Assignment Numerical Analysis 07 and 08

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Exercise 01:

Use the indicated method to approximate the solutions to the initial-value problems

$$f'(x) = y' = t^{-2}(sin2t - 2ty) \text{ ,}$$
$$1 \le t \le 2, y(1) = 2, h = 0.1 \text{ , } has\ actual\ solution\ y(t) = \frac{1}{2}t^{-2}(4 + cos2 - cos2t)$$

and compare the results to the actual values.

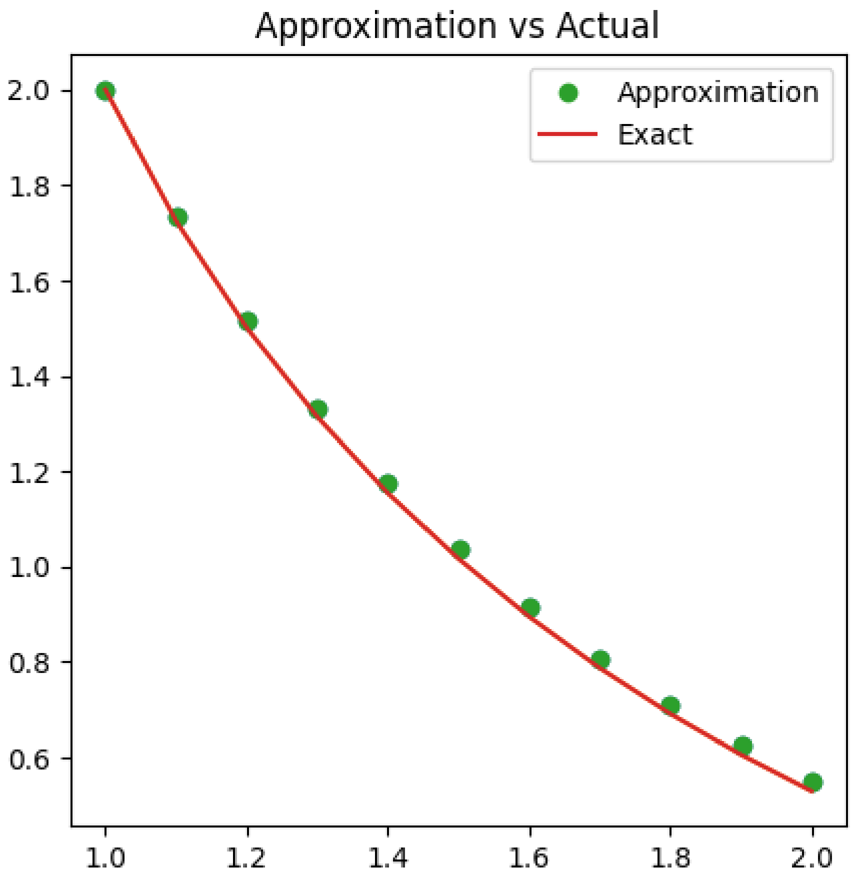
1. Runge-Kutta third-order method.
2. Heun’s third-order method.
3. Ralston’s third-order method.
4. Third-order Strong Stability Preserving Runge-Kutta.

1. Runge-Kutta third-order method

In [35]:

```
from collections.abc import Callable
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

def RungKutta3(f:Callable[[np.float64,np.float64],np.float64],
               t_span:np.ndarray,
               y_int:np.float64,
               n:np.int64)->pd.DataFrame:

    h = (t_span[1]-t_span[0])/n
    t = np.linspace(start=t_span[0],stop=t_span[1],num=n+1,dtype=np.float64)
    y = np.full_like(a=t,fill_value=np.nan,dtype=np.float64)
    y[0] = y_int

    for i in range(0,n,1):
        k1 = f(t[i],y[i])
        k2 = f(t[i] + 0.5*h , y[i] + 0.5*h*k1)
        k3 = f(t[i] + h , y[i] + h*k2)
        y[i+1] = y[i] + h*(k1 + 2.0*k2 + 2.0*k3 + k3)/6.0

    df = pd.DataFrame(data={'t':t,'y':y},dtype=np.float64)
    return df

if __name__ == '__main__':
    def f(t:np.float64,y:np.float64)->np.float64:
        return t**(-2) * (np.sin(2*t) - 2*t*y)
    t_span = np.array(object=[1,2],dtype=np.float64)
    y_init = 2
    n = 10
    df = RungKutta3(f=f,t_span=t_span,y_int=y_init,n=n)
    def y(t:np.float64)->np.float64:
        return 0.5*t**(-2) * (4+np.cos(2)-np.cos(2*t))
    df.loc[:,'exact'] = df.loc[:,'t'].apply(func=y)
    df.loc[:,'error'] = abs(df.loc[:,'y']-df.loc[:,'exact'])
    pd.options.display.float_format = '{:.10f}'.format
    print(df)

    fig = plt.figure(figsize=(5,5))
    ax = fig.add_subplot(1,1,1)
    ax.plot(df.loc[:,'t'],df.loc[:,'y'],'o')
    ax.plot(df.loc[:,'t'],df.loc[:,'exact'],'-')
    plt.title('Approximation vs Actual')
    ax.plot(df.loc[:, 't'], df.loc[:, 'y'], 'o', label='Approximation')
    ax.plot(df.loc[:, 't'], df.loc[:, 'exact'], '-', label='Exact')
    ax.legend()
    plt.show()
```

	t	y	exact	error
0	1.0000000000	2.0000000000	2.0000000000	0.0000000000
1	1.1000000000	1.7338435396	1.7241133391	0.0097302005
2	1.2000000000	1.5155183404	1.5004329441	0.0150853963
3	1.3000000000	1.3317766865	1.3138289695	0.0179477171
4	1.4000000000	1.1739821038	1.1546110980	0.0193710058
5	1.5000000000	1.0363727697	1.0164101467	0.0199626230
6	1.6000000000	0.9150304861	0.8949507694	0.0200797167
7	1.7000000000	0.8072445159	0.7873099232	0.0199345927
8	1.8000000000	0.7111058418	0.6914524043	0.0196534375
9	1.9000000000	0.6252410020	0.6059308692	0.0193101328
10	2.0000000000	0.5486333967	0.5296870980	0.0189462986

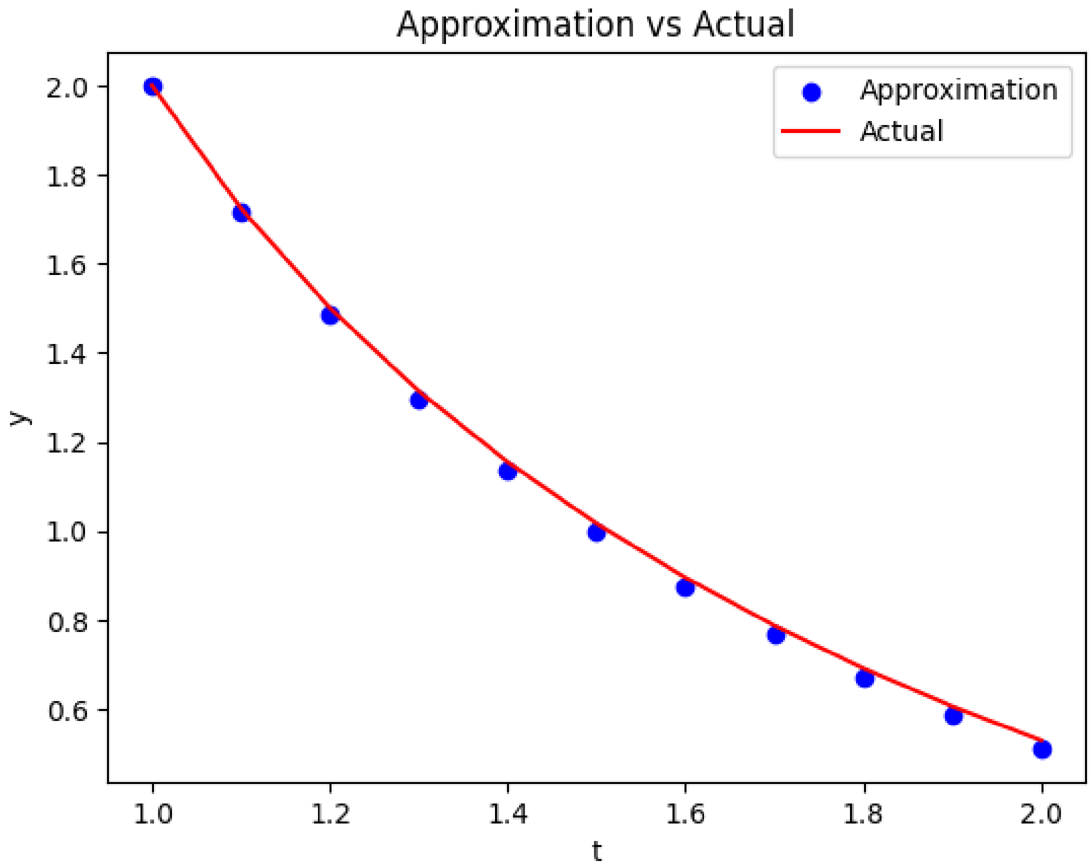


2. Heun's third-order method.

In [12]:

```
import numpy as np
def heuns_method(t0, y0, h, N):
    t_values = [t0]
    y_values = [y0]
    for i in range(1, N+1):
        t = t_values[i-1]
        y = y_values[i-1]
        k1 = h * f(t, y)
        k2 = h * f(t + h/3, y + k1/3)
        k3 = h * f(t + 2*h/3, y + 2*k2/3)
        t_next = t + h
        y_next = y + (k1 + 4*k2 + k3) / 6
        t_values.append(t_next)
        y_values.append(y_next)
    return t_values, y_values

if __name__ == '__main__':
    def f(t, y):
       return t**(-2) * (np.sin(2 * t) - 2 * t * y)
    # Initial conditions
    t0 = 1
    y0 = 2
    h = 0.1
    N = 10
    # Apply Heun's method
    t_values, y_values = heuns_method(t0, y0, h, N)

    # Compare with actual solution
    def actual_solution(t):
        return 1/(2 * t**2) * (4 + np.cos(2) - np.cos(2 * t))
    actual_values = [actual_solution(t) for t in t_values]
    print("\tt\t\tApproximation\t\t\tActual")
    for i in range(len(t_values)):
        print(f"{t_values[i]:.10e}\t{y_values[i]:.10e}\t{actual_values[i]:.10e}")

    plt.scatter(t_values, y_values, color='blue', label='Approximation')
    plt.plot(t_values, actual_values, color='red', label='Actual')
    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('Approximation vs Actual')
    plt.legend()
    plt.show()
```

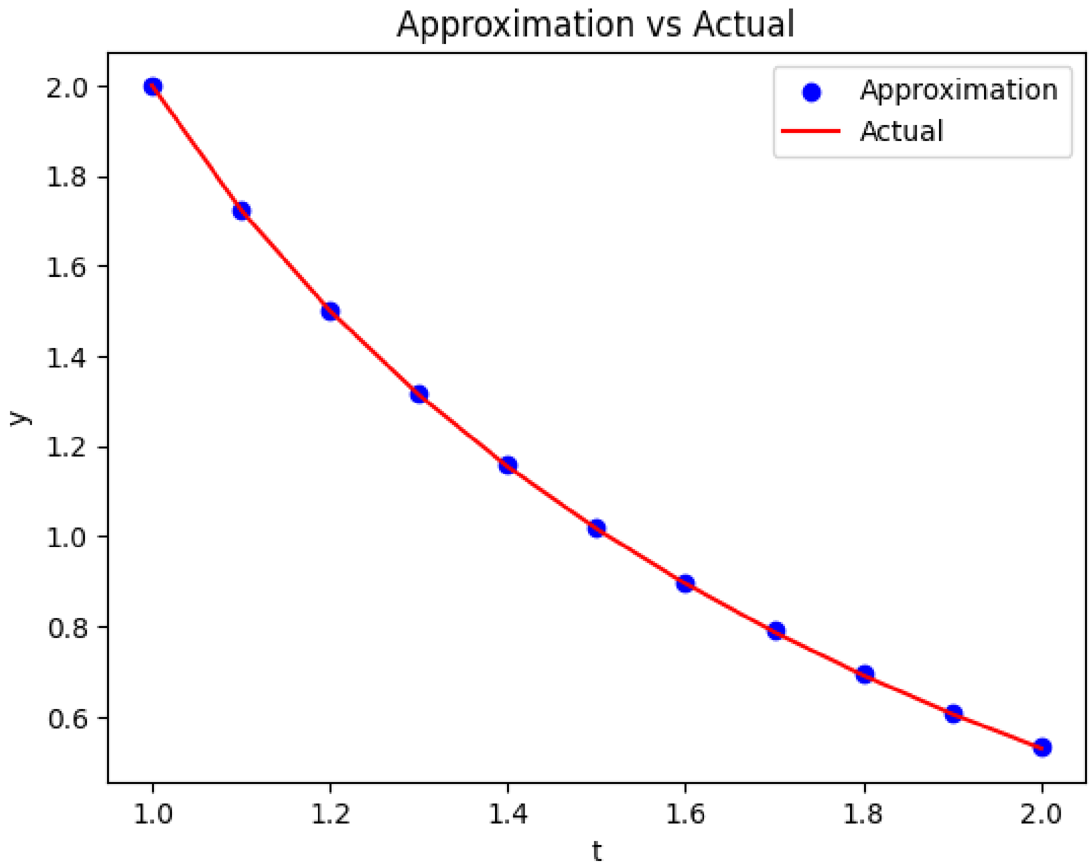
```
        t               Approximation                   Actual
1.0000000000e+00        2.0000000000e+00        2.0000000000e+00
1.1000000000e+00        1.7144562882e+00        1.7241133391e+00
1.2000000000e+00        1.4855980264e+00        1.5004329441e+00
1.3000000000e+00        1.2963182631e+00        1.3138289695e+00
1.4000000000e+00        1.1358413867e+00        1.1546110980e+00
1.5000000000e+00        9.9718345091e-01        1.0164101467e+00
1.6000000000e+00        8.7571321999e-01        8.9495076938e-01
1.7000000000e+00        7.6829921807e-01        7.8730992319e-01
1.8000000000e+00        6.7278423343e-01        6.9145240429e-01
1.9000000000e+00        5.8765073217e-01        6.0593086916e-01
2.0000000000e+00        5.1180189027e-01        5.2968709804e-01
```


Approximation vs Actual
Approximation
Actual
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
y
1.0
1.2
1.4
1.6
1.8
2.0
t

In [19]:

```python
def ralstons_third_order(f, t0, y0, h, num_steps):
    t_values = [t0]
    y_values = [y0]

    t = t0
    y = y0

    for _ in range(num_steps):
        k1 = f(t, y)
        y_temp = y + (3/4) * h * k1
        k2 = f(t + (2/3) * h, y_temp)

        y = y + (1/3) * h * (k1 + 2 * k2)
        t = t + h
        t_values.append(t)
        y_values.append(y)
    return t_values, y_values

if __name__ == '__main__':
    def f(t, y):
        return t**(-2) * (np.sin(2 * t) - 2 * t * y)
    # Initial conditions
    t0 = 1
    y0 = 2
    h = 0.1
    num_steps = 10

    t_values, y_values = ralstons_third_order(f, t0, y0, h, num_steps)
    def actual_solution(t):
        return 1/(2 * t**2) * (4 + np.cos(2) - np.cos(2 * t))
    actual_values = [actual_solution(t) for t in t_values]

    print("\tt\tApproximation\tActual")
    for i in range(len(t_values)):
        print(f"{t_values[i]:.10f}\t{y_values[i]:.10f}\t{actual_values[i]:.10f}")

    plt.scatter(t_values, y_values, color='blue', label='Approximation')
    plt.plot(t_values, actual_values, color='red', label='Actual')
    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('Approximation vs Actual')
    plt.legend()
    plt.show()
```

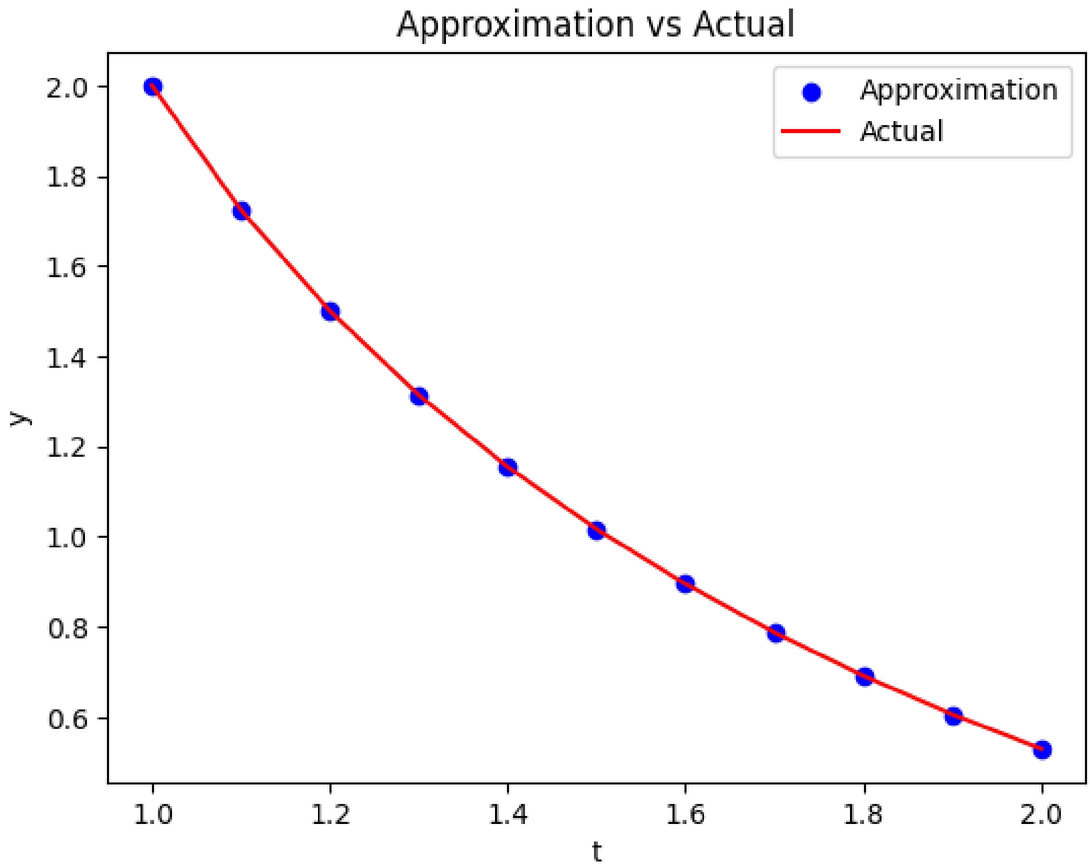
```
        t       Approximation   Actual
1.0000000000    2.0000000000    2.0000000000
1.1000000000    1.7255166497    1.7241133391
1.2000000000    1.5025909811    1.5004329441
1.3000000000    1.3163937292    1.3138289695
1.4000000000    1.1573827195    1.1546110980
1.5000000000    1.0192653830    1.0164101467
1.6000000000    0.8978079685    0.8949507694
1.7000000000    0.7901113287    0.7873099232
1.8000000000    0.6941550085    0.6914524043
1.9000000000    0.6085016196    0.6059308692
2.0000000000    0.5321003646    0.5296870980
```


Approximation vs Actual
Approximation
Actual
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
y
1.0
1.2
1.4
1.6
1.8
2.0
t

In [26]:

```
import numpy as np

def f(t, y):
    return t**(-2) * (math.sin(2 * t) - 2 * t * y)
def ssp_runge_kutta_third_order(t0, y0, h, N):
    t_values = [t0]
    y_values = [y0]

    for i in range(1, N+1):
        t = t_values[i-1]
        y = y_values[i-1]
        k1 = h * f(t, y)
        k2 = h * f(t + h/2, y + k1/2)
        k3 = h * f(t + h, y - k1 + 2*k2)
        t_next = t + h
        y_next = y + (k1 + 4*k2 + k3) / 6
        t_values.append(t_next)
        y_values.append(y_next)
    return t_values, y_values

if __name__ == '__main__':
    # Initial conditions
    t0 = 1
    y0 = 2
    h = 0.1
    N = 10

    t_values, y_values = ssp_runge_kutta_third_order(t0, y0, h, N)
    def actual_solution(t):
        return 1/(2 * t**2) * (4 + np.cos(2) - np.cos(2 * t))
    actual_values = [actual_solution(t) for t in t_values]
    print("\tt\tApproximation\tActual")
    for i in range(len(t_values)):
        print(f"{t_values[i]:.10f}\t{y_values[i]:.10f}\t{actual_values[i]:.10f}")

    plt.scatter(t_values, y_values, color='blue', label='Approximation')
    plt.plot(t_values, actual_values, color='red', label='Actual')
    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('Approximation vs Actual')
    plt.legend()
    plt.show()
```

```
        t       Approximation   Actual
1.0000000000    2.0000000000    2.0000000000
1.1000000000    1.7240640003    1.7241133391
1.2000000000    1.5003629904    1.5004329441
1.3000000000    1.3137519776    1.3138289695
1.4000000000    1.1545335466    1.1546110980
1.5000000000    1.0163350623    1.0164101467
1.6000000000    0.8948794604    0.8949507694
1.7000000000    0.7872428360    0.7873099232
1.8000000000    0.6913895581    0.6914524043
1.9000000000    0.6058720812    0.6059308692
2.0000000000    0.5296321004    0.5296870980
```



Use the Runge-Kutta-Fehlberg method with tolerance $TOL = 10^{-6}, h_{max} = 0.5, and\ h_{min} = 0.05$

to approximate the solutions to the following initial-value problems. Compare the results to the actual values.

$$1.)\ y' = \frac{y}{t} - (\frac{y}{t})^2,\ 1 \le t \le 4,\ y(1) = 1;\ has\ actual\ solution\ y(t) = \frac{1}{(1 + lnt)}$$

$$2.)\ y' = (2 + 2t^3)y^3 - ty,\ 0 \le t \le 2, y(0) = \frac{1}{3},\ has\ actual\ solution\ y(t) = (3 + 2t^2 + 6e^{t^2}) - \frac{1}{2}$$

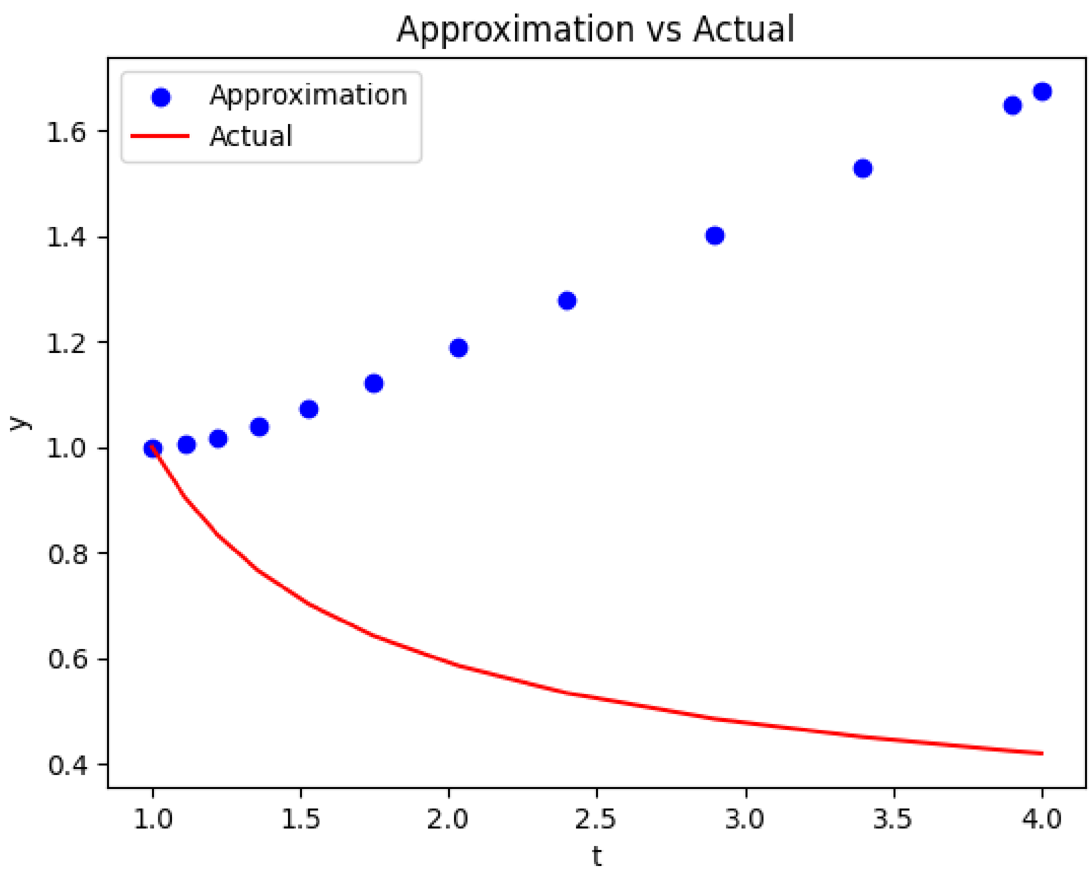
1.) $y' = \frac{y}{t} - (\frac{y}{t})^2,\ 1 \le t \le 4,\ y(1) = 1\ ;\ has\ actual\ solution\ y(t) = \frac{1}{(1+lnt)}$

In [30]:

```
import math
def rkf45(f, t0, y0, h, TOL, hmax, hmin):
    t_values = [t0]
    y_values = [y0]
    t = t0
    y = y0
    while t < 4:
        if t + h > 4:
            h = 4 - t
        k1 = h * f(t, y)
        k2 = h * f(t + h/4, y + k1/4)
        k3 = h * f(t + 3*h/8, y + 3*k1/32 + 9*k2/32)
        k4 = h * f(t + 12*h/13, y + 1932*k1/2197 - 7200*k2/2197 + 7296*k3/2197)
        k5 = h * f(t + h, y + 439*k1/216 - 8*k2 + 3680*k3/513 - 845*k4/4104)
        k6 = h * f(t + h/2, y - 8*k1/27 + 2*k2 - 3544*k3/2565 + 1859*k4/4104 - 11*

        R = abs(k1/360 - 128*k3/4275 - 2197*k4/75240 + k5/50 + 2*k6/55) / h

        if R <= TOL:
            t = t + h
            y = y + 25*k1/216 + 1408*k3/2565 + 2197*k4/4104 - k5/5
            t_values.append(t)
            y_values.append(y)

        delta = 0.84 * (TOL/R)**0.25
        if delta <= 0.1:
            h = 0.1 * h
        elif delta >= 4:
            h = 4 * h
        else:
            h = delta * h
        if h > hmax:
            h = hmax
        elif h < hmin:
            print("Error: Step size below minimum.")
            break
    return t_values, y_values

if __name__ == '__main__':
    def f(t, y):
        return y/t - (y/t)**2
    # Initial conditions
    t0 = 1
    y0 = 1
    h = 0.5
    TOL = 1e-6
    hmax = 0.5
    hmin = 0.05
    t_values, y_values = rkf45(f, t0, y0, h, TOL, hmax, hmin)

    def actual_solution(t):
        return 1/(1 + math.log(t))
    actual_values = [actual_solution(t) for t in t_values]
    print("\tt\tApproximation\tActual")
    for i in range(len(t_values)):
        print(f"{t_values[i]:.10f}\t{y_values[i]:.10f}\t{actual_values[i]:.10f}")

    plt.scatter(t_values, y_values, color='blue', label='Approximation')
    plt.plot(t_values, actual_values, color='red', label='Actual')
```

```
    plt.xlabel('t')
    plt.ylabel('y')
    plt.title('Approximation vs Actual')
    plt.legend()
    plt.show()
```

```
          t          Approximation      Actual
1.0000000000     1.0000000000     1.0000000000
1.1101945703     1.0051237268     0.9053581269
1.2191313820     1.0175212074     0.8346279643
1.3572694376     1.0396749316     0.7660047608
1.5290111802     1.0732756720     0.7019409599
1.7470583595     1.1213948500     0.6418759472
2.0286415975     1.1881701537     0.5856973623
2.3994349613     1.2795395769     0.5332669873
2.8985147448     1.4041842624     0.4844495453
3.3985147448     1.5285639134     0.4497740680
3.8985147448     1.6514963160     0.4236218946
4.0000000000     1.6762392508     0.4190597842
```



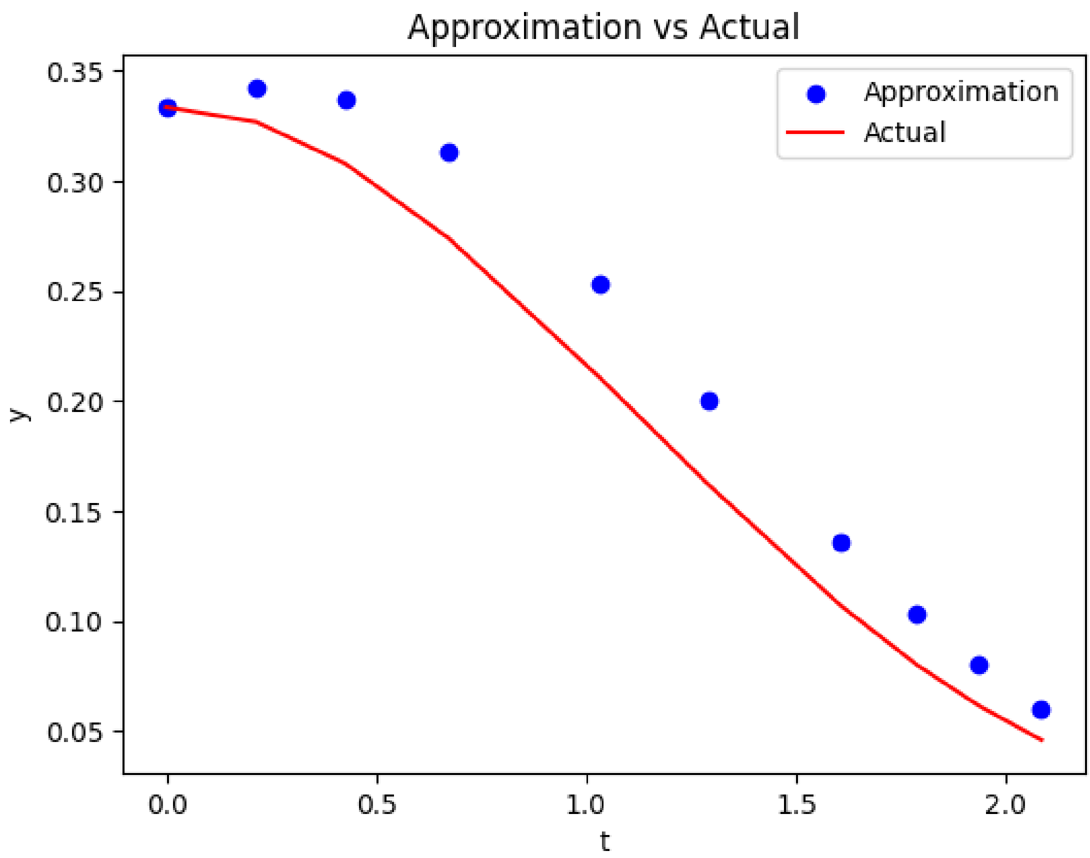
2.) $y' = (2 + 2t^3)y^3 - ty,\ 0 \le t \le 2, y(0) = \frac{1}{3},\ \textit{has actual solution } y(t) = (3 + 2t^2 + 6e^{t^2}) - \frac{1}{2}$

In [31]:

```python
import math
def rkf45(f, t0, y0, h, TOL, hmax, hmin):
    t_values = [t0]
    y_values = [y0]
    t = t0
    y = y0
    while t < 2:
        if t + h > 4:
            h = 4 - t
        k1 = h * f(t, y)
        k2 = h * f(t + h/4, y + k1/4)
        k3 = h * f(t + 3*h/8, y + 3*k1/32 + 9*k2/32)
        k4 = h * f(t + 12*h/13, y + 1932*k1/2197 - 7200*k2/2197 + 7296*k3/2197)
        k5 = h * f(t + h, y + 439*k1/216 - 8*k2 + 3680*k3/513 - 845*k4/4104)
        k6 = h * f(t + h/2, y - 8*k1/27 + 2*k2 - 3544*k3/2565 + 1859*k4/4104 - 11*
        R = abs(k1/360 - 128*k3/4275 - 2197*k4/75240 + k5/50 + 2*k6/55) / h
        if R <= TOL:
            t = t + h
            y = y + 25*k1/216 + 1408*k3/2565 + 2197*k4/4104 - k5/5
            t_values.append(t)
            y_values.append(y)
        delta = 0.84 * (TOL/R)**0.25
        if delta <= 0.1:
            h = 0.1 * h
        elif delta >= 4:
            h = 4 * h
        else:
            h = delta * h
        if h > hmax:
            h = hmax
        elif h < hmin:
            print("Error: Step size below minimum.")
            break

    return t_values, y_values

if __name__ == '__main__':
    def f(t, y):
        return (2 + 2*t**3) * y**3 - t*y
    # Initial conditions
    t0 = 0
    y0 = 1/3
    h = 0.5
    TOL = 1e-6
    hmax = 0.5
    hmin = 0.05
    t_values, y_values = rkf45(f, t0, y0, h, TOL, hmax, hmin)

    def actual_solution(t):
        return (3 + 2*t**2 + 6*np.exp(t**2))**(-1/2)
    actual_values = [actual_solution(t) for t in t_values]
    print("\tt\tApproximation\tActual")
    for i in range(len(t_values)):
        print(f"{t_values[i]:.10f}\t{y_values[i]:.10f}\t{actual_values[i]:.10f}")

    plt.scatter(t_values, y_values, color='blue', label='Approximation')
    plt.plot(t_values, actual_values, color='red', label='Actual')
    plt.xlabel('t')
    plt.ylabel('y')
```

```python
    plt.title('Approximation vs Actual')
    plt.legend()
    plt.show()
```

```
          t        Approximation    Actual
0.0000000000     0.3333333333     0.3333333333
0.2149834562     0.3422604378     0.3265768396
0.4261550791     0.3365185930     0.3077566831
0.6713359616     0.3126714249     0.2740212440
1.0324605269     0.2527853768     0.2105661835
1.2934689059     0.1998841964     0.1615492673
1.6079133021     0.1359637743     0.1067331514
1.7884145944     0.1031321013     0.0799716903
1.9350933798     0.0799256375     0.0615179940
2.0847792474     0.0599462327     0.0458904593
```



The End