I2-TD1
(JORDAN CANONICAL FORMS)

1. Find all eigenvalues and their corresponding eigenspaces of the following matrix.

(a) $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$

2. Find the characteristics and the minimal polynomial of the following matrices over $\mathbb{R}$, then deduce the their corresponding Jordan Canonical Form J.

(a) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 2 & -5 & -1 & 6 \\ 1 & -3 & -2 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$

3. The following are the Jordan Canonical Form of linear transformations. Find the characteristic polynomial, minimal polynomials, the algebraic multiplicity, geometric multiplicity and the index of each of the eigenvalues of L

(a) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

4. Determine all possible Jordan canonical forms for a linear operator L whose characteristic polynomial is $p(\lambda) = (\lambda - 3)^3(\lambda - 4)^2$.

5. Determine all possible Jordan canonical forms J for a matrix of order 6 whose minimal polynomial is $m(\lambda) = (\lambda - 1)^3(\lambda - 3)^2$.

6. Find all possible Jordan canonical forms for those matrices whose characteristics polynomial $p(\lambda)$ and minimal polynomial $m(\lambda)$ are as follows:

(a) $p(\lambda) = (\lambda - 3)^4(\lambda - 2)^2$ and $m(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$

(b) $p(\lambda) = (\lambda - 5)^7$ and $m(\lambda) = (\lambda - 5)^2$

(c) $p(\lambda) = (\lambda - 3)^4(\lambda - 2)^4$ and $m(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$

7. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $m_A(\lambda)$ be its minimal polynomial. Let f be a polynomial satisfies $f(A) = 0$. Show that $f(\lambda)$ is divisible by $m_A(\lambda)$.

8. Let $A \in \mathcal{M}_6(\mathbb{R})$ be an invertible matrix satisfies $A^3 - 4A^2 + 3A = 0$ and $\operatorname{tr}(A) = 8$. Find the characteristics polynomial of A.

9. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ (no need distinct) be eigenvalues of A. Show that

   (a) $\displaystyle\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A)$.

   (b) $\displaystyle\prod_{i=1}^{n} \lambda_i = |A|$.

10. Let $A \in \mathcal{M}_3(\mathbb{R})$ with $\operatorname{tr}(A) = 9$. Suppose that $\lambda = 2$ is an eigenvalue of A and E_2 spanned by $(1, 0, 1)$ and $(-1, 1, 3)$. Find $|A|$.

11. Find $A \in \mathcal{M}_n(\mathbb{R})$ such that
$$A^3 - 4A^2 + 4A = 0 \quad \text{and} \quad \operatorname{tr}(A) = 0.$$

12. Let A be a square matrix defined by $A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$. Find the minimal polynomial of A. Then express A^4 and A^{-1} in terms of A and I.

13. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. One can show that $A^2 \neq 0$ and $A^3 = 0$. Find the nilpotent matrix M in canonical form which is similar to A.

14. Let $a \in \mathbb{R}^*$ and $A = \begin{pmatrix} 0 & a & a^2 \\ a^{-1} & 0 & a \\ a^{-2} & a^{-1} & 0 \end{pmatrix}$.

   (a) Show that $A^2 = A + 2I$.

   (b) Deduce that A is diagonalizable.

15. Determine the value of a so that $\lambda = 2$ is an eigenvalue of
$$A = \begin{pmatrix} 1 & -1 & 0 \\ a & 1 & 1 \\ 0 & 1+a & 3 \end{pmatrix}$$
then show that A is diagonalizable and diagonalize it.

16. Let A be a 2×2 matrix defined by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that A is diagonalizable if $(a - d)^2 + 4bc \neq 0$.

17. Let A be a square matrix defined by $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$.

    (a) Find the characteristic polynomial of A.
    (b) Show that A is diagonalizable then diagonalize it.
    (c) Write A^n in term of n.

18. Let A be a square matrix defined by $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$.

    (a) Find the characteristic polynomial of A.
    (b) Show that A is diagonalizable then diagonalize it.
    (c) Write A^n in term of n.

19. Let A be a square matrix defined by $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$.

    (a) Find the characteristic polynomial of A.
    (b) Show that A is diagonalizable then diagonalize it.
    (c) Write A^n in term of n.

20. Let A be a square matrix defined by $A = \begin{pmatrix} -8 & -3 & -6 \\ 4 & 0 & 4 \\ 4 & 2 & 2 \end{pmatrix}$.

    (a) Find the characteristic polynomial of A.
    (b) Find the eigenvalues and eigenspaces of A.
    (c) Show that A is not diagonalizable, but it is triangularizable, then triangularize A.
    (d) Write A^n in terms of I, A, A^2 and n.

21. Let A be a square matrix defined by $A = \begin{pmatrix} -2 & -1 & -5 \\ 2 & 2 & 3 \\ 4 & 2 & 6 \end{pmatrix}$.

    (a) Find the characteristic polynomial of A.
    (b) Find the eigenvalues and eigenspaces of A.
    (c) Show that A is not diagonalizable, but it is triangularizable, then triangularize A.
    (d) Find the three real sequences $(a)_n, (b)_n, (c)_n$ satisfying

$$\begin{cases} a_{n+1} = -2a_n - b_n - 5c_n, & a_0 = 1 \\ b_{n+1} = 2a_n + 2b_n + 3c_n, & b_0 = 0 \\ c_{n+1} = 4a_n + 2b_n + 6c_n, & c_0 = 1 \end{cases}$$

22. Let A be a square matrix defined by $A = \begin{pmatrix} -1 & -2 & -1 & 3 \\ -6 & -5 & 1 & 6 \\ -6 & -4 & 0 & 6 \\ -6 & -7 & 1 & 8 \end{pmatrix}$ and its characteristics polynomial $p(\lambda) = (\lambda + 1)^2(\lambda - 2)^2$.

(a) Find the minimal polynomial of A.

(b) Deduce that A is not diagonalizable, but it is triangularizable, then triangularize A.

(c) Write A^n in terms of n.

23. Let A be a square matrix defined by $A = \begin{pmatrix} 6 & 2 & 3 \\ -3 & -1 & -1 \\ -5 & -2 & -2 \end{pmatrix}$ and L be a map from $\mathbb{R}^3$ into $\mathbb{R}^3$ by $L(v) = Av$.

(a) Show that L is a linear operator on $\mathbb{R}^3$.

(b) Find the characteristic polynomial of L with respect to standard basis for $\mathbb{R}^3$. Derive the determinant of L then deduce that L is invertible.

(c) Find the eigenvalues and eigenspaces of L.

(d) Show that L is not diagonalizable, but it is triangularizable, then triangularize L.

(e) Write L^n in term of n, where $L^n = L(L(\ldots(L)\ldots))$, the n compositions of L.

24. Let $B_1 = \{(2,1,1,1),(1,1,1,1),(1,1,2,1)\}$ and $B_2 = \{(2,1,2,2)\}$ be two subsets of $\mathbb{R}^4$, E_1 be a subspace spanned by B_1, E_2 be a subspace spanned by B_2, and L be a linear operator on $\mathbb{R}^4$ defined by

$$L(v) = (-w + 4x - y + z, -w + 3x, -w + 2x + y, -w + 2x + z), v = (w, x, y, z).$$

(a) Show that B_1 is a basis for E_1 and B_2 is a basis for E_2.

(b) Show that E_1 and E_2 are L-invariant. Find the matrices $A_1 = [L_{E_1}]_{B_1}$ and $A_2 = [L_{E_2}]_{B_2}$

(c) Show that $\mathbb{R}^4$ is a direct sum of E_1 and E_2.

(d) Find the characteristic and minimal polynomials of A_1 and A_2.

(e) Let A be the matrix representation of L with respect to the standard basis for $\mathbb{R}^4$. Show that A is similar to a block diagonal matrix to be specified.

(f) Deduce the characteristic and minimal polynomials of L.

25. Let $T \in \mathcal{L}(\mathbb{R}^3)$ defined by $T(x_1, x_2, x_3) = (-4x_1 - x_2 - 2x_3, 4x_1 + x_2 + 3x_3, -x_2 - x_3)$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A. Show that A is not diagonalizable over $\mathbb{R}$.

(c) Show that A is diagonalizable over $\mathbb{C}$. Find the eigenspaces. Diagonalize A.

(d) Express T^n in the form of $a_nT^2 + b_nT + c_nI$ where a_n, b_n and c_n are real sequences to be specified, and $T^n = T(T(\ldots(T)\ldots))$, the n compositions of T.

26. Let A be a square matrix defined by $A = \begin{pmatrix} -3 & -1 & -3 \\ 5 & 2 & 5 \\ -1 & -1 & -1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A. Show that A is not diagonalizable over $\mathbb{R}$.

(c) Show that A is diagonalizable over $\mathbb{C}$. Find the eigenspaces. Diagonalize A.

(d) Express A^n in the form of $a_nA^2+b_nA+c_nI_n$ where (a_n), (b_n) and (c_n) are real sequences to be specified.

27. Let A be a symmetric matrix defined by $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and S be the surface

$$x^2 + 2y^2 + z^2 - 2xy - 2yz + 2x + y = 1.$$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues and eigenspaces of A. Show that A is diagonalizable over $\mathbb{R}$.

(c) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

(d) Does the surface S have a center? If any, specify its coordinates.

(e) Determine the type of the surface S.

28. Let A be a symmetric matrix defined by $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 3 \end{pmatrix}$ and S be the surface of

$$2x^2 + 3y^2 + 3z^2 + 2xy + 4yz - 2xz + 2x + 14y = 2.$$

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues and eigenspaces of A. Show that A is diagonalizable over $\mathbb{R}$.

(c) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

(d) Does the surface S have a center? If any, specify its coordinates.

(e) Determine the type of the surface S.

29. Determine the nature of the surface (S)

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = \lambda, \quad \lambda \in \mathbb{R}.$$

30. Given the following matrices and their corresponding characteristics polynomials, Jordanize them over $\mathbb{C}$.

(a) $\begin{pmatrix} 5 & 4 & -5 \\ 3 & 6 & -5 \\ 5 & 8 & -7 \end{pmatrix}$ and $p(\lambda) = -\lambda^3 + 4\lambda^2 - 6\lambda + 4$

(b) $\begin{pmatrix} 3 & 2 & 2 & -4 \\ 3 & 4 & 2 & -5 \\ -2 & 7 & 7 & -9 \\ 1 & 5 & 4 & -6 \end{pmatrix}$ and $p(\lambda) = (\lambda^2 - 4\lambda + 5)^2$

(c) $\begin{pmatrix} 3 & 1 & 2 & -3 \\ 4 & 1 & 1 & -2 \\ -2 & 6 & 7 & -8 \\ 2 & 2 & 3 & -3 \end{pmatrix}$ and $p(\lambda) = (\lambda^2 - 4\lambda + 5)^2$