Date: 24-12-2022
Duration: 120 minutes Name:__________________Signature:_______________

AMS-I3
Midterm Exam
Mathematical Modeling

Problem 1 (40 pts).

The **Ricker model** (Ricker 1954) is a population dynamic model that is often used in fishery management. It can be expressed by the following recursion equation:

$$N_{t+1} = N_t e^{r\left(1 - \frac{N_t}{K}\right)} \qquad (1)$$

where:

N_t is the number of fish that will be present in a fishery at time t;

$r > 0$ is interpreted as an intrinsic growth rate; and

$K > 0$ as the carrying capacity of the environment.

Question.

(a) If $N_{t+1} = f\left(N_t\right)$, determine the function $f(x)$.

(b) Find $\lim\limits_{x \to +\infty} f(x)$.

(c) Calculate $f'(x)$.

(d) Using the result of part **(c)**, for what values of x is f increasing? decreasing? At what value of x does f have an absolute maximum?

(e) Calculate $f''(x)$.

(f) Using the result in part **(e)**, for what values of x is f concave upward? concave downward? At what value of x does f have an inflection point?

(g) Draw the graph of $f(x)$.

(h) Determine equilibrium point of the dynamical system **(1)** above.

(i) According to the value of r and K, tell when the equilibrium point found in part **(h)** is stable and unstable.

(j) When $K = 1{,}000$ and $r = 1.5$, and initial population is 200, find the population for the next three years. Round the population each year to the nearest integer.

(k) When $K = 1{,}000$ and $r = 1.5$, and initial population is 200, determine equilibrium point of the dynamical system **(1)**. Is the equilibrium stable or unstable?

> **STABILITY OF EQUILIBRIUM POINTS**
> Suppose a function $y = f(x)$ has an equilibrium point at $x = a$. Then the equilibrium point is stable if $|f'(a)| < 1$ and unstable if $|f'(a)| > 1$.

Note: Calculator is allowed.

Date: 24-12-2022

Duration: 120 minutes Name:__________________Signature:_______________

Problem 2 (60 pts). ***In any calculation, round your answer to 3 decimal places*.**
In a research laboratory, a researcher wanted to study how a particular type of bacteria evolves by growing them in the petri dish. The daily recorded number of living bacteria (in million) for the first 5 days is given in **Table 1** below.

Number of days (n)	0	1	2	3	4	5
Number of bacteria X_n ($\times 10^6$)	0.30	0.36	0.39	0.41	0.42	0.42

Question.

(a) Draw the scatterplot of the above data, using *x*-axis for Number of days (n) and *y*-axis for Number of living bacteria (X_n). Does the graph suggest any trend? Provide your comment.

(b) If $X_{n+1} = f\left(X_n\right)$, where f is some function to be estimated. Fill in the blank of following table (do it in your paper).

Number of days, n	0	1	2	3	4	5
Number of bacteria, X_n ($\times 10^6$)	0.30	0.36	0.39	0.41	0.42	0.42
$X_{n+1} = f\left(X_n\right)$						

(c) Draw the scatterplot of X_n and X_{n+1}, using *x*-axis for X_n and *y*-axis for X_{n+1}. Does the graph suggest any trend? Provide your comment.

(d) Suppose a linear model: $f(x) = b_0 + b_1 x$ is proposed. Using the available data in part (b), estimate the coefficient b_0 and b_1 by means least squares method. What is the value of the minimized least squares?

(e) What is the value of estimated equilibrium point of the model in part **(d)**? Describe in word the long-run behavior of the number of living bacteria in this case.

(f) In real situation, the ***logistic model*** of the form: $f(x) = a_1 x + a_2 x^2$ is usually proposed. Using the available data in part **(b)**, estimate the coefficient a_1 and a_2 by means of least squares method. What is the value of the minimized least squares?

(g) What is the value of estimated equilibrium point of the model in part **(f)**? Describe in word the long-run behavior of the number of living bacteria in this case.

(h) Are the values of minimized least squares of the model in part **(d)** and part **(f)**, respectively, comparable? Justify your answer.

(i) If the model in part **(f)** is written in the form: $f(x) = rx(M - x)$. Determine the value r and M.

(j) Suppose now the researcher is interested in experimental model to fit the data in part **(b)**. He proposes a polynomial of 4th degree: $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$. To find this polynomial, he uses the Lagrange's method by writing $f(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x) + f(x_3)\,L_3(x) + f(x_4)\,L_4(x)$, where L_0, L_1, L_2, L_3, L_4 are Lagrangian polynomials corresponding to x_0, x_1, x_2, x_3, x_4, respectively.

1. Determine L_0, L_1, L_2, L_3, L_4.
2. Determine the values of $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$.

(k) Estimate the value of X_5 using models in part **(d)**, in part **(f)** and in part **(j)**, respectively. Which model provide the closest estimate to X_5? Note that the exact value of X_5 is 0.42 (See Table 1).

Note: Calculator is allowed.