Modeling Change

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Course outline

What is Mathematical Modeling?

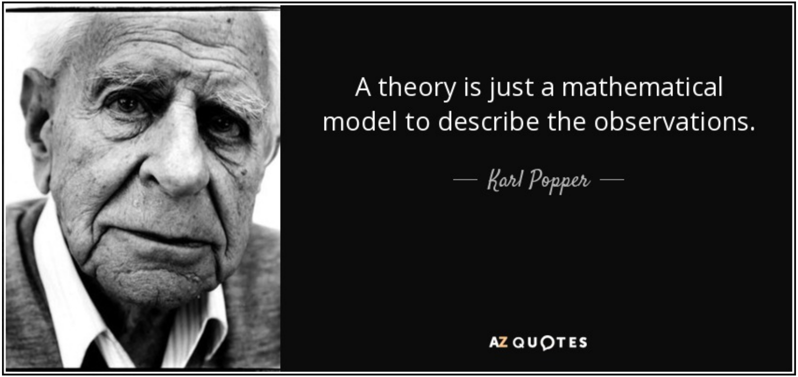
Definition of Mathematical Model

Mathematical model is an idealization of the real-world phenomenon using mathematical function or equation.

- Mathematical model is never a completely accurate representation of the real-world problem.

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain they do not refer to reality.
Albert Einstein

What is Mathematical Modeling?



What is Mathematical Modeling

Observed real-world behavior

- What is behind the behavior?
- How can we measure what is happening?
- What data can we collect?
- How can we convey that our reasoning is plausible?

Steps in Mathematical Modeling

4 Steps in Mathematical Modeling

The steps involved in mathematical modeling are

1. **Formulation** (stating the question, identifying factors and describing mathematically)
2. **Mathematical Manipulation** (calculations, solving an equations, proving a theorem, etc.)
3. **Interpretation**
4. **Validation**

Steps of the Modeling Process

Step 1: Formulation.

- **State the question.** If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- **Identify factors.** Decide which quantities influence the behavior. Determine relationships between the quantities.
- **Describe mathematically.** Assign each quantity a variable. Represent each relationship with an equation.

Steps of the Modeling Process

Motivating Eaxmple: Gravity by Galileo

In Galileo's time the question changed from:
Why do objects fall? (Philosophical question) to
How do objects fall? (How to describe a falling object's velocity?)
Step 1: Formulation.

- **State the question:** What formula describes how an object gains velocity as it falls?
- **Identify factors:** Galileo chose only distance, time, and velocity. Assumption: Velocity is proportional to the distance fallen.
- **Describe mathematically:** Assign variables to distance (x), time (t), and velocity (v). Relationships give equations: velocity and distance satisfy $v = \frac{dx}{dt}$. Proportionality implies $v = ax$ for some constant a.

Steps of the Modeling Process

After formulation, we have some variables and equations and now we have to do some sort of analysis of these equations in order to develop some sort of **mathematical conclusions.**

Motivating Eaxmple: Gravity by Galileo, Continue

Step 2: Mathematical Manipulation. This may entail one or more of:

- Calculations,
- Proving a theorem,
- Solving an equation,
- Other....

Since $v = \frac{dx}{dt}$ and $v = ax$, we set equal the two equations.
This gives the (differential) equation: $\frac{dx}{dt} = ax$.
Solving gives that $x(t) = ke^{at}$ for some constants a and k.
Something is not quite right...

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a "right answer"? Perhaps the most important, but least considered step of the modeling process is:

Motivating Eaxmple: Gravity by Galileo, Continue

Step 3: Evaluation. Translate the mathematical results back to the real-world situation and ask the following questions:

- Has the model explained the real-world observations?
- Are the answers we found accurate enough?
- Were our assumptions good assumptions?
- What are the strengths and weaknesses of our model?
- Did we make any mistakes in our mathematical manipulations?

  If there are any problems, we need to return to the formulation step and return through the modeling process.

Steps of the Modeling Process

Motivating Eaxmple: Gravity by Galileo, Continue

Step 3: Evaluation. Through our mathematical calculations, we have determined that the position of a falling object is given by the function $x(t) = ke^{at}$.

The real-world situation we are modeling is starting from rest at time zero. That is, $x(0) = 0$. This implies that
$0 = ke^{a0} = ke^{0} = k$, and therefore, $x(t) = 0$.
In words, this means that the object stays at rest for all t.
This is absurd; perhaps the proportionality assumption is incorrect.

Steps of the Modeling Process

Motivating Example: Gravity by Galileo, Continue

Step 1: Formulation.

Q. What formula describes how an object gains velocity as it falls?

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by $32\text{ft}/\text{sec}$.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Step 2: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$. Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

Step 3: Evaluation.

This function agrees well with observations in many instances. (Although not all!)

Steps of Modeling Process

Example: 1

Suppose you have a room of length 6 m and breadth 5 m. You want to cover the floor of the room with square mosaic tiles of side 30 cm. How many tiles will you need? Solve this by constructing a mathematical model.

Solution: Step 1: Formulation We have to consider the area of the room and the area of a tile for solving the problem. The side of the tile is 0.3 m. Since the length is 6 m, we can fit in $\frac{1}{0.3} = 20$ tiles along the length of the room in one row (see Fig. A2.1.).

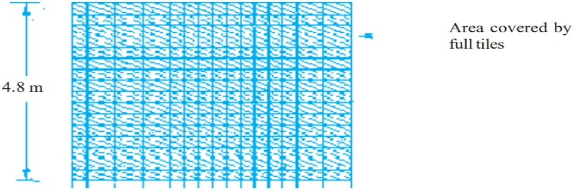


Fig. A2.1

Steps of Modeling Process

Since the breadth of the room is 5 metres, we have $\dfrac{5}{0.3} = 16.67$, we can fit in 16 tiles in a column.
Since $16 \times 0.3 = 4.8$, $5 - 4.8 = 0.2$ metres along the breadth will not be covered by tiles. This part will have to be covered by cutting the other tiles. The breadth of the floor left uncovered, 0.2 metres, is more than half the length of a tile, which is 0.3 m. So we cannot break a tile into two equal halves and use both the halves to cover the remaining portion.
Step 2: Mathematical Description. We have:
Total number of tiles required = (Number of tiles along the length $\times$ Number of tiles along the breadth) + Number of tiles along the uncovered area (1)

Steps of Modeling Process

Solution: As we said above, the number of tiles along the length is 20 and the number of tiles along the breadth is 16. We need 20 more tiles for the last row. Substituting these values in (1), we get
$(20 \times 16) + 20 = 320 + 20 = 340$
Step 3: Interpretation. We need 340 tiles to cover the floor .
Step 4: Vaidation. In real-life, your mason may ask you to buy some extra tiles to replace those that get damaged while cutting them to size. This number will of course depend upon the skill of your mason! But, we need not modify Equation (1) for this. This gives you a rough idea of the number of tiles required. So, we can stop here.

Modeling Change with Difference Equations

Definition of Propostionality

Two voriables y and x are **propostisual** (to each other) if one is always a constant multiple of the other, that is, if $\exists k \neq 0$ s.t. $y = kx$. We write $y \propto x$

Modeling change

$$\textbf{Future value} = \textbf{present value} + \textbf{change}$$

Then

$$\textbf{Change} = \textbf{futuer value - present value}$$

- If time is measured in discrete steps: **Diference Equation**
- If time is measured continuously: **Differential Equation**

Definition of difference change

For a sequence of numbers $A = \{a_0, a_1, a_2, a_3, \ldots\}$ the first differences are

$$\Delta a_0 = a_1 - a_0$$

$$\Delta a_1 = a_2 - a_1$$

$$\Delta a_2 = a_3 - a_2$$

$$\Delta a_3 = a_4 - a_3$$

For each positive integer n, the **nth first difference** is

$$\Delta a_n = a_{n+1} - a_n$$

- $Change = \Delta a_n = somefunction$
  or
- $Change = \Delta a_n = a_{n+1} - a_n = f($terms in the sequence, external terms$)$

Modeling change with Difference Equations

Example 1.

Consider the value of a savings certificate initially worth $\$1000$ that accumulates interest paid each month at 1% per month. The following sequence of numbers represents the value of the certificate month by month.

$$A = (1000, 1010, 1020.10, 1030.30, \ldots)$$

Example 2.

Example.2 Six years ago your parents purchased a home by financing $\$80,000$ for 20 years, paying monthly payments of $\$880.87$ with a monthly interest of 1%. They have made 72 payments and wish to know how much they owe on the mortgage.

Modeling change with Difference Equations

Definition

- A **sequence** is a function whose domain is the set of all nonnegative integers and whose range is a subset of the real numbers.
- A **dynamical system** is a relationship among terms in a sequence.
- A **numerical solution** is a table of values satisfying the dynamical system.

Approximating Change with Difference Equations

Example 1: Growth of a Yeast Culture

The data in Figure 1.7 were collected from an experiment measuring the growth of a yeast culture. The graph represents the assumption that the change in population is proportional to the current size of the population. That is, $\Delta p_n = (p_{n+1} - p_n) = kp_n$, where p_n represents the size of the population biomass after n hours, and k is a positive constant.

The value of k depends on the time measurement.

Time in hours n	Observed yeast biomass p_n	Change in biomass $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	

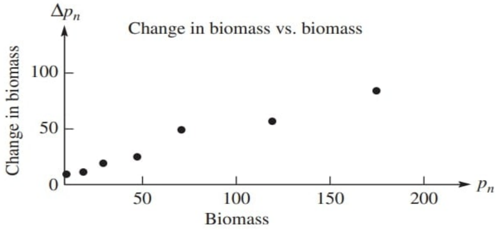


Figure 1.7

Approximating Change with Difference Equations

Example: Decay of Digoxin in the Bloodstream

Digoxin is used in the treatment of heart disease. Doctors must prescribe an amount of medicine that keeps the concentration of digoxin in the bloodstream above an **effective level** without exceeding a **safe level** (there is variation among patients). For an initial dosage of 0.5 mg in the bloodstream, Table 1.2 shows the amount of digoxin a_n remaining in the bloodstream of a particular patient after n days, together with the change Δa_n each day.

Table 1.2 The change a_n in digoxin in a patient's bloodstream

n	0	1	2	3	4	5	6	7	8
a_n	0.500	0.345	0.238	0.164	0.113	0.078	0.054	0.037	0.026
Δa_n	−0.155	−0.107	−0.074	−0.051	−0.035	−0.024	−0.017	−0.011	

Solutions to Dynamical System

The Method of Cojecture

The method of conjecture is a powerful mathematical technique to hypothesize the form of a solution to a dynamical system and then to accept or reject the hypothesis. It is basedon exploration and observation from which we attempt to discern a pattern to the solution.

The Method of Conjecture Process

1. Look for a pattern
2. Conjecture
3. Test The Conjecture
4. Conclusion

Solutions to Dynamical System

Example 1: A savings certificate Revisited

In the savings certificate example (Example 1, Section 1.1), a savings certificate initially worth $\$1000$ accumulated interest paid each month at 1% of the balance. No deposits or withdrawals occurred in the account, determining the dynamical system

$$\begin{aligned} a_{n+1} &= 1.01a_n \\ a_0 &= 1000 \end{aligned} \tag{4.1}$$

Theroem 1.

The solution of the linear dynamical system $a_{n+1} = ra_n$ for r any nonzero constant is

$$a_k = r^k a_0$$

where a_0 is a given initial value.

Solutions to Dynamical System

Example 2: Sewage Treatment

A sewage treatment plant processes raw sewage to produce usable fertilizer and clean water by removing all other contaminants. The process is such that each hour 12% of remaining contaminants in a processing tank are removed. What percentage of the sewage would remain after 1 day? How long would it take to lower the amount of sewage by half? How long until the level of sewage is down to 10% of the original level?

Solutions to Dynamical System

Definition of equilibrium

A number a is called an **equilibrium value** or **fixed point** of a dynamical system $a_{n+1} = f(a_n)$ if $a_k = a$ for all is a constant solution to the dynamical system.

Example 3: Prescription for Digoxin

digoxin is used in the treatment of heart patients. The objective of the problem is to consider the decay of digoxin in the bloodstream to prescribe a dosage that keeps the concentration between acceptable levels (so that it is both safe and effective). Suppose we prescribe a daily drug dosage of 0.1 mg and know that half the digoxin remains in the system at the end of each dosage period. This results in the dynamical system

$$a_{n+1} = 0.5a_n + 0.1$$

Solutions to Dynamical Systems

Now consider three starting values, or initial does:

$$A : a_0 = 0.1$$
$$B : a_0 .2$$
$$C : a_0 = 0.3$$

In Figure 1.18 we compute the numerical solutions for each case

	A	B	C
n	a_n	a_n	a_n
0	0.1	0.2	0.3
1	0.15	0.2	0.25
2	0.175	0.2	0.225
3	0.1875	0.2	0.2125
4	0.19375	0.2	0.20625
5	0.196875	0.2	0.203125
6	0.1984375	0.2	0.2015625
7	0.19921875	0.2	0.20078125
8	0.19960938	0.2	0.20039063
9	0.19980469	0.2	0.20019531
10	0.19990234	0.2	0.20009766
11	0.19995117	0.2	0.20004883
12	0.19997559	0.2	0.20002441
13	0.19998779	0.2	0.20001221
14	0.1999939	0.2	0.2000061
15	0.19999695	0.2	0.20000305

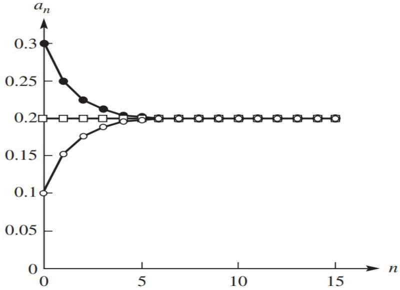


Figure 1.18

Three initial digoxin doses

Splutions to Dynamical Systems

Theorem 2

The equilibrium value for the dynamical system

$$a_{n+1} = ra_n + b, \ r \neq$$

is

$$a = \frac{1}{1-r}$$

If $r = 1$ and $b = 0$, every number is an equilibrium value. If $r = 1$ and $n \neq 0$, no equilibrium value exists.

Noted: Dynamical System $a_{n+1} = ra_n + b, b \neq 0$

Value of r	Long-term behavior observed
$\|r\| < 1$	Stable equilibrium
$\|r\| > 1$	Ubstable equilibrium
r=1	Graph is a line with no equilibrium

Solutions to Dynamical System

Theorem 3

The solution of the dynamical system $a_{n+1} = ra_n + b, r \neq 0$ is

$$a_k = r^k c + \frac{b}{1-r}$$

for some constant c (which depends on the initial condition).

Systems of Differnce Equations

Example 1: A car Rental Company

A car rental company has distributorships in Orlando and Tampa. The company specializes in catering to travel agents who want to arrange tourist activities in both cities. Consequently, a traveler will rent a car in one city and drop the car off in the second city. Travelers may begin their itinerary in either city. The company wants to determine how much to charge for this drop-off convenience. Because cars are dropped off in both cities, will a sufficient number of cars end up in each city to satisfy the demand for cars in that city? If not, how many cars must the company transport from Orlando to Tampa or from Tampa to Orlando? The answers to these questions will help the company figure out its expected costs. The historical records reveal that 60% of the cars rented in Orlando are returned to Orlando, whereas 40% end up in Tampa. Of the cars rented from the Tampa office, 70% are returned to Tampa, whereas 30% end up in Orlando. Figure 1.22 is helpful in summarizing the situation.

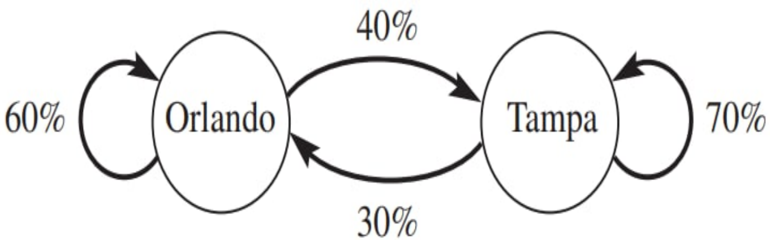


Figure 1.22 Car rental office in Oriando and Tampa
Hint: Find the Dynamical System Model and Find Equilibrium Values