CHAPTER I
JORDAN CANONICAL FORM

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Contents

Notatoins.

Notatoins. In this whole chapter, we will use the following notations.

1. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
2. $\mathbb{K}^{m\times n} = \mathcal{M}_{m\times n}(\mathbb{K}) = \{A = (a_{ij})_{m\times n} : a_{ij} \in \mathbb{K}\}$.
3. $\mathcal{M}_n(\mathbb{K}) = \mathcal{M}_{n\times n}(\mathbb{K})$
4. $V^{\mathbb{K}}$ = the set of vector spaces over field $\mathbb{K}$.
5. $V_n^{\mathbb{K}}$ = the set of n-dimensional vector spaces over $\mathbb{K}$.
6. I = identity matrix or identity mapping.
7. $L : V \to V$ is a map. L_S = the restriction of L on $S \subseteq V$.
8. $\mathbb{K}[X]$ = the set of polynomials with coefficients in $\mathbb{K}$.
9. $\mathbb{K}_n[X]$ = the set of polynomials in $\mathbb{K}[X]$ with degree at most n.

Eigenvalues and Eigenvectors

Definition 1

Let $A \in \mathcal{M}_n(\mathbb{K})$.

1. A scalar $\lambda \in \mathbb{K}$ is said to be an **eigenvalue** of A if there exists a nonzero vector $x \in \mathbb{K}^n$ such that $Ax = \lambda x$. The nonzero vector x is said to be an **eigenvector** associated to λ.
2. The set of all eigenvalues of A is called **spectrum** of A, we write $\mathrm{spect(A)}$.
3. The **eigenspace** corresponding to an eigenvalue λ is defined as

$$E_\lambda = \mathrm{Ker}\,(A - \lambda I_n) = \{x \in \mathbb{K}^n : \,(A - \lambda I_n)\, x = 0\,\}.$$

Definition 2

A subspace S of $\mathbb{K}^n$ is called an A-**invariant subspace** of $\mathbb{K}^n$ if $\{Ax : x \in S\} \subseteq S$.

Ex: Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$.

Find all eigenvalue and their corresponding eigenspace.

Proof: Let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Then we have

$$Ax = \lambda x \ ; \ \forall x \in \mathbb{R}^2, \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \ x \neq 0$$

$$\Leftrightarrow \ Ax - \lambda x = 0$$

$$\Leftrightarrow \ (A - \lambda I)x = 0 \ , \quad x \neq 0$$

$$\Rightarrow \ |A - \lambda I| = 0$$

$$\Leftrightarrow \ \left| \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Leftrightarrow \ \begin{vmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \ (2-\lambda)(5-\lambda) + 2 = 0$$

$\Leftrightarrow \lambda^2 - 7\lambda + 12 = 0$

$AX = 0$

$\Rightarrow \lambda_1 = 3, \ \lambda_2 = 4.$ eigenvalue

+ If A^{-1} exist $\Rightarrow X = 0$
+ If $X \neq 0 \Rightarrow |A| = 0$

So, $\text{spect}(A) = \{3, 4\}$.

+ For $\lambda_1 = 3$: $(A - \lambda_1 I)x = 0$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (\mathbb{R}^*)^2$

$\Leftrightarrow (A - 3I)x = 0$

$\Leftrightarrow \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} x = 0$

$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Let $x_2 = t \Rightarrow x_1 = -t$

$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix}$; $t \in \mathbb{R}^*$. x is an eigenvector corresponding to $\lambda_1 = 3$.

The eigenspace corresponding to $\lambda_1 = 3$ is

$E_3 = \left\{ \begin{pmatrix} -t \\ t \end{pmatrix} \in \mathbb{R}^2 / t \in \mathbb{R}^* \right\}$

+ For $\lambda_2 = 4$: $(A - \lambda_2 I)x = 0$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow (A - 4I)x = 0 \Leftrightarrow \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} = 0$

$\rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0$

Let $x_1 = t \in \mathbb{R}^*$, $x_2 = -2x_1 = -2t$

$\Rightarrow E_4 = \left\{ \begin{pmatrix} t \\ -2t \end{pmatrix} \in \mathbb{R}^2 \,/\, t \in \mathbb{R}^* \right\}$.

<u>Ex</u>: $A = \begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

<u>Proof</u>: $\underbrace{|A - \lambda I|}_{P_A(\lambda)=} = 0 \Leftrightarrow \begin{vmatrix} -8-\lambda & -3 & -6 \\ 4 & -\lambda & 4 \\ 4 & 2 & 2-\lambda \end{vmatrix} = 0$

We have, $S_1 = tr(A) = -8+0+2 = -6$

$$S_2 = \begin{vmatrix} -8 & -3 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} -8 & -6 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} = 12+8-8 = 12$$

$$S_3 = \begin{vmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -8 & -3 & 2 \\ 4 & 0 & 0 \\ 4 & 2 & -2 \end{vmatrix} = -4 \begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix} = -8$$

$$p_A(\lambda) = p_L(\lambda) = (-1)^n \left(\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \cdots + (-1)^n S_n\right)$$

$$\Rightarrow p_A(\lambda) = -\lambda^3 + S_1\lambda^2 - S_2\lambda + S_3$$
$$= -\lambda^3 - 6\lambda^2 - 12\lambda - 8$$

$$p_A(\lambda) = 0 \iff -(\lambda^3 + 6\lambda^2 + 12\lambda + 8) = 0$$
$$\iff -(\lambda+2)^3 = 0 \Rightarrow \lambda = -2 \text{ is an eigenvalue}$$

$$Spect(A) = \{-2\}$$

+ Consider $(A - \lambda I)x = 0$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow (A + 2I)x = 0$

$$\Leftrightarrow \begin{pmatrix} -6 & -3 & -6 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -6 & -3 & -6 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $x_1 = t$, $x_3 = s$, $x_2 = -2x_1 - 2x_3 = -2t - 2s$

$$\Rightarrow E_{-2} = \left\{ \begin{pmatrix} t \\ -2t-2s \\ s \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; t, s \in \mathbb{R} \right\}.$$

$$\Rightarrow \quad = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim E_{-2} = 2. \checkmark$$

or $gm(-2) = 2$

Eigenvalues and Eigenvectors

Theorem 1

Let $\lambda \in \mathbb{K}$ be an eigenvalue of $A \in \mathcal{M}_n(\mathbb{K})$. Then the eigenspace E_λ is a subspace of $\mathbb{K}^n$ and it is A-invariant. The dimension of this eigenspace is called **geometric multiplicity** of λ denoted $\mathrm{gm}(\lambda)$. That is,

$$\mathrm{gm}(\lambda) = \dim(E_\lambda)$$

Theorem 2

Let $A \in \mathcal{M}_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The following statements are equivalent.

1. λ is an eigenvalue of A.
2. $(A - \lambda I)\,x = 0$ has a nontrivial solution.
3. The eigenspace $E_\lambda \neq \{0\}$.
4. $\det(A - \lambda I) = 0$.

$|A - \lambda I| = 0$ ✓

Eigenvalues and Eigenvectors

Definition 3

Let $L \in \mathcal{L}(V)$. $L: V \longrightarrow V$ is linear transform.

1. A scalar $\lambda \in \mathbb{K}$ is said to be an **eigenvalue** of L if there exists a nonzero vector $x \in V$ such that $L(x) = \lambda x$. The nonzero vector x is called an **eigenvector** corresponding to the eigenvalue λ. The pair (λ, x) is called an eigenpair for L.
2. The set of all eigenvalues of L is called the **spectrum** of L.
3. The **eigenspace** corresponding to an eigenvalue λ is defined as

$$E_\lambda = \mathrm{Ker}\,(L - \lambda I) = \{v \in V : \; L(v) = \lambda v\}.$$

Definition 4

Let $V \in V^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, $A \in \mathcal{M}_n(\mathbb{K})$. A subspace S of V is called an ***L*-invariant subspace** of V if $L(S) \subseteq S$.

Ex: Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$; $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Find all eigenvalues and their corresponding eigenspace.

Proof: Let $\lambda \in \mathbb{R}$. Consider eq:

$$L(x) = \lambda x \; ; \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \, , \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda)(5-\lambda) + 2 = 0$$

$$\Leftrightarrow \lambda^2 - 7\lambda + 12 = 0 \Rightarrow \lambda_1 = 3, \ \lambda_2 = 4$$

$\Rightarrow \text{Spect}(L) = \{3, 4\}.$

$+$ For $\lambda_1 = 3$; $E_3 = \left\{ \begin{pmatrix} -t \\ t \end{pmatrix} \in \mathbb{R}^2 \,/\, t \in \mathbb{R}^* \right\} \Rightarrow gm(3) = 1.$

$+$ For $\lambda_2 = 4$; $E_4 = \left\{ \begin{pmatrix} t \\ -2t \end{pmatrix} \,/\, t \in \mathbb{R}^* \right\}. \Rightarrow gm(4) = 1$

Eigenvalues and Eigenvectors

Theorem 3

Let $V \in V^{\mathbb{K}}$ and $\lambda \in \mathbb{K}$ be an eigenvalue of $L \in \mathcal{L}(V)$. Then the eigenspace E_λ is a subspace of V and it is L-invariant. If $V \in V_n^{\mathbb{K}}$, then the eigenspace E_λ is a finite dimensional subspace of V. The dimension of this eigenspace is called **geometric multiplicity** of λ denoted $\mathrm{gm}\,(\lambda)$.

Theorem 4

Let $L \in \mathcal{L}(V)$. If $\dim V < \infty$ and B is an ordered basis for V. Then λ is an eigenvalue of L if and only if λ is an eigenvalue of $[L]_B$.

Theorem 5

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, B be a basis for V, $A = [L]_B$, and $\lambda \in \mathbb{K}$ be a scalar. The following statements are equivalent.

1. λ is an eigenvalue of L.
2. $(A - \lambda I)[x]_B = 0$ has a nontrivial solution.
3. The eigenspace $E_\lambda \neq \{0\}$.
4. $\det(L - \lambda I) = 0$.

Eigenvalues and Eigenvectors

Definition 5

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, B be a basis for V, $A = [L]_B$, and $\lambda \in \mathbb{K}$. The polynomial $p_L(\lambda) = \det(L - \lambda I) = \det(A - \lambda I)$ does not depend on a basis for V. This polynomial is called **characteristic polynomial** of L (also of A) and $p_A(\lambda) = p_L(\lambda) = \det(A - \lambda I) = 0$ is called **characteristic equation** from L (also from A).

Theorem 6

Let $A, B \in \mathcal{M}_n(\mathbb{K})$. If $B \sim A$, then the two matrices have the same characteristic polynomial and consequently both have the same spectrum.

Theorem 7

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, B be a basis for V, and $A = [L]_B$. Then

$$p_A(\lambda) = p_L(\lambda) = (-1)^n \left(\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \cdots + (-1)^n S_n\right).$$

where S_i is the sum of the principal minors of order i of matrix A.

Theorem 8

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, $A \in \mathcal{M}_n(\mathbb{K})$ and p be a polynomial.

1. If λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$.
2. If λ is an eigenvalue of L then $p(\lambda)$ is an eigenvalue of $p(L)$.

Theorem 9

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$. Let λ be an eigenvalue of L. Then

$$1 \leq \mathrm{gm}(\lambda) \leq \mathrm{am}(\lambda).$$

Definition 6

The minimal polynomial $m_A(\lambda)$ of an $n \times n$ matrix A over a field $\mathbb{K}$ is the monic polynomial of least degree such that $m_A(A) = 0$.

Theorem 10

Let $p_A(\lambda)$ be the characteristic polynomial and $m_A(\lambda)$ be the minimal polynomial of A.

1. The polynomial $m_A(\lambda)$ divides every polynomial that has A as a zero.
2. The polynomials $p_A(\lambda)$ and $m_A(\lambda)$ have the same irreducible factors over $\mathbb{K}$. Consequently, $p_A(\lambda)$ and $m_A(\lambda)$ have the same zeros in $\mathbb{K}$.
3. If $p_A(\lambda) = (\lambda_1 - \lambda)^{n_1} \cdots (\lambda_k - \lambda)^{n_k}$ where $\lambda_1, \ldots, \lambda_k$ are distinct, then there exist integers $m_1, \ldots, m_k$ such that $1 \leq m_j \leq n_j$, $j = 1, 2, \ldots, k$ and
$$m_A(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}.$$

Eigenvalues and Eigenvectors

Theorem 11

Let $A \in \mathcal{M}_n(\mathbb{K})$. Suppose that A is a block diagonal matrix with diagonal blocks $A_1, \dots, A_b$. Then the minimal polynomial of A equals the least common multiple of the minimal polynomial of the diagonal blocks A_i.

Theorem 12

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, $A = [L]_B$ and $p_L(\lambda)$ be the characteristic polynomial of L. Suppose that $V = S_1 \oplus \cdots \oplus S_k$ where S_i is an L-invariant subspace of V for each $i = 1, 2, \dots, k$ with bases $B_1, \dots, B_k$, respectively and $A_i = [L_{S_i}]_{B_i}$, and $p_{L_{S_i}}(\lambda)$ is the characteristic polynomial for L_{S_i} $(i = 1, 2, \dots, k)$. Then

1. A is similar to the block diagonal matrix with diagonal blocks $A_1, \dots, A_k$.
2. $p_L(\lambda) = p_{L_{S_1}}(\lambda) \dots p_{L_{S_k}}(\lambda)$.

Eigenvalues and Eigenvectors

Theorem 13

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$. If the minimal polynomial of L is

$$m(\lambda) = (p_1(\lambda))^{n_1} \ldots (p_k(\lambda))^{n_k}$$

where $p_i(\lambda)$ are relatively prime, distinct monic irreducible polynomials, then V is the direct sum of the L-invariant subspaces $S_1, \ldots, S_k$ where S_i is the kernel of $(P_i)(L)^{n_i}$. Moreover, $(p_i(L))^{n_i}$ is the minimal polynomial of L_{S_i}.

Eigenvalues and Eigenvectors

Theorem 14

Let $\lambda_1, \ldots, \lambda_k$ be k different eigenvalues of $L \in \mathcal{L}(V)$ (also of $A \in \mathcal{M}_n(\mathbb{K})$) and $x_1, \ldots, x_k$ be eigenvectors, respectively. Then the vectors $x_1, \ldots, x_k$ are linearly independent.

Theorem 15

Let $\lambda_1, \ldots, \lambda_k$ be k different eigenvalues of $L \in \mathcal{L}(V)$ (also of $A \in \mathcal{M}_n(\mathbb{K})$). Then

$$E_{\lambda_1} + \cdots + E_{\lambda_k} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}.$$

Theorem 16

Let λ be an eigenvalue of $A \in \mathcal{M}(\mathbb{K})$ and k be the multiplicity of λ. Then

$$1 \leq \dim E_\lambda \leq k.$$

Diagonalization and Triangularization

Definition 7

Let $A \in \mathcal{M}_n(\mathbb{K})$.

1. A is said to be **diagonalizable** in $\mathbb{K}$ if $A \sim D$, where D is a diagonal matrix.
2. A is said to be **triangularizable** in $\mathbb{K}$ if $A \sim T$, where T is a triangular matrix T.

Definition 8

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$.

1. L is said to be **diagonalizable** in $\mathbb{K}$ if there is an ordered basis B such that $[L]_B$ is diagonal.
2. L is said to be **triangularizable** in $\mathbb{K}$ if there is an ordered basis B such that $[L]_B$ is a triangular matrix.

Diagonalization and Triangularization

Theorem 17

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, B be an ordered basis for V, and $A = [L]_B$. Then

1. L is diagonalisable over $\mathbb{K}$ if and only if A is diagonalisable over $\mathbb{K}$.
2. L is triangularizable over $\mathbb{K}$ if and only if A is triangularizable over $\mathbb{K}$.

Theorem 18

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, and $E_{\lambda_1}, \ldots, E_{\lambda_k}$ be eigenspaces of L. The following assertions are equivalent.

1. L is diagonalizable over $\mathbb{K}$.
2. There exists a basis of V formed by the eigenvectors of L.
3. $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.
4. $n = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_k}$

Diagonalization and Triangularization

Theorem 19

Let $A \in \mathcal{M}_n(\mathbb{K})$ and $\lambda_1, \lambda_2, \ldots, \lambda_k$ be k distinct eigenvalues of A. The following assertions are equivalent.

1. A is diagonalizable in $\mathbb{K}$.
2. There exists a basis of $\mathbb{K}^n$ formed by the eigenvectors of A.
3. $\mathbb{K}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.
4. $n = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_k}$

Theorem 20

$A \in \mathcal{M}_n(\mathbb{K})$ is diagonalizable over $\mathbb{K}$ if and only if

1. The characteristic polynomial $p_A(\lambda)$ splits over $\mathbb{K}$.
2. Every eigenvalue λ of A, $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda)$.

Diagonalization and Triangularization

Theorem 21

Let $V \in V_n^{\mathbb{K}}$. $L \in \mathcal{L}(V)$ is diagonalizable over $\mathbb{K}$ if and only if

1. The characteristic polynomial $p_L(\lambda)$ splits over $\mathbb{K}$.
2. Every eigenvalue λ of L, $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda)$.

Theorem 22

Let $V \in V_n^{\mathbb{K}}$. $L \in \mathcal{L}(V)$ is diagonalisable over $\mathbb{K}$ if and only if its minimal polynomial is a product of district linear polynomials.

Theorem 23

If $A \in \mathcal{M}_n(\mathbb{R})$ and A is symmetric, then there exists an orthogonal matrix P such that $P^{-1}AP$ is diagonal (i.e. A can be diagonalized by an orthogonal matrix).

Diagonalization and Triangularization

Theorem 24

Let $A \in \mathcal{M}_n(\mathbb{K})$ with the characteristic polynomial $p_A(\lambda)$. The following assertions are equivalent.

1. A is triagularizable over $\mathbb{K}$.
2. $p_A(\lambda)$ splits over $\mathbb{K}$.

Theorem 25

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$ with the characteristic polynomial $p_L(\lambda)$. The following assertions are equivalent.

1. L is triagularizable over $\mathbb{K}$.
2. $p_L(\lambda)$ splits over $\mathbb{K}$.

Cayley-Hamilton Theorem

Definition 9

Let $V \in V^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, $A \in \mathcal{M}_n(\mathbb{K})$, and $k > 0$ be an integer.

1. The subspace C_x spanned by $\left\{x, L(x), L^2(x), \dots\right\}$ is called the L-**cyclic subspace** of V generated by $x \in V$.
2. The subspace C_x spanned by $\left\{x, Ax, A^2x, \dots\right\}$ is called the A-**cyclic subspace** of $\mathbb{K}^n$ generated by $x \in \mathbb{K}^n$.

Theorem 26

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$ and S be an L-invariant subspace of V. Then the characteristic polynomial of L_S divides the characteristic polynomial $p_L(\lambda)$ of L.

Cayley-Hamilton Theorem

Theorem 27

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, and C_x be an L-cyclic subspace of V generated by a nonzero vector x with $\dim C_x = k$. Then

1. $\left\{x, L(x), L^2(x), \ldots, L^{k-1}(x)\right\}$ is a basis for C_x.
2. If $a_0 x + a_1 L(x) + \cdots + a_{k-1} L^{k-1}(x) + L^k(x) = 0$, then the characteristic polynomial of L_{C_x} is

$$p_{C_x}(\lambda) = (-1)^k \left(a_0 + a_1 \lambda + \cdots + a_{k-1} \lambda^{k-1} + \lambda^k\right).$$

Cayley-Hamilton Theorem

Theorem 28 (Cayley-Hamilton)

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$. If $p_L(\lambda)$ is the polynomial characteristic of L then $p_L(L) = 0$.

Theorem 29 (Cayley-Hamilton)

Let $A \in \mathcal{M}_n(\mathbb{K})$. If $p_A(\lambda)$ is the polynomial characteristic of A then $p_A(A) = 0$.

Jordan Canonical Form

Definition 10

Let $V \in V^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, $A \in \mathcal{M}_n(\mathbb{K})$, and $k > 0$ be an integer.

1. The linear operator L is said to be **nilpotent** if $L^k = 0$ for some positive integer k.
2. The linear operator L is said to be **nilpotent of index** k if $L^k = 0$ and $L^{k-1} \neq 0$.
3. The matrix A is said to be **nilpotent** if $A^k = 0$ for some positive integer k.
4. The matrix A is said to be **nilpotent of index** k if $A^k = 0$ and $A^{k-1} \neq 0$.

Jordan Canonical Form

Definition 11

Let $V \in V_n^{\mathbb{K}}$ and λ be an eigenvalue of $L \in \mathcal{L}(V)$.

1. The **index of** λ, denoted index (λ), is the smallest positive integer k such that $\operatorname{rank}(L - \lambda I)^k = \operatorname{rank}(L - \lambda I)^{k+1}$
2. A nonzero vector $x \in V$ is called a **generalized eigenvector of** L corresponding to the eigenvalue λ if $(L - \lambda I)^p (x) = 0$ for some positive integer p.
3. The **generalized eigenspace of** L corresponding to λ is defined by

$$G_\lambda (L) = \{x \in V : (L - \lambda I)^p (x) = 0, \text{ for some } p \in \mathbb{N}\}.$$

Jordan Canonical Form

Definition 12

Let λ be an eigenvalue of $A \in \mathcal{M}_n(\mathbb{K})$.

1. The **index of** λ, denoted $\text{index}(\lambda)$, is the smallest positive integer k such that $\mathrm{rank}(A - \lambda I_n)^k = \mathrm{rank}(A - \lambda I_n)^{k+1}$
2. A nonzero vector $x \in \mathbb{K}^n$ is called a **generalized eigenvector of** A corresponding to the eigenvalue λ if $(A - \lambda I_n)^p x = 0$ for some positive integer p.
3. The **generalized eigenspace of** A corresponding to λ is defined by

$$G_\lambda(A) = \{x \in \mathbb{K}^n : (A - \lambda I_n)^p x = 0, \text{ for some } p \in \mathbb{N}\}.$$

Jordan Canonical Form

Theorem 30

Suppose the characteristic polynomial of $L \in \mathcal{L}(V)$ splits over $\mathbb{K}$ and λ is an eigenvalue with algebraic multiplicity k. Then

1. $\dim\left(G_\lambda\left(L\right)\right) \leq k$.
2. $G_\lambda\left(L\right) = \mathrm{N}\left(\left(L - \lambda I\right)^k\right)$.

Theorem 31

Suppose the characteristic polynomial of $A \in \mathcal{M}_n\left(\mathbb{K}\right)$ splits over $\mathbb{K}$ and λ is an eigenvalue with algebraic multiplicity k. Then

1. $\dim\left(G_\lambda\left(A\right)\right) \leq k$.
2. $G_\lambda\left(A\right) = \mathrm{N}\left(\left(A - \lambda I_n\right)^k\right)$.

Jordan Canonical Form

Theorem 32

Suppose the characteristic polynomial of $L \in \mathcal{L}(V)$ splits over $\mathbb{K}$, $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues with algebraic multiplicities $k_1, \ldots, k_m$, respectively. Then for every $x \in V \in V_n^{\mathbb{K}}$ there exist elements $x_i \in G_{\lambda_i}(L)$, $1 \leq i \leq m$ such that

$$x = x_1 + \cdots + x_m.$$

Theorem 33

Suppose the characteristic polynomial of $A \in \mathcal{M}_n(\mathbb{K})$ splits over $\mathbb{K}$, $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues with algebraic multiplicities $k_1, \ldots, k_m$, respectively. Then for every $x \in \mathbb{K}^n$ there exist elements $x_i \in G_{\lambda_i}(A)$, $1 \leq i \leq m$ such that

$$x = x_1 + \cdots + x_m.$$

Jordan Canonical Form

Theorem 34

Suppose the characteristic polynomial of $L \in \mathcal{L}(V)$ splits over $\mathbb{K}$, $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues with algebraic multiplicities $k_1, \dots, k_m$, respectively, and B_i is an ordered basis for $G_{\lambda_i}(L)$ for $i = 1, 2, \dots, m$. Then

1. $B_i \cap B_j = \varnothing$, $i \neq j$.
2. $B = B_1 \cup \cdots \cup B_m$ is an ordered basis for $V \in V_n^{\mathbb{K}}$
3. $\dim\left(G_{\lambda_i}(L)\right) = k_i$ for $i = 1, 2, \dots, m$.

Jordan Canonical Form

Theorem 35

Suppose the characteristic polynomial of $A \in \mathcal{M}_n(\mathbb{K})$ splits over $\mathbb{K}$, $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues with algebraic multiplicities $k_1, \ldots, k_m$, respectively, and B_i is an ordered basis for $G_{\lambda_i}(A)$ for $i = 1, 2, \ldots, m$. Then

1. $B_i \cap B_j = \varnothing$, $i \neq j$.
2. $B = B_1 \cup \cdots \cup B_m$ is an ordered basis for $\mathbb{K}^n$
3. $\dim(G_{\lambda_i}(A)) = k_i$ for $i = 1, 2, \ldots, m$.

Jordan Canonical Form

Definition 13

The matrix

$$J_b(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in \mathcal{M}_b(\mathbb{K})$$

is called **Jordan block of order** b corresponding to λ.

Jordan Canonical Form

Definition 14

The matrix

$$J_s(\lambda) = \begin{pmatrix} J_{b_1}(\lambda) & O & \cdots & O \\ O & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & J_{b_k}(\lambda) \end{pmatrix} \in \mathcal{M}_{b_1+\cdots+b_k}(\mathbb{K})$$

is called **Jordan segment** corresponding to λ.

Jordan Canonical Form

Definition 15

A matrix $J \in \mathcal{M}_n(\mathbb{K})$ is said to be in **Jordan canonical form** or a **Jordan matrix** if it is made of Jordan segment (along the diagonal). Namely,

$$J = \begin{pmatrix} J_{s_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{s_l}(\lambda_l) \end{pmatrix}$$

Jordan Canonical Form

Definition 16

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$. The linear operator L is said to be **jordanizable over** $\mathbb{K}$ if there is a basis B for V such that $[L]_B = J$ is a matrix in Jordan canonical form. The basis B is called **Jordan canonical basis for** V. $[L]_B = J$ is a Jordan is called Jordan canonical form for the linear operator L.

Definition 17

A matrix $A \in \mathcal{M}_n(\mathbb{K})$ is said to be **jordanizable over** $\mathbb{K}$ if it is similar to a matrix in Jordan canonical form, that is, there is a nonsingular matrix S such that $J = S^{-1}AS$ is a Jordan matrix.

Jordan Canonical Form

Definition 18

Let $V \in V^{\mathbb{K}}$, $L \in \mathcal{L}(V)$, and $x \in G_\lambda(L)$ be a generalized eigenvector corresponding to the eigenvalue λ. Suppose that k is the smallest positive integer for which $(L - \lambda I)^k(x) = 0$. Then the ordered set

$$\left\{(L-\lambda I)^{k-1}(x), (L-\lambda I)^{k-2}(x), \ldots, (L-\lambda I)(x), x\right\}$$

is called a **cycle of generalized eigenvectors of** L corresponding toλ. The elements $(L-\lambda I)^{k-1}(x)$ and x are called the **initial vector** and **end vector** of the cycle, respectively. We say that the length of the cycle is k.

Jordan Canonical Form

Definition 19

Let $A \in \mathcal{M}_n(\mathbb{K})$ and $x \in G_\lambda(A)$ be a generalized eigenvector corresponding to the eigenvalue λ. Suppose that k is the smallest positive integer for which $(A - \lambda I)^k x = 0$. Then the ordered set

$$\left\{(A - \lambda I)^{k-1}x, (A - \lambda I)^{k-2}x, \ldots, (A - \lambda I)\, x, x\right\}$$

is called a **cycle of generalized eigenvectors** of A corresponding toλ. The elements $(A - \lambda I)^{k-1}x$ and x are called the **initial vector** and **end vector** of the cycle, respectively. We say that the length of the cycle is k.

Jordan Canonical Form

Theorem 36

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$. Suppose that the characteristic polynomial $p_L(\lambda)$ splits over $\mathbb{K}$ and B is a basis for V such that B is a disjoint union of cycles of generalized eigenvectors of L. Then

1. For each cycle of generalized eigenvectors C contained in B, $S = \operatorname{span}(C)$ is L-invariant, and $[L_S]_C$ is a Jordan block.
2. B is a Jordan canonical basis for V.

Theorem 37

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$, and λ be an eigenvalue of L. Suppose that $B_1, \ldots, B_k$ are cycles of generalized eigenvectors of L corresponding to λ such that the initial vectors of the B_i's are distinct and form a linearly independent set. Then the B_i's are disjoint and the union $B_1 \cup \cdots \cup B_k$ is linearly independent.

Jordan Canonical Form

Theorem 38

Let $V \in V_n^{\mathbb{K}}$ and $L \in \mathcal{L}(V)$, and λ be an eigenvalue of L. Then the generalized eigenspace $G_\lambda(L)$ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ. Consequently, if the characteristic polynomial $p_L(\lambda)$ splits over $\mathbb{K}$ then L is jordanizable over $\mathbb{K}$.

Theorem 39

Let $V \in V_n^{\mathbb{K}}$, $L \in \mathcal{L}(V)$ and $A = [L]_B$. Suppose the characteristics and minimal polynomials of L are, respectively
$p_L(\lambda) = (\lambda_1 - \lambda)^{n_1} \cdots (\lambda_k - \lambda)^{n_k}$ and
$m_L(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}$, where $\lambda_1, \ldots, \lambda_k$ are distinct. Then L is jordanizable whose Jordan canonical form J consists of k Jordan segments $J_s(\lambda)$ which have the following properties:

1. There is at least one $m_j \times m_j$ Jordan block which is the largest in $J_{s_j}(\lambda_j)$; the number of $i \times i$ Jordan blocks in $J_{s_j}(\lambda_j)$ is given by $\nu_i(\lambda_j) = r_{i-1}(\lambda_j) - 2r_i(\lambda_j) + r_{i+1}(\lambda_j)$, where $r_i(\lambda_j) = \operatorname{rank}\left((A - \lambda_j I)^i\right)$.
2. The sum of all orders of the Jordan blocks in the Jordan segment $J_{s_j}(\lambda_j)$ equals n_j.
3. The number of Jordan blocks in Jordan segment $J_{s_j}(\lambda_j)$ equals the geometric multiplicity of λ_j.

Jordan Canonical Form

Theorem 40

Let $V \in V_n^{\mathbb{C}}$, $L \in \mathcal{L}(V)$, $A = [L]_B$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_l$. Then there is a Jordan canonical basis forming a nonsingular matrix P such that:

1. $J = P^{-1}AP = \begin{pmatrix} J_{s_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{s_l}(\lambda_l) \end{pmatrix}$
2. J has one Jordan segment $J_{s_j}(\lambda_j)$ for each eigenvalue $\lambda_j, j = 1, 2, \ldots, l$.
3. Each Jordan segment $J_{s_j}(\lambda_j)$ is made up of $\beta_j = \dim \mathrm{N}(A - \lambda_j I)$ Jordan blocks $J_b(\lambda_j)$.
4. The largest Jordan block in $J_{s_j}(\lambda_j)$ is $k_j \times k_j$, where $k_j = \mathrm{index}(\lambda_j)$.

Jordan Canonical Form

(5) The number of $i \times i$ Jordan blocks in $J_{s_j}(\lambda_j)$ is given by $\nu_i(\lambda_j) = r_{i-1}(\lambda_j) - 2r_i(\lambda_j) + r_{i+1}(\lambda_j)$ where $r_i(\lambda_j) = \mathrm{rank}\left((A - \lambda_j I)^i\right)$.

(6) The structure of J is unique in the sense that the number of Jordan segments as well as the number and the sizes of the Jordan blocks is uniquely determined by the entries in A.

Definition 20

A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said to be in **real Jordan canonical form** if it is made of Jordan blocks (along the diagonal) of the forms $J_b(\lambda)$, with $\lambda \in \mathbb{R}$, and $\hat{J}_{2s}(\mu)$, with $\mu = a + bi \in \mathbb{C}$ $(b \neq 0)$ where

$$\hat{J}_{2s}(\mu) = \begin{pmatrix} D & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ & & & D \end{pmatrix} \in \mathcal{M}_{2s}(\mathbb{R})$$

$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 41

If $A \in \mathcal{M}_n(\mathbb{R})$, then A is similar to a real Jordan canonical form.