Numerical Analysis

Boundary-Value Problems for Ordinary Differential Equations

OL Say

ol.say@itc.edu.kh

Institute of Technology of Cambodia

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Outline

Consider the ODE $y'' = p(t)y' + q(t)y + r(t)$ on the domain $a \leq t \leq b$.

1. First type, or Dirichlet, boundary conditions specify fixed values of y at the boundaries: $y(a) = \alpha$ and $y(b) = \beta$.
2. Second type, or Neumann, boundary conditions specify values of the derivative at the boundaries: $y'(a) = \alpha$ and $y'(b) = \beta$.
3. Third type, or Robin, boundary conditions specify a linear combination of the function value and its derivative at the boundaries: $c_1 y(a) + c_2 y'(a) = \alpha$ and $c_1 y(b) + c_2 y'(b) = \beta$ for some constants c_1, c_2, α, β.
4. Mixed boundary conditions, which combine any of these three at the different boundaries. For example, we could have $y(a) = \alpha$ and $y'(b) = \beta$.

Theorem 1

Suppose the function f in the boundary-value problem

$$y'' = f(t, y, y'), \text{ for } a \leq t \leq b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta,$$

is continuous on the set

$$D = \{(t, y, y') \mid \text{for } a \leq t \leq b, \text{ with } y, y' \in \mathbb{R}\},$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

1. $f_y(t, y, y') > 0$, *for all* $(t, y, y') \in D$, *and*
2. *a constant M exists, with*

$$|f_{y'}(t, y, y')| \leq M, \text{ for all } (t, y, y') \in D,$$

then the boundary-value problem has a unique solution.

Corollary 2

Suppose the linear boundary-value problem

$$(*):\ y'' = p(t)y' + q(t)y + r(t),\ \textit{for } a \leq t \leq b,$$

with $y(a) = \alpha$ *and* $y(b) = \beta$*, satifies*

1. $p(t), q(t)$, *and* $r(t)$ *are continuous on* $[a, b]$,
2. $q(t) > 0$ *on* $[a, b]$.

Then the boundary-value problem has a unique solution.

1. To approximate the unique solution to this linear problem, we first consider the initial value problems

$$(i):\ y'' = p(t)y' + q(t)y + r(t),\ a \le t \le b,\ y(a) = \alpha,\ y'(a) = 0,$$
$$(ii):\ y'' = p(t)y' + q(t)y,\ a \le t \le b,\ y(a) = 0,\ y'(a) = 1.$$

2. Let $y_1(t)$ and $y_2(t)$ be the solutions to (i) and (ii) respectively and assume that $y_2(b) \neq 0$.
3. Let $(0):\ y(t) = y_1(t) + \dfrac{\beta - y_1(b)}{y_2(b)} y_2(t).$
4. Then $y(t)$ is the solution to the boundary problem $(*)$.

5. To verify this, first note that

$$(1):\ y_1'' = p(t)y_1' + q(t)y_1 + r(t),\ a \le t \le b,\ y_1(a) = \alpha,\ y_1'(a) = 0,$$

$$(2):\ y_2'' = p(t)y_2' + q(t)y_2,\ a \le t \le b,\ y_2(a) = 0,\ y_2'(a) = 1,$$

$$(3):\ y' = y_1' + \frac{\beta - y_1(b)}{y_2(b)}y_2'$$

$$(4):\ y'' = y_1'' + \frac{\beta - y_1(b)}{y_2(b)}y_2''$$

6. From $(1), (2)$ and (4), we get

$$\begin{aligned} y'' &= [p(t)y_1' + q(t)y_1 + r(t)] + \frac{\beta - y_1(b)}{y_2(b)}[p(t)y_2' + q(t)y_2] \\ &= p(t)\left[y_1' + \frac{\beta - y_1(b)}{y_2(b)}y_2'\right] + q(t)\left[y_1 + \frac{\beta - y_1(b)}{y_2(b)}y_2\right] + r(t) \end{aligned}$$

7. From (0) and (3), we derive $y'' = p(t)y' + q(t)y + r(t)$.
8. In addition,

$$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} y_2(a) = \alpha$$

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} y_2(b) = \beta$$

9. Thus, the procedure for solving $(*)$ is as the following:
   a. Solve (i) and (ii) by any desire method.
   b. Get $y_1(b)$ and $y_2(b)$ and then compute

   $y(t) = y_1(t) + \dfrac{\beta - y_1(b)}{y_2(b)} y_2(t)$ for $a \leq t \leq b$.

Example 3

Apply the Linear Shooting technique and the Runge-Kutta method of order 4 with $N = 10$ to the boundary-value problem

$$y'' = -\frac{2}{t}y' + \frac{2}{t^2}y + \frac{\sin(\ln t)}{t^2}, \text{ for } 1 \le t \le 2, \text{ with } y(1) = 1, \; y(2) = 2;$$

actual solution

$$y(t) = c_1 t + c_2/t^2 - (3/10)\sin(\ln t) - (1/10)\cos(\ln t)$$
$$c_2 = (8 - 12\sin(\ln 2) - 4\cos(\ln 2))/70, \; c_1 = 11/10 - c_2$$

1. For $1 \le t \le 2$, apply Runge-Kutta method of order to

$$y'' = -\frac{2}{t}y' + \frac{2}{t^2}y + \frac{\sin(\ln t)}{t^2}, \text{ with } y(1) = 1, \; y'(1) = \boxed{0};$$
$$y'' = -\frac{2}{t}y' + \frac{2}{t^2}y, \text{ with } y(1) = \boxed{0}, \; y'(1) = \boxed{1}.$$

2. For $i = 0, 1, \ldots, 10$, compute $w_i = w_{1,i} + \dfrac{2 - w_{1,10}}{w_{2,10}} w_{2,i}$.

i	t	w_1	w_2	w	y	$\lvert y - w \rvert$
0	1.0	1.00000000	0.00000000	1.00000000	1.00000000	0.00000000
1	1.1	1.00896058	0.09117986	1.09262916	1.09262930	0.00000013
2	1.2	1.03245472	0.16851175	1.18708471	1.18708484	0.00000013
3	1.3	1.06674375	0.23608704	1.28338227	1.28338236	0.00000010
4	1.4	1.10928795	0.29659067	1.38144589	1.38144595	0.00000006
5	1.5	1.15830000	0.35184379	1.48115939	1.48115942	0.00000003
6	1.6	1.21248371	0.40311695	1.58239245	1.58239246	0.00000001
7	1.7	1.27087454	0.45131840	1.68501396	1.68501396	0.00000000
8	1.8	1.33273851	0.49711137	1.78889854	1.78889853	0.00000001
9	1.9	1.39750618	0.54098928	1.89392951	1.89392951	0.00000000
10	2.0	1.46472815	0.58332538	2.00000000	2.00000000	0.00000000

1. The shooting technique for the nonlinear second-order boundary-value problem

$$(1):\ y'' = f(t, y, y'),\ for\ a \le t \le b,\ \text{with } y(a) = \alpha \text{ and } y(b) = \beta,$$

is similar to the linear technique, except that the solution to a nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems.

2. Instead, we approximate the solution to the boundary-value problem by using the solutions to asequence of initial-value problems involving a parameter p_k,

$$(2):\ y'' = f(t, y, y'),\ \text{for } a \le t \le b,\ \text{with } y(a) = \alpha \text{ and } y'(a) = p_k$$

in a manner that $\lim\limits_{k\to\infty} y(b, p_k) = y(b) = \beta$, where $y(t, p_k)$ denotes the solution to the initial-value problem above.

3. We start with a parameter p_0 that determines the initial elevation at which the object is fired from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(t, y, y'), \text{ for } a \leq t \leq b, \text{ with } y(a) = \alpha \text{ and } y'(a) = p_0$$

4. If $y(b, p_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations p_1, p_2, and so on, until $y(b, p_k)$ is sufficiently close to "hitting" β.
5. If $y(t, p)$ denotes the solution to the initial-value problem above, we next determine p with $g(p) = y(b, p) - \beta = 0$.
6. To solve this equation for p, we use the Secant iterations

$$p_k = p_{k-1} - \frac{(y(b, p_{k-1}) - \beta)(p_{k-1} - p_{k-2})}{y(b, t_{k-1}) - y(b, t_{k-2})}$$

Example 4

Apply the Shooting method with Secant Method to the boundary-value problem

$$y'' = \frac{1}{8}(32 + 2t^3 - yy'), \text{ for } 1 \le t \le 3, \text{ with } y(1) = 17 \text{ and } y(3) = \frac{43}{3}.$$

Use $n = 20, m = 10$, and $TOL = 10^{-10}$, and compare the results with the exact solution $y(t) = t^2 + 16/t$.

i	t	w_1	w_2	y	$\lvert y - w_1 \rvert$
0	1.0	17.0000000000	−14.0001919171	17.0000000000	0.0000000000
1	1.1	15.7554961579	−11.0233384939	15.7554545455	0.0000416125
2	1.2	14.7733911821	−8.7112938360	14.7733333333	0.0000578487
3	1.3	13.9977543159	−6.8676174573	13.9976923077	0.0000620082
4	1.4	13.3886318130	−5.3634063645	13.3885714286	0.0000603844
5	1.5	12.9167227424	−4.1112334078	12.9166666667	0.0000560757
6	1.6	12.5600506483	−3.0501059819	12.5600000000	0.0000506483
7	1.7	12.3018096101	−2.1364241774	12.3017647059	0.0000449042
8	1.8	12.1289281414	−1.3383516559	12.1288888889	0.0000392525
9	1.9	12.0310865274	−0.6322027862	12.0310526316	0.0000338959
10	2.0	12.0000289268	−0.0000610175	12.0000000000	0.0000289268
11	2.1	12.0290719981	0.5718286994	12.0290476190	0.0000243790
12	2.2	12.1127475278	1.0941681519	12.1127272727	0.0000202550
13	2.3	12.2465382803	1.5753844678	12.2465217391	0.0000165412
14	2.4	12.4266798825	2.0221865514	12.4266666667	0.0000132158
15	2.5	12.6500102540	2.4399689581	12.6500000000	0.0000102540
16	2.6	12.9138537834	2.8331092018	12.9138461538	0.0000076296
17	2.7	13.2159312426	3.2051894624	13.2159259259	0.0000053167
18	2.8	13.5542890043	3.5591638899	13.5542857143	0.0000032900
19	2.9	13.9272429048	3.8974862436	13.9272413793	0.0000015255
20	3.0	14.3333333333	4.2222082621	14.3333333333	0.0000000000

The linear and nonlinear Shooting methods for boundary-value problems can present problems of instability. The methods in this section have better stability characteristics, but they generally require more computation to obtain a specified accuracy.

1. The finite difference method for the linear second-order boundary-value problem

   $$y'' = p(t)y' + q(t)y + r(t), \text{ for } a \leq t \leq b,$$

   requires that difference-quotient approximations be used to approximate both y' and y''.
2. First,we select an integer $n > 0$ and divide the interval $[a, b]$ into $(n+1)$ equal subintervals whose endpoints are the mesh points $t_i = a + ih$, for $i = 0, 1, \ldots, n+1$, where $h = (b-a)/(n+1)$.
3. At the interior mesh points, x_i, for $i = 1, 2, \ldots, n$, the differential equation to be approximated is

   $$y''(t_i) = p(t_i)y'(t_i) + q(t_i)y(t_i) + r(t_i).$$

4. The use of these centered-difference formulas

$$y'(x_i) = \frac{1}{2h}[y(t_{i+1}) - y(t_{i-1})] - \frac{h^2}{6}y'''(\eta_i),\ \eta_i \in (t_{i-1}, t_{i+1})$$

$$y''(t_i) = \frac{1}{h^2}[y(t_{i+1}) - 2y(t_i) + y(t_{i-1})] - \frac{h^2}{12}y^{(4)}(\xi_i),\ \xi_i \in (t_{i-1}, t_{i+1})$$

we get

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = p(t_i)\left[\frac{y(t_{i+1}) - y(t_{i-1})}{2h}\right] + q(t_i)y(t_i) + r(t_i)$$
$$- \frac{h^2}{12}[2p(t_i)y'''(\eta_i) - y^{(4)}(\xi_i)].$$

which is simplified to

$$\left[1 + \tfrac{h}{2}p(t_i)\right]y_{i-1} - \left[2 + h^2(t_i)\right]y_i + \left[1 - \tfrac{h}{2}p(t_i)\right]y_{i+1} = h^2r(t_i) + O(h^2).$$

5. With truncation error of order $O(h^2)$ and the Dirichlet boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ we derive

$$w_0 = \alpha, w_{n+1} = \beta$$
$$c_i w_{i-1} + d_i w_i + e_i w_{i+1} = b_i, \text{ for } i = 1, 2, \ldots, n,$$

where

$$c_i = 1 + \frac{h}{2}p(t_i), d_i = -2 - h^2(t_i), e_i = 1 - \frac{h}{2}p(t_i), b_i = h^2 r(t_i).$$

6. The system of equations is expressed as $AW = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & & (0) \\ c_1 & d_1 & e_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_n & d_n & e_n \\ (0) & & & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} \alpha \\ b_1 \\ \vdots \\ b_n \\ \beta \end{pmatrix}, \begin{cases} c_i = 1 + \dfrac{h}{2}p(t_i), \\ d_i = -2 - h^2 q(t_i), \\ e_i = 1 - \dfrac{h}{2}p(t_i), \\ b_i = h^2 r(t_i), \end{cases}$$

$$W = (w_0, w_1, \cdots, w_n, w_{n+1})^T.$$

7. with Neumann boundary conditions $y'(a) = \alpha$ and $y'(b) = \beta$,

$$\begin{cases} \dfrac{y(a+h) - y(a-h)}{2h} + O(h^2) = \alpha \\ \dfrac{y(b+h) - y(b-h)}{2h} + O(h^2) = \beta \end{cases} \Rightarrow \begin{cases} w_{-1} = w_1 - 2h\alpha \\ w_{n+2} = w_n + 2h\beta \end{cases}$$

and the equations

$$\begin{cases} c_0 w_{-1} + d_0 w_0 + e_0 w_1 = b_0 \\ c_{n+1} w_n + d_{n+1} w_{n+1} + e_{n+1} w_{n+2} = b_{n+1} \end{cases}$$

we obtain

$$\begin{cases} d_0 w_0 + (e_0 + c_0) w_1 = b_0 + c_0(2h\alpha) \\ (c_{n+1} + e_{n+1}) w_n + d_{n+1} w_{n+1} = b_{n+1} - e_{n+1}(2h\beta) \end{cases}$$

8. So the equations for the Neumann boundary conditions are

$$
\begin{aligned}
d_0 w_0 + (e_0 + c_0) w_1 &= b_0 + c_0(2h\alpha) \\
c_i w_{i-1} + d_i w_i + e_i w_{i+1} &= b_i, \text{ for } i = 1, 2, \dots, n \\
(c_{n+1} + e_{n+1}) w_n + d_{n+1} w_{n+1} &= b_{n+1} - e_{n+1}(2h\beta)
\end{aligned}
$$

9. In matrix form,

$$
\begin{pmatrix}
d_0 & e_0 + c_0 & 0 & & (0) \\
c_1 & d_1 & e_1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & c_n & d_n & e_n \\
(0) & & & c_{n+1} + e_{n+1} & d_{n+1}
\end{pmatrix}
\begin{pmatrix}
w_0 \\ w_1 \\ \vdots \\ w_n \\ w_{n+1}
\end{pmatrix}
=
\begin{pmatrix}
b_0 + c_0(2h\alpha) \\ b_1 \\ \vdots \\ b_n \\ b_{n+1} - e_{n+1}(2h\beta)
\end{pmatrix}
$$

10. We do in the same manner for Robin and mixed boundary conditions.

Theorem 5

Suppose that p, q, and r are continuous on $[a, b]$. If $q(x) \geq 0$ on $[a, b]$, then the tridiagonal linear system above has a unique solution provided that $h < 2/L$, where $L = \max\limits_{a \leq t \leq b} |p(x)|$.

Example 6

Use finite difference method with $N = 9$ to approximate the solution to the linear boundary-value problem

$$y'' = -\frac{2}{t}y' + \frac{2}{t^2}y + \frac{\sin(\ln t)}{t^2}, \text{ for } 1 \leq t \leq 2, \text{ with } y(1) = 1, \; y(2) = 2;$$

actual solution

$$y(t) = c_1 t + c_2/t^2 - (3/10)\sin(\ln t) - (1/10)\cos(\ln t)$$
$$c_2 = (8 - 12\sin(\ln 2) - 4\cos(\ln 2))/70, \; c_1 = 11/10 - c_2$$

1. The equation is linear with $p(t) = -\frac{2}{t}, q(t) = \frac{2}{t^2}, r(t) = \frac{\sin(\ln(t))}{t^2}$,
2. and Dirichlet boundary conditions $y(1) = 1, y(2) = 2$.

i	t_i	w_i	y_i	$\lvert y_i - w_i \rvert$
0	1.00	1.00000000	1.00000000	0.00000000
1	1.10	1.09260052	1.09262930	0.00002878
2	1.20	1.18704313	1.18708484	0.00004171
3	1.30	1.28333687	1.28338236	0.00004549
4	1.40	1.38140205	1.38144595	0.00004391
5	1.50	1.48112026	1.48115942	0.00003915
6	1.60	1.58235990	1.58239246	0.00003257
7	1.70	1.68498902	1.68501396	0.00002494
8	1.80	1.78888175	1.78889853	0.00001679
9	1.90	1.89392110	1.89392951	0.00000841
10	2.00	2.00000000	2.00000000	0.00000000

Example 7

Use finite difference method with $N = 9$ to approximate the solution to the linear boundary-value problem

$$y'' = -y + 1 + 2t, \text{ for } 0 \le t \le \pi/2, \text{ with } y'(0) = \pi \text{ and } y'(\pi/2) = \pi.$$

actual solution $y(t) = -1 + t^2 + \pi \sin(t)$.

i	t_i	w_i	y_i	$\|y_i - w_i\|$
0	0.00000000	-1.00510329	-1.00000000	0.00510329
1	0.15707963	-0.48688610	-0.48387262	0.00301348
2	0.31415927	0.06862734	0.06950156	0.00087423
3	0.47123890	0.64955671	0.64831932	0.00123739
4	0.62831853	1.24461218	1.24136601	0.00324617
5	0.78539816	1.84337299	1.83829174	0.00508125
6	0.94247780	2.43654459	2.42986624	0.00667834
7	1.09955743	3.01618791	3.00820609	0.00798182
8	1.25663706	3.57591532	3.56696887	0.00894645
9	1.41371669	4.11104821	4.10150933	0.00953888
10	1.57079633	4.61873241	4.60899375	0.00973865

Example 8

Use finite difference method with $N = 9$ to approximate the solution to the linear boundary-value problem

$$y'' = -4y + 4t, \text{ for } 0 \le t \le \pi/2, \text{ with } y(0) = 0 \text{ and } y'(\pi/2) = 0;$$

actual solution $y(t) = t + 0.5\sin(2t)$.

i	t_i	w_i	y_i	$\lvert y_i - w_i \rvert$
0	0.00000000	0.00000000	0.00000000	0.00000000
1	0.15707963	0.31417267	0.31158813	0.00258454
2	0.31415927	0.61284088	0.60805189	0.00478899
3	0.47123890	0.88203040	0.87574740	0.00628300
4	0.62831853	1.11067642	1.10384679	0.00682963
5	0.78539816	1.29171563	1.28539816	0.00631746
6	0.94247780	1.42278330	1.41800605	0.00477725
7	1.09955743	1.50644672	1.50406593	0.00238080
8	1.25663706	1.54995178	1.55052969	0.00057791
9	1.41371669	1.56450783	1.56822519	0.00371736
10	1.57079633	1.56418140	1.57079633	0.00661493

1. Our objective is to solve the simultaneous, nonlinear equations

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \Leftrightarrow \begin{cases} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ \qquad \vdots & \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{cases}$$

   for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ using Newton-Raphson method.

2. We start with the Taylor series expansion of $f_i(\mathbf{x})$ about $\mathbf{x}$:

$$f_i(\mathbf{x} + \mathbf{\Delta x}) = f_i(\mathbf{x}) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{x})\Delta x_j + O(\Delta x^2), \text{ for } i = 1, 2, \ldots, n.$$

3. Truncate the terms of order Δx^2, we get

$$\mathbf{f}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\boldsymbol{\Delta}\mathbf{x}$$

where $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix defined by

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{pmatrix}, \text{ and } \boldsymbol{\Delta}\mathbf{x} = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{pmatrix}.$$

4. Assume that $\mathbf{x}$ is the current approximation of the solution of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, and let $\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}$ be the improved solution.

5. To find the correction $\mathbf{\Delta x}$, we set $\mathbf{f}(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{0}$ and so

$$\mathbf{J}(\mathbf{x})\mathbf{\Delta x} \approx -\mathbf{f}(\mathbf{x}).$$

6. The analytical derivation of each $\dfrac{\partial f_i}{\partial x_j}(\mathbf{x})$ can be impractical for large value of n, we can approximate it by the finite difference

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}) \approx \frac{f_i(\mathbf{x} + \mathbf{e}_j h_j) - f_i(\mathbf{x})}{h_j}, \text{ for } i = 1, 2, \dots, n.$$

where $\mathbf{e}_j$ and h_j is the unit vector in the direction of x_j and h_j is a small increment.

7. Then Newton-Raphson method for simultaneous, nonlinear equations can be described as

$$\mathbf{J}(\mathbf{x}^{(k)})(\mathbf{\Delta x})^{(k)} = -\mathbf{f}(\mathbf{x}^{(k)})$$
$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + (\mathbf{\Delta x})^{(k)}$$

where the first equation can be solved by Gaussian eliminatin with scaled-row pivoting.

8. To illustrate the method, consider a system of nonlinear equations

$$\begin{cases} (x_1 - 1)^2 + (x_2 - 2)^3 = 4 \\ x_1^2(x_2 - 3)^3 + 9 = 0 \end{cases}, \mathbf{x}^{(0)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{h} = \begin{pmatrix} 10^{-4} \\ 10^{-4} \end{pmatrix}$$

9. Write the exact jacobian matrix

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} 2(x_1 - 1) & 3(x_2 - 2)^2 \\ 2x_1(x_2 - 3)^3 & 3x_1^2(x_2 - 3)^2 \end{pmatrix}$$

10. First iteration

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{pmatrix} -4.0000000000 & 3.0000000000 \\ 16.0000000000 & 12.0000000000 \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}^{(0)}) = \begin{pmatrix} -1.0000000000 \\ 1.0000000000 \end{pmatrix}$$

$$(\mathbf{\Delta x})^{(0)} = \begin{pmatrix} -0.1562500000 \\ 0.1250000000 \end{pmatrix}$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} -1.1562500000 \\ 1.1250000000 \end{pmatrix}$$

11. Second iteration

$$\mathbf{J}(\mathbf{x}^{(1)}) = \begin{pmatrix} -4.3125000000 & 2.2968750000 \\ 15.2435302734 & 14.1002655029 \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}^{(1)}) = \begin{pmatrix} -0.0205078125 \\ 0.1873340607 \end{pmatrix}$$

$$(\mathbf{\Delta x})^{(1)} = \begin{pmatrix} -0.0075083431 \\ -0.0051687258 \end{pmatrix}$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} -1.1637583431 \\ 1.1198312742 \end{pmatrix}$$

12 Third iteration

$$\mathbf{J}(\mathbf{x}^{(2)}) = \begin{pmatrix} -4.3275166861 & 2.3240909575 \\ 15.4697493321 & 14.3628464859 \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}^{(2)}) = \begin{pmatrix} -0.0000138917 \\ -0.0015249253 \end{pmatrix}$$

$$(\boldsymbol{\Delta}\mathbf{x})^{(2)} = \begin{pmatrix} 0.0000340902 \\ 0.0000694541 \end{pmatrix}$$

$$\mathbf{x}^{(3)} = \begin{pmatrix} -1.1637242529 \\ 1.1199007283 \end{pmatrix}$$

13. Fourth iteration

$$\mathbf{J}(\mathbf{x}^{(3)}) = \begin{pmatrix} -4.3274485057 & 2.3237241841 \\ 15.4675819136 & 14.3609439754 \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}^{(3)}) = \begin{pmatrix} -0.0000000116 \\ -0.0000001030 \end{pmatrix}$$

$$(\boldsymbol{\Delta}\mathbf{x})^{(3)} = \begin{pmatrix} 0.0000000007 \\ 0.0000000064 \end{pmatrix}$$

$$\mathbf{x}^{(4)} = \begin{pmatrix} -1.1637242521 \\ 1.1199007347 \end{pmatrix}$$

14. The approximated solutions with exact jacobian are

k	$x_1^{(k)}$	$x_2^{(k)}$
0	-1.0000000000	1.0000000000
1	-1.1562500000	1.1250000000
2	-1.1637583431	1.1198312742
3	-1.1637242529	1.1199007283
4	-1.1637242521	1.1199007347

15. The approximated solutions with approximated jacobian are

k	$x_1^{(k)}$	$x_2^{(k)}$
0	-1.0000000000	1.0000000000
1	-1.1562582034	1.1250067709
2	-1.1637584182	1.1198307345
3	-1.1637242533	1.1199007340
4	-1.1637242521	1.1199007347

1. The objective of this section is to approximate solution to

$$y'' = f(t, y, y'), \text{ for } a \le t \le b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta.$$

2. By the approximate centered-difference formula, for each $i = 1, 2, \dots, N$,

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = f\left(t_i, y(t_i), \frac{y(t_{i+1}) - y(t_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta_i)\right) + \frac{h^2}{12}y^{(4)}(\xi_i),$$

for some η_i and ξ_i in the interval (x_{i-1}, x_{i+1}).

3. Truncate the terms of order $O(h^2)$, we get

$$w_0 = \alpha, \ w_{N+1} = \beta,$$

and for $i = 1, 2, \dots, N$:

$$-\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + f\left(t_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0$$

4. We obtain $(N+2)\times(N+2)$ nonlinear, simultaneous equations,

$$\begin{cases} -w_0 + \alpha & = 0, \\ -w_0 + 2w_1 - w_2 + h^2 f\left(t_1, w_1, \dfrac{w_2 - w_0}{2h}\right) & = 0, \\ -w_1 + 2w_2 - w_3 + h^2 f\left(t_2, w_2, \dfrac{w_3 - w_1}{2h}\right) & = 0, \\ \vdots & \vdots \\ -w_{N-1} + 2w_N - w_{N+1} + h^2 f\left(t_N, w_N, \dfrac{w_{N+1} - w_{N-1}}{2h}\right) & = 0, \\ -w_{N+1} + \beta & = 0 \end{cases}$$

where $w_0, w_1, \ldots, w_{N+1}$ ar the unknowns and $\alpha, \beta, h, t_1, \ldots, t_N$ are constants.

5. Let $g_0(\mathbf{w}) = -w_0 + \alpha, g_{N+1}(\mathbf{w}) = -w_{N+1} + \beta$, and for $i = 1, 2, \ldots, N:$

$$g_i(w_0, \ldots, w_N) = -w_{i-1} + 2w_i - w_{i+1} + h^2 f\left(t_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right)$$

6. Then the jacobian can be written as

$$\mathbf{J}(\mathbf{w}) = \begin{pmatrix} d_0 & e_0 & & & (0) \\ c_1 & d_1 & e_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_N & d_N & e_N \\ (0) & & & c_{N+1} & d_{N+1} \end{pmatrix}$$

where $d_0 = -1, e_0 = 0, c_{N+1} = 0, d_{N+1} = -1$ and for $i = 1, \dots, N$:

$$c_i = \frac{\partial g_i}{\partial w_{i-1}}(\mathbf{w}) = -1 - \frac{h}{2}\frac{\partial f}{\partial y'}\left(t_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right),$$

$$d_i = \frac{\partial g_i}{\partial w_i}(\mathbf{w}) = 2 + h^2\frac{\partial f}{\partial y}\left(t_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right),$$

$$e_i = \frac{\partial g_i}{\partial w_{i+1}}(\mathbf{w}) = -1 + \frac{h}{2}\frac{\partial f}{\partial y'}\left(t_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right).$$

7. Then Newton-Raphson iteration becomes

$$\mathbf{J}(\mathbf{w}^{(k)})(\boldsymbol{\Delta}\mathbf{w})^{(k)} = -\mathbf{g}(\mathbf{w}^{(k)})$$
$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + (\boldsymbol{\Delta}\mathbf{w})^{(k)}$$

Example 9

Solve the boundary value problem

$$y'' = -3yy', \text{ for } 0 \le t \le 2, \text{ with } y(0) = 0, \text{ and } y(2) = 1.$$

Use $n = 9$.

1. We have $n = 9, h = (b - a)/(n + 1) = (1 - 0)/(9 + 1) = 0.1$;
2. the $(n + 2) = 11$ nodes of t are $0.0, 0.2, \ldots, 2.0$;
3. $f(t, y, y') = -3yy'$ then $f_y(t, y, y') = -3y'$ and $f_{y'}(t, y, y') = -3y$.

i	t	y
0	0.0	0.0000000000
1	0.2	0.3024044000
2	0.4	0.5545035096
3	0.6	0.7346912471
4	0.8	0.8497946415
5	1.0	0.9181319565
6	1.2	0.9569534841
7	1.4	0.9784566376
8	1.6	0.9902005480
9	1.8	0.9965651428
10	2.0	1.0000000000