38. Determine whether the following vector spaces equipped with real mappings are inner product spaces. If any, define the norm and the distance associated with each of these inner products.

(a) $\mathbb{R}^4$ equipped with $\langle x\,;y\rangle = 4x_1y_1 + 2x_2y_2 + 3x_3y_3 + 7x_4y_4$.

(b) $\mathbb{R}^3$ equipped with $\langle x\,;y\rangle = x_1y_1 + x_2y_2 - 2x_3y_3$.

(c) $C^0\,[0;1]$ equipped with $\langle f_1\,;f_2\rangle = \int_0^1 t f_1(t)\, f_2(t)\, dt$.

(d) P_3 equipped with

$$\left\langle a_0 + a_1t + a_2t^2 + a_3t^3\,; b_0 + b_1t + b_2t^2 + b_3t^3\right\rangle = a_0b_0 + a_1b_1 + 2a_2b_2 + a_3b_3.$$

ⓐ $\mathbb{R}^4$, $\langle x; y\rangle = 4x_1y_1 + 2x_2y_2 + 3x_3y_3 + 7x_4y_4$

(i) $\langle x, x\rangle = 4x_1^2 + 2x_2^2 + 3x_3^2 + 7x_4^2 \geq 0$

$\langle x, x\rangle = 0 \iff x_1 = x_2 = x_3 = x_4 = 0 \Rightarrow x = (x_1; x_2, x_3, x_4) = \vec{0}$

(ii) $\langle x, y\rangle = 4x_1y_1 + 2x_2y_2 + 3x_3y_3 + 7x_4y_4$

$= 4y_1x_1 + 2y_2x_2 + 3y_3x_3 + 7y_4x_4$

$= \langle y, x\rangle = \overline{\langle y, x\rangle}$ because $\bar{a} = a, \forall a \in \mathbb{R}$

(iii) $\langle kx, y\rangle = 4(kx_1)y_1 + 2(kx_2)y_2 + 3(kx_3)y_3 + 7(kx_4y_4)$

$= k(4y_1x_1 + 2y_2x_2 + 3y_3x_3 + 7y_4x_4)$

$= k\langle x, y\rangle$

(iv) $\langle x+y, z\rangle = 4(x_1+y_1)z_1 + 2(x_2+y_2)z_2 + 3(x_3+y_3)z_3 + 7(x_4+y_4)z_4$

$= (4x_1z_1 + 2x_2z_2 + 3x_3z_3 + 7x_4z_4)$

$+ (4y_1z_1 + 2y_2z_2 + 3y_3z_3 + 7y_4z_4)$

$= \langle x, z\rangle + \langle y, z\rangle$

By definition, $\langle \cdot, \cdot\rangle$ is an inner product on $\mathbb{R}^4$.

- Norm: $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{4x_1^2 + 2x_2^2 + 3x_3^2 + 7x_4^2}$

- Distance: $d(x,y) = \|x-y\| = \sqrt{4(x_1-y_1)^2 + 2(x_2-y_2)^2 + 3(x_3-y_3)^2 + 7(x_4-y_4)^2}$

(b) $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, $\langle x, y\rangle = x_1y_1 + x_2y_2 - 2x_3y_3$

(i) For $x = (0,0,1)$, $\langle x, x\rangle = 0^2 + 0^2 - 2 \cdot 1^2 = -2 < 0$

Thus, $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is not an inner product space.

(c) $(C^0([0,1]), \langle \cdot, \cdot \rangle)$, $\langle f_1, f_2 \rangle = \int_0^1 t f_1(t) f_2(t)\, dt$

(i) . $f \in C^0([0,1])$, $\langle f, f\rangle = \int_0^1 t f^2(t)\, dt \geq 0$

because $\forall t \in [0,1]$, $t \geq 0$ and $f^2(t) \geq 0$

. $\langle f, f\rangle = 0 \iff f(t) = 0, \ \forall t \in [0,1]$

(ii) $\langle f, g\rangle = \int_0^1 t f(t) g(t)\, dt = \int_0^1 t g(t) f(t)\, dt$

$= \langle g, f\rangle = \overline{\langle g, f\rangle}$, because $\langle g, f\rangle \in \mathbb{R}$

(iii) $\langle kf, g\rangle = \int_0^1 t (kf)(t) g(t)\, dt$

$= k \int_0^1 t f(t) g(t)\, dt = k\langle f, g\rangle$

(iv) $\langle f+g, h\rangle = \int_0^1 t (f+g)(t) h(t)\, dt$

$= \int_0^1 t f(t) h(t)\, dt + \int_0^1 t g(t) h(t)\, dt$

$= \langle f, h\rangle + \langle g, h\rangle$

Therefore $(C^0([0,1]), \langle \cdot, \cdot \rangle)$ is an inner product space.

- Norm: $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^1 t f^2(t)\, dt}$

- Distance: $d(f,g) = \|f-g\| = \sqrt{\int_0^1 t (f-g)(t)^2\, dt}$.

ⓓ $(P_3, \langle \cdot, \cdot \rangle)$,

$$\langle a(t), b(t)\rangle = \langle a_0 + a_1 t + a_2 t^2 + a_3 t^3,\ b_0 + b_1 t + b_2 t^2 + b_3 t^3 \rangle$$

$$= a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

(i) $\langle a(t), a(t)\rangle = a_0^2 + a_1^2 + a_2^2 + a_3^2 \geqslant 0$

$$\langle a(t), a(t)\rangle = 0 \iff a_0^2 + a_1^2 + a_2^2 + a_3^2 = 0$$

$$\iff a_0 = a_1 = a_2 = a_3 = 0$$

$$\iff a(t) = 0$$

(ii) $\langle a(t), b(t)\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$

$$= b_0 a_0 + b_1 a_1 + b_2 a_2 + b_3 a_3$$

$$= \langle b(t), a(t)\rangle = \langle b(t), a(t)\rangle$$

(iii) $\langle k a(t), b(t)\rangle = (ka_0) b_0 + (ka_1) b_1 + (ka_2) b_2 + (ka_3) b_3$

$$= k(a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3) = k\langle a(t), b(t)\rangle$$

(iv) $\langle a(t) + b(t), c(t)\rangle = (a_0 + b_0)c_0 + (a_1 + b_1)c_1 + (a_2 + b_2)c_2 + (a_3 + b_3)c_3$

$$= (a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$+ (b_0 c_0 + b_1 c_1 + b_2 c_2 + b_3 c_3)$$

$$= \langle a(t), c(t)\rangle + \langle b(t), c(t)\rangle$$

Therefore, $(P_3, \langle \cdot, \cdot \rangle)$ is an inner product space.

Norm: $\|a(t)\| = \sqrt{\langle a(t), a(t)\rangle} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$

Distance: $d(a(t), b(t)) = \|a(t) - b(t)\| = \sqrt{(a_0 - b_0)^2 + (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.

40. Let $\langle ; \rangle_1, \langle ; \rangle_2$ be two mappings from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ defined by

$$\langle x ; y \rangle_1 = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad \text{and} \quad \langle x ; y \rangle_2 = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3.$$

Let $v_1 = (1,1,1)$, $v_2 = (1,1,-2)$, $v_3 = (1,1,-1)$ three vectors in $\mathbb{R}^3$.

(a) Show that $\langle ; \rangle_1, \langle ; \rangle_2$ are inner products. Find the norm of v_1 for $\langle ; \rangle_1$ and for $\langle ; \rangle_2$. Find the distance between v_1 and v_2 for $\langle ; \rangle_1$ and for $\langle ; \rangle_2$.

• Show that $\langle \cdot,\cdot \rangle_1$ and $\langle \cdot,\cdot \rangle_2$ are inner product spaces.

(i) $\langle x, x \rangle_1 = x_1^2 + x_2^2 + x_3^2 \geq 0$ and $\langle x, x \rangle_1 = 0 \Leftrightarrow x = (x_1, x_2, x_3) = 0$.

(ii) $\langle x, y \rangle_1 = x_1 y_1 + x_2 y_2 + x_3 y_3 = y_1 x_1 + y_2 x_2 + y_3 x_3 = \langle y, x \rangle_1 = \overline{\langle y, x \rangle_1}$

(iii) $\langle kx, y \rangle_1 = (kx_1) y_1 + (kx_2) y_2 + (kx_3) y_3 = k(x_1 y_1 + x_2 y_2 + x_3 y_3)$
$= k \langle x, y \rangle_1$

(iv) $\langle x+y, z \rangle = (x_1+y_1) z_1 + (x_2+y_2) z_2 + (x_3+y_3) z_3$
$= (x_1 z_1 + x_2 z_2 + x_3 z_3) + (y_1 z_1 + y_2 z_2 + y_3 z_3)$
$= \langle x, z \rangle_1 + \langle y, z \rangle$

Therefore, $\langle \cdot,\cdot \rangle_1$ is an inner product on $\mathbb{R}^3$.

(i) $\langle x, x \rangle_2 = x_1^2 + 2x_2^2 + 3x_3^2 \geq 0$, $\langle x, x \rangle_2 = 0 \Leftrightarrow x = 0$

(ii) $\rightarrow$ (iv) are similar to $\langle \cdot,\cdot \rangle_1$.

Therefore, $\langle \cdot,\cdot \rangle_2$ is also an inner product on $\mathbb{R}^3$.

• Norm: $\|x\|_1 = \sqrt{\langle x, x \rangle_1} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$\|x\|_2 = \sqrt{\langle x, x \rangle_2} = \sqrt{x_1^2 + 2x_2^2 + 3x_3^2}$

For $V_1 = (1,1,1)$, we have

$\|v_1\|_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|v_1\|_2 = \sqrt{1^2 + 2 \cdot 1^2 + 3 \cdot 1^2} = \sqrt{6}$

- Distance:

$$d_1(x,y) = \|x,y\|_1 = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$$

$$d_2(x,y) = \|x-y\|_2 = \sqrt{(x_1-y_1)^2+2(x_2-y_2)^2+3(x_3-y_3)^2}$$

For $v_1=(1,1,1)$ and $v_2=(1,1,-2)$, we get

- $d_1(v_1,v_2) = \sqrt{(1-1)^2+(1-1)^2+(1+2)^2} = 3$
- $d_2(v_1,v_2) = \sqrt{(1-1)^2+2(1-1)^2+3(1+2)^2} = 3\sqrt{3}$

(b) Show that v_1 and v_2 are orthogonal with respect to $\langle ; \rangle_1$ but are not orthogonal with respect to $\langle ; \rangle_2$.

We have $v_1=(1,1,1)$ and $v_2=(1,1,-2)$, then

$$\langle v_1, v_2\rangle_1 = (1)(1)+(1)(1)+(1)(-2) = 0$$

$$\langle v_1, v_2\rangle_2 = (1)(1)+2(1)(1)+3(1)(-2) = -3 \neq 0$$

Therefore, v_1 and v_2 are orthogonal w.r.t. $\langle \cdot,\cdot\rangle_1$ but not w.r.t. $\langle \cdot,\cdot\rangle_2$.

(c) Show that v_1 and v_3 are orthogonal with respect to $\langle ; \rangle_2$ but are not orthogonal with respect to $\langle ; \rangle_1$.

$v_1=(1,1,1)$, $v_3=(1,1,-1)$.

$$\langle v_1, v_3\rangle_1 = (1)(1)+(1)(1)+(1)(-1) = 1 \neq 0$$

$$\langle v_1, v_3\rangle_2 = (1)(1)+2(1)(1)+3(1)(-1) = 0$$

Therefore, v_1 and v_3 are orthogonal w.r.t. $\langle \cdot,\cdot\rangle_2$ but not w.r.t. $\langle \cdot,\cdot\rangle_1$.

41. Let $\mathbb{R}^3$ be a vector space equipped with the inner product

$$\langle x\,;y\rangle = 4x_1y_1 + 3x_2y_2 + 5x_3y_3.$$

Let $v_1 = (1,1,1)$, $v_2 = (1,2,-2)$, $v_3 = (-5,5,1)$ be three vectors in $\mathbb{R}^3$.

(a) Show that $B = \{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Derive an orthonormal basis B_0 for $\mathbb{R}^3$ from B.

(b) Let $v = (2,1,2)$. Determine the coordinates of v with respect to B then with respect to B_0.

(c) Find the scalar and vector projections of v onto v_2.

(a) Show that $B = \{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$

$$\left([v_1]_{B_0}, [v_2]_{B_0}, [v_3]_{B_0}\right) = \begin{pmatrix} 1 & 1 & -5 \\ 1 & 2 & 5 \\ 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & 10 \\ 0 & -3 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & 10 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & 10 \\ 0 & 0 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow B = \{v_1, v_2, v_3\}$ is linearly independent.

Moreover, $\# B = 3 = \dim \mathbb{R}^3 \Rightarrow B$ is a basis for $\mathbb{R}^3$

$$\langle v_1, v_2\rangle = 4(1)(1) + 3(1)(2) + 5(1)(-2) = 0$$

$$\langle v_1, v_3\rangle = 4(1)(-5) + 3(1)(5) + 5(1)(1) = 0$$

$$\langle v_2, v_3\rangle = 4(1)(-5) + 3(2)(5) + 5(-2)(1) = 0$$

Since B is an orthogonal set, we conclude that B is an orthogonal basis for $\mathbb{R}^3$.

Let $u_1 = \dfrac{1}{\|v_1\|} \cdot v_1 = \dfrac{1}{\sqrt{4\cdot1^2+3\cdot1^2+5\cdot1^2}} \cdot (1,1,1) = \left(\dfrac{\sqrt{2}}{10}, \dfrac{\sqrt{2}}{10}, \dfrac{\sqrt{2}}{10}\right)$

$u_2 = \dfrac{1}{\|v_2\|} \cdot v_2 = \dfrac{1}{\sqrt{4\cdot1^2+3\cdot2^2+5\cdot(-2)^2}} (1,2,-2) = \left(\dfrac{1}{6}, \dfrac{1}{3}, -\dfrac{1}{3}\right)$

$u_3 = \dfrac{1}{\|v_3\|} \cdot v_3 = \dfrac{1}{\sqrt{4\cdot(-5)^2+3\cdot5^2+5\cdot1}} (-5,5,1) = \left(-\dfrac{\sqrt{5}}{6}, \dfrac{\sqrt{5}}{6}, \dfrac{\sqrt{5}}{30}\right)$

Therefore, $B_0 = \{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $v = (2, 1, 2)$. Determine the coordinates of v with respect to B then with respect to B_0.

$$v = \frac{\langle v, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2\rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle v, v_n\rangle}{\|v_n\|^2} v_n.$$ Theorem 23

$V = (2,1,2)$

$V_1 = (1,1,1)$

$V_2 = (1,2,-2)$

$V_3 = (-5,5,1)$

$$c_1 = \frac{\langle v, v_1\rangle}{\|v_1\|^2} = \frac{4(2)(1)+3(1)(1)+5(2)(1)}{4(1)^2+3(1)^2+5(1)^2}$$

$$= \frac{8+3+10}{4+3+5} = \frac{21}{12} = \frac{7}{4}$$

$$c_2 = \frac{\langle v, v_2\rangle}{\|v_2\|^2} = \frac{4(2)(1)+3(1)(2)+5(2)(-2)}{4(1)^2+3(2)^2+5(-2)^2}$$

$$= \frac{8+6-20}{4+12+20} = \frac{-6}{36} = -\frac{1}{6}$$

$$c_3 = \frac{\langle v, v_3\rangle}{\|v_3\|^2} = \frac{4(2)(-5)+3(1)(5)+5(2)(1)}{4(-5)^2+3(5)^2+5(1)^2}$$

$$= \frac{-40+15+10}{100+75+5} = \frac{15}{180} = \frac{1}{12}$$

Therefore, $[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \left(\frac{7}{4}, -\frac{1}{6}, \frac{1}{12}\right).$

Theorem 23 $\quad v = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \cdots + \langle v, v_n\rangle v_n$

$B_0 = \{u_1, u_2, u_3\} = \{(\quad), (\quad), (\quad)\}$

$c_1 = \langle v, u_1\rangle = 4(\quad)(\quad) + 3(\quad)(\quad) + 5(\quad)(\quad) = \cdots$

$c_2 = \langle v, u_2\rangle = 4(\quad)(\quad) + 3(\quad)(\quad) + 5(\quad)(\quad) = \cdots$

$c_3 = \langle v, u_3\rangle = 4(\quad)(\quad) + 3(\quad)(\quad) + 5(\quad)(\quad) = \cdots$

Therefore, $[v]_{B_0} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}.$