Pre-Final Exam

1. What is the Lagrange Form of the Polynomial?
2. Suppose the following data have been collected

X	-1	0	1	2
y	3	-4	5	-6

   Using Lagrange interpolation to find the unique polynomial $P_3(x)$ with given data.
3. Suppose we have four points of $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4).
   (i). what is natural cubic spline $S(x)$ that pass through above data points? (ii) what are properties of natural cubic spline?
4. Find the natural Cubic Spline that pass through the given data points:
$(-1,3), (0,-4), (1,5)$ and $(2,-6)$.
5. Write the system of equations $f(x) = \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ into the matrix form.
6. Solve the given linear programming problems by using graphical method and Simplex method:

Maximize: $Z = 50x + 15y$

Subject to: $5x + y \leq 100,$

$x + y \leq 50\,,$

$x \geq 0,$

$y \geq 0$

Solution:

1. The Lagrange Form of the Polynomial: is the polynomial $P(x)$ of degree $\leq n+1$ that pass through the n points $(x_1, y_1), (x_2, y_2), \ldots., (x_n, y_n)$ and is given by

$$P_n(x) = \sum_{i=1}^{n} y_i L_i(x) = y_1 L_1(x) + \cdots + y_n L_n(x)$$

Where

$$L_k(x) = \prod_{\substack{k=1 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

2. Using Lagrange interpolation to find the unique polynomial $P_3(x)$ with given data:

X	-1	0	1	2
y	3	-4	5	-6

The Lagrange Form of the Polynomial $P(x)$ pass through points $(-1,3), (0,4), (1,5)$ & $(2,-6)$ is

$$P_3(x) = \sum_{i=1}^{4} y_i L_i(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x)$$

We obtain:

$$L_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} = \frac{(x - 0)(x - 1)(x - 2)}{(-1 - 0)(-1 - 1)(-1 - 2)}$$
$$= -\frac{1}{6}(x^3 - 3x^3 + 2x)$$

$$L_2(x) = \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} = \frac{(x + 1)(x - 1)(x - 2)}{(0 + 1)(0 - 1)(0 - 2)}$$
$$= \frac{1}{2}(x^3 - 2x^2 - x + 2)$$

$$L_3(x) = \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} = \frac{(x + 1)(x - 0)(x - 2)}{(1 + 1)(1 - 0)(1 - 2)} = -\frac{1}{2}(x^3 - x^2 - 2x)$$

$$L_4(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} = \frac{(x + 1)(x - 0)(x - 1)}{(2 + 1)(2 - 0)(2 - 1)} = \frac{1}{6}(x^3 - x)$$

Therefore,

$$P_3(x) = 3\left(-\frac{1}{6}\right)(x^3 - 3x^3 + 2x) + (-4)\left(\frac{1}{2}\right)(x^3 - 2x^2 - x + 2) + 5\left(-\frac{1}{2}\right)(x^3 - x^2 - 2x)$$
$$+ (-6)\,\frac{1}{6}(x^3 - x)$$

Therefore , $\quad P_3(x\,) = 6x^3 + 8x^2 + 7x - 4$

3. (i) The natural cubic spline $S(x)$ that pass-through data point $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) :

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \; x \in [x_i, x_{i+1}]$$

(ii) The properties of natural cubic spline as following:

- $S_i(x_i) = y_i$ & $S_i(x_{i+1}) = y_{i+1}, \quad i = 1,2,3$
- $S'_i(x_{i+1}) = S'_{i+1}(x_{i+1}), \quad i = 1,2$
- $S"_i(x_{i+1}) = S"_{i+1}(x_{i+1}), \;\; i = 1,2$

- $S_1''(x_1) = S_{n-1}''(x_n), \quad n = 4$
- $S_1''(x_1) = 0, \; S_{n-1}''(x_n) = 0, \quad n = 4$

4. ind the natural Cubic Spline that pass through the given data points: $(-1,3), (0,-4), (1,5)$ and $(2,-6)$.
   - The natural cubic Spline $S(x)$ such as:

$$S_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3, \qquad x \in [-1,0]$$
$$S_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3 \qquad , x \in [0,1]$$
$$S_3(x) = a_3 + b_3x + c_3x^2 + d_3x^3 \quad , x \in [1,2]$$

   - $S_1(x)$ pass through $(-1,3)$ & $(0,4)$ requires that $S_1(-1) = 3$ & $S_1(0) = 4$, we obtain :

$$a_1 - b_1 + c_1 - d_1 = 3 \qquad (1)$$
$$a_1 + 0b_1 + 0c_1 + 0d_1 = -4 \qquad (2)$$

   - $S_2(x)$ pass through points $(0,-4)$&$(1,5)$ requires that $S_2(0) = -4$ & $S_2(1) = 5$, we obtain:

$$a_2 + 0b_2 + 0c_2 + 0d_2 = -4 \qquad (3)$$
$$a_2 + b_2 + c_2 + d_2 = 5 \qquad (4)$$

   - $S_3(x)$ pass through points $(1,5)$&$(2,-6)$ requires that $S_3(1) = 5$ & $S_3(2) = -6$, we obtain:

$$a_3 + b_3 + c_3 + d_3 = 5 \qquad (5)$$
$$a_3 + 2b_3 + 4c_3 + 8d_3 = -6 \qquad (6)$$

   - $S_1'(x)$, $S_2'(x)$ & $S_3'(x)$ are forced to match at $x_2 = 0: S_1'(0) = S_2'(0)$ and $x_3 = 1: S_2'(1) = S_3'(1)$. We have :

$$S_1'(x) = b_1 + 2c_1x + 3d_1x^2$$
$$S_2'(x) = b_2 + 2c_2x + 3d_2x^2$$
$$S_3'(x) = b_3 + 2c_3x + 3d_3x^2$$

   Then

$$b_1 = b_2 \qquad (7)$$
$$b_2 + 2c_2 + 3d_2 = b_3 + 2c_3 + 3d_3 \qquad (8)$$

   - $S_1''(x), S_2''(x)$ & $S_3''(x)$ are forced to match at $x_2 = 0: S_1''(0) = S_2''(0)$ and $x_3 = 1: S''_2(1) = S_3''(1)$. We have :

$$S_1''(x) = 2c_1 + 6d_1x$$
$$S_2''(x) = 2c_2 + 6d_2x$$
$$S_3''(x) = 2c_3 + 6d_3x$$

   Then $$2c_1 = 2c_2 \Longrightarrow c_1 = c_2 \qquad (9)$$
$$2c_2 + 6d_2 = 2c_3 + 6d_6 \Longrightarrow c_2 + 3d_2 = c_3 + 2d_3 \qquad (10)$$

   - A natural spline is built by requiring that $S_1''(-1) = 0$ & $S_3''(2) = 0$ then

$$2c_2 - 6d_2 = 0 \;\Rightarrow c_2 - 3d_2 = 0 \qquad (11)$$
$$2c_3 + 12d_3 = 0 \;\Rightarrow c_3 + 3d_3 = 0 \qquad (12)$$

By (1),….., (12), we obtain equation system:

$$
\begin{cases}
a_1 - b_1 + c_1 - d_1 = 3 & (1)\\
a_1 + 0b_1 + 0c_1 + 0d_1 = -4 & (2)\\
a_2 + 0b_2 + 0c_2 + 0d_2 = -4 & (3)\\
a_2 + b_2 + c_2 + d_2 = 5 & (4)\\
a_3 + b_3 + c_3 + d_3 = 5 & (5)\\
a_3 + 2b_3 + 4c_3 + 8d_3 = -6 & (6)\\
b_1 - b_2 = 0 & (7)\\
b_2 + 2c_2 + 3d_2 - b_3 - 2c_3 - 3d_3 = 0 & (8)\\
c_1 - c_2 = 0 & (9)\\
c_2 + 3d_2 - c_3 - 2d_3 = 0 & (10)\\
c_2 - 3d_2 = 0 & (11)\\
c_3 + 3d_3 = 0 & (12)
\end{cases}
$$

Write the system of equation in the matrix form that solve for $a_i, b_i, c_i, \& \ d_i, \quad i = 1, 2, 3.$
Matrix Form

$$
\begin{pmatrix}
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & -1 & -2 & -3\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & -1 & -2\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3
\end{pmatrix}
\begin{pmatrix} a_1\\ b_1\\ c_1\\ d_1\\ a_2\\ b_2\\ c_2\\ d_3\\ a_3\\ b_3\\ c_3\\ d_3 \end{pmatrix}
=
\begin{pmatrix} 3\\ -4\\ -4\\ 5\\ 5\\ -6\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}
$$

5. Write the system of equations $f(x) = \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ into the matrix form.

   Matrix form

$$
\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
=
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}
$$

6. Please check this link to solve exercise number (6)

   https://www.geeksforgeeks.org/graphical-solution-of-linear-programming-problems/