CHAPTER II
Ordinary Differential Equations

Second Year of Engineering Program
Department of Foundation Year

LIN Mongkolsery
sery@itc.edu.kh

2021-2022

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Definition and Terminology

Definition 1

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

CLASSIFICATION BY TYPE

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE)**.

For example,

$$\frac{dy}{dt} + 5y = e^t, \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} + 6y = 0, \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

are ordinary differential equations.

Definition and Terminology

CLASSIFICATION BY TYPE

An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.

For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

are partial differential equations.

Definition and Terminology

CLASSIFICATION BY ORDER

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.
For example,

$$\frac{d^2y}{dt^2} + 5\left(\frac{dy}{dt}\right)^3 - 4y = e^t$$

is a second-order ordinary differential equation.

Definition 2

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Definition and Terminology

General and Normal Form

A **general form** of mth-order ordinary differential equation in one dependent variable is of the form

$$F\left(t, y, y', \ldots, y^{(m)}\right) = 0, \tag{1}$$

where F is a real-valued function of $m+2$ variables $t, y, y', \ldots, y^{(m)}$ and defines on U.

The **normal form** is defined by

$$\frac{d^m y}{dt^m} = f\left(t, y, y', \ldots, y^{(m-1)}\right) \tag{2}$$

where f is a real-valued continuous function.

Definition and Terminology

CLASSIFICATION BY LINEARITY

An mth-order ODE is said to be **linear** if F is linear in $y, y', \ldots, y^{(m)}$. This means that an mth-order ODE is linear when

$$a_m(t)y^{(m)} + a_{m-1}(t)y^{(m-1)} + \cdots + a_0(t)y = g(t) \tag{3}$$

An mth-order linear ODE is **homogeneous** if $g\,(t) = 0$, that is the equation

$$a_m\,(t)\,y^{(m)} + \cdots + a_1\,(t)\,y' + a_0\,(t)\,y = 0 \tag{4}$$

Otherwise it is said to be **nonhomogeneous**.

Initial-Value Problems

Definition 3

On some interval I containing x_0 the problem is

Solve:

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

Subject to:

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem (IVP)**.

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

are called **initial conditions**.

Contents

First Order Differential Equations

Definition 4

The **first order differential equation** is of the form

$$\frac{dy}{dt} = f\left(t, y\right)$$

or

$$F(t, y, y') = 0$$

Definition 5

A differential equation $F\left(t, y, y'\right) = 0$ is said to be **separable** or to have **separable variables** if it can be written in the form

$$\alpha\left(t\right) dt = \beta\left(y\right) dy$$

Example 6

Solve the initial value problem

$$y' = \frac{dy}{dt} = \frac{t}{y + t^2 y}, \quad y(0) = -1$$

Solution: The equation can be written as a seperable variable equation

$$\begin{aligned} y\,dy &= \frac{t}{(1+t^2)}\,dt \\ \int y dy &= \int \frac{t}{1+t^2} dt \\ \frac{y^2}{2} &= \frac{1}{2}\int \frac{1}{1+t^2} d\left(1+t^2\right) \\ y^2 &= \ln\left(1+t^2\right) + C \end{aligned}$$

since $y(0) = -1$, then $C = 1$. Thus

$$y^2 = 1 + \ln\left(1+t^2\right) \quad \text{or} \quad y = -\sqrt{1 + \ln\left(1+t^2\right)}$$

First Order Differential Equations

Definition 7

A differential equation of the form

$$M(t,y)\,dt + N(t,y)\,dy = 0$$

is said to be **exact** if the expression on the first side corresponds to the differential of some real function of two real variables.

Theorem 1

Let $M(t,y)$ and $N(t,y)$ be continuous and have continuous partial derivatives on a rectangular region. Then a necessary and sufficient condition that $M(t,y)\,dt + N(t,y)\,dy = 0$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

First Order Differential Equations

Definition 8

A real function $I = I(t, y)$ for which the differential equation

$$IM(t, y)\, dt + IN(t, y)\, dy = 0$$

is exact is called an **integrating factor**.

Definition 9

Let f be a real function defined on region $D \subset \mathbb{R}^m$. f is said to be a **homogeneous function** of degree $\alpha \in \mathbb{R}$ if

$$f(\lambda x_1, \ldots, \lambda x_m) = \lambda^\alpha f(x_1, \ldots, x_m)$$

for all $x_1, \ldots, x_m, \lambda$ for which $(\lambda x_1, \ldots, \lambda x_m) \in D$.

Theorem 2

Let $M(t,y)$ and $N(t,y)$ be continuous and have continuous partial derivatives in a rectangular region (more general simply connected region).

- If $\dfrac{M(t,y)}{N(t,y)}$ is homogeneous of degree zero then $\dfrac{1}{tM(t,y)+yN(t,y)}$ is an integrating factor.
- If $M(t,y)=y\alpha(ty)$ and $N(t,y)=t\beta(ty)$ then $\dfrac{1}{tM(t,y)-yN(t,y)}$ is an integrating factor.
- If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial t}\right)=\alpha(t)$ then $e^{\int \alpha(t)dt}$ is an integrating factor.
- If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial t}-\dfrac{\partial M}{\partial y}\right)=\beta(y)$ then $e^{\int \beta(y)dy}$ is an integrating factor.
- If $\dfrac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{N-M}=\beta(u), u=t+y$ then $e^{\int \beta(u)du}$ is an integrating factor.

- If $\dfrac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{N+M}=\beta\left(u\right), u=t-y$ then $e^{\int \beta(u)du}$ is an integrating factor.
- If $\dfrac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{2tN-2yM}=\beta\left(u\right), u=t^2+y^2$ then $e^{\int \beta(u)du}$ is an integrating factor.
- If $\dfrac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{2tN+2yM}=\beta\left(u\right), u=t^2-y^2$ then $e^{\int \beta(u)du}$ is an integrating factor.
- If $\dfrac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{yN-tM}=\beta\left(u\right), u=ty$ then $e^{\int \beta(u)du}$ is an integrating factor.
- If $\dfrac{t^2\left(\frac{\partial N}{\partial t}-\frac{\partial M}{\partial y}\right)}{tM+yN}=\beta\left(u\right), u=\dfrac{y}{t}$ then $e^{\int \beta(u)du}$ is an integrating factor.

- If $\dfrac{t^2\left(\frac{\partial N}{\partial t}-\frac{\partial M}{\partial y}\right)+ntN}{tM+yN}=\beta(u),\ u=\dfrac{y}{t}$ then $t^n e^{\int \beta(u)du}$ is an integrating factor.
- If $\dfrac{y^2\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)}{tM+yN}=\beta(u),\ u=\dfrac{t}{y}$ then $e^{\int \beta(u)du}$ is an integrating factor.
- A function of the form $I=t^a y^b$, for some values of a, b, is an integrating factor of $y\left(c_1 t^p y^q+c_2 t^r y^s\right)dt+t\left(c_3 t^p y^q+c_4 t^r y^s\right)dy$.
- If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{N\frac{\partial u}{\partial t}-M\frac{\partial u}{\partial y}}=\beta(u),\ u=u(t,y)$ then $e^{\int \beta(u)du}$ is an integrating factor.
- If I_1 is an integrating factor and G_1 is a solution of M_1dx+N_1dy and I_2 is an integrating factor and G_2 is a solution of M_2dx+N_2dy then we can find functions f_1, f_2 such that $I=I_1f_1(G_1)=I_2f_2(G_2)$ is an integrating factor of $(M_1+M_2)dx+(N_1+N_2)dy$.

Etact Equations and Non-exact Equations

Example 10

Solve

$$y' = \frac{dy}{dt} = \frac{e^y + t}{e^{2y} - te^y}$$

Solution: Rewrite the equation as

$$\left(e^y + t\right) dt + \left(te^y - e^{2y}\right) dy = 0$$

since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^y + t\right) = e^y = \frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(te^y - e^{2y}\right)$$

the equation is etact. Set

$$F(t, y) = \int \left(e^y + t\right) dt = te^y + \frac{1}{2}t^2 + g(y)$$

We want

$$\frac{\partial F}{\partial y} = te^y - e^{2y}$$

$$te^y + g'(y) = te^y - e^{2y}$$

$$g'(y) = -e^{2y}$$

Therefore we may choose $g(y) = -\frac{1}{2}e^{2y}$ and the solution is

$$te^y + \frac{1}{2}t^2 - \frac{1}{2}e^{2y} = C$$

Example 11

Solve

$$\frac{dy}{dt} = \frac{ty - y^2}{t^2 + 3y^2}$$

Solution: Rewrite the equation as

$$(y^2 - ty)dt + (t^2 + 3y^2)dy = 0$$

since

$$\frac{\partial}{\partial y}(y^2 - ty) = 2y - t \neq \frac{\partial}{\partial t}(t^2 + 3y^2) = 2t$$

Therefore, the equation is not etact. However,

$$f(t, y) = \frac{M(t, y)}{N(t, y)} = \frac{y^2 - ty}{t^2 + 3y^2}$$

is an homogeneous function of degree zero. Then, an integrating factor is

$$I(t, y) = \frac{1}{tM(t, y) + yN(t, y)} = \frac{1}{ty^2 + 3y^3}$$

Multiply the equation by $I(t, y)$, we have

$$\frac{y-t}{ty+3y^2}dt + \frac{t^2+3y^2}{ty^2+3y^3}dy = 0$$

this is an exact equation. We have

$$f(t,y) = \int \frac{y-t}{ty+3y^2}dt = 4\ln(t+3y) - \frac{t}{y} + C(y)$$

we want

$$\Longrightarrow \quad \frac{\partial f}{\partial y}(t,y) = \frac{12}{t+3y} - \frac{t}{y^2} + C'(y) = \frac{12y^2+t^2+3ty}{ty^2+3y^3} + C'(y)$$

$$\Longrightarrow \quad \frac{\partial f}{\partial y}(t,y) = \frac{t^2+3y^2}{ty^2+3y^3} + \frac{3}{y} + C'(y)$$

$$\Longrightarrow \quad C'(y) = -\frac{3}{y} \quad \Longrightarrow \quad C(y) = -3\ln y + k$$

Thus the solution of the equation is

$$4\ln(t+3y) - \frac{t}{y} - 3\ln y = k$$

Example 12

$$\left(3t^2y + 2ty + y^3\right) dt + \left(t^2 + y^2\right) dy = 0$$

Solution. Here

$$M(t, y) = 3t^2y + 2ty + y^3$$
$$N(t, y) = t^2 + y^2$$

since

$$\frac{\partial M}{\partial y} = 3t^2 + 2t + 3y^2 \neq 2t = \frac{\partial N}{\partial t}$$

this equation is not etact.

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = \frac{3t^2 + 2t + 3y^2 - 2t}{t^2 + y^2} = \frac{3\left(t^2 + y^2\right)}{t^2 + y^2} = 3$$

it does not depend on y. The integrating factor is

$$I(t) = e^{\int 3dt} = e^{3t}$$

Multiply the original equation by I, then it will transform into an etact equation. Therefore, the general solution is

$$e^{3t}t^2y + e^{3t}y^3 = C$$

Let $y = tu$

Theorem 13

If the equation

$$\frac{dy}{dt} = f(t, y)$$

whre f is a homogeneous function of degree zero, then the equation can be transfromed into a separable equation by substitute $y = tu$.

Example 14

Solve

$$\frac{dy}{dt} = \frac{ty - y^2}{t^2 + 3y^2}$$

Solution: The function $F(t, y) = \dfrac{ty - y^2}{t^2 + 3y^2} = \dfrac{y/t - y^2/t^2}{1 + 3y^2/t^2}$ is a homogeneous function of degree 0. Now, let $y = tu$, then

$$\frac{dy}{dt} = u + t\frac{du}{dt}$$

Substitute this into the equation, we have

$$u + t\frac{du}{dt} = \frac{u - u^2}{1 + 3u^2} \iff \frac{1 + 3u^2}{u^2 + 3u^3}du = -\frac{1}{t}dt$$

$$4\ln(3u + 1) - 3\ln u - \frac{1}{u} = -\ln t + C$$

$$4\ln(3y + t) - 3\ln y - \frac{t}{y} = C$$

Theorem 15

The equation

$$\frac{dy}{dt} = F\left(\frac{y}{t}\right)$$

can be transfromed into a separable equation by substitute $y = tu$.

Example 16

Solve

$$\left(y + 2te^{-y/t}\right) dt - tdy = 0$$

Solution: Rewrite the equation as

$$\frac{dy}{dt} = \frac{y + 2te^{-y/t}}{t} = \frac{y}{t} + 2e^{-y/t}$$

Let $u = y/t$, we have

$$u + t\frac{du}{dt} = u + 2e^{-u}$$

$$e^u du = 2\frac{dt}{t}$$

$$\int e^u du = \int 2\frac{dt}{t}$$

$$e^u = 2\ln t + C$$

$$e^{y/t} - 2\ln t = C$$

Linear Fraction equation

Study the first order ODE of the form

$$y' = \frac{at + by + c}{At + By + C}$$

- Solve the system

$$\begin{cases} at + by + c = 0 \\ At + By + C = 0 \end{cases}$$

(1) If the system has a solution $(t, y) = (t_0, y_0)$. Then the oginal equation can be written as

$$z' = \frac{dz}{dx} = \frac{ax + bz}{Ax + Bz} = \frac{a + b\frac{z}{x}}{A + B\frac{z}{x}}$$

where $x = t - t_0$ and $z = y - y_0$

Linear Fraction equation

- Let $z = xu$, then $\dfrac{dz}{dx} = u + x\dfrac{du}{dx}$. The last equation, can be written as

$$u + x\frac{du}{dx} = \frac{a + bu}{A + Bu}$$

$$\frac{A + Bu}{a + (b - A)u - Bu^2}du = \frac{1}{x}dx$$

this is a separable variable equation.

(2) If the system has no a unique solution, then there is a constant k such that

$$at + by = k(At + By)$$

Let $z = at + by$. Then $\dfrac{dz}{dt} = a + b\dfrac{dy}{dt}$. The original equation will be transformed in to

$$z' - a = bk\frac{z + c}{z + kC}$$

this is a separable variable equation.

Linear Fraction equation

Example 17

Solve

$$y' = \frac{-t + 3y - 3}{t + y - 1}$$

Example 18

Solve

$$y' = \frac{2t + y - 3}{4t + 2y - 1}$$

Definition 19

A differential equation of the form

$$a_1(t)\,y' + a_0(t)\,y = b(t)$$

is said to be a **first order linear equation** in the dependent variable y.

To solve **first order linear ode**, we first transform it into a standard form

$$y' + p(t)y = f(t)$$

The integration factor of the last equation is

$$I(t) = e^{\int p(t)dt}$$

After multimlying the standard equation by I, we have

$$(I(t)y)' = I(t)f(t) \quad \Longrightarrow \quad I(t)y = \int I(t)f(t)dt$$

Example 20

Solve

$$\left(t^2 - 1\right) y' + ty = 2t, t > 1$$

Solution: Dividing both sides by $t^2 - 1$, the equation becomes

$$\frac{dy}{dt} + \frac{t}{t^2-1}y = \frac{t}{t^2-1}$$

Now

$$-\int \frac{t}{t^2-1}dt = \frac{1}{2}\ln\left(t^2-1\right) + C$$

Thus we multiply both sides of the equation by

$$\exp\left(\frac{1}{2}\ln\left(t^2-1\right)\right) = \left(t^2-1\right)^{\frac{1}{2}}$$

and get

$$\begin{aligned}
\left(t^2-1\right)^{\frac{1}{2}}\frac{dy}{dt} + \frac{t}{(t^2-1)^{\frac{1}{2}}}y &= \frac{2t}{(t^2-1)^{\frac{1}{2}}} \\
\frac{d}{dt}\left(\left(t^2-1\right)^{\frac{1}{2}}y\right) &= \frac{2t}{(t^2-1)^{\frac{1}{2}}} \\
y &= \left(t^2-1\right)^{-\frac{1}{2}}\left(2\left(t^2-1\right)^{\frac{1}{2}} + C\right)
\end{aligned}$$

Definition 21

A differential equation of the form

$$y' + a(t)y = b(t)y^n, \quad n \neq 0, 1$$

is called **Bernoulli's equation**.

The **Bernoulli's equation** is a non-linear equation and $y(t) = 0$ is always a solution when $n > 0$. To find a non-trivial solution, we use the substitution $u = y^{1-n}$. Then

$$\begin{aligned} \frac{du}{dt} &= (1-n)y^{-n}\frac{dy}{dt} = (1-n)y^{-n}\left(-a(t)y + b(t)y^n\right) \\ \frac{du}{dt} &+ (1-n)a(t)y^{1-n} = (1-n)b(t) \\ \frac{du}{dt} &+ (1-n)a(t)u = (1-n)b(t) \end{aligned}$$

which is a linear differential equation of u

Note: Don't forget that $y(t) = 0$ is always a solution to the Bernoulli's equation when $n > 0$.

Example 22

Solve

$$t\frac{dy}{dt} + y = ty^3$$

Solution: Let $u = y^{1-3} = y^{-2}$

$$\begin{aligned}
\frac{du}{dt} &= -2y^{-3}\frac{dy}{dt} \\
\frac{du}{dt} &= -\frac{2y^{-3}}{t}\left(-y + ty^3\right) \\
\frac{du}{dt} - \frac{2y^{-2}}{t} &= -2 \\
\frac{du}{dt} - \frac{2u}{t} &= -2
\end{aligned}$$

Bernoulli's equation

which is a linear equation of u. To solve it, multiply both side by $\exp\left(-\int 2t^{-1}dt\right) = t^{-2}$, we have

$$\begin{aligned}
t^{-2}\frac{du}{dt} - 2t^{-3}u &= -2t^{-2} \\
\frac{d}{dt}\left(t^{-2}u\right) &= -2t^{-2} \\
t^{-2}u &= 2t^{-1} + C \\
u &= 2t + Ct^2 \\
y^{-2} &= 2t + Ct^2 \\
y^2 &= \frac{1}{2t + Ct^2} \text{ or } y = 0
\end{aligned}$$

Definition 23

A differential equation of the form

$$y' = \alpha(t) + \beta(t)\, y + \chi(t)\, y^2$$

is called **Ricatti equation**.

Theorem 3

Suppose that $y(t) = y_1(t)$ is a particular solution of the **Ricatti equation**, then the equation can be transformed, using the substitution

$$y = y_1 + \frac{1}{u}$$

to a linear equation of u.
The substitution $y = y_1 + u$ reduces the equation to a Bernoulli's equation.

Example 24

Solve the Riccati's equation given that $y = t$ is a particular solution.

$$y' - \frac{y}{t} = 1 - \frac{y^2}{t^2}$$

Solution: Let

$$y = t + \frac{1}{u}$$

We have

$$\begin{aligned}
\frac{dy}{dt} &= 1 - \frac{1}{u^2}\frac{du}{dt} \\
\frac{1}{t}y + 1 - \frac{1}{t^2}y^2 &= 1 - \frac{1}{u^2}\frac{du}{dt} \\
\frac{1}{u^2}\frac{du}{dt} &= \frac{1}{t^2}\left(t + \frac{1}{u}\right)^2 - \frac{1}{t}\left(t + \frac{1}{u}\right) \\
\frac{1}{u^2}\frac{du}{dt} &= \frac{1}{tu} + \frac{1}{t^2u^2}
\end{aligned}$$

$$\frac{du}{dt} - \frac{1}{t}u = \frac{1}{t^2}$$

which is a linear equation of u. An integrating factor is

$$\exp\left(-\int \frac{1}{t}dt\right) = \exp(-\ln t) = t^{-1}$$

Thus

$$\begin{aligned} t^{-1}\frac{du}{dt} - t^{-2}u &= t^{-3} \\ \frac{d}{dt}\left(t^{-1}u\right) &= t^{-3} \\ t^{-1}u &= -\frac{1}{2t^2} + C'' \\ u &= -\frac{1}{2t} + C't \end{aligned}$$

Therefore the general solution is

$$y = t + \frac{2t}{Ct^2 - 1} \text{ or } y = t$$

Lagrange's equation

Definition 25

A differential equation of the form

$$y = t\alpha\left(y'\right) + \beta\left(y'\right)$$

is called **Lagrange equation**.

To solve this equation, we let $p = y'$, then $dy = p\,dt$. The original equation transforms to

$$y = t\alpha(p) + \beta(p)$$

Differentiate both sides of the original equation with respect to p, we have

$$\frac{dy}{dp} = \frac{dt}{dp}\alpha(p) + t\alpha'(p) + \beta'(p)$$

$$p\frac{dt}{dp} = \frac{dt}{dp}\alpha(p) + t\alpha'(p) + \beta'(p)$$

Lagrange's equation

$$(p-\alpha(p))\frac{dt}{dp}-\alpha'(p)t-\beta'(p)=0$$

This is the first order liner equation of t.

Example 26

Solve

$$y = ty' + (y')^2$$

Solution. Let $p = y'$. The original equation can be written as

$$y = tp + p^2 \quad \Longrightarrow \quad p\frac{dt}{dp} = p\frac{dt}{dp} + t + 2p$$

$$t + 2p = 0 \quad \Longleftrightarrow \quad \frac{dy}{dt} = -\frac{1}{2}t \quad \Longrightarrow y = -\frac{1}{4}t^2 + C$$

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Higher Order Differential Equations

Definition 27

The linear mth-order differential equation is of the form

$$a_m(t)y^{(m)} + a_{m-1}(t)y^{(m-1)} + \cdots + a_0(t)y = g(t) \tag{5}$$

and its homogenous equation is

$$a_m(t)\, y^{(m)} + \cdots + a_1(t)\, y' + a_0(t)\, y = 0 \tag{6}$$

Higher Order Differential Equations

Definition 28

Let $a_m, \ldots, a_1, a_0$ be constants with $a_m \neq 0$. The **characteristic equation** of

$$a_m y^{(m)} + \cdots + a_1 y' + a_0 y = 0$$

is given by

$$a_m \lambda^m + \cdots + a_1 \lambda + a_0 = 0.$$

Definition 29

A set of function $f_1(t), \ldots, f_m(t)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, \ldots, c_m$, not all zero, such that

$$c_1 f_1(t) + \cdots + c_m f_m(t) = 0$$

for every t in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Higher Order Differential Equations

Definition 30

Let $f_1(t), \ldots, f_m(t)$ be m functions being at least $m-1$ times differentiable on an interval I. The determinant

$$W\left(f_1, \ldots, f_m\right) = \begin{vmatrix} f_1 & f_2 & \ldots & f_m \\ f_1' & f_2' & \ldots & f_m' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \ldots & f_m^{(m-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions $f_1, \ldots, f_m$.

Theorem 4

Let $y_1, \ldots, y_m$ be an m solutions of linear mth-order differential equation (6) on an interval I, Then the set of solutions **linear independent** on I if $W\left(y_1, \ldots, y_m\right) \neq 0$ for every t in the interval.

Example 31

Given $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ be two solutions of solutions *of*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

Find the general solution of the equation.

Solution: It is easy to check that y_1 and y_2 are solutions to the equation. Now

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}$$

is not identically zero. Thus the general solution is

$$y = c_1y_1 + c_2y_2 = c_1t^{1/2} + c_2t^{-1}$$

Theorem 5

The set of all solutions of a homogeneous differential equation

$$a_m(t)\, y^{(m)} + \cdots + a_1(t)\, y' + a_0(t)\, y = 0$$

is a vector space of dimension m.

Definition 32

Let $y_1, \ldots, y_m$ are m linearly independent solutions of

$$a_m(t)\, y^{(m)} + \cdots + a_1(t)\, y' + a_0(t)\, y = 0.$$

The **general solution** of this homogeneous ODE is

$$y_h(t) = c_1 y_1(t) + \cdots + c_m y_m(t). \tag{7}$$

where $c_i,\ i = 1, 2, \ldots, m$ are arbitrary constants.

Higher Order Differential Equations

Definition 33

Any function $y_p(t)$, free of arbitrary parameters, that satisfies (5), is said to be a **particular solution** of the equation.

Theorem 6

Let $y_h(t)$ be the general solution of homogeneous ode given in (7). If $y_p(t)$ is a particular solution of the nonhomogeneous differential equation

$$a_m(t)\, y^{(m)} + \cdots + a_1(t)\, y' + a_0(t)\, y = g(t)$$

then

$$y(t) = y_h(t) + y_p(t)$$

is the general solution of the equation above.

Higher Order Differential Equations

Theorem 7 (Superposition Principle)

Let $y_{p_1}, \ldots, y_{p_k}$ be k particular solution of the nonhomogeneous linear nth-order differential equation (5) on an interval I corresponding, in tern, to k distinct $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_m(t)\,y^{(m)} + \cdots + a_1(t)\,y' + a_0(t)\,y = g_i(t)$$

where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(t) + \cdots + y_{p_k}(t)$$

is a particular solution of

$$a_m(t)\,y^{(m)} + \cdots + a_1(t)\,y' + a_0(t)\,y = g_1(t) + g_2(t) + \cdots + g_k(t)$$

Higher Order Differential Equations

Now we assume that the coefficients are constants and consider

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, t \in I$$

where $a_0, a_1, \ldots, a_n$ are constants. The characteristic equation of the differential equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Root with multiplicity m	Solutions
Real number λ	$e^{\lambda t}, te^{\lambda t}, \ldots, t^{m-1} e^{\lambda t}$
Imaginary number μi	$\cos \mu t, t \cos \mu t, \ldots, t^{m-1} \cos \mu t$ $\sin \mu t, t \sin \mu t, \ldots, t^{m-1} \sin \mu t$
Complex number $\lambda + \mu i$	$e^{\lambda t} \cos \mu t, te^{\lambda t} \cos \mu t, \ldots, t^{m-1} e^{\lambda t} \cos \mu t$ $e^{\lambda t} \sin \mu t, te^{\lambda t} \sin \mu t, \ldots, t^{m-1} e^{\lambda t} \sin \mu t$

Second order with constant coeffcients

We consider homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0$$

where a, b, c are constants. The equation

$$ar^2 + br + c = 0$$

is called the **characteristic equation** of the differential equation.

Discriminant	Nature of roots	General solution
$b^2 - 4ac > 0$	$r_1, r_2 \in \mathbb{R}$	$y = Ae^{r_1 t} + Be^{r_2 t}$
$b^2 - 4ac = 0$	$r_1 = r_2$ are equal	$y = (A + Bt)e^{r_1 t}$
$b^2 - 4ac < 0$	$r_1, r_2 = \lambda \pm i\mu$	$y = e^{\lambda t}[A\cos(\mu t) + B\sin(\mu t)]$

Homogeneous equations with constant coeffcients

Example 34

Solve:

$$y'' - y' - 6y = 0$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - r - 6 &= 0 \\ r &= 3, -2 \end{aligned}$$

Thus the general solution is

$$y = c_1 e^{3t} + c_2 e^{-2t}$$

Example 35

Solve:

$$\begin{cases} y'' - 4y' + 4y = 0 \\ y(0) = 3, y'(0) = 1 \end{cases}$$

Solution: The characteristic equation

$$r^2 - 4r + 4 = 0$$

has a double root $r_1 = r_2 = 2$. Thus the general solution is

$$y = c_1 e^{2t} + c_2 te^{2t}$$

Now

$$y' = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 te^{2t} = (2c_1 + c_2)\, e^{2t} + 2c_2 te^{2t}$$

Thus

$$\begin{cases} y(0) = c_1 = 3 \\ y'(0) = 2c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -5 \end{cases}$$

Therefore

$$y = 3e^{2t} - 5te^{2t}$$

Example 36

Solve the initial value problem

$$\begin{cases} y'' - 6y' + 25y = 0, & t \in I \\ y(0) = 3, y'(0) = 1 \end{cases}$$

Solution: The roots of the characteristic equation is

$$r_1, r_2 = 3 \pm 4i$$

Thus the general solution is

$$y = e^{3t}\left(c_1 \cos 4t + c_2 \sin 4t\right)$$

Now

$$\begin{aligned} y' &= 3e^{3t}\left(c_1 \cos 4t + c_2 \sin 4t\right) + e^{3t}\left(-4c_1 \sin 4t + 4c_2 \cos 4t\right) \\ &= e^{3t}\left((3c_1 + 4c_2)\cos 4t + (3c_2 - 4c_1)\sin 4t\right) \end{aligned}$$

Thus

$$\begin{cases} y(0) = c_1 = 3 \\ y'(0) = 3c_1 + 4c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -2 \end{cases}$$

Homogeneous equations with constant coeffcients

Therefore

$$y = e^{3t}(3\cos 4t - 2\sin 4t)$$

Example 37

Solve

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Solution:

$$y(t) = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{t} + c_4 e^{2t}$$

Example 38

Solve

$$y^{(3)} - y'' + 4y' - 4y = 0$$

Example 39

Solve

$$y^{(4)} + 2y'' + y = 0$$

Solution: The characteristic equation is

$$r^4 + 2r^2 + 1 = (r - i)^2(r + i)^2 = 0$$

and its roots are

$$r = i, -i$$

Thus the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

or

$$y(t) = (a_0 + a_1 t) \cos t + (b_0 + b_1 t) \sin t$$

Contents

Non-homogeneous Equations

Example 40

Solve

$$y'' - 3y' - 4y = 18e^{2t}$$

Solution: The roots of the characteristic equation $r^2 - 3r - 4 = 0$ is $r = 4, -1$. So the homogeneous solution is

$$y_h = c_1 e^{4t} + c_2 e^{-t}$$

since 2 is not a root of $r^2 - 3r - 4 = 0$, we let $y_p = Ae^{2t}$, where A is a constant to be determined. Now

$$\begin{cases} y_p' = 2Ae^{2t} \\ y_p'' = 4Ae^{2t} \end{cases}$$

By comparing coefficients of

$$
\begin{aligned}
y_p'' - 3y_p' - 4y_p &= 18e^{2t} \\
(4A - 3(2A) - 4A)e^{2t} &= 18e^{2t} \\
-6Ae^{2t} &= 18e^{2t}
\end{aligned}
$$

we get $A = -3$ and a particular solution is $y_p = -3e^{2t}$. Therefore the general solution is

$$y = y_h + y_p = c_1 e^{4t} + c_2 e^{-t} - 3e^{-3t}$$

Example 41

Solve

$$y'' - 3y' - 4y = 34 \sin t$$

Solution: since $\pm i$ are not roots of $r^2 - 3r - 4 = 0$, we let

$$y_p = A \cos t + B \sin t$$

Then

$$\begin{cases} y_p' = B \cos t - A \sin t \\ y_p'' = -A \cos t - B \sin t \end{cases}$$

By comparing the coefficients of

$$\begin{aligned} y_p'' - 3y_p' - 4y_p &= 34 \sin t \\ (-A \cos t - B \sin t) - 3(B \cos t - A \sin t) - 4(A \cos t + B \sin t) &= 34 \sin t \\ (-A - 3B - 4A) \cos t + (-B + 3A - 4B) \sin t &= 34 \sin t \end{aligned}$$

we have

$$\begin{cases} -A - 3B - 4A & = & 0 \\ -B + 3A - 4B & = & 34 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = -5 \end{cases}$$

Hence a particular solution is $y_p = 3\cos t - 5\sin t$. Therefore the general solution is

$$y = y_h + y_p = c_1 e^{4t} + c_2 e^{-t} + 3\cos t - 5\sin t$$

Example 42

Solve

$$y'' - 3y' - 4y = 52e^t \sin 2t$$

Solution: since $1 \pm 2i$ are not roots of $r^2 - 3r - 4 = 0$, we let

$$y_p = e^t(A\cos 2t + B\sin 2t)$$

Then

$$\begin{cases} y_p' = e^t((A+2B)\cos 2t + (B-2A)\sin 2t) \\ y_p'' = e^t((-3A+4B)\cos 2t + (-4A-3B)\sin 2t) \end{cases}$$

By comparing coefficients

$$y_p'' - 3y_p' - 4y_p = 52e^t \sin 2t$$

$$e^t \left[\begin{array}{c} ((-3A+4B) - 3(A+2B) - 4A)\cos 2t \\ +((-4A-3B) - 3(B-2A) - 4B)\sin 2t \end{array} \right] = 52e^t \sin 2t$$

$$(-10A - 2B)\cos 2t + (2A - 10B)\sin 2t = 52\sin 2t$$

we have $(A, B) = (1, -5)$ and a particular solution is
$y_p = e^t(\cos 2t - 5\sin 2t)$

Example 43

Solve

$$y'' - 3y' - 4y = 10e^{-t}$$

Solution: since -1 is a (simple) root of the characteristic equation $r^2 - 3r - 4 = 0$, we let

$$y_p = Ate^{-t}$$

Then

$$\begin{cases} y_p' = (-At + A)e^{-t} \\ y_p'' = (At - 2A)e^{-t} \end{cases}$$

Now we want

$$\begin{aligned} y_p'' - 3y_p' + 4y_p &= 10e^{-t} \\ ((At - 2A) - 3(-At + A) - 4At)e^{-t} &= 10e^{-t} \\ -5Ae^{-t} &= 10e^{-t} \end{aligned}$$

Hence we take $A = -2$ and a particular solution is $y_p = -2te^{-t}$

Example 44

Solve

$$y'' + 4y = 4\cos 2t$$

Solution: since $\pm 2i$ are roots of the characteristic equation $r^2 + 4 = 0$, we let

$$y_p = At\cos 2t + Bt\sin 2t$$

Then

$$\begin{cases} y_p' = (2Bt + A)\cos 2t + (-2At + B)\sin 2t \\ y_p'' = (-4At + 4B)\cos 2t + (-4Bt - 4A)\sin 2t \end{cases}$$

By comparing coefficients of

$$\begin{aligned} y_p'' + 4y_p &= 4\cos 2t \\ (-4At + 4B)\cos 2t - (4Bt + 4A)\sin 2t + 4(At\cos 2t + Bt\sin 2t) &= 4\cos 2t \\ 4B\cos 2t - 4A\sin 2t &= 4\cos 2t \end{aligned}$$

we take $A = 0, B = 1$ and a particular solution is $y_p = t\cos 2t$

Example 45

Solve

$$y'' + 2y' + y = 6te^{-t}$$

Solution: The characteristic equation $r^2 + 2r + 1 = 0$ has a double root -1. So the complementary function is

$$y_h = c_1 e^{-t} + c_2 te^{-t}$$

since -1 is a double root of the characteristic equation, we let $y_p = t^2(At + B)e^{-t}$, where A and B are constants to be determined. Now

$$\begin{cases} y_p' = \left(-At^3 + (3A - B)t^2 + 2Bt\right) e^{-t} \\ y_p'' = \left(At^3 + (-6A + B)t^2 + (6A - 4B)t + 2B\right) e^{-t} \end{cases}$$

By comparing coefficients of

$$y_p'' + 2y_p' + y_p = 6te^{-t}$$

we take $A = 1, B = 0$ and a particular solution is $y_p = t^3 e^{-t}$. Therefore the general solution is

$$y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + t^3 e^{-t}$$

Example 46

Determine the form of the particular solution of the differential equation

$$y'' + y' - 2y = 3t - \sin 4t + 3t^2 e^{2t}$$

Solution: The characteristic equation $r^2 + r - 2 = 0$ has roots $r = 2, -1$. So the complementary function is

$$y_c = c_1 e^{2t} + c_2 e^{-t}$$

A particular solution takes the form

$$y_p = (A_1 t + A_0) + (B_1 \cos 4t + B_2 \sin 4t) + t\left(C_2 t^2 + C_1 t + C_0\right) e^{2t}$$

Example 47

Determine the form of the particular solution of the differential equation

$$y'' + 2y' + 5y = te^{3t} - t\cos t + 2te^{-t}\sin 2t$$

Solution: The characteristic equation $r^2 + 2r + 5 = 0$ has roots $r = -1 \pm 2i$. So the complementary function is

$$y_c = e^{-t}\left(c_1 \cos 2t + c_2 \sin 2t\right)$$

A particular solution takes the form

$$y_p = \left(A_1 t + A_0\right) e^{3t} + \left(B_1 t + B_2\right)\cos t + \left(B_3 t + B_4\right)\sin t + te^{-t}\left(\left(C_1 t + C_2\right)\cos\right.$$

Method of Undetermined Coefficients

Definition 48

Let $D^k : C^k(I) \to C^0(I)$ be defined by

$$D^k = D(D^{k-1}), \quad k = 1, 2, \ldots$$

so that

$$D^k(f) = \frac{d^k f}{dt^k}$$

The symbol D is called **differential operator**. In general, **a linear** m**th order differential operator** is

$$L_m = a_m(t)\, D^m + a_{m-1}(t)\, D^{m-1} + \cdots + a_1(t)\, D + a_0(t)\,.$$

and

$$L_m(y) = a_m(t)\, y^m + a_{m-1}(t)\, y^{m-1} + \cdots + a_1(t)\, y + a_0(t)\,.$$

Method of Undetermined Coefficients

Definition 49

If L is a linear differential operator with constant coefficients and f is a sufficient differentiable function such that

$$L(f(t)) = 0$$

then L is said to be **annihilator** of the function.

Theorem 8

The differential operator L is a linear operator.

Method of Undetermined Coefficients

Theorem 9

1. If all $a_j(t) = a_j$, $j = 0, 1, \ldots, m$ are constant and
$$L_m(y_1) = 0,\ L_n(y_2) = 0$$
then
$$L_m L_n(\alpha y_1 + \beta y_2) = L_n L_m(\alpha y_1 + \beta y_2) = 0.$$
2. $(D-\alpha)^k\left[\left(b_0 + b_1 t + \cdots + b_{k-1} t^{k-1}\right) e^{\alpha t}\right] = 0.$
3. $\left(D^2 - 2\alpha D + \alpha^2 + \beta^2\right)^k (u+v) = 0$
where
$$\begin{aligned} u &= e^{\alpha t}\left(c_{10} + c_{11} t + \cdots + c_{1k-1} t^{k-1}\right)\cos\beta t \\ v &= e^{\alpha t}\left(c_{20} + c_{21} t + \cdots + c_{2k-1} t^{k-1}\right)\sin\beta t. \end{aligned}$$

Method of Undetermined Coefficients

Theorem 10

If the characteristic equation

$$
\begin{aligned}
a_m\lambda^m + \cdots + a_1\lambda + a_0 = a_m\left(\lambda-\lambda_1\right)^{m_1}\cdots\left(\lambda-\lambda_2\right)^{m_1} \\
\left(\lambda^2 - 2\alpha_1\lambda + \alpha_1^2 + \beta_1^2\right)^{m_3}\cdots \\
\left(\lambda^2 - 2\alpha_2\lambda + \alpha_2^2 + \beta_2^2\right)^{m_4} = 0
\end{aligned}
$$

such that $m_1 + \cdots + m_2 + 2m_3 + \cdots + 2m_4 = m$, then the general solution of

$$a_m y^{(m)} + \cdots + a_1 y' + a_0 y = 0$$

is a linear combination of $t^k e^{\lambda_1 t}$, $k = 0, 1, \ldots, m_1 - 1, \ldots, t^k e^{\lambda_2 t}$, $k = 0, 1, \ldots, m_2 - 1, t^k e^{\alpha_1 t}\cos\beta_1 t,\ k = 0, 1, \ldots, m_3 - 1, t^k e^{\alpha_1 t}\sin\beta_1 t,\ k = 0, 1, \ldots, m_3 - 1, \ldots, t^k e^{\alpha_2 t}\cos\beta_2 t,\ k = 0, 1, \ldots, m_4 - 1,$ $t^k e^{\alpha_2 t}\sin\beta_2 t,\ k = 0, 1, \ldots, m_4 - 1.$

Contents

Variation-of-Parameters Method

Theorem 11

Let $y_1, \ldots, y_m$ be m linearly independent solutions of the homogeneous differential equation

$$y^{(m)} + b_{m-1}(t)\, y^{(m-1)} + \cdots + b_1(t)\, y' + b_0(t)\, y = 0$$

Then a particular solution of

$$y^{(m)} + b_{m-1}(t)\, y^{(m-1)} + \cdots + b_1(t)\, y' + b_0(t)\, y = f(t)$$

is

$$y_p(t) = u_1(t)\, y_1(t) + \cdots + u_m(t)\, y_m(t)$$

Definition 50

The differential equation of the form

$$a_m (at+b)^m y^{(m)} + \cdots + a_1 (at+b) y' + a_0 y = f(t)$$

is called **Cauchy-Euler equation**.

Theorem 12

The change of variable $at+b=e^p$, the Cauchy-Euler equation will be transformed into a linear ODE with constant coefficients.

Variation-of-Parameters Method

where u_k', $k = 1, 2, \ldots, m$ are determined by the m equations

$$\begin{cases} y_1 u_1' + \cdots + y_m u_m' = 0 \\ y_1' u_1' + \cdots + y_m' u_m' = 0 \\ \qquad\quad \vdots \\ y_1^{(m-2)} u_1' + \cdots + y_m^{(m-2)} u_m' = 0 \\ y_1^{(m-1)} u_1' + \cdots + y_m^{(m-1)} u_m' = f\left(t\right) \end{cases}$$

Variational Parameters

Example 51

Solve

$$y'' + 4y = \frac{3}{\sin t}$$

Solution: Solving the corresponding homogeneous equation, we let

$$y_1 = \cos 2t, \quad y_2 = \sin 2t$$

We have

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$$

So

$$\begin{cases} u_1' = -\dfrac{gy_2}{W} = -\dfrac{\left(\frac{3}{\sin t}\right)\sin 2t}{2} = -\dfrac{3(2\cos t \sin t)}{2\sin t} = -3\cos t \\ u_2' = \dfrac{gy_1}{W} = \dfrac{\left(\frac{3}{\sin t}\right)\cos 2t}{2} = \dfrac{3\left(1 - 2\sin^2 t\right)}{2\sin t} = \dfrac{3}{2\sin t} - 3\sin t \end{cases}$$

Hence,

$$\begin{cases} u_1 = -3\sin t + c_1 \\ u_2 = \dfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2 \end{cases}$$

and the general solution is

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (-3\sin t + c_1)\cos 2t + \left(\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2\right)\sin 2t \\ &= c_1\cos 2t + c_2\sin 2t - 3\sin t\cos 2t + \frac{3}{2}\sin 2t\ln|\csc t - \cot t| + 3\cos t\sin \\ &= c_1\cos 2t + c_2\sin 2t + \frac{3}{2}\sin 2t\ln|\csc t - \cot t| + 3\sin t \end{aligned}$$

where c_1, c_2 are constants.

Example 52

Solve

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1}$$

Solution: Solving the corresponding homogeneous equation, we let

$$y_1 = e^t, y_2 = e^{2t}$$

We have

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

So

$$\begin{cases} u_1' = -\dfrac{gy_2}{W} = -\left(\dfrac{e^{3t}}{e^t+1} e^{2t}\right) / e^{3t} = -\dfrac{e^{2t}}{e^t+1} \\ u_2' = \dfrac{gy_1}{W} = \left(\dfrac{e^{3t}}{e^t+1} e^t\right) / e^{3t} = \dfrac{e^t}{e^t+1} \end{cases}$$

Thus

$$u_1 = -\int \frac{e^{2t}}{e^t + 1} dt = \ln\left(e^t + 1\right) - \left(e^t + 1\right) + c_1$$

and

$$\begin{aligned} u_2 &= \int \frac{e^t}{e^t + 1} dt \\ &= \int \frac{1}{e^t + 1} d\left(e^t + 1\right) \\ &= \ln\left(e^t + 1\right) + c_2 \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(\ln\left(e^t + 1\right) - \left(e^t + 1\right) + c_1\right) e^t + \left(\ln\left(e^t + 1\right) + c_2\right) e^{2t} \\ &= c_1 e^t + c_2 e^{2t} + \left(e^t + e^{2t}\right) \ln\left(e^t + 1\right) \end{aligned}$$