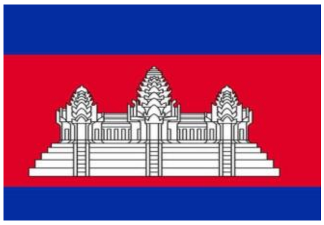
National Religion King



Institute of Technology of Cambodia

Department of Applied Mathematics and Statistic

(Option Data Science)

Assignment of Probability

Probability, Discrete RV, Continuous RV, Joint Probability Distribution

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I2–TD1
(Probability)

1. Suppose that vehicles taking a particular freeway exit can turn right (R), turn left (L), or go straight (S). Consider observing the direction for each of three successive vehicles.

   (a) List all outcomes in the event A that all three vehicles go in the same direction.

   (b) List all outcomes in the event B that all three vehicles take different directions. I

   (c) List all outcomes in the event C that exactly two of the three vehicles turn right. (d) List all outcomes in the event D that exactly two vehicles go in the same direction. (e) List outcomes in D^0, $C \cup D$, and $C \cap D$.

2. An engineering construction firm is currently working on power plants at three different sites. Let A_i denote the event that the plant at site i is completed by the contract date. Use the operations of union, intersection, and complementation to describe each of the following events in terms of A_1, A_2, and A_3, draw a Venn diagram, and shade the region corresponding to each one.

   (a) At least one plant is completed by the contract date.

   (b) All plants are completed by the contract date.

   (c) Only the plant at site 1 is completed by the contract date.

   (d) Exactly one plant is completed by the contract date.

   (e) Either the plant at site 1 or both of the other two plants are completed by the contract date.

3. An academic department has just completed voting by secret ballot for a department head. The ballot box contains four slips with votes for candidate A and three slips with votes for candidate B. Suppose these slips are removed from the box one by one.

   (a) List all possible outcomes.

   (b) Suppose a running tally is kept as slips are removed. For what outcomes does A remain ahead of B throughout the tally?

4. Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a MasterCard. Suppose that $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.25$.

   (a) Compute the probability that the selected individual has at least one of the two types of cards (i.e., the probability of the event $A \cup B$).

   (b) What is the probability that the selected individual has neither type of card?

   (c) Describe, in terms of A and B, the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.

5. Suppose that 55% of all adults regularly consume coffee, 45% regularly consume carbonated soda, and 70% regularly consume at least one of these two products.

   (a) What is the probability that a randomly selected adult regularly consumes both coffee and soda?

   (b) What is the probability that a randomly selected adult doesn’t regularly consume at least one of these two products?

6. Consider the type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.

(a) If the probability that at most one of these purchases an electric dryer is .428, what is the probability that at least two purchase an electric dryer?

(b) If and P(all five purchase gas) =0.116 and P(all five purchase electric) =0.005, what is the probability that at least one of each type is purchased?

7. An individual is presented with three different glasses of cola, labeled C,D, and P. He is asked to taste all three and then list them in order of preference. Suppose the same cola has actually been put into all three glasses.

(a) What are the simple events in this ranking experiment, and what probability would you assign to each one?

(b) What is the probability that C is ranked first?

(c) What is the probability that C is ranked first and D is ranked last?

8. Let A denote the event that the next request for assistance from a statistical software consultant relates to the SPSS package, and let B be the event that the next request is for help with SAS. Suppose that P(A) = .30 and P(B) = 0.50.

(a) Why is it not the case that P(A) + P(B) = 1?

(b) Calculate $P(A^0)$.

(c) Calculate $P(A \cup B)$.

(d) Calculate $P(A^0 \cap B^0)$.

9. The three most popular options on a certain type of new car are a built-in GPS (A), a sunroof (B), and an automatic transmission (C). If 40% of all purchaser's request A, 55request B, 70% request C, 63% request A or B, 77% request A or C, 80% request B or C, and 85% request A or B or C, determine the probabilities of the following events. [Hint: "A or B" is the event that at least one of the two options is requested; try drawing a Venn diagram and labeling all regions.]

(a) The next purchaser will request at least one of the three options.

(b) The next purchaser will select none of the three options.

(c) The next purchaser will request only an automatic transmission and not either of the other two options.

(d) The next purchaser will select exactly one of these three options.

10. An academic department with five faculty members— Anderson, Box, Cox, Cramer, and Fisher—must select two of its members to serve on a personnel review committee. Because the work will be time-consuming, no one is anxious to serve, so it is decided that the representative will be selected by putting the names on identical pieces of paper and then randomly selecting two.

(a) What is the probability that both Anderson and Box will be selected? [Hint: List the equally likely outcomes.]

(b) What is the probability that at least one of the two members whose name begins with C is selected?

(c) If the five faculty members have taught for 3, 6, 7, 10, and 14 years, respectively, at the university, what is the probability that the two chosen representatives have a total of at least 15 years teaching experience there?

11. Certain system can experience three different types of defects. Let A_i (i = 1,2,3) denote the event that the system has a defect of type i. Suppose that $P(A_1) = 0.12$, $P(A_2) = 0.07$, $P(A_3) = 0.05$, $P(A_1 \cup A_2) = 0.13$, $P(A_1 \cup A_3) = 0.14$, $P(A_2 \cup A_3) =$
0.10, $P(A_1 \cap A_2 \cap A_3) = 0.01$.

(a) What is the probability that the system does not have a type 1 defect?

(b) What is the probability that the system has both type 1 and type 2 defects?

(c) What is the probability that the system has both type 1 and type 2 defects but not a type 3 defect?

(d) What is the probability that the system has at most two of these defects?

12. Pollution of the rivers in Cambodia has been a problem for many years. Consider the following events: A: the river is polluted; B: a sample of water tested detects pollution; C: fishing is permitted.
Assume $P(A) = 0.3$, $P(B|A) = 0.75$, $P(B|A^0) = 0.20$,$P(C|A \cap B) = 0.20$,$P(C|A^0 \cap B) = 0.15$,$P(C|A \cap B^0) = 0.80$, and $P(C|A^0 \cap B^0) = 0.90$.

(a) Find $P(A \cap B \cap C)$.

(b) Find $P(B^0 \cap C)$.

(c) Find $P(C)$.

(d) Find the probability that the river is polluted, given that fishing is permitted and the sample tested did not detect pollution.

13. A production facility employs 20 workers on the day shift, 15 workers on the swing shift, and 10 workers on the graveyard shift. A quality control consultant is to select 6 of these workers for in-depth interviews. Suppose the selection is made in such a way that any particular group of 6 workers has the same chance of being selected as does any other group (drawing 6 slips without replacement from among 45).

(a) How many selections result in all 6 workers coming from the day shift? What is the probability that all 6 selected workers will be from the day shift?

(b) What is the probability that all 6 selected workers will be from the same shift?

(c) What is the probability that at least two different shifts will be represented among the selected workers?

(d) What is the probability that at least one of the shifts will be unrepresented in the sample of workers?

14. An academic department with five faculty members narrowed its choice for department head to either candidate A or candidate B. Each member then voted on a slip of paper

for one of the candidates. Suppose there are actually three votes for A and two for B. If the slips are selected for tallying in random order, what is the probability that A remains ahead of B throughout the vote count (e.g., this event occurs if the selected ordering is AABAB, but not for ABBAA)?

15. In five-card poker, a straight consists of five cards with adjacent denominations (e.g., 9 of clubs, 10 of hearts, jack of hearts, queen of spades, and king of clubs). Assuming that aces can be high or low, if you are dealt a five-card hand, what is the probability that it will be a straight with high card 10? What is the probability that it will be a straight? What is the probability that it will be a straight flush (all cards in the same suit)?

16. Suppose an individual is randomly selected from the population of all adult males living in the United States. Let A be the event that the selected individual is over 6 ft in height, and let B be the event that the selected individual is a professional basketball player. Which do you think is larger, $P(A|B)$ or $P(B|A)$? Why?

17. A certain shop repairs both audio and video components. Let A denote the event that the next component brought in for repair is an audio component, and let B be the event that the next component is a compact disc player (so the event B is contained in A). Suppose that $P(A) = 0.6$ and $P(B) = 0.05$. What is $P(B|A)$?

18. Deer ticks can be carriers of either Lyme disease or human granulocytic ehrlichiosis (HGE). Based on a recent study, suppose that 16% of all ticks in a certain location carry Lyme disease, 10% carry HGE, and 10% of the ticks that carry at least one of these diseases in fact carry both of them. If a randomly selected tick is found to have carried HGE, what is the probability that the selected tick is also a carrier of Lyme disease?

19. A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, he needs to answer all three questions. What is the probability that the student will pass the examination if he knows the answers to 90 questions on the list?

20. The Reviews editor for a certain scientific journal decides whether the review for any particular book should be short (1–2 pages), medium (3–4 pages), or long (5–6 pages). Data on recent reviews indicates that 60% of them are short, 30are medium, and the other 10% are long. Reviews are submitted in either Word or LaTeX. For short reviews, 80% are in Word, whereas 50% of medium reviews are in Word and 30% of long reviews are in Word. Suppose a recent review is randomly selected.

    (a) What is the probability that the selected review was submitted in Word format?

    (b) If the selected review was submitted in Word format, what are the posterior probabilities of it being short, medium, or long?

21. We have two dice, a red and a white one. The red die is unfair so the probability of getting 6 is $\frac{1}{3}$, the remaining outcomes being equally likely among themselves. The white die is fair. We can pick one die, leaving the other one to the opponent. The dice are rolled and the player with the highest result wins. In case of equal results, the player with the white die wins. Which die does one choose to have a better chance of winning the game?

22. Suppose that you have three coins, two fair ones but the third biased with probability of heads p and tails $1 - p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins?

23. If A and B are independent events, show that A^0 and B are also independent. [Hint: First establish a relationship between $P(A^0 \cap B)$, $P(B)$, and $P(A \cap B)$.]

24. An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.

    (a) If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.

    (b) What is the probability that at least one of the two projects will be successful?

    (c) Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

25. Consider independently rolling two fair dice, one red and the other green. Let A be the event that the red die shows 3 dots, B be the event that the green die shows 4 dots, and C be the event that the total number of dots showing on the two dice is 7. Are these events pairwise independent (i.e., are A and B independent events, are A and C independent, and are B and C independent)? Are the three events mutually independent?

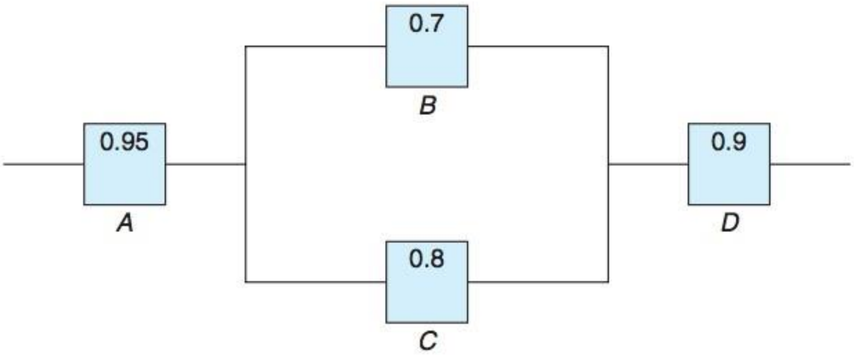
26. A fair coin is tossed three times and the following events are considered:

    A = toss 1 and toss 2 produce different outcomes,

    B = toss 2 and toss 3 produce different outcomes, C = toss 3 and toss 1 produce different outcomes.

    Show that $P(A) = P(A|B) = P(A|C)$, but $P(A)$ 6= $P(A|B \cap C)$.

27. Suppose the diagram of an electrical system is as given in Figure. What is the probability that the system works? Assume the components fail independently.



Solution

I2-TD-1.

(Probability).

1. Let : R turn right , L turn Left , S go straight.

(a). List all out come even A all vehicle go in the same direction

$A = \{(RRR), (L,LL), (SSS)\}$

(b) Ourt come event B that all three vehicle go different direction

$B = \{(RSL)(SLR), (LSR), (RLS), (LRS), (SRL)\}$

(c) event C. two of three turn right.

$C = \{(RRS), (RRL), (RSR), (RLR), (SRR), (LRR)\}$

(d). event D exacly two go the same direction.

$D = \{(RRS)(RRL)(RSR), (RLR)(SRR), (LRR), (LLS)(LLR), (LSL)$

$(LRL), (RLL), (SLL), (SSR)(SSL), (SRS), (SLS), (RSS), (LSS)\}$

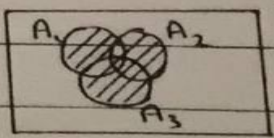
(e). List out come in D', $C \cup D$, $C \cap D$

. $C \cup D = D$. $C \cap D = C$

2. A_i de note the event that the plan at site i is completed.

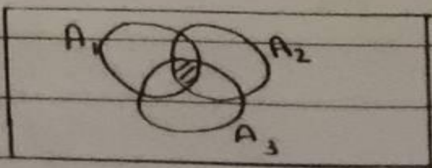
(a). at least one plan is completed by the contract date.

$A = A_1 \cup A_2 \cup A_3$



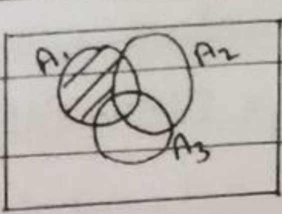
(b) All plan are completed by the contract date.

$B = A_1 \cap A_2 \cap A_3$



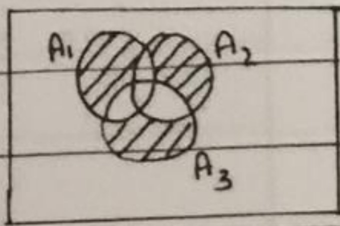
(c). Only the plan at site 1 is complete by the contract date.

$C = A_1 \cap \bar{A}_2 \cap \bar{A}_3$



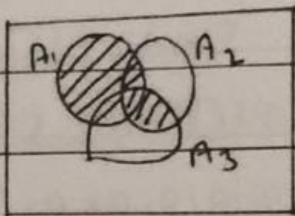
(d). exacly one plan is complete by the contract.

$D = (A_1 \cap \bar{A}_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap A_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap \bar{A}_2 \cap A_3)$



(e) Either the plant at site 1 or both of the orther two plans are completed by the contract date.

$E = A_1 \cup (A_2 \cap A_3)$



3. (a). List all outcomes.

$n(A) = \dfrac{7!}{3!\,4!} = 35$

$\Rightarrow A = \{(AAAABBB), (AAABBBA) \ldots (BBBAAAA)\}$

(b). Out come that A remain ahead of B.

$A' = \{(AAAABBB), (AAABABB), (AAABBAB), (AABABAB)$

$(AABAABB)\}$

4. A has a visa card

B has a muster card.

(a). Compute. $P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0.25 = 0.65$

(b). $P(\overline{A \cup B})$ has neither type of card.

$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.65 = 0.35.$

(c). Event that selected student has a visa card but not a master card.

$$P(A\cap\bar{B}) = P(A) - P(A\cap B) = 0.5 - 0.25 = 0.25.$$

5. (a). Calculate $P(C\cap S)$.

we have $P(C\cup S) = P(C) + P(S) - P(C\cap S)$

$\Rightarrow P(C\cap S) = P(C) + P(S) - P(C\cup S)$

$= 0.55 + 0.45 - 0.7. = 0.4.$

(b). Find $P(\overline{C\cup S})$

$P(\overline{C\cup S}) = 1 - P(C\cup S) = 1 - 0.7 = 0.3.$

6. (a). Find the probability that at least two purchase Electric.

$P(E\geq 2) = 1 - P(E\leq 1) = 1 - 0.428 = 0.572$

(b). Find the probability that at least one of each type is purchased

$P(A) = 1 - P(E=5) - P(G=5) = 1 - 0.116 - 0.005 = 0.879$

7. (a) Find P(each one)

$S = \{(CDP), (CPD), (PCD), (PDC), (DPC), (DCP)\} = 6.$

$\Rightarrow P(\text{each one}) = \frac{1}{6}$

(b). $P(C^{1st})$.

$\Rightarrow P(C^{1st}) = \frac{2}{6} = \frac{1}{3}$

(c). $P(C^{1st} \text{ and } D^{3rd})$

$\Rightarrow P(C^{1st} \text{ and } D^{3rd}) = \frac{1}{6}$

8. A: SPSS . B: SAS , $P(A)=0.30$, $P(B)=0.50$.

(a). $P(A)+P(B) \neq 1$ Because there're more services.

(b). Caculate $P(A') = P(\bar{A})$

$P(A') = 1 - P(A) = 1 - 0.3 = 0.7$

(c) Caculate $P(A \cup B)$.

$P(A \cup B) = P(A) + P(B) = 0.8$.

(d) Calculate $P(A' \cap B')$

$P(A' \cap B') = 1 - P(\overline{A \cup B}) = 1 - 0.8 = 0.2$.

9. Given: $P(A)=0.4$, $P(B)=0.55$, $P(C)=0.70$, $P(A \cup B)=0.63$

$P(A \cup C) = 77\% = 0.77$, $P(B \cup C) = 0.80$, $P(A \cup B \cup C) = 0.85$

(a) at least one of three options.

$\Rightarrow P(A \cup B \cup C) = 0.85$

(b) non-of three opction.

$\Rightarrow P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\overline{A \cup B \cup C}) = 1 - 0.85 = 0.15$

(c) only (C) and not either of two opction.

$\Rightarrow P(C \cap \bar{A} \cap \bar{B}) = P(C \cap \overline{A \cup B}) = P(C) - P(C \cap A \cup B)$

$= P(C) - P(C) - P(A \cup B) + P(C \cup A \cup B)$

$= P(C \cup A \cup B) - P(A \cup B)$

$\Rightarrow P(C \cap \bar{A} \cap \bar{B}).$ $= 0.85 - 0.63 = 0.22$

(d) exacly one of three opction.

$\Rightarrow P(\text{one of three options}) = P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$

$= 3P(A \cup B \cup C) - P(A \cup B) - P(B \cup C) - P(A \cup C)$

$= 3(0.85) - 0.63 - 0.77 - 0.8 = 0.35$

10. A: Anderson, B: Box, C: Cox, D: Cramer, E: Fishermust

(a) probability both A and B will be seleted.

$P(A) = \frac{1}{10}$

(b). at least one of two member name begen with C.

$B = \{CA, CB, CF, CE, CD, DA, DB, DC\} = 7.$

$\Rightarrow P(B) = \frac{7}{10}$

(c). The probability that total 15 years :

$C = \{(AF), (BE), (BF), (CE)(CF), (EF)\}$

$\Rightarrow P(C) = \frac{6}{10} = \frac{3}{5}.$

11. Find the probability

(a) the system doesn't have type 1 defet

$P(A_1') = 1 - P(A_1) = 1 - 0.12 = 0.88.$

(b). The system have both type 1 and 2 defets.

$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = 0.12 + 0.07 - 0.13 = 0.06$

(c). The probability has type 1 & 2 defet but not type 3.

$P(A_1 \cap A_2 \cap \bar{A}_3) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$

$= 0.06 - 0.01 = 0.05$

(d). At most two of this defets.

$P(D) = 1 - P(A_1 \cap A_2 \cap A_3)$

$= 1 - 0.01 = 0.99.$

12. (a). Find $P(A\cap B\cap C)$.

we have $P(A|B) = \dfrac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(B)P(A|B)$

$\Rightarrow P(A\cap B\cap C) = P(A)P(B|A)P(C|A\cap B)$

Note $P(A_1\cap A\cap \dots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\dots P(A_1|A_1\cap\dots\cap A_{n-1})$

$\Rightarrow P(A\cap B\cap C) = (0.3)(0.75)(0.20) = 0.045$

(b). Find $P(B'\cap C)$

$P(B'\cap C) = P(C) - P(B\cap C)$.

$= P(\bar{B}\cap C\cap A) + P(\bar{B}\cap C\cap \bar{A})$

$= P(A)P(\bar{B}|A)P(C|\bar{B}\cap A) + P(\bar{A})P(\bar{B}|\bar{A})P(C|\bar{A}\cap\bar{B})$

$= (0.3)(0.25)(0.8) + (0.7)(0.8)(0.9) = 0.564$.

(c). Find $P(C)$.

$P(C) = P(C\cap B) + P(C\cap\bar{B})$

- $P(C\cap B) = P(C\cap B\cap A) + P(C\cap B\cap\bar{A})$

$= P(A\cap B\cap C) + P(\bar{A})P(B|\bar{A})P(C|\bar{A}\cap B)$.

$= 0.045 + (0.7)(0.2)(0.15) = 0.066$.

(d). Find $P(A|C\cap\bar{B})$.

$P(A|C\cap\bar{B}) = \dfrac{P(A\cap C\cap\bar{B})}{P(C\cap\bar{B})} = \dfrac{P(A)P(\bar{B}|A)P(C|A\cap\bar{B})}{P(C\cap\bar{B})}$

$= \dfrac{(0.3)(0.8)(0.8)}{0.8} = 0.24$.

13. (a). Find the selecting result from the day shift. And Find the probability A of it.

$n(\text{day shift}) = \binom{20}{6} = 38760$ ways

$$P(A) = \frac{\binom{20}{6}}{\binom{45}{6}} = 0.0048.$$

(b). B: All 6 workers from the same shift

$$P(B) = \frac{\binom{20}{6}\binom{15}{6}\binom{10}{6}}{\binom{45}{6}}$$

(c). C. At least 2 shifts are selected.

$$P(C) = 1 - P(\text{At most 1 shift})$$

$$= 1 - P(B)$$

$$= 1 - \frac{\binom{20}{6} + \binom{15}{6} + \binom{10}{6}}{\binom{45}{6}}$$

(d). D At least one of shifts will be unrepresented.

$$P(D) = \frac{C_{20}^6 + C_{15}^6 + C_{10}^6 + (C_{35}^6 - C_{20}^6 - C_{15}^6) + (C_{30}^6 - C_{10}^6 - C_{20}^6) + (C_{25}^6 - C \ldots}{C_{45}^6}$$

$$= \frac{C_{35}^6 + C_{30}^6 + C_{25}^6 - C_{20}^6 - C_{15}^6 - C_{10}^6}{C_{45}^6}$$

14. Find the probability that A remain ahade of B.

$n(S) = \frac{5!}{2!3!} = 10$ Let: $D = \{(AABAB), (AAABB)\}$

$$\Rightarrow P(D) = \frac{2}{10} = \frac{1}{5}.$$

15 Find probability:

+ A: It will straight with high card 10 ~~de~~

$P(A) = \frac{4^5}{C_{52}^5}$

+ B. It will be straight

$P(B) = \frac{4^5}{C_{52}^5}$

+ C: It will be straight flush.

$P(C) = \frac{4 \times 10}{C_{52}^5}$

16 It's exactly $P(A|B) > P(B|A)$. because we actully know that the Pro- basketball is higher than 6ft.

17. Find : $P(B|A)$.

$P(B|A) = \frac{P(B \cap A)}{P(A)}$ since $B \subset A \Rightarrow P(B \cap A) = P(B)$.

$\Rightarrow P(B|A) = \frac{P(B)}{P(A)} = \frac{0.05}{0.6} = 0.083$.

18. Let : L : Lyme disease

H : HGE , $P(L) = 0.16$, $P(H) = 0.1$, $P(L \cap H | L \cup H) = 0.1$

$\Rightarrow P(L|H) = \frac{P(L \cap H)}{P(H)}$

$\cdot P(L \cap H | L \cup H) = \frac{P[(L \cap H) \cap (L \cup H)]}{P(L \cup H)} = \frac{P(L \cap H)}{P(L \cup H)} = 0.1$

Page

$\Rightarrow P(L \cap H) = 0.1\,(P(L) + P(H) - P(L \cap H))$

$\rightarrow P(L | H) = \frac{0.036}{0.16} = 0.225.$

19. Find the probability that the student can pass.

$$P(A \cap B \cap C) = \frac{\binom{90}{1} \times \binom{89}{1} \times \binom{88}{1}}{\binom{100}{1} \times \binom{99}{1} \times \binom{98}{1}}$$

20. Let $P(A) = 60\%$, $P(B) = 30\%$, $P(C) = 10\%$, $P(\text{word}|A) = 80\%$

$P(\text{word}|B) = 50\%$, $P(\text{word}|C) = 30\%$

a) the selected review in word

$P(\text{word}) = P(\text{word}|A) \times P(A) + P(\text{word}|B)P(B) + P(\text{word}|C)P(C)$

$= 0.8 \times 0.6 + 0.3 \times 0.1 + 0.5 \times 0.3 = 0.66.$

b). what is the probability of being short, medium, Long?

$+ P(A|\text{word}) = \frac{P(W|A) \cdot P(A)}{P(\text{word})} = \frac{0.8 \times 0.6}{0.66} = 0.727$

$+ P(B|\text{word}) = \frac{P(\text{word}|B) \times P(B)}{P(\text{word})} = \frac{0.5 \times 0.3}{0.66} = 0.227.$

$+ P(C|\text{word}) = \frac{P(\text{word}|C) \times P(C)}{P(\text{word})} = \frac{0.3 \times 0.1}{0.66} = 0.045$

21. which die does the one has more chance to win?

Let W be the event that the white die win the game

Let R_i be the event that the red die showing result i $[1,6]$

$P(R_6) = \frac{1}{3}$, $P(R_i) = \frac{2}{15}$; $i = 1 \dots 5$

$$\Rightarrow P(W) = \sum_{i=1}^{6} P(W|R_i)P(R_i)$$

$$= \left(\frac{6}{6}\right)\left(\frac{2}{15}\right) + \left(\frac{5}{6}\right)\left(\frac{2}{15}\right) + \left(\frac{4}{6}\right)\left(\frac{2}{15}\right) + \left(\frac{3}{6}\right)\left(\frac{2}{15}\right) + \left(\frac{2}{6}\right)\left(\frac{2}{15}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{3}\right)$$

$$= 0.5.$$

Thus They have the equal Chance to win.

22. H = Head , T = Tail , F = fair , U = unfair.

$$P(F|H) = \frac{P(F\cap H)}{P(H)} = \frac{P(F)P(H|F)}{P(H|F)P(F) + P(H|U)P(U)}$$

$$= \frac{\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + P\left(\frac{1}{3}\right)} = \frac{1}{1+P}$$

23. show that $\bar{A}$ & B are independant.

$P(\bar{A}\cap B) = P(B) - P(A\cap B)$, A & B are independent.

$= P(B) - P(A)P(B)$

$= P(B)(1 - P(A))$

$= P(B)P(\bar{A})$

Thus B & $\bar{A}$ are independant.

24. (a) Find the probability that the euro project not succesful.

$$P(\bar{B}|\bar{A}) = \frac{P(\bar{B}\cap\bar{A})}{P(\bar{A})} = \frac{P(\bar{B})P(\bar{A})}{P(\bar{A})} = P(\bar{B}) = 1-P(B)$$
$$= 1-0.7 = 0.3.$$

(b). At least one of two project is succes

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$
$$= P(A)+P(B)-P(A)P(B)$$
$$= 0.4+0.7-0.4\times 0.7$$
$$= 0.89.$$

(c) At least one of two project is succesful, Find $P(A\cap\bar{B}|A\cup B)$.

$$\Rightarrow P(A\cap\bar{B}|A\cup B) = \frac{P[(A\cap\bar{B})\cap(A\cup B)]}{P(A\cup\bar{B})}$$

$$= \frac{P[(A\cap\bar{B}\cap A)\cup(A\cap\bar{B}\cap B)]}{P(A\cup B)}$$

$$= \frac{P(A\cap\bar{B})}{P(A\cup B)} = \frac{0.3\times 0.4}{0.89} = 0.146.$$

25. we have $P(A)=\frac{1}{6}$, $P(B)=\frac{1}{6}$

+ $P(C) = P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = 6.$

$$= \frac{6}{36} = \frac{1}{6}.$$

+ $P(A\cap B) = P(\{3,4\}) = \frac{1}{36} = P(A)P(B)$

$\Rightarrow$ A & B is independent.

+ $P(B\cap C) = P(B)\cdot P(C) = \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$

=) ~~Pts~~ B & C are independant.

+ $P(A\cap B\cap C) = P(\{(3,4)\}) \neq P(A)P(B)P(C)$

Thus A, B & C are not independant.

26. Show that $P(A) = P(A|B) = P(A|C)$ but $P(A) \neq P(A|B\cap C)$

$S = \{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT\}$

$A = \{HTH, HTT, THH, THT\}$

$B = \{HHT, THT, HTH, TTH\}$

$C = \{HHT, HTT, THH, TTH\}$

+ $P(A) = \frac{4}{8} = \frac{1}{2}$

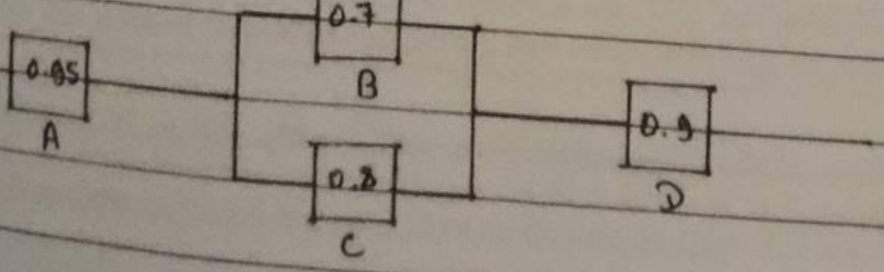
+ $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2/8}{4/8} = \frac{1}{2}$

+ $P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{2/8}{4/8} = \frac{1}{2}$

+ $P(A\cap B\cap C) = 0$

∴ Therefore $P(A) = P(A|B) = P(A|C) \neq P(A\cap B\cap C)$

27. Find the probability that system work.



$\Rightarrow P[A\cap(B\cup C)\cap D] = P[A\cap D\cap(B\cup C)]$

$= P[(A\cap D\cap B)\cup(A\cap D\cap C)]$

$= P(A\cap D\cap B)+P(A\cap D\cap C)-P(A\cap B\cap C\cap D)$

or $= P(A)\times P(B\cup C)\times P(D)$

$= 0.95\times(1.5-0.56)\times 0.9$

$= 0.8037.$

I2–TD2
(Discrete R.V and Probability Distribution)

1. A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the pmf of X is as given in the accompanying table.

x	0	1	2	3	4	5	6
p(x)	0.10	0.15	0.20	0.25	0.20	0.06	0.04

Calculate the probability of each of the following events.

(a) At most three lines are in use. are in use.

(b) Fewer than three lines are in use. (e) Between two and four lines, inclusive, (c) At least three lines are in use. are not in use.

(d) Between two and five lines, inclusive, (f) At least four lines are not in use.

2. A group of four components is known to contain two defectives. An inspector tests the components one at a time until the two defectives are located. Let X denote the number of the test on which the second defective is found. Find the probability distribution for X.

3. A problem in a test given to small children asks them to match each of three pictures of animals to the word identifying that animal. If a child assigns the three words at random to the three pictures, find the probability distribution for Y , the number of correct matches.

4. In order to verify the accuracy of their financial accounts, companies use auditors on a regular basis to verify accounting entries. The company's employees make erroneous entries 5% of the time. Suppose that an auditor randomly checks three entries.

   (a) Find the probability distribution for X , the number of errors detected by the auditor.

   (b) Find the probability that the auditor will detect more than one error.

5. A new battery's voltage may be acceptable (A) or unacceptable (U). A certain flashlight requires two batteries, so batteries will be independently selected and tested until two acceptable ones have been found. Suppose that 90% of all batteries have acceptable voltages. Let Y denote the number of batteries that must be tested.

   (a) What is p(2), that is, P(Y = 2)?

   (b) What is p(3)? [Hint: There are two different outcomes that result in Y = 3.]

   (c) To have Y = 5, what must be true of the fifth battery selected? List the four outcomes for which Y = 5 and then determine p(5).

   (d) Use the answers for parts (a)-(c) to obtain a general formula for p(y).

6. An insurance company offers its policyholders a number of different premium payment options. For a randomly selected policyholder, let X be the number of months between successive payments. The cdf

of X $F(x) = \begin{cases} 0, & x < 1 \\ 0.10, & 1 \le x < 3 \\ 0.40, & 3 \le x < 7 \\ 0.80, & 7 \le x < 12 \\ 1, & 12 \le x. \end{cases}$

What is the pmf of X?

(a) Using just the cdf, compute $P(3 \le X \le 6)$ and $P(6 \le X)$.

7. An Internet survey estimates that, when given a choice between English and French, 60% of the population prefers to study English. Three students are randomly selected and asked which of the two languages they prefer.

   (a) Find the probability distribution for Y , the number of students in the sample who prefer English.

   (b) What is the probability that exactly one of the three students prefer English?

   (c) What are the mean and standard deviation for Y ?

   (d) What is the probability that the number of students prefer English falls within 2 standard deviations of the mean?

8. In a gambling game a person draws a single card from an ordinary 52-card playing deck. A person is paid \$15 for drawing a jack or a queen and \$5 for drawing a king or an ace. A person who draws any other card pays \$4. If a person plays this game, what is the expected gain?

9. An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks for breeding. Assume yaks are loners and roam about Himalayas at random. The probability $p \in (0,1)$ that a given trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a pair is obtained.

   (a) Show that the expected value of N is $1 + p/q + q/p$, where $q = 1 - p$.

   (b) Find the variance of N.

10. A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has four identical components, each with a probability of 0.2 of failing in less than 1000 hours. The subsystem will operate if any two of the four components are operating. Assume that the components operate independently. Find the probability that

   (a) exactly two of the four components last longer than 1000 hours.

   (b) the subsystem operates longer than 1000 hours.

11. Many utility companies promote energy conservation by offering discount rates to consumers who keep their energy usage below certain established subsidy standards. A recent report notes that 70% of the people live in Phnom Penh have reduced their electricity usage sufficiently to qualify for discounted rates. If five residential subscribers are randomly selected from Phnom Penh, find the probability of each of the following events:

   (a) All five qualify for the favorable rates.

   (b) At least four qualify for the favorable rates.

   (c) At least two do not qualify the favorable rates.

12. Knowing that 80% of the volunteers donating blood in a clinic have type A blood.

(a) If five volunteers are randomly selected, what is the probability that at least one does not have type A blood?

(b) If five volunteers are randomly selected, what is the probability that at most four have type A blood?

(c) What is the smallest number of volunteers who must be selected if we want to be at least 90% certain that we obtain at least five donors with type A blood?

13. An electronics store has received a shipment of 20 table radios that have connections for an iPod or iPhone. Twelve of these have two slots (so they can accommodate both devices), and the other eight have a single slot. Suppose that six of the 20 radios are randomly selected to be stored under a shelf where the radios are displayed, and the remaining ones are placed in a storeroom. Let X= the number among the radios stored under the display shelf that have two slots.

    (a) What kind of a distribution does X have (name and values of all parameters)?

    (b) Compute $P(X = 2)$,$P(X \leq 2)$, and $P(X \geq 2)$.

    (c) Calculate the mean value and standard deviation of X.

14. A manufacturing company uses an acceptance scheme on items from a production line before they are shipped. The plan is a two-stage one. Boxes of 25 items are readied for shipment, and a sample of 3 items is tested for defectives. If any defectives are found, the entire box is sent back for 100% screening. If no defectives are found, the box is shipped.

    (a) What is the probability that a box containing 3 defectives will be shipped?

    (b) What is the probability that a box containing only 1 defective will be sent back for screening?

15. In an assembly-line production of industrial robots, gearbox assemblies can be installed in one minute each if holes have been properly drilled in the boxes and in ten minutes if the holes must be redrilled. Twenty gearboxes are in stock, 2 with improperly drilled holes. Five gearboxes must be selected from the 20 that are available for installation in the next five robots.

    (a) Find the probability that all 5 gearboxes will fit properly.

    (b) Find the mean, variance, and standard deviation of the time it takes to install these 5 gearboxes.

16. Suppose that $p = P(\text{male birth}) = 0.3$. A couple wishes to have exactly two female children in their family. They will have children until this condition is fulfilled.

    (a) What is the probability that the family has x male children?

    (b) What is the probability that the family has four children?

    (c) What is the probability that the family has at most four children?

    (d) How many male children would you expect this family to have? How many children would you expect this family to have?

17. In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

    (a) What is the probability that team A will win the series in 5 games?

    (b) What is the probability that team A will win the series?

    (c) What is the probability that team B will win the series in 6 games?

18. Ten percent of the engines manufactured on an assembly line are defective. What is the probability that the third nondefective engine will be found

    (a) on the fifth trial?

    (b) on or before the fifth trial?

    (c) Given that the first two engines tested were defective, what is the probability that at least two more engines must be tested before the first nondefective is found?

    (d) Find the mean and variance of the number of the trial on which a the first nondefective engine is found.

    (e) Find the mean and variance of the number of the trial on which the third nondefective engine is found.

19. A large stockpile of used pumps contains 20% that are in need of repair. A maintenance worker is sent to the stockpile with three repair kits. She selects pumps at random and tests them one at a time. If the pump works, she sets it aside for future use. However, if the pump does not work, she uses one of her repair kits on it. Suppose that it takes 10 minutes to test a pump that is in working condition and 30 minutes to test and repair a pump that does not work. Find the mean and variance of the total time it takes the maintenance worker to use her three repair kits.

20. Suppose small aircraft arrive at a certain airport according to a Poisson process with rate $\alpha = 8$ per hour, so that the number of arrivals during a time period of t hours is a Poisson rv with parameter $\lambda = 8t$.

    (a) What is the probability that exactly 6 small aircraft arrive during a 1-hour period? At least 6? At least 10?

    (b) What are the expected value and standard deviation of the number of small aircraft that arrive during a 90-min period?

    (c) What is the probability that at least 20 small aircraft arrive during a 2.5-hour period? That at most 10 arrive during this period?

21. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. If it takes approximately ten minutes to serve each customer, find the mean and variance of the total service time for customers arriving during a 1-hour period. (Assume that a sufficient number of servers are available so that no customer must wait for service.) Is it likely that the total service time will exceed 2.5 hours?

22. Suppose that trees are distributed in a forest according to a two-dimensional Poisson process with parameter α, the expected number of trees per acre, equal to 80.

    (a) What is the probability that in a certain quarter-acre plot, there will be at most 16 trees?

    (b) If the forest covers 85,000 acres, what is the expected number of trees in the forest?

    (c) Suppose you select a point in the forest and construct a circle of radius .1 mile. Let X= the number of trees within that circular region. What is the pmf of X? [Hint: 1 sq mile = 640 acres.]

23. In proof testing of circuit boards, the probability that any particular diode will fail is 0.01. Suppose a circuit board contains 200 diodes.

    (a) How many diodes would you expect to fail, and what is the standard deviation of the number that are expected to fail?

    (b) What is the (approximate) probability that at least four diodes will fail on a randomly selected board?

(c) If five boards are shipped to a particular customer, how likely is it that at least four of them will work prop- erly? (A board works properly only if all its diodes work.)

24. **[Markov inequality]** If $X \geq 0$, i.e. X takes only nonnegative values, then for an $a > 0$, we have

$$P(X \geq a) \leq \frac{E(X)}{a}$$

25. **[Chebyshev inequality]** For any random variable X and any $a > 0$, we have

$$P(|X - E(X)| \geq a) \leq \frac{V(X)}{a^2}$$

26. The number of customers per day at a sales counter, Y , has been observed for a long period of time and found to have mean 20 and standard deviation 2. The probability distribution of Y is not known. What can be said about the probability that, tomorrow, Y will be greater than 16 but less than 24?

27. Let X be a random variable with mean 11 and variance 9. Using Tchebysheff's theorem, find

(a) a lower bound for $P(6 < X < 16)$.

(b) the value of C such that $P(|X - 11| \geq C) \leq 0.09$.

Solution

I2-TD2.

(Discrete R.V and probability Distribution).

1. Let x denote the number of line in use at specif time.

x	0	1	2	3	4	5	6
$P(x)$	0.10	0.15	0.20	0.25	0.20	0.06	0.04

Compute the probability

(a). At most three line are inuse.

$P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

$= 0.10 + 0.15 + 0.20 + 0.25$

$= 0.7$

Therefore $P(x \leq 3) = 0.7$

(b). Fewer than three line are in use.

$P(x < 3) = P(x \leq 3) - P(3)$

$= 0.7 - 0.25 = 0.45$

Therefore $P(x < 3) = 0.45$

(c). at least three lines are in use.

$P(x \geq 3) = 1 - P(x < 3)$

$= 1 - 0.45 = 0.55$

Therefore $P(x \geq 3) = 0.55$.

(d). Between two and five line, inclusive are in use.

$P(2 \leq x \leq 5) = P(2) + P(3) + P(4) + P(5)$

$= 0.20 + 0.25 + 0.20 + 0.06$

$= 0.71.$

(e) between two and four line, includsive are not in use.

Let x is the number of Lines are in used.

=) 6-x is the number of line are not in use.

=) $P(2 \leq 6-x \leq 4) = 1 - P(2 \leq x \leq 4)$

$= 1 - 0.65$

Therefore $P(2 \leq 6-x \leq 4) = 0.35.$

(f) at least four line are not in use

$P(\bar{x} \geq 4) = P(5) + P(6).$

$= 0.06 + 0.04$

Therefor $P(6-x \geq 4) = 0.1.$

2. Find the probability distrubution of x.

Let D: ~~Non~~ Defective , N= Non-defective.

the posibility of x are x = 2, 3, 4.

. $x = 2$ {(DD)}

$\Rightarrow P(x=2) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$

. $x = 3$ {(DND), (NDD)}

$\Rightarrow P(x=3) = 2 \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

. $x = 4$ {(DNND), (NDND), (NNDD)}

$\Rightarrow P(x=4) = 3 \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$

Therefore.

x	2	3	4
$P(x)$	$1/6$	$1/3$	$1/2$

3. Find the probability distrubution of Y, The number of correct matches.

Y be the number of correct match Y = 0, 1, 3 C: Correct, I: incorrect

. Y = 0 : {(III)} : $P(y=0) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

. Y = 1 : {(CII), (ICI), (IIC)} , $P(y=1) = 3 \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$

. Y = 3 : {(CCC)} , $P(y=3) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

Thus :

x	0	1	3
y	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

4(a) Find the pmf of x: the number of errors

$x \sim Bin(3, 0.05)$, $p = 0.05$.

so $P(X = x) = C_n^x p^x (1-p)^{n-x}$, $n = 3$

$= C_3^x p^x (1-p)^{3-x}$

. $P(x=0) = (1-p)^3 = (0.95)^3 = 0.857375$

. $P(x=1) = 3(0.05)(0.95)^2 = 0.135375$

. $P(x=2) = 3(0.05)^2(0.95) = 0.007125$

. $P(x=3) = (0.05)^3 = 0.000125$.

Thus

x	0	1	2	3
$P(x)$	0.857375	0.135375	0.007125	0.000125

(b). Find $P(x>1)$.

$P(x>1) = P(2) + P(3) = 0.00712$.

Thus $P(x>1) = 0.00712$.

5. let Y denoted the number of batteries that must be tested.

(a). what is $P(2)$?

$P(2) = 0.9 \times 0.9 = 0.81$

Thus $P(2) = 0.81$

(b). what is $P(3)$. ?

$Y = 3 : \{(AUA), (UAA)\}$

$P(Y=3) = 2 \times (0.9) \times (0.1) \times (0.9) = 0.162$.

(c). List the outcomes and Find p(5).

$Y=5$ {(AUUUA), (UAUUA), (UUAUA), (UUUAA)}

$P(y=5) = 4 \times 0.9 \times 0.9 \times 0.1 \times 0.1 \times 0.1 = 0.0032.$

Thus $P(y=5) = 0.0032$

(d). General formular

$P(Y=y) = C_n^y (0.9)^2 (0.9)^{y-2}$

6. Let x: The number of month between succesive paymen[t]

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.10 & 1 \leq x < 3 \\ 0.40 & 3 \leq x < 7 \\ 0.80 & 7 \leq x < 12 \\ 1 & x \leq 12 \end{cases}$$

(a). Find pmf of x

x	1	3	7	12
$p(x)$	0.1	0.3	0.4	0.2

(b). Compute $P(3 \leq x \leq 6)$, $P(6 \leq x)$.

. $P(3 \leq x \leq 6) = P(x \leq 6) - P(x < 3) = P(x \leq 6) - P(1)$

$= F(3) - F(1) = 0.3.$

. $P(6 \leq x) = P(x \geq 6) = 1 - P(x < 6) = 1 - F(6^-) = 1 - 0.4 = 0.6$

Thus $P(3 \leq x \leq 6) = 0.3$, $P(x \geq 6) = 0.6$.

7. Let Y be the number of students prefer English.

(a). Find ~~P(Y=3)~~. pmf for Y

. $P(Y=y) = C_3^y (0.6)^y (0.4)^{3-y}$

. $P(Y=0) = 0.064$

. $P(y=1) = 3 \times 0.6 \times (0.4)^2 = 0.288$

. $P(y=2) = 3 \times 0.6^2 \times 0.4 = 0.432$

. $P(y=3) = 0.6^3 = 0.216$.

Thus

y	0	1	2	3
P(y)	0.064	0.288	0.432	0.216

(b). Find $P(y=1)$.

Thus: $P(y=1) = 0.288$

(c). ... mean standard devision for y

$E(y) = np = 3 \times 0.6 = 1.8$.

$V(y) = npq = 0.6 \times 3 \times 0.4 = 0.72$.

$\delta(y) = \sqrt{V(y)} = \sqrt{0.72}$

(d) Probability that the number of student prefer E fall within $2\sigma(Y)$ of the mean.

$P(|y - E(y)| < 2\delta(y)) = P(E(y) - 2\delta(y) < Y < E(Y) + 2\delta(Y))$

$E(y) - 2\delta(y) = 0.12$, $E(y) + 2\delta(y) = 3.48$

$\Rightarrow P(0.12 \le y \le 3.48) = P(1 \le y \le 3) = 0.936$.

8. what is the expected gain.?

Let y be the money which paid ~~p~~ or pay

$$\begin{cases} y = 15 & \text{for jack ~~of~~ or queen.} \\ y = 5 & \text{for king or ace} \\ y = -4 & \text{for orther.} \end{cases}$$

$P(J \text{ or } Q) = \frac{8}{52}$, $P(K \text{ or } A) = \frac{8}{52}$, $P(\text{other}) = \frac{32}{52}$

Then $E(y) = 15 \times \frac{8}{52} + 5 \times \frac{8}{52} - 4 \times \frac{32}{52} = 0.3\$$

Thus $E(y) = 0.3\$$.

9. (a) show that the expected value of N is $1 + \frac{p}{q} + \frac{q}{p}$.

Let N be number of yaks that must be cutch until the pair ob

For $n = 2$ ... we have

$p(n) = P(N=n) = p^{n-1}q + pq^{n-1}$.

$\Rightarrow E(N) = \sum_{n=2}^{\infty} nP(n) = q\sum_{n=2}^{\infty} np^{n-1} + p\sum_{n=2}^{\infty} nq^{n-1}$

$= pq\left[\frac{1}{1-p} - \frac{1}{1-q} + \frac{1}{(1-p)^2} + \frac{1}{(1-q)^2}\right]$

$= 1 + \frac{p}{q} + \frac{q}{p}$.

(b) Find $V(N)$

$V(N) = E(N^2) - [E(N)]^2$

$E(N^2) = q\sum_{n=2}^{\infty} n^2p^{n-1} + p\sum_{n=2}^{\infty} n^2q^{n-1}$

we have $\sum_{n=2}^{n} np^{n-1} = \frac{1-q^2}{q^2} = \frac{1}{q^2} - 1 = \frac{1}{(1-p)^2} - 1$.

$\Rightarrow \sum_{n=2}^{\infty} np^{n-2}(n-1) = \frac{2(1-p)}{(1-p)^4} = \frac{2}{(1-p)^3}$

$$\sum_{n=2}^{\infty} n^2 p^{n-1} = \frac{2p}{(1-p)^3} + \sum_{n=2}^{\infty} n p^{n-1}$$

$$= \frac{2p}{q^3} + \frac{1-q^2}{q^2} = \frac{2p+q-q^3}{q^3}$$

Similary $\sum_{n=2}^{\infty} n^2 q^{n-1} = \frac{2q+p-p^3}{p^3}$

$$\Rightarrow E(N^2) = \frac{2p+q-q^3}{q^3} + \frac{2q+p-p^3}{p^3}$$

$$[E(N)]^2 = \left(\frac{1}{q} + \frac{q}{p}\right)^2 = \left(\frac{1}{q^2} + 2\frac{1}{p} + \frac{q^2}{p^2}\right)$$

$$\Rightarrow V(N) = \frac{2p-1}{q^2} + \frac{1}{pq} - 1 + \frac{2q-q^2}{p^2}$$

10. Let x be the number of components last longer that 1000h

(a) Find $P(x=2)$. , we have $p(x)=0.8$.

$x \sim Bin(4, 0.8)$.

$P(x=2) = C_4^2 (0.8)^2 (0.2)^2 = 0.1536$.

Thus ~~P(x)~~ $P(x=2) = 0.1536$.

(b). The subsystem work longer than 1000h.

$P(x \geq 2) = \sum_{n=2}^{4} C_4^n (0.8)^n (0.2)^{4-n} = 0.9728$.

Thus $P(x \geq 2) = 0.9728$.

11. Let x be the number residuntail subcriber who quality for the favorable rates: Find the probability.

(a). $P(x=5)$.

$x \sim Bin(5, 0.70)$.

$P(x=5) = C_5^5 (0.70)^5 (0.3)^{5-5} = 0.16807$.

Thus $P(x=5) = 0.16807$.

(b). at least four favorutable.

$P(x \geq 4) = P(4) + P(5) = 5(0.7)^4(0.3) + 0.16807 = 0.52822$

(c) at least two do not qulity favorable

$P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

$= \sum_{n=0}^{3} C_3^n (0.7)^n (0.3)^{3-n} = 0.78048$.

12. Let x be the number of volunteers who has type A blood

(a). Find the probability : at least one has no A blood type.

$\Rightarrow P(x \leq 4) = 1 - P(x=5) = 1 - (0.8)^5 = 0.67232$

Thus $P(x \leq 4) = 0.67232$.

(b). Find the probability : At most 4 have type A blood.

$P(x \leq 4) = 0.67232$.

(c). Find the smallest number of volunteers. if $P(x \geq 5) \geq 90\%$

we have $P(x \geq 5) = \cancel{\sum_{x=5}^{n} C_n^5 (0.9)^5 (0.1)^5}$ $P(x=5) + \dots + P(x=n)$

For $n=5$: $P(5) = 0.32768$.

$n=6$: $P(x \geq 5) = P(5) + P(6) = 0.65536$

$n=7$: ... $\Rightarrow P(x \geq 5) = 0.852$

$n=8$: ... $\Rightarrow P(x \geq 5) = 0.9437$

Thus $n=8$.

13. (a). what kind of distribution does x have.

~~X = bin~~ $X \sim Hp(n, M, N)$, $n=6$, $M=12$, $N=20$

$\Rightarrow X \sim Hp(6, 12, 20)$.

(b). Compute $P(X=2)$, $P(X\leq 2)$, $P(X\geq 2)$

. $P(X=x) = \dfrac{C_{12}^{x} C_{8}^{6-x}}{C_{20}^{x}}$

. $P(X=2) = \dfrac{C_{12}^{2} C_{8}^{4}}{C_{20}^{2}}$

. $P(X\leq 2) = P(X=1) + P(X=2) + P(X=0)$

$= \dfrac{C_{12}^{0} C_{8}^{6}}{C_{20}^{6}} + \dfrac{C_{12}^{1} C_{8}^{5}}{C_{20}^{6}} + \dfrac{C_{12}^{2} C_{8}^{4}}{C_{20}^{6}} = a,\ a\in\mathbb{R}.$

. $P(X\geq 2) = 1 - P(X\leq 2) + P(X=2) = 1 - a + \dfrac{C_{12}^{2} C_{8}^{4}}{C_{20}^{2}}$

(c). Calculate

. $E(X) = n\times\dfrac{M}{N} = 6\times\dfrac{12}{20} = 3.6.$

. $V(X) = n\times\dfrac{M}{N}\left(1-\dfrac{M}{N}\right)\left(\dfrac{N-n}{N-1}\right) = 1.06$

. $\sigma(X) = \sqrt{V(X)} = \sqrt{1.06} = 1.02.$

14. Let X be the number of defective item.

(a). The probability that a box containing 3 defective will be shiped. ~~X ~ Hp(3,~~

$X=0$: $P(X=0) = \dfrac{C_{22}^{3} C_{3}^{0}}{C_{25}^{3}} = 0.669$

Thus $P(X=0) = 0.669$

(b). Only one defective will be sent back.

$$P(X=1) = \frac{C_{24}^{1}}{C_{25}^{3}} = 0.12.$$

Thus $P(X=1) = 0.12$

15. (a). Find the probability that all 5 gearboxes will fit

Let x be the number of gearbox fitting probabily

$X \sim Hp(5, 18, 20)$.

$$P(X=x) = \frac{C_{18}^{x} C_{2}^{5-x}}{C_{20}^{x}}$$

$$P(X=5) = \frac{C_{18}^{5}}{C_{20}^{5}} = 0.55.$$

(b). Find the mean and standard deviation.

Let T be the table time it take to install these 5 gearbox

Then $T = 10X + (1)(5-.) = 9X+5$

Thus $E(T) = 9E(X) + 5 = 9\left[n.\frac{M}{N}\right] + 5 = 9\left(5\frac{2}{20}\right) + 5 = 9.5.$

$V(T) = V(5+9X) = 81 \times V(M)$

$= 81 \times 0.355 = 28.755$

$\sigma(T) = \sqrt{V(T)} = \sqrt{28.755} = 5.36.$

16. Let x be the number of male children

Then $X \sim Nb(r,p)$ where $r=2$, $p=0.7$.

$X \sim Nb(2,0.7)$.

-) The pmf of X $P(X=x) = C^{1}_{x+1}(0.3)^{x}(0.7)^{2}$

$= (x+1)(0.3)^{x}(0.7)^{2}$

(a). Family has x male children.

$p(x) = P(X=x) = (x+1)(0.3)^{x}(0.7^{2})$, $x = 0,1,2 \ldots$

(b). Family has 4 children.

$p(X=2) = 3(0.3)^{2}(0.7)^{2}$

(c). Family has at most 4 children.

$P(X \leq 2) = \sum_{x=0}^{2} p(x) = P(0) + P(1) + P(2)$

$= (0.7)^{2} + 2(0.3)(0.7)^{2} + 3(0.3)^{2}(0.7)^{2}$

(d). How many male children would expect this family to have

$E(X) = \cancel{rp/q} \; \frac{r(1-p)}{p} = \frac{2(0.3)}{0.7} = 0.85714$.

How many children whould we expect this family to have

$E(X+2) = E(X) + 2 = 2.85714$.

17. $P(\text{winning}) = 0.55$.

Let X be the number of lossing process 4 wins.

(a). team A will win in 5 games.

$P(X=1) = C_4^3 (0.55)^4 (0.45)$

$= 0.16$.

(b). A will win the series.

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$.

$= 0.6$.

(c). B will win the series in 6 games.

$P(\text{B will win in 6 games}) = C_5^2 (0.55)^2 (0.45)^4$

18. Let X be the number of defective process 3 nondefective.

~~(a)~~. Find the Proba. third nondefective will be found.

(a). on the fifth trial

$X \sim Nb(3, 0.9)$.

$P(X=2) = C_4^2 (0.9)^3 (0.1)^2 = 0.043$.

(b). on or before the 5^{th} trial.

$P(X \leq 2) = P(0) + P(1) + P(2)$.

$= (0.9)^3 + 3(0.9)^3(0.1) + C_4^2 (0.9)^3 (0.1)^2$

$= 0.99144$.

Therefore $P(X \leq 2) = 0.99144$.

(c). Find the probability that at least two more egines must be tested before the first nondefective is found

let Y be the number of defetive egines process 1 nondefective.

$$P(y \geq 3 \mid y \geq 2) = \frac{P(y \geq 3)}{P(y \geq 2)}$$

$Y \sim Nb(1, 0.9)$.

$P(Y) = C_y^0 (0.9)^1 (0.1)^y$

$P(y \geq 3) = 0.0009$

$P(Y \geq 2) = 0.009$

$$\Rightarrow P(Y \geq 3 \mid Y \geq 2) = \frac{0.0009}{0.009} = 0.1$$

(d). First nondefective is found

$E(X+1) = E(X) + 1 = 1 + \frac{1 \times 0.1}{0.9} = 1.1$

$V(X+1) = V(X) = 0.123$.

(e). Third non-defetive is found.

$E(X+3) = 3.1$

$V(X+3) = V(X) = 0.123$.

19. Find the mean and variance of total time.

. 10 min for checking and 30 min to check and repair

. Let x be number of pumps that works precede 3 defetive pumps.

$\Rightarrow X \sim Nb(3, 0.2)$.

$\Rightarrow E(X) = 3 \times \frac{0.8}{0.2} = 12$.

The total time $T(X) = 10 \times x + 3 \times 30 = 10x + 90$

$\Rightarrow E(T(X)) = 10(12) + 90 = 210$ min.

$V(T(X)) = 100 \times V(X) = 100 \times 3 \times \frac{0.8}{0.4} = 6000$.

20. Let x be the number of small aircraft arrive $X \sim P_o(\lambda)$.

(a). Find the probability.

$\lambda = 8t$, $t = 1$, $\Rightarrow \lambda = 8$

$X \sim P_o(\lambda)$.

$P(X = x) = \frac{e^{-8} 8^x}{x!}$

. $x = 6$ $\Rightarrow P(X = 6) = \frac{e^{-8} 8^6}{6!}$

. $x \geq 6$ $\Rightarrow P(X \geq 6) = \cancel{1 - \frac{e^{-8} 8^6}{6!}}$ $1 - \sum_{x=0}^{5} \frac{e^{-8} 8^x}{x!}$

. $x \geq 10$ $\Rightarrow P(X \geq 10) = 1 - \sum_{x=0}^{9} \frac{e^{-8} 8^x}{x!}$

(b). Expect value & standard deviation of x for $t = \frac{3}{2}h$

$E = \lambda = \frac{3}{2} \times 8 = 12$

$\delta(x) = \sqrt{\lambda} = \sqrt{12}$.

$\Rightarrow E(x) = 12, \delta = \sqrt{12}$.

(c). Find the probability at least 20 small aircraft for 2.5h

$\lambda = 8 \times 2.5 = 20 \Rightarrow P(x \geq 20) = 1 - \sum_{x=0}^{19} \frac{e^{-20} 20^{x}}{x!}$

The most 10 arrive.

$P(x \leq 10) = \sum_{x=0}^{10} \frac{e^{-20} 20^{x}}{x!}$

21. $\lambda = 7t$, Find the mean and variance of the total time

The total time denoted by $T(\lambda) = 10\lambda$

during 1-hours $T(\lambda) = 70$

Thus ~~$T(\lambda) =$~~ $E(T) = 70$, $V(T) = 700$.

22. Let Y be the trees in t acre.

Then $Y = P_o(\lambda = 80t)$.

$$P(y) = P(Y = y) = \frac{e^{-80t} \cdot (80t)^{y}}{y!}$$

(a). $t = \frac{1}{4} \Rightarrow \lambda = 20$ Find the probability there will be at most 16 trees.

$P(y \leq 16) = \sum_{y=0}^{16} \frac{e^{-20} 20^{y}}{y!}$

(b). Find $E(y)$ if $A = 85000$ acres

$E(y) = \lambda = 85000 \times 80$

$= 68 \times 10^5$

(c) Find the pmf of y

$\lambda = 80 \times A$, $A = \pi r^2$, $r = 1$ mille. $= 640$ acres.

$\Rightarrow \lambda = 51200\pi$

$$\Rightarrow P(Y=y) = \frac{e^{-51200\pi}(51200\pi)^y}{y!}$$

23. (a) Find $E(x)$ and $\sigma(x)$.

x number of failing diodes.

$x \sim Bin(200, 0.01)$.

$\Rightarrow E(x) = 0.01 \times 200 = 2$.

$V(x) = 2 \times (0.99) = 1.98$.

$\sigma(x) = \sqrt{V(x)} = \sqrt{1.98} = 1.4$.

(b). at least four bords will fail

$P(x \geq 4) = 1 - P(x \leq 3)$

$$= 1 - \sum_{x=0}^{3} C_{200}^{x} (0.01)^x (0.99)^{200-x}$$

But since $n = 200$ and $p \to 0$ and $np = 2$

$\Rightarrow \lambda = 2$

$X \sim P_o(\lambda) \Rightarrow P(X=x) = \dfrac{e^{-2}2^x}{x!}$

$P(X \geq 4) = 1 - \sum_{x=0}^{3} \dfrac{e^{-2}2^x}{x!}$

(c). Find $P(x \geq 4)$.

if All five are work $P(w) = (0.99)^5 = 0.134$

Then $X = Bin(5, 0.134)$

$P(x \geq 4) = P(4) + P(5)$

~~Thus I~~ $= 0.001$.

Thus It is unlikely happend.

24. [Markov inequality] $P(x \geq a) \leq \dfrac{E(X)}{a}$

$E(X) = \sum_{x=0}^{n} x P(x) = \sum_{x=0}^{a} x P(x) + \sum_{x=a}^{n} x P(x) \geq \sum_{x=a}^{n} x P(x)$

$\geq a \sum_{x=a}^{n} P(x) = a P(x \geq a)$

Thus $P(x \geq a) \leq \dfrac{E(x)}{a}$

25. [Chebyshev inequality] $P(|x - E(x)| \geq a) \leq \dfrac{V(x)}{a^2}$

$P(|x - E(x)| \geq a) = P(|x - E(x)|^2 \geq a^2)$

$\leq \dfrac{E(|x - E(x)|^2)}{a^2} = \dfrac{V(x)}{a^2}$

Thus $P(|x - E(x)| \geq a) \leq \dfrac{V(x)}{a^2}$

Page

26. Y: The number of costomer per day

$E(Y)=20$, $\delta(Y)=2$.

+ Probability that tomorrow $16 < m < 24$

by using Tchebycheff theorem , $K>0$

$P(|x-E(x)|<K\delta) \geq 1-\frac{1}{K^2}$

or $P(E(x)-K\delta < m < E(x)+K\delta) \geq 1-\frac{1}{K^2}$

Choose $K=2$.

Then $\frac{3}{4} \leq P(16<m<24) \leq 1$.

27. Let X be the rendom variable with $E(X)=11$.

$V(X)=9$. $\Rightarrow \delta=3$.

(a). Find a lower bounded for $P(6<\dots<16)$

Tchebyshoff's Theorem.

$P(E(x)-K\delta < x < E(x)+K\delta) \geq 1-\frac{1}{K^2}$

$P(6<m<16) = P(6-E(x)<x-E(x)<16-E(x))$

$= P(6-11 < x-E(x) < 16-11)$

$= P(\cancel{E(x)}|x-E(x)|<5)$.

$= 1-P(|x-E(x)|\geq 5)$

$\geq \frac{16}{25}$

Thus The lower bounded $= \frac{16}{25}$.

(b) Find the value c

$P(|X-11| \geq c) \leq 0.99$.

by chebyshev's theorem $P(|M-11| \geq c) \leq \frac{V(M)}{c^2}$

Then $\frac{V(M)}{c^2} = 0.09 \Rightarrow c^2 = \frac{V(M)}{0.09} \Rightarrow c = 10$

Thus $c = 10$.

I2–TD3
(Continuous RV and Probability Distribution)

1. Let X be a random variable whose pdf is

$$f(x) = \begin{cases} 0.2, & -1 \leq x \leq 0 \\ 0.2 + kx, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value k (d) Find P(X < 1.5) (b) Find the cdf $F_X(x)$. (e) Find P(X > 0.5|X < 1)

(c) Deduce the values $F_X(-1)$ and $F_X(1)$ (f) Find E(X) and V (X).

2. Let X be a random variable whose cdf is

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{8}, & 0 < x < 2 \\ \frac{x^2}{16}, & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

(a) Find the probability distribution function f(x). (c) Find P(1 < X < 3) (d) Find P(X > 0.5|X < 1) (e) Find E(X) and V (X).

(b) Find P(X > 3)

3. Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(x) = \begin{cases} \frac{3}{64}x^2(4-x), & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases},$$

(a) Find the expected value and variance of weekly CPU time.

(b) The CPU time costs the firm $200 per hour. Find the expected value and variance of the weekly cost for CPU time.

(c) Would you expect the weekly cost to exceed $600 very often? Why?

4. The grade point averages (GPAs) for graduating seniors at a college are distributed as a continuous rv X with pdf

$$f(x) = \begin{cases} k(1-(x-3)^2), & 2 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of k.

(b) Find the probability that a GPA exceeds 3.

(c) Find the probability that a GPA is within 0.25 of 3.

(d) Find the probability that a GPA differs from 3 by more than 0.5.

5. Suppose that X is a continuous random variable with density f(x) that is positive only if $x \geq 0$. If $F_X(x)$ is the cumulative distribution function of X, show that

$$E(X) = \int_0^{\infty} (1 - F_X(x))\, dx$$

6. If X is a continuous random variable such that $E\left[(X-a)^2\right] < \infty$ for all a, show that $E\left[(X-a)^2\right]$ is minimized when a = E(X).

7. Suppose that X ~ U(a,b). Show that

(a) $E(X) = \dfrac{a+b}{2}$ (b) $V(X) = \dfrac{(b-a)^2}{12}$ (c) $M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

8. Beginning at 12:00 midnight, a computer center is up for one hour and then down for two hours on a regular cycle. A person who is unaware of this schedule dials the center at a random time between 12:00 midnight and 5:00 AM. What is the probability that the center is up when the person's call comes in?

9. In commuting to work, a professor must first get on a bus near her house and then transfer to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with A = 0 and B = 5, then it can be shown that the total waiting time Y has the pdf

$$f(y) = \begin{cases} \frac{1}{25}y, & 0 \leq y < 5, \\ \frac{2}{5} - \frac{1}{25}y, & 5 \leq y \leq 10, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch a graph of the pdf of Y .

(b) Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$.

(c) What is the probability that total waiting time is at most 3 min? (d) What is the probability that total waiting time is at most 8 min?

(e) What is the probability that total waiting time is between 3 and 8 min?

(f) What is the average of the total waiting time.

10. Suppose that X ~ N(μ,σ^2). Show that

(a) E(X) = μ (b) V (X) = σ^2 (c) $M(t) = \exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$.

11. If X is a random variable such that

$$\begin{cases} E(X^{2n}) = (2n)!/2^n n!, \\ E(X^{2n-1}) = 0, \end{cases} \quad \text{, for } n = 1, 2, \ldots.$$

Find the moment generating function and the distribution of X.

12. The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8.

(a) If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?

(b) How many percent of students will possess a GPA in excess of 3.0.

(c) Suppose that 5 students are randomly selected from the student body. What is the probability that only three will possess a GPA in excess of 3.0

13. Scores on an examination are assumed to be normally distributed with mean 65 and variance 36.

(a) What is the probability that a person taking the examination scores higher than 70?

(b) Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?

(c) What must be the cutoff point for passing the examination if the examiner wants only the top 75% of all scores to be passing?

(d) Approximately what proportion of students have scores 5 or more points above the score that cuts off the lowest 15%?

(e) If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

14. The distribution of 1,000 examinees according to marks percentage is given below:

% Score	< 40%	40%-75%	> 75%	Total
No. of examinees	430	420	150	1,000

Assuming the marks percentage to follow a normal distribution.

(a) Calculate the mean and standard deviation of marks.

(b) If not more than 300 examinees are to fail, what should be the passing marks?

15. Suppose the diameter at breast height (in.) of trees of a certain type is normally distributed with $\mu = 8.8$ and $\sigma = 2.8$, as suggested in the article "Simulating a Harvester Forwarder Softwood Thinning" (Forest Products J., May 1997: 36–41).

(a) What is the probability that the diameter of a randomly selected tree will be at least 10 in.? Will exceed 10 in.?

(b) What is the probability that the diameter of a randomly selected tree will exceed 20 in.?

(c) What is the probability that the diameter of a randomly selected tree will be between 5 and 10 in.?

(d) What value c is such that the interval $(8.8-c, 8.8+c)$ includes 98% of all diameter values?

(e) If four trees are independently selected, what is the probability that at least one has a diameter exceeding 10 in.?

16. A machine that produces ball bearings has initially been set so that the true average diameter of the bearings it produces is 0.500 in. A bearing is acceptable if its diameter is within 0.004 in. of this target value. Suppose, however, that the setting has changed during the course of production, so that the bearings have normally distributed diameters with mean value 0.499 in. and standard deviation 0.002 in. What percentage of the bearings produced will not be acceptable?

17. Suppose only 70% of all drivers in a state regularly wear a seat belt. A random sample of 500 drivers is selected. What is the probability that

    (a) Between 320 and 370 (inclusive) of the drivers in the sample regularly wear a seat belt?

    (b) Fewer than 325 of those in the sample regularly wear a seat belt? Fewer than 315?

18. Let X ~ Exp(λ). Show that

    (a) $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{\lambda}}, & x \geq 0. \end{cases}$,

    (b) E(X) = λ

    (c) V (X) = λ^2

    (d) $E(X^n) = \lambda^n n!$, for $n = 1, 2, \ldots$

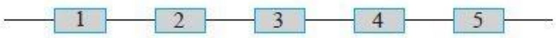
    (e) $M(t) = \dfrac{1}{1 - \lambda t}$, $t < \dfrac{1}{\lambda}$

19. A survey evidence indicates that the accidents on a particular road #4 in the country ABC have an approximately exponential distribution. Assume that the mean time between accidents is 4 hours. If one of the accidents occurred on 7:00 AM of a randomly selected day in the study period.

    (a) What is the probability that another accident occurred that same day?

    (b) What time that the next accident occurred if the probability that the next accident occurred is 0.8

20. A system consists of five identical components connected in series as shown



    As soon as one component fails, the entire system will fail. Suppose each component has a lifetime that is exponentially distributed with λ = 0.01 and that components fail independently of one another. Define events A_i = {ith component lasts at least t hours}, i = 1,...,5, so that the A_is are independent events. Let X = the time at which the system fails-that is, the shortest (minimum) lifetime among the five components.

    (a) The event {X ≥ t} is equivalent to what event involving $\{A_1, A_2, \ldots, A_5\}$?

    (b) Using the independence of the $\{A_i\}$'s, compute P(X ≥ t). Then obtain F(t) = P(X ≤ t) and the pdf of X. What type of distribution does X have?

    (c) Suppose there are n components, each having exponential lifetime with parameter λ. What type of distribution does X have?

21. Suppose that a system contains a certain type of component whose time, in years, to failure is given by T . The random variable T is modeled nicely by the exponential distribution with mean time to failure λ = 5. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

22. Let X ~ Gam(α,β). Show that

    (a) E(X) = αβ

    (b) V (X) = $\alpha\beta^2$

    (c) $M(t) = \dfrac{1}{(1 - \beta t)^{\alpha}}$, $t < \dfrac{1}{\beta}$

    (d) $F(x) = \Gamma\left(\dfrac{x}{\beta}, \alpha\right)$

23. Suppose that when a transistor of a certain type is subjected to an accelerated life test, the lifetime X (in weeks) has a gamma distribution with mean 24 weeks and standard deviation 12 weeks.

    (a) What is the probability that a transistor will last between 12 and 24 weeks?

    (b) What is the probability that a transistor will last at most 24 weeks? Is the median of the lifetime distribution less than 24? Why or why not?

    (c) What is the 99th percentile of the lifetime distribution?

    (d) Suppose the test will actually be terminated after t weeks. What value of t is such that only 0.5% of all transistors would still be operating at termination?

24. Let $X \sim Gam(\alpha,\beta)$. Show that $X \sim \chi^2(\nu)$ if $\alpha = \nu/2$ and $\beta = 2$.

25. If $X \sim \chi^2(\nu)$. Show that

    (a) $E(X) = \nu$

    (b) $V(X) = 2\nu$

    (c) $E(X^n) = \dfrac{2^n\Gamma(\frac{\nu}{2}+n)}{\Gamma(\frac{\nu}{2})}, \quad n > -\nu/2$

    (d) $M(t) = \dfrac{1}{(1-2t)^{\frac{\nu}{2}}}, \quad t < \dfrac{1}{2}$

26. Suppose $X \sim N(\mu,\sigma^2)$,$\sigma > 0$ and $Y = \dfrac{(X-\mu)^2}{\sigma^2}$. Show that $Y \sim \chi^2(1)$.

27. Suppose that a random variable Y has a probability density function given by

    (ky3e−y/2, y > 0, f(y) =
    0, otherwise

    (a) Find the value of k.

    (b) Does Y have a χ^2 distribution? If so, how many degrees of freedom?

    (c) What are the mean and standard deviation of Y ?

    (d) What is the probability that Y lies within 2 standard deviations of its mean? 28. Let $X \sim Bet(\alpha,\beta)$.

    Show that

$$E(X) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

29. A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

30. Let $X \sim Log(\mu,\sigma)$. Show that

    E(X) = eμ+σ2/2 and V (X) = e2μ+σ2(eσ2 − 1)

31. Concentrations of pollutants produced by chemical plants historically are known to exhibit behavior that resembles a lognormal distribution. This is important when one considers issues regarding compliance with government regulations. Suppose it is assumed that the concentration of a certain

pollutant, in parts per million, has a lognormal distribution with parameters $\mu = 3.2$ and $\sigma = 1$. What is the probability that the concentration exceeds 8 parts per million?

32. The life, in thousands of miles, of a certain type of electronic control for locomotives has an approximately lognormal distribution with $\mu = 5.149$ and $\sigma = 0.737$. Find the 5th percentile of the life of such an electronic control.

33. Let $X \sim \text{Wei}(\alpha, \beta)$. Show that

$$E(X) = \alpha^{-1/\beta}\Gamma\left(1+\frac{1}{\beta}\right), \quad \text{and} \quad V(X) = \alpha^{-2/\beta}\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}$$

34. Let $X \sim \text{Wei}(\alpha,\beta)$. Show that

$$F(x) = 1 - e^{\alpha x\beta} \qquad \text{for } x \geq 0$$

for $\alpha > 0, \beta > 0$.

35. Suppose that the service life, in years, of a hearing aid battery is a random variable having a Weibull distribution with $\alpha = 1/2$ and $\beta = 2$.

(a) How long can such a battery be expected to last?

(b) What is the probability that such a battery will be operating after 2 years?

36. The authors of the article "A Probabilistic Insulation Life Model for Combined ThermalElectrical Stresses" (IEEE Trans. on Elect. Insulation, 1985: 519–522) state that "the Weibull distribution is widely used in statistical problems relating to aging of solid insulating materials subjected to aging and stress." They propose the use of the distribution as a model for time (in hours) to failure of solid insulating specimens subjected to AC voltage. The values of the parameters depend on the voltage and temperature; suppose $\alpha = 2.5$ and $\beta = 200$(values suggested by data in the article).

(a) What is the probability that a specimen's lifetime is at most 250? Less than 250? More than 300?

(b) What is the probability that a specimen's lifetime is between 100 and 250?

(c) What value is such that exactly 50% of all specimens have lifetimes exceeding that value?

Solution:

I2 - TD3

① Let X be a random variable, whose pdf is

$$f(x) = \begin{cases} 0.2 & -1 \le x \le 0 \\ 0.2 + kx & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value K.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-1}^{0} 0.2dx + \int_{0}^{1} (0.2 + kx)dx = 1$$

$$+0.2 + 0.2 + \frac{k}{2} = 1$$

$$\underline{k = 1.2}$$

(b) Find the cdf $F_X(x) = \int_{-\infty}^{x} f(t)dt$

- If $x < -1$ then $F_X(x) = \int_{-\infty}^{-1} 0dt = 0$
- If $-1 \le x < 0$ then $F_X(x) = \int_{-1}^{x} 0.2\,dt = 0.2x + 0.2$
- If $0 \le x < 1$ then $F_X(x) = \int_{-1}^{0} (0.2)dt + \int_{0}^{x} (0.2 + kt)dt$

$$= 0.2 + 0.2x + \frac{kx^2}{2}$$

$$= 0.2 + 0.2x + 0.6x^2$$

- If $x \ge 1$ then

$$F_X(x) = \int_{-1}^{0} 0.2dt + \int_{0}^{1} (0.2 + kt)dt$$

$$= 0.2 + 0.2 + \frac{1.2}{2} = 1$$

(c) Deduce the values $F_X(-1)$ and $F_X(1)$

$F_X(-1) = 0.2(-1) + 0.2 = 0$

$F_X(1) = 0.2 + 0.2(1) + 0.2(1)^2 = 0.6$

(d) Find $P(X < 1.5)$

$P(X<1.5) = F_X(1.5) = F_X(x \geq 1) = 1$

(e). Find $P(X > 0.5 \mid X < 1)$

$$P(X>0.5 \mid X<1) = \frac{P(0.5<x<1)}{P(X<1)} = \frac{F_X(1) - F_X(0.5)}{F_X(1)}$$

$$= \frac{0.6 - 0.3}{0.6} = \underline{0.5}$$

f). Find $E(X)$ and $V(X)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{0} 0.2x\,dx + \int_{0}^{1} (0.2x + kx^2)\,dx$$

$$= -0.1 + 0.1 + \frac{1.2}{3} = \underline{0.4}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-1}^{0} 0.2x^2\,dx + \int_{0}^{1} (0.2x^2 + kx^3)\,dx$$

$$= \frac{0.2}{3} + \frac{0.2}{3} + \frac{1.2}{4} = 0.43$$

$$V(X) = E(X^2) - [E(X)]^2 = 0.43 - 0.16 = \underline{0.27}$$

② Let X be a random variable whose cdf is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{8}, & 0 < x < 2 \\ \frac{x^2}{16}, & 2 \le x < 4 \\ 1, & x \geq 4 \end{cases}$$

ⓐ Find the probability distribution function $f(x)$

The PDF $f(x) = \frac{d}{dt} F_X(x) = \begin{cases} \frac{1}{8}, & 0 < x < 2 \\ \frac{x}{8}, & 2 \le x < 4 \\ 0, & \text{otherwise} \end{cases}$

ⓑ Find $P(X>3)$

$P(X>3) = 1 - P(X \le 3) = 1 - F_X(3) = 1 - \frac{9}{16} = 0.4375$

ⓒ Find $P(1 \le x < 3)$

$P(1 < x < 3) = F_X(3) - F_X(1) = \frac{9}{16} - \frac{1}{8} = 0.4375$

ⓓ Find $P(X > 0.5 \mid X < 1)$

$$P(X > 0.5 \mid X < 1) = \frac{P(0.5 < x < 1)}{P(X < 1)} = \frac{F_X(1) - F_X(0.5)}{F_X(1)}$$

$$= \frac{1/8 - \frac{0.5}{8}}{\frac{1}{8}} = \underline{0.5}$$

(e) Find $E(x)$ and $V(x)$

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx = \frac{1}{8}\int_0^2 x dx + \frac{1}{8}\int_2^4 x^2 dx$$

$$= \frac{1}{8}\left(2 + \frac{64}{3} - \frac{8}{3}\right) = \frac{1}{4} + \frac{8}{3} - \frac{1}{3} = \frac{1}{4} + \frac{7}{3} = 2.58$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{8}\left[\int_0^2 x^2 dx + \int_2^4 x^3 dx\right]$$

$$= \frac{1}{8}\left(\frac{8}{3} + 64 - 2\right) = \frac{1}{3} + 8 - \frac{1}{4} = 8.08$$

$$V = E(x^2) - [E(x)]^2 = 8.08 - 6.65 = \underline{1.43}$$

③ Let x be the weekly CPU time used by an accounting firm

$$f(x) = \begin{cases} \frac{3}{64}x^2(4-x), & 0 \le x < 4 \\ 0, & \text{otherwise} \end{cases}$$

ⓐ Find the expected value and variance of weekly CPU time.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^4 \frac{3}{64}x^3(4-x)\,dx$$

$$= \frac{3}{64}\left(4 \cdot \frac{4^4}{4} - \frac{4^5}{5}\right) = 3\left(4 - \frac{16}{5}\right) = 2.4$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{3}{64}\int_0^4 x^4(4-x)\,dx = \frac{3}{64}\left(4 \cdot \frac{4^5}{5} - \frac{4^6}{6}\right) = 6.4$$

$$V(X) = E(X^2) - [E(X)]^2 = 6.4 - 5.76 = \underline{0.64}$$

ⓑ The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

Let T denote the cost, then

$$T = 200X$$

$$E(T) = E(200X) = 200E(X) = 200 \times 2.4 = \underline{480}$$

$$V(T) = V(200X) = 200\,V(X) = 200 \times 0.64 = \underline{128}$$

ⓒ Would you expect the weekly cost to exceed \$600 very often? why?

$$P(T>600) = P(X>3) = \int_3^4 f(x)\,dx$$

$$= \frac{3}{64}\int_3^4 x^2(4-x)\,dx$$

$$= \frac{3}{64}\left(4\cdot\frac{4^3}{3} - \frac{4^4}{4} - 4\cdot\frac{3^3}{3} + \frac{3^4}{4}\right)$$

$$= 0.26 = 26.17\%$$

The expect weekly cost to exceed \$600 is occasionally.

(4) Let X denote the GPA for graduating senior at a college.

The PDF of X is $f(x) = \begin{cases} K(1-(x-3)^2), & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$

(a) Find the value of K

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$\Leftrightarrow \int_2^4 K(1-(x-3)^2) = 1$$

$$\Leftrightarrow K\left(4-2 + \frac{(x-3)^3}{3}\Big|_2^4\right) = 1$$

$$\Leftrightarrow \frac{16}{9}K = 1 \Rightarrow K = \frac{9}{16}$$

(b) Find the probability that GPA is within 0.25 of 3

$$P(X>3) = \int_3^{\infty} f(x) = \int_3^4 K(1-(x-3)^2)dx$$

$$= \frac{9}{16}\left(4-3 - \frac{(x-3)^3}{3}\Big|_3^4\right)$$

$$= \frac{9}{16} \times \frac{8}{9} = \frac{1}{2}$$

(c). Find the probability that a GPA is with 0.25 of 3

$$P(x \text{ is within } 0.25 \text{ of } 3) = P(3-0.25 < X < 3+0.25)$$

$$= \int_{2.75}^{3.25} f(x)dx$$

$$= \frac{9}{16}\left(0.5 - \frac{(x-3)^3}{3}\Big|_{2.75}^{3.25}\right)$$

$$= \frac{9}{16}\left(\frac{143}{288}\right) = \underline{0.27}$$

(d). Find the probability that a GPA differs from 3 by more than 0.5

$$P(x \text{ is more than } 0.5) = P(|X-3| > 0.5)$$

$$= 1 - P(|X-3| \le 0.5)$$

$$= 1 - P(-0.5 \le x-3 \le 0.5)$$

$$= 1 - P(2.5 \le x \le 3.5)$$

$$= 1 - \int_{2.5}^{3.5} K(1-(x-3)^2)dx$$

$$= 1 - \frac{9}{16}\left(1 - \frac{(x-3)^3}{3}\Big|_{2.5}^{3.5}\right)$$

$$= 1 - \frac{35}{64} = \underline{0.45}$$

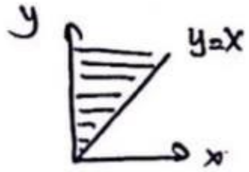
(5) let X be a crv with pdf $f(x)$ that is positive only if $x \geq 0$. If $F_X(x)$ is the cdf of X, show that

$$E(X) = \int_0^\infty (1 - F_X(x))\,dx$$

we have $F_X(x) = P(X \leq x)$

$$\Rightarrow 1 - F_X(x) = P(X > x) = \int_x^\infty f(y)\,dy$$

Then $\displaystyle\int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty \int_x^\infty f(y)\,dy\,dx$



$$= \int_0^\infty \int_0^y f(y)\,dx\,dy$$

$$= \int_0^\infty y f(y)\,dy = \int_0^\infty x f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x f(x)\,dx = E(X)$$

Thus $\displaystyle E(X) = \int_0^\infty (1 - F_X(x))\,dx$

⑥ If X is a continuous random variable such that $E[(X-a)^2] < \infty$ for all a, show that $E[(X-a)^2]$ is minimized when $a = E(X)$.

$$[E(X-a)^2] = E[(X-\mu+\mu-a)^2] = E[(X-\mu)^2 + 2(X-\mu)(\mu-a) + (\mu-a)^2]$$

$$= E[(X-\mu)^2] + 2[E(X)-\mu](\mu-a) + (\mu-a)^2$$

$$= \sigma^2 + (\mu-a)^2 \geq \sigma^2 \quad \forall a \in \mathbb{R}$$

Thus $E[(X-a)^2]$ is minimized to σ^2 when $a=\mu$

⑦ Suppose that $X \sim U(a,b)$. Show that

(a) $E(X) = \frac{a+b}{2}$

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{b} x f(x)\,dx + \int_{b}^{+\infty} x f(x)\,dx$$

$$= 0 + \int_a^b \frac{x}{b-a}\,dx + 0 = \frac{1}{2}\cdot\frac{1}{b-a}\cdot(b^2-a^2)$$

$$= \frac{a+b}{2}$$

Thus $E(X) = \frac{a+b}{2}$

(b) $V(X) = E(X^2) - [E(X)]^2$

$$= \int_a^b x^2 f(x)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{3}\cdot\frac{b^3-a^3}{b-a} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \frac{(b-a)^2}{12}$$

Thus $V(X) = \frac{(b-a)^2}{12}$

(c) $M(t) = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$

$$M_X(t) = E(e^{tx}) = \int_a^b \frac{e^{tx}}{b-a}\,dx$$

$$= \begin{cases} \dfrac{1}{t(b-a)} e^{tx}\Big|_a^b & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

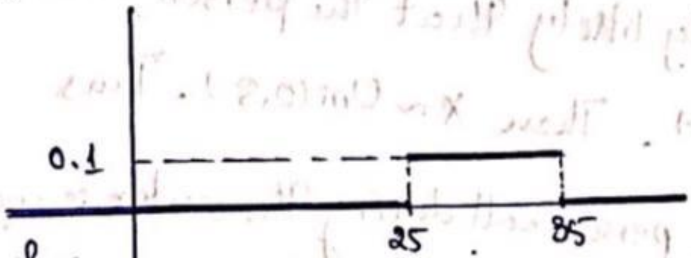
Thus $M(t) = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & , t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

⑧ $X \sim \text{Uni}(25, 35)$

ⓐ Determine the pdf of X and sketch the corresponding density curve.

$$f(x) = \begin{cases} \frac{1}{35-25} = \frac{1}{10}, & 25 \le x \le 35 \\ 0, & \text{otherwise} \end{cases}$$

Figure:



Grape of $f(x)$

ⓑ What is the probability that preparation time exceed 33 min?

$$P(X>33) = P(33<X<35) = \int_{33}^{35} \frac{1}{10}dx = \frac{35-33}{10} = \frac{1}{5}$$

ⓒ What is the probability that preparation time is within 2 mn of the mean time?

we have $\mu = E(X) = \frac{25+35}{2} = 30$

$$P(|X-\mu|) = P(\mu-2 \le X \le \mu+2) = P(28 \le X \le 32)$$

$$= \int_{28}^{32} \frac{1}{10}dx = \frac{32-28}{10} = \frac{2}{5}$$

ⓓ For any a such that $25 < a+2 < 35$, what is the probability that preparation time is between a and a+2

$$P(a \le X \le a+2) = \int_{a}^{a+2} \frac{1}{10}dx = \frac{a+2-a}{10} = \frac{1}{5}$$

③ What is the probability that the center is up when the person's call come in ?

- From 12:00AM to 1:00AM the computer is up
- from 1:00AM to 2:00AM the computer is down
- From 2:00AM to 3:00AM the computer is up
- From 3:00AM to 4:00AM the computer is down

Let X be the time when the person call. It is equally likely that the person call from 12:00AM to 5:00AM. Then $X \sim Unif(0,5)$. Thus

P(The person call during the center is up)

$= P(0 \leq X \leq 1) + P(3 \leq X \leq 4)$

$= \int_0^1 \frac{1}{5} dx + \int_2^3 \frac{1}{5} dx = \frac{1}{5} + \frac{3}{5} - \frac{2}{5} = \frac{2}{5}$

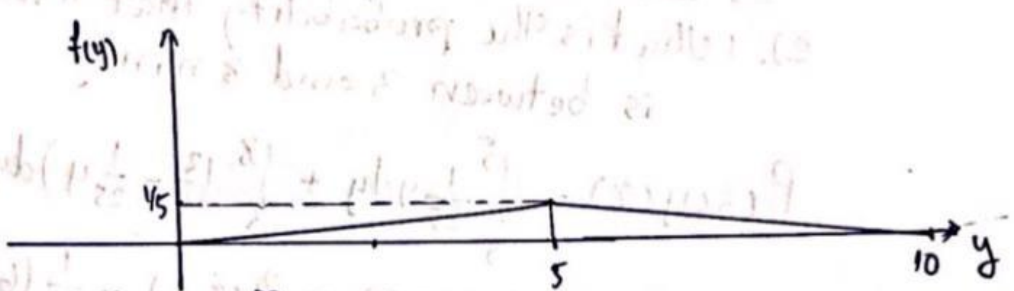
Ex10 let y be the total waitting time

$Y \sim V(A,B)$ which $A=0$, $B=5$

and the pdf of f is

$$f(y) = \begin{cases} \frac{y}{25} & \text{if } 0 \le y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a). Sketch a graph of the pdf of y



b). Verify that $\int_{-\infty}^{\infty} f(y)dy = 1$

we have $\int_{-\infty}^{+\infty} f(y)dy = \int_0^5 \frac{y}{25}dy + \int_5^{10} \left(\frac{2}{5} - \frac{1}{25}y\right)dy$

$= \frac{1}{2} \cdot \frac{25}{25} + \frac{2}{5}(10-5) - \frac{1}{50}(100-25)$

$= \frac{1}{2} + 2 - \frac{3}{2} = 1$

Thus $\int_{-\infty}^{+\infty} f(y)\,dy = 1$

c). What is the probability that total waiting time is at most 3 min ?

$P(y \le 3) = \int_{-\infty}^{3} f(y)dy = \int_0^3 \frac{y}{25}dy = \frac{1}{2} \cdot \frac{9}{25} = \frac{9}{50}$

Thus $P(y \le 3) = \frac{9}{50}$

d)- What is the probability that total waiting time is at most 8 min ?

$$P(y \le 8) = \int_{-\infty}^{8} f(y)dy = \int_0^5 \frac{1}{25} y\,dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25}y\right)dy$$

$$= \frac{1}{2} \times \frac{25}{25} + \frac{2}{5}(8-5) - \frac{1}{50}(64-25)$$

$$= \frac{1}{2} + \frac{6}{5} - \frac{39}{50} = \frac{66}{50} = \frac{23}{25}$$

Thus $P(y \le 8) = \frac{66}{50} = \frac{23}{25}$

e). What is the probability that total waiting is between 3 and 8 min ?

$$P(3 < y < 8) = \int_3^5 \frac{1}{25} y\,dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25}y\right)dy$$

$$= \frac{1}{50}(25-9) + \frac{2}{5}(8-5) - \frac{1}{50}(64-25)$$

$$= -\frac{23}{50} + \frac{6}{5} = \frac{37}{50}$$

$P(3 < y < 8) = \frac{37}{50}$

f)- What is the average of the total waiting time ?

$$E(y) = \int_{-\infty}^{+\infty} y f(y)dy = \int_0^5 \frac{1}{25} y^2 dy + \int_5^{10} \left(\frac{2}{5}y - \frac{1}{25}y^2\right)dy$$

$$= \frac{1}{75}(125-0) + \frac{1}{5}(100-25) - \frac{1}{75}(1000-125)$$

$$= -\frac{750}{75} + 15 = 5$$

$E(y) = 5$

(11) If $X \sim N(\mu, \sigma^2)$, then : Show that

(a) $E(X) = \mu$ (b) $V(X) = \sigma^2$ (c) $M(t) = \exp(\mu t + \frac{\sigma^2 t^2}{2})$

$$M(t) = E(e^{tx}) = \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma}(x-\mu-\sigma^2 t^2)} e^{\mu t + \frac{1}{2}\sigma^2 t^2} dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2 t)^2} dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

The last integral integrates to 1 because the integrand is the pdf of normal random variable whose mean is $\mu + \sigma^2 t$ and variance σ^2, that is $N(\mu + \sigma^2 t, \sigma^2)$

$E(X) = M'(0)$ and $E(X^2) = M''(0)$

Since $M'(t) = (\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$M''(t) = (\sigma^2 + (\mu + \sigma^2 t)^2) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Then $E(X) = M'(0) = \mu$

$V(X) = E(X^2) - [E(X)]^2 = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2$

Thus $M(t) = \exp(\mu t + \frac{\sigma^2 t^2}{2})$

$E(X) = \mu$

$V(X) = \sigma^2$

(12) If X is a random variable such that

$$\begin{cases} E(X^{2n}) = (2n)!/2^n n! \\ E(X^{2n-1}) = 0 \end{cases}, \text{ for } n = 1, 2, \dots$$

Find the moment generating function and distribution of X.

use $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ (Maclaurin series)

$$M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right] = E\left[\sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\right]$$

$$= E\left[1 + \frac{t}{1!}X + \frac{t^2}{2!}X^2 + \dots + \frac{t^{2n}}{(2n)!}X^{2n} + \dots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \dots + \frac{t^{2n}}{(2n)!}E(X^{2n}) + \dots$$

$$= 1 + t(0) + \frac{t^2}{2!} \cdot \frac{2!}{2 \cdot 1!} + \dots + \frac{t^{2n}}{(2n)!} \cdot \frac{(2n)!}{2^n n!} + \dots$$

$$= 1 + \frac{t^2}{2^2 2!} + \dots + \frac{t^{2n}}{2^n n!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} = e^{t^2/2}$$

Recall that when $X \sim N(\mu, \sigma^2)$, then

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Thus $X \sim N(0, 1)$ and

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < +\infty$$

(13) Let X denote the GPA of a college student

$X \approx N(\mu, \sigma^2)$, $\mu = 2.4$, $\sigma = 0.8$

X is a CRV

(a) If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?

$$P(X<1.9) = \Phi\left(\frac{1.9-\mu}{\sigma}\right) = \Phi\left(\frac{1.9-2.4}{0.8}\right) = \Phi(-0.625)$$

$$= 1-\Phi(0.625) = 1-0.7357 = 0.2643$$

(b) How many percent of student will possess a GPA in excess of 3.00

$$P(X>3) = 1-P(X<3) = 1-\Phi\left(\frac{3-\mu}{\sigma}\right)$$

$$= 1-\Phi\left(\frac{3-2.4}{0.8}\right) = 1-\Phi(0.75)$$

$$= 1-0.7734 = 0.2266$$

(c). Suppose that 5 students are randomly selected from the student body. What is the probability that only three will possess a GPA in excess of 3.00?

let Y = # of students that have GPA excess of 3.00 among 5 selected student

$Y \sim Bin(n,p)$, $n=5$, $p = P(X>3) = 0.2266$

$$P(Y=3) = C_5^3 (p)^3 (1-p)^2 = \frac{5!}{3!\,2!}(0.2266)^3(1-0.2266)^2$$

$$= 0.0696$$

(14) Let X denote the resistante of a wire produced by the company A

$X \sim N(\mu, \sigma^2)$, $\mu = 0.13$, $\sigma = 0.005$

ⓐ What is the probability that a randomly selected wire from company A's production will meet the specifications?

$$P(0.12 \leq X \leq 0.14) = \phi\left(\frac{0.14-\mu}{\sigma}\right) - \phi\left(\frac{0.12-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{0.14-0.13}{0.005}\right) - \phi\left(\frac{0.12-0.13}{0.005}\right)$$

$$= \phi(2) - \phi(-2) = \phi(2) - (1-\phi(2))$$

$$= -1 + 2\phi(2) = -1 + 2(0.9773) = \underline{0.9546}$$

ⓑ If four of these wires are used in each computer systeme and all are selected from company A, what is the propability that all four in randomly selected system will meet the specifications?

Let Y = # of wires that meet the specification among 4 selected wires.

$Y = Bin(n, p)$, $n = 4$, $p = P(0.12 \leq X \leq 0.04)$

$$P(Y=4) = C_4^4 p^4 (1-p)^0$$

$$= 1 \times (0.9546)^4 (1-0.9546)^0 = \underline{0.8304}$$

(15) Let X be the score on an examination

$X \sim N(\mu, 6^2)$, $\mu = 65$, $6 = 6$

a)- What is the probability that a person taking the examination scores higher than 70?

$P(X > 70) = 1 - P(X \leq 70) = 1 - \phi\left(\frac{70-\mu}{6}\right)$

$= 1 - \phi\left(\frac{70-65}{6}\right) = 1 - \phi(0.83)$

$= 1 - \phi(0.8\bar{3}) = 1 - 0.7977 = \underline{0.2023}$

b)- Suppose that students scoring in the top 10% of this distribution are to receive an A grade.
What is the minimum score a student must achieve to earn an A grade?

Let c be the minimum score that a student must achieve to get an A grade

find P such that $P(X > C) = 10\% = 0.1$

$1 - P(X \leq C) = 0.1 \Rightarrow P(X \leq C) = 0.9$

$\phi\left(\frac{C-65}{6}\right) = \phi(1.29)$

$C = 65 + 6(1.29) = \underline{72.68}$

c)- What must be the cutoff point for passing the examination if the examiner wants only the top 75% of all scores to be passing?

Let d be the minimum passing score

find d such that $P(X > d) = 75\% = 0.75$

$\phi\left(\frac{65-C}{6}\right) = 0.75 = \phi(0.67)$

$C = 65 - 6 \times 0.67 = \underline{60.98}$

d). Approximately what proportion of sudent have scores 5 or more points above the score that cuts off the lowest 15%?

$P(X \le e) = 0.15 \Leftrightarrow \phi\left(\frac{e-65}{6}\right) = 0.15$

$\Rightarrow \phi\left(\frac{65-e}{6}\right) = 0.85 = \phi(1.036)$

$e = 65 - 6(1.036) = 58.784$

So the score that cuts off the lowest 15% is 58.784.

Thus $P(X \ge 5 + 58.784) = P(X \ge 63.784) = 1 - \phi\left(\frac{65-63.784}{6}\right)$

$= 1 - \phi(0.2026)$

$= 1 - 0.5793 = 42.07\%$

e). If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

$P(X > 84 \mid X > 72) = \frac{P(X>84)}{P(X>72)} = \frac{1-\phi\left(\frac{84-65}{6}\right)}{1-\phi\left(\frac{72-65}{6}\right)}$

$= \frac{1-\phi(3.1667)}{1-\phi(1.1667)} =$

$= \frac{1-0.9992}{1-0.8786} = \frac{0.0008}{0.1214} = 6.6 \times 10^{-3}$

(16) The distribution of 1000 examinees according to marks percentage is given below:

% Score	< 40%	40% - 75%	> 75%	Total
No. of examinees	430	420	150	1000

a)- Calculate the mean and standard deviation of marks.

$X \sim N(\mu, \sigma^2)$, $\mu = ?$, $\sigma = ?$

Given $\begin{cases} P(X < 40) = \frac{430}{1000} = 0.43 \\ P(X > 75) = \frac{150}{1000} = 0.15 \end{cases}$

$\begin{cases} \phi\left(\frac{40-\mu}{\sigma}\right) = 0.43 = \phi(-0.1764) \\ \phi\left(\frac{75-\mu}{\sigma}\right) = 0.85 = \phi(1.0364) \end{cases}$

$\begin{cases} 40-\mu = -0.1746 \\ 75-\mu = 1.03646 \end{cases}$

$\mu = 45.09$, $\sigma = 28.86$

Thus $X \sim N(\mu, \sigma^2)$, $\mu = 44.91$, $\sigma = 28.92$

b)- If not more than 300 examinees are to fail what should the passing mark ?

Let c be the minimum passing mark

$P(X \le c) \le \frac{300}{1000} = 0.3$

$\phi\left(\frac{45.09 - c}{28.86}\right) \geqslant 0.7 = \phi(0.53)$

$\frac{45.09 - c}{28.86} \geqslant 0.53 \Rightarrow c \le 29.9558 \approx 30$

(17) Let x : the amount of time a book on two-hour reserve is actually cheaked out

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{4} & , 0 \le x < 2 \\ 1 & , 2 \le x \end{cases}$$

a Calcute $P(X \le 1)$

$P(X \le 1) = F(1) = \frac{1^2}{4} = \frac{1}{4}$

Thus $P(X \le 1) = 0.25$

b Calculate $P(0.5 \le X \le 1)$

$P(0.5 \le X \le 1) = F(1) - F(0.5)$

$= \frac{1}{4} - \frac{0.25}{4} = 0.1875$

Thus $P(0.5 \le X \le 1) = 0.1875$

c Calculate $P(X > 1.5)$

$P(X > 1.5) = 1 - F(1.5)$

$= 1 - \frac{(1.5)^2}{4} = 1 - 0.5625$

$= 0.4375$

Thus $P(X > 1.5) = 0.4375$

d The median cheakout duration $\tilde{\mu}$

$F(\tilde{\mu}) = \frac{1}{2}$

$\Rightarrow \frac{(\tilde{\mu})^2}{4} = \frac{1}{2}$

$\Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.4142$

Thus $\tilde{\mu} = 1.4142$

e Obtain the density function $f(x)$

$$f(x) = F'(x) = \begin{cases} \frac{x}{2}, & 0 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

f Calculate $E(x)$

$$E(x) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$= \int_0^2 x\left(\frac{x}{2}\right)dx = \frac{1}{6}x^3\Big|_0^2 = \frac{1}{6}(8) = \frac{4}{3} = 1.33$$

Thus $E(x) = 1.33$

(18) a What is the probability that the diameter of a randomly selected tree will be at least 10 in? will exceed 10 in?

$X \sim N(\mu, \sigma^2)$, $\mu = 8.8$, $\sigma = 2.8$

$P(X \geq 10) = 1 - P(X \leq 10) = 1 - \Phi\left(\frac{10-8.8}{2.8}\right)$

$= 1 - \Phi(0.428) = 1 - 0.6628 = 0.3372$

Thus $P(X \geq 10) = 0.3372$

For continuous distribution $P(X > 10) = P(X \geq 10) = 0.3336$

Thus $P(X > 10) = 0.3372$

b). What is the probability that the diameter of a randomly selected tree will exceed 20 in?

$P(X > 20) = 1 - \Phi\left(\frac{20-8.8}{2.8}\right) = 1 - \Phi(4) \approx 0$

Thus $P(X > 20) \simeq 0$

c). What is the probability that the diameter of a randomly selected tree will be between 5 and 10 in?

$P(5 \leq X \leq 10) = \Phi\left(\frac{10-8.8}{2.8}\right) - \Phi\left(\frac{5-8.8}{2.8}\right)$

$= \Phi(0.428) - (1 - \Phi(1.357))$

$= 0.6628 - 1 + 0.9115 = 0.5743$

Thus $P(5 \leq X \leq 10) = 0.5743$

d)- What value c is such that the interval (8.8-c, 8.8+c) includes 98% of all diameter values ?

$P(8.8-c \le X \le 8.8+c) = 0.98$

$\Phi\left(\frac{c}{2.8}\right) - \Phi\left(\frac{-c}{2.8}\right) = 0.98$

$2\Phi\left(\frac{c}{2.8}\right) = 1.98$

$\Phi\left(\frac{c}{2.8}\right) = 0.99 = \Phi(2.33)$

$c = 2.8 \times 2.33 = \underline{6.524}$

e)- If four trees are independently selected, what is the probability that at least one has a dimeter exceeding 10 in ?

Let y denote the number of tree that have a diameter exceed 10 in. among 4 seeted trees

$Y \sim Bin(n,p)$, $n=4$, $p = P(X>10) = 0.334$

$P(y \ge 1) = P(y=1) + P(y=2) + P(y=3) + P(y=4)$
$= \binom{4}{1}pq^3 + \binom{4}{2}p^2q^2 + \binom{4}{3}p^3q + \binom{4}{4}p^4$
$= 0.39 + 0.29 + 0.09 + 0.01 = 0.78$

Thus $P(y \ge 1) = 0.78$.

(19) What percentage of the bearing produced will not be ceptable ?

A bearing is aceptable if its diameter is with in 0.004 in or $P(0.496 \leq X \leq 0.504)$

Then a bearing is not acceptable :

$$P(x) = P(X \leq 0.496) + P(X \geq 0.504)$$

$$= \phi\left(\frac{0.496-\mu}{6}\right) + 1 - \phi\left(\frac{0.504-\mu}{6}\right)$$

where $\mu = 0.499$ and $6 = 0.002$

$$P(x) = \phi\left(\frac{0.496-0.499}{0.002}\right) + 1 - \phi\left(\frac{0.504-0.499}{0.002}\right)$$

$$= \phi(-1.5) + 1 - \phi(2.5)$$

$$= 1 - \phi(1.5) + 1 - \phi(2.5)$$

$$= 2 - 0.9332 - 0.9938 = 0.073$$

$$P(x) = 0.073$$

(20) Let X = # of driver who wear a seat belt among 500 selected drivers.

$X \sim Bin(n, p)$, $n = 500$, $p = 0.7$

Because n is large, then

$X \sim N(\mu, \sigma^2)$, $\mu = np = 500 \times 0.7 = 350$

$\sigma = \sqrt{np(1-p)} = \sqrt{500 \times 0.7 \times 0.3}$
$= 10.24$

a). Between 320 and 370 (inclusive) of the drivers in the sample regularly wear a seat belt?

$P(320 < X < 370) = \phi\left(\frac{370+0.5-350}{10.24}\right) - \phi\left(\frac{320-0.5-350}{10.24}\right)$

$= \phi(2.001) - \phi(-2.97)$

$= 0.9772 - 1 + 0.9985$

$= \underline{0.9757}$

b). Fewer than 325 in the sample regularly wear a seat belt? Fewer than 315?

$P(X < 325) = P(X \le 324) \approx \phi\left(\frac{324+0.5-350}{10.24}\right)$

$= 1 - \phi(2.49)$

$= 1 - 0.9936 = \underline{6.4 \times 10^{-3}}$

$P(X < 315) = P(X \le 314) = \phi\left(\frac{314+0.5-350}{10.24}\right)$

$= 1 - \phi(3.46)$

$= 1 - 0.9997$

$= \underline{0.0003}$

(21) Let $X \sim Exp(\lambda)$ Show that

(a) $F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x/\lambda} & , x \geq 0 \end{cases}$

(b) $E(X) = \lambda$

(c) $V(X) = \lambda^2$

(d) $E(X^n) = \lambda^n n!$ for $n = 1, 2, \ldots$

(e) $M(t) = \frac{1}{1 - \lambda t}$, $t < 1/\lambda$

we have the pdf of $X \sim Exp(\lambda)$ is $f(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/\lambda}, & x \geq 0 \end{cases}$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

• If $x < 0$, then $F(x) = 0$

• If $x \geq 0$, then $F(x) = \underbrace{\int_{-\infty}^{0} f(y)dy}_{0} + \int_{0}^{x} f(y)dy + \int_{0}^{x} \frac{1}{\lambda} e^{-y/\lambda}dy$

$$= [-e^{-y/\lambda}]_{0}^{x} = 1 - e^{-x/\lambda}$$

Thus $F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x/\lambda} & , x \geq 0 \end{cases}$

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{0}^{+\infty} \frac{1}{\lambda} e^{tx - \frac{x}{\lambda}} dx$$

$$= \int_{0}^{+\infty} \frac{1}{\lambda} e^{(t - \frac{1}{\lambda})x} dx = \left[\frac{1}{\lambda} \cdot \frac{1}{t + \frac{1}{\lambda}} e^{(t - \frac{1}{\lambda})x}\right]_{0}^{+\infty}$$

$$= \frac{1}{1 - \lambda t} , t < \frac{1}{\lambda}$$

Thus $M(t) = \frac{1}{1 - \lambda t}$, $t < \frac{1}{\lambda}$

$M'(t) = ((1 - \lambda t)^{-1})' = \lambda (1 - \lambda t)^{-2}$

$M''(t) = 2\lambda^2 (1 - \lambda t)^{-3}$

$M'''(t) = 6\lambda^3 (1 - \lambda t)^{-4}$

$M^{(n)}(t) = n! \lambda^n (1 - \lambda t)^{-n}$

$\Rightarrow E(X) = M'(0) = \lambda$

$\Rightarrow E(X^n) = (M(0))^{(n)} = n!\lambda^n$ for $n = 1, 2, \ldots$

$V(X) = M''(0) - [M'(0)]^2 = 2\lambda^2 - \lambda^2 = \lambda^2$

(22) Suppose that X has an exponential probability density function. Show that if $x, y > 0$

$$P(X > x+y \mid X > x) = P(X > y)$$

Suppose that $X \sim Exp(\lambda)$, $\lambda > 0$

$$P(X > x+y \mid X > x) = \frac{P(X > x+y \text{ and } X > x)}{P(X > x)}$$

$$= \frac{P(X > x+y)}{P(X > x)}$$

Since $P(X > x+y) = \int_{x+y}^{\infty} f(t)\,dt = \int_{x+y}^{\infty} \frac{1}{\lambda} e^{-\frac{t}{\lambda}} dt$

$$= \left[-e^{-t/\lambda}\right]_{x+y}^{\infty} = e^{-\frac{x+y}{\lambda}}$$

Similarly $P(X > x) = e^{-x/\lambda}$

$$P(X > x+y \mid X > x) = \frac{e^{-\frac{x+y}{\lambda}}}{e^{-x/\lambda}} = e^{-y/\lambda} = P(X > y)$$

Thus $P(X > x+y \mid X > x) = P(X > y)$

(23) Let X be the time past 7:00 AM to wait another accidents accured (in hour)

$X \sim Exp(\lambda)$, $\lambda = 4$

(a). What is the probability that another accident occurred that same day ?

$P(x \le 17) = \int_0^{17} \frac{1}{4} e^{-x/4} dx = (-e^{-17/4} + 1) = 0.9857$

(b). What time that the next accident occurred if the probability that the next accident occurrent is 0.8

$P(x \le t) = 0.8$

$\int_0^t \frac{1}{4} e^{-\frac{x}{4}} dx = 0.8$

$\left[-e^{-x/4}\right]_0^t = 0.8$

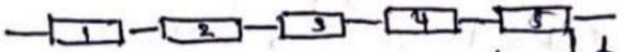
$(1 - e^{-t/4}) = 0.8$

$e^{-t/4} = 0.2 \Rightarrow -\frac{t}{4} = \ln 0.2 \Rightarrow t = -4\ln 0.2$
$= 6.43$

$t = 6h\ 26mn$

So the other accident perhaps occurrent at 1.26 PM

(24) A systeme consists of five identical components connected in series as shown



(a) The event $\{X \geq t\}$ is equivalent to what event involving $\{A_1, A_2, \ldots, A_5\}$?

The event $\{u \geq t\}$ is equivalent to the event

$$\bigcap_{i=1}^{5} A_i = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$$

(b) Using the independence of the $\{A_i\}$'s, compute $P(X \geq t)$.
Then obtain $F(t) = P(X \leq t)$ and the pdf of X.
What type of distribution does X have?

$$P(X \geq t) = P\left(\bigcap_{i=1}^{5} A_i\right) \quad \text{(independent of the } \{A_i\}\text{'s)}$$

$$= \prod_{i=1}^{5} P(A_i)$$

$$P(A_i) = P(X_i \geq t), \quad X_i \sim Exp(\lambda), \lambda = 100$$

$$= \int_t^{\infty} \frac{1}{100} e^{-\frac{u}{100}} du = \left[-e^{\frac{u}{100}}\right]_t^{\infty} = e^{-t/100}$$

Thus $P(X \geq t) = \left(e^{-t/100}\right)^5 = e^{-t/20}$

$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-t/20}, t \geq 0$
is the cdf of $Exp(20)$

Thus $X \sim Exp(20)$

(c) Suppose there are n component, each having exponential lifetime with parameter λ.
What type of distribution does X have?

From (b), if there are n components,

$$P(X \geq t) = \left(e^{-t/100}\right)^n = e^{-\frac{n}{100}t}$$

and $F(t) = 1 - e^{-\frac{n}{100}t}$, $t \geq 0$

Thus $X \sim Exp\left(\frac{100}{n}\right)$

(25) What is the probability that at least 2 are still functioning at the end of 8 years?

Let X = # of the component having lifetime at least 8 years among 5 selected component

$X \sim Bin(n, p)$, $n = 5$, $p = P(T > 8)$

$T \sim Exp(\lambda)$, $\lambda = 5$

$$p = \int_8^\infty e^{-t/5} dt = \left[-e^{-t/5}\right]_8^\infty = e^{-8/5} = 0.2$$

$$P(y \geq 2) = 1 - P(y \leq 2)$$

$$= 1 - \phi\left(\frac{2 + 0.05 - np}{\sqrt{np(1-p)}}\right)$$

$$= 1 - \phi\left(\frac{2 + 0.05 - 5 \times 0.2}{\sqrt{5 \times 0.2 \times 0.8}}\right) = 1 - \phi(1.159)$$

$$\approx 1 - \phi(1.16)$$

$$= 1 - 0.887 = 0.113$$

(26) Let $X \sim Gam(\alpha, \beta)$. Show that

(a) $E(X) = \alpha\beta$

we have $E(X) = \int_0^{+\infty} x \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{x/\beta} dx$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha} e^{-x/\beta} dx$$

let $t = \frac{x}{\beta} \Rightarrow dx = \beta dt$

$$\Rightarrow E(X) = \frac{1}{\beta^{\alpha-1}\Gamma(\alpha)} \int_0^{+\infty} (t\beta)^{\alpha} e^{-t} dt$$

$$= \frac{\beta^{\alpha}}{\beta^{\alpha-1}\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha} e^{-t} dt$$

$$= \frac{\beta}{\Gamma(\alpha)} \times \Gamma(\alpha+1) = \frac{\beta}{\Gamma(\alpha)} \times (\alpha+1-1) \times \Gamma(\alpha+1-1) = \alpha\beta$$

Thus $E(X) = \alpha\beta$

(b). $V(X) = \alpha\beta^2$

$$E(X^2) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-x/\beta} dx$$

let $t = \frac{x}{\beta} \Rightarrow dx = \beta dt$

$$E(X^2) = \frac{\beta^{\alpha+1}}{\beta^{\alpha-1}\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha+1} e^{-t} dt$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \times \Gamma(\alpha+2)$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \times (\alpha+1)\Gamma(\alpha+1) = \frac{\beta^2}{\Gamma(\alpha)} \times (\alpha+1)(\alpha) \times \Gamma(\alpha) = \beta^2(\alpha+1)\alpha$$

$V(x) = E(x^2) - [E(x)]^2$

$= \beta^2(\alpha+1)\alpha - \beta^2\alpha^2 = \beta^2\alpha^2 + \beta^2\alpha - \beta^2\alpha^2 = \beta^2\alpha$

Thus $V(x) = \alpha\beta^2$

(c). $M(t) = \frac{1}{(1-\beta t)^\alpha}$, $t < \frac{1}{\beta}$

$$M(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} e^{tx} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{x(t-\frac{1}{\beta})} dx$$

let $-U = (-\frac{1}{\beta}+t)\, x \Rightarrow dx = \frac{1}{\frac{1}{\beta}-t} dU$

and $x = \frac{1}{\frac{1}{\beta}-t} \times U$

$$M(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \times \frac{1}{(\frac{1}{\beta}-t)^\alpha} \times \int_0^{+\infty} U^{\alpha-1} e^{-U} dU$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \times \frac{1}{\frac{(1-\beta t)^\alpha}{\beta^\alpha}} \times \Gamma(\alpha) = \frac{1}{(1-\beta t)^\alpha}, t < \frac{1}{\beta}$$

Thus $M(t) = \frac{1}{(1-\beta t)^\alpha}$, $t < \frac{1}{\beta}$

(d). $F(x) = \dfrac{\Gamma\left(\frac{x}{\beta}, \alpha\right)}{\Gamma(\alpha)}$

$$F(x) = P(X \le x) = \int_0^x f(x)\,dx$$

$$= \int_0^x \frac{1}{\beta^\alpha \Gamma(\alpha)} t^{\alpha-1} e^{-t/\beta} dt$$

let $u = \frac{t}{\beta} \Rightarrow dt = \beta du$

$$F(x) = \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} du = \frac{1}{\Gamma(\alpha)} \int_0^{x/\beta} u^{\alpha-1} e^{-u} du$$

$$= \frac{\Gamma\left(\frac{x}{\beta}, \alpha\right)}{\Gamma(\alpha)}$$

Thus $F(x) = \dfrac{\Gamma\left(\frac{\alpha}{\beta}, \alpha\right)}{\Gamma(\alpha)}$

(27) If X is a random variable such that $E(X^n) = (n+1)!2^n$, for $n = 1, 2, \ldots$

Find the moment generating function and the distribution of X

$$M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}(n+1)!2^n = \sum_{n=0}^{\infty} (2t)^n (n+2)$$

$$= \sum_{n=0}^{\infty} n(2t)^n + \sum_{n=0}^{\infty} (2t)^n$$

$$= 2t\sum_{n=1}^{\infty} n(2t)^{n-1} + \sum_{n=0}^{\infty} (2t)^n$$

$$= \frac{2t}{(1-2t)^2} + \frac{1}{1-2t}, \quad |2t| < 1$$

$$= \frac{2t+1-2t}{(1-2t)^2} = \frac{1}{(1-2t)^2}, \quad |t| < \frac{1}{2}$$

$X \sim Gam(2, 2)$

Thus $M_X(t) = \frac{1}{(1-2t)^2}$, $|t| < 1/2$

$X \sim Gam(2, 2)$

28 Let X be a rv such that

$$E(X^n) = \frac{(n+3)!}{3!} 3^n \quad , \text{for } n=1,2,\cdots$$

Find $M_X(t)$ and the distribution of X

$$M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} E(X^n)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{(n+3)!}{3!} \cdot 3^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{6} (n+1)(n+2)(n+3)(3t)^n$$

$$= 1 - 1 + \sum_{n=0}^{\infty} \frac{1}{6} (n+1)(n+2)(n+3)(3t)^n$$

$$= \frac{1}{6} \times 2 \, \frac{1+2(3t)}{(1-3t)^4}$$

$$= \frac{1}{3} \cdot \frac{1+6t}{(1-3t)^4} \quad , |t| < 1/3$$

(29) Let X denote the lifetime (in weeks) of a transistor

$X \sim Gam(\alpha, \beta)$ where $\mu = 24$, $\sigma = 12$

Since $\begin{cases} \mu = \alpha\beta \\ \sigma^2 = \alpha\beta^2 \end{cases} \Rightarrow \begin{cases} \alpha\beta = 24 \\ \alpha\beta^2 = 144 \end{cases} \Rightarrow \begin{cases} \alpha = 4 \\ \beta = 6 \end{cases}$

Recall that if $X \sim Gam(\alpha, \beta)$, then

$P(X \le a) = P(X < a) = \Gamma(\frac{a}{\beta}, \alpha) = F(\frac{a}{\beta}, \alpha)$

where F is the cdf of incomplete gamma distribution or standard gamma.

(a) What is the probability that a transistor will last between 12 and 24 weeks?

$P(12 \le X \le 24) = P(X \le 24) - P(X < 12)$

$= F(\frac{24}{6}, 4) - F(\frac{12}{6}, 4)$

$= F(4, 4) - F(2, 4)$

$= 0.567 - 0.143 = 0.424$

(b) What is the probability that a transistor will last at most 24 weeks? Is the median of the lifetime less than 24? Why or why not?

$P(X \le 24) = F(\frac{24}{6}, 4) = F(4, 4) = 0.567$

Let $\tilde{\mu}$ be the median, then

$P(X \le \tilde{\mu}) = 0.5$

Thus $\tilde{\mu} < 24$

(c) What is the 99th percentile of the lifetime distribution?

$P(X \leq c) = 0.99$

$F(\frac{c}{6}, 4) = 0.99 = F(10, 4)$

$\frac{c}{6} = 10 \Rightarrow c = 60$

(d) Suppose the test will actually be terminated after t weeks. What value of t is such that only 0.5% of all transistors would still be operating at termination?

$P(T \geq t) = 0.5\% = 0.005$

$P(T < t) = 0.995$

$F_X(\frac{t}{6}, 4) = 0.995 = F(11, 4)$

$\frac{t}{6} = 11 \Rightarrow t = 66$

Thus $t = 66$

(30) Let $X \sim Gam(\alpha, \beta)$. Show that $X \sim \chi^2(v)$ if $\alpha = v/2$, $\beta = 2$

$$X \sim \chi^2(v) \Leftrightarrow X \sim Gam(\alpha, \beta) = Gam\left(\frac{v}{2}, 2\right)$$

$$f(x) = \begin{cases} \frac{1}{2^{v/2}\Gamma\left(\frac{v}{2}\right)} x^{\frac{v}{2}-1} e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

if $\alpha = \frac{v}{2}$ & $\beta = 2$

Thus $X \sim \chi^2(v)$ if $\alpha = v/2$, $\beta = 2$

(31) If $X \sim \chi^2(v)$. Show that

(a) $E(X) = v$

$X \sim Gam(\alpha = \frac{v}{2}, \beta = 2)$

$E(X) = \alpha\beta = \frac{v}{2} \times 2 = v$

(b) $V(X) = 2v$

$X \sim Gam(\alpha = \frac{v}{2}, \beta = 2)$

$V(X) = \alpha\beta^2 = \frac{v}{2} \times 2^2 = 2v$

(c) $E(X^n) = \frac{2^n \Gamma(\frac{v}{2} + n)}{\Gamma(\frac{v}{2})}, \quad n \geq -v/2$

$$E(X^n) = \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} \int_0^{+\infty} u^{\frac{v}{2}+n} e^{-u/2} du$$

let $t = \frac{u}{2} \Rightarrow du = 2dt$

$$\Rightarrow E(X^n) = \frac{2^{v/2+n}}{2^{v/2}\Gamma(\frac{v}{2})} \int_0^{+\infty} t^{\frac{v}{2}+n} e^{-t} dt$$

$$E(X)^n = \frac{2^n}{\Gamma(\frac{v}{2})} \times \Gamma(\frac{v}{2}+n)$$

(d). $M(t) = \frac{1}{(1-2t)^{v/2}}$

we have $M(t) = \frac{1}{(1-\beta t)^\alpha}$ which $\alpha = \frac{v}{2}, \beta = 2$

$$M(t) = \frac{1}{(1-2t)^{v/2}}$$

(32) Suppose $X \sim N(\mu, \sigma^2)$, $\sigma > 0$ and $Y = \frac{(X-\mu)^2}{\sigma^2}$

Show that $Y \sim \chi^2(1)$

To show that $Y \sim \chi^2(1)$, we will prove that the pdf of Y is

$$g(v) = \begin{cases} \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} v^{-\frac{1}{2}} e^{-\frac{v}{2}}, & v \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let G be the cdf of Y, then we have

$$G(y) = P(Y \leq y) = P\left[\left(\frac{X-\mu}{\sigma}\right)^2 < y\right] = P\left(-\sqrt{y} < \frac{X-\mu}{\sigma} < \sqrt{y}\right)$$

$= P(-\sqrt{y} < Z < \sqrt{y})$, where $Z = \frac{X-\mu}{\sigma}$, $N(0,1)$

$= \int_{-\sqrt{y}}^{\sqrt{y}} f(z)\,dz$ where $f(z)$ is the pdf of Z

the pdf of Y is

$$g(y) = \frac{d}{dy} G(y) = \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} f(z)\,dz$$

$$= f(\sqrt{y}) \frac{1}{2\sqrt{y}} - f(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} v^{-1/2} e^{-v/2} \quad \text{since } \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} v^{-1/2} e^{-v/2}, \quad v \geq 0$$

Thus $Y = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$

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(33) Suppose that a random variable Y has a probability density function given by

$$f(y) = \begin{cases} ky^3 e^{-y/2}, & y > 0 \\ 0. & \text{otherwise} \end{cases}$$

(a). Find the value of k.

$\int_{-\infty}^{+\infty} f(y)\,dy = 1 \Leftrightarrow \int_0^{+\infty} ky^3 e^{-y/2} = 1$

Since $\int y^3 e^{-y/2} dy = 16\left(-\frac{1}{8}e^{-\frac{y}{2}}y^3 - 3\left(\frac{1}{4}e^{-y/2}y^2 + e^{-\frac{y}{2}}y + 2e^{-y/2}\right)\right)$

then $k[0 - (16(-3)] = 1$

$\Rightarrow \boxed{k = \frac{1}{96}}$

(b). Does y have χ^2 distribution? If so, how many degree of freedom?

$$Y \sim \chi^2(v) \Leftrightarrow f(y) = \begin{cases} \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} y^{\frac{v}{2}-1} e^{-y/2}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

we get $\frac{v}{2} - 1 = 3$ then $v = 8$

and $2^{\frac{v}{2}}\Gamma(\frac{v}{2}) = 2^4 \times \Gamma(4) = 16 \times 3! = 96$

Thus $Y \sim \chi^2(v = 8)$

(c). What are the mean and standard deviation of Y?

$\mu = E(y) = v = 8$ and $\sigma_Y = \sqrt{2v} = \sqrt{16} = 4$

(d) what is the probability that Y lies within 2 standard deviations of its mean?

$P(\mu - 2\sigma \leq y \leq \mu + 2\sigma) = P(0 \leq y \leq 16) = P(y \leq 16) - P(y)$

$P(0 \leq y \leq 16) = \frac{16}{96}\left(-\frac{1}{8}e^{-8}16^3 - 3\left(\frac{1}{4}e^{-8}16^2 + e^{-8}16 + 2e^{-8}\right)\right) = \underline{15.32}$

(85) Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

$X \sim Bet(\alpha, \beta)$, $\alpha = 4$, $\beta = 2$

Find $P(X \geq 0.90) = \int_{0.9}^{\infty} x f(x)\,dx$

Since $f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$P(X \geq 90) = \int_{0.9}^{1} \frac{1}{B(4,2)} x^3 (1-x)\,dx$

$B(4,2) = \frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} = \frac{\Gamma(4)\Gamma(2)}{6.5.\Gamma(4)} = \frac{2}{6,5} = \frac{1}{15}$

$P(X \geq 90) = \frac{1}{15}\int_{0.9}^{1} (x^3 - x^4)\,dx$

$= \frac{1}{15}\left(\frac{1}{4} - \frac{1}{5} - \frac{0.9^4}{4} + \frac{0.9^5}{5}\right) = \underline{0.000271533}$

(36) Let $X \sim Log(\mu, \sigma^2)$ Show that

$E(x) = e^{\mu + \sigma^2/2}$ and $V(x) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$

let t be a positive integer we have

$$E(x^t) = \int_{-\infty}^{+\infty} x^t f(x)\,dx = \int_0^{\infty} x^t \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} dx$$

Let $y = \ln(x)$, we get

$$E(x^t) = \int_{-\infty}^{+\infty} e^{ty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy = M_y(t)$$

where $M_y(t)$ is the mgf of $Y \sim N(\mu, \sigma^2)$

Thus $M_y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

let $t = 1$, we get $E(x) = e^{\mu + \frac{1}{2}\sigma^2}$

$t = 2$, we get $E(x^2) = e^{2\mu + 2\sigma^2}$

$V(x) = E(x^2) - [E(x)]^2 = [e^{\sigma^2} - 1]e^{2\mu + \sigma^2}$

Thus $E(x) = e^{\mu + \frac{\sigma^2}{2}}$

$V(x) = [e^{\sigma^2} - 1]e^{2\mu + \sigma^2}$

(37) What is the probability that the concentration exceeds 8 parts per million ?

Let $X \sim Log(\mu, \sigma^2)$, $\mu = 3.2$, $\sigma = 1$

Find $P(X>8)$

<u>Recall</u> when $X \sim Log(\mu, \sigma)$ then

$\ln X \sim N(\mu, \sigma^2)$

$$P(X>8) = 1 - P(X \le 8) = 1 - P(\ln X \le \ln 8)$$
$$= 1 - \phi\left(\frac{\ln 8 - \mu}{\sigma}\right)$$
$$= 1 - \phi\left(\frac{\ln 8 - 3.2}{1}\right) = 1 - \phi(-1.1205)$$
$$= 1 - (1 - \phi(1.1205))$$
$$\approx 0.8686$$

<u>Thus</u> $P(X>8) \approx 0.8686$

(88) Find the 5^{th} percentile of the life of such an electronic control.

Let X denote the life of such electronic control.

$X \sim \log(\mu, \sigma)$ $\mu = 5.149$, $\sigma = 0.737$

Let c such that $P(X \leq c) = 0.05$

$$\phi\left(\frac{\ln c - \mu}{\sigma}\right) = 0.05$$

$$\Leftrightarrow \phi\left(\frac{\mu - \ln c}{\sigma}\right) = 0.95 = \phi(1.65)$$

$$\frac{5.149 - \ln c}{0.737} = 1.65$$

$$\Rightarrow \ln c = -0.737 \times 1.65 + 5.149 = 3.93295$$

$$c = e^{3.93295} = 51.05737$$

Thus $c = 51.05737$

(39) Let $X \sim Wei(\alpha, \beta)$. Show that

$E(X) = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$ and $V(X) = \alpha^{-2/\beta}\left\{\Gamma(1+\frac{2}{\beta}) - [\Gamma(1+\frac{1}{\beta})]^2\right\}$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^{+\infty} x \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx$$

$$= \alpha\beta \int_0^{+\infty} x^\beta e^{-\alpha x^\beta} dx$$

Let $t = \alpha x^\beta \Rightarrow x = \left(\frac{t}{\alpha}\right)^{1/\beta} \Rightarrow dx = \frac{\left(\frac{t}{\alpha}\right)^{\frac{1}{\beta}-1}}{\alpha\beta} dt = \frac{\left(\frac{t}{\alpha}\right)^{1/\beta}}{\beta t} dt$

$$\Rightarrow E(X) = \beta \int_0^{+\infty} \frac{t e^{-t} \left(\frac{t}{\alpha}\right)^{1/\beta}}{\beta t} dt = \int_0^{+\infty} e^{-t} \left(\frac{t}{\alpha}\right)^{1/\beta} dt$$

$$= \int_0^{+\infty} \frac{t^{1/\beta}}{\alpha^{1/\beta}} e^{-t} dt = \alpha^{-1/\beta} \int_0^{+\infty} t^{(1+\frac{1}{\beta}-1)} e^{-t} dt$$

$$E(X) = \alpha^{-1/\beta} \Gamma(1+\frac{1}{\beta})$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx$$

$$= \alpha\beta \int_0^{+\infty} x^{\beta+1} e^{-\alpha x^\beta} dx$$

Let $t = \alpha x^\beta \Rightarrow x = \left(\frac{t}{\alpha}\right)^{1/\beta} \Rightarrow dx = \frac{\left(\frac{t}{\alpha}\right)^{1/\beta}}{\beta t}$

$$E(X^2) = \beta \int_0^{\infty} \alpha x^\beta x e^{-\alpha x^\beta} dx = \beta \int_0^{\infty} t \left(\frac{t}{\alpha}\right)^{1/\beta} e^{-t} \frac{\left(\frac{t}{\alpha}\right)^{1/\beta}}{\beta t} dt$$

$$= \alpha^{(-2/\beta)} \int_0^{\infty} t^{1+\frac{2}{\beta}-1} e^{-t} dt = \alpha^{(-2/\beta)} \Gamma(1+\frac{2}{\beta})$$

$$V(X) = \alpha^{(-2/\beta)} \Gamma(1+\frac{2}{\beta}) - \alpha^{(-1/\beta)} [\Gamma(1+\frac{1}{\beta})]^2$$

$$V(X) = \alpha^{(-2/\beta)} \left\{ \Gamma(1+\frac{2}{\beta}) - [\Gamma(1+\frac{1}{\beta})]^2 \right\}$$

(40) Let $X \sim Wei(\alpha, \beta)$. Show that

$$F(x) = 1 - e^{-\alpha x^{\beta}}, \text{ for } x \geq 0$$

for $\alpha > 0$, $\beta > 0$

we have $F(x) = \int_{-\infty}^{x} f(y)dy$, $y \geq 0$

For Weibull distribution: $f(y, \alpha, \beta) = \alpha\beta y^{\beta-1} e^{-\alpha y^{\beta}}$

$\Rightarrow F(x) = \int_0^x \alpha\beta y^{\beta-1} e^{-\alpha y^{\beta}} dy$ (1)

Let $z = y^{\beta} \Rightarrow \ln z = \beta \ln y \Rightarrow y = e^{\frac{\ln z}{\beta}}$

$\Rightarrow dy = \frac{1}{z\beta} e^{\frac{\ln z}{\beta}} dz$

(1) $F(x) = \int_0^{x^{\beta}} \frac{\alpha\beta z}{e^{\frac{\ln z}{\beta}}} \times \frac{1}{z\beta} e^{\frac{\ln z}{\beta}} e^{-\alpha z} dz$

$= \int_0^{x^{\beta}} \alpha e^{-\alpha z} dz = -e^{-\alpha z}\Big|_0^{x^{\beta}} = 1 - e^{-\alpha x^{\beta}}$

Thus $F(x) = 1 - e^{-\alpha x^{\beta}}$

(41) Let X be the lifetime of hearing and battery

$X \sim Wei(\alpha, \beta)$, $\alpha = \frac{1}{2}$, $\beta = 2$

(a). How long can such a battery be expected to last?

$$E(X) = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta}) = 2^{1/2}\Gamma(\frac{3}{2})$$

$$= \sqrt{2}\,\Gamma(1+\frac{1}{2}) = \sqrt{2}(1+\frac{1}{2}-1)\Gamma(1+\frac{1}{2}-1)$$

$$= \sqrt{2}\times\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{2}}{2}\times\sqrt{\pi} = \sqrt{\frac{\pi}{2}} \approx 1.25$$

Thus a battery be expected to last a year and three moths.

(b). What is the probability that such a battery will be operating after 2 years?

$$P(X>2) = 1 - P(X\le 2) = 1 - F(2)$$

$$= 1-(1-e^{-\alpha 2^{\beta}}) = 1+e^{-\frac{1}{2}\times 2^2}-1 = e^{-2}$$

Thus There is $P(X>2) = e^{-2}$ that a battery will be operating after 2 years.

(42) $X \sim \text{weibull}\ (\alpha = 2.5\ ,\ \beta = 200)$

(a) What is the probability that a specimen's lifetime is at most 250? less than 250? More than 250?

$P(X \leq 250) = F(250) = 1 - e^{2.5(250)^{200}}$

$P(X > 300) = 1 - P(X < 300) = 1 - (1 - e^{2.5(300)^{200}})$

$= e^{2.5(300)^{200}}$

$P(X < 250) = P(X \leq 250) = 1 - e^{2.5(250)^{200}}$

(b). What is the probability that a specimen's life time is between 100 and 250?

$P(100 \leq X \leq 250) = P(X \leq 250) - P(X < 100)$

$= 1 - e^{2.5(250)^{200}} - (1 - e^{2.5(100)^{200}})$

$= e^{2.5(100)^{200}} - e^{2.5(250)^{200}}$

(c). What value is such that exactly 50% of all specimens have lifetimes exceeding that value?

$P(X > 50) = 1 - P(X < 50) = 1 - e\ 1 - e^{2.5(50)^{200}})$

$= e^{2.5(50)^{200}}$

I2–TD4
(Joint Probability Distribution and Random Samples)

1. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

$p(x,y)$		y = 0	1	2
	0	.10	.04	.02
x	1	.08	.20	.06
	2	.06	.14	.30

(a) What is $P(X = 1 \text{ and } Y = 1)$?

(b) Compute $P(X \leq 1 \text{ and } Y \leq 1)$.

(c) Give a word description of the event $\{X$ 6= 0 and Y 6= 0 $\}$, and compute the probability of this event.

(d) Compute the marginal pmf of X and of Y . Using $P_X(x)$, what is $P(x \leq 1)$?

(e) Are X and Y independent rv's? Explain.

2. The number of customers waiting for gift-wrap service at a department store is an rv X with possible values 0, 1, 2, 3, 4 and corresponding probabilities .1, .2, .3, .25, .15. A randomly selected customer will have 1, 2, or 3 packages for wrapping with probabilities .6, .3, and .1, respectively. Let Y = the total number of packages to be wrapped for the customers waiting in line (assume that the number of packages submitted by one customer is independent of the number submitted by any other customer).

(a) Determine $P(X = 3, Y = 3)$, i.e,. $p(3,3)$ (b) Determine $p(4,11)$.

3. Let X denote the number of Canon digital cameras sold during a particular week by a certain store. The pmf of X is

x	0	1	2	3	4
$p_X(x)$	.1	.2	.3	.25	.15

Sixty percent of all customers who purchase these cameras also buy an extended warranty. Let Y denote the number of purchasers during this week who buy an extended warranty.

(a) What is $P(X = 4, Y = 2)$? [Hint: This probability equals $P(Y = 2|X = 4).P(X = 4)$; now think of the four purchases as four trials of a binomial experiment, with success on a trial corresponding to buying an extended warranty.] (b) Calculate $P(X = Y)$.

(c) Determine the joint pmf of X and Y and then the marginal pmf of Y .

4. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable-X for the right tire and Y for the left tire, with joint pdf

$$f(x) = \begin{cases} K(x^2 + y^2), & 20 \leq x \leq 30, 20 \leq y \leq 30 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of K?

(b) What is the probability that both tires are undefiled?

(c) What is the probability that the difference in air pressure between the two tires is at most 2 psi?

(d) Determine the (marginal) distribution of air pressure in the right tire alone.

(e) Are X and Y independent rv's?

5. Two different professors have just submitted final exams for duplication. Let X denote the number of typographical errors on the first professor's exam and Y denote the number of such errors on the second exam. Suppose X has a Poisson distribution with parameter μ_1, Y has a Poisson distribution with parameter μ_2, and X and Y are independent.

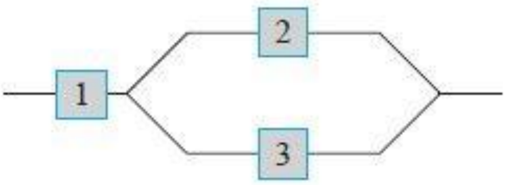
(a) What is the joint pmf of X and Y ?

(b) What is the probability that at most one error is made on both exams combined?

(c) Obtain a general expression for the probability that the total number of errors in the two exams is m (where m is a nonnegative integer). [Hint: $A = \{(x,y) : x + y = m\} = \{(m,0),(m-1,1),...,(1,m-1),(0,m)\}$. Now sum the joint pmf over$(x,y) \in A$ and use the binomial theorem, which says that

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} = (a+b)^m$$ for any $[a,b]$

6. Consider a system consisting of three components as pictured. The system will continue to function as long as the first component functions and either component 2 or component 3 functions. Let X_1,X_2, and X_3 denote the lifetimes of components 1, 2, and 3, respectively. Suppose the X_i s are independent of one another and each X_i has an exponential distribution with parameter l.



(a) Let Y denote the system lifetime. Obtain the cumulative distribution function of Y and differentiate to obtain the pdf. [Hint: $F(y) = P(Y \leq y)$; express the event $\{Y \leq y\}$ in terms of unions and/or intersections of the three events
$\{X_1 \leq y\}$, $\{X_2 \leq y\}$, and $\{X_3 \leq y\}$]

(b) Compute the expected system lifetime.

7. An ecologist wishes to select a point inside a circular sampling region according to a uniform distribution (in practice this could be done by first selecting a direction and then a distance from the center in that direction). Let X = the x coordinate of the point selected and Y = the y coordinate of the point selected. If the circle is centered at (0,0) and has radius R, then the joint pdf of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the probability that the selected point is within $R/2$ of the center of the circular region? [Hint: Draw a picture of the region of positive density D. Because $f(x,y)$ is constant on D, computing a probability reduces to computing an area.]

(b) What is the probability that both X and Y differ from 0 by at most $R/2$?

(c) Answer part (b) for $R/2$ replacing $R/2$.

(d) What is the marginal pdf of X? Of Y ? Are X and Y independent?

8. The joint pdf of pressures for right and left front tires is given in Exercise 4.

(a) Determine the conditional pdf of Y given that $X = x$ and the conditional pdf of X given that $Y = y$.

(b) If the pressure in the right tire is found to be 22 psi, what is the probability that the left tire has a pressure of at least 25 psi? Compare this to $P(Y \le 25)$.

(c) If the pressure in the right tire is found to be 22 psi, what is the expected pressure in the left tire, and what is the standard deviation of pressure in this tire?

9. An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

		y			
$p(x,y)$		0	5	10	15
	0	.02	.06	.02	.10
x	5	.04	.15	.20	.10
	10	.01	.15	.14	.01

(a) If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score $E(X + Y)$?

(b) If the maximum of the two scores is recorded, what is the expected recorded score?

10. Annie and Alvie have agreed to meet for lunch between noon (0:00 P.M.) and 1:00 P.M. Denote Annie's arrival time by X, Alvie's by Y , and suppose X and Y are independent with pdf's

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person? [Hint: $h(X,Y) = |X - Y|$.]

11. (a) Compute the covariance between X and Y in Exercise 4.

(b) Compute the correlation coefficient ρ for this X and Y .

12. (a) Use the rules of expected value to show that $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X,Y)$.

(b) Use part (a) along with the rules of variance and standard deviation to show that $\text{Corr}(aX + b, cY + d) = \text{Corr}(X,Y)$ when a and c have the same sign.

(c) What happens if a and c have opposite signs?

13. A health-food store stocks two different brands of a certain type of grain. Let X be the amount of brand A on hand and Y be the amount of brand B on hand. Suppose the joint pdf of X and Y is

$$f(x,y) = \begin{cases} kxy, & x \geq 0, y \geq 0, 20 \leq x + y \leq 30 \\ 0, & \text{otherwise} \end{cases}$$

(a) Draw the region of positive density and determine the value of k.

(b) Are X and Y independent? Answer by first deriving the marginal pdf of each variable.

(c) Compute $P(X + Y \leq 25)$.

(d) What is the expected total amount of this grain on hand?

(e) Compute $\text{Cov}(X,Y)$ and $\text{Corr}(X,Y)$.

(f) What is the variance of the total amount of grain on hand?

14. Tow real-valued rv's, X and Y, have joint pdf

$$f(x,y) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}\left(x^2 - 2rxy + y^2\right)\right]$$

where $-1 < r < 1$.

(a) Prove that each of X and Y is normally distributed with mean 0 and variance 1.

(b) Prove that the number r is the correlation coefficient of X and Y.

15. (a) Continuous rv's X and Y have a joint pdf

$$f(x,y) = \frac{(m+n+2)!}{m!n!}(1-x)^m y^n$$

for $0 < y \leq x < 1$, where m,n are given positive integers. Check that f is a proper pdf. Find the marginal distributions of X and Y. Hence calculate

$$P\left(Y \leq \frac{1}{3} \middle| X = \frac{2}{3}\right).$$

(b) Let X and Y be rv's. Check that

$$\text{Cov}(X,Y) = E(1-X) \times E(Y) - E[(1-X)Y]$$

(c) Let X,Y be as in (a). Use the form of $f(x,y)$ to express the expectations $E(1-X)$, $E(Y)$ and $E[(1-X)Y]$ in terms of factorials. Using (b), or otherwise, show that

$$\text{Cov}(X,Y) = \frac{(m+1)(n+1)}{(m+n+3)^2(m+n+4)}$$

16. There are 40 students in an elementary statistics class. On the basis of years of experience, the instructor knows that the time needed to grade a randomly chosen first examination paper is a random variable with an expected value of 6 min and a standard deviation of 6 min.

    (a) If grading times are independent and the instructor begins grading at 6 : 50 p.m. and grades continuously, what is the (approximate) probability that he is through grading before the 11 : 00 p.m. TV news begins?

    (b) If the sports report begins at 11 : 10, what is the probability that he misses part of the report if he waits until grading is done before turning on the TV?

17. The number of parking tickets issued in a certain city on any given weekday has a Poisson distribution with parameter $\lambda = 50$.

    (a) Calculate the approximate probability that between 35 and 70 tickets are given out on a particular day.

    (b) Calculate the approximate probability that the total number of tickets given out during a 5-day week is between 225 and 275.

    (c) Use software to obtain the exact probabilities in (a) and (b) and compare to their approximations.

18. Let X_1,X_2, and X_3 represent the times necessary to perform three successive repair tasks at a certain service facility. Suppose they are independent, normal rv's with expected values μ_1,μ_2, and μ_3 and variances σ_1^2, σ_2^2 and σ_3^2, respectively.

    (a) If $\mu_i = 60$ and $\sigma_i = 15; i = 1,2,3$. Calculate $P(T_0 \leq 200)$ and $P(150 \leq T_0 \leq 200)$

    (b) Using the μ_i's and σ_i's given in part (a), calculate $P(55 \leq \overline{X})$ and $P(58 \leq \overline{X} \leq 62)$

    (c) Using the μ_i's and σ_i's given in part (a), calculate $P(-10 \leq X_1 - .5X_2 - .5X_3 \leq 5)$

    (d) If $\mu_1 = 40, \mu_2 = 50, \mu_3 = 60, \sigma_1^2 = 10, \sigma_2^2 = 12$ and $\sigma_3^2 = 14$; calculate $P(X_1 + X_2 + X_3 \leq 160)$ and $P(X_1 + X_2 \geq 2X_3)$

19. Manufacture of a certain component requires three different machining operations. Machining time for each operation has a normal distribution, and the three times are independent of one another. The mean values are 15,30, and 20 min, respectively, and the standard deviations are 1,2, and 1.5 min, respectively. What is the probability that it takes at most 1 hour of machining time to produce a randomly selected component?

Solution

I2 - TD4.

(Joint Probability Distribution and Random samples.)

1. The joint pmf of x and Y appears in the table:

$p(x,y)$		0	1	2
x	0	0.10	0.04	0.02
	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

(a). $P(X=1 \text{ and } Y=1) = \underline{0.2}$

(b) compute $P(X \leq 1 \text{ and } Y \leq 1)$.

$P(X\leq 1 \text{ and } Y \leq 1) = P(X=0, y=0) + P(X=1, Y=1) + P(X=1, Y=0)$
$+ P(X=0, Y=1) = \underline{0.42}$

(c) $P(x \neq 0, Y \neq 0)$.

event $\{x \neq 0, Y \neq 0\}$ is the event that at Least one hose being used in self-service or full service

and $P(X \neq 0, Y \neq 0) = 1 - P(X=0, Y=0) = \underline{0.9}$

(d). Compute the marginal pmf of x and Y. using $P_X(x)$ and find $P(x \leq 1)$.

$P(X=x) = \sum_{y \in y} P(x,y) = \sum_{i=0}^{3} P(x,i)$

Then $P(X=x) = \begin{cases} 0.16 & , x=0 \\ 0.34 & , x=1 \\ 0.5 & , x=2 \\ 0 & , \text{otherwise.} \end{cases}$

+ $P(X \leq 1) = P(X=0) + P(X=1) = \underline{0.5}$

(e). Are X and Y independent ?

we have $P(X=0, Y=0) = 0.10$

and $P(X=0) \times P(Y=0) = 0.0384 \neq P(X=0, Y=0)$

Therefore X and Y are not independent.

2. (a) Determine $P(X=3, Y=3)$.

+ X the number of costomer waiting for gift wrap

+ Y the number of packet to be wrapped.

$P(3,3) = P_X(3) \times P'_{Y|X}(3|3)$

$= 0.25 \times (0.6 \times 0.6 \times 0.6) = \underline{0.054}$

(b) Detemine $P(4,11)$.

$P(4,11) = P_X(4) \times P_{Y|X}(11,4) = 0.15 \times 4 (0.1 \times 0.1 \times 0.1 \times 0.3)$

$= \underline{0.00018}$

3. Let X denote the number of Cannon digital cameras sold during a particular week by certain store:

x	0	1	2	3	4
$P_X(x)$	0.1	0.2	0.3	0.25	0.15

(a) Find $P(X=4, Y=2)$.

$P(X=4, Y=2) = P(X=4) \times P(Y=2 \mid X=4)$.

$P(Y=2 \mid X=4)$ is the event that 4 cameras has been sold and 2 of 4 have extended warranty.

then $P(Y=2 \mid X=4) = C_4^2 (0.6)^2 (0.4)^2 = 0.3456$.

Thus $P(X=4, Y=2) = 0.3456 \times 0.15 = \underline{0.05184}$

(b). Calculate $P(X=Y)$.

Let Y be the cameras with warraty

so $Y = Bin(0.6, n)$, $n = 0, 1, 2, 3, 4$.

$P(Y=n) = (0.6)^n$

$\Rightarrow P(X=Y) = 0.1 + (0.2 \times 0.6) + (0.3 \times 0.6^2) + (0.25 \times 0.6^3) + (0.15 \times 0.6^4) = 0.4014$

Therefore $P(X=Y) = 0.4014$.

(c). Determine joint pmf of X and Y & marginal p

+ $P(X=x, Y=y) = P(X=x) \times P(Y=y \mid X=x) = \underline{C_x^y (0.6)^x (0.4)^{x-y}} P(x)$

+ $P_Y(y) = \sum_{x=0}^{4} P(X=x, Y=y)$

Then $\underline{P_Y(y) = \sum_{x=0}^{4} P(X=x) C_x^y (0.6)^x (0.4)^{x-y}}$

Let x is the right tire and Y is the Left tire

$$f(x) = \begin{cases} k(x^2+y^2) & , 20 \le x \le 30, 20 \le y \le 30 \\ 0 & , \text{otherwise.} \end{cases}$$

(a) Find K.

we have $\int_{20}^{30}\int_{20}^{30} k(x^2+y^2)\,dx\,dy = 1$

$\Leftrightarrow$ $k = \frac{3}{380\,000}$

(b). Find the probability that both tires are not defined.

$P(X<26, Y<26) = \int_{20}^{26}\int_{20}^{26} k(x^2+y^2)\,dx\,dy = 0.3024$

(c) Probability that the difference in air preseur between the two tires at most 2. or $|x-y| < 2$.

so $P(y-2 < x < y+2) = 1 - P(x > y+2) - P(x < y-2)$

$$= 1 - \int_{20}^{26}\int_{y+2}^{30} k(x^2+y^2)\,dx\,dy - \int_{22}^{30}\int_{0}^{y-2} k(x^2+y^2)\,dx\,dy$$

$= 0.35932$

Therefore $P(|x-y|<2) = 0.35922$

(d). Determine (marginal) distribution of x.

$f(x) = \int_{20}^{30} k(x^2+y^2)\,dy = \frac{3}{380\,000}x^2 + \frac{1}{20}$

Therefore $f(x) = \frac{3}{380000}x^2 + \frac{1}{20}$

(e). Are X and Y independend r.v's ?

Since $f(x) = \frac{3}{380000} x^2 + 0.05$, ~~x <~~

Similary $f(y) = \frac{3}{380000} y^2 + 0.05$

$\Rightarrow f(x,y) = f(x) \cdot f(y)$

Therefore X and Y are indepundent.

5. Let X be the number of typoraphical erros on the first professor' exam.

Let Y be the number of secound exam.

(a). Find joint pmf of X & Y.

we have $f(x) = \frac{e^{-\mu_1} \mu_1^x}{x!}$ and $f(y) = \frac{e^{-\mu_2} \mu_2^y}{y!}$

Since X and Y are dependent

$\Rightarrow f(x,y) = f(x) \cdot f(y) = \frac{e^{-\mu_1 - \mu_2} \mu_1^x \mu_2^y}{x! \, y!}$

(b) at most one erros in both exam $X+Y \leq 1$

Thus $P(x+y \leq 1) = P(0,0) + P(0,1) + P(1,0) =$

$= e^{-\mu_1-\mu_2} + e^{-\mu_1-\mu_2} \mu_1 + e^{-\mu_1-\mu_2} \mu_2$

$= e^{-\mu_1-\mu_2} (1 + \mu_1 + \mu_2)$.

Therefore : $P(X+Y \leq 1) = e^{-\mu_1-\mu_2} (1 + \mu_1 + \mu_2)$

(c) General formular:

$$P(x+y=m) = \sum_{k=1}^{m} f(x=k, y=m-k)$$

$$= \sum_{k=1}^{m} e^{-\mu_1-\mu_2} \frac{\mu_1^k \times \mu_2^{m-k}}{k!(m-k)!}$$

$$= \frac{1}{m!} \sum_{k=1}^{m} e^{-\mu_1-\mu_2} \frac{m!}{k!(m-k)!} \mu_1^k \times \mu_2^{m-k}$$

$$= \frac{e^{-\mu_1-\mu_2}}{m!} (\mu_1+\mu_2)^m$$

Thus $P(x+y=m) = \dfrac{e^{-\mu_1-\mu_2}}{m!}(\mu_1+\mu_2)^m$

6. X_1, X_2, X_3 denote the Lifetime of components 1, 2, 3

X_i is independent aind $X_i \sim exp(1)$.

(a) obtain cdf of Y and differentate to obtuin pdf.

$\{Y \le y\} = (x_1 \le y) \cap [(x_2 \le y) \cup (x_3 \le y)]$

or $\{y \le y\} = \{(x_1 \cap (x_2 \cup x_3)) \le y\}$

we have $P(X_1 \cup X_2 \le Y) = P(X_1 \le y, x_2 \le y)$

$= P(X_1 \le Y) \cdot P(X_2 \le Y)$

$= (1-e^{-y})^2$ (1).

we have $P(X_1 \cap (X_2 \cup X_3) \ge Y) = P(X_1 \ge y) P(X_2 \cup X_3 \ge Y)$

~~$= e^{-y}(1-e^{-y}$~~

$= 1 - e^{-y}(1-(1-e^{-y})^2)$

Thuse $F_y(y) = 1 - e^{-y}(1-(1-e^{-y})^2)$

since $f(y) = F_Y'(y) = e^{-y}[1-(1-e^{-y})^2] - e^{-y}[2(e^{-y})(1-e^{-y})]$

$= e^{-y} - e^{-y}(1-e^{-y})^2 + 2e^{-2y}(1-e^{-y})$

$= 4e^{-2y} - 3e^{-3y}$

Therefore $f(y) = 4e^{-2y} - 3e^{-3y}$

(b). Find $E(Y)$.

$E(Y) = \int_0^{+\infty} y f(y)\,dy = \int_0^{+\infty} y(4e^{-2y} - 3e^{-3y})\,dy$

$= \frac{2}{3}$

Therefore $E(Y) = \frac{2}{3}$.

7. $f(x,y) = \begin{cases} \frac{1}{\pi R^2} & , x^2+y^2 \leq R^2 \\ 0 & , \text{otherwise.} \end{cases}$

(a). Find the probability that the select point is within $R/2$

$P\left(X \leq R/2, Y \leq \sqrt{x^2 - \frac{R^2}{4}}\right) = P\left(0 < \theta < 2\pi, r < \frac{R}{2}\right)$

$= \int_0^{2\pi}\int_0^{R/2} \frac{1}{\pi R^2} r\,dr\,d\theta = \frac{1}{4}$

Therefore: $P\left(\text{point within } \frac{R}{2}\right) = \frac{1}{4}$.

(b). probability that both x and y differ from 0 by at most $R/2$?

the bounnary line is $y = \pm \frac{R}{2}$, $x = \pm \frac{R}{2}$

so the region is the square is R^2

Then: P(both x, y differ from 0 by at most $\frac{R}{2}$) $= \frac{1}{\pi}$

(c). Answer from part (b) $\frac{R}{\sqrt{2}}$ replace $\frac{R}{2}$

$\Rightarrow A_{square} = \frac{R^2}{2}$

Then $P(x, y$ by $\frac{R}{\sqrt{2}}) = \frac{1}{2\pi}$

Therefore $P(x, y$, by $\frac{R}{\sqrt{2}}) = \frac{1}{2\pi}$

(d). Find the maginal pdf of x. and y

$$f(x) = \int_{-\sqrt{x^2-R^2}}^{\sqrt{x^2-R^2}} \frac{1}{\pi R^2}\, dy = \frac{1}{\pi R^2}\left(2\sqrt{x^2-R^2}\right) = \frac{2\sqrt{x^2-R^2}}{\pi R^2}$$

Similary $f(y) = \frac{2\sqrt{y^2-R^2}}{\pi R^2}$

since $f(x)\cdot f(y) \neq f(x, y)$

Therefore x, y are dependent

8. Given from ex 4. $f(x,y) = \begin{cases} K(x^2+y^2), & 20 \le x \le 30,\ 20 \le y \\ 0 & , \text{otherwise} \end{cases}$

with $K = \frac{1}{380000}$

(a). Determine the conditional pdf of Y given that $X = x$ and pdf of X, $Y = y$

$$f(Y=y \mid X=x) = \frac{f(x,y)}{f(x=x)} = \frac{K(x^2+y^2)}{K(10x^2 + \frac{19000}{3})} = \frac{x^2+y^2}{10x^2+\frac{19000}{3}}$$

$$f(X=x \mid Y=y) = \frac{f(x,y)}{f(Y=y)} = \frac{x^2+y^2}{10y^2+\frac{19000}{3}}$$

(b). find $P(Y \ge 25 \mid X=22) = \frac{\int_{25}^{30} (22^2+y^2)\,dy}{10 \times 22^2 + \frac{19000}{3}} = 0.556$

since $P(Y \ge 25) = \int_{25}^{30} K\left(10y^2 + \frac{19000}{3}\right) dy$

$= 0.549.$

Thus $P(Y \ge 25 \mid X=22) > P(Y \ge 25)$

(c). Find $\sigma_{Y|X=22}$

$$f(Y=y \mid X=22) = \frac{y^2+22^2}{\frac{19000}{3}+10 \times 484} = \frac{484+y^2}{11173.33}$$

Then $E(Y=y \mid X=22) = \int_{20}^{30} \frac{484y+y^3}{11173.33}\,dy = 25.37$

$\sigma(Y=y \mid X=22) = \sqrt{E(Y=y^2 \mid X=22) - (25.37)^2}$

$E(Y=y^2 \mid X=22) = \int_{20}^{30} \frac{y^2 484 + y^5}{11173.33} dy \doteq 652.0286$

$\Rightarrow P(Y=y \mid X=22) = \cancel{289}\ \underline{2.89}$

9. Let X be the ~~number of~~ point earned in the firstpath.

Y be the point earn in secound path.

$p(x,y)$	0	5	10	15
0	0.02	0.06	0.02	0.1
5	0.04	0.15	0.2	0.1
10	0.01	0.15	0.14	0.01

(a). Find $E(X+Y)$.

$E(X+Y) = E(X) + E(Y) = \sum\sum x\,p(x,y) + \sum\sum y\,p(x,y)$

$= 5(0.04+0.15+0.2+0.1) + 10(0.01+0.15+0.14+0.01)$

$+ 5(0.06+0.15+0.15) + 10(0.02+0.2+0.14)$

$+ 15(0.1+0.1+0.01) = 14.1.$

Therefore $\underline{E(X+Y) = 14.1}$

(b). Find $E(\max(x,y))$.

$E(\max(x,y)) = \sum_x\sum_y \max(x,y)\,p(x,y).$

$= \max(0,5)p(0,5) + \max(5,0) + p(5,0)$

$+ \max(0,10)p(0,10) + \max(10,10)p(10,0)$

$+ \ldots \max(10,10)p(10,10) = 9.6.$

$\Rightarrow \underline{E(\max(x,y)) = 9.6.}$

10. X = Annie Arrive time , Y = Alvie arrive time.
X and Y are indepodent.

$f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$, $f_Y(x) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

$E(|X-Y|) = ?$

we have $f(x,y) = f(x) \cdot f(y) = 6x^2y \quad 0 \le x, y \le 1$

and $|x-y| = x-y$ if $x > y$

$|x-y| = -x+y$ if $y > x$.

$\Rightarrow E(|x-y|) = \int_0^1 \int_y^1 (x-y) f(x,y)\,dx\,dy + \int_0^1 \int_x^1 (y-x) f(x,y)\,dy\,dx$

$= \frac{1}{4}$

Therefore $\underline{E(|x-y|) = \frac{1}{4}}$

11. (a). From 4. compute the covariance between X & Y

$Cov(X,Y) = E(XY) - E(X)E(Y)$.

we have $f(x,y) = \begin{cases} K(x^2+y^2), & 20 \le x; y \le 30 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow E(XY) = \frac{3}{380\,000} \int_{20}^{30} \int_{20}^{30} xy(x^2+y^2)\,dx\,dy = 8.7$

we have $f(x) = K\left(10x^2 + \frac{3}{19000}\right)$

$\Rightarrow E(X) = \int_{20}^{30} Kx\left(10x^2 + \frac{3}{19000}\right) = 0.55$

similary $E(y) = 0.55$.

Therefore $Cov(x,y) = 8.7 - 0.55^2 = \underline{8.3975}$

(b). Compute the correlation coefficient ρ for x and Y

$$\rho_{x,y} = \frac{Cov(x,y)}{\delta_x \cdot \delta_y}$$

we have $E(x^2) = \int_{20}^{30} k\,x^2\left(10x^2 + \frac{3}{19000}\right)dx = 333.$

and $E(x) = 055 \Rightarrow V(x) = E(x^2) - E^2(x) = 332.69.$

$\Rightarrow \delta_x(x) = \sqrt{V(x)} = 18.23.$

Similary $\delta_y(x) = 18.23.$

$\Rightarrow \rho_{x,y} = \frac{Cov(x,y)}{\delta_x \delta_y} = \frac{8.3975}{18.23^2} = 0.025$

Therefore $\underline{\rho_{x,y} = 0.025.}$

12. (a) show that $Cov(ax+b, cy+d) = ac\,Cov(x,y).$

we have $Cov(ax+b, cy+d) = E((ax+b)(cy+d))$

$- E(ax+b)E(cy+d)$

$= E(acxy + adx + bcy + bd) - (aE(x)+b)(cE(x)+d)$

$= ac\,E(xy) - acE(x)E(y) = ac\,Cov(x,y).$

Therefore $Cov(ax+b, cy+d) = ac\,Cov(x,y).$

(b). use part (a). show that $corr(ax+b, cy+d) = corr(x,y)$

by (a) we will show that $\delta(ax+b)\delta(cy+d) = ac\,\delta(x)\delta(y)$

by property of δ : $\delta(ax+b)\delta(cy+d) = |ac|\delta(x)\delta(y)$

Therefore : $corr(ax+b, cy+d) = corr(x,y).$

(c). what happend if a & c have opposite sign ?

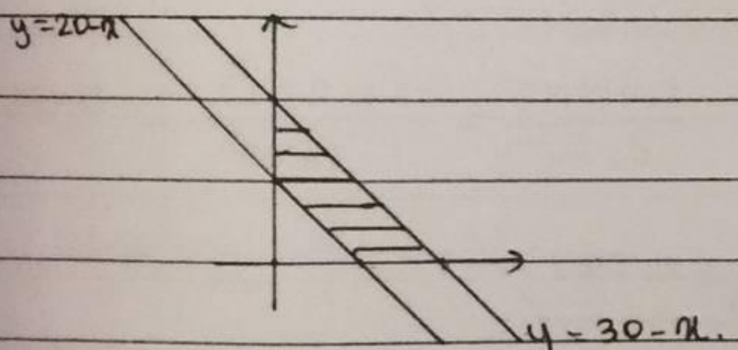
If $ac < 1$ then $|ac| = -ac$.

so $\text{corr}(ax+b, cy+d) = -\text{corr}(x,y)$.

13. Let x: amound of brand A, y amound of brand B

$$f(x,y) = \begin{cases} kxy & , x \geq 0, y \geq 0, 20 \leq x+y \leq 30 \\ 0 & , \text{Otherwise.} \end{cases}$$

(a). Draw region and find k.



we have $\iint kyx\,dy\,dx = 1$

$$\Leftrightarrow \int_{20}^{30}\int_{0}^{30-x} kxy\,dy\,dx + \int_{0}^{20}\int_{20-x}^{30-x} kxy\,dy\,dx = 1$$

$$\Leftrightarrow k\frac{81250}{3} = 1 \Rightarrow k = \frac{3}{81250}$$

Therefore $k = \frac{3}{81250}$

(b). Are X & Y independent ?

we have $f(x) = \int_{y \in D} f(x,y)\,dy$

$= \int_0^{30-x} k f(x,y)\,dy - \int_0^{20-x} k f(x,y)\,dy$

$= k\left[\frac{x}{2}(30-x)^2 - \frac{x}{2}(20-x)^2\right]$

Similary $f(y) = k\left[\frac{y}{2}(30-y)^2 - \frac{y}{2}(20-y)^2\right]$

$f(x) \cdot f(y) \neq f(x,y)$

Therefore $\underline{X \text{ and } Y \text{ are dependent}}$

(c) Compute $P(X+Y \leq 25)$

$P(X+Y \leq 25) = k\int_0^{25}\int_0^{25-x} xy\,dy\,dx - k\int_0^{20}\int_0^{20-x} xy\,dy\,dx$

$= 0.354.$

Therefore $\underline{P(X+Y \leq 25) = 0.354}$

(d). Find $E(X+Y)$.

$E(X+Y) = E(X) + E(Y) = 2\int_0^{30} x f(x)\,dx - 2\int_0^{20} x f(x)\,dx$

$= 2\int_0^{30} y f(y)\,dy - 2\int_0^{20} y f(y)\,dy$

we have $f(x) = k(250x^2 - 10x^3)$.

$\Rightarrow E(X+Y) = \underline{25.969}$

(e). compute $cov(x,y)$ and $corr(x,y)$.

we have $E(x) = E(y) = 12.9845$.

+ $E(XY) = \iint_D k x^2 y^2 \, dx\,dy$

$$= \int_0^{30}\int_0^{30-x} k x^2 y^2 \, dy\,dx - \int_0^{20}\int_0^{20-x} k x^2 y^2 \, dy\,dx$$

$$= 136.4103.$$

$\Rightarrow cov(x,y) = 136.4103 - (12.9845)^2 = \underline{-32.1869}$

and we have $E(x^2) = \int_0^{30} x^2 f(x)\,dx - \int_0^{20} x^2 f(x)\,dx$

$= 204.61$.

Then $\delta_X(x) = \delta_Y(y) = \sqrt{V(x)} = \sqrt{V(Y)} = \sqrt{204.61 - (12.98)^2}$

$= \sqrt{36.0182}$

$\Rightarrow corr(x,y) = \dfrac{-32.19}{36.0182} = \underline{-0.894}$

(f). the variance total.

+ $V(X+Y) = E[(x+y)^2] - E^2(x+y)$

$= E(x^2) + E(y^2) + 2E(xy) - E^2(x+y)$

$= \underline{7.66}$ (The value of $E(x^2) = E(y^2)$

$E^2(x+y)$, $E(xy)$ we found above

14. Giving Joint pdf : $f(x,y) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)\right]$

(a). Prove that each of X or Y is $N(0,1)$.

we have $f(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$.

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)} dy$$

$$= \frac{e^{-\frac{x^2}{2(1-r^2)}}}{2\pi\sqrt{1-r^2}} \int_{-\infty}^{+\infty} e^{\frac{2rxy-y^2}{2(1-r^2)}} dy$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

Therefore $X \sim N(0,1)$.

Similary for $Y \sim N(0,1)$.

(b). prove that r is $corr(X,Y)$

we have $corr(X,Y) = \frac{E(XY) - E(X)E(Y)}{\delta_X \delta_Y}$

Since $X \sim N(0,1) \Rightarrow E(X) = 0$, $\delta(X) = 1$.

$Y \sim N(0,1) \Rightarrow E(Y) = 0$, $\delta(Y) = 1$.

$\Rightarrow corr(X,Y) = E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f(x,y)\,dx\,dy$

$= \ldots$ (solve Integral) ...

$= r$

Therefore $corr(X,Y) = r$.

15. (a). $f(x,y) = \frac{(m+n+2)!}{m!n!}(1-x^m)y^n$, $0 \le y \le x \le 1$

check if f is pdf.

we have $f(x,y) = \frac{(m+n+2)!}{m!n!}(1-x)^m y^n$

Then $\int_0^1\int_y^1 \frac{(m+n+2)!}{m!n!}(1-x)^m y^n \,dx\,dy = \frac{(m+n+2)!}{m!n!}\int_0^1\int_y^1 (1-x)^m y^n$

$= \cdots$ (solve it) $\cdots$

$= 1$

Then f is the pdf.

+ Find maginal distribution.

$f(x) = \int_0^x \frac{(m+n+2)!}{m!n!}(1-x)^m y^n \,dy = \frac{(m+n+2)!}{m!(n+1)!}(1-x)^m x^{n+1}$

Similary $f(y) = \frac{(m+n+2)!}{(1+m)!n!}(1-y)^{m+1} y^n$

+ Find $P\left(Y \le \frac{1}{3} \mid X = \frac{2}{3}\right)$

we have $f(y|x) = (n+1)\frac{y^n}{x^{n+1}}$

and $P\left(Y \le \frac{1}{3} \mid X = \frac{2}{3}\right) = (n+1)\int_0^{\frac{1}{3}} \frac{y^n}{\left(\frac{2}{3}\right)^{n+1}}\,dy = \frac{1}{2^{n+1}}$

Thus $P\left(Y \le \frac{1}{3} \mid X = \frac{2}{3}\right) = \frac{1}{2^{n+1}}$

(b). check that: $\text{Cov}(X,Y) = E(1-X) \times E(Y) - E[(1-X)Y]$

$\text{Cov}(X,Y) = E(1-X)E(Y) - E[(1-X)Y]$

$= E(Y) - E(X)E(Y) - E(Y) + E(X*Y)$

$= E(XY) - E(X)E(Y)$ True.

(c). show that $\text{Cov}(X,Y) = \frac{(m+1)(n+1)}{(m+n+3)^2(m+n+4)}$.

we have $f(x,y) = \frac{(m+n+2)!}{m!n!}(1-x)^m y^n$

$\Leftrightarrow E(1-x) = \int_0^1\int_0^x \frac{(m+n+2)!}{m!n!}(1-x)^{m+1}y^n\,dy\,dx$

$= \frac{m+1}{n+m+3}$

$+ E(y) = \int_0^1\int_0^x \frac{(m+n+2)!}{m!n!}(1-x)^m y^{n+1}\,dy\,dx$

$= \frac{n+1}{m+n+3}$

$+ E[(1-x)y] = \int_0^1\int_0^x \frac{(m+n+2)!}{m!n!}(1-x)^{m+1}y^{n+1}\,dy\,dx.$

$= \frac{(m+1)!(n+2)!}{(m+n+4)!(n+2)!}$

Then: $\text{Cov}(X,Y) = \frac{(m+1)(n+1)}{(m+n+3)^2(m+n+4)}$ True

16. Let X_j = grading time of j-th examination with $\mu_{X_j} = 6$ min and $\sigma_{X_i} = 6$ min

(a). Find P(through grading before 11:PM)

P(through granding before 11PM) = $P(X_1 + \dots + X_{40} < 250\text{min})$.

$= P\left(\frac{X_1 + \dots X_{40}}{40} < \frac{25}{4}\right)$

$= P(\bar{X} < \frac{25}{4})$

Since $n = 40$ then $\bar{X} \sim N_b(\mu_{\bar{X}}, \sigma^2_{\bar{X}})$

where $\bar{X} \sim N_b(6, \frac{36}{40})$

$\Rightarrow P(\bar{X} < \frac{25}{4}) = P\left(\frac{\bar{X}-6}{6/\sqrt{40}} < \frac{25/4 - 6}{6/\sqrt{40}}\right)$, $Z \sim N(0,1)$.

$= P(Z < 0.26) = \Phi(0.26) = 0.6026$.

Therefore $\underline{P(\text{through grunding before 11 PM}) = 0.6026}$.

(b). Probability that he missed part of report if he wait grad is done before turning on T.V.

$P(X_1 + \dots X_{40} > 260 \text{ min}) = P(\bar{X} > \frac{260}{40})$

$= P(Z > 0.52)$

$= 1 - P(Z < 0.52)$

$= 0.30153$.

Therefore $P(X_1 + \dots + X_{40} > 260) = \underline{0.30153}$

7. Let X be the number of parking ticket

$X \sim Poi(50)$.

(a). Find $P(35 < x < 70)$

$$P(35 < x < 70) = \sum_{x=35}^{70} \frac{e^{-50} 50^x}{x!} = e^{-50} \sum_{x=35}^{70} \frac{50^x}{x!}$$

Using appoximate method.

$\mu = 50, \quad \sigma = \sqrt{50}$

then $P(35 < x < 70) = P\left(\frac{35-50}{\sqrt{50}} < x < \frac{70-50}{\sqrt{50}}\right)$

$= P(2.05 < z < 2.89)$

$= 0.99807 - 0.2018$

$= 0.79627$

(b) during 5 day a week is between 225 and 275

Let Y = tickets in 5 day

$Y = poi(250)$ since $\lambda \to \infty \Rightarrow Y \sim N(250, 250)$

$\Rightarrow P(225 < Y < 275)$ with $\lambda = 250$

$\Rightarrow P(225 < Y < 275) = P\left(\frac{225-250}{\sqrt{250}} < Z < \frac{275-250}{\sqrt{250}}\right)$

$= \Phi(1.58) - \Phi(-1.58)$

$= 0.37265$

18. X_1, X_2, X_3, represent the times necessary to perform the succesive repair task at a certain service facility

(a). If $\mu_i = 60$ and $\sigma_i = 15$, calculate $P(T_0 \le 200)$.

$T_0 = X_1 + X_2 + X_3 \Rightarrow E(T_0) = 180$, $\sigma(T_0) = 15\sqrt{3}$

Then $T_0 \sim N(180, (15\sqrt{3})^2)$.

$\Rightarrow P(T_0 \le 200) = P\left(\frac{T_0 - 180}{15\sqrt{3}} \le \frac{200-180}{15\sqrt{3}}\right)$, $Z \sim N(0,1)$

$= P(Z \le 0.78) = \underline{0.78230}$

$+ P(150 \le T_0 \le 200) = 0.78230 - 1 + P(Z < -1.17)$

$= \underline{0.6613}$

(b). Calculate $P(55 \le \bar{X})$ & $P(58 \le \bar{X} \le 62)$.

$\bar{X} = 60$, $\mu_{\bar{X}} = \mu_i = 60$, $\sigma_{\bar{X}} = \frac{15}{\sqrt{3}}$

$\Rightarrow \bar{X} \sim N\left(60, \left(\frac{15}{\sqrt{3}}\right)^2\right)$

$+ P(55 < \bar{X}) = 1 - P\left(Z < \frac{55-60}{15/\sqrt{3}}\right) = 1 - P(-0.58)$

$= \underline{0.71904}$

$P(58 \le \bar{X} \le 62) = \underline{0.97358}$

(c) Calculate $P(-10 \le X_1 - 0.5X_2 - 0.5X_3 \le 5)$.

Let $Y = X_1 - \frac{1}{2}X_2 - \frac{1}{2}X_3$. $\Rightarrow E(Y) = 0$

and $\sigma(Y) = 15\sqrt{1 + \frac{1}{2}} = 15\sqrt{\frac{3}{2}}$

$\Rightarrow Y \sim N(0, \sigma^2)$

$\Rightarrow P(-10 < Y < 5) = P\left(\frac{-10}{15\sqrt{\frac{3}{2}}} < Y < \frac{5}{15\sqrt{\frac{3}{2}}}\right)$

$= P(-0.54 < Z < 0.40)$.

$= \underline{0.35736}$

(d) Calculate $P(X_1+X_2+X_3 \le 160)$ & $P(X_1+X_2 \ge 2X_3)$.

Let $X_1+X_2+X_3 = Y \Rightarrow E(Y) = 150$

and $\sigma_Y = \sqrt{10+12+14} = 6$.

Then $Y \sim N(150, 36)$ so $P(Y \le 160) = P(Z < \frac{10}{6})$

$= \underline{0.95154}$

+ $P(X_1+X_2 \ge 2X_3)$ or $P(X_1+X_2-2X_3 \ge 0)$.

Let $Y = X_1+X_2-2X_3$ the $E(Y) = 90-120 = -30$

$\sigma(Y) = \sqrt{10+12+4\times14} = 8.83$.

$\Rightarrow Y \sim N(-30, 78) \Rightarrow P(Y \ge 0) = P(Z \ge \frac{30}{8.83})$

$= 1 - \Phi(3.39)$

$= 0.00035$

Therefore $\underline{P(X_1+X_2 \ge 2X_3) = 0.00035}$

19. Probability that it takes most 1 hour.

Let X_i be the number of matching time.

X_1 : $E(X_1) = 15$, $\delta(X_1) = 1$.

X_2 : $E(X_2) = 30$, $\delta(X_2) = 2$

X_3 : $E(X_3) = 20$, $\delta(X_3) = 1.5$.

Let $Y = X_1 + X_2 + X_3 \Rightarrow Y \sim (E(Y), \delta^2(Y))$

$E(Y) = 65$, $\delta(Y) = \sqrt{1 + 4 + 2.25} = 2.69$.

So: $P(Y \leq 60) = P\left(Z < \frac{60-65}{2.69}\right) = P(Z < -1.85)$

Therefore $\underline{P(Y \leq 60) = 0.03216.}$