Wednesday, November 10, 2021 9:40 AM

$$A.X = b \iff \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\iff \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \text{---} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \quad (S)$$

Gauss Elimination Method

The **Gauss Elimination** procedure for solving the linear system $AX = b$ is as follows.

- Form the augmented matrix $(A|b)$.
- Transform $(A|b)$ to row echelon form by using elementary row operations ERO REF
- Use back substitution to obtain the solution

$$a) \begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 - 2x_3 = -6 \end{cases}$$

(1) $$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right) \begin{array}{l} L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - L_1 \end{array}$$

(2) $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array}\right) L_3 \to L_3 + 2L_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array}\right) L_3 \to -\frac{1}{5} L_3$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right) \qquad \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_2 - x_3 = -1 \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_2 - x_3 = -1 \\ x_3 = 2 \end{cases}$$

③ $x_3 = 2$; $x_2 = -1 + x_3 = -1 + 2 = 1$; $x_1 = 2 - x_2 - x_3 = -1$

So, $(x_1, x_2, x_3) = \underline{(-1; 1; 2)}$.

☆ There are three possibility for solution of (S)

① unique solution ($2x = 3 \Rightarrow x = 3/2$)

② no solution ($0.x = 3$) ✓

③ infinitely many solution. ($0.x = 0$)

<u>Ex</u>: Solve the system of linear eq.

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

$$+ \ (A|b) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 0 \end{array}\right) \begin{array}{l} L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - L_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & 3 & -1 \end{array}\right) \begin{array}{l} L_2 \to -\frac{1}{3} L_2 \\ L_3 \to L_3 - L_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -4 \end{array}\right) \Rightarrow \text{system has no solution}$$

(b)

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \\ x_1 - x_2 + 2x_3 = 4 \end{cases} , \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 4 \end{array}\right) \begin{array}{l} L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - L_1 \end{array}$$

$$\begin{cases} 2x_1+x_2+x_3=5 \\ x_1-x_2+2x_3=4 \end{cases} \quad ; \quad \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 4 \end{array}\right) \quad L_3 \to L_3 - L_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & 3 & 3 \end{array}\right) \begin{array}{l} L_2 \to -\frac{1}{3}L_2 \\ L_3 \to L_3 - L_2 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1+2x_2-x_3=1 \\ x_2-x_3=-1 \end{cases}$$

Let $x_3 = t \in \mathbb{R}$; $x_2 = -1 + x_3 = -1 + t$; $x_1 = 1 - 2x_2 + x_3 = 1 + 2 - 2t + t = 3 - t$

So, $(x_1, x_2, x_3) = \{(3-t, -1+t, t) \mid t \in \mathbb{R}\}$. $\quad t=0: (3,-1,0)$

$t=1: (2,0,1)$

★ If # variables are more than # eq in REF, then the system has infinitely many solution.

★ To find all solution, we introduce parameters to the variable that more that # eq.

Ex: solve the system
$$\begin{cases} x_1+2x_2-x_3+x_4=1 \\ 2x_1-x_2+x_3 \quad\quad =0 \\ x_1-3x_2+2x_3-x_4=-1 \end{cases}$$

$+ \ (A|b) = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & -1 & -1 \end{array}\right) \begin{array}{l} L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - L_1 \end{array}$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & -5 & 3 & -2 & -2 \\ 0 & -5 & 3 & -2 & -2 \end{array}\right) \begin{array}{l} L_2 \to -\frac{1}{5}L_2 \\ L_3 \to L_3 - L_2 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -3/5 & 2/5 & 2/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$+$ Let $\underline{x_3 = t}$; $\underline{x_4 = s}$, $t, s \in \mathbb{R}$.

$x_2 = \frac{2}{5} + \frac{3}{5}t - \frac{2}{5}s$

$$x_2 = \frac{2}{5} + \frac{3}{5}t - \frac{2}{5}s$$

$$x_1 = 1 - 2x_2 + x_3 - x_4 = 1 - \frac{4}{5} - \frac{6}{5}t + \frac{4}{5}s + t - s = \frac{1}{5} - \frac{1}{5}t - \frac{1}{5}s$$

So, $(x_1, x_2, x_3, x_4) = \left\{ \left(\frac{1}{5} - \frac{1}{5}t - \frac{1}{5}s \,;\, \frac{2}{5} + \frac{3}{5}t - \frac{2}{5}s \,;\, t, s \right) / \, t, s \in \mathbb{R} \right\}$.

<u>Def</u>: A^{-1} is the inverse matrix of A if

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

<u>Ex</u>: Find A^{-1}, where $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

<u>Proof</u>: $A^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$. because

$$AA^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A^{-1}A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

<u>TH</u>: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; $ad - bc \neq 0$.

<u>Ex</u>: $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$. Find A^{-1} ?

$$(A|I) = \left(\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) \xrightarrow{L_2 \leftrightarrow L_1} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array} \right) \xrightarrow{L_2 \to L_2 - 3L_1}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right) \begin{array}{l} L_2 \to -L_2 \\ L_1 \to L_1 + 2L_2 \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right)$$

$$\underbrace{\quad\quad}_{I} \quad \underbrace{\quad\quad}_{A^{-1}}$$

So, $A^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

Ex: Find rank (A) & null(A) where $A = \begin{pmatrix} 1 & -1 & 4 & 3 \\ 1 & 2 & -2 & 0 \\ 2 & 5 & -8 & -3 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 4 & 3 \\ 1 & 2 & -2 & 0 \\ 2 & 5 & -8 & -3 \end{pmatrix} \begin{array}{l} L_2 \to L_2 - L_1 \\ L_3 \to L_3 - 2L_1 \end{array} \begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 3 & -6 & -3 \\ 0 & 7 & -16 & -9 \end{pmatrix} L_2 \to \frac{1}{3} L_2$$

$$\begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 7 & -16 & -9 \end{pmatrix} L_3 \to L_3 - 7L_2 \begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix} L_3 \to -\frac{1}{2} L_3$$

$$\begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \text{rank}(A) = 3 \; ; \; \text{null}(A) = 1$$

TH: $A = (a_{ij})_{m \times n}$. $\quad \text{rank}(A) + \text{null}(A) = n$.