1. Determine if the following mappings are linear transformations.

(a) $L : \mathbb{R}^2 \to \mathbb{R}^2, L(x_1, x_2) = (x_1 - 2x_2, 3 + x_1 + 3x_2)$

- $\theta = (0,0) \in \mathbb{R}^2$ but $L(\theta) = (0 - 2(0), 3 + (0) + 3(0)) = (0,3) \neq \theta$

By theorem 1.), the mapping L is not a linear transformation.

(b) $L : \mathbb{R}^3 \to \mathbb{R}^2, L(\underbrace{x_1, x_2, x_3}_{x}) = (x_1 + 2x_2 + x_3, x_1 - x_2)$

- $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in V = \mathbb{R}^3$, $\alpha \in \mathbb{R}$

$$L(x + \alpha y) = ((x_1 + \alpha y_1) + 2(x_2 + \alpha y_2) + (x_3 + \alpha y_3), (x_1 + \alpha y_1) - (x_2 + \alpha y_2))$$

$$= (x_1 + 2x_2 + x_3, x_1 - x_2) + \alpha(y_1 + 2y_2 + y_3, y_1 - y_2)$$

$$= L(x) + \alpha L(y)$$

By theorem 1.), L is a linear transformation.

(c) $L : \mathbb{R}^2 \to \mathbb{R}^2, L(x_1, x_2) = (x_1 + 2x_2, x_1 x_2)$

$x = (x_1, x_2), y = (y_1, y_2) \in V = \mathbb{R}^2$, $\alpha \in \mathbb{R}$

$$L(x + \alpha y) = ((x_1 + \alpha y_1) + 2(x_2 + \alpha y_2), (x_1 + \alpha y_1)(x_2 + \alpha y_2))$$

$$= (x_1 + 2x_2 + \alpha y_1 + 2\alpha y_2, x_1 x_2 + \alpha x_1 y_2 + \alpha x_2 y_1 + \alpha^2 y_1 y_2)$$

$$= (x_1 + 2x_2, x_1 x_2) + \alpha(y_1 + 2y_2, x_1 y_2 + x_2 y_1 + \alpha y_1 y_2)$$

$$= L(x) + \alpha(y_1 + 2y_2, x_1 y_2 + x_2 y_1 + \alpha y_1 y_2)$$

$$\neq L(x) + \alpha L(y)$$

For instance, $x = (1,0), y = (0,1), \alpha = 1$

$$\Rightarrow L(x + \alpha y) = L(1,1) = ((1) + 2(1), (1)(1)) = (3,1)$$

$$L(x) + \alpha L(y) = L(1,0) + 1 \cdot L(0,1)$$

$$= ((1) + 2(0), (1)(0)) + ((0) + 2(1), (0)(1))$$

$$= (1,0) + (2,0) = (3,0)$$

Since $L(x+\alpha y) \neq L(x) + \alpha L(y)$, L is not a linear tran.

(d) Let $A \in \mathbb{R}^{n\times n}$ and $L : \mathbb{R}^n \to \mathbb{R}^n, L(x) = Ax$

- $\forall x, y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$,

$$
\begin{aligned}
L(x+\alpha y) &= A(x+\alpha y) \\
&= Ax + A(\alpha y), \quad A(B+C) = AB + AC \\
&= Ax + \alpha Ay, \quad A(\alpha B) = \alpha AB \\
&= L(x) + \alpha L(y)
\end{aligned}
$$

(e) $\varphi : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R}), \varphi(f) = 2tf''(t) + \cos t f'(t) - (t^2-1)f(t)$.

$$\begin{cases} (f+g)' = f' + g' \\ (\alpha f)' = \alpha f' \\ \hookrightarrow (f+\alpha g)' = f' + \alpha g' \end{cases}$$

- $f, g \in C^2(\mathbb{R})$, $\alpha \in \mathbb{R}$,

$$
\begin{aligned}
\varphi(f+\alpha g) &= 2t\,(f+\alpha g)''(t) + \cos t\,(f+\alpha g)'(t) - (t^2-1)(f+\alpha g)(t) \\
&= 2t\,(f''(t) + \alpha g''(t)) + \cos t\,(f'(t) + \alpha g'(t)) \\
&\qquad - (t^2-1)(f(t) + \alpha g(t)) \\
&= [2tf''(t) + \cos t\, f'(t) - (t^2-1)f(t)] \\
&\qquad + \alpha[2tg''(t) + \cos t\, g'(t) - (t^2-1)g(t)] \\
&= \varphi(f) + \alpha\varphi(g)
\end{aligned}
$$

Therefore, φ is a linear transformation.

(f) $\varphi : \mathcal{C}^0(\mathbb{R}) \to \mathbb{R}$, $\varphi(f) = \int_0^1 (f(x) + 3f'(x))\,dx$

- $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

$f, g \in C^1(\mathbb{R})$, $\alpha \in \mathbb{R}$

$$
\begin{aligned}
\varphi(f+\alpha g) &= \int_0^1 [(f+\alpha g)(x) + 3(f+\alpha g)'(x)]\,dx \\
&= \int_0^1 [(f(x) + 3f'(x)) + \alpha(g(x) + 3g'(x))]\,dx
\end{aligned}
$$

$$= \int_0^1 [(f(x)+3f'(x)) + \alpha(g(x)+3g'(x))]dx$$

$$= \int_0^1 (f(x)+3f'(x))\,dx + \alpha \int_0^1 (g(x)+3g'(x))\,dx$$

$$= \varphi(f) + \alpha\varphi(g).$$

Therefore, φ is a linear transformation.

2. Let $L \in \mathcal{L}(V, W)$. Show that

(a) if $v_1, v_2, \ldots, v_n$ span a vector space V, then $L(v_1), L(v_2), \ldots, L(v_n)$ span $\mathrm{Im}(L)$.

(b) if $L(v_1), L(v_2), \ldots, L(v_n)$ are linearly independent, then $v_1, v_2, \ldots, v_n$ linearly independent.

(a) If $V = \mathrm{span}\{v_1, \ldots, v_n\}$ then

for any $v \in V$, there exists $k_1, \ldots, k_n \in \mathbb{K}$,

$$v = k_1 v_1 + \cdots + k_n v_n = \sum_{i=1}^{n} k_i v_i$$

Since $L \in \mathcal{L}(V, W)$, then

$$L(v) = L\left(\sum_{i=1}^{n} k_i v_i\right) = \sum_{i=1}^{n} k_i L(v_i)$$

$$= k_1 \underline{L(v_1)} + k_2 \underline{L(v_2)} + \cdots + k_n \underline{L(v_n)}$$

Therefore,

$$\mathrm{Im}(L) = \{L(v) : v \in V\} = \mathrm{span}\{L(v_1), \ldots, L(v_n)\}$$

(b) $\underline{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0}$ $\rightsquigarrow 0$ Theorem 2

$L(\alpha_1 v_1 + \cdots + \alpha_n v_n) = L(0)$ $\longrightarrow$ 1)

$\alpha_1 L(v_1) + \cdots + \alpha_n L(v_n) = 0$ $\longrightarrow$ 3)

We know that $L(v_1), \ldots, L(v_n)$ is linearly independent.

$\Rightarrow \underline{\alpha_1 = \alpha_2 = \cdots = \alpha_n}$. Therefore, $v_1, v_2, \ldots, v_n$ is L.I.

4. Let $L \in \mathcal{L}(V, W)$.

(a) Show that L is one-to-one if and only if $\mathrm{Ker}(L) = \{0\}$.

(b) Show that L is onto if and only if $\mathrm{Im}(L) = W$.

(c) Suppose $\dim V = \dim W < \infty$. Show that L is one-to-one if and only if it is onto.

(a) • If L is one-to-one show that $\mathrm{Ker}(L) = \{0\}$

$\mathrm{Ker}(L) = \{ v \in V : L(v) = 0 \}$

If $v \neq 0$, then $L(v) \neq L(0)$ because L is one-to-one.

But $L \in \mathcal{L}(V, W)$, by theorem 1, $L(0) = 0$.

$\Rightarrow L(v) \neq 0$.

Therefore, $\ker(L) = \{0\}$. $\rightarrow \{v \in V, L(v) = 0\} = \{0\}$

• If $\mathrm{Ker}(L) = \{0\}$ show that L is one-to-one.

$u \neq v \Rightarrow u - v \neq 0 \Rightarrow u - v \notin \ker(L) = \{0\}$

$\Rightarrow L(u - v) \neq 0 \Rightarrow L(u) - L(v) \neq 0$

$\Rightarrow L(u) \neq L(v)$.

Therefore, L is one-to-one.

$\mathrm{Im}(L) = \{ L(v) : v \in V \}$

(b) Show that L is onto if and only if $\mathrm{Im}(L) = W$.

$L : V \to W$

• If L is onto show that $\mathrm{Im}(L) = W$

$\forall w \in W$, $\exists v \in V$ s.t. $w = L(v) \in \mathrm{Im}(L)$

$\Rightarrow W \subseteq \mathrm{Im}(L)$. But $\mathrm{Im}(L) \subseteq W$.

Therefore, $\mathrm{Im}(L) = W$.

• If $\mathrm{Im}(L) = W$ show that L is onto.

$\forall w \in W \Rightarrow w \in \mathrm{Im}(L) \Rightarrow \exists v \in L$ s.t. $w = L(v)$

Therefore, L is onto.

(c) Suppose $\dim V = \dim W < \infty$. Show that L is one-to-one if and only if it is onto.

• If L is one-to-one show that it is onto.

By exc. 3.c) $\dim Ker(L) + \dim Im(L) = \dim V$

$$\underbrace{\dim \{0\}}_{0} + \dim Im(L) = \dim V$$

$$\dim Im(L) = \dim V$$

But $\dim V = \dim W$, then $\dim Im(L) = \dim W$

By the fact that $Im(L) \leq W$, we get

$Im(L) = W$.

By exc (4.b.), L is onto.

. If L is onto show that L is one-to-one

By exc 4.b) $Im(L) = W \Rightarrow \dim Im(L) = \dim W$.

By exc 3.c) $\dim Ker(L) + \dim Im(L) = \dim V$

$$\dim Ker(L) + \cancel{\dim W} = \cancel{\dim V}$$

But $\dim W = \dim V \Rightarrow \dim Ker(L) = 0$

$\Rightarrow Ker(L) = \{0\}$

By exc 4.a), L is one-to-one.

6. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (z, x - y, y + z)$. Show that f is an automorphism.

one-to-one

onto

$Ker(f) = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = (0,0,0)\}$

$f(x,y,z) = 0 \iff (z, x-y, y+z) = (0,0,0)$

$$\begin{cases} z = 0 \\ x - y = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \Rightarrow Ker(f) = \{(0,0,0)\}$$

Thus, by exc 4.a), f is one-to-one (*)

- $\forall (a,b,c) \in W = \mathbb{R}^3$, $\exists (x,y,z) = (b+c-a, c-a, a) \in V = \mathbb{R}^3$ s.t.

$f(x,y,z) = (a,b,c)$ because

$(z, x-y, y+z) = (a,b,c)$

$$\begin{cases} z = a \\ x-y = b \\ y+z = c \end{cases} \Rightarrow \begin{cases} x = b+c-a \\ y = c-a \\ z = a \end{cases}$$

This means that f is onto $(**)$.

By $(*)$ & $(**)$, thus, f is an <u>automorphism</u>.

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x-y \\ y+z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f(v) = Av, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1) \cdot (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$