I2-TD4
(Systems of Ordinary Differential Equations)

1. Solve the following homogeneous system of ode

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad x(t) = (x_1(t), x_2(t), \dots, x_n(t));\ n = 2, 3, \dots$$

and the matrix A is given below:

(a) $A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} -18 & 9 \\ -49 & 24 \end{pmatrix}$

(c) $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

(d) $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

(e) $A = \begin{pmatrix} -1 & 1 & -1 \\ -10 & 6 & -5 \\ -6 & 3 & -2 \end{pmatrix}$

(f) $A = \begin{pmatrix} 22 & -5 & 1 & -5 \\ 18 & -2 & 2 & -5 \\ 6 & -4 & 3 & -4 \\ 62 & -16 & 2 & -13 \end{pmatrix}$

(g) $A = \begin{pmatrix} 5 & 2 & -1 & -1 \\ 3 & 10 & -3 & -3 \\ 4 & 8 & 0 & -4 \\ 3 & 6 & -3 & 1 \end{pmatrix}$

(h) $A = \begin{pmatrix} 4 & 1 & 2 & -2 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 5 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

(i) $A = \begin{pmatrix} 5 & -1 & 2 & 4 \\ 5 & -1 & 3 & 4 \\ 8 & -5 & 3 & 4 \\ -4 & 1 & -1 & -3 \end{pmatrix}$

(j) $A = \begin{pmatrix} 2 & -3 & -2 & 3 \\ -4 & 2 & 4 & -4 \\ -4 & 0 & 4 & -3 \\ -8 & 4 & 8 & -8 \end{pmatrix}$

(k) $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

(l) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(m) $A = \begin{pmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix}$

(n) $A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

(o) $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

2. Let A be a square matrix defined by

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 4 & -3 & -2 \\ 5 & -1 & -5 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenspaces of A.

(b) Show that A is diagonalizable. Diagonalize A.

(c) Solve the system of linear differential equations $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3)^T$. Then solve the initial value problem $\frac{dx}{dt} = Ax + B(t), x(0) = (0, 0, 1)^T$ where $B(t) = (e^{2t}, 3e^{2t}, -5e^{2t})^T$.

3. Let A be a square matrix defined by

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -2 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenspaces of A.

(b) Show that A is not diagonalizable but triangularizable. Triangularize A.

(c) Solve the system of linear differential equations $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3)^T$. Then solve the initial value problem $\frac{dx}{dt} = Ax + B(t), x(0) = (1,0,1)^T$ where $B(t) = (-6e^t, 5e^t, -3e^t)^T$.

4. Let A be a square matrix defined by

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(a) Find the eigenvalues and eigenspaces of A in $\mathbb{C}$.

(b) Solve the system of linear differential equations $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3, x_4)^T$. Then solve the initial value problem $\frac{dx}{dt} = Ax + B(t), x(0) = (1,0,1,0)^T$ where $B(t) = (-6e^t, 5e^t, -3e^t, 0)^T$.

5. Solve the following systems of ordinary differential equations:

(a) $$\begin{cases} x'(t) = -5x(t) + 12y(t) + 4z(t) + 9e^{-3t}, x(0) = 1 \\ y'(t) = -4x(t) + 11y(t) + 4z(t) - 5e^{-3t}, y(0) = 0 \\ z'(t) = 4x(t) - 12y(t) - 5z(t) - 7e^{-3t}, z(0) = 1 \end{cases}$$

(b) $$\begin{cases} x'(t) = -3x(t) - y(t) + z(t) + 9\cos t, x(0) = -1 \\ y'(t) = -3x(t) - 5y(t) + 3z(t) + 4\cos t, y(0) = 0 \\ z'(t) = -5x(t) - 5y(t) + 3z(t) + 7\cos t, z(0) = 1 \end{cases}$$

(c) $$\begin{cases} x'(t) = 8x(t) + 16y(t) + 6z(t) + 12\sin t \\ y'(t) = -6x(t) - 14y(t) - 6z(t) + 5\cos t \\ z'(t) = 9x(t) + 24y(t) + 11z(t) + \cos t \end{cases}$$

(d) $$\begin{cases} w'(t) = -17w(t) + 6x(t) + 3y(t) + 3z(t) + 12e^{-2t} \\ x'(t) = -18w(t) + 7x(t) + 3y(t) + 3z(t) + 6e^{-2t} \\ y'(t) = -15w(t) + 6x(t) + y(t) + 3z(t) + 8e^{-2t} \\ z'(t) = -39w(t) + 12x(t) + 9y(t) + 7z(t) + 10e^{-2t} \end{cases}$$

6. Solve the following systems of ordinary differential equations:

(a) $$\begin{cases} x'(t) = 2x(t) + y(t) - z(t) + 9e^{-2t}\sin t \\ y'(t) = x(t) + 3y(t) - 2z(t) + 7e^{-2t}\sin t \\ z'(t) = -x(t) + 2y(t) + z(t) - 4e^{-2t}\sin t \end{cases}$$

(b) $\begin{cases} x'(t) = -5x(t) + 7y(t) + 2z(t) - 3e^{2t}\cos t, x(0) = 0 \\ y'(t) = -2x(t) + 2y(t) + z(t) - 2e^{2t}\cos t, y(0) = 0 \\ z'(t) = 2x(t) - 4y(t) - 3z(t) + 4e^{2t}\cos t, z(0) = 0 \end{cases}$

(c) $\begin{cases} x'(t) = -5x(t) + 3y(t) - 2z(t) - 6e^{-2t}\cos t \\ y'(t) = -2x(t) - 2y(t) - 3z(t) - 8e^{-2t}\cos t \\ z'(t) = 2x(t) + 4y(t) + 5z(t) + 6e^{-2t}\cos t \end{cases}$

(d) $\begin{cases} w'(t) = -4w(t) + 3x(t) + y(t) - z(t) - 2e^{-t} \\ x'(t) = -6w(t) + 5x(t) + y(t) - z(t) + 6e^{-t} \\ y'(t) = -2w(t) + 3x(t) - 2z(t) - 8e^{-t} + 2e^{-t}\cos t \\ z'(t) = -4w(t) + 3x(t) + 2y(t) - 2z(t) + 4e^{-t} \end{cases}$

(e) $\begin{cases} w'(t) = 13w(t) - 2x(t) - 3y(t) - 5z(t) - 2e^{2t}\sin t, w(0) = -1 \\ x'(t) = 15w(t) - x(t) - 4y(t) - 6z(t) + 6e^{2t}\sin t, x(0) = 1 \\ y'(t) = 15w(t) - 2x(t) - 4y(t) - 6z(t) + 2e^{-t}\cos t, y(0) = 0 \\ z'(t) = 13w(t) - 2x(t) - 2y(t) - 6z(t) + 4e^{-t}\cos t, z(0) = 0 \end{cases}$

(f) $\begin{cases} w'(t) = 6w(t) - 2x(t) - 2z(t) - 2e^{-t}\sin t, w(0) = 0 \\ x'(t) = 12w(t) - 4x(t) - y(t) - 3z(t) + 6e^{-t}\sin t, x(0) = 0 \\ y'(t) = 9w(t) - 2x(t) - y(t) - 3z(t) + 2e^{-t}\cos t, y(0) = 0 \\ z'(t) = 13w(t) - 4x(t) + y(t) - 5z(t) + 4e^{-t}\cos t, z(0) = 0 \end{cases}$

7. Solve the following systems of ordinary differential equations:

(a) $\begin{cases} x'(t) = -2x(t) + y(t) + 2z(t) + 9e^{2t} \\ y'(t) = -5x(t) + 3y(t) + 3z(t) + 7e^{2t} \\ z'(t) = -4x(t) + y(t) + 4z(t) - 4e^{2t} \end{cases}$

(b) $\begin{cases} x'(t) = 4x(t) + 18y(t) + 8z(t) + 2e^{-2t} \\ y'(t) = -5x(t) - 16y(t) - 6z(t) - 2e^{-2t} \\ z'(t) = 7x(t) + 19y(t) + 6z(t) - 4e^{-2t} \end{cases}$

(c) $\begin{cases} x'(t) = 8x(t) + 18y(t) + 8z(t) + 4e^{2t}, x(0) = 0 \\ y'(t) = -5x(t) - 12y(t) - 6z(t) - 6e^{2t}, y(0) = 0 \\ z'(t) = 7x(t) + 19y(t) + 10z(t) + 8e^{2t}, z(0) = 1 \end{cases}$

(d) $\begin{cases} w'(t) = -19w(t) + 7x(t) + 4y(t) + 3z(t) - 2e^{t}, w(0) = 0 \\ x'(t) = -20w(t) + 8x(t) + 4y(t) + 3z(t) + 6e^{-2t}, x(0) = 0 \\ y'(t) = -17w(t) + 7x(t) + 2y(t) + 3z(t) + 2e^{t}, y(0) = 0 \\ z'(t) = -44w(t) + 15x(t) + 11y(t) + 7z(t) + 4e^{-2t}, z(0) = 0 \end{cases}$

(e) $\begin{cases} w'(t) = -10w(t) - 6x(t) + 2y(t) + 7z(t) - 2e^{2t} \\ x'(t) = 4w(t) + x(t) - 3z(t) + 6e^{-t} \\ y'(t) = 7w(t) + 5x(t) - 4y(t) - 6z(t) + 2e^{-t} \\ z'(t) = -15w(t) - 12x(t) + 8y(t) + 12z(t) \end{cases}$

(f) $\begin{cases} w'(t) = -5w(t) + x(t) + y(t) + z(t) - 2e^{t} \\ x'(t) = -5w(t) + 2y(t) + z(t) + 6e^{-t} \\ y'(t) = -6w(t) + x(t) + 2z(t) + 2e^{t} \\ z'(t) = -8w(t) + 2x(t) + 2y(t) + z(t) + 4e^{-t} \end{cases}$

(g) $\begin{cases} w'(t) = 2w(t) - 2x(t) + y(t) - 2z(t) - 2e^{t} + 4e^{-t}\cos t, w(0) = 0 \\ x'(t) = 6w(t) - 4x(t) - z(t) - 2e^{t} - 4e^{-t}\cos t, x(0) = 0 \\ y'(t) = 6w(t) - 4x(t) + 2y(t) - 6z(t) + 6e^{t} + 4e^{-t}\cos t, y(0) = 0 \\ z'(t) = 2w(t) - 2x(t) + 2y(t) - 4z(t) + 6e^{t} - 4e^{-t}\cos t, z(0) = 0 \end{cases}$