I2-TD4
(Linear Transformations)

1. Determine if the following mappings are linear transformations.

   (a) $L : \mathbb{R}^2 \to \mathbb{R}^2, L(x_1, x_2) = (x_1 - 2x_2, 3 + x_1 + 3x_2)$

   (b) $L : \mathbb{R}^3 \to \mathbb{R}^2, L(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 - x_2)$

   (c) $L : \mathbb{R}^2 \to \mathbb{R}^2, L(x_1, x_2) = (x_1 + 2x_2, x_1x_2)$

   (d) Let $A \in \mathbb{R}^{n\times n}$ and $L : \mathbb{R}^n \to \mathbb{R}^n, L(x) = Ax$

   (e) $\varphi : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R}), \varphi(f) = 2tf''(t) + \cos t f'(t) - (t^2 - 1)f(t)$.

   (f) $\varphi : \mathcal{C}^0(\mathbb{R}) \to \mathbb{R},\ \varphi(f) = \int_0^1 (f(x) + 3f'(x))\, dx$

2. Let $L \in \mathcal{L}(V, W)$. Show that

   (a) if $v_1, v_2, \ldots, v_n$ span a vector space V, then $L(v_1), L(v_2), \ldots, L(v_n)$ span $\mathrm{Im}(L)$.

   (b) if $L(v_1), L(v_2), \ldots, L(v_n)$ are linearly independent, then $v_1, v_2, \ldots, v_n$ linearly independent.

3. Let $L \in \mathcal{L}(V, W)$. Show that

   (a) $\mathrm{Ker}(L)$ is a subspace of V.

   (b) $\mathrm{Im}(L)$ is a subspace of W.

   (c) if $\dim V < \infty$, then $\mathrm{null}(L) + \mathrm{rank}(L) = \dim V$.

4. Let $L \in \mathcal{L}(V, W)$.

   (a) Show that L is one-to-one if and only if $\mathrm{Ker}(L) = \{0\}$.

   (b) Show that L is onto if and only if $\mathrm{Im}(L) = W$.

   (c) Suppose $\dim V = \dim W < \infty$. Show that L is one-to-one if and only if it is onto.

5. Let U, V and W be vector spaces over the same field. Let f be a linear transformation from U to V and g be a linear transformation from V to W. Show that

   (a) $g \circ f$ is also a linear transforation.

   (b) $\mathrm{rank}(g \circ f) \leq \min\{\mathrm{rank}(f); \mathrm{rank}(g)\}$

   (c) if f is surjective then $\mathrm{rank}(g \circ f) = \mathrm{rank}(g)$

   (d) if g is injective then $\mathrm{rank}(g \circ f) = \mathrm{rank}(f)$

6. Let $f : \mathbb{R}^3 \to \mathbb{R}^3,\ f(x, y, z) = (z, x - y, y + z)$. Show that f is an automorphism.

7. Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   (a) Find the kernel of L and one of its bases. Is L one-to-one?

(b) Find the range of L and one of its bases. Is L onto?

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

   (a) Find the kernel of L and one of its bases. Is L one-to-one?

   (b) Find the range of L and one of its bases. Is L onto?

9. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

   (a) Find the kernel of L and one of its bases. Is L one-to-one?

   (b) Find the range of L and one of its bases. Is L onto?

10. Let T be the mapping from $\mathbb{R}^4$ into $\mathbb{R}_3[X]$ defined by, for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

$$T(x) = x_1 + 2x_2 + x_3 + (x_2 - x_4)\,t + (x_1 - 3x_3)\,t^2 + (x_1 + x_2 + x_3 - x_4)\,t^3.$$

   (a) Determine a basis and the dimension of $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

   (b) Is T an isomorphism from $\mathbb{R}^4$ into $\mathbb{R}_3[X]$?

11. Let $T : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^2(\mathbb{R})$; $T(f) = 2f'' - 5f' + 3f$. Show that T is an endomorphism and determine its kernel.

12. Let $n \geq 2$ and $f : \mathbb{R}_n[X] \to \mathbb{R}_2[X]$ defined by

$$f(P) = x^2P(0) + (x-2)P(1)$$

   (a) Show that f is linear.

   (b) Find the kernel and image of f.

13. Let $\varphi : \mathbb{R}_n[X] \to \mathbb{R}_n[X]$ defined by

$$\varphi(P) = P(x+1) - P(x)$$

   (a) Show that φ is an endomorphism.

   (b) Find the kernel and image of φ.

14. Let f be an endomorphism on vector space V satisfies $f^2 - 5f + 6\mathrm{Id} = 0$. Show that $V = \operatorname{Ker}(f - 2\mathrm{Id}) \oplus \operatorname{Ker}(f - 3\mathrm{Id})$. (Note: $f^2 = f \circ f$)

15. Show that there is a unique linear transformation f such that $f(1, 2) = 2$ and $f(-2, 1) = 5$. Find the nullity and image of f.

16. Let $L \in \mathcal{L}(\mathbb{R}^3)$ verifies

$$L\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad L\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(a) Find the image of $e_1 = (1, 0, 0)$ under the linear transformation L.

(b) Find the linear transformation L.

17. Consider $L \in \mathcal{L}(\mathbb{R}^3), L(x, y, z) = (x + y + z; x - y, 2z)$. Let

$$B_1 = \{(1, 1, 1); (1, 1, 0), (1, 0, 0)\} \quad \text{and} \quad B_2 = \{(-1, 0, 1); (0, 1, 0), (1, 2, 3)\}$$

(a) Find the matrix representation L relative to the standard basis.

(b) Find the transition matrix from B_2 to B_1, denoted by S.

(c) Find the matrix representation L relative to bases B_1 and B_2.

(d) Verify that $[L]_{B_1}[v]_{B_1} = [L(v)]_{B_1}$ for any vector $v \in \mathbb{R}^3$.

(e) Verify that $[L]_{B_2} = S^{-1}[L]_{B_1}S$.

18. Let T be a linear transformation from $\mathbb{R}^4$ into $\mathbb{R}^3$ defined by, for $x = (x_1, x_2, x_3, x_4)$,

$$T(x) = (x_1 + x_2 + x_3 + x_4, 2x_1 + 3x_2 - x_3 + 3x_4, x_1 + 2x_2 - 2x_3 + 2x_4).$$

(a) Let B_1 and B_2 be the standard basis for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Find the matrix representation of T with respect to the bases B_1 and B_2.

(b) Let $B_3 = \{(1, 1, 1, 1), (1, 2, 1, 3), (1, 1, 1, 2), (1, 3, 2, 2)\}$ be a subset of $\mathbb{R}^4$ and $B_4 = \{(1, 1, 2), (0, 1, 2), (1, 1, 3)\}$ be a subset of $\mathbb{R}^3$. Show that B_3 is a basis for $\mathbb{R}^4$ and B_4 is a basis for $\mathbb{R}^3$. Find the matrix representation of T with respect to the bases B_3 and B_4.

(c) Verify that for any $v \in \mathbb{R}^4$, we have $[T]_{B_3}^{B_4}[v]_{B_4} = [T(v)]_{B_3}$.

19. The set $S = \{e^{2t}, te^{2t}, t^2e^{2t}\}$ is a basis of a vector space V of functions $f : \mathbb{R} \to \mathbb{R}$. Let D be the differential operator on V; that is $D(f) = df/dt$. Find the matrix representation of D relative to the basis S.

20. Let $L : \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}_1[X]$, $L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d)x + b + c$.

(a) Show that L is a linear transformation.

(b) Determine the matrix represents L in the canonical bases of $\mathcal{M}_2(\mathbb{R})$ and $\mathbb{R}_1[X]$.

(c) Find $\text{Ker}(L)$ and $\text{Im}(L)$.

21. Let $f : \mathbb{R}_n[X] \to \mathbb{R}_n[X]; f(P) = xP' - P$.

(a) Show that $f \in \mathcal{L}(\mathbb{R}_n[X])$

(b) Determine the matrix represents f in the canonical bases of $\mathbb{R}_n[X]$.

(c) Find $\text{Ker}(f)$ and $\text{Im}(f)$.

22. Let $T : \mathbb{R}_3[X] \to \mathbb{R}_3[X]$ defined by $T(f) = xf'' - 2xf' + f$. Show that T is a linear transformation and then find the matrix represents T with respect to the standard basis of $\mathbb{R}_3[X]$.

23. Let $f : \mathbb{R}_n[X] \to \mathbb{R}_{n+1}[X]; f(P) = e^{x^2}\left(Pe^{-x^2}\right)'$.

(a) Show that $f \in \mathcal{L}(\mathbb{R}_n[X], \mathbb{R}_{n+1}[X])$

(b) Determine the matrix represents f in the canonical bases of $\mathbb{R}_n[X]$ and $\mathbb{R}_{n+1}[X]$.

(c) Find $\text{Ker}(f)$ and $\text{Im}(f)$.