Chapter 1 Modeling Change (TD1)

1.1. Modeling Change with Difference Equations.

1.2. Approximating Change with Difference Equations.

Example. Six years ago your parents purchased a home by financing \$80,000 for 20 years, paying monthly payments of \$880.87 with a monthly interest of 1%. They have made 72 payments and wish to know how much they owe on the mortgage, which they are considering paying off with an inheritance they received. Or they could be considering refinancing the mortgage with several interest rate options, depending on the length of the payback period. The change in the amount owed each period increases by the amount of interest and decreases by the amount of the payment:

$$\Delta b_n = b_{n+1} - b_n = 0.01 b_n - 880.87$$

Solving for b_{n+1} and incorporating the initial condition gives the dynamical system model

$$\begin{aligned} b_{n+1} &= b_n + 0.01 b_n - 880.87 \\ b_0 &= 80000 \end{aligned}$$

where b_n represents the amount owed after n months. Thus,

$$\begin{aligned} b_1 &= 80000 + 0.01(80000) - 880.87 = 79919.13 \\ b_2 &= 79919.13 + 0.01(79919.13) - 880.87 = 79837.45 \end{aligned}$$

yielding the sequence

$$B = (80000, 79919.13, 79837.45, \ldots)$$

Calculating b_3 from b_2, b_4 from b_3, and so forth in turn, we obtain $b_{72} = \$71,523.11$.
This sequenc is graphed in **Figure 1.6**.

Months n	Amount owed b_n
0	80000.00
1	79919.13
2	79837.45
3	79754.96
4	79671.64
5	79587.48
6	79502.49
7	79416.64
8	79329.94
9	79242.37
10	79153.92
11	79064.59
12	78974.37

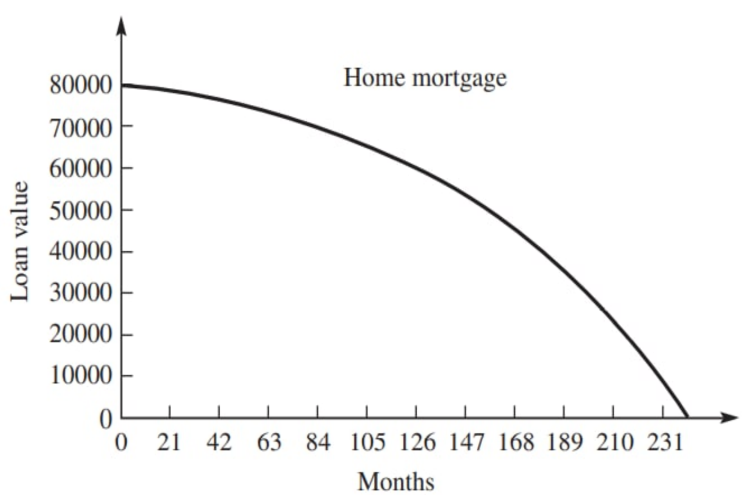

Figure 1.6

Exercise in class for 13-Octomber, 2022

1). Write out the first five terms $a_0 - a_4$ of the sequences:
a. $a_{n+1} = 3a_n,\ a_0 = 1$ b. $a_{n+1} = 2a_n(a_n + 3),\ a_0 = 4$

2). Find a formula for the nth term of the sequence :
a. $\{1, 4, 16, 64, 256, ...\}$ b. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ...\right\}$

3). By examining the following sequences, write a difference equation to represent the change during the nth interval as a function of the previous term in the sequence.
a. {2, 4, 16, 256,...} b. {1, 2, 5, 11, 23,...}

4). Write out the first five terms of the sequence satisfying the following difference equations:
a. $\Delta a_n = \frac{1}{2}a_n,\ a_0 = 1$ b. $\Delta p_n = 0.001(500 - p_n),\ p_0 = 10$

5). By substituting $n = 0, 1, 2, 3$ write out the first four algebraic equations represented by the following dynamical systems:
a. $a_{n+1} = 3a_n,\ a_0 = 1$ b. $a_{n+1} = 2a_n(a_n + 3),\ a_0 = 4$

***T*he problem 6 and 7, formulate a dynamical system that models change eactly for the described situation.**

6). You owe \$500 on a credit card that charges 1.5% interest each month. You pay \$50 each month and you make no new charges.

7). Your grandparents have an annuity. The value of the annuity increases each month by an automatic deposit of 1% interest on the previous month's balance. Your grandparents withdraw \$1000 at the beginning of each month for living expenses. Currently, they have $\$50,000$ in the annuity. Model the annuity with a dynamical system. Will the annuity run out of money? When? *Hint* : What value will an have when the annuity is depleted?

8). The following data were obtained for the growth of a sheep population introduced into a new environment on the island of Tasmania.

Year	1814	1824	1834	1844	1854	1864
Population	125	275	830	1200	1750	1650

Plot the data. Is there a trend? Plot the change in population versus years elapsed after 1814. Formulate a discrete dynamical system that reasonably approximates the change you have observed.

9). Assume that we are considering the survival of whales and that if the number of whales falls below a minimum survival level m, the species will become extinct. Assume also that the population is limited by the carrying capacity M of the environment. That is, if the whale population is above M, it will experience a decline because the environment cannot sustain that large a population level. In the following model, a_n represents the whale population after n years. Discuss the model.

$$a_{n+1} - a_n = k(M - a_n)(a_n - m)$$

10). A certain drug is effective in treating a disease if the concentration remains above $100mg/L$. The initial concentration is $640mg/L$. It is known from laboratory experiments that the drug decays at the rate of 20% of the amount present each hour.
a. Formulate a model representing the concentration at each hour.
b. Build a table of values and determine when the concentration reaches $100mg/L$.

Assignmet

Deadline: 20-October, 2022.

1). With the price of gas continuing to rise, you wish to look at cars that get better gas mileage. You narrow down your choices to the following 2012 models: Ford Fiesta, Ford Focus, Chevy Volt, Chevy Cruz, Toyota Camry, Toyota Camry Hybrid, Toyota Prius and Toyota Corolla. Each company has offered you their ‘‘best deal” as listed in the following table. You are able to allocate approximately \$500 for a car payment each month up to 60 months, although less time would be preferable. Use dynamical systems to determine which new car you can afford.

2012 Model	Best Deal Price	Cash Down	Interest and Duration
Ford Fiesta	$\$14,200$	\$500	4.5% APR for 60 months
Ford Focus	$\$20,705$	\$750	4.38% APR for 60 months
Chevy Volt	$\$39,312$	$\$1,000$	3.28% APR for 48 months
Chevy Cruz	$\$16,800$	\$500	4.4% APR for 60 months
Toyota Camry	$\$22,955$	0	4.8% APR for 60 months
Toyota Camry Hybrid	$\$26,500$	0	3% APR for 48 months
Toyota Corolla	$\$16,500$	\$900	4.25% for 60 months
Toyota Prius	$\$19,950$	$\$1,000$	44.3% for 60 months

2). You are considering a 30−year mortgage that charges 0.4% interest each month to pay off a $\$250,000$ mortgage.

a. Determine the monthly payment p that allows the loan to be paid off at 360 months.

b. Now assume that you have been paying the mortgage for 8 years and now have an opportunity to refinance the loan. You have a choice between a 20-year loan at 4% per year with interest charged monthly and a 15-year loan at 3.8% per year with interest charged monthly. Each of the loans charges a closing cost of \$2500. Determine the monthly payment p for both the 20-year loan and the 15-year loan. Do you think refinancing is the right thing to do? If so, do you prefer the 20-year or the 15-year option?

3). Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. Formulate a dynamical system to approximate the change in the number of people in the population who have the disease.

4). The data in the accompanying table show the speed n (in increments of 5mph) of an automobile and the associated distance a_n in feet required to stop it once the brakes are applied. For instance, $n = 6$ (representing $6 \times 5 = 30$ mph) requires a stopping distance of $a_6 = 47$ ft.

a. Calculate and plot the change Δa_n an versus n. Does the graph reasonably approximate a linear relationship?

b. Based on your conclusions in part (a), find a difference equation model for the stopping distance data. Test your model by plotting the errors in the predicted values against n. Discuss the appropriateness of the model.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	3	6	11	21	32	47	65	87	112	140	171	204	241	282	325	376

1.3 Solutions to Dynamical Systes
1.4. Systems of Difference Equations
Exercise in Class for 20-Octobe, 2022

1. Find the solution to the difference equations in the following problems:

a. $a_{n+1} = \dfrac{3a_n}{4}, \ a_0 = 1$ b. $a_{n+1} = 0.1a_n + 3.2, \ a_0 = 1.3$

2. For the following problems, find an equilibrium value if one exists. Classify the equilibrium value as stable or unstable.

a. $a_{n+1} = 0.9a_n$ $b.a_{n+1} = -1.2a_n + 50$ $c.a_{n+1} = 08a_n + 100$

3. Build a numerical solution for the following initial value problems. Plot your data to observe patterns in the solution. Is there an equilibrium solution? Is it stable or unstable?

a.

$$a_{n+1} = 0.8a_n - 100, a_0 = -500$$

4. You owe \$500 on a credit card that charges 1.5% interest each month. You can pay \$50 each month with no new charges. What is the equilibrium value? What does the equilibrium value mean in terms of the credit card? Build a numerical solution. When will the account be paid off? How much is the last payment?

5. Your grandparents have an annuity. The value of the annuity increases each month as 1% interest on the previous month's balance is deposited. Your grandparents withdraw \$1000 each month for living expenses. Currently, they have \$50,000 in the annuity. Model the annuity with a dynamical system. Find the equilibrium value. What does the equilibrium value represent for this problem? Build a numerical solution to determine when the annuity is depleted.

6. Suppose the spotted owls' primary food source is a single prey: mice. An ecologist wishes to predict the population levels of spotted owls and mice in a wildlife sanctuary. Letting M_n represent the mouse population after n years and O_n the predator owl population, the ecologist has suggested the model.

$$M_{n+1} = 1.2M_n - 0.001O_nM_n$$
$$O_{n+1} = 0.7O_n + 0.002O_nM_n$$

The ecologist wants to know whether the two species can coexist in the habitat and whether the outcome is sensitive to the starting populations. Find the equilibrium values of the dynamical system for this predator-prey model.

a. Compare the signs of the coefficients of the preceding model with the signs of the coefficients of the owls–hawks model in Example 3. Explain the sign of each of the four coefficients $1.2, -0.001, 0.7$, and 0.002 in terms of the predator-prey relationship being modeled.

b. Test the initial populations in the following table and predict the long-term outcome:

	Owls	Mice
Case A	150	200
Case B	150	300
Case C	100	200
Case D	10	20

c. Now experiment with different values for the coefficients using the starting values given. Then try different starting values. What is the long-term behavior? Do your experimental results indicate that the model is sensitive to the coefficients? Is it sensitive to the starting values?

5. An economist is interested in the variation of the price of a single product. It is observed that a high price for the product in the market attracts more suppliers. However, increasing the quantity of the product supplied tends to drive the price down. Over time, there is an interaction between price and supply. The economist has proposed the following model, where P_n represents the price of the product at year n, and Q_n represents the quantity. Find the equilibrium values for this system.

$$P_{n+1} = P_n - 0.1(Q_n - 500)Q_{n+1} = Q_n + 0.2(P_n - 100)$$

a. Does the model make sense intuitively? What is the significance of the constants 100 and 500? Explain the significance of the signs of the constants -0.1 and 0.2.
b. Test the initial conditions in the following table and predict the long-term behavior

	Price	Quantity
Case A	100	500
Case B	200	500
Case C	100	600
Case D	100	400

Assignmet

Deadline: 27-October, 2022.

1. You plan to invest part of your paycheck to finance your children's education. You want to have enough in the account to draw \$1000 a month every month for 8 years beginning 20 years from now. The account pays 0.5% interest each month.
a. How much money will you need 20 years from now to accomplish the financial objective? Assume you stop investing when your first child begins college—a safe assumption.
b. How much must you deposit each month during the next 20 years?
2. Assume we are considering the survival of whales and that if the number of whales falls below a minimum survival level m, the species will become extinct. Assume also that the population is limited by the carrying capacity M of the environment. That is, if the whale population is above M, then it will experience a decline because the environment cannot sustain that large a population level. In the following model, a_n represents the whale population after n years. Build a numerical solution for $M = 5000, m = 100, k = 0.0001$, and $a_0 = 4000$.

$$a_{n+1} - a_n = k(M - a_n)(a_n - m)$$

Now experiment with different values for M, m, and k. Try several starting values for a_0. What does your model predict?
3. Mercury in Fish—Public officials are worried about the elevated levels of toxic mercury pollution in the reservoirs that provide the drinking water to your city. They have asked for your assistance in analyzing the severity of the problem. Scientists have known about the adverse affects of mercury on the health of humans for more than a century. The term mad as a hatter stems from the nineteenth-century use of mercuric nitrate in the making of felt hats. Human activities are responsible for most mercury emitted into the environment. For example, mercury, a by-product of coal, comes from the smokestack emissions of old, coal-fired power plants in the Midwest and South and is disseminated by acid rain. Its particles rise on the

smokestack plumes and hitch a ride on prevailing winds, which often blow northeast. After colliding with mountains, the particles drop to earth. Once in the ecosystem, microorganisms in the soil and reservoir sediment break down the mercury and produce a very toxic chemical known as methyl mercury.

Mercury undergoes a process known as bioaccumulation. This occurs when organisms take in contaminants more rapidly than their bodies can eliminate them. Therefore, the amount of mercury in their bodies accumulates over time. Humans can eliminate mercury from their system at a rate proportional to the amount remaining. Methyl mercury decays 50% every 65 to 75 days (known as the half-life of mercury) if no further mercury is ingested during that time.

Officials in your city have collected and tested 2425 samples of largemouth bass from the reservoirs and provided the following data. All fish were contaminated. The mean value of the methyl mercury in the fish samples was $0.43\mu g$ (microgram) per gram. The average weight of the fish was 0.817 kg.

a. Assume the average adult person (70 kg) eats one fish (0.817 kg) per day. Construct a difference equation to model the accumulation of methyl mercury in the average adult. Assume the half-life is approximately 70 days. Use your model to determine the maximum amount of methyl mercury that the average adult human will accumulate in her or his lifetime.

b. You find out that there is a lethal limit to the amount of mercury in the body; it is 50 mg/kg. What is the maximum number of fish per month that can be eaten without exceeding this lethal limit?

4. Complete the modules ‘‘The Growth of Partisan Support I: Model and Estimation’’ (UMAP 304) and ‘‘The Growth of Partisan Support II: Model Analytics’’ (UMAP 305), by Carol Weitzel Kohfeld. UMAP 304 presents a simple model of political mobilization, refined to include the interaction between supporters of a particular party and recruitable nonsupporters. UMAP 305 investigates the mathematical properties of the first-order quadratic-difference equation model. The model is tested using data from three U.S. counties.