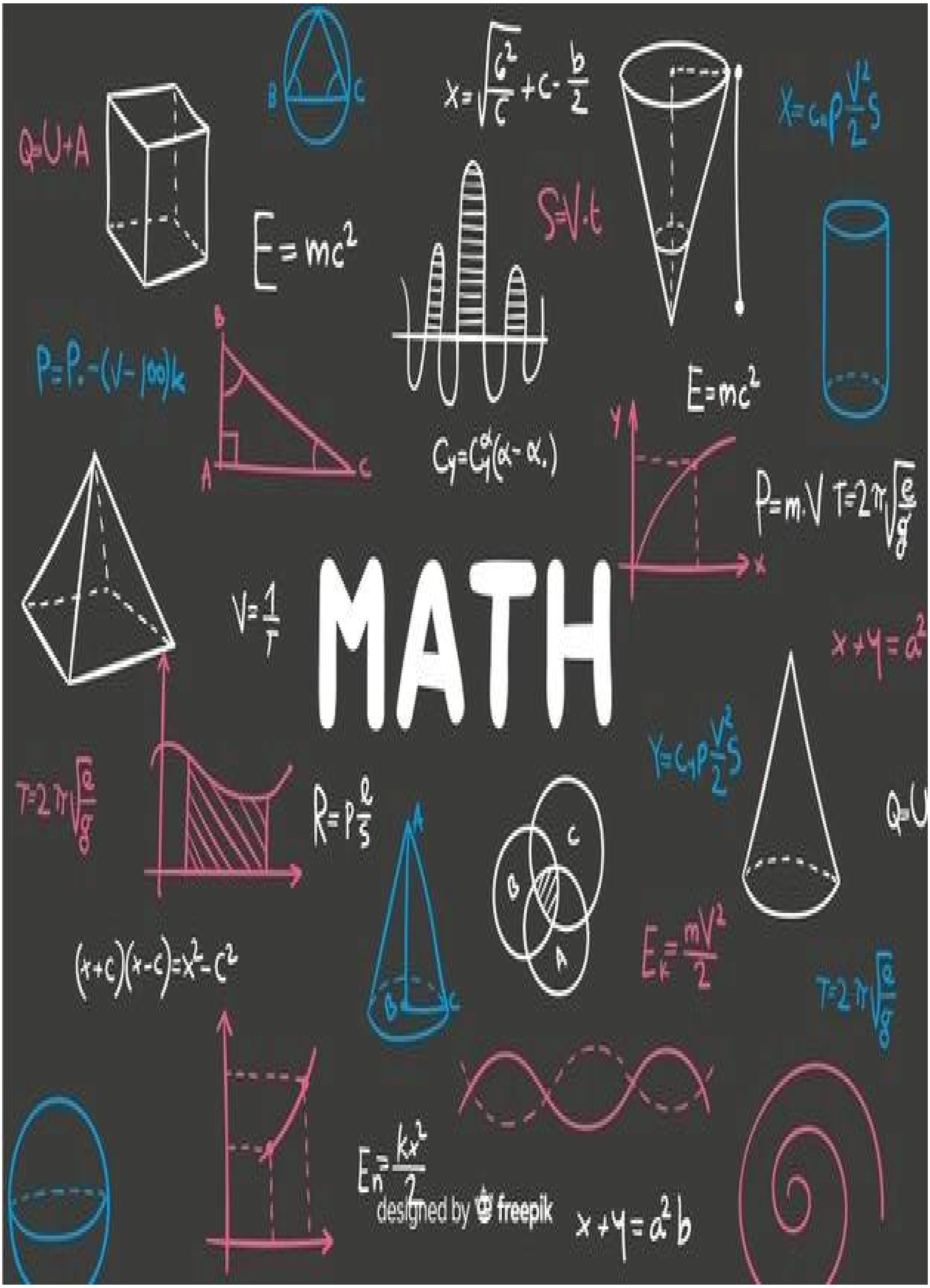


Taing Kimmeng
CalculusIII & Linear Algebra
TD3&4 (Function of Several Varaiable)

I2–TD3
(Function of Several Variables)

1. Sketch the level curves $f(x, y) = c$ and the level surfaces $f(x, y, z) = c$ of the functions for the indicated values of c.

   (a) $f(x, y) = y^2 - x^2;\quad c = 0, \pm 1, \pm 2, \pm 3$　　(b) $f(x, y, z) = 4x^2 + 4y^2 - z^2;\quad c = 0, 1$

2. Find the limit (if it exists) of the following functions.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - \cos(xy)}{xy^2}$

   (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2 + yz^2}{x^2 + 2y^2 + 3z^2}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sinh x^2 + \sinh y^2}$

   (e) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\ln\sqrt{1 + \sqrt{x^2 + (y-1)^4}}}{\sin\sqrt{x^2 + (y-1)^4}}$

   (f) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{\sin x}{x}(y + 1), \frac{\sin(x^2) + \sin(y^2)}{\sqrt{x^2 + y^2}}\right)$

3. Let $\alpha \in \mathbb{R}$. Determine the value of α so that the function f has limit at $(0, 0)$.

$$f(x, y) = \frac{x^\alpha y}{x^2 + y^2}.$$

4. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ be positive real numbers and $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \dfrac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1} + |y|^{\beta_2})^\gamma}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

   Show that f is continuous at $(0, 0)$ if and only if $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} > \gamma$.

5. Let $\alpha > 0$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{|x|^\alpha y}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (0, 0) \end{cases}$$

   (a) Show that $\forall (x, y) \neq (0, 0)$:

$$|f(x, y)| \leq (x^2 + y^4)^{\frac{2\alpha - 3}{4}}$$

   (b) Compute $\displaystyle\lim_{y\to 0} |f(y^2, y)|$.

   (c) Study the continuity of f at $(0, 0)$.

   (d) Show that

$$\forall (x, y) \neq (0, 0) : \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} \leq |x|^{\alpha - 2}$$

   (e) Calculate $\displaystyle\lim_{x\to 0} \frac{|f(x, x)|}{\sqrt{2}|x|}$

(f) Study the differentiability of f at $(0,0)$.

6. Determine the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\frac{\partial f}{\partial x}(x,y) = \frac{2-y}{x+y+1} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{1+x}{x+y+1}$$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} (x^2+y^2)^x, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (0,0) \end{cases}$$

(a) Is f continuous at $(0,0)$?

(b) Find the partial derivatives of f at the point different from the origin.

(c) Do the partial derivatives of f exist at the point $(0,0)$?

8. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is in class C^1

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$

10. Suppose that $w = f(u,v)$ is a differentiable function of $u = \dfrac{y-x}{xy}$ and $v = \dfrac{z-x}{xz}$. Show then that:

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0.$$

11. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \arcsin\left(\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}\right) \quad \text{and} \quad g(x,y) = \arctan x - \arctan y.$$

(a) Calculate the partial derivative of f and g.

(b) Simplify f.

12. Find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ by using the appropriate Chain Rule.

(a) $f(x,y) = y^2x + 2x^2$, $x(u,v) = u + v$ and $y(u,v) = 2u - v$.

(b) $f(x,y) = y^2 \ln(1 + y + x^2)$, $x(u,v) = uv$ and $y(u,v) = u^2 + v^2$.

13. Let $g(r,\theta) = f(x = r\cos\theta, y = r\sin\theta)$. Verify that for all $(r,\theta) \in \mathbb{R}_+^* \times \mathbb{R}$:

$$\Delta f(x,y) = \Delta f(r\cos\theta, r\sin\theta) = \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)$$

where $\Delta f(x,y) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$.

14. Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\alpha \in \mathbb{R}$. We say that f is an homogenous function of degree α if

$$\forall (x,y) \in \mathbb{R}^2,\ \forall \lambda > 0 :\ f(\lambda x, \lambda y) = \lambda^\alpha f(x,y).$$

(a) Show that if f is homogenous of degree α, then its partial derivatives are homogenous of degree $\alpha - 1$.

(b) Show that f is homogenous of degree α if and only if

$$\forall (x,y) \in \mathbb{R}^2 :\ x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$$

(c) Suppose that $f \in C^2$. Show that

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y) + 2xy\frac{\partial^2 f}{\partial x \partial y}(x,y) + y^2\frac{\partial^2 f}{\partial y^2}(x,y) = \alpha(\alpha-1)f(x,y)$$

15. Differentiate implicitly to find the first partial derivatives of z

(a) $x^2 + 2yz + z^2 = 1$

(b) $\tan(x+y) + \tan(y+z) = 1$

(c) $z = e^x \sin(y+z)$

(d) $x\ln y + y^2 z + z^2 = 0$.

16. Determine the gradient of the following functions:

(a) $f(x,y) = \arctan\dfrac{x+y}{x-y}$

(b) $f(x,y) = (x+y)\ln(2x-y)$

(c) $f(x,y,z) = \sin(x+y)\cos(y-z)$

(d) $f(x,y,z) = (x+y)^z$.

17. Let $f : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ be a function defined by

$$f(x,y) = \int_0^{\frac{\pi}{2}} \ln\left(x^2\sin^2 t + y^2\cos^2 t\right) dt.$$

(a) Show that for all $x, y > 0$: $\nabla f(x,y) = \left(\dfrac{\pi}{x+y}, \dfrac{\pi}{x+y}\right)$.

(b) Deduce that for all $x, y > 0$: $f(x,y) = \pi \ln\left(\dfrac{x+y}{2}\right)$.

18. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = \left(\int_0^x e^{-t^2}\,dt\right)^2 \quad \text{and} \quad g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\,dt$$

(a) Show that for all $x \in \mathbb{R}$:

$$f(x) + g(x) = \frac{\pi}{4}$$

(b) Deduce that
$$\int_0^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

19. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function in class C^1 and let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by
$$g(x,y,z) = f(x-y, y-z, z-x).$$
Show that
$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0.$$

20. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be in class C^1 such that $f(1,0,2) = 1$, $\frac{\partial f}{\partial x}(1,0,2) = 1$, $\frac{\partial f}{\partial y}(1,0,2) = -2$ and $\frac{\partial f}{\partial z}(1,0,2) = 1$. Compute
$$\lim_{t\to 0} \frac{f(e^t, \sin t, 2e^t)}{f(\cos t, t, 2-t)}.$$

21. Determine $f : \mathbb{R} \to \mathbb{R}$ twice differentiable such that for function φ defined by
$$\varphi(x,y) = f\left(\frac{x}{y}\right) \quad \text{satisfies} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

22. Compute the directional derivative along u, at the indicated points:

(a) $f(x,y) = x\sqrt{y-3}$ $\quad u = (-1,6)$ $\quad a = (2,12)$.

(b) $f(x,y,z) = \dfrac{1}{x+2y-3z}$, $\quad u = (12,-9,-4)$ $\quad a = (1,1,-1)$.

23. Then determine the Jacobbien matrix at a given point a.

(a) $f(x,y,z) = \left(\frac{1}{2}(x^2 - z^2), \sin x \sin y\right)$, $D = \mathbb{R}^3$, $a = (1,1,0)$.

(b) $f(x,y) = \left(xy, \frac{1}{2}x^2 + y, \ln(1+x^2)\right)$, $D = \mathbb{R}^2$, $a = (1,1)$.

24. Suppose a hill is described mathematically by using the model $z = f(x,y) = 300 - 0.01x^2 - 0.005y^2$, where x, y and z are measured in feet. If you are at the point $(50, 100, 225)$ on the hill, in what direction should you aim your toboggan if you want to achieve the quickest descent? What is the maximum rate of decrease of the hight of the hill at this point?

25. Compute the Hessian matrix of f at the given point.

(a) $f(x,y) = xy^2 + x^2 + y^2 + 1$ at $(0,0)$.

(b) $f(x,y,z) = x^3 + x^2z + y^2z + z^3$ at $(1,0,-1)$

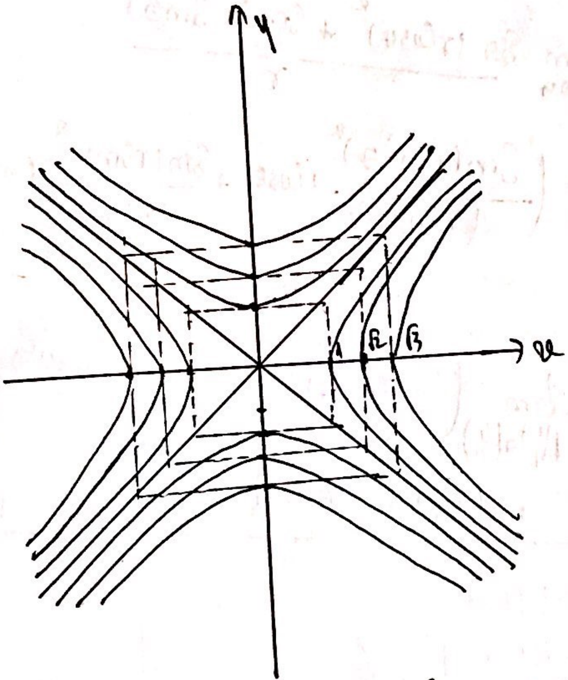
26. Given the symmetric matrix $A = (a_{ij})_n$, a vector $b \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$. We define the map $f : \mathbb{R}^n \to \mathbb{R}$ by
$$f(x) = x.Ax + b.x + c = \sum_{p=1}^{n} x_p \left(\sum_{q=1}^{n} a_{pq} x_q\right) + \sum_{p=1}^{n} b_p x_p + c.$$
Determine the Hessian matrix of f.

I_2 TD3

1 Sketch the level $f(x,y)$ and the level surface $f(x,y,z) = C$ of the function for the indicated values of c.

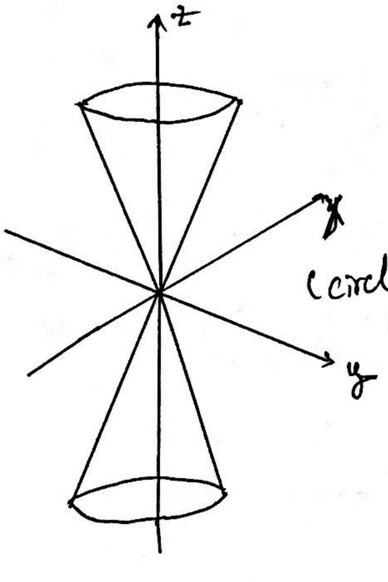
(a). $f(x,y) = y^2 - x^2$

$C = 0, \pm 1, \pm 2, \pm 3$



b). $f(x,y,z) = 4x^2 + 4y^2 - z^2$, $C = 0, 1$

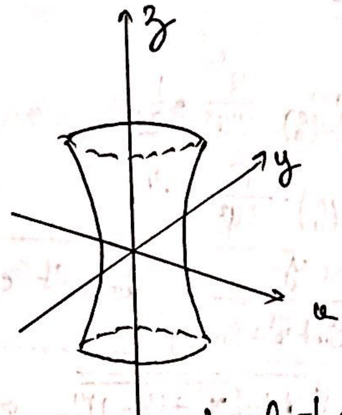
if $C = 0 \Rightarrow 4x^2 + 4y^2 = z^2$



(circle double cone)

if $C = 1$

then $4x^2 + 4y^2 - z^2 = 1$



2 Find the limit (if it exists) of the following function

(a) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$

if $x = y \Rightarrow \lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{x^2}{2x^2} = \frac{1}{2}$ (*)

$x = -y \Rightarrow \lim\limits_{\binom{-x}{x} \to \binom{0}{0}} \frac{-x^2}{2x^2} = -\frac{1}{2}$ (**)

by (*) & (**) $\lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{xy}{x^2+y^2}$ doesn't exist

(b) $\lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{1 - \cos(xy)}{xy^2} = A$

• $\cos x \sim 1 - \frac{x^2}{2} \Rightarrow \cos(xy) \sim 1 - \frac{x^2y^2}{2}$

then $\lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{1 - \cos(xy)}{xy^2} = \lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{x^2y^2}{2xy^2}$

$= \lim\limits_{\binom{x}{y} \to \binom{0}{0}} \frac{x}{2} = 0$

$\boxed{\text{Therefore } A = 1/2 \text{ (exist)}}$

d). $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sinh x^2 + \sinh y^2}$

$= \lim_{\binom{x}{y}\to\binom{0}{0}} \dfrac{xy}{x^2+y^2}$

if $x=y \Rightarrow \lim_{\binom{x}{x}\to\binom{0}{0}} \dfrac{x^2}{2x^2} = \frac{1}{2}$

$x=-y \Rightarrow \lim_{\binom{-y}{y}\to\binom{0}{0}} -\dfrac{y^2}{2y^2} = -1/2$

<u>Thus</u> $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sinh x^2 + \sinh y^2}$ doesn't exist

e). $\lim_{(x,y)\to(0,0)} \dfrac{\ln\sqrt{1+\sqrt{x^2+(y-1)^4}}}{\sin\sqrt{x^2+(y-1)^4}} = D$

let $t = \sqrt{x^2+(y-1)^4}$

if $\binom{x}{y}\to\binom{0}{0} \Rightarrow t\to 0$

$\Rightarrow \lim_{t\to 0} \frac{1}{2}\dfrac{\ln(1+t)}{\sin t} = \frac{1}{2}\lim_{t\to 0}\left(\dfrac{t}{\sin t}\cdot\dfrac{\ln(1+t)}{t}\right)$

$= 1/2$

Thus $D = 1/2$ (exist)

(c) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy^2+y\cancel{z}^2}{x^2+2y^2+3z^2} = K$

if $y=0$, $x \,\&\, z \neq 0$

Then $\lim_{y\to 0} \dfrac{xy+yz}{x^2+2y^2+3z^2} = 0$

if $x=y=z$

$\lim_{\binom{x}{x}\to\binom{0}{0}} \dfrac{2x^2}{6x^2} = 1/3$

<u>Thus</u> K doesn't exist

f. $\lim_{\binom{x}{y}\to\binom{0}{0}} \left(\dfrac{\sin x}{x}(y+1),\ \dfrac{\sin x^2+\sin y^2}{\sqrt{x^2+y^2}}\right)$

- $\lim_{\binom{x}{y}\to\binom{0}{0}} \dfrac{\sin x}{x}(y+1) = 1$
- $\lim_{\binom{x}{y}\to\binom{0}{0}} \dfrac{\sin x^2+\sin y^2}{\sqrt{x^2+y^2}}$

let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

if $\binom{x}{y}\to\binom{0}{0} \Rightarrow r\to 0$

$\Rightarrow \lim_{r\to 0} \dfrac{\sin(r\cos\theta)^2 + \sin(r\sin\theta)^2}{r}$

$= \lim_{r\to 0}\left(\dfrac{\sin(r\cos\theta)^2}{(r\cos\theta)^2}\cdot r\cos\theta + \dfrac{\sin(r\sin\theta)^2}{(r\sin\theta)^2}\cdot r\sin\theta\right)$

$= 0$

<u>Thus</u> $\lim_{\binom{x}{y}\to\binom{0}{0}} \left(\dfrac{\sin x}{x}(y+1),\ \dfrac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}\right)$

$= (0,1)$ exist

③ Let $\alpha \in \mathbb{R}$. Determine the value of α so that the function f has limit at $(0,0)$

$$f(x,y) = \frac{x^{\alpha} y}{x^2+y^2}$$

let $x = r\cos\theta$, $y = r\sin\theta$

we get $f(r,\theta) = \dfrac{r^{\alpha+1}\cos^{\alpha}x \sin y}{r^2}$

• If f has limit at $(0,0)$ that is $\binom{x}{y} \to \binom{0}{0}$

Then $f(r,\theta)$ has limit as $r \to 0$

So $\lim_{r\to 0} f(r,\theta)$ exist if and only if $\alpha+1>2 \Rightarrow \alpha>1$, $\forall \alpha$

Therefore $\alpha>1$ so that the function f has limit $(0,0)$

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④ let $\alpha_1, \alpha_2, \beta_1, \beta_2$ and r be positive real numbers and $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined

$$f(x,y) = \begin{cases} \dfrac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } \begin{pmatrix} x=0 \\ y=0 \end{pmatrix} \end{cases}$$

Show that f is continuous at $(0,0)$ if and only if $\dfrac{\alpha_1}{\beta_1}+\dfrac{\alpha_2}{\beta_2}>r$

* concept: we need to show that f is continuous at $(0,0)$

That is $\lim_{\binom{x}{y}\to\binom{0}{0}} f(x,y) = f(0,0) = 0$

Firstly, $|f(x,y)-0| = \left|\dfrac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}\right|$

$$= \frac{(|x|^{\beta_1})^{\alpha_1/\beta_1}(|y|^{\beta_2})^{\alpha_2/\beta_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}$$

$$\leq \frac{(|x|^{\beta_1}+|y|^{\beta_2})^{\alpha_1/\beta_1}\cdot(|x|^{\beta_1}+|y|^{\beta_2})^{\alpha_2/\beta_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^r}$$

$$= (|x|^{\beta_1}+|y|^{\beta_2})^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-r} = g(x,y)$$

$\Rightarrow \lim_{\binom{x}{y}\to\binom{0}{0}} g(x,y) = 0$ because of $\dfrac{\alpha_1}{\beta_1}+\dfrac{\alpha_2}{\beta_2}>r$.

That is $\lim_{\binom{x}{y}\to\binom{0}{0}} f(x,y) = f(0,0) = 0$

showing that f is continuous at $(0,0)$ (1)

(**) Suppose that f is continuous at $(0,0)$

That is $\lim_{\binom{x}{y}\to\binom{0}{0}} f(x,y) = f(0,0) = 0$

Concept We need to show that $\dfrac{\alpha_1}{\beta_1}+\dfrac{\alpha_2}{\beta_2}>r$

Choose the path $(z^{1/\beta_1}, z^{1/\beta_2})$

$(x,y)\to(0,0) \Rightarrow z\to 0$

We have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\binom{z^{1/\beta_1}}{z^{1/\beta_2}}\to\binom{0}{0}} f(z^{1/\beta_1}, z^{1/\beta_2})$

$$\Leftrightarrow \lim_{z\to 0} \frac{|z|^{\alpha_1/\beta_1} * |z|^{\alpha_2/\beta_2}}{(|z|+|z|)^r} = 0$$

$$\Leftrightarrow \lim_{z\to 0} \frac{1}{2^r}|z|^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-r} = 0$$

Obviously $\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}-r>0 \Leftrightarrow \frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}>r$ (2)

(1) & (2) $\Rightarrow$ f is continuous if and only if $\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}>r$

<u>Thus</u> f is continuous at (0, 0) if and only if $\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}>r$

(5) Let $\alpha>0$ and $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{|x|^\alpha y}{x^2+y^4} & ; \text{ if } (x,y)\neq(0,0)\\ 0 & ; \text{ if } (x,y)=(0,0)\end{cases}$$

(a). Show that $|f(x,y)|\le (x^2+y^4)^{\frac{2\alpha-3}{4}}$ $\forall(x,y)\neq(0,0)$

$$|f(x,y)|=\frac{|x|^\alpha y}{x^2+y^4}=\frac{(|x|^2)^{\alpha/2}(y^4)^{1/4}}{x^2+y^4}$$

$$\le\frac{(|x|^2+|y|^4)^{\alpha/2}(y^4+x^2)^{1/4}}{x^2+y^4}$$

$$=(|x|^2+|y|^4)^{\frac{\alpha}{2}+\frac{1}{4}-1}$$

$$=(|x|^2+|y|^4)^{\frac{2\alpha-3}{4}} \text{ true}$$

<u>Thus</u> $|f(x,y)|\le(|x|^2+|y|^4)^{\frac{2\alpha-3}{4}}$

(b) Compute $\lim_{y\to0}|f(y^2,y)|$

we have

$$0\le f(y^2,y)\le(y^4+y^4)^{\frac{2\alpha-3}{4}}$$

$$=2^{\frac{2\alpha-3}{4}}y^{2\alpha-3}$$

let $g(y)=y^{2\alpha-3}$

- if $2\alpha-3>0\Rightarrow\alpha>3/2\Rightarrow\lim_{y\to0}g(y)=0$
- if $\alpha=\frac{3}{2}\Rightarrow$ for $(x,y)\neq(0,0)$
- if $\alpha<3/2\Rightarrow\lim_{y\to0}g(y)$ doesn't exist

we have

$$0\le|f(y^2,y)|\le2^{\frac{2\alpha-3}{4}}g(y)$$

- if $\alpha>3/2$

$\Rightarrow\lim_{y\to0}|f(y^2,y)|=0$

- if $\alpha<3/2$

$\Rightarrow\lim_{y\to0}|f(y^2,y)|$ doesn't exist

- if $\alpha=3/2$

$\Rightarrow\lim_{y\to0}|f(y^2,y)|=\lim_{y\to0}\frac{y^4}{2y^4}=1/2$

(c). Study the continuity of f at (0, 0)

$$\lim_{\binom{x}{y}\to\binom{0}{0}}f(x,y)=\lim_{\binom{x}{y}\to\binom{0}{0}}\frac{|x|^\alpha y}{x^2+y^2}\begin{cases}=0 \text{ ; if } \alpha>3/2\\ =1/2 \text{ if } \alpha=3/2\\ \text{doesn't exist if } \alpha<3/2\end{cases}$$

<u>Thus</u> f is continuous if $\alpha>3/2$

d). Show that $\forall(x,y)\neq(0,0)$: $\frac{|f(x,y)|}{\sqrt{x^2+y^2}}\le|x|^{\alpha-2}$

$$|f(x,y)|=\frac{|x|^\alpha y}{x^2+y^4}\le\frac{|x|^\alpha|y|}{x^2}=|x|^{\alpha-2}|y|$$

$$\frac{|f(x,y)|}{\sqrt{x^2+y^2}}\le\frac{|y||x|^{\alpha-2}}{\sqrt{x^2+y^2}}\le\frac{|y||x|^{\alpha-2}}{|y|}$$

$$=|x|^{\alpha-2} \text{ true}$$

<u>Thus</u> $\frac{|f(x,y)|}{\sqrt{x^2+y^2}}\le|x|^{\alpha-2}$

e)- Calculate $\lim_{x\to 0} \frac{|f(x,x)|}{\sqrt{2}|x|}$.

we have $\frac{|f(x,x)|}{\sqrt{x^2+y^2}} \le |x|^{\alpha-2}$

Then $0 \le \frac{|f(x,x)|}{\sqrt{2}|x|} \le |x|^{\alpha-2}$

. let $h(x) = |x|^{\alpha-2}$

+ if $\alpha > 2 \Rightarrow \lim_{x\to 0} h(x) = 0$

Then $\lim_{x\to 0} \frac{|f(x,x)|}{\sqrt{2}|x|} = 0$

+ if $\alpha < 2 \Rightarrow \lim_{x\to 0} h(x)$ don't exist

Then $\lim_{x\to 0} \frac{|f(x,x)|}{\sqrt{2}|x|}$ doesn't exist

+ if $\alpha = 2$:

$\Rightarrow \frac{|f(x,x)|}{} = \frac{|x|^2|x|}{x^2+x^4}$

then $\frac{|f(x,x)|}{\sqrt{2}|x|} = \frac{|x|^2}{\sqrt{2}(x^2+x^4)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{1+x^2} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{2}}$

$\Rightarrow \lim_{x\to 0} \frac{|f(x,x)|}{\sqrt{2}|x|} = \frac{\sqrt{2}}{2}$

f- Study the differentiability of f at (0,0)

. If $\alpha \le \frac{3}{2} \Rightarrow$ f is not continuous at (0,0)

That is $f \notin C^1(0,0)$

. If $\alpha > 3/2$

Since $\lim_{\binom{x}{y}\to\binom{0}{0}} f(x,y) = 0 = f(0,0)$

and also $|f(x,y)| \le (x^2+y^4)^{\frac{2\alpha-3}{4}} \to 0$ as $\binom{x}{y}\to\binom{0}{0}$

Showing that f, is continuous at (0,0)

On the other hand, $\forall (x,y) \neq (0,0)$ f is in class C^1 where

$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|^\alpha(0) - (0)^\alpha 0}{h(h^2+0^4)} = 0$

$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \left(\frac{|0|^\alpha h}{h^2(0^2+h^4)} - \frac{0^\alpha(0)}{h(0^2+0^4)} \right) = 0$

Showing that f has a partial derivative

Then, we need to show that $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous at (0,0)

$.\frac{\partial f}{\partial x}(x,y) = \frac{\alpha|x|^{\alpha-1}y(x^2+y^4) - 2x|x|^\alpha y}{(x^2+y^4)^2}$

$= \frac{\alpha|x|^{\alpha+1}y + \alpha|x|^{\alpha-1}y^5 \mp 2|x|^{\alpha+1}y}{(x^2+y^4)^2}$

$= \frac{(\alpha\mp 2)|x|^{\alpha+1}y}{(x^2+y^4)^2} + \frac{\alpha|x|^{\alpha-1}y^5}{(x^2+y^4)^2}$

$\le (\alpha\mp 2)(x^2+y^4)^{\frac{\alpha+1}{2}+\frac{1}{4}-2} + \alpha(x^2+y^4)^{\frac{\alpha-1}{2}+\frac{5}{4}-2}$

$= (\alpha\mp 2)(x^2+y^4)^{\frac{2\alpha-5}{4}} + \alpha(x^2+y^4)^{\frac{2\alpha-5}{4}}$

we get $\frac{\partial f}{\partial x}(x,y) \xrightarrow[\binom{x}{y}\to\binom{0}{0}]{} 0$ where $\frac{2\alpha-5}{4} > 0$ (1)

$.\frac{\partial f}{\partial y}(x,y) = \frac{|x|^\alpha(x^2+y^4) - 4y^3|x|^\alpha y}{(x^2+y^4)^2}$

$= \frac{|x|^{\alpha+2}y^0}{(x^2+y^4)^2} - \frac{3(x^2)^{\alpha/2}y^4}{(x^2+y^4)^2}$

$\le (x^2+y^2)^{\frac{\alpha+2}{2}-2} - 3(x^2+y^4)^{\frac{\alpha}{2}+1-2}$

let $g(x,y) = (x^2+y^2)^{\frac{\alpha-2}{2}} - 3(x^2+y^4)^{\frac{\alpha-2}{2}}$

$\lim_{\binom{x}{y}\to\binom{0}{0}} g(x) = 0$ where $\frac{\alpha-2}{2} > 0$

Then $\frac{\partial f}{\partial y}(x,y) \xrightarrow[\binom{x}{y}\to\binom{0}{0}]{} 0$ (2)

Then following (1) & (2) : $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous at (0,0) for $\alpha > 5/2$

Therefore f is differentiability at (0,0) where $\alpha > 5/2$

⑥ Determine the function $f: \mathbb{R}^2 \to \mathbb{R}$ satisfies

$\frac{\partial f}{\partial x}(x,y) = \frac{2-y}{x+y+1}$ and $\frac{\partial f}{\partial y}(x,y) = \frac{1+x}{x+y+1}$

we have

$\frac{\partial f}{\partial y} = \frac{1+x}{x+y+1} \Rightarrow f(x,y) = \int \frac{1+x}{1+y+x} dy$

let $t = 1+x+y \Rightarrow dt = dy$

$\Rightarrow f(t) = \int \frac{1+x}{t} dt$

$= (1+x)\ln(1+x+y) + C(x)$ (1)

by using (1)

$\frac{\partial f}{\partial x}(x,y) = \ln(1+x+y) + \frac{1+x}{1+x+y} + C'(x)$

$\Leftrightarrow \frac{2-y}{x+y+1} = \ln(1+x+y) + \frac{1+x}{1+x+y} + C'(x)$

$\frac{1-y-x}{x+y+1} - \ln(1+x+y) = C'(x)$

$C(x) = \int \left(\frac{1-y-x}{x+y+1} - \ln(1+x+y) \right) dx$

$= 2\ln(1+x+y) - (1+x+y)\ln(1+x+y) + K$

$= (1-x-y)\ln(1+x+y) + K \;/\; K \in \mathbb{R}$

<u>Thus</u> $f(x) = (2-y)\ln(1+x+y) + K \;/\; K \in \mathbb{R}$

⑦ Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$f(x,y) = \begin{cases} (x^2+y^2)^x, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (0,0) \end{cases}$

(a). Is f continuous at (0,0)?

$f(x,y) = (x^2+y^2)^x$

$\Rightarrow \ln f(x,y) = x\ln(x^2+y^2)$

$\lim_{(x,y)\to(0,0)} \ln f(x,y) = \lim_{(x,y)\to(0,0)} x\ln(x^2+y^2) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} \ln f(x,y) = 0$

$\Rightarrow \ln f(0,0) = 0 \Rightarrow f(0,0) = 1$

<u>Thus</u> $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 1$

then f is continuous at (0,0)

b). Find the partial derivative of f at the point different from the origin

$\frac{\partial f}{\partial x}(x,y) = (x^2+y^2)^x \left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} \right)$

$\frac{\partial f(x,y)}{\partial y} = 2xy(x^2+y^2)^{x-1}$

c). Do the partial derivative of f exist at the point (0,0)?

$\frac{\partial f}{\partial x}(0,0)$ doesn't exist

$\frac{\partial f}{\partial y}(0,0) = 0$

So f doesn't exist at the point (0,0)

⑧ Show that the function $f:\mathbb{R}^2\to\mathbb{R}$ is in class C^1

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $\theta\in\mathbb{R}$

Then $\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{x^2y^3}{x^2+y^2}=\lim\limits_{r\to 0}\dfrac{r^5\sin^3\theta\cos^2\theta}{r^2}=0$

Showing that $\lim\limits_{\binom{x}{y}\to\binom{0}{0}} f(x,y)=f(0,0)=0$

Then f is continuous at point $(0,0)$

So f is continuous $\forall(x,y)\in\mathbb{R}^2$

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{2xy^3(x^2+y^2)-(2x)(x^2y^3)}{(x^2+y^2)^2}=\dfrac{2xy^5}{x^2+y^2}$
- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{3x^2y^2(x^2+y^2)-2y(x^2y^3)}{(x^2+y^2)^2}=\dfrac{y^4x^2+3x^4y^2}{(x^2+y^2)^2}$

$\Rightarrow \lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{\partial f}{\partial x}(x,y)=\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{2xy^5}{x^2+y^2}=0$

$\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{\partial f}{\partial y}(x,y)=\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{y^4x^2+3x^4y^2}{(x^2+y^2)^2}=0$

Then $\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{\partial f(x,y)}{\partial x}=\lim\limits_{\binom{x}{y}\to\binom{0}{0}}\dfrac{\partial f(x,y)}{\partial y}=\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0$

Then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ is continuous at its limit exists.

Consequently, f is differential at point on $D\in\mathbb{R}$

Therefore f is in class C^1

⑨ let $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^2} & \text{, if } (x,y)\neq(0,0)\\ 0 & \text{, if } (x,y)=(0,0)\end{cases}$$

Show that $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)\neq\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$

- $\dfrac{\partial f}{\partial x}(0,t)=\lim\limits_{h\to 0}\dfrac{f(0+h,t)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{ht^3}{h^2+t^2}-0}{h}=t$
- $\dfrac{\partial f}{\partial y}(0,t)=\lim\limits_{h\to 0}\dfrac{f(t,0+h)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{th^3}{t^2+h^2}-0}{h}=0$
- $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)=\lim\limits_{h\to 0}\dfrac{\frac{\partial f}{\partial x}(0,0+h)-\frac{\partial f}{\partial x}(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\frac{h-0}{h}-0}{h}=1\quad(1)$
- $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)=\lim\limits_{h\to 0}\dfrac{\frac{\partial f}{\partial y}(0+h,0)-\frac{\partial f}{\partial y}(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{0-0}{h}=0\quad(2)$

by (1) & (2)

Therefore $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)\neq\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$

(10) Suppose that $w = f(U,V)$ is a differentible function of $U = \frac{y-x}{xy}$ and $V = \frac{z-x}{xz}$. Show then that:

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0$$

by using chain rule:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial U}\cdot\frac{\partial U}{\partial x} + \frac{\partial w}{\partial V}\cdot\frac{\partial V}{\partial x}$$

$$= \left(\frac{-xy - y(y-x)}{(xy)^2}\right)\frac{\partial w}{\partial U} + \frac{\partial w}{\partial V}\left(\frac{-xz - z(z-x)}{(xz)^2}\right)$$

$$= -\frac{1}{x^2}\cdot\frac{\partial w}{\partial U} - \frac{1}{x^2}\frac{\partial w}{\partial V} = -\frac{1}{x^2}\left(\frac{\partial w}{\partial U} + \frac{\partial w}{\partial V}\right) \quad (1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial U}\cdot\frac{\partial U}{\partial y} + \frac{\partial w}{\partial V}\cdot\frac{\partial V}{\partial y}$$

$$= \frac{\partial w}{\partial U}\left(\frac{xy - x(y-x)}{(xy)^2}\right) + 0 = \frac{1}{y^2}\left(\frac{\partial w}{\partial y}\right) \quad (2)$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial U}\cdot\frac{\partial U}{\partial z} + \frac{\partial w}{\partial V} + \frac{\partial V}{\partial z}$$

$$= 0 + \frac{xz - x(z-x)}{(xz)^2}\frac{\partial w}{\partial V}$$

$$= \frac{1}{z^2}\frac{\partial w}{\partial V} \quad (3)$$

by (1), (2) & (3)

$$\Rightarrow \frac{\partial w}{\partial x}x^2 + \frac{\partial w}{\partial y}\cdot y^2 + z^2\frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial w}{\partial U} - \frac{\partial w}{\partial V} + \frac{\partial w}{\partial U} + \frac{\partial w}{\partial V} = 0 \text{ true}$$

Thus $x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0$

(11) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$f(x,y) = \arcsin\left(\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}\right)$ and

$g(x,y) = \arctan x - \arctan y$

(a) Calculate the partical derivative of f and g

. $g(x,y) = \arctan x - \arctan y$

$$\Rightarrow \frac{\partial g}{\partial x}(x,y) = \frac{1}{1+x^2}, \quad \frac{\partial g(x,y)}{\partial y} = \frac{1}{1+y^2}$$

. $f(x,y) = \arcsin\left(\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}\right)$

$$\sin(f(x,y)) = (1+xy)(1+x^2)^{-1/2}(1+y^2)^{-1/2} \quad (*)$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x}\cos(f(x,y))$$

$$= (1+y^2)^{-1/2}\left[y(1+x^2)^{-1/2} - x(1+xy)(1+x^2)^{-3/2}\right]$$

$$= \left((1+y^2)(1+x^2)\right)^{-1/2}\left[y - \frac{x(1+xy)}{1+x^2}\right]$$

$$= (1+y^2)^{-1/2}(1+x^2)^{-1/2}\frac{y-x}{1+x^2} \quad (1)$$

from (*)

$$\begin{cases} \sin(f(x,y)) = (1+xy)(1+x^2)^{-1/2}(1+y^2)^{-1/2} \\ \sin^2(f(x,y)) + \cos^2(f(x,y)) = 1 \end{cases}$$

Then $\cos(f(x,y)) = \sqrt{1 - (1+xy)^2\left[(1+x^2)(1+y^2)\right]^{-1}}$

$$= \sqrt{1 - \frac{(1+xy)^2}{(1+x^2)(1+y^2)}}$$

$$= \sqrt{\frac{\cancel{1}+y^2+x^2+\cancel{x^2y^2}-\cancel{1}-2xy-\cancel{x^2y^2}}{(1+x^2)(1+y^2)}}$$

$$= \sqrt{\frac{(x-y)^2}{(1+x^2)(1+y^2)}} = \frac{|x-y|}{\sqrt{(1+x^2)(1+y^2)}} \quad (2)$$

following by (1) & (2)

$$\frac{\partial f}{\partial x}(x,y)\,\cos(f(x,y)) = \frac{y-x}{\sqrt{(1+x^2)(1+y)^2}}$$

$$\frac{\partial f}{\partial x}(x,y)\cdot\frac{1(x-y)}{\sqrt{\cancel{(1+x^2)(1+y^2)}}} = \frac{y-x}{(1+x^2)\sqrt{\cancel{(1+x^2)(1+y)^2}}}$$

$$\frac{\partial f}{\partial x}(x,y) = \pm\frac{1}{1+x^2} \quad (3)$$

Similarly, we have

$$\frac{\partial f}{\partial y}(x,y) = \mp\frac{1}{1+y^2} \quad (4)$$

Thus $\frac{\partial f}{\partial x}(x,y) = \pm\frac{1}{1+x^2}$, $\frac{\partial f}{\partial y}(x,y) = \mp\frac{1}{1+y^2}$

$\frac{\partial g(x,y)}{\partial x} = \frac{1}{1+x^2}$, $\frac{\partial g}{\partial y}(x,y) = -\frac{1}{1+y^2}$

b). Simplify f

we have (3): $\frac{\partial f}{\partial x} = \pm\frac{1}{1+x^2} \Rightarrow f(x,y) = \pm\int\frac{1}{1+x^2}dx$

$= \pm\arctan x + C(y)$

$\Rightarrow \frac{\partial f}{\partial y}(x,y) = \frac{\partial C}{\partial y}(y) \quad (6)$

by following (4) & (6) :

$C'(y) = \mp\frac{1}{1+y^2}$

$\Rightarrow C(y) = \mp\arctan y + K$

we get $f(x,y) = \pm\arctan x \mp \arctan y + K / K\in\mathbb{R}$

We have $f(0,0) = K$ and $f(0,0) = \arcsin 1 = \pi/2$

$\Rightarrow K = \pi/2$

Therefore $f(x,y) = \pm g(x,y) + \frac{\pi}{2}$

(12) Find $\frac{\partial f}{\partial U}$ and $\frac{\partial f}{\partial V}$ by using the appropriate Chain Rule.

(a) $f(x,y) = y^2x + 2x^2$, $\begin{cases} x(U,V) = U+V \\ y(U,V) = 2U-V \end{cases}$

$$\frac{\partial f}{\partial U}(x,y) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial U} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial U}$$

$= (y^2+4x) + 4xy$

$= y^2 + 4xy + 4x$

$= (2U-V)^2 + 4(U+V)(2U-V) + 4(U+V)$

$= \boxed{12U^2 - 3V^2 + 4U + 4V}$

$$\frac{\partial f}{\partial V}(x,y) = \frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial V} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial V}$$

$= y^2 + 4x - 2x$

$= y^2 - 2x$

$= (2U-V)^2 - 2(U+V)$

$= \boxed{4U^2 - 4UV + V^2 - U - 2}$

(b) $f(x,y) = y^2\ln(1+y+x^2)$

$\begin{cases} x(U,V) = UV \\ y(U,V) = U^2+V^2 \end{cases}$

$\cdot\ \frac{\partial f}{\partial U} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial U} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial U}$

$= y^2\cdot\frac{2x}{1+y+x}(V) + \left(2y\ln(1+y+x^2) + \frac{y^2}{1+y^2+x}\right)$

$\cdot\ \frac{\partial f}{\partial V} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial V} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial V}$

$= \frac{2xy^2}{1+x^2+y}U + \left((2y\ln(1+y+x^2) + \frac{y^2}{2+x+y}\right)2V$

(13) let $g(r,\theta) = f(x = r\cos\theta,\ y = r\sin\theta)$.

Verify that for all $(r,\theta) \in \mathbb{R}_+^* \times \mathbb{R}$:

$$\Delta f(x,y) = \Delta f(r\cos\theta, r\sin\theta)$$
$$= \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)$$

where $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

• $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta$

$\Rightarrow \frac{\partial^2 g}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial g}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial g}{\partial y}\right)$

$= \cos^2\theta\frac{\partial^2 g}{\partial x^2} + \sin\theta\cos\theta\frac{\partial^2 g}{\partial x\partial y} + \sin\theta\cos\theta\frac{\partial^2 g}{\partial y\partial x} + \sin^2\theta\frac{\partial^2 g}{\partial y^2}$

• $\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$= \frac{\partial g}{\partial x}(-r\sin\theta) + \frac{\partial g}{\partial y}(r\cos\theta)$

$\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial g}{\partial x} + r\cos\theta\frac{\partial g}{\partial y}\right)$

$= -r\cos\theta\frac{\partial g}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}\right) - r\sin\theta\frac{\partial g}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial y}\right)$

$= -r\cos\theta\frac{\partial^2 g}{\partial x} - r\sin\theta\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)\frac{\partial x}{\partial \theta} - r\sin\theta\frac{\partial}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)\frac{\partial y}{\partial \theta}$

$- r\sin\theta\frac{\partial g}{\partial y} + r\cos\theta\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)\frac{\partial x}{\partial \theta} + r\cos\theta\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right)\frac{\partial y}{\partial \theta}$

$= -r\cos\theta\frac{\partial g}{\partial x} + r^2\sin^2\theta\frac{\partial^2 g}{\partial x^2} - r^2\sin\theta\cos\theta\frac{\partial^2 g}{\partial y\partial x}$

$- r\sin\theta\frac{\partial g}{\partial y} - r^2\sin\theta\cos\theta\frac{\partial^2 g}{\partial x\partial y} + r^2\cos\theta\frac{\partial^2 g}{\partial y^2}$

• $\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = -\frac{1}{r}\cos\theta\frac{\partial g}{\partial x} + \sin^2\theta\frac{\partial^2 g}{\partial x^2} - \sin\theta\cos\theta\frac{\partial^2 g}{\partial y\partial x}$

$- \frac{1}{r}\sin\theta\frac{\partial g}{\partial y} - \sin\theta\cos\theta\frac{\partial^2 g}{\partial x\partial y} + \cos\theta\frac{\partial^2 g}{\partial y^2}$

Then

$\Delta f(x,y) = \cos^2\theta\frac{\partial^2 g}{\partial x^2} + \cancel{\sin\theta\cos\theta\frac{\partial^2 g}{\partial x\partial y}} + \cancel{\sin\theta\cos\theta\frac{\partial^2 g}{\partial y\partial x}} + \sin^2\theta\frac{\partial^2 g}{\partial y^2} + \frac{1}{r}\cos\theta\frac{\partial g}{\partial x} + \frac{1}{r}\cancel{\sin\theta\frac{\partial g}{\partial y}}$

$\cancel{-\frac{1}{r}\cos\theta\frac{\partial g}{\partial x}} + \sin^2\theta\frac{\partial^2 g}{\partial x^2} - \cancel{\sin\theta\cos\theta\frac{\partial^2 g}{\partial y\partial x}} - \cancel{\frac{1}{r}\sin\theta\frac{\partial g}{\partial y}}$

$\cancel{-\sin\theta\cos\theta\frac{\partial^2 g}{\partial x\partial y}} + \cos\theta\frac{\partial^2 g}{\partial y^2}$

$\Delta f(x,y) = \frac{\partial^2 g}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 g}{\partial y^2}(\sin^2\theta + \cos^2\theta)$

$= \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$ true

Thus $\Delta f(x,y) = \Delta f(r\cos\theta, r\sin\theta) = \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)$ where $\Delta f(x,y) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$

(14) let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\alpha \in \mathbb{R}$.
We say that f is an homogeneous function of degree α if
$\forall (x,y) \in \mathbb{R}^2, \forall \lambda > 0;\ f(\lambda x, \lambda y) = \lambda^\alpha f(x,y)$

(a) Show that if f is homogeneous of degree α, then its partial derivative are homogeneous of degree $\alpha - 1$

. f is homogeneous of degree α, if
$f(\lambda x, \lambda y) = \lambda^\alpha f(x,y)$
$\lambda \frac{\partial f}{\partial x}(\lambda x, \lambda y) = \lambda^\alpha \frac{\partial f}{\partial y}(x,y)$
$\frac{\partial f}{\partial x}(\lambda x, \lambda y) = \lambda^{\alpha-1} \frac{\partial f}{\partial y}(x,y)$
Then $\frac{\partial f}{\partial x}$ is homogenous of degree $(\alpha-1)$
Similarly, we have $\frac{\partial f}{\partial y}$ is also an homogenous of degree $(\alpha-1)$

Thus f is homogenous of degree α, then its partial derivatives are homogenous of degree $(\alpha-1)$

(b) Show that f is homogenous of degree α if and only if
$\forall (x,y) \in \mathbb{R}^2 : x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$

Suppose that f is homogenous of degree α
$\forall (x,y) \in \mathbb{R}^2, \forall \lambda > 0$
$f(\lambda x, \lambda y) = \lambda^\alpha f(x,y)$
then differentiate both side with respect to λ, we get
(*) $x\frac{f(\lambda x, \lambda y)}{\partial x} + \frac{y(f\lambda x, \lambda y)}{\partial y} = \alpha \lambda^{\alpha-1} f(x,y)$
we have from (a) $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are homogenous of degree $(\alpha-1)$

Since (*) we can written to:
$x\lambda^{\alpha-1} \frac{f(x,y)}{\partial x} + y\lambda^{\alpha-1}\frac{f(x,y)}{\partial y} = \alpha\lambda^{\alpha-1} f(x,y)$

$\frac{x f(x,y)}{\partial x} + \frac{y f(x,y)}{\partial x} = \alpha f(x,y)$ (true)

. Conversly, suppose that
$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$ (1)
concept: we need to show that f is homogenous of degree α.
change $x \to \lambda x$, $y \to \lambda y$ we get
$x\lambda\frac{\partial f}{\partial x}(x\lambda, y\lambda) + \lambda y\frac{\partial f}{\partial y}(\lambda x, \lambda y) = \alpha f(\lambda x, \lambda y)$
consider function $\varphi(\lambda) = f(x\lambda, \lambda y)$
$\Rightarrow \varphi'(\lambda) = \frac{x\partial f(\lambda x, \lambda y)}{\partial x} + \frac{y\partial f(\lambda x, \lambda y)}{\partial y}$ (2)
following by (1) & (2) $\varphi'(\lambda) = \frac{\alpha}{\lambda}\varphi(\lambda)$
$\int \frac{\varphi'(\lambda)}{\varphi(\lambda)} d\lambda \int \frac{\alpha}{\lambda} d\lambda$
$\ln\varphi(\lambda) = \alpha\ln\lambda + K$
$= \alpha\ln\lambda + \ln c$
$= \ln c\lambda^\alpha$
$\Rightarrow \varphi(\lambda) = C\lambda^\alpha$
$\lambda = 1 \Rightarrow \varphi(1) = C = f(x,y)$
we get $\varphi(\lambda) = f(\lambda x, \lambda y) = \lambda^\alpha f(x,y)$ (true)

Therefore f is homogenous of degree α if and only if $\forall (x,y) \in \mathbb{R}^2$:
$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$

c)- Suppose that $f \in C^2$. Show that

$$x^2 \frac{\partial^2 f}{\partial x^2}(x,y) + 2xy \frac{\partial^2 f}{\partial x \partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y) = \alpha(\alpha-1) f(x,y)$$

we have by (b): $\forall (x,y) \in \mathbb{R}^2$:

$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = \alpha f(x,y)$$

Differentiate both side with respect to x and y, we get:

$$\begin{cases} (1) & \frac{\partial f}{\partial x}(x,y) + x^2 \frac{\partial^2 f}{\partial x^2}(x,y) + yx \frac{\partial^2 f}{\partial x \partial y}(x,y) = \alpha \frac{\partial f}{\partial x}(x,y) \\ (2) & yx \frac{\partial^2 f}{\partial y \partial x}(x,y) + \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y) = \alpha \frac{\partial f(x,y)}{\partial x} \end{cases} +$$

$$\begin{cases} (1) & x^2 \frac{\partial^2 f}{\partial x^2}(x,y) + yx \frac{\partial^2 f}{\partial x \partial y} = (\alpha-1) \frac{\partial f}{\partial x}(x,y) \\ (2) & yx \frac{\partial^2 f}{\partial y \partial x}(x,y) + y^2 \frac{\partial^2 f(x,y)}{\partial y^2} = (\alpha-1) \frac{\partial f}{\partial x}(x,y) \end{cases} +$$

$$x^2 \frac{\partial^2 f}{\partial x^2}(x,y) + 2xy \frac{\partial^2 f}{\partial x \partial y}(x,y) + y^2 \frac{\partial^2 f(x,y)}{\partial y^2} = (\alpha-1)\left[\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\right]$$

$$= (\alpha-1)\alpha f(x,y)$$

Thus $x^2 \frac{\partial^2 f}{\partial x^2}(x,y) + 2xy \frac{\partial^2 f}{\partial x \partial y}(x,y) + y^2 \frac{\partial^2 f(x,y)}{\partial y^2} = (\alpha-1)\alpha f(x,y)$

(16) Determine the gardient of the following functions:

(a) $f(x,y) = \arctan \frac{x+y}{x-y}$

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

- $$\frac{\partial f}{\partial x}(x,y) = \left(\frac{x+y}{x-y}\right)' \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}$$

$$= \frac{\cancel{x}-y+\cancel{x}+y}{(x-y)^2} \times \frac{(x-y)^2}{(x-y)^2+(x+y)^2}$$

$$= \frac{-2y}{(x-y)^2+(x+y)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{x+y}{x-y}\right)' \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}$$

$$= \frac{2x}{(x-y)^2+(x+y)^2}$$

Thus $\nabla f = \operatorname{grad} f = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

b)- $f(x,y) = (x+y)\ln(2x-y)$

$$\frac{\partial f}{\partial x} = \ln(2x-y) + \frac{2(x+y)}{2x-y}$$

$$\frac{\partial f}{\partial y} = \ln(2x-y) - \frac{(x+y)}{2x-y}$$

Thus $\nabla f = \operatorname{grad} f = \begin{pmatrix} \ln(2x-y) + \frac{2(x+y)}{2x-y} \\ \ln(2x-y) - \frac{x+y}{2x-y} \end{pmatrix}$

c)- $f(x,y,z) = \sin(x+y)\cos(y-z)$

$$\frac{\partial f}{\partial x} = \cos(x+y)\cos(y-z)$$

$$\frac{\partial f}{\partial y} = \cos(x+2y-z)$$

$$\frac{\partial f}{\partial z} = \sin(x+y)\sin(y-z)$$

Thus $\nabla f = \operatorname{grad} f = \begin{pmatrix} \cos(x+y)\cos(y-z) \\ \cos(x+2y-z) \\ \sin(x+y)\sin(y-z) \end{pmatrix}$

d)- $f(x,y,z) = (x+y)^z$

$$\frac{\partial f}{\partial x} = z(x+y)^{z-1}, \quad \frac{\partial f}{\partial y} = z(x+y)^{z-1}$$

$$\frac{\partial f}{\partial z} = \left(\ln(x+y)\right)(x+y)^z$$

Thus $\nabla f = \operatorname{grad} f = \begin{pmatrix} z(x+y)^{z-1} \\ z(x+y)^{z-1} \\ (x+y)^z \ln(x+y) \end{pmatrix}$

(15) Differentiate implicitly to find the first partial derivatives of z

(a) $x^2+2xy+z^2=1$

$\Leftrightarrow x^2+2xy+z^2-1=0$

let $f(x,y,z)=x^2+2xy+z^2-1$

$$\frac{\partial z}{\partial x}=-\left(\frac{2z}{2y+2x}\right)^{-1}=-\left(\frac{z}{y+x}\right)^{-1}=-\frac{x+y}{z}$$

$$\frac{\partial z}{\partial y}=-\left(\frac{2z}{2y}\right)^{-1}=-\left(\frac{z}{y}\right)^{-1}=-\frac{y}{z}$$

(b) $\tan(x+y)+\tan(y+z)=1$

$\Leftrightarrow \tan(x+y)+\tan(y+z)-1=0$

let $f(x,y,z)=\tan(x+y)+\tan(y+z)-1$

$$\frac{\partial z}{\partial x}=-\left(\frac{1+\tan^2(y+z)}{1+\tan^2(x+y)}\right)^{-1}=-\frac{1+\tan^2(x+y)}{1+\tan^2(y+z)}$$

$$\frac{\partial z}{\partial y}=-\frac{2+\tan^2(y+z)+\tan^2(x+y)}{1+\tan^2(y+z)}$$

(c) $z=e^x\sin(y+z)$

$f(x,y,z)=e^x\sin(y+z)-z$

$$\frac{\partial z}{\partial x}=-\frac{e^x\sin(y+z)}{e^x\cos(y+z)-1}$$

$$\frac{\partial z}{\partial y}=-\frac{e^x\cos(y+z)}{e^x\cos(y+z)-1}$$

(d). $x\ln y+y^2z+z^2=0$

$f(x,y,z)=x\ln y+y^2z+z^2$

$$\frac{\partial z}{\partial x}=-\frac{\ln y}{y^2+2z}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{x}{y}+2yz}{y^2+2z}$$

(17) Let $f: \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ be a function defined by:

$$f(x,y)=\int_0^{\pi/2}\ln(x^2\sin^2t+y^2\cos^2t)\,dt$$

(a). Show that for $x,y>0$: $\nabla f(x,y)=\left(\frac{\pi}{x+y},\frac{\pi}{x+y}\right)$

$$\frac{\partial f}{\partial x}=\int_0^{\pi/2}\frac{2x\sin^2t}{x^2\sin^2t+y^2\cos^2t}\,dt \quad (1)$$

$$\frac{\partial f}{\partial y}=\int_0^{\pi/2}\frac{2y\cos^2t}{x^2\sin^2t+y^2\cos^2t}\,dt \quad (2)$$

we take $(1)x+(2)y$

then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\int_0^{\pi/2}\frac{2x^2\sin^2t+2y^2\cos^2t}{x^2\sin^2t+y^2\cos^2t}\,dt$

$$=\int_0^{\pi/2}2\,dt=\pi \quad (*)$$

Similarly, we take $y(1)+x(2)$

$$y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=2xy\int_0^{\pi/2}\frac{1}{x^2\sin^2t+y^2\cos^2t}\,dt$$

$$=2xy\int_0^{\pi/2}\frac{\frac{1}{\cos^2t}}{x^2\tan^2t+y^2}\,dt$$

let $u=\tan t \Rightarrow du=\frac{1}{\cos^2t}$

when $\begin{cases}t=0 \Rightarrow u=0\\ t=\pi/2 \Rightarrow u=+\infty\end{cases}$

Then $y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=2xy\int_0^{+\infty}\frac{du}{x^2u^2+y^2}$

$$=\frac{2xy}{y^2}\int_0^{+\infty}\frac{du}{1+\left(\frac{x}{y}\right)^2u^2}$$

$$=\frac{2xy}{y^2}\cdot\frac{y}{x}\arctan\Big|_0^{+\infty}=2\times\frac{\pi}{2}=\pi$$

$$\begin{cases}x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\pi\\ y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial x}=\pi\end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\pi}{x+y}$$

Thus $\nabla f(x,y)=\frac{\pi}{x+y}$

(b). Deduce that for all $x, y > 0$:
$f(x,y) = \pi \ln\left(\frac{x+y}{2}\right)$

$\frac{\partial f}{\partial x} = \frac{\pi}{x+y}$

$\Rightarrow \int \frac{\partial f}{\partial x} = \int \frac{\pi}{x+y} dx = \pi \ln(x+y) + C(y)$ (I)

But $\frac{\partial f}{\partial y} = \frac{\pi}{x+y} + \frac{dC(y)}{dy} = \frac{\pi}{x+y}$

we get $\frac{dC(y)}{dy} = 0 \Rightarrow C(y) = K / K \in \mathbb{R}$

In case $x = y = 1$

$\Rightarrow f(1,1) = \int_0^{\pi/2} \ln 1 \, dt = 0$

we take $x = y = 1$ substitute into (I)

Then $\pi \ln 2 + C(1) = 0$

$\Rightarrow C(1) = -\pi \ln 2$

Thus $f(x,y) = \pi \ln\left(\frac{x+y}{2}\right)$ for $x, y > 0$

(18) Let f, g $\mathbb{R} \to \mathbb{R}$ be two function defined by

$f(x) = \left(\int_0^x e^{-t^2} dt\right)^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$

(a) Show that for all $x \in \mathbb{R}$:

$f(x) + g(x) = \pi/4$

- $f(x) = \left(\int_0^x e^{-t^2} dt\right)^2$

$\Rightarrow f'(x) = 2e^{-x^2} \int_0^x e^{-t^2} dt$

- $g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$

$\Rightarrow g'(x) = -2x \int_0^1 e^{-x^2(1+t^2)} dt$

$= -\int_0^1 2x e^{-x^2(1+t^2)} dt$

$= -2e^{-x^2} \int_0^x e^{-t^2} dt$

we get $f'(x) + g'(x) = 0$ for all $x \in \mathbb{R}$. Let $h(x) = f(x) + g(x)$

Showing that $h(x)$ is constance function

consequenly $h(t) = \int_0^1 \frac{dt}{1+t^2} = \arctan 1 = \pi/4$

Thus $f(x) + g(x) = \pi/4$ for all $x \in \mathbb{R}$

b). Deduce that

$\int_0^{+\infty} e^{-t^2} dt = \sqrt{\pi}/2$

we have $f(x) + g(x) = \pi/4$

$f(x) = \frac{\pi}{4} - g(x)$

$\Rightarrow \int_0^x e^{-t^2} dt = \sqrt{\frac{\pi}{4} - g(x)}$

when $x \to +\infty$ then

$g(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t} \xrightarrow[x \to \infty]{} 0$

we get $\int_0^{+\infty} e^{-t^2} dt = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}$

Thus $\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

(19) let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function in class C^1 and let $g: \mathbb{R} \to \mathbb{R}$ be a function defined by
$g(x,y,z) = f(x-y, y-z, z-x)$

Show that $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0$

let $g(x,y,z) = f(x-y, y-z, z-x)$
where $x' = x-y$, $y' = y-z$, $z' = z-x$

Since $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial x} + \frac{\partial f}{\partial z'} \cdot \frac{\partial z'}{\partial x}$

$+ \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial y} + \frac{\partial f}{\partial z'} \cdot \frac{\partial z'}{\partial y} + \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial z} +$

$\frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial z} + \frac{\partial f}{\partial z'} \cdot \frac{\partial z'}{\partial z}$

$= \frac{\partial f}{\partial x'} - \frac{\partial f}{\partial z'} - \frac{\partial f}{\partial x'} + \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial z'} = 0$

by using chain rule

$$\boxed{\text{Thus } \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0}$$

(20) let $: \mathbb{R}^3 \to \mathbb{R}$ be in class C^1 such that
$f(1,0,2) = 1$, $\frac{\partial f}{\partial x}(1,0,2) = 1$, $\frac{\partial f}{\partial y}(1,0,2) = -2$
and $\frac{\partial f}{\partial z}(1,0,2) = 1$. Compute

$$\lim_{t \to 0} \frac{f(e^t, \sin t, 2e^t)}{f(\cos t, t, 2-t)}$$

by using l'hospital Rule, we get

$$\lim_{t \to 0} \frac{f(e^t, \sin t, 2e^t)}{f(\cos t, t, 2-t)}$$

$$= \lim_{t \to 0} \frac{\frac{\partial f}{\partial t}(e^t, \sin t, 2e^t)}{\frac{\partial f}{\partial t}(\cos t, t, 2-t)}$$

- $\frac{\partial f}{\partial t} = e^t \frac{\partial f}{\partial x} + \cos t \frac{\partial f}{\partial y} + 2e^t \frac{\partial f}{\partial z}$

$\Rightarrow \frac{\partial f}{\partial t}(1,0,2) = 1 \times 1 + (-2)(1) + 2(1) = 1$

- $\frac{\partial f}{\partial t} = -\sin t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}$

$\Rightarrow \frac{\partial f}{\partial t}(1,0,2) = 0 - 2 - 1 = -3$

$$\boxed{\text{Thus } \lim_{t \to 0} \frac{f(e^t, \sin t, 2e^t)}{f(\cos t, t, 2-t)} = -\frac{1}{3}}$$

(21) Determine $f: \mathbb{R} \to \mathbb{R}$ twice differentiable such that for function φ defined by $\varphi(x,y) = f(\frac{x}{y})$ satisfies $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

we have $\varphi(x,y) = f(\frac{x}{y})$

- $\frac{\partial \varphi}{\partial x} = \frac{1}{y} f'(\frac{x}{y})$
- $\frac{\partial^2 \varphi}{\partial^2 x} = \frac{1}{y} f''(\frac{x}{y})$
- $\frac{\partial \varphi}{\partial y} = -\frac{x}{y^2} f'(\frac{x}{y})$
- $\frac{\partial^2 \varphi}{\partial y^2} = \frac{2x}{y^3} f'(\frac{x}{y}) + \frac{x^2}{y^4} f''(\frac{x}{y})$

we get

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

$\Leftrightarrow \frac{1}{y} f''(\frac{x}{y}) + \frac{2x}{y^3} f'(\frac{x}{y}) + \frac{x^2}{y^4} f''(\frac{x}{y}) = 0$

$\frac{1}{y}\left[(1 + \frac{x^2}{y^2}) f''(\frac{x}{y}) + \frac{2x}{y} f'(\frac{x}{y}) \right] = 0$

$\Leftrightarrow (1 + \frac{x^2}{y^2}) f''(\frac{x}{y}) + \frac{2x}{y} f'(\frac{x}{y}) = 0$

let $t = \frac{x}{y}$ we have

$(1+t^2) f''(t) + 2t f'(t) = 0$

$\left((1+t^2) f'(t) \right)' = 0$

$\Rightarrow (1+t^2) f'(t) = C$

$f'(t) = \frac{C}{1+t^2}$

$f(t) = C \tan^{-1}(t) + K \quad / C, K \in \mathbb{R}$

$$\text{Thus } f(\tfrac{x}{y}) = C \tan^{-1}(\tfrac{x}{y}) + K \ / \ C, K \in \mathbb{R}$$

(22) Compute the directional derivative along u, at the indicated points.

(a) $f(x,y) = x\sqrt{y-3}$ $\quad U=(-1,6)$, $a=(2,12)$

- $\frac{\partial f}{\partial x} = \sqrt{y-3} \Rightarrow \frac{\partial f}{\partial x}(2,12) = \sqrt{12-3} = 3$
- $\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y-3}} \Rightarrow \frac{\partial f}{\partial y}(2,12) = \frac{2}{2\sqrt{12-3}} = \frac{1}{3}$

let $V = \frac{1}{\|U\|}U = \frac{1}{\sqrt{37}}(-1,6) = (V_1,V_2)$

Then $\|V\|=1$, so V is unit vector

we get $D_uf(a) = D_vf(a)$

$= V_1\frac{\partial f}{\partial x}(a) + V_2\frac{\partial f}{\partial y}(a)$

$= -\frac{3}{\sqrt{37}} + \frac{2}{\sqrt{37}} = \frac{-1}{\sqrt{37}}$

Done $D_uf(2,12) = -\frac{\sqrt{37}}{37}$

(b) $f(x,y,z) = \frac{1}{x+2y-3z}$ $\quad U=(12,-9,-4)$, $a=(1,1,-1)$

- $\frac{\partial f}{\partial x} = -\frac{1}{(x+2y-3z)^2} \Rightarrow \frac{\partial f}{\partial x}(1,1,-1) = \frac{-1}{36}$
- $\frac{\partial f}{\partial y} = -\frac{2}{(x+2y-3z)^2} \Rightarrow \frac{\partial f}{\partial y}(1,1,-1) = \frac{-1}{18}$
- $\frac{\partial f}{\partial z} = -\frac{3}{(x+2y-3z)^2} \Rightarrow \frac{\partial f}{\partial z}(1,1,-1) = \frac{1}{12}$

let $V = \frac{1}{\|U\|}\cdot U = \frac{1}{\sqrt{241}}(12,-9,-4) = (V_1,V_2,V_3)$

then $\|V\|=1$, so V is unit vector

we get $D_uf(a) = D_vf(a)$

$= V_1\frac{\partial f}{\partial x}(a) + V_2\frac{\partial f}{\partial y}(a) + V_3\frac{\partial f}{\partial z}(a)$

$= \frac{1}{36\sqrt{241}}(-12+18-12) = \frac{-1}{6\sqrt{241}}$

Done $D_uf(a) = \frac{-1}{6\sqrt{241}}$

(23) Then determine the Jacobbien matrix at a given point a.

(a) $f(x,y,z) = \left(\frac{1}{2}(x^2-z^2), \sin x\sin y\right)$,

$D = \mathbb{R}^3$, $a=(1,1,0)$

- $\frac{\partial f}{\partial x}(x,y,z) = (x, \cos x\sin y)$
- $\frac{\partial f}{\partial y}(x,y,z) = (0, \cos y\sin x)$
- $\frac{\partial f}{\partial z}(x,y,z) = (-z, 0)$

Since $(x,y,z) \in \mathbb{R}^3$. Showing that $f(x,y,z)$ is differentiable on $D=\mathbb{R}^3$

- Determine the Jacobbien matrix at given point a.

we have, $D_f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y}, & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y}, & \frac{\partial f_2}{\partial z} \end{pmatrix}$

we have, $f(x,y,z) = \left(\frac{1}{2}(x^2-z^2), \sin x\sin y\right)$

That is $\begin{cases} f_1(x,y,z) = \frac{1}{2}(x^2-z^2) \\ f_2(x,y,z) = \sin x\sin y \end{cases}$

we get $D_f(a) = \begin{pmatrix} x & 0 & -z \\ \cos x\sin y & \cos y\sin x & 0 \end{pmatrix}$

Consequently $D_f(1,1,0) = \begin{pmatrix} 1 & 0 & 0 \\ \cos1\sin1 & \cos1\sin1 & 0 \end{pmatrix}$

Therefore

$D_f(1,1,0) = D_f(a) = \begin{pmatrix} 1 & 0 & 0 \\ \cos1\sin1 & \cos1\sin1 & 0 \end{pmatrix}$

(b). $f(x,y) = \left(xy, \frac{1}{2}x^2+y, \ln(1+x^2)\right)$

$D = \mathbb{R}^2$, $a = (1,1)$

- $\frac{\partial f}{\partial x}(x,y) = \left(y, x, \frac{2x}{1+x^2}\right)$
- $\frac{\partial f}{\partial y}(x,y) = (x, 1, 0)$

Since $(x,y) \in \mathbb{R}^2$, then $f(x,y)$ is deferentiable on $D = \mathbb{R}^2$ by knowing every partials derivative are continous and f also continous

- Detertermine the Jacobbien matrix at a given point a.

we have $D_f(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}_{3\times 2}$

$D_f(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

Therefore $D_f(a) = D_f(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

(24) we have

$z = f(x,y) = 300 - 0.01x^2 - 0.005y^2$

The gradient of the height function is

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-0.02x, 0.01y)$

Therefore, the direction of quickest descent when you are at the point $(50, 100, 225)$ is given by the direction of:

$-\nabla f(50,100) = (1, 1)$

Thus maximum rate of deCrease of the hight of mountain at the $(100, 50, 225)$ is $\|-\nabla f(50,100)\| = \sqrt{1+1} = \sqrt{2}$

(25) Compute the Hessian matrix of f at the given point

(b). $f(x,y,z) = x^3 + 2x^2z + y^2z + z^3$ at $(1,0,-1)$

By using hessian matrix

$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$

- $\frac{\partial f}{\partial x} = 3x^2 + 2xz$ • $\frac{\partial^2 f}{\partial x^2} = 6x + 2z$
- $\frac{\partial f}{\partial y} = 2yz$ • $\frac{\partial^2 f}{\partial y^2} = 2z$
- $\frac{\partial f}{\partial z} = x^2 + 3z^2 + y^2$ • $\frac{\partial^2 f}{\partial z^2} = 6z$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$ • $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0$
- $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 2x$ • $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 2y$
- $\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 2x$ • $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 2y$

Then $H_f(x,y) = \begin{pmatrix} 6x+2z & 0 & 2x \\ 0 & 2z & 2y \\ 2x & 2y & 6z \end{pmatrix}$

Then, we compute the hessian matrix of f at $(1,0,-1)$

$H_f(1,0,-1) = \begin{pmatrix} 4 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & 6 \end{pmatrix}$

(a). $f(x,y) = xy^2 + x^2 + y^2 + 1$ at $(0,0)$

By using hessian matrix

$H_f(x,y) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

- $\frac{\partial f}{\partial x} = y^2 + 2x$ $\quad \frac{\partial^2 f}{\partial x^2} = 2xy + 2y$
- $\frac{\partial f}{\partial y} = 2xy + 2y$ $\quad \frac{\partial^2 f}{\partial y^2} = 2 + 2x$
- $\frac{\partial^2 f}{\partial x \partial y} = 2y$ • $\frac{\partial^2 f}{\partial y \partial x} = 2y$

Then, we compute the hessian matrix of f at $(0,0)$

Thus $H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(26) Given the symmetric matrix $A=(a_{ij})_n$, a vector $b \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$. We define the map $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = x.Ax + b.x + c$$
$$= \sum_{p=1}^{n} x_p\left(\sum_{q=1}^{n} a_{pq}x_q\right) + \sum_{p=1}^{n} b_p x_p + c$$

Determine the Hessian matrix of f.

$$f(x) = x.Ax + bx + c = \sum_{p=1}^{n} x_p\left(\sum_{q=1}^{n} a_{pq}x_q\right) + \sum_{p=1}^{n} b_p x_p + c$$

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sum_{p=1}^{n} x_p\left(\sum_{q=1}^{n} a_{pq}x_q\right) + \sum_{p=1}^{n} b_p x_p\right)$$

let $\begin{cases} A = \dfrac{\partial}{\partial x_i}\left(\sum_{p=1}^{n} x_p\left(\sum_{q=1}^{n} a_{pq}x_q\right)\right) \\ B = \dfrac{\partial}{\partial x_i}\left(\sum_{p=1}^{n} b_p x_p\right) \end{cases}$

$$A = \frac{\partial}{\partial x_i}\left[\sum_{p=1}^{n} x_p(a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pn}x_n)\right]$$

$$= \frac{\partial}{\partial x_i}\left[x_1(a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n) + \dots + x_n(a_{n1}x_1 + \dots + a_{nn}x_n)\right]$$

$$= x_1a_{1i} + x_2a_{2i} + \dots + x_{i-1}a_{(i-1)i} + x_{(i+1)}a_{(i+1)i} + \dots + x_na_{ni} + (a_{i1}x_1 + \dots + a_{in}x_n) + x_ia_{ii}$$

$$= x_1a_{1i} + \dots + x_na_{ni} + a_{i1}x_1 + \dots + a_{in}x_n$$

$$= \sum_{k=1}^{n} x_k a_{ki} + \sum_{k=1}^{n} x_k a_{ik}$$

$$B = \frac{\partial}{\partial x_i}\left(\sum_{p=1}^{n} b_p x_p\right)$$
$$= \frac{\partial}{\partial x_i}(b_1x_1 + b_2x_2 + \dots + b_nx_n)$$
$$= b_i$$

we get

$$\frac{\partial f}{\partial x_i}(x) = \sum_{k=1}^{n} x_k a_{ki} + \sum_{k=1}^{n} x_k a_{ik} + b_i$$

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = a_{ij} + a_{ji}$$

Consequently

$$H_f = (a_{ij} + a_{ji})_{n\times n}$$
$$= 2(a_{ij})_n = 2A$$

Therefore

$$H_f = 2(a_{ij})_n = 2A$$

A is the symmetric matrix

I2–TD4
(Differentiation in Several Variables)

1. Find the equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

   (a) $f(x,y) = xy + \ln \sqrt[3]{1+x^2+2y^4}$ at $(1,0)$.

   (b) $F(x,y,z) = y \ln xz^2 - 2 = 0$ at $(e,2,1)$.

2. Show that the tangent plane to the Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) is given by $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \int_{xy}^{x^2} \left(x^2 yt + xt^2\right) dt$$

   Find the equation of tangent plane to the surface $z = f(x,y)$ at the point $(1,0)$.

4. Find the global extrema of the function over the region R.

   (a) $f(x,y) = x^2 + y^2 + x + 1, \quad R = \{(x,y) \in \mathbb{R}^2 \mid y \leq \sqrt{9-x^2}; -3 \leq x \leq 3\}$.

   (b) $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1, \quad R = \{(x,y) \in [-2,0] \times [0,1]\}$.

5. Determine the local extrema and/or saddle points of the following functions.

   (a) $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$

   (b) $f(x,y) = 2x^3 + (x-y)^2 - 6y$

   (c) $f(x,y) = \cos x + \cos y + \sin(x+y)$

   (d) $f(x,y,z) = x^2 - xy + z^2 - 2xz + 6z$

   (e) $f(x,y,z) = (x^2 + 2y^2 + 1)\cos z$

   (f) $f(x,y,z) = x^4 + x^2 y + y^2 + z^2 + xz + 1$.

6. Show that $(0,0)$ is the critical point of the function f defined by

$$f(x,y) = 10x^2 \cos y + \int_{x^2}^{y^2} \ln \sqrt{2 + x^4 + \cos(ty)}\, dt$$

   then study the nature of the critical point.

7. Study the extrema of the function $f : \mathbb{R}^n \to \mathbb{R}$ where

$$f(x_1, x_2, \ldots, x_n) = e^{-(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

8. A company manufacturers two type of sneakers, running shoes and basketball shoes. The total revenue from x_1 units of running shoes and x_2 units of basketball shoes is $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$, where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximise the revenue.

9. For each of the following functions, use Lagrange Multipliers to find all extrema of f subject to the given constraints.

   (a) $f(x,y) = x^2 + y \quad$ constrain : $x^2 + 2y^2 = 1$

(b) $f(x, y, z) = xyz$ contrain: $2x + 3y + z = 6$

(c) $f(x, y, z) = x^2 + y^2 + z^2$, constraint $5x^2 + 9y^2 + 6z^2 + 4yz - 1 = 0$.

(d) $f(x, y, z) = x^2 + y^2 + z^2$, constraint $x^2 + y^2 + 2z^2 - 4 = 0$ and $xyz - 1 = 0$.

(e) $f(x, y, z) = x + y + z$, constrains: $y^2 - x^2 = 1$; and $x + 2z = 1$

(f) $f(x, y, z, w) = 3x + y + w$, constraints $3x^2 + y + 4z^3 = 1$ and $-x^3 + 3z^4 + w = 0$.

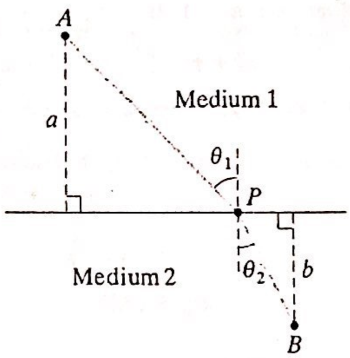
10. Find the extrema of the function f defined by

$$f(x, y) = \int_x^y te^{-t^2} dt$$

subjects to the constrain $g(x, y) = e^{x^2} + e^{y^2} = 8$.

11. A cargo container (in the shape of a rectangular solid) must have a volume of $480m^3$. The bottom will cost 50$ per square meter to construct and sides and the top will cost 30$ per square meter to construct. Use Lagrange Multiplier to find the dimensions of the container of this size that has minimum cost.

(a) Figuer 1



(b) Figuer 2

12. Your company must design a storage tank for Super Suds liquid laundry detergent. The customer's specifications call for a cylindrical tank with hemispherical ends (see Figure 1), and the tank is to hold $8000\,m^3$ of detergent. Suppose that it costs twice as much (per square foot of sheet metal used) to machine the hemispherical ends of the tank as it does to make the cylindrical part. What is the minimum cost of manufacturing the tank?

13. A ray of light travels at a constant speed in a uniform medium, but in different media (such as air and water) light travels at different speeds. For example, if a ray of light passes from air to water, it is bent (or refracted) as shown in Figure 2. Suppose the speed of light in medium 1 is v_1 and in medium 2 is v_2. Then, by Fermat's principle of least time, the light will strike the boundary between medium 1 and medium 2 at a point P so that the total time the light travels is minimized.

(a) Determine the total time the light travels in going from point A to point B via point P as shown in Figure 2.

(b) Use the method of Lagrange multipliers to establish **Snell's law of refraction:** that the total travel time is minimized when

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

(Hint: The horizontal and vertical separations of A and B are constant.)

14. Find the largest sphere centered at the origin that can be inscribed in the ellipsoid

$$3x^2 + 2y^2 + z^2 = 6.$$

15. Use Lagrange multiplier to show that the distance from a point (x_0, y_0) to the line $ax+by = c$ is defined by

$$D = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2+b^2}}.$$

16. Find the distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 - 225 = 0$.

17. Find the distance from between the ellipsoid (E) and plane (P), where

$$(E):\ 2x^2 + y^2 + 2z^2 - 8 = 0 \quad \text{and} \quad (P):\ x + y + z - 10 = 0.$$

18. Find the vertices of the ellipsoide $4x^2 + 9y^2 + 6z^2 + 4yz - 4 = 0$

19. Find the point closest to the point $(2, 5, -1)$ and on the line of intersection of the planes $x - 2y + 3z = 8$ and $2z - y = 3$.

20. Find the nearest and farthest distance from the origin to the conic section obtained by intersecting of the cone $z^2 = x^2 + y^2$ and the plane $x + y - z + 2 = 0$

21. (a) Find the maximum of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} x_k^2$$

subject to the constraint $\sum_{k=1}^{n} x_k^2 = 1$.

(b) Deduce that for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$: $\left|\prod_{k=1}^{n} x_k\right| \le \left(\frac{\|x\|}{\sqrt{n}}\right)^n$

22. Consider the problem of finding extrema of $f(x, y, z) = x^2 + y^2$ subject to the constraint $z = c$, where c is any constant.

   (a) Use the method of Lagrange multipliers to identify the critical points of f subject to the constraint given above.

   (b) Find the Hessian of the function L, where $L(\lambda, x, y, z) = f(x, y, z) - \lambda(z - c)$. for each critical point you found in part (a). Try to use the second derivative test for constrained extrema to determine the nature of the critical points you found in part (a). What happens?

   (c) Repeat part (b), this time using the variable ordering (z, y, x) What does the second derivative test tell you now?

   (d) Without making any detailed calculations, discuss why f must attain its minimum value at the point $(0, 0, c)$.

I_2 TD4

1) Differentiation in several numbers)

① - Find the equation of tangent place and find symmetric equation of the normal line to the surface at the given point.

(a) $f(x,y) = xy + \ln\sqrt[3]{1+x^2+2y^4}$ at $(1,0)$

$= xy + \frac{1}{3}\ln(1+x^2+2y^4)$

By using equation of tangent form

$F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)=0$

$\Rightarrow \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x,y_0)(y-y_0)=0$

$\frac{\partial f}{\partial x_0} = y_0 + \frac{1}{3}\left(\frac{2x_0}{1+x_0^2+4y_0^4}\right)$

$\Rightarrow \frac{\partial f}{\partial x}(1,0) = 0 + \frac{1}{3}\left(\frac{2}{1+1}\right) = \frac{1}{3}$

$\frac{\partial f}{\partial y_0} = x_0 + \frac{1}{3}\left(\frac{16y_0^3}{1+x_0^2+4y_0^4}\right)$

$\Rightarrow \frac{\partial f}{\partial y}(1,0) = 1+0 = 1$

we get, $\frac{1}{3}(x-1) + (y-0) = 0$

$\Rightarrow \frac{1}{3}x + y - \frac{1}{3} = 0$

Therefore the equation of the tangent plane is $\frac{1}{3}x+y-\frac{1}{3}=0$ and symmetric equation of normal line to the surface $\frac{x-1}{1/3} = \frac{y}{1}$

b)- $F(x,y,z) = y\ln xz^2 - 2 = 0$ at $(e,2,1)$

By using equation of tangent place

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)=0$

$\frac{\partial f}{\partial x_0}(x_0,y_0,z_0) = y_0 \times \frac{z_0^2}{x_0 z_0^2}$

$\Rightarrow \frac{\partial f}{\partial x}(e,2,1) = \frac{2}{e}$

• $\frac{\partial f}{\partial y_0}(x_0,y_0,z_0) = \ln x_0 z_0^2$

$\Rightarrow \frac{\partial f}{\partial y_0}(e,2,1) = \ln e = 1$

• $\frac{\partial f}{\partial z_0}(x_0,y_0,z_0) = y_0 \frac{2z_0x_0}{x_0 z_0^2}$

$\Rightarrow \frac{\partial f}{\partial z_0}(e,2,1) = 2\,\frac{2\cdot 1\cdot e}{e\cdot 1^2} = 4$

Then $\frac{2}{e}(x-e) + (y-2) + 4(z-1) = 0$

$\frac{2}{e}x - 2 + y - 2 + 4z - 4 = 0$

$\frac{2}{e}x + y + 4z - 8 = 0$

Therefore the equation of the tangent place is $\frac{2}{e}x+y+4z-8=0$. and symmetric equation of the normal line to surface is $\frac{x-e}{2/e} = \frac{y-2}{1} = \frac{(z-1)}{4}$

② Show that the tangent plance to the Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at point (x_0,y_0,z_0) is given by $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$

let $F(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$

By using equation of tangent form

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)=0$

$\Rightarrow \frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z}{c^2}(z-z_0)=0$

$\Rightarrow \frac{x_0x}{a^2} - \frac{x_0^2}{a^2} + \frac{y_0y}{b^2} - \frac{y_0^2}{b^2} + \frac{zz_0}{c^2} - \frac{z_0^2}{c^2} = 0$

$\Rightarrow \left(\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{zz_0}{c^2}\right) - \left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=0$

$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{zz_0}{c^2}-1=0$ true

$$\boxed{\text{Thus } \frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{zz_0}{c^2}-1=0}$$

(3) Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$f(x,y)=\int_{xy}^{x^2}(x^2yt+xt^2)\,dt$$

Find the equation of tangent plane to the surface $z=f(x,y)$ at the point $(1,0)$

we have $f(x,y)=\int_{xy}^{x^2}(x^2yt+xt^2)\,dt$

$$=\left[x^2y\frac{t^2}{2}+x\frac{t^3}{3}\right]_{xy}^{x^2}$$

$$=\frac{x^6y}{2}+\frac{x^7}{3}-\frac{x^4y^3}{2}-\frac{x^4y^3}{3}$$

$$=\frac{x^6y}{2}+\frac{x^7}{3}-\frac{5}{3}x^4y^3$$

by using Equation of tangent plane we get

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$

- $f_x(x_0,y_0)=3x_0^5y_0+\frac{7}{3}x_0^6-\frac{20}{3}x_0^3y_0^3$

$\Rightarrow f_x(1,0)=\frac{7}{3}$

- $f_y(x_0,y_0)=\frac{x_0^6}{2}-5x_0^4y_0^2$

$\Rightarrow f_y(1,0)=\frac{1}{2}-5\times0=\frac{1}{2}$

we get

$$\frac{7}{3}(x-1)+\frac{1}{2}(y-0)=0$$

$$\Leftrightarrow \frac{7}{3}x+\frac{1}{2}y-7/3=0$$

Therefore $\frac{7}{3}x+\frac{1}{2}y-7/3=0$ is the equation of the tangent plane.

(4)- Find the global extrema of the function over the region R.

(a) $f(x,y)=x^2-xy+y^2+x-2y+1$

$R=\{(x,y)\in y\le\sqrt{9-x^2},\ -3\le x\le 3\}$

$f_x(x,y)=2x-y$

$f_y(x,y)=-x+2y-2$

So $\nabla f(x,y)=(2x-y,\,-x+2y-2)=0$

$$\begin{cases}2x-y=0\\-x+2y-2=0\end{cases}$$

$\Rightarrow x=0\ ;\ y=1$

$f(0,1)=1-2+1=0$

and $f(3,0)=9+3+1=13$

$f(-3,0)=9-3+1=7$

Thus global minimum $f(0,1)=0$

global maximum $f(3,0)=13$

b)- $f(x,y)=x^2-xy+y^2+3x-2y+1$

$R=\{(x,y)\in[-2,0]\times[0,1]\}$

$f_x(x,y)=2x-y+3$

$f_y(x,y)=-x+2y-2$

$\nabla f(x,y)=0\Rightarrow\begin{cases}2x-y+3=0\\-x+2y-2=0\end{cases}$

$\Rightarrow x=-4/3\ ;\ y=1/3$

Since $f(-2,0)=4-6+1=-1$, $f(0,0)=1$, $f(-4/3,1/3)$
$f(-2,1)=4+2+1-6-2+1=0$

$=\frac{16}{9}+\frac{4}{9}+\frac{1}{9}-\frac{2}{3}-4+1$

$=-4/3$

Thus the global maximum is $f(0,0)=1$

global minimum is $f(-4/3,1/3)=-\frac{4}{3}$

5. Determine the local extrema and/or saddle points of the following functions -

(a) $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$

we have

$$\nabla f(x,y) = 0 \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$$

then $\begin{cases} 2y - 4x = 0 \\ 2x - 10y = -4 \quad \times 2 \end{cases}$

$$y = \frac{4\times 2}{18} = 4/9$$

and $x = 2/9$

then $a(2/9, 4/9)$ is critical point of f

And $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -10 \end{pmatrix}$

$\Rightarrow Hf(4/9, 2/9) = \begin{pmatrix} -4 & 2 \\ 2 & -10 \end{pmatrix}$

- $H_1 = |-4| = -4 < 0$
- $H_2 = \begin{vmatrix} -4 & 2 \\ 2 & -10 \end{vmatrix} = 40 - 4 = 36 > 0$

Thus $Hf(2/9, 4/9)$ is negative definite (ND)

$\therefore$ then the $a(2/9, 4/9)$ is the local maximum of point f.

b) $f(x,y) = 2x^3 + (x-y)^2 - 6y$

we have

$$\nabla f(x,y) = 0 \Leftrightarrow \begin{cases} 6x^2 + 2(x-y) = 0 \\ -2(x-y) - 6 = 0 \end{cases}$$

$$\Leftrightarrow + \begin{cases} 6x^2 + 2x - 2y = 0 \\ -2x + 2y - 6 = 0 \end{cases}$$

$$6x^2 - 6 = 0 \Rightarrow x = \pm 1$$

if $x = 1 \Rightarrow y = \frac{8}{2} = 4$

if $x = -1 \Rightarrow y = \frac{4}{2} = 2$

we get $a_1(1,4)$ & $a_2(-1,2)$ are cp of f

The Hessian of function f

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x+2 & -2 \\ -2 & 2 \end{pmatrix}$$

$Hf(1,4) = \begin{pmatrix} 14 & -2 \\ -2 & 2 \end{pmatrix}$ extrema

$H_1 = |14| = 14 > 0$

$H_2 = \begin{vmatrix} 14 & -2 \\ -2 & 2 \end{vmatrix} = 28 - 4 = 24 > 0$

Thus $Hf_{(1,4)}$ is positive definite, the $a_1(1,4)$ is the local minimum of f

$Hf(-1,2) = \begin{pmatrix} -10 & -2 \\ -2 & 2 \end{pmatrix}$

$H_1 = |-10| = -10 < 0$

$H_2 = \begin{vmatrix} -10 & -2 \\ -2 & 2 \end{vmatrix} = -20 - 4 = -24 < 0$

Thus $Hf(-1,2)$ is neither PD nor ND

then $a_2(-1,2)$ is a saddle point of f

~~(c). $f(x,y) = \cos x + \cos y + \sin(x+y)$~~

~~we have~~

~~$\nabla f(x,y) = 0$~~ $\cancel{\begin{cases} -\sin x + \cos(x+y) = 0 \\ \sin y + \cos(x+y) = 0 \end{cases}}$

~~$-\sin x + \cos(x+y) = 0$~~

~~$\Leftrightarrow \cos(x+y) = \sin x = \cos(\pi/2 - x)$~~

~~$\Leftrightarrow \cos(x+y) = \cos(\pi/2 - x)$~~

$\cancel{\begin{cases} x+y = \pi/2 - x \\ x+y = -\pi/2 + x \end{cases} \Rightarrow \begin{cases} y = -\pi/2 \\ x = \pi/2 \end{cases}}$

~~$\sin y + \cos(x+y) = 0$~~

~~$\Leftrightarrow \cos(x+y) = \sin y = \cos(y - \pi/2)$~~

$\cancel{\begin{cases} x+y = y - \pi/2 \\ x+y = -y + \pi/2 \end{cases} \Rightarrow \begin{cases} x = -\pi/2 \\ y = \frac{\pi}{2} \end{cases}}$

~~Then $a_1(-\pi/2, \pi/2)$ & $a_2(\pi/2, -\pi/2)$ are the critical point of f~~

$\cancel{Hf = \begin{pmatrix} -\cos x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & \cos y - \sin(x+y) \end{pmatrix}}$

$\cancel{Hf(-\pi/2, \pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$

c)- $f(x,y) = \cos x + \cancel{\cos} y + \sin(x+y)$

$\nabla f(x,y)=0 \Rightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} -\sin x + \cos(x+y) = 0 \\ -\sin y + \cos(x+y) = 0 \end{cases}$

$-\sin x + \sin y = 0$

$\Rightarrow \begin{cases} y = x \\ y = \pi - x \end{cases}$

+ for $y = x$

$f(x,y) = \cos x + \cos x + \sin 2x$
$= 2\cos x + \sin 2x$

$f_x(x,y) = 0 \Rightarrow -2\sin x + 2\cos 2x = 0$

$\sin x = \cos 2x$

$\begin{cases} \pi/2 - x = 2x \\ \pi/2 - x = -2x \end{cases} \Rightarrow \begin{cases} x = \pi/6 = y \\ x = -\pi/2 = y \end{cases}$

+ for $y = \pi - x$

$f(x, \pi - x) = \cancel{\cos x} - \cancel{\cos x} + \sin(\cancel{x} + \pi - \cancel{x})$

$= 0$

Then $(\pi/6, \pi/6)$ & $(-\pi/2, -\pi/2)$ are CP of f

The Hessian of function f

$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -\cos x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\cos y - \sin(x+y) \end{pmatrix}$

+ $a_1(\pi/6, \pi/6)$

$H_1 = -\cos\pi/6 - \sin\frac{\pi}{3} = -\sqrt{3} < 0$

$H_2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

Thus $H_f(\pi/6, \pi/6)$ is ND then CP $(\pi/6, \pi/6)$ local maximum.

+ $a_2(-\pi/2, -\pi/2)$

$H_1 = 0$

Thus $H_f(-\pi/2, -\pi/2)$ is neither PD nor ND

Then CP $(-\pi/2, -\pi/2)$ is saddle point

$\Rightarrow H_f(0,-6,-3) = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{vmatrix}$

$H_1 = |2| = 2 > 0$

$H_2 = \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1 < 0$

$H_3 = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 0 & 0 \\ -2 & 0 & 2 \end{vmatrix} = \cancel{2} \cancel{<} 0$

Thus $Hf(0,6,-3)$ is neither PD nor ND, and $(0,-6,-3)$ is saddle point.

e)- $f(x,y,z) = (x^2 + 2y^2 + 1)\cos z$

$f_x = 2x\cos z$

$f_y = 4y\cos z$

$f_z = -\sin z(x^2 + 2y^2 + 1)$

then $\nabla f(x,y,z) = 0$ $\begin{cases} 2x\cos z & (1) \\ 2y\cos z & (2) \\ -\sin z(x^2+2y^2+1) = 0 & (3) \end{cases}$

Following by (3)

$\sin z = 0$ or $x^2 + 2y^2 + 1 = 0$ impossible

- for $\sin z = 0$

we have $\cos^2 z + \sin^2 z = 1$
$\Rightarrow \cos z = \pm 1$

substitute into (1) & (2)

we get $x = 0$ & $y = 0$

follow by (1) & (2) if $\cos z = 1$ that is $z = 2k\pi$, $k \in \mathbb{Z}$

we get $x = 0$, $y = 0$

follow by (1) & (2) if $\cos z = -1$ that is $z = \pi + 2p\pi$, $p \in \mathbb{Z}$

we get $x = 0$, $y = 0$

Then $a_1(0,0,2k\pi)$ & $a_2(0,0,2p\pi + \pi)$ are CP of f with $k, p \in \mathbb{Z}$

$H_f = \begin{pmatrix} 2\cos z & 0 & -2x\sin z \\ 0 & 2\cos z & -2y\sin z \\ -2x\sin z & -2y\sin z & -(x^2+2y^2+1)\cos z \end{pmatrix}$

- for $a_1 = (0,0,2k\pi)$

$Hf(a_1) = \begin{pmatrix} 2\cos 2k\pi & 0 & 0 \\ 0 & 2\cos 2k\pi & 0 \\ 0 & 0 & 2\cos 2k\pi \end{pmatrix}$

$H_1 = |2\cos(2k\pi)| = 2\cos(2k\pi)$ can be <0 or $0>0$

$H_2 = 4\cos^2(2k\pi) > 0$

$H_3 = -4\cos^3(2k\pi)$ can be <0 or $0>0$

Therefore $Hf(a_1)$ is neither PD nor ND for $k \in \mathbb{Z}$ That is $(0,0,2k\pi)$ is a saddle point of f.

- for $a_2 = (0,0,\pi+2p\pi)$, $p \in \mathbb{Z}$

$$Hf(a_2) = \begin{pmatrix} 2\cos(\pi+2p\pi) & 0 & 0 \\ 0 & 2\cos(\pi+2p\pi) & 0 \\ 0 & 0 & 2\cos(\pi+2p\pi) \end{pmatrix}$$

$H_1 = 2\cos(\pi+2p\pi)$ can be >0 or $0<0$

$H_2 = 2\cos^2(\pi+2p\pi) > 0$

$H_3 = -4\cos^3(\pi+2p\pi)$ — " —

Therefor $Hf(a_2)$ is neither PD nor ND for $p \in \mathbb{Z}$ That is $(0,0,\pi+2p\pi)$ is a saddle point.

f. $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xz + 1$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 4x^3 + 2xy + z = 0 & (1) \\ x^2 + 2y = 0 \Rightarrow y = -\frac{x^2}{2} & (2) \\ 2z + x = 0 \Rightarrow z = -\frac{x}{2} & (3) \end{cases}$$

Then (1) become

$4x^3 + x(-x^2) - \frac{x}{2} = 0$

$8x^3 - 2x^3 - x = 0$

$x(8x^2 - 2x^2 - 1) = 0$

$x(6x^2-1) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 1/\sqrt{6} \\ x = -1/\sqrt{6} \end{cases}$

if $x = 0$ then $y = 0$ & $z = 0$

if $x = 1/\sqrt{6}$ then $y = -\frac{1}{12}$ & $z = -\frac{1}{2\sqrt{6}}$

if $x = -1/\sqrt{6}$ then $y = -\frac{1}{12}$ & $z = \frac{1}{2\sqrt{6}}$

Then $\left.\begin{array}{l} a_1(0,0,0) \\ a_2(1/\sqrt{6}, -1/12, -\frac{1}{2\sqrt{6}}) \\ a_3(-1/\sqrt{6}, -1/12, \frac{1}{2\sqrt{6}}) \end{array}\right\}$ is CP of point f

and $$Hf_{(a_i)} = \begin{pmatrix} 12x^2+2y & 2x & 1 \\ 2x & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

- for $a_1(0,0,0)$

$$Hf(a_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$H_1 = |0| = 0$

Therefore $Hf_{(a_1)}$ is neither PD nor ND That is $(0,0,0)$ is saddle point

- for $a_2(1/\sqrt{6}, -1/12, -\frac{1}{2\sqrt{6}})$

$$Hf(a_2) = \begin{pmatrix} 11/6 & 2/\sqrt{6} & 1 \\ 2/\sqrt{6} & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$H_1 = |11/6| = 11/6 > 0$

$H_2 = \frac{22}{6} - \frac{4}{6} = \frac{18}{6} = 3 > 0$

$$H_3 = \begin{vmatrix} 11/6 & 2/\sqrt{6} & 1 \\ 2/\sqrt{6} & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} \begin{matrix} 11/6 & 2/\sqrt{6} \\ 2/\sqrt{6} & 2 \\ 1 & 0 \end{matrix}$$

$= \frac{44}{6} - 2 + \frac{8}{6} = \frac{52}{6} - 2 = \frac{40}{6} = \frac{20}{3} > 0$

Therefore $Hf(a_2)$ is PD and $(1/\sqrt{6}, -1/12, -\frac{1}{2\sqrt{6}})$ is local maximum

- for $a_3(1/\sqrt{6}, -1/12, -\frac{1}{2\sqrt{6}})$

$$Hf(a_3) = \begin{pmatrix} 11/6 & 2/\sqrt{6} & 1 \\ 2/\sqrt{6} & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$H_1 = 11/6 > 0$

$H_2 = 3 > 0$

$H_3 = 20/3 > 0$

There fore $Hf(a_3)$ is PD and $(1/\sqrt{6}, -1/12, \frac{1}{2\sqrt{6}})$ is local maximum.

⑥ Show that $(0,0)$ is the critical point of the function f defined by

$$f(x,y) = 10x^2\cos y + \int_{x^2}^{y^2} \ln\sqrt{2+x^2+\cos(ty)}\,dt$$

$$= 10x^2\cos y + \frac{1}{2}\int_{x^2}^{y^2} \ln(2+x^2+\cos ty)\,dt$$

$$f_x(x,y) = 20x\cos y + \frac{1}{2}\int_{x^2}^{y^2} \frac{2x}{2+x^2+\cos yt}\,dt$$
$$- x\ln(2+x^2+\cos x^2 y)$$

$$f_y(x,y) = -10x^2\cos y + \frac{1}{2}\int_{x^2}^{y^2} \frac{-\sin yt\, t}{2+x^2+\cos yt}\,dt$$
$$+\frac{1}{2}\sin y\ln(2+x^2+\cos y^3)$$

Then $f_x(0,0) = 0$
$f_y(0,0) = 0$

Thus $\nabla f(0,0) = 0 \Rightarrow (0,0)$ is the critical point

The Hessian of a function f

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Since $f_{xx}(x,y) = 20\cos y - \ln(2+x^2+\cos x^2 y) - \dfrac{x(2x-2xy\sin x^2 y)}{2+x^2+\cos x^2 y}$

$\Rightarrow f_{xx}(0,0) = 20 - \ln 3$

$$f_{yy} = -\sin y \cdot 10x^2 + \frac{1}{2}\cos y\ln(2+x^2+\cos y^3) + \frac{\frac{1}{2}\sin y(\quad +3y^2\sin y^3)}{2+x^2+\cos y^3}$$

$\Rightarrow f_{yy}(0,0) = \frac{1}{2}\ln 3$

$$f_{xy} = -\sin y\, 20x + \frac{-x(-\sin x^2 y)}{2+x^2+\cos x^2 y}$$

$\Rightarrow f_{xy}(0,0) = 0$

then $H_f(0,0) = \begin{pmatrix} 20-\ln 3 & 0 \\ 0 & \frac{1}{2}\ln 3 \end{pmatrix}$

$H_1 = |20-\ln 3| = 20 - \ln 3 > 0$

$H_2 = \frac{1}{2}\ln 3(20-\ln 3) = 10\ln 3 - \frac{1}{2}\ln 3 > 0$

Thus $H_f(0,0)$ is PD
That is $(0,0)$ is local minimum of f

⑦ - Study the extrema of the function $f: \mathbb{R}^n \to \mathbb{R}$ where

$$f(x_1, x_2, \dots, x_n) = e^{-(x_1^2+x_2^2+\dots+x_n^2)}$$

we have $f(x_1, x_2, \dots, x_n) = e^{-(x_1^2+\dots+x_n^2)}$

$0 < f(x) \le 1$, $\forall x \in \mathbb{R}^n$

So $\dfrac{\partial f}{\partial x_i} = -x_i e^{-(x_1^2+\dots+x_n^2)}$; $\forall i = 1,2,\dots,n$

$\nabla f(x) = 0 \Rightarrow \dfrac{\partial f}{\partial x_i} = 0$; $\forall i = 1,2,\dots,n$

$\Leftrightarrow -x_i e^{-(x_1^2+\dots+x_n^2)} = 0$

$\Rightarrow x_i = 0$; $\forall i = 1,2,\dots,n$

Then $x = (0,0,\dots,0)$ is the critical point of f

So $f(0,0,\dots,0) = 1$

Thus f has a maximum at $(0,\dots,0)$ and $f(0,\dots,0) = 1$

⑧ Find x_1 & x_2 so as to maximise the revenue

we have $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$\nabla R(x_1,x_2) = 0 \Rightarrow \begin{cases} -10x_1 - 2x_2 + 42 = 0 & (1) \\ -16x_2 - 2x_1 + 102 = 0 & (2) \end{cases}$$

$(1) - 5(2)$: $-2x_2 + 80x_2 + 42 - 510 = 0$

$\Rightarrow x_2 = \dfrac{510-42}{-78} = 6$

$\Rightarrow x_1 = 3$

Thus $(x_1, x_2) = (3,6)$ is the CP of f

Hessian $H_f = \begin{pmatrix} -10 & -2 \\ -2 & -16 \end{pmatrix} \Rightarrow H_f(3,6) = \begin{pmatrix} -10 & -2 \\ -2 & -16 \end{pmatrix}$

$H_1 = -10 > 0$
$H_2 = 160 - 4 = 156 > 0$

Thus $x_1 = 3$ & $x_2 = 6$ are the local maximum of f

⑨ For each of the following functions, use Lagrang Multipliers to find the extrema of f subject to the given constraints.

(a) $f(x,y) = x^2+y$ constrain $x^2+2y^2=1$

Let $L(\lambda, x, y) = f(x,y) - \lambda(g(x,y)-c)$
$= x^2+y-\lambda(x^2+2y^2-1)$

$\nabla L(\lambda,x,y) = 0 \Leftrightarrow \left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}\right) = (0,0,0)$

we get $\begin{cases} -x^2-2y^2+1=0 & (1) \\ 2x-2\lambda x = 0 & (2) \\ 1-4\lambda y = 0 & (3) \end{cases}$

Following by (2) : $x(2-2\lambda) = 0 \Rightarrow x=0$ or $\lambda=1$

substitute into (1) and (3) :

For $x=0$: $\begin{cases} 2y^2-1=0 & (4) \\ 1-4\lambda y = 0 & (5) \end{cases}$

Following by (4) : $2y^2=1 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

substitute into (5)

if $y=\frac{1}{\sqrt{2}} \Rightarrow (5)$: $1-4\lambda\left(\frac{1}{\sqrt{2}}\right)=0 \Rightarrow \lambda = \frac{\sqrt{2}}{4}$

if $y=-\frac{1}{\sqrt{2}} \Rightarrow (5)$: $1-4\lambda\left(-\frac{1}{\sqrt{2}}\right)=0 \Rightarrow \lambda = -\frac{\sqrt{2}}{4}$

we got $a_1 = (0, \frac{1}{\sqrt{2}})$ is a CP of f with $\lambda = \frac{\sqrt{2}}{4}$

$a_2 = (0, -\frac{1}{\sqrt{2}})$ is a CP of f with $\lambda = -\frac{\sqrt{2}}{4}$

For λ : $\begin{cases} x^2+2y^2-1=0 & (*) \\ 1-4y=0 \Rightarrow y = 1/4 \end{cases}$

follow by (*) : $x^2+2\times\frac{1}{16}-1=0$

$\Rightarrow x^2 = \frac{7}{8} \Rightarrow x = \pm\frac{\sqrt{7}}{2\sqrt{2}}$

we got $a_3 = \left(\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ is a CP of f with $\lambda=1$

$a_4 = \left(\frac{-\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ is a CP of f with $\lambda=1$

$H_L(\lambda,x,y) = \begin{pmatrix} 0 & -2x & -4y \\ -2x & 2-2\lambda & 0 \\ -4y & 0 & -4\lambda \end{pmatrix}$

- For $a_1 = (0, \frac{1}{\sqrt{2}})$ with $\lambda = \frac{\sqrt{2}}{4}$

$H_L\left(\frac{\sqrt{2}}{4}, 0, \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 0 & 0 & -4/\sqrt{2} \\ 0 & 2-\frac{\sqrt{2}}{2} & 0 \\ -\frac{4}{\sqrt{2}} & 0 & -\sqrt{2} \end{pmatrix}$

$-d_3 = -\begin{vmatrix} 0 & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 2-\frac{\sqrt{2}}{2} & 0 \\ -\frac{4}{\sqrt{2}} & 0 & -\sqrt{2} \end{vmatrix} = \frac{4}{\sqrt{2}}\begin{vmatrix} 0 & -\frac{4}{\sqrt{2}} \\ 2-\frac{\sqrt{2}}{2} & 0 \end{vmatrix}$

$= \frac{4}{\sqrt{2}}\left(\frac{8}{\sqrt{2}}-2\right) = \frac{-8}{\sqrt{2}}+16 > 0$

Thus $a_1 = (0, \frac{1}{\sqrt{2}})$ is a local minimum point of f subjects to the constrain $x^2+2y^2=1$ of f

- For $a_2 = (0, \frac{-1}{\sqrt{2}})$ with $\lambda = -\frac{\sqrt{2}}{4}$

$H_L\left(-\frac{\sqrt{2}}{4}, 0, -\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 0 & 0 & \frac{4}{\sqrt{2}} \\ 0 & 2+\frac{\sqrt{2}}{2} & 0 \\ \frac{4}{\sqrt{2}} & 0 & \sqrt{2} \end{pmatrix}$

$-d_3 = -\begin{vmatrix} 0 & 0 & \frac{4}{\sqrt{2}} \\ 0 & 2+\frac{\sqrt{2}}{2} & 0 \\ \frac{4}{\sqrt{2}} & 0 & \sqrt{2} \end{vmatrix} = \frac{4}{\sqrt{2}}\left(\frac{8}{\sqrt{2}}+2\right)$

$= 16+\frac{8}{\sqrt{2}} > 0$

Thus $a_2 = (0, -\frac{1}{\sqrt{2}})$ is a local minimum point of f subjects to the constrain $x^2+2y^2=1$

- For $a_3 = \left(\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ with $\lambda = 1$

$$H_L\left(1, \frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right) = \begin{pmatrix} 0 & -\frac{\sqrt{7}}{\sqrt{2}} & -1 \\ -\frac{\sqrt{7}}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & -4 \end{pmatrix}$$

$$-d_3 = -\begin{vmatrix} 0 & -\frac{\sqrt{7}}{\sqrt{2}} & -1 \\ -\frac{\sqrt{7}}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & -4 \end{vmatrix} = -\frac{\sqrt{7}}{\sqrt{2}}\left(\frac{4\sqrt{7}}{\sqrt{2}}\right) = -14 < 0$$

Thus $a_3 = \left(\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ is a local maximum point of f subjects to the constrain $x^2+2y^2=1$

for $a_4 = \left(-\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ with $\lambda = 1$

$$H_L = \left(1, -\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right) = \begin{pmatrix} 0 & \frac{\sqrt{7}}{2\sqrt{2}} & -1 \\ \frac{\sqrt{7}}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

$$-d_3 = -\begin{vmatrix} 0 & \frac{\sqrt{7}}{\sqrt{2}} & -1 \\ \frac{\sqrt{7}}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & -4 \end{vmatrix} = \frac{\sqrt{7}}{\sqrt{2}}\left(-\frac{4\sqrt{7}}{\sqrt{2}}\right) = -14 < 0$$

Thus $a_4 = \left(-\frac{\sqrt{7}}{2\sqrt{2}}, \frac{1}{4}\right)$ is a local maximum point of f subjects to the constrain $x^2+2y^2=1$

(b) $f(x,y,z) = xyz$ constraint $2x+3y+z=6$

(i) Let $L(\lambda, x, y, z) = f(x,y,z) - \lambda(g(x,y,z) - 6)$

Then $L(\lambda, x, y, z) = xyz - \lambda(2x+3y+z-6)$

(ii) $\nabla L(\lambda, x, y, z) = (-2x-3y-z+6, yz-2\lambda, xz-3\lambda, xy-\lambda)$

$$\nabla L(\lambda, x, y, z) = 0 \Rightarrow \begin{cases} -2x-3y-z+6=0 & (1) \\ yz-2\lambda=0 & (2) \\ xz-3\lambda=0 & (3) \\ xy-\lambda=0 & (4) \end{cases}$$

Following by (4): $\lambda = xy$ substitute into (1), (2) and (3) we get

$$\begin{cases} -2x-3y-z+6=0 & (5) \\ yz-2xy=0 & (6) \\ xz-3yx=0 & (7) \end{cases}$$

by following (6): $yz - 2xy = 0 \Rightarrow y(z-2x) = 0$

Then $y=0$ or $z-2x=0 \Rightarrow z=2x$

For $y=0$ $\begin{cases} -2x-z+6=0 & (8) \\ xz=0 & (9) \end{cases}$

Follow by (9) $x=0$ or $z=0$ substitute into (8)

⊕ If $x=0 \Rightarrow (8): z=6$
we get $a_1 = (0,0,6)$ is a CP of f with $\lambda=0$

⊕ If $z=0 \Rightarrow (8): x = \frac{6}{2} = 3$
we get $a_2 = (3,0,0)$ is a CP of f with $\lambda = 0$

For $z=2x$ $\begin{cases} -2x-3y-2x+6=0 & (10) \\ 2x^2-3xy=0 & (11) \end{cases}$

Follow by (11) $2x^2 - 3xy = 0 \Rightarrow x(2x-3y)=0$

Then $x=0$ or $2x-3y=0 \Rightarrow x = \frac{3}{2}y$

⊕ If $x=0 \Rightarrow (10): -3y=-6 \Rightarrow y=2$
we get $a_3 = (0,2,0)$ is a CP of f with $\lambda = 0$

⊕ If $x = \frac{3}{2}y \Rightarrow (10): -4 \times \frac{3}{2}y - 3y = -6$
$y = \frac{6}{9} = \frac{2}{3}$

we get $a_4 = (1, \frac{2}{3}, 2)$ is a CP of f with $\lambda = \frac{2}{3}$

(iii) For $a_1 = (0,0,6)$ is a CP of f with $\lambda=0$

$$H_L(\lambda, x, y, z) = \begin{pmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & z & y \\ -3 & z & 0 & x \\ -1 & y & x & 0 \end{pmatrix}$$

$$H_L(0,0,0,6) = \begin{pmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 6 & 0 \\ -3 & 6 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(iv) (s): $-d_3, -d_4$ because $n=3$ and $k=1$

$$-d_3 = \begin{vmatrix} 0 & -2 & -3 \\ -2 & 0 & 6 \\ -3 & 6 & 0 \end{vmatrix} = -2(18)+3(-12) = -72<0$$

$$-d_4 = \begin{vmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 6 & 0 \\ -3 & 6 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -2 & -3 & -1 \\ 0 & 6 & 0 \\ 6 & 0 & 0 \end{vmatrix} = -6(6) = -36<0$$

Thus (s) doesn't satisfy the two conditions then f has a constrained saddle point at $a_1=(0,0,6)$

(iii)$_2$ For $a_2=(3,0,0)$ is a CP of f with $\lambda=0$

$$H_L=(0,3,0,0) = \begin{pmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 3 \\ -1 & 0 & 3 & 0 \end{pmatrix}$$

$$-d_3 = \begin{vmatrix} 0 & -2 & -3 \\ -2 & 0 & 0 \\ -3 & 0 & 0 \end{vmatrix} = -3(0)=0$$

Thus we clearly see that (s) doesn't satisfy the two conditions $a_2=(3,0,0)$ is a saddle point of f

(iii)$_3$ For $a_3=(0,2,0)$ with $\lambda=0$

$$H_L(0,0,2,0) = \begin{pmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 0 & 2 \\ -3 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix}$$

$$-d_3 = \begin{vmatrix} 0 & -2 & -3 \\ -2 & 0 & 0 \\ -3 & 0 & 0 \end{vmatrix} = 0$$

Thus (s) doesn't not satisfy the two condictions, then f has a constrained saddle point at a_3

(iii)$_4$ For $a_4=\left(1,\frac{2}{3},2\right)$ with $\lambda=\frac{2}{3}$

$$H_L\left(\tfrac{2}{3},1,\tfrac{2}{3},2\right) = \begin{pmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 2 & 2/3 \\ -3 & 2 & 0 & 1 \\ -1 & 2/3 & 1 & 0 \end{pmatrix}$$

$$-d_3 = \begin{vmatrix} 0 & -2 & -3 \\ -2 & 0 & 2 \\ -3 & 2 & 0 \end{vmatrix} = -[2(6)-3(-4)]$$

$$= -24<0$$

$$-d_4 = -\begin{vmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 2 & 2/3 \\ -3 & 2 & 0 & 1 \\ -1 & 2/3 & 1 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & -2 & -3 & -1 \\ -2 & 0 & 2 & 2/3 \\ -3 & 0 & -3 & 0 \\ -1 & 0 & 0 & -1/3 \end{vmatrix}$$

$$= -2\begin{vmatrix} -2 & 2 & 2/3 \\ -3 & -3 & 0 \\ -1 & 0 & -1/3 \end{vmatrix} = -4\begin{vmatrix} -2 & 1 & 0 \\ -3 & -3 & 0 \\ -1 & 0 & -1/3 \end{vmatrix}$$

$$= \frac{4}{3}(6+3) = 12>0$$

That is, (s) begin with negative sign and after it changes alternatively by seeing $-d_3<0$ and then $-d_4>0$

Thus $a_4=\left(1,\frac{2}{3},2\right)$ is a local maximum point of f subjects to constran $g(x,y,z)=2x+3y+z=6$

(c) $f(x,y,z) = x^2+y^2+z^2$ constraint $5x^2+9y^2+6z^2+4yz-1=0$

① Let $L(\lambda,x,y,z) = f(x,y,z) - \lambda(g(x,y,z)-0)$

Then $L(\lambda,x,y,z) = x^2+y^2+z^2 - \lambda(5x^2+9y^2+6z^2+4yz-1)$

$\nabla L(\lambda,x,y,z) = 0 \Rightarrow \begin{cases} -5x^2-9y^2-6z^2-4yz+1=0 & (1) \\ 2x-10\lambda x=0 & (2) \\ 2y-18\lambda y-4\lambda z=0 & (3) \\ 2z-12\lambda z-4\lambda y=0 & (4) \end{cases}$

Following by (2) : $2x(1-5\lambda)=0 \Rightarrow x=0$ or $\lambda=1/5$

For $\lambda=1/5 \Rightarrow \begin{cases} -5x^2-9y^2-4yz+1=0 & (5) \\ 2y-\frac{18}{5}y-\frac{4}{5}z=0 & (6) \\ 2z-\frac{12}{5}z-\frac{4}{5}y=0 & (7) \end{cases}$

Following by (6) : $-\frac{8}{5}y = -\frac{4}{5}z \Rightarrow y=\frac{1}{2}z$

then (7): $-\frac{2}{5}z-\frac{4}{5}\left(\frac{1}{2}z\right)=0 \Rightarrow z=0$

$\Rightarrow y=0$

(5): $-5x^2+1=0 \Rightarrow x^2=\frac{1}{5} \Rightarrow x=\pm\frac{1}{\sqrt{5}}$

we get $a_1=\left(\frac{1}{\sqrt5},0,0\right)$ and $a_2=\left(-\frac{1}{\sqrt5},0,0\right)$ are the CP of f with $\lambda=1/5$

For $x=0$: $\begin{cases} -9y^2-6z^2-4yz+1=0 & (8) \\ 2y-18\lambda y-4\lambda z=0 & (9) \\ 2z-12\lambda z-4\lambda z=0 & (10) \end{cases}$

Following by (9) and (10):

$\begin{cases} (1-9\lambda)y-2\lambda z=0 & (9) \\ -2\lambda y+(1-6\lambda)z=0 & (10) \end{cases}$

or $\begin{cases} (1-9\lambda)y-2\lambda z=0 & (9) \\ -2\lambda y+(1-6\lambda)z=0 & (10) \end{cases}$

we can write as $\begin{pmatrix} 1-9\lambda & -2\lambda \\ -2\lambda & 1-6\lambda \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

It seem, $Ax=0$

Since $(y,z)\neq(0,0)$ because of (8),

we have $\det\begin{pmatrix} 1-9\lambda & -2\lambda \\ -2\lambda & 1-6\lambda \end{pmatrix}=0$

$\Rightarrow (1-6\lambda)(1-9\lambda)-4\lambda^2=0$

$\Rightarrow 1-6\lambda-9\lambda+54\lambda^2-4\lambda^2=0$

$\Rightarrow 50\lambda^2-15\lambda+1=0$

$\Rightarrow \Delta = 225-200=25$

$\lambda=\frac{15\pm5}{100} = \begin{cases} 1/5 \\ 1/10 \end{cases}$ substitute into (9) & (10)

For $\lambda=1/5$: $\begin{cases} \left(1-\frac{9}{5}\right)y-\frac{2}{5}z=0 & (9) \\ -\frac{2}{5}y+\left(1-\frac{6}{5}\right)z=0 & (10) \end{cases}$

or $\begin{cases} \frac{4}{5}y+\frac{2}{5}z=0 & (9) \\ \frac{2}{5}y+\frac{1}{5}z=0 & (10) \end{cases}$

$\Rightarrow y=-\frac{1}{2}z$ substitute into (8) we have

Following by (8) : $-9\left(\frac{1}{4}z^2\right)-6z^2+4\left(\frac{1}{2}z^2\right)+1=0$

$-\frac{9}{4}z^2-6z^2+2z^2+1=0$

$-\frac{25}{4}z^2=-1 \Rightarrow z=\pm\frac{2}{5}$

If $z=\frac{2}{5} \Rightarrow y=-\frac{1}{2}\left(\frac{2}{5}\right)=-1/5$

if $z=-2/5 \Rightarrow y=-\frac{1}{2}\left(-\frac{2}{5}\right)=1/5$

we got $a_3=(0,-1/5,\frac{2}{5})$ is CP of f with $\lambda=1/5$

$a_4=(0,1/5,-2/5)$ is CP of f with $\lambda=1/5$

• For $\lambda=1/10$: $\begin{cases} \left(1-\frac{9}{10}\right)y-1/5\,z=0 & (9) \\ -\frac{1}{5}y+\left(1-\frac{3}{5}\right)z=0 & (10) \end{cases}$

or $\begin{cases} \frac{1}{10}y-\frac{1}{5}y=0 & (9) \\ -\frac{1}{5}y+\frac{2}{5}z=0 & (10) \end{cases}$

$\Rightarrow y=2z$ substitute into (8)

Follow by (8): $-9(4z^2)-6z^2-4(2z)z+1=0$

$-50z^2=-1$

$z=\pm\frac{1}{5\sqrt2}$

If $z = \frac{1}{5\sqrt{2}} \Rightarrow y = 2\frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{5}$

If $z = -\frac{1}{5\sqrt{2}} \Rightarrow y = 2\left(\frac{-1}{5\sqrt{2}}\right) = -\frac{\sqrt{2}}{5}$

we got $a_5 = (0, \frac{\sqrt{2}}{5}, \frac{1}{5\sqrt{2}})$ is a CP of f with $\lambda = 1/10$

$a_6 = (0, -\frac{\sqrt{2}}{5}, \frac{1}{5\sqrt{2}})$ is a CP of f with $\lambda = 1/10$

(iii) The Hessian of a Function:

$$H_L(\lambda, x, y, z) = \begin{pmatrix} 0 & -10x & -18y-4z & -12z-4y \\ -10x & 2-10\lambda & 0 & 0 \\ -18y-4z & 0 & 2-18\lambda & -4\lambda \\ -12z-4y & 0 & -4\lambda & 2-12\lambda \end{pmatrix}$$

-For $a_1 = (1/\sqrt{5}, 0, 0)$ with $\lambda = 1/5$

$$H_L(1/5, 1/\sqrt{5}, 0, 0) = \begin{pmatrix} 0 & -10/\sqrt{5} & 0 & 0 \\ -10/\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & -8/5 & -4/5 \\ 0 & 0 & -4/5 & -2/5 \end{pmatrix}$$

(iv) (S): $-d_3$, $-d_4$ because $n=3$, $k=1$

$$-d_3 = -\begin{vmatrix} 0 & -10/\sqrt{5} & 0 \\ -10/\sqrt{5} & 0 & 0 \\ 0 & 0 & -8/5 \end{vmatrix} = \frac{8}{5}\left(-\frac{100}{5}\right) = -32<0$$

$$-d_4 = \begin{vmatrix} 0 & -10/\sqrt{5} & 0 & 0 \\ -10/\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & -8/5 & -4/5 \\ 0 & 0 & -4/5 & -2/5 \end{vmatrix}$$

$$= -\frac{10}{\sqrt{5}} \begin{vmatrix} -\frac{10}{\sqrt{5}} & 0 & 0 \\ 0 & -8/5 & -4/5 \\ 0 & -4/5 & -2/5 \end{vmatrix} = 20\left(\frac{16}{25} - \frac{16}{25}\right) = 0$$

Therefore (S) doesn't satisfy the two conditions then f has a constrained saddle point at $a_1 = (1/\sqrt{5}, 0, 0)$

For $a_2 = (-\frac{1}{\sqrt{5}}, 0, 0)$ with $\lambda = 1/5$

$$H_L(1/5, -\frac{1}{\sqrt{5}}, 0, 0) = \begin{pmatrix} 0 & 10/\sqrt{5} & 0 & 0 \\ 10/\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & -8/5 & -4/5 \\ 0 & 0 & -4/5 & -2/5 \end{pmatrix}$$

$$-d_3 = -\begin{vmatrix} 0 & 10/\sqrt{5} & 0 \\ 10/\sqrt{5} & 0 & 0 \\ 0 & 0 & -8/5 \end{vmatrix} = \frac{8}{5}\left(-\frac{100}{5}\right) = -32<0$$

$$-d_4 = -\begin{vmatrix} 0 & 10/\sqrt{5} & 0 & 0 \\ 10/\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Thus (S) doesn't satisfy the two conditions, then f has a constrained saddle point at $a_2 = (-\frac{1}{\sqrt{5}}, 0, 0)$

For $a_3 = (0, \frac{1}{5}, \frac{2}{5})$ with $\lambda = 1/5$

$$H_L(1/5, 0, 1/5, 2/5) = \begin{pmatrix} 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -8/5 & -4/5 \\ -4 & 0 & -4/5 & -2/5 \end{pmatrix}$$

$-d_3 = 0$ and $-d_4 = 0$

Thus (S) doesn't satisfy the two conditions, then f has a constrained saddle point at $a_3 = (0, 1/5, 2/5)$

for $a_4 = (0, 1/5, -2/5)$ with $\lambda = 1/5$

Clearly, (S) does not satisfy the two condition because the CP of f has $x = 0$ like $a_3 = (0, 1/5, 2/5)$ with $\lambda = 1/5$

Thus f has constrained saddle point $a_4 = (0, 1/5, -2/5)$

For $a_5 = (0, \frac{\sqrt{2}}{5}, \frac{1}{5\sqrt{2}})$ with $\lambda = 1/10$

$$H_L\left(\frac{1}{10}, 0, \frac{\sqrt{2}}{5}, \frac{1}{5\sqrt{2}}\right) = \begin{pmatrix} 0 & 0 & -8/\sqrt{2} & -4/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ -8/\sqrt{2} & 0 & 1/5 & -2/5 \\ 4/\sqrt{2} & 0 & -2/5 & 4/5 \end{pmatrix}$$

$$-d_3 = -\begin{vmatrix} 0 & 0 & -8/\sqrt{2} \\ 0 & 1 & 0 \\ -8/\sqrt{2} & 0 & 1/5 \end{vmatrix} = \begin{vmatrix} 0 & -8/\sqrt{2} \\ -8/\sqrt{2} & 1/5 \end{vmatrix} = 32 > 0$$

$$-d_4 = -\begin{vmatrix} 0 & 0 & -8/\sqrt{2} & -4/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ -8/\sqrt{2} & 0 & 1/5 & -2/5 \\ -4/\sqrt{2} & 0 & -2/5 & 4/5 \end{vmatrix} = 40 > 0$$

Showing that (s) are strictly possitive

Thus $a_5 (0, \frac{\sqrt{2}}{5}, \frac{1}{5\sqrt{2}})$ is a local minimum point of f subject to the constrain $g(x,y,z) = 5x^2 + 9y^2 + 6z^2 + 4yz - 1 = 0$

- For $a_6 = (0, -\frac{\sqrt{2}}{5}, -\frac{1}{5\sqrt{2}})$ with $\lambda = 1/10$

$$H_L = \left(\frac{1}{10}, 0, \frac{-\sqrt{2}}{5}, \frac{-1}{5\sqrt{2}}\right) = \begin{pmatrix} 0 & 0 & 8/\sqrt{2} & 4/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ \frac{8}{\sqrt{2}} & 0 & 1/5 & -2/5 \\ 4/\sqrt{2} & 0 & -2/5 & 4/5 \end{pmatrix}$$

$$-d_3 = -\begin{vmatrix} 0 & 0 & 8/\sqrt{2} \\ 0 & 1 & 0 \\ \frac{8}{\sqrt{2}} & 0 & 1/5 \end{vmatrix} = -\frac{8}{\sqrt{2}}\left(-\frac{8}{\sqrt{2}}\right) = 32 > 0$$

- d_4 also strictly positive like a_5

Thus $a_6 = (0, -\frac{\sqrt{2}}{5}, -\frac{1}{5\sqrt{2}})$ is a local minimum point of f subject to the constrain $g(x,y,z) = 5x^2 + 9y^2 + 6z^2 + 4yz - 1 = 0$

(d) $f(x,y,z) = x^2 + y^2 + z^2$ constraint $x^2 + y^2 + 2z^2 - 4 = 0$ and $xyz - 1 = 0$

let $g_1 = x^2 + y^2 + 2z^2 - 4 = 0$

$g_2 = xyz - 1 = 0$

$L(\mu, \lambda, x, y, z) = f(x,y,z) + \mu g(x,y,z) + \lambda g(x,y,z)$

if $\nabla L(\mu, \lambda, x, y, z) = 0$

$$\Rightarrow \begin{cases} 2x(1+\mu) + \lambda yz = 0 & (1) \\ 2y(1+\mu) + \lambda xz = 0 & (2) \\ 2z(1+\mu) + \lambda xy = 0 & (3) \end{cases}$$

$(1) \times (2) \Rightarrow \begin{cases} 2x^2(1+\mu) + \lambda = 0 \\ 2y^2(1+\mu) + \lambda = 0 \end{cases}$

for $x, y, z \neq 0$

mean $x^2 = y^2 \Rightarrow z^2 = 2 - x^2$

since $xyz = 1$

$x^2y^2z^2 = 1$

$\Rightarrow 2x^4 - x^2 - 1 = 0 \Rightarrow x = \pm 1$ and $x^2 = \frac{1+\sqrt{5}}{2}$

for $x = \pm 1 \Rightarrow z^2 = 1 \Rightarrow x^2 = y^2 = z^2 = 1$

$x^2 = y^2 = \frac{1-\sqrt{5}}{2} \Rightarrow z^2 = \frac{3-\sqrt{5}}{2}$

$\Rightarrow f_1(x,y,z) = 3$

$f_2(x,y,z) = \frac{5+\sqrt{5}}{2}$

Thus $\max f(x,y,z) = \frac{5+\sqrt{5}}{2}$

$\min f(x,y,z) = 3$

(e) $f(x,y,z) = x+y+z$ constrains $y^2 - x^2 = 1$ and $x + 2z = 1$

let $L(\lambda, s, x) = f(x) - \lambda(g_1(x) - c_1) - s(g_2(x) - c_2)$

$$\nabla L(\lambda,s,x,y,z) = 0 \Rightarrow \begin{cases} -y^2 + x^2 + 1 = 0 & (1) \\ -x - 2z + 1 = 0 & (2) \\ 1 + 2\lambda x - s = 0 & (3) \\ 1 - 2\lambda y = 0 & (4) \\ 1 - 2s = 0 & (5) \end{cases}$$

following by (5) $s = 1/2$

follow by (3) & (4)
$$+\begin{cases} 1 + 2\lambda x - \frac{1}{2} = 0 & (\times y) \\ 1 - 2\lambda y = 0 & (\times x) \end{cases}$$
$$\overline{\tfrac{1}{2}y + x = 0 \Rightarrow y = -2x}$$

Follow by (1): $4x^2 - x^2 - 1 = 0$

$x^2 = 1/3 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

If $x = \frac{1}{\sqrt{3}} \Rightarrow y = -\frac{2}{\sqrt{3}}$

if $x = -\frac{1}{\sqrt{3}} \Rightarrow y = \frac{2}{\sqrt{3}}$

For $x = \frac{1}{\sqrt{3}}$: (2) $\Rightarrow \frac{1}{\sqrt{3}} + 2z - 1 = 0$

$\Rightarrow z = \frac{\sqrt{3}-1}{2\sqrt{3}}$, $\lambda = -\frac{\sqrt{3}}{4}$

For $x = -\frac{1}{\sqrt{3}}$: (2) $\Rightarrow -\frac{1}{\sqrt{3}} + 2z - 1 = 0$

$\Rightarrow z = \frac{\sqrt{3}+1}{\sqrt{3}}$, $\lambda = \frac{\sqrt{3}}{4}$

we get $a_1 = \left(\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right)$ and

$a_2 = \left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$

are the CP of L with $\lambda = -\frac{\sqrt{3}}{4}$, $s = 1/2$ and $\lambda = \frac{\sqrt{3}}{4}$, $s = 1/2$ respectively

$$H_L(\lambda,s,x,y,z) = \begin{pmatrix} 0 & 0 & 2x & -2y & 0 \\ 0 & 0 & -1 & 0 & -2 \\ 2x & -1 & 2\lambda & 0 & 0 \\ -2y & 0 & 0 & -2\lambda & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

For $a_1 = \left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right)$ with $\lambda = -\frac{\sqrt{3}}{4}$, $s = 1/2$

$$H_L\left(-\frac{\sqrt{3}}{4}, \frac{1}{2}, a_1\right) = \begin{pmatrix} 0 & 0 & 2/\sqrt{3} & 4/\sqrt{3} & 0 \\ 0 & 0 & -1 & 0 & -2 \\ 2/\sqrt{3} & -1 & -\sqrt{3}/2 & 0 & 0 \\ 4/\sqrt{3} & 0 & 0 & \sqrt{3}/2 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

(S): d_5 because $n = 3$ and $k = 2$

$$d_5 = \begin{vmatrix} 0 & 0 & 2/\sqrt{3} & 4/\sqrt{3} & 0 \\ 0 & 0 & -1 & 0 & -2 \\ 2/\sqrt{3} & -1 & -\sqrt{3}/2 & 0 & 0 \\ 4/\sqrt{3} & 0 & 0 & \sqrt{3}/2 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 0 & \frac{2}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -1 & -\frac{\sqrt{3}}{2} & 0 \\ 4/\sqrt{3} & 0 & 0 & \sqrt{3}/2 \\ 0 & -2 & 0 & 0 \end{vmatrix}$$

$$= -4\begin{vmatrix} 0 & 2/\sqrt{3} & 4/\sqrt{3} \\ 2/\sqrt{3} & -\sqrt{3}/2 & 0 \\ 4/\sqrt{3} & 0 & \sqrt{3}/2 \end{vmatrix}$$

$$= -\frac{16}{\sqrt{3}}\begin{vmatrix} 2/\sqrt{3} & -\sqrt{3}/2 \\ 4/\sqrt{3} & 0 \end{vmatrix} - 2\sqrt{3}\begin{vmatrix} 0 & 2/\sqrt{3} \\ 2/\sqrt{3} & -\sqrt{3}/2 \end{vmatrix}$$

$$= -\frac{32\sqrt{3}}{3} + \frac{8\sqrt{3}}{3} < 0$$

Thus $a_1 = \left(1/\sqrt{3}, -2/\sqrt{3}, \frac{\sqrt{3}-1}{2\sqrt{3}}\right)$ is a local maximum point of f subject to constrain $y^2 - x^2 = 1$ and $x + 2z = 1$

For $a_2 = \left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{\sqrt{3}+2}{\sqrt{3}}\right)$ with $\lambda = \frac{\sqrt{3}}{4}$ & $s = 1/2$

$$H_L\left(\frac{\sqrt{3}}{4}, \frac{1}{2}, a_2\right) = \begin{pmatrix} 0 & 0 & -2/\sqrt{3} & -4/\sqrt{3} & 0 \\ 0 & 0 & -1 & 0 & -2 \\ -2/\sqrt{3} & -1 & \sqrt{3}/2 & 0 & 0 \\ 4/\sqrt{3} & 0 & 0 & -\sqrt{3}/2 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

$$d_5 = \frac{32\sqrt{3}}{3} - \frac{8\sqrt{3}}{3} > 0$$

Thus $a_2 = \left(-\frac{1}{3}, \frac{2}{\sqrt{3}}, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$ is a local minimum point of f subjects to constrain $y^2 - x^2 = 1$ and $x + 2z = 1$

(7) $f(x,y,z,w) = 3x+y+w$, constraints
$3x^2+y+4z^3=1$ and $-x^3+3z^4+w=0$

Let $L(\lambda,s,x) = f(x) - \lambda(g_1(x)-c_1) - s(g_2(x)-c_2)$

$$\nabla L(\lambda,s,x,y,z,w)=0 \Rightarrow \begin{cases} 3x^2+y+4z^3-1=0 & (1)\\ -x^3+3z^4+w=0 & (2)\\ 3-6\lambda x+3sx^2=0 & (3)\\ 1-\lambda=0 & (4)\\ -12\lambda z^2-12sz^3=0 & (5)\\ 1-s=0 & (6)\end{cases}$$

Follow by (4) and (6) $\lambda=1$, $s=1$

Follow by (3)&(5): $\begin{cases} 3-6x+3x^2=0 & (7)\\ -12z^2-12z^3=0 & (8)\end{cases}$

Follow by (7) $x^2-2x+1=0 \Rightarrow x=1$.

Follow by (8) $z^2(z+1)=0 \Rightarrow z=0$ or $z=-1$

For $x=1$ and $z=0$: $\begin{cases} 3+y-1=0\\ -1+w=0\end{cases}$ from (1)&(2)

we get $y=-2$ and $w=1$

consequenly $a_1=(1,-2,1,0)$ is the CP of L with $\lambda=1$ and $s=1$

For $x=1$ and $z=-1$: $\begin{cases} y+3-4-1=0\\ -1+3+w=0\end{cases}$ from (1)&(2)

we get $y=2$ and $w=-2$

consequenly $a_2=(1,2,-1,-2)$ is the CP of L with $\lambda=1$ and $s=1$

$$H_L=\begin{pmatrix} 0 & 0 & 6x & 1 & 12z^2 & 0\\ 0 & 0 & -3x^2 & 0 & 12z^3 & 1\\ -6x & 3x^2 & -6\lambda+6sx & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ -12z^2-12z^3 & 0 & 0 & 0 & -24\lambda z-36sz^2 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{pmatrix}$$

For $a_1=(1,-2,1,0)$ with $\lambda=1$, $s=1$

$$H_L(1,1,1,-2,1,0)=\begin{pmatrix} 0 & 0 & 6 & 1 & 0 & 0\\ 0 & 0 & -3 & 0 & 0 & 1\\ -6 & 3 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{pmatrix}$$

(S): d_5, d_6 because $n=4$ and $k=2$

$d_5=0$

clearly, (S) is not satisfy the two conditions

Thus f has a constrained saddle point at $a_1=(1,-2,1,0)$

---

for $a_2=(1,2,-1,-2)$ with $\lambda=1$, $s=1$

$$H_L(1,1,1,2,-1,-2)=\begin{pmatrix} 0 & 0 & 6 & 1 & 12 & 0\\ 0 & 0 & -3 & 0 & -12 & 1\\ -6 & 3 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -12 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{pmatrix}$$

$d_5=-108<0$

$d_6=0$

we see that $d_5<0$, and $d_6=0$.

That is, (S) is not satisfy the two condition

Thus f has a constrained saddle point at $a_2=(1,2,-1,-2)$

(10) Find the extrema of the function f defined by:

$f(x,y)=\int_x^y te^{-t^2}dt$

subjects to the constraing $g(x,y)=e^{x^2}+e^{y^2}=8$

$$f(x,y)=\int_x^y te^{-t^2}dt$$
$$=-\frac{1}{2}(e^{-x^2}-e^{-y^2})$$

Equation de lagrange $\nabla(f+\lambda g)(x,y)=(0,0)$

Since $\nabla(f+\lambda g)=(-xe^{-x^2}+\lambda x * 2e^{x^2}, ye^{-y^2}+2\lambda ye^{y^2})$

$$\nabla(f+\lambda g)=0 \quad \begin{cases} -xe^{-x^2}*\lambda x+2e^{x^2}=0 \\ ye^{-y^2}+2\lambda ye^{y^2}=0 \end{cases}$$

$$\Rightarrow \begin{cases} x(-e^{-x^2}+2\lambda e^{x^2})=0 \Rightarrow xy=0 \\ y(e^{-y^2}+2\lambda e^{y^2})=0 \end{cases}$$

we have $e^{x^2}+e^{y^2}=8$ if $x=0 \Rightarrow e^{y^2}=7$

Then $f(0,y)=-\frac{1}{2}(\frac{1}{7}-1)=3/7$

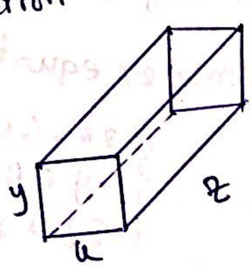
If $y=0 \Rightarrow e^{x^2}=7$

$f(x,0)=-\frac{1}{2}(1-1/7)=-3/7$

Thus $\min f(x,y)=-3/7$, $\max f(x,y)=3/7$

(11) Use the lagrange Multiplier to find the dimensions of the container of this size that has minimum cost.

+ The optional function is the cost of the container:



$f(x,y,z)=50xz+30xz+2(30xy)+2\times30(yz)$

$f(x,y,z)=80xz+60xy+60yz$ $(x,y,z>0)$

$g(x,y,z)=xyz=480$ (*) (The valume of the metagular)

$f(x,y,z)=80xz+60xy+60yz$

let $L(\lambda,x,y,z)=f(x)-\lambda(g(x)-c)$

$$\nabla L(\lambda,x,y,z)=0 \Rightarrow \begin{cases} xyz=0 & (1) \\ 80z+60y-\lambda yz=0 & (2) \\ 60x+60z-\lambda xz=0 & (3) \\ 80x+60y-\lambda yx=0 & (4) \end{cases}$$

Then $80xz+60yx-\lambda xyz=60yz+60yz-\lambda xyz$
$=80xz+60yz-\lambda xyz$

$$\Rightarrow \begin{cases} 80xz=60yz \Rightarrow x=\frac{3}{4}y \\ 60xy=80xz \Rightarrow z=\frac{3}{4}y \end{cases}$$

Then (*) $y(\frac{3}{4}y)(\frac{3}{4}y)=480$

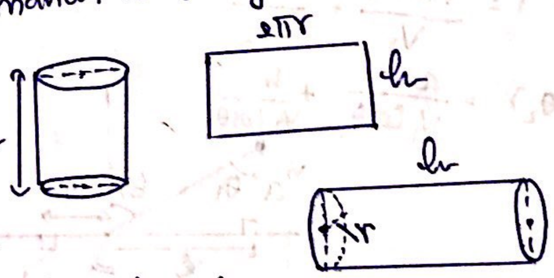
$y^3=\frac{480}{9}\times16=2560$

$\Rightarrow y=\sqrt[3]{2560}=8\sqrt[3]{5}$

$\Rightarrow x=z=\frac{3}{4}\times8\sqrt[3]{5}=6\sqrt[3]{5}$

$$\boxed{\text{Thus } x=z=6\sqrt[3]{5}\ (m) \quad y=8\sqrt[3]{5}\ cm}$$

(12) What is the minimum cost of manufacturing to the tank.



• The optional function is the cost of tank

$f(r,h)=2(4\pi r^2)+2r\pi h=8\pi r^2+2\pi rh$

• The constrain $g(r,h)=\frac{4}{3}r^3\pi+r^2\pi h=8000$

− Minimize: $f(r,h)=8r^2\pi+2\pi rh$

constrain: $g(r,h)=\frac{4}{3}r^3\pi+r^2\pi h=8000$

let $L(\lambda,r,h)=f(r,h)-\lambda(g(r,h)-8000)$

$$\nabla L(\lambda, r, h) = 0 \iff \begin{cases} \frac{4}{3} r^3 \pi h + r^2 \pi h = 8000 & (1) \\ 16 r \pi + 2\pi h - 4r^2 \pi \lambda - 2\pi \lambda r h = 0 & (2) \\ 2\pi r - r^2 \pi \lambda = 0 & (3) \end{cases}$$

$$\iff \begin{cases} 4r^3\pi + 3r^2\pi h = 24000 & (1) \\ 8r + h - 2\lambda r^2 = \lambda r h & (2) \\ r(2 - \lambda r) = 0 & (3) \end{cases}$$

by (3) $2 - \lambda r = 0 \Rightarrow \lambda = \frac{2}{r}$

then (2): $8r + h - 4r - 2h = 0 \Rightarrow h = 4r$

then (1): $4r^3\pi + 12r^3\pi = 24000$

$\Rightarrow r^3 = \frac{24000}{16\pi} \Rightarrow r = \sqrt[3]{\frac{3000}{2\pi}} = 10\sqrt[3]{\frac{3}{2\pi}}$

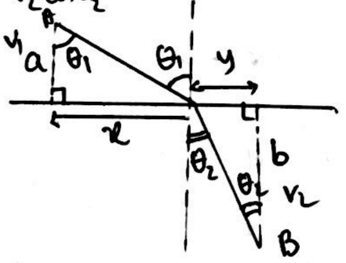
$\Rightarrow h = 40\sqrt[3]{\frac{3}{2\pi}}$

Thus the cost minimum of tank is $(r, h) = \left(10\sqrt[3]{\frac{3}{2\pi}}, 40\sqrt[3]{\frac{3}{2\pi}}\right)$

13 a)_ Determine the total time the light travels in going from point A to point B via point P as shown in Figure 2.

we have $T = \frac{\sqrt{a^2 + x^2}}{V_1} + \frac{\sqrt{y^2 + b^2}}{V_2}$

$T(\theta_1, \theta_2) = \frac{a}{V_1 \cos\theta_1} + \frac{b}{V_2 \cos\theta_2}$



b)_ use the method of lagrange multiplier to establish snell's law of refraction that the totale travel time is minimized where $\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

we need to study the minimize of T subject to $G(\theta_1, \theta_2) = x + y = a\tan\theta_1 + b\tan\theta_2 = K$

By using Lagrang multiplier

Let $L(\lambda, \theta_1, \theta_2) = T(\theta_1, \theta_2) - \lambda(G(\theta_1, \theta_2) - k)$

$$\nabla L(\lambda, \theta_1, \theta_2) = 0 \Rightarrow \begin{cases} \frac{a}{V_2} \cdot \frac{\sin\theta_1}{\cos^2\theta_1} - \lambda a \frac{1}{\cos^2\theta_1} = 0 \\ \frac{b}{V_2} \frac{\sin\theta_2}{\cos^2\theta_2} - \lambda b \frac{1}{\cos^2\theta_2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\sin\theta_1}{V_1} = \lambda \\ \frac{\sin\theta_2}{V_2} = \lambda \end{cases} \Rightarrow \begin{cases} \sin\theta_1 = \lambda V_1 & (1) \\ \sin\theta_2 = \lambda V_2 & (2) \end{cases}$$

$\Rightarrow \frac{(1)}{(2)} \Rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}$

Therefore $\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{V_1}{V_2}}$

14 Find the largest sphere centered at the origin that can be inscribed in the ellipsoid.

$3x^2 + 2y^2 + z^2 = 6$

let $f(x,y,z) = 4\pi R^2$

or $f(x,y,z) = x^2 + y^2 + z^2 = R^2$

be the sphere equation in $\mathbb{R}^3$

Find the maximum of f subject to ellipsoid $3x^2 + 2y^2 + z^2 = 6$

$(f + \lambda g)(x,y,z) = x^2 + y^2 + z^2 + \lambda(3x^2 + 2y^2 + z^2 - 6) - R^2 = 0$

consider equation $\nabla(f + \lambda g) = 0$

$$\begin{cases} 2x + 6\lambda x = 0 \\ 2y + 4\lambda y = 0 \\ 2z + 2z\lambda = 0 \end{cases} \Rightarrow \begin{cases} x(1 + 3\lambda) = 0 \\ y(1 + 2\lambda) = 0 \\ z(1 + \lambda) = 0 \end{cases}$$

- If $x \neq 0 \Rightarrow \lambda = -1/3$, $y = 0$, $z = 0$
  $g(x,0,0) = 3x^2 - 6 = 0 \Rightarrow x = \pm\sqrt{2}$
- If $y \neq 0 \Rightarrow \lambda = -1/2$, $x = 0$, $z = 0$
  $g(0,y,0) = 2y^2 = 6 \Rightarrow y = \pm\sqrt{3}$

If $z \neq 0 \Rightarrow \lambda = -1$, $x=0, y=0$

$g(0,0,z) = z^2 = 6 \Rightarrow z = \pm\sqrt{6}$

$$\Rightarrow \begin{cases} f(\pm\sqrt{2},0,0) = 2 \\ f(0,\pm\sqrt{3},0) = 3 \\ f(0,0,\pm\sqrt{6}) = 6 \end{cases}$$

Therefore the largest sphere inscribed the ellipsoid is the sphere whose radius $=\sqrt{6}$

(15) Use Lagrange multiplier to show that the distance from a point (x_0,y_0) to the line $ax+by=c$ is defined by

$$D = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

let $M(x,y) \in L$ and $P(x_0,y_0)$ is a point

The optimize function is distance from P to M

So $d(x,y) = |PM| = \sqrt{(x-x_0)^2+(y-y_0)^2}$,

the constrain $g(x,y) = ax+by=c$ is the same to the point that minimize $f(x,y) = (x-x_0)^2+(y-y_0)^2$ subject to the constrain $g(x,y)$

Let $L(\lambda,x,y) = f(x) - \lambda(g(x)-c)$

$= (x-x_0)^2+(y-y_0)^2+(z-z_0)^2 - \lambda(ax+by+c)$

$$\nabla L(\lambda,x,y)=0 \begin{cases} ax+by-c=0 & (1) \\ 2(x-x_0)-a\lambda=0 & (2) \\ 2(y-y_0)-b\lambda=0 & (3) \end{cases}$$

by (2): $2(x-x_0)-a\lambda=0 \Rightarrow x = \frac{a\lambda}{2}+x_0$

(3): $2(y-y_0)-b\lambda=0 \Rightarrow y = \frac{b\lambda}{2}+y_0$

we substitute x and y into (1):

(1): $\frac{a^2\lambda}{2}+ax_0+\frac{b^2\lambda}{2}+by_0 = c$

$\frac{\lambda}{2}(a^2+b^2)+ax_0+by_0 = c$

$$\Rightarrow \lambda = \frac{2(c-ax_0-by_0)}{a^2+b^2}$$

then we get

$$x = \frac{a(c-x_0a-by_0)}{a^2+b^2}+x_0$$

$$y = \frac{b(c-ax_0-by_0)}{a^2+b^2}+y_0$$

So $D = d(x,y) = \sqrt{\frac{(c-ax_0-by_0)^2(a^2+b^2)}{(a^2+b^2)^2}}$

$$= \frac{|c-ax_0-by_0|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

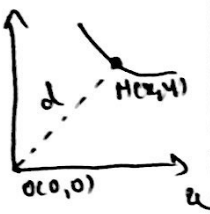
Therefore

The distance from a point (x_0,y_0) to the line $ax+by-c=0$ is defined by $D = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$

(16) Find the distance from the origin to the hyperbola $x^2+8xy+7y^2-225=0$

we have $x^2+8xy+7y^2-225=0$

let $M \in$ hyperbola



The optimize function is distance from $(0,0)$ to $M(x,y)$

So $d(x,y) = |OM| = \sqrt{x^2+y^2}$, $O(0,0)$

constrain $g(x,y) = x^2+8xy+7y^2-225=0$

The point that minimize $f(x,y) = x^2+y^2$ subject to $g(x,y)$

let $L(\lambda,x,y) = f(x) - \lambda(g(x)-0)$

$$\nabla L(\lambda,x,y)=0 \begin{cases} x^2+8xy+7y^2-225=0 & (1) \\ 2x-\lambda(8y+2x)=0 & (2) \\ 2y-\lambda(14y+8x)=0 & (3) \end{cases}$$

by (2)&(3) $\begin{cases} x-\lambda x-4\lambda y=0 \\ y-7\lambda y-4\lambda x=0 \end{cases}$

$\Leftrightarrow \begin{cases} x(1-\lambda)-4\lambda y=0 \\ -4\lambda x+y(1-7\lambda)=0 \end{cases}$

Since $(x,y) \neq 0$, $A\cancel{x}=0$

$$|A| = \begin{vmatrix} 1-\lambda & -4\lambda \\ -4\lambda & 1-7\lambda \end{vmatrix} = 0$$

$\Rightarrow 9\lambda^2+8\lambda-1=0$

$\Rightarrow \lambda_1=-1 \;,\; \lambda_2=1/9$

for $\lambda_1=-1$

then $\begin{cases} 2x+4y=0 \\ 4x+8y=0 \end{cases} \Rightarrow x=-2y$

we substitute $x=-2y$ into (1)

(1) $4y^2-16y^2+7y^2=225$

$-5y^2=225 \Rightarrow y^2=-225$

$\Rightarrow y=\phi$

$\Rightarrow x=\phi$

for $\lambda=1/9$

then $\begin{cases} 8x-4y=0 \\ -4x+2y=0 \end{cases} \Rightarrow y=2x$

then (1): $x^2+16x^2+28x^2=225$

$45x^2=225$

$\Rightarrow x=\pm\sqrt5$

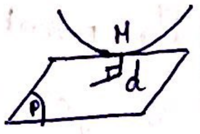
Thus the distance from the origin to the hyperbola is $D=d(\pm\sqrt5,\pm2\sqrt5)=5\sqrt5$

(7) Find the distance from between the ellipsoid(E) and place (P), where

(E): $2x^2+y^2+2z^2-8=0$ & (P): $x+y+z-10=0$

we draw (E): $2x^2+y^2+2z^2-8=0$

(P): $x+y+z-10=0$



let $M(x,y,z)\in(E)$

The optimize function is the distance from M to Place

$$d(x,y,z)=d(M,(P))=\frac{|x+y+z-10|}{\sqrt{1+1+1}}=\frac{\sqrt3}{3}|x+y+z-10|$$

The constrain $g(x)=2x^2+y^2+2z^2-8=0$

we need to prove that the minimize $f(x)=x+y+z-10$ subject to $g(x,y,z)=$.

let $L(\lambda,x,y)=f(x)-\lambda(g(x)-0)$

$\nabla L(\lambda,x,y,z)=0 \Leftrightarrow \begin{cases} 2x^2+y^2+2z^2=8 \; (1) \\ 1-4x\lambda=0 \Rightarrow x=\frac{1}{4\lambda} \\ 1-2y\lambda=0 \Rightarrow y=1/2\lambda \\ 1-4z\lambda=0 \Rightarrow z=1/(4\lambda) \end{cases}$

then $y=2x=\frac{1}{2\lambda}$ and $x=z=1/(4\lambda)$

then (1): $2x^2+(2x)^2+2x^2=8 \Rightarrow x=\pm1$

then $y=\pm2$, $z=x=\pm1$

+ For $x=1$, $y=2$, $z=1$

So $D=d(1,2,1)=\frac{|1+2+1-10|}{\sqrt3}=2\sqrt3$

+ For $x=-1$, $y=-2$, $z=-1$

So $D=d(-1,-2,-1)=\frac{|-1-2-1-10|}{\sqrt3}=\frac{14\sqrt3}{3}$

(18) Find the vertices of the ellipsoide $4x^2+9y^2+6z^2+4yz-4=0$

The center of ellipsoide is (x_0,y_0,z_0), where $\nabla f(x_0,y_0,z_0)=0$

Since $\nabla f(x_0,y_0,z_0)=0$

$\Rightarrow \begin{cases} 8x=0 \\ 18y+4z=0 \\ 12z+4y=0 \end{cases} \Rightarrow x=y=z=0$

So $O(0,0,0)$ is the center of ellipsoide

The vertice of the ellipsoide is the point $M(x,y,z)$ on ellipsoid that has logest distance from O.

That is the function is

$d(O,M)=\sqrt{x^2+y^2+z^2}$, $O(0,0,0)$

we need to prove the point that minimize function $f(x,y,z)=x^2+y^2+z^2$ subject constrain $g(x,y,z)$

By using Langrang mutiplier.

Let $L(\lambda,x)=f(x)-\lambda(g(x))$

$\nabla L(\lambda,x,y,z) \Leftrightarrow \begin{cases} 4x^2+9y^2+6z^2+4yz-4=0 & (1) \\ 2y-18y\lambda+4\lambda z=0 & (2) \\ 2x-8\lambda x=0 & (3) \\ 2z-12\lambda z+4\lambda y=0 & (4) \end{cases}$

(3): $2x - 8\lambda x = 0 \Leftrightarrow x(2-8\lambda) = 0 \Rightarrow x = 0$ & $\lambda = 1/4$

If $x = 0$

we get
$$\begin{cases} 9y^2 + 6z^2 - 4yz - 4 = 0 \ (*) \\ 2y - 18\lambda y + 4\lambda z = 0 \Leftrightarrow y(1-9\lambda) + 2\lambda z = 0 \ (**) \\ 2z - 12\lambda z + 4\lambda y = 0 \Leftrightarrow z(1-6\lambda) + 2\lambda y = 0 \ (***) \end{cases}$$

Since $(y,z) \neq 0$

Then $\begin{vmatrix} 1-9\lambda & 2\lambda \\ 2\lambda & 1-6\lambda \end{vmatrix} = 0$

$\Rightarrow 1 - 6\lambda - 9\lambda + 54\lambda^2 - 4\lambda^2 = 0$

$50\lambda^2 - 15\lambda + 1 = 0$

$\Delta = 225 - 200 = 25 \Rightarrow \lambda_1 = 1/5, \lambda_2 = 1/10$

+ For $\lambda_1 = 1/5$

So $\begin{cases} -\frac{4}{5}y + \frac{2}{5}z = 0 \\ \frac{2}{5}y - \frac{1}{5}z = 0 \end{cases} \Rightarrow y = \frac{1}{2}z$

(*) $9 \times \frac{z^2}{4} + 6z^2 + 2z^2 - 4 = 0$ by substitute y into (*)

$\Leftrightarrow \frac{9}{4}z^2 + 8z^2 = 4 \times 4 \Leftrightarrow 41z^2 = 16 \Leftrightarrow z = \pm\frac{4\sqrt{41}}{41}$

for $z = \pm\frac{4\sqrt{41}}{41} \Rightarrow y = \pm\frac{2\sqrt{41}}{41}$

+ For $\lambda_2 = 1/10$

So $\begin{cases} \frac{1}{10}y + \frac{1}{5}z = 0 \\ \frac{1}{5}y + \frac{2}{5}z = 0 \end{cases} \Rightarrow y = -2z$

we substitute y into (*)

So $z = \pm\frac{2\sqrt{34}}{34} = \pm\frac{\sqrt{34}}{17} \Rightarrow y = \mp\frac{2\sqrt{34}}{17}$

+ For $\lambda = 1/4$

So $\begin{cases} 4x^2 + 9y^2 + 6z^2 - 6yz - 4 = 0 \ (7) \\ 2y - \frac{9}{2}y + z = 0 \Leftrightarrow -\frac{5}{2}y + z = 0 \ (5) \\ 2z - 3z + y = 0 \qquad y = z \ (6) \end{cases}$

by (5) & (6)

Then $y = z = 0$

Substitute $y = z = 0$ into (7)

$4x^2 = 4 \Rightarrow x = \pm 1$

Thus the vertices of ellipsoide are $M_{1,2}\left(0, \pm\frac{2\sqrt{41}}{41}, \pm\frac{2\sqrt{41}}{41}\right)$

$M_3(1,0,0) \quad M_4(-1,0,0)$

$M_{5,6}\left(0, \mp\frac{2\sqrt{34}}{17}, \mp\frac{\sqrt{34}}{17}\right)$

(19) Find the point closest to the point $(2,5,-1)$ and on the line of intersection of the planes $x - 2y + 3z = 8$ and $2z - y = 3$

we have

$x - 2y + 3z = 8$ & $2z - y = 3$

The optimize function is $f(x,y,z) = (x-2)^2 + (y-5)^2 + (z+1)^2$

$g_1(x,y,z) = 8 \Leftrightarrow x - 2y + 3z = 8$

$g_2(x,y,z) = 3 \Leftrightarrow 2z - y = 3$

we need to prove that the point of minimize $f(x) = (x-2)^2 + (y-5)^2 + (z+1)^2$ subject to $g_1(x,y,z)$ & $g_2(x,y,z)$

let $L(\lambda, x) = f(x) - \lambda_1(g_1(x,y,z) - c_1) - \lambda_2(g_1(x,y,z) - c_2)$

$$\nabla L(\lambda,x,y,z) = 0 \begin{cases} x - 2y + 3z - 8 = 0 \ (1) \\ 2z - y - 3 = 0 \ (2) \Rightarrow y = 2z - 3 \ (6) \\ 2x - 4 - \lambda_1 = 0 \Rightarrow \lambda_1 = 2x - 4 \ (3) \\ 2y - 10 + 2\lambda_1 + \lambda_2 = 0 \ (4) \\ 2z + 2 - 3\lambda_1 - 2\lambda_2 = 0 \ (5) \end{cases}$$

we substitute (3) into (4) & (5)

So $\begin{cases} 2y - 10 + 4x - 8 + \lambda_2 = 0 \Rightarrow \lambda_2 = -(2y + 4x + 18) \\ 2z + 2 - 6x + 12 - 2\lambda_2 = 0 \qquad = z - 3x + 7 \end{cases}$

Then $2y + 4x + 18 = 3x - z - 7$

$2y + x - 7 + 25 = 0$ (*)

$2y + x + 18 = 0$ (*)

we substitute (6) into (*) and (1)

Then $\begin{cases} x - 4z + 6 + 3z - 8 = 0 \\ x + 4z - 6 + z + 25 = 0 \end{cases}$

$-6z + 29 = 0 \Rightarrow z = 29/6 \Rightarrow$

$y = 2\left(\frac{29}{6}\right) - 3 = \frac{20}{3}$, $x = \frac{41}{6}$

Thus $\left(\frac{41}{6}, \frac{20}{3}, \frac{29}{6}\right)$ is the closest point to the point $(2,5,-1)$

20 Find the nearest and farest distance from the origin to the conic section obtained by intersecting of the cone.

we have $z^2 = x^2 + y^2$ and plane (P): $x+y-z+2=0$

let $M(x,y,z) \in (C)$ & (P)

The optimize function is the distance from O to M

$d = \sqrt{x^2+y^2+z^2}$, $g_1(x,y,z) = -x^2-y^2+z^2 = 0$ and $g_2(x,y,z) = x+y-z+2=0$

we need to prove the extreme of the

$f(x,y,z) = x^2+y^2+z^2$ subject to $g_1(x,y,z)$ & $g_2(x,y,z)$.

let $L(\lambda, x) = f(x) - \lambda_1(g_1(x,y,z) - c_1) - \lambda_2(g_2(x,y,z) - c_2)$

$$\nabla L(\lambda, x, y, z) = 0 \quad \begin{cases} z^2 - x^2 - y^2 = 0 & (1) \\ x+y-z+2 = 0 & (2) \\ 2x + 2\lambda_1 x - \lambda_2 = 0 & (3) \\ 2y + 2\lambda_1 y - \lambda_2 = 0 & (4) \\ 2z - 2\lambda_1 z + \lambda_2 = 0 & (5) \end{cases}$$

by (3) & (4) $\Rightarrow \lambda_1 = -1$, $x = y$

- If $\lambda_1 = -1 \Rightarrow \lambda_2 = 0$ & $z = 0 \Rightarrow x = y = 0$
- If $x = y$

So $\begin{cases} z^2 - 2x^2 = 0 & (6) \\ 2x - z + 2 = 0 \Rightarrow z = 2x+2 & (7) \end{cases}$

we substitute (7) & (6)

$4x^2 + 8x + 4 - 2x^2 = 0 \Leftrightarrow x^2 + 4x + 2 = 0$

$x^2 + 4x + 4 = 2$

$(x+2)^2 = 2$

$x = \pm\sqrt{2} - 2$

For $x = -2+\sqrt{2}$

$\Rightarrow y = -2+\sqrt{2}$

$\Rightarrow z = -2+2\sqrt{2}$

$\Rightarrow d = \sqrt{(-2+\sqrt{2})^2 + (-2+\sqrt{2})^2 + (-2+2\sqrt{2})^2} = \sqrt{8(3-2\sqrt{2})}$

For $x = -(2+\sqrt{2})$, $y = -(2+\sqrt{2})$, $z = -2(1+\sqrt{2})$

$d = \sqrt{(2+\sqrt{2})^2 + (2+\sqrt{2})^2 + 4(1+\sqrt{2})^2} = \sqrt{8(3-2\sqrt{3})}$

Thus $d_{min} = 2\sqrt{6-4\sqrt{3}}$

$d_{max} = 2\sqrt{6+4\sqrt{3}}$

22 Consider the problem of the finding extrema of $f(x,y,z)$ subject to the constraint $z=c$, where c is any constant.

we have $f(x,y,z) = x^2+y^2$

$g(x,y,z) = z - c = 0$

a)- Use the method of Lagrange multipliers to identify the critical points of f subject to the constraint given above

Let $L(\lambda, x) = f(x) - \lambda(g(x,y,z) - c]$

$$\nabla L(\lambda, x, y, z) = 0 \Rightarrow \begin{cases} z - c = 0 \Rightarrow z = c \\ 2x = 0 \Rightarrow x = 0 \\ 2y = 0 \Rightarrow y = 0 \\ \lambda = 0 \end{cases}$$

Then $(x,y,z) = (0,0,c)$ is the critical point of L

Thus $(0,0,c)$ is the critical point of f subject to $g(x,y,z)$

b)- Find the Hessian of function L

we have $L(\lambda, x, y, z) = f(x,y,z) - \lambda(z-c)$

Since $H_L(\lambda, x, y, z) = H_L(0,0,0,c)$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and $\frac{\partial g}{\partial x}(0,0,c) = 0 \Rightarrow$ second derivate test fail.

Thus the nature of the CP in point (a) does not exist.