Modeling Change

Assignment
GROUP 6

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1. Consider an automobile suspension system. Build a model that relates the spring's stretch (or compression) to the mass it supports. If possible, obtain a car spring and collect data by measuring the change in spring size to the mass supported by the spring. Graphically test your proportionality argument. If it is reasonable, find the constant of proportionality.
   Solution.

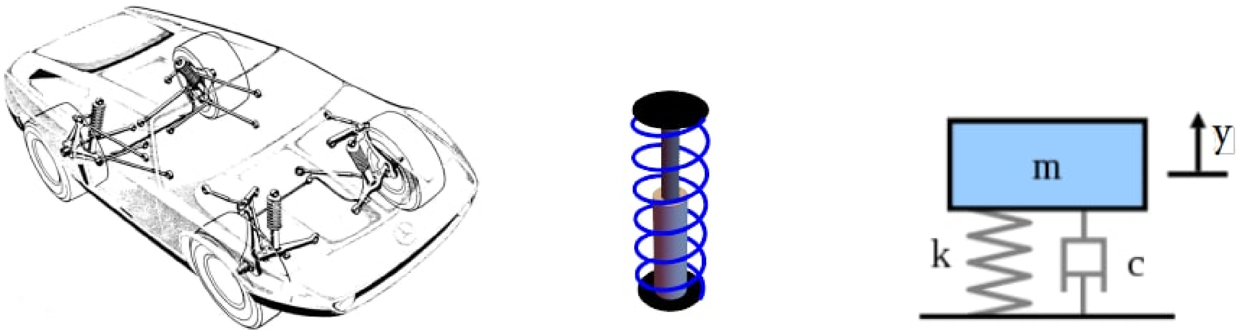


Figure 1. (a) Suspension (b) Spring Assembly (c) Diagram

A stylized version of an uninstalled spring assembly is shown in Figure 1(b). A simplified version of the spring and shock absorber that directly relates to our differential equation is shown in Figure 1(c).
We model the vertical displacement $y(t)$ from its equilibrium position, meaning when the vehicle is at rest. The differential equation, Equation 1, for the displacement in a standard spring-mass-dashpot situation shows a sum of the three forces in Figure 1(c). Our righthand side is zero for no external forces, such as for a car on a smooth road.

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0, \qquad y(0) = y_0, \quad y'(0) = 0. \qquad (1)$$

Here are the parameters of our system:

- m is the mass of one wheel. Let us consider one-quarter of the mass of a car.
- k is the spring constant. A spring restores motion proportional to the amount of displacement. Here is a statement of Hooke's Law in a car context: the more a car spring is stretched or compressed, the more it acts to return to an equilibrium position of rest.
- c is the damping coefficient for the reduction in movement by the shock absorber. Here is a statement of the velocity relationship of damping in a car context: a shock absorber resists motion more when the suspension moves faster. "All modern shock absorbers are velocity sensitive - the

faster the suspension moves, the more resistance the shock absorber provides. This enables shocks to adjust to road conditions and control all the unwanted motions that can occur in a moving vehicle, including bounce, sway, brake dive, and acceleration squat."

- $y(0) = y_0, \quad y'(0) = 0$. We begin our clock, $t_0 = 0$, when the car is at a small displacement, y0, with initial velocity zero. That could be from a person hopping onto the hood of a car and compressing the spring for a negative initial displacement, or at some instant in a lab test.

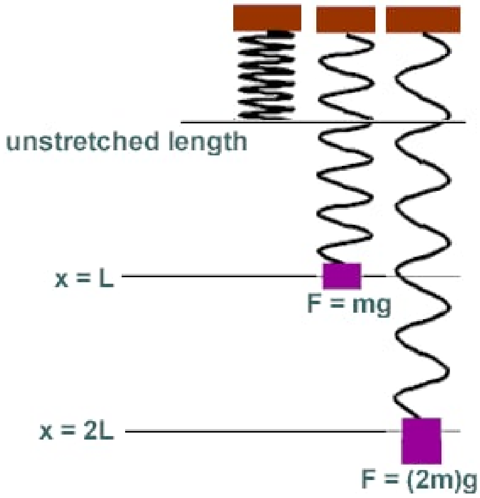
2. Research and prepare a 10-minute report on Hooke's law.
Research: Any motion that repeats itself in equal intervals of time is called periodic motion. A special form of periodic motion is called Simple Harmonic Motion (SHM). Simple Harmonic Motion is defined as oscillatory motion in which the resultant force on the oscillating body at any instant is directly proportional to its displacement from the rest position and opposite in direction to its motion.

For a spring system, this can be written as $F = -kx$ where F is the resultant force on the object attached to the spring, x is the displacement of the object from equilibrium, and k is a constant called the spring constant. The force is a restoring force because it tends to restore the object back to its original position. This relationship is called Hooke's Law.

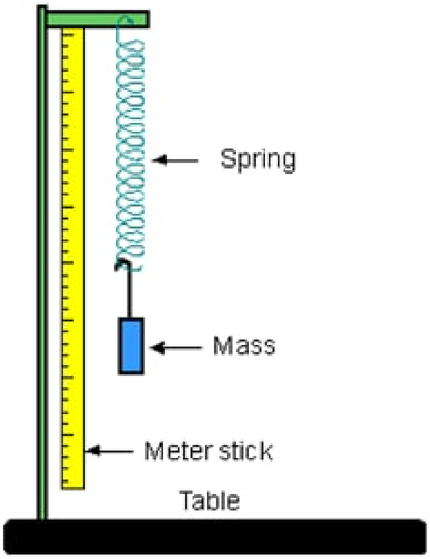
If a mass is attached to a spring and then displaced from its rest position and released, it will oscillate around that rest position in simple harmonic motion. The period T of the oscillating system does not depend on the displacement from rest as long as the spring is not overstretched. The period is the time it takes for a system to go through one full oscillation and return to its starting position.

In this lab, we will study Hooke's Law for a mass connected to a spring and then investigate the SHM of the mass on the spring. We will find the spring constant in each case and compare the results.

- Mount the spring so that it hangs vertically. Attach a mass hanger or a small mass and allow the system to stretch to an equilibrium state (the $x = L$ situation in the figure).



- Place the bottom of the mass hanger or small mass even with a reference point on a meter stick as shown in the figure. This will be your zero for measurements.



3. Consider the models $W \propto l^2 G$ and $W \propto g^3$. Interpret each of these models geometrically. Explain how these two models differ from Models (2.11) and (2.13), respectively. In what circumstances, if any, would the four models coincide? Which model do you think would do the the best job of predicting W? Why? In Chapter 3 you will be asked to compare the four models analytically.

   a. Let $A(x)$ denote a typical cross-sectional area of bass, $0 \leq x \leq l$, Where l denotes the length of the fish. Use the mean value theorem from to show that the volume V of the vish is given by

   $$V = l \cdot \bar{A}$$

   where A is the average value of $A(x)$.

b. Assuming that A is proportional to the square of the girth g and that weight density for the bass is constant, establish that

$$W \propto lg^2$$

Soluton Firstly, consider the following model:

$$w \propto l^2 g$$

Where

	W	is the weight of the fish.
	l	is the effective length.
And	g	is the measured girth (circumference of the fish at its widest point).

(Note: The symbols will mean the same throughout the problem) This model predicts that the weight of the fish (W) is proportional to the length of the fish (l) squared and multiplied by the girth(g). Hence it assumes that the lengthwise cross-sections are geometrically similar but treat the girth separately. This model will be effective in predicting the weight of lengthy rectangular fish which is fairly thick in the middle. Moving on to the second model:

$$w \propto g^3$$

This model only takes the girth of the fish into consideration (neglecting the length). This model will predict accurately thick and round fish of smallish lengths. Models 2.11 and 2.13 have been analyzed below:

$W \propto l^3 \ldots\ldots (2.11)$
$W \propto g^2 l \ldots\ldots (2.13)$

Model(2.11) predicts that the weight of the fish is directly proportional to the length cubed and is fairly accurate for skinny and lengthy fish with ignorable girth.
Model (2.13) predicts that the weight of the fish is proportional to the girth squared multiplied by the length.
This model will be fairly accurate for smaller fish that are dense in the middle. Hence the differences between the former and the latter models predicted above lie in their utility in predicting the weights of the different kinds of fish caught. The four models coincide if the fish are geometrically similar, that is $l \propto g$ If the length is proportional to the girth then the analogy can be extended to all of the four models and they will coincide. The best-suited model will be that which defines the fish best, which, in turn, will be dependent on the statistical distribution of the fish in the

water. The model $W \propto g^2 l$ will do the best job in predicting W as the majority of the fish are of considerable lengths and bulging in the middle. Hence the average area of the fish determined by the girth supersedes the importance of the length playing a detrimental role.

(a) The average value of a function $f(x)$ over the interval $[a, b]$ is given by

$$\int_{ab} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This integral characteristic will be exploited here to determine the volume v of the fish. The mean value Theorem for integration is stated below:

If $f(x)$ is a continuous function on $[a, b]$, then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x)\,dx = (b-a)f(x)$$

In this problem $A(x)$ denotes the typical cross-sectional area of the interval $[0, I]$. The average value of $A(x)$ will be as follows:

$$\bar{A} = \frac{1}{I-0}\int_0^I A(x)\,dx$$

Using the Mean value Theorem for integrals as stated above, the following is obtained:

$$\int_0^I A(x)\,dx = A(c)(I-0)$$

$$A(c) = \frac{1}{I-0}\int_0^I A(x)\,dx$$

$$A(c) = \bar{A}$$

Substituting, the Equation becomes:

$$\int_0^I A(x)\,dx = \bar{A}I$$

But $\int_0^I A(x)\,dx = V$ (Area integrated with differential length dx yields volume)

Therefore, the result obtained is:

$$V = I \cdot \bar{A} \text{ (This is required proof)}$$

(b) Since, $\bar{A} \propto g^3$ and weight density for the bass is given to be constant. Since **Weight=(weight density)$\times$ Volume** and the weight density is said to be constant, it can be said that

$$V \propto W$$

Using the relationship we obtained in part **(a)**, the following is obtained:

$$W \propto lg^2$$

4. Heart Rate of Mammals—The following data relate the weights of some mammals to their heart rate in beats per minute. Based on the discussion relating blood flow through the heart to body weight, as presented in Project 2, construct a model that relates heart rate to body weight. Discuss the assumptions of your model. Use the data to check your model.

Mammal	Body weight(g)	Pulse rate (beats/min)
Vespergo pipistrellas	4	660
Mouse	25	670
Rat	200	420
Guinea pig	300	300
Rabbit	2000	205
Little dog	5000	120
Big dog	30000	85
Sheep	50000	70
Man	70000	72
Horse	450000	38
Ox	500000	40
Elephant	3000000	48

Solution The objective of the model will be to relate blood flow through the heart (The pulse rate) to the weight of the mammals.

Let H be the volume of the blood pumped by the heart in one stroke. The volume H Pumped will be inversely proportional to the Pulse rate P (As the greater the pulse rate, the lesser volume of blood will be pumped in each pulse to compensate for the greater frequency of the pulse.)

Hence the relationship obtained above is:

$$H \propto \frac{1}{p}$$

As the amount of lost energy E_{lost} is directly proportional to the volume of the blood pumped per second by the heart, the following proportionality relationship can be derived:

$$E_{lost} \propto \frac{1}{p}$$

The amount of available energy will be equal to the sum total of kinetic Energy (caused by movement) and Potential Energy (energy required to maintain the body temperature):

Assuming the Kinetic Energy to be 0, only Potential Energy will be dealt with here, The energy required to keep the body temperature is kept fixed ar 0 is given by:

$$E_{gain} = m\theta_h$$

Where m is the mass of the body and θ_h is the heat required to change the fixed temperature of the body. As θ_h is a constant entity for each mammal

$$E_{gain} \propto m$$

It is known to us that Energy lost is equal to the Energy Gained. Hence from the above statement, the final model obtained is given below:

$$m \propto \frac{1}{p}$$

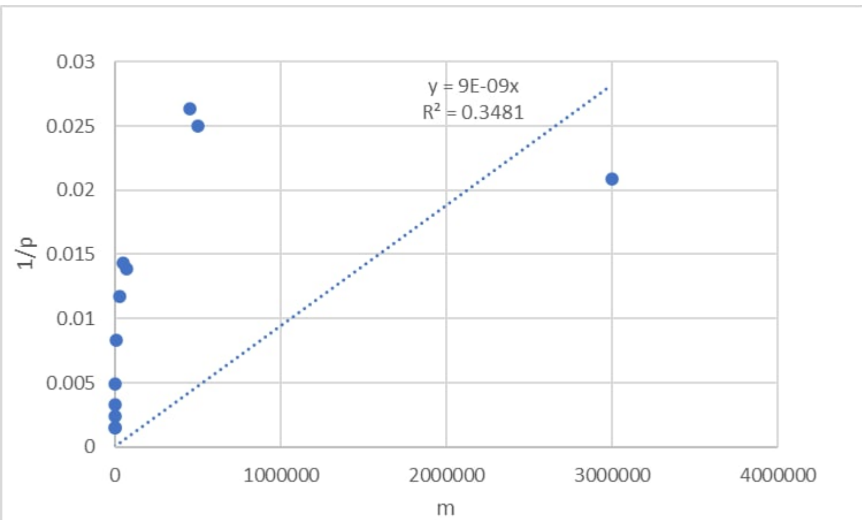
Assumptons made:

1- The body is taken to be at rest.

2- The Pulse rate is considered to be uniform.

3- Other Energy dissipation factors are neglected.

Using the above-obtained model to check the given data, the graph has been plotted below:



From the above graph, the slope obtained for the model is $k = 5.7362 \times 10^{-8}$
The final relationship can be framed as:

$$\frac{1}{p} = 5.7362 \times 10^{-8}$$

Or,

$$p = \frac{1}{5.7362 \times 10^{-8}}$$

Checking the model with the given data:

Observation:	200g	$\rightarrow$	420 beats/min
Prediction:	200g	$\rightarrow$	87,165 beats/min
Observation:	5,000g	$\rightarrow$	120 beats/min
Prediction:	5,000g	$\rightarrow$	3,486 beats/min
Observation:	450,00g	$\rightarrow$	38 beats/min
Prediction:	450,000g	$\rightarrow$	38 beats/min

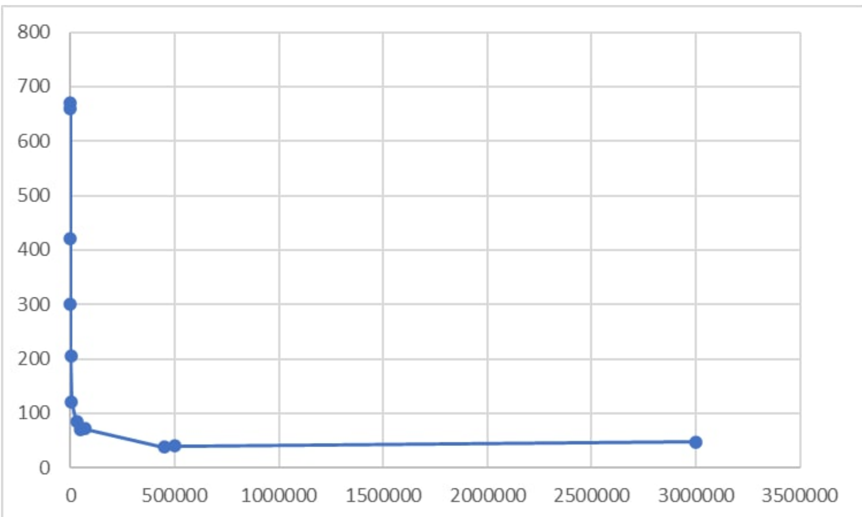
The model is inaccurate, as the best-fit straight line eludes the majority of the plotted points. The graph is incapable of predicting the PUlse Rate of the mammals with lower body masses, whereas the results obtained are satisfactory for the larger mammals. Refinement is definitely needed and three different models have been proposed (due to the nature of the available data) to map Pulse rate to body masses.

First Proposed Model:

The first proposed model is as follows:

$$P = \left(\frac{1}{5.7362 \times 10^{-8} \times m}\right)^{0.6} \qquad \text{for } 4g \leq m \leq 5,000g$$

The graph is as follows:



Testing the model:

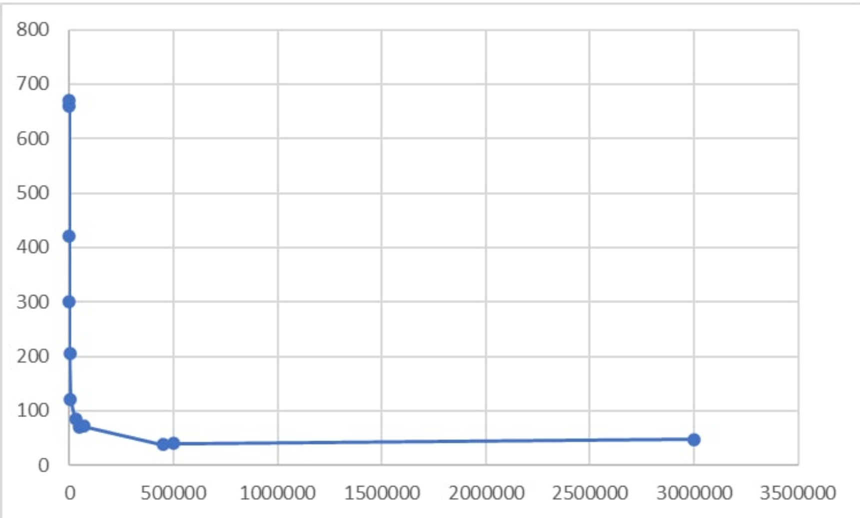
Observation:	300g	$\rightarrow$	300 beats/min
Prediction:	300g	$\rightarrow$	722 beats/min
Observation:	2,000g	$\rightarrow$	205 beats/min
Prediction:	2,000g	$\rightarrow$	231 beats/min
Observation:	5,000g	$\rightarrow$	120 beats/min
Prediction:	5,000g	$\rightarrow$	133 beats/min

Second Proposed Model:

The second proposed model is as follows:

$$P = \left(\frac{1}{5.7362 \times 10^{-8} \times m}\right)^{0.75} \qquad \text{for } 30,000g \leq m \leq 70,000g$$

The graph is as follows:



Testing the model:

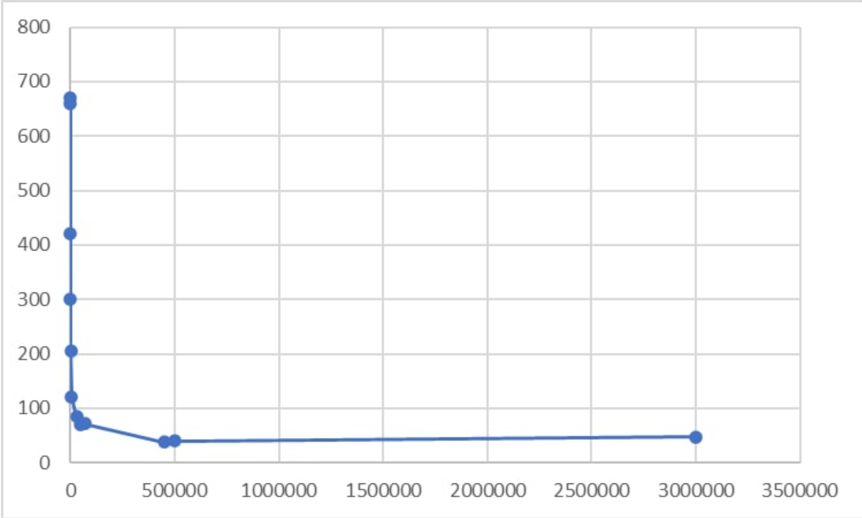
Observation:	3,000g	$\rightarrow$	85 beats/min
Prediction:	3,000g	$\rightarrow$	118 beats/min
Observation:	50,000g	$\rightarrow$	70 beats/min
Prediction:	50,000g	$\rightarrow$	231 beats/min
Observation:	70,000g	$\rightarrow$	72 beats/min
Prediction:	70,000g	$\rightarrow$	63 beats/min

Third Proposed Model(Same as the original model):

The third proposed model is as follows:

$$P = \left(\frac{1}{5.7362 \times 10^{-8} \times m}\right) \qquad \text{for } m \geq 450,000g$$

The graph is as follows:



Testing the model:

Observation:	450,000g	$\rightarrow$	38 beats/min
Prediction:	450,000g	$\rightarrow$	38 beats/min
Observation:	500,000g	$\rightarrow$	40 beats/min
Prediction:	500,000g	$\rightarrow$	35 beats/min

Conclusion:

Divide and Rule methodology has been applied to correctly predict the Pulse rate of the masses and it has been quite effective as the results suggest.

5. Lumber Cutters—Lumber cutters wish to use readily available measurements to estimate the number of board feet of lumber in a tree. Assume they measure the diameter of the tree in inches at waist height. Develop a model that predicts board feet as a function of diameter in inches. Use the following data for your test:

x	17	19	20	23	25	28	32	38	39	41
y	19	25	32	57	71	113	123	252	259	294

The variable x is the diameter of ponderosa pine in inches, and y is the number of board feet divided by 10.

(a) Consider two separate assumptions, allowing each to lead to a model. Completely analyze each model.

i. Assume that all trees are right-circular cylinders and are approximately the same height.

ii. Assume that all trees are right-circular cylinders and that the height of the tree is proportional to the diameter.

(b) Which model appears to be better? Why? Justify your conclusions.

Solution The objective is to determine the number of board feet of lumber in a tree and construct a model function.

(a) Consider the data of the lumber cutters in the table shown below:

x	17	19	20	23	25	28	32	38	39	41
y	19	25	32	57	71	113	123	252	259	294

Here, the variable x is the diameter of a ponderosa pine inches and y is the number of the board feet which is divisible by 10.

i. The volume of the right circular cylinder trees with a diameter (x **in inches**) and with height h is given by:

$$V = \frac{\pi x^2 h}{4} (\text{in})^3$$

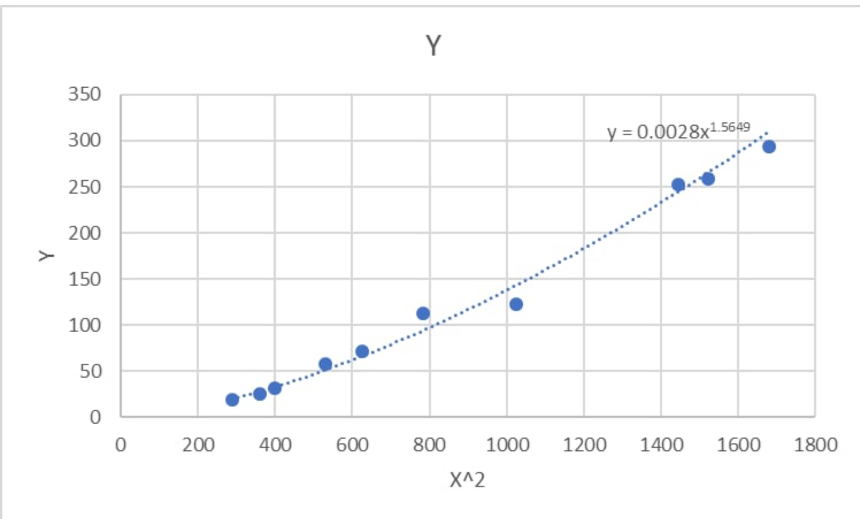
Therefore, the following holds

Number of board feet

$$(y) = \frac{V(\text{in})}{144 \times 10}$$
$$y = \frac{\pi x^2 h}{144 \times 4 \times 10}$$

As all the other terms are a constant entity, the following model can be proposed: $y \propto x^2$

Plot the graph in order to find out the slope k, which will ensure the completion of the model.



The slope has been found to be $k = 0.0028$.
Therefore, the final model becomes

$$\begin{aligned} y &= 0.0028 \times x^{1.58} \\ &\approx 0.0028 \times x^2 \end{aligned}$$

The model is fairly accurate in mapping larger diameters to their respective board feet but deviates at lower values.

ii. Suppose the height of the tree is proportional to the diameter, then

$$h \propto d^2$$

Remove the proportionality sign to get

$$h = kd^2$$

Here k is an arbitrary constant.

Therefore, the volume is:

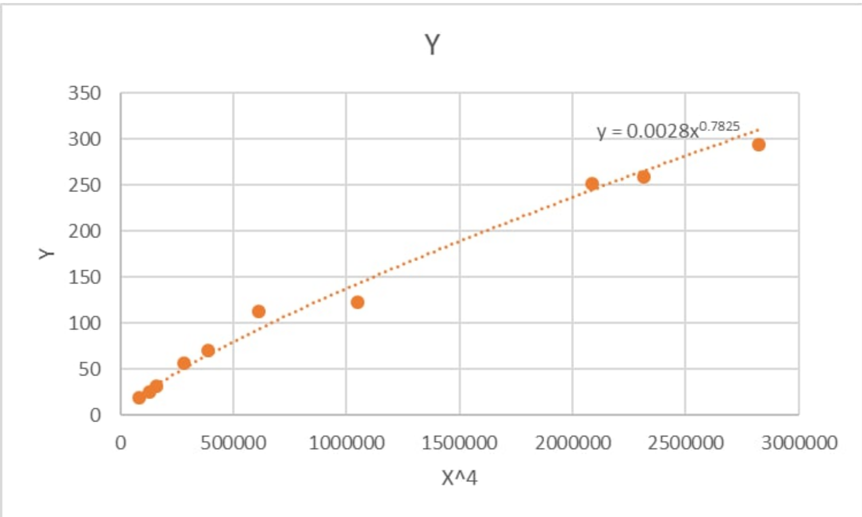
$$\begin{aligned} V &= \frac{\pi x^2 h}{4} (\text{in})^3 \\ &= \frac{k\pi x^4}{4} (\text{in})^3 \end{aligned}$$

Thus, the numbers of boards are as follows:
<u>**Number of board feet**</u>

$$\begin{aligned} (y) &= \frac{V(\text{in})^3}{144 \times 10} \\ y &= \frac{k\pi x^4}{144 \times 4 \times 10} \end{aligned}$$

As all the other terms are a constant entity, the following model can be proposed: $y \propto x^4$

The graph has been plotted below:



The slope has been found to be $k = 0.0028$.
Therefore, the final model becomes

$$\begin{aligned} y &= 0.0028 \times x^{0.782} \\ &\approx 0.0028 \times x^{1} \\ &\approx 0.0028 \times x \end{aligned}$$

Hence, the function of the models is $y \approx 0.0028x$

(b) As is clear from both graphs, the second model does a far better job of predicting the board length com[ared to the first model.

The reason why the second model is better is that the second model takes height into consideration, whereas the first one does not.

The first model only depends on the diameter and is mapped to the diameter squared. No distinguishing is done between short and tall trees.

Whereas in the latter model the model is mapped to a diameter raised to the power 4, and the height

Hence, the **Second model is better than first model.**