I2–TD2
(Determinants)

1. Find the signatures of the following permutations.

   (a) 45312 (b) 38562147 (c) 397264581.

2. In S_8, write the following permutations into cyclic form, then determine their signature.

   (a) 85372164 (b) 87651234 (c) 12435687.

3. In S_7, write the following permutations into normal form, then determine their signature.

   (a) (6437) (b) (465)(735) (c) (241)(5416).

4. For $n \in \mathbb{N}^*$, compute the signature of the following permutations.

   (a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & \dots & 2 & 1 \end{pmatrix}$

   (b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n & n+1 & n+2 & \dots & 2n-1 & 2n \\ 1 & 3 & 5 & \dots & 2n-1 & 2 & 4 & \dots & 2n-2 & 2n \end{pmatrix}$

5. Prove that the transposition is an odd permutation.

6. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Assuming that $|A| = 3$. Find

   (a) $|3A|$ (b) $|A^{-1}|$ (c) $|2A^{-1}|$ (d) $|(2A)^{-1}|$ (e) $\begin{vmatrix} -a & 2g & 3d \\ -b & 2h & 3e \\ -c & 2i & 3f \end{vmatrix}$.

7. Let A and B be invertible matrices. Show that

   (a) $\mathrm{adj}(A^{-1}) = (\mathrm{adj}(A))^{-1}$ (b) $\mathrm{adj}(A^t) = (\mathrm{adj}(A))^t$ (c) $\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)$.

8. Let A be an $n \times n$ matrix and $\alpha \neq 0$. Show that

   (a) $|\mathrm{adj}(A)| = |A|^{n-1}$ (b) $\mathrm{adj}(\alpha A) = \alpha^{n-1}\mathrm{adj}(A)$.

9. If $A^{-1} = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & -2 \end{pmatrix}$. Compute $|\mathrm{adj}(A)|$ and $|2A^{-1} + 3\mathrm{adj}(2A)|$.

10. Let $A = (a_{ij})_{2021} \in \mathcal{M}_{2021}(\mathbb{R})$ where $a_{ij} = i - j$ for $i, j = 1, 2, \dots, 2021$. Compute $\det(A)$.

11. Prove that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33}+b_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix}$$

12. Compute the determinant
$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$
Then deduce the value of the determinant
$$\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$$

13. Compute the following determinants.

(a) $\begin{vmatrix} 7 & 4 & -5 \\ 10 & 3 & 21 \\ 23 & -2 & 11 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 2 & 3 & 5 \\ 6 & 3 & -3 & 2 \\ 8 & 10 & 0 & 11 \\ 11 & 23 & 2 & -4 \end{vmatrix}$

(c) $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix}$

(d) $\begin{vmatrix} a & b & ab \\ a & c & ac \\ b & c & bc \end{vmatrix}$

(e) $\begin{vmatrix} a & c & c & b \\ c & a & b & c \\ c & b & a & c \\ b & c & c & a \end{vmatrix}$

(f) $\begin{vmatrix} 2a & a-b-c & 2a \\ b-c-a & 2b & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

14. Compute determinants of $n \times n$ matrices below:

(a) $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{vmatrix}$

(b) $\begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \dots & n \end{vmatrix}$

(c) $\begin{vmatrix} 5 & 2 & & & (0) \\ 3 & 5 & 2 & & \\ & 3 & \ddots & \ddots & \\ & & \ddots & \ddots & 2 \\ (0) & & & 3 & 5 \end{vmatrix}_n$

(d) $\begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ a & b & \ddots & \ddots & \vdots \\ a^2 & ab & \ddots & \ddots & 0 \\ \vdots & \vdots & & b & -1 \\ a^n & a^{n-1}b & \dots & ab & b \end{vmatrix}$

(e) $\begin{vmatrix} a+b & b & \dots & b \\ a & a+b & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ a & \dots & a & a+b \end{vmatrix}_{n\times n}$

(f) $\begin{vmatrix} x+a_1 & a_1 & a_1 & \dots & a_1 \\ a_2 & x+a_2 & a_2 & \dots & a_2 \\ a_3 & a_3 & x+a_3 & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \dots & x+a_n \end{vmatrix}$

15. $\Delta_n(x) = \begin{vmatrix} \alpha_1+x & a+x & \dots & a+x \\ b+x & \alpha_2+x & \ddots & \vdots \\ \vdots & \ddots & \ddots & a+x \\ b+x & \dots & b+x & \alpha_n+x \end{vmatrix}_n, (a \neq b).$

(a) Show that $\Delta_n(x)$ is of the form $\Delta_n(x) = \alpha x + \beta$.
(b) Compute $\Delta_n(x)$, then deuce the value of $\Delta_n(0)$.

16. Let $n \in \mathbb{N} - \{0, 1\}$ and

$$A = \begin{pmatrix} 2 & & & \\ & 2 & & (\mathbf{1}) \\ (\mathbf{1}) & & \ddots & \\ & & & 2 \end{pmatrix}_n$$

(a) Show that A is invertible and express A^{-1} in terms of n, I and A.
(b) Calculate $|A|$
(c) Determine $\operatorname{adj}(A)$ and $|\operatorname{adj}(A)|$.
(d) Calculate A^m, $m \in \mathbb{N}$.

17. Let $A, B, C, D \in \mathcal{M}_n(\mathbb{K})$. Suppose that D is invertible and $CD = DC$ commutent. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

18. Let $A, B, C, D \in \mathcal{M}_n(\mathbb{K})$ with D is invertible. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \times \det(A - BD^{-1}C)$$

19. Compute the following determinants.

(a) $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 1 & 5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{vmatrix}$

20. Solve the following system of linear equations by using Cramer's rule.

(a) $\begin{cases} 2x_1 - x_2 + 3x_3 = 9 \\ -x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 - 2x_3 = -1 \end{cases}$ (b) $\begin{cases} mx_1 + x_2 + x_3 = 1 \\ x_1 + mx_2 + x_3 = m \\ x_1 + x_2 + mx_3 = m^2 \end{cases}$

21. Let $a, b, c \in \mathbb{C}$ and $P(X) = X^3 - (x + yX + zX^2)$. Solve the following system by using polynomial P.

(a) $\begin{cases} x + ay + a^2z & = a^3 \\ x + by + b^2z & = b^3 \\ x + cy + c^2z & = c^3 \end{cases}$ (b) $\begin{cases} x + ay + a^2z & = a^4 \\ x + by + b^2z & = b^4 \\ x + cy + c^2z & = c^4 \end{cases}$

22. Solve the system $AX = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & n \\ 1 & 2^2 & 3^2 & \dots & n^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \dots & n^{n-1} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$