# I2–TD4
# (Joint Probability Distribution and Random Samples)

1. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let $X$ denote the number of hoses being used on the self-service island at a particular time, and let $Y$ denote the number of hoses on the full-service island in use at that time. The joint pmf of $X$ and $Y$ appears in the accompanying tabulation.

| $p(x,y)$ | | $y$ = 0 | 1 | 2 |
|---|---|---|---|---|
| | 0 | .10 | .04 | .02 |
| $x$ | 1 | .08 | .20 | .06 |
| | 2 | .06 | .14 | .30 |

   (a) What is $P(X = 1 \text{ and } Y = 1)$?
   (b) Compute $P(X \leq 1 \text{ and } Y \leq 1)$.
   (c) Give a word description of the event $\{X \neq 0 \text{ and } Y \neq 0\}$, and compute the probability of this event.
   (d) Compute the marginal pmf of $X$ and of $Y$. Using $P_X(x)$, what is $P(x \leq 1)$?
   (e) Are $X$ and $Y$ independent rv's? Explain.

2. The number of customers waiting for gift-wrap service at a department store is an rv $X$ with possible values 0, 1, 2, 3, 4 and corresponding probabilities .1, .2, .3, .25, .15. A randomly selected customer will have 1, 2, or 3 packages for wrapping with probabilities .6, .3, and .1, respectively. Let $Y$ = the total number of packages to be wrapped for the customers waiting in line (assume that the number of packages submitted by one customer is independent of the number submitted by any other customer).

   (a) Determine $P(X = 3, Y = 3)$, i.e,. $p(3, 3)$
   (b) Determine $p(4, 11)$.

3. Let $X$ denote the number of Canon digital cameras sold during a particular week by a certain store. The pmf of $X$ is

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_X(x)$ | .1 | .2 | .3 | .25 | .15 |

   Sixty percent of all customers who purchase these cameras also buy an extended warranty. Let $Y$ denote the number of purchasers during this week who buy an extended warranty.

   (a) What is $P(X = 4, Y = 2)$? [Hint: This probability equals $P(Y = 2|X = 4).P(X = 4)$; now think of the four purchases as four trials of a binomial experiment, with success on a trial corresponding to buying an extended warranty.]
   (b) Calculate $P(X = Y)$.
   (c) Determine the joint pmf of $X$ and $Y$ and then the marginal pmf of $Y$.

4. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable-$X$ for the right tire and $Y$ for the left tire, with joint pdf

$$f(x) = \begin{cases} K(x^2+y^2), & 20 \le x \le 30, 20 \le y \le 30 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of $K$?
   (b) What is the probability that both tires are undefiled?
   (c) What is the probability that the difference in air pressure between the two tires is at most 2 psi?
   (d) Determine the (marginal) distribution of air pressure in the right tire alone.
   (e) Are $X$ and $Y$ independent rv's?

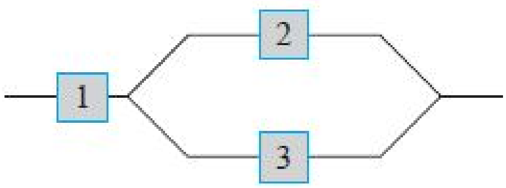
5. Two different professors have just submitted final exams for duplication. Let $X$ denote the number of typographical errors on the first professor's exam and $Y$ denote the number of such errors on the second exam. Suppose X has a Poisson distribution with parameter $\mu_1$, $Y$ has a Poisson distribution with parameter $\mu_2$, and $X$ and $Y$ are independent.

   (a) What is the joint pmf of $X$ and $Y$?
   (b) What is the probability that at most one error is made on both exams combined?
   (c) Obtain a general expression for the probability that the total number of errors in the two exams is m (where m is a nonnegative integer). [Hint: $A = \{(x, y) : x + y = m\}= \{(m, 0),(m - 1, 1), ...,(1, m - 1),(0, m)\}$. Now sum the joint pmf over$(x, y) \in A$ and use the binomial theorem, which says that

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} = (a+b)^m$$

   for any $a, b$]

6. Consider a system consisting of three components as pictured. The system will continue to function as long as the first component functions and either component 2 or component 3 functions. Let $X_1, X_2$, and $X_3$ denote the lifetimes of components 1, 2, and 3, respectively. Suppose the $X_i$ s are independent of one another and each $X_i$ has an exponential distribution with parameter l.




   (a) Let $Y$ denote the system lifetime. Obtain the cumulative distribution function of Y and differentiate to obtain the pdf. [Hint: $F(y) = P(Y \le y)$; express the event $\{Y \le y\}$ in terms of unions and/or intersections of the three events $\{X_1 \le y\}, \{X_2 \le y\}$, and $\{X_3 \le y\}$]

(b) Compute the expected system lifetime.

7. An ecologist wishes to select a point inside a circular sampling region according to a uniform distribution (in practice this could be done by first selecting a direction and then a distance from the center in that direction). Let $X$ = the $x$ coordinate of the point selected and $Y$ = the y coordinate of the point selected. If the circle is centered at $(0,0)$ and has radius $R$, then the joint pdf of $X$ and $Y$ is

$$f(x,y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the probability that the selected point is within $R/2$ of the center of the circular region? [Hint: Draw a picture of the region of positive density $D$. Because $f(x,y)$ is constant on $D$, computing a probability reduces to computing an area.]

(b) What is the probability that both $X$ and $Y$ differ from 0 by at most $R/2$?

(c) Answer part (b) for $R/\sqrt{2}$ replacing $R/2$.

(d) What is the marginal pdf of $X$? Of $Y$? Are $X$ and $Y$ independent?

8. The joint pdf of pressures for right and left front tires is given in Exercise 4.

(a) Determine the conditional pdf of $Y$ given that $X = x$ and the conditional pdf of $X$ given that $Y = y$.

(b) If the pressure in the right tire is found to be 22 psi, what is the probability that the left tire has a pressure of at least 25 psi? Compare this to $P(Y \leq 25)$.

(c) If the pressure in the right tire is found to be 22 psi, what is the expected pressure in the left tire, and what is the standard deviation of pressure in this tire?

9. An instructor has given a short quiz consisting of two parts. For a randomly selected student, let $X$ = the number of points earned on the first part and $Y$ = the number of points earned on the second part. Suppose that the joint pmf of $X$ and $Y$ is given in the accompanying table.

| | | $y$ | | | |
|---|---|---|---|---|---|
| $p(x,y)$ | | 0 | 5 | 10 | 15 |
| | 0 | .02 | .06 | .02 | .10 |
| $x$ | 5 | .04 | .15 | .20 | .10 |
| | 10 | .01 | .15 | .14 | .01 |

(a) If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score $E(X+Y)$?

(b) If the maximum of the two scores is recorded, what is the expected recorded score?

10. Annie and Alvie have agreed to meet for lunch between noon (0:00 P.M.) and 1:00 P.M. Denote Annie's arrival time by $X$, Alvie's by $Y$, and suppose $X$ and $Y$ are independent with pdf's

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person? [Hint: $h(X, Y) = |X - Y|$.]

11. (a) Compute the covariance between $X$ and $Y$ in Exercise 4.
    (b) Compute the correlation coefficient $\rho$ for this $X$ and $Y$.

12. (a) Use the rules of expected value to show that $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$.
    (b) Use part (a) along with the rules of variance and standard deviation to show that $\text{Corr}(aX + b, cY + d) = \text{Corr}(X, Y)$ when $a$ and $c$ have the same sign.
    (c) What happens if $a$ and $c$ have opposite signs?

13. A health-food store stocks two different brands of a certain type of grain. Let $X$ be the amount of brand $A$ on hand and $Y$ be the amount of brand $B$ on hand. Suppose the joint pdf of $X$ and $Y$ is

$$f(x, y) = \begin{cases} kxy, & x \ge 0, y \ge 0, 20 \le x + y \le 30 \\ 0, & \text{otherwise} \end{cases}$$

    (a) Draw the region of positive density and determine the value of $k$.
    (b) Are $X$ and $Y$ independent? Answer by first deriving the marginal pdf of each variable.
    (c) Compute $P(X + Y \le 25)$.
    (d) What is the expected total amount of this grain on hand?
    (e) Compute $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$.
    (f) What is the variance of the total amount of grain on hand?

14. Tow real-valued rv's, $X$ and $Y$, have joint pdf

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - r^2}} \exp\left[-\frac{1}{2(1 - r^2)}\left(x^2 - 2rxy + y^2\right)\right]$$

where $-1 < r < 1$.

    (a) Prove that each of $X$ and $Y$ is normally distributed with mean 0 and variance 1.
    (b) Prove that the number $r$ is the correlation coefficient of $X$ and $Y$.

15. (a) Continuous rv's $X$ and $Y$ have a joint pdf

$$f(x, y) = \frac{(m + n + 2)!}{m!n!}(1 - x)^m y^n$$

for $0 < y \le x < 1$, where $m, n$ are given positive integers. Check that $f$ is a proper pdf. Find the marginal distributions of $X$ and $Y$. Hence calculate

$$P\left(Y \le \frac{1}{3} \middle| X = \frac{2}{3}\right).$$

(b) Let $X$ and $Y$ be rv's. Check that

$$\text{Cov}(X,Y) = E(1-X) \times E(Y) - E[(1-X)Y]$$

(c) Let $X, Y$ be as in (a). Use the form of $f(x,y)$ to express the expectations $E(1-X)$, $E(Y)$ and $E[(1-X)Y]$ in terms of factorials. Using (b), or otherwise, show that

$$\text{Cov}(X,Y) = \frac{(m+1)(n+1)}{(m+n+3)^2(m+n+4)}$$

16. There are 40 students in an elementary statistics class. On the basis of years of experience, the instructor knows that the time needed to grade a randomly chosen first examination paper is a random variable with an expected value of 6 min and a standard deviation of 6 min.

    (a) If grading times are independent and the instructor begins grading at $6:50$ p.m. and grades continuously, what is the (approximate) probability that he is through grading before the $11:00$ p.m. TV news begins?

    (b) If the sports report begins at $11:10$, what is the probability that he misses part of the report if he waits until grading is done before turning on the TV?

17. The number of parking tickets issued in a certain city on any given weekday has a Poisson distribution with parameter $\lambda = 50$.

    (a) Calculate the approximate probability that between 35 and 70 tickets are given out on a particular day.

    (b) Calculate the approximate probability that the total number of tickets given out during a 5-day week is between 225 and 275.

    (c) Use software to obtain the exact probabilities in (a) and (b) and compare to their approximations.

18. Let $X_1, X_2$, and $X_3$ represent the times necessary to perform three successive repair tasks at a certain service facility. Suppose they are independent, normal rv's with expected values $\mu_1, \mu_2$, and $\mu_3$ and variances $\sigma_1^2, \sigma_2^2$ and $\sigma_3^2$, respectively.

    (a) If $\mu_i = 60$ and $\sigma_i = 15; i = 1,2,3$. Calculate $P(T_0 \leq 200)$ and $P(150 \leq T_0 \leq 200)$

    (b) Using the $\mu_i$'s and $\sigma_i$'s given in part (a), calculate $P(55 \leq \overline{X})$ and $P(58 \leq \overline{X} \leq 62)$

    (c) Using the $\mu_i$'s and $\sigma_i$'s given in part (a), calculate $P(-10 \leq X_1 - .5X_2 - .5X_3 \leq 5)$

    (d) If $\mu_1 = 40, \mu_2 = 50, \mu_3 = 60, \sigma_1^2 = 10, \sigma_2^2 = 12$ and $\sigma_3^2 = 14$; calculate $P(X_1 + X_2 + X_3 \leq 160)$ and $P(X_1 + X_2 \geq 2X_3)$

19. Manufacture of a certain component requires three different machining operations. Machining time for each operation has a normal distribution, and the three times are independent of one another. The mean values are $15, 30$, and 20 min, respectively, and the standard deviations are $1, 2$, and 1.5 min, respectively. What is the probability that it takes at most 1 hour of machining time to produce a randomly selected component?