I2–TD3
(Continuous RV and Probability Distribution)

1. Let X be a random variable whose pdf is

$$f(x) = \begin{cases} 0.2, & -1 \leq x \leq 0 \\ 0.2 + kx, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value k
   (b) Find the cdf $F_X(x)$.
   (c) Deduce the values $F_X(-1)$ and $F_X(1)$
   (d) Find $P(X < 1.5)$
   (e) Find $P(X > 0.5 | X < 1)$
   (f) Find $E(X)$ and $V(X)$.

2. Let X be a random variable whose cdf is

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{8}, & 0 < x < 2 \\ \frac{x^2}{16}, & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

   (a) Find the probability distribution function $f(x)$.
   (b) Find $P(X > 3)$
   (c) Find $P(1 < X < 3)$
   (d) Find $P(X > 0.5 | X < 1)$
   (e) Find $E(X)$ and $V(X)$.

3. Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(x) = \begin{cases} \frac{3}{64}x^2(4 - x), & 0 \leq x \leq 4, \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the expected value and variance of weekly CPU time.
   (b) The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.
   (c) Would you expect the weekly cost to exceed \$600 very often? Why?

4. The grade point averages (GPAs) for graduating seniors at a college are distributed as a continuous rv X with pdf

$$f(x) = \begin{cases} k(1 - (x - 3)^2), & 2 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of k.
   (b) Find the probability that a GPA exceeds 3.
   (c) Find the probability that a GPA is within 0.25 of 3.
   (d) Find the probability that a GPA differs from 3 by more than 0.5.

5. Suppose that X is a continuous random variable with density $f(x)$ that is positive only if $x \geq 0$. If $F_X(x)$ is the cumulative distribution function of X, show that

$$E(X) = \int_0^\infty (1 - F_X(x))\, dx$$

6. If X is a continuous random variable such that $E\left[(X-a)^2\right] < \infty$ for all a, show that $E\left[(X-a)^2\right]$ is minimized when $a = E(X)$.

7. Suppose that $X \sim \mathrm{U}(a,b)$. Show that

   (a) $E(X) = \dfrac{a+b}{2}$ (b) $V(X) = \dfrac{(b-a)^2}{12}$ (c) $M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

8. Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 AM. What is the probability that the center is up when the person's call comes in?

9. In commuting to work, a professor must first get on a bus near her house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with $A = 0$ and $B = 5$, then it can be shown that the total waiting time Y has the pdf

$$f(y) = \begin{cases} \frac{1}{25}y, & 0 \leq y < 5, \\ \frac{2}{5} - \frac{1}{25}y, & 5 \leq y \leq 10, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Sketch a graph of the pdf of Y.
   (b) Verify that $\displaystyle\int_{-\infty}^{\infty} f(y)dy = 1$.
   (c) What is the probability that total waiting time is at most 3 min?
   (d) What is the probability that total waiting time is at most 8 min?
   (e) What is the probability that total waiting time is between 3 and 8 min?
   (f) What is the average of the total waiting time.

10. Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that

   (a) $E(X) = \mu$ (b) $V(X) = \sigma^2$ (c) $M(t) = \exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$.

11. If X is a random variable such that

$$\begin{cases} E(X^{2n}) = (2n)!/2^n n!, \\ E(X^{2n-1}) = 0, \end{cases} \text{, for } n = 1, 2, \ldots.$$

   Find the moment generating function and the distribution of X.

12. The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8.

(a) If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?

(b) How many percent of students will possess a GPA in excess of 3.0.

(c) Suppose that 5 students are randomly selected from the student body. What is the probability that only three will possess a GPA in excess of 3.0

13. Scores on an examination are assumed to be normally distributed with mean 65 and variance 36.

    (a) What is the probability that a person taking the examination scores higher than 70?

    (b) Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?

    (c) What must be the cutoff point for passing the examination if the examiner wants only the top 75% of all scores to be passing?

    (d) Approximately what proportion of students have scores 5 or more points above the score that cuts off the lowest 15%?

    (e) If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

14. The distribution of $1,000$ examinees according to marks percentage is given below:

% Score	$< 40\%$	40%-75%	$> 75\%$	Total
No. of examinees	430	420	150	1,000

    Assuming the marks percentage to follow a normal distribution.

    (a) Calculate the mean and standard deviation of marks.

    (b) If not more than 300 examinees are to fail, what should be the passing marks?

15. Suppose the diameter at breast height (in.) of trees of a certain type is normally distributed with $\mu = 8.8$ and $\sigma = 2.8$, as suggested in the article "Simulating a Harvester-Forwarder Softwood Thinning" (Forest Products J., May 1997: 36–41).

    (a) What is the probability that the diameter of a randomly selected tree will be at least 10 in.? Will exceed 10 in.?

    (b) What is the probability that the diameter of a randomly selected tree will exceed 20 in.?

    (c) What is the probability that the diameter of a randomly selected tree will be between 5 and 10 in.?

    (d) What value c is such that the interval $(8.8-c, 8.8+c)$ includes 98% of all diameter values?

    (e) If four trees are independently selected, what is the probability that at least one has a diameter exceeding 10 in.?

16. A machine that produces ball bearings has initially been set so that the true average diameter of the bearings it produces is 0.500 in. A bearing is acceptable if its diameter is within 0.004 in. of this target value. Suppose, however, that the setting has changed during the course of production, so that the bearings have normally distributed diameters with mean value 0.499 in. and standard deviation 0.002 in. What percentage of the bearings produced will not be acceptable?

17. Suppose only 70% of all drivers in a state regularly wear a seat belt. A random sample of 500 drivers is selected. What is the probability that

    (a) Between 320 and 370 (inclusive) of the drivers in the sample regularly wear a seat belt?

    (b) Fewer than 325 of those in the sample regularly wear a seat belt? Fewer than 315?

18. Let $X \sim \text{Exp}(\lambda)$. Show that

    (a) $F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\frac{x}{\lambda}}, & x \geq 0. \end{cases}$

    (b) $E(X) = \lambda$

    (c) $V(X) = \lambda^2$

    (d) $E(X^n) = \lambda^n n!$, for $n = 1, 2, \ldots$

    (e) $M(t) = \dfrac{1}{1 - \lambda t}$, $t < \dfrac{1}{\lambda}$

19. A survey evidence indicates that the accidents on a particular road #4 in the country ABC have an approximately exponential distribution. Assume that the mean time between accidents is 4 hours. If one of the accidents occurred on 7:00 AM of a randomly selected day in the study period.

    (a) What is the probability that another accident occurred that same day?

    (b) What time that the next accident occurred if the probability that the next accident occurred is 0.8

20. A system consists of five identical components connected in series as shown

As soon as one component fails, the entire system will fail. Suppose each component has a lifetime that is exponentially distributed with $\lambda = 0.01$ and that components fail independently of one another. Define events $A_i = \{i\text{th component lasts at least } t \text{ hours}\}$, $i = 1, \ldots, 5$, so that the $A_i s$ are independent events. Let $X =$ the time at which the system fails-that is, the shortest (minimum) lifetime among the five components.

    (a) The event $\{X \geq t\}$ is equivalent to what event involving $\{A_1, A_2, \ldots, A_5\}$?

    (b) Using the independence of the $\{A_i\}$'s, compute $P(X \geq t)$. Then obtain $F(t) = P(X \leq t)$ and the pdf of X. What type of distribution does X have?

    (c) Suppose there are n components, each having exponential lifetime with parameter λ. What type of distribution does X have?

21. Suppose that a system contains a certain type of component whose time, in years, to failure is given by T . The random variable T is modeled nicely by the exponential distribution with mean time to failure $\lambda = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

22. Let $X \sim \text{Gam}(\alpha, \beta)$. Show that

    (a) $E(X) = \alpha\beta$

    (b) $V(X) = \alpha\beta^2$

    (c) $M(t) = \dfrac{1}{(1-\beta t)^{\alpha}}, \quad t < \dfrac{1}{\beta}$

    (d) $F(x) = \Gamma\left(\dfrac{x}{\beta}, \alpha\right)$

23. Suppose that when a transistor of a certain type is subjected to an accelerated life test, the lifetime X (in weeks) has a gamma distribution with mean 24 weeks and standard deviation 12 weeks.

    (a) What is the probability that a transistor will last between 12 and 24 weeks?

    (b) What is the probability that a transistor will last at most 24 weeks? Is the median of the lifetime distribution less than 24? Why or why not?

    (c) What is the 99th percentile of the lifetime distribution?

    (d) Suppose the test will actually be terminated after t weeks. What value of t is such that only 0.5% of all transistors would still be operating at termination?

24. Let $X \sim \text{Gam}(\alpha, \beta)$. Show that $X \sim \chi^2(\nu)$ if $\alpha = \nu/2$ and $\beta = 2$.

25. If $X \sim \chi^2(\nu)$. Show that

    (a) $E(X) = \nu$

    (b) $V(X) = 2\nu$

    (c) $E(X^n) = \dfrac{2^n\Gamma(\frac{\nu}{2}+n)}{\Gamma(\frac{\nu}{2})}, \quad n > -\nu/2$

    (d) $M(t) = \dfrac{1}{(1-2t)^{\frac{\nu}{2}}}, \quad t < \dfrac{1}{2}$

26. Suppose $X \sim \mathcal{N}(\mu, \sigma^2), \sigma > 0$ and $Y = \dfrac{(X-\mu)^2}{\sigma^2}$. Show that $Y \sim \chi^2(1)$.

27. Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} ky^3e^{-y/2}, & y > 0, \\ 0, & \text{otherwise} \end{cases}$$

    (a) Find the value of k.

    (b) Does Y have a χ^2 distribution? If so, how many degrees of freedom?

    (c) What are the mean and standard deviation of Y ?

    (d) What is the probability that Y lies within 2 standard deviations of its mean?

28. Let $X \sim \text{Bet}(\alpha, \beta)$. Show that

$$E(X) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

29. A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

30. Let $X \sim \text{Log}(\mu, \sigma)$. Show that

$$E(X) = e^{\mu+\sigma^2/2} \quad \text{and} \quad V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

31. Concentrations of pollutants produced by chemical plants historically are known to exhibit behavior that resembles a lognormal distribution. This is important when one considers issues regarding compliance with government regulations. Suppose it is assumed that the concentration of a certain pollutant, in parts per million, has a lognormal distribution with parameters $\mu = 3.2$ and $\sigma = 1$. What is the probability that the concentration exceeds 8 parts per million?

32. The life, in thousands of miles, of a certain type of electronic control for locomotives has an approximately lognormal distribution with $\mu = 5.149$ and $\sigma = 0.737$. Find the 5th percentile of the life of such an electronic control.

33. Let $X \sim \text{Wei}(\alpha, \beta)$. Show that

$$E(X) = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right), \quad \text{and} \quad V(X) = \alpha^{-2/\beta}\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

34. Let $X \sim \text{Wei}(\alpha, \beta)$. Show that

$$F(x) = 1 - e^{\alpha x^\beta} \qquad \text{for } x \geq 0$$

for $\alpha > 0, \beta > 0$.

35. Suppose that the service life, in years, of a hearing aid battery is a random variable having a Weibull distribution with $\alpha = 1/2$ and $\beta = 2$.
    (a) How long can such a battery be expected to last?
    (b) What is the probability that such a battery will be operating after 2 years?

36. The authors of the article "A Probabilistic Insulation Life Model for Combined Thermal-Electrical Stresses" (IEEE Trans. on Elect. Insulation, 1985: 519–522) state that "the Weibull distribution is widely used in statistical problems relating to aging of solid insulating materials subjected to aging and stress." They propose the use of the distribution as a model for time (in hours) to failure of solid insulating specimens subjected to AC voltage. The values of the parameters depend on the voltage and temperature; suppose $\alpha = 2.5$ and $\beta = 200$(values suggested by data in the article).
    (a) What is the probability that a specimen's lifetime is at most 250? Less than 250? More than 300?
    (b) What is the probability that a specimen's lifetime is between 100 and 250?
    (c) What value is such that exactly 50% of all specimens have lifetimes exceeding that value?