = e^x + xy^{\cancel{2}}$, $C : r(t) = (t, 3t^{\cancel{2}})$, $t \in [0,1]$.

We have, $r(t) = (t, 3t^{\cancel{2}}) \Rightarrow r'(t) = (1,3) \Rightarrow \|r'(t)\| = \sqrt{1+9} = \sqrt{10}$

Then, $f(r(t)) = f(t, 3t^{\cancel{2}}) = e^t + 9t^3$

Follow by $\int_C f\,ds = \int_0^1 f(r(t))\|r'(t)\|\,dt$

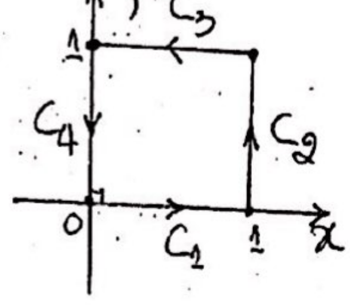
$$= \int_0^1 \sqrt{10}(e^t + 9t^3)\,dt$$

$$= \sqrt{10}\left(e - \frac{5}{4}\right)$$

<u>Therefore</u>, $\boxed{\int_C f\,ds = \sqrt{10}\left(e - \frac{5}{4}\right).}$

b). $f(x,y) = x+y$, C is the perimeter of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

Follow by $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds + \int_{C_4} f\,ds$.



- $C_1 : r_1(t) = (t, 0)$, $t \in [0,1]$

$\Rightarrow r_1'(t) = (1, 0) \Rightarrow \|r_1'(t)\| = 1$

- $C_2 : r_2(t) = (1, t)$, $t \in [0,1]$

$\Rightarrow r_2'(t) = (0, 1) \Rightarrow \|r_2'(t)\| = 1$

- $C_3 : r_3(t) = (1-t, 1)$, $t \in [0,1]$

$\Rightarrow r_3'(t) = (-1, 0) \Rightarrow \|r_3'(t)\| = 1$

- $C_4 : r_4(t) = (0, t)$, $t \in [0,1]$

$\Rightarrow r_4'(t) = (0, 1) \Rightarrow \|r_4'(t)\| = 1$.

We get, $\int_C f\,ds = \int_0^1 t\,dt + \int_0^1 (1+t)\,dt + \int_0^1 (2-t)\,dt + \int_0^1 t\,dt$

$$= \frac{1}{2} + 1 + \frac{1}{2} + 2 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = 4$$

Therefore, $\boxed{\int_C f\,ds = 4 .}$

c). $f(x,y) = x$, Let C be the union of the paraboloid arc $y = 4 - x^2$ going form the point $A = (-2, 0)$ to the point $B = (2, 0)$, and the circle $x^2 + y^2 = 4$ from B to A.

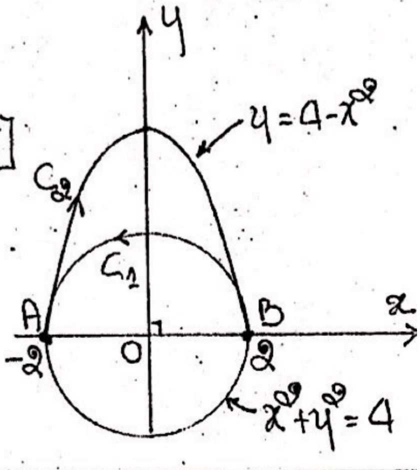
Follow by $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$.

$C_1 : r_1(t) = (2\cos t, 2\sin t)$, $t \in [0, \pi]$

$\Rightarrow r_1'(t) = (-2\sin t, 2\cos t) \Rightarrow \|r'(t)\| = 2$.

$C_2 : r_2(t) = (t, 4 - t^2)$, $t \in [-2, 2]$

$\Rightarrow r_2'(t) = (1, -2t) \Rightarrow \|r'(t)\| = \sqrt{1 + 4t^2}$



We get, $\int_C f\,ds = \int_0^{\pi} 2\cos t \times 2\,dt + \int_{-2}^{2} t\sqrt{1+4t^2}\,dt$

$= 4\left[\sin t\right]_0^{\pi} + \frac{1}{8}\left[(1+4t^2)^{3/2}\right]_{-2}^{2} \times \frac{2}{3}$

$= 0$

<u>Therefore</u>, $\boxed{\int_C f\,ds = 0}$

d). $f(x,y,z) = xyz$; $C : r(t) = (t, 2t, 3t)$, $t \in [0,2]$.

Follow by $\int_C f\,ds = \int_0^2 (6t^3 \times \sqrt{14})\,dt$

$= 6\sqrt{14} \times \frac{2^4}{4} = 24\sqrt{14}$.

<u>Therefore</u>, $\boxed{\int_C f\,ds = 24\sqrt{14}}$

e). $f(x,y,z) = \frac{z}{x^2+y^2}$; $C, r(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t, e^{2t})$
$t \in [0,5]$.

We have, $C : r(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t, e^{2t})$

$\Rightarrow r'(t) = (2e^{2t}\cos 3t - 3\sin 3t\, e^{2t}, 2e^{2t}\sin 3t + 3\cos 3t\, e^{2t}, 2e^{2t})$.

$\Rightarrow \|r'(t)\| = \sqrt{(2e^{2t})^2 + (3e^{2t})^2 + (2e^{2t})^2} = \sqrt{17}\, e^{2t}$

Follow by $\int_C f\,ds = \int_0^5 \frac{e^{2t}}{e^{4t}} \times \sqrt{17}\, e^{2t} = \underline{5\sqrt{17}}$.

f). $f(x,y,z) = 2x - y^{1/2} + 2z^2$, $C = C_1 + C_2$, where

$C_1 : r_1(t) = (t, t^2, 0)$; $t \in [0,1]$ and $C_2 : r_2(t) = (1, 1, t-1)$, $t \in [1,3]$.

Follow by $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$

- $C_1 : r_1(t) = (t, t^2, 0)$, $t \in [0,1]$

$\Rightarrow r_1(t) = (1, 2t, 0) \Rightarrow \|r'(t)\| = \sqrt{1+4t^2}$

- $C_2 : r_2(t) = (1, 1, t-1)$, $t \in [1,3]$

$\Rightarrow r_2'(t) = (0, 0, 1) \Rightarrow \|r'(t)\| = 1$.

We get, $\int_C f\,ds = \int_0^1 (2t - t)\sqrt{1+4t^2}\,dt + \int_1^3 \left(1 + 2(t-1)^2\right)dt$

$$= \frac{1}{8} \times \frac{2}{3}\left[(1+4t^2)^{3/2}\right]_0^1 + \left[3t + \frac{2}{3}t^3 - 2t^2\right]_1^3$$

$$= \frac{1}{12}\left(5\sqrt{5} - 1\right) + \frac{22}{3}$$

$$= \frac{5\sqrt{5}}{12} + \frac{87}{12} = \frac{5\sqrt{5}+87}{12}$$

Therefore, $\boxed{\int_C f\,ds = \frac{5\sqrt{5}+87}{12}}$

g). $f(x,y,z) = x+y+z$; C is the line segment from $(1,1,2)$ to $(3,-1,1)$.

We have, $r(a) = (1,1,2)$ and $r(b) = (3,-1,1)$

Then $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \Leftrightarrow \begin{cases} x = 1 + 2t \\ y = 1 - 2t \\ z = 2 - t \end{cases}$, $t \in [0,1]$.

We get, $\int_C f\, ds = \int_0^1 (1 + \cancel{2t} + 1 - \cancel{2t} + 2 - t)\sqrt{4+4+1}\, dt$

$= 11$.

Therefore, $\boxed{\int_C f\, ds = 11}$

h). $f(x,y,z) = \sqrt{1+yz^2}$, C is the intersection of the surface $x^2 + z^2 = 4$ and $y = x^2$.

We get, $r(t) = (2\cos t, 4\cos^2 t, 2\sin t)$, $t \in [0, 2\pi]$.

$\Rightarrow r'(t) = (-2\sin t, -8\sin t\cos t, 2\cos t)$

We get, $\int_C f\, ds = \int_0^{2\pi} (\sqrt{1+16\sin^2 t\cos^2 t}) \times (\sqrt{1+16\sin^2 t\cos^2 t})\, dt$

$= 2\int_0^{2\pi} (3 - 2\cos t)\, dt$

$= 2\left[3t - 2\sin t\right]_0^{2\pi} = 2(6\pi - 0) = 12\pi$

Therefore, $\boxed{\int_C f\, ds = 12\pi.}$

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19.MECH BEALY 20.MENG LYEONG

i). $f(x,y,z) = 2x - \sqrt{y} + 2z^2$, $C : r(t) = \begin{cases} (t, t^2, 0) , & 0 \leq t \leq 1 \\ (1, 1, t-1) , & 1 \leq t \leq 3 \end{cases}$.

We have, $\begin{cases} r_1'(t) = (1, 2t, 0) , & t \in [0,1] \\ r_2'(t) = (0, 0, 1) , & t \in [1,3] \end{cases}$

$$\Rightarrow \begin{cases} \|r_1'(t)\| = \sqrt{1 + 4t^2} \\ \|r_2'(t)\| = 1 \end{cases}$$

and $\begin{cases} f(r_1(t)) = f(t, t^2, 0) = 2t - t + 0 = t \\ f(r_2(t)) = f(1, 1, t-1) = 2 - 1 + 2(t-1)^2 = 3 - 4t + 2t^2 \end{cases}$

We get, $\int_{C_1} f(x,y,z)\, ds = \int_0^1 t\sqrt{1+4t^2}\, dt = \frac{1}{12} 5^{3/2} - \frac{1}{12}$

and $\int_{C_2} f(x,y,z)\, ds = \int_1^3 (3 - 4t + 2t^2)\, dt$

$$= \left[3t - 2t^2 + \frac{2}{3}t^3\right]_1^3$$

$$= 9 - 18 - 18 - 3 + 2 - \frac{2}{3}$$

$$= -\frac{86}{3}$$

<u>Therefore</u>, $\boxed{\int_C f(x,y,z)\, ds = \begin{cases} \frac{1}{12}\left(5^{3/2} - 1\right) \\ -\frac{86}{3} \end{cases}}$

4. Compute the vetor line integral $\int_C F\cdot dr$, where F and C are as indicated.

a). $F(x,y)=(\sin y, x)$; $C: r(t)=(t^2-1, t)$, $t\in[0,\pi]$.

We have, $C, r(t)=(t^2-1,t) \Rightarrow r'(t)=(2t,1)$

and $F(r(t))=F(t^2-1,t)=(\sin t, t^2-1)$.

Follow by $\int_C F\,dr=\int_0^\pi(\sin t, t^2-1)(2t,1)\,dt$

$$=\int_0^\pi(2t\sin t+t^2-1)\,dt$$

$$=-2\left[t\cos t\right]_0^\pi+2\left[\sin t\right]_0^\pi+\left[\frac{t^3}{3}-t\right]_0^\pi$$

$$=-2(-\pi)+\frac{\pi^3}{3}-\pi$$

$$=\frac{3\pi-\pi^3}{3}$$

Therefore, $\boxed{\int_C F\,dr=\frac{3\pi-\pi^3}{3}}$

b). $F(x,y)=(x,y+1)$; $C: r(t)=(1-\sin t, 1-\cos t)$, $t\in[0,2\pi]$.

We have, $C: r(t)=(1-\sin t, 1-\cos t) \Rightarrow r'(t)=(-\cos t, \sin t)$.

and $F(r(t)) = F(1-\sin t, 1-\cos t) = (1-\sin t, 2-\cos t)$

Follow by $\int_C F\,dr = \int_0^{2\pi}(1-\sin t, 2-\cos t)(-\cos t, \sin t)\,dt$

$$= \int_0^{2\pi}(-\cos t + \cancel{\cos t \sin t} + 2\sin t - \cancel{\cos t\sin t})\,dt$$

$$= \left[-\sin t - 2\cos t\right]_0^{2\pi} = 0$$

Therefore, $\boxed{\int_C F\,dr = 0}$

c). $F(x,y)=(xy, y-x)$; C is the curve $y=x^2$ from $(1,1)$ to $(3,9)$.

We have, $C: r(t)=(t,t^2)$, $t\in[1,3]$

$\Rightarrow r'(t)=(1,2t)$

and $F(r(t)) = F(t,t^2) = (t^3, t^2-t)$

Follow by $\int_C F\,dr = \int_1^3 (t^3, t^2-t)(t,t^2)\,dt$

$$= \int_1^3 (t^4+t^4-t^3)\,dt = \frac{384}{5}$$

Therefore, $\boxed{\int_C F\,dr = \frac{384}{5}}$

d). $F(x,y,z) = (yz^2, xz^2, 2xyz)$; $C = C_1 + C_2$, where C_1 is the helix $r(t) = (\cos t, \sin t, t)$, $t \in (0, 2\pi)$, and C_2 is the line segment from $(1, 0, 2\pi)$ to $(2\pi, 2\pi, 2\pi)$.

Follow by $\int_C F\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr$.

- C_1 : $r_1(t) = (\cos t, \sin t, t)$, $t \in (0, 2\pi)$

$\Rightarrow r_1'(t) = (-\sin t, \cos t, 1) \Rightarrow F(r_1(t)) \times r_1'(t) = 0$

- C_2 : $r_2(t) = ((2\pi - 1)t + 1, 2\pi t, 0)$

$\Rightarrow r_2'(t) = (2\pi - 1, 2\pi, 0) \Rightarrow F(r_2(t))\, r_2'(t) = 0$

We get, $\int_C F\,dr = 0 + 0 = 0$

<u>Therefore</u>, $\boxed{\int_C F\,dr = 0}$

$\rightarrow$ <u>TH</u> : (Line Integral of a vector Field).

Let $C : r(t)$, $t \in [a, b]$ $F : \mathbb{R}^n \to \mathbb{R}^n$.

The line integral of vector field F is denoted or defined by $\int_C F\,dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$.

e). $F(x,y,z) = \left(\sqrt{x^3+y^3+5},\ z,\ x^2\right)$; C is the intersection of the elliptical cylinder $y^2+2z^2=1$ with the plane $x=-1$, oriented in the counterclockwise direction when viewed from far out the positive x-axis.

We have, $y^2+2z^2=1 \Leftrightarrow \begin{cases} y = \cos t \\ z = \frac{1}{\sqrt{2}}\sin t \end{cases}$, $t \in [0, 2\pi]$.

We get, $C: r(t) = \left(-1, \cos t, \frac{1}{\sqrt{2}}\sin t\right)$

$\Rightarrow r'(t) = \left(0, -\sin t, \frac{1}{\sqrt{2}}\cos t\right)$.

and $F(r(t)) = \left(\sqrt{-1+\cos^3 t+5},\ \frac{1}{\sqrt{2}}\sin t,\ 1\right)$

Follow by $\int_C F\,dr = \int_0^{2\pi}\left(-\frac{1}{\sqrt{2}}\sin^2 t + \frac{1}{\sqrt{2}}\cos t\right)dt$

$$= \int_0^{2\pi}\left(-\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\cos t\right)dt$$

$$= \left[-\frac{1}{2\sqrt{2}}t + \frac{1}{4\sqrt{2}}\sin 2t + \frac{1}{\sqrt{2}}\sin t\right]_0^{2\pi}$$

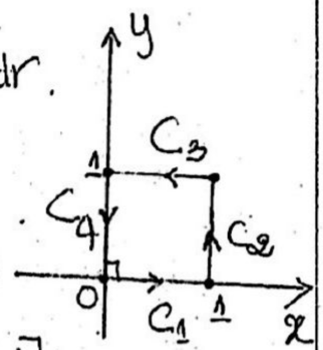
$$= \frac{-1}{\sqrt{2}}\pi = -\frac{\sqrt{2}}{2}\pi$$

Therefore, $\boxed{\int_C F\,dr = \frac{-\sqrt{2}}{2}\pi}$

5. Evaluate the differential form of the vector field.

a). $\int_C xy\,dx + (x^2+y^2)\,dy$, C is the perimeter of the square with $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, oriented in the counterclockwise direction.

Follow by $\int_C F\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr + \int_{C_3} F\,dr + \int_{C_4} F\,dr$.



- $C_1 : r_1(t) = (t, 0) \Rightarrow r_1'(t) = (1, 0)$, $t \in [0,1]$
  and $F(x,y) = (xy, x^2+y^2) \Rightarrow F(r_1(t)) = (0, t^2)$
- $C_2 : r_2(t) = (1, t) \Rightarrow r_2'(t) = (0, 1)$, $t \in [0,1]$.
  and $F(r_2(t)) = F(1,t) = (t, 1+t^2)$
- $C_3 : r_3(t) = (1-t, 1) \Rightarrow r_3'(t) = (-1, 0)$, $t \in [0,1]$
  and $F(r_3(t)) = F(1-t, 1) = (1-t, 2-2t+t^2)$
- $C_4 : r_4(t) = (0, 1-t) \Rightarrow r_4'(t) = (0, -1)$, $t \in [0,1]$
  and $F(r_4(t)) = F(0, 1-t) = (0, (1-t)^2)$.

We get; $\int_C F\,dr = \int_0^1 0\,dt + \int_0^1 (1+t^2)\,dt + \int_0^1 (-1+t)\,dt - \int_0^1 (1-t)^2\,dt$

$$= \cancel{1} + \cancel{\tfrac{1}{3}} - \cancel{1} + \cancel{1} - \cancel{\tfrac{1}{3}} - \cancel{1} + \frac{1}{2} = 1/2$$

Therefore, $\boxed{\int_C F\,dr = 1/2}$.

b). $\int_C x^2y\,dx - xy\,dy$, C is the curve with equation $y^2 = x^3$, from $(+1,-1)$ to $(1,1)$.

Let $F(x,y) = (x^2y, -xy)$.

We have, $r(t) = (t, t^{3/2})$, $t \in [-1, 1]$.

$\Rightarrow r'(t) = (1, \frac{3}{2}t^{1/2})$ and $F(r(t)) = (t^{7/2}, -t^{5/2})$.

Follow by $\int_C F dr = \int_{-1}^{1}(t^{7/2} - \frac{3}{2}t^3)\,dt$

$= \left[\frac{2}{9}x^{9/2} - \frac{3}{2}\frac{t^4}{4}\right]_{-1}^{1}$

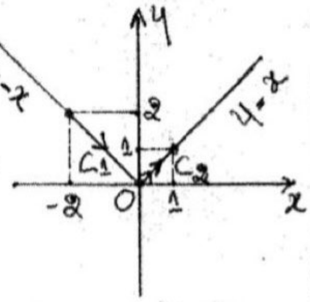
$= \frac{2}{9} - \cancel{\frac{3}{8}} + \frac{2}{9} + \cancel{\frac{3}{8}} = \frac{4}{9}$.

Therefore, $\boxed{\int_C F dr = \frac{4}{9}}$

c). $\int_C (x-y)^2dx - (x+y)^2dy$, C is the portion of $y=|x|$, from $(-2,2)$ to $(1,1)$.

Let $F(x,y) = ((x-y)^2, -(x+y)^2)$.

Follow by $\int_C F dr = \int_{C_1} F dr + \int_{C_2} F dr$.



- $C_1 : r_1(t) = (t, -t) \Rightarrow r_1'(t) = (1, -1)$, $t \in [-2, 0]$.
  and $F(r_1(t)) = F(t,-t) = (4t^2, 0)$.
- $C_2 : r_2(t) = (t, t) \Rightarrow r_2'(t) = (1, 1)$, $t \in [0, 1]$.
  and $F(r_2(t)) = F(t,t) = (0, 4t^2)$

We get, $\int_C F dr = \int_{-2}^{0} 4t^2dt + \int_0^1 4t^2dt = [\frac{4}{3}t^3]_{-2}^{0} + [\frac{4}{3}t^3]_0^1 = \frac{28}{3}$.

d). $\int_C yz\,dx - xz\,dy + xy\,dz$, C is the line segment from $(1,1,2)$ to $(5,3,1)$.

Let $F(x,y,z)=(yz,-xz,xy)$

We have, $C: r(t)=(1+4t,\,1+2t,\,2-t)$, $t\in[0,1]$

$\Rightarrow r'(t)=(4,2,-1)$ and $F(r(t))=(2+3t-2t^2,\,-2-7t+4t^2,\,1+6t+8t^2)$.

Follow by $\int_C F\,dr = \int_0^1 (8+12t-\cancel{8t^2}-4-14t+\cancel{8t^2}-1-6t-8t^2)\,dt$

$$=\int_0^1(-8t^2-8t+3)\,dt = \frac{-11}{3}$$

Therefore, $\boxed{\int_C F\,dr = -\frac{11}{3}}$

e). $\int_C z\,dx + x\,dy + y\,dz$, C is the curve obtained by intersecting the surface $z=x^2$ and $x^2+y^2=4$ and oriented counterclockwise around the z-axis.

Let $F(x,y,z)=(z,x,y)$ and $x=2\cos t$, $y=2\sin t$, $t\in[0,2\pi]$

We have, $C: r(t)=(2\cos t,\,2\sin t,\,4\cos^2 t)$

$\Rightarrow r'(t)=(-2\sin t,\,2\cos t,\,-8\sin t\cos t)$

and $F(r(t))=(4\cos^2 t,\,2\cos t,\,2\sin t)$.

Follow by $\int_C F\,dr = \int_0^{2\pi}(-8\sin t\cos^2 t + 4\cos^2 t - 16\cos t\sin^2 t)\,dt$

$$=\left[\frac{8}{3}\cos^3 t + 2t + \sin 2t - 16/3\,\sin^3 t\right]_0^{2\pi} = \underline{4\pi}.$$

7). $\int_C dx + (x+y)\,dy + (x^2+xy+y^2)\,dz$, C is the intersection of $z = x^2+y^2$ and $x^2+y^2+z^2 = 1$, oriented in the counterclockwise direction when view from high up the positive z-axis.

We have, $\begin{cases} z = x^2+y^2 \\ x^2+y^2+z^2 = 1 \end{cases} \Rightarrow z^2+z-1=0$

$\Rightarrow z = \dfrac{-1+\sqrt{5}}{2} \Rightarrow x^2+y^2 = \sqrt{\dfrac{-1+\sqrt{5}}{2}}^{\,2}$

Let $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$, $t \in [0, 2\pi]$

$\Rightarrow r(t) = (r\cos t, r\sin t, r^2) \Rightarrow r'(t) = (-r\sin t, r\cos t, 2r)$.

We get, $\int_C F\,dr = \int_0^{2\pi} (-r\sin t + r^2\cos^2 t + r^2\sin t\cos t)\,dt$

$= \pi\left(\dfrac{-1+\sqrt{5}}{2}\right)$

Therefore, $\boxed{\int_C F\,dr = \left(\dfrac{-1+\sqrt{5}}{2}\right)\pi}$.

6. Show that F is conservative and find the potential function of F. Compute $\int_C F\,dr$ for the given F and curve C from points A to B.

a). $F(x,y) = 2xyi + (1+x^2-y^2)j$; $A=(1,0)$ and $B=(2,3)$.

We can be written as, $F(x,y) = (2xy, 1+x^2-y^2)$.

Let $M(x,y) = 2xy$ and $N(x,y) = 1+x^2-y^2$

$\Rightarrow \frac{\partial M}{\partial y}(x,y) = 2x$ and $\frac{\partial N}{\partial x}(x,y) = 2x$

Showing that, $\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$

That is, F is conservative.

$\Rightarrow \exists f$ such that $F = \nabla f \Leftrightarrow (2xy, 1+x^2-y^2) = (f_x(x,y), f_y(x,y))$

$$\Rightarrow \begin{cases} f_x(x,y) = 2xy & (1) \\ f_y(x,y) = 1+x^2-y^2 & (2) \end{cases}$$

Follow by (1): $\int f_x\,dx = \int 2xy\,dx = x^2y + C(y) \Rightarrow f(x,y) = x^2y + C(y)$

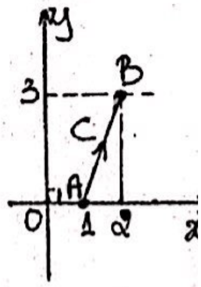
Then, $f_y(x,y) = x^2 + C'(y) = 1+x^2-y^2 \Rightarrow C'(y) = 1-y^2$

We get, $\int C'(y)\,dy = \int (1-y^2)\,dy \Rightarrow C(y) = y - \frac{y^3}{3} + k, k \in \mathbb{R}$.

That is, the potential function of F is

$$f(x,y) = x^2y + y - \frac{y^3}{3} + k,\ k \in \mathbb{R}.$$

1^{st} method:



• We have, $C: y = 3x-3$, $x \in [1,2]$

Or $r(t) = (t, 3t-3) \Rightarrow r'(t) = (1,3)$, $t \in [1,2]$.

and $F(r(t)) = F(t, 3t-3) = \left(2t(3t-3),\ 1+t^2-(3t-3)^2\right)$.

Follow by $\int_C F dr = \int_1^2 \left(2t(3t-3), 1+t^2-(3t-3)^2\right)(1,3)\, dt$

$$= \int_1^2 (6t^2-6t+3+3t^2-27t^2+54t-27)\, dt$$

$$= \int_1^2 (-18t^2+48t-24)\, dt$$

$$= \left[-\frac{18}{3}t^3 + 24t^2 - 24t\right]_1^2$$

$$= -\frac{144}{3} + 96 - 48 + \frac{18}{3} - 24 + 24 = 6$$

- 2^{nd} method:

$$\int_C F dr = f(r(b)) - f(r(a))$$

$$= f(2,3) - f(1,0)$$

$$= 12 + 3 - \frac{27}{3} + k - k$$

$$= 6 \ \checkmark$$

Therefore; $\boxed{\int_C F dr = 6.}$

b). $F(x,y) = (x + \tan^{-1} y)\,i + \dfrac{x+y}{1+y^2}\,j$, $A = (0,0)$ and $B = (1,1)$.

We can be written as, $F(x,y) = \left((x + \tan^{-1} y), \dfrac{x+y}{1+y^2}\right)$.

Let $M(x,y) = (x + \tan^{-1} y)$ and $N(x,y) = \dfrac{x+y}{1+y^2}$

$\Rightarrow \dfrac{\partial M}{\partial y}(x,y) = \dfrac{1}{1+y^2}$ and $\dfrac{\partial N}{\partial x}(x,y) = \dfrac{1}{1+y^2}$

Showing that, $\dfrac{\partial M}{\partial y}(x,y) = \dfrac{\partial N}{\partial x}(x,y)$

That is, F is conservative.

$\Rightarrow \exists f$ such that $\nabla f = F \Leftrightarrow \left(f_x(x,y), f_y(x,y)\right) = \left((x + \tan^{-1} y), \dfrac{x+y}{1+y^2}\right)$

$$\Rightarrow \begin{cases} f_x(x,y) = x + \tan^{-1} y & (1) \\ f_y(x,y) = \dfrac{x+y}{1+y^2} & (2) \end{cases}$$

Follow by (1): $\int f_x\,dx = \int (x + \tan^{-1} y)\,dx = \dfrac{x^2}{2} + x\tan^{-1} y + C(y)$

$\Rightarrow f(x,y) = \dfrac{x^2}{2} + x\tan^{-1} y + C(y) \Rightarrow f_y(x,y) = \dfrac{x}{1+y^2} + C'(y)$ (3)

Follow by ( ) and (3): $\dfrac{x+y}{1+y^2} = \dfrac{x}{1+y^2} + C'(y) \Rightarrow C'(y) = \dfrac{y}{1+y^2}$

Then, $\int C'(y)\,dy = \int \dfrac{y}{1+y^2}\,dy \Leftrightarrow C(y) = \dfrac{1}{2}\ln(1+y^2) + K,\ K \in \mathbb{R}$

That is, the potential function of F is

$$f(x,y) = \frac{x^2}{2} + x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) + K,\ K \in \mathbb{R}.$$

- $\displaystyle\int_C F\,dr = f(r(b)) - f(r(a))$

$$= f(1,1) - f(0,0)$$

$$= \frac{1}{2} + \frac{\pi}{4} + \frac{1}{2}\ln(2) + \cancel{K} - \cancel{K}$$

$$= \frac{1+\ln 2}{2} + \frac{\pi}{4}$$

<u>Therefore</u>, $\boxed{\displaystyle\int_C F\,dr = \frac{1+\ln 2}{2} + \frac{\pi}{4}.}$

c). $F(x,y,z) = 2xyz^2\,i + x^2z^2\,j + 2x^2zy\,k$,

$A = (1,0,1)$ and $B = (1,2,-1)$.

We can be written F as, $F(x,y,z) = (M, N, P)$

- $\dfrac{\partial P}{\partial y}(x,y,z) = 2x^2z$ and $\dfrac{\partial N}{\partial z}(x,y,z) = 2x^2z \Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$ (1)
- $\dfrac{\partial P}{\partial x}(x,y,z) = 4xzy$ and $\dfrac{\partial M}{\partial z}(x,y,z) = 4xyz \Rightarrow \dfrac{\partial P}{\partial x} = \dfrac{\partial M}{\partial z}$ (2)
- $\dfrac{\partial N}{\partial x}(x,y,z) = 2xz^2$ and $\dfrac{\partial M}{\partial y}(x,y,z) = 2xz^2 \Rightarrow \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ (3)

Follow by (1), (2) and (3) : $\text{curl}\,F(x,y,z) = 0$.

That is, F is conservative.

$\Rightarrow \exists f$ such that $\nabla f = F \Leftrightarrow (f_x, f_y, f_z) = (2xyz^2, x^2z^2, 2x^2zy)$.

$$\Rightarrow \begin{cases} f_x(x,y,z) = 2xyz^2 & (1) \\ f_y(x,y,z) = x^2z^2 & (2) \\ f_z(x,y,z) = 2x^2zy & (3) \end{cases}$$

Follow by (1): $f(x,y,z) = x^2yz^2 + C(y,z) \Rightarrow f_y = x^2z^2 + C'(y,z)$ (4)

Follow by (2) and (4): $x^2z^2 = x^2z^2 + C'(y,z) \Rightarrow C'(y,z) = 0$

Then, $\int C'(y,z)\,dy = \int 0\,dy \Rightarrow C(y,z) = C(z)$

$\Rightarrow f(x,y,z) = x^2yz^2 + C(z) \Rightarrow f_z = 2x^2yz + C'(z)$ (5)

Follow by (3) and (5): $2x^2zy = 2x^2yz + C'(z)$

$\Rightarrow C'(z) = 0 \Rightarrow C(z) = K,\ K \in \mathbb{R}$

That is, the potential function of F is

$$f(x,y,z) = x^2yz^2 + K,\ K \in \mathbb{R}.$$

- $\int_C F\,dr = f(r(b)) - f(r(a))$

$= f(1,2,-1) - f(1,0,1)$

$= 2 + \cancel{K} - \cancel{K} = 2.$

Therefore, $\boxed{\int_C F\,dr = 2}$

d). $F(x,y,z) = e^y i + (xe^y + \ln z) j + \frac{y}{z} k$, $A = (0,1,1)$ and $B = (1,0,2)$

We can be written as, $F(x,y,z) = \left(e^y, xe^y + \ln z, \frac{y}{z}\right) = (M, N, P)$

• $\frac{\partial P}{\partial y}(x,y,z) = \frac{1}{z}$ and $\frac{\partial N}{\partial z}(x,y,z) = \frac{1}{z} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ (1)

• $\frac{\partial P}{\partial x}(x,y,z) = 0$ and $\frac{\partial M}{\partial z}(x,y,z) = 0 \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$ (2)

• $\frac{\partial N}{\partial x}(x,y,z) = e^y$ and $\frac{\partial M}{\partial y}(x,y,z) = e^y \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ (3)

Follow by (1), (2), (3) , $\operatorname{curl} F(x,y,z) = 0$

Showing that, F is conservative.

$\Rightarrow \exists f$ such that $F = \nabla f \Leftrightarrow \left(e^y, xe^y + \ln z, \frac{y}{z}\right) = (f_x, f_y, f_z)$.

$$\Rightarrow \begin{cases} f_x(x,y,z) = e^y & (4) \\ f_y(x,y,z) = xe^y + \ln z & (5) \\ f_z(x,y,z) = \frac{y}{z} & (6) \end{cases}$$

Follow by (4) : $\int f_x(x,y,z)\,dx = xe^y + C(y,z) \Rightarrow f(x,y,z) = xe^y + C(y,z)$

Then, $f_y(x,y,z) = xe^y + C'(y,z)$ (7)

Follow by (5) and (7) : $xe^y + \ln z = xe^y + C'(y,z) \Rightarrow C'(y,z) = \ln z$

Then, $C(y,z) = y\ln z + C(z) \Rightarrow f(x,y,z) = xe^y + y\ln z + C(z)$ (*)

$\Rightarrow f_z(x,y,z) = \frac{y}{z} + C'(z)$ (8)

Follow by (6) and (8) : $\frac{y}{z} = \frac{y}{z} + C'(z) \Rightarrow C'(z) = 0 \Rightarrow C(z) = K, K \in$

Follow by (*), $f(x,y,z) = xe^y + y\ln z + K$, $K \in \mathbb{R}$.

Showing that, $f(x,y,z) = xe^y + y\ln z + K$, $K \in \mathbb{R}$ is a potential function of $F(x,y,z)$.

- $\int_C F\,dr = f(r(b)) - f(r(a))$

$= f(1,0,2) - f(0,1,1)$

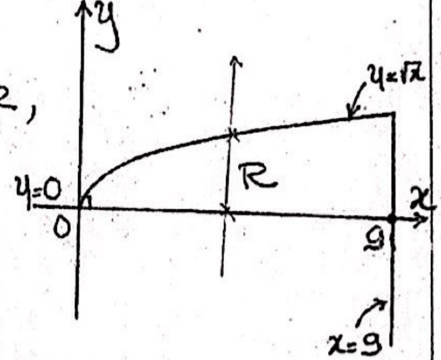
$= 1 + K - K = 1$

Therefore, $\boxed{\int_C F\,dr = 1}$.

7. Use Green's Theorem to evaluate the line integrals.

a). $\int_C 2xy^2 dx - x^2 y dy$, C is the boundary of the region lying between the graph of $y=0$, $y=\sqrt{x}$ and $x=9$.

Since, (C) is a simple closed curve, then by using Green's Theorem,



$$\int_C F dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Let $M(x,y) = 2xy^2$ and $N = -x^2 y$, we get

$$\int_C F dr = \int_0^9 \int_0^{\sqrt{x}} (-2xy - 4xy)\, dy\, dx$$

$$= \int_0^9 \int_0^{\sqrt{x}} (-6xy)\, dy\, dx$$

$$= \int_0^9 \left[-6x \frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= \int_0^9 -3x^2 dx = -729$$

Therefore, $\boxed{\int_C F dr = -729}$

Or we can use "the line integral of vector field".

b). $\int_C 2\arctan\frac{y}{x}\,dx + \ln(x^2+y^2)\,dy$, $C: x = 4+2\cos\theta, y = 4\sin\theta$.

Let $F(x,y) = \left(2\arctan\frac{y}{x}, \ln(x^2+y^2)\right)$.

and $M(x,y) = 2\arctan\frac{y}{x}$ and $N = \ln(x^2+y^2)$.

By using Green's Theorem, we get

$$\int_C F\,dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} - \frac{2\times 1/x}{1+y^2/x^2}\right) dA = 0$$

<u>Therefore</u>, $\boxed{\int_C F\,dr = 0}$

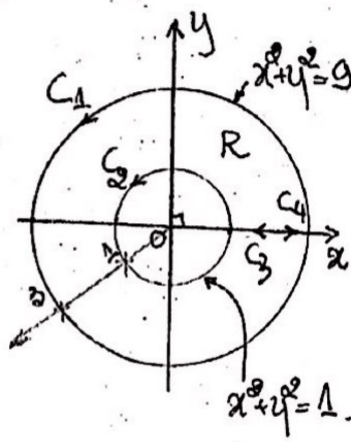
c). $\int_C (x-3y)\,dx + (x+y)\,dy$, C is the region lying between the graphs of $x^2+y^2 = 1$ and $x^2+y^2 = 9$.

By using Green's Theorem,

Let $C = (C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4) - (C_3 \sqcup C_4)$

But, $\int_{C_3 \sqcup C_4} F\,dr = \int_{C_3} F\,dr + \int_{C_4} F\,dr = \int_{C_3} - \int_{C_4} = 0$

$\Rightarrow \int_C F\,dr = \int_{C_1 \sqcup \ldots \sqcup C_4} F\,dr$.



Since, $C_1 \cup \dots \cup C_4$ is closed and R is the region bounded by $C_1 \cup \dots \cup C_4$, we get

$$\int_{C_1 \cup \dots \cup C_4} F\,dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (1+3)\, dA$$

$$= 4\iint_R dA$$

$$= 4\int_0^{2\pi}\int_1^3 r\,dr\,d\theta$$

$$= 32\pi$$

Therefore, $\boxed{\int_C F\,dr = 32\pi}$

* TH (Green's Theorem):

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (M(x,y), N(x,y))$.
Let (C) be a simple closed curve in $\mathbb{R}^2$ and R is a region bounded by (C). Then

$$\oint_C F.dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

8. Find the equation of the tangent plane to the parametric surface represented by r at the specified point.

a). $r(u,v) = (u-v)i + (u-v)j + v^2k$, $(2,0,1)$.

We can be written $r(u,v)$ as $r(u,v) = (u-v, u-v, v^2)$.

- $r_u(u,v) = (1, 1, 0)$ and $r_v(u,v) = (-1, -1, 2v)$

$\Rightarrow r_u(u_0, v_0) = r_u(2,1) = (1,1,0)$

and $r_v(u_0,v_0) = r_v(2,1) = (-1,-1,2)$.

Then, $n = r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = (2,-2,0)$.

Therefore, $\boxed{\text{the equation of a plane is } 2(x-2) - 2y = 0}$.

b). $r(u,v) = u\cos v\, i + 2u\sin v\, j + u^2k$, $(u,v) = (1, \pi)$

We can be written as, $r(u,v) = (u\cos v, 2u\sin v, u^2)$.

- $r_u(u,v) = (\cos v, 2\sin v, 2u)$ and $r_v(u,v) = (-u\sin v, 2u\cos v, 0)$.

$\Rightarrow r_u(u_0, v_0) = r_u(1,\pi) = (-1, 0, 2)$ and $r_v(u_0,v_0) = (0,-2,0)$.

Then, $n = r_u(u_0,v_0) \times r_v(u_0 \times v_0) = \begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{vmatrix} = (4,0,2)$.

Therefore, $\boxed{\text{the equation of a plane is } 4(x+1) + 2(z-1) = 0}$.

9. Find the area of the surface.

a). The part of the plane $r(u,v) = (u+2v-1)i + (2u+3v+1)j + (u,v,2)k$, $0 \leq u \leq 1$; $0 \leq v \leq 2$.

• $r_u(u,v) = (1, 2, 1)$ and $r_v(u,v) = (2, 3, 1)$.

$\Rightarrow r_u \times r_v = (-1, 1, -1) \Rightarrow \| r_u , r_v \| = \sqrt{1+1+1} = \sqrt{3}$.

We get, $\text{Area}(S) = \iint_D \sqrt{41}\, dA = \int_0^1 \int_0^2 \sqrt{3}\, dv du = 2\sqrt{3}$

Therefore, $\boxed{\text{Area}(S) = 2\sqrt{3} \text{ unit area.}}$

b). The part of the plane $z = 8-2x-3y$ that lies inside the cylinder $x^2+y^2 = 4$.

Follow by $\iint_S f\, dS = \iint_R (f(x,y,g(x,y))) \sqrt{1+g_x^2+g_y^2}\, dA$

Let $(S): z = g(x,y) = 8-2x-3y$

• $g_x(x,y) = -2$ and $g_y(x,y) = -3$

$\Rightarrow \sqrt{1+g_x^2+g_y^2} = \sqrt{1+4+9} = \sqrt{14}$

We get, $\text{Area}(S) = \int_0^{2\pi} \int_0^2 \sqrt{14}\, r dr d\theta$

$= \sqrt{14} \times 2\pi \times 2 = 4\pi\sqrt{14}$

Therefore, $\boxed{\text{Area}(S) = 4\pi\sqrt{14} \text{ unit area.}}$

c). The surface $r(u,v) = \sin u \cos v\, i + \sin u \sin v\, j + u k$,

$0 \le u \le \pi$; $0 \le v \le 2\pi$.

- $r_u(u,v) = (\cos u \cos v, \cos u \sin v, 1)$
- $r_v(u,v) = (-\sin u \sin v, \sin u \cos v, 0)$.

$$\Rightarrow r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & 1 \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix}$$

$$= (-\sin u \cos v, -\sin u \sin v, \cos u \sin u)$$

We get, $\text{Area}(S) = \int_0^{2\pi}\int_0^{\pi} \sqrt{\sin^2 u + \sin^2 u \cos^2 u}\, du\, dv$

$$= 2\pi \int_0^{\pi} \sin u \sqrt{1 + \cos^2 u}\, du$$

Let $M = \cos u \Rightarrow dM = -\sin u\, du$

Consequently, $\text{Area}(S) = 2\pi \int_{-1}^{1} \sqrt{1 + M^2}\, dM$

$$\text{Area}(S) = 2\pi\left(\ln(1+\sqrt{2}) + \sqrt{2}\right)$$

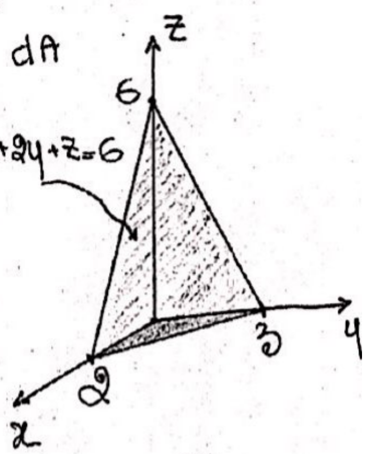
Therefore, $\boxed{\text{Area}(S) = 2\pi\left(\ln(1+\sqrt{2}) + \sqrt{2}\right) \text{ unit area.}}$

$$* \int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left(x + \sqrt{x^2 + a^2}\right) + C.$$

10. Find $\iint_S f(x,y,z)\,dS$.

a). $f(x,y,z) = x+y$, S is the part of the plane $3x+2y+z=6$ in the first octant.

We get, $\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dA$



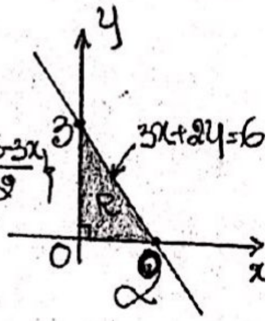
Let $(S): z = g(x,y) = 6-3x-2y$

Since, $g_x = -3$ and $g_y = -2$

$\Rightarrow \sqrt{1+g_x^2+g_y^2} = \sqrt{1+9+4} = \sqrt{14}$

Then, $\iint_S f(x,y,z)\,dS = \iint_R (x+y)\sqrt{14}\,dA$

The region R is given by $R = \{(x,y)\in\mathbb{R}^2 ; 0<x<2,\ 0<y<\frac{6-3x}{2}\}$



$$\Rightarrow \iint_S f(x,y,z)\,dS = \int_0^2\int_0^{\frac{6-3x}{2}} \sqrt{14}\,(x+y)\,dy\,dx$$

$$= \int_0^2 \left[\sqrt{14}\,xy + \frac{\sqrt{14}}{2}y^2\right]_0^{\frac{6-3x}{2}} dx$$

$$= \int_0^2 \sqrt{14}\left(3x-\frac{3}{2}x^2\right) + 2\sqrt{14}\,(6-3x)^2\,dx$$

$$= 5\sqrt{14}$$

Therefore, $\boxed{\iint_S f(x,y,z)\,dS = 5\sqrt{14}.}$

b). $f(x,y,z) = xz$; S is part of the plane $y+z=4$ inside the cylinder $x^2+y^2=4$.

Let $(S): g(x,y) = z = 4-y$

Since; $g_x(x,y) = 0$ and $g_y(x,y) = -1$

$\Rightarrow \sqrt{1+g_x^2+g_y^2} = \sqrt{1+0+1} = \sqrt{2}$

And $f(x,y,g(x,y)) = x(4-y) = 4x - xy$.

We get, $\iint_S f(x,y,z)\,dS = \iint_R (4x-xy)\sqrt{2}\,dA$

By using Polar coordinates, we have

$$\iint_S f(x,y,z)\,dS = \sqrt{2}\int_0^{2\pi}\int_0^2 (4r\cos\theta - r^2\cos\theta\sin\theta)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{32}{3}\cos\theta - 4\cos\theta\sin\theta\right)d\theta$$

$$= \sqrt{2}\left[\frac{32}{3}\sin\theta - 2\sin^2\theta\right]_0^{2\pi}$$

$$= \sqrt{2}\times 0 = 0$$

Therefore, $\boxed{\iint_S f(x,y,z)\,dS = 0}$

c). $f(x,y,z) = x + \dfrac{y}{\sqrt{4z+5}}$; S is the surface with vector representation $r(u,v) = ui + vj + (v^2-1)k$, $0 \le u \le 1$, $-1 \le v \le 1$.

The surface integral of $f(x,y,z)$ over S is given by

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \| r_u(u,v) \times r_v(u,v) \| \, dA.$$

We have, $r(u,v) = (u, v, v^2-1)$

$\Rightarrow r_u(u,v) = (1, 0, 0)$ and $r_v(u,v) = (0, 1, 2v)$

Then, $r_u(u,v) \times r_v(u,v) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = (0, -2v, 1)$

$\Rightarrow \| r_u(u,v) \times r_v(u,v) \| = \sqrt{4v^2+1}$

We get,
$$\iint_S f(x,y,z)\,dS = \iint_D \left(u + \frac{v}{\sqrt{4(v^2-1)+5}} \right) \sqrt{4v^2+1}\, dA$$
$$= \int_{-1}^{1} \int_0^1 \left(u\sqrt{4v^2+1} + v \right) du\,dv$$
$$= \int_{-1}^{1} \left(\frac{1}{2}\sqrt{4v^2+1} + v \right) dv$$
$$= \frac{1}{2} \int_{-1}^{1} \sqrt{4v^2+1}\, dv.$$
$$= \int_0^1 \sqrt{4v^2+1}\, dv.$$

Let $\alpha = 2v \Rightarrow d\alpha = 2dv$

$$\Rightarrow \iint_S f(x,y,z)\,dS = \int_0^2 \frac{1}{2}\sqrt{\alpha^2+1}\,d\alpha.$$

$$= \frac{1}{2}\left[\frac{1}{2}\alpha\sqrt{\alpha^2+1} + \frac{1}{2}\ln\left(\alpha+\sqrt{\alpha^2+1}\right)\right]_0^2$$

$$= \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right)$$

<u>Therefore</u>, $\boxed{\iint_S f(x,y,z)\,dS = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right).}$

d). $f(x,y,z) = z\sqrt{1+x^2+y^2}$, S is the helicoid with vector representation $r(u,v) = u\cos v\, i + u\sin v\, j + vk$, $0 \le u \le 1$, $0 \le v \le 2\pi$

- $r_u(u,v) = (\cos v, \sin v, 0)$ and $r_v(u,v) = (-u\sin v, u\cos v, 1)$.

$\Rightarrow r_u(u,v) \times r_v(u,v) = (\sin v, -\cos v, u\cos^2 v + u\sin^2 v)$.

$\Rightarrow \|r_u(u,v) \times r_v(u,v)\| = \sqrt{1+u^2}$

We get, $\iint_S f(x,y,z)\,dS = \int_0^{2\pi}\int_0^1 \left(v\sqrt{1+u^2} \times \sqrt{1+u^2}\right) du\,dv$

$$= \int_0^{2\pi} \frac{4}{3} v\,dv = \frac{2}{3} \times 4\pi^2 = \frac{8}{3}\pi^2$$

<u>Therefore</u>, $\boxed{\iint_S f(x,y,z)\,dS = \frac{8}{3}\pi^2}$

11. Find the mass of the surface S having the given density mass function.

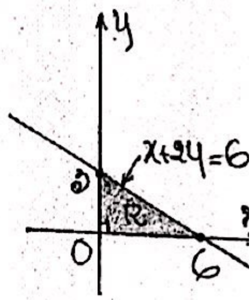
a). S is the part of the plane $x+2y+3z=6$ in the first octant; the density at any point of S is directly proportional to the square of distance between the point to yz-plane.

We get, $\frac{\delta(x,y,z)}{x^2} = k$, $k \in \mathbb{R}$.

$\Rightarrow \delta(x,y,z) = kx^2$. That is, the density of (S).

Then, $\text{mass}(S) = \iint_S \delta(x,y,z)\, dS$

$\Rightarrow \text{mass}(S) = \iint_R kx^2 \sqrt{g_x^2+g_y^2+1}\, dA$



Let $(S): g(x,y) = z = \frac{6-x-2y}{3}$

Since, $g_x = -\frac{1}{3}$ and $g_y = -\frac{2}{3}$

$\Rightarrow \sqrt{1+g_x^2+g_y^2} = \sqrt{1+ 1/9 + 4/3^2} = \frac{\sqrt{14}}{3}$

The region R is given by $R = \{(x,y) \in \mathbb{R}^2 : 0<x<6, 0<y<\frac{6-x}{2}\}$

We get, $\text{mass}(S) = \int_0^6 \int_0^{\frac{6-x}{2}} kx^2 \times \frac{\sqrt{14}}{3}\, dy\, dx$

$= \frac{\sqrt{14}}{3} k \int_0^6 \frac{1}{2}(x^2 \times 6 - x^3)\, dx = \underline{18\sqrt{14}\, k}$, $k \in \mathbb{R}$

b). S is the hemisphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$, the density at any point on S is directly proportional to the distance between the point to xy-plan.

By using hypothesis above, $\frac{\delta(x,y,z)}{z}=k$, $k\in\mathbb{R}$.

$\Rightarrow \delta(x,y,z)=kz$ (density of (S)).

Then, $\text{mass}(S)=\iint_S \delta(x,y,z)\,dS$

$\Rightarrow \text{mass}(S)=\iint_R kz\sqrt{g_x^2+g_y^2+1}\,dA$

Let (S): $g(x,y)=z=\sqrt{1-x^2-y^2}$

$$\Rightarrow g_x(x,y)=\frac{-x}{\sqrt{1-x^2-y^2}} \text{ and } g_y(x,y)=\frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \sqrt{g_x^2+g_y^2+1}=\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}=\frac{1}{\sqrt{1-x^2-y^2}}$$

We have, $\left\{\begin{array}{l} z=\sqrt{x^2+y^2} \\ z^2+x^2+y^2=1 \end{array}\right) \Rightarrow x^2+y^2=\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2$.

So, the region R is given by $R=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2\leq\left(\frac{1}{\sqrt{2}}\right)^2\}$

By using Polar Coordinates, we have

$$\text{mass}(S)=\int_0^{2\pi}\int_0^{1/\sqrt{2}} kr\,dr\,d\theta=2\pi\cdot k\tfrac{1}{2}\cdot\tfrac{1}{2}=\frac{\pi}{2}k,\ k\in\mathbb{R}.$$

Therefore, $\boxed{\text{mass}(S)=\frac{\pi}{2}k,\ k\in\mathbb{R}.}$

12. Find $\iint_S F\cdot N\,dS$, that is, find the flux of F across S. If S is closed, use the positive (outward) orientation.

a). $F(x,y,z)=(2x,2y,z)$; S is the part of the paraboloid $z=4-x^2-y^2$ above the xy-plane; N point upward.

Follow by $\iint_S F\cdot N\,dS=\iint_R(-Mg_x-Ng_y+P)\,dA$

We have, $F(x,y,z)=(2x,2y,z)$ where $M=2x$, $N=2y$ and $P=z$.

Let $(S): g(x,y)=z=4-x^2-y^2$

Since, $g_x(x,y)=-2x$ and $g_y(x,y)=-2y$.

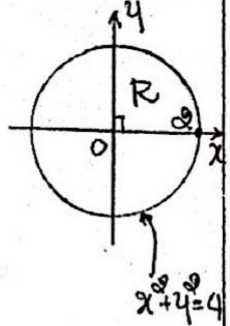
We get, $\iint_S F\cdot N\,dS=\iint_R(4x^2+4y^2+4-x^2-y^2)\,dA$

$$=\iint_R(3x^2+3y^2+4)\,dA$$

$$=\int_0^{2\pi}\int_0^2(3r^2+4)\,r\,dr\,d\theta$$

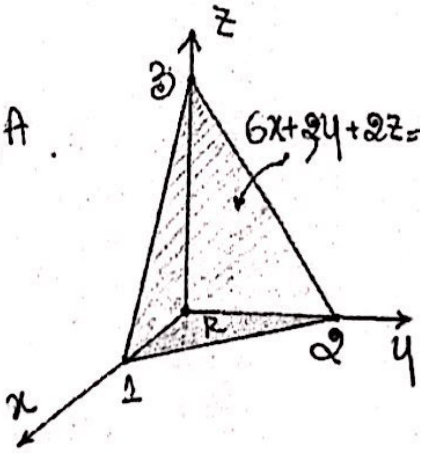
$$=2\pi\left[\frac{3}{4}r^4+2r^2\right]_0^2$$

$$=40\pi.$$



Therefore, $\boxed{\iint_S F\cdot N\,dS=40\pi.}$

b). $F(x,y,z) = (x^2, xy, xz)$; S is the surface of the surface of tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.



We get, $\iint_S F\cdot N\,dS = \iint_R (-Mg_x - Ng_y + P)\,dA$.

We have, $F(x,y,z) = (x^2, xy, xz)$.

Let $M(x,y,z) = x^2$, $N = xy$ and $P = xz$

Let $g(x,y) = z = \frac{1}{2}(6-6x-3y)$

$\Rightarrow g_x(x,y) = -3$ and $g_y(x,y) = -\frac{3}{2}$

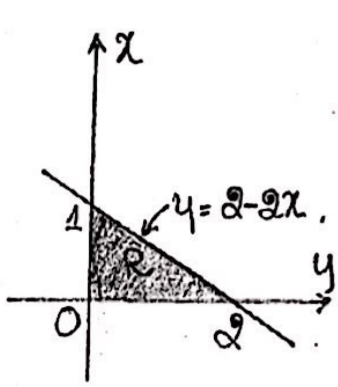
We get, $\iint_S F\cdot N\,dS = \iint_R \left(3x^2 + \frac{3}{2}xy + x\left(3-3x-\frac{3}{2}y\right)\right) dA$

$\Rightarrow \iint_S F\cdot N\,dS = \iint_R 3x\,dA$

The region R is given by $R = \{(x,y)\in\mathbb{R}^2 : 0<x<1 ; 0<y<2-2x\}$

We get, $\iint_S F\cdot N\,dS = \int_0^1\int_0^{2-2x} 3x\,dy\,dx$

$= \int_0^1 3/2\,[x^2]_0^{2-2x}\,dx$



$= \int_0^1 3/2\,(4-8x+4x^2)\,dx$

$= 3/2\left[4x - 4x^2 + \frac{4}{3}x^3\right]_0^1$

$= 2$.

Therefore, $\boxed{\iint_S F\cdot N\,dS = 2.}$

13. [Electric Charge] Find the total charge on the part of the hemisphere $z = \sqrt{25-x^2-y^2}$ that lies directly above the plane region $R = \{(x,y) : x^2+y^2 \leq 9\}$ If the charge density at any point on the surface is directly proportional to the distance between the point and xy-plane.

By using the above hypothesis : $\dfrac{\delta(x,y,z)}{z} = k$, $k \in \mathbb{R}$.

Then, $\delta(x,y,z) = kz$ (charge density).

Let $Q = \iint_S f(x,y,z)\,dS = \iint_R kz\sqrt{1+g_x^2+g_y^2}\,dA$

Let $g(x,y) = z = \sqrt{25-x^2-y^2}$

Then, $g_x(x,y) = \dfrac{-x}{\sqrt{25-x^2-y^2}}$ and $g_y(x,y) = \dfrac{-y}{\sqrt{25-x^2-y^2}}$

$\Rightarrow \sqrt{1+g_x^2+g_y^2} = \sqrt{1+\dfrac{x^2}{25-x^2-y^2}+\dfrac{y^2}{25-x^2-y^2}} = \dfrac{5}{g(x,y)}$

we get, $Q = \iint_R k\cancel{z} \times \dfrac{5}{\cancel{g(x,y)}}\,dA$

$= 5k\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$ (Polar coordinates).

$= 5k \times \cancel{2}\pi \times \dfrac{1}{\cancel{2}}[r^2]_0^3 = 45\pi k$, $k \in \mathbb{R}$.

Therefore, $\boxed{Q = 45\pi k,\ k \in \mathbb{R}}$

14. [Flow of a Fluid] The flow of a fluid is described by the vector field $F(x,y,z) = (2x, 2y, 3z)$. Find the rate of flow of the fluid upward through the surface S that is the part of the plane $x+3y+z=6$ in the first octant.

Let $R = \iint_S F \cdot N \, dS = \iint_R (-Mg_x - Ng_y + P)\, dA$.

We have, $F(x,y,z) = (2x, 2y, 3z)$

Let $M(x,y,z) = 2x$, $N = 2y$ and $P = 3z$.

Let $g(x,y) = z = 6 - x - 3y$.

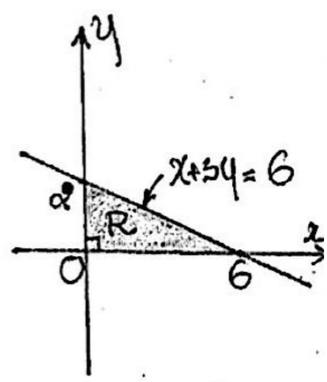
$\Rightarrow g_x(x,y) = -1$ and $g_y(x,y) = -3$.

We get; $R = \iint_R (2x + 6y + 3(6 - x - 3y))\, dA$

The region R is given by $R = \{(x,y) \in \mathbb{R}^2;\ 0 \leqslant x \leqslant 6,\ 0 \leqslant y \leqslant 2 - \frac{x}{3}\}$

$$\Rightarrow R = \int_0^6 \int_0^{2-x/3} (-x - 3y + 18)\, dy\, dx$$

$$= \int_0^6 \left[-xy - \frac{3}{2}y^2 + 18y\right]_0^{2-x/3} dx$$



$$= \int_0^6 \left(\cancel{-2x} + \frac{x^2}{3} - 6 + \cancel{2x} - \frac{x^2}{6} + 36 - 6x\right) dx$$

$$= \left[\frac{1}{18}x^3 - 3x^2 + 30x\right]_0^6 = 84$$

Therefore, $\boxed{R = 84}$

19. [Flow of a Liquid] The flow of a liquid is described by the vector field $F(x,y,z) = (x, y, 3z)$. If the mass density of the fluid is 1000 (in appropriate units), find the rate of flow (mass per unit time) upward of the liquid through the surface S that is part of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.

Let $R = \iint_S \delta F.N\,dS = \iint_R \delta(-Mg_x - Ng_y + P)\,dA$, $\delta = 1000$

Let $g(x,y) = z = 9 - x^2 - y^2 \Rightarrow g_x(x,y) = -2x$ and $g_y(x,y) = -2y$.

We have, $F(x,y,z) = (x, y, 3z) = (M, N, P)$.

We get, $R = \iint_R 1000\left(2x^2 + 2y^2 + 3(9 - x^2 - y^2)\right) dA$

The region R is given by $R = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq 9\}$

By using Polar coordinates,

$$R = 1000 \int_0^{2\pi}\int_0^3 (-r^2 + 27)\, r\,dr\,d\theta .$$

$$= 1000 \times 2\pi \left[-\frac{r^4}{4} + \frac{27}{2}r^2\right]_0^3$$

$$= 2000\pi\left(\frac{405}{4}\right) = 202\,500\pi$$

Therefore, $\boxed{R = 202\,500\pi.}$

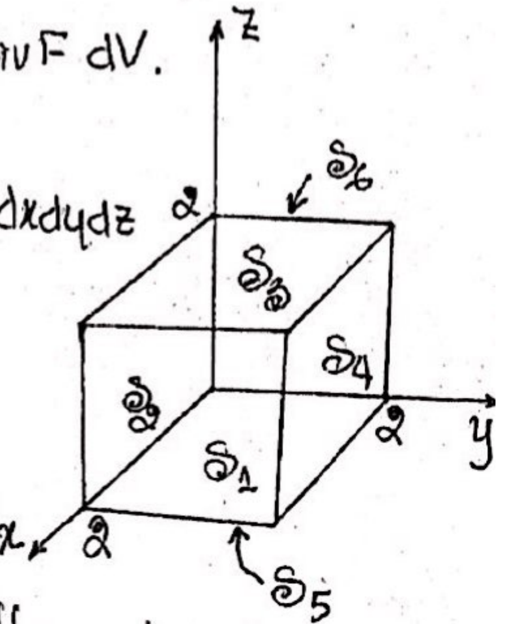
16. Verify the Divergence Theorem by evaluating $\iint_S F.n\,dS$.

a). $F(x,y,z) = (2xy, -y^2, 3yz)$; S is the cube bounded by the plane $x=0$, $x=2$, $y=0$, $y=2$, $z=0$ and $z=2$.

• Divergence Theorem : $\iint_S F.n\,dS = \iiint_Q \operatorname{div} F\,dV$.

$$\iiint_Q \operatorname{div} F\,dV = \int_0^2\int_0^2\int_0^2 (\cancel{2y}-\cancel{2y}+3z)\,dx\,dy\,dz$$

$$= 2\times 2\times 3\times \frac{4}{2} = 24.$$



• Evaluate $\iint_S F.n\,dS$.

We get, $\iint_S F.n\,dS = \iint_{S_1} F.n_1\,dS_1 + \iint_{S_2} F.n_2\,dS_2 + \iint_{S_3} F.n_3\,dS_3 + \iint_{S_4} F.n_4\,dS_4$

$+ \iint_{S_5} F.n_5\,dS_5 + \iint_{S_6} F.n_6\,dS_6$

⊕ For $\iint_{S_1} F.n_1\,dS_1$

We have, (S_1): $z = g_1(x,y) = 0$, $n_1 = (0,0,-1)$

$\Rightarrow \iint_{S_1} F.n_1\,dS_1 = 0$

⊕ For $\iint_{S_2} F\cdot n_2\,dS_2$

We have, (S_2): $y = g_2(x,z) = 0$, $n_2 = (0,-1,0)$

$\Rightarrow \iint_{S_1} F\cdot n_2\,dS_2 = 0$

⊕ For $\iint_{S_5} F.N_5\, dS_5$.

We have, $(S_5): z = g_5(x,y) = 0$, $N_5 = (0,0,-1)$

$\Rightarrow \iint_{S_5} F.N\, dS_5 = 0$

⊕ For $\iint_{S_4} F.N_4\, dS_4$

We have, $(S_4): z = g(x,y) = 0$, $N_4 = (0,0,-1)$

$\Rightarrow \iint_{S_4} F.N_4\, dS_4 = 0$

⊕ For $\iint_{S_3} F.N_3\, dS_3$

We have, $(S_3): z = g_3(x,y) = 1$, $N_3 = (0,0,1)$

$\Rightarrow \iint_{S_3} F.N_3\, dS_3 = \iint_R 3y\, dA = \int_0^2\int_0^2 3y\, dx\, dy = 12$.

⊕ For $\iint_{S_6} F.N_6\, dS_6$.

We have, $(S_6): z = g_6(x,y) = 1$, $N_6 = (0,0,1)$

$\Rightarrow \iint_{S_6} F.N\, dS_6 = \iint_R 3y\, dA = \int_0^2\int_0^2 3y\, dx\, dy = 12$.

Consequently, $\iint_S F.N\, dS = 12 + 12 = 24$

<u>Therefore</u>, $\boxed{\iint_S F.N\, dS = 24}$.

b). $F(x,y,z) = (xz, zy, 2z^2)$; S is the surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

- Divergence Theorem : $\iint_S F \cdot N\, dS = \iiint_Q \operatorname{div} F\, dV$

$$\iiint_Q \operatorname{div} F\, dV = \iint_R \int_0^{1-x^2-y^2} 6z\, dz\, dA .$$

By using Cylindrical coordinates, we have

$$\iiint_Q \operatorname{div} F\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6z\, r\, dz\, dr\, d\theta .$$

$$= 2\pi \int_0^1 r \times 3(1-r^2)^2\, dr$$

$$= 2\pi \times 3 \left[(1-r^2)^3\right]_0^1 \left(-\tfrac{1}{6}\right) = \pi$$

- Evaluate $\iint_S F \cdot N\, dS$

Let (S) : $z = g(x,y) = 1 - x^2 - y^2$

$\Rightarrow g_x(x,y) = -2x$ and $g_y(x,y) = -2y$.

We get, $\iint_S F \cdot N\, dS = \iint_R \left(2x^2(1-x^2-y^2) + 2y^2(1-x^2-y^2) + 2(1-x^2-y^2)^2\right) dA$

By using Polar coordinates, we have

$$\iint_S F \cdot N\, dS = \int_0^{2\pi}\int_0^1 (2 - 2r^2)\, r\, dr\, d\theta = 2\pi\left(1 - \frac{1}{2}\right) = \pi .$$

Therefore, $\boxed{\iint_S F \cdot N\, dS = \iiint_Q \operatorname{div} F\, dV = \pi .}$

17. Use the Divergence Theorem to find the flux of F across S, that is, calculate $\iint_S F.N\,dS$.

a). $F(x,y,z) = (xy^2, 2yz, -3x^2y^3)$; S is the surface of the cube bounded by the planes $x = \pm 1, y = \pm 1, z = \pm 1$.

We have, $\operatorname{div} F = y^2 + 2z$

Follow by
$$\iint_S F.N\,dS = \iiint_Q \operatorname{div} F\,dV$$
$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(y^2+2z)\,dx\,dy\,dz$$
$$= \int_{-1}^{1}\int_{-1}^{1} 2(y^2+2z)\,dy\,dz$$
$$= \int_{-1}^{1}\left[\frac{2}{3}y^3+4zy\right]_{-1}^{1} dz$$
$$= \int_{-1}^{1}\left(\frac{2}{3}+4z+\frac{2}{3}+4z\right)dz$$
$$= \left[\frac{4}{3}z+4z^2\right]_{-1}^{1}$$
$$= \frac{4}{3}+\cancel{4}+\frac{4}{3}-\cancel{4} = \frac{8}{3}$$

Therefore, $\boxed{\iint_S F.N\,dS = \frac{8}{3}}$.

b). $F(x,y,z) = \left(x+1,\ yz^2+\cos xz,\ 2y^2z+e^{\tan x}\right)$;

S is the sphere $x^2+y^2+z^2=1$.

We have, $\operatorname{div} F = 1+z^2+2y^2$

We get, $\iint_S F.n\,dS = \iiint_Q \operatorname{div} F\, dV$.

By using Spherical Coordinates,

We have $\iint_S F.n\,dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(1+\rho^2\cos^2\varphi+2\rho^2\sin^2\varphi\cos^2\theta\right)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$.

$$\Rightarrow \iint_S F.n\,dS = \int_0^{2\pi}\int_0^{\pi}\left(\tfrac{1}{3}\sin\varphi+\frac{1}{5}\cos^2\varphi\sin\varphi+\frac{2}{5}\sin^3\varphi\cos^2\theta\right)d\varphi\, d\theta.$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\cos\varphi-\frac{1}{15}\cos^3\varphi+\frac{2}{5}\cos^2\theta\left(-\cos\varphi+\frac{\cos^3\varphi}{3}\right)\right]_0^{\pi}d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{3}+\frac{1}{15}+\frac{4}{15}\cos^2\theta+\frac{1}{3}+\frac{1}{15}+\frac{4}{15}\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{4}{5}+\frac{8}{15}\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{4}{5}+\frac{4}{15}+\frac{4}{15}\cos 2\theta\right)d\theta$$

$$= \left[\frac{16}{15}\theta+\frac{2}{15}\sin 2\theta\right]_0^{2\pi} = \frac{16}{15}2\pi = \frac{32}{15}\pi$$

Therefore, $\boxed{\iint_S F.n\,dS = \frac{32}{15}\pi}$

c). $F(x,y,z)=(xz, x^2y, y^2z+1)$; S is the surface region that lies between the cylinder $x^2+y^2=1$ and $x^2+y^2=4$ and between the planes $z=1$ and $z=3$.

We have, $\operatorname{div} F = z + x^2 + y^2$

We get, $\iint_S F.n\, dS = \iiint_Q \operatorname{div} F\, dV = \iint_R \int_1^3 (z+x^2+y^2)\, dz\, dA$.

The region R is given by $R=\{(x,y)\in\mathbb{R}^2 : 1 \leq x^2+y^2 \leq 4\}$

By using Polar coordinates, we have

$$\iint_S F\cdot n\, dS = \int_0^{2\pi}\int_1^2\int_1^3 (z+r^2)\, r\, dz\, dr\, d\theta$$

$$= 2\pi \int_1^2 \left[\frac{z^2}{2}r + r^3 z\right]_1^3 dr$$

$$= 2\pi \int_1^2 \left(\frac{9}{2}r + 3r^3 - \frac{1}{2}r - r^3\right) dr$$

$$= 2\pi \left[2r^2 + \frac{1}{2}r^4\right]_1^2$$

$$= 2\pi\left(8+8-2-\frac{1}{2}\right) = -27\pi$$

Therefore, $\boxed{\iint_S F.n\, dS = -27\pi}$

18. a). Use the Divergence Theorem to verify that the solid bounded by a surface S is

$$\iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy .$$

- Let $f(x,y,z) = (x,0,0)$ and $n = (1,0,0)$.

We get, $\iint_S F.n\,dS = \iint_S x\,dy\,dz = \iiint_Q \text{div}F\,dv = V \quad (1)$

- Let $f(x,y,z) = (0,y,0)$ and $n = (0,1,0)$,

We get, $\iint_S F.n\,dS = \iint_S y\,dx\,dz = \iiint_Q \text{div}F\,dv = V \quad (2)$

- Let $f(x,y,z) = (0,0,z)$ and $n = (0,0,1)$.

We get, $\iint_S F.n\,dS = \iint_S z\,dx\,dy = \iiint_Q \text{div}F\,dv = V \quad (3)$.

Follow by (1), (2) and (3): $\iint_S x\,dy\,dz = \iint_S y\,dx\,dz = \iint_S z\,dx\,dy$.

<u>Therefore</u>, $\boxed{\iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy.}$

b). Verify that the result of part (a) for the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$ and $z=a$.

By using the result of part (a) for the cube:

- Follow by (1): $V=\iint_S x\,dy\,dz=\int_0^a\int_0^a a\,dy\,dz=a^3$.
- Follow by (2): $V=\iint_S y\,dx\,dz=\int_0^a\int_0^a a\,dx\,dz=a^3$.
- Follow by (3): $V=\iint_S z\,dx\,dy=\int_0^a\int_0^a a\,dx\,dy=a^3$.

And also, $V_{cube}=a\times a\times a=a^3$.

Therefore $\boxed{\text{the result of part (a) is verified.}}$

19. Verify that $\iint_S \text{curl}\,F \cdot N \, dS = 0$ for any closed surface S.

By using Divergence Theorem:

$$\iint_S \text{curl}\,F \cdot N \, dS = \iiint_Q \text{div}(\text{curl}\,F) \, dV.$$

Since, $\text{curl}\,F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

$$\Rightarrow \text{curl}\,F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\frac{\partial P}{\partial x} + \frac{\partial M}{\partial z}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right).$$

$$\Rightarrow \text{div}(\text{curl}\,F) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial N}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$= 0$$

Therefore, $\boxed{\iint_S \text{curl}\,F \cdot N \, dS = 0.}$

20. Given the vector field $F(x,y,z)=(x,y,z)$, verify that $\iint_S F\cdot n\,ds = 3V$, where V is the volume of the solid bounded by the closed surface S.

By using Divergence Theorem :

$$\iint_S F.n\,ds = \iiint_Q \operatorname{div} F\,dV = \iiint_Q 3\,dV = 3V.$$

Therefore, $\boxed{\iint_S F.n\,ds = 3V}$

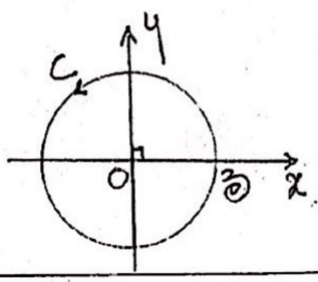
21. Verify Stokes's Theorem by evaluating $\oint_C F\cdot T\,ds = \oint_C F.dr$ as a line integral and as a double integral.

a). $F(x,y,z) = (-y+z,\ x-z,\ x-y)$; S is part of the paraboloid $z = 9-x^2-y^2$ and $z \geq 0$.

• First, we find $\oint_C F\,dr$.

We have, $z = 9-x^2-y^2$

Since, $z=0 \Rightarrow x^2+y^2 = 9$.



We get, $r(t) = (3\cos t, 3\sin t, 0)$, $t \in [0, 2\pi]$.

$\Rightarrow r'(t) = (-3\sin t, 3\cos t, 0)$

and $F(r(t)) = (-3\sin t, +3\cos t, -3\sin t - 3\cos t)$.

Follow by, $\oint_C F dr = \int_0^{2\pi} (9\sin^2 t + 9\cos^2 t)\, dt$

$$= 9 \times 2\pi = 18\pi \quad (a)$$

- Second, we find $\oint_C F.T ds$ as finding $\iint_S (\text{curl} F).N ds$.

We have, $F(x, y, z) = (-y + z, x - z, x - y)$.

Since, $\text{curl} F = \nabla \times F(x, y, z) = (-1+1, -1+1, 1+1) = (0, 0, 2)$.

$\Rightarrow \text{curl} F.N = 2$, $N = (2x, 2y, 1)$.

We get, $\iint_S (\text{curl} F).N ds = \int_0^{2\pi} \int_0^3 2r dr d\theta$. (Polar Coordinates)

$$= 2\pi \times 2/2 [r^2]_0^3$$

$$= 2\pi \times 9 = 18\pi \quad (b)$$

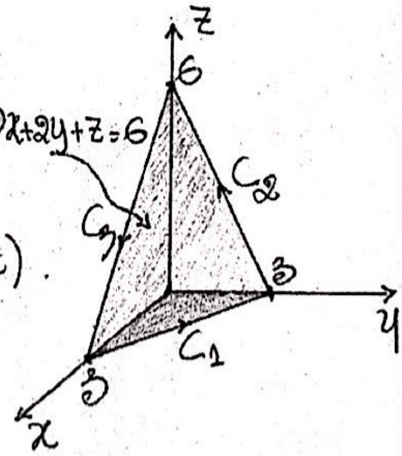
Follow by (a) and (b) : Stokes's Theorem is verified.

Therefore, $\boxed{\text{Stokes's Theorem is verified.}}$

b). $F(x,y,z)=(y,z,x)$, S is part of the plane $2x+2y+z=6$ lying in the first octant.

⊙ First, we find $\oint_C F dr$.

We get, $\oint_C F dr = \int_{C_1} F dr + \int_{C_2} F dr + \int_{C_3} F dr$.



• $C_1 : r_1(t) = (t, 3-t, 0)$, $t \in [3,0]$.

$\Rightarrow r_1'(t) = (1, -1, 0)$ and $F(r_1(t)) = (3-t, 0, t)$.

We get, $\int_{C_1} F dr = \int_3^0 (3-t)\, dt = \frac{-9}{2}$

• $C_2 : r_2(t) = (0, t, 6-2t)$, $t \in [3,0]$

$\Rightarrow r_2'(t) = (0, 1, -2)$ and $F(r_2(t)) = F(0, t, 6-2t) = (t, 6-2t, 0)$

We get, $\int_{C_2} F dr = \int_3^0 (6-2t)\, dt = [6t - t^2]_3^0 = -(18-9) = -9$.

• $C_3 : r_3(t) = (t, 0, 6-2t)$, $t \in [0,3]$.

$\Rightarrow r_3'(t) = (1, 0, -2)$ and $F(r_3(t)) = F(t, 0, 6-2t) = (0, 6-2t, t)$.

We get, $\int_{C_3} F dr = \int_0^3 -2t\, dt = -[t^2]_0^3 = -9$.

Consequently, $\oint_C F dr = \frac{-9}{2} - 9 - 9 = \frac{-45}{2}$ (a).

⊙ Second, we find $\oint F.T\, ds$ as finding $\iint_S (\text{curl } F).N\, ds$.

We have, $F(x,y,z) = (y, z, x)$

Since, $\text{curl} F = \nabla \times F(x,y,z) = (-1,-1,-1)$.

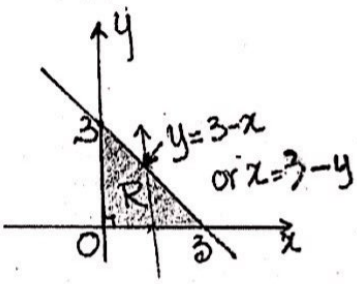
and $r(x,y) = (x, y, 6-2x-2y)$.

$\Rightarrow r_x(x,y) = (1,0,-2)$ and $r_y(x,y) = (0,1,-2)$.

We get, $n = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = (2,2,1)$

Then, $\iint_S (\text{curl} F)\cdot n \, dS = \iint_R (-2-2-1)\, dA$

$$= \int_0^3 \int_0^{3-y} (-5)\, dx\, dy$$



$$= \int_0^3 -5(3-y)\, dx$$

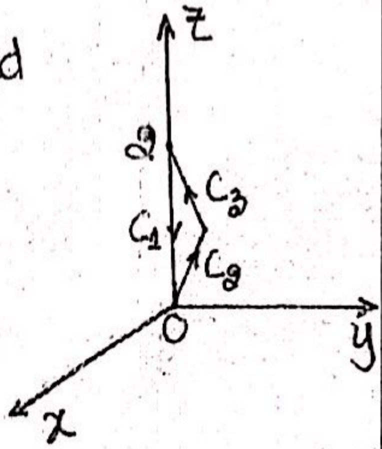
$$= -\frac{45}{2} \quad (b).$$

Follow by (a) and (b): Stokes's Theorem is verified.

Therefore, $\boxed{\text{Stokes's Theorem is verified.}}$

22. Use Stokes's Theorem to evaluate $\int_C F\,dr$.

a). $F(x,y,z) = \left(\arctan\frac{y}{x}\,;\ln x^2+y^2\,;1\right)$; C is a triangle with vertices $(0,0,0)$, $(1,1,1)$ and $(0,0,2)$.



Since, $C = C_1 \cup C_2 \cup C_3$ is a simple closed curve, then by using Stokes's Theorem,

we have $\oint_C F\,dr = \iint_S (\text{curl } F)\cdot n\, ds$

- $\text{curl } F = \nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan y/x & \ln x^2+y^2 & 1 \end{vmatrix}$

$\Rightarrow \text{curl } F = \left(0\,,\,0\,,\,\frac{2x}{x^2} - \frac{x}{x^2+y^2}\right)$

We know that the equation of plane is $x-y=0 \Leftrightarrow y-x$

Let (S): $y = g(x,z) = x$, $(x,z)\in\mathbb{R}^2$

$\Rightarrow n = (-g_x, 1, -g_z) = (-1, 1, 0)$.

We get, $\oint_C F\,dr = \iint_S 0\, ds = 0$

Therefore, $\boxed{\oint_C F\,dr = 0.}$

c). $F(x,y,z) = \left(-\ln\sqrt{x^2+y^2}, \arctan\frac{x}{y}, 1\right)$, S is the surface of $z = 9-2x-3y$ over $r = 2\sin\theta$ in the first octant.

- $\text{curl} F = \nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\ln\sqrt{x^2+y^2} & \arctan\frac{x}{y} & 1 \end{vmatrix} = \left(0, 0, \frac{2y}{x^2+y^2}\right)$.

Let (S): $g(x,y) = z = 9-2x-3y$

$\Rightarrow N = (-g_x, -g_y, 1) = (2, 3, 1)$

By using Stoke's Theorem, we have

$$\oint_C F\,dr = \iint_S (\text{curl} F)\cdot N\,dS$$

$$= \iint_R \frac{2y}{x^2+y^2}\,dA$$

By using Polar coordinates, we have

$$\oint_C F\,dr = \int_0^{\pi/2}\int_0^{2\sin\theta} 2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} 4\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/2} (2 - 2\cos 2\theta)\,d\theta$$

$$= \pi - 1$$

Therefore, $\boxed{\oint_C F\,dr = \pi - 1}$

d) $F(x,y,z) = (xyz, y, z)$; S is the surface of $z = x^2$, $0 \le x \le a$, $0 \le y \le a$. N is the downward unit normal to the surface.

• $\text{curl} F = \nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = (0, xy, -xz)$

Let (S): $g(x,y) = z = x^2$

$\Rightarrow N = (-g_x, -g_y, -1) = (-2x, 0, -1)$

By using Stoke's Theorem, we have

$$\oint_C F\,dr = \iint_S (\text{curl} F) \cdot N\,dS$$

$$= \iint_R x^3\,dA$$

$$= \int_0^a \int_0^a x^3\,dy\,dx$$

$$= \int_0^a x^3 a\,dx$$

$$= \frac{a}{4}\left[x^4\right]_0^a = \frac{a^5}{4}$$

<u>Therefore</u>; $\boxed{\oint_C F\,dr = \frac{a^5}{4}}$