CHAPTER I
PROBABILITY

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Contents

The Sample Space of an Experiment

Definition 1

The **sample space** of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of that experiment.

An **experiment** is any activity or process whose outcome is subjects to uncertainty.

Example 2

In the toss of a coin, let the outcome tails be denoted by T and let the outcome heads be denoted by H. Then the sample space is

$$\mathcal{S} = \{T, H\}.$$

If we toss two coins, then the sample space is

$$\mathcal{S} = \{TT, HH, TH, HT\}.$$

Definition 3

An **event** is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$. That is, if A is an event then

$$A = \{\omega : \ \omega \in \mathcal{S}\}.$$

Example 4

Consider an experiment in which each of three vehicles taking a particular freeway exit turns left (L) or right (R) at the end of the exit ramp. Then, the sample space is

$$\mathcal{S} = \{LLL, RLL, LRL, LLR, LRR, RLR, RRL, RRR\}.$$

- $A = \{LLL\}$ = the event that all the vehicles turn left.
- $B = \{RLL, LRL, LLR\}$ = the event that exactly one of the three vehicles turns right.

Definition 5

1. The **complement** of an event A, denoted by $\overline{A}$ or A' or A^c, is the set of all outcomes in S that are not contained in A.
$$\overline{A} = \{\omega \in S : \omega \notin A\}$$
2. The **union** of two events A and B, denoted by $A \cup B$ and read "A or B" where
$$A \cup B = \{\omega \in S : \omega \in A \text{ or } \omega \in B\}$$
3. The **intersection** of two events A and B, denoted by $A \cap B$ and read "A and B" where
$$A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$$

Definition 6

The **null event**, denoted by $\varnothing$, is the event consisting no outcomes. Two events A and B are said to be **mutually exclusive** or **disjoint** if $A \cap B = \varnothing$.

Some Relations from Set Theory

Properties

Let $\mathcal{S}$ be a sample space and let $A, B, C \in \mathcal{S}$. Then,

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Some Relations from Set Theory

Example 7

Four universities 1, 2, 3, and 4 are participating in a holiday basketball tournament. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will play for the championship, and the two losers will also play. One possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).

(a) List all outcomes in $\mathcal{S}$.

(b) Let A denote the event that 1 wins the tournament. List outcomes in A.

(c) Let B denote the event that 2 gets into the championship game. List outcomes in B.

(d) What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A'?

Definition 8

Let $\mathcal{A}$ be a collection of subsets of S. We say $\mathcal{A}$ is a σ-field if

1. $\emptyset \in \mathcal{A}$, ($\mathcal{A}$ is not empty).
2. if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$, ($A$ is closed under complements).
3. If the sequence of sets $\{A_1, A_2, \ldots\}$ is in $\mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, ($\mathcal{A}$ is closed under countable unions).

Note by (1) and (2), a σ-field always contains $\emptyset$ and S. By (2)and (3), it follows that a σ-field is closed under countable intersections, besides countable unions. This is what we need for our collection of events.

Example 9

1. Let $\mathcal{S}$ be any set and let $A \in \mathcal{S}$. Then $\mathcal{A} = \{A, \overline{A}, \emptyset, \mathcal{S}\}$ is a a-field.
2. Let $\mathcal{S}$ be any set and let $\mathcal{A}$ be the power set of $\mathcal{S}$, (the collection of all subsets of $\mathcal{S}$). Then $\mathcal{A}$ is a σ-field.

The Probability Set Function

Definition 10

Let $\mathcal{S}$ be a sample space and let $\mathcal{A}$ be a σ-field on $\mathcal{S}$. Let P be a real valued function defined on $\mathcal{A}$. Then P is a **probability set function** if P satisfies the following three conditions:

1. $P(A) \geq 0$, for all $A \in \mathcal{A}$.
2. $P(\mathcal{S}) = 1$.
3. If $\{A_n\}$ is a sequence of sets in $\mathcal{A}$ and $A_m \cap A_n = \emptyset$ for all $m \neq n$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

In the following, $\mathcal{A}$ will be denoted a σ-field on a sample space $\mathcal{S}$. Otherwise, it will be mentioned.

Theorem 1

Let $A, B \in \mathcal{A}$. Then,

1. $P(A) = 1 - P(\overline{A})$
2. $P(\emptyset) = 0$
3. if $A \subset B$, then $P(A) \leq P(B)$
4. $0 \leq P(A) \leq 1$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 11

In a certain residential suburb, 60% of all households get Internet service from the local cable company, 80% get television service from that company, and 50% get both services from that company. If a household is randomly selected, what is the probability that it gets at least one of these two services from the company, and what is the probability that it gets exactly one of these services from the company?

More Probability Properties

Theorem 2

Let $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots, n$. Then

$$P(A_1 \cup A_2 \cdots \cup A_n) = p_1 - p_2 + \cdots + (-1)^{n+1} p_n$$

where p_i equals the sum of the probabilities of all possible intersections involving i sets.

In particular, if $n = 3$, then

$$P(A_1 \cup A_2 \cup A_3) = p_1 - p_2 + p_3$$

where

$$\begin{aligned} p_1 &= P(A_1) + P(A_2) + P(A_3) \\ p_2 &= P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3) \\ p_3 &= P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

More Probability Properties

Example 12

A family consisting of three persons A, B, and C goes to a medical clinic that always has a doctor at each of stations 1, 2, and 3. During a certain week, each member of the family visits the clinic once and is assigned at random to a station. The experiment consists of recording the station number for each member. One outcome is $(1, 2, 1)$ for A to station 1, B to station 2, and C to station 1. Suppose that any incoming individual is equally likely to be assigned to any of the three stations irrespective of where other individuals have been assigned. What is the probability that

(a) All three family members are assigned to the same station?

(b) At most two family members are assigned to the same station?

(c) Every family member is assigned to a different station?

Permutations and Combinations

Definition 13

An ordered arrangement is called a **permutation**. An unordered arrangement is called a **combination**.

Theorem 3

- The number of permutations of n distinct objects taken r at a time is

$$P_n^r = \frac{n!}{(n-r)!}$$

- The number of combination of n distinct objects taken r at a time is

$$C_n^r = \frac{n!}{n!(n-r)!}$$

Permutations and Combinations

Example 14

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) A will serve only if he is president;

(c) B and C will serve together or not at all;

(d) D and E will not serve together?

Answers: a) 2450 b) 2401 c) 2258 d) 2448

Example 15

A box contains 10 black balls, 8 red balls and 5 blue balls. How many ways are there that a randomly select 7 balls from the box can get 3 black 2 red and 2 blue balls?

Permutations and Combinations

Theorem 4

If in an n objects containing k kinds which n_1 of first kind, n_2 of a second kind, $\ldots$, n_k of a kth kind. The number of permutation of n objects is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Theorem 5

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$.

Permutations and Combinations

Example 16

How many different letter arrangements can be made from the letters in the word *PROBABILITY*?

Example 17

In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Answer: $\dfrac{10!}{1!2!4!3!} = 12600.$

Example 18

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

The Definition of Conditional Probability

Definition 19

For any two events A and B with $P(B) > 0$, the **conditional probability of** A **given that** B **has occurred** is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Theorem 6 (The Multiplication Rule)

The probability that two events A and B, both occur is given by the multiplication rule

$$P(A \cap B) = P(A)P(B|A), \quad \text{if } P(A) > 0$$

or

$$P(A \cap B) = P(B)P(A|B), \quad \text{if } P(B) > 0.$$

The Definition of Conditional Probability

Example 20

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

(a) arrives on time, given that it departed on time,

(b) departed on time, given that it has arrived on time.

Answers: a) 0.94 b) 0.95

Example 21

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Bayes' Theorem

Theorem 7 (The Law of Total Probability)

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k)$$

Theorem 8 (Bayes's Theorem)

Let $A_1, A_2, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events with *prior* probabilities $P(A_i), i = 1, 2, \ldots, k$. Then for any other event B for which $P(B) > 0$, the *posterior* probability of A_j given that B has occurred is

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}, \quad j = 1, \ldots, k$$

Bayes' Theorem

Example 22

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective.

(a) Suppose that a finished product is randomly selected. What is the probability that it is defective?

(b) If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Answers: a) 0.0245 b) 10/49

Independence

Definition 23

Let $A, B \in \mathcal{A}$. A and B are **independent** if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Theorem 9

Let $A, B \in \mathcal{A}$. A and B are independent if and only if

$$P(A \cap B) = P(A).P(B)$$

Definition 24

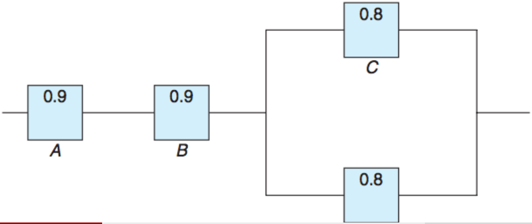
Events $A_1, \ldots, A_n$ are **mutually independent** if for every $k (k = 2, 3, \ldots, n)$ and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}).P(A_{i_2}) \ldots P(A_{i_k})$$

Independence of More Than Two Events

Example 25

An electrical system consists of four components as illustrated in Figure. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in Figure. Find the probability that (a) the entire system works and (b) the component C does not work, given that the entire system works. Assume that the four components work independently.



Independence of More Than Two Events

Theorem 10

If, in an experiment, the events $A_1, A_2, \ldots, A_n$ can occur, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_n)$$

If the events $A_1, A_2, \ldots, A_n$ are independent, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n).$$

Example 26

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.