14. Find the inverse of each matrix (if exists) below:

(a) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$

$$(A \mid I_3) \longrightarrow (I_3 \mid A^{-1})$$
$$(A \mid v) \longrightarrow (I_3 \mid \underbrace{A^{-1} v}) \checkmark$$

$$(A \mid v) = \left(\begin{array}{ccc|c} 2 & -1 & 1 & v_1 \\ 1 & 0 & 0 & v_2 \\ -1 & 2 & 1 & v_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_2 \\ 2 & -1 & 1 & v_1 \\ -1 & 2 & 1 & v_3 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_2 \\ 0 & -1 & 1 & v_1 - 2v_2 \\ 0 & 2 & 1 & v_2 + v_3 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + 2R_2]{R_2 \leftarrow (-1)R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_2 \\ 0 & 1 & -1 & -v_1 + 2v_2 \\ 0 & 0 & 3 & 2v_1 - 3v_2 + v_3 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_2 \\ 0 & 1 & -1 & -v_1 + 2v_2 \\ 0 & 0 & 1 & \frac{2}{3}v_1 - v_2 + \frac{1}{3}v_3 \end{array}\right)$$

$0v_1 + 1 \cdot v_2 + 0 \cdot v_3$

$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_2 \\ 0 & 1 & 0 & -\frac{1}{3}v_1 + v_2 + \frac{1}{3}v_3 \\ 0 & 0 & 1 & \frac{2}{3}v_1 - v_2 + \frac{1}{3}v_3 \end{array}\right) = (I_3 \mid A^{-1} v)$$

$$\Rightarrow A^{-1} v = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}.$$

(c) $\begin{pmatrix} 1 & 1 & \dots & 1 \\ & \ddots & \ddots & \vdots \\ & (0) & \ddots & 1 \\ & & & 1 \end{pmatrix}_n$. Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Then,

$$(C \mid v) = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & \dots & & 1 & v_1 \\ 0 & 1 & 1 & \dots & & 1 & v_2 \\ 0 & 0 & 1 & \ddots & & 1 & v_3 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & & 1 & v_n \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & v_1 \\ 0 & 1 & 1 & \cdots & 1 & v_2 \\ 0 & 0 & 1 & \cdots & 1 & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v_n \end{array}\right)$$

For $i = 1, 2, 3, \dots, n-1$

$$R_i \leftarrow R_i - R_{i+1}$$

$i=1$, $R_1 \leftarrow R_1 - R_2$

$i=2$, $R_2 \leftarrow R_2 - R_3$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & v_1 - v_2 \\ 0 & 1 & 0 & \cdots & 0 & v_2 - v_3 \\ 0 & 0 & 1 & \cdots & 0 & v_3 - v_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v_n \end{array}\right) = \left(I_n \mid C^{-1} v\right)$$

$$\Rightarrow C^{-1} v = \begin{pmatrix} v_1 - v_2 \\ v_2 - v_3 \\ v_3 - v_4 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

Therefore, $C^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) $\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & 2 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}_n$

$$D_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad 4$$

R_1 ✓

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_2$

$R_4 \leftarrow R_4 - R_3$

$$(D \mid v) = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & & 1 & v_1 \\ 1 & 2 & 2 & \cdots & & 2 & v_2 \\ 1 & 2 & 3 & \cdots & & 3 & v_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & & n & v_n \end{array}\right)$$

For $i = n, n-1, n-2, \dots, 2$

$$R_i \leftarrow R_i - R_{i-1}$$

$$\left(\begin{array}{cccccc|c} 1&1&1&\cdots&\cdots&1 & v_1\\ 1&2&2&\cdots&\cdots&2&v_2\\ 1&2&3&\cdots&\cdots&3&v_3\\ \vdots&\vdots&\vdots&\ddots&&\vdots&\vdots\\ \vdots&\vdots&\vdots&&\ddots&\vdots&\vdots\\ 1&2&3&\cdots&\cdots&n&v_n\end{array}\right)$$

For $i=n,n-1,n-2,\dots,2$

$R_i \leftarrow R_i - R_{i-1}$

$$\rightarrow \left(\begin{array}{cccccc|c} 1&1&1&\cdots&\cdots&1 & v_1\\ 0&1&1&\cdots&\cdots&1&v_2-v_1\\ 0&0&1&\cdots&\cdots&1&v_3-v_2\\ \vdots&\vdots&\vdots&\ddots&&\vdots&\vdots\\ \vdots&\vdots&\vdots&&\ddots&\vdots&\vdots\\ 0&0&0&\cdots&\cdots&1&v_n-v_{n-1}\end{array}\right)$$

For $i=1,2,3,\dots,n-1$

$R_i \leftarrow R_i - R_{i+1}$

v_4-v_3

$(v_2-v_1)-(v_3-v_2)$

$= -v_1+2v_2-v_3$

$(v_3-v_2)-(v_4-v_3) = -v_2+2v_3-v_4$

$$\rightarrow \left(\begin{array}{cccccc|c} 1&0&0&\cdots&\cdots&0 & 2v_1-v_2\\ 0&1&0&\cdots&\cdots&0&-v_1+2v_2-v_3\\ 0&0&1&\cdots&\cdots&0&-v_2+2v_3-v_4\\ \vdots&\vdots&\vdots&\ddots&&\vdots&\vdots\\ \vdots&\vdots&\vdots&&\ddots&\vdots&\\ 0&0&0&\cdots&\cdots&1&-v_{n-1}+v_n\end{array}\right) = (I_n \mid D^{-1}v)$$

$$\Rightarrow D^{-1}v = \begin{pmatrix} 2v_1-v_2 \\ -v_1+2v_2-v_3 \\ -v_2+2v_3-v_4 \\ \vdots \\ -v_{n-1}+v_n \end{pmatrix} = \begin{pmatrix} 2&-1&0&\cdots&0\\ -1&2&-1&\cdots&0\\ 0&-1&2&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\\ \vdots\\ v_n\end{pmatrix}$$

Therefore, $$D^{-1} = \begin{pmatrix} 2&-1&0&\cdots&0\\ -1&2&-1&\cdots&0\\ 0&-1&2&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\end{pmatrix}.$$

(e) $$\begin{pmatrix} 1 & \alpha & & (0)\\ & 1 & \ddots & \\ & & \ddots & \alpha\\ (0) & & & 1\end{pmatrix}_n$$

$$E_4 = \begin{pmatrix} 1&\alpha&0&0\\ 0&1&\alpha&0\\ 0&0&1&\alpha\\ 0&0&0&1\end{pmatrix}_4$$

$R_3 \leftarrow R_3 - \alpha R_4$

$$(E|V) = \left(\begin{array}{cccccc|c} 1 & \alpha & 0 & \cdots & & 0 & v_1 \\ 0 & 1 & \alpha & \cdots & & 0 & v_2 \\ 0 & 0 & 1 & \cdots & & 0 & v_3 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & 1 & v_n \end{array}\right) \quad R_{n-1} \leftarrow R_{n-1} - \alpha R_n$$

$$R_{n-1} : 0\ 0\ 0 \cdots 1\ \alpha \mid v_{n-1}$$

$$R_n : 0\ 0\ 0 \cdots 0\ 1 \mid v_n$$

$$\text{new } R_{n-1} : 0\ 0\ 0 \cdots 1\ 0 \mid v_{n-1} - \alpha v_n$$

- $R_{n-2} \leftarrow R_{n-2} - \alpha R_{n-1}$

$$R_{n-2} : 0\ 0\ 0 \cdots 1\ \alpha\ 0 \mid v_{n-2}$$

$$R_{n-1} : 0\ 0\ 0 \cdots 0\ 1\ 0 \mid v_{n-1} - \alpha v_n$$

$$\text{new } R_{n-2} : 0\ 0\ 0 \cdots 1\ 0\ 0 \mid v_{n-2} - \alpha v_{n-1} + \alpha^2 v_n$$

Continue in this way, we obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & v_1 - \alpha v_2 + \alpha^2 v_3 - \cdots + (-1)^{n-1}\alpha^{n-1} v_n \\ 0 & 1 & 0 & \cdots & 0 & v_2 - \alpha v_3 + \alpha^2 v_4 - \cdots + (-1)^{n-2}\alpha^{n-2} v_n \\ 0 & 0 & 1 & \cdots & 0 & v_3 - \alpha v_4 + \alpha^2 v_5 - \cdots + (-1)^{n-3}\alpha^{n-3} v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v_n \end{array}\right) = (I_n \mid E^{-1} v)$$

$$\Rightarrow E^{-1} v = \begin{pmatrix} v_1 - \alpha v_2 + \alpha^2 v_3 - \cdots + (-1)^{n-1}\alpha^{n-1} v_n \\ v_2 - \alpha v_3 + \alpha^2 v_4 - \cdots + (-1)^{n-2}\alpha^{n-2} v_n \\ v_3 - \alpha v_4 + \alpha^2 v_5 - \cdots + (-1)^{n-3}\alpha^{n-3} v_n \\ \vdots \\ v_n \end{pmatrix}$$

$$E^{-1}v = \begin{pmatrix} v_1 - \alpha v_2 + \alpha^2 v_3 - \cdots + (-1)^{n-1}\alpha^{n-1} v_n \\ v_2 - \alpha v_3 + \alpha^2 v_4 - \cdots + (-1)^{n-2}\alpha^{n-2} v_n \\ v_3 - \alpha v_4 + \alpha^2 v_5 - \cdots + (-1)^{n-3}\alpha^{n-3} v_n \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & -\alpha & \alpha^2 & \cdots & (-1)^{n-1}\alpha^{n-1} \\ 0 & 1 & -\alpha & \cdots & (-1)^{n-2}\alpha^{n-2} \\ 0 & 0 & 1 & \cdots & (-1)^{n-3}\alpha^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

Therefore, $E^{-1} = \begin{pmatrix} 1 & -\alpha & \alpha^2 & \cdots & (-1)^{n-1}\alpha^{n-1} \\ 0 & 1 & -\alpha & \cdots & (-1)^{n-2}\alpha^{n-2} \\ 0 & 0 & 1 & \cdots & (-1)^{n-3}\alpha^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

(f) $\begin{pmatrix} -1 & 2 & \cdots & 2 \\ 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \cdots & 2 & -1 \end{pmatrix}_n$

$$F_4 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 2 & -1 & 2 & 2 \\ 2 & 2 & -1 & 2 \\ 2 & 2 & 2 & -1 \end{pmatrix}_4$$

$2(4-1)-1 = 5$ ✓

$2(n-1)-1 = 2n-3$

$$(F \mid V) = \left(\begin{array}{ccccc|c} -1 & 2 & 2 & \cdots & 2 & v_1 \\ 2 & -1 & 2 & \cdots & 2 & v_2 \\ 2 & 2 & -1 & \cdots & 2 & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & -1 & v_n \end{array}\right)$$

$R_1 \leftarrow R_1 + (R_2 + R_3 + \cdots + R_n)$

$$\to \left(\begin{array}{ccccc|c} 2n-3 & 2n-3 & 2n-3 & \cdots & 2n-3 & \sum_{i=1}^{n} v_i \\ 2 & -1 & 2 & \cdots & 2 & v_2 \\ 2 & 2 & -1 & \cdots & 2 & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & -1 & v_n \end{array}\right)$$

$R_1 \leftarrow \frac{1}{2n-3} R_1$

$$\frac{1}{2n-3}\sum_{i=1}^{n} v_i \rightsquigarrow u$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & \frac{1}{2n-3}\sum_{i=1}^{n} v_i \\ 2 & -1 & 2 & \cdots & 2 & v_2 \\ 2 & 2 & -1 & \cdots & 2 & v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & -1 & v_n \end{array}\right)$$

For $i = 2, 3, 4, \dots, n$

$R_i \leftarrow R_i - 2R_1$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & u \\ 0 & -3 & 0 & \cdots & 0 & -2u + v_2 \\ 0 & 0 & -3 & \cdots & 0 & -2u + v_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -3 & -2u + v_n \end{array}\right)$$

For $i = 2, 3, 4, \dots, n$

$R_i \leftarrow \frac{1}{-3} R_i$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & u \\ 0 & 1 & 0 & \cdots & 0 & -\frac{1}{3}(-2u + v_2) \\ 0 & 0 & 1 & \cdots & 0 & -\frac{1}{3}(-2u + v_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{3}(-2u + v_n) \end{array}\right)$$

$R_1 \leftarrow R_1 - (R_2 + R_3 + \cdots + R_n)$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & -\frac{1}{3}(-2u + v_1) \\ 0 & 1 & 0 & \cdots & 0 & -\frac{1}{3}(-2u + v_2) \\ 0 & 0 & 1 & \cdots & 0 & -\frac{1}{3}(-2u + v_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{3}(-2u + v_n) \end{array}\right) = (I_n \mid F^{-1} v)$$

$$\Rightarrow F^{-1} v = \begin{pmatrix} -\frac{1}{3}(-2u + v_1) \\ -\frac{1}{3}(-2u + v_2) \\ -\frac{1}{3}(-2u + v_3) \\ \vdots \\ -\frac{1}{3}(-2u + v_n) \end{pmatrix} = \frac{2}{3}\begin{pmatrix} u \\ u \\ u \\ \vdots \\ u \end{pmatrix} - \frac{1}{3}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

$$F^{-1}V = \frac{2}{3}\begin{pmatrix} u \\ u \\ u \\ \vdots \\ u \end{pmatrix} - \frac{1}{3}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

$$u = \frac{1}{2n-3}(v_1+v_2+v_3+\cdots+v_n)$$

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \frac{2}{3(2n-3)}\begin{pmatrix} v_1+v_2+v_3+\cdots+v_n \\ v_1+v_2+v_3+\cdots+v_n \\ v_1+v_2+v_3+\cdots+v_n \\ \vdots \\ v_1+v_2+v_3+\cdots+v_n \end{pmatrix} - \frac{1}{3}V$$

$$= \frac{2}{3(2n-3)}\underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}_{J}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} - \frac{1}{3}V$$

$$= \frac{2}{3(2n-3)}JV - \frac{1}{3}I_nV = \underbrace{\left(\frac{2}{3(2n-3)}J - \frac{1}{3}I_n\right)}_{F^{-1}}V$$

Therefore, $\underline{F^{-1} = \frac{2}{3(2n-3)}J - \frac{1}{3}I_n}$.