CHAPTER II: DETERMINANT

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Contents

Permutations

Definition 1 (Permutations)

Let $S = \{1, 2, \dots, n\}$. A one-to-one mapping $\sigma : S \to S$ is called a **permutation of** S. That is,

$$\sigma(i) = j_i, \quad i = 1, 2, \dots, n \quad \text{and} \quad j_i \neq j_k \text{ if } i \neq k$$

We also use the following notation for simple form of permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$$

or just

$$\sigma = j_1 j_2 \dots j_n.$$

Permutations

Definition 2 (Cyclic permutation (cycle))

Let σ be a permutation on S. $\sigma = (j_1 j_2 \ldots j_k)$, $k \leq n$ is called a **cyclic permutation** (or cycle) of length k. By definition,

$$\begin{cases} \sigma(j_i) = j_{i+1}, & \text{for } i = 1, 2, \ldots, k-1 \\ \sigma(j_k) = j_1, & \\ \sigma(j_i) = j_i, & \text{if } i \neq 1, 2, \ldots, k \end{cases}$$

That is,

$$j_1 \mapsto j_2 \mapsto j_3 \mapsto \cdots \mapsto j_{k-1} \mapsto j_k \mapsto j_1$$

and

$$j_i \mapsto j_i, \quad \text{if } i \neq 1, 2, \ldots, k$$

Permutations

Remark.

The set of all permutations of S is denoted by S_n. That is,

$$S_n = \{\sigma \mid \sigma \text{ is a permuation of } S\}.$$

We have the following:

- $\#S_n = n!$ (the number of elements of S_n).
- If $\sigma \in S_n$, then the inverse $\sigma^{-1} \in S_n$.
- If $\sigma, \tau \in S_n$, then the composition mapping $\sigma \circ \tau \in S_n$.
- The identity permutation of S is denoted by ϵ where $\epsilon = 12 \ldots n$ and for $\sigma \in S_n : \sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \epsilon$.

Permutations

Definition 3

- Let $\sigma = j_1 j_2 \dots j_n$ of S_n. We say that $(\sigma(i), \sigma(k)) = (j_i, j_k)$ is an **inversion** of σ if

$$i < k \quad \text{and} \quad j_i > j_k \quad (\sigma(i) > \sigma(k))$$

- A permutation is called **even** or **odd** according to whether the total number of inversions in it is even or odd.
- The **signature** of a permutation σ, denoted $\operatorname{sgn}(\sigma)$, is defined as

$$\operatorname{sgn}(\sigma) = (-1)^{\#\text{inversions}} = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

Permutations

Theorem 1

Let $\sigma_1, \sigma_2 \in S_n$. Then,

- $\mathrm{sgn}(\sigma_1 \circ \sigma_2) = \mathrm{sgn}(\sigma_1).\mathrm{sgn}(\sigma_2)$
- $\mathrm{sgn}(\sigma_1^{-1}) = \mathrm{sgn}(\sigma_1)$

Definition 4

A **Transposition** is a permutation that is obtained from $1, 2, \ldots, n$ by interchange just two integers.

Theorem 2

The Transpositions are odd permutations.

Determinants

Definition 5

Let $A = (a_{ij})_n$. The **determinant of** A (written $\det(A)$ or $|A|$) is defined by:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1j_1} a_{2j_2} \dots a_{nj_n}.$$

Properties of Determinants

Theorem 3

If B is a matrix resulting from matrix A:

- by interchanging two rows of A; that is $B = E_{ij}A$, then $|B| = -|A|$.
- by multiplying a row of A by a scalar λ; that is $B = E_i(\lambda)A$, then $|B| = \lambda|A|$.
- by adding to row i of A a constant λ times row j of A with $i \neq j$; that is $B = E_{ij}(\lambda)A$, then $|B| = |A|$.

Theorem 4

Let $A = (a_{ij})_n$.

- If a row(column) of A consists entirely of zeros, then $|A| = 0$.
- If two rows (columns) of A are equal, then $|A| = 0$.
- If A is triangular matrix, then $|A| = a_{11}a_{22}\ldots a_{nn}$.

Properties of Determinants

Theorem 5

Let $A, B \in \mathcal{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We have

1. $|A^t| = |A|$
2. $|AB| = |A|.|B|$
3. $|A^{-1}| = \dfrac{1}{|A|}$
4. $|\lambda A| = \lambda^n |A|$

Example 6

Suppose that $A = (a_{ij})_3$ with $|A| = 7$. Compute the following determinants

$$a).|2A| \qquad b).|3A^{-1}| \qquad c).|(3A)^{-1}| \qquad d).\begin{vmatrix} a_{11} & a_{31} & a_{21} \\ a_{12} & a_{32} & a_{22} \\ a_{13} & a_{33} & a_{23} \end{vmatrix}$$

Determinant of Block Matrix

Theorem 6

Let $A, B, C, D \in \mathcal{M}_n(\mathbb{R})$. Suppose that D is invertible and $CD = DC$. We have

$$\det \begin{pmatrix} A & O \\ C & D \end{pmatrix} = \det \begin{pmatrix} A & B \\ O & D \end{pmatrix} = \det(A) \det(D)$$

where $O = (0)_n$ is the zero matrix of order n.

Theorem 7

Let $A, B, C, D \in \mathcal{M}_n(\mathbb{R})$. Suppose that D is invertible and $CD = DC$. We have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

Determinant of Block Matrix

Theorem 8

Let $A, B, C, D \in \mathcal{M}_n(\mathbb{R})$ with D is invertible. We have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \times \det(A - BD^{-1}C)$$

Theorem 9

Let $A, B, C, D \in \mathcal{M}_n(\mathbb{R})$ with A is invertible. We have

$$\det \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) = \det(A) \times \det \left(D - CA^{-1}B \right)$$

Minor and Cofactor

Definition 7

Let $A = (a_{ij})_n$ and let M_{ij} be the $(n-1) \times (n-1)$ sub matrix of A, obtained by deleting ith row and jth column of A.

- The **minor of** a_{ij} **of** A is defined by M_{ij}.
- The **cofactor of** a_{ij} is defined by $A_{ij} = (-1)^{i+j}|M_{ij}|$.

Example 8

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 2 \\ 3 & -1 & 0 \end{pmatrix}$. Then

$$M_{13} = \begin{pmatrix} -1 & 4 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad A_{13} = (-1)^{1+3}|M_{13}| = 11$$

Theorem 10 (Laplace expansion)

Let $A = (a_{ij})_n$. We have

$$|A| = \sum_{k=1}^{n} a_{ik}A_{ik} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

$$|A| = \sum_{k=1}^{n} a_{kj}A_{kj} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

For each $i = 1, 2, \ldots n$ or for $j = 1, 2, \ldots n$.

Example 9

Find the determinant of the following matrix using Laplace expansion

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 2 \\ 3 & -1 & 0 \end{pmatrix} \Longrightarrow |A| = 1(2) - 2(-6) + 3(-11) = -19$$

Minor and Cofactor

Definition 10 (Adjoint)

Let $A = (a_{ij})_n$. The **adjoint** of A is denoted and defined by

$$\mathrm{adj}(A) = (A_{ij})^t = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^t$$

Theorem 11

Let A be a square matrix. Then,

$$A.\mathrm{adj(A)} = (\mathrm{adj}(A))A = |A|I.$$

If $|A| \neq 0$, then $A^{-1} = \dfrac{1}{|A|}(\mathrm{adj}(A))$.

Minor and Cofactor

Theorem 12

Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be invertible matrices and $\alpha \neq 0$. Then

1. $\mathrm{adj}(A^{-1}) = (\mathrm{adj}(A))^{-1}$
2. $\mathrm{adj}(A^t) = (\mathrm{adj}(A))^t$
3. $\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)$.
4. $|\mathrm{adj}(A)| = |A|^{n-1}$
5. $\mathrm{adj}(\alpha A) = \alpha^{n-1}\mathrm{adj}(A)$.

Example 11

Find inverse matrix of the below matrix

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix}.$$

Find
(a) $\mathrm{adj}(A)$, (b) $|\mathrm{adj}(A)|$, (c) A^{-1}, (d) $|\mathrm{adj}(3A)|$, (e) $|\mathrm{adj}(2A^{-1})|$

Cramer's rule

Considers a system of n linear equations in the n unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

This system is equivalent to the matrix form

$$AX = b$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer's rule

Theorem 13

The above system has a unique solution if and only if $\Delta \neq 0$. In this case, the unique solution is given by

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \ldots, x_n = \frac{\Delta_n}{\Delta}.$$

where

- $\Delta = |A|$.
- $\Delta_i, i = 1, 2, \ldots, n$, be the determinant of the matrix obtained by replacing the ith column of A by b.

Cramer's rule

Theorem 14

Let A be a square matrix. The following statement are equivalent.

- A is invertible.
- $AX = 0$ has only the trivial solution.
- The determinant of A is not zero.

Theorem 15

The homogeneous system $AX = 0$ has a nontrivial solution if and only if $\Delta = |A| = 0$.