National Religion King

Institute of Technology of Cambodia

Department of Applied Mathematics and Statistic

(Option Data Science)

Assignment of Differential Equation

Jordan Canonical Form, Ordinary Differential Equation, System of Ordinary Differential Equation

Lecture: Dr. Lin Mongkolsery (Cours)

Mr. Pen Chentra (TD)

Prepared and Researched by : Group I2-AMS1 Work Hard Team

Name	:	ID
Kry SengHort	:	e20200706
Lab Thavrith	:	e20200014
Lang Yanut	:	e20200164
Leat Seanglong	:	e20200971

Academic Year : 2021-2022

♣♣♣♣♣

I2-TD1
(JORDAN CANONICAL FORMS)

1. Find all eigenvalues and their corresponding eigenspaces of the following matrix.

(a) $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

2. Find the characteristics and the minimal polynomial of the following matrices over $\mathbb{R}$, then deduce the their corresponding Jordan Canonical Form J.

(a) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 2 & -5 & -1 & 6 \\ 1 & -3 & -2 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$

3. The following are the Jordan Canonical Form of linear transformations. Find the characteristic polynomial, minimal polynomials, the algebraic multiplicity, geometric multiplicity and the index of each of the eigenvalues of L

(a) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

4. Determine all possible Jordan canonical forms for a linear operator L whose characteristic polynomial is $p(\lambda) = (\lambda - 3)^3(\lambda - 4)^2$.

5. Determine all possible Jordan canonical forms J for a matrix of order 6 whose minimal polynomial is $m(\lambda) = (\lambda - 1)^3(\lambda - 3)^2$.

6. Find all possible Jordan canonical forms for those matrices whose characteristics polynomial $p(\lambda)$ and minimal polynomial $m(\lambda)$ are as follows:

(a) $p(\lambda) = (\lambda - 3)^4(\lambda - 2)^2$ and $m(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$

(b) $p(\lambda) = (\lambda - 5)^7$ and $m(\lambda) = (\lambda - 5)^2$

(c) $p(\lambda) = (\lambda - 3)^4(\lambda - 2)^4$ and $m(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$

7. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $m_A(\lambda)$ be its minimal polynomial. Let f be a polynomial satisfies $f(A) = 0$. Show that $f(\lambda)$ is divisible by $m_A(\lambda)$.

8. Let $A \in \mathcal{M}_6(\mathbb{R})$ be an invertible matrix satisfies $A^3 - 4A^2 + 3A = 0$ and $\operatorname{tr}(A) = 8$. Find the characteristics polynomial of A.

9. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ (no need distinct) be eigenvalues of A. Show that

(a) $\displaystyle\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A)$.

(b) $\displaystyle\prod_{i=1}^{n} \lambda_i = |A|$.

10. Let $A \in \mathcal{M}_3(\mathbb{R})$ with $\operatorname{tr}(A) = 9$. Suppose that $\lambda = 2$ is an eigenvalue of A and E_2 spanned by $(1, 0, 1)$ and $(-1, 1, 3)$. Find $|A|$.

11. Find $A \in \mathcal{M}_n(\mathbb{R})$ such that

$$A^3 - 4A^2 + 4A = 0 \quad \text{and} \quad \operatorname{tr}(A) = 0.$$

12. Let A be a square matrix defined by $A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$. Find the minimal polynomial of A. Then express A^4 and A^{-1} in terms of A and I.

13. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. One can show that $A^2 \neq 0$ and $A^3 = 0$. Find the nilpotent matrix M in canonical form which is similar to A.

14. Let $a \in \mathbb{R}^*$ and $A = \begin{pmatrix} 0 & a & a^2 \\ a^{-1} & 0 & a \\ a^{-2} & a^{-1} & 0 \end{pmatrix}$.

(a) Show that $A^2 = A + 2I$.

(b) Deduce that A is diagonalizable.

15. Determine the value of a so that $\lambda = 2$ is an eigenvalue of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ a & 1 & 1 \\ 0 & 1+a & 3 \end{pmatrix}$$

then show that A is diagonalizable and diagonalize it.

16. Let A be a 2×2 matrix defined by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that A is diagonalizable if $(a - d)^2 + 4bc \neq 0$.

17. Let A be a square matrix defined by $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Show that A is diagonalizable then diagonalize it.
(c) Write A^n in term of n.

18. Let A be a square matrix defined by $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Show that A is diagonalizable then diagonalize it.
(c) Write A^n in term of n.

19. Let A be a square matrix defined by $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Show that A is diagonalizable then diagonalize it.
(c) Write A^n in term of n.

20. Let A be a square matrix defined by $A = \begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Find the eigenvalues and eigenspaces of A.
(c) Show that A is not diagonalizable, but it is triangularizable, then triangularize A.
(d) Write A^n in terms of I, A, A^2 and n.

21. Let A be a square matrix defined by $A = \begin{pmatrix} -2 & -1 & -5 \\ 2 & 2 & 3 \\ 4 & 2 & 6 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Find the eigenvalues and eigenspaces of A.
(c) Show that A is not diagonalizable, but it is triangularizable, then triangularize A.
(d) Find the three real sequences $(a)_n, (b)_n, (c)_n$ satisfying

$$\begin{cases} a_{n+1} = -2a_n - b_n - 5c_n, & a_0 = 1 \\ b_{n+1} = \ \ 2a_n + 2b_n + 3c_n, & b_0 = 0 \\ c_{n+1} = \ \ 4a_n + 2b_n + 6c_n, & c_0 = 1 \end{cases}$$

22. Let A be a square matrix defined by $A = \begin{pmatrix} -1 & -2 & -1 & 3 \\ -6 & -5 & 1 & 6 \\ -6 & -4 & 0 & 6 \\ -6 & -7 & 1 & 8 \end{pmatrix}$ and its characteristics polynomial $p(\lambda) = (\lambda + 1)^2(\lambda - 2)^2$.

(a) Find the minimal polynomial of A.

(b) Deduce that A is not diagonalizable, but it is triangularizable, then triangularize A.

(c) Write A^n in terms of n.

23. Let A be a square matrix defined by $A = \begin{pmatrix} 6 & 2 & 3 \\ -3 & -1 & -1 \\ -5 & -2 & -2 \end{pmatrix}$ and L be a map from $\mathbb{R}^3$ into $\mathbb{R}^3$ by $L(v) = Av$.

(a) Show that L is a linear operator on $\mathbb{R}^3$.

(b) Find the characteristic polynomial of L with respect to standard basis for $\mathbb{R}^3$. Derive the determinant of L then deduce that L is invertible.

(c) Find the eigenvalues and eigenspaces of L.

(d) Show that L is not diagonalizable, but it is triangularizable, then triangularize L.

(e) Write L^n in term of n, where $L^n = L(L(\ldots(L)\ldots))$, the n compositions of L.

24. Let $B_1 = \{(2,1,1,1),(1,1,1,1),(1,1,2,1)\}$ and $B_2 = \{(2,1,2,2)\}$ be two subsets of $\mathbb{R}^4$, E_1 be a subspace spanned by B_1, E_2 be a subspace spanned by B_2, and L be a linear operator on $\mathbb{R}^4$ defined by

$$L(v) = (-w + 4x - y + z, -w + 3x, -w + 2x + y, -w + 2x + z), v = (w, x, y, z).$$

(a) Show that B_1 is a basis for E_1 and B_2 is a basis for E_2.

(b) Show that E_1 and E_2 are L-invariant. Find the matrices $A_1 = [L_{E_1}]_{B_1}$ and $A_2 = [L_{E_2}]_{B_2}$

(c) Show that $\mathbb{R}^4$ is a direct sum of E_1 and E_2.

(d) Find the characteristic and minimal polynomials of A_1 and A_2.

(e) Let A be the matrix representation of L with respect to the standard basis for $\mathbb{R}^4$. Show that A is similar to a block diagonal matrix to be specified.

(f) Deduce the characteristic and minimal polynomials of L.

25. Let $T \in \mathcal{L}(\mathbb{R}^3)$ defined by $T(x_1, x_2, x_3) = (-4x_1 - x_2 - 2x_3, 4x_1 + x_2 + 3x_3, -x_2 - x_3)$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A. Show that A is not diagonalizable over $\mathbb{R}$.

(c) Show that A is diagonalizable over $\mathbb{C}$. Find the eigenspaces. Diagonalize A.

(d) Express T^n in the form of $a_n T^2 + b_n T + c_n I$ where a_n, b_n and c_n are real sequences to be specified, and $T^n = T(T(\ldots(T)\ldots))$, the n compositions of T.

26. Let A be a square matrix defined by $A = \begin{pmatrix} -3 & -1 & -3 \\ 5 & 2 & 5 \\ -1 & -1 & -1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A. Show that A is not diagonalizable over $\mathbb{R}$.

(c) Show that A is diagonalizable over $\mathbb{C}$. Find the eigenspaces. Diagonalize A.

(d) Express A^n in the form of $a_n A^2 + b_n A + c_n I_n$ where (a_n), (b_n) and (c_n) are real sequences to be specified.

27. Let A be a symmetric matrix defined by $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and S be the surface

$$x^2 + 2y^2 + z^2 - 2xy - 2yz + 2x + y = 1.$$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues and eigenspaces of A. Show that A is diagonalizable over $\mathbb{R}$.

(c) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

(d) Does the surface S have a center? If any, specify its coordinates.

(e) Determine the type of the surface S.

28. Let A be a symmetric matrix defined by $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 3 \end{pmatrix}$ and S be the surface of

$$2x^2 + 3y^2 + 3z^2 + 2xy + 4yz - 2xz + 2x + 14y = 2.$$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues and eigenspaces of A. Show that A is diagonalizable over $\mathbb{R}$.

(c) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

(d) Does the surface S have a center? If any, specify its coordinates.

(e) Determine the type of the surface S.

29. Determine the nature of the surface (S)

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = \lambda, \quad \lambda \in \mathbb{R}.$$

30. Given the following matrices and their corresponding characteristics polynomials, Jordanize them over $\mathbb{C}$.

(a) $\begin{pmatrix} 5 & 4 & -5 \\ 3 & 6 & -5 \\ 5 & 8 & -7 \end{pmatrix}$ and $p(\lambda) = -\lambda^3 + 4\lambda^2 - 6\lambda + 4$

(b) $\begin{pmatrix} 3 & 2 & 2 & -4 \\ 3 & 4 & 2 & -5 \\ -2 & 7 & 7 & -9 \\ 1 & 5 & 4 & -6 \end{pmatrix}$ and $p(\lambda) = (\lambda^2 - 4\lambda + 5)^2$

(c) $\begin{pmatrix} 3 & 1 & 2 & -3 \\ 4 & 1 & 1 & -2 \\ -2 & 6 & 7 & -8 \\ 2 & 2 & 3 & -3 \end{pmatrix}$ and $p(\lambda) = (\lambda^2 - 4\lambda + 5)^2$

Solution

Differential   I2-TD1: Jordan Canonical Forms.

1.) Find all eigenvalues and their corresponding eigenspace of following matrix:

a.) $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ Let $A = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ then consider on equality

$Ax = \lambda x$; where $\lambda \in \mathbb{R}$; $x = \mathcal{M}_2 \in \mathbb{R} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$

$\Leftrightarrow Ax - \lambda x = 0$ or $x(A - \lambda I) = 0 \,(1) \Rightarrow |A - \lambda I| = 0.$

$\begin{vmatrix} 3-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) = 6 - 5\lambda + \lambda^2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3.$

$\Rightarrow \text{Spect}(A) = \{2;3\}$. are all eigenvalues.

* For $\lambda_1 = 2$: substitute into (1) we have:

$\begin{bmatrix} 3-2 & 0 \\ 1 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$; Let $x_2 = t \in \mathbb{R}$

and $x_1 = 0 \Rightarrow E_{\lambda_1} = E_2 = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} \in \mathbb{R}^2 / t \in \mathbb{R}^* \right\}$ are eigenspace.

* For $\lambda_2 = 3$: substitute into (1) we have:

$\begin{bmatrix} 3-3 & 0 \\ 1 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$; let $x_2 = t \in \mathbb{R}^*$.

$\Rightarrow x_1 = x_2 = t$

$\Rightarrow E_{\lambda_2} = E_3 = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \in \mathbb{R}^2 / t \in \mathbb{R}^* \right\}$ are eigenspace corresponding.

b.) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$ Consider equality: $Ax = \lambda x$.

$\Rightarrow |A - \lambda I_3| = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$

$+(2-\lambda)\left[(2+\lambda)(\lambda-2)+3\right] + 3\left[2-\lambda-1\right] + 1(-3+2+\lambda)$

$= (2-\lambda)\left[\lambda^2 - 4 + 3\right] + 3(1-\lambda) + (\lambda - 1) = (2-\lambda)(\lambda^2 - 1) + 2(1-\lambda)$

$= (+\lambda - 1)\left[(2-\lambda)(\lambda+1) - 2\right] = (\lambda - 1)\left[3\lambda - \lambda^2 - 4 \times 0\right] = (1-\lambda)(\lambda^2 - 3\lambda + 0) = 0$

$\Leftrightarrow (1-\lambda)(\lambda^2 - 3\lambda + 0) = 0$ then $\lambda_1 = 1$; $\lambda_2 = 0$, $\lambda_3 = 1$.

$\Leftrightarrow (1-\lambda)(\lambda^2 - \lambda) = 0 \Leftrightarrow \lambda(\lambda - 1)(1-\lambda) = 0$

(1) $\Rightarrow \text{Spect}(A) = \{1, 0, 1\}$

* For $\lambda_1 = 1$: Consider (1): $\begin{bmatrix} 2-\lambda_1 & -3 & 1 \\ 1 & -2-\lambda_1 & 1 \\ 1 & -3 & 2-\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 0 \Leftrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Let $s = x_2$; $x_3 = t \Rightarrow x_1 = 3x_2 - x_3 = 3s - t$; $s;t \in \mathbb{R}$

$\Rightarrow E_{\lambda_1} = E_{\lambda_3} = E_1 = \left\{ \begin{pmatrix} 3s-t \\ s \\ t \end{pmatrix} \in \mathbb{R}^3 \middle/ s;t \in \mathbb{R}^* \right\} = \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is eigenspace.

* For $\lambda_2 = 0$: Consider (1): $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Let $x_3 = t \in \mathbb{R} \Rightarrow x_2 = x_3 = t$; $x_1 = x_2 = t$.

$\Rightarrow E_{\lambda_2} = E_0 = \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} \in \mathbb{R}^3 \middle/ t \in \mathbb{R}^* \right\}$ is eigenspace.

c.) $\begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$; $S_1 = tr(A)$

<u>Proof:</u> Consider equality: $Ax = \lambda x$ since $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq 0$

$\Rightarrow |A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} -8-\lambda & -3 & -6 \\ 4 & -\lambda & 4 \\ 4 & 2 & 2-\lambda \end{vmatrix} = ?\ P_A(\lambda)$.

$= (-1)^n(\lambda^n - S_1\lambda^{n-1} + \dots + (-1)^n S_n)$

$+ S_1 = \begin{vmatrix} -8 & -3 \\ 4 & 0 \end{vmatrix} = -12+6$; $= -6$; $S_2 = \begin{vmatrix} -8 & -3 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} -8 & -6 \\ 4 & 2 \end{vmatrix} = 12 - 8 - 16 + 24$

$\Rightarrow S_2 = 12$. ; $S_3 = \begin{vmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{vmatrix} = -8(-8) + 3(8-16) - 6(8-0) = 64 - 24 - 48$

$\Rightarrow S_3 = -8$

$\Rightarrow P_A(\lambda) = (-1)^3(\lambda^3 + \frac{12}{2}\lambda^2 + 12\lambda + 8) = -\lambda^3 + \frac{12}{-2}\lambda^2 - 12\lambda - 8 = 0$

(2)

$P_A(\lambda) = -\lambda^3 - 6\lambda^2 - 12\lambda - 8 = -(\lambda+2)^3 = 0 \Rightarrow \lambda = -2.$

$\Rightarrow$ Spect $(A) = \{-2\}$.

Consider & substitute into (1): $\begin{bmatrix} -8+2 & -3 & -6 \\ 4 & +2 & 4 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Leftrightarrow \begin{bmatrix} -6 & -3 & -6 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Leftrightarrow \begin{bmatrix} -2 & -1 & -2 \\ 4 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Leftrightarrow \begin{bmatrix} -2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$; Let $x_3 = t \in \mathbb{R}$; $x_2 = s \in \mathbb{R}$

$\Rightarrow x_1 = -x_2 - 2x_3 = -s - 2t$.

$\Rightarrow E_\lambda = E_{-2} = \left\{ \begin{pmatrix} -s-2t \\ s \\ t \end{pmatrix} \in \mathbb{R}^3 / \ s; t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

2.) Find the characteristics and the minimal polynomial of the following matrix over $\mathbb{R}$, then deduce their corresponding Jordan Canonical Form J.

Proof: a.) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

* Characteristic polynomial : $P_A(\lambda) = |A - \lambda I|$.

$= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & -\lambda & -1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[-\lambda(2-\lambda) - (-1)(1)\right]$

$= (1-\lambda)(1 - 2\lambda + \lambda^2) = (1-\lambda)(\lambda-1)^2$

$= -(\lambda-1)^3 = -(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = \underline{-\lambda^3 + 3\lambda^2 - 3\lambda + 1}$

* minimal polynomial :

* $A - 1I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \neq 0 \Rightarrow m_A(\lambda) \neq (\lambda - 1)$

* $(A-I)^2 = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(3)

(b). $\begin{pmatrix} 2 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 2 & -5 & -1 & 6 \\ 1 & -3 & -2 & 6 \end{pmatrix} = B$

$P(\lambda) = \det|B - \lambda I| = (\lambda-2)^4$

and since $(A-2I) \neq 0$ $(A-2I)^2 = 0 \Rightarrow m(\lambda) = (\lambda-2)^2$

Therefore $J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(c). $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix}$

$P(\lambda) = \det|C - \lambda I| = (\lambda-1)^3(\lambda+1)$.

since $(A-1)(A+1) \neq 0 \Rightarrow (A-1)^2(A+1) = 0 \Rightarrow m(\lambda) = (\lambda-1)^2(\lambda+1)$

$\Rightarrow J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

3 Determine the posible jordan canonical form.

(a). $P(\lambda) = (\lambda-3)^3(\lambda-4)^2$.

$J_1 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$, $J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$, $J_3 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

$J_3 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $J_5 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$, $J_6 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

Page...

4 Determine all posible Jodan canonical J. M_6

$m(\lambda) = (\lambda-1)^3(\lambda-3)^2$

$$J_1 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

5 Find all posible Jordan canonical matrix

(a). $p(\lambda) = (\lambda-3)^4(\lambda-2)^2$ and $m(\lambda) = (\lambda-3)^2(\lambda-2)^2$

$$J_1 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

(b). $p(\lambda) = (\lambda-5)^7$, $m(\lambda) = (\lambda-5)^2$.

$$J_1 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

(c). $p(\lambda) = (\lambda-3)^4(\lambda-2)^4$ and $m(\lambda) = (\lambda-3)^2(\lambda-2)^2$

$$J_1 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

6 Let $A \in M_n(\mathbb{R})$ and $m_A(\lambda)$ be its minimal polynormal

Let $f : f(A) = 0$, show that $f(\lambda)$ is divisible by $m_A(\lambda)$

we have $f(\lambda) = m_A(\lambda)Q(\lambda) + g(\lambda)$ since $m_A(\lambda)$ is the polynormal of A : $m_A(A) = 0$ and $g(A) \neq 0$.

So $f(A) = m_A(A)Q(A) + g(A) \Rightarrow g(A) = 0$.

so $g(\lambda) = 0 \quad \forall (\lambda \in Sp\{A\})$.

Therefore $f(\lambda)$ divisible by $m_A(\lambda)$.

7 Let $A \in M_6(\mathbb{R})$ be the inversible matrix $A^3 - 4A^2 + 3A = 0$ and $tra(A) = 8$ Find the $P_A(A)$.

we have $A^3 - 4A^2 + 3A = 0$, $f(\lambda) = \lambda^3 - 4\lambda^2 + 3\lambda = \lambda(\lambda-1)(\lambda-3)$

Then we have $P_A(\lambda) = \lambda^{K_1}(\lambda-1)^{K_2}(\lambda-3)^{K_3}$, where $K_1+K_2+K_3=$

Let $g(x) = (x-1)^{K_2} = a_{K_2}x^{K_2} + \dots a_{01}$

$q(x) = (x-3)^{K_3} = a_{K_3}x^{K_3} + \dots + a_{02}$.

The cofficient of $x^{K_2+K_3-1}$ is $a_{K_2}a_{K_3-1} + a_{K_3}a_{K_2-1}$

where $a_{K_2} = 1$, $a_{K_3} = 1$, $a_{K_2-1} = -K_2$, $a_{K_3-1} = -3K_3$

Then $a_{K_2}a_{K_3-1} + a_{K_3}a_{K_2-1} = -K_2 - 3K_3$.

but since $P(\lambda) = \lambda^8 - tra(A)\lambda^7 + \dots$

Then : $\begin{cases} K_1 + K_2 + K_3 = 6 \\ K_2 + 3K_3 = 6 \end{cases} \Rightarrow K_3 = -1 + \dfrac{K_1}{2}$

$\Rightarrow K_1 = 2, 4, 6. \Rightarrow (2,6,0), (4,3,1), (6,0,2)$.

Therefore $P(\lambda) = \lambda^2(\lambda-1)^6$

$P(\lambda) = \lambda^4(\lambda-1)^3(\lambda-3)$

$P(\lambda) = \lambda^6(\lambda-3)^2$

8 Let $A \in M_n(\mathbb{R})$, $\lambda_1, \dots \lambda_n$ be eigenvalue of A, show that:

(a). $\sum_{i=1}^{n} \lambda_i = tr(A)$. (b). $\prod_{i=1}^{n} \lambda_i = |A|$.

we have $P(\lambda) = (-1)^n(\lambda^n - s_1\lambda^{n-1} + \dots + (-1)^{n-1}s_{n-1}\lambda + (-1)^n s_n)$ (3)

since $\lambda_1, \dots \lambda_n \in sp(A)$: then: $P(\lambda) = (\lambda-\lambda_1)\times \dots (\lambda-\lambda_n)(-1)^n$ (4)

by (3) & (4)

Therefore $\sum_{i=1}^{n} \lambda_i = tr(A)$ and $\prod_{i=1}^{n} \lambda_i = |A|$.

9 Let $A \in M_n(\mathbb{R})$ with $tr(A) = 9$, suppose that $\lambda = 2$ is an eigenvalue of A and E_2 spanned by $(1,0,1)$ and $(-1,1,3)$ Find $|A|$.

since $E_2 = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\3\end{pmatrix}\right\} \Rightarrow gm(2) = 2$.

and we have $am(2) \geq gm(2) = 2$

since $tr(A) = 9 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 9$.

obviously, we have $am(2) = 2$, so $\lambda_1 = \lambda_2 = 2 \Rightarrow \lambda_3 = 5$.

Then $\det|A| = 2\times2\times5 = 20$.

10 Find $A \in M_n(\mathbb{R})$ such that $A^3 - 4A^2 + 4A = 0$ and $tr(A) = 0$.

we have $f(x) = x^3 - 4x^2 + 4 = x(x-2)^2$ cancel A. $f(A) = 0$

$\Rightarrow \lambda_1 = 0$, $\lambda_2 = 2$, $Tra(A) = 0$ so $\lambda_2 \in sp\{A\}$

$\Rightarrow (A - 2I)$ is invertable., we have $A(A-2I) = 0 \Rightarrow A = 0$

Therefore $A = 0$.

Page...

11 Let $A=\begin{pmatrix}4&-2&1\\2&0&1\\2&-2&3\end{pmatrix}$, Find $P(\lambda)$ and A^4, A^{-1} Interm A & I

we have $A=\begin{pmatrix}4&-2&1\\2&0&1\\2&-2&3\end{pmatrix}$

$\Rightarrow P(\lambda)=|(A-\lambda I)|=-\lambda^3+7\lambda^2-16\lambda+12=-(\lambda-2)^2(\lambda-3)$.

since $(\lambda-2)(\lambda-3)=0 \Rightarrow m(\lambda)=\cancel{\theta}=(\lambda-2)(\lambda-3)$.

we have $\lambda^2-5\lambda+6=0$ then $\lambda^4=q(\lambda)(\lambda^2-5\lambda-6)+a\lambda+b$

and we have $\begin{cases}2^4=2a+b\\3^4=3a+b\end{cases} \Rightarrow a=3^4-2^4, b=3^4-3(3^4-$

Thus $A^4=65A-114I$.

and $A^{-1}=-\frac{1}{144}(A^3-65I)$.

12 Let $A=\begin{pmatrix}0&1&1&0&1\\0&0&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$, show that $A^2\neq 0$ and $A^3=0$

Find the nilpotent matris M in canonical form witch is sim-

Since $A^3=0 \Rightarrow \text{Index}(0)=3$.

$\Rightarrow$ block matrix with Jordan form $\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$.

Futher $gm(0)=3$ then there are 3 blocks matrix

Therefore $M=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$

13 Let $a \in \mathbb{R}^*$ and $A = \begin{pmatrix} 0 & a & a^2 \\ a^{-1} & 0 & a \\ a^{-2} & a^{-1} & 0 \end{pmatrix}$

(a). Show that $A^2 = A + 2I$

we have $A^2 = \begin{pmatrix} 0 & a & a^2 \\ a^{-1} & 0 & a \\ a^{-2} & a^{-1} & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & a & a^2 \\ a^{-1} & 2 & a \\ a^{-2} & a^{-1} & 2 \end{pmatrix} = A + 2I.$

Therefore $A^2 = A + 2I$.

(b). Deduce that A is diagonalizble:

we have $A^2 - A - 2I = 0$ then $f(\lambda) = (\lambda+1)(\lambda-2)$ is $m(\lambda)$

since $m(\lambda)$ issplitted with simple roots.

Therefore A is diagonalizable.

14 Determine the value of a so that $\lambda = 2$ is the eig- of A then show that A is diagonalizable

we have $A = \begin{pmatrix} 1 & -1 & 0 \\ a & 1 & 1 \\ 0 & 1+a & 3 \end{pmatrix}$

we have $\lambda = 2 \in Sp(A)$ then $|A - 2I| = 0 \Rightarrow a = -1$ so

$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow P(\lambda) = \det(A - \lambda I) = \lambda(\lambda - 2)(3 - \lambda)$

$\Rightarrow Sp(0, -3, 2)$.

- $\lambda = 0 \Rightarrow E_0 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$
- $\lambda = 2 \Rightarrow E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$
- $\lambda = -3 \Rightarrow E_3 = \left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

Page...

Therefore $A = PDP^{-1}$, where $P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \\ -\frac{1}{2} & \frac{1}{2} & \\ 0 & 0 & \end{pmatrix}$

15 Let A be the 2×2 matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ show that A is diagonalizable if $(a-d)^2 + 4bc \neq 0$.

we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow P(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$

Then $\Delta = (a+d)^2 + 4(ad-bc) = (a-d)^2 + 4bc \neq 0$

Therefore A is diagonalizable if $(a-d)^2 + 4ac \neq 0$.

16 (a) Find $P(\lambda)$

$P(\lambda) = \det(A - \lambda I) = \lambda^2 + \lambda - 12 = (\lambda - 4)(\lambda + 3)$.

$\Rightarrow P(\lambda) = (\lambda - 4)(\lambda + 3)$

(b) show that A is digonalizable.

since $P(\lambda)$ is splitted. $\Rightarrow$ A is digonalizable.

Then eigenspace

$\lambda = 4$ is $E_4 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$

$\lambda = -3$ is $E_{-3} = \text{span}\left\{\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\}$

Therefore $A = PDP^{-1}$, $P = \begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{3}{7} & \\ -\frac{1}{7} & \end{pmatrix}$

(c). write A^n interm of n.

we have $A = PDP^{-1} \Rightarrow A^n = (PDP^{-1})^n = PD^nP^{-1}$

$$= P\begin{pmatrix} 4^n & 0 \\ 0 & (-3)^n \end{pmatrix}P^{-1}.$$

Page...

17 a). Find $P(\lambda)$.

we have $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$

$P(\lambda) = \det(A - \lambda I) = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2$.

Thus $P(\lambda) = -\lambda(\lambda-1)^2$.

(b). show that A is diagonalizable and find it digonalize

we have $P(\lambda) = -\lambda(\lambda-1)^2$

Then the eigenvalue of A is 0 and 1.

Then for: $\lambda = 1$: $E_1 = span\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow am(1) = gm(1) = 2$.

and for $\lambda = 0$: $E_0 = span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow am(0) = gm(0) \Rightarrow A$ is diagonalizable:

Then $A = PDP^{-1}$ where $P = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c). write A^n in term of n.

we have $A^n = (PDP^{-1})^n = PD^nP^{-1} = A \Rightarrow$ ~~A~~

Therefor $A^n = A$.

Page...

18 (a). Find $P(\lambda)$.

we have $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$

Then $P(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2$

(b). Find out that A is diagonalizable and diagonalize it.

we have $P(\lambda) = -(\lambda - 1)(\lambda - 2)^2$

for. $\lambda = 2$: $E_2 = \left\{ \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right\}$

$gm(2) = am(2) = 2$

for. $\lambda = 1$: $E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$gm(1) = am(1) = 1$

Thus: A is diagonalizable.

$\Rightarrow A = PDP^{-1}$

$P = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c). Find A^n interm of n.

we have $A^n = (PDP^{-1})^n = PD^nP^{-1} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}P^{-1}$.

Therefore $A^n = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}P^{-1}$.

(a) Find $P(A)$.

We have $A = \begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

Then $P(\lambda) = \det(A - \lambda I) = -\lambda^3 - 6\lambda^2 - 12\lambda - 8 = -(\lambda+2)^3$

(b). Find the eignevalue and eigenespace of A.

Thus eignevalue of A is $\lambda = -2$

$$E_2 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(c). since $am(2) \neq gm(-2)$ Thus: A is not diagonalizable.

Trtangulalize it, That is.

$$A = PJP^{-1}, \text{ where } J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Then, we have $(A+2I)^2 = 0$

Then $G_\lambda = \{x \in \mathbb{R}^3 : (A+2I)^2 x = 0\}$

or $G_\lambda = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Let $V_2 \in G_\lambda \setminus E_\lambda$ then $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

so, we have $V_1 = (A+2I)V_2 = \begin{pmatrix} -6 & -3 & -6 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 4 \end{pmatrix}$

Thus $P = \begin{pmatrix} -6 & 1 & 1 \\ 4 & 0 & -2 \\ 4 & 0 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 1 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Therefore $A = PJP^{-1}$

Page...

(d). write A in term of I, A, A^2 and n.

we have $P(\lambda) = -(\lambda+2)^3 \Rightarrow P(x) = -(x+2)^3$ and $P(A)=0$,

Let $f(x) = x^n$ then.

$$x^n = P(x)q(x) + ax^2+bx+c \quad \text{where } a,b,c \in \mathbb{R}$$

so $$nx^{n-1} = P'(x)q(x) + q'(x)P(x) + 2a+b.$$

$$n(n+1)x^{n-2} = P''(x)q(x) + q''(x)p(x) + 2p'(x)q(x) + 2a.$$

Then $\begin{cases} 2^n = 4a+2b+c \\ n2^{n-1} = 4a+b \\ n(n-1)2^{n-2} = 2a \end{cases}$

Therefore $A^n = n(n-1)2^{n-3}A^2 + (2n-n^2)^{n-1}A + (n^2-3n+2)2^{n-1}$

20 (a) Find $P(\lambda)$.

we have $A = \begin{pmatrix} -2 & -1 & -5 \\ 2 & 2 & 3 \\ 4 & 2 & 6 \end{pmatrix}$

Then $P(\lambda) = |A-\lambda I| = -\lambda^3+6\lambda^2-12\lambda+8 = -(\lambda-2)^3$

(b). Find the eigenvalue and eigenspace of A.

we have $P(\lambda) = -(\lambda-2)^3$

Thus $E_2 = \text{span}\left\{\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}\right\}$

(c). show that A is not diagonalizable.

by $am(2) \neq gm(2)$

Thus A is not diagonalizable.

Page...

• triangularize A:

we have $A = PJP^{-1}$, where $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

we have $(A-2I)^3 = 0$

Then we have $G_\lambda = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Let $V_3 = G_\lambda \setminus E_\lambda^2$ so $V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then.

$$V_2 = (A-2I)V_3 = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

and $V_1 = (A-2I)V_2 = \begin{pmatrix} -6 \\ 4 \\ 4 \end{pmatrix}$

Therefore $A = PJP^{-1}$ where $P = \begin{pmatrix} -6 & -4 & 1 \\ 4 & 2 & 0 \\ 4 & 4 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(d) Find a_n, b_n, c_n

we have $\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} -2 & -1 & -5 \\ 2 & 2 & 3 \\ 4 & 2 & 6 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$

$$= A^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$$

Page...

21

$A = \begin{pmatrix} -1 & -2 & -1 & 3 \\ -6 & -5 & 1 & 6 \\ -6 & -4 & 0 & 6 \\ -6 & -7 & 1 & 8 \end{pmatrix}$, $P(\lambda) = (\lambda+1)^2(\lambda-2)^2$

(a). Find $m(\lambda)$.

we have $\lambda_1 = -1$, $\lambda_2 = 2$

• For $\lambda_1 = -1 \Rightarrow E_{\lambda_1} = \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \Rightarrow gm(\lambda_1) = 1$.

• For $\lambda_2 = 2 \Rightarrow E_{\lambda_2} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow gm(\lambda_2) = 2$.

Thus $m(\lambda) = (\lambda+1)^2(\lambda-2)$

(b). deduce that A is not digonalizable.

by $am(\lambda_1) \neq gm(\lambda_1) \Rightarrow$ A is not diagonalizable.

+ traingonalize it.

we have $A = PJP^{-1}$, where $J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$(A+I)^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Then $E_1^2 = \text{span}\left\{ \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Let $V_2 \in E_{-1}^2 \setminus E_{-1}$ then $V_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix}$

so $V_1 = (A+I)V_2 = \begin{pmatrix} 0 & -2 & -1 & 3 \\ -6 & -4 & 1 & 6 \\ -6 & -4 & 1 & 6 \\ -6 & -7 & 1 & 9 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \\ -2 \end{pmatrix}$

Then $P = \begin{pmatrix} 1 & -\frac{1}{3} & -1 & \frac{1}{2} \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix}$

Page...

Therefore $A = PJP^{-1}$ where P and J we solve above.

(c). write A^n in term of n.

we have $A^n = (PJP^{-1})^n = PJ^nP^{-1}$

$$J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = D + M$$

$$J^n = D^n + nD^{n-1}M = \begin{pmatrix} 2^n & & & (0) \\ & 2^n & & \\ & & (-1)^n & n(-1)^{n-1} \\ (0) & & & (-1)^n \end{pmatrix}$$

$$\text{Therefore } A = P \begin{pmatrix} 2^n & 0 & 0 & 0 \\ 0 & 2^n & 0 & 0 \\ 0 & 0 & (-1)^n & n(-1)^{n-1} \\ 0 & 0 & 0 & (-1)^n \end{pmatrix}.$$

22 Let $A = \begin{pmatrix} 6 & 2 & 3 \\ -3 & -1 & -1 \\ -5 & -2 & -2 \end{pmatrix}$, $L(v) = Av$

(a). Show that L is Linear operator on $\mathbb{R}^3$.

Let $u, v \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ so that:

$$L(u + \alpha v) = L(u) + \alpha L(v).$$

Therefore L is Linear operator on $\mathbb{R}^3$.

(b). Find $P_L(\lambda)$

$$P_L(\lambda) = |A - \lambda I| = \begin{vmatrix} 6-\lambda & 2 & 3 \\ -3 & -1-\lambda & -1 \\ -5 & -2 & -2-\lambda \end{vmatrix} = -(\lambda - 1)^3$$

So $\det(L) = P_L(0) = 1 \Rightarrow L$ is inversible:

Page...

(c). Find the eigenvalues and eigenspace of L.

$P_L(\lambda) = 0 \Leftrightarrow \lambda = 1$ with multiplicity 3:

Then $E_1 = \ker(A - I) = \text{span}\left\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$

(d). Show that L is not diagonalizable but L is traingonalizable and traingonalize it:

Since $gm(1) \neq am(1)$, but $P_L(\lambda)$ is splited so that L is traingonalizable:

That is $A = PJP^{-1}$. where

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we have $(A - I)^3 = 0$ then

$$G_\lambda = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

Let $V_3 \in G_\lambda \setminus E_\lambda^2 \Rightarrow V_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ so

$$V_2 = (A - I)V_3 = \begin{pmatrix} 5 & 2 & 3 \\ -3 & -2 & -1 \\ -5 & -2 & -3 \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}5\\-3\\-5\end{pmatrix}$$

and $V_1 = (A - I)V_2 = \begin{pmatrix} 5 & 2 & 3 \\ -3 & -2 & -1 \\ -5 & -2 & -3 \end{pmatrix}\begin{pmatrix}5\\-3\\-5\end{pmatrix} = \begin{pmatrix}4\\-4\\-4\end{pmatrix}$

Then $P = \begin{pmatrix} 4 & 5 & 1 \\ -4 & -3 & 0 \\ -4 & -5 & 0 \end{pmatrix}$

Therefore: $A = PJP^{-1}$, where P and J are solved above (e). write L^n in term of n

we have $L^n = A^n V$ and $A^n = P J^n P^{-1}$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow J^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore $L^n = P \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$.

23 Let $B_1 = \{(2,1,1,1), (1,1,1,1), (1,1,2,1)\}$ and $B_2 = \{(2,1,2,2)\} \subset \mathbb{R}^4$

E_1 a subspace spanned by B_1, & E_2 be a subspace spanned by B_2 : $L(v) = (-w + 4x - y + z, -w + 3x, -w + 2x + y, -w + 2x + z)$

(a). show that B_1 is the basis of E_1 and B_2 is the basis E_2

$E_1 = span\{(2,1,1,1), (1,1,1,1), (1,1,2,1)\}$

checke $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow L.I.$

Therefore B_1 is the basis of E_1 and logically B_2 is the basis of E_2.

(b). show that E_1 and E_2 are stable by L.

we have $E_1 = span\{(2,1,1,1), (1,1,1,1), (1,1,2,1)\}$

Let $x \in E_1 \Rightarrow x = \begin{pmatrix} 2\alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 + 2\alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \end{pmatrix}$

So $L(x) = (x_1, x_2, x_3, x_4)$

where $x_1 = 2\alpha_1 + \alpha_2 + \alpha_3$, $x_2 = \alpha_1 + \alpha_2 + \alpha_3$

$x_3 = \alpha_1 + \alpha_2 + 2\alpha_3$, $x_4 = \alpha_1 + \alpha_2 + \alpha_3$

and $L(E_1) = \text{span}\left\{\begin{pmatrix}2\\1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\\2\end{pmatrix}, \begin{pmatrix}-2\\2\\1\\2\end{pmatrix}\right\}$

we will show that $L(E_1) \subseteq E_1$

Let $\alpha = \begin{pmatrix}u\\v\\w\end{pmatrix} \in L(E_1) \Rightarrow \begin{pmatrix}u\\v\\w\end{pmatrix} = \alpha_1\begin{pmatrix}2\\1\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}3\\2\\1\\2\end{pmatrix} + \alpha_3\begin{pmatrix}-2\\2\\1\\2\end{pmatrix}$

If $\alpha \in E_1$ so $\alpha = \beta_1\begin{pmatrix}2\\1\\1\\1\end{pmatrix} + \beta_2\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \beta_3\begin{pmatrix}1\\1\\2\\1\end{pmatrix}$

we will get $\begin{cases} 2\alpha_1 + 3\alpha_2 - 2\alpha_3 = 2\beta_1 + \beta_2 + \beta_3 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 = \beta_1 + \beta_2 + \beta_3 \\ \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 \end{cases}$

$\Rightarrow$ There are exist β_1, β_2 & β_3

Since $\alpha \in L(E_1)$ and $L(E_1) \in E_1$

Therefore E_1 is stable by L.

+ Show that E_2 is L-invariant.

$E_2 = \left\{\begin{pmatrix}2\\1\\2\\2\end{pmatrix}\right\} \Rightarrow L(2t, t, 2t, 2t) = (2t, t, 2t, 2t). \Rightarrow L(\alpha) =$

so that $L(E_2) \subseteq E_2$

Therefore E_2 is L-invariant.

+ Find $A_1 = [L_{E_1}]_{B_1}$ and $B = [L_{E_2}]_{B_2}$.

Let $\alpha \in \mathbb{R}^4 \Rightarrow [\alpha]_{B_1} = (\alpha_1, \alpha_2, \alpha_3)$.

where $[\alpha]_{B_1} = \alpha_1\begin{pmatrix}2\\1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\1\\2\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$\Rightarrow \alpha_1 = a-b,\ \alpha_3 = -b+c,\ \alpha_2 = 3b-a-c.$

Then $[\alpha]_{B_1} = \cancel{\begin{pmatrix}a-b\\-b+c\\3b-a-c\end{pmatrix}}\ \begin{pmatrix}a-b\\3b-a-c\\-b+c\end{pmatrix}$

we have $L\begin{pmatrix}2\\1\\1\\1\end{pmatrix}=(2,1,1,1)$, $L\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=(3,2,2,2)$, $L\begin{pmatrix}1\\1\\2\\1\end{pmatrix}=(2,2,3,2)$

$\Rightarrow [L_{E_1}]_{B_1}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$

Let $x\in\mathbb{R}^4 \Rightarrow [x]_{B_2}=\alpha_1$ such that $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\alpha_1\begin{pmatrix}2\\1\\2\\2\end{pmatrix}$

$\Rightarrow [L_{E_2}]_{B_1}=1$.

Then $A_2=(1)$

(c). Show that $\mathbb{R}^4$ is direct sum of E_1 and E_2

$E_1+E_2=\text{span}\{(2,1,1,1),(1,1,1,1),(1,1,2,1),(2,1,2,2)\}$

check $\begin{pmatrix}2&1&1&1\\1&1&1&1\\1&1&2&1\\2&1&2&2\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$\Rightarrow \dim(E_1+E_2)=4=\dim E_1+\dim E_2 \Rightarrow \dim(E_1\cap E_2)=0$

$\Rightarrow E_1\oplus E_2=E_1+E_2$ and $\dim(E_1+E_2)=\dim\mathbb{R}^4=4$

Therefore $\mathbb{R}^4=E_1\oplus E_2$

(d). Find $P_{A_1}(\lambda)$ ~~and~~, $m_{A_1}(\lambda)$, $P_{A_2}(\lambda)$, $m_{A_2}(\lambda)$.

$P_{A_2}(\lambda)=1-\lambda$, $m_{A_2}(\lambda)=-1+\lambda$.

$A_1=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix} \Rightarrow P_{A_1}(\lambda)=-(\lambda-1)^3$, $m_{A_1}(\lambda)=(\lambda-1)^3$

(e). show that A is similar to block matrix.

$A=[L]_B$, $B=\{e_1,e_2,e_3,e_4\}$

since $\mathbb{R}^4=E_1\oplus E_2$

so $A\sim J$ where $J=\text{diag}\left\{\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix},(1)\right\}$

Page...

(f). deduce $P_A(\lambda)$, $m_A(\lambda)$

. $P_A(\lambda) = P_{A_1}(\lambda) \times P_{A_2}(\lambda) = (\lambda-1)^4$

. & $m_A(\lambda) = m_{A_1}(\lambda) \times m_{A_2}(\lambda) = (\lambda-1)^4$.

Therefore:

$$P_A(\lambda) = (\lambda-1)^4$$

$$m_A(\lambda) = (\lambda-1)^4.$$

24 Let $T \in \mathcal{L}(\mathbb{R}^3)$, $T(x_1,x_2,x_3) = (-4x_1 - x_2 - 2x_3, 4x_1 + x_2 + 3x_3, -x_2 -$

(a). Find $P(\lambda)$.

We have $T(x_1,x_2,x_3) = \begin{pmatrix} -4 & -1 & -2 \\ 4 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ax.$

$\Rightarrow P(\lambda) = |A - \lambda I| = -(\lambda+2)(\lambda^2+2\lambda+2).$

Thus $P(\lambda) = -(\lambda+2)(\lambda^2+2\lambda+2)$.

(b). Find the eigenvalue of A, show that A is not diagonal on $\mathbb{R}$ if $P(\lambda)=0 \Rightarrow \lambda = 2$, $P(\lambda)$ is not splited.

Thus A is not diagonalizable over $\mathbb{R}$.

(c) show that A is diagonalizable over $\mathbb{C}$

Since $P(\lambda) = -(\lambda+2)(\lambda^2+2\lambda+2) = -(\lambda+2)(\lambda+1-i)(\lambda+1-i)$

since $P(\lambda)$ is splited with simple roots.

Therefore A is diagonalizable over $\mathbb{C}$

(d). Express T^n in term of $a_nT^2 + b_nT + c_nI$ where a_n, b_n, c_n are real sequenes to be specified.

Page.

and $T^n = (T(T(T\ldots(T)\ldots))$, The n composition T.

we have $x^n = \ell(x)q(x) + a_n x^2 + b_n x + c_n$

and $\ell(-2) = 0$, $\ell(-1-i) = 0$, $\ell(-1+i) = 0$

$$\Rightarrow \begin{cases} (-1-i)^n = a_n(-1-i)^2 + b_n(-1-i) + c_n \\ (-1+i)^n = a_n(-1+i)^2 + b_n(-1+i) + c_n \\ (-2)^n = 4a_n - 2b_n + c_n \end{cases}$$

$$\text{or} \begin{pmatrix} (-1-i)^2 & (-1-i) & 1 \\ (-1+i)^2 & (-1+i) & 1 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} (-1-i)^n \\ (-1+i)^n \\ (-2)^n \end{pmatrix}$$

$$\sim \left(\begin{array}{ccc|c} 2i & -1-i & 1 & (-1-i)^n \\ -2i & -1+i & 1 & (-1+i)^n \\ 4 & -2 & 1 & (-2)^n \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2i & -1-i & 1 & (-1-i)^n \\ 0 & -2 & 2 & (-1-i)^n + (-1+i)^n \\ 4 & -2 & 1 & (-2)^n \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -4 & 2 & 0 & 2i(-1-i)^n - i[(-1+i)^n + (-1-i)^n] \\ 0 & -2 & 2 & (-1+i)^n + (-1-i)^n \\ 4 & 0 & -1 & (-2)^n - (-1+i)^n - (-1-i)^n \end{array}\right)$$

$$\Rightarrow \begin{cases} -4a_n + 2b_n = 2i(-1-i)^n - i[(-1+i)^n + (-1-i)^n] \\ -2b_n + 2c_n = (-1+i)^n + (-1-i)^n \\ 4a_n - c_n = (-2)^n - (-1+i)^n - (-1-i)^n \end{cases}$$

Therefore $T^n = a_n T^2 + b_n T + c_n I$

where $a_n = \frac{1}{4}[2(-2)^n + 2i(-1-i)^n + (i-1)[(-1-i)^n + (-1+i)^n]]$

$b_n = (-2)^n + 2i(-1-i)^n - (\frac{1}{2} + i)[(-1-i)^n + (-1+i)^n]$

$c_n = 2i(1-i)^n + (-2)^n - i[(-1-i)^n + (-1+i)^n]$

Page...

25 Let $A \in M_3(\mathbb{R})$, $A = \begin{pmatrix} -3 & -1 & -3 \\ 5 & 2 & 5 \\ -1 & -1 & -1 \end{pmatrix}$

(a). Find $P(\lambda)$.

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} -3-\lambda & -1 & -3 \\ 5 & 2-\lambda & 5 \\ -1 & -1 & -1-\lambda \end{vmatrix} = -\lambda(\lambda^2+2\lambda+2)$$

(b). Find eigenvalue of A, show that A is not diagonalizable

since A is not splited

Therefore: A is not diagonalizable over $\mathbb{R}$.

(c). Show that A is diagonalizable over $\mathbb{C}$, Find eigenspace and Digonalize A.

we have $P(\lambda) = -\lambda(\lambda^2+2\lambda+2) = -\lambda(\lambda+1+i)(\lambda+1-i)$

and $am(\lambda_i) = gm(\lambda_i)$, $i = 1, 2, 3$.

Therefore: A is diagonalizable over $\mathbb{C}$

+ Digonalizable A:

$A = PDP^{-1}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{pmatrix}$

+ $\lambda_1 = 0$: $E_{\lambda_1} = \{x \in \mathbb{R}^3 : Ax = 0\}$

$$A = \begin{pmatrix} -3 & -1 & -3 \\ 5 & 2 & 5 \\ -1 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_3 = 1 \Rightarrow x_1 = -1$, $x_2 = 0$

Then $E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

+ For $\lambda_2 = -1-i$

$$A + (1+i)I = \begin{pmatrix} -2+i & -1 & -3 \\ 5 & 3+i & 5 \\ -1 & -1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1+2i \\ 0 & 1 & -1-3i \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $E_2 = \text{span}\left\{\begin{pmatrix} -1-2i \\ 1+3i \\ 1 \end{pmatrix}\right\}$

+ For $\lambda_3 = -1+i$

$$A + (1-i)I = \begin{pmatrix} -2-i & -1 & -3 \\ 5 & 3-i & 5 \\ -1 & -1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1-2i \\ 0 & 1 & -1+3i \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $E_3 = \text{span}\left\{\begin{pmatrix} -1+2i \\ 1-3i \\ 1 \end{pmatrix}\right\}$

(d). Express A^n interm of $a_nA^2 + b_nA + c_nI$.

we have $x^n = P(x)q(x) + a_nx^2 + b_nx + c_n$

as $P(0) = 0$, $P(-1-i) = 0$, $P(-1+i) = 0$

$\Rightarrow c_n = 0$

$b_n = \frac{1}{2}[(-1-i)^n - (-1+i)^n]$

$a_n = -2i(-1-i)^n - [(-1-i)^n - (-1+i)^n]$.

Therefore $A^n = a_nA^2 + b_nA + c_nI$

where: $a_n = -2i(-1-i)^n - [(-1-i)^n - (-1+i)^n]$

$b_n = \frac{1}{2}[(-1-i)^n - (-1+i)^n]$

$c_n = 0$.

26

$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, $(S): x^2+2y^2+z^2-2xy-2yz+2x+y=1$.

(a). Find $P(\lambda)$.

$P(\lambda) = |A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = -\lambda(\lambda-1)(\lambda-3)$.

Therefore $P(\lambda) = -\lambda(\lambda-1)(\lambda-3)$.

(b). Find eigenvalue and eigenspace of A.

$P(\lambda) = -\lambda(\lambda-1)(\lambda-3) \Rightarrow sp(A) = \{0,1,3\}$

- For $\lambda_1 = 0$ $E_1 = Ax$.

  $\Rightarrow E_{\lambda_1} = \text{span}\{(1,1,1)\}$

- For $\lambda_2 = 1$. $E_2 = (A-I)x$.

  $\Rightarrow E_{\lambda_2} = \text{span}\{(1,0,-1)\}$

- For $\lambda_3 = 3$. $E_3 = (A-3I)x$.

  $\Rightarrow E_{\lambda_3} = \text{span}\{(1,-2,1)\}$

since $P(\lambda)$ is splited and $am(\lambda_i) = gm(\lambda_i)$

Therefore A is diagonalizable.

(c). Find matrix P such that $P^{-1}AP$ is a digonal

we have $A = PDP^{-1}$ where $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

and $P_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ orthogonal matrix $(PP^T = P^T P$

we have to divide each colum by its norm.

$\Rightarrow P = \begin{pmatrix} \frac{1}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{6} \\ \frac{1}{\sqrt{3}} & 0 & -2/\sqrt{6} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 1/\sqrt{6} \end{pmatrix}$

(d). check for center.

Let $F(x,y,z) = x^2+2y^2+z^2-2xy-2yz+2x+y-1$

$F_x = 2x-2y+2$

$F_y = 2y-2x-2z$

$F_z = 2z-2y$

if $F_x = F_y = F_z = 0 \Rightarrow \begin{cases} x-y+1=0 \\ -x+2y-z=0 \\ z-y \quad =0 \end{cases}$

$\Rightarrow$ There have no roots.

Then : (S) doesn't have center.

(e) Determine the type of surface (S):

$(S): x^2+2y^2+z^2-2xy-2yz+2x+y=1.$

$$(x \; y \; z)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2x+y = 1.$$

consider XAX^T : where $X = (x,y,z)$.

$$(x, y, z)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = XP\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}P^{-1}X^T$$

Let $Y = XP \Rightarrow Y^T = (XP)^T = P^TX^T = P^{-1}X^T$

$$\Rightarrow Y\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}Y^T$$

Then : (S) is Elliptic parabolic.

27 $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 3 \end{pmatrix}$, $(S): 2x^2 + 3y^2 + 3z^2 + 2xy + 4yz - 2xz + 2x + 14y$

(a). Find $P(\lambda)$

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 3-\lambda & 2 \\ -1 & 2 & 3-\lambda \end{vmatrix} = -\lambda(\lambda^2 + 8\lambda + 15)$$

Therefore $P(\lambda) = -\lambda(\lambda^2 + 8\lambda + 15). = -\lambda(\lambda - 5)(\lambda - 3).$

(b). Find eigenvalue and eigenspace of A.

$P(\lambda) = 0 \iff sp(A) = \{0, 3, 5\}$

- $\lambda_1 = 0 \Rightarrow E_0 = span\{(1, -1, 1)\}$
- $\lambda_2 = 3 \Rightarrow E_3 = span\{(-2, -1, 1)\}$
- $\lambda_3 = 5 \Rightarrow E_5 = span\{(0, 1, 1)\}$

. Since $P(\lambda)$ is splited and $am(\lambda_i) = gm(\lambda_i)$

Therefore A is diagonalizable.

(c) Find the othogonal matrix P. Such that $P^{-1}AP$ is an orthogonal matrix.

We have $A = PDP^{-1}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $P_1 = \begin{pmatrix} 1 & -2 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Then $P = \begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}$

(d). check for center.

Let $F(x, y, z) = 2x^2 + 3y^2 + 3z^2 + 2xy + 4yz - 2xz + 2x + 14y - 2.$

[redacted]

$$\Rightarrow \begin{cases} F_x = 4x + 2y - 2z + 2 \\ F_y = 6y + 2x + 4z + 14 \\ F_z = 6z + 4y - 2x \end{cases}$$

If $F_x = F_y = F_z = 0 \Rightarrow$ there are no roots for system:

Therefor (S) doesn't have center.

(e). Determiné the type of surface S.

$$(S):\ (x\ y\ z)\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2x + 14y - 2 = 0$$

Let $X = (x, y, z)$

then $XPDP^{-1}X^T + 2x + 14y - 2 = 0$

Let $Y = XP \Rightarrow Y^T = P^{-1}X^T$

we got $YDY^T + 2x + 14y - 2 = 0$

Thus (S) is elliptic parabolic.

28 Determine the type of surface.

$$(S).\ (x-y)^2 + (y-z)^2 + (z-x)^2 = \lambda\ ,\ \lambda \in \mathbb{R}.$$

$$2x^2 \cancel{+y} + 2y^2 + 2z^2 - 2xy - 2xz - 2yz = \lambda.$$

we can write

$$(x\ y\ z)\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda.$$

Let $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $X = (x, y, z)$.

$\Rightarrow P(\lambda) = -\lambda(\lambda-3)^2 \Rightarrow sp(A) = \{0, 3\}$

. for $\lambda_1 = 0 \Rightarrow E_{\lambda_1} = span\{(1,1,1)\}$

. For $\lambda_2 = 3 \Rightarrow E_{\lambda_2} = span\{(-1,1,0), (-1,0,1)\}$

$\Rightarrow am(\lambda_i) = gm(\lambda_i)$, $\lambda = 1, 2$.

Then A is diagonalizable.

Let $v_1 = (1,1,1)$, $v_2 = (-1,1,0)$, $v_3 = (-1,0,1)$.

since $\langle v_2, v_3 \rangle \neq 0$ then $\{v_1, v_2, v_3\}$ not orthgonal basis

. By gramm smrict process

Let $w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = (-1, 1, 0)$.

$w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 = (-\frac{1}{2}, -\frac{1}{2}, 1)$

$\Rightarrow \{w_1, w_2, w_3\}$ is an ortogonal basis for $\mathbb{R}^3$

Let $u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{3}}(1,1,1)$, $u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(-1,1,0)$, $u_3 = \frac{1}{\sqrt{2}}(-$

$\Rightarrow \{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$

since A is diagonalizable then $D = PAP^{-1}$ where

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{2} \end{pmatrix}$

Let $x = PY$, $x^T = PY^T = P^{-1}Y^T$

$\Rightarrow Y^T P^{-1} A P Y = \lambda$

$$(a\ b\ c)\begin{pmatrix}0&0&0\\0&3&0\\0&0&3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\lambda$$

$$3b^2+3c^2=\lambda$$

$$b^2+c^2=\frac{\lambda}{3}$$

Therefore If :

- $\lambda>0$ (S) is elliptic cylinder
- $\lambda=0$ (S) is a line.
- $\lambda<0$ (S) not exits.

I2-TD2
(Ordinary Differential Equations)

1. State the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear; homogeneous or non-nomogeneous.

   (a) $ty^{(3)} + \cos ty'' + y = \sin t$

   (b) $t^3y'' + (1-t^2)(y')^3 + y - \tanh t = 0$

   (c) $y''' - t^3y'' - y.y' = 0$

   (d) $y'' + (2+t)^3y = \sin(y^{(4)})$

2. Wite a differential equation in which $y^2 = 4(t+1)$ is a solution.

3. Solve the following problems.

   (a) $\dfrac{dy}{dt} = \dfrac{t+2\sqrt{t}}{\sin y + ye^y}$

   (b) $ty\dfrac{dy}{dt} = \sqrt{t^2y^2 + t^2 + y^2 + 1}$

   (c) $\cos(2t+y)\,dy = dt$

   (d) $\dfrac{dy}{dt}\tan y = \sin(t+y) + \sin(t-y)$

4. Solve the following problems.

   (a) $\dfrac{dy}{dt} = 2 + \dfrac{y}{t}$

   (b) $3t + 2y\dfrac{dy}{dt} = y$

   (c) $\dfrac{dy}{dt} + \dfrac{t^2+3y^2}{3t^2+y^2} = 0$

   (d) $\left(2t^2y + y^3\right)dt + \left(ty^2 - 2t^3\right)dy = 0$

   (e) $\dfrac{dy}{dt} = \dfrac{y}{t} + t\sin\dfrac{y}{t}$

   (f) $t\cos\left(\dfrac{y}{t}\right)(ydt + tdy) = y\sin\left(\dfrac{y}{t}\right)(tdy - ydt)$

   (g) $(2t - y - 1)\,dy = (3t + y - 4)\,dt$

   (h) $(2t + y - 3)\,dy = (t + 2y - 3)\,dt$

5. The rate at which a body cools is proportional to the difference between the temperature of the body and that of the surrounding air. If a body in air at 25°C will cool from 100° to 75° in one minute, find its temperature at the end of three minutes.

6. The rate at which the ice melts is proportional to the amount of ice at the instant. Find the amount of ice left after 2 hours if half the quantity melts in 30 minutes.

7. If the population of a country doubles in 50 years, in how many years will it treble, assuming that the rate of increase is proportional to the number of inhabitants?

8. The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hours, determine

   (a) The number present after 12 hours.

   (b) The doubling time.

9. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, and assume that no one leaves the campus throughout the duration of the disease, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

10. Solve the following problems.

(a) $(t+2y-11)\,dt+(2t+y-4)\,dy=0$

(b) $\left(1+e^{t/y}\right)+e^{t/y}\left(1-\frac{t}{y}\right)\frac{dy}{dt}=0$

(c) $\left(2ty+3y^2\right)dt+\left(2t^2+y^2\right)dy=0$

(d) $\left(2ty^2+t^2y^3\right)dt+\left(2t+3t^4y^3\right)dy=0$

(e) $(3ty+y+4)\,dt+\frac{t}{2}dy=0$

(f) $3t^2ydt+\left(3t^3-2\right)dy=0, y(0)=1$

(g) $\left(2y^2+3ty\right)dt+\left(t^2+2ty\right)dy=0$

(h) $\left(t^2y-y^4\right)dt+\left(2ty^3-t^3\right)dy=0$

(i) $tdt+ydy+t\,(tdy-ydt)=0$

(j) $\left(t^2+y^2+1\right)dt-2tydy=0$

11. Find the general solutions of the following differential equations

(a) $\left(t^3-t\right)\frac{dy}{dt}-\left(3t^2-1\right)y=t^5-2t^3+t$

(b) $\sin t\frac{dy}{dt}+2y=\tan^3\left(\frac{t}{2}\right)$

(c) $\frac{dy}{dt}+\frac{y}{t}=t^2y^6$

(d) $t^2\frac{dy}{dt}+ty=-y^{-3/2}, y(1)=1$

(e) $t^3\frac{dy}{dt}+2t^2y=y^{-3}, y(1)=1$

(f) $t\frac{dy}{dt}-y^2+(2t+1)\,y=t^2+2t$

(g) $\frac{dy}{dt}+e^{-t}y^2=y+e^t, y(0)=6$

(h) $t\frac{dy}{dt}+y\log y=tye^t$; (Hint: let $u=\ln y$)

(i) $\frac{dy}{dt}=\frac{y^2}{t^2}-\frac{y}{t}+1, y(1)=3$

12. Consider the differential equation

$$y^{-1}y'+p(t)\ln y=q(t), \tag{1}$$

where $p(t)$ and $q(t)$ are continuous functions on some interval (a,b). Show that the change of variables $u=\ln y$ reduces Equation (1) to the linear differential equation

$$u'+p(t)u=q(t),$$

and hence show that the general solution to Equation (1) is

$$y(t)=\exp\left\{I^{-1}\left[\int I(t)q(t)dt+c\right]\right\}$$

where

$$I=e^{\int p(t)dt}$$

and c is an arbitrary constant.

13. Use the technique derived in the previous problem to solve the initial-value problem

$$y^{-1}y'-2t^{-1}\ln y=t^{-1}(1-2\ln t)$$
$$y(1)=e$$

14. Under certain conditions, cane sugar is converted into dextrose at a rate, which is proportional to the amount unconverted at any time. If out of 75 grams of sugar at $t=0$, 8 grams are converted during the first 3 minutes, find the amount converted in 1.5 hours.

15. Integrate the following differential equations.

(a) $3y'' - 2y' - 8y = 0$

(b) $y'' + 2y' + y = 0$

(c) $y'' - 4y' + 3y = 0, y(0) = 6, y'(0) = 10$

(d) $y''' + 3y'' + 7y' + 5y = 0$

(e) $y^{(4)} + 10y''' + 35y'' + 50y' + 24y = 0$

(f) $y^{(4)} + 5y''' + 13y'' + 19y' + 10y = 0$

16. By not solving for a solution, determine the form of particular solution, y_p, of the following differential equations.

(a) $y'' + 2y' + y = 2te^t + e^{2t}$

(b) $y'' + 2y' + y = 2te^{-t}$

(c) $y'' + 3y' + 2y = 3t^2e^{-2t} + (t+1)e^{2t}$

(d) $y'' - 2y' + 2y = 5t^2e^t \sin t + 4e^{-t}$

(e) $y^{(3)} + 3y'' + 4y' + 2y = 2te^{-t} + 3t^2e^t - 5t^3e^{-t}\cos t$

(f) $y^{(3)} + 3y'' + 7y' + 5y = t^{2m}e^{-t} + 6t^m e^{-t}\sin t\cos t, m = 0, 1, 2, 3, ...$

17. Integrate the following differential equations.

(a) $y'' + 2y' + y = 2te^{-t}$

(b) $y'' + 3y' + 2y = 2te^t + 3e^{-2t}$

(c) $y'' + 2y' + 5y = 2e^{-t}\sin 2t$

(d) $y^{(3)} + y'' - 5y' + 3y = 2e^{-3t} + e^t, y(0) = -2, y'(0) = 1, y''(0) = 1$

(e) $y^{(3)} - 2y'' + y' - 2y = t^2 - 2t + 4 - 3\cos t$

(f) $y^{(3)} + 6y'' + 12y' + 8y = e^t - 3\sin t - 8e^{-2t}$

(g) $y^{(3)} - 5y'' + y' + 7y = -\cos 3t, y(0) = -2, y'(0) = 1, y''(0) = 0$

18. Find the general solution of the following differential equations if given y_1 is a solution.

(a) $t^2y'' - 3ty' + 4y = 0; y_1 = t^2$

(b) $y'' + y' + e^{-2t}y = 0; y_1 = \cos(e^{-t})$

(c) $t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0; y_1 = \frac{\cos t}{\sqrt{t}}$

(d) $(1-t^2)y'' + 2ty' - 2y = 0; y_1 = t$

(e) $(2t^2+1)y'' - 4ty' + 4y = 0; y_1 = t$

(f) $t^2y'' - (2t^2\tan t + 2t)y' + (2t\tan t + 2)y = 0; y_1 = t$

19. Integrate the following differential equations.

(a) $y'' + 2y' + y = t^{-1}e^{-t}$

(b) $y'' + y = (\sin t)^{-1}$

(c) $y'' + 2y' + 2y = e^{-t}(\sin t)^{-1}$

(d) $t^2y'' - 4ty' + 4y = t^4 + t^2, t > 0$

(e) $y'' + y' + e^{-2t}y = e^{-3t}$

(f) $t^3y^{(3)} - 3t^2y'' + 7ty' - 8y = 8\ln t + 12, y(1) = -2, y'(1) = 3, y''(1) = 0$

(g) $y^{(3)} + 2y'' - y' - 2y = 4e^{-t^2}(-4t^3 + 4t^2 + 7t - 3)$

(h) $y^{(3)} + y'' + 4y' + 4y = 2e^{t^2}(4t^4 + 2t^3 + 16t^2 + 5t + 5)$

Solution

①. Determine whether the equation is linear or nonlinear homogeneous or non-homogeneous.

a). $t\,y^{(3)} + \cos t\, y'' + y = \sin t$

3^{th} order, linear and non-homogeneous

b). $t^3 y'' + (1-t^2)(y')^3 + y - \tanh t = 0$

$\cancel{3}\ 2^{th}$ order, nonlinear and non-homogeneous.

c). $y''' - t^3 y'' - y \cdot y' = 0$

3^{th} order, nonlinear and homogeneous

d). $y'' + (2+t)^3 y = \sin(y^{(4)})$

2^{th} order, ~~non~~linear, homogeneous

②. Write a differential equation

We have $y^2 = 4(t+1)$

$$\frac{dy^2}{dt} = \frac{d}{dt}4(t+1)$$

$$2y'y' = 4 \Rightarrow y' - \frac{2}{y} = 0$$

$$y\frac{dy}{dt} = 2$$

$$\int y\,dy = \int 2\,dt$$

$$\frac{y^2}{2} = 2t + C$$

$$y^2 = 4t + C \ ; \ C \in \mathbb{R}$$

Take $t = 0 \Rightarrow y^2 = 4 \Rightarrow y = 2$

$$2^2 = C = 4$$

$$\Rightarrow y^2 = 4t + 4$$

$$y^2 = 4(t+1)$$

Thus: $\boxed{\text{The differential equation is } y' - \frac{2}{y} = 0 \ ; \ y^2(0) = 4}$

③. Solve the following problem

a). $\dfrac{dy}{dt} = \dfrac{t + 2\sqrt{t}}{\sin y + ye^y}$

$$\int (\sin y + ye^y)\, dy = \int (t + 2\sqrt{t})\, dt$$

$$-\cos y + (y-1)e^y = \frac{t^2}{2} + \frac{1}{\sqrt{t}} + C$$

Thus $(y-1)e^y - \cos y = \dfrac{t^2}{2} + \dfrac{1}{\sqrt{t}} + C$

b). $ty\dfrac{dy}{dt} = \sqrt{t^2y^2 + t^2 + y^2 + 1}$

$$\cancel{t^2(y^2+1) + (y^2+1)}$$

$$ty\frac{dy}{dt} = \sqrt{(t^2+1)(y^2+1)}$$

$$\int \frac{y}{\sqrt{y^2+1}}\, dy = \int \frac{\sqrt{t^2+1}}{t}\, dt$$

$$\boxed{\sqrt{y^2+1} = \sqrt{t^2+1} - \arctan\sqrt{t^2+1} + C} \quad C \in \mathbb{R}$$

c). $\cos(2t+y)\,dy = dt$

$$\frac{dy}{dt} = \frac{1}{\cos(2t+y)} \quad (1)$$

Let $u = 2t + y \Rightarrow \dfrac{dy}{dt} = \dfrac{du}{dt} - 2$

(1): $\dfrac{du}{dt} = 2 + \dfrac{1}{\cos u}$

$$\frac{du}{dt} = \frac{2\cos u + 1}{\cos u}$$

$$\int dt = \int \frac{\cos u}{2\cos u + 1}\, du$$

$$t = \frac{1}{2}u - \frac{1}{2}\int \frac{1}{2\cos u + 1}\, du = \frac{1}{2}u - \frac{1}{2\sqrt{3}}\left[\ln\left(\frac{\tan u/2}{\sqrt{3}} + 1\right) - \ln\left(\frac{\tan u/2}{\sqrt{3}}\right)\right] +$$

$$\boxed{t = \frac{1}{6}\left[\sqrt{3}\ln\frac{(\sqrt{3} - \tan\frac{2t+y}{2})}{(\tan\frac{2t+y}{2} + \sqrt{3})} + 6\tan^{-1}\tan\frac{2t+y}{2}\right] + C}$$

d). $\frac{dy}{dt}\tan y = \sin(t+y) + \sin(t-y)$

$= 2\sin t\cos y$

$$\int \frac{\tan y}{\cos y}\,dy = \int 2\sin t\,dt$$

$$\boxed{\frac{1}{\cos y} = -2\cos t + C \; ; \; C \in \mathbb{R}}$$

4. Solve the following problem

a). $\frac{dy}{dt} = 2 + \frac{y}{t}$ (1)

Let $u = \frac{y}{t} \Rightarrow y = ut$

$dy = t\,du + u\,dt$

$\frac{dy}{dt} = t\frac{du}{dt} + u$

(1): $t\frac{du}{dt} + u = 2 + u$

$t\,du = 2\,dt$

$\int du = 2\int \frac{1}{t}\,dt$

$u = 2\ln|t| + C \; ; C \in \mathbb{R}$

$\frac{y}{t} = 2\ln|t| + C$

$$\boxed{y = 2t\ln|t| + C \; ; \; C \in \mathbb{R}}$$

b). $3t + 2y\frac{dy}{dt} = y$

$\frac{dy}{dt} = \frac{1}{2} - \frac{3t}{2y}$ (1)

Let $u = \frac{y}{t} \Rightarrow \frac{dy}{dt} = t\frac{du}{dt} + u$

(1): $t\frac{du}{dt} = \frac{1}{2} - \frac{3t}{2y} - u = \frac{1}{2} - \frac{3}{2u} - u = \frac{-4u^2 + u - 6}{2u}$

$$\int \frac{2u}{-4u^2 + u - 6}\,du = \int \frac{1}{t}\,dt = \ln|t| + C \; ; \; C \in \mathbb{R}$$

$$\Rightarrow \ln|t| + C = -\frac{1}{4}\ln|4u^2 \mp u + 6| + \frac{2}{\sqrt{95}}\arctan\left(\frac{8u-1}{\sqrt{95}}\right)$$

$$\boxed{\ln|t| + C = -\frac{1}{4}\ln\left|4\left(\frac{y}{t}\right)^2 - \frac{y}{t} + 6\right| + \frac{2}{\sqrt{95}}\arctan\left(\frac{2y-t}{t\sqrt{95}}\right)}$$

c) $\frac{dy}{dt} = \frac{t^2+3y^2}{3t^2+y^2} = 0$

$$\frac{dy}{dt} = -\frac{t^2+3y^2}{3t^2+y^2} = -\frac{3\left(\frac{y}{t}\right)^2+1}{3+\left(\frac{y}{t}\right)^2}$$

Let $y/t = u \Rightarrow y = ut$

$$dy = u\,dt + t\,du$$

$$\frac{dy}{dt} = u + t\frac{du}{dt}$$

$$\Rightarrow \quad u + t\frac{du}{dt} = -\frac{3u^2+1}{3+u^2}$$

$$t\frac{du}{dt} = \frac{-3u^2-1-3u-u^3}{3+u^2}$$

$$\frac{3+u^2}{-3u^2-1-3u-u^3}\,du = t^{-1}dt$$

$$\frac{dt}{\cancel{d}t} = -\frac{(u^2+3)}{(u+1)^3}\,du$$

$$\int -\frac{dt}{t} = \int\left(\frac{1}{u+1} - \frac{2}{(u+1)^2} + \frac{4}{(u+1)^3}\right)du$$

$$-\ln|t| + C = \ln|u+1| - \frac{2}{u+1} + \frac{2}{(u+1)^2}$$

$$\boxed{-\ln|t| + C = \ln\left|\frac{y}{t}+1\right| - \frac{2t}{y+t} + \frac{2}{\left(\frac{y}{t}+1\right)^2}}$$

d). $(2t^2y+y^3)dt+(y^2t-2t^3)dy=0$

$$\frac{dy}{dt}=-\frac{2t^2y+y^3}{ty^2-2t^3}=-\frac{2\left(\frac{y}{t}\right)+\left(\frac{y}{t}\right)^3}{\left(\frac{y}{t}\right)^2-2}$$

Let $u=\frac{y}{t}\Rightarrow y=ut$

$$dy=u\,dt+t\,du$$

$$\frac{dy}{dt}=u+t\frac{du}{dt}$$

$$\Rightarrow\quad u+t\frac{du}{dt}=-\frac{2u+u^3}{u^2-2}$$

$$t\frac{du}{dt}=\frac{-2u-u^3-u^3+2u}{u^2-2}=\frac{-2u^3}{u^2-2}$$

$$\int\frac{u^2-2}{2u^3}du=\int-\left(\frac{dt}{t}\right)$$

$$\frac{1}{2}\ln|u|+\frac{1}{2u^2}=-\ln|t|+C$$

$$\frac{1}{2}\ln\left|\frac{y}{t}\right|+\frac{1}{2}\left(\frac{t}{y}\right)^2=-\ln|t|+C$$

$$\boxed{\ln|yt|+\frac{t^2}{y^2}=C\ ;\ C\in\mathbb{R}}$$

e). $\dfrac{dy}{dt}=\dfrac{y}{t}+t\sin\dfrac{y}{t}$

Let $u=\frac{y}{t}\Rightarrow\frac{dy}{dt}=u+t\frac{du}{dt}$

$$\Rightarrow\quad u+t\frac{du}{dt}=u+t\sin u$$

$$\int\frac{1}{\sin u}du=\int dt$$

$$t=\ln\left(\tan\left(\frac{u}{2}\right)\right)+C\ ;\ C\in\mathbb{R}$$

$$\boxed{t=\ln\left(\tan\left(\frac{y}{2t}\right)\right)+C\ ;C\in\mathbb{R}}$$

f). $t\cos\left(\frac{y}{t}\right)(y\,dt+t\,dy)=y\sin\left(\frac{y}{t}\right)(t\,dy-y\,dt)$

$$t\cos\left(\frac{y}{t}\right)y\,dt+y^2\sin\left(\frac{y}{t}\right)dt+t^2\cos\left(\frac{y}{t}\right)-ty\sin\left(\frac{y}{t}\right)dy=0$$

multiply bothside by $t^2\cos\left(\frac{y}{t}\right)$

$$\Rightarrow \left(\frac{y}{t}+\left(\frac{y}{t}\right)^2\tan\frac{y}{t}\right)\cancel{dy}\,dt+dy\left(1-\frac{y}{t}\tan\frac{y}{t}\right)=0$$

$$\frac{dy}{dt}=\frac{\frac{y}{t}+\frac{y^2}{t^2}\tan(y/t)}{(y/t)\tan y/t-1}$$

Let $u=\frac{y}{t} \Rightarrow \frac{du}{dt}=u+t\frac{du}{dt}$

$$\Rightarrow u+t\frac{du}{dt}=\frac{u+u^2\tan u}{u\tan u-1}$$

$$\frac{t\,dy}{dt}=\frac{u+u^2\tan u-u^2\tan u+u}{u\tan u-1}$$

$$\int\frac{1}{t}\,dt=\int\frac{u\tan u-1}{2u}\,du$$

$$\ln|t|+c=-\ln\left(\frac{\cos u}{2}\right)-\ln\left(\frac{u}{2}\right)$$

$$\ln|t|+c=-\ln\left(\frac{\cos\frac{y}{t}}{2}\right)-\ln\left(\frac{y}{2t}\right)$$

$$\boxed{y\,t\cos\left(\frac{y}{t}\right)=C\ ;\ C\in\mathbb{R}}$$

g). $(2t-y-1)dy=(3t+y-4)\,dt$

$$\frac{dy}{dt}=\frac{3t+y-4}{2t-y-1}$$

by $\begin{vmatrix}3 & 1\\ 2 & -1\end{vmatrix}=-3-2=-5\neq 0$

choose α,β such that $\left\{\begin{array}{l}3\alpha+\beta-4=0\\ 2\alpha-\beta-1=0\end{array}\right. \Rightarrow \alpha=\beta=1$

Let $\left.\begin{array}{l}t=u+1\\ y=v+1\end{array}\right| \Rightarrow \begin{array}{l}dt=du\\ dy=dv\end{array}$

$$\frac{dy}{dt}=\frac{dv}{du}=\frac{3u+3+v+1-4}{2u+2-v-1-1}=\frac{3u+v}{2u+v}$$

Let $K = \frac{V}{u} \Rightarrow V = Ku$

$dv = Kdu + udK$

$\frac{dv}{du} = K + u\frac{dK}{du}$

$\Rightarrow K + u\frac{dK}{du} = \frac{3K+1}{2K-1}$

$u\frac{dK}{du} = \frac{3K+1-2K^2-K}{2K-1} = \frac{-2K^2+4K+1}{2K-1}$

$\int \frac{2K-1}{-2K^2+4K+1}\,dK = \int \frac{1}{u}\,du$

$\Rightarrow \ln|u| + C = -\frac{1}{2}\ln|2K^2-4K+1| - \frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}K-\sqrt{2}-1}{\sqrt{2}K-\sqrt{2}+1}\right|$

by $K = \frac{V}{u}$ but $V = y-1$; $u = t-1$

$\Rightarrow K = \frac{y-1}{t-1}$

$\Rightarrow \ln\left|\frac{y-1}{t-1}\right| + C = -\frac{1}{2}\ln\left|2\left(\frac{y-1}{t-1}\right)^2 - 4\left(\frac{y-1}{t-1}\right)+1\right| - \frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}\left(\frac{y-1}{t-1}\right)-\sqrt{2}-1}{\sqrt{2}\left(\frac{y-1}{t-1}\right)-\sqrt{2}+1}\right|$

h). $(2t+y-3)dy = (t+2y-3)dt$

$\frac{dy}{dt} = \frac{t+2y-3}{2t+y-3}$

by $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3 \neq 0$

choose α, β such that $\begin{cases} \alpha+2\beta-3=0 \\ 2\alpha+\beta-3=0 \end{cases} \Rightarrow \alpha = \beta = 1$

Let $\begin{cases} t = u+1 \\ y = V+1 \end{cases} \Rightarrow \begin{cases} dt = du \\ dy = dv \end{cases} \Rightarrow \frac{dy}{dt} = \frac{dv}{du}$

$\Rightarrow \frac{dv}{du} = \frac{u+1+2(V+1)-3}{2(u+1)+V+1-3} = \frac{u+2V}{2u+V}$

Let $K = \frac{V}{u} \Rightarrow V = Ku$

$dv = Kdu + udv$

$\frac{dv}{du} = K + u\frac{dK}{du}$

$$\Rightarrow K + u\frac{dK}{du} = \frac{2K+1}{K+2}$$

$$u\frac{dK}{du} = \frac{1-K^2}{K+2}$$

$$\frac{K+2}{1-K^2}dK = \frac{1}{u}du$$

$$\ln|u| + C = -\frac{3}{2}\ln(1-K) + \frac{1}{2}\ln(1+K)$$

$$\boxed{\ln|t-1| + C = -\frac{3}{2}\ln\left|1 - \frac{y-1}{t-1}\right| + \frac{1}{2}\ln\left|1 + \frac{y-1}{t-1}\right|}$$

6. Find the amout of ice left after 2 hour

Let m be the amout of ice at anytime t

$$\frac{dm}{dt} = Km \Rightarrow \frac{dm}{m} = kdt$$

$$\int\frac{dm}{m} = \int kdt$$

$$\ln|m| = Kt + C \,;\, C \in \mathbb{R}$$

At $t = 0$, $m = M$

$$\ln|m| = C$$

$$\Rightarrow \ln|m| = Kt + \ln|m|$$

At $t = \frac{1}{2}h$; $m = \frac{M}{2}$

$$\ln\left|\frac{m}{2}\right| = \frac{K}{2} + \ln|M|$$

$$\Rightarrow \ln\left|\frac{M}{2M}\right| = \frac{K}{2} \rightarrow K = 2\ln\left|\frac{1}{2}\right)$$

We have $\ln|m| = 4\ln\frac{1}{2} + \ln|M|$

$$\ln\left(\frac{m}{M}\right) = \ln\frac{1}{16}$$

$$\frac{m}{M} = \frac{1}{16}$$

Thus: After 2 hour amount of ice left $= \frac{1}{16}$ of the amount of ice at the begin.

⑤. Given $\frac{dT}{dt} = k(T_t - T_s)$

where $\frac{dT}{dt}$ is rate of cooling

T_t is temperative at time t

T_s is temperative of surrounding

$$T(t) = T_s + (T_0 - T_s).\, e^{-kt}$$

where T_0 is initial temperative of body

We have $T_s = 20°C$; $T(t) = 60°C$; $t = 20$ mins

Substituting the value in equation

$$60°C = 20°C + (100-20)e^{-k\times 20}$$

$$e^{-20k} = \frac{1}{2} \Rightarrow k = \frac{\ln 2}{20}$$

Then the temperature drops to $30°C$

$$30 = 20 + (60 - 20)\, e^{-kt}$$

$$e^{-kt} = \frac{1}{4}$$

$$-kt = \ln 4^{-1}$$

$$t = 20 \frac{\ln 4^{-1}}{\ln 2} = 40 \text{mins}$$

Therefore
$$\boxed{\begin{array}{l}\text{it takes 40 minute more} \\ \text{Total time} = 20+40 = 60\text{min}\end{array}}$$

⑦ Let y = numbere of the population at time t years

$t = 1$; $y = y_0$

$t = 50$; $y = 2y_0$

$t = ?$; $y = 3y_0$

Then, the rate of increase of population is proportional to y

$\frac{dy}{dt} = ky$; k is constant

For $t = 0$; $y = y_0$ and $t = 50$; $y = 2y_0$

$$\int_{y_0}^{2y_0} \frac{1}{y} dy = k\int_{10}^{50} dt$$

$$\ln\left|\frac{2y_0}{y_0}\right| = 50k \Rightarrow k = \frac{\ln 2}{50}$$

. For $t = 0$; $y = y_0$ and $t = ?$; $y = 3y_0$

$$\int_{y_0}^{3y_0} \frac{1}{y} dy = \frac{\ln 2}{50} \int_0^t dt$$

$$\ln \left| \frac{3y_0}{y_0} \right| = \frac{\ln 2}{50} t$$

$$t = \frac{\ln 3}{\ln 2} \times 50 = 79.24 \simeq 79 \text{ years}$$

$$\boxed{t = 79 \text{ years}}$$

8.) Determine

a.) the number present after 12h.

We have $\frac{dp}{dt} = kp \Leftrightarrow \frac{dp}{p} = kdt$

$\Leftrightarrow \int \frac{dp}{p} = \int kdt \Leftrightarrow \ln|P| = kt + C$

$\Rightarrow P = \pm e^{kt+C} \times P_0 = P_0 e^{kt}$; let $P_0 = \pm e^{C}$.

- for $t=0$; we have $P_0 = P(t=0) = 500$

$\Rightarrow P(t) = 500e^{kt}$

- for $t = 2s$; we have $P(t=2s) = 500 \times e^{2k} = 2000$, then $e^{2k} = 4 = e^{\ln 4}$

$\Rightarrow k = \ln 2$.

then, $P(t) = 500e^{t\ln 2}$

$\Rightarrow P(t=12h) = 500\,e^{12\ln 2} = 500 \times 2^{12}$

$= 2048000$

b.) determine doubling time:

$P(t) = 500e^{t\ln 2}$

$\Rightarrow P_d(t) = 1000\,e^{t\ln 2} = 2000$

$\Rightarrow e^{t\ln 2} = 2 = e^{\ln 2}$

$\Rightarrow t = 1h$

9

$$\frac{dx}{dt} = kx(1000-x)$$

$$\frac{dx}{x(1000-x)} = k\,dt \quad -(\text{I})$$

Put $\frac{1}{x(1000-x)} = \frac{A}{x} + \frac{B}{1000-x}$

$\Rightarrow \quad 1 = A(1000-x) + Bx$

$\Rightarrow \quad 1 = A(1000) + x(B-A)$

On Comparing

$A = \frac{1}{1000}$ & $B - A = 0$

$B = A = \frac{1}{1000}$

So, equation (I) becomes -

$$\Rightarrow \left[\frac{1}{1000x} + \frac{1}{1000(1000-x)}\right] = k\,dt$$

On integrating -

$$\Rightarrow \frac{1}{1000}\left[\ln(x) - \ln(1000-x)\right] = kt + C$$

$$\Rightarrow \ln\left(\frac{x}{1000-x}\right) = 1000(kt + C) - (\text{II})$$

Put $t = 0, x = 1$

$$\ln\left(\frac{1}{999}\right) = 1000\,C$$

$$\Rightarrow C = \frac{1}{1000}\ln\left(\frac{1}{999}\right)$$

Put $t = 4, x = 50$ in (II)

$$\Rightarrow \ln\left(\frac{1}{19}\right) = 1000\left(4k + \frac{1}{1000}\ln\left(\frac{1}{999}\right)\right)$$

$\Rightarrow k = 0.00099057$

So, (II) becomes -

$$\ln\left(\frac{x}{1000-x}\right) = 1000\left(0.00099057\,t + \frac{1}{1000}\ln\left(\frac{1}{999}\right)\right)$$

Now, Put $t = 6$

$$\ln\left(\frac{x}{1000-x}\right) = -0.96328$$

Taking exponential -

$$\Rightarrow \frac{x}{1000-x} = e^{-0.96328} = 0.38163$$

$\Rightarrow x = 276.2$

or $\boxed{x \approx 276 \text{ students}}$

10 Solve the following problems:

(a). $\frac{dy}{dt} = \frac{-2t\,8y^3-2}{3t^2y^2+e^y}$, $y(-1)=0$

$\Leftrightarrow (3t^2y^2+e^y)\,dy = (-2ty^3-2)dt$ (1)

$M(t,y) = 3t^2y^2+e^y$, $N(t,y) = (-2ty^3-2)$

(1) is exact

$\Leftrightarrow \begin{cases} \frac{\partial F}{\partial m} = M = 3t^2y^2+e^y & (1) \\ \frac{\partial F}{\partial y} = N = -2ty^3-2. & (2) \end{cases}$

by (1) and (2)

Therefore: $\underline{t^2y^3+2t+e^y = c \quad , c\in\mathbb{R}}$

(b). $(2ty+3y^2)\,dt + (2t^2+y^2)\,dy = 0$

Let $M = 2ty+3y^2$, $N = 2t^2+y^2$

Let $y = ut \Rightarrow \frac{dy}{dt} = u + t\frac{du}{dt}$

so : $(2ut^2+3u^2t^2)\left(u+t\frac{du}{dt}\right) + (2t^2+u^2t^2) = 0$

$\Leftrightarrow \frac{1}{t}\,dt = \frac{2u+3u^2}{-2-3u^2-3u^3}$

$-\ln|t| = \int \frac{2u+3u^2}{3u^3+3u^2+2}\,du = \frac{1}{3}\ln\left|u^3+u^2+\frac{2}{3}\right| + c$

Therefore $\underline{-\ln|t| = \frac{1}{3}\ln\left|\left(\frac{y}{t}\right)^3+\left(\frac{y}{t}\right)^2+\frac{2}{3}\right| + c \quad , c\in\mathbb{R}}$

(c). $(2ty^2 + t^2y^3)dt + (2t + 3t^4y^3)\,dy = 0$

Let $M = 2ty^2 + t^2y^3$, $N = (2t + 3t^4y^3)$

but $M(t,y) = y(2ty + t^2y^2) = y\alpha(ty)$

$N(t,y) = t(2 + 3t^3y^3) = t\beta(ty)$.

Then $I(t,y) = \dfrac{1}{tM(t,y) + yN(t,y)} = \dfrac{1}{2t^2y^2 + t^3y^3 - 2yt - 3\cdot}$

so $IM(t,y)dt + IN(t,y)dy = 0$ is said to be exact

$$\text{so} \begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{2y + ty^2}{2ty + t^2y^2 - 2 - 3t^3y^3} & (1) \\[2ex] \dfrac{\partial F}{\partial y} = \dfrac{2 + 3t^3y^3}{2ty^2 + t^2y^3 - 2y - 2t^3y^4} & (2). \end{cases}$$

(d) $(3ty+y+4)dt + \frac{t}{2}dy = 0.$

Let $\begin{cases} M(t,y) = 3ty+y+4 \\ N(t,y) = t/2 \end{cases} \Rightarrow \begin{cases} M_y = 3t+1 \\ N_t = \frac{1}{2} \end{cases}$

$\frac{M_y - N_t}{N} = \frac{3t + 1/2}{t/2} = \frac{6t+1}{t} = d(t).$

So $I(t) = e^{\int d(t)dt} = e^{\int(6+1/t)dt} = te^{6t}$

then: $te^{6t}(3ty+y+4)dt + \frac{t^2e^{6t}}{2}dy = 0$ is exact

$\Rightarrow \begin{cases} \frac{\partial F}{\partial x} = (3ty+y+4)te^{6t} & (1) \\ \frac{\partial F}{\partial y} = \frac{t^2e^{6t}}{2} & (2) \end{cases}$

by (1) and (2).

Therefore: $\underline{\frac{t^2}{2}e^{6t}y + \frac{2}{3}te^{6t} - \frac{1}{6}e^{6t} + c, \ c \in \mathbb{R}.}$

(e). $3t^2y\,dt + (3t^3-2)dy = 0, \ y(0) = 1.$

Let $M = 3t^2y$, $N = 3t^3-2.$

$\begin{cases} M_y = 3t^2 \\ N_t = 6t \end{cases} \Rightarrow \frac{M_y - N_t}{N} = \frac{3t^2-6t}{3t^2-2} = d(t).$

$I = e^{\int d\,dt}$ ... It's abit hard :(

(i). $(t^2+y^2+1)dt - 2ty\,dy = 0$.

$M = t+y+1$, $N = -2ty$

$\Rightarrow M_y = 2t$, $N_t = -2y$

$M_y - N_t = 2(t+y)$.

$-M+N = -(t+y)^2 - 1$.

So: $\dfrac{M_y - N_t}{N-M} = \dfrac{2(t+y)}{-(t+y)^2-1} = \dfrac{2u}{-u^2-1}$, $u = t+y$

Then $I(t,y) = e^{\int B(u)du}$

11 Find the general solution of the following equations.

(a) $\frac{dy}{\partial t} + 2ty = e^{-t^2}$

$dy + 2tydt - e^{-t^2} dt = 0$

$(2ty - e^{-t^2})dt + dy = 0$

$\begin{cases} M = 2ty - e^{-t^2} \\ N = 1 \end{cases} \Rightarrow \begin{cases} M_y = 2t \\ N_t = 0 \end{cases} \Rightarrow M_y - N_t = 2t.$

So $\frac{M_y - N_t}{N} = 2t = d(t) \Rightarrow I(t,y) = e^{t^2}$

Then $(2tye^{t^2} - 1)dt + e^{t^2}dy = 0$ is said to be exact

$\Rightarrow \begin{cases} F_t = 2tye^{t^2} - 1 & (1) \\ F_y = e^{t^2} & (2) \end{cases}$

by (1) and (2).

Therefore $\underline{ye^{t^2} - t + c = 0 \quad , c \in \mathbb{R}}$.

(b) $\frac{dy}{dt} + \frac{y}{t-2} = 3t \quad , y(3) = 1.$ (1).

$(t-2)dy + ydt = (3t^2 - 6t)dt$

$(y + 6t - 3t^2)dt + (t-2)dy = 0$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1. \Rightarrow$ (1) is exact

$\Rightarrow \begin{cases} F_t = y + 6t - 3t^2 & (i) \\ F_y = t - 2 & (ii) \end{cases}$

by (1) and (2)

Therefore: $\underline{(t-2)y + 3t^2 - t^3 + C = 0, C \in \mathbb{R}}$.

(c). $\frac{dy}{dt} + \frac{y}{t} \cancel{+ 2ty} = t^n$ (*)

$t\,dy + y\,dt \cancel{+ 2t^2 y\,dt} = t^{n+1} dt$

$t\,dy + (y - t^{n+1})\,dt = 0$

$\begin{cases} M = y - t^{n+1} \\ N = t \end{cases} \Rightarrow \begin{cases} M_y = 1 \\ N_t = 1 \end{cases} \Rightarrow *$ is exact.

$\Leftrightarrow \begin{cases} F_y = t & (1) \\ F_t = y - t^{n+1}. & (2) \end{cases}$

by (1) and (2)

Therefore: $ty + c = \frac{t^{n+2}}{n+2}$, $c \in \mathbb{R}$.

(d). $(t^2 - 1)\frac{dy}{dt} + 2ty = 1.$

$(t^2-1)\,dy + 2ty\,dt = dt$

$(t^2-1)\,dy + (2ty - 1)\,dt = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = 2t$

$\Rightarrow \exists F \Leftrightarrow \begin{cases} \frac{\partial F}{\partial y} = t^2 + 1 & (1) \\ \frac{\partial F}{\partial t} = 2ty - 1 & (2) \end{cases}$

by (1) and (2)

Therefore: $t^2 y + (-t + c) = 0$, $c \in \mathbb{R}$.

(e). $(2t - 10y^3)\frac{dy}{dt} + y = 0$

$\Leftrightarrow (2t - 10y^3)dy + y\,dt = 0$

$\begin{cases} M = y \\ N = 2t - 10y^3 \end{cases} \Rightarrow \begin{cases} M_y = 1 \\ N_t = 2 \end{cases} \Rightarrow I = e^{\int \frac{1}{y} dy} = y$

$y^2 dt + (2ty - 10y^4)dy = 0$ is exact.

$\Rightarrow \begin{cases} f_t = y^2 & (1) \\ f_y = 2ty - 10y^4 & (2) \end{cases}$

by (1) and (2)

Therefore: $2ty - 2y^5 + c = 0 \quad , c \in \mathbb{R}$

(f). $\frac{dy}{dt} + \frac{y}{t} = t^2 y^6$

$y^{-6}\frac{dy}{dt} + \frac{1}{t}y^{-5} = t^2 \quad$, Let $v = y^{-5}$, $v' = -5y'y^{-6}$

$\Leftrightarrow \quad v' - \frac{1}{5t}v = t^2\left(-\frac{1}{5}\right)$

$\Rightarrow I(t, v) = e^{\int -\frac{1}{5t} dt} = \frac{1}{t^5}$

Then $v(t) = \dfrac{\int\left(-\frac{t^5}{5} \times \frac{1}{t^5}\right)dt}{1/t^5} + \dfrac{c}{1/t^5} = ct^5 + t^5\left(\frac{1}{10t^2}\right) = ct^5 + \frac{t^3}{10}$

Therefore $\frac{1}{y^5} = \frac{t^3}{10} + ct^5 \quad , c \in \mathbb{R}$

Page...

(g) $t^2 \frac{dy}{dt} + ty = -y^{-3/2}$, $y(1) = 1$.

$y' + \frac{1}{t} y = -\frac{1}{t^2} y^{-3/2}$

$y^{3/2} y' + \frac{1}{t} y^{5/2} = -\frac{1}{t^2}$

Let $y^{5/2} = v \Rightarrow v' = \frac{5}{2} y^{3/2} y' \Rightarrow y^{3/2} y' = \frac{2}{5} v'$

$\Rightarrow \frac{2}{5} v' + \frac{1}{t} v = -\frac{1}{t^2} \Rightarrow v' + \frac{5}{2t} v = -\frac{5}{2t^2}$

then $y^{5/2} = \dfrac{\int \left(-\frac{5}{2t^2}\right)\left(e^{\int 5/2t\, dt}\right) dt}{\int \frac{5}{2t} dt} + \dfrac{c}{e^{\int 5/2t\, dt}}$

Therefore: $\underline{y^{5/2} = t^{-5/2}\left(-\frac{5}{3} t^{3/2}\right) + c\, t^{-5/2}}$, $t \in \mathbb{R}$

(h). $t \frac{dy}{dt} + y = t^2 y^4$.

or $y' + \frac{1}{t} y = t y^4$

$y^{-4} y' + \frac{1}{t} y^{-3} = t \Leftrightarrow v' - \frac{1}{4t} v = -\frac{1}{4} t$

or $-4v' + v = t$.

$v' - \frac{1}{4t} v = -\frac{1}{4} t$

$\Rightarrow y^{-3} = \dfrac{\int \left[\left(-\frac{1}{4} t\right) e^{\int \left(-\frac{1}{4} t\right) dt}\right] dt}{e^{\int \left(-\frac{1}{4t}\right) dt}} + \dfrac{c}{e^{\int -1/4t\, dt}}$

(i) $t^2 \frac{dy}{dt} - ty = y^3 e^t$. $y(0) = 1$. (*)

$y^{-3} \frac{dy}{dt} - \frac{1}{t} y^{-2} = t^{-2} e^t$

Let $u = y^{-2} \Rightarrow \frac{du}{dt} = -2y^{-3} \frac{dy}{dt} \Leftrightarrow -\frac{1}{2} \frac{du}{dt} = y^{-3} \frac{dy}{dt}$

$\Rightarrow -\frac{1}{2} \frac{dy}{dt} - t^{-1} u = t^{-2} e^t$

$\Leftrightarrow \frac{du}{dt} + \frac{2}{t} u = -\frac{2}{t^2} e^t$

$\Rightarrow I(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$

Mutiply $I(t) = t^2$ to (*)

$\Leftrightarrow (t^2 u') = -2e^t$

$t^2 u = -2e^t + C \Rightarrow u = \frac{-2e^t + c}{t^2}$

$\Leftrightarrow y^{-2} = \frac{-2e^t + c}{t^2} \Leftrightarrow y = \frac{t}{\sqrt{-2e^t + c}}$

$y(0) = 1 \Rightarrow c = 2$.

Therefore $y(t) = \frac{t}{\sqrt{-2e^t + 2}}$.

(j) $t^3 \frac{dy}{dt} + 2t^2 y = y^{-3}$, $y(1) = 1$.

$y^3 \frac{dy}{dt} + \frac{2}{t} y^4 = t^{-3}$

Let $u = y^4 \Rightarrow du = 4y^3 \frac{dy}{dt}$

$\frac{1}{4} \frac{du}{dt} + \frac{2}{t} u = t^{-3}$

$I(t) = t^2$

$\Leftrightarrow \frac{t^2}{4} u' + 2tu = \frac{1}{t}$

$(t^2 u)' = \frac{1}{t}$

$t^2 u = \ln|t|$

$y^4 = \frac{\ln t}{t^2} + c$, $c \in \mathbb{R}$

12 Show that the change of $u = \ln y$ reduce equation (1) to the linear differential equation:

$$u' + p(t)u = q(t)$$

Proof: we have $y'y^{-1} + p(t)\ln y = q(t)$

Consider $u = \ln y \Rightarrow \frac{du}{dt} = \frac{d(\ln y)}{dy} \times \frac{dy}{dt}$

$= \frac{1}{y} \times y' = y^{-1}y'$ then substitute onto (1)

we have $u' + p(t)u = q(t)$

* show that the general solution to equation is

$$y(t) = \exp\left\{ I^{-1}\left(\int I(t)q(t)\,dt + c \right) \right\}$$

where $I = e^{\int p(t)dt}$

we have $u' + p(t)u = q(t)$

$I = \int e^{\int p(t)dt}$

then $y(t) = \exp\left\{ I^{-1}\left(\int I(t)q(t)\,dt + c \right) \right\}$.

13) Solve problem $y^{-1}y' - 2t^{-1}y\ln y = t^{-1}(1-2\ln t)$

$y(1) = e.$

Proof: Since from ex (12) we have

$$y(t) = \exp\left\{I^{-1}\left[\int I(t)g(t)\,dt + c\right]\right\}.$$

where $I = e^{\int p(t)dt}$

Since $p(t) = -2t^{-1} = -\frac{2}{t}$; $g(t) = \frac{1-2\ln t}{t}$

$$\Rightarrow y(t) = \exp\left\{\frac{1}{e^{\int \frac{2}{t}dt}}\left[\int e^{\int -\frac{2dt}{t}}\frac{(1-2\ln t)}{t}dt + c\right]\right\}$$

$$= \exp\left\{\frac{1}{t^2}\left[\int \frac{(1-2\ln t)}{t^3}dt + c\right]\right\}$$

$$= \exp\left\{\frac{1}{t^2}\left(-\frac{2}{t^2} - \frac{2\ln t + 1}{4t^2} + K + C\right)\right\}$$

$$= \exp\left(-\frac{2}{t^4} - \frac{2\ln t + 1}{4t^4} + \frac{m}{t^2}\right)$$

where $m = K + C$.

$$\boxed{\text{Therefore } y(t) = \exp\left(-\frac{2}{t^4} - \frac{2\ln t+1}{4t^4} + \frac{m}{t^2}\right)}$$

14

$$C - is\ converted\ sugar\ cane \rightarrow \begin{cases} \frac{dC}{dt}\ is\ conversion\ rate \\ (75 - C)\ is\ unconverted\ sugar\ cane \end{cases}$$

Then, by the condition of the problem

$$\frac{dC}{dt} \sim (75 - C) \rightarrow \frac{dC}{dt} = k(75 - C), \quad where\ k\ is\ the\ coefficient\ of\ proportionality$$

Then,

$$\frac{dC}{dt} = k(75 - C)\Big| \times \left(\frac{dt}{75 - C}\right) \rightarrow \frac{dC}{75 - C} = kdt \rightarrow \int \frac{dC}{75 - C} = \int kdt \rightarrow$$

$$-\ln|75 - C| = kt - \ln|A|| \times (-1) \rightarrow \ln|75 - C| = -kt + \ln|A| \rightarrow$$

$$e^{\ln|75-C|} = e^{-kt+\ln|A|} \rightarrow 75 - C = A \cdot e^{-kt} \rightarrow \boxed{C(t) = 75 - A \cdot e^{-kt}}$$

By the condition of the problem:

$$C(0) = 0 = 75 - A \cdot e^{-k \cdot 0} \rightarrow 75 - A = 0 \rightarrow \boxed{A = 75}$$

$$C(t) = 75 - 75 \cdot e^{-kt} \rightarrow C(t) = 75 \cdot (1 - e^{-kt})$$

$$C(30) = 8 = 75 \cdot (1 - e^{-k \cdot 30}) \rightarrow 1 - e^{-k \cdot 30} = \frac{8}{75} \rightarrow e^{-30k} = 1 - \frac{8}{75} \rightarrow$$

$$e^{-30k} = \frac{75 - 8}{75} \rightarrow \ln|e^{-30k}| = \ln\left|\frac{67}{75}\right| \rightarrow -30k = \ln\left|\frac{67}{75}\right| \rightarrow k = -\frac{1}{30} \cdot \ln\left|\frac{67}{75}\right| \rightarrow$$

$$k = \ln\left|\left(\frac{67}{75}\right)^{-1/30}\right| = \ln\left|\left(\frac{75}{67}\right)^{1/30}\right| \rightarrow \boxed{k = \ln\left|\left(\frac{75}{67}\right)^{1/30}\right|}$$

Then,

$$C(t) = 75 \cdot \left(1 - e^{-t \cdot \ln\left|\left(\frac{75}{67}\right)^{\frac{1}{30}}\right|}\right) = 75 \cdot \left(1 - e^{\ln\left|\left(\frac{75}{67}\right)^{\frac{-t}{30}}\right|}\right) = 75 \cdot \left(1 - \left(\frac{75}{67}\right)^{\frac{-t}{30}}\right) \rightarrow$$

Conclusion,

$$\boxed{C(t) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{t}{30}}\right)}$$

The amount converted in one and half hours is

$$C(90) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{90}{30}}\right) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{3}\right) = \frac{121112}{5625} \approx 21.53(gm)$$

ANSWER:

$$C(t) = 75 \cdot \left(1 - \left(\frac{67}{75}\right)^{\frac{t}{30}}\right)$$

$$C(90) = \frac{121112}{5625} \approx 21.53(gm)$$

(b). Find $W(y_1, y_2)$.

$W(y_1, y_2) = 0$ on $\mathbb{R}$.

we can't say that y_1 and y_2 are L.I.

Conversly If $|W| = 0$ we can't say that

y_1 & y_2 are L.I.

15 Integral this following equations.

(a). $3y'' - 2y' - 8y = 0$

cha. equation $3r^2 - 2r - 8 = 0 \Rightarrow r_1 = 2, r_2 = -\frac{4}{3}$

so $y(t) = Ae^{2t} + Be^{-4/3\, t}$, $A, B \in \mathbb{R}$.

(b) $y'' + 2y' + y = 0$

$\Rightarrow y(t) = (A + Bt)e^{-t}$, $A, B \in \mathbb{R}$.

(c). $y'' - 4y' + 3y = 0 \Rightarrow y(t) = Ae^{t} + Be^{3t}$

$\begin{cases} y(0) = 6 \\ y'(0) = 10 \end{cases} \Rightarrow \begin{cases} A = 4 \\ B = 2 \end{cases}$

Therefore $y(t) = 4e^{t} + 2e^{3t}$

(d). $y''' + 3y'' + 7y' + 5y = 0$

char-equation $r^3 + 3r^2 + 7r + 5 = 0$, $r_1 = -1$, $r_2 = r_3 = -1\pm$

Therefore: $y(t) = Ae^{-t} + e^{-t}(A_1\cos 2t + A_2\sin 2t)$

A, A_1 and A_2 $\in \mathbb{R}$.

(e). $y^{(4)} + 5y''' + 13y'' + 19y' + 10y = 0$

Char-equation: $r^4 + 10r^3 + 35r^2 + 50r + 24 = 0$

$\Leftrightarrow (r+1)(r+2)(-1+2i)(-1-2i)$.

Then: $y(t) = Ae^{-t} + Be^{-2t} + e^{-t}(C\cos 2t + D\sin 2t)$.

Where A, B, C and $D \in \mathbb{R}$.

(f). $y^{(4)} + 10y''' + 35y'' + 50y' + 24y = 0$.

Char-equation $r^4 + 10r + 35r^2 + 50r + 24 = 0$

$\Leftrightarrow (r+1)(r+2)(r+3)(r+4) = 0$

Then: $y(t) = Ae^{-t} + Be^{-2t} + Ce^{-3t} + De^{-4t}$

A, B, C, and $D \in \mathbb{R}$.

16 By not solving for a solution, Determinie the form of particular solution of the following Differential equation.

(a). $y'' + 2y' + y = 2te^{t} + e^{2t}$.

$$y_p = (At + B)e^{t} + Ce^{2t} \quad , A, B, C \in \mathbb{R}.$$

(b). $y'' + 2y' + y = 2te^{-t}$, rooths of cha- equation , $r_{1,2} = -1$.

Then $y_p(t) = t^2(At + B)e^{-t} \quad , A, B \in \mathbb{R}.$

(c). $y'' + 3y' + 2y = 3t^2e^{-2t} + (t+1)e^{2t}$

Char- equation : $r^2 + 3r + 2 = 0$, $r_1 = +1$, $r_2 = +2$.

Then $y_p(t) = (A_1t + B_1)e^{2t} + t(A_2t^2 + B_2t + C)e^{-2t}$.

(d). $y'' - 2y' + 2y = 5t^2e^{t}\sin t + 4e^{-t}$

Cha- equation $r^2 - 2r + 2 = 0$, $r_{1,2} = 1 \pm i$

~~Then~~ $\cancel{y_p(t) = Ae^{-t} + te^{t}[(A_1t^2 + B_1t + C)e^{-2t}]}$

Then $y_p(t) = Ae^{-t} + te^{t}[(A_1t^2 + B_1t + C_1)\sin t + (A_2t^2 + B_2t + C_2)\cos t]$

(e) $y^{(3)} + 3y'' + 4y' + 2y = 2te^{-t} + 3t^2e^{t} - 5t^3e^{-t}\cos t$.

Chur- equation : $r^3 + 3r^2 + 4r + 2 = 0 \Leftrightarrow (r+1)(r+1+i)(r-1+i) = 0$

Therefore $y_p(t) = t(At + B)e^{-t} + (at^2 + B + c)e^{t}$

$$+ te^{-t}[(a_1t^3 + a_2t^2 + a_3t + a_4)\cos t$$

$$+ (b_1t^3 + b_2t^2 + b_3t + b_4)\sin t]$$

(f) $y^{(3)} + 3y'' + 7y' + 5y = t^{2m}e^{-t} + 6t^m e^{-t}\sin t\cos t$, $m = 1, 2, 3 \ldots$

Chu- equation : $r^3 + 3r^2 + 7r + 5 = 0 \Leftrightarrow (r+1)(r+1+2i)(r-1+2i) = 0$

then : $y_p(t) = t(a_{2m}t^{2m} + \ldots + a_0)e^{-t} + t(b_m t^m + \ldots + b_0)e^{-t}\ldots$

$+ t(c_m t^m + \ldots c_0)e^{-t}\sin t.$

17 Integral the following equiition

(a). $y'' + 2y' + y = 2te^{-t}$.

$(D^2 + 2D + 1) = 2te^{-t}$

$(D+1)^2(D^2+2D+1) = 2t(D^2+1)e^{-t}$

$(D+1)^2(D^2+2D+1) = 0 \quad \Leftrightarrow (D+1)^4 = 0$

so : $y(t) = (a_3t^3 + a_2t^2 + a_1t + a_0)e^{-t}$, $a_1 \ldots a_3 \in \mathbb{R}$.

(b). $y'' + 3y' + 2y = 2te^t + 3e^{-2t}$

$\Rightarrow y(t) = Ae^{-t} + Be^{-2t} + (at+b)e^t$, $a, b, A, B \in \mathbb{R}$.

(c). $y'' + 2y' + 5y = 2e^{-t}\sin t$

$y_h = e^{-t}(A\sin 2t + B\cos 2t)$

$y_p(t) = e^{-t}[(A_1t + B_1)\sin 2t + (A_2t + B_2)\cos 2t]$

where $A_1 = 0$, $B_1 = 0$, $A_2 = \frac{-1}{2}$, $B_2 = 0$

(d): $y^{(3)} + y'' - 5y' + 3y = 2e^{-3t} + e^t$, $y(0) = -2$, $y'(0) = 1$, $y''(0) = 1$

homogenous equation : $y^{(3)} + y'' - 5y' + 3y = 0$

$r^3 + r^2 - 5r + 3 = 0$

~~$(r-2)(r^2+1) = 0$~~

$(r-1)^2(r+3)=0 \Rightarrow r_{1,2}=1,\ r_3=-3.$

$y_h(t)=Ae^t+Bte^t+Ce^{-3t}$

Let $y_p(t)=a_2te^{-3t}+a_1t^2e^t+a_3te^t$

where $a_1=\frac{1}{8},\ a_2=\frac{1}{8},\ a_3=c,\ c\in\mathbb{R}.$

(e). $y^{(3)}-2y''+y'-2y=t^2-2t+4-3\cos t$

homogenous equation: $r^3-2r^2+r-2=0$

$(r-2)(r^2+1)=0$

$\Rightarrow y_h(t)=Ae^{2t}+a\cos t+b\sin t,\ A,a,b\in\mathbb{R}.$

Let $y_p(t)=m_1t^2+m_2t+m_3+m_4t\cos t+m_5t\sin t$

Therefore $y(t)=y_p+y_h$, where $m_1=-\frac{1}{2},\ m_2=\frac{1}{2},\ m_3=-\frac{3}{4}$

$m_4=\frac{3}{10},\ m_5=\frac{3}{5},\ A,a,b\in\mathbb{R}.$

(g). $y^{(3)}-5y''+12y'+7y=-\cos 3t$

homogenous equation $y^{(3)}-5y''+12y'+7y=0$

or $r^3-5r^2+12r+7=0$

$(r+1)(r^2-6r+7)=0$

$\Rightarrow r_1=-1,\ r_{2,3}=3\pm\sqrt{2}$

$\Rightarrow y_h(t)=A_1e^{-t}+A_2e^{(3+\sqrt{2})t}+A_3e^{(3-\sqrt{2})t},\ A_1,A_2,A_3\in\mathbb{R}.$

$y_p(t)=A_4\cos 3t+A_5\sin 3t,\ A_4=\frac{3}{410},\ A_5=\frac{-13}{820}$

(8). $y^{(3)}+6y''+12y'+8y=e^{t}-3\sin t-8e^{-2t}$ (1).

homogenous equation: $y^{(3)}+6y''+12y'+8y=0$

$$r^3+6r^2+12r+8=0$$

$$(r+2)^3=0$$

Then, $r_{1,2,3}=-2$

$\Rightarrow y_h(t)=(At^2+Bt+C)e^{-2t}$, $A,B,C\in\mathbb{R}$

Let $y_p(t)=A_1e^{t}+A_2\sin t+A_3\cos t+(a_3t^3+a_2t^2+a_1t+a_0)e^{-}$

Therefore $y(t)=y_p(t)+y_h(t)$

where $A_1=\frac{1}{27}$, $A_2=\frac{6}{125}$, $A_3=\frac{33}{125}$, $a_3=-$

18 Find the general solution of the following diff- equation

(a). $t^2y'' - 3ty' + 4y = 0$, $y_1 = t^2$

Let $y = y_1 u(t)$

$y = t^2 u(t)$

$y' = 2tu(t) + t^2 u'(t)$

$y'' = 2u + 2tu' + 2tu' + t^2u'' = 2u + 4tu' + t^2u''$

$t^2(2u + 4tu' + t^2u'') - 3t(2tu + t^2u') + 4t^2u = 0$

$2t^2u + 4t^3u' + t^4u'' - 6t^2u - 3t^3u' + 4t^2u = 0$

$t^4u'' + t^3u' = 0$

$u'' + \frac{1}{t}u' = 0$

Let $w = u' \Rightarrow w' = u''$

then $w' + \frac{1}{t}w = 0$

Integral factor: $I = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$\Rightarrow tw' + w = 0$

$d(tw) = 0$

$tw = c_1 \Rightarrow w = \frac{c_1}{t} = u'$

$\Rightarrow u = c_1 \ln|t| + c_2 = \frac{y}{t^2}$

$\Rightarrow y = t^2(c_1 \ln|t| + c_2)$

Therefore $\underline{y = t^2(c_1 \ln|t| + c_2)\ ,\ c_1, c_2 \in \mathbb{R}.}$

(b) $y'' + y' + e^{-2t}y = 0$, $y_1 = \cos(e^{-t})$

Let $y = y_1 u$

$y' = y_1' u + y_1 u'$

$y'' = y_1'' u + 2y_1' u' + y_1 u''$

$\Leftrightarrow$ Integral $y'' + P(t)y' + Q(t)y = 0$

$\Leftrightarrow u(y_1'' + Py_1' + Qy_1) + y_1 u'' + (2y_1' + Py_1)u' = 0$

Since y_1 is the solution of $y_1'' + Py_1' + Qy_1 = 0$

$\Leftrightarrow y_1 u'' + (2y_1' + Py_1)u' = 0$

Let $w = u' \Rightarrow y_1 w' + (2y_1' + Py_1)w = 0$

$\frac{w'}{w} + \left(2\frac{y_1'}{y_1} + P\right) = 0$

$\ln(wy_1^2) = -\int P\,dt + C$

$wy_1^2 = C_1 e^{-\int P dt}$

$u = C_1 \int \frac{e^{-\int P dt}}{y_1^2}\,dt + C_2$

by choosing $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, $y = y_1 u$.

then the anothe solution of the problem is:

$$y_2(t) = y_1(t) \int \frac{e^{-\int P dt}}{y_1^2}\,dt$$

and $y = C_1 y_1 + C_2 y_2$, C_1 & $C_2 \in \mathbb{R}$.

for (b) we have $P(t) = 1$, $y_1(t) = \cos(e^{-t})$

$$\Rightarrow y_2(t) = \cos e^{-t} \int \frac{e^{-\int dt}}{\cos^2(e^{-t})}\,dt = -\sin e^{-t}$$

Therefore $y(t) = C_1 \cos e^{-t} + C_2 \sin e^{-t}$.

(c) $t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0$, $y_1 = \cos t \times \frac{1}{\sqrt{t}}$

from (b). $y_2(t) = y_1(t)\int \frac{e^{-\int p\,dt}}{y_1^2(t)}\,dt$

by $y'' + \frac{1}{t}y' + \left(1 - \frac{1}{4t^2}\right)y = 0$

$\Rightarrow p(t) = \frac{1}{t}$, $y_1(t) = \frac{\cos t}{\sqrt{t}}$

$\Rightarrow y_2(t) = \frac{\cos t}{\sqrt{t}}\int \frac{\sqrt{t}^2 e^{-\int \frac{1}{t}dt}}{\cos^2 t}\,dt = \frac{\cos t}{\sqrt{t}}\int \frac{t \times \frac{1}{t}}{\cos t}\,dt = \frac{\sin t}{\sqrt{t}}$

Therefore , $y(t) = c_1\frac{\cos t}{\sqrt{t}} + c_2\frac{\sin t}{\sqrt{t}}.$

(d) $(1-t^2)y'' + 2ty' - 2y = 0$, $y_1 = t$

from (b). $y_2 = y_1\int \frac{e^{-\int p\,dt}}{y_1^2}\,dt.$

we have $y'' + \frac{2t}{1-t^2}y' - \frac{2}{1-t^2}y = 0$

$\Rightarrow p = \frac{2t}{1-t^2}$, $y_1 = t.$

$\Rightarrow y_2 = t\int \frac{e^{-\int \frac{2t}{1-t^2}dt}}{t^2}\,dt = t\int \frac{1-t^2}{t^2}\,dt = t\left(-\frac{1}{t} - t\right)$

Thus $y(t) = c_1 t + c_2(-1-t^2)$

(e). $(2t^2+1)y'' - 4ty' + 4y = 0$, $y_1 = t$

$\Leftrightarrow y'' - \frac{4t}{(2t^2+1)}y' + \frac{4}{2t^2+1}y = 0$ $\Rightarrow p(t) = \left(\frac{-4t}{2t^2+1}\right)$

from (b) : $y_2(t) = y_1(t)\int \frac{e^{-\int p\,dt}}{y_1^2(t)}\,dt$

$\Rightarrow y_2(t) = t\int \frac{e^{\int \frac{4t}{2t^2+1}}}{t^2}\,dt = t\int \frac{2t^2+1}{t^2}\,dt = 2t^2 - 1.$

Page.

Therefore $y(t) = c_1 t + c_2(2t^2+1)$.

(f). $t^2y'' - (2t^2\tan t + 2t)y' + (2t\tan t + 2)y = 0$, $y_1 = t$

$\Leftrightarrow y'' - \left(\tan t + \frac{2}{t}\right)y' + \left(\frac{2t\tan t + 2}{t^2}\right)y = 0$

$\Rightarrow p(t) = -\left(\tan t + \frac{2}{t}\right)$

from part (b)

$\Rightarrow y_2(t) = t\int \frac{e^{\int(\tan t + \frac{2}{t})dt}}{t^2}dt = t\int \frac{e^{2\ln t - 2\ln\dots}}{t^2}$

$= t\int \frac{e^{\ln\left(\frac{t^2}{\cos^2 t}\right)}}{t}dt = t\int \frac{t^2}{t^2\cos^2 t}dt = \dots$

Thus $y(t) = c_1 t + c_2 t\tan t$, $c_1, c_2 \in \mathbb{R}$.

19

Phyrun Thanaksith

Integral the following diff- equation.

(a) $y'' + 2y' + y = t^{-1}e^{-t}$

$y_h = (C_1 t + C_2)e^{-t}$

Let $y_1 = te^{-t}$, $y_2 = e^{-t}$

then $y_p = y_1 u_1 + y_2 u_2$

where $\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + u_2' y_2' = t^{-1}e^{-t} \end{cases}$

by $y_1 = te^{-t} \Rightarrow y_1' = e^{-t} - te^{-t}$

$y_2 = e^{-t} \Rightarrow y_2' = -e^{-t}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} te^{-t} & e^{-t} \\ e^{-t} - te^{-t} & -e^{-t} \end{vmatrix} = -e^{-2t}$$

$$W_1(y_1, y_2) = \begin{vmatrix} 0 & e^{-t} \\ t^{-1}e^{-t} & -e^{-t} \end{vmatrix} = -\frac{e^{-2t}}{t}$$

$$W_2(y_1, y_2) = \begin{vmatrix} te^{-1} & 0 \\ e^{-t} - te^{-t} & t^{-1}e^{-t} \end{vmatrix} = e^{-2t}$$

$\Rightarrow u_1' = \frac{W_1}{W} = \frac{1}{t} \Rightarrow u_1(t) = \ln|t|$

$\Rightarrow u_2' = \frac{W_2}{W} = -1 \Rightarrow u_2(t) = t.$

Then $y_p = \ln t\,(te^{-t}) + t(e^{-t})$.

Therefore $y = e^{-t}(C_1 t + C_2) + e^{-t}(t\ln t + t)$.

(6) $y'' + y = (\sin t)^{-1}$

$\Rightarrow y_h = c_1 \cos t + c_2 \sin t$

Let $y_1 = \cos t$; $y_2 = \sin t$ $\Rightarrow y_p = u_1 y_1 + u_2 y_2$

where $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = (\sin)^{-1}. \end{cases}$

by $y_1' = -\sin t$, $y_2' = \cos t$

$\Rightarrow W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$, $W_1 = \begin{vmatrix} 0 & \sin t \\ (\sin t)^{-1} & \cos t \end{vmatrix} = -1$

$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & (\sin t)^{-1} \end{vmatrix} = \cot(t).$

$\Rightarrow$

$\Rightarrow u_1' = \dfrac{W_1}{W} = -1$, $\Rightarrow u_1 = -t$

$u_2' = \dfrac{W_2}{W} = \cot(t)$ $\Rightarrow u_2 = \ln(\sin t).$

$\Rightarrow y_p(t) = -t(\cos t) + \ln|\sin t|(\sin t)$

Therefore $y = c_1 \cos t + c_2 \sin t + \sin t \ln|\sin t| - t \cos t$

where c_1 & $c_2 \in \mathbb{R}$.

(c) $y'' + 2y' + 2y = e^{-t}(\sin t)^{-1}$.

$y_h = e^{-t}(c_1\cos t + c_2\sin t) = e^{-t}c_1\cos t + e^{-t}c_2\sin t$

Let $y_1 = e^{-t}\cos t$, $y_2 = e^{-t}\sin t$

then $y_p = u_1\cos t\, e^{-t} + u_2 e^{-t}\sin t$.

where $\begin{cases} \cancel{u_1' + y_1}\ u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = e^{-t}(\sin t)^{-1} \end{cases}$

$$W = \begin{vmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \end{vmatrix} = e^{-2t}$$

$$W_1 = \begin{vmatrix} 0 & e^{-t}\sin t \\ e^{-t}(\sin t)^{-1} & -e^{-t}\sin t - e^{-t}\cos t \end{vmatrix} = -e^{-2t}\ \cancel{\cot(t)}.$$

$$W_2 = \begin{vmatrix} e^{-t}\cos t & 0 \\ -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}(\sin t)^{-1} \end{vmatrix} = -e^{-2t}\cot(t).$$

$\therefore u_1' = \frac{W_1}{W} = -1 \Rightarrow u_1 = -t$, $u_2' = \frac{W_2}{W} = \cot(t) \Rightarrow u_2' =$ $\ln|\sin t|$

Therefore. $y(t) = e^{-t}(c_1\cos t + c_2\sin t) + e^{-t}\sin t\ln|\sin t| - t\cos$

where c_1 & $c_2 \in \mathbb{R}$.

(d) $t^2y'' - 4ty' + 4y = t^4 + t^2,\ t > 0$

~~$y_h = t^2y'' + 6ty' + 6y = 0$~~

$y_h = c_1 t^4 + c_2 t.$

Let $y_p = u_1 y_1 + u_2 y_2$

where $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = t^2 + 1 \end{cases}$. $y_1' = 4t^3$, $y_2' = 1$

$$W = \begin{vmatrix} t^4 & t \\ 4t^3 & 1 \end{vmatrix} = -3t^4$$

$$W_1 = \begin{vmatrix} 0 & t \\ t^2+1 & 1 \end{vmatrix} = -t^3 - t \quad , \quad W_2 = \begin{vmatrix} t^4 & 0 \\ 4t^3 & t^2+1 \end{vmatrix} = t^4$$

- $u_1' = \dfrac{W_1}{W} \Rightarrow$ ~~w_1~~ $u_1 = \dfrac{1}{3}\left(\ln t - \dfrac{1}{2t^2}\right)$

- $u_2' = \dfrac{W_2}{W} \Rightarrow u_2 = -\dfrac{1}{3}(t^3 + t)$

Therefore $y = y_p + y_h.$

(e). $y'' + y' + e^{-2t}y = e^{-3t}$

we have $y_h = c_1\cos(e^{-t}) + c_2\sin(e^{-t})$.

Let $y_1 = \cos(e^{-t})$, $y_2 = \sin(e^{-t})$

we get $y_p = u_1y_1 + u_2y_2$ where $\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = e^{-3t} \end{cases}$

$$W = \begin{vmatrix} \cos e^{-t} & \sin e^{-t} \\ e^{-t}\sin e^{-t} & -e^{-t}\cos(e^{-t}) \end{vmatrix} = -e^{-t}$$

$$W_1 = \begin{vmatrix} 0 & \sin e^{-t} \\ e^{-3t} & -e^{-t}\cos(e^{-t}) \end{vmatrix} = -e^{-3t}\sin(e^{-t}).$$

$$W_2 = \begin{vmatrix} \cos e^{-t} & 0 \\ e^{-t}\sin e^{-t} & e^{-3t} \end{vmatrix} = e^{-3t}\cos(e^{-t}).$$

$\Rightarrow u_1' = \dfrac{W_1}{W} \Rightarrow u_1 = e^{-t}\cos(e^{-t}) - \sin(e^{-t})$

$\Rightarrow u_2' = \dfrac{W_2}{W} \Rightarrow u_2 = e^{-t}\sin(e^{-t}) + \cos(e^{-t})$.

$\Rightarrow y_p = e^{-t}$.

Therefore: $c_1\cos(e^{-t}) + c_2\sin(e^{-t}) + e^{-t}$.

where $c_1, c_2 \in \mathbb{R}$.

(f). $t^3y^{(3)} - 3t^2y'' + 7ty' - 8y = 8\ln t + 12$, $y(1) = -2$, $y'(1) = 3$

calculate y_h. (the solution of $t^3y^{(3)} - 3t^2y'' + 7ty' - 8y$)

$\Rightarrow y_h = c_1t^2 + c_2t^2\ln|t| + c_3t^2(\ln|t|)^2$

Let $y_1 = t^2$, $y_2 = t^2\ln|t|$, $y_3 = t^2(\ln|t|)^2$

we get $y_p = u_1y_1 + u_2y_2 + u_3y_3$

$$\bullet\ W = \begin{vmatrix} t^2 & t^2\ln t & t^2(\ln t)^2 \\ 2t & 2t\ln t + t & 2t(\ln t)^2 + 2(\ln t)t \\ 2 & 2\ln t + 3 & 2(\ln t)^2 + 6\ln t + 2 \end{vmatrix} = 2t^3$$

$$\bullet\ W_1 = \begin{vmatrix} 0 & t^2\ln t & t^2(\ln t)^2 \\ 0 & 2t\ln t + t & 2t(\ln t)^2 + 2(\ln t)t \\ \frac{8\ln t + 12}{t^3} & 2\ln t + 3 & 2(\ln t)^2 + 6\ln t + 2 \end{vmatrix}$$

$= 8(\ln t)^3 + 12(\ln t)^2$

Similary:

$\bullet\ W_2 = -[(8\ln t)^2 + 12\ln t]2$

$\bullet\ W_3 = 8\ln t + 12$

$\bullet\ u_1' = \frac{W_1}{W} = \frac{8(\ln t)^3 + 12(\ln t)^2}{2t^3} \Rightarrow u_1 = -\frac{2\ln^3 t + 6\ln^2 t + 6\ln t}{t^2}$

$\bullet\ u_2 = \frac{4\ln^2 t + 10\ln t + 5}{t^2}$, $u_3 = -\frac{2\ln t + 4}{t^2}$

Then: $y = c_1t^2 + c_2t^2\ln t + c_3t^2(\ln t)^2 - \ln t - 3.$

I2-TD3
(Systems of Ordinary Differential Equations)

1. Solve the following homogeneous system of ode

$$\frac{dx}{dt} = Ax, \quad \text{where} \quad x(t) = (x_1(t), x_2(t), \ldots, x_n(t));\ n = 2, 3, \ldots$$

and the matrix A is given below:

(a) $A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} -18 & 9 \\ -49 & 24 \end{pmatrix}$

(c) $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

(d) $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

(e) $A = \begin{pmatrix} -1 & 1 & -1 \\ -10 & 6 & -5 \\ -6 & 3 & -2 \end{pmatrix}$

(f) $A = \begin{pmatrix} 22 & -5 & 1 & -5 \\ 18 & -2 & 2 & -5 \\ 6 & -4 & 3 & -4 \\ 62 & -16 & 2 & -13 \end{pmatrix}$

(g) $A = \begin{pmatrix} 5 & 2 & -1 & -1 \\ 3 & 10 & -3 & -3 \\ 4 & 8 & 0 & -4 \\ 3 & 6 & -3 & 1 \end{pmatrix}$

(h) $A = \begin{pmatrix} 4 & 1 & 2 & -2 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 5 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

(i) $A = \begin{pmatrix} 5 & -1 & 2 & 4 \\ 5 & -1 & 3 & 4 \\ 8 & -5 & 3 & 4 \\ -4 & 1 & -1 & -3 \end{pmatrix}$

(j) $A = \begin{pmatrix} 2 & -3 & -2 & 3 \\ -4 & 2 & 4 & -4 \\ -4 & 0 & 4 & -3 \\ -8 & 4 & 8 & -8 \end{pmatrix}$

(k) $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

(l) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(m) $A = \begin{pmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix}$

(n) $A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

(o) $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

2. Let A be a square matrix defined by

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 4 & -3 & -2 \\ 5 & -1 & -5 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenspaces of A.

(b) Show that A is diagonalizable. Diagonalize A.

(c) Solve the system of linear differential equations $\dfrac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3)^T$. Then solve the initial value problem $\dfrac{dx}{dt} = Ax + B(t)$, $x(0) = (0, 0, 1)^T$ where $B(t) = (e^{2t}, 3e^{2t}, -5e^{2t})^T$.

3. Let A be a square matrix defined by

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -2 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenspaces of A.

(b) Show that A is not diagonalizable but triangularizable. Triangularize A.

(c) Solve the system of linear differential equations $\dfrac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3)^T$. Then solve the initial value problem $\dfrac{dx}{dt} = Ax + B(t), x(0) = (1, 0, 1)^T$ where $B(t) = (-6e^t, 5e^t, -3e^t)^T$.

4. Let A be a square matrix defined by

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(a) Find the eigenvalues and eigenspaces of A in $\mathbb{C}$.

(b) Solve the system of linear differential equations $\dfrac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3, x_4)^T$. Then solve the initial value problem $\dfrac{dx}{dt} = Ax + B(t), x(0) = (1, 0, 1, 0)^T$ where $B(t) = (-6e^t, 5e^t, -3e^t, 0)^T$.

5. Solve the following systems of ordinary differential equations:

(a) $\begin{cases} x'(t) = -5x(t) + 12y(t) + 4z(t) + 9e^{-3t}, x(0) = 1 \\ y'(t) = -4x(t) + 11y(t) + 4z(t) - 5e^{-3t}, y(0) = 0 \\ z'(t) = 4x(t) - 12y(t) - 5z(t) - 7e^{-3t}, z(0) = 1 \end{cases}$

(b) $\begin{cases} x'(t) = -3x(t) - y(t) + z(t) + 9\cos t, x(0) = -1 \\ y'(t) = -3x(t) - 5y(t) + 3z(t) + 4\cos t, y(0) = 0 \\ z'(t) = -5x(t) - 5y(t) + 3z(t) + 7\cos t, z(0) = 1 \end{cases}$

(c) $\begin{cases} x'(t) = 8x(t) + 16y(t) + 6z(t) + 12\sin t \\ y'(t) = -6x(t) - 14y(t) - 6z(t) + 5\cos t \\ z'(t) = 9x(t) + 24y(t) + 11z(t) + \cos t \end{cases}$

(d) $\begin{cases} w'(t) = -17w(t) + 6x(t) + 3y(t) + 3z(t) + 12e^{-2t} \\ x'(t) = -18w(t) + 7x(t) + 3y(t) + 3z(t) + 6e^{-2t} \\ y'(t) = -15w(t) + 6x(t) + y(t) + 3z(t) + 8e^{-2t} \\ z'(t) = -39w(t) + 12x(t) + 9y(t) + 7z(t) + 10e^{-2t} \end{cases}$

6. Solve the following systems of ordinary differential equations:

(a) $\begin{cases} x'(t) = 2x(t) + y(t) - z(t) + 9e^{-2t}\sin t \\ y'(t) = x(t) + 3y(t) - 2z(t) + 7e^{-2t}\sin t \\ z'(t) = -x(t) + 2y(t) + z(t) - 4e^{-2t}\sin t \end{cases}$

(b) $\begin{cases} x'(t) = -5x(t) + 7y(t) + 2z(t) - 3e^{2t}\cos t, x(0) = 0 \\ y'(t) = -2x(t) + 2y(t) + z(t) - 2e^{2t}\cos t, y(0) = 0 \\ z'(t) = 2x(t) - 4y(t) - 3z(t) + 4e^{2t}\cos t, z(0) = 0 \end{cases}$

(c) $\begin{cases} x'(t) = -5x(t) + 3y(t) - 2z(t) - 6e^{-2t}\cos t \\ y'(t) = -2x(t) - 2y(t) - 3z(t) - 8e^{-2t}\cos t \\ z'(t) = 2x(t) + 4y(t) + 5z(t) + 6e^{-2t}\cos t \end{cases}$

(d) $\begin{cases} w'(t) = -4w(t) + 3x(t) + y(t) - z(t) - 2e^{-t} \\ x'(t) = -6w(t) + 5x(t) + y(t) - z(t) + 6e^{-t} \\ y'(t) = -2w(t) + 3x(t) - 2z(t) - 8e^{-t} + 2e^{-t}\cos t \\ z'(t) = -4w(t) + 3x(t) + 2y(t) - 2z(t) + 4e^{-t} \end{cases}$

(e) $\begin{cases} w'(t) = 13w(t) - 2x(t) - 3y(t) - 5z(t) - 2e^{2t}\sin t, w(0) = -1 \\ x'(t) = 15w(t) - x(t) - 4y(t) - 6z(t) + 6e^{2t}\sin t, x(0) = 1 \\ y'(t) = 15w(t) - 2x(t) - 4y(t) - 6z(t) + 2e^{-t}\cos t, y(0) = 0 \\ z'(t) = 13w(t) - 2x(t) - 2y(t) - 6z(t) + 4e^{-t}\cos t, z(0) = 0 \end{cases}$

(f) $\begin{cases} w'(t) = 6w(t) - 2x(t) - 2z(t) - 2e^{-t}\sin t, w(0) = 0 \\ x'(t) = 12w(t) - 4x(t) - y(t) - 3z(t) + 6e^{-t}\sin t, x(0) = 0 \\ y'(t) = 9w(t) - 2x(t) - y(t) - 3z(t) + 2e^{-t}\cos t, y(0) = 0 \\ z'(t) = 13w(t) - 4x(t) + y(t) - 5z(t) + 4e^{-t}\cos t, z(0) = 0 \end{cases}$

7. Solve the following systems of ordinary differential equations:

(a) $\begin{cases} x'(t) = -2x(t) + y(t) + 2z(t) + 9e^{2t} \\ y'(t) = -5x(t) + 3y(t) + 3z(t) + 7e^{2t} \\ z'(t) = -4x(t) + y(t) + 4z(t) - 4e^{2t} \end{cases}$

(b) $\begin{cases} x'(t) = 4x(t) + 18y(t) + 8z(t) + 2e^{-2t} \\ y'(t) = -5x(t) - 16y(t) - 6z(t) - 2e^{-2t} \\ z'(t) = 7x(t) + 19y(t) + 6z(t) - 4e^{-2t} \end{cases}$

(c) $\begin{cases} x'(t) = 8x(t) + 18y(t) + 8z(t) + 4e^{2t}, x(0) = 0 \\ y'(t) = -5x(t) - 12y(t) - 6z(t) - 6e^{2t}, y(0) = 0 \\ z'(t) = 7x(t) + 19y(t) + 10z(t) + 8e^{2t}, z(0) = 1 \end{cases}$

(d) $\begin{cases} w'(t) = -19w(t) + 7x(t) + 4y(t) + 3z(t) - 2e^{t}, w(0) = 0 \\ x'(t) = -20w(t) + 8x(t) + 4y(t) + 3z(t) + 6e^{-2t}, x(0) = 0 \\ y'(t) = -17w(t) + 7x(t) + 2y(t) + 3z(t) + 2e^{t}, y(0) = 0 \\ z'(t) = -44w(t) + 15x(t) + 11y(t) + 7z(t) + 4e^{-2t}, z(0) = 0 \end{cases}$

(e) $\begin{cases} w'(t) = -10w(t) - 6x(t) + 2y(t) + 7z(t) - 2e^{2t} \\ x'(t) = 4w(t) + x(t) - 3z(t) + 6e^{-t} \\ y'(t) = 7w(t) + 5x(t) - 4y(t) - 6z(t) + 2e^{-t} \\ z'(t) = -15w(t) - 12x(t) + 8y(t) + 12z(t) \end{cases}$

(f) $\begin{cases} w'(t) = -5w(t) + x(t) + y(t) + z(t) - 2e^{t} \\ x'(t) = -5w(t) + 2y(t) + z(t) + 6e^{-t} \\ y'(t) = -6w(t) + x(t) + 2z(t) + 2e^{t} \\ z'(t) = -8w(t) + 2x(t) + 2y(t) + z(t) + 4e^{-t} \end{cases}$

(g) $\begin{cases} w'(t) = 2w(t) - 2x(t) + y(t) - 2z(t) - 2e^{t} + 4e^{-t}\cos t, w(0) = 0 \\ x'(t) = 6w(t) - 4x(t) - z(t) - 2e^{t} - 4e^{-t}\cos t, x(0) = 0 \\ y'(t) = 6w(t) - 4x(t) + 2y(t) - 6z(t) + 6e^{t} + 4e^{-t}\cos t, y(0) = 0 \\ z'(t) = 2w(t) - 2x(t) + 2y(t) - 4z(t) + 6e^{t} - 4e^{-t}\cos t, z(0) = 0 \end{cases}$

TD3: System of ODE

1. Solve the homogeneous ~~system~~ system of ode

c). $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

The characteristic polynomial

$$p(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+3)^2$$

$p(\lambda) = 0 \Rightarrow -(\lambda-1)(\lambda+3)^2 = 0$

$\Rightarrow \lambda_1 = 1 \; ; \; \lambda_2 = \lambda_3 = -3$

. For $\lambda_1 = 1 \Rightarrow E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

. For $\lambda_2 = \lambda_3 = -3 \Rightarrow E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Since $p(\lambda) = -(\lambda-1)(\lambda+3)^2$ is ~~stipt~~ split

$am(\lambda_1) = gm(\lambda_1) = 1$

$gm(\lambda_2) = am(\lambda_2) = 2$

Thus, A is diagonalizable

$$\Rightarrow X(t) = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2 + C_3 e^{\lambda_3 t} V_3$$

$$= C_1 e^{t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{X(t) = C_1 e^{t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} -2C_2 - C_3 \\ C_3 \\ C_2 \end{pmatrix}}$$

e). $A = \begin{pmatrix} -1 & 1 & -1 \\ -10 & 6 & -5 \\ -6 & 3 & -2 \end{pmatrix}$

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & -1 \\ -10 & 6-\lambda & -5 \\ -6 & 3 & -2-\lambda \end{vmatrix} = -(\lambda-1)^3$$

$p(\lambda) = 0 \Rightarrow \lambda = 1$

For $\lambda = 1$: $E_{\lambda} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} ; \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

since $p(\lambda) = -(\lambda-1)^3$ is splited

$am(\lambda) = 3 \neq gm(\lambda) = 2$

A is not diagonalizable, but triangularizable

. Find index (λ)

$E_\lambda^2 : (A-\lambda I)^2 v = 0$

$$(A-\lambda I)^2 = \begin{pmatrix} -2 & 1 & -1 \\ -10 & 5 & -5 \\ -6 & 3 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 \\ -10 & 5 & -5 \\ -6 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow E_\lambda^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow \dim E_\lambda^2 = am(\lambda) = 3$

$\Rightarrow \text{index}(\lambda) = 2$

Choose $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in E_\lambda^2 \setminus E_\lambda$

$$V_1 = (A-\lambda I)V_2 = \begin{pmatrix} -1 & 1 & -1 \\ -10 & 5 & -5 \\ -6 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \\ -6 \end{pmatrix}$$

$$X_1 = V_1 e^{\lambda t} = \begin{pmatrix} -2 \\ -10 \\ -6 \end{pmatrix} e^t$$

$$X_2 = (V_2 + V_1 t)e^{\lambda t} = e^t \begin{pmatrix} 1-2t \\ -10t \\ -6t \end{pmatrix}$$

$$X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} e^t$$

$$X(t) = C_1 X_1 + C_2 X_2 + C_3 X_3$$

$$= C_1 e^t \begin{pmatrix} -2 \\ -10 \\ -6 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1-2t \\ -10t \\ -6t \end{pmatrix} \cancel{+ C_3 e^t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}$$

$$\cancel{= e^t} \begin{pmatrix} -2C_1 + C_2 - 2C_2 t \\ -10C_1 - 10C_2 t \\ -6C_1 - 6C_2 t \end{pmatrix}$$

$$\boxed{X(t) = C_1 e^t \begin{pmatrix} -2 \\ -10 \\ -6 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1-2t \\ -10t \\ -6t \end{pmatrix} + C_3 e^t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}$$

d) $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

$S_1 = tr(A) = 0$

$S_2 = -3\begin{vmatrix} 5 & -1 \\ 6 & -2 \end{vmatrix} - \begin{vmatrix} -7 & -1 \\ -6 & -2 \end{vmatrix} - \begin{vmatrix} -7 & 5 \\ -6 & 6 \end{vmatrix} = 12 - 8 + 12 = -8$

$S_3 = \begin{vmatrix} -3 & 1 \\ -7 & 5 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 6 & -2 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -6 & -2 \end{vmatrix} = 8 - 4 = 4$

$p(\lambda) = -(\lambda^3 + (-8)\lambda - 4) = -\lambda^3 + 8\lambda + 4$

$p(\lambda) = -(\lambda-1)(\lambda+3)^2$

$p(\lambda) = 0 \Rightarrow \lambda_1 = 1 \,;\, \lambda_2 = -3$

$E_{\lambda_1} = Span\left\{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right\}$

$E_{\lambda_2} = Span\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

$am(\lambda_1) = gm(\lambda_1) = 1$

$am(\lambda_2) = gm(\lambda_2) = 2$

$p(\lambda) = -(\lambda-1)(\lambda+3)^2$ is splite

Therefore, A is diagonalizable where

$X_1(t) = e^t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$X_2 t = e^{3t}$

d). $A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$

$p(\lambda) = (\lambda+2)^2(\lambda-6)$ is splite

$p(\lambda) = 0 \Rightarrow p(\lambda) = (\lambda+2)^2(\lambda-4) = 0$

$\lambda_1 = -2 ; \lambda_2 = 4$

For $\lambda_1 = -2$: $E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

For $\lambda_2 = 4 \Rightarrow E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$am(\lambda_2) = gm(\lambda_2) = 1$

$am(\lambda_1) \neq gm(\lambda_1) =$

Find index (λ_1)

$(A - \lambda_1 I)^2 = (A + 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -36 & 36 & 0 \\ -36 & 36 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E^2_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim E^2_{\lambda} = am(\lambda_1) = 2$

Let $V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow V_1 = (A + 2I)V_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$$\boxed{X(t) = Ae^{4t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + Ce^{-2t}\begin{pmatrix} 0-t \\ 0-t \\ 1 \end{pmatrix} ; A, B, C \in \mathbb{R}}$$

f). $A = \begin{pmatrix} 22 & -5 & 1 & -5 \\ 18 & -2 & 2 & -5 \\ 6 & -4 & 3 & -4 \\ 62 & 16 & 2 & -13 \end{pmatrix}$

$p(\lambda) = \lambda^4 - 10\lambda^3 + 47\lambda^2 - 110\lambda + 96$

$p(\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = \lambda_4 = \frac{1}{2}(5 + i\sqrt{39})$

For $\lambda_1 = 2$: $E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -48 \\ 50 \\ 62 \end{pmatrix} \right\} \Rightarrow am(\lambda_1) = gm(\lambda_1) = 1$

For $\lambda_2 = 3$: $E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow am(\lambda_2) = gm(\lambda_2) = 1$

$\lambda_3 = \lambda_4 = \frac{1}{2}(5 \pm i\sqrt{39})$: $E_{\lambda_3} = \text{Span}\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 1/4\,(1 \pm i\sqrt{39}) \\ 1 \end{pmatrix} \right\} \Rightarrow gm(\lambda_3) = 1 \neq am(\lambda_3) = 2$

$$X_3(t) = A e^{(1+\sqrt{39}i)t} \begin{pmatrix} 1/2 \\ 1 \\ 1/4\,(1+\sqrt{39}i) \\ 1 \end{pmatrix} + B e^{(1-i\sqrt{39})t} \begin{pmatrix} 1/2 \\ 1 \\ 1/4\,(1-i\sqrt{39}) \\ 1 \end{pmatrix}$$

$$X_2(t) = C e^{3t} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$X_2(t) = D e^{2t} \begin{pmatrix} 1 \\ -48 \\ 50 \\ 62 \end{pmatrix}$$

where $X(t) = X_1(t) + X_2(t) + X_3(t)$

g). $A = \begin{pmatrix} 5 & 2 & -1 & -1 \\ 3 & 10 & -3 & -3 \\ 4 & 8 & 0 & -4 \\ 3 & 6 & -3 & 1 \end{pmatrix}$

$p(\lambda) = \lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = \cancel{(\lambda+}(\lambda-4)^4$ is split

$p(\lambda) = 0 \Rightarrow \lambda = 4$

$E_\lambda = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow gm(\lambda) = 3 \neq am(\lambda) = 4$

$E_\lambda^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow E_\lambda^2 = am(\lambda) = 4$

Let $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$V_1 = (A - \lambda I)V_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 3 \end{pmatrix}$

$= (A - 4I)V_2$

$$X(t) = \left\{ A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + C\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} e^{4t} + D e^{4t}\left[t\begin{pmatrix} 1 \\ 3 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right] + E e^{4t}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 3 \end{pmatrix} ; A, B, C, E, D \in \mathbb{R}$$

h). $\begin{pmatrix} 4 & 1 & 2 & -2 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 5 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

$p(\lambda) = (\lambda - 3)^4$ is split

$p(\lambda) = 0 \Rightarrow \lambda = 3$; $am(\lambda) = 4$

$$E_\lambda = \text{Span}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\right\} \Rightarrow \dim E_\lambda \neq am(\lambda)$$

Find index(λ)

$$(A-3I)^2 = \begin{pmatrix}-2 & 2 & 4 & 0\\0 & 0 & 0 & 0\\-1 & -1 & 2 & 0\\-2 & -2 & 4 & 0\end{pmatrix}$$

$$\Rightarrow E_\lambda^2 = \text{Span}\left\{\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\right\} \Rightarrow \dim E_\lambda^2 \neq am(\lambda)$$

$$(A-3I)^3 = 0I_4 \Rightarrow$$

$$\Rightarrow E_\lambda^3 = \text{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\} \Rightarrow \dim E_\lambda^3 = am(\lambda)$$

$\Rightarrow$ index$(\lambda) = 3$

So Let $V_3 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$V_2 = (A-\lambda I)V_3 = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}$$

$$V_1 = (A-\lambda I)V_2 = \begin{pmatrix}-2\\0\\-1\\-2\end{pmatrix}$$

$$X(t) = Ae^{3t}\begin{pmatrix}2\\0\\1\\2\end{pmatrix} + Be^{3t}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + Ce^{3t}\left[\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + t\begin{pmatrix}-2\\0\\-1\\-2\end{pmatrix}\right]$$

$$+ De^{3t}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + \frac{t^2}{2}\begin{bmatrix}-2\\0\\-1\\-2\end{bmatrix}\right] \quad ; A, B, C, D \in \mathbb{R}$$

7). $A = \begin{pmatrix}5 & -1 & 2 & 4\\5 & -1 & 3 & 4\\8 & -5 & 3 & 4\\-4 & 1 & -1 & -3\end{pmatrix}$

$p(\lambda) = (\lambda-1)^4$

$p(\lambda) = 0 \Rightarrow \lambda = 1$, $am(\lambda) = 4$

$$E_\lambda = \text{Span}\left\{\begin{pmatrix}-4\\-4\\0\\3\end{pmatrix}\right\} \Rightarrow \dim E_\lambda \neq am(\lambda)$$

- Find index(λ)

$$(A-I)^2 = \begin{pmatrix} 11 & -8 & 5 & 4 \\ 18 & -12 & 6 & 8 \\ 7 & -4 & 1 & 4 \\ -3 & 3 & -3 & 0 \end{pmatrix}$$

$$\Rightarrow E_\lambda^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -4 \\ 0 \\ 3 \end{pmatrix} \right\} \Rightarrow \dim E_\lambda^2 \neq am(\lambda)$$

$$(A-I)^3 = \begin{pmatrix} 28 & -16 & 4 & 16 \\ 28 & -16 & 4 & 16 \\ 0 & 0 & 0 & 0 \\ -21 & 12 & -3 & -12 \end{pmatrix} \Rightarrow \dim E_\lambda^3 \neq am(\lambda)$$

$$\cancel{E_\lambda^3 = \text{Span}}$$

$$(A-I)^4 = 0I_4$$

$$E_\lambda^4 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim E_\lambda^4 = am(\lambda)$$

$$\Rightarrow \text{index}(\lambda) = 4$$

Let $V_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$V_3 = (A-I)V_4 = \begin{pmatrix} 4 \\ 5 \\ 3 \\ -4 \end{pmatrix}$$

$$V_4 = (A-I)V_3 = \begin{pmatrix} 11 \\ 18 \\ 7 \\ -3 \end{pmatrix}$$

$$V_5 = (A-I)V_4 = \begin{pmatrix} 28 \\ 28 \\ 0 \\ -21 \end{pmatrix}$$

$$X(t) = Ae^t \begin{pmatrix} 28 \\ 28 \\ 0 \\ -21 \end{pmatrix} + Be^t \left[\begin{pmatrix} 11 \\ 18 \\ 7 \\ -3 \end{pmatrix} + t \begin{pmatrix} 28 \\ 28 \\ 0 \\ -21 \end{pmatrix} \right] + Ce^t \left[\begin{pmatrix} 4 \\ 5 \\ 8 \\ -4 \end{pmatrix} + \begin{pmatrix} 11 \\ 18 \\ 7 \\ -3 \end{pmatrix} t \right] + \frac{t^2}{2} \begin{pmatrix} 28 \\ 28 \\ 0 \\ -21 \end{pmatrix} \Bigg]$$

$$+ \frac{t^2}{2} \Bigg(+ De^t \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ 8 \\ -4 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 11 \\ 18 \\ 7 \\ -3 \end{pmatrix} + \frac{t^3}{6} \begin{pmatrix} 28 \\ 28 \\ 0 \\ -21 \end{pmatrix} \right]$$

J). $A = \begin{pmatrix} 2 & -3 & -2 & 3 \\ -4 & 2 & 4 & -4 \\ -4 & 0 & 4 & -3 \\ -5 & 4 & 8 & -3 \end{pmatrix}$

$p(\lambda) = \lambda^4$

$p(\lambda) = 0 \Rightarrow \lambda = 0 \quad , am(\lambda) = 4$

For $\lambda = 0$: $E_\lambda = \text{Span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim E_\lambda \neq am(\lambda)$

$A^2 = 0 \quad : E_\lambda^2 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim E_\lambda^2 = am(\lambda)$

$\Rightarrow \text{index}(\lambda) = 2$

Let $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$V_1 = (A - \lambda I) V_2 = \begin{pmatrix} 2 \\ -4 \\ -4 \\ -8 \end{pmatrix}$

$X(t) = A \begin{pmatrix} -3 \\ 2 \\ 0 \\ 4 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + C \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -4 \\ -4 \\ 8 \end{pmatrix} \right] + D \begin{pmatrix} 2 \\ -4 \\ -4 \\ -8 \end{pmatrix}]$

K). $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

$p(\lambda) = \lambda^2 + 4$ is not split

$p(\lambda) = 0 \Rightarrow \lambda_{1,2} = \pm 2i$

$\lambda_1 = 2i \; : E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$

$\lambda_2 = -2i \; : E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$

$X(t) = Ae^{2it} \begin{pmatrix} -i \\ 1 \end{pmatrix} + Be^{-2it} \begin{pmatrix} i \\ 1 \end{pmatrix}$

L). $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$p(\lambda) = (1-\lambda)^2 + 1$

$p(\lambda) = 0 \Rightarrow \lambda_{1,2} = 1 + i$

$E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\} ; \; E_{\lambda_2} = \left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$

$$X(t) = a_1 e^{\lambda_1 t}\begin{pmatrix} i \\ 1 \end{pmatrix} + a_2 e^{\lambda_2 t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$X(t) =$$

m). $A = \begin{pmatrix} -1 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix}$

$p(\lambda) = -\lambda^3 + 3\lambda^2 - 4\lambda + 2 = -(\lambda-1)(\lambda-1-i)(\lambda-1+i)$ is split

$p(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 1+i \\ \lambda_3 = 1-i \end{cases}$

$E_{\lambda_1} = \text{Span}\left\{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow \dim E_{\lambda_1} = am(\lambda_1) = 1$

$E_{\lambda_2} = \text{Span}\left\{\begin{pmatrix} -5+i \\ -1+i \\ 2 \end{pmatrix}\right\} \Rightarrow \dim E_{\lambda_2} = am\ \lambda_2 = 1$

$E_{\lambda_3} = \text{Span}\left\{\begin{pmatrix} -5-i \\ -1-i \\ 2 \end{pmatrix}\right\} \Rightarrow \dim E_{\lambda_3} = am\ \lambda_3 = 1$

Then, A is diagonalizable

$$X(t) = C_1 e^t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{(1+i)t}\begin{pmatrix} -5+i \\ -1+i \\ 2 \end{pmatrix} + C_3 e^{(1-i)t}\begin{pmatrix} -5-i \\ -1-i \\ 2 \end{pmatrix} \ ; C_1, C_2, C_3 \in \mathbb{R}$$

n). $A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

$p(\lambda) = (\lambda-3)^2(\lambda^2+1)$ is split

$p(\lambda) = 0 \Rightarrow \lambda_1 = 3 \ , am(\lambda_1) = 2$

$\lambda_2 = i \ ; \lambda_3 = -i \ , \ am(\lambda_2) = am(\lambda_2) = 1$

$E_{\lambda_1} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\} \Rightarrow gm(\lambda_1) \neq am(\lambda_1)$

$E_{\lambda_2} = \text{Span}\left\{\begin{pmatrix} 1+7i \\ -11-2i \\ -25i \\ 25 \end{pmatrix}\right\} \Rightarrow gm(\lambda_2) = am(\lambda_2)$

$E_{\lambda_3} = \text{Span}\left\{\begin{pmatrix} 1-7i \\ -11+2i \\ 25i \\ 25 \end{pmatrix}\right\} \Rightarrow gm(\lambda_3) = am(\lambda_2)$

Find index(λ_1)

$$(A-3I)^2 = \begin{pmatrix} 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 8 & -6 \\ 0 & 0 & 6 & 8 \end{pmatrix}$$

$$E^2_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim E^2_{\lambda_1} = am(\lambda_1)$$

$\Rightarrow$ index$(\lambda_1) = 2$

Let $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$V_1 = (A-3I)V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$X(t) = a_1 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + a_2 e^{-t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right] + a_3 e^{it}\begin{pmatrix} 1+7i \\ -11+2i \\ -25i \\ 25 \end{pmatrix} + a_4 e^{-it}\begin{pmatrix} 1-7i \\ -11-2i \\ 25i \\ 25 \end{pmatrix}$$

$a_1, a_2, a_3, a_4 \in \mathbb{R}$

o). $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

$p(\lambda) = (\lambda^2+1)^2$ is split , $\cancel{am(\lambda_1\lambda_2) = 2}$

$p(\lambda) = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$E_{\lambda_1} = \text{Span}\left\{\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$E_{\lambda_2} = \text{Span}\left\{\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$D = \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad P = \begin{pmatrix} -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$A = PDP^{-1}$

$$x(t) = c_1 e^{-t}\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-t}\left[\begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix} + t\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right] + c_3 e^{t}\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{t}\left[\begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix} + t\begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$\cancel{a_1, a}$ $c_1, c_2, c_3, c_4 \in \mathbb{R}$

2. Let A Be a square matrix

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 4 & -3 & -2 \\ 5 & -1 & -5 \end{pmatrix}$$

(a). Find the eigenvalue and eigenspaces of A.

$P(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3)$, $P_\lambda = 0$

Therefore $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

* Find eigenspace:

for: $\lambda_1 = -1$

$$\Leftrightarrow A+I = \begin{pmatrix} 3 & -1 & -2 \\ 4 & -2 & -2 \\ 5 & -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & -1 & 2 \\ 1 & -1 & 0 \\ 4 & 0 & -4 \end{pmatrix}$$

Then $E_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

* for $\lambda_2 = -2 \Rightarrow E_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$
* for $\lambda_3 = -3 \Rightarrow$ $E_{\lambda_3} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(b). Show that A is diagonalizable

by $am(\lambda_i) = gm(\lambda_i)$, $i = 1, 2, 3$

therefroe A is diagonalizable.

* Diaganalize A:

$A = PDP^{-1}$ then $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, $P \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(c). solve: $\frac{dx}{dt} = Ax$.

$\Rightarrow x(t) = A_1e^{-t}\begin{pmatrix}1\\1\\1\end{pmatrix} + A_2e^{-2t}\begin{pmatrix}1\\2\\1\end{pmatrix} + A_3e^{-3t}\begin{pmatrix}1\\1\\2\end{pmatrix}$, $A_1, A_2, A_3 \in \mathbb{R}$

+ Solve $\frac{dx}{dt} = Ax + B(t)$. , $x(0) = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $B(t) = \begin{pmatrix}e^{2t}\\3e^{2t}\\-5e^{2t}\end{pmatrix}$

we have $\Phi(t) = \begin{pmatrix}e^{-t} & e^{-2t} & e^{-3t}\\ e^{-t} & 2e^{-2t} & e^{-3t}\\ e^{-t} & e^{-2t} & 2e^{-3t}\end{pmatrix}$

$$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix}3e^{-t} & -e^{-t} & -e^{-t}\\ -e^{2t} & e^{-2t} & 0\\ -e^{3t} & 0 & -e^{-3t}\end{pmatrix}$$

$$\Rightarrow \Phi^{-1}B(t) = \begin{pmatrix}3e^{-t} & -e^{-t} & -e^{-t}\\ -e^{2t} & e^{-2t} & 0\\ -e^{3t} & 0 & -e^{-3t}\end{pmatrix}\begin{pmatrix}e^{2t}\\3e^{2t}\\-5e^{2t}\end{pmatrix}$$

$$= \begin{pmatrix}5e^{t}\\2\\4e^{-t}\end{pmatrix}$$

then $x_p = \Phi\int\begin{pmatrix}5e^{t}\\2\\4e^{-t}\end{pmatrix}ds = \begin{pmatrix}e^{-t} & e^{-2t} & e^{-3t}\\ e^{-t} & 2e^{-2t} & e^{-3t}\\ e^{-t} & e^{-2t} & 2e^{-3t}\end{pmatrix}\begin{pmatrix}5e^{t}\\2t\\-4e^{-t}\end{pmatrix}$

$$x_p = \begin{pmatrix}5 + 2te^{-2t} - 4e^{-4t}\\ 5 + 4te^{-2t} - 4e^{-4t}\\ 5 + 2te^{-2t} - 8e^{-4t}\end{pmatrix}$$

therefore: $x(t) = A_1e^{-t}\begin{pmatrix}1\\1\\1\end{pmatrix} + A_2e^{-2t}\begin{pmatrix}1\\2\\1\end{pmatrix} + A_3e^{-3t}\begin{pmatrix}1\\1\\2\end{pmatrix} + x_p$

3. Let A be a square matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

(a). Find the eignvalue and eigenspace of A.

we have $p(\lambda) = (\lambda - 1)^3$, then $\lambda = 1$

for $\lambda = 1 \Leftrightarrow E_\lambda = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

fore $(A - \lambda I)^2 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

$(A - \lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E^3_\lambda = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

(b). show that A is not diagonalizable but triangularizable

Since $am(1) \neq gm(1) \Rightarrow A$ is not diagonalizable.

(c). solve for $\frac{dx}{dt} = Ax$.

we have $E^3_\lambda = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

Let $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}, v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

~~Therefore $x_h(t) = Ae^t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + Be^t$~~ $\Big[$

Therefore $x_h(t) = a_1 e^t \left[\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-2\\-3\\-2\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right] + a_2 e^t\left[\begin{pmatrix}-2\\-3\\-2\end{pmatrix} + t\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right]$

$$+ a_3 e^t \begin{pmatrix}-1\\-1\\-1\end{pmatrix} \quad , a_1, a_2, a_3 \in \mathbb{R}$$

+ Solve for $Ax + B(t) = \frac{dx}{dt}$, $x(0) = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $B(t) = \begin{pmatrix}-6e^t\\5e^t\\-3e^t\end{pmatrix}$

we have $x_1 = \begin{pmatrix}-e^t\\-e^t\\-e^t\end{pmatrix}$

$$x_2 = \begin{pmatrix}-2e^t - te^t\\-3e^t - te^t\\-2e^t - te^t\end{pmatrix}$$

$$x_3 = \begin{pmatrix}e^t - 2te^t - \frac{1}{2}t^2e^t\\-3te^t - \frac{1}{2}t^2e^t\\-2te^t - \frac{1}{2}t^2e^t\end{pmatrix}$$

$$\Rightarrow \Phi(t) = \begin{pmatrix}-e^t & -2e^t - te^t & e^t - 2te^t - \frac{1}{2}t^2e^t\\ -e^t & -3e^t - te^t & -3te^t - \frac{1}{2}t^2e^t\\ -e^t & -2e^t - te^t & -2te^t - \frac{1}{2}t^2e^t\end{pmatrix}$$

Then $x_p = \Phi(t)\int \Phi^{-1}(t)\cdot b(t)\,dt$

Thus $x(t) = x_h + x_p$.

$$\Phi^{-1}(t) = \begin{pmatrix}\frac{t^2}{2e^t} & \frac{t+2}{e^t} & -\frac{t^2+2t+6}{2e^t}\\ -\frac{t}{e^t} & -\frac{1}{e^t} & -\frac{-1-t}{e^t}\\ \frac{1}{e^t} & 0 & -\frac{1}{e^t}\end{pmatrix}$$

Page...

4. Let A Be a square matrix

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(a). Find the eigenvalues and eigenspace of A in $\mathbb{C}$

we have $P(\lambda) = (\lambda-3)^2(\lambda-i)(\lambda+i)$

Then the eigenvalues $\lambda_1 = 3$, $\lambda_2 = i$, $\lambda_3 = -i$

. Find eigenspace

For $\lambda_1 = 3$ $\quad E_{\lambda_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = i \quad E_{\lambda_2} = \begin{pmatrix} 1+7i \\ -11-2i \\ -25i \\ 25 \end{pmatrix}$

$\lambda_3 = -i \quad E_{\lambda_3} = \begin{pmatrix} 1-7i \\ -11+2i \\ 25i \\ 25 \end{pmatrix}$

(b). solve for $Ax = \frac{dx}{dt}$

. For $\lambda_1 = 3$

$$(A-3I)^2 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow E_\lambda^2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Choose $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

Then $x_1(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $x_2 = e^{3t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right]$

Page

$x_3(t) = \begin{pmatrix} 1 \\ -11 \\ 0 \\ 25 \end{pmatrix}\cos t - \sin(t)\begin{pmatrix} 7 \\ -2 \\ -25 \\ 0 \end{pmatrix}$, $x_4(t) = \begin{pmatrix} 7 \\ -2 \\ -25 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ -11 \\ 0 \\ 25 \end{pmatrix}\sin t$

Therefore : $x(t) = a_1 x_1(t) + a_2(x_2(t)) + a_3 x_3(t) + a_4 x_4(t)$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$.

+ Solve for $\frac{dx}{dt} = Ax + B(t)$, $x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} -6e^t \\ 5e^t \\ -3e^t \\ 0 \end{pmatrix}$

we have $\Phi(t) = \begin{pmatrix} e^{3t} & e^{3t} + te^{3t} & \cos t - 7\sin t & 7\cos t + \sin t \\ -e^{3t} & -te^{3t} & -11\cos t - 2\sin t & -2\cos t - 11\sin t \\ 0 & 0 & 25\sin t & -25\cos t \\ 0 & 0 & 25\cos t & 25\sin t \end{pmatrix}$

$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix} -\frac{t}{e^{3t}} & -\frac{1+t}{e^{3t}} & -\frac{-2+5t}{25e^{3t}} & -\frac{11+10t}{25e^{3t}} \\ \frac{1}{e^{3t}} & \frac{1}{e^{3t}} & \frac{1}{5e^{3t}} & \frac{2}{5e^{3t}} \\ 0 & 0 & \frac{\sin t}{25(\sin^2(t) + \cos^2(t))} & \frac{\cos t}{25(\cos^2 t + \sin^2 t)} \\ 0 & 0 & \frac{-\cos t}{25(\cos^2 t + \sin^2 t)} & \frac{+\sin t}{25(\cos^2 t + \sin^2 t)} \end{pmatrix}$

Then $x_p(t) = \Phi \cdot \int \Phi(t) B(t)\, dt$

Therefore $x(t) = x_h(t) + x_p(t)$.

Page...

5. Solve the following systems of ordinary diff.

a. $\begin{cases} x'(t) = -5x(t) + 12y(t) + 4z(t) + 9e^{-3t} \\ y'(t) = -4x(t) + 11y(t) + 4z(t) - 5e^{-3t} \\ z'(t) = 4x(t) - 12y(t) - 5z(t) - 7e^{-3t} \end{cases}$

Let $x(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

Then we get $x'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$

so $\dfrac{dx}{dt} = \begin{pmatrix} -5 & 12 & 4 \\ -4 & 11 & 4 \\ 4 & -12 & -5 \end{pmatrix} x(t) + b(t)$.

Let $A = \begin{pmatrix} -5 & 12 & 4 \\ -4 & 11 & 4 \\ 4 & -12 & -5 \end{pmatrix}$

$$p(\lambda) = -\lambda^3 + \lambda^2 + 5\lambda + 3$$
$$= -(\lambda - 3)(\lambda + 1)^2$$

Then $\lambda_1 = 3$, $\lambda_2 = -1$

where $E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Then $x_h(t) = a_1 e^{3t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + a_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_3 e^{-t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Page:

Let $x_p(t) = \begin{pmatrix} b_1 e^{-3t} \\ b_2 e^{-3t} \\ b_3 e^{-3t} \end{pmatrix}$

and $\frac{dx_p}{dt} = \begin{pmatrix} -3b_1 \\ -3b_2 \\ -3b_3 \end{pmatrix}$

$$\rightarrow \begin{cases} -3b_1 = -5b_1 + 12b_2 + 4b_3 + 9 \\ -3b_2 = -4b_1 + 11b_2 + 4b_3 - 5 \\ -3b_3 = 4b_1 - 12b_2 - 5b_3 - 7 \end{cases}$$

$$\Rightarrow \begin{cases} -2b_1 + 12b_2 + 4b_3 = -9 \\ -4b_1 + 14b_2 + 4b_3 = +5 \\ 4b_1 - 12b_2 - 2b_3 = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} -2 & 12 & 4 & -9 \\ -4 & 14 & 4 & 5 \\ 4 & -12 & -2 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -2 & 12 & 4 & -9 \\ 0 & -10 & -4 & 23 \\ 0 & -36 & -10 & 25 \end{array}\right)$$

Then we get b_1, b_2, b_3

so $x(t) = x_p(t) + x_h(t)$.

where a_1, a_2, a_3 are determined by initiale value.

b. $\begin{cases} x'(t) = -3x(t) - y(t) + z(t) + 9\cos t \\ y'(t) = -3x(t) - 5y(t) + 3z(t) + 4\cos t \\ z'(t) = -5x(t) - 5y(t) + 3z(t) + 7\cos t \end{cases}$

or $x'(t) = \begin{pmatrix} -3 & -1 & 1 \\ -3 & -5 & 3 \\ -5 & -5 & 3 \end{pmatrix} x(t) + b(t)$.

$A = \begin{pmatrix} -3 & -1 & 1 \\ -3 & -5 & 3 \\ -5 & -5 & 3 \end{pmatrix}$

with $P(\lambda) = -(\lambda+2)^2(\lambda+1)$

Then $\lambda_1 = -1, \lambda_2 = -2$.

Then $E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}\right\}$

$E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

Then $x_h(t) = a_1 e^{-t}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + a_2 e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_3 e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Let $x_p(t) = \begin{pmatrix} a_4\cos t + a_5 \sin t \\ a_6 \cos t + a_7 \sin t \\ a_8 \cos t + a_9 \sin t \end{pmatrix}$

and $\frac{dx_p(t)}{dt} = A x_p(t) + b(t) \to a_4, \ldots, a_9$ are determined.

Thus $x(t) = x_h(t) + x_p(t)$ where a_1, a_2, a_3 are found by initiale values.

c. $\begin{cases} x'(t) = 8x(t) + 16y(t) + 6z(t) + 12\sin t \\ y'(t) = -6x(t) - 14y(t) - 6z(t) + 5\cos t \\ z'(t) = 9x(t) + 24y(t) + 11z(t) + \cos t. \end{cases}$

$$A = \begin{pmatrix} 8 & 16 & 6 \\ -6 & -14 & -6 \\ 9 & 24 & 11 \end{pmatrix}$$

Then $P(\lambda) = -(\lambda-2)^2(\lambda-1)$

so $\lambda_1 = 2,\ \lambda_2 = 1$

and $E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 3 \\ 0 \end{pmatrix}\right\}$

$E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}\right\}$

Then $x_h(t) = a_1 e^{2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + a_2 e^{2t}\begin{pmatrix} -8 \\ 3 \\ 0 \end{pmatrix} + a_3 e^{t}\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

where $a_1, a_2, a_3 \in \mathbb{R}$.

Let $x_p(t) = \begin{pmatrix} a_4\cos t + a_5\sin t \\ a_6\cos t + a_7\sin t \\ a_8\cos t + a_9\sin t \end{pmatrix}$

Then $x_p(t)$ is satisfied $\frac{dx_p}{dt} = Ax_p + b(t)$

Then $a_4, \ldots, a_9$ are found.

Thus $\underline{x(t) = x_h(t) + x_p(t).}$

d. $x'(t) = \begin{pmatrix} -17 & 6 & 3 & 3 \\ -18 & 7 & 3 & 3 \\ -15 & 6 & 1 & 3 \\ -39 & 12 & 9 & 7 \end{pmatrix} x(t) + b(t)$

$$A = \begin{pmatrix} -17 & 6 & 3 & 3 \\ -18 & 7 & 3 & 3 \\ -15 & 6 & 1 & 3 \\ -39 & 12 & 9 & 7 \end{pmatrix}$$

$$\begin{aligned} P(\lambda) &= \lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4 \\ &= (\lambda - 1)(\lambda^3 + 3\lambda^2 - 4) \\ &= (\lambda - 1)^2(\lambda^2 + 4\lambda + 4) = (\lambda - 1)^2(\lambda + 2)^2 \end{aligned}$$

Then $\lambda_1 = 1$, $\lambda_2 = -2$.

and $E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \\ 0 \end{pmatrix} \right\}$

$E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\}$.

Then $x_h(t) = a_1 e^{t} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix} + a_2 e^{t} \begin{pmatrix} 2 \\ 5 \\ 2 \\ 0 \end{pmatrix} + a_3 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix} + a_4 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}$

By using variation of parameters

$x_p(t) = \Phi(t) \int \Phi^{-1}(t) b(t)\, dt$.

where $\Phi(t) = \begin{pmatrix} 0 & 2e^{t} & e^{-2t} & e^{-2t} \\ -e^{t} & 5e^{t} & e^{-2t} & e^{-2t} \\ 0 & 2e^{t} & 0 & 3e^{-2t} \\ 2e^{t} & 0 & 3e^{-2t} & 0 \end{pmatrix}$

Then $\underline{x(t) = x_h(t) + x_p(t)}$

6. solve the following systems of the differential equations

a. $\begin{cases} x'(t) = 2x(t) + y(t) - z(t) + 9e^{-2t}\sin t \\ y'(t) = x(t) + 3y(t) - 2z(t) + 7e^{-2t}\sin t \\ z'(t) = -x(t) + 2y(t) + z(t) - 4e^{-2t}\sin t \end{cases}$

or $\dfrac{dx}{dt} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ -1 & 2 & 1 \end{pmatrix} x + b(t).$ (1)

$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ -1 & 2 & 1 \end{pmatrix}$ $P(\lambda) = -\lambda^3 + 6\lambda^2 - 13\lambda + 10$

Then $\lambda_1 = 2,\ \lambda_2 = 2+i,\ \lambda_3 = 2-i$

and $E_{\lambda_1} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \right\}$

$E_{\lambda_2} = \operatorname{span}\left\{ \begin{pmatrix} 3+i \\ 4+3i \\ 5 \end{pmatrix} \right\},\ E_{\lambda_3} = \operatorname{span}\left\{ \begin{pmatrix} 3-i \\ 4-3i \\ 5 \end{pmatrix} \right\}$

Then $x_h(t) = a_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_2 e^{\lambda_2 t} \begin{pmatrix} 3+i \\ 4+3i \\ 5 \end{pmatrix} + a_3 e^{\lambda_3 t} \begin{pmatrix} 3-i \\ 4-3i \\ 5 \end{pmatrix}.$

$a_1, a_2, a_3 \in \mathbb{R}.$

Let $x_p(t) = \begin{pmatrix} e^{-2t}(a_4\cos t + a_5 \sin t) \\ e^{-2t}(a_6 \cos t + a_7 \sin t) \\ e^{-2t}(a_8 \cos t + a_9 \sin t) \end{pmatrix}$

where the constants are determined by substitution $x_p(t)$ into (1).

Thus $\underline{x(t) = x_h(t) + x_p(t)}$

* For the rest of exercises :

1st step: we have to write them in matrix form where $\frac{dx}{dt} = Ax + b(t)$ where $b(t)$ is the nonhomogeneous.

2st step : find the all eigenvalue and eigenvect to solve homogeneous system.

In this step there are many kinds of solution :

- If we get k distinct eigenvalues and k Linearly independent eigenvectors then :

$$x_h(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_k e^{\lambda_k t} v_k.$$

- k_1 distincts and k_2 repeated eigenvalues:

we will calculate them seperately

(i.e) since there are $k_1 + k_2$ eigenvalues (count as the sums of am). so the system of homogeneous equations must have $k_1 + k_2$ Linearly independent solutions.

- suppose for distinct eigenvalues there are $\{\lambda_1, \dots, \lambda_{k_1}\}$ and corresponding eigenvectors are $\{v_1, \dots, v_{k_1}\}$.

Then the k_1 solutions are :

$$a_1 e^{\lambda_1 t} v_1 + \cdots + a_{k_1} e^{\lambda_{k_1} t} v_{k_1}, \quad a_1, \dots, a_{k_1} \in \mathbb{R}.$$

- Talk about k_2 repeated ~~eigen~~ eigenvalues.

Let's me talk about 2 repeated eigenvalues.

There are two cases :

- suppose we have λ_1 is a repeated root of charac. polynomial $(am(\lambda_1) = 2)$.
  - If $gm(\lambda_1) = am(\lambda_1) = 2$ (diagonalizable)

    Then the solution is $e^{\lambda_1 t}(a_1 v_1 + a_2 v_2)$.
  - If $gm(\lambda_1) \neq am(\lambda_1)$ (triangularizable)

    Then we have to find its generalized eigenspace (In this case $E^2_{\lambda_1}$ is generalized eigenspace) Then let $v_2 \in E^2_{\lambda_1} - E_{\lambda_1}$

    we will get $v_1 = (A - \lambda_1 I) v_2$

    Then the solution will be :

$$x(t) = a_1 e^{\lambda_1 t} v_2 + a_2 e^{\lambda_1 t}(v_1 + t v_2)$$

as you can see : for the higher order (3, 4. ..) we solve it in the same method. Just the Larger repearted root the Longer we will suffer.

- 3rd step: If the system is nonhomogeneous. we can use Undetermined coefficients and variation of parameters to find the particular solution.

  + If the homogeneous solution is trivial we can guess the particular solution and use undetermined coefficient.

- If the homogeneous solution is disorganized or similar to b(t).
  we should use variation of parameters to avoid from getting wrong answer.

$$x_p(t) = \Phi(t) \int \Phi^{-1}(t)\, b(t)\, dt.$$

Remark: This integral is unnecessarily to plus constants because it is the particular solution.

4th step: If the given problem is IVP (initiale values problem) we have to find the constants by using those initiale values.

⑦ Solve the following systems of ordinary differential equation

(a). $\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ -5 & 3 & 3 \\ -4 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 9e^{2t} \\ 7e^{2t} \\ -4e^{2t} \end{pmatrix}$

Let $X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$, $A = \begin{pmatrix} -2 & 1 & 2 \\ -5 & 3 & 3 \\ -4 & 1 & 4 \end{pmatrix}$, $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $B(t) = \begin{pmatrix} 9e^{2t} \\ 7e^{2t} \\ -4e^{2t} \end{pmatrix}$

for Jordanize A

$P(\lambda) = \begin{vmatrix} -2-\lambda & 1 & 2 \\ -5 & 3-\lambda & 3 \\ -4 & 1 & 4-\lambda \end{vmatrix} = -(\lambda-2)^2(\lambda-1)$

For $\lambda_1 = 2$: $(A-\lambda_1 I)u = 0$

$\begin{pmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -4 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$

For $\lambda_2 = 1$: $(A-\lambda_1 I)^2 u = 0$

$\begin{pmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E^2_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$

Let $V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 1$: $(A-\lambda_1 I)u = 0$

$\begin{pmatrix} -3 & 1 & 2 \\ -5 & 2 & 3 \\ -4 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$\Rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Then $X_h(t) = c_1 e^{2t}\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} + c_2 e^{2t}\left[\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right] + c_3 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Let $\Phi(t) = \begin{pmatrix} -e^{t} & -te^{t} & e^{t} \\ -2e^{t} & -2te^{2t} & e^{t} \\ -e^{2t} & te^{2t} - e^{2t} & e^{t} \end{pmatrix}$

$$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix} -e^{-2t}(t-2) & e^{-2t} & e^{-2t}(t-1) \\ e^{-2t} & 0 & e^{-2t} \\ 3e^{-t} & e^{-t} & -e^{-t} \end{pmatrix}$$

$$\Rightarrow \Phi^{-1}(t)B(t) = \begin{pmatrix} -13t+29 \\ 5 \\ 24e^{t} \end{pmatrix} \Rightarrow \int \Phi^{-1}(t)B(t)dt = \begin{pmatrix} -\frac{13}{2}t^2+29t \\ 5t \\ 24e^{t} \end{pmatrix}$$

$$X_p(t) = \Phi(t)\int \Phi^{-1}(t)B(t)dt = \begin{pmatrix} -e^{2t} & -te^{2t} & e^{t} \\ -2e^{2t} & -2te^{2t}+e^{2t} & e^{t} \\ -e^{2t} & -te^{2t}-e^{2t} & e^{t} \end{pmatrix}\begin{pmatrix} -\frac{13}{2}t^2+29t \\ 5t \\ 24e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{23}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix} t^2e^{2t} + \begin{pmatrix} -29 \\ -54 \\ -34 \end{pmatrix} te^{2t} + \begin{pmatrix} 24 \\ 24 \\ 24 \end{pmatrix} e^{t}$$

$$\boxed{\begin{aligned} \underline{\text{Thus}} \quad X(t) &= X_h(t) + X_p(t) \\ &= c_1e^{2t}\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} + c_2e^{2t}\left[\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right] + c_3e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{23}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix} t^2e^{2t} + \begin{pmatrix} -24 \\ -54 \\ -34 \end{pmatrix} te^{2t} + \begin{pmatrix} 24 \\ 24 \\ 24 \end{pmatrix} e^{2t} \\ &\quad c_1, c_2, c_3 \in \mathbb{R} \end{aligned}}$$

(b) $\begin{pmatrix}x'(t)\\y'(t)\\z'(t)\end{pmatrix} = \begin{pmatrix}4&18&8\\-5&-16&-6\\7&19&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}2\\-2\\-4\end{pmatrix}e^{-2t}$

Let $x' = \begin{pmatrix}x'(t)\\y'(t)\\z'(t)\end{pmatrix}$, $A = \begin{pmatrix}4&18&8\\-5&-16&-6\\7&19&6\end{pmatrix}$, $x(t) = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $B = \begin{pmatrix}2\\-4\\-4\end{pmatrix}e^{-2t}$

Jordanize A

$P(\lambda) = \begin{vmatrix}4-\lambda&18&8\\-5&-16-\lambda&-6\\7&19&6\end{vmatrix} = -(\lambda+2)^3$

For $\lambda=-2 \Rightarrow (A+2I)u = 0$

$\begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix} \sim \begin{pmatrix}1&0&-2/3\\0&1&2/3\\0&0&0\end{pmatrix} \Rightarrow E_\lambda = \text{span}\left\{\begin{pmatrix}2\\-2\\3\end{pmatrix}\right\}$

For $\lambda=-2 \Rightarrow (A+2I)^2 u = 0$

$\begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix}\begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix} = \begin{pmatrix}2&8&4\\-2&-8&-4\\3&12&6\end{pmatrix} \sim \begin{pmatrix}1&4&2\\0&0&0\\0&0&0\end{pmatrix}$

$\Rightarrow E_\lambda^2 = \text{span}\left\{\begin{pmatrix}-4\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$

$(A+2I)^3 u = 0$

$\begin{pmatrix}2&8&4\\-2&-8&-4\\3&12&6\end{pmatrix}\begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$\Rightarrow E_\lambda^3 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

Let $v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}6\\-5\\7\end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix}6&18&8\\-5&-14&-6\\7&19&8\end{pmatrix}\begin{pmatrix}6\\-5\\7\end{pmatrix} = \begin{pmatrix}2\\-2\\3\end{pmatrix}$

Then $x_h(t) = c_1\begin{pmatrix}2\\-2\\3\end{pmatrix}e^{-2t} + c_2\left[t\begin{pmatrix}2\\-2\\3\end{pmatrix} + \begin{pmatrix}6\\-5\\7\end{pmatrix}\right]e^{-2t} +$

$c_3\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}6\\-5\\7\end{pmatrix}t + \begin{pmatrix}2\\-2\\3\end{pmatrix}\frac{t^2}{2}\right]e^{-2t}$

Let $\Phi(t)=\begin{pmatrix} 2e^{-2t} & 2te^{-2t}+6e^{-2t} & t^2e^{-2t}+6te^{-2t}+e^{-2t} \\ -2e^{-2t} & -2te^{-2t}-5e^{-2t} & -t^2e^{-2t}-5te^{-2t} \\ 3e^{-2t} & 3te^{-2t}+7e^{-2t} & \frac{3}{2}t^2e^{-2t}+7te^{-2t} \end{pmatrix}$

$\Rightarrow \Phi^{-1}(t)=\begin{pmatrix} \frac{t^2}{2}e^{2t} & (t^2+3t+7)e^{2t} & (t^2+2t+5)e^{2t} \\ -te^{2t} & (-4t+3)e^{2t} & (-2t-2)e^{2t} \\ e^{2t} & 4e^{2t} & 2e^{2t} \end{pmatrix}$

$\Rightarrow \Phi^{-1}(t)B(t)=\begin{pmatrix} \frac{t^2}{2}e^{2t} & (t^2+3t+7)e^{2t} & (t^2+2t+5)e^{2t} \\ -te^{2t} & (-4t+3)e^{2t} & (-2t-2)e^{2t} \\ e^{2t} & 4e^{2t} & 2e^{2t} \end{pmatrix}\begin{pmatrix} 2e^{-2t} \\ -2e^{-2t} \\ -4e^{-2t} \end{pmatrix}$

$=\begin{pmatrix} -14t-34-5t^2 \\ 14t+14 \\ -14 \end{pmatrix}$

$\Rightarrow \int \Phi^{-1}(t)B(t)\,dt=\begin{pmatrix} -7t^2-34t-\frac{5t^3}{3} \\ 7t^2+14t \\ -14t \end{pmatrix}$

Then $X_p(t)=\Phi(t)\int\Phi^{-1}(t)B(t)\,dt$

$=\begin{pmatrix} 2e^{-2t} & 2te^{-2t}+6e^{-2t} & t^2e^{-2t}+6te^{-2t}+e^{-2t} \\ -2e^{-2t} & -2te^{-2t}-5e^{-2t} & -t^2e^{-2t}-5te^{-2t} \\ 3e^{-2t} & 3te^{-2t}+7e^{-2t} & \frac{3}{2}t^2e^{-2t}+7te^{-2t} \end{pmatrix}\begin{pmatrix} -7t^2-34t-\frac{5t^3}{3} \\ 7t^2+14t \\ -14t \end{pmatrix}$

$=\begin{pmatrix} 14t(t+2)(t+3)e^{-2t}-14t(t^2+6t+1)e^{-2t}-\frac{2}{3}t(5t^2+21t+102)e^{-2t} \\ 14t^2(t+5)e^{-2t}-7t(t+2)(2t+5)e^{-2t}+\frac{2}{3}t(5t^2+21t+102)e^{-2t} \\ 7t(3t+7)(t+2)e^{-2t}-7t(3t^2+14)e^{-2t}-t(5t^2+21t+102)e^{-2t} \end{pmatrix}$

Thus $X(t)=X_h(t)+X_p(t)$

(c) $\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 8 & 18 & 8 \\ -5 & -12 & -6 \\ 7 & 19 & 10 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 4e^{2t} \\ -6e^{2t} \\ 8e^{2t} \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

let $X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$, $A = \begin{pmatrix} 8 & 18 & 8 \\ -5 & -12 & -6 \\ 7 & 19 & 10 \end{pmatrix}$, $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $B(t) = \begin{pmatrix} 4e^{2t} \\ -6e^{2t} \\ 8e^{2t} \end{pmatrix}$

Jordanize A

$$P_\lambda : \begin{vmatrix} 8-\lambda & 18 & 8 \\ -5 & -12-\lambda & -6 \\ 7 & 19 & 10-\lambda \end{vmatrix} = -(\lambda-2)^3$$

- For $\lambda = 2$, $(A-2I)v = 0$

$$\begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_\lambda = \text{span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right\}$$

- $(A-2I)^2 v = 0$

$$\begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix} \cdot \begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 4 \\ -2 & -8 & -4 \\ 3 & 12 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_\lambda^2 = \text{span}\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- $(A-2I)^3 v = 0$

$$\begin{pmatrix} 2 & 8 & 4 \\ -2 & -8 & -4 \\ 3 & 12 & 6 \end{pmatrix} \begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_\lambda^3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Let $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix}$

$$\Rightarrow V_1 = \begin{pmatrix} 6 & 18 & 8 \\ -5 & -14 & -6 \\ 7 & 19 & 8 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$X_h(t) = c_1\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}e^{2t} + c_2\left(\begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}\right)e^{2t} + c_3\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix}t + \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}\frac{t^2}{2}\right)e^{2t}$$

$c_1, c_2, c_3 \in \mathbb{R}$

$$\text{let } \Phi(t) = \begin{pmatrix} 2e^{2t} & (6+2t)e^{2t} & (1+6t+t^2)e^{2t} \\ -2e^{2t} & (-5-2t)e^{2t} & (-5t-t^2)e^{2t} \\ 3e^{2t} & (7+3t)e^{2t} & (7t+\frac{3}{2}t^2)e^{2t} \end{pmatrix}$$

$$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix} -\frac{t^2}{2} & (t^2+3t+7)e^{-2t} & (t^2+2t+5)e^{-2t} \\ -t^2 e^{-2t} & (-4t-3)e^{-2t} & (-2t-2)e^{-2t} \\ e^{-2t} & 4e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$\Rightarrow \Phi^{-1}(t)B(t) = \begin{pmatrix} \frac{t^2}{2} & (t^2+3t+7)e^{-2t} & (t^2+2t+5)e^{-2t} \\ -t^2 e^{-2t} & (-4t+3)e^{-2t} & (-2t-2)e^{-2t} \\ e^{-2t} & 4e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} 4e^{2t} \\ -6e^{2t} \\ 8e^{2t} \end{pmatrix} = \begin{pmatrix} -2t-2+4t^2 \\ 4t-2 \\ -4 \end{pmatrix}$$

$$\Rightarrow \int \Phi^{-1}(t)B(t)dt = \begin{pmatrix} -t^2-2t+\frac{4t^3}{3} \\ 2t^2-2t \\ -4t \end{pmatrix}$$

$$\Rightarrow X_p(t) = \Phi(t)\int \Phi^{-1}(t)B(t)dt$$

$$= \begin{pmatrix} 2e^{2t} & (6+2t)e^{2t} & (1+6t+t^2)e^{2t} \\ -2e^{2t} & (-5-2t)e^{2t} & (-5t-t^2)e^{2t} \\ -3e^{2t} & (7+3t)e^{2t} & (7t+\frac{3}{2}t^2)e^{2t} \end{pmatrix}\begin{pmatrix} -t^2-2t+\frac{4}{3}t^3 \\ 2t^2-2t \\ -4t \end{pmatrix}$$

$$= \begin{pmatrix} 4t(t-1)(t+3)e^{2t} - 4t(t^2+6t+1)e^{2t} - \frac{2}{3}t(4t^2+3t+6)e^{2t} \\ 4t(t+5)e^{2t} - 2t(t-1)(2t+5)e^{2t} + \frac{2}{3}t(4t^2+3t+6)e^{2t} \\ 2t(t-1)(3t+7)e^{-2t} - 2t(3t^2+14)e^{2t} + t(4t^2+3t+6)e^{2t} \end{pmatrix}$$

for $\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ & $X(t) = X_h(t) + X_p(t)$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}C_1 + C_2\begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix} + C_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} C_1 = 5/29 \\ C_2 = 2/29 \\ C_3 = -22/29 \end{cases}$$

<u>Thus</u>

$$X(t) = \frac{5}{29}\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}e^{2t} + \frac{2}{29}\left[\begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}\right]e^{2t} + \left(-\frac{22}{29}\right)\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -5 \\ 7 \end{pmatrix}t + \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}\frac{t^2}{2}\right]e^{2t}$$

$$+ \begin{pmatrix} 4t(t-1)(t+3)e^{2t} - 4t(t^2+6t+1)e^{2t} - \frac{2}{3}t(4t^2+3t+6)e^{2t} \\ 4t(t+5)e^{2t} - 2t(t-1)(2t+5)e^{2t} + \frac{2}{3}t(4t^2+3t+6)e^{2t} \\ 2t(t-1)(3t+7)e^{-2t} - 2t(3t^2+14)e^{2t} + t(4t^2+3t+6)^{2t} \end{pmatrix}$$