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AMS-I3
Midterm Exam-Solution
Mathematical Modeling

Problem 1 (40 pts).

The **Ricker model** (Ricker 1954) is a population dynamic model that is often used in fishery management. It can be expressed by the following recursion equation:

$$N_{t+1} = N_t e^{r\left(1 - \frac{N_t}{K}\right)} \qquad (1)$$

where:

N_t is the number of fish that will be present in a fishery at time t;

$r > 0$ is interpreted as an intrinsic growth rate; and

$K > 0$ as the carrying capacity of the environment.

Question.

(a) $f(x) = xe^{r(1-x/K)}$.

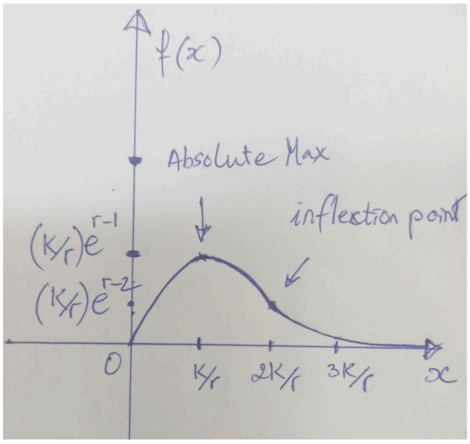
(b) Find $\lim_{x\to+\infty} f(x) = 0$.

(c) $f'(x) = e^{r(1-x/K)} - (r/K)xe^{r(1-x/K)} = \left(1 - (r/K)x\right) e^{r(1-x/K)}$.

(d) $f'(x) = \left(1 - (r/K)x\right) e^{r(1-x/K)} = 0$, then $x = K/r$. f is increasing when $x < K/r$ as $f'(x) > 0$ and is decreasing when $x > K/r$ as $f'(x) < 0$. The function f have an absolute maximum at $x = K/r$.

(e) $f''(x) = -(r/K)\left(2 - (r/K)x\right) e^{r(1-x/K)}$. f is concave upward if $f''(x) > 0$ or $x > 2K/r$ and f is concave downward if $f''(x) < 0$ or $x < 2K/r$. Moreover, $f''(x) = 0$ at $x = 2K/r$. Thus, f has an inflection point at $x = 2K/r$.

(f) Draw the graph of $f(x)$.



Note: Calculator is allowed.

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(g) Determine equilibrium point of the dynamical system **(1)** above. Equilibrium occurs at the point $N = f(N)$ or $N = Ne^{r(1-N/K)}$. Then solving the equation, we have $N = 0$ (not interesting as we start at 0, the process will be always 0) and $N = K$.

(h) According to the value of r and K, tell when the equilibrium point found in part **(h)** is stable and unstable.

At $N = 0$, then $f'(N) = f'(0) = e^r$. Hence at this point, we have stable equilibrium if $|f'(0)| < 1$ or at $r < 0$ and we have unstable equilibrium if $|f'(0)| > 1$ or $r > 0$.

At $N = K$, then $f'(N) = f'(K) = 1 - r$. Hence at this point, we have stable equilibrium if $|f'(K)| < 1$ or at $|1 - r| < 1$. Solving this inequality, we have $0 < r < 2$.

The system has unstable equilibrium if $|f'(K)| > 1$or $r > 2$.

(i) When $K = 1{,}000$ and $r = 1.5$, and initial population is 200, find the population for the next three years. Round the population each year to the nearest integer.

$N_0 = 200, \quad K = 1{,}000$ and $r = 1.5$, then

$N_{t+1} = N_t e^{1.5(1-N_t/1000)}$.

t	$N_{t+1} = N_t e^{1.5(1-N_t/1000)}$
0	200
1	664
2	1099
3	947

(j) When $K = 1{,}000$ and $r = 1.5$, and initial population is 200, determine equilibrium point of the dynamical system **(1)**. Is the equilibrium stable or unstable?

$N_0 = 200, \quad K = 1{,}000$ and $r = 1.5$, then

$N_{t+1} = N_t e^{1.5(1-N_t/1000)}$. The equilibrium point is $N = 1000$. It is stable as $0 < r < 2$.

<u>**Problem 2 (60 pts).**</u> ***In any calculation, round your answer to 3 decimal places.***
In a research laboratory, a researcher wanted to study how a particular type of bacteria evolves by growing them in the petri dish. The daily recorded number of living bacteria (in million) for the first 5 days is given in **Table 1** below.

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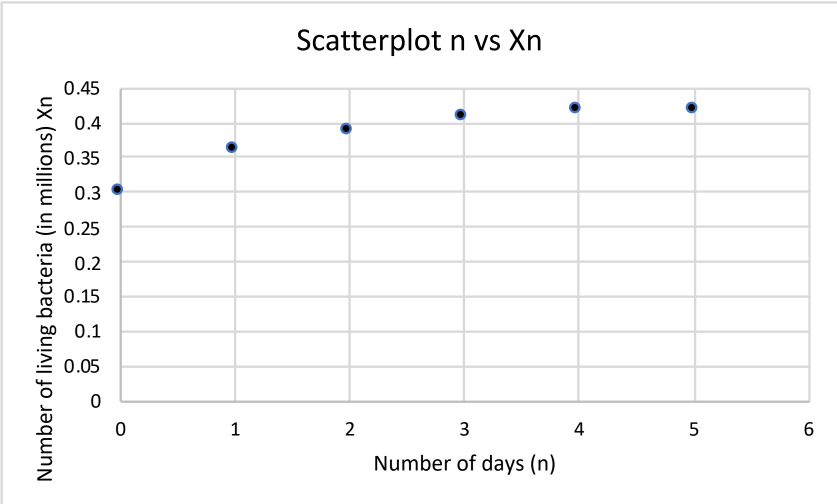
Number of days (n)	0	1	2	3	4	5
Number of bacteria X_n ($\times 10^6$)	0.30	0.36	0.39	0.41	0.42	0.42

Question.

(a) Draw the scatterplot of the above data, using *x*-axis for Number of days (n) and *y*-axis for Number of living bacteria (X_n). Does the graph suggest any trend? Provide your comment.

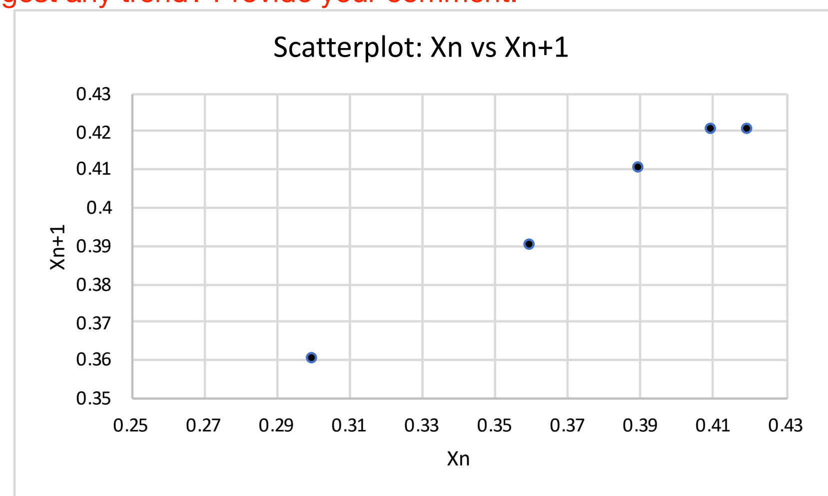


Yes, the graph apparently suggests a positive trend, but it grows slowly. It looks nonlinear.

(b) If $X_{n+1} = f\left(X_n\right)$, where f is some function to be estimated. Fill in the blank of following table (do it in your paper).

Number of days, n	0	1	2	3	4	5
Number of bacteria, X_n ($\times 10^6$)	0.30	0.36	0.39	0.41	0.42	0.42
$X_{n+1} = f\left(X_n\right)$	0.36	0.39	0.41	0.42	0.42	

(c) Draw the scatterplot of X_n and X_{n+1}, using *x*-axis for X_n and *y*-axis for X_{n+1}. Does the graph suggest any trend? Provide your comment.



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From the graph, it seems that as X_n increases, X_{n+1} also increases. The trend may be linear or nonlinear (we need more data to decide which one).

(d) Suppose a linear model: $f(x) = b_0 + b_1 x$ is proposed. Using the available data in part (b), estimate the coefficient b_0 and b_1 by means least squares method. What is the value of the minimized least squares?

Denote $y_i = f(x_i) = x_{i+1}$. We have $\bar{x} = 0.38$, $\bar{y} = 0.40$

$$b_1 = \frac{n\sum(x_i y_i) - (\sum x_i)(\sum y_i)}{n\sum(x_i^2) - (\sum x_i)^2} = 0.526$$

$$b_0 = \bar{y} - b_1\bar{x} = 0.40 - (0.526)(0.38) = 0.202$$

Thus, $f(x) = 0.202 + 0.526\,x$

$\min S = 2.38 \times 10^{-5}$.

(e) What is the value of estimated equilibrium point of the model in part **(d)**? Describe in word the long-run behavior of the number of living bacteria in this case.

The equilibrium point: $f(x) = x$ or $x = 0.202 + 0.526\,x$. Solving the equation, we have $x = \dfrac{0.202}{1 - 0.526} = 0.426$.

Following the model, in the long run, starting with 0.3×10^6 bacteria, the number of bacteria will eventually reach 0.426×10^6 bacteria.

(f) In real situation, the ***logistic model*** of the form: $f(x) = a_1 x + a_2 x^2$ is usually proposed. Using the available data in part **(b)**, estimate the coefficient a_1 and a_2 by means of least squares method. What is the value of the minimized least squares?

Denote $y_i = f(x_i) = x_{i+1}$

We define $S = \sum_{i=0}^{4}(y_i - a_1 x_i - a_2 x_i^2)^2$. We minimize S with respect to a_1, a_2

$$\frac{\partial S}{\partial a_1} = -2\sum_{i=0}^{4} x_i(y_i - a_1 x_i - a_2 x_i^2) = 0 \quad \text{or} \quad a_1\sum_{i=1}^{4} x_i^2 + a_2\sum_{i=1}^{4} x_i^3 = \sum_{i=1}^{4} x_i y_i$$

$$\frac{\partial S}{\partial a_2} = -2\sum_{i=0}^{4} x_i^2(y_i - a_1 x_i - a_2 x_i^2) = 0 \quad \text{or} \quad a_1\sum_{i=1}^{4} x_i^3 + a_2\sum_{i=1}^{4} x_i^4 = \sum_{i=1}^{4} x_i^2 y_i$$

Calculating all the sums, and then solving the system of equations, we have $a_1 = 1.667$ and $a_2 = -1.584$

The model equation is $f(x) = 1.776\,x - 1.584\,x^2$.

The value of minimized S is $\min S = 3.84 \times 10^{-5}$.

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(g) What is the value of estimated equilibrium point of the model in part **(f)**? Describe in word the long-run behavior of the number of living bacteria in this case.

The equilibrium point: $f(x) = x$ or $x = 1.776\,x - 1.584\,x^2$. Solving the equation, we $x = 0$ or $x = 1.776/1.584 = 1.121$

(h) Are the values of minimized least squares of the model in part **(d)** and part **(f)**, respectively, comparable? Justify your answer.

The minimized least squares of the model in part (d): $\min S = 2.38 \times 10^{-5}$
The minimized least squares of the model in part (e): $\min S = 3.84 \times 10^{-5}$

The value found in (d) is relatively lower.

(i) If the model in part **(f)** is written in the form: $f(x) = rx(M - x)$. Determine the value r and M.

$f(x) = 1.776\,x - 1.584\,x^2 = 1.584x(1.121 - x)$. Hence, $r = 1.584$ and $M = 1.121$.

(j) Suppose now the researcher is interested in experimental model to fit the data in part **(b)**. He proposes a polynomial of 4th degree: $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$. To find this polynomial, he uses the Lagrange's method by writing
$f(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x) + f(x_3)\,L_3(x) + f(x_4)\,L_4(x)$, where L_0, L_1, L_2, L_3, L_4 are Lagrangian polynomials corresponding to x_0, x_1, x_2, x_3, x_4, respectively.

1. Determine L_0, L_1, L_2, L_3, L_4.

Number of days, n	0	1	2	3	4	5
Number of bacteria, X_n ($\times 10^6$)	0.30	0.36	0.39	0.41	0.42	0.42
$X_{n+1} = f(X_n)$	0.36	0.39	0.41	0.42	0.42	

$$L_0(x) = \frac{(x - 0.36)(x - 0.39)(x - 0.41)(x - 0.42)}{(0.3 - 0.36)(0.3 - 0.39)(0.3 - 0.41)(0.3 - 0.42)}$$

$$L_0(x) = 14029.181(x - 0.36)(x - 0.39)(x - 0.41)(x - 0.42)$$

$$L_1(x) = \frac{(x - 0.3)(x - 0.39)(x - 0.41)(x - 0.42)}{(0.36 - 0.3)(0.36 - 0.39)(0.36 - 0.41)(0.36 - 0.42)}$$

$$L_1(x) = -\,185185.1852(x - 0.3)(x - 0.39)(x - 0.41)(x - 0.42)$$

$$L_2(x) = \frac{(x - 0.3)(x - 0.36)(x - 0.41)(x - 0.42)}{(0.39 - 0.3)(0.39 - 0.36)(0.39 - 0.41)(0.39 - 0.42)}$$

$$L_2(x) = 617283.951(x - 0.3)(x - 0.36)(x - 0.41)(x - 0.42)$$

Note: Calculator is allowed.

$$L_3(x) = \frac{(x-0.3)(x-0.36)(x-0.39)(x-0.42)}{(0.41-0.3)(0.41-0.36)(0.41-0.39)(0.41-0.42)}$$

$$L_3(x) = -909090.909(x-0.3)(x-0.36)(x-0.39)(x-0.42)$$

$$L_4(x) = \frac{(x-0.3)(x-0.36)(x-0.39)(x-0.41)}{(0.42-0.3)(0.42-0.36)(0.42-0.39)(0.42-0.41)}$$

$$L_4(x) = 462962.963(x-0.3)(x-0.36)(x-0.39)(x-0.41)$$

2. Determine the values of $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$.

We have
$f(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x) + f(x_3)\,L_3(x) + f(x_4)\,L_4(x)$

Expanding this, $f(x) = -22.8 + 262.2x - 1110x^2 + 2083x^3 - 1459.03x^4$

We the get

$\alpha_0 = -22.8$
$\alpha_1 = 262.2$
$\alpha_2 = -1110$
$\alpha_3 = 2083$
$\alpha_4 = -1459.03$

(k) Estimate the value of X_5 using models in part **(d)**, in part **(f)** and in part **(j)**, respectively. Which model provide the closest estimate to X_5? Note that the exact value of X_5 is 0.42 (See Table 1).

Part	Model	Predicted	Error $\lvert X_5 - X_5^{pred} \rvert$
(d)	$f(x) = 0.202 + 0.526\,x$	0.423	0.003
(f)	$f(x) = 1.776\,x - 1.584\,x^2$	0.467	0.047
(j)	$f(x) = -22.8 + 262.2x - 1110x^2 + 2083x^3 - 1459.03x^4$	0.445	0.025

Model in part (d) provides the closest estimate to X_5.