Notations.

- $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$
- $\mathbf{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t))^T$

Definition. A system of first order ordinary linear differential equation is of the form

$$\begin{cases} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t) \end{cases}$$

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \qquad (*)$$

where

$$A(t) = (a_{ij}(t))_n \quad \text{and} \quad \mathbf{b}(t) = (b_1(t), b_2(t), \ldots, b_n(t))^T$$

If $\mathbf{b}(t) = 0$, then the system is called a homogeneous, otherwise

it is called nonhomogeneous.

Theorem. If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly independent solution of

of a homogeneous equation

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \qquad (**)$$

then the general solution of equation $(**)$ is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + \mathbf{x}_n(t)$$

where $c_1, c_2, \ldots, c_n$ are costants.

Definition. Let $\mathbf{x}_1(t),\ \mathbf{x}_2(t),\ldots,\mathbf{x}_n(t)$ be n vectors defined by

$$\mathbf{x}_i(t)=(x_{i1}(t),x_{2i}(t),\ldots,x_{ni}(t))^T,\quad \forall\, i=1,2,\ldots,n$$

The Wronskian of $\mathbf{x}_1(t),\mathbf{x}_2(t),\ldots,\mathbf{x}_n(t)$ is denoted and defined by

$$W(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)=\det(x_{ij}(t))_n=\begin{vmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t)\\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t)\end{vmatrix}$$

Theorem. Let $\mathbf{x}_1(t),\ \mathbf{x}_2(t),\ldots,\mathbf{x}_n(t)$ be n vectors defined as in the above definiton.

Then $\mathbf{x}_1(t),\ \mathbf{x}_2(t),\ldots,\mathbf{x}_n(t)$ are linearly independent on I if

$$W(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\neq 0,\ \text{for every } t\in I.$$

Theorem. If $W(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)\neq 0,$ for some $t_0\in I,$ then $\mathbf{x}_1(t),\mathbf{x}_2(t),\ldots,\mathbf{x}_n(t)$

are linearly independent.

Homogeneous linear systems with constant coefficients

$$\mathbf{x}'(t)=A\mathbf{x}(t)\qquad (***)$$

where $\quad\mathbf{x}(t)=(x_1(t),x_2(t),\ldots,x_n(t))\quad$ and $\quad A=(a_{ij})_n$

Suppose that the system has a solution of the form

$$\mathbf{x}(t)=e^{\lambda t}v$$

where v is a non $-$ zero constant vector. Then

$$\mathbf{x}'(t)=te^{\lambda t}v$$

put it into the equation $(***)$ we have of A related to eigenvalue λ. Conversely, if λ is an eigenvalue of A and v is an eigenvector of A associated with λ, then $e^{\lambda t}v$ is a solution.

Example 1. Solve the system of equation

$$\begin{cases} x_1'(t) = x_1(t) + x_2(t) \\ x_2'(t) = 4x_1(t) + x_2(t) \end{cases}$$

$X(t) = c_1 X_1(t) + c_2 X_2(t)$

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

Solution. Rewrite the system as

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where

$$\mathbf{x}(t) = (x_1(t), x_2(t))^T, \text{ and } \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

The characteristics polynomial of A is

$$p(\lambda) = \lambda^2 - 2\lambda - 3$$

$$p(\lambda) = 0 \quad \Leftrightarrow \quad \lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \quad \lambda = 3, -1$$

we find that the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

- For $\lambda_1 = 3$: $\quad (A - \lambda_1 I)v = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $v_1 = \alpha$. Then $v_2 = 2v_1 = 2\alpha$. Thus an eigenvector of A associated with eigenvalue $\lambda_1 = 3$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $X_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

- For $\lambda_2 = -1$: We can show that $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is its corresponding eigenvector. Therefore, the general solution is

- For $\lambda_2 = -1$: We can show that $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is its corresponding , $X_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

  eigenvector. Therefore, the general solution is

$$\mathbf{x}_1(t) = e^{\lambda_1 t} v_1$$

$$\mathbf{x}_2(t) = e^{\lambda_2 t} v_2$$

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Theorem. Suppose that λ is an eigenvalue of matrix $A = (a_{ij})_n$ and v_k is its associated eigenvector of index k ($\text{index}(\lambda) = k$). We denote . $v_k \in E_\lambda^k \setminus E_\lambda^{k-1}$

$$v_i = (A - \lambda I)^{k-i} v_k, \quad i = 1, 2, \ldots, k-1. \qquad v_{k-1} = (A - \lambda I) v_k$$

Then

$$\mathbf{x}_1(t) = e^{\lambda t} v_1$$

$k = \text{index}(\lambda)$

$$\mathbf{x}_2(t) = e^{\lambda t}(v_2 + t v_1)$$

if $\dim(E_\lambda^k) = \text{am}(\lambda)$

$$\mathbf{x}_3(t) = e^{\lambda t}\left(v_3 + t v_2 + \frac{t^2}{2!} v_1\right)$$

$$\vdots$$

$$\mathbf{x}_k(t) = e^{\lambda t}\left(v_k + t v_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!} v_1\right)$$

are linearly independent solution of the equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

Example 2. Solve the system

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad ; \quad X(t) = c_1 X_1(t) + c_2 X_2(t)$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))^T$, and $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$

Proof. The characteristics equation of A is

$$\lambda^2 - 8\lambda + 16 = 0 \quad \Leftrightarrow \quad (\lambda - 4)^2 = 0$$

Thus $\lambda = 4$ is the only eigenvalue of A with $\text{am}(\lambda) = 2$.

To find the associated eigenvector, we consider the equation

$$(A - \lambda I)v = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad ; \quad X_1(t) = e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector associated with } \lambda = 4.$$

Therefore, $E_\lambda = \text{Span}\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\} \quad \Rightarrow \quad \text{gm}(\lambda) = \dim(E_\lambda) = 1 \neq \text{am}(\lambda) = 2$

To find E_λ^2, we consider the equation $(A - \lambda I)^2 v = 0$.

$$(A - \lambda I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}^2 = 0$$

Thus

index(λ)

$$E_\lambda^2 = \text{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \quad \Rightarrow \quad \dim(E_\lambda^{2}) = 2 = \text{am}(\lambda) \quad \Rightarrow \quad \text{index}(\lambda) = 2$$

We have index$(\lambda) = 2$. So, choose $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in E_\lambda^2 - E_\lambda$. Then

$$v_1 = (A - \lambda I)v_2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

Hence, the solutions of the system are

$$\mathbf{x}_1(t) = e^{\lambda t} v_1 = e^{4t}\begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{\lambda t}(v_2 + t v_1) = e^{4t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right]$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{4t}\begin{pmatrix} -3 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right]$$

$$\text{or} \quad \begin{cases} x_1(t) = e^{4t}(-3c_1 + (1 - 3t)c_2) \\ x_2(t) = e^{4t}(3c_1 + 3tc_2) \end{cases}, \qquad c_1, c_2 \in \mathbb{R}.$$

Ex $X'(t) = AX(t)$. Suppose that $A = PJP^{-1}$

where $J = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$; $P = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$.

Then the general solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$
$$= c_1 e^{3t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \qquad c_1, c_2 \in \mathbb{R}.$$

Ex: $X'(t) = AX(t)$. Suppose that $A = PJP^{-1}$

where $J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$; $P = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = (v_1 \ v_2)$

Then the general solution is $\qquad v_2 \in E_\lambda^2 - E_\lambda$

$\qquad v_1 = (A - \lambda I) v_2$

$$X_1(t) = e^{\lambda t} v_1 = e^{3t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$X_2(t) = e^{\lambda t}(v_2 + t v_1) = e^{3t}\left[\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right].$$

Therefore, the general solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$
$$= c_1 e^{3t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + c_2 e^{3t} \left[\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right].$$

Ex $X'(t) = AX(t)$. Suppose that $A = PJP^{-1}$

where $J = \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \end{array}\right)$; $P = \begin{array}{ccc} v_1 & v_2 & \\ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \end{array}$.

$X_1(t) = e^{\lambda t} v_1 = e^{3t} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$; $X_2(t) = e^{\lambda t}(v_2 + t v_1) = e^{3t}(v_2 + t v_1)$

$v_2 \in E_\lambda^{2} - E_\lambda$. $\quad v_1 = (A - \lambda I) v_2$.

$X_3(t) = e^{3t} u$; $u \in E_\lambda$

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$$

$$\underline{X(t) = c_1 e^{3t} v_1 + c_2 e^{3t}(v_2 + t v_1) + c_3 e^{3t} u}$$

Ex: $X'(t) = A X(t)$. Suppose that $A = P J P^{-1}$

where
$$J = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} ; \quad P = \begin{pmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{u} & \underline{w_1} & \underline{w_2} \end{pmatrix}$$

* $X_1(t) = e^{3t} v_1$; $X_2(t) = e^{3t}(v_2 + t v_1)$; $X_3(t) = e^{3t}\left(v_3 + t v_2 + \frac{t^2}{2} v_1\right)$

$v_3 \in E_3^3 \setminus E_3^2$. $v_2 = (A - 3I) v_3$; $v_1 = (A - 3I) v_2$.

* $X_4(t) = e^{3t} u$; $u \in E_3$

* $X_5(t) = e^{2t} w_1$; $X_6(t) = e^{2t}(w_2 + t w_1)$; $w_2 \in E_2^2 - E_2$

$w_1 = (A - 2I) w_2$.

The general solution is

$$X(t) = c_1 X_1(t) + - \; - + c_6 X_6(t)$$

Example 3. Solve the system of equation $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{pmatrix}$$

Proof. The characteristics polynomial of A is

$$p(\lambda) = -(\lambda - 2)^3$$

$\implies \quad \lambda = 2$ is an eigenvalue of A wtith $\mathrm{am}(\lambda) = 3$

We have

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$E_\lambda = \mathrm{Span}\{\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}\} \implies \mathrm{gm}(\lambda) = \dim E_\lambda = 1 \neq \mathrm{am}(\lambda)$$

$$E_\lambda^2 = \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\} \implies \dim(E_\lambda^2) = 2 \neq \mathrm{am}(\lambda)$$

$$E_\lambda^3 = \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\} \implies \dim(E_\lambda^3) = 3 = \mathrm{am}(\lambda)$$

Choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in E_\lambda^3 - E_\lambda^2$. Then,

$$v_2 = (A - \lambda I)v_3 = (A - 2I)v_3 = \begin{pmatrix} 3 & 0 & 1 \\ 1 & -1 & 0 \\ -7 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}$$

$$v_1 = (A - \lambda I)v_2 = (A - 2I)v_2 = \begin{pmatrix} 3 & 0 & 1 \\ 1 & -1 & 0 \\ -7 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}$$

Thus the solutions to the system are

Thus the solutions to the system are

$$\mathbf{x}_1(t) = e^{\lambda t} v_1 = e^{2t} \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{\lambda t}(v_2 + t v_1) = e^{2t} \left[\begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 3+2t \\ 1+2t \\ -7-6t \end{pmatrix}$$

$$\mathbf{x}_3(t) = e^{\lambda t}(v_3 + t v_2 + \frac{t^2}{2!} v_1) = e^{2t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 1+3t+t^2 \\ t+t^2 \\ -7t-3t^2 \end{pmatrix}$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t)$$

$$= e^{2t} \left[c_1 \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} 3+2t \\ 1+2t \\ -7-6t \end{pmatrix} + c_3 \begin{pmatrix} 1+3t+t^2 \\ t+t^2 \\ -7t-3t^2 \end{pmatrix} \right]$$

Example4. Solve the system $\mathbf{x}'(t) = A\mathbf{x}(t),$ where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 8 & -1 & 6 \\ 7 & -3 & 8 \end{pmatrix}$$

Proof. The characteristics equation of A is

$$-(\lambda - 3)(\lambda - 2)^2 = 0$$

We have $\lambda_1 = 3$ and $\lambda_2 = 2$ are eigenvalues of A with $\mathrm{am}(\lambda_1) = 1,\ \mathrm{am}(\lambda_2) = 2.$

We have

$$E_{\lambda_1} = E_3 = \mathrm{Span}\{(-1, 1, 2)^T\} \quad \Rightarrow \quad \mathrm{gm}(\lambda_1) = 1 = \mathrm{am}(\lambda_1)$$

$$E_{\lambda_2} = E_2 = \mathrm{Span}\{(0, 2, 1)^T\} \quad \Rightarrow \quad \mathrm{gm}(\lambda_2) = 1 \neq \mathrm{am}(\lambda_2) = 2$$

Thus a solution of the system is

$$\mathbf{x}_3(t) = e^{\lambda_1 t} v = e^{3t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Note that

$$E^2_{\lambda_2} = E^2_2 = \mathrm{Span}\{(0, 2, 1)^\mathrm{T}, (1, 0, -1)^\mathrm{T}\} \quad \Rightarrow \quad \dim(E^2_2) = 2 = \mathrm{am}(\lambda_2)$$

Choose $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in E^2_2 - E_2.$ Then

$$v_1 = (A - \lambda_2 I) v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

We have the other two solutoins of the systm are

$$\mathbf{x}_1(t) = e^{\lambda_2 t} v_1 = e^{2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{\lambda_2 t}(v_2 + t v_1) = e^{2t} \left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right]$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t)$$

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right] + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Example5. Solve the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$

$$A = \begin{pmatrix} 0 & -4 & 1 \\ 2 & -6 & 1 \\ 4 & -8 & 0 \end{pmatrix}$$

Proof. The characteristics polynomial of A is

$$p(\lambda) = -(\lambda + 2)^3 \quad \Longrightarrow \quad \lambda = -2 \text{ is an eigenvalue of } A \quad \text{wtith } \operatorname{am}(\lambda) = 3$$

The eigenspace is

$$E_\lambda = \operatorname{Span}\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \} \quad \Rightarrow \quad \dim(E_\lambda) = \operatorname{gm}(\lambda) = 2 \neq \operatorname{am}(\lambda)$$

$$E_\lambda^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \dim(E_\lambda^2) = 3 = \text{am}(\lambda)$$

Choose $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in E_\lambda^2 - E_\lambda$. Then,

$$v_1 = (A - \lambda I)v_2 = (A + 2I)v_2 = \begin{pmatrix} 2 & -4 & 1 \\ 2 & -4 & 1 \\ 4 & -8 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Thus the two linearly independent solutions of the system are

$$\mathbf{x}_1(t) = e^{\lambda t} v_1 = e^{-2t} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{\lambda t}(v_2 + t v_1) = e^{-2t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \right]$$

Choose $u = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \in E_\lambda$ such that $\{u, v_1, v_2\}$ is linearly independent.

Then another solution to the system is

$$\mathbf{x}_3(t) = e^{\lambda t} u = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t)$$

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + c_2 e^{-2t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \right] + c_3 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Homogeneous linear systems with constant coefficients

$$\mathbf{x}'(t) = A\mathbf{x}(t) \qquad (***)$$

Complex eigenvalues

- Suppose that the system has a solution of the form

$$\mathbf{x}(t) = e^{\lambda t} v$$

where v is a non − zero constant vector. Then

$$\mathbf{x}'(t) = te^{\lambda t} v$$

put it into the equation $(***)$ we have

$$te^{\lambda t} v = Ae^{\lambda t} v \quad \Leftrightarrow \quad (A - \lambda I)v = 0$$

Since $v \neq 0$, then λ is an eigenvalue of A and v is an eigenvector of A related to eigenvalue λ. Conversely, if λ is an eigenvalue of A and v is an eigenvector of A associated with λ, then $e^{\lambda t} v$ is a solution.

- Suppose that $\lambda = \alpha \pm \beta i$ are eigenvalues and $\mathbf{a} \pm \mathbf{b}i$ be the associated eigenvectors respectively, then the real part and imaginary part of

$$e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b}) = e^{\alpha t}(\cos(\beta t + i\sin(\beta t))(\mathbf{a} + i\mathbf{b})$$

$$= e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) + e^{\alpha t}(\mathbf{b}\cos\beta t + \mathbf{a}\sin\beta t)i$$

give two linearly independent solutions of the system. We have

Theorem. Suppose that $\lambda = \alpha \pm \beta i$ are eigenvalues and $\mathbf{a} \pm \mathbf{b}i$ be the associated eigenvectors respectively, then

$$\mathbf{x}_1(t) = e^{\alpha t}(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)) \quad \text{and} \quad \mathbf{x}_2(\mathrm{t}) = e^{\alpha t}(\mathbf{b}\cos(\beta t) + \mathbf{a}\sin(\beta t))$$

are two linearly independent solutions of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Example 1. Solve the system of equation

$$\begin{cases} x_1'(t) = -3x_1(t) - 2x_2(t) \\ x_2'(t) = 4x_1(t) + x_2(t) \end{cases}$$

Solution. Rewrite the system as

$$\mathbf{x}'(t) = A\mathbf{x}(t) \qquad ; \; X(t) = c_1 X_1(t) + c_2 X_2(t)$$

where $A = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$

The characteristics equation is

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = -1 \pm 2i$$

For $\lambda = -1 + 2i$: $(A - \lambda I)v = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$$\begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} \to \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix}$$

Let $v_1 = s \quad \Rightarrow \quad v_2 = -(1+i)s$. Thus the associated eigenvector is

$$v = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \mathbf{a} + i\mathbf{b}$$

The two linearly independent solutions are

$$\mathbf{x}_1(t) = e^{-t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin 2t\right] = e^{-t}\begin{pmatrix} \cos 2t + \sin 2t \\ -\cos 2t \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{-t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t\right] = e^{-t}\begin{pmatrix} -\cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix}$$

Therefore, the general solution of the system is

$$\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\
&= e^{-t}\left[c_1\begin{pmatrix} \cos 2t + \sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} -\cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix}\right] \\
&= e^{-t}\begin{pmatrix} (c_1 - c_2)\cos 2t + (c_1 + c_2)\sin 2t \\ -c_1\cos 2t - c_2\sin 2t \end{pmatrix}
\end{aligned}$$

Theorem. Suppose that λ is an eigenvalue of matrix $A = (a_{ij})_n$ and v_k is its associated eigenvector of index k (index$(\lambda) = k$). We denote

$$v_i = (A - \lambda I)^{k-i} v_k, \quad i = 1, 2, \ldots, k-1. \qquad v_{k-1} = (A - \lambda I) v_k$$

Then

$$\mathbf{x}_1(t) = e^{\lambda t} v_1$$

$$\mathbf{x}_2(t) = e^{\lambda t}(v_2 + t v_1)$$

$$\mathbf{x}_3(t) = e^{\lambda t}(v_3 + t v_2 + \frac{t^2}{2!} v_1)$$

$$\vdots$$

$$\mathbf{x}_k(t) = e^{\lambda t}(v_k + t v_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!} v_1)$$

are linearly independent solution of the equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

Example 3. Solve the system $\mathbf{x}'(t) = A\mathbf{x}(t)$

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 3 & 2 & 3 & -5 \\ 1 & 5 & 4 & -5 \end{pmatrix}$$

Proof. The characteristics equation of A is

$$(\lambda^2 + 2\lambda + 5)^2 = 0 \quad \Leftrightarrow \quad \lambda = -1 \pm 2i$$

Thus $\lambda = -1 + 2i$ and $-1 - 2i$ are eigenvalues of A with $\mathrm{am}(\lambda) = 2$.

For $\lambda = -1 + 2i$: by solving the equation $(A - \lambda I)v = 0$, we have the associated eigenvector is

$$v = \begin{pmatrix} 0 \\ 0 \\ 2+i \\ 2 \end{pmatrix}$$

Therefore, $E_\lambda = \text{Span}\{\begin{pmatrix} 0 \\ 0 \\ 2+i \\ 2 \end{pmatrix}\} \quad \Rightarrow \quad \text{gm}(\lambda) = \dim(E_\lambda) = 1 \neq am(\lambda) = 2$

To find E_λ^2, we consider the equation $(A - \lambda I)^2 v = 0$.

Therefore, $E_\lambda^2 = \text{Span}\{\begin{pmatrix} -4+8i \\ 4+12i \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} -8i \\ -8-8i \\ 7 \\ 0 \end{pmatrix}\} \quad \Rightarrow \quad \dim(E_\lambda^2) = 2 = \text{am}(\lambda)$

Choose $v_2 = \begin{pmatrix} -4+8i \\ 4+12i \\ 0 \\ 7 \end{pmatrix} \in E_\lambda^2 - E_\lambda$

$$v_1 = (A - \lambda I)v_2 = \begin{pmatrix} 0 \\ 0 \\ -39+48i \\ -12+54i \end{pmatrix}$$

The complex solutions are

$$\mathbf{z}_1(t) = e^{\lambda t} v_1 = e^{(-1+2i)t} \begin{pmatrix} 0 \\ 0 \\ -39+48i \\ -12+54i \end{pmatrix} = e^{(-1+2i)t} \left[\begin{pmatrix} 0 \\ 0 \\ -39 \\ -12 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 48 \\ 54 \end{pmatrix} \right]$$

$$\mathbf{z}_2(t) = e^{\lambda(t)}(v_2 + tv_1) = e^{(-1+2i)t}\left[\begin{pmatrix} -4+8i \\ 4+12i \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ -39+48i \\ -12+54i \end{pmatrix}\right]$$

$$= e^{(-1+2i)t}\left[\begin{pmatrix} -4 \\ 4 \\ -39t \\ 7-12t \end{pmatrix} + i\begin{pmatrix} 8 \\ 12 \\ 48t \\ 54t \end{pmatrix}\right]$$

The real solutions are

$$\mathbf{x}_1(t) = e^{-t}\left[\begin{pmatrix} 0 \\ 0 \\ -39 \\ -12 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ 0 \\ 48 \\ 54 \end{pmatrix}\sin(2t)\right] = e^{-t}\begin{pmatrix} 0 \\ 0 \\ -39\cos(2t) - 48\sin(2t) \\ -12\cos(2t) - 54\sin(2t) \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{-t}\left[\begin{pmatrix} 0 \\ 0 \\ -48 \\ 54 \end{pmatrix}\cos(2t) + \begin{pmatrix} 0 \\ 0 \\ -39 \\ -12 \end{pmatrix}\sin(2t)\right] = e^{-t}\begin{pmatrix} 0 \\ 0 \\ -48\cos(2t) - 39\sin(2t) \\ 54\cos(2t) - 12\sin(2t) \end{pmatrix}$$

$$\mathbf{x}_3(t) = e^{-t}\left[\begin{pmatrix} -4 \\ 4 \\ -39t \\ 7-12t \end{pmatrix}\cos(2t) - \begin{pmatrix} 8 \\ 12 \\ 48t \\ 54t \end{pmatrix}\sin(2t)\right] = e^{-t}\begin{pmatrix} -4\cos(2t) - 8\sin(2t) \\ 4\cos(2t) - 12\sin(2t) \\ -39t\cos(2t) - 48t\sin(2t) \\ (7-12t)\cos(2t) - 54t\sin(2t) \end{pmatrix}$$

$$\mathbf{x}_4(t) = e^{-t}\left[\begin{pmatrix} 8 \\ 12 \\ 48t \\ 54t \end{pmatrix}\cos(2t) + \begin{pmatrix} -4 \\ 4 \\ -39t \\ 7-12t \end{pmatrix}\sin(2t)\right] = e^{-t}\begin{pmatrix} 8\cos(2t) - 4\sin(2t) \\ 12\cos(2t) + 4\sin(2t) \\ 48t\cos(2t) - 39t\sin(2t) \\ 54t\cos(2t) + (7-12t)\sin(2t) \end{pmatrix}$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3(t)\mathbf{x}_3(t) + c_4\mathbf{x}_4(t)$$

Example2. Solve the system $\mathbf{x}'(t) = A\mathbf{x}(t),$ where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$

$$A = \begin{pmatrix} 15 & -16 & 8 \\ 10 & -10 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

Proof. The characteristics equation of A is

$$-(\lambda - 5)(\lambda^2 - 2\lambda + 5) = 0$$

We have $\lambda_1 = 5$ and $\lambda_{2,3} = 1 \pm 2i$ are eigenvalues of A with $\mathrm{am}(\lambda_1) = 1,\ \mathrm{am}(\lambda_{2,3}) = 1.$

We have

$$E_{\lambda_1} = E_5 = \mathrm{Span}\{\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}\} \quad \Rightarrow \quad \mathrm{gm}(\lambda_1) = 1 = \mathrm{am}(\lambda_1)$$

Then a solution of the system is

$$\mathbf{x}_1(t) = e^{\lambda_1 t} v = e^{5t} \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 + 2i$:

$$E_{\lambda_2} = E_2 = \mathrm{Span}\{\begin{pmatrix} -2+2i \\ -1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\} \quad \Rightarrow \quad \mathrm{gm} = 1 = \mathrm{am}(\lambda_2)$$

We have the other two solutoins of the systm are

$$\mathbf{x}_2(t) = e^t\left[\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\cos(2t) - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\sin(2t)\right] = e^t\begin{pmatrix} -2\cos(2t) - 2\sin(2t) \\ -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

$$\mathbf{x}_3(t) = e^t\left[\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\cos(2t) + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\sin(2t)\right] = e^t\begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ 2\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}$$

Therefore, the general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t)$$

$$\mathbf{x}(t) = c_1e^{5t}\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + e^t\left[c_2\begin{pmatrix} -2\cos(2t) - 2\sin(2t) \\ -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_3\begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ 2\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}\right]$$

Definition. **Nonhomogeneous linear systems with constant coefficients**

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_n(t) \end{cases}$$

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t) \qquad (*)$$

where

$$A = (a_{ij})_n \quad \text{and} \quad \mathbf{b}(t) = (b_1(t), b_2(t), \ldots, b_n(t))^T \neq (0, 0, \ldots, 0)$$

Theorem. The general solution of equation $(*)$ is of the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

- $\mathbf{x}_h(t)$ is the solution of a homogeneous equation.
- $\mathbf{x}_p(t)$ is a particular solution of equation $(*)$.

Method 1 : Undetermined Coefficients

Suppose $\mathbf{b}(t)$ is a linear combination of the following functions

$$(a_0 + a_1 t + \cdots + a_k t^k)e^{mt}; \; (b_0 + b_1 t + \cdots + b_p t^p)\cos \mu t; \; (c_0 + c_1 t + \cdots + c_q t^q)\sin \lambda t$$

$$(d_0 + d_1 t + \cdots + d_l t^l)e^{\alpha t}\cos \mu t; \; (e_0 + e_1 t + \cdots + e_m t^m)e^{\beta t}\sin \lambda t$$

where $a_i, \; b_i, \; c_i, \; d_i, \; e_i,$ are constants vectors.

Example1. Solve the system of ODEs

$$\begin{cases} x_1'(t) = x_1(t) + x_2(t) + 1 \\ x_2'(t) = 4x_1(t) + x_2(t) + 2t \end{cases}$$

Solution. Rewrite the system as

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

where

$$\mathbf{x}(t) = (x_1(t), x_2(t))^T, \ A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} t = a_0 + a_1 t$$

The characteristics equation is

$$p(\lambda) = 0 \quad \Leftrightarrow \quad \lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \quad \lambda = 3, -1$$

The $\lambda_1 = 3, \ \lambda_2 = -1$ are eigenvalues of A. The associated eigenspaces are

$$E_{\lambda_1} = E_3 = \text{Span}\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\} \quad \Rightarrow \quad \mathbf{x}_1(t) = e^{\lambda_1 t} v = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$E_{\lambda_2} = E_{-1} = \text{Span}\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\} \quad \Rightarrow \quad \mathbf{x}_2(t) = e^{\lambda_2 t} u = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Thus the solution of the homogeneous equation is

$$\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Let $\mathbf{x}_p(t) = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} t = \begin{pmatrix} a_0 + a_1 t \\ b_0 + b_1 t \end{pmatrix}$. Then $\mathbf{x}_p'(t) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

Substitute into the original equation, we obtain

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} a_0 + a_1 t \\ b_0 + b_1 t \end{pmatrix} + \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 + a_1 t + b_0 + b_1 t \\ 4a_0 + 4a_1 t + b_0 + b_1 t \end{pmatrix} + \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 + a_0 + b_0 + (a_1 + b_1) t \\ 4a_0 + b_0 + (4a_1 + b_1 + 2) t \end{pmatrix}$$

$$\Rightarrow \begin{cases} a_1 = 1 + a_0 + b_0 + (a_1 + b_1) t \\ b_1 = 4a_0 + b_0 + (4a_1 + b_1 + 2) t \end{cases}$$

$$\Leftrightarrow \begin{cases} a_1 = 1 + a_0 + b_0 \\ 0 = a_1 + b_1 \\ b_1 = 4a_0 + b_0 \\ 0 = 4a_1 + b_1 + 2 \end{cases}$$

Example3. Solve the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$

$$A = \begin{pmatrix} 15 & -16 & 8 \\ 10 & -10 & 5 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{pmatrix} 2t^2 \\ 3te^{5t} + te^t\sin(2t) \\ te^t \end{pmatrix}$$

Proof. The solution of the homogeneous equation is

$$\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t)$$

$$\mathbf{x}_h(t) = c_1 e^{5t}\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + e^t\left[c_2\begin{pmatrix} -2\cos(2t) - 2\sin(2t) \\ -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_3\begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ 2\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}\right]$$

$$\mathbf{x}_h(t) = c_1\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}e^{5t} + \begin{pmatrix} -2c_2 + 2c_3 \\ -c_2 + 2c_3 \\ c_2 \end{pmatrix}e^t\cos(2t) + \begin{pmatrix} -2c_2 - 2c_3 \\ -2c_2 - c_3 \\ c_3 \end{pmatrix}e^t\sin(2t)$$

We have $$\mathbf{b}(t) = \begin{pmatrix} 2t^2 \\ 3te^{5t} + te^t\sin(2t) \\ te^t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}t^2 + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}te^{5t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^t\sin(2t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}te^t$$

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}t + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}t^2$$

Let $$\mathbf{x}_p(t) = \begin{pmatrix} a_0 + a_1t + a_2t^2 \\ b_0 + b_1t + b_2t^2 \\ c_0 + c_1t + c_2t^2 \end{pmatrix} + \left[\begin{pmatrix} e_0 \\ f_0 \\ g_0 \end{pmatrix} + \begin{pmatrix} e_1 \\ f_1 \\ g_1 \end{pmatrix}t + \begin{pmatrix} e_2 \\ f_2 \\ g_2 \end{pmatrix}t^2\right]e^{5t}$$

$$+\left[\begin{pmatrix} h_0 \\ i_0 \\ j_0 \end{pmatrix} + \begin{pmatrix} h_1 \\ i_1 \\ j_1 \end{pmatrix}t + \begin{pmatrix} h_2 \\ i_2 \\ j_2 \end{pmatrix}t^2\right]e^t\cos(2t) + \left[\begin{pmatrix} k_0 \\ l_0 \\ m_0 \end{pmatrix} + \begin{pmatrix} k_1 \\ l_1 \\ m_1 \end{pmatrix}t + \begin{pmatrix} k_2 \\ l_2 \\ m_2 \end{pmatrix}t^2\right]e^t\sin(2t) + + \left[\begin{pmatrix} n_0 \\ o_0 \\ p_0 \end{pmatrix} + \begin{pmatrix} n_1 \\ o_1 \\ p_1 \end{pmatrix}t\right]e^t$$

Method 2 : Variation of Parameters

Definition. Assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of a homogeneous equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where $\quad \mathbf{x}_i(t) = (x_{i1}(t), x_{2i}(t), \ldots, x_{ni}(t))^T, \quad \forall\, i = 1, 2, \ldots, n$

We denote the matrix

$$\Phi(t) = (x_{ij}(t))_n = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t) \end{pmatrix}$$

Φ is called the fundamental matrix of the homogeneous linear differential equation.

Theorem. Assume that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of a homogeneous equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

The particular solution of the nonhomogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$ is

$$\mathbf{x}_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(x)\mathbf{b}(x)\, dx$$

Example. Solve the system of ODEs

$$\begin{cases} x_1'(t) = x_1(t) + x_2(t) + 1 \\ x_2'(t) = 4x_1(t) + x_2(t) + 2t \end{cases}$$

Solution. Note that

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

Thus the solution of the homogeneous equation is

$$\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

The fundamental matrix is

$$\Phi(t) = \begin{pmatrix}e^{3t} & e^{-t}\\ 2e^{3t} & -2e^{-t}\end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(t) = -\frac{1}{4}e^{-2t}\begin{pmatrix}-2e^{-t} & -e^{-t}\\ -2e^{3t} & e^{3t}\end{pmatrix}$$

$$\Rightarrow \quad \Phi^{-1}(t)\mathbf{b}(t) = -\frac{1}{4}\begin{pmatrix}-2e^{-3t} & -e^{-3t}\\ -2e^{t} & e^{t}\end{pmatrix}\begin{pmatrix}1\\2t\end{pmatrix} = -\frac{1}{4}\begin{pmatrix}-2e^{-3t}-2te^{-3t}\\ -2e^{t}+2te^{t}\end{pmatrix}$$

$$\Rightarrow \quad \int\Phi^{-1}(t)\mathbf{b}(t)\,dt = \frac{1}{2}\int\begin{pmatrix}(1+t)e^{-3t}\\ (1-t)e^{t}\end{pmatrix}dt = \begin{pmatrix}(-\frac{2}{9}-\frac{t}{6})e^{-3t}\\ (1-\frac{t}{2})e^{t}\end{pmatrix} + \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\Rightarrow \quad \mathbf{x}_p(t) = \Phi(t)\int\Phi^{-1}(t)\mathbf{b}(t)\,dt = \begin{pmatrix}e^{3t} & e^{-t}\\ 2e^{3t} & -2e^{-t}\end{pmatrix}\begin{pmatrix}(-\frac{2}{9}-\frac{t}{6})e^{-3t}\\ (1-\frac{t}{2})e^{t}\end{pmatrix} = \begin{pmatrix}\frac{7}{9}-\frac{2}{3}t\\ -\frac{22}{9}+\frac{2}{3}t\end{pmatrix} = \frac{1}{9}\begin{pmatrix}7\\-22\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-2\\2\end{pmatrix}t$$

Thus the general solution of the system is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

$$= c_1e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2e^{-t}\begin{pmatrix}1\\-2\end{pmatrix} + \frac{1}{9}\begin{pmatrix}7\\-22\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-2\\2\end{pmatrix}t.$$

Remark:

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{b}(t)\,dt = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} (-\frac{2}{9} - \frac{t}{6})e^{-3t} \\ (1 - \frac{t}{2})e^{t} \end{pmatrix} + \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Example. Solve the system of equation

$$\begin{cases} x_1'(t) = x_1(t) + x_2(t) + 3te^{t} \\ x_2'(t) = 4x_1(t) + x_2(t) + 2e^{3t} \end{cases}$$

Solution. We have

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{pmatrix} 3te^{t} \\ 2e^{3t} \end{pmatrix}$$

We have the solution of the homogeneous equation is

$$\mathbf{x}_h(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

The fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(t) = -\frac{1}{4}e^{-2t}\begin{pmatrix} -2e^{-t} & -e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix}$$

$$\Rightarrow \quad \Phi^{-1}(t)\mathbf{b}(t) = -\frac{1}{4}\begin{pmatrix} -2e^{-3t} & -e^{-3t} \\ -2e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} 3te^{t} \\ 2e^{3t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + 3te^{-2t} \\ 3te^{2t} - e^{4t} \end{pmatrix}$$

$$\Rightarrow \quad \int \Phi^{-1}(t)\mathbf{b}(t)\,dt = \frac{1}{2}\int \begin{pmatrix} 1+3te^{-2t} \\ e^{4t}-3te^{2t} \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}\,t - (\frac{3}{8}+\frac{3}{4}\,t)e^{-2t} \\ (-\frac{3}{8}+\frac{3}{4}\,t)e^{2t} - \frac{1}{8}\,e^{4t} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{b}(t)\,dt = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\,t - (\frac{3}{8}+\frac{3}{4}\,t)e^{-2t} \\ (-\frac{3}{8}+\frac{3}{4}\,t)e^{2t} - \frac{1}{8}\,e^{4t} \end{pmatrix} = \begin{pmatrix} (-\frac{1}{8}+\frac{1}{2}\,t)e^{3t} - \frac{3}{4}\,e^{t} \\ (\frac{1}{4}+t)e^{3t} - 3te^{t} \end{pmatrix}$$

$$= \left[\begin{pmatrix} -\frac{1}{8} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} t\right] e^{3t} + \left[\begin{pmatrix} -\frac{3}{4} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} t\right] e^{t}$$

Thus the general solution of the system is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = \ldots$$

Example. Solve the nonhomogeneous system of ode

$$\begin{cases} x_1'(t) = -3x_1(t) - 2x_2(t) \\ x_2'(t) = 4x_1(t) + x_2(t) + \dfrac{1}{e^t \cos(2t)} \end{cases}$$

Solution. We have

$$A = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{pmatrix} 0 \\ \dfrac{1}{e^t \cos(2t)} \end{pmatrix}$$

The eigenalues of A are $\lambda = -1 \pm 2i$ and the associated eigenvectors are

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \pm i \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \mathbf{a} \pm i\mathbf{b}$$

Thus the two linearly independent solution of the homogeneous system are

$$\mathbf{x}_1(t) = e^{-t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 2t\right] = e^{-t}\begin{pmatrix} \cos 2t + \sin 2t \\ -\cos 2t \end{pmatrix}$$

$$\mathbf{x}_2(t) = e^{-t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t\right] = e^{-t}\begin{pmatrix} -\cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix}$$

The fundamental matrix is

$$\Phi(t) = e^{-t}\begin{pmatrix} \cos(2t) + \sin(2t) & -\cos(2t) + \sin(2t) \\ -\cos(2t) & -\sin(2t) \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(t) = -e^{t}\begin{pmatrix} -\sin(2t) & \cos(2t) - \sin(2t) \\ \cos(2t) & \cos(2t) + \sin(2t) \end{pmatrix}$$

$$\Rightarrow \quad \Phi^{-1}(t)\mathbf{b}(t) = -e^{t}\begin{pmatrix} -\sin(2t) & \cos(2t) - \sin(2t) \\ \cos(2t) & \cos(2t) + \sin(2t) \end{pmatrix}\begin{pmatrix} 0 \\ \dfrac{1}{e^t \cos(2t)} \end{pmatrix} = \begin{pmatrix} -1 + \tan(2t) \\ -1 - \tan(2t) \end{pmatrix}$$

$$\Rightarrow \quad \int \Phi^{-1}(t)\mathbf{b}(t)\,dt = \int \begin{pmatrix} -1+\tan(2t) \\ -1-\tan(2t) \end{pmatrix} dt = \begin{pmatrix} -t - \dfrac{1}{2}\ln[\cos(2t)] \\ -t + \dfrac{1}{2}\ln[\cos(2t)] \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Thus the particular solution of the system is

$$\mathbf{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{b}(t)\,dt = e^{-t}\begin{pmatrix} \cos(2t)+\sin(2t) & -\cos(2t)+\sin(2t) \\ -\cos(2t) & -\sin(2t) \end{pmatrix}\begin{pmatrix} -t - \dfrac{1}{2}\ln[\cos(2t)] \\ -t + \dfrac{1}{2}\ln[\cos(2t)] \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} -2t\sin(2t) - \cos(2t)\ln[\cos(2t)] \\ t(\cos(2t)+\sin(2t)) - \dfrac{1}{2}(\sin(2t)-\cos(2t))\ln[\cos(2t)] \end{pmatrix}$$