14. Compute determinants of $n \times n$ matrices below:

(a) $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \dots & n \end{vmatrix}$

$$\longrightarrow \begin{matrix} 1 & 2 & 3 & \dots & n-2 & n-2 & n-2 \\ 1 & 2 & 3 & \dots & n-2 & n-1 & n-1 \\ 1 & 2 & 3 & \dots & n-2 & n-1 & n \end{matrix}$$

$n-(n-1)=1$

- $R_n \leftarrow R_n - R_{n-1}$
  $R_{n-1} \leftarrow R_{n-1} - R_{n-2}$
  $\vdots$
  $R_2 \leftarrow R_2 - R_1$

For $i = n, n-1, \dots, 2$: $R_i \leftarrow R_i - R_{i-1}$

- $$a = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = \underbrace{1 \cdot 1 \cdot \dots \cdot 1}_{n \text{ factors}} = 1$$

(b) $\begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \dots & n \end{vmatrix}$

$$\begin{vmatrix} 1 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix}_{4 \times 4}$$

For $i = 1, 2, \dots, n-1$: $R_i \leftarrow R_i - R_n$

$$b = \begin{vmatrix} (1-n) & 0 & 0 & \dots & 0 \\ 0 & (2-n) & 0 & \dots & 0 \\ 0 & 0 & (3-n) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \dots & n \end{vmatrix} = (1-n)(2-n)(3-n)\cdots\underbrace{((n-1)-n)}_{(-1)}\, n$$

$$= (-1)^{n-1}(n-1)(n-2)(n-3)\cdots 1 \cdot n = (-1)^{n-1} \cdot n!$$

Date. No.

LUN TOLA

$$= (1-n)(2-n)(3-n)\dots(-1)n$$

$$= (-1)^{n-1}\, n(n-1)(n-2)\dots 1$$

$$= (-1)^{n-1}\, n!$$

(c)

$$\begin{vmatrix} 5 & 2 & & (0) \\ 3 & 5 & 2 & \\ & 3 & \ddots & 2 \\ (0) & & 3 & 5 \end{vmatrix}$$

Laplace expansion along the first row

$$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}| + 0 + \dots + 0$$

$$= 5 \begin{vmatrix} 5 & 2 & & \\ 3 & 5 & \ddots & \\ & \ddots & \ddots & 2 \\ & & 3 & 5 \end{vmatrix}_{n-1} - 2 \begin{vmatrix} 3 & 2 & & (0) \\ 0 & 5 & \ddots & \\ & \ddots & \ddots & 2 \\ (0) & & 3 & 5 \end{vmatrix}_{n-1}$$

$(-1)^{1+2}$

$$= 5C_{n-1} - 2 \times 3 \begin{vmatrix} 5 & 2 & & (0) \\ 3 & 5 & \ddots & \\ & \ddots & \ddots & 2 \\ (0) & & 3 & 5 \end{vmatrix}_{n-2}$$

$$= 5C_{n-1} - 6C_{n-2}$$

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$\Rightarrow C_n - 5C_{n-1} + 6C_{n-2} = 0$

ch eq: $r^2 - 5r + 6 = 0$ ✓

$\Delta = 1$ ✓

$\Rightarrow r_1 = 2,\ r_2 = 3$ ✓

$C_1 = |5|$

$C_2 = \begin{vmatrix} 5 & 2 \\ 3 & 5 \end{vmatrix} = 25 - 6 = 19$

$C_n = Ar_1^n + Br_2^n = A2^n + B3^n$ ✓

. for $n = 1$: $C_1 = 5 \iff 2A + 3B = 5$ (1) ✓

. for $n = 2$: $C_2 = 19 \iff 4A + 9B = 19$ (2)

(1) & (2)

$$\begin{cases} 2A + 3B = 5 \\ 4A + 9B = 19 \end{cases}$$ ✓

$$+\begin{cases} -6A - 9B = -15 \\ 4A + 9B = 19 \end{cases}$$

$-2A = 4$

$A = -2$

$\Rightarrow B = 3$ ✓

Thus

$C_n = -2 \times 2^n + 3 \times 3^n = -2^{n+1} + 3^{n+1}$

$C_1 = 9 - 4 = 5$ ✓

$C_2 = 27 - 8 = 19$ ✓

NGIM PHANHA

14) Compute determinants of $n \times n$ matrices below:

a) $\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix} \underset{L_i \to L_i - L_{i-1}}{=} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \; L_i \to L_i - L_{i+1}$

$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = \boxed{1}$

b) $\begin{vmatrix} 1 & n & n & \cdots & n \\ n & 2 & n & \cdots & n \\ n & n & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{vmatrix} \underset{(i=1,2,\ldots,n-1)}{L_i \to L_i - L_n} = \begin{vmatrix} 1-n & 0 & 0 & \cdots & 0 \\ 0 & 2-n & 0 & \cdots & 0 \\ 0 & 0 & 3-n & \cdots & 0 \\ \vdots & \vdots & \vdots & -1 & \vdots \\ n & n & n & \cdots & n \end{vmatrix}$ is triangular matrix

$= (1-n)(2-n)(3-n)\cdots(-1)n = \left(n(n-1)(n-2)(n-3)\cdots 1\right)(-1)^{n-1} = (-1)^{n-1} n!$

c) $\begin{vmatrix} 5 & 2 & & & (0) \\ 3 & 5 & 2 & & \\ & 3 & \ddots & \ddots & \\ & & \ddots & \ddots & 2 \\ (0) & & & 3 & 5 \end{vmatrix} = 5 \begin{vmatrix} 5 & 2 & & (0) \\ 3 & 5 & 2 & \\ & 3 & \ddots & 2 \\ (0) & & 3 & 5 \end{vmatrix}_{(n-1)} - 2 \begin{vmatrix} 3 & 2 & & & (0) \\ 0 & 5 & 2 & & \\ 0 & 3 & 5 & 2 & \\ \vdots & (0) & & \ddots & 2 \\ 0 & & & 3 & 5 \end{vmatrix}_{(n-1)}$

$= 5 \begin{vmatrix} 5 & 2 & & (0) \\ 3 & 5 & 2 & \\ & 3 & \ddots & 2 \\ (0) & & 3 & 5 \end{vmatrix}_{(n-1)} - 6 \begin{vmatrix} 5 & 2 & & (0) \\ 3 & 5 & 2 & \\ & (0) & \ddots & 2 \\ & & 3 & 5 \end{vmatrix}_{(n-2)}$

Let $U_n = 5U_{n-1} - 6U_{n-2} \Rightarrow U_n - 5U_{n-1} + 6U_{n-2} = 0$

+ cha Eq: $r^2 - 5r + 6 = 0$, $\Delta = (-5)^2 - 4 \times 6 = 1$

$r_1 = \frac{5-1}{2} = 2$, $r_2 = \frac{5+1}{2} = 3$

$\Rightarrow U_n = A r_1^n + B r_2^n = A 2^n + B.3^n$

We have the Hypothesis.

+ $\begin{cases} U_1 = 5 \\ U_2 = 19 \end{cases} \Rightarrow \begin{cases} 2A + 3B = 5 \quad (1) \times(-2) \\ 4A + 9B = 19 \quad (2) \end{cases}$

$3B = 9 \Rightarrow B = 3$

$\Rightarrow A = -2$

Thus $\boxed{U_n = -2^{n+1} + 3^{n+1}}$ ✓

LIM SENGLY

(d) $\begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ a & b & \ddots & \ddots & \vdots \\ a^2 & ab & \ddots & \ddots & 0 \\ \vdots & \vdots & & b & -1 \\ a^n & a^{n-1}b & \cdots & ab & b \end{vmatrix}$ $R_i \to R_i - aR_{i-1}$, $\boxed{i = 2, \ldots, n}$

$i = n, n-1, \ldots, 2$

$= \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & b+a & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & (0) & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & b+a \end{vmatrix} = (b+a)^{n-1}$ ✓

$$d_4 = \begin{vmatrix} 1 & -1 & 0 & 0 \\ a & b & -1 & 0 \\ a^2 & ab & b & -1 \\ a^3 & a^2b & ab & b \end{vmatrix}$$

(e) $\begin{vmatrix} a+b & b & \cdots & b \\ a & a+b & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ a & \cdots & a & a+b \end{vmatrix}_{n\times n} = e_n \quad (\text{by exercise 11})$

(a) (b)

$$\begin{matrix} 0+b \\ 0+b \\ \vdots \\ 0+b \\ a+b \end{matrix}$$

$$e_n = \begin{vmatrix} a+b & b & \cdots & 0 \\ a & a+b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & a \end{vmatrix}_n + \begin{vmatrix} a+b & b & \cdots & b \\ a & a+b & \cdots & b \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & b \end{vmatrix}_n$$

$$= a(-1)^{n+n} \begin{vmatrix} a+b & b & \cdots & b \\ a & a+b & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{vmatrix}_{n-1} + b \begin{vmatrix} a+b & b & \cdots & 1 \\ a & a+b & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 1 \end{vmatrix}_n$$

For $i = 1, 2, \ldots, n-1$: $C_i \leftarrow C_i - a \cdot C_n$

$$= a \cdot e_{n-1} + b \begin{vmatrix} b & b-a & \cdots & 1 \\ 0 & b & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}_n$$

$$= a e_{n-1} + b\,(\underbrace{b \cdots b}_{(n-1)\text{ factors}} \cdot 1) = a e_{n-1} + b^n$$

$\Rightarrow \quad e_n - a e_{n-1} = b^n, \quad e_1 = a+b, \quad e_2 = a^2+ab+b^2$

* If $a = 0$, $e_n = b^n$
* If $a \neq 0$, $\dfrac{e_n}{a^n} - \dfrac{e_{n-1}}{a^{n-1}} = \left(\dfrac{b}{a}\right)^n$

$$\sum_{k=2}^{n} \left(\frac{e_k}{a^k} - \frac{e_{k-1}}{a^{k-1}} \right) = \sum_{k=2}^{n} \left(\frac{b}{a}\right)^k$$

$$\sum_{k=2}^{n}\left(\frac{e_k}{a^k}-\frac{e_{k-1}}{a^{k-1}}\right)=\sum_{k=2}^{n}\underbrace{\left(\frac{b}{a}\right)^k}_{\to r}$$

$$\frac{e_n}{a^n}-\frac{e_1}{a}=\sum_{k=2}^{n}\left(\frac{b}{a}\right)^k \qquad (\#)$$

* In case $b=a$,

$$\frac{e_n}{a^n}-\frac{a+a}{a}=n-1 \Rightarrow e_n=(n+1)a^n$$

* In case $b\neq a$,

$$\frac{e_n}{a^n}-\frac{a+b}{a}=\frac{1}{\frac{b}{a}-1}\left[\left(\frac{b}{a}\right)^{n+1}-\left(\frac{b}{a}\right)^2\right]$$

$$\begin{cases} S=r^2+r^3+\dots+r^n \\ rS=r^3+\dots+r^n+r^{n+1} \end{cases}$$

$$(r-1)S=r^{n+1}-r^2$$

$$=\frac{a}{b-a}\left(\frac{b^{n+1}}{a^{n+1}}-\frac{b^2}{a^2}\right)$$

$$\Rightarrow e_n=(a+b)a^{n-1}+\frac{a^{n+1}}{b-a}\left(\frac{b^{n+1}}{a^{n+1}}-\frac{b^2}{a^2}\right)$$

$$=(a+b)a^{n-1}+\frac{1}{b-a}\left(b^{n+1}-b^2a^{n-1}\right)$$

$$=\frac{(b^2-a^2)a^{n-1}+b^{n+1}-b^2a^{n-1}}{b-a}$$

$$=\frac{\cancel{b^2a^{n-1}}-a^{n+1}+b^{n+1}-\cancel{b^2a^{n-1}}}{b-a}$$

$$=\frac{b^{n+1}-a^{n+1}}{b-a}=\sum_{k=0}^{n}a^kb^{n-k}$$

- $a=0$, $e_n=b^n$
- $a\neq 0, a=b$, $e_n=(n+1)a^n$
- $a\neq 0, a\neq b$, $e_n=\sum_{k=0}^{n}a^kb^{n-k}$

Therefore,
$$e_n=\begin{cases} b^n & \text{if } a=0 \\ \sum_{k=0}^{n}a^kb^{n-k} & \text{if } a\neq 0 \end{cases}=\boxed{\sum_{k=0}^{n}a^kb^{n-k}}\ \checkmark$$

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\lambda \neq 0$

$$\lambda A = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \\ \lambda g & \lambda h & \lambda i \end{bmatrix}$$

$$\Rightarrow |\lambda A| = \begin{vmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \\ \lambda g & \lambda h & \lambda i \end{vmatrix} = \lambda^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \lambda^3 |A|$$

$$|B| = \begin{vmatrix} a & b & \lambda c \\ d & e & \lambda f \\ g & h & \lambda i \end{vmatrix} = \lambda \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \lambda |A|$$

f.

$$\begin{vmatrix} x+a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & x+a_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & x+a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \\ a_n & a_n & a_n & & x+a_n \end{vmatrix} \qquad C_i = C_i - C_1, \quad i = 2, 3, \dots, n$$

$$= \begin{vmatrix} x+a_1 & -x & -x & \cdots & -x \\ a_2 & x & 0 & & 0 \\ a_3 & 0 & -x & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_n & 0 & 0 & & x \end{vmatrix} \qquad L_i = L_i + L_2 + L_3 + \cdots + L_n$$

$$= \begin{vmatrix} \left(x+\sum_{k=1}^{n} a_k\right) & 0 & 0 & 0 \\ a_2 & x & 0 & 0 \\ a_3 & 0 & x & 0 \\ \vdots & \vdots & \vdots & \ddots \\ a_n & 0 & 0 \cdots & x \end{vmatrix} = x^{n-1}\left(x+\sum_{k=1}^{n} a_k\right)$$

there fore $\boxed{\det(f) = x^{n-1}\left(x+\sum_{k=1}^{n} a_k\right)}$

$$\text{(f)}\quad \begin{vmatrix} x+a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & x+a_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & x+a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & x+a_n \end{vmatrix}$$

For $i = 2, 3, \ldots, n$:

$R_1 \leftarrow R_1 + R_i$

$$f_n = \begin{vmatrix} x+\sum_{k=1}^{n} a_k & x+\sum_{k=1}^{n} a_k & x+\sum_{k=1}^{n} a_k & \cdots & x+\sum_{k=1}^{n} a_k \\ a_2 & x+a_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & x+a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & x+a_n \end{vmatrix}_n$$

$$= \left(x+\sum_{k=1}^{n} a_k\right) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_2 & x+a_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & x+a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & x+a_n \end{vmatrix}$$

For $i = 2, 3, \ldots, n$: $R_i \leftarrow R_i - a_i R_1$

$$\Rightarrow f_n = \left(x+\sum_{k=1}^{n} a_k\right) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x & 0 & \cdots & 0 \\ 0 & 0 & x & \cdots & 0 \\ \vdots & \vdots \; (0) & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x \end{vmatrix} = \underline{x^{n-1}\left(x-\sum_{k=1}^{n} a_k\right)}$$