I2-TD3
(VECTOR SPACE)

1. For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, we define an addition in $\mathbb{R}^3$ and scalar multiplication with element in $\mathbb{R}$ by

$$\begin{aligned} x + y &= (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ \alpha.x &= \alpha (x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3). \end{aligned}$$

   Show that $(\mathbb{R}^3, +, .)$ is a vector space over $\mathbb{R}$.

2. For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we define two operations by

$$x \oplus y = (x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 2x_2 + y_2) \quad \text{and} \quad \alpha * x = \alpha * (x_1, x_2) = (\alpha x_1, \alpha x_2).$$

   Is $(\mathbb{R}^2, \oplus, *)$ a vector space over $\mathbb{R}$? Justify your answer.

3. For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we define two operations by

$$x \oplus y = (x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \text{and} \quad \alpha * x = \alpha * (x_1, x_2) = (2x_1, \alpha x_2).$$

   Is $(\mathbb{R}^2, \oplus, *)$ a vector space over $\mathbb{R}$? Justify your answer.

4. Let $V = \{(x, 1) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. For any $u = (x, 1), v = (y, 1) \in V$ and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} u \oplus v &= (x + y, 1) \\ \alpha * u &= (\alpha x, 1) \end{aligned}$$

   Show that $(V, \oplus, *)$ is a vector space over $\mathbb{R}$.

5. Show that $(\mathbb{R}_+, \oplus, *, \mathbb{R})$ is a vector space where

$$x \oplus y = xy \quad \text{and} \quad \alpha * x = x^\alpha$$

   for $x, y \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$.

6. For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, let us define

$$\begin{aligned} x \oplus y &= (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1) \\ \alpha * x &= (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1, \alpha x_3 + \alpha - 1) \end{aligned}$$

   Show that $(\mathbb{R}^3, \oplus, *)$ is a vector space over $\mathbb{R}$.

7. Determine if the following sets are subspaces of $\mathbb{R}^3$ under the usual operations.
   (a) $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 + x_3 = 0\}$
   (b) $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = 2x_2 \text{ and } x_3 = -x_2\}$
   (c) $W_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = x_3^2\}$

8. Determine if the following sets are subspaces of $\mathcal{F}_{\mathbb{R}}^{\mathbb{R}} = \{f : \ f : \mathbb{R} \to \mathbb{R}\}$ under the usual operations.

(a) $\mathcal{F}_1 = \{f \in \mathcal{C}^1 \ : f'(1) + f(2) = 0\}$

(b) $\mathcal{F}_2 = \{f : \ f(0) + f(1) = 3\}$.

(c) $\mathcal{F}_3 = \{f : \ f(0) = 3\}$.

(d) $\mathcal{F}_4 = \{f : \ f(1) = 2f'(1)\}$.

9. Determine if the following sets are subspaces of $\mathbb{R}^{n\times n}$ under the usual operations.

   (a) The set of all upper triangular matrices.

   (b) The set of all symmetric matrices.

   (c) The set of all orthogonal matrices.

   (d) The set of all matrices A such that $A^2 = 0$.

10. Let G and H be two subspaces of V. Show that $G \cap H$ is also a subspace of V.

11. Let G and H be two subspaces of V. Show that

$$(G \cup H \text{ subspace of } V) \Longleftrightarrow (G \subset F \text{ or } F \subset G)$$

12. Let G and H be two subspaces of V. Show that $G + H$ is also a subspace of V.

13. Let G and H be subspaces of V. Show that

$$G + H = G \cap H \quad \Longleftrightarrow \quad G = H.$$

14. Let F, G and H be three subspaces of V satisfy

$$F \cap G = F \cap H, \quad F + G = F + H \quad \text{and} \quad G \subset H$$

   Show that $G = H$.

15. If $S = \{v_1, v_2, \ldots, v_n\}$ is a subset of vector space V. Show that $\mathrm{Span}(S)$ is the smallest subspace of V containing set S.

16. Determine the values of x and y so that the vector $(2, 3, x, y)$ is an element of the subspace of $\mathbb{R}^4$ spanned by $(2, -1, 3, 5)$ and $(1, 3, 7, 2)$.

17. Which of the following are spanning sets for $\mathbb{R}^4$? Justify your answers.

   (a) $\{(1, 1, 1, 2),\ (1, 0, 1, 0), (2, 1, -2, 3)\}$

   (b) $\{(1, 1, 2, 1),\ (2, 3, 1, 2), (2, 1, 2, 1), (1, 2, 1, 2)\}$

   (c) $\{(0, 2, 1, 0), (1, -1, 0, 1), (0, 0, -2, 1), (1, 1, -1, 2)\}$

   (d) $\{(2, 1, 2, 1),\ (2, 3, 1, 2), (3, 1, 2, -1), (1, -2, 1, -3), (1, 0, 0, -2)\}$

   (e) $\{(1, 1, -1, 1),\ (2, 3, -1, 2), (3, 1, 2, 1), (1, -2, 1, 3), (1, -1, 1, -2)\}$

18. Determine whether the following vectors are linearly independent.

   (a) $(1, 1, 0),\ (2, 1, 0),\ (2, 3, 4)$

   (b) $(1, 1, -1, 2),\ (1, 2, 1, 1), (2, 1, 2, 3)$

   (c) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

   (d) $e^t,\ e^{2t},\ e^{3t}$

(e) $\cos^2 t,\ \sin^2 t,\ \cos 2t$

19. Let $f(x) = \ln(1+x), x \in \mathbb{R}_+$. Let $f_1 = f; f_2 = f \circ f$ and $f_3 = f \circ f \circ f$. Show that the set $\{f_1, f_2, f_3\}$ is linearly independent.

20. Let $\{x_1, x_2, \dots, x_n\}$ be a linearly independent set of V and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Suppose that

$$u = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \quad \text{and} \quad \forall 1 \le i \le n,\ y_i = x_i + u$$

What condition on α_i, for the set $\{y_1, y_2, \dots, y_n\}$ is linearly independent.

21. Let $\{x_1, x_2, \dots, x_n\}$ be a linearly independent set of V. For $k = 1, 2, \dots, n$, we let $y_k = x_k + x_{k+1}$ and $y_n = x_n + x_1$. Study the linearly independent of $\{y_1, y_2, \dots, y_n\}$.

22. Determine which of the following sets are bases for $\mathbb{R}^3$ if the vectors in $\mathbb{R}^3$, or for $\mathbb{R}^4$ if the vectors in $\mathbb{R}^4$, or for $P_2(\mathbb{R})$ if the vectors in $P_2(\mathbb{R})$.

   (a) $(1, -1, 2),\ (2, 1, 0),\ (2, 3, 4)$
   (b) $(2, -1, 2),\ (2, -1, 1),\ (0, 1, 1),\ (5, 2, 7)$
   (c) $(1, 1, -1, 1), (2, 3, -1, 2), (3, 1, -2, 1), (1, 2, -1, 3)$
   (d) $(1, 1, -1, 1), (2, 2, -1, 2), (1, 1, -2, 1)$
   (e) $1 + 2x + x^2,\ 3 + x^2,\ x + x^2$
   (f) $1 - 2x - 2x^2,\ -2 + 3x - x^2,\ 1 - x - 6x^2$

23. Let F be a subspace of $\mathbb{R}^4$ defined by

$$F = \{(w, x, y, z) \in \mathbb{R}^4 \mid w = 2x - y \text{ and } z = w + x + y\}$$

Find a basis and dimension of F.

24. Let F and G be two subset of $\mathbb{R}^4$ defined by

$$\begin{aligned} F &= \{(x, y, z, t) \in \mathbb{R}^4 \mid x - 2y = 0 \text{ and } y - 2z = 0\} \\ G &= \{(x, y, z, t) \in \mathbb{R}^4 \mid x + z = 0 \text{ and } y + t = 0\} \end{aligned}$$

   (a) Show that F and G are subspaces of $\mathbb{R}^4$.
   (b) Find a basis for F, a basis for G and a basis for $F + G$.
   (c) Deduce that $\mathbb{R}^4 = F + G$.

25. Let V be the vector space of n-square matrices over a field $\mathbb{R}$. Show that $V = U \oplus W$, where U and W are the subspaces of symmetric and antisymmetric matrices, respectively.

26. Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

If $V = W_1 + W_2$, then deduce that

$$V = W_1 \oplus W_2 \quad \Longleftrightarrow \quad \dim V = \dim W_1 + \dim W_2.$$

27. Let $S_1, S_2, \ldots, S_p$ be subspaces of V. Suppose that

$$V = \sum_{1\leq i\leq p} S_i \quad \text{and} \quad \sum_{1\leq i\leq p} \dim(S_i) = \dim(V).$$

Show that

$$V = S_1 \oplus S_2 \oplus \cdots \oplus S_p$$

28. In the vector space $\mathbb{R}^4$, consider the vectors

$$a = (1,2,-1,3), \quad b = (2,4,1,-2), \quad c = (3,6,3,-7), \quad d = (1,2,-4,11), \quad e = (2,4,-5,14)$$

Let F be a subspace of $\mathbb{R}^4$ spanned by $\{a,b,c\}$ and G be a subspace of $\mathbb{R}^4$ spanned by $\{d,e\}$.

(a) Determine the bases for F and G, then deduce the dimension of F and G.

(b) Show that $F = G$.

29. In the vector space $\mathbb{R}^4$, consider the vectors

$$a = (1,3,0,3), \quad b = (2,1,3,-2), \quad c = (0,1,0,0), \quad d = (1,0,-1,-1)$$

Let F be a subspace of $\mathbb{R}^4$ spanned by $\{a,b\}$ and G be a subspace of $\mathbb{R}^4$ spanned by $\{c,d\}$.

(a) Determine the bases for F and G, then deduce the dimension of F and G.

(b) Determine $F \cap G$.

(c) Deduce $F + G$. What can we conclude?

30. For $n \geq 2$, we define $\mathbb{R}_n[X]$, the set of polynomial of degree less than or equal n with coefficients in $\mathbb{R}$. Let

$$\mathcal{F} = \{P \in \mathbb{R}_n[X] \mid P(1) = P'(1) = 0\}.$$

(a) Show that $\mathcal{F}$ is a subspace of $\mathbb{R}_n[X]$.

(b) Show that P belongs to $\mathcal{F}$ if and only if $(X-1)^2$ divides P.

(c) Determine the basis and dimension of $\mathcal{F}$.

31. Let $\mathbb{R}_n[X]$ be the set of all real polynomials of degree at most n and let

$$P_k(x) = (x+1)^{k+1} - x^{k+1}, \quad k = 0,1,\ldots,n$$

Show that $\{P_0, P_1, \ldots, P_n\}$ is a basis for $\mathbb{R}_n[X]$.

32. Let S_1, S_2, S_3, S_4 be four subspaces of $\mathbb{R}^4$ and $v_1 = (1,1,1,2)$, $v_2 = (2,1,2,1)$, $v_3 = (1,0,1,-1)$, $v_4 = (3,-1,-1,4)$, $v_5 = (1,3,1,2)$, $v_6 = (1,-2,-1,1)$, $v_7 = (2,1,1,4)$, $v_8 = (-1,3,-1,2)$. Let S_1 be spanned by v_1, v_2, v_3, S_2 be spanned by v_4, v_5, v_6, S_3 be spanned by v_7, S_4 be spanned by v_8.

(a) Find a basis and the dimension of each of the subspaces $S_1, S_2, S_3, S_4, S_1+S_2, S_1+S_2+S_3$, $S_2+S_3+S_4$, and $S_1+S_3+S_4$.

(b) Show that $\mathbb{R}^4 = S_1 \oplus S_2$, $\mathbb{R}^4 = S_1 \oplus S_3 \oplus S_4$, and $\mathbb{R}^4 = S_2 \oplus S_3 \oplus S_4$.

(c) Show that $\mathbb{R}^4$ is the sum of S_1, S_2, S_3, S_4. Show that $\mathbb{R}^4$ is not the direct sum of the four subspaces S_1, S_2, S_3, S_4.

33. Let S_1 be a subspace of $\mathbb{R}^5$ spanned by $(1,1,1,2,1),(2,1,1,3,2),(1,0,0,1,1),(1,2,2,3,3)$ and S_2 be a subspace of $\mathbb{R}^5$ spanned by $(1,-1,-1,0,-1),(3,1,1,4,3),(2,2,2,3,4)$.

    (a) Find a basis B_1 and the dimension of S_1. Find the coordinates of $(2,-1,-1,1,0)$ with respect to B_1.

    (b) Find a basis and the dimension of S_2, S_1+S_2, and $S_1\cap S_2$.

34. Consider the vector space $P_3(x)$ of real polynomials in x of degree ≤ 3.

    (a) Show that $B=\{1,1-t,(1-t)^2,(1-t)^3\}$ is a basis of $P_3(x)$.

    (b) Find the coordinates of vector $v(t)=2+3t-t^2+t^3$ with respect to basis B.

35. Let P_m be the set of all real polynomials of degrees at most m.

    (a) Show that $B=\{1,t-1,\dots,(t-1)^m\}$ is a basis for P_m.

    (b) Determine the coordinates of $(t+2)^m$ with respect to the basis B.

36. Let $n\in\mathbb{N}$ and $a,b\in\mathbb{R}$ with $a\neq b$. Let $\mathbb{R}_{2n}[X]$ be the set of all real polynomials of degree at most $2n$.

    (a) Show that $B=\{\left((x-a)^k\right)_{0\leq k\leq 2n}\}$ is a basis for $\mathbb{R}_{2n}[X]$.

    (b) Determine the coordinates of $(x-a)^n(x-b)^n$ with respect to basis B.

37. Let $B_1=\{(1,1,1),\ (1,1,0),\ (1,0,0)\}$ and $B_2=\{(1,1,-1),\ (1,-1,0),\ (2,0,0)\}$ be two ordered bases for $\mathbb{R}^3$. Find a transition matrix from ordered basis B_1 to ordered basis B_2 and also a transition matrix from ordered basis B_2 to ordered basis B_1. Then find the coordinates of $v=(1,-1,0)$ in ordered basis B_2.

38. Determine whether the following vector spaces equipped with real mappings are inner product spaces. If any, define the norm and the distance associated with each of these inner products.

    (a) $\mathbb{R}^4$ equipped with $\langle x\,;y\rangle=4x_1y_1+2x_2y_2+3x_3y_3+7x_4y_4$.

    (b) $\mathbb{R}^3$ equipped with $\langle x\,;y\rangle=x_1y_1+x_2y_2-2x_3y_3$.

    (c) $C^0\,[0;1]$ equipped with $\langle f_1\,;f_2\rangle=\displaystyle\int_0^1 tf_1(t)\,f_2(t)\,dt$.

    (d) P_3 equipped with

$$\left\langle a_0+a_1t+a_2t^2+a_3t^3\,;b_0+b_1t+b_2t^2+b_3t^3\right\rangle=a_0b_0+a_1b_1+2a_2b_2+a_3b_3.$$

39. Let $\varphi:\mathcal{M}_n(\mathbb{R})\to\mathbb{R}$ defined by

$$\varphi(A,B)=\operatorname{tr}(A^tB)$$

Show that φ is an inner product in $\mathcal{M}_n(\mathbb{R})$.

40. Let $\langle;\rangle_1,\langle;\rangle_2$ be two mappings from $\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}$ defined by

$$\langle x\,;y\rangle_1=x_1y_1+x_2y_2+x_3y_3\quad\text{and}\quad\langle x\,;y\rangle_2=x_1y_1+2x_2y_2+3x_3y_3.$$

Let $v_1=(1,1,1)$, $v_2=(1,1,-2)$, $v_3=(1,1,-1)$ three vectors in $\mathbb{R}^3$.

(a) Show that $\langle ; \rangle_1, \langle ; \rangle_2$ are inner products. Find the norm of v_1 for$\langle ; \rangle_1$ and for $\langle ; \rangle_2$. Find the distance between v_1 and v_2 for $\langle ; \rangle_1$ and for $\langle ; \rangle_2$.

(b) Show that v_1 and v_2 are orthogonal with respect to $\langle ; \rangle_1$ but are not orthogonal with respect to $\langle ; \rangle_2$.

(c) Show that v_1 and v_3 are orthogonal with respect to $\langle ; \rangle_2$ but are not orthogonal with respect to $\langle ; \rangle_1$.

41. Let $\mathbb{R}^3$ be a vector space equipped with the inner product

$$\langle x\,; y\rangle = 4x_1y_1 + 3x_2y_2 + 5x_3y_3.$$

Let $v_1 = (1, 1, 1)$, $v_2 = (1, 2, -2)$, $v_3 = (-5, 5, 1)$ be three vectors in $\mathbb{R}^3$.

(a) Show that $B = \{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Derive an orthonormal basis B_0 for $\mathbb{R}^3$ from B.

(b) Let $v = (2, 1, 2)$. Determine the coordinates of v with respect to B then with respect to B_0.

(c) Find the scalar and vector projections of v onto v_2.

42. Find an orthogonal basis for the subspace W of $\mathbb{R}^4$ where

$$W = \{(x, y, z, w) \in \mathbb{R}^4 : \ x - y - z = 0 \text{ and } x + z = 0\}.$$

43. Let $\mathbb{R}^4$ be a vector space equipped with the inner product

$$\langle x\,; y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4y_4.$$

Let $v_1 = (1, 2, 1, 2)$, $v_2 = (2, 1, 2, 1)$, $v_3 = (1, 1, 2, 2), v_4 = (2, 2, 1, 1), v_5 = (1, 2, 1, 3)$ be five vectors in $\mathbb{R}^4$, S_1 be a subspace of $\mathbb{R}^4$ spanned by v_1, v_2, v_3, v_4, and S_2 be a subspace of $\mathbb{R}^4$ spanned by v_1, v_2, v_3, v_5.

(a) Determine a basis and the dimension for S_1, denoted by B_1. Use Gram-Schmidt process to transform B_1 to an orthogonal and orthonormal basis for S_1.

(b) Show that $B_2 = \{v_1, v_2, v_3, v_5\}$ is a basis for S_2, then deduce that $S_2 = \mathbb{R}^4$. Use Gram-Schmidt process to transform B_2 to an orthogonal basis for S_2.

44. Find $W^\perp$, the orthogonal complement of W, where

$$W = \{(x, y, z) \in \mathbb{R}^3 : \ x + 3y - z = 0\}.$$

45. Let $V = \mathcal{C}[-1, 1]$. Suppose that S_e and S_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $S_e^\perp = S_o$, where the inner product on V is defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

46. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp \quad \text{and} \quad (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$$

47. Let W be a subspace of a finite-dimensional vector space V. Show that

(a) $(W^\perp)^\perp = W$

(b) $V = W \oplus W^\perp$