17. Let $f(x) = x^3 - 2x^2 + x$ and $g(x) = x^{2020} - 10x^{1000} + 3x - 1$. Let

$$A = \begin{pmatrix} 1 & -1 & -5 \\ 1 & 3 & 7 \\ 1 & 0 & -2 \end{pmatrix}$$

Compute $f(A)$ and $g(A)$.

(17) $f(A) = A^3 - 2A^2 + A = A(A^2 - 2A + I) = A(A-I)^2$

- $(A-I)^2 = \begin{pmatrix} 0 & -1 & -5 \\ 1 & 2 & 7 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 & -5 \\ 1 & 2 & 7 \\ 1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -6 & -2 & 8 \\ 9 & 3 & -12 \\ -3 & -1 & 4 \end{pmatrix}$

$\Rightarrow f(A) = A(A-I)^2 = \begin{pmatrix} 1 & -1 & -5 \\ 1 & 3 & 7 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} -6 & -2 & 8 \\ 9 & 3 & -12 \\ -3 & -1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_3$

- $g(x) = f(x)q(x) + r(x)$, $\deg r(x) < \deg f(x)$

$x^{2020} - 10x^{1000} + 3x - 1 = x(x-1)^2 q(x) + ax^2 + bx + c$ (1)

$2020x^{2019} - 10000x^{999} + 3 = (x-1)^2 q(x) + 2x(x-1)q(x) + x(x-1)^2 q'(x)$
$+ 2ax + b$ (2)

- Substitute $x = 0$ in (1): $-1 = c \Rightarrow c = -1$ (i)
- substitute $x = 1$ in (1): $1 - 10 + 3 - 1 = a + b + c$

$\Rightarrow a + b + c = -7$ (ii)

- substitute $x = 1$ in (2): $2020 - 10000 + 3 = 2a + b$

$\Rightarrow 2a + b = -7977$ (iii)

$$\begin{cases} c = -1 & \text{(i)} \\ a + b = -6 & \text{(ii)} \\ 2a + b = -7977 & \text{(iii)} \end{cases}$$

$$\begin{cases} c = -1 & (i) \\ a+b=-6 & (ii) \\ 2a+b=-7977 & (iii) \end{cases}$$

$(iii)-(ii)$: $a = -7971$

(ii): $b = -6-a = -6+7971 = 7965$

Then $g(x) = f(x)q(x) - 7971x^2 + 7965x - 1$

Therefore, $g(A) = \underbrace{f(A)}_{0}q(A) - 7971A^2 + 7965A - I$

$= -7971A^2 + 7965A - I = \ldots$

Ex. $f(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$

Explanation

$g(x) = f(x)q(x) + \underbrace{ax^2+bx+c}_{r(x)}$ (1)

Take $x=0$, (i)

$x=1$, (ii)

$x=2$, (iii)

$x^n = (x+1)^3 q(x) + ax^2+bx+c$

$g(x) = f(x)q(x) + r(x)$

Take $x=-1$, $(-1)^n = a-b+c$ (i)

$nx^{n-1} = 3(x+1)^2 q(x) + (x+1)^3 q'(x) + 2ax+b$ (2)

Take $x=-1$ (ii)

19. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Try it yourself

(a) Show that A is invertible and find its inverse

(b) Find A^n, $n \in \mathbb{N}$.

(c) Let $(u_n), (v_n)$ and (w_n) be real sequences defined by

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{cases} u_{n+1} = 2u_n - v_n - w_n \\ v_{n+1} = -u_n + 2v_n - w_n \\ w_{n+1} = -u_n - v_n + 2w_n \end{cases}$$

Find the general terms of $(u_n), (v_n)$ and (w_n) in terms of n.

$A^2 = bA + cI$

$A^2 - bA - cI = 0$ date

$x^2 - bx + c = 0$

$x^n = (x^2 - bx - c)q(x) + r(x)$

ⓐ show that A is invertible and find its inverse

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{J}$$

$$A = 3I - J \quad (*)$$

$$A^2 = (3I - J)(3I - J) = 9I^2 - 3IJ - 3JI + J^2$$

$$= 9I - 6J + 3J = 9I - 3J = 3(3I - J) = 3A$$

$$J^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3J$$

20. Given $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$.

(a) Find the smallest positive integer k such that $B^k = 0$.

(b) Find A^n in terms of $n \in \mathbb{N}$.

ⓐ Find the smallest positive integer K such that $B^k = O_3$

- $B^1 = B \neq O_3$
- $B^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq O_3$
- $B^3 = B^2 B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_3$

Therefore, $K = 3$.

ⓑ Find A^n in terms of $n \in \mathbb{N}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B + I$$

$A^n = (B+I)^n$. The corresponding polynomial is

$$a^n = (b+1)^n$$

$$a^n = \sum_{i=0}^{n} C(n,i)\, b^i\, 1^{n-i} = \sum_{i=0}^{n} C(n,i)\, b^i$$

$$a^n = C(n,0) + C(n,1)\,b + C(n,2)\,b^2 + C(n,3)\,b^3 + \dots + C(n,n)\,b^n$$

Substitute $a = A$ and $b = B$, we obtain

$$A^n = \frac{n!}{0!\,(n-0)!} I + \frac{n!}{1!\,(n-1)!} B + \frac{n!}{2!\,(n-2)!} B^2$$

ⓐ $B^3 = 0$

$$A^n = I + nB + \frac{n(n-1)}{2} B^2$$

$$A^n = I + nB + \frac{n(n-1)}{2} B^2$$

$$A^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} + \frac{n(n-1)}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -n & 1 & 0 \\ 2n + \frac{n(n-1)}{2} & -n & 0 \end{pmatrix}$$

$$(a+b)^n = \sum_{i=0}^{n} C(n,i)\, a^i\, b^{n-i}$$

same

$$(a+b)^n = \sum_{i=0}^{n} C(n,i)\, a^{n-i}\, b^i$$

$C(n,i) = C(n, n-i)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^2 = b^2 + 2ab + a^2$

$$(x+1)^n = \sum_{i=0}^{n} C(n,i)\, x^i \underbrace{1^{n-i}}_{1} = \sum_{i=0}^{n} C(n,i)\, x^i$$

22. Let $A \in \mathcal{M}_n(\mathbb{K})$ such that $A + A^{-1} = I$. Calculate $A^k + A^{-k}$, $\quad k \in \mathbb{N}$.

- $A^1 + A^{-1} = A + A^{-1} = I = u_1 \cdot I \quad (1) \rightarrow u_1 = 1$
- $A^2 + A^{-2} = ?$

$(1): \quad (A + A^{-1})^2 = I^2$

$(A + A^{-1})(A + A^{-1}) = I$

$A^2 + I + I + A^{-2} = I$

$\Rightarrow A^2 + A^{-2} = -I = u_2 \cdot I \quad (2) \rightarrow u_2 = -1$

• $A^3 + A^{-3} = ?$

(1) & (2): $(A^1 + A^{-1})(A^2 + A^{-2}) = I(-I)$

$\Rightarrow A^3 + A^{-1} + A^1 + A^{-3} = -I$ because $A^1 + A^{-1} = I$ by (1)

$A^3 + A^{-3} = -2I = u_3 I \quad (3)$

We claim that $A^k + A^{-k} = u_k I \quad (*)$

where $u_1 = 1,\ u_2 = -1,\ u_3 = -2,\ \ldots \quad u_{k+1} = ?$

(1) & $(*)$: $(A^1 + A^{-1})(A^k + A^{-k}) = I(u_k I)$

$$\underline{A^{k+1}} + A^{-(k-1)} + A^{k-1} + \underline{A^{-(k+1)}} = u_k I$$

$(*)$: $(A^{k+1} + A^{-(k+1)}) + (A^{k-1} + A^{-(k-1)}) = u_k I$

$$u_{k+1} I + u_{k-1} I = u_k I$$

$$(u_{k+1} + u_{k-1}) I = u_k I$$

$$u_{k+1} + u_{k-1} = u_k$$

$$u_{k+1} - u_k + u_{k-1} = 0, \quad u_1 = 1, \ u_2 = -1 \quad (\#)$$

Characteristic equation of $(\#)$,

$r - 1 + r^{-1} = 0 \iff r^2 - r + 1 = 0$

$\Delta = b^2 - 4ac = (-1)^2 - 4(1)(1) = 1 - 4 = -3 = (i\sqrt{3})^2$

$r_1 = \dfrac{-b - \sqrt{\Delta}}{2a} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i, \quad r_2 = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i$

$r_1 = e^{-\frac{\pi}{3} i}, \quad r_2 = e^{\frac{\pi}{3} i}$

Then $u_k = c_1 r_1^k + c_2 r_2^k$

$$= c_1 e^{-\frac{k\pi}{3}i} + c_2 e^{\frac{k\pi}{3}i}$$

$$= \left[c_1 \cos\left(\frac{k\pi}{3}\right) - i\, c_1 \sin\left(\frac{k\pi}{3}\right)\right]$$

$$+ \left[c_2 \cos\left(\frac{k\pi}{3}\right) + i\, c_2 \sin\left(\frac{k\pi}{3}\right)\right]$$

$$= (c_1+c_2)\cos\left(\frac{k\pi}{3}\right) + i(c_2-c_1)\sin\left(\frac{k\pi}{3}\right)$$

$$u_k = d_1 \cos\left(\frac{k\pi}{3}\right) + d_2 \sin\left(\frac{k\pi}{3}\right) \qquad (**)$$

where d_1 and d_2 are constants to be determined.

$$\begin{cases} u_1 = 1 \\ u_2 = -1 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2}d_1 + \frac{\sqrt{3}}{2}d_2 = 1 & (i) \\ -\frac{1}{2}d_1 + \frac{\sqrt{3}}{2}d_2 = -1 & (ii) \end{cases}$$

$(i)+(ii)$: $\sqrt{3}\,d_2 = 0 \Rightarrow d_2 = 0$

(i): $\frac{1}{2}d_1 = 1 \Rightarrow d_1 = 2$

$\Rightarrow (**)$: $u_k = 2\cos\left(\frac{k\pi}{3}\right)$ $\qquad A^k + A^{-k} = u_k I \qquad (*)$

By $(*)$: $\boxed{A^k + A^{-k} = 2\cos\left(\frac{k\pi}{3}\right) I.}$ ✓

$A^1 + A^{-1} = I$, $A^2 + A^{-2} = -I$, $A^3 + A^{-3} = -2I$ ✓