TP3 Numerical Differentiations and Integrations

1 Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following tables.

a

x	0.5	0.6	0.7
$f(x)$	0.4794	0.5646	0.6442
$f'(x)$			

b

x	0.0	0.2	0.4
$f(x)$	0.00000	0.74140	1.3718
$f'(x)$			

2 Use the most accurate three-point formula to determine each missing entry in the following tables.

a

x	1.1	1.2	1.3	1.4
$f(x)$	9.025013	11.02318	13.46374	16.44465
$f'(x)$				

b

x	8.1	8.3	8.5	8.7
$f(x)$	16.94410	17.56492	18.19056	18.82091
$f'(x)$				

3 The following data can be used to approximate the integral $I^* = \int_0^{3\pi/2} \cos x \, dx$.

$$A_0(h) = 2.356194,\ A_0(h/2) = -0.4879837,$$
$$A_0(h/4) = -0.8815732,\ A_0(h/4) = -0.9709157.$$

Assuming $I^* = A_0(h) + a_0h^2 + a_1h^4 + a_2h^6 + a_3h^8 + O(h^{10})$, construct an extrapolation table todetermine $A_3(h)$.

4 Suppose that $A(h)$ is an approximation to A^* for every $h > 0$ and that

$$A^* = A(h) + a_0h^2 + a_1h^4 + a_2h^6 + \cdots,$$

for some constants $a_0, a_1, a_2, \dots$. Use the values $A(h), A(h/3)$, and $A(h/9)$ to produce an $O(h^6)$ approximation to A^*.

5 Approximate the following integrals using the Trapezoidal rule, Simpson's rule and Midpoint rule.

a $\int_1^{1.5} x^2 \ln x \, dx$

b $\int_0^{\pi/4} e^{3x} \sin 2x \, dx$

6 Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)\, dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

7 Let $h = (b-a)/3, x_0 = a, x_1 = a+h$, and $x_2 = b$. Find the degree of precision of the quadrature formula

$$\int_a^b f(x)\,dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

8 Given the function f at the following values,

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

approximate $\int_{1.8}^{2.6} f(x)\,dx$ using all appropriate quadrature formulas of this chapter.

9 Write a general purpose algorithm for Composite Trapezoidal rule.

- **a** Describe your algorithm objective?
- **b** Describe required input parameter(s)?
- **c** Describe output value(s)?
- **d** Write your algorithm body here.
- **e** Implement your developed algorithm with Python, or any other programming language.

10 Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

- **a** $\displaystyle\int_1^2 x\ln x\,dx, \quad n = 4$
- **b** $\displaystyle\int_1^3 \frac{x}{x^2+4}\,dx, \quad n = 8$

11 Write a general purpose algorithm for Composite Simpson's rule.

- **a** Describe your algorithm objective?
- **b** Describe required input parameter(s)?
- **c** Describe output value(s)?
- **d** Write your algorithm body here.
- **e** Implement your developed algorithm with Python, or any other programming language.

12 Use the Composite Simpson's rule with the indicated values of n to approximate the following integrals.

- **a** $\displaystyle\int_{-2}^2 x^3e^x\,dx, \quad n = 4$
- **b** $\displaystyle\int_1^3 x^2\cos x\,dx, \quad n = 6$

13 Write a general purpose algorithm for Composite Midpoint rule.

- **a** Describe your algorithm objective?
- **b** Describe required input parameter(s)?
- **c** Describe output value(s)?
- **d** Write your algorithm body here.
- **e** Implement your developed algorithm with Python, or any other programming language.

14 Use the Composite Midpoint rule with $n+2$ subintervals to approximate the following integrals.

- **a** $\displaystyle\int_0^2 \frac{2}{x^2+4}\,dx, \quad n=6$
- **b** $\displaystyle\int_0^2 e^{2x}\sin 3x\,dx, \quad n=8$

15 Approximate $\displaystyle\int_0^2 x^2e^{-x^2}\,dx$. Use

- **a** Composite Trapezoidal rule.
- **b** Composite Simpson's rule.
- **c** Composite Midpoint rule.

16 Determine the values of n and h required to approximate

$$\int_0^2 \frac{1}{x+4}\,dx$$

to within 10^{-5} and compute the approximation. Use

- **a** Composite Trapezoidal rule.
- **b** Composite Simpson's rule.
- **c** Composite Midpoint rule.

17 Let f be defined by

$$f(x)=\begin{cases} 1+x^3, & 0.0\le x\le 0.1\\ 1.001+0.03(x-0.1)+0.3(x-0.1)^2+2(x-0.1)^3, & 0.1\le x\le 0.2\\ 1.009+0.15(x-0.2)+0.9(x-0.2)^2+2(x-0.2)^3, & 0.2\le x\le 0.3.\end{cases}$$

- **a** Investigate the continuity of the derivatives of f.
- **b** Use the Composite Trapezoidal rule with $n=6$ to approximate $\displaystyle\int_0^{0.3} f(x)\,dx$, and estimate the error using the error bound.
- **c** Use the Composite Simpson's rule with $n=6$ to approximate $\displaystyle\int_0^{0.3} f(x)\,dx$. Are the results more accurate than in part (b)?

18 A car laps a race track in 84 seconds. The speed of the car at each 6-second interval is determined by using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table.

t	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
$v(t)$	124	134	148	156	147	133	121	109	99	85	78	89	104	116	123

How long is the track?

19 Write a general purpose algorithm for Romberg integration.

- **a** Describe your algorithm objective?
- **b** Describe required input parameter(s)?
- **c** Describe output value(s)?
- **d** Write your algorithm body here.
- **e** Implement your developed algorithm with Python, or any other programming language.

20 Use Romberg integration to compute $R_{3,3}$ for the following integrals.

- **a** $\displaystyle\int_{-1}^{1} (\cos x)^2 \, dx$
- **b** $\displaystyle\int_{e}^{2e} \frac{1}{x \ln x} \, dx$

21 Use Romberg integration to approximate the integral $\operatorname{erf}(1) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^1 e^{-x^2} \, dx$ to within 10^{-7}. Compute the Romberg table until either $|R_{n,n} - R_{n-1,n-1}| < 10^{-7}$, or $n = 12$.

22 Write a general purpose algorithm for Adaptive Simpson's quadrature.

- **a** Describe your algorithm objective?
- **b** Describe required input parameter(s)?
- **c** Describe output value(s)?
- **d** Write your algorithm body here.
- **e** Implement your developed algorithm with Python, or any other programming language.

23 Use Adaptive Simpson's quadrature to approximate the following integrals to within 10^{-5}.

- **a** $\displaystyle\int_{1}^{2} (x + \sin(4x)) \, dx$
- **b** $\displaystyle\int_{1}^{3} e^{2x} \sin(3x) \, dx$

24 Sketch the graphs of $\sin(1/x)$ and $\cos(1/x)$ on $[0.1, 2]$. Use adaptive quadrature to approximate the following integrals to within 10^{-3}.

a $\displaystyle\int_{0.1}^{2} \sin\frac{1}{x}\,dx$

b $\displaystyle\int_{0.1}^{2} \cos\frac{1}{x}\,dx$

25 The differential equation

$$mu''(t) + ku(t) = F_0\cos(\omega t)$$

describes a spring-mass system with mass m, spring constant k, and no applied damping. The term $F_0 cos(\omega t)$ describes a periodic external force applied to the system. The solution to the equation when the system is initially at rest $(u'(0) = u(0) = 0)$ is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos(\omega t) - \cos(\omega_0 t)), \quad \omega_0 = \sqrt{\frac{k}{m}} \neq \omega.$$

Sketch the graph of u when $m = 1, k = 9, F_0 = 1, \omega = 2$, and $t \in [0, 2\pi]$. Approximate $\displaystyle\int_0^{2\pi} u(t)\,dt$ to within 10^{-4}.

26 The period of a simple pendulum of length L is $\tau = 4\sqrt{L/g}\,h(\theta_0)$, where g is the gravitational acceleration, θ_0 represents the angular amplitude, and

$$h(\theta_0) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\theta}}$$

Compute $h(15^\circ)$, $h(30^\circ)$, and $h(45^\circ)$, and compare these values with $h(0) = \pi/2$ to within 10^{-4}.

27 Approximate $\displaystyle\int_a^b f(x)dx$ using Gaussian quadrature with $n = 1$ assuming that f is defined and continuous on the closed interval $[a, b]$.

28 Approximate the following integrals using Gaussian quadrature with $n = 2$, and compare your results to the exact values of the integrals.

a $\displaystyle\int_{-1}^{1} \cosh(x)\,dx$

b $\displaystyle\int_{-1}^{0} \frac{1}{1+x^2}\,dx$

c $\displaystyle\int_{0}^{1} \frac{x}{1+x^2}\,dx$

d $\displaystyle\int_{0}^{\ln 2} \frac{1}{1+\exp(x)}\,dx$

29 Write a general purpose algorithm for Gaussian quadrature that work for $n = 1, 2, 3, 4, 5$ with the weights and abscissa provided in the course.

a Describe your algorithm objective?

b Describe required input parameter(s)?

c Describe output value(s)?

d Write your algorithm body here.

e Implement your developed algorithm with Python, or any other programming language.

30 Composite Gaussian Quadrature routine to approximate $\int_{-1}^{1} x^2 e^x \, dx$ in the following manner.

a Use Gaussian Quadrature with $n = 4$ on the interval $[-1, 0]$ and $[0, 1]$.

b Use Gaussian Quadrature with $n = 2$ on the interval $[-1, -0.5], [-0.5, 0], [0, 0.5]$ and $[0.5, 1]$.

31 Legendre's polynomials of degree $n = 6, 7, 8, 9, 10$ are listed below

$$
\begin{aligned}
p_6(x) &= -\frac{5}{231} + \frac{5}{11}x^2 - \frac{15}{11}x^4 + x^6, \\
p_7(x) &= -\frac{35}{429}x + \frac{105}{143}x^3 - \frac{21}{13}x^5 + x^7, \\
p_8(x) &= \frac{7}{1287} - \frac{28}{143}x^2 + \frac{14}{13}x^4 - \frac{28}{15}x^6 + x^8, \\
p_9(x) &= \frac{63}{2431}x - \frac{84}{221}x^3 + \frac{126}{85}x^5 - \frac{36}{17}x^7 + x^9, \\
p_{10}(x) &= -\frac{63}{46189} + \frac{315}{4199}x^2 - \frac{210}{323}x^4 + \frac{630}{323}x^6 - \frac{45}{19}x^8 + x^{10},
\end{aligned}
$$

a Use Müller's Method and Horn's Method to find all roots of each of the polynomials with 10^{-10}.

b Use Adaptive Simpson's Quadrature to find the corresponding weights for use in the Gaussian Quadrature.

c Extend the program in exercise **29** to $n = 1, 2, \ldots, 10$.

d Use Gaussian Quadrature with $n = 8$ on the single interval $[-1, 1]$ to approximate $\int_{-1}^{1} x^2 e^x \, dx$.