11. Find the rank of matrix A where

(a) $A = \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 2 & 1 \\ a & b & c \\ -1 & 3 & 0 \end{pmatrix}$ (c) $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & \lambda & \lambda \end{pmatrix}$

(a) Transform A into REF (or RREF).

$$A \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{pmatrix} \xrightarrow{L_3 \leftarrow L_3 - L_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{pmatrix} \xrightarrow[L_2 \leftarrow (-1)L_2]{L_3 \leftarrow L_3 + L_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Therefore, } \operatorname{Rank}(A) = 2.$$

14. Find the inverse of each matrix (if exists) below:

(a) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & \dots & 1 \\ & \ddots & \ddots & \vdots \\ & (0) & \ddots & 1 \\ & & & 1 \end{pmatrix}_n$ (e) $\begin{pmatrix} 1 & \alpha & & (0) \\ & 1 & \ddots & \\ & & \ddots & \alpha \\ (0) & & & 1 \end{pmatrix}_n$

(b) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & \dots & \dots & 1 \\ \vdots & 2 & \dots & 2 \\ \vdots & \vdots & & \vdots \\ 1 & 2 & \dots & n \end{pmatrix}_n$ (f) $\begin{pmatrix} -1 & 2 & \dots & 2 \\ 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \dots & 2 & -1 \end{pmatrix}_n$

$AB = BA = I$, $A^{-1} = B$ or $B^{-1} = A$

$(A \mid I) \longrightarrow (A^{-1}A \mid A^{-1}I) \longrightarrow (I \mid A^{-1})$

(a) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 \leftrightarrow L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[L_3 \leftarrow L_3 + L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow[L_2 \leftarrow L_2(-1)]{L_3 \leftarrow L_3 + 2L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right) \xrightarrow{L_3 \leftarrow \frac{1}{3}L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow L_2 + L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right) = (I \mid A^{-1})$$

Therefore, $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 1 & \frac{1}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix}$ (Find its inverse) Method: $(B \mid I) \to (I \mid B^{-1})$ ✓

$$(B \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - 3L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & -1 & -6 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[L_2 \leftarrow L_2(-1)]{L_3 \leftarrow L_3 - L_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right).$$ Therefore, B has no inverse.

cannot $\to I$

15. Solve the system of linear equation unknow

(a) $\begin{cases} x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 4 \\ x_1 + 4x_3 &= 3 \end{cases}$ (b) $\begin{cases} x_1 + x_2 + x_3 &= 1 \\ -x_1 + mx_2 + 2x_3 &= 2 \\ 2x_1 + 2x_2 - x_3 &= -1 \end{cases}$ (c) $\begin{cases} x_1 + x_3 - x_4 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ x_1 - x_2 - 2x_4 &= 0 \end{cases}$

(d) $\begin{cases} x_1 + x_2 + x_3 + \cdots + x_n &= 1 \\ x_1 + 2x_2 + 2x_3 + \cdots + 2x_n &= 1 \\ x_1 + 2x_2 + 3x_3 + \cdots + 3x_n &= 1 \\ \vdots & \vdots \\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n &= 1 \end{cases}$ (e) $\begin{cases} x_1 - x_2 - x_3 - \cdots - x_n &= 1 \\ -x_1 + x_2 - x_3 - \cdots - x_n &= 2 \\ -x_1 - x_2 + x_3 - \cdots - x_n &= 3 \\ \vdots & \vdots \\ -x_1 - x_2 - x_3 - \cdots + x_n &= n \end{cases}$

(a) $\begin{cases} x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 4 \\ x_1 + 4x_3 &= 3 \end{cases}$ Solve for x_1, x_2 and x_3

$$\Leftrightarrow \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$$

$$\Leftrightarrow AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$$

$$2x = 4$$
$$2^{-1} \cdot 2x = 2^{-1} \cdot 4$$
$$x = 2$$

$$(A \mid B) \rightarrow (A^{-1}A \mid A^{-1}B)$$
$$\rightarrow (I \mid X)$$

We solve this problem using Gauss-Jordan elimination,

$$(A \mid B) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & 1 & 2 & 4 \\ 1 & 0 & 4 & 3 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - L_1]{L_2 \leftarrow L_2 + L_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$\xrightarrow[L_2 \leftrightarrow L_3]{L_2 \leftarrow \frac{1}{4}L_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[L_2 \leftarrow L_2 - 2L_3]{L_1 \leftarrow L_1 - 2L_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{L_1 \leftarrow L_1 + L_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) = (I \mid A^{-1}B) = (I \mid X).$$

Therefore, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ or $(x_1, x_2, x_3) = (-1, 1, 1)$

or $x_1 = -1, \; x_2 = 1, \; x_3 = 1.$