I2-TD2
(Ordinary Differential Equations)

1. State the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear; homogeneous or non-nomogeneous.

   (a) $ty^{(3)} + \cos ty'' + y = \sin t$

   (b) $t^3y'' + (1-t^2)(y')^3 + y - \tanh t = 0$

   (c) $y''' - t^3y'' - y.y' = 0$

   (d) $y'' + (2+t)^3y = \sin(y^{(4)})$

2. Wite a differential equation in which $y^2 = 4(t+1)$ is a solution.

3. Solve the following problems.

   (a) $\dfrac{dy}{dt} = \dfrac{t + 2\sqrt{t}}{\sin y + ye^y}$

   (b) $ty\dfrac{dy}{dt} = \sqrt{t^2y^2 + t^2 + y^2 + 1}$

   (c) $\cos(2t + y)\, dy = dt$

   (d) $\dfrac{dy}{dt} \tan y = \sin(t+y) + \sin(t-y)$

4. Solve the following problems.

   (a) $\dfrac{dy}{dt} = 2 + \dfrac{y}{t}$

   (b) $3t + 2y\dfrac{dy}{dt} = y$

   (c) $\dfrac{dy}{dt} + \dfrac{t^2 + 3y^2}{3t^2 + y^2} = 0$

   (d) $\left(2t^2y + y^3\right) dt + \left(ty^2 - 2t^3\right) dy = 0$

   (e) $\dfrac{dy}{dt} = \dfrac{y}{t} + t \sin \dfrac{y}{t}$

   (f) $t \cos\left(\dfrac{y}{t}\right)(ydt + tdy) = y \sin\left(\dfrac{y}{t}\right)(tdy - ydt)$

   (g) $(2t - y - 1)\, dy = (3t + y - 4)\, dt$

   (h) $(2t + y - 3)\, dy = (t + 2y - 3)\, dt$

5. The rate at which a body cools is proportional to the difference between the temperature of the body and that of the surrounding air. If a body in air at 25°C will cool from 100° to 75° in one minute, find its temperature at the end of three minutes.

6. The rate at which the ice melts is proportional to the amount of ice at the instant. Find the amount of ice left after 2 hours if half the quantity melts in 30 minutes.

7. If the population of a country doubles in 50 years, in how many years will it treble, assuming that the rate of increase is proportional to the number of inhabitants?

8. The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hours, determine

   (a) The number present after 12 hours.

   (b) The doubling time.

9. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, and assume that no one leaves the campus throughout the duration of the disease, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

10. Solve the following problems.

(a) $(t+2y-11)\,dt+(2t+y-4)\,dy=0$

(b) $\left(1+e^{t/y}\right)+e^{t/y}\left(1-\dfrac{t}{y}\right)\dfrac{dy}{dt}=0$

(c) $\left(2ty+3y^2\right)dt+\left(2t^2+y^2\right)dy=0$

(d) $\left(2ty^2+t^2y^3\right)dt+\left(2t+3t^4y^3\right)dy=0$

(e) $(3ty+y+4)\,dt+\dfrac{t}{2}dy=0$

(f) $3t^2ydt+\left(3t^3-2\right)dy=0, y\left(0\right)=1$

(g) $\left(2y^2+3ty\right)dt+\left(t^2+2ty\right)dy=0$

(h) $\left(t^2y-y^4\right)dt+\left(2ty^3-t^3\right)dy=0$

(i) $tdt+ydy+t\left(tdy-ydt\right)=0$

(j) $(t^2+y^2+1)\,dt-2tydy=0$

11. Find the general solutions of the following differential equations

(a) $\left(t^3-t\right)\dfrac{dy}{dt}-\left(3t^2-1\right)y=t^5-2t^3+t$

(b) $\sin t\dfrac{dy}{dt}+2y=\tan^3\left(\dfrac{t}{2}\right)$

(c) $\dfrac{dy}{dt}+\dfrac{y}{t}=t^2y^6$

(d) $t^2\dfrac{dy}{dt}+ty=-y^{-3/2}, y\left(1\right)=1$

(e) $t^3\dfrac{dy}{dt}+2t^2y=y^{-3}, y\left(1\right)=1$

(f) $t\dfrac{dy}{dt}-y^2+(2t+1)\,y=t^2+2t$

(g) $\dfrac{dy}{dt}+e^{-t}y^2=y+e^t, y\left(0\right)=6$

(h) $t\dfrac{dy}{dt}+y\log y=tye^t$; (Hint: let $u=\ln y$)

(i) $\dfrac{dy}{dt}=\dfrac{y^2}{t^2}-\dfrac{y}{t}+1, y\left(1\right)=3$

12. Consider the differential equation

$$y^{-1}y'+p(t)\ln y=q(t), \qquad (1)$$

where $p(t)$ and $q(t)$ are continuous functions on some interval (a,b). Show that the change of variables $u=\ln y$ reduces Equation (1) to the linear differential equation

$$u'+p(t)u=q(t),$$

and hence show that the general solution to Equation (1) is

$$y(t)=\exp\left\{I^{-1}\left[\int I(t)q(t)dt+c\right]\right\}$$

where

$$I=e^{\int p(t)dt}$$

and c is an arbitrary constant.

13. Use the technique derived in the previous problem to solve the initial-value problem

$$y^{-1}y'-2t^{-1}\ln y=t^{-1}(1-2\ln t)$$
$$y(1)=e$$

14. Under certain conditions, cane sugar is converted into dextrose at a rate, which is proportional to the amount unconverted at any time. If out of 75 grams of sugar at $t=0$, 8 grams are converted during the first 3 minutes, find the amount converted in 1.5 hours.

15. Integrate the following differential equations.

(a) $3y'' - 2y' - 8y = 0$

(b) $y'' + 2y' + y = 0$

(c) $y'' - 4y' + 3y = 0, y(0) = 6, y'(0) = 10$

(d) $y''' + 3y'' + 7y' + 5y = 0$

(e) $y^{(4)} + 10y''' + 35y'' + 50y' + 24y = 0$

(f) $y^{(4)} + 5y''' + 13y'' + 19y' + 10y = 0$

16. By not solving for a solution, determine the form of particular solution, y_p, of the following differential equations.

(a) $y'' + 2y' + y = 2te^t + e^{2t}$

(b) $y'' + 2y' + y = 2te^{-t}$

(c) $y'' + 3y' + 2y = 3t^2e^{-2t} + (t+1)e^{2t}$

(d) $y'' - 2y' + 2y = 5t^2e^t \sin t + 4e^{-t}$

(e) $y^{(3)} + 3y'' + 4y' + 2y = 2te^{-t} + 3t^2e^t - 5t^3e^{-t}\cos t$

(f) $y^{(3)} + 3y'' + 7y' + 5y = t^{2m}e^{-t} + 6t^m e^{-t}\sin t \cos t, m = 0, 1, 2, 3, ...$

17. Integrate the following differential equations.

(a) $y'' + 2y' + y = 2te^{-t}$

(b) $y'' + 3y' + 2y = 2te^t + 3e^{-2t}$

(c) $y'' + 2y' + 5y = 2e^{-t}\sin 2t$

(d) $y^{(3)} + y'' - 5y' + 3y = 2e^{-3t} + e^t, y(0) = -2, y'(0) = 1, y''(0) = 1$

(e) $y^{(3)} - 2y'' + y' - 2y = t^2 - 2t + 4 - 3\cos t$

(f) $y^{(3)} + 6y'' + 12y' + 8y = e^t - 3\sin t - 8e^{-2t}$

(g) $y^{(3)} - 5y'' + y' + 7y = -\cos 3t, y(0) = -2, y'(0) = 1, y''(0) = 0$

18. Find the general solution of the following differential equations if given y_1 is a solution.

(a) $t^2y'' - 3ty' + 4y = 0; y_1 = t^2$

(b) $y'' + y' + e^{-2t}y = 0; y_1 = \cos(e^{-t})$

(c) $t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0; y_1 = \frac{\cos t}{\sqrt{t}}$

(d) $(1 - t^2)y'' + 2ty' - 2y = 0; y_1 = t$

(e) $(2t^2 + 1)y'' - 4ty' + 4y = 0; y_1 = t$

(f) $t^2y'' - (2t^2 \tan t + 2t)y' + (2t\tan t + 2)y = 0; y_1 = t$

19. Integrate the following differential equations.

(a) $y'' + 2y' + y = t^{-1}e^{-t}$

(b) $y'' + y = (\sin t)^{-1}$

(c) $y'' + 2y' + 2y = e^{-t}(\sin t)^{-1}$

(d) $t^2y'' - 4ty' + 4y = t^4 + t^2, t > 0$

(e) $y'' + y' + e^{-2t}y = e^{-3t}$

(f) $t^3y^{(3)} - 3t^2y'' + 7ty' - 8y = 8\ln t + 12, y(1) = -2, y'(1) = 3, y''(1) = 0$

(g) $y^{(3)} + 2y'' - y' - 2y = 4e^{-t^2}(-4t^3 + 4t^2 + 7t - 3)$

(h) $y^{(3)} + y'' + 4y' + 4y = 2e^{t^2}(4t^4 + 2t^3 + 16t^2 + 5t + 5)$