Modeling Change

Assignment
GROUP 6

October 23, 2022

1. With the price of gas continuing to rise, you wish to look at cars that get better gas mileage. You narrow down your choices to the following 2012 models: Ford Fiesta, Ford Focus, ChevyVolt, Chevy Cruz, Toyota Camry, Toyota Camry Hybrid, Toyota Prius and Toyota CorollaEach company has offered you their “best deal” as listed in the following table. You are able to allocate approximately \$500 for a car payment each month for up to 60 months, although less time would be preferable. Use dynamic systems to determine which new car you can afford.

2012 Model	Best Deal Price	Cash Down	Interest and Duration
Ford Fiesta	\$14,200	\$500	4.5% APR for 60 months
Ford Focus	\$20,705	\$750	4.38% APR for 60 months
Chevy Volt	\$39,312	\$1,000	3.28% APR for 48 months
Chevy Cruz	\$16,800	\$500	4.4% APR for 60 months
Toyota Camry	\$22,955	0	4.8%APR for 60 months
Toyota Camry Hybrid	\$26,500	0	3% APR for 48 months
Toyota Corolla	\$16,500	\$900	4.25%for 60 months
Toyota Prius	\$19,950	1,000	44.3%for 60 months

Solution.

To find the best model of a new car that can be afforded, we consider about less time would be preferable and we can pay each month. We have 8 models of cars and each model has the Best Price, Cash Down, Interest, and Duration.

$$\begin{aligned}
\text{Let} \qquad & a_n \text{ be the amount after n months} \\
\Rightarrow \qquad & a_0 = \text{Deal Price - Cash Down} \\
& r \text{ is the Interest per month} \\
& P \text{ represents the monthly payment} \\
\Rightarrow \qquad & P \leq \$500
\end{aligned}$$

The dynamical system:

$$\begin{aligned}
a_1 &= (1+r)a_0 - p \\
a_2 &= (1+r)^2 a_0 - (1+r)p - p \\
a_3 &= (1+r)^3 a_0 - (1+r)^2 - (1+r)p - p \\
&\vdots \\
a_n &= (1+r)^n a_0 - p\left(\frac{1\cdot(1+r)^n - 1}{(1+r)-1}\right)
\end{aligned}$$

Therefore we get:

$$P = \frac{r\left[(1+r)^n a_0 - a_n\right]}{(1+r)^n - 1}$$

For the Ford Fiesta model:

- The initial price

$$a_0 = 14,200 - 500 = \$13,700$$

- The amount after 60^{th} month should be 0
- The Interest per month

$$I = \frac{0.045}{12} = 0.00375\%$$

$$\Rightarrow P = \frac{0.00375\left[(1+0.00375)^{60}13,700 - 0\right]}{(1+0.00375)^{60}-1} = \$255.41$$

This model may we can afford by the price of this car. After applying the Dynamical System to another model, then

Model	Monthly payment(P)	Affordable
Ford Fiesta	\$255.41	affordablle
Ford Focus	\$370.93	affordablle
Chevy Volt	\$852.76	unaffordablle
Chevy Cruz	\$303.14	affordablle
Toyota Camry	\$431.09	affordablle
Toyota Camry Hybrid	\$586.56	unaffordablle
Toyota Corolla	\$289.06	affordablle
Toyota Pruis	\$370.12	affordablle

Based on the data above the Ford Fiesta is the best choice because we can pay less than other models in the least amount of time.

2. You are considering a 30-year mortgage that charges 0.4% interest each month to pay off a \$250,000 mortgage.

   a. Determine the monthly payment p that allows the loan to be paid off at 360 months.

   b. Now assume that you have been paying the mortgage for 8 years and now have an opportunity to refinance the loan. You have a choice between a 20-year loan at 4% per year with interest charged monthly and a 15-year loan at 3.8% per year with interest charged monthly. Each of the loans charges a closing cost of \$2500. Determine the monthly payment p for both the 20-year and 15-year loans. Do you think refinancing is the right thing to do? If so, do you prefer the 20-year or the 15-year option?

Solution.

(a) Determine the monthly payment p that allows the loan to be paid off at 360 months.
Let a_n denote the Standing Capital at the end of n^{th} month. The change (decrement) in the amount owed at the end of each period (each month) increases by the amount of interest and decreases by the amount of monthly payment.

$$\Delta a_n = a_{n+1} - a_n = ra_n - p$$

Here, The monthly rate of interest is $r = 0.4\%$
Initial amount $a_0 = \$250,000$
p denotes the monthly payment that allows the loan to be paid off.
Rearrange the equation:

$$\begin{aligned} a_{n+1} &= a_n + ra_n - p \\ &= (1+r)a_n - p \end{aligned}$$

The loan is to be paid off at 360 in months. So the Standing capital at the end of 360^{th} month or a_{360} should be 0
To obtain a formula for a_n to equate a_{360} to 0
Put $n = 0$, above expression gives

$$\begin{aligned} a_1 &= (1+r)a_0 - p \\ a_2 &= (1+r)^2 a_0 - (1+r)p - p \\ a_3 &= (1+r)^3 a_0 - (1+r)^2 - (1+r)p - p \\ &\vdots \\ a_n &= (1+r)^n a_0 - p\left(\frac{1 \cdot (1+r)^n - 1}{(1+r) - 1}\right) \end{aligned}$$

Substitute the values of a_o and r, above expression, is equivalent to

$$(1+0.004)^{360} \cdot 250000 - p\left(\frac{(1+0.004)^{360} - 1}{0.004}\right) = 0$$

Solve this expression for the value of p

$$p = \$1,311.663386$$

(b) Determine the monthly payment p for both the 20-year and 15-year loans.
The mortgage for 8 years has been paid, so the amount is

$$a_{96} = \$213,611.6374$$

We do the same as a question (a), and we get:

$$p = \frac{r\left[(1+r)^n a_0 - a_n\right]}{(1+r)^n - 1}$$

- Here, the monthly rate of interest is

$$r = 0.3333\%$$

  Total amount to refinance

$$a_0 = 2500 + 213,611.6374 = \$216,111.6374$$

  The loan is to be paid off at 20 years or 240 months. So the standing capital at the end of 240^{th} month or a_{240} should be 0. Substitute the values of a_0 and r, the above expression, is equivalent to

$$p = \frac{0.003333\left[(1+0.003333)^{240} \cdot 216,111.6374 - 0\right]}{(1+0.0033)^{240} - 1} = \$1,309.594012$$

- For the monthly payment p for the 15-year loan and charges closing cost of \$2500 $\Rightarrow a_0 = 2500 + 213,611.6374 = \$216,111.6374$ with the monthly rate of interest is $r = 0.003166\%$, the p given by:

$$p = \frac{0.003166\left[(1+0.003166)^{180} \cdot 216,111.6374 - 0\right]}{(1+0.003166)^{180} - 1} = \$1,576.978528$$

Based on the monthly payment between the 20-year and the 15-year refinance, we can make a conclusion that the refinance is the right thing and we prefer to refinance for 15 years with the rate interest of 3.8% because the total amount we have to pay less than the 20-year refinance.

3. Consider the spreading of a highly communicable disease on an isolated island with a population size of N. A portion of the population travels abroad and returns to the island infected with the disease. Formulate a dynamic system to approximate the change in the number of people in the population who have the disease.

   Solution.

   Formulate a dynamic system to approximate the change in the number of people.

   Let i_n be the person who has been infected with the disease after n time periods. The population size N, then $(N - i_n)$ represents those susceptible but not yet infected. If those infected remain contagious, we can model the change of those infected as a proportionality to the product of those infected by those susceptible but not yet infected, or

$$\Delta i_n = i_{n+1} - i_n = k i_n (N - i_n)$$

In this model the product $i_n(N - i_n)$ represents the number of possible interactions between those infected and those not infected at time n. A fraction k of these interactions would cause additional infection, represented by Δi_n

4. The data in the accompanying table show the speed n (in increments of 5mph) of an automobile and the associated distance in feet required to stop it once the brakes are applied.For instance, $n = 6$ (representing $6 \times 5 = 30$ mph) requires a stopping distance of $a_6 = 47$ ft.

   a. Calculate and plot the change Δa_n versus n. Does the graph reasonably approximate a linear relationship?

   b. Based on your conclusions in part (a), find a different equation model for the stopping distance data. Test your model by plotting the errors in the predicted values against n. Discuss the appropriateness of the model.

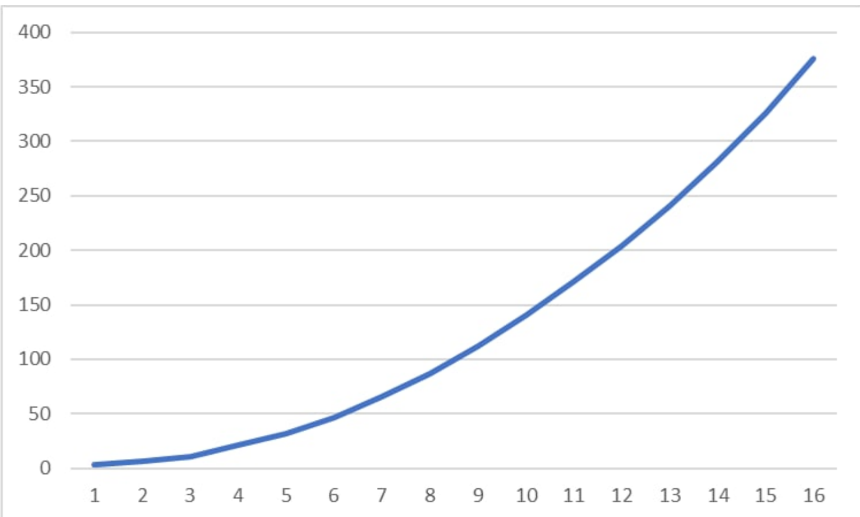
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	3	6	11	21	32	47	65	87	112	140	171	204	241	282	325	376

Solution.

(a) Let a_n be the distance in the feed of an automobile. Let us plot the change Δa_n versus n

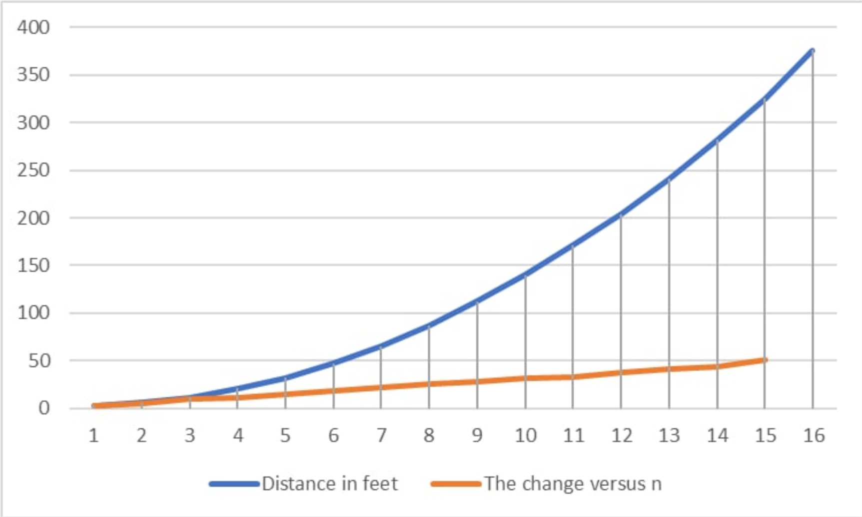
The given data is as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	3	6	11	21	32	47	65	87	112	140	171	204	241	282	325	376
Δa_n	3	5	10	11	15	18	22	25	28	31	33	37	41	43	51	



Hence, observe that, from the above graph, the speed and change in the distance are almost linear, to

(b) Plot the graph of the above table as follows:



to find the differential equation model for stopping distance data, note that the differential equation is

$$\Delta a_n = k a_n$$

So, the slope

$$k = \frac{\Delta a_n}{a_n}$$

Obtain the following:

$$\Delta a_n = a_n$$

Further, as

$$a_{n+1} - a_n = a_n$$

Hence, the difference equation is

$$a_{n+1} = 2a_n$$