Mathematical Modeling

Chapter 2.
The Modeling Process, Proportionality, and Geometric Similarity

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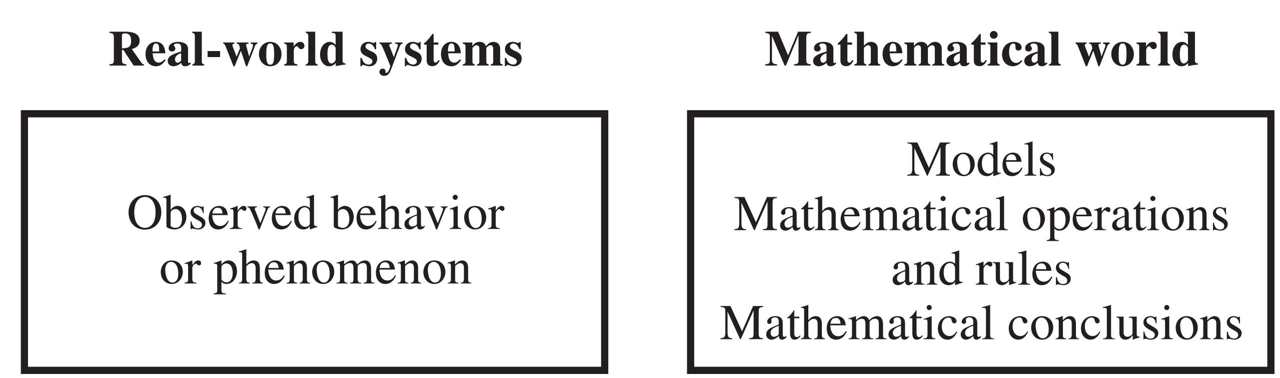
- Automobile Gasoline Mileage
- Body Weight and Height, Strength and Agility

Introduction

- We want to understand some behavior or phenomenon in the real world.
- We may wish to make predictions about that behavior in the future and analyze the effects that various situations have on it. (See Figure 2.1)

Figure 2.1

The real and mathematical worlds



Introduction

- A **system** is an assemblage of objects joined in some regular interaction or interdependence.

  The modeler is interested in

- understanding how a particular system works, what causes changes in the system, and how sensitive the system is to certain changes.

- predicting what changes might occur and when they occur. How might such information be obtained?

Introduction

Goal. Draw conclusions about an observed phenomenon in the real world.

Two approaches to conclusions about the real world :

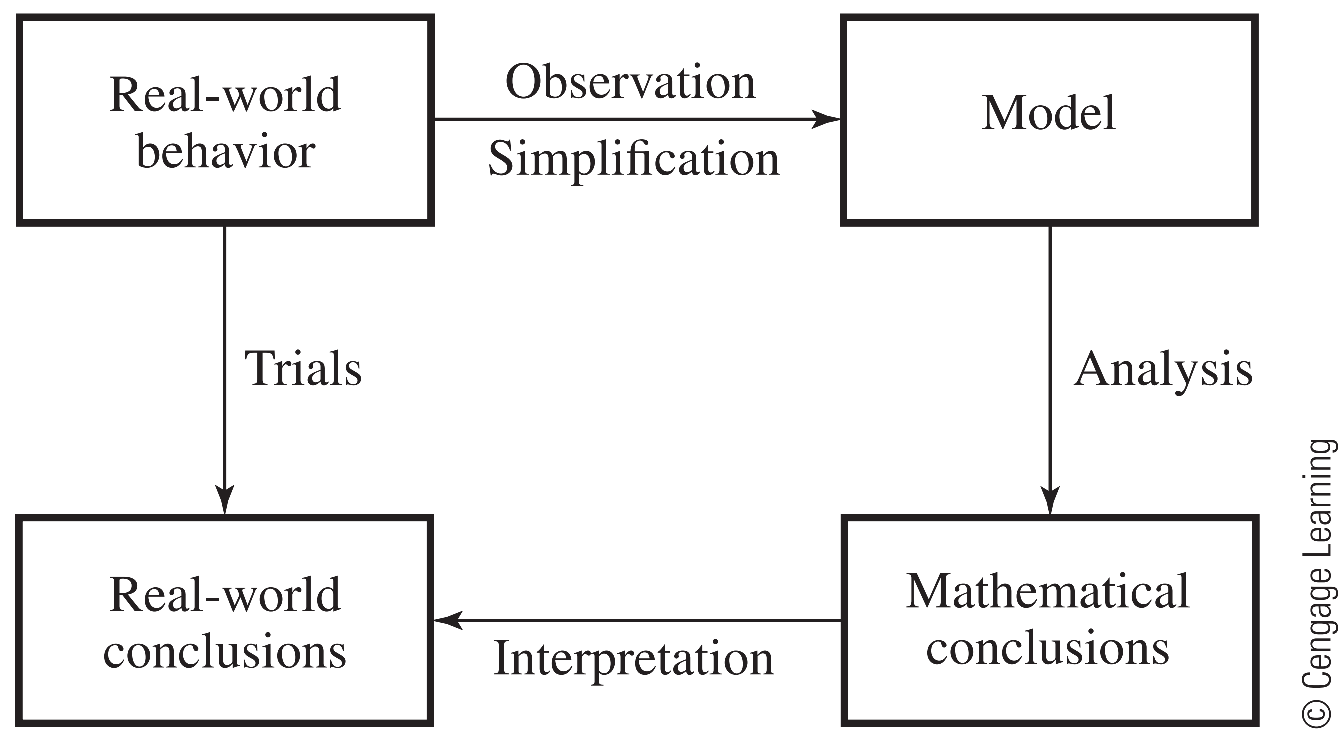
Approach 1. One procedure would be to conduct some real-world behavior trials or experiments and observe their effect on the real-world behavior (**See Figure 2.2**).

Drawback. This approach can be undoable, or too costly, or too harmful.

Introduction

Figure 2.2

Reaching conclusions about the behavior of real-world systems



Introduction

Approach 2. Use mathematical modeling (**See Figure 2.3**).

- We conduct the following rough modeling procedure:

1. Through observation, identify the primary factors involved in the real-world behavior, possibly making simplifications.

2. Conjecture tentative relationships among the factors.

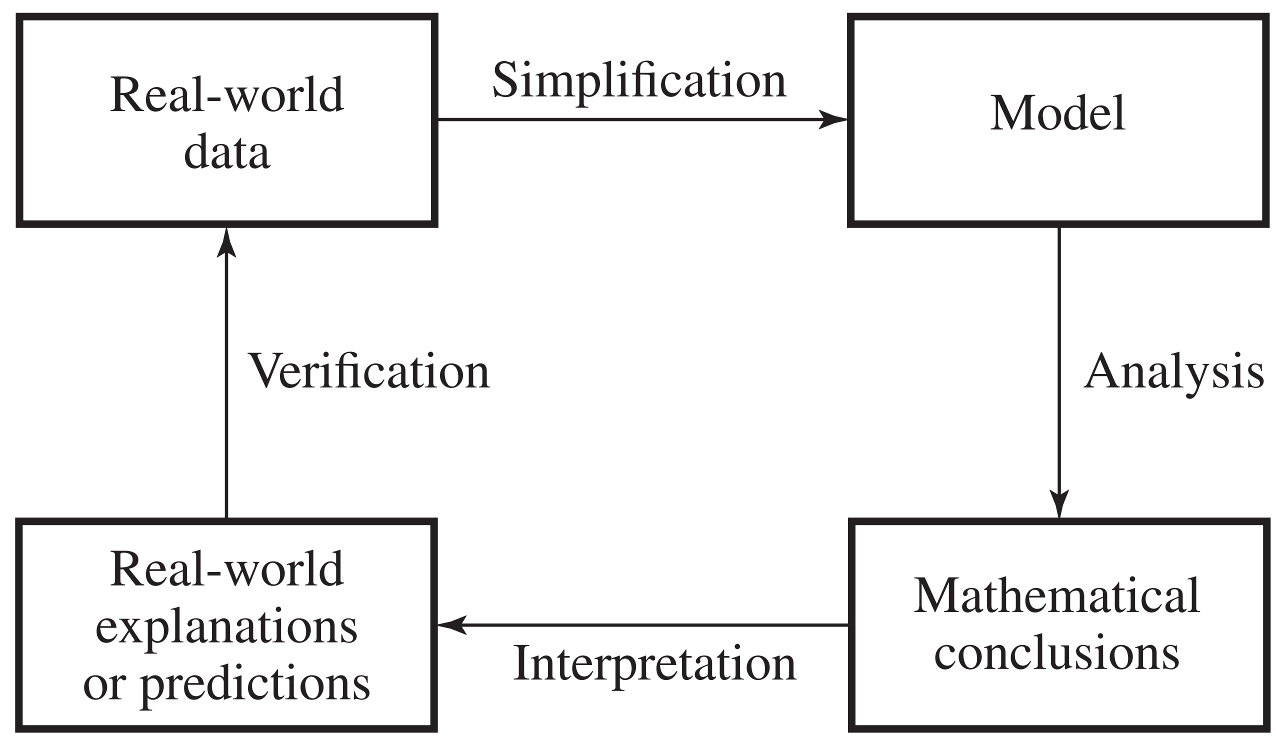
3. Apply mathematical analysis to the resultant model.

4. Interpret mathematical conclusions in terms of the real-world problem.

Introduction

Figure 2.3

The modeling process as a closed system



Mathematical Models

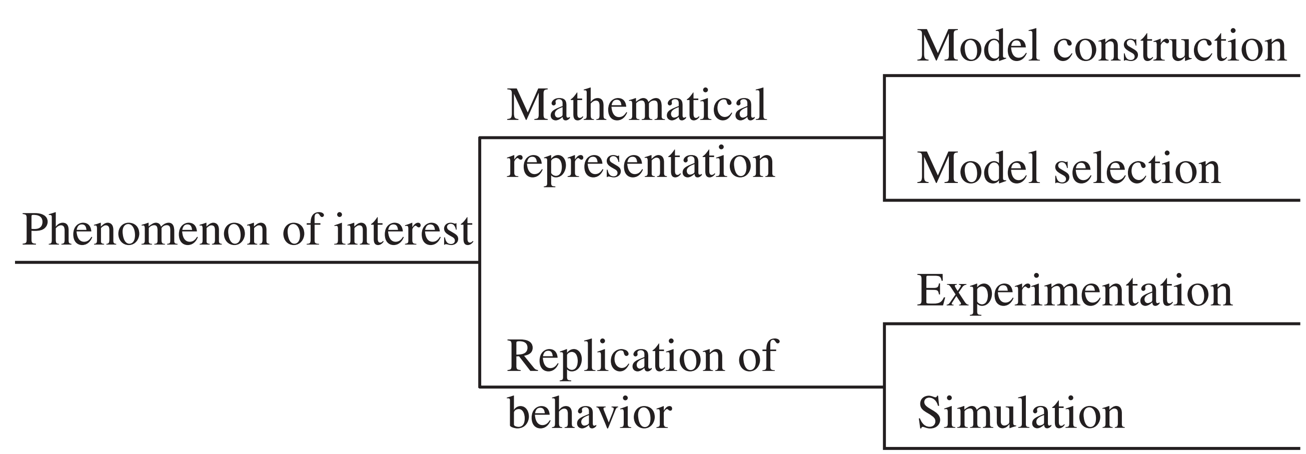
Definition. **Mathematical model** is a mathematical construct designed to study a particular real-world system or phenomenon. We include graphical, symbolic, simulation, and experimental constructs.

- Existing mathematical models that can be identified with some particular real-world phenomenon and used to study it.

- New mathematical models that we construct specifically to study a special phenomenon. See **Figure 2.4**.

Figure 2.4

The nature of the model



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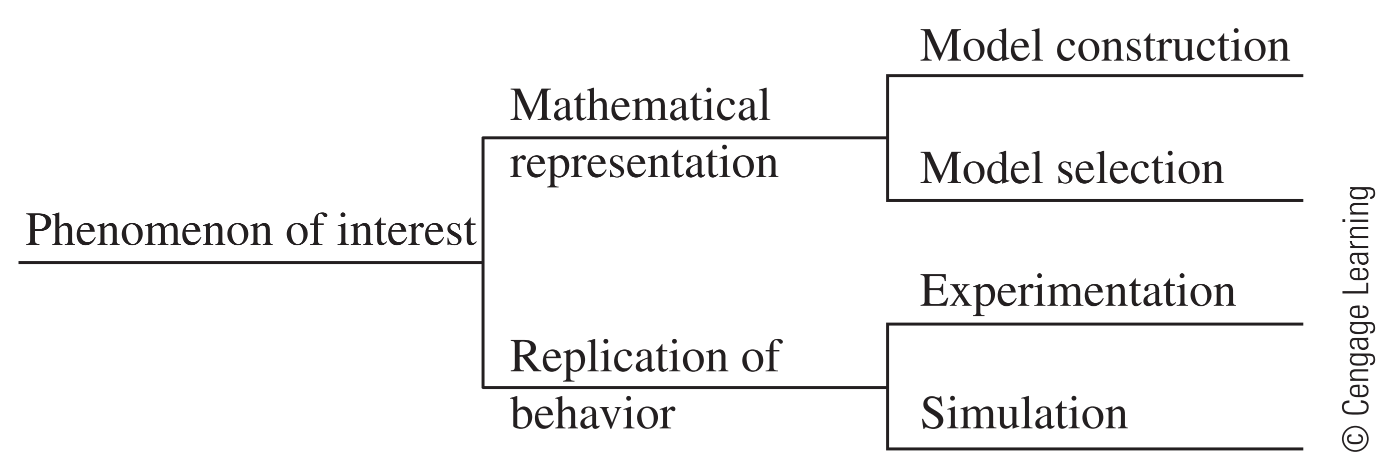

Mathematical Models

There are **three characteristics that a Mathematical model** can possess.

- **Fidelity:** The preciseness of a model's representation of reality

- **Costs:** The total cost of the modeling process

- **Flexibility:** The ability to change and control conditions affecting the model as required data are gathered

Mathematical Models

Comparison of Characteristics of Mathematical Model.

Vertical axis = Degree of Effectiveness

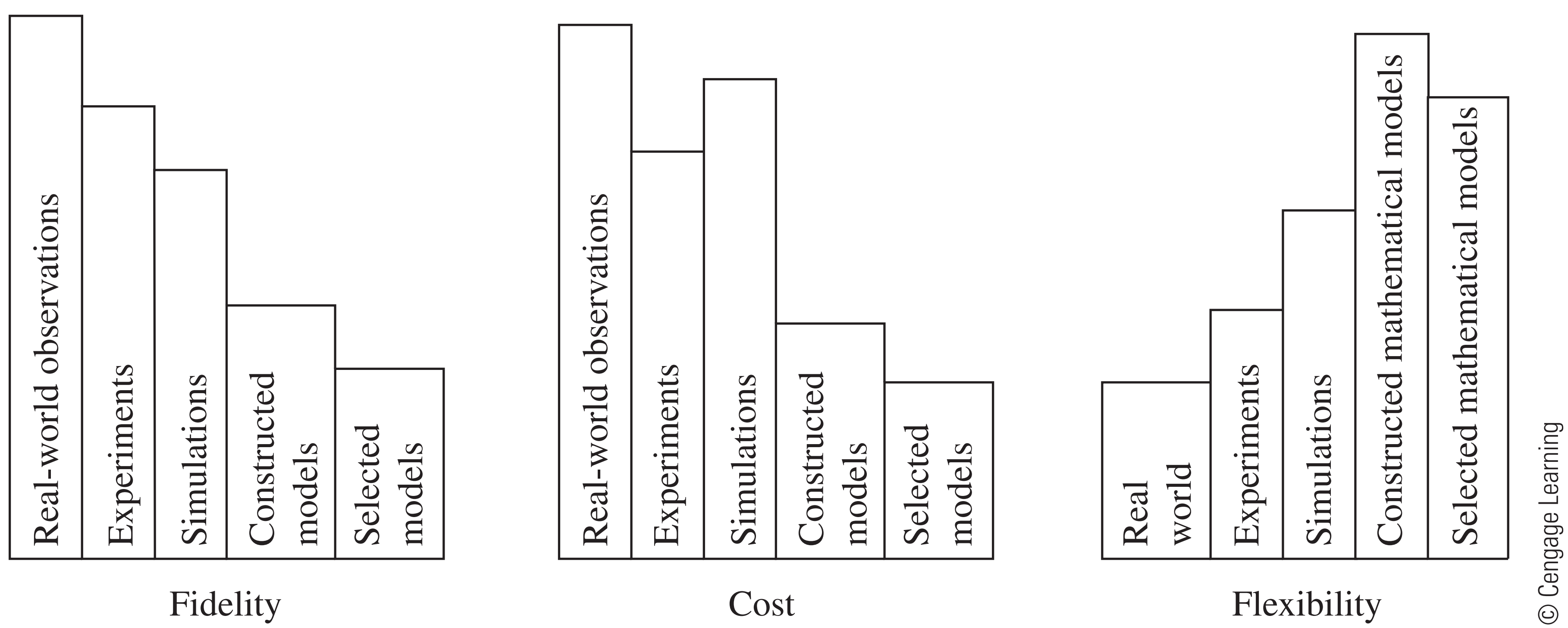

Figure 2.5

Comparisons among the model types

Mathematical Models

Construction of Models

Step 1: Identify the problem. What is the problem you would like to explore?

- Typically this is a difficult step because in real-life situations no one simply hands you a mathematical problem to solve.
- Usually you have to sort through large amounts of data and identify some particular aspect of the situation to study.
- Moreover, it is imperative to be sufficiently precise (ultimately) in the formulation of the problem to allow for translation of the verbal statements describing the problem into mathematical symbols. This translation is accomplished through the next steps.
- It is important to realize that the answer to the question posed might not lead directly to a usable problem identification.

Mathematical Models

Construction of Models

Step 2: Make assumptions.

- Generally, we cannot hope to capture in a usable mathematical model all the factors influencing the identified problem.

- The task is simplified by reducing the number of factors under consideration. Then, relationships among the remaining variables must be determined.

- Again, by assuming relatively simple relationships, we can reduce the complexity of the problem.

Mathematical Models

Construction of Models

Step 2: Make assumptions. The assumptions fall into two main activities:

- **Classifying the variables.** What things influence the behavior of the problem identified in Step 1? List these things as variables. The variables the model seeks to explain are the dependent variables, and there may be several of these. The remaining variables are the independent variables. Each variable is classified as dependent, independent, or neither.

- **Determine relationships among variables selected for study.** Before we can hypothesize a relationship among the variables, we generally must make some additional simplifications. The problem may be so complex that we cannot see a relationship among all the variables initially. In such cases it may be possible to study submodels. That is, we study one or more of the independent variables separately. Eventually we will connect the submodels together. Studying various techniques, such as proportionality, will aid in hypothesizing relationships among the variables.

Mathematical Models

Construction of Models

Step 3: Solve or interpret the model.

- Now put together all the submodels to see what the model is telling us.

- In some cases the model may consist of mathematical equations or inequalities that must be solved to find the information we are seeking.

- Often, a problem statement requires a best solution, or *optimal solution*, to the model. Models of this type are discussed later.

Mathematical Models

Construction of Models

Step 4: Verifying the model.

- Before we can use the model, we must test it out.
- There are several questions to ask before designing these tests and collecting data—a process that can be expensive and time-consuming.
- First, does the model answer the problem identified in Step 1, or did it stray from the key issue as we constructed the model?
- Second, is the model usable in a practical sense? That is, can we really gather the data necessary to operate the model?
- Third, does the model make common sense?

Mathematical Models

Construction of Models

Step 5: Implement the model.

- We will want to explain our model in terms that the decision makers and users can understand if it is ever to be of use to anyone.

- Furthermore, unless the model is placed in a user-friendly mode, it will quickly fall into disuse. Expensive computer programs sometimes suffer such a demise. Often the inclusion of an additional step to facilitate the collection and input of the data necessary to operate the model determines its success or failure.

Mathematical Models

Construction of Models

Step 6: Maintain the model.

- Remember that the model is derived from a specific problem identified in Step 1 and from the assumptions made in Step 2. Has the original problem changed in any way, or have some previously neglected factors become important? Does one of the submodels need to be adjusted?

Mathematical Models

Construction of Models

Summary

Figure 2.6

Construction of a mathematical model

Step 1. Identify the problem.
Step 2. Make assumptions.
  a. Identify and classify the variables.
  b. Determine interrelationships between the variables and submodels.
Step 3. Solve the model.
Step 4. Verify the model.
  a. Does it address the problem?
  b. Does it make common sense?
  c. Test it with real-world data.
Step 5. Implement the model.
Step 6. Maintain the model.

Mathematical Models

Construction of Models

EXAMPLE 1 *Vehicular Stopping Distance*

Scenario Consider the following rule often given in driver education classes:

Allow one car length for every 10 miles of speed under normal driving conditions, but more distance in adverse weather or road conditions. One way to accomplish this is to use the 2-second rule for measuring the correct following distance no matter what your speed. To obtain that distance, watch the vehicle ahead of you pass some definite point on the highway, like a tar strip or overpass shadow. Then count to yourself "one thousand and one, one thousand and two;" that is 2 seconds. If you reach the mark before you finish saying those words, then you are following too close behind.

The preceding rule is implemented easily enough, but how good is it?

Read the text

Mathematical Models

Iterative Nature of Model Construction

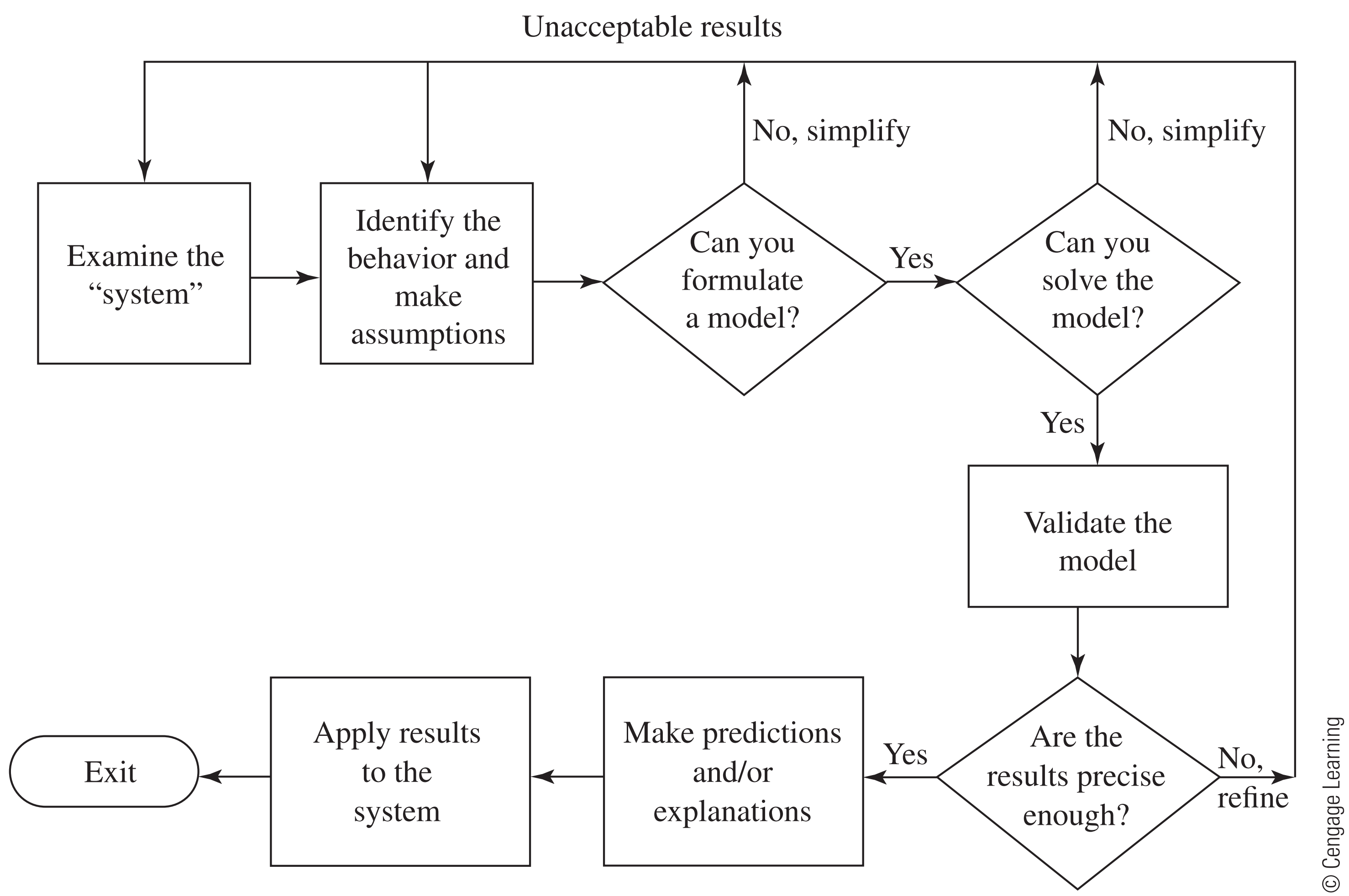

Figure 2.7

The iterative nature of model construction

Mathematical Models

Iterative Nature of Model Construction

Table 2.1 The art of mathematical modeling: simplifying or refining the model as required

Model simplification	Model refinement
1. Restrict problem identification.	1. Expand the problem.
2. Neglect variables.	2. Consider additional variables.
3. Conglomerate effects of several variables.	3. Consider each variable in detail.
4. Set some variables to be constant.	4. Allow variation in the variables.
5. Assume simple (linear) relationships.	5. Consider nonlinear relationships.
6. Incorporate more assumptions.	6. Reduce the number of assumptions.

Modeling Using Proportionality

We introduced the concept of proportionality in Chapter 1 to model change. Recall that

$$y \propto x \text{ if and only if } y = kx \text{ for some constant } k \neq 0 \tag{2.1}$$

Of course, if $y \propto x$, then $x \propto y$ because the constant k in Equation (2.1) is not equal to zero and then $x = (\frac{1}{k})y$. The following are other examples of proportionality relationships:

$$y \propto x^2 \text{ if and only if } y = k_1 x^2 \text{ for } k_1 \text{ a constant} \tag{2.2}$$

$$y \propto \ln x \text{ if and only if } y = k_2 \ln x \text{ for } k_2 \text{ a constant} \tag{2.3}$$

$$y \propto e^x \text{ if and only if } y = k_3 e^x \text{ for } k_3 \text{ a constant} \tag{2.4}$$

In Equation (2.2), $y = kx^2, k \neq 0$, so we also have $x \propto y^{1/2}$ because $x = (\frac{1}{\sqrt{k}})y^{1/2}$. This leads us to consider how to link proportionalities together, a transitive rule for proportionality:

$$y \propto x \quad \text{and} \quad x \propto z, \text{ then} \quad y \propto z$$

Modeling Using Proportionality

Table 2.2 Famous proportionalities

Hooke's law: $F = kS$, where F is the restoring force in a spring stretched or compressed a distance S.

Newton's law: $F = ma$ or $a = \frac{1}{m}F$, where a is the acceleration of a mass m subjected to a net external force F.

Ohm's law: $V = iR$, where i is the current induced by a voltage V across a resistance R.

Boyle's law: $V = \frac{k}{p}$, where under a constant temperature k, the volume V is inversely proportional to the pressure p.

Einstein's theory of relativity: $E = c^2M$, where under the constant speed of light squared c^2, the energy E is proportional to the mass M of the object.

Kepler's third law: $T = cR^{\frac{3}{2}}$, where T is the period (days) and R is the mean distance to the sun.