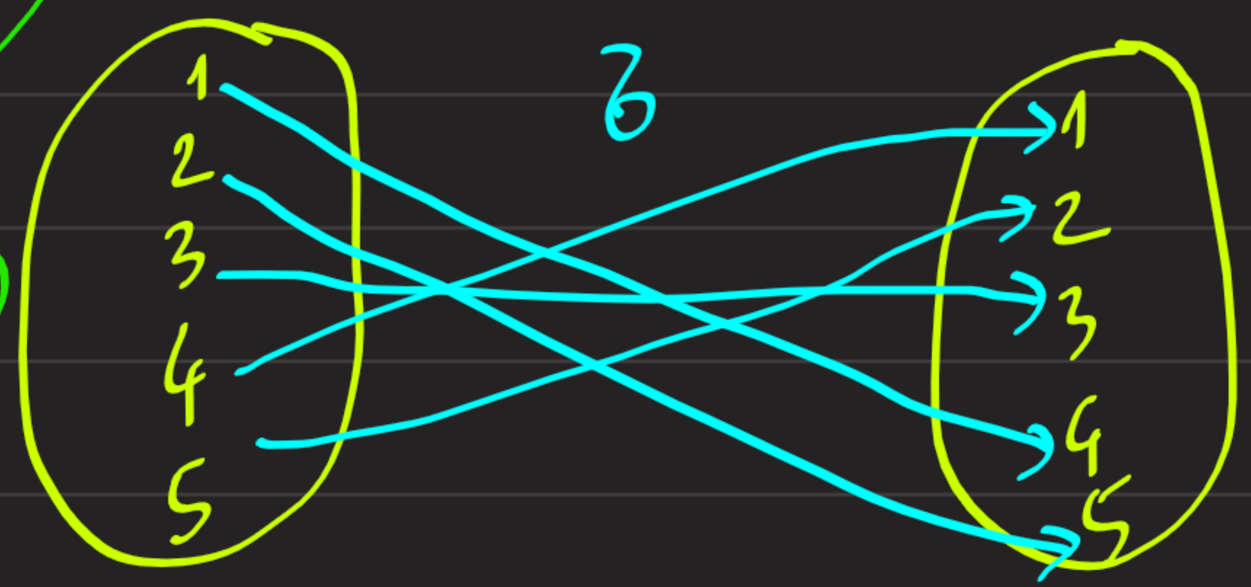
1. Find the signatures of the following permutations.

(a) 45312 (b) 38562147 (c) 397264581.

ⓐ $\sigma : S \to S$ where $S = \{1,2,3,4,5\}$



or $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$ or $\sigma = 45312$

$\sigma = 45312 \Rightarrow \#\mathrm{inv}(\sigma) = 3+3+2 = 8$

Therefore, $\mathrm{sgn}(\sigma) = (-1)^{\#\mathrm{inv}(\sigma)} = (-1)^{8} = \underline{1}$

ⓑ $\sigma = 38562147$

$\#\mathrm{inv}(\sigma) = 2+6+3+3+1 = 15$

$\mathrm{sgn}(\sigma) = (-1)^{\#\mathrm{inv}(\sigma)} = (-1)^{15} \underline{= -1}$

ⓒ $\sigma = 397264581$

$\#\mathrm{inv}(\sigma) = 2+7+5+1+3+1+1+1 = 21$

$\mathrm{sgn}(\sigma) = (-1)^{21} \underline{= -1}$

2. In S_8, write the following permutations into cyclic form, then determine their signature.

(a) 85372164 (b) 87651234 (c) 12435687.

ⓐ $\sigma = 85372164 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 3 & 7 & 2 & 1 & 6 & 4 \end{pmatrix}$

$= (18476)(25)(3) = (18476)(25) = \sigma_1 \circ \sigma_2$

$$sgn(\sigma) = sgn(\sigma_1)\cdot sgn(\sigma_2) = (-1)^{length(\sigma_1)-1}(-1)^{length(\sigma_2)-1}$$

$$= (-1)^{5-1}(-1)^{2-1} = (1)(-1) = -1$$

ⓑ $\sigma = 87651234 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

$$= (1\,8\,4\,5)(2\,7\,3\,6) = \sigma_1 \circ \sigma_2$$

$$sgn(\sigma) = sgn(\sigma_1)\cdot sgn(\sigma_2) = (-1)^{4-1}(-1)^{4-1} = (-1)(-1) = 1$$

ⓒ $\sigma = 12435687 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 3 & 5 & 6 & 8 & 7 \end{pmatrix}$

$$= (1)(2)(3\,4)(5)(6)(7\,8) = (3\,4)(7\,8) = \sigma_1 \circ \sigma_2$$

$$sgn(\sigma) = (-1)^{2-1}(-1)^{2-1} = (-1)(-1) = 1$$

Verification by the total number of inversions.

$\sigma = 12435687 \Rightarrow \#inv(\sigma) = 1+1 = 2 \Rightarrow sgn(\sigma) = (-1)^2 = 1$

3. In S_7, write the following permutations into normal form, then determine their signature.

(a) (6437) (b) (465)(735) (c) (241)(5416).

ⓐ $\sigma = (6437) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 3 & 5 & 4 & 6 \end{pmatrix} = 1273546$

$\#inv(\sigma) = 4+1 = 5 \Rightarrow sgn(\sigma) = (-1)^{\#inv(\sigma)} = (-1)^5 = -1$

ⓑ $\sigma = (465)(735) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 6 & 4 & 5 & 7 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 6 & 7 & 5 & 3 \end{pmatrix} = 1246753$$

$\sigma = 1246753 \Rightarrow \#inv(\sigma) = 1+2+2+1 = 6$

$\Rightarrow sgn(\sigma) = (-1)^{\#inv(\sigma)} = (-1)^{6} = \underline{1}$

ⓒ $\sigma = (241)(5416) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 3 & 1 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 2 & 3 & 1 & 4 & 5 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 3 & 2 & 1 & 5 & 7 \end{pmatrix} = 6432157$

$\#inv(\sigma) = 5+3+2+1 = 11 \Rightarrow sgn(\sigma) = (-1)^{\#inv(\sigma)} = (-1)^{11} = \underline{-1}$

4. For $n \in \mathbb{N}^*$, compute the signature of the following permutations.

(a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & \dots & 2 & 1 \end{pmatrix}$

(b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n & n+1 & n+2 & \dots & 2n-1 & 2n \\ \boxed{1} & \boxed{3} & \boxed{5} & 7 \dots & \boxed{2n-1} & 2 & 4 & \dots & 2n-2 & 2n \end{pmatrix}$

$2n-2$

$2, 4, 6, \dots, 2n-2$

$\underline{2\ 4\ 6}$

?

ⓐ $\sigma = \boxed{n}, n-1, n-2, \dots, 2, 1$

$S_n = \frac{n}{2}(u_1 + u_n)$

$\#inv(\sigma) = (n-1) + (n-2) + (n-3) + \dots + 1 = \frac{n-1}{2}\{(n-1)+1\}$

$= \frac{n(n-1)}{2}$

Therefore, the signiture of σ is $sgn(\sigma) = (-1)^{\frac{n(n-1)}{2}}$.

ⓑ $\#inv(\sigma) = 0 + 1 + 2 + 3 + \dots + (n-1) = \frac{n(n-1)}{2}$

Therefore, $sgn(\sigma) = (-1)^{\#inv(\sigma)} = (-1)^{\frac{n(n-1)}{2}}$.

5. Prove that the transposition is an odd permutation.

Ex: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} = (1)(25)(3)(4) = (25)$

- $\operatorname{sgn}(\sigma) = (-1)^{\text{length}(25)-1} = (-1)^{2-1} = (-1)^1 = -1$

  This means that σ is an odd permutation.

- $\#\operatorname{inv}(\sigma) = 3+1+1 = 5 \Rightarrow \operatorname{sgn}(\sigma) = (-1)^5 = -1$

5. Let $\sigma \in S_n$ be a transposition by interchanging two position i-th and j-th.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & j & \cdots & n-2 & n-1 & n \\ 1 & 2 & 3 & \cdots & \boxed{j} & \cdots & \boxed{i} & \cdots & n-2 & n-1 & n \end{pmatrix}$$

$i < j$

$j,\ i+1,\ i+2,\ i+3,\ \ldots,\ j-3,\ j-2,\ j-1,\ i$

$i + \underbrace{(j-i-1)}$

$$\#\operatorname{inv}(\sigma) = (j-i) + \underbrace{1 + 1 + 1 + \cdots + 1}_{(j-i-1)} = (j-i) + (j-i-1)$$

$= 2(j-i) - 1$ is always odd integer.

$\Rightarrow \operatorname{sgn}(\sigma) = (-1)^{\#\operatorname{inv}(\sigma)} = (-1)^{2(j-i)-1} = -1$

Therefore, the transposition is an odd permutation.