National Religion King

Institute of Technology of Cambodia

Department of Applied Mathematics and Statistic

(Option Data Science)

Assignment of Linear Algebra

Matrices, Determinant, Vector Space

Lecture: Dr. Lin Mongkolsery (Cours)

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TD1: Matrices

TD1 : (Matrices)

1.) write the following into row echelon form.

a.) $A = \begin{pmatrix} 1 & 1 & 3 & 2 \\ -2 & 2 & 1 & 0 \\ 0 & 4 & 7 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 4 & 7 & 4 \\ 0 & 4 & 7 & 4 \end{pmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 4 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{4}} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & \frac{7}{4} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b.) $B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 0 & 4 \\ 2 & 3 & 4 \end{pmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & -9 & 1 \\ 0 & -3 & 2 \end{pmatrix} \xrightarrow{R_2 \to -\frac{R_2}{9}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -3 & 2 \end{pmatrix}$ $R_3 \to R_3 + 3R_2$

$\longrightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$ is row echelon form.

c.) $C = \begin{pmatrix} 1 & 2 & 1 & 4 & 5 & 7 \\ 2 & 1 & 0 & 1 & 2 & 1 \\ 3 & 3 & 1 & 5 & 7 & 8 \end{pmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ $\longrightarrow$

$\begin{pmatrix} 1 & 2 & 1 & 4 & 5 & 7 \\ 0 & -3 & -2 & -7 & -8 & -13 \\ 0 & -3 & -2 & -7 & -8 & -13 \end{pmatrix}$ $R_2 \to \frac{-R_2}{3}$, $R_3 \to R_2 + R_3$ $\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 4 & 5 & 7 \\ 0 & 1 & \frac{3}{2} & \frac{7}{3} & \frac{8}{3} & \frac{13}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

d.) $D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & \lambda & \lambda \end{pmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 + R_1$ $\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 1+\lambda & \lambda+1 \end{pmatrix}$

$R_2 \to -\frac{R_2}{2}$, $R_3 \to \frac{R_3}{2}$, $R_4 \to R_4 + R_2$ $\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1+\lambda & \lambda-1 \end{pmatrix}$ $R_4 \to R_4 - (\lambda+1)R_1$.

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$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} R_4 \longrightarrow -\frac{R_4}{2} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.) Let X be a 2×2 matrix. Find the solution of the equation $X^2 - 3X = I$.

proof: we have $X^2 - 3X = I \Leftrightarrow X(X-3) = I$ or $X(X-3I) = I$

Let $X = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow X \cdot X = X^2 = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+cb & ac+cd \\ ab+bd & bc+d^2 \end{pmatrix}$

$$X^2 = \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & abc/a + d^2 \end{pmatrix} = \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & bc+d^2 \end{pmatrix} ; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow X^2 - 3XI = I \Leftrightarrow \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & bc+d^2 \end{pmatrix} - 3\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow a^2 + bc - 3a = 1$ (1) ; $ac + cd - 3c = 0$ (2) ; $b(a+d) - 3b = 0$ (3)

$bc + d^2 - 3d = 1$ (4) : form (2): $c(a+d-3) = 0 \Rightarrow c = 0 ; a+d = 3$ (α)

(3): $b(a+d-3) = 0 \Rightarrow b = 0 ; a+d = 3$ (α) :

from: (1): $a^2 - 3a = 1 \Leftrightarrow a^2 - 3a - 1 = 0$; $\Delta = 9 - 4(-1) = 9 + 4$

$= 13 \Rightarrow \sqrt{\Delta} = \sqrt{13}$; $a_1 = \frac{3+\sqrt{13}}{2}$; $a_2 = \frac{3-\sqrt{13}}{2}$

then $d_1 = 3 - a_1 = 3 - \frac{3+\sqrt{13}}{2} = \frac{3-\sqrt{13}}{2}$; $d_2 = \frac{9-6+\sqrt{13}}{2} = \frac{3+\sqrt{13}}{2}$

$$\boxed{\text{therefore; } X_1 = \begin{pmatrix} \frac{3+\sqrt{13}}{2} & 0 \\ 0 & \frac{3-\sqrt{13}}{2} \end{pmatrix} ; X_2 = \begin{pmatrix} \frac{3-\sqrt{13}}{2} & 0 \\ 0 & \frac{3+\sqrt{13}}{2} \end{pmatrix}}$$

3.) Let $X \in M_3(\mathbb{R})$. Solve the equation $X - 2X^t = tr(X) I_3$

proof: Let $X = \begin{pmatrix} a & d & i \\ b & e & g \\ c & f & h \end{pmatrix} \Rightarrow X^t = \begin{pmatrix} a & b & c \\ d & e & f \\ i & g & h \end{pmatrix}$

$$\Rightarrow X - 2X^t = tr(X) I_3 \Leftrightarrow \begin{pmatrix} a & d & i \\ b & e & g \\ c & f & h \end{pmatrix} - 2\begin{pmatrix} a & b & c \\ d & e & f \\ i & g & h \end{pmatrix} = tr(X) I_3$$

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then $tr(x) - 2tr(x^t) = tr(x)\,tr(I_3)$; $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow tr(x) - 2tr(x^t) = 3tr(x)$; since $tr(x) = tr(x^t)$

$\Rightarrow tr(x) = tr(x^t) = 0 \Rightarrow$ (equation above): $x - 2x^t = 0$

$\Leftrightarrow x = 2x^t \Leftrightarrow \begin{pmatrix} a & d & i \\ b & e & g \\ c & f & h \end{pmatrix} = 2\begin{pmatrix} a & b & c \\ d & e & f \\ i & g & h \end{pmatrix}$.

$\Rightarrow a = 2a$ (1) ; $d = 2b$; $i = 2c$; $b = 2d$; $e = 2e$; $g = 2f$

$c = 2i$ (2) ; $f = 2g$; $h = 2h \Rightarrow h = 0$; $a = 0$; $e = 0$; $g = 0$

$f = 0$; $b = 0$, $d = 0 \Rightarrow \boxed{x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is solution.}}$

4.) Find the matrix $x \in M_2(\mathbb{R})$ satisfies the equation.

$x^3 = A$; where $A = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$

proof: Let $x = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ be $M_2(\mathbb{R})$ such that with A

$x = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \Rightarrow x^2 = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} \Rightarrow x^3 = x^2 \cdot x$

$= \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} \cdot \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab^2 \\ a^2b & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & ab^2 \\ a^2b & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$.

$\Rightarrow \begin{cases} ab^2 = -3 & (1) \\ a^2b = 2 & (2) \end{cases} \Rightarrow \frac{(1)}{(2)} : \frac{ab^2}{a^2b} = -\frac{3}{2} \Leftrightarrow \frac{b}{a} = -\frac{3}{2} \Rightarrow a = -\frac{2}{3}b$.

Substitute $a = -\frac{2}{3}b$ into (2): we have $\left(-\frac{2}{3}\right)^2 b^2 \cdot b = 2$

$\Rightarrow b^3 = \frac{2 \times 3^2}{2^2} = \frac{3^2}{2} \Rightarrow b = \frac{\sqrt[3]{9}}{\sqrt[3]{2}} = \frac{\sqrt[3]{18}\,\sqrt[3]{2}}{2} \Rightarrow a = -\frac{2}{3} \times \frac{\sqrt[3]{18 \times 2}}{2} = -\frac{\sqrt[3]{18}}{3} \times \sqrt[3]{2}$

$b = \frac{\sqrt[3]{36}}{2}$; $a = -\frac{\sqrt[3]{36}}{3}$.

$\boxed{\text{Therefore; } x = \begin{pmatrix} 0 & \frac{\sqrt[3]{36}}{2} \\ -\frac{\sqrt[3]{36}}{3} & 0 \end{pmatrix}}$.

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5.) Let $A, B \in M_n(\mathbb{R})$ such that $AB = 2A + 3B$. Show that

a.) $(A - 3I_n)(B - 2I_n) = 6I_n$

proof: we have $(A-3I_n)(B-2I_n) = AB - 2AI_n - 3BI_n + 6I_n^2$

$= AB - I_n(2A+3B) + 6I_n^2 = AB(-I_n+1) + 6I_n^2 = (1-1)AB + 6I_n = 6I_n$

b.) show that $AB = BA$

first way: we have: $AB = 2A + 3B$, $A, B \in M_n(\mathbb{R})$

then , $A(AB) = A(3B + 2A) \Leftrightarrow A^2B = 3AB + 2A^2$

then $A^2B \cdot B = (3AB + 2A^2) \cdot B = 3AB^2 + 2A^2B = (2A^2 + 6A + 9B) \cdot B$.

$\Leftrightarrow A^2B^2 = 2A^2B + 6AB + 9B^2$ (1) ; On the other hand, we have

$A^2B^2 = (AB)^2 = (2A+3B)^2 = 4A^2 + 12AB + 9B^2 - (2)\, 6AB + 6BA$

Compare (1) with (2) : $2A^2B + 6AB + 9B^2 = 4A^2 + \frac{12AB}{2} + 9B^2 + 6BA$

$\Leftrightarrow 6BA + \frac{6AB}{6AB} + 4A^2 = 2A^2B = 2A \cdot (AB) = 2A(2A+3B) = 4A^2 + 6AB$.

$\Rightarrow 6BA = 6AB \Rightarrow \boxed{BA = AB \text{ true}}$

Second way: we have $(A - 3I_n)(B - 2I_n) = 6I_n \Rightarrow$

$\frac{1}{6}(A - 3I_n)$; $(B - 2I_n)$ are invertible

we consider ; we have $(B - 2I_n)(A - 3I_n) = BA - 3B - 2A + 6I_n$

$= 6I_n \Rightarrow BA = 2A + 3B = AB$.

$\boxed{\text{Therefore } BA = AB. \text{ and } A, B \text{ is invertible}}$

6.) (a.) Are there matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I$

put trace on both side of equality ; we have :

$tr(AB - BA) = tr(I) \Leftrightarrow tr(AB) - tr(BA) = tr(I)$

Since $tr(AB) = tr(BA)$; where $AB \neq BA$; $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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$\Rightarrow tr(I) = 0$ not true because $tr(I) = tr(I_n) = n$.

Therefore; the matrices $A, B \in M_n(\mathbb{R})$ does not such that $AB - BA = I$.

b.) Suppose that $A, B \in M_n(\mathbb{R})$ such that $(AB-BA)^2 = AB-BA$. Show that A, B are commuatable.

proof: we have $(AB-BA)^2 = AB-BA \Leftrightarrow (AB-BA)(AB-BA-I) = 0$

$\Rightarrow AB - BA = 0$, then $AB = BA$; Therefore; A, B are commuatable

7.) show that $A+I$ is invertible and then find its inverse.

proof: we have $x^5+1 = (x+1)(x^4-x^3+x^2-x+1)$ or

$x^5 = (x+1)(x^4-x^3+x^2-x+1) - 1$., then take $x = A$ we have:

$A^5 = (A+I)(A^4-A^3+A^2-A+I) - I \Rightarrow (A+I)(A^4-A^3+A^2-A+I) = I$.

$\Rightarrow A+I$ is invertible, where $A^{-1} = A^4-A^3+A^2-A+I$.

Therefore; $A^{-1} = A^4-A^3+A^2-A+1$ is inverse of $A+I$.

8.) show that A^2+A+I is invertible, then find its inverse:

proof: we have $x^5+x = 1 \Leftrightarrow x^5+x-1 = 0$; then we divided the polynomial between x^5+x-1 with x^2+x+1 we get result

Such that: $x^5+x-1 = (x^2+x+1)(x^3-x^2+1) - 2I$.

$\Rightarrow A^5+A-I = (A^2+A+I)(A^3-A^2+I) - 2I = 0$

$\Leftrightarrow (A^2+A+I)(A^3-A^2+I)\frac{1}{2} = I \Rightarrow A^2+A+I$ is invertible.

Therefore; A^2+A+I is invertible and $A^{-1} = \frac{1}{2}(A^3-A^2+I)$

9.) a.) Show that $B = (I-A)^{-1}(A+I)^{-1} = (I-A)(I+A)^{-1} = (I+A)^{-1}(I-A)$

proof: we have $(I+A)(I-A) = I - AI + AI - A^2 = I - A^2$ (1)

$(I-A)(I+A) = I + IA - AI - A^2 = I - A^2$ (2)

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$\Rightarrow (I+A)(I-A) = (I-A)(I+A)$

$\Leftrightarrow (I+A)(I-A)(I+A)^{-1} = (I-A)(I+A)(I+A)^{-1}$

$\Leftrightarrow (I+A)B = (I-A) \Leftrightarrow (I+A)^{-1}(I+A)B = (I+A)^{-1}(I-A)$

$\Rightarrow B = (I+A)^{-1}(I-A)$; $\boxed{\text{Therefore; } B = (I+A)^{-1}(I-A)}$

b.) Show that $I+B$ is invertible and express A in terms of B:

proof: we have $I+B = I + (I+A)^{-1}(I-A)$

$\Leftrightarrow (I+A)(I+B) = (I+A)(I + (I+A)^{-1}(I-A))$

$\Leftrightarrow (I+A)(I+B) = I+A+I-A = 2I \Leftrightarrow \frac{1}{2}(I+A)(I+B) = I.$

or $(I+B)\frac{1}{2}(I+A) = I$ or $(I+B)(I+B)^{-1} = I$, then we have $(I+B)^{-1} = \frac{1}{2}(I+A)$; $\boxed{\text{Therefore; } (I+B)^{-1} = \frac{1}{2}(I+A)}$

On the others hand:

$(I+B)\frac{1}{2}(I+A) = I \Leftrightarrow (I+B)^{-1}(I+B)\frac{1}{2}(I+A) = (I+B)^{-1}I$

$\Leftrightarrow I+A = 2(I+B)^{-1} \Rightarrow A = 2(I+B)^{-1} - I$; $\boxed{\text{Therefore; } A = 2(I+B)^{-1} - I}$

10.) a.) Compute A^2:

proof: we have $A = \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix}_n$, then $A^2 = A \cdot A$

$$= \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix}\begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} = \begin{pmatrix} a_1(a_1+a_2+\cdots+a_n) & \cdots & a_1(a_1+\cdots+a_n) \\ a_2(a_1+\cdots+a_n) & \cdots & a_2(a_1+\cdots+a_n) \\ \vdots & & \vdots \\ a_n(a_1+\cdots+a_n) & \cdots & a_n(a_1+\cdots+a_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_1\alpha & a_1\alpha & \cdots & a_1\alpha \\ a_2\alpha & a_2\alpha & \cdots & a_2\alpha \\ \vdots & \vdots & & \vdots \\ a_n\alpha & a_n\alpha & \cdots & a_n\alpha \end{pmatrix} = \alpha A \text{ ; Therefore } A^2 = \begin{pmatrix} a_1\alpha & a_1\alpha & \cdots & a_1\alpha \\ a_2\alpha & a_2\alpha & \cdots & a_2\alpha \\ \vdots & \vdots & & \vdots \\ a_n\alpha & a_n\alpha & \cdots & a_n\alpha \end{pmatrix}$$

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b.) Show that B is invertible and then find its inverse:

proof: we have $B = (b_{ij})_n$ where $b_{ij} = 2a_i$ if $i \neq j$ and

$$b_{ii} = a_i - \sum_{j \neq i}^{n} a_j \Rightarrow B = \begin{pmatrix} 2a_1-\alpha & 2a_1-\alpha\times 0 & \cdots & 2a_1-\alpha\times 0 \\ 2a_2-\alpha\times 0 & 2a_2-\alpha & \cdots & 2a_2-\alpha\times 0 \\ \vdots & & & \vdots \\ 2a_n-\alpha\times 0 & 2a_n-\alpha\times 0 & \cdots & 2a_n-\alpha \end{pmatrix}$$

$$= 2\begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & & & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} - \alpha\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = 2A - \alpha I.$$

again expression, we have: $B = \begin{pmatrix} 2a_1-\alpha & 2a_1 & \cdots & 2a_1 \\ 2a_2 & 2a_2-\alpha & \cdots & 2a_2 \\ \vdots & & & \vdots \\ 2a_n & 2a_n & \cdots & 2a_n-\alpha \end{pmatrix}$

$$\Leftrightarrow B = 2A - \alpha I \Leftrightarrow B^2 = 4A^2 - 4A\alpha I + \alpha^2 I^2$$

$$\Leftrightarrow B^2 = 4A^2 - 4\alpha A + \alpha^2 I. = 4\alpha A - 4\alpha A + \alpha^2 I = \alpha^2 I.$$

$$\Rightarrow \frac{1}{\alpha^2}B^2 = I \text{ or } \frac{1}{\alpha^2}B.B = I. \Rightarrow B^{-1} = \frac{1}{\alpha^2}B.$$

$\boxed{\text{Therefore, } B \text{ is invertible and } B^{-1} = \frac{1}{\alpha^2}B}$.

11.) Find the rank of matrix A where:

a.) $A = \begin{pmatrix} 0 & -1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{pmatrix} \begin{matrix} L_2 \to -L_2 \\ L_3 \to L_3 - L_1 \end{matrix}$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{pmatrix} \xrightarrow{L_3 \to L_3 - L_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{3\times 4}$$

$\Rightarrow \boxed{\text{Rank}(A) = 2 \text{ ; nullity}(A) = 2}$

b.) $B = \begin{pmatrix} 1 & 2 & 1 \\ a & b & c \\ -1 & 3 & 0 \end{pmatrix} \begin{matrix} L_2 \to L_2 - aL_1 \\ L_3 \to L_3 + L_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & b-2a & c-a \\ 0 & 5 & 1 \end{pmatrix} \xrightarrow{L_3 \leftrightarrow L_2}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & b-2a & c-a \end{pmatrix} \begin{matrix} L_2 \to \frac{1}{5}L_2 \\ L_3 \to L_3 + 0 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & b-2a & c-a \end{pmatrix} L_3 \to L_3 - (b-2a)L_2$$

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$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & c-a-\frac{1}{5}(b-2a) \end{pmatrix}_{3\times 3} L_3 \to \frac{1}{c-a-\frac{1}{5}(b-2a)} L_3 \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{pmatrix}$$

- if $c-a-\frac{1}{5}(b-2a) = 0 \Rightarrow \text{rank}(B) = 2$; $\text{nullity}(B) = 1$
- if $c-a-\frac{1}{5}(b-2a) \neq 0 \Rightarrow \text{rank}(B) = 3$. ; $\text{null}(B) = 0$

c.) $$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & \lambda & \lambda \end{pmatrix} \begin{matrix} L_3 \to L_3 + L_1 \\ L_2 \to L_2 - L_1 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & \lambda+1 & \lambda+1 \end{pmatrix}$$

$$\begin{matrix} L_2 \leftrightarrow L_3 \\ L_3 \to \frac{1}{2} L_3 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2}(\lambda+1) & \frac{1}{2}(\lambda+1) \\ 0 & 0 & -2 & -2 \end{pmatrix} \begin{matrix} L_3 \to -\frac{1}{2} L_3 \\ L_2 \to L_2 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{\lambda+1}{2} & \frac{\lambda+1}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$L_2 \to L_2 - \left(\frac{\lambda+1}{2}\right) L_3 \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}_{3\times 4} \Rightarrow \boxed{\begin{matrix} \text{rank}(C) = 3. \\ \text{nullity}(C) = 1. \end{matrix}}$$

12.) Find rank (A) :

Proof: we have $A = (a_{ij})_n \in M_n(\mathbb{R})$ where $a_{ij} = i-j+1$ for $i, j = 1, 2, \dots, n$. then we have : $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$

$$\Leftrightarrow A = \begin{pmatrix} 1 & 0 & -1 & \cdots & (2-n) \\ 2 & 1 & 0 & \cdots & (3-n) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \cancel{(2-n)}\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}_n \begin{matrix} R_n \to R_n - R_{n-1} \\ R_{n-1} \to R_{n-1} - R_{n-2} \\ \vdots \\ R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \end{matrix}$$

or $R_k \to R_k - R_{k-1}$ $(k = 2, 3, \dots n)$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & -1 & \cdots & (2-n) \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & 1 & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \\ \vdots \\ R_n \to R_n - R_2 \end{matrix}$$

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$$\sim \begin{pmatrix} 1 & 0 & -1 & \cdots & -n+2 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_n \quad R_2 \to R_2 - R_1$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & \cdots & -n+2 \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_n$$

$\boxed{\text{Therefore, } \operatorname{Rank}(A) = 1 \text{ if } n = 1 \text{ and } \operatorname{Rank}(A) = 2 \text{ if } n \geq 2.}$

13.) let $A = (a_{ij})_n \in M_n(\mathbb{R})$ where $a_{ij} = \cos(i+j)$ for $i \,\&\, j = 1, 2, \ldots, n$.

Proof: we have $A = (a_{ij})_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}_n$

$$= \begin{pmatrix} \cos(2) & \cos(3) & \cdots & \cos(n+1) \\ \cos(3) & \cos(4) & \cdots & \cos(n+2) \\ \vdots & \vdots & & \vdots \\ \cos(n+1) & \cos(n+2) & \cdots & \cos(n+n) \end{pmatrix}_n.$$

we observe that $\cos(n+1) = \cos(n)\cos(1) - \sin(n)\sin(1)$

$= 2\cos(n)\cos(1) - \cos(n)\cos(1) - \sin(n)\sin(1)$

$= (2\cos(1))\cos(n) - (\cos(n)\cos(1) + \sin(n)\sin(1))$

$= (2\cos(1))\cos(n) - \cos(n-1) \Rightarrow \cos(n+1) = \cos(n)\,2\cos(1) - \cos(n-1)$

$\Rightarrow \cos(n+2) = \cos(n+1)\,2\cos(1) - \cos(n)$; $\cos(n+3) = 2\cos(1)\cos(n+2) - \cos(n+1)$

$\cdots\cdots\cdots$

$\Rightarrow \cos(n+n) = (2\cos(1))\cos(n+n-1) - \cos(n+n-2)$

For $K = n, n-1, \ldots, 3, 2, 1$: $R_K \to R_K - 2\cos(1)R_{K-1}$

$R_K \to R_K + R_{K-2}$

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$$\rightarrow \begin{pmatrix} \cos(2) & \cos(3) & \cos(4) & \cdots & \cos(n+1) \\ \cos(3) & \cos(4) & \cos(5) & \cdots & \cos(n+2) \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \begin{matrix} R_1 \to \frac{1}{\cos(2)} R_1 \\ R_2 \to \frac{1}{\cos 3} R_2 \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & \frac{\cos(3)}{\cos(2)} & \frac{\cos(4)}{\cos(2)} & \cdots & \frac{\cos(n+1)}{\cos(2)} \\ 1 & \frac{\cos(4)}{\cos(3)} & \frac{\cos(5)}{\cos(3)} & \cdots & \frac{\cos(n+2)}{\cos(3)} \\ 0 & 0 & 0 & \cdots & \frac{\cos(n+2)}{\cos(3)} \times 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad R_2 \to R_2 - R_1$$

$$\rightarrow \begin{pmatrix} 1 & \frac{\cos(3)}{\cos(2)} & \frac{\cos(4)}{\cos(2)} & \cdots & \frac{\cos(n+1)}{\cos(2)} \\ 0 & \frac{\cos 4}{\cos 3} - \frac{\cos 3}{\cos 2} & \frac{\cos(5)}{\cos(3)} - \frac{\cos(4)}{\cos(2)} & \cdots & \frac{\cos(n+2)}{\cos(3)} - \frac{\cos(n+1)}{\cos(2)} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$R_2 \to \frac{1}{\frac{\cos(4)}{\cos(3)} - \frac{\cos(3)}{\cos(2)}} R_2 \rightarrow \begin{pmatrix} 1 & b_2 & b_3 & \cdots & b_n \\ 0 & 1 & c_3 & \cdots & c_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\boxed{\text{Therefore; } \operatorname{Rank}(A) = 1 \text{ if } n = 1 \text{ and } \operatorname{Rank}(A) = 2 \text{ if } n \geq 2}$

14.) Find the inverse matrix if it exist below.

a.) $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix} \Rightarrow (A|I) = \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$

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16_ Let $n \in \mathbb{N} - \{0,1\}$ and $A = \begin{pmatrix} 2 & & (1) \\ & 2 & \\ (1) & & 2 \end{pmatrix}$

a_ Show that A is invertible and express A^{-1} in terms of n, I and A

Let $U = (1)_n \Rightarrow U^2 = nU$

$\Rightarrow A = U + I$ or $U = A - I$

$\Rightarrow A^2 = (U+I)^2 = U^2 + 2U + I = nU + 2U + I = (n+2)U + I$

$\Rightarrow A^2 = (n+2)(A-I) + I \Rightarrow A^2 = (n+2)A - (n+2)I + I$

$\Rightarrow A^2 - (n+2)A = -(n+1)I \Rightarrow A(A - (n+2)I) = -(n+1)I$

$\Rightarrow -\frac{1}{n+1} A(A - (n+2)I) = I$

Thus A is invertible and $A^{-1} = -\frac{1}{n+1}(A - (n+2)I)$

b_ Calculate $|A|$

$$|A| = \begin{vmatrix} 2 & & (1) \\ & 2 & \\ (1) & & 2 \end{vmatrix}_n \overset{L_1 \to L_1 + (L_2 + \dots + L_n)}{=} \begin{vmatrix} n+1 & n+1 & \dots & n+1 \\ 1 & 2 & & (1) \\ & (1) & & 2 \end{vmatrix}$$

$$|A| = (n+1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & & (1) \\ \vdots & (1) & & \\ 1 & & & 2 \end{vmatrix}_n \quad C_j = C_j - C_1,\ j = 2, 3, \dots, n$$

$$= (n+1) \begin{vmatrix} 1 & & (0) \\ 1 & 1 & \\ \vdots & & \\ 1 & (0) & 1 \end{vmatrix}_n = \underline{n+1}$$

c. Determine $adj(A)$ and $|adj(A)|$

$adj(A) = |A| \times A^{-1} = (n+1)\left[-\frac{1}{n+1}(A - (n+2)I)\right] = -(A - (n+2)I)$

$|adj(A)| = |A|^{n-1} = \underline{(n+1)^{n-1}}$

d_ Calculate A^m, $m \in \mathbb{N}$

We have $A = U + I \Rightarrow A^m = (U+I)^m$

$\Rightarrow A^m = \sum_{k=0}^{m} C_m^k U^k$, where $U^k = n^{k-1}U$; $k \geq 1$

Remark: $C_m^k = \frac{m!}{k!(m-k)!}$; $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$

$(U+I)^m = \sum_{k=0}^{m} C_m^k U^k \underbrace{I^{m-k}}_{I} = \sum_{k=0}^{m} C_m^k U^k$

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$\Rightarrow A^m = I + \sum_{k=1}^{m} C_m^k n^{k-1} U \quad ; C_m^0 = 1 \quad , U^0 = I$

$\Rightarrow A^m = I + \frac{1}{n} U \sum_{k=1}^{m} C_m^k n^k \quad ; (n+1)^m = \sum_{k=0}^{m} C_m^k n^k$

$\Rightarrow A^m = I + \frac{1}{n} U \left(\sum_{k=0}^{m} C_m^k n^k - 1 \right)$

Thus $A^m = I + \frac{1}{n} \left[(n+1)^m - 1 \right] U$

(17). Let $A, B, C, D \in M_n(\mathbb{K})$. Suppose that D is invertible and $CD = DC$ commutant. Show that $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC)$

Solution: we have $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D & 0 \\ -C & I \end{bmatrix} = \begin{bmatrix} AD - BC & B \\ 0 & D \end{bmatrix}$ Because $CD = DC$

$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \det \begin{bmatrix} D & 0 \\ -C & I \end{bmatrix} = \det \begin{bmatrix} AD - BC & B \\ 0 & D \end{bmatrix}$

$\Rightarrow \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \det [D] = \det(AD - BC) \det(D)$

Since D is invertible, It mean that $\det(D) \neq 0$. Then we can divide $\det(D)$ at the both side.

Thus $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC)$

(18). Let $A, B, C, D \in M_n(\mathbb{K})$. Suppose that D is invertible. Show that $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \times \det(A - BD^{-1}C)$

Proof we have $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$

Because $C - DD^{-1}C = C - C = 0$

By determinant on both side: $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \det \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} = \det \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}$

$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \times \det(D) = \det(D) \times \det(A - BD^{-1}C)$ true

Thus $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \times \det(A - BD^{-1}C)$

(15) - Compute the following determinants

a. $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 1 & 5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{vmatrix}$ b. $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{vmatrix}$

Proof:

(a) $|X| = \begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 1 & 5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$

we have:

$C = \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix}$; $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \Rightarrow |D| = 1$; $CD = DC$

$|X| = |AD - BC| = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -8 & 11 \\ -10 & 18 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} -8 & 11 \\ -10 & 18 \end{pmatrix} \right| = \begin{vmatrix} a+8 & b-11 \\ c+10 & d-18 \end{vmatrix} = (a+8)(d-18) - (c+10)(b-11)$

$= ad - 18a + 8d - 144 - cb + 11c - 10b + 110$

$= ad - bc - 18a - 10b + 11c + 8d - 34$

Thus $|X| = ad - bc - 18a - 10b + 11c + 8d - 34$

(b). $|Y| = \begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$

we have $D = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\Rightarrow |D| = 1 \Rightarrow D$ is invertible and $D^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

By Determinant of Block Matrix

we have: $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \times \det(A - BD^{-1}C)$

$\Rightarrow |Y| = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 11 & 10 \\ 16 & 14 \end{pmatrix} \right)$

$= \det\begin{pmatrix} a-11 & b-10 \\ c-16 & d-14 \end{pmatrix} = ad - bc - 14a + 16b + 10c - 11d - 6$

Thus $|Y| = ad - bc - 14a + 16b + 10c - 11d - 6$

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(20). Solve the following system of linear equations by using Cramer's rule.

a). $\begin{cases} 2x_1 - x_2 + 3x_3 = 9 \\ -x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 - 2x_3 = -1 \end{cases}$, $A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 9 \\ 4 \\ -1 \end{pmatrix}$

$$\Delta = |A| = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 1 \\ 1 & 2 & -2 \end{vmatrix}\begin{matrix} 2 & -1 \\ -1 & 1 \\ 1 & 2 \end{matrix} = (-4-1-6)-(3+4-2) = -16$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 4 & 1 & 1 \\ -1 & 2 & -2 \end{vmatrix}\begin{matrix} 9 & -1 \\ 4 & 1 \\ -1 & 2 \end{matrix} = (-18+1+24)-(-3+18+8) = -24$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ -1 & 4 & 1 \\ 1 & -1 & -2 \end{vmatrix}\begin{matrix} 2 & 9 \\ -1 & 4 \\ 1 & -1 \end{matrix} = (-16+9+3)-(12-2+18) = -32$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ -1 & 1 & 4 \\ 1 & 2 & -1 \end{vmatrix}\begin{matrix} 2 & -1 \\ -1 & 1 \\ 1 & 2 \end{matrix} = (-2-4-18)-(9+16-1) = -48$$

Thus $x_1 = \frac{\Delta_1}{\Delta} = \frac{-24}{-16} = \frac{3}{2}$; $x_2 = \frac{\Delta_2}{\Delta} = \frac{-32}{-16} = 2$; $x_3 = \frac{\Delta_3}{\Delta} = \frac{-48}{-16} = 3$

(b). $\begin{cases} mx_1 + x_2 + x_2 = 1 \\ x_1 + mx_2 + x_3 = m \\ x_1 + x_2 + mx_3 = m^2 \end{cases}$; $A = \begin{pmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{pmatrix}$; $B = \begin{pmatrix} 1 \\ m \\ m^2 \end{pmatrix}$

$$\Delta = |A| = \begin{vmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{vmatrix}\begin{matrix} m & 1 \\ 1 & m \\ 1 & 1 \end{matrix} = m^3+1+1-m-m-m = m^3-1-3m+3$$

$$= (m-1)(m^2+m+1) - 3(m-1) = (m-1)(m^2+m-2) = (m-1)^2(m-2)$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ m & m & 1 \\ m^2 & 1 & m \end{vmatrix}\begin{matrix} 1 & 1 \\ m & m \\ m^2 & 1 \end{matrix} = -m^3+m^2+m-1 = -m^2(m-1)+(m-1) = (m-1)(-m^2+1)$$

$$= -(m-1)^2(m+1)$$

$$\Delta_2 = \begin{vmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & m^2 & m \end{vmatrix}\begin{matrix} m & 1 \\ 1 & m \\ 1 & m^2 \end{matrix} = m^3+1+m^2-m-m^3-m = m^2-2m+1 = (m-1)^2$$

$$\Delta_3 = \begin{vmatrix} m & 1 & 1 \\ 1 & m & m \\ 1 & 1 & m^2 \end{vmatrix}\begin{matrix} m & 1 \\ 1 & m \\ 1 & 1 \end{matrix} = m^4+m+1-m-m^2-m^2 = m^4-2m^2+1$$

$$= (m^2-1)^2 = (m-1)^2(m+1)^2$$

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Thus for $m \neq \{1,-2\}$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-(m-1)^2(m+1)}{(m-1)^2(m+2)} = -\frac{m+1}{m+2}$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{(m-1)^2}{(m-1)^2(m+2)} = \frac{1}{m+2}$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{(m-1)^2(m+1)^2}{(m-1)^2(m+2)} = \frac{(m+1)}{m+2}$$

if $m=1$: $x_1+x_2+x_3=1$ Take $x_2=\alpha$, $x_3=\beta$

$\Rightarrow x_1 = 1-\alpha-\beta$

The system of equation has infinite solution of the form $x = \begin{pmatrix} 1-\alpha-\beta \\ \alpha \\ \beta \end{pmatrix}$

• $m=-2$ $\text{Rank}(A) < \text{Rank}(A|b)$ $\Rightarrow$ no solution

(21) Let $a,b,c \in \mathbb{C}$ and $P(X) = X^3-(x+yX+zX^2)$. Solve the following systeme by using polynomial P.

(a) $\begin{cases} x+ay+a^2z=a^3 \\ x+by+b^2z=b^3 \\ x+cy+c^2z=c^3 \end{cases}$

By using polynomial $P(X) = X^3-(x+yX+zX^2)$

we have $P(a) = a^3-(x+ay+a^2z)=0$

$P(b) = b^3-(x+by+b^2z)=0$

$P(c) = c^3-(x+cy+c^2z)=0$

$\Rightarrow a,b,c$ are the roots of P since $\deg(P)=3$. So it has at most 3 roots

Thus a,b,c are the roots of P

we have $\begin{cases} x = abc \\ y = -(ab+bc+ac) \\ z = a+b+c \end{cases}$

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(b) $\begin{cases} x + ay + a^2z = a^4 \\ x + by + b^2z = b^4 \\ x + cy + c^2z = c^4 \end{cases}$

Let $P(X) = X^4 - (x + yX + zX^2)$ then $P(a) = P(b) = P(c) = 0$.

we have $P(X) = (X-a)(X-b)(X-c)(X-d)$

Since d is a root of $P(X)$: so we can say

$P(X) = (X^2 - (a+b)X + ab)(X^2 - (c+d)X + cd)$

$= X^4 - (c+d)X^3 + cdX^2 - (a+b)X^3 + (a+b)(c+d)X^2 - cd(a+b)X + abX^2 - ab(c+d)X + abcd$

$= X^4 - (a+b+c+d)X^3 + (ab+ac+ad+bc+bd+cd)X^2 - (abc+abd+acd+bcd)X + abcd$

Since $P(X) = X^4 - zX^2 - yX - x$ so we get

$a+b+c+d = 0 \Rightarrow d = -(a+b+c)$

$x = -abcd = abc(a+b+c)$

$y = abc - (ab+ac+bc)(a+b+c)$

$z = -[ab+ac+bc - (a+b+c)^2]$

(22) Solve the system $Ax = b$, where

$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Apply crammer's rule

$\Delta = |A|_{n\times n} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{n-1} \\ 1 & 3 & 3^2 & \cdots & 3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^{n-1} \end{vmatrix} = |A^t|$

• Vandermonde matrix

$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = A^t$

$$|V| = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} \quad \begin{matrix} C_i \to C_i - x_1 C_{i-1} \\ i = 2, 3, \ldots, n \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-1} - x_1 x_2^{n-2} \\ 1 & x_3 - x_1 & x_3^2 - x_1 x_3 & \cdots & x_3^{n-1} - x_1 x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{n-2}(x_2 - x_1) \\ 1 & x_3 - x_1 & x_3(x_3 - x_1) & \cdots & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & x_n(x_n - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)\cdots(x_n - x_1) \begin{vmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} \\ 1 & x_4 & x_4^2 & \cdots & x_4^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{vmatrix}$$

$$= \prod_{k=2}^{n} (x_k - x_1) V(x_2, x_3, \ldots, x_n) = V(x_1, x_2, x_3, \ldots, x_n)$$

Since $V(x_1, x_2, \ldots, x_n) = \prod_{k=2}^{n} (x_k - x_1) V(x_2, x_3, \ldots, x_n)$

$$\Rightarrow V(x_2, x_3, \ldots, x_n) = \prod_{k=3}^{n} (x_k - x_1) V(x_3, x_4, \ldots, x_n)$$

$$\vdots$$

$$V(x_{n-2}, x_{n-1}, x_n) = \prod_{k=n-1}^{n} (x_k - x_{n-2}) V(x_{n-1}, x_n)$$

and $V(x_{n-1}, x_n) = \begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix} = x_n - x_{n-1} = \prod_{k=n}^{n} (x_k - x_{n-1})$

$$\Rightarrow |V| = \prod_{k=2}^{n} (x_k - x_1) \prod_{k=3}^{n} (x_k - x_2) \prod_{k=4}^{n} (x_k - x_3) \cdots \prod_{k=n-1}^{n} (x_k - x_{n-2}) \prod_{k=n}^{n} (x_k - x_{n-1})$$

$$\Leftrightarrow |V| = \prod_{1 \le i < j \le n} (x_j - x_i) = V(x_1, x_2, \ldots, x_n)$$

So $\Delta = |A| = \prod_{1 \le i < j \le n}^{n} (j - i) = V(1, 2, \ldots, n)$

$$= \prod_{k=2}^{n} (k-1) \prod_{k=3}^{n} (k-2) \cdots \prod_{k=n}^{n} (k - (n-1)) = (n-1)!(n-2)! \ldots 1!$$

$$= \prod_{k=1}^{n-1} k! = (n-1)! V(2, 3, \ldots, n) = \ldots = (n-1)! \ldots 2! V(n-1, n)$$

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$$\cdot\ D_1=\begin{vmatrix}1&1&1&\cdots&1\\0&2&3&\cdots&n\\0&2^2&3^2&\cdots&n^2\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&2^{n-1}&3^{n-1}&\cdots&n^{n-1}\end{vmatrix}=\begin{vmatrix}1&0&0&\cdots&0\\1&2&2^2&\cdots&2^{n-1}\\1&3&3^2&\cdots&3^{n-1}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&n&n^2&\cdots&n^{n-1}\end{vmatrix}$$

$$=2\times3\times4\times\cdots\times n\begin{vmatrix}1&2&2^2&\cdots&2^{n-2}\\1&3&3^2&\cdots&3^{n-2}\\1&4&4^2&\cdots&4^{n-2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&2^{n-1}&3^{n-1}&\cdots&n^{n-1}\end{vmatrix}=\frac{n!}{1}V(2,3,\dots,n)$$

$$\cdot\ D_2=\begin{vmatrix}1&1&1&\cdots&1\\1&0&3&\cdots&n\\1&0&3^2&\cdots&n^2\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&0&3^{n-1}&\cdots&n^{n-1}\end{vmatrix}=-\begin{vmatrix}1&1&1&\cdots&1\\0&1&3&\cdots&n\\0&1&3^2&\cdots&n^2\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&3^{n-1}&\cdots&n^{n-1}\end{vmatrix}$$

$$=-\begin{vmatrix}1&0&0&\cdots&0\\1&1&1&\cdots&1\\1&3&3^2&\cdots&3^{n-1}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&n&n^2&\cdots&n^{n-1}\end{vmatrix}=-1\times3\times\cdots\times n\begin{vmatrix}1&1&1&\cdots&1\\1&3&3^2&\cdots&3^{n-2}\\1&4&4^2&\cdots&4^{n-2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&n&n^2&\cdots&n^{n-2}\end{vmatrix}\quad \begin{matrix}C_i\to C_i-C_{i-1}\\ i=2,3,\dots,n\end{matrix}$$

$$=-\frac{n!}{2}\begin{vmatrix}1&0&0&\cdots&0\\1&(3-1)&3(3-1)&\cdots&3^{n-3}(3-1)\\1&(4-1)&4(4-1)&\cdots&4^{n-3}(4-1)\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&(n-1)&n(n-1)&\cdots&n^{n-3}(n-1)\end{vmatrix}$$

$$=-\frac{n!}{2}\underbrace{(3-1)(4-1)\cdots(n-1)}_{2\times3\times\cdots\times4\times\cdots\times(n-1)}\begin{vmatrix}1&3&3^2&\cdots&3^{n-3}\\1&4&4^2&\cdots&4^{n-3}\\1&5&5^2&\cdots&5^{n-3}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&n&n^2&\cdots&n^{n-3}\end{vmatrix}=-\frac{n!}{2}\frac{(n-1)!}{1}V(3,4,\dots,n)$$

$$\cdot\ D_3=\begin{vmatrix}1&1&1&\cdots&1\\1&2&0&\cdots&n\\1&2^2&0&\cdots&n^2\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&2^{n-1}&0&\cdots&n^{n-1}\end{vmatrix}=\begin{vmatrix}1&1&1&1&\cdots&1\\0&1&2&4&\cdots&n\\0&1&2^2&4^2&\cdots&n^2\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&2^{n-1}&4^{n-1}&\cdots&n^{n-1}\end{vmatrix}$$

$$=1\times2\times4\times5\times\cdots\times n\begin{vmatrix}1&1&1&\cdots&1\\1&2&2^2&\cdots&2^{n-2}\\1&4&4^2&\cdots&4^{n-2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&n&n^2&\cdots&n^{n-2}\end{vmatrix}\quad \begin{matrix}C_i\to C_i-C_{i-1}\\ i=2,3,\dots,n\end{matrix}$$

$$=\frac{n!}{3}\begin{vmatrix}1&0&0&\cdots&0\\1&(2-1)&2(2-1)&\cdots&2^{n-3}(3-1)\\1&(4-1)&4(4-1)&\cdots&4^{n-3}(4-1)\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&(n-1)&n(n-1)&\cdots&n^{n-3}(n-1)\end{vmatrix}$$

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$$= \frac{n!}{3}(2-1)(4-1)\dots(n-1)\begin{vmatrix} 1 & 2 & 2^2 & \dots & 2^{n-3} \\ 1 & 4 & 4^2 & \dots & 4^{n-3} \\ 1 & 5 & 5^2 & \dots & 5^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \dots & n^{n-3} \end{vmatrix} \quad \begin{array}{l} C_i \to C_i - 2C_{i-1} \\ i = 2,3,\dots,n \end{array}$$

$$= \frac{n!}{3}\,\frac{(n-1)!}{2}\begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (4-2) & 4(4-2) & \dots & 4^{n-4}(4-2) \\ 1 & (5-2) & 5(5-2) & \dots & 5^{n-4}(5-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (n-2) & n(n-2) & \dots & n^{n-4}(n-2) \end{vmatrix}$$

$$= \frac{n!}{3}\,\frac{(n-1)!}{2}(4-2)(5-2)\dots(n-2)\begin{vmatrix} 1 & 4 & 4^2 & \dots & 4^{n-4} \\ 1 & 5 & 5^2 & \dots & 5^{n-4} \\ 1 & 6 & 6^2 & \dots & 6^{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \dots & n^{n-4} \end{vmatrix}$$

$$= \frac{n!}{3}\,\frac{(n-1)!}{2}\,\frac{(n-2)!}{1}\,V(4,5,\dots,n)$$

$$\Rightarrow x_1 = \frac{\Delta_1}{\Delta} = \frac{\frac{n!}{1}V(2,3,\dots,n)}{(n-1)!\,V(2,3,\dots,n)} = n$$

$$\Rightarrow x_2 = \frac{\Delta_2}{\Delta} = \frac{-\frac{n!}{2}\frac{(n-1)!}{1}V(3,4,\dots,n)}{(n-1)!\,(n-2)!\,V(3,4,\dots,n)} = -\frac{n(n-1)}{1\times 2}$$

$$\Rightarrow x_3 = \frac{\Delta_3}{\Delta} = \frac{\frac{n!}{3}\frac{(n-1)!}{2}\frac{(n-2)!}{1}V(4,5,\dots,n)}{(n-1)!\,(n-2)!\,(n-3)!\,V(4,5,\dots,n)} = \frac{n(n-1)(n-2)}{1\times 2\times 3}$$

we get $x_i = (-1)^{i+1}\dfrac{n!}{i!\,(n-i)!} = (-1)^{i+1}C_n^i$

Therefore $x_i = (-1)^{i+1}C_n^i$.

TD2: Determinant

I2 TD2

(Determinants)

1.) Find the signatures of the following permutations.

(a). 45312

Proof: we have $\sigma_1 = 45312$; in permutation

$(4,3)$; $(4,1)$; $(4;2)$; $(5,3)$; $(5,1)$; $(5,2)$; $(3,1)$; $(3,2)$

$\Rightarrow$ # inversion $= 8$ then $\text{Sign}(\sigma_1) = (-1)^{\#\text{inversion}} = (-1)^8 = \boxed{1}$

(b). 38562147

Proof: we have $\sigma_2 = 38562147$; in permutation

$(3,2)$; $(3,1)$; $(8,5)$; $(8,6)$; $(8,2)$; $(8,1)$; $(8,4)$; $(8,7)$

$(5,2)$; $(5,1)$; $(5,4)$; $(6,2)$; $(6,1)$; $(6,4)$; $(2,1)$

$\Rightarrow$ # inversion $= 15$ then $\text{Sign}(\sigma_2) = (-1)^{\#\text{inversion}} = (-1)^{15} = \boxed{-1}$

(c). 397264581

Proof: we have $\sigma_3 = 397264581$; in permutation

$(3,2)$; $(3,1)$; $(9,7)$; $(9,2)$; $(9,6)$; $(9,4)$; $(9,5)$; $(9,8)$

$(9,1)$; $(7,2)$; $(7,6)$; $(7,4)$; $(7,5)$; $(7,1)$; $(2,1)$; $(6,4)$

$(6,5)$; $(6,1)$; $(4,1)$; $(5,1)$; $(8,1)$ $\Rightarrow$ # inversion $= 21$

then $\text{sgn}(\sigma_3) = (-1)^{\#\text{inversion}} = (-1)^{21} = \boxed{-1}$

2.) In S_8 ; write the following permutations into cyclic form, then determine their signature.

(a); 85372164

Proof: Let $\sigma_1 = 85372164 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 3 & 7 & 2 & 1 & 6 & 4 \end{pmatrix}$

$= (18476)\circ(25)\circ(3) = (18476)\circ(25) = \sigma_1' \circ \sigma_1''$

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$\Rightarrow \operatorname{Sgn}(\sigma_1) = \operatorname{Sgn}(\sigma_1') \cdot \operatorname{Sgn}(\sigma_1'') = (-1)^{\text{length}(\sigma_1)-1} \times (-1)^{\text{length}-1}$

$= (-1)^{5-1} \times (-1)^{2-1} = \boxed{-1}$.

(b). 87651234

Proof: Let $\sigma_2 = 87651234 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

$= (1845)\circ(2736) = \sigma_2' \circ \sigma_2'' \Rightarrow \operatorname{Sgn}(\sigma_2) = \operatorname{Sgn}(\sigma_2') \times \operatorname{Sgn}(\sigma_2'')$

$= (-1)^{4-1}(-1)^{4-1} = (-1)(-1) = \boxed{1}$

(C.) 12435687

Proof: Let $\sigma_3 = 12435687 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 3 & 5 & 6 & 8 & 7 \end{pmatrix}$

$= (34)\circ(78) = \sigma_3' \circ \sigma_3'' \Rightarrow \operatorname{Sgn}(\sigma_3) = \operatorname{Sgn}(\sigma_3') \times \operatorname{Sgn}(\sigma_3'')$.

$\Leftrightarrow \operatorname{Sgn}(\sigma_3) = (-1)^{2-1}(-1)^{2-1} = \boxed{1}$.

3.) In S_7, write the following permutations into normal form, then determine signature.

(a). (6437)

Proof: Let $\sigma_1 = (6437) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 3 & 5 & 4 & 6 \end{pmatrix}$.

$= 1273546$; # inversion $= 4+1 = 5$

$\Rightarrow \operatorname{Sgn}(\sigma_1) = (-1)^{\#\text{inversion}} = (-1)^5 = \boxed{-1}$

(b). (465)(735)

Proof: Let $\sigma_2 = (465)(735) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 6 & 4 & 5 & 7 \end{pmatrix}$

$\circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 6 & 7 & 5 & 3 \end{pmatrix}$

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$\Rightarrow \sigma_2 = 1246753$; # inversion $= 1+2+2+1 = 6$.

$\Rightarrow sgn(\sigma_2) = (-1)^6 = \boxed{1}$

c.) (241)(5416)

Proof: Let $\sigma_3 = (241)(5416) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 1 & 5 & 6 & 7 \end{pmatrix} \circ$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 1 & 4 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 3 & 2 & 1 & 5 & 7 \end{pmatrix} = 6432157$

inversion $= 4+3+2+2 = 11 \Rightarrow Sgn(\sigma_3) = (-1)^{11} = \boxed{-1}$

4.) For $n \in \mathbb{N}^*$, compute the signature of the following permutations.

(a). $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}$.

Proof: $\sigma = n; n-1; n-2; \cdots; 2; 1$.

$\Rightarrow$ # inversion $= (n-1)+(n-2)+(n-3)+\cdots+3+2+1 = \frac{n(n+1)}{2} - n$

$S_n =$ # inversion $= \frac{n^2+n-2n}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$

$\Rightarrow Sgn(\sigma) = (-1)^{\# inversion} = \boxed{(-1)^{\frac{n(n-1)}{2}}}$

(b). $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 & n+2 & \cdots & 2n-1 & 2n \\ 1 & 3 & 5 & \cdots & 2n-1 & 2 & 4 & \cdots & 2n-2 & 2n \end{pmatrix}$.

Proof: $\sigma = 1, 3, 5, (\ldots), 2n-1, 2, 4, \cdots, 2n-2, 2n$

inversion $= 1+2+3+\cdots+(n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$

$\Rightarrow sgn(\sigma) = (-1)^{\# inversion} = (-1)^{\frac{n(n-1)}{2}}$

5.) Prove that the transposition is an odd permutation

Ex: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} = (25)$.

$\Rightarrow sgn(\sigma) = (-1)^{length-1} = (-1)^{2-1} = -1$

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This means that σ is an odd permutation.

$\# \text{inversion}(\sigma) = 3+1+1 = 5 \Rightarrow sgn(\sigma) = (-1)^5 = -1$.

* In general properties :

Let $\sigma \in S_n$ be an transposition by interchanging two positions i-th and j-th.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots i & \cdots j & \cdots n-2 & n-1 & n \\ 1 & 2 & 3 & \cdots j & \cdots i & \cdots n-2 & n-1 & n \end{pmatrix}$$

$$\# \text{inversion}(\sigma) = (j-i) + \underbrace{1 + \cdots + 1}_{j-i-1} = (j-i) + (j-i-1) = 2(j-1)-1$$

$$j, i+1, i+2, i+3, \cdots, j-3, j-2, j-1, i \quad (i<j)$$

$\Rightarrow \# \text{inversion}(\sigma) = 2(j-1)-1$ is alway odd integer $\forall j$

Therefore $Sgn(\sigma) = (-1)^{2(j-1)-1} = -1$; then the transposition is an odd permutation.

6.) Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. We are assuming that $|A| = 3$

find $|A| = 3$

(a). $|3A| = \begin{vmatrix} 3a & 3b & 3c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{vmatrix} = 3\times3\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$

$= 3\times3\times3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3^3 \times |A| = 27\times 3 = \underline{81}$.

(b). $|A^{-1}| = \frac{1}{|A|} = \frac{1}{3}$

(c). $|2A^{-1}| = 2^3 |A^{-1}| = \frac{2^3}{|A|} = \frac{2^3}{3} = \frac{8}{3}$.

(d). $|(2A)^{-1}| = \frac{1}{|2A|} = \frac{1}{8\times3} = \frac{1}{24}$

(e).

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e.) $$\begin{vmatrix} -a & 2g & 3d \\ -b & 2h & 3e \\ -c & 2i & 3f\end{vmatrix} = \begin{vmatrix} -a & -b & -c \\ 2g & 2h & 2i \\ 3d & 3e & 3f\end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ 2g & 2h & 2i \\ 3d & 3e & 3f\end{vmatrix}$$

$$= (-2)(3)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f\end{vmatrix} = (-6)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i\end{vmatrix} = 6|A| = 6\times 3 = \underline{18}$$

7. Let A and B be invertible matrice. Show that

(a) $adj(A^{-1}) = (adj(A))^{-1}$

From $A^{-1} = \frac{1}{|A|} adj(A) \Rightarrow adj(A^{-1}) = (A^{-1})^{-1}|A^{-1}| = \frac{1}{|A|}.A$ (1)

and $adj(A) = |A|.A^{-1}$

$\Rightarrow adj(A).A = |A|I$

$\Rightarrow (adj(A))^{-1} = \frac{A}{|A|}$ (2)

From (1) and (2), we have $adj(A^{-1}) = (adj(A))^{-1}$

Thus $adj(A^{-1}) = (adj(A))^{-1}$

(b). $adj(A^t) = (adj(A))^t$

we have $adj(A^t) = |A^t|.(A^t)^{-1} = |A|.(A^{-1})^t$ (1)

and $(adj(A))^t = (|A|.A^{-1})^t = |A|.(A^{-1})^t$ (2)

From (1) and (2) $\Rightarrow adj(A^t) = (adj(A))^t$

Thus $adj(A^t) = (adj(A))^t$

(c). $adj(AB) = adj(B)adj(A)$

$adj(AB) = |AB|.(AB)^{-1} = |A|.|B|.A^{-1}.B^{-1}$

$= |B|.B^{-1}.|A|.A^{-1} = adj(B).adj(A)$

Thus $adj(AB) = adj(B)adj(A)$

9. If $A^{-1} = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & -2 \end{pmatrix}$. Compute $|adj(A)|$ and $|2A^{-1}+3adj(2A)|$.

We have $adj(A) = |A|.A^{-1}$

$$|A^{-1}| = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = -8-6-3 = -17$$

but $|A||A^{-1}| = |I| = 1 \Rightarrow |A| = \frac{1}{|A^{-1}|} = -\frac{1}{17}$

By Theorem 12 : $|adj(A)| = |A|^{n-1} = |A|^{3-1} = |A|^2 = \frac{1}{17^2}$

$|2A^{-1}+3adj(2A)| = |2A^{-1}+3\times 2^2 adj(A)| = |2A^{-1}+12.|A|.A^{-1}| = |A^{-1}(2+12\times\frac{-1}{17})|$

$= |A^{-1}|.[2(1-\frac{6}{17})]^3 = (-17)\times\frac{11^3}{17^3}\times 8 = -8\times\frac{11^3}{17^2}$

Thus. $|adj(A)| = \frac{1}{17^2}$ and $|2A^{-1}+3adj(2A)| = -8\times\frac{11^3}{17^2}$

11. Prove that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33}+b_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix}$$

By using Laplace expansion:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33}+b_{33} \end{vmatrix} = (a_{13}+b_{13})(a_{21}a_{32}-a_{22}a_{31}) - (a_{23}+b_{23})(a_{11}a_{32}-a_{12}a_{31})$$

$$+ (a_{33}+b_{33})(a_{11}a_{22}-a_{12}a_{12})$$

$$= a_{13}(a_{21}a_{32}-a_{22}a_{31}) - a_{23}(a_{11}a_{32}-a_{12}a_{31}) + a_{33}(a_{11}a_{22}-a_{12}a_{21})$$

$$+ b_{13}(a_{21}a_{32}-a_{22}a_{31}) - b_{23}(a_{11}a_{32}-a_{12}a_{31}) + b_{33}(a_{11}a_{22}-a_{12}a_{21})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix} \quad \text{(true)}$$

13. Compute the following determinants.

(a). $\begin{vmatrix} 7 & 4 & -5 \\ 10 & 3 & 21 \\ 23 & -2 & 11 \end{vmatrix} = [(7)(3)(11)+(4)(21)(23)+(-5)(10)(-2)] - [(23)(3)(-5)+(-2)(21)(7)+(11)(10)(4)]$

$= 2462$

(b). $\begin{vmatrix} 4 & 4 & 3 & 5 \\ 6 & 3 & +3 & 2 \\ 8 & 10 & 0 & 11 \\ 11 & 23 & 2 & -4 \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_3 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 5 \\ 3 & 3 & 3 & 2 \\ 8 & 10 & 0 & 11 \\ 9 & 23 & 2 & -4 \end{vmatrix} \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \\ C_4 \to C_4 - 5C_1 \end{matrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & -15 & -30 & -13 \\ 8 & -6 & -24 & -29 \\ 9 & 5 & -25 & -49 \end{vmatrix} = \begin{vmatrix} -15 & -30 & -13 \\ -6 & -24 & -29 \\ 5 & -25 & -49 \end{vmatrix}$

$= -5205$

(c). $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \begin{matrix} R_i \to R_i - R_{i-1} \\ i = 2,3,4 \end{matrix} = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c \\ 0 & 0 & 0 & d-c \end{vmatrix} = a(b-a)(c-b)(d-c)$

(d). $\begin{vmatrix} a & b & ab \\ a & c & ac \\ b & c & bc \end{vmatrix} = c\begin{vmatrix} a & b & \frac{ab}{c} \\ a & c & a \\ b & c & b \end{vmatrix} = \begin{vmatrix} a & b & \frac{ab}{c}-a \\ a & c & 0 \\ b & c & 0 \end{vmatrix} c = c\left(\frac{ab}{c}-a\right)\begin{vmatrix} a & c \\ b & c \end{vmatrix} = ac(b-c)(a-b)$

(e). $\begin{vmatrix} a & c & c & b \\ c & a & b & c \\ c & b & a & c \\ b & c & c & a \end{vmatrix} \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \\ C_4 \to C_4 - C_1 \end{matrix} = \begin{vmatrix} a & c-a & c-a & b-a \\ c & a-c & b-c & 0 \\ c & b-c & a-c & 0 \\ b & c-b & c-b & a-b \end{vmatrix} C_3 \to C_3 - C_2 = \begin{vmatrix} a & c-a & 0 & b-a \\ c & a-c & b-a & 0 \\ c & b-c & a-b & 0 \\ b & c-b & 0 & a-b \end{vmatrix}$

$= (b-a)^2 \begin{vmatrix} a & c-a & 0 & 1 \\ c & a-c & \cancel{b-a}\,1 & 0 \\ c & b-c & \cancel{a-b}\,-1 & 0 \\ b & c-b & 0 & -1 \end{vmatrix} R_4 \to R_4 + R_1 = (b-a)^2 \begin{vmatrix} a & c-a & 0 & 1 \\ c & a-c & 1 & 0 \\ c & b-c & -1 & 0 \\ a+b & c-b & 0 & 0 \end{vmatrix} = (b-a)^2 \begin{vmatrix} c & a-c & 1 \\ c & b-c & -1 \\ a+b & 2a-b-a & 0 \end{vmatrix} R_2 \to R_2 + R_1$

$= (b-a)^2 \begin{vmatrix} c & a-c & 1 \\ 2c & a+b-2c & 0 \\ a+b & 2c-a-b & 0 \end{vmatrix} = -(b-a)^2(-a^2-b^2-2ab+4c^2)$

15. $\Delta_n(x) = \begin{vmatrix} a_1+x & a+x & \cdots & a+x \\ b+x & a_2+x & \ddots & \vdots \\ \vdots & \ddots & \ddots & a+x \\ b+x & \cdots & b+x & a_n+x \end{vmatrix}_n, \ (a \neq b)$

(a). Show that $\Delta_n(x)$ is of the form $\Delta_n(x) = \alpha x + \beta$

$$\Delta_n(x) = \begin{vmatrix} \alpha_1 + x & a+x & \cdots & a+x \\ b+x & \alpha_2 + x & \ddots & \vdots \\ \vdots & \ddots & \ddots & a+x \\ b+x & \cdots & b+x & \alpha_n + x \end{vmatrix}_n \begin{matrix} L_i \to L_i - L_1 \\ i = 2,3,\dots,n \end{matrix} = \begin{vmatrix} \alpha_1 + x & a+x & \cdots & a+x \\ b-\alpha_1 & \alpha_2 - a & & (0) \\ \vdots & (b-a) & \ddots & \\ b-\alpha_1 & & & \alpha_n - a \end{vmatrix}_n$$

since x at the first row then by deutimation we have $\Delta_n(x) = \alpha x + \beta$ (*)

Thus : $\Delta_n(x)$ is of the form $\Delta_n(x) = \alpha x + \beta$

(b). Compute $\Delta_n(x)$ then deuce the value of $\Delta_n(0)$

we have $\Delta_n(-a) = \prod_{n=1}^{k} (\alpha_n - a) = P$, $\Delta_n(-b) = \prod_{n=1}^{k} (\alpha_n - b) = q$

Substitute $n = -a, -b$ in (*)

$$\begin{cases} \Delta_n(-a) = -\alpha a + \beta = P \ (1) \\ \Delta_n(-b) = -\alpha b + \beta = q \ (2) \end{cases} \Rightarrow \alpha(a-b) = q - P \Rightarrow \alpha = \frac{q-P}{(a-b)}$$

By (1) $\Rightarrow \beta = P + \alpha a = P + \frac{a}{a-b}(q-P)$

Thus. $\Delta_n(x) = \frac{x}{a-b}(q-P) - \frac{b}{a-b}P + \frac{a}{a-b}q$

17 Let $A, B, C, D \in M_n(K)$ with D is invertible and $CD = DC$ commutent. Show that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

we have $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D & 0 \\ -C & I \end{pmatrix} = \begin{pmatrix} AB - BC & B \\ 0 & D \end{pmatrix}$

$\Rightarrow \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det\begin{pmatrix} D & 0 \\ -C & I \end{pmatrix} = \det\begin{pmatrix} AB - BC & B \\ 0 & D \end{pmatrix}$

$\Rightarrow \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}\det(D) = \det(AB - BC)\det(D)$

since D is invertible. It mean that $\det(D) \neq 0$. That can divide $\det(D)$ at the both side.

Thus : $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AB - BC)$

19. Compute the following determinants.

(a). $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 1 & 5 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = X$, $C = \begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \Rightarrow CD = DC$

$\Rightarrow |X| = |AD - BC| = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -3 & 2 \end{pmatrix} \right| = \begin{vmatrix} a+8 & b-11 \\ c+10 & d-18 \end{vmatrix}$

$= (a+8)(d-18) - (c+10)(b-11)$

(b). $\begin{vmatrix} a & b & 1 & 3 \\ c & d & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{vmatrix} \begin{matrix} C_1 \to C_1 - 2C_3 \\ C_3 \to C_3 - C_1 \end{matrix} = \begin{vmatrix} a-2 & b-1 & 1 & 3 \\ c-4 & d-2 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} a-2 & b-1 & 3 \\ c-4 & d-2 & 4 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 4R_3 \end{matrix}$

$= \begin{vmatrix} a-2 & b-7 & 0 \\ c-6 & d-8 & 0 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a-5 & b-7 \\ c-6 & d-8 \end{vmatrix} = (a-5)(d-8) - (c-6)(b-7)$

21. Let $a,b,c \in \mathbb{C}$ and $P(X) = X^3 - (x + yX + zX^2)$. Solve the following system by using polynomial P.

(a). $\begin{cases} x + ay + a^2z = a^3 \\ x + by + b^2z = b^3 \\ x + cy + c^2z = c^3 \end{cases}$

we have $P(a) = a^3 - (x + ay + a^2z) = 0$, $P(b) = b^3 - (x + by + b^2z) = 0$, $P(c) = c^3 - (x + cy + c^2z) = 0$

$\Rightarrow a,b,c$ are the roots of P. since $\deg(P) = 3$, so it has at most 3 roots

Thus, a,b,c are the roots of P, we has $\begin{cases} x = abc \\ y = -(ab + bc + ac) \\ z = a + b + c \end{cases}$

(b). $\begin{cases} x + ay + a^2z = a^4 \\ x + by + b^2z = b^4 \\ x + cy + c^2z = c^4 \end{cases}$

Let $P(x) = -(x + yX + zX^2) + X^4$

we have $P(a) = a^4 - (x + ay + a^2z) = 0$, $P(b) = b^4 - (x + by + b^2z) = 0$, $P(c) = c^4 - (x + cy + c^2z) = 0$

showing a,b,c are the roots of P since $\deg(P) = 4$. So it has at most 4 roots,

Hence a,b,c are the roots of P

let d be another roots of P

We have $\begin{cases} a + b + c + d = 0 \Rightarrow d = -(a+b+c) \\ ab + ac + ad + bc + bd + cd = -z \quad (1) \\ abc + abd + acd + bcd = y \quad (2) \\ abcd = -x \quad (3) \end{cases}$

Thus: $\begin{cases} (3): x = abc(a+b+c) \\ y = abc - (a+b+c)(ab+ac+bc) \\ z = -(ab+ac+bc) + (a+b+c)^2 \end{cases}$

TD3: Vector Space

I2-TD1

(Vector Space)

1. For any $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$, we define and additional in $\mathbb{R}^3$ and scalar multiplication with element in $\mathbb{R}$ by:

- $x+y=(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$
- $\alpha x=\alpha(x_1,x_2,x_3)=(\alpha x_1,\alpha x_2,\alpha x_3)$.

① $x+y=(x_1+y_1,x_2+y_2,x_3+y_3)\in\mathbb{R}^3$

② $x+y=(x_1+y_1,x_2+y_2,x_3+y_3)=y+x$

③ $(x+y)+z=x+(y+z)$

④ $\exists\ 0\in\mathbb{R}^3:0+x=x$

⑤ $\exists\ x\in\mathbb{R}^3:-x\in\mathbb{R}^3:x+(-x)=0$

⑥ $\alpha x=(\alpha x_1,\alpha x_2,\alpha x_3)$

⑦ $\alpha(x+y)=\alpha x+\alpha y$

⑧ $(\alpha+\beta)\times x=\alpha x+\beta x$, $\beta\in\mathbb{R}$

⑨ $(\alpha\beta)x=\alpha(\beta x)$

⑩ $1\times x=x$, $1\in\mathbb{R}$

Since $\mathbb{R}^3$ follow ten condition above:

Therefore $\mathbb{R}^3$ is a vector space.

2. We have $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$

and we have $x\oplus y=(x_1,x_2)\oplus(y_1,y_2)=(x_1+y_1,2x_2+y_2)$

and $\alpha * x=\alpha*(x_1,x_2)=(\alpha x_1,\alpha x_2)$

Is $(\mathbb{R}^2,\oplus,*)$ a vector space over $\mathbb{R}$?

Let $z=(z_1,z_2)$

we have

① $x \oplus y = (x_1+y_1, 2x_2+y_2) \in \mathbb{R}^2$

② $x \oplus y = y \oplus x$

③ $(x \oplus y) \oplus z = (x_1+y_1, 2x_2+y_2) \oplus (z_1, z_2)$

$= (x_1+y_1+z_1, 4x_2+2y_2+z_2) \quad (1)$

$x \oplus (y \oplus z) = (x_1, x_2) \oplus (y_1+z_1, 2y_2+z_2)$

$= (x_1+y_1+z_1, 2x_2+2y_2+z_2) \quad (2)$

by $1 \neq 2 \iff (x \oplus y) \oplus z \neq x \oplus (y \oplus z)$

Thus $(\mathbb{R}^2, \oplus, *)$ is not a vector space.

3. we have $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$

$x \oplus y = (x_1+y_1, x_2+y_2) = (x_1, x_2) \oplus (y_1, y_2)$

~~$x * y = (x_1, x_2) * (y_1, y_2)$~~

$\alpha * x = \alpha * (x_1, x_2) = (2x_1, \alpha x_2)$

Let $z = (z_1, z_2)$

① $x \oplus y = (x_1+y_1, x_2+y_2) \in \mathbb{R}^2$

② $x \oplus y = (x_1+y_1, x_2+y_2) = y \oplus x$

③ $(x \oplus y) \oplus z = (x_1+y_1+z_1, x_2+y_2+z_2) = x \oplus (y \oplus z)$

④ $\exists\, 0 \in \mathbb{R}^3 : 0 + x = x$

⑤ $\exists\, x \in \mathbb{R}^2$ & $-x \in \mathbb{R}^2$ $x + (-x) = 0$

⑥ $\alpha * x = \alpha(x_1, x_2) = (2x_1, \alpha x_2) \in \mathbb{R}^2$

⑦ $\alpha(x \oplus y) = \dots (2x_1+2y_1, \alpha x_2+\alpha y_2) \quad (1)$

$\alpha x + \alpha y = (2x_1+2y_1, \alpha x_2+\alpha y_2) \quad (2)$

by (1) & (2) $\iff \alpha * (x \oplus y) = \alpha * x \oplus \alpha * y$

⑧. $(\alpha \oplus \beta) * x = (4x_1, \alpha x_1 + \alpha x_2)$ (1)

~~$(\alpha) * (\beta$~~

$\alpha x \oplus \beta * x = (4x_1, \alpha x_1 + \beta x_2)$ (2)

by (1) & (2) $(\alpha \oplus \beta) * x = \alpha x \oplus \beta * x$

⑨. $(\alpha\beta) * x = \alpha * (\beta * x) = (4x_1, \alpha\beta x_2)$

⑩. $1 * x = x$

by $\mathbb{R}^2$ follow ten conditions above

Thus $(\mathbb{R}^2, \oplus, *$ is avector space)

4. Let $V = \{(x,1) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ For any $u = (x,1), v(y,1) \in V$ and $\alpha \in \mathbb{R}$, we define

$$u \oplus v = (x+y, 1)$$

$$\alpha * v = (\alpha x, 1)$$

show that $(V, \oplus, *)$ is a vector space.

Answer:

let $w = (z,1)$

①. $u \oplus v = (x+y, 1) \in V$

②. $u \oplus v = v \oplus u$

③. $(u \oplus v) \oplus w = (x+y+z, 1) = u \oplus (v \oplus w)$

④. $\exists 0 \in V, u \in V, 0 \oplus u = U$

⑤. $\exists v \in V, \exists -v \in V, v + (-v) = 0$

⑥. $\alpha * v = (\alpha, 1)$

⑦. $\alpha * (u \oplus v) = (\alpha + \alpha, 1) = \alpha * u \oplus \alpha * v$

⑧ $(\alpha+\beta) * v = ((\alpha+\beta)x, 1) = \alpha * v \oplus \beta * v$

⑨. $(\alpha\beta) * v = \alpha * (\beta * v)$

⑩. $1 * v = v$

by V follow 10 conditions above

Thus V is a vector space.

5 Show that $(\mathbb{R}_+, \oplus, *, \mathbb{R})$ is a vector space where

$$x \oplus y = xy \qquad \alpha * x = x^\alpha$$

Answer

Let $z \in \mathbb{R}$

①. $x \oplus y = xy \in \mathbb{R}$.

②. $x \oplus y = xy = y \oplus x$.

③. $(x \oplus y) \oplus z = xyz = x \oplus (y \oplus z)$

④ $\exists \theta^\alpha \in \mathbb{R}.\ \exists x \in \mathbb{R} \quad 0 \oplus x = x$

⑤ $\exists x \in \mathbb{R}.\ \exists -x \in \mathbb{R} \quad x \oplus (-x) = x\frac{1}{x} = 1 = \theta^\alpha$

⑥. $\alpha * x = x^\alpha \in \mathbb{R}$

⑦. $\alpha * (x \oplus y) = \alpha * (xy) = (xy)^\alpha = x^\alpha . y^\alpha = \alpha * x \oplus \alpha * y$

⑧. $(\alpha+\beta) * x = x^{\alpha+\beta} = x^\alpha x^\beta = \alpha * x \oplus \beta * x$

⑨. $(\alpha\beta) * x = x^{\alpha\beta} = \alpha * (\beta * x)$

⑩. $1 * x = x^1 = x$

by $(\mathbb{R}_+, \oplus, *, \mathbb{R})$ follow 10 conditions above

Thus $(\mathbb{R}_+, \oplus, *, \mathbb{R})$ is a vector space.

6. For any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ and let us define:

$x \oplus y = (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1)$

$\alpha * x = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1, \alpha x_3 + \alpha - 1)$

Show that $(\mathbb{R}^3, \oplus, *)$ is a vector space.

Answer.

Let $z = (z_1, z_2, z_3)$

① $x \oplus y = (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1) \in \mathbb{R}^3$

② $x \oplus y = y \oplus x$

③ $+ (x \oplus y) \oplus z = (x_1 + y_1 + z_1 + 2, x_2 + y_2 + z_2 + 2, x_3 + y_3 + z_3 + 2)$ (1)

$+ x \oplus (y \oplus z) = (x_1 + y_1 + z_1 + 2, x_2 + y_2 + z_2 + 2, x_3 + y_3 + z_3 + 2)$ (2)

(1) & (2) $\Rightarrow (x \oplus y) \oplus z = x \oplus (y \oplus z)$

④ $\exists E \in \mathbb{R}^3$, $E = (-1, -1, -1)$

$0 \oplus x = (x_1, x_2, x_3) \oplus (-1, -1, -1) = x$

⑤ $\exists x \in \mathbb{R}^3$, $\exists -v \in \mathbb{R}^3$, $v = (-x_1 - 2, -x_2 - 2, -x_3 - 2) \in \mathbb{R}^3$

$x \oplus (-v) = (-1, -1, -1) = e.$

⑥ $\alpha * x = \alpha * (x_1, x_2, x_3) = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1, \alpha x_3 + \alpha - 1) \in \mathbb{R}^3$

⑦ $\alpha * (x \oplus y) = \alpha * (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1)$

$= (\alpha(x_1 + y_1 + 1) + \alpha - 1, \alpha(x_2 + y_2 + 1) + \alpha - 1, \alpha(x_3 + y_3 + 1) + \alpha - 1)$

$= (\alpha x_1 + \alpha y_1 + 2\alpha - 1, \alpha x_2 + \alpha y_2 + 2\alpha - 1, \alpha x_3 + \alpha y_3 + 2\alpha - 1)$

$= (\alpha x_1 + \alpha - 1 + \alpha y_1 + \alpha - 1 + 1, \alpha x_2 + \alpha - 1 + \alpha y_2 + \alpha - 1 + 1, \alpha x_3 + \alpha - 1 + \alpha y_3 + \alpha - 1 + 1)$.

$= (\alpha * (x_1, x_2, x_3)) \oplus (\alpha * (y_1, y_2, y_3))$

$= (\alpha * (x)) \oplus (\alpha * y)$

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⑧. $(\alpha+\beta)*x = (\alpha+\beta)*(x_1, x_2, x_3)$

$= ((\alpha+\beta)x_1 + \alpha+\beta-1, (\alpha+\beta)x_2 + \alpha+\beta-1, (\alpha+\beta)x_3 + \alpha+\beta-1)$

$= (\alpha * x) \oplus (\beta * x)$

⑨. $(\alpha\beta)*x = (\alpha\beta x_1 + \alpha\beta - 1, \alpha\beta x_2 + \alpha\beta - 1, \alpha\beta x_3 + \alpha\beta - 1)$

$= \alpha*(\beta * x)$.

⑩. $\exists\, 1 \in \mathbb{R}^3$

$1 * x = (x_1 + 1 - 1, x_2 + 1 - 1, x_3 + 1 - 1) = x$.

Thus $(\mathbb{R}^3, \oplus, *)$ is a vector space.

7. Determine if the following set are subspace of $\mathbb{R}^3$ under the usual operation:

(a) $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

Since $0 \in W_1$ $(0+0+0=0)$ (1)

Let $x \in W_1$ and $y \in W_1$, $\alpha \in \mathbb{R}$

Then $x + \alpha y = (x_1 + \alpha y_1, x_2 + \alpha y_2, x_3 + \alpha y_3)$

and $x_1 + \alpha y_1 + x_2 + \alpha y_2 + x_3 + \alpha y_3 = 0$

Thus W_1 is the subspace of $\mathbb{R}^3$

(b) $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 2x_2 \text{ and } x_3 = -x_2\}$

Let $x, y \in W_2$, $\alpha \in \mathbb{R}$

Then $x + \alpha y = (x_1 + \alpha y_1, x_2 + \alpha y_2, x_3 + \alpha y_3)$

$x_1 + \alpha y_1 + x_2 + \alpha y_2 + x_3 + \alpha y_3 = \cancel{2x_2 + 2\alpha y_2}\ 0$

Thus W_2 is the subspace of $\mathbb{R}^3$

(c). $W_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_3^2\}$

Let $x_1 = (1, 0, 1)$, $x_2 = (3, 0, 5)$

$x_1 + x_2 = (4, 0, 6)$ where $6^2 \neq 4$.

Thus W_3 is not subspace of $\mathbb{R}^3$.

8. Determine if the following set if is the subspace of $F_{\mathbb{R}}^{\mathbb{R}} = \{f : f : \mathbb{R} \to \mathbb{R}\}$ under the usual operations.

(a). $F_1 = \{f \in C^1 : f'(1) + f'(2) = 0\}$

F_1 contain to $\vec{0}$ (1).

Let $f, g \in F_1$, $\alpha \in \mathbb{R}$.

So $(f + \alpha g)' = f'(1) + \alpha g'(1) = -f'(2) - \alpha g'(2)$

$\Rightarrow (f + \alpha g)'(1) + (f + \alpha g)'(2) = 0$ (2)

by (1) & (2)

Therefore F_1 is the subspace of $F_{\mathbb{R}}^{\mathbb{R}}$.

(b) $F_2 = \{f : f(0) + f(1) = 3\}$

$0 \notin F_2$

Therefore F_2 is not the subspace of $F_{\mathbb{R}}^{\mathbb{R}}$

(c). $F_3 = \{f : f(0) = 3\}$

$0 \notin F_3$

Therefore F_3 is not the subspace of $F_{\mathbb{R}}^{\mathbb{R}}$.

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(d). $F_4 = \{f : f(1) = 2f'(1)\}$

Let $f, g \in F_4$, $\alpha \in \mathbb{R}$

$(f + \alpha g)(1) = f(1) + \alpha g(1) = 2f'(1) + 2\alpha g'(1)$

$= 2(f + \alpha g)'(1)$

$\Rightarrow (f + \alpha g) \in F_4$ (1)

$0 \in F_4$ (2)

Therefore F_4 is the subspace of $F_{\mathbb{R}}^{\mathbb{R}}$

9 Determine if the following sets are the subsp

of $\mathbb{R}^{n\times n}$ under the usual operations:

(a). The set of all upper triangular matrix:

Let $S = \{A \in M_n(\mathbb{R}) : A = (a_{ij})_n, a_{ij} = 0 \ \forall \ i > j\}$

Then $0 \in S$ (1)

Let $A, B \in S$, $\alpha \in \mathbb{R}$

Then $A + \alpha B = (a_{ij} + \alpha B_{ij})_n \in S$ (2)

by (1) & (2)

Therefor S is the subspace of $\mathbb{R}^{n\times n}$.

(b). The set of symetric matrice.

Let $S = \{A \in M_n(\mathbb{R}) : A^T = A\}$

$0 \in S$ (1) Let $A, B \in S$, $\alpha \in \mathbb{R}$

Then $(A + \alpha B)^T = A^T + \alpha B^T = A + \alpha B \in S$ (2)

by (1) & (2)

Therefore S is the subspace of $\mathbb{R}^n$

(c). The set of orthogonal matrices.

Let $S = \{A \in M_n(\mathbb{R}) : A^{-1} = A^T\}$

we have $A^T = A^{-1} \Rightarrow |A| \neq 0 \Rightarrow 0 \notin S$

Therefore S is not the subspace of $\mathbb{R}^{n\times n}$

(d). The set of all matrices A such that $A^2 = 0$.

Let $S = \{A \in M_n(\mathbb{R}) : A^2 = 0\}$

Let $A, B \in S$, $\alpha \in \mathbb{R} \Rightarrow (A + \alpha B)^2 = A^2 + \alpha^2 B^2 + \alpha AB + \alpha BA$

$\neq 0$ from some A, B.

Thus S is not the subspace of ~~A~~ $\mathbb{R}^{n\times n}$.

10). Let G and H be the two subspaces of V.

show that $G \cap H$ is also a subspace of V

Answer:

since $G, H < V \Rightarrow 0 \in G, 0 \in H$ then $0 \in G \cap H$ (1)

Let $u, v \in G \cap H$, $\alpha \in \mathbb{R}$.

$\Rightarrow u + \alpha v \in G \cap H$ because $\begin{cases} u + \alpha v \in G \\ u + \alpha v \in H \end{cases}$

Therefor $G \cap H$ is also a subspace of V.

11 Let G and H is the two subspace of V.
show that $(G \cup H < V) \Leftrightarrow (G \subset F$ or $F \subset G)$

+ If $G \cup H = H \Rightarrow G \subset H$ (1)
+ If $G \cup H = G \Rightarrow H \subset G$

If $G \subset H$ or $H \subset G \Rightarrow \begin{cases} G \cup H = H < V \\ G \cup H = G < V \end{cases}$ (2)

From (1) and (2).
Therefore $(G \cup H < V) \Leftrightarrow (G \subset F$ or $F \subset G)$

12. Let $G, H < V$ show that $G + H < V$

Since $\begin{cases} 0 \in G \\ 0 \in H \end{cases} \Rightarrow 0 \in G + H$ (1)

Let $u, v \in G$, $w, z \in H$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.
$\Rightarrow (u + \alpha v) + (w + \alpha z)\beta \in G + H$.
where $u + \alpha w \in G + H$, $v + \alpha z \in G + H$
Therefore $G + H < V$

13. Let $G, H < V$ show that: $G + H = G \cap H \Leftrightarrow G = H$.

If: $\begin{cases} G \subset G + H = G \cap H \Rightarrow G \subset H \\ H \subset G + H = G \cap H \Rightarrow H \subset G \end{cases} \Rightarrow G = H$. (1)
If $G = H \Rightarrow G + H = G \cap H$ (2)
by (1) & (2)
Therefore $G + H = G \cap H = G = H$.

14. Let $F, G, H \leq V$ satisfy:

$F \cap G = F \cap H$, $F+G = F+H$, and $G \subset H$.

show that $G = H$.

Answer

We have $F+G = F+H$, Let $x \in H$, $G \subset H$ (I)

$\Rightarrow x \in F+G = F+H$ or $x = f+g$ $f \in F$, $g \in G$

or $f = x - g$ (1).

but $G \subset H \Rightarrow g \in H$. (2).

from (1) & (2) $\Rightarrow f \in G$

Since $x = f+g \Rightarrow x \in G$

Then $H \subset G$ (II).

(I) (II) $H = G$.

Therefore $H = G$.

15) If $S = \{v_1, v_2, \dots v_n\}$ is a subset of vector space V, show That span(S) is the smallest subspace of V counting S's

Answer:

Let $S' \subset V$ then S' is closed under Linear combination

suppose that $S \subset S'$

$\Rightarrow$ exist a bigger subset than span(S).

Thus span(S) is the smallest supspace.

16). Determine the values of x and y so that the vector $(2,3,x,y)$ is an element of the subspace of $\mathbb{R}^4$ span by $(2,-1,3,5)$ and $(1,3,7,2)$

Answer:

We have:

$$\begin{pmatrix}2\\3\\x\\y\end{pmatrix} = \alpha_1\begin{pmatrix}2\\-1\\3\\5\end{pmatrix} + \alpha_2\begin{pmatrix}1\\3\\7\\2\end{pmatrix}$$

$$\Rightarrow \begin{cases}2\alpha_1 + \alpha_2 = 2\\ -\alpha_1 + 3\alpha_2 = 3\\ x = 3\alpha_1 + 7\alpha_2\\ y = 5\alpha_1 + 2\alpha_2\end{cases} \Rightarrow \begin{cases}\alpha_1 = 3/7\\ \alpha_2 = 8/7\\ x = 65/7\\ y = 31/7.\end{cases}$$

Therefore: $x = \frac{65}{7}$, $y = \frac{31}{7}$.

17) Which of the following are spanning set of $\mathbb{R}^4$?

(a) $\{(1,1,1,2),(1,0,1,0),(2,1,-2,3)\} = S$

$\dim(S) = 3 < \dim\mathbb{R}^4 \Rightarrow S$ can not span $\mathbb{R}^4$

Thus S does not span $\mathbb{R}^4$

(b). $\{(1,1,2,1),(2,3,1,2),(2,1,2,1),(1,2,1,2)\} = S$

consider $\begin{pmatrix}1&1&2&1\\2&3&1&2\\2&1&2&1\\1&2&1&2\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&2&1\end{pmatrix}$

Then S is LI.

Thus S is spaining set for $\mathbb{R}^4$.

(c). $\{(0,2,1,0),(1,-1,0,1),(0,0,-2,1),(1,1,-1,2)\}$

consider $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

since $\mathbb{R}^4$ span at least 4 L.I vectors.

Thus S does not span.

(d). $\{(2,1,2,1),(2,3,1,2),(3,1,2,-1),(1,-2,1,-3),(1,0,0,-2)\}$

consider $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & -1 \\ 1 & -2 & 1 & -3 \\ 1 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & -7 & -1 & 8 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 0 & -7 & -1 & 8 \\ 0 & -2 & 0 & -1 \\ 0 & 2 & -1 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

There are 3 L.I vectors.

Therefore S does not spun in $\mathbb{R}^4$

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e. $\{(1,1,-1,1),(2,3,-1,2),(3,1,2,1),(1,-2,1,3),(1,-1,1,-2)\}$

Consider: $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -1 & 2 \\ 3 & 1 & 2 & 1 \\ 1 & -2 & 1 & 3 \\ 1 & -1 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & -3 & 2 & 2 \\ 0 & -2 & 2 & -3 \end{pmatrix}$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -5 & 2 \\ 0 & -5 & 0 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & -7 & 2 \\ 0 & -5 & 0 & 2 \end{pmatrix}$$

Thereare more then 4 L.I. vectors.

Therefore S is spanning set for $\mathbb{R}^4$.

18. Determine weather the following vectors are L.I

(a). $(1,1,0),(2,1,0),(2,3,4)$

consider $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 4 \end{pmatrix}$

Thus : These vectors are L.I.

(b). $(1,1,-1,2),(1,2,1,1),(2,1,2,3)$.

consider $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 4 & -1 \end{pmatrix}$

There are non-zero 3rd row.

Therefore They are L. I.

(c) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

consider $\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Therefore These vectors are L.I.

(d). e^t, e^{2t}, e^{3t}

consider equation $\alpha_1 e^t + \alpha_2 e^{2t} + \alpha_3 e^{3t} = 0$

$\alpha_1 e^t + 2\alpha_2 e^{2t} + 3\alpha_3 e^{3t} = 0$

$\alpha_1 e^t + 4\alpha_2 e^{2t} + 9\alpha_3 e^{3t} = 0$

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consider : $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Thus e^t, e^{2t}, e^{3t} are L.I.

(e). $\cos^2 t$, $\sin^2 t$, $\cos 2t$

consider equation $\alpha_1 \cos^2 t + \alpha_2 \sin^2 t + \alpha_3 \cos 2t = 0$

$\alpha_1 \cos^2 t + \alpha_2 \sin^2 t + \alpha_3 \cos^2 t - \alpha_3 \sin^2 t = 0$

Thus $\alpha_1 = -\alpha_3$, $\alpha_2 = +\alpha_3$ are solution

Therefore $\cos^2 t$, $\sin^2 t$, $\cos 2t$ are L.D.

19. Let $f(x) = \ln(1+x)$, $x \in \mathbb{R}_+$ Let $f_1 = f$, $f_2 = f \circ f$, $f_3 = f \circ f \circ f$

show that the set $\{f_1, f_2, f_3\}$ is L.I.

Answer.

we have $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^4)$.

Then $f \circ f(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^2 + \frac{1}{3}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^3 + o$

$= x - x^2 + \frac{5}{6}x^3 + o(x^4)$.

$f \circ f \circ f(x) = x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + o(x^4)$.

consider $|A| = \begin{vmatrix} 1 & -\frac{1}{2} & \frac{1}{3} \\ 1 & -1 & 5/6 \\ 1 & -\frac{3}{2} & 1/6 \end{vmatrix} \neq 0$

Then $\{f_1, f_2, f_3\}$ is L.I.

20. Condition on α_i for the set $\{y_1 \dots y_n\}$ is L.I.

Consider $\sum_{i=1}^{n} \beta_i y_i = 0 \Rightarrow \sum_{i=1}^{n} \beta_i (x_i + u) = 0$

$$\Rightarrow \sum_{i=1}^{n} \beta_i \alpha_1 + \sum_{i=1}^{n} \beta_i \left(\sum_{i=1}^{n} a_i x_i\right) = 0$$

$$(\beta_1 x_1 + \beta_1 \alpha_1 x_1) + \beta_1 x_1 + \beta_2 x_2 + \alpha_1 \beta_1 x_1 + \alpha_2 \beta_2 x_2 + \dots + \dots = 0$$

$$\Rightarrow \alpha_i = -\frac{\beta_i}{\sum_{i=1}^{n} \beta_i}$$

Therefore $\alpha_i = -\dfrac{\beta_i}{\sum_{i=1}^{n} \beta_i}$, $1 \leq i \leq n$.

21. Study the Linear independent set of ~~V and~~ ~~$\alpha_1 \dots, \alpha_n \in \mathbb{R}$.~~ $\{y_1, y_2 \dots y_n\}$

Let $\sum_{k=1}^{n} \alpha_k y_k = 0$

$$\Leftrightarrow \sum_{k=1}^{n-1} \alpha_k (x_k + x_{k+1}) + \alpha_n (x_n + x_1) = 0$$

$$(\alpha_1 + \alpha_n) x_1 + (\alpha_1 + \alpha_2) x_2 + \dots (\alpha_{n-1} + \alpha_n) x_n = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_n = 0 \\ \alpha_1 + \alpha_2 = 0 \\ \vdots \\ \alpha_{n-1} + \alpha_n = 0 \end{cases}$$

Consider $\begin{vmatrix} 1 & & & 1 \\ 1 & 1 & (0) & \\ & 1 & 1 & \\ (0) & & \ddots & \\ & & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{vmatrix} + (-1)^{n-1} \begin{vmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{vmatrix}$

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$= 1+(-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

Thus $\{y_1, \dots y_n\}$ is $\begin{cases} L.D & \text{if } n \text{ is odd} \\ L.I & \text{if } n \text{ is even.} \end{cases}$

22. Determine wich of following sets are bases for $\mathbb{R}^3$ if the vectors in $\mathbb{R}^3$, or for $\mathbb{R}^4$ if the vector in $\mathbb{R}^4$, or for $\mathbb{R}^5$ $P_2(\mathbb{R})$ if the vectors in $P_2(\mathbb{R})$

(a). $(1,-1,2), (2,1,0), (2,3,4)$.

Consider: $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 2 & 3 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Thus The set is L.I $\Rightarrow$ They are spaning of $\mathbb{R}^3$

(b). $(2,-1,2), (2,-1,1), (0,1,1), (5,2,7)\}$

The set is L.D.

$\Rightarrow$ Therefore: It's not basis for $\mathbb{R}^3$

(c). $(1,1,-1,1), (2,3,-1,2), (3,1,-2,1), (1,2,-1,3)$.

Check $|A| = \begin{vmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -1 & 2 \\ 3 & 1 & -2 & 1 \\ 1 & 2 & -1 & 3 \end{vmatrix} \neq 0$

$\Rightarrow$ The set is L.I $\Rightarrow$ It span $\mathbb{R}^4$

Thus it is a basis for $\mathbb{R}^4$.

(e) $1+2x+x^2, 3+x^2, x+x^2$

check determinant of coeffients.

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(-1)-2(3)+1(3) = -4 \neq 0$$

$\Rightarrow$ The set is L.I

Therefore The set is basic of $P_2(\mathbb{R})$

(f). $1-2x-2x^2, \cancel{22}\ -2+3x-x^2, 1-x-6x^2$

Consider $\begin{vmatrix} 1 & -2 & -2 \\ -2 & 3 & -1 \\ 1 & -1 & -6 \end{vmatrix} \neq 0$

$\Rightarrow$ The set is L.I.

Therefore The set is basis for $P_2(\mathbb{R})$

23. Find the basis and dimension of F.

we have $F = \{(w,x,y,z) \in \mathbb{R}^4 \mid w = 2x - y \text{ and } z = w+x+y\}$

$$= x(2,1,0,3) + y(-1,0,1,0)$$

$$= \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\} = \text{span } S.$$

Then S is a basis of $\mathbb{R}^4$ and $\dim F = 2$.

2b. (a). show that F and G are the subspace of $\mathbb{R}^4$

we have F and $G \subset \mathbb{R}^4$

$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid x - 2y = 0$ and $y - 2z = 0\}$

$G = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + z = 0$ and $y + t = 0\}$

we have $0 \in F, G$.

for F let $U = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_1 = 2\alpha_2$, $\alpha_2 = 2\alpha_4$

$V = (\beta_1, \beta_2, \beta_3, \beta_4)$, $\beta_1 = 2\beta_2$, $\beta_2 = 2\beta_4$

$\alpha \in \mathbb{R}$

Then $U + \alpha V = (\alpha_1 + \alpha\beta_1, \alpha_2 + \alpha\beta_2, \alpha_3 + \alpha\beta_3, \alpha_4 + \alpha\beta_4)$

Then $\alpha_1 + 4\beta_4 = 4(\alpha_4 + \alpha\beta_4)$

$\alpha_1 + \alpha\beta_2 = 2(\alpha_4 + \alpha\beta_4)$.

Then F is the subspace of $\mathbb{R}^4$

and similarly G is the subspace of $\mathbb{R}^4$.

(b). Find the basis for F, G and F+G

$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid x - 2y = 0, y = 2z\}$

$= \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, $\dim(F) = 2$.

$G = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + z = 0$ and $y + t = 0\}$

$= \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right\}$ $\dim(G) = 2$.

$F + G = \left\{\begin{pmatrix} 2 \\ 1 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

(c) since $\dim(F+G) = 4 \Rightarrow F + G = \mathbb{R}^4$.

25. let V be the vector space of n-square matrices over a field $\mathbb{R}$. show that $V = U \oplus W$ where V and W are the subspaces of symmetric and antisymmetric matrices, respectively.

Answer:

we have $U = \{A \in M_n(\mathbb{R}) : A^T = A\}$

$V = \{A \in M_n(\mathbb{R}) : A = -A^T\}$

obviouly $U \cap V = \{0\}$ (1).

and we have $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Let $B = A + A^T \Rightarrow B^T = A + A^T = B$.

$C = A - A^T \Rightarrow C^T = -A^T + A = -C$.

Then $V = U + W$ (2).

Therefore $V = U \oplus W$

26. Prove that $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Let $\{y_1 \ldots y_p\}$ be a basis of $W_1 \cap W_2$

Than $\{u_1 \ldots u_m, y_1 \ldots y_p\}$ is a basis of W_1

$\{v_1 \ldots v_n, y_1 \ldots y_p\}$ is a basis of W_2

Then show that $\{u_1 \ldots u_m, v_1 \ldots v_n, y_1 \ldots y_p\}$ is the basis of W_1+W_2

consider $\alpha_1 u_1 + \ldots + \alpha_m u_m + \beta_1 v_1 + \ldots \beta_n v_n + \gamma_1 y_1 + \ldots \gamma_n y_n = 0$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i u_i + \sum_{i=1}^{n} \beta_i v_i + \sum_{i=1}^{p} \gamma_i y_i = 0$$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i u_i = -\left(\sum_{i=1}^{n} \beta_i v_i + \sum_{i=1}^{p} \gamma_i y_i\right) \quad (1).$$

Showing that $\{u_1 \ldots u_m\} \in W_2$ but $\{u_1 \ldots u_m\} \in W_1$

Then $\{u_1 \ldots u_m\} \in W_1 \cap W_2$

Then $\exists\, a_1 \ldots a_p$ such that

$\alpha_1 u_1 + \ldots \alpha_m u_m + a_1 y_1 + \ldots a_p y_p = 0$

that implies $(\gamma_i, \beta_i) = (0,0)$ $\forall$ posible $i \in \mathbb{N}$

Thus $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Since if $W_1 \oplus W_2 \Leftrightarrow W_1 \cap W_2 = \{0\}$

Then $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$.

27. Show That $V = S_1 \oplus S_2 \oplus \dots \oplus S_p$

If $V = S_1 \oplus S_2 \oplus \dots \oplus S_p$

$\Rightarrow \exists ! \, v \in V : v = u_1 + \dots + u_p$ where $u_i \in S_i$

Let suppose v can be written as two forms.

$v = u_1 + \dots + u_p$ (1)

$v = w_1 + \dots + w_p$ (2)

We will show that $u_i = w_i$

$(1) - (2) = 0 = \sum_{i=1}^{p} (u_i - w_i)$

Since we have $\dim V = \sum_{i=1}^{p} \dim S_i$

Then we get $S_i \cap S_j = \{0\}$, $i \neq j$

Therefore $u_i = w_i$, because $u_i \neq u_j$, $i \neq j$

Therefore $v = u_1 + \dots + u_p$

So: $V = S_1 \oplus S_2 \oplus \dots \oplus S_n$.

28. (a). Determine the dimation of F and G. and also basis.

We have $F = \text{span}\{(1,2,-1,3),(2,4,1,-2),(3,6,3,-7)\}$

Consider $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then $F = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ -2 \end{pmatrix} \right\}$, $\dim F = 2$

and $G = \text{span}\{(1,2,-4,11), e(2,4,-5,14)\}$ is a basis.

Consider $\begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \Rightarrow \dim G = 2$.

Then $\dim G = 2$.

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(b). Show that $F = G$.

Let $x \in F$ then $x = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 4 \\ 1 \\ 2 \end{pmatrix}$

Suppose that $x \in G$

$$\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 4 \\ 1 \\ 2 \end{pmatrix} - \alpha_3 \begin{pmatrix} 1 \\ 2 \\ -4 \\ 11 \end{pmatrix} - \alpha_4 \begin{pmatrix} 2 \\ 4 \\ -5 \\ 14 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 - \alpha_3 - 2\alpha_4 = 0 \\ -\alpha_1 - \alpha_2 + 4\alpha_3 + 5\alpha_4 = 0 \\ 3\alpha_1 - 2\alpha_2 - 11\alpha_3 - 14\alpha_4 = 0 \end{cases}$$

Then $\exists\ \alpha_3, \alpha_4 \in \mathbb{R}$. Then $x \in G \Rightarrow F \subset G$ (1)

Similarly we get $G \subset F$ (2).

Thus $F = G$

29. (a). Determine the bases of F & G, then deduce $\dim F$ & $\dim$

$F = \text{span}\{(1,3,0,3), (2,1,3,2)\}$ these two vectors are L.I.

Then there is the basis of F, $\dim F = 2$.

$G = \text{span}\{(0,1,0,0), (1,0,-1,-1)\}$ Similarly there is the basis of G and $\dim G = 2$.

(b) Determine $F \cap G$.

Consider $\begin{pmatrix} 1 & 3 & 0 & 3 \\ 2 & 1 & 3 & -2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 3 & 1 & 4 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix}$

Therefore these four vectors are L.I. imply $F \cap G$

(c) Deduce F + G,

$$F+G = \text{span}\left\{\begin{pmatrix}1\\3\\0\\3\end{pmatrix}, \begin{pmatrix}2\\1\\3\\-2\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}\right\}$$

Thus we can say $F+G = F\oplus G$.

30. $F = \{P \in \mathbb{R}_n[x] \mid P(1) = P'(1) = 0\}$

(a). Show that F is a subspace of $\mathbb{R}_n[x]$.

- $\vec{0} \in F$ (1).

Let $f, g \in F$, $\alpha \in \mathbb{R}$, then $f + \alpha g \in F$ (2).

Thus F_1 is a subspace of $\mathbb{R}_n[x]$.

(b). Show that P belongs to F iff $(x-1)^2$ divides P.

Suppose that $P(1) = P'(1) = 0$

Then $P(x) = (x-1)^2 Q(x)$.

So $P \mid (x-1)^2$ (1).

Suppose $P \mid (x-1)^2$ then $P(x)$ can be written as:

$P(x) = (x-1)^2 Q(x)$ where $P(1) = P'(1) = 0$

So $P \in F$ (2).

Therefore $P \in F \Leftrightarrow (x-1)^2$ divides P.

(c) Determine basis and dimension of F

We have $P \in F$: $P(x) = (x-1)^2 Q(x)$, $Q \in \mathbb{R}_{n-2}[x]$

Then. $F = \text{span}\{(x-2)^2, x(x-2)^2 \ldots x^{n-2}(x-2)^2\}$

Therefore $\dim F = n-1$.

31. Show that $\{P_0, P_1, \ldots P_n\}$ is a basis for $\mathbb{R}_n[x]$

$P_k(m) = (m+1)^{k+1} - m^{k+1}$

$= (m+1)^k + m(m+1)^{k-1} + \ldots m^{k-1}(m+1) + m^k$.

Since $\{P_0 \ldots P_n\}$ is the set of echolon function (Polynomial with different degrees then the set is L.I.

32 (a). Find a basis and dimention of each of subspace $S_1, S_2, S_3, S_4, S_1+S_2, S_1+S_2+S_3, S_2+S_3+S_4$ and $S_1+S_3+S_4$

- $S_1 = \text{span}\{(1,1,1,2),(2,1,2,1),(1,0,1,-1)\} = \text{span}\{(1,1,1,2),(2,1,2,1)\}$

$\Rightarrow \dim(S_1) = 2$.

- $S_2 = \text{span}\{(3,-1,-1,4),(1,3,1,2),(1,-2,1,1)\} = \text{span}\{(3,-1,-1,4),(1,3,1,2)\}$

$\Rightarrow \dim(S_2) = 2$.

- $S_3 = \{(2,1,1,4)\} \Rightarrow \dim(S_3) = 1$.
- $S_4 = \text{span}\{(-1,3,-1,2)\} \Rightarrow \dim(S_4) = 1$.
- $S_1+S_2 = \text{span}\{(1,1,1,2),(2,1,2,1),(3,-1,-1,4),(1,3,1,2)\} \Rightarrow \dim(S_1+S_2) = 4$
- $S_1+S_2+S_3 = \text{span}\{(1,1,1,2),(2,1,2,1),(3,-1,-1,4),(1,3,1,2),(2,1,1,4)\}$

consider: $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(S_1+S_2+S_3) = 4$

- $S_2+S_3+S_4 = \text{span}\{(3,-1,-1,4),(1,3,1,2),(2,1,1,4),(-1,3,-1,2)\}$ is the basis and $\dim(S_1+S_2+S_3) = 4$, $\dim(S_1 \cap S_2 \cap S_3) = 0$.

• $S_1+S_3+S_4 = \text{span}\{(1,1,1,2),(2,1,2,1),(2,1,1,4),(-1,3,-1,2)\}$

is the basis and $\dim(S_1+S_3+S_4) = 4$. $\dim(S_1 \cap S_3 \cap S_4) = 0$

(b). show that $\mathbb{R}^4 = S_1 \oplus S_2$, $\mathbb{R}^4 = S_1 \oplus S_3 \oplus S_4$ and $\mathbb{R}^4 = S_2 \oplus S_3 \oplus S_4$

since $\dim(S_1+S_2) = 4 = \dim(\mathbb{R}^4) \Rightarrow \mathbb{R}^4 = S_1 \oplus S_4$

$\dim(S_1+S_3+S_4) = 4 = \dim(\mathbb{R}^4) \Rightarrow \mathbb{R}^4 = S_1 \oplus S_3 \oplus S_4$

$\dim(S_2 \oplus S_3 \oplus S_4) = 4$ $\dim(\mathbb{R}^4) \Rightarrow \mathbb{R}^4 = S_2 \oplus S_3 \oplus S_4$

(c). show that $\mathbb{R}^4 = S_1+S_2+S_3+S_4$

$S_1+S_2+S_3+S_4 = \text{span}\{(1,1,1,2)(2,1,2,1),(1,3,1,2),(1,-2,-1,1),(-1,3,-1,2)\}$

consider $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & -2 & -1 & 1 \\ -1 & 3 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(S_1+\dots S_4) = 4.$

Thus: $\mathbb{R}^4 = S_1+S_2+S_3+S_4$.

33. (a) Find the basic B_1 and dimmension of S_1.

we have $S_1 = span\{(1,1,1,2,1),(2,1,1,3,2),(1,0,0,1,1),(1,2,2,3,3)\}$

$= span\{(1,1,1,2,1),(2,1,1,3,2),(1,2,2,3,3)\}$

$\Rightarrow dim(S_1) = 3$

Find coordinate of B_1. $[2,-1,-1,1,0]_{B_1}$.

$$\Rightarrow \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$\Rightarrow \alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = -1.$

Therefore $[2,-1,-1,1,0]_{B_1} = (1,2,-1)$

(b) Find the basis and dimension of S_2, S_1+S_2, $S_1 \cap S_2$.

we have $S_2 = span\{(1,-1,-1,0,-1),(3,1,1,4,3),(2,2,2,3,4)\}$

$= span\{(1,-1,-1,0,-1),(0,1,1,1,0),(0,0,0,-1,2)\}$

is the basis of S_2 and so $dim(S_2) = 3$.

$\Rightarrow S_1+S_2 = span\{(1,1,1,2,1),(2,1,1,3,2),(1,2,2,3,3),(1,-1,-1,0,-1),(0,1,1,1,0),$

$(0,0,0,-1,2)\}$

$= span\{(1,1,1,2,1),(1,0,0,1,1),(1,2,2,3,3),(2,2,2,3,4)\}$ is a

basis of S_1+S_2 and $dim(S_1+S_2) = 4$.

$\Rightarrow dim(S_1 \cap S_2) = 2$

34. (a). show that $B=\{1, 1-t, (1-t)^2, (1-t)^3\}$ is a basis of $P_3(\mathbb{R})$.

Since they have different degree

Therefore B is a basis of $P_3(\mathbb{R})$.

(b). Find the coordinat of vector $V(t)=2+3t-t^2+t^3$

we have $v(t)=2+3t-t^2+t^3$

$= t^3-3t^2+3t-1+3+2t^2$

$= -(1-t)^3+2(t^2-2t+1)+6t-1.$

$= -(1-t)^3-2(t-1)^2-6(1-t)+5$

Then $[v(t)]=(5,-6,-2,-1)$.

35. (a) show that $B=\{1, t-1, \dots, (t-1)^m\}$ is the basis of P_m.

since they have different degree

Therefore B is the basis of P_m.

(b) determinate the coordinate of $(t+2)^m$ with respace to basis B.

we have $(t+2)^m=\sum_{k=0}^{m}\binom{m}{k}(t-1)^k(3)^{m-k}$

Therefore $[(t+2)^m]_B=(C_m^0 3^m, \dots, 3^0)$

36. (a). show that $B = \{(x-a)^k\}_{0\le k\le 2n}$ is a basis of $\mathbb{R}_{2n}[x]$

since it has different degree.

Therefore B is a basis of $\mathbb{R}_{2n}[x]$.

(b). Determine the coordinate of $(x-a)^n(x-b)^n$ respace to B.

we have $(x-a)^n(x-b)^n = (x-a)^n \times \sum_{k=0}^{n} \binom{n}{k}(x-a)^k(a-b)^{n-k}$

$$= \sum_{k=0}^{n}\binom{n}{k}(x-a)^{k+n}(a-b)^{n-k}.$$

Therefore $[(x-a)^n(x-b)^n]_B = \left(\underbrace{0,0,\dots,0}_{n\text{-time}}, \binom{n}{0}(a-b)^n \dots \binom{n}{n}\right)$

37. Find $P_{B_2\leftarrow B_1}$ and $P_{B_1\leftarrow B_2}$.

Let $v\in\mathbb{R}_3$ so $[v]_{B_2} = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$

$\Rightarrow \alpha_1\begin{pmatrix}1\\1\\-1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + \alpha_3\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow \alpha_1 = z,\ \alpha_2 = -y-z,\ \alpha_3 = \frac{1}{2}(x+y+2z).$

Then $[v]_{B_2} = \begin{pmatrix}-z\\ -y-z\\ \frac{1}{2}(x+y+2z)\end{pmatrix}$

$\Rightarrow [v_1]_{B_2} = \begin{bmatrix}1\\1\\1\end{bmatrix}_{B_2} = \begin{pmatrix}-1\\-2\\2\end{pmatrix}, [v_2]_{B_2} = \begin{bmatrix}1\\1\\0\end{bmatrix}_{B_2} = \begin{pmatrix}0\\-1\\1\end{pmatrix} [v_3]_{B_2} =$

Therefore $P_{B_2 \leftarrow B_1} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & 1/2 \end{pmatrix}$

Since $P_{B_2 \leftarrow B_1} \times P_{B_1 \leftarrow B_2} = I$

$\Rightarrow P_{B_1 \leftarrow B_2} = P^{-1}_{B_2 \leftarrow B_1}$

we have $\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right)$

$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 \end{array}\right)$

Therefore $P_{B_1 \leftarrow B_2} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$

- Find $[(1,-1,0)]_{B_2}$

$[v]_{B_2} = P_{B_2 \leftarrow B_1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

38. Determine whether the following vector space are inner produce space equipped with real mappings

(a). $\mathbb{R}^4$ equipped with $\langle x, y\rangle = 4x_1y_1 + 2x_2y_2 + 3x_3y_3 + 7x_4y_4$

Let $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$, $z = (z_1, z_2, z_3, z_4)$.

① $\langle x, x\rangle = 4x_1^2 + 2x_2^2 + 3x_3^2 + 7x_4^2$

$\langle x, x\rangle = 0 \Leftrightarrow x_1 = x_2 = x_3 = x_4 = 0$

② $\langle x, y\rangle = 4x_1y_1 + 2x_2y_2 + 3x_3y_3 + 7x_4y_4$

$= 4y_1x_1 + 2y_2x_2 + 3y_3x_3 + 7y_4x_4 = \overline{\langle y, x\rangle}$

③ $k\langle x, y\rangle = 4kx_1y_1 + 2kx_2y_2 + 3kx_3y_3 + 7kx_4y_4 = \langle kx, y\rangle$

④ $\langle x+y, z\rangle = 4(x_1+y_1)z_1 + 2(x_2+y_2)z_2 + 3(x_3+y_3)z_3 + 7(x_4+y_4)z_4$

$= \langle xz, yz\rangle$

Therefor $\mathbb{R}^4$ is an inner produce.

(b). $\langle x, y\rangle = x_1y_1 + x_2y_2 - 2x_3y_3$

similarly

Therefor it is an innerproduce.

(c). $C^0[0,1]$ equipped with $\langle f_1 : f_2\rangle = \int_0^1 t f_1(t) f_2(t) dt$

Let $f_1, f_2, f_3 \in C^0[0,1]$.

① $\langle f_1, f_1\rangle \int_0^1 t f^2(t) dt \geq 0$

$\int_0^1 t f^2(t) dt = 0 \Rightarrow t = 0, f(t) = 0$ not satify.

Therefore $\langle f_1, f_2\rangle$ is not innerproduct.

(d). P_3 equipped with.

$$\langle a_0 + a_1 t + a_2 t^2 + a_3 t^3, b_0 + b_1 t + b_2 t^2 + b_3 t^3 \rangle$$

$$= a_0 b_0 + a_1 b_1 + 2a_2 b_2 + a_3 b_3$$

Let ~~$f(t) = a_0 + a_1 t + a_2 t + a_3 t$~~

$a = (a_0, a_1, a_2, a_3)$

$b = (b_0, b_1, b_2, b_3)$

$c = (c_0, c_1, c_2, c_3)$

$f(t) = (1, t, t^2, t^3)$.

- $\langle af(t), af(t) \rangle = a_0^2 + a_1^2 + 2a_2^2 + a_3^2 \geq 0$

  $\langle af(t), af(t) \rangle = 0 \iff a_0 = a_1 = a_2 = a_3 = 0$.

- $\langle af(t), bf(t) \rangle = a_0 b_0 + a_1 b_1 + 2a_2 b_2 + a_3 b_3$

  $= b_0 a_0 + b_1 a_1 + 2b_2 a_2 + b_3 a_3 = \overline{\langle bf(t), af(t) \rangle}$

- $\langle kaf(t), bf(t) \rangle = ka_0 b_0 + ka_1 b_1 + k2a_2 b_2 + ka_3 b_3$

  $= k\langle af(t), bf(t) \rangle$

- $\langle af(t) + bf(t), cf(t) \rangle = \langle af(t) cf(t), bf(t) cf(t) \rangle$

Therefore $\langle ; \rangle$ is an inner product.

39. show that φ is an inner product in $M_n(\mathbb{R})$

We have $\varphi(A,B) = tr(A^t B)$

1. $tra(A^T A) = \varphi(A,A) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij}^2\right) \geq 0$

and $\varphi(A,A) = 0$ iff: $a_{ij} = 0$

2. $\varphi(A,B) = tra(A^T B) = tra(B^T A)$

3. $\varphi(kA,B) = tra(kA^T B) = k\varphi(A,B)$

4. $\varphi(A+C,B) = tra((A+C)^T \times B) = tra(A^T B) + tra(C^T B)$

$= \varphi(A,B) + \varphi(C,B)$

Therefore φ is inner product.

40. (a) Show that $\langle ; \rangle_1$, $\langle ; \rangle_2$ are inner produce.

. $\langle x,y\rangle_1 = x_1y_1 + x_2y_2 + x_3y_3$

① $\langle x,x\rangle_1 = x_1^2 + x_2^2 + x_3^2$ if $\langle x,x\rangle = 0 \Rightarrow x_1 = x_2 = x_3 = 0$

② $\langle x,y\rangle_1 = x_1y_1 + x_2y_2 + x_3y^3 = \langle y,x\rangle$

③ $\langle kx,y\rangle_1 = k\langle x,y\rangle_1$

④ $\langle x+y,z\rangle = \langle xy, yz\rangle$

. $\langle x,y\rangle_2 = x_1y_1 + 2x_2y_2 + 3x_3y_3$

similarly :

Therefore $\langle ; \rangle_1$ and $\langle ; \rangle$ are innerproduct.

+ Find the norm of v_1 for $\langle ; \rangle_1$ and v_2 for $\langle ; \rangle_2$

. v_1 for $\langle ; \rangle_1$

$\cancel{\|\langle v_1, \rangle\|_1 = \sqrt{1^2+1^2+1^2} = \sqrt{3}}$

--- $\|\langle v_1, \rangle\|_1 = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

+ v_2 for $\langle ; \rangle_2$

$\|v_2\| = \sqrt{\langle v_2, v_2 \rangle_2} = \sqrt{1 + 2 + 3(-2)^2} = 3$.

Therefore $\|v_1\| = \sqrt{3}$, $\|v_2\| = 3$.

+ Find distance $d(v_1, v_2)_1$ and $d(v_1, v_2)_2$.

(b). show that v_1 and v_2 are orthogonal with respect to $\langle ; \rangle_1$ and not for $\langle ; \rangle_2$

$\langle v_1, v_2 \rangle_1 = (1)(1) + (1)(1) + (1)(-2) = 0$

$\langle v_1, v_2 \rangle_2 = (1)(1) + 2(1)(1) + 3(-2)(1) \neq 0$ True.

(c). show that v_1 and v_3 ortogonal with respect to $\langle ; \rangle_2$ not for $\langle ; \rangle_1$.

. $\langle v_1, v_3 \rangle_2 = 0$

. $\langle v_1, v_3 \rangle_1 \neq 0$ True.

41. (a). show that $B\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$

since $\langle v_1, v_2\rangle = 4(1)(1)$ ~~...~~

we have $\langle x, y\rangle = 4x_1y_1 + 3x_2y_2 + 5x_3y_3$

since $\langle v_1, v_2\rangle = 4(1)(1) + 3(1)(2) + 5(1)(-2) = 0$ (1)

$\langle v_1, v_3\rangle = 4(1)(-5) + 3(1)(5) + 5(1)(1) = 0$ (2)

$\langle v_2, v_3\rangle = 4(1)(-5) + 3(2)(5) + 5(-2)(1) = 0$ (3)

by (1), (2)(3) B is an orthogonal basis.

+ Derive orthonomal basis B_0

we have $B_0 = \left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}$

. $\|v_1\| = \sqrt{4+3+5} = \sqrt{12}$, $\|v_2\| = \sqrt{4+12+20} = 6$

. $\|v_3\| = 6\sqrt{5}$

Therefore $B_0 = \left\{\frac{1}{2\sqrt{3}}(1,1,1), \frac{1}{6}(1,2,-2), \frac{1}{6\sqrt{5}}(-5,5,1)\right\}$

(b). Determine $[v]_B$ and $[v]_{B_0}$

+ $[v]_B$

Since B is orthogonal basis $v = c v_1 + c v_2 + c v_3$

$$\Rightarrow \begin{cases} c_1 = \dfrac{\langle v, v_1\rangle}{\|v_1\|^2} = \dfrac{7}{4} \\ c_2 = \dfrac{\langle v, v_2\rangle}{\|v_2\|^2} = -\dfrac{1}{6} \\ c_3 = \dfrac{\langle v, v_3\rangle}{\|v_3\|^2} = -\dfrac{1}{12} \end{cases}$$

Therefore $[v]_B = \left(\frac{7}{4}, \frac{1}{6}, -\frac{1}{12}\right)$.

. $[V]_{B_o}$

since B_o is ordhogonal basis.

$$\Rightarrow \begin{cases} c_1' = \langle v, v_1' \rangle = \frac{21}{\sqrt{12}} \\ c_2' = \langle v, v_2' \rangle = -1 \\ c_3' = \langle v, v_3' \rangle = \frac{-15}{6\sqrt{5}} \end{cases}$$

Thus $[V]_{B_o} = \left(\frac{21}{\sqrt{12}}, -1, -\frac{15}{6\sqrt{15}} \right)$

(c). Find the scalar and vector projection.

. $k = \frac{\langle v, v_2 \rangle}{\|v_2\|} = \frac{8+6-20}{6} = -1$

. $\ell = k \left(\frac{v_2}{\|v_2\|} \right) = -\frac{1}{6} v_2 = -\frac{1}{6}(1, 2, -2)$.

42. Find the orthogonal basis for the subspace of $\mathbb{R}^4$ where

$W = \{(x, y, z, w) \in \mathbb{R}^4 : x - y - z = 0 \text{ and } x + z = 0\}$

Answer.

we have

$\begin{cases} y = z - x \\ z = -x \end{cases}$

$\Rightarrow \begin{cases} z = -x \\ y = +2x \end{cases}$

$\Rightarrow (x, 2x, -x, w) = x(1, 2, -1, 0) + w(0, 0, 0, 1)$

Thus $W = \text{span}\{(1, 2, -1, 0), (0, 0, 0, 1)\}$

and $(1, 2, -1, 0)(0, 0, 0, 1) = 0$

Therefore. $W = \text{span}\{(1, 2, -1, 0), (0, 0, 0, 1)\}$ is the orthogonal basis

43. Let $\mathbb{R}^4$ be a vector space equipped with inner product

$$\langle x,y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4y_4$$

Let: $v_1 = (1,2,1,2)$, $v_2 = (2,1,2,1)$, $v_3 = (1,1,2,2)$, $v_4 = (2,2,1,1)$.

$v_5 = (1,2,1,3)$.

and. $S_1 = \{v_1, v_2, v_3, v_4\}$, $S_2 = \text{span}\{(1,2,1,2),(2,1,2,1),(2,2,1,1),$

$(1,2,1,3)\}$

(a). Determine basis of S_1 and $\dim(S_1)$. denote by B_1.

consider $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow S_1 = \text{span}\{(1,2,1,2),(2,1,2,1),(1,1,2,2)\}$ is a basis and $\dim(S_1) = 3$.

+ Use gram-schmidt process the transform B_1 to an orthogonal and orthonormal basis for S_2.

Page...

we have $S_1 = \text{Span}\{v_1, v_2, v_3\}$

~~by Gram~~

Let $\{u_1, u_2, u_3\}$ be an athogonal basis.

by ram-schmidt

$u_1 = v_1 = (1,2,1,2).$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2} \cdot u_1$$

$$= (2,1,2,1) - \frac{2+2+2+4}{1+4+1+8}(1,2,1,2)$$

$$= \left(\frac{9}{7}, -\frac{3}{7}, \frac{9}{7}, -\frac{3}{7}\right)$$

$$u_3 = v_3 - \frac{\langle v_3, u_1\rangle}{\|u_1\|^2}u_1 - \frac{\langle v_3, u_2\rangle}{\|u_2\|^2}u_2$$

$$= (1,1,2,2) - \frac{1+2+2+8}{14}(1,2,1,2) - \frac{12/7}{\frac{189}{9}}\left(\frac{9}{7}, -\frac{3}{7}, \frac{9}{7},\right.$$

$$= (1,1,2,2) - \frac{637}{686}(1,2,1,2) - \frac{8}{686}(9,-3,9,-3)$$

$$= (1,1,2,1) - \frac{1}{686}(709, 1250, 709, 1250).$$

$$= \left(\frac{686-709}{686}, \frac{686-1250}{686}, \frac{1372-709}{686}, \frac{686-1250}{686}\right)$$

$$= \left(-\frac{23}{686}, -\frac{564}{686}, \frac{663}{686}, -\frac{564}{686}\right).$$

Thus $\{u_1, u_2, u_3\}$ is an orthogonal basis

+ orthonormal basis.

Let $\{w_1, w_2, w_3\}$ be an orthonormal basis.

$\Rightarrow$. $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{14}}(1, 2, 1, 2)$.

. $w_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{49}}{\sqrt{39}}\left(\frac{2}{7}, -\frac{2}{7}, \frac{9}{7}, -\frac{3}{7}\right)$.

. $w_3 = \frac{v_3}{\|v_3\|}$

+ $\|v_3\| = \frac{1}{686}\sqrt{(-23)^2 + (-564)^2 + (663)^2 + (-564)^2} = a$.

$\Rightarrow w_3 = \frac{1}{a}\left(-\frac{23}{686}, -\frac{564}{686}, \frac{663}{686}, -\frac{564}{686}\right)$

(b) Show that $B_2 = \{v_1, v_2, v_5, v_3\}$ is a basis of S_2 then deduce $S_2 = \mathbb{R}^4$.

we consider

$$\det\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 3 \end{pmatrix} \neq 0$$

Therefore $B_2 = \text{span}\{v_1, v_2, v_3, v_5\}\}$ is a basis of S_2

and $\dim(S_2) = 4$.

$\Rightarrow S_2 = \mathbb{R}^4$.

→ By using Gram-Schmidt

- orthogonal basis

Let $\{u_1, u_2, u_3, u_5\}$ Be an orthogonal basis of S_2

$\Rightarrow u_1 = v_1 = (1, 2, 1, 2)$.

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = \left(\frac{9}{7}, \frac{-3}{7}, \frac{9}{7}, \frac{-3}{7}\right).$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} = \left(-\frac{23}{686}, -\frac{566}{686}, \frac{663}{686}, \ldots\right.$$

$$u_5 = v_5 - \frac{\langle v_5, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_5, u_2 \rangle}{\|u_2\|^2} u_2 - \frac{\langle v_5, u_3 \rangle}{\|u_3\|^2} u_3$$

$$= (m, n, p, q) \quad , m, n, p, q \in \mathbb{R}$$

Therefore $\{u_1, u_2, u_3, u_5\}$ is an orthogonal basis for S_2.

44. Find $W^{\perp}$, the orthogonal complement of W, where

$$W = \{(x,y,z) \in \mathbb{R}^3 : x+3y-z=0\}$$

Answer

$W = \{(x,y,z) \in \mathbb{R}^3 : x+3y-z=0\}$

$\Rightarrow \{(x, y, x+3y) = x(1,0,1), y(0,1,3)\}$

$\Rightarrow W = \text{span}\{(1,0,1),(0,1,3)\}$

Then let $v = \{v_1, v_2, v_3\}$ be an orthogonal vector $\in W^{\perp}$

Thus $\begin{cases} v \perp v_1 \\ v \perp v_2 \end{cases} \Rightarrow \begin{cases} v_1 + v_3 = 0 \\ v_2 + 3v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_3 = -v_1 \\ v_2 = +3v_1 \end{cases}$

$\Rightarrow v = v_1(1,3,1)$

Thus $W^{\perp} = \text{span}\{(1,3,1)\}$

45. Prove that $S_e^{\perp} = S_o$, where the inter product only is defined by $\langle f,g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$

we have : S_e : even function

S_o : odd function

$\forall v \in [-1,1]$ = $f(x) = \frac{1}{2}(f(x)+g(-x)) + \frac{1}{2}(f(x)-g(-x))$

+ prove that $f(x)+g(-x)$ is even function.

$f(-x)+g(x) = g(x)+f(-x)$. replace x by $-x$

Thus $f(x)+g(-x)$ is even.

. prove that $f(x)-g(-x)$ is odd function

replace x by $-x$

$\Rightarrow f(x)-g(-x) = -(g(x)-f(-x))$ (odd)

Thus everyfunction can be writtend.

$S_e \oplus S_o$.

now prove that $S_e \perp S_o$ so we got $S_e^{\perp} = S_o$

since we defined $\langle f,g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$

Let $f \in S_e$, $g \in S_o$ $\Rightarrow \int_{-1}^{1} f(t)g(t)\,dt = 0$

$\Rightarrow \langle f,g \rangle = 0 \Leftrightarrow S_e \perp S_o$

Therefore $S_e^{\perp} = S_o$

46. Let W_1 and W_2 be a subspace of finite-dimensional inner produce space.

prove $+ (W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$.

for $x \in W_1 \Rightarrow x \in W_1+W_2 \Rightarrow \langle x, y\rangle = 0 \Rightarrow v \in W_1^\perp$ (1)

$y \in W_2 \Rightarrow y \in W_1+W_2 \Rightarrow \langle y, v\rangle = 0 \Rightarrow v \in W_2^\perp$ (2)

from (1) & (2) $\Rightarrow v \in W_1^\perp \cap W_2^\perp$ or $(W_1+W_2)^\perp \subset W_1^\perp \cap W_2^\perp$ (I)

now let $v \in W_1^\perp \cap W_2^\perp \Rightarrow \begin{cases} v \in W_1^\perp \\ v \in W_2^\perp \end{cases}$

for $u_1 \in W_1 \Rightarrow \langle v, u_1\rangle = 0$

$u_2 \in W_2 \Rightarrow \langle v, u_2\rangle = 0$

$\Rightarrow \langle v, u_1\rangle + \langle v, u_2\rangle = 0$.

$\langle v, u_1+u_2\rangle = 0 \Rightarrow v \in (W_1+W_2)^\perp$

$W_1^\perp \cap W_2^\perp \subset (W_1+W_2)^\perp$ (II).

from (I) & (II) $\Rightarrow (W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$.

$+ (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

we have $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$

$((W_1+W_2)^\perp)^\perp = (W_1^\perp \cap W_2^\perp)^\perp$

$W_1+W_2 = (W_1^\perp + W_2^\perp)^\perp$

$\Rightarrow W_1^\perp \cap W_2^\perp = W_1^\perp + W_2^\perp$

47. let W be a subspace of finite-dimensional vector V

show that:

(a). $(W^{\perp})^{\perp} = W$

Let $u \in (W^{\perp})^{\perp}$, $v \in W^{\perp} \Rightarrow \langle u,v\rangle = 0 \Rightarrow u \in W$ (1)

Let $u \in W$, $v \in W^{\perp} \Rightarrow \langle u,v\rangle = 0 \Rightarrow u \in (W^{\perp})^{\perp}$ (2)

by (1) & (2)

Therefore $(W^{\perp})^{\perp} = W$.

(b). $V = W \oplus W^{\perp}$

Since $V = W \oplus W^{\perp} \Leftrightarrow \begin{cases} W \cap W^{\perp} = \{0\} \\ \dim V = \dim W + \dim W^{\perp} \end{cases}$

Suppose $u \in W \cap W^{\perp} \Rightarrow \begin{cases} u \in W \\ u \in W^{\perp} \end{cases}$

Then $\langle u,u\rangle = \|u^2\| = 0 \Leftrightarrow u = 0$

then $W \cap W^{\perp} = \{0\}$ (1)

let $W = \{x_1, \dots x_n\}$ be an orthogonal basis

let $\{x_1, \dots, x_n, v_1, \dots v_p\}$ —— .. —— of V

then $\langle x_i, v_j\rangle = 0$, $\forall i \in [1,n]$, $j \in [1,p]$

thus $\{v_1, \dots v_p\} \in W^{\perp}$

$\Rightarrow \dim V = n + p = \dim W + \dim W^{\perp}$

Thus $V = W \oplus W^{\perp}$.