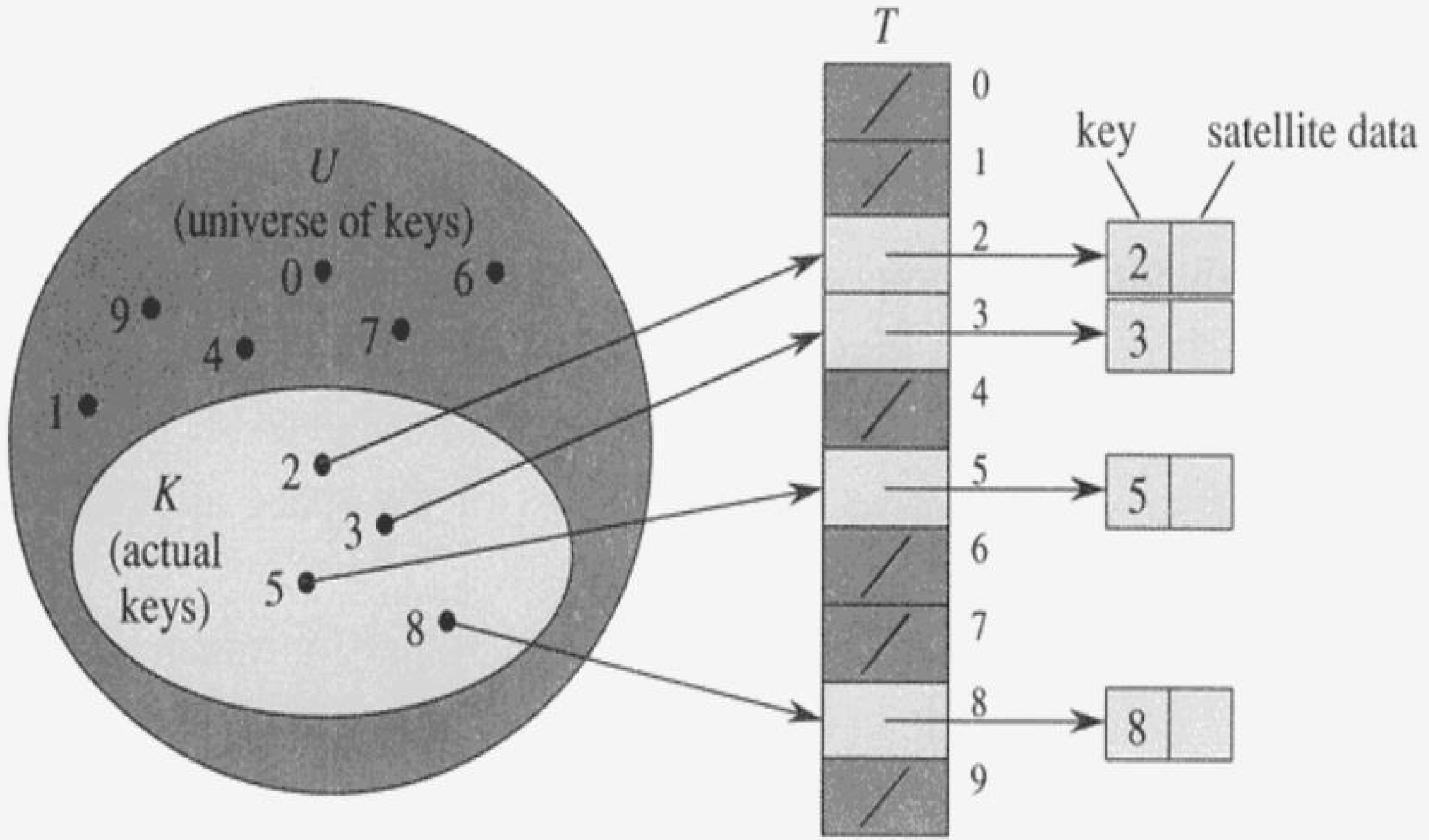
Hash Table

Introduction

- Dictionary: a dynamic set that supports INSERT, SEARCH, and DELETE
- Hash table: an effective data structure for dictionaries
  - O(n) time for search in worst case
  - O(1) expected time for search

Direct-Address Tables

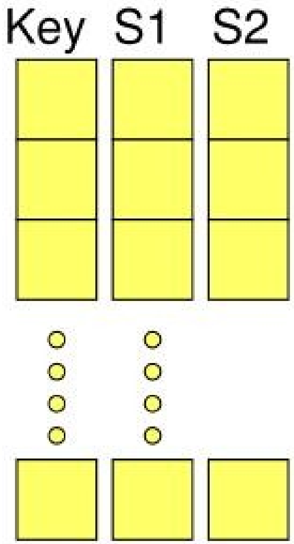
- An Array is an example of a direct-address table
  - Given a key *k*, the corresponding is stored in T[*k*]
    - Assume no two elements have the same key
  - Suitable when the universe U of keys is reasonably small
    - U={0, 1, 2,…, m-1} ➔ T[0..m-1] ➔ Θ(m)
    - What if the no. of keys actually occurred is small
- Operations of direct-address tables: all Θ(1) time
  - DIRECT-ADDRESS-SEARCH(T, k): return T[k]
  - DIRECT-ADDRESS-INSERT(T, x): T[key[x]] ← x
  - DIRECT-ADDRESS-DELETE(T, x): T[key[x]] ← NIL



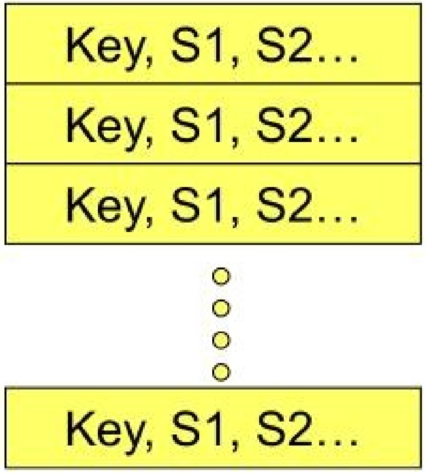
If the set contains no element with key k, T[k] = NIL

Implementing a dynamic set by a direct-address table T. Each key in the universe $U = \{0, 1, \ldots, 9\}$ corresponds to an index in the table. The set $K = \{2, 3, 5, 8\}$ of actual keys determines the slots in the table that contain pointers to elements. The other slots, heavily shaded, contain NIL.

Alternative Illustration of Direct-Address Tables



Use separate arrays for key and satellite data if the programming language does not support objects (parallel array)



If the programming language can support object-element arrays

Hash Tables

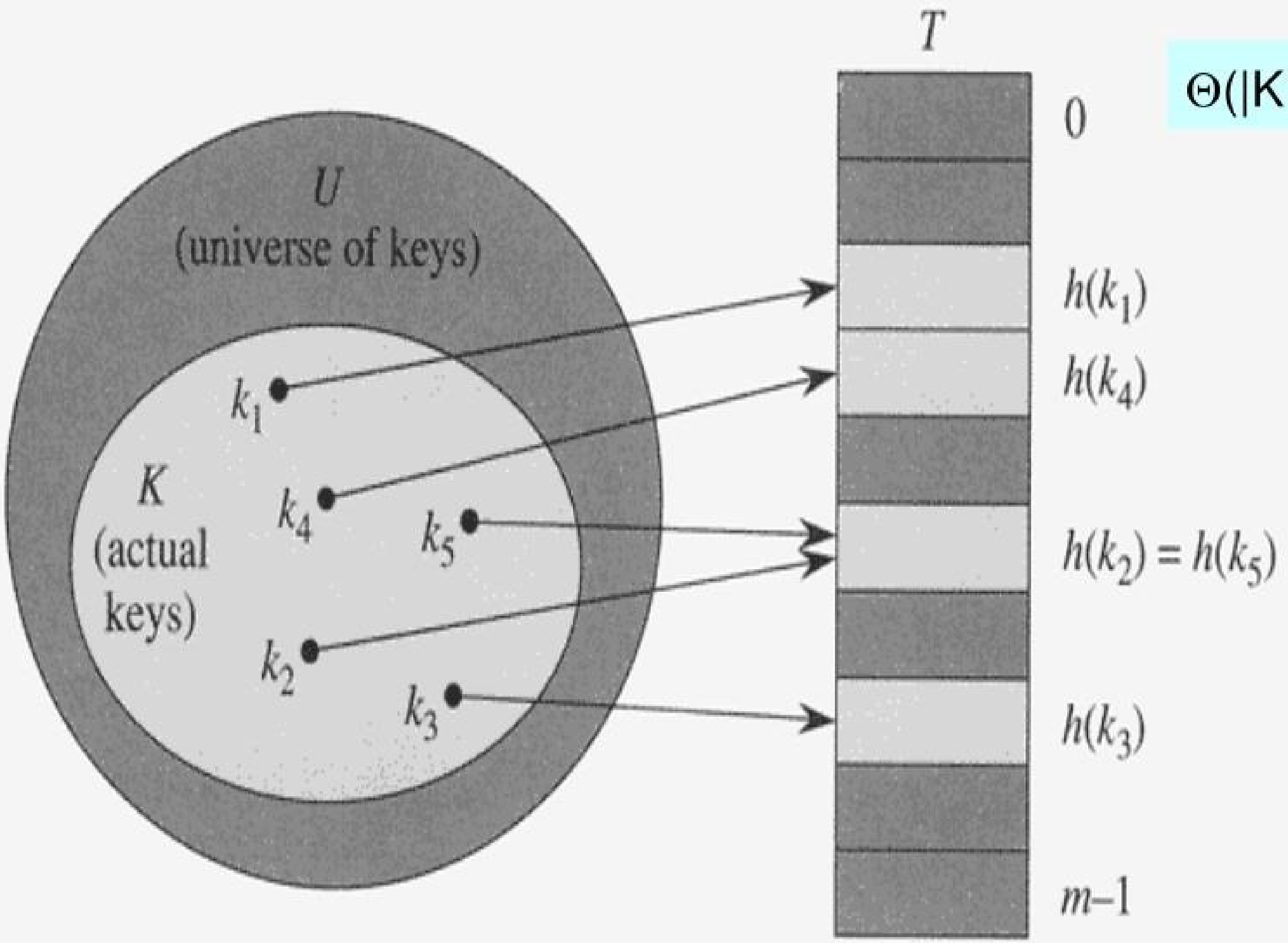
Overview

- Difficulties with direct addressing
  - If U is large, storing a table T of size |U| may be impractical
  - The set K of keys actually stored may be small relative to U
- When the set K of keys stored in a dictionary is much smaller than U of all possible keys, a hash table requires much less storage than a direct-address table
  - $\Theta(|K|)$ storage
  - O(1) average time to search for an element
    - $\Theta(1)$ worst case to search for an element in direct-address table

Hash Table

- Direct addressing: an element with key k → T[k]
- Hash table: an element with key k → T[***h(k)***]
  - h(k): hash function; usually can be computed in O(1) time
    - h: U → {0, 1, .., m-1}
    - An element with key k hashes to slot h(k)
    - h(k) is the hash function of key k
    - Instead of |U| values, we need to handle only m values
  - Collision may occur
    - Two different keys hash to the same slot ($h(k_1) = h(k_2) = x$)
    - Eliminating all collisions is impossible
    - But a well-designed "random"-looking hash table can minimize collisions

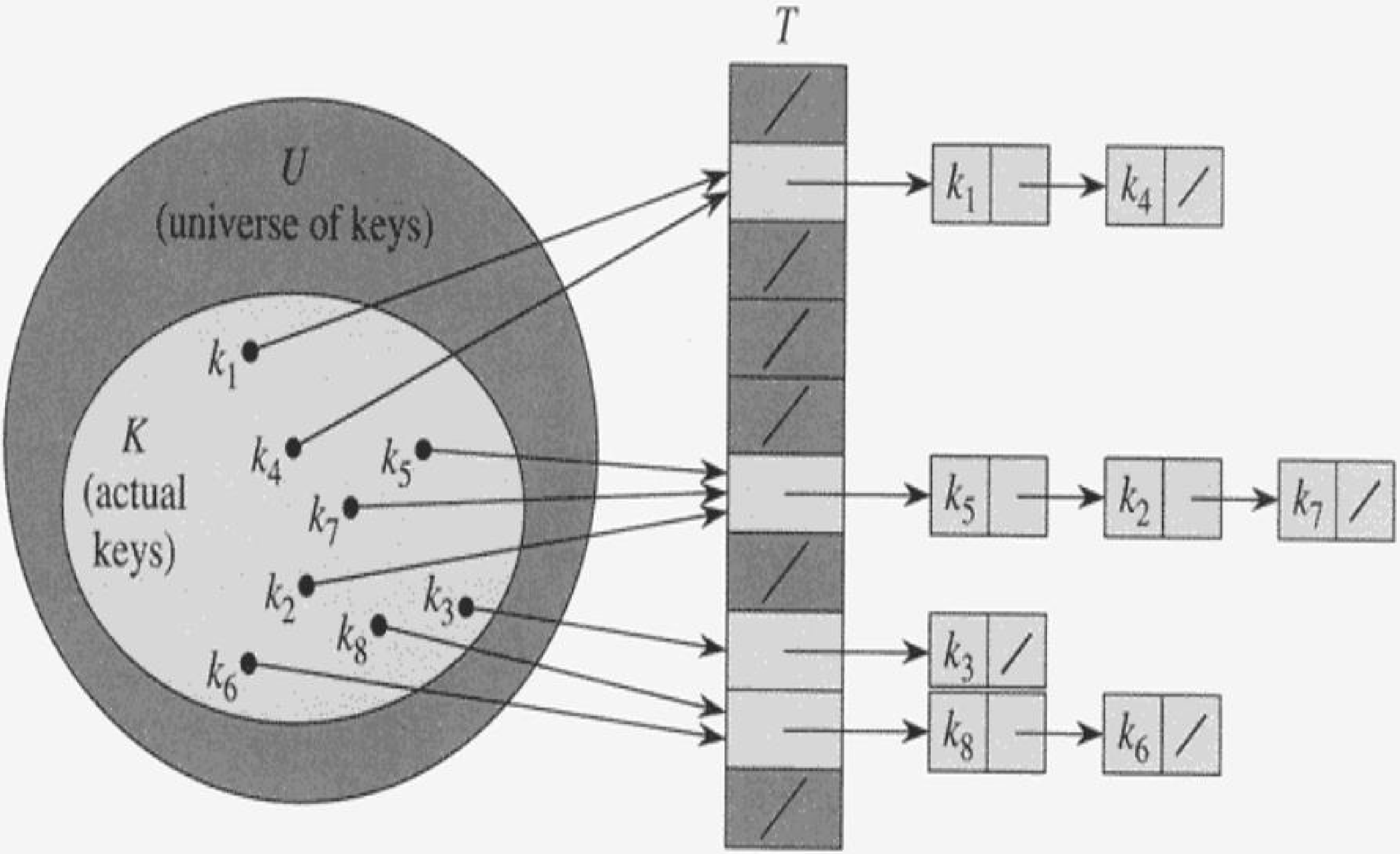
$\Theta(|K|) \approx \mathbf{m} << |\mathbf{U}|$



Using a hash function h to map keys to hash-table slots. Keys k_2 and k_5 map to the same slot, so they collide.

Collision Resolution by Chaining

- Idea
  - Put all the elements that hash to the same slot in a linked list
  - Slot j contains a pointer to the head of the list of all stored elements that hash to j
- Operations
  - CHAINED-HASH-INSERT(T, x): O(1) (no check for duplication)
    - Insert x at the head of list T[***h(key[x])***]
  - CHAINED-HASH-SEARCH(T, k): proportional to the list length
    - Search for an element with key k in list T[***h(k)***]
  - CHAINED-HASH-DELETE(T, x): O(1) for doubly linked list
    - Delete x from the list T[***h(key[x])***]



另一種方式:open addressing

Collision resolution by chaining. Each hash-table slot $T[j]$ contains a linked list of all the keys whose hash value is j. For example, $h(k_1) = h(k_4)$ and $h(k_5) = h(k_2) = h(k_7)$.

Analysis of Hashing with Chaining for Searching

- Load factor α = n elements/m slots (負載因子)
  - Average number of elements stored in a chain
- Worst case: $\Theta(n)$ + time to compute the hash function
  - all n keys hash into the same slot
- Average performance
  - How well the hash function h distributes the set of keys to be stored among the m slots, on the average
  - Assume simple uniform hashing
    - Any given element is equally likely to hash into any of the m slots, independently of where any other elements has hashed to
  - The time required for a successful or unsuccessful search is $\Theta(1+\alpha)$
    - $n = O(m) \rightarrow \alpha = n/m = O(m)/m = O(1)$

Hash Functions

Overview

- Interpreting keys as natural numbers N={0, 1, 2,…}
  - If the keys are not natural number → convert

    For example "pt": p=112,t=116,using radix-128 integer→pt becomes 112*128+116=14452
- What makes a good hash function?
  - Each key is equally likely to hash to any of the m slots, independently of where any other key has hashed to
    - Rarely knows the probability distribution
    - Keys may not be drawn independently
  - Hash by division: heuristic
  - Hash by multiplication: heuristic
  - Universal hashing: randomization to provide provably good performance

Hash By Division

- h(k) = k mod m
  - m = 12 and k=100 → h(k) = 4
  - Must avoid certain values of m ➔ rely on characteristics of k values
    - m should not be a power of 2 → h(k) is the lowest-order bits of k
  - Good values for m are primes not too close to an 2^P
  - For example: n=2000 character string, we don'n mind examing 3 elements in unsuccessful search➔allocate a hash table of size 701

    701 is prime near 2000/3, not near any power of 2

    h(k)=k mod 701

Hash By Multiplication

- $h(k) = \lfloor m*(k*A \bmod 1) \rfloor$
  - multiply the key k by a $A \in (0, 1)$ and extract the fractional part of kA
  - multiply the fractional part of kA by m and take the floor of the result
- The value of m is not critical
  - Typically, $m = 2^P$ (for easy implementation on computers)
  - (Knuth) $A \approx \left(\sqrt{5}-1\right)/2 = 0.6180339887...$
  - For example:k=123456,m=10000,a=0.618

    H(k)=floor(10000*(123456*0.618… mod 1))

    =floor(10000*(76300.004151..mod 1))

    =floor(10000*0.$\underline{0041}$151…..)=41.

Universal Hashing

- Idea:
  - Choose the hash function randomly in a way that is independent of the keys that are actually going to be stored
  - Select the hash function at random from a carefully designed class of functions at the beginning of execution
    - The algorithm can behave differently on each execution, even for the same input

Universal Hashing (Cont.)

- Let H be a finite collection of hash functions that map a given universe U of keys into the range {0, 1,…, m-1}.
- H is ***universal*** if for each pair of distinct keys k, l $\in$ U, the number of hash functions h $\in$H for which k(h) = k(l) is at most |H|/m
  - Collision chance = 1/m
- Theorem 11.3.
  - Suppose that a hash function h is chosen from a universal collection of hash function and is used to hash n keys into a table T of size m, using chaining to resolve collisions. If key k is not in the table, then the expected length $E[n_{h(k)}]$ of the list that key k hashes to is at most α. If key k is in the table, then the expected length $E[n_{h(k)}]$ of the list containing key k is at most 1+ α

Designing A Universal Class of Hash Functions

- Steps to design a universal class of hash functions
  - Choose a prime number p, so that $k \in [0, p-1]$ and $p > m$
  - $Z_p=\{0, 1,\ldots, p-1\}$
  - $Z^*_p=\{1, 2,\ldots, p-1\}$
  - $h_{a,b}(k) =$ ***((ak+b) mod p) mod m***, for any $a \in Z^*_p$ and $b \in Z_p$
- The family of all such hash functions is
  - $H_{p,m} = \{h_{a,b}: a \in Z^*_p \text{ and } b \in Z_p\}$
  - Total: *p(p-1)* hash functions in $H_{p,m}$

Other hash function(1)

- Mid-square: 先將數值平方在取中間部分位元

  $(39)^2=(100111)^2=(101\underline{11110}001)_2$

  $F(39)=(11110)_2=(29)_{10}$

- Digital analysis:位數分析,對資料每一位數加以分析,剔除不均勻分布之位數,剩下位數作為hash位址

  適合static data 且位數相同

  For example

x	F(x)
1510327	5107
1857384	8574
2621439	6219
1796333	7963
1038420	0380
7142481	1421
7203326	2036
2385425	3855

Other hash function(2)

- Folding(摺疊法):
- (1)shift folding
- (2)folding at boundaries

1214	3412	1511	2351	20

$$\begin{array}{rr} & 1214 \\ & 3412 \\ & 1511 \\ & 2351 \\ + & 20 \\ \hline & 8508 \end{array} \qquad \begin{array}{rr} & 1214 \\ & 2143 \\ & 1511 \\ & 1532 \\ + & 20 \\ \hline & 6420 \end{array}$$

Open Addressing

Overview

- All elements are stored in the hash table itself
  - Each table entry contains either an set element or NIL
  - Search: systematically examine table slots until the desired element is found or it is clear that the element is not in the table
  - No lists and no elements are stored outside the table
    - Load factor $\alpha \leq 1$
  - At most m elements can be stored in the hash table
    - The extra memory freed by not storing pointers provides the hash table with a larger number of slots for the same amount of memory, potentially yielding fewer collisions and faster retrieval

Probe

- Insertion: successively examine (***probe***) the hash table until we find an empty slot in which to put the key
  - The sequence of positions probed relies on the key being inserted
    - *Probe sequence* for every key k must be a permutation of <0,1,..,m-1>
  - Extended hash function → h: U * {0,1,…,m-1} → {0,1,…, m-1}
    - <h(k,0), h(k,1),…,h(k,m-1)>
- Assume uniform hashing
  - Each key is equally likely to have any of the m! permutations of <0,1,…,m-1> as its probe sequence

INSERT and SEARCH in Open Addressing

```
HASH-INSERT(T, k)
i ← 0
repeat j ← h(k, i)
        if T[j] = NIL
           then T[j] ← k
                return j
           else i ← i + 1
   until i = m
error "hash table overflow"
```

```
HASH-SEARCH(T, k)
i ← 0
repeat j ← h(k, i)
          if T[j] = k
             then return j
          i ← i + 1
     until T[j] = NIL or i = m
return NIL
```

Assume keys are not deleted from the table

Example

h'(k)=k mod 5
h(k,i)=(h'(k) + i) mod 5
For k=9 → h(k,i) = <4, 0, 1, 2, 3>
For k=3 → h(k,i) = <3, 4, 0, 1, 2>

0	14
1	
2	2
3	
4	9

HASH-INSERT(T,2)

HASH-INSERT(T,9)

HASH-INSERT(T,14)

0	14
1	
2	2
3	
4	DELETED

HASH-DELETE(T, 9)

HASH-SEARCH(T, 14)

We have to mark a deleted slot, instead of letting it be NIL

DELETE in Open Addressing

- Difficult !!
  - Cannot simply mark the deleted slot *i* as empty by storing NIL
    - Impossible to retrieve any key k during whose insertion we had probed slot *i* and found it occupied
  - Solution: mark the slot by storing in it a special value DELETED
    - Modify HASH-INSERT: treat a DELETED slot as a NIL slot
    - No modification of HASH-SEARCH is needed
      - Search times are no longer dependent on α
      - Therefore, chaining is more commonly selected as a collision resolution technique when keys must be deleted

Linear Probing

- h(k,i) = (h'(k) + i) mod m (i = 0, 1,…, m-1)
  - h': U → {0,1,…,m-1} is called an auxiliary hash function
  - T[h'(k)] →T[h'(k)+1]→…→T[m-1]→T[0]→…→T[h'(k)-1]
  - Only m distinct probe sequences
  - Easy to implement
  - Problem: Primary clustering
    - Long runs of occupied slots tend to get longer, and the average search time increases
  - Secondary clustering
    - If two keys have the same initial probe position, then their probe sequences are the same
  - Example:f(x)=x, table size=19(0..18), data={1.0.5.1.18.3.8.9.14.7.5.5.1.13.12.5}

Quadratic Probing

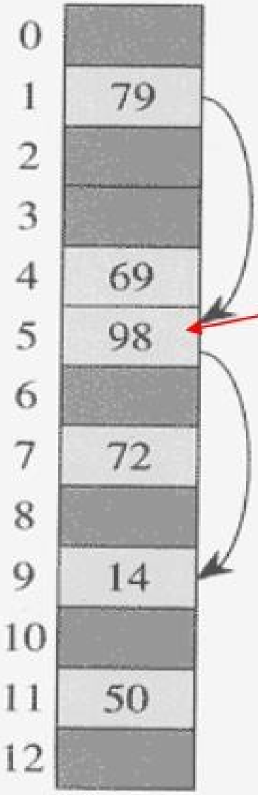
- $h(k,i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$ ($i = 0, 1, \ldots, m-1$)
  - h': auxiliary hash function; $c_1, c_2 \neq 0$ (auxiliary constants)
  - The values of c_1, c_2, m are constrained (See Problem 11-3)
  - Work much better than linear probing (alleviate primary clustering)
  - Secondary clustering
    - If two keys have the same initial probe position, then their probe sequences are the same
  - Only m distinct probe sequences

Double Hashing: the best

- $h(k,i) = (h_1(k) + i*h_2(k)) \bmod m$ ($i = 0, 1, \ldots, m-1$)
  - h_1 and h_2 are auxiliary hash functions
  - Depends in two ways upon the key i, since the initial probe position, the offset, or both, may vary ➔ $\Theta(m^2)$ distinct probing sequences
  - The value of $h_2(k)$ must be relative prime to the hash-table size m for the entire hash table to be searched (Exercise 11.4-3)
    - Let $m = 2^P$ and design h_2 so that it always produces an odd number
    - Let m be prime and design h_2 so that it always produces a positive integer less than m
      - $h_1(k) = k \bmod 701$
      - $h_2(k) = 1 + (k \bmod 700)$

Example of Double Hashing

Data sequence: 69, 72, 50,79,98,14



$h_1(14) = 1$
$h_2(14) = 4$

H1(98)=98 mod 13=7
H2(98)=1+(98 mod 11)=11
H(98,1)=(7+11) mod 13=5

$h(k,i) = (h_1(k) + i^*h_2(k)) \bmod m$

h(14,0) = 14 mod 13=1……..1st collision

h(14,1) = (1+1*(1+(14 mod 11)))mod 13=5….2nd collision

h(14,2) = (1+2*(4)) mod 13=9

$H_1(k)$=k mod 13
$H_2(k)$=1+(k mod 11)

Insertion by double hashing. Here we have a hash table of size 13 with $h_1(k) = k \bmod 13$ and $h_2(k) = 1 + (k \bmod 11)$. Since $14 \equiv 1 \pmod{13}$ and $14 \equiv 3 \pmod{11}$, the key 14 is inserted into empty slot 9, after slots 1 and 5 are examined and found to be occupied.

Analysis of Open-Address Hashing

- Theorem 11.6
  - Given an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$, assuming uniform hashing
    - If α is a constant → an unsuccessful search runs in O(1) time
- Corollary 11.7
  - Inserting an element into an open-address hash table with load factor α requires at most $1/(1-\alpha)$ probes on average, assuming uniform hashing
- Theorem 11.8
  - Given an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an successful search is at most $1/\alpha * \ln(1/(1-\alpha))$, assuming uniform hashing and that each key in the table is equally likely to be searched for
    - If α is a constant → an unsuccessful search runs in O(1) time

Self-Study

- Proof of Theorems
- Section 11.5 Perfect Hashing
  - The worst-case number of memory accesses required to perform a search is O(1)

example

- A hashed table is constructed using the division hash algorithm function with 5 buckets (a bucket at most 4 records) . If the following key field values are to be placed in buckets: 3 ,5, 24, 22, 109, 10, 8, 6, 23, 28, 100, 103, 9, 39, 27, 0.
  - Identify the number of records in each bucket.
  - Which bucket overflows?

no

$H(k)=k \bmod 5$

0	5	10	100	0
1	6			
2	22	7	27	
3	3	8	23	28
4	24	109	9	29

103

•**Bucket 3 overflow.**