TD1: (Matrices)

1. Write the following matrices into row echelon form.

(a) $A = \begin{pmatrix} 1 & 1 & 3 & 2 \\ -2 & 2 & 1 & 0 \\ 0 & 4 & 7 & 4 \end{pmatrix}$ (c) $C = \begin{pmatrix} 1 & 2 & 1 & 4 & 5 & 7 \\ 2 & 1 & 0 & 1 & 2 & 1 \\ 3 & 3 & 1 & 5 & 7 & 8 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix}$ (d) $D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & \lambda & \lambda \end{pmatrix}$

2. Let X be a 2×2 matrix. Find the solutions of the equation $X^2 - 3X = I$.

3. Let $X \in \mathcal{M}_3(\mathbb{R})$. Solve the equation $X - 2X^t = \operatorname{tr}(X) I_3$.

4. Find the matrix $X \in \mathcal{M}_2(\mathbb{R})$ satisfies the equation

$$X^3 = A, \quad \text{where} \quad A = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}.$$

5. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $AB = 2A + 3B$. Show that

(a) $(A - 3I_n)(B - 2I_n) = 6I_n$

(b) $AB = BA$.

6. (a) Are there matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $AB - BA = I$.

(b) Suppose that $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $(AB - BA)^2 = AB - BA$. Show that A and B are commutable.

7. Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ and $A^5 = 0$. Show that $A + I$ is invertible and then find its inverse.

8. Given $A \in \mathcal{M}_n(\mathbb{K})$ such that $A^5 + A = I$. Show that $A^2 + A + I$ is invertible then find its inverse.

9. Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $I + A$ is invertible. Suppose that

$$B = (I - A)(I + A)^{-1}$$

(a) Show that $B = (I + A)^{-1}(I - A)$

(b) Show that $I + B$ is invertible and express A in terms of B.

10. Let $A = \begin{pmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \dots & a_n \end{pmatrix}_n$ where $\sum_{i=1}^{n} a_i = \alpha \neq 0$. The matrix B is defined by

$$B = (b_{ij})_n \quad \text{where } b_{ij} = 2a_i \quad \text{if } i \neq j \text{ and } b_{ii} = a_i - \sum_{j \neq i} a_j.$$

(a) Compute A^2

(b) Show that B is invertible and then find its inverse.

11. Find the rank of matrix A where

(a) $A = \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 2 & 1 \\ a & b & c \\ -1 & 3 & 0 \end{pmatrix}$ (c) $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & \lambda & \lambda \end{pmatrix}$

12. Let $A = (a_{ij})_n \in \mathcal{M}_n(\mathbb{R})$ where $a_{ij} = i - j + 1$ for $i, j = 1, 2, \dots, n$. Find $\operatorname{rank}(A)$.

13. Let $A = (a_{ij})_n \in \mathcal{M}_n(\mathbb{R})$ where $a_{ij} = \cos(i + j)$ for $i, j = 1, 2, \dots, n$. Find $\operatorname{rank}(A)$.

14. Find the inverse of each matrix (if exists) below:

(a) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & \dots & 1 \\ & \ddots & \ddots & \vdots \\ & (0) & \ddots & 1 \\ & & & 1 \end{pmatrix}_n$ (e) $\begin{pmatrix} 1 & \alpha & & (0) \\ & 1 & \ddots & \\ & & \ddots & \alpha \\ (0) & & & 1 \end{pmatrix}_n$

(b) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & \dots & \dots & 1 \\ \vdots & 2 & \dots & 2 \\ \vdots & \vdots & & \vdots \\ 1 & 2 & \dots & n \end{pmatrix}_n$ (f) $\begin{pmatrix} -1 & 2 & \dots & 2 \\ 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2 & \dots & 2 & -1 \end{pmatrix}_n$

15. Solve the system of linear equation unknow

(a) $\begin{cases} x_1 - x_2 + 2x_3 & = 0 \\ -x_1 + x_2 + 2x_3 & = 4 \\ x_1 + 4x_3 & = 3 \end{cases}$ (b) $\begin{cases} x_1 + x_2 + x_3 & = 1 \\ -x_1 + mx_2 + 2x_3 & = 2 \\ 2x_1 + 2x_2 - x_3 & = -1 \end{cases}$ (c) $\begin{cases} x_1 + x_3 - x_4 & = 1 \\ x_1 + x_2 + 2x_3 & = 2 \\ x_1 - x_2 - 2x_4 & = 0 \end{cases}$

(d) $\begin{cases} x_1 + x_2 + x_3 + \cdots + x_n & = 1 \\ x_1 + 2x_2 + 2x_3 + \cdots + 2x_n & = 1 \\ x_1 + 2x_2 + 3x_3 + \cdots + 3x_n & = 1 \\ \quad\vdots & \quad\vdots \\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n & = 1 \end{cases}$ (e) $\begin{cases} x_1 - x_2 - x_3 - \cdots - x_n & = 1 \\ -x_1 + x_2 - x_3 - \cdots - x_n & = 2 \\ -x_1 - x_2 + x_3 - \cdots - x_n & = 3 \\ \quad\vdots & \quad\vdots \\ -x_1 - x_2 - x_3 - \cdots + x_n & = n \end{cases}$

16. Let $A = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$ and $p(x) = x^2 - 6x - 7$.

 (a) Find $P(A)$. Then deduce that A is invertible and find its inverse .
 (b) Find A^n, $n \in \mathbb{N}$.
 (c) Let (u_n) and (v_n) be real sequences defined by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{cases} u_{n+1} = 5u_n + 4v_n \\ v_{n+1} = 3u_n + v_n \end{cases}$$

 Find the general terms of (u_n) and (v_n) in terms of n.

17. Let $f(x) = x^3 - 2x^2 + x$ and $g(x) = x^{2020} - 10x^{1000} + 3x - 1$. Let

$$A = \begin{pmatrix} 1 & -1 & -5 \\ 1 & 3 & 7 \\ 1 & 0 & -2 \end{pmatrix}$$

Compute $f(A)$ and $g(A)$.

18. Let
$$A=\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$$
   (a) Compute $(A+I)^3$
   (b) Deduce that A is invertible and find its inverse
   (c) Find A^n, $n \in \mathbb{N}$.
   (d) Let $(u_n), (v_n)$ and (w_n) be real sequences defined by
$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{cases} u_{n+1} = 2u_n - v_n + 2w_n \\ v_{n+1} = 5u_n - 3v_n + 3w_n \\ w_{n+1} = -u_n - 2w_n \end{cases}$$
   Find the general terms of $(u_n), (v_n)$ and (w_n) in terms of n.

19. Let
$$A=\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$
   (a) Show that A is invertible and find its inverse
   (b) Find A^n, $n \in \mathbb{N}$.
   (c) Let $(u_n), (v_n)$ and (w_n) be real sequences defined by
$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{cases} u_{n+1} = 2u_n - v_n - w_n \\ v_{n+1} = -u_n + 2v_n - w_n \\ w_{n+1} = -u_n - v_n + 2w_n \end{cases}$$
   Find the general terms of $(u_n), (v_n)$ and (w_n) in terms of n.

20. Given $A=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$.
   (a) Find the smallest positive integer k such that $B^k = 0$.
   (b) Find A^n in terms of $n \in \mathbb{N}$.

21. Let J and A be two square matrices satisfy
$$J^2 = I, \quad \text{and} \quad A = \alpha I + \beta J, \quad \alpha, \beta \in \mathbb{R}$$
   Show that $A^n = \alpha_n I + \beta_n J$, then determine α_n and β_n.

22. Let $A \in \mathcal{M}_n(\mathbb{K})$ such that $A + A^{-1} = I$. Calculate $A^k + A^{-k}, \quad k \in \mathbb{N}$.

23. Let $A_n = \begin{pmatrix} 1 & -\frac{a}{n} \\ \frac{a}{n} & 1 \end{pmatrix}$, $a \in \mathbb{R}$. Calculate $\lim\limits_{n\to\infty} A_n^n$.