TP2: Polynomial Interpolation

Numerical Analysis

March 19, 2023

1. Let $P_3(x)$ be the interpolating polynomial for the data $(0,0), (0.5, y), (1,3)$, and $(2,2)$. The coefficient of x^3 in $P_3(x)$ is 6. Find y.

2. Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.

   (a) $f(x) = e^{2x}\cos(3x), \quad x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$

   (b) $f(x) = \sin(\ln x), \quad x_0 = 2.0, x_1 = 2.4, x_2 = 2.6, n = 2$

3. Let $f(x) = e^x$, for $0 \le x \le 2$.

   (a) Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.

   (b) Approximate $f(0.75)$ using linear interpolation with $x_0 = 0.5$ and $x_1 = 1$.

   (c) Approximate $f(0.25)$ and $f(0.75)$ using second interpolating polynomial with $x_0 = 0, x_1 = 1$ and $x_2 = 2$.

   (d) Which approximations are better and why?

4. The error function defined by $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$.

   (a) Integrate the Maclaurin series for e^{-x^2} to show that $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$.

   (b) Use the Maclaurin series to construct a table for $\operatorname{erf}(x)$ that is accurate to within 10^{-4} for $\operatorname{erf}(x_i)$, where $x_i = 0.2i$, for $i = 0, 1, \ldots, 5$.

   (c) Use both linear interpolation and quadratic interpolation to obtain an approximation to $\operatorname{erf}(\frac{1}{3})$. Which approach seems most feasible?

5. Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

   (a) $f(0.43)$ if $f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.48169$.

   (b) $f(0)$ if $f(-0.5) = 1.93750, f(-0.25) = 1.33203, f(0.25) = 0.800781, f(0.5) = 0.687500$.

6. Use Neville's method to approximate $\sqrt{3}$ with $f(x) = 3^x$ and the values $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$, and $x_4 = 2$.

7. Let $P_3(x)$ be the interpolating polynomial for the data $(0,0), (0.5, y), (1,3)$, and $(2,2)$. Use Neville's method to find y if $P_3(1.5) = 0$.

8. Neville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$			
$x_1 = 0.25$	$P_1 = 2$	$P_{0,1} = 2.6$		
$x_2 = 0.5$	P_2	$P_{1,2}$	$P_{0,1,2}$	
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$

9. Suppose $f \in C^1[a,b], f'(x) \neq 0$ on $[a,b]$ and f has one zero p in $[a,b]$. Let $x_0, \dots, x_n$, be $n+1$ distinct numbers in $[a,b]$ with $f(x_k) = y_k$, for each $k = 0, 1, \dots, n$. To approximate p construct the interpolating polynomial of degree n on the nodes $y_0, \dots, y_n$ for f^{-1}. Since $y_k = f(x_k)$ and $0 = f(p)$, it follows that $f^{-1}(y_k) = x_k$ and $p = f^{-1}(0)$. Using iterated interpolation to approximate $f^{-1}(0)$ is called iterated inverse interpolation. Use iterated inverse interpolation to find an approximation to the solution of $x - e^{-x} = 0$, using the data

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

10. Use the Newton forward-difference formula to construct interpolating polynomials of degree one,two, and three for the following data. Approximate the specified value using each of the polynomials.

    (a) $f(0.43)$ if $f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.48169.$

    (b) $f(0.18)$ if $f(0.1) = -0.29004986, f(0.2) = -0.56079734, f(0.3) = -0.81401972,$ $f(0.4) = -1.0526302.$

11. (a) Us the Newton Divided-Difference method to construct the interpolating polynomial of degree three for the unequally spaced points given in the following table:

x	$f(x)$
-0.1	5.30000
0.0	2.00000
0.2	3.19000
0.3	1.00000

    (b) Add $f(0.35) = 0.97260$ to the table, and construct the interpolating polynomial of degree four.

12. Show that the polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

13. (a) Show that the following two cubic polynomials

$$P(x) = 3 - 2(x+1) + 0(x+1)(x) + (x+1)(x)(x-1)$$
$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)(x)$$

both interpolate the data

x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

(b) Why does part (a) not violate the uniqueness property of interpolating polynomials?

14. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3
$P(x)$	4	9	15	18

15. For a function f, the Newton divided-difference formula gives the interpolating polynomial

$$P_3(x) = 1 + 4x + 4x(x - 0.25) + \frac{16}{3}x(x - 0.25)(x - 0.5),$$

on the nodes $x_0 = 0, x_1 = 0.25, x_2 = 0.5$ and $x_3 = 0.75$. Find $f(0.75)$.

16. For a function f, the forward-divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
$x_1 = 0.4$	$f[x_1]$	$f[x_0, x_1]$	
$x_2 = 0.7$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	$f[x_0, x_1, x_2] = \frac{50}{7}$

17. A census of the population of the United States is taken every 10 years. The followingtable lists the population, in thousands of people, from 1950 to 2000, and the data are alsorepresented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population (thousands)	151326	179323	203302	226542	249633	281422

(a) Use appropriate divided differences to approximate the population in the years 1940, 1975, and 2020.

(b) The population in 1940 was approximately 132165000. How accurate do you think your 1975 and 2020 figures are?

18. Determine the natural cubic spline S that interpolates the data $f(0) = 0, f(1) = 1$, and $f(2) = 2$.

19. Determine the clamped cubic spline s that interpolates the data $f(0) = 0, f(1) = 1, f(2) = 2$ andsatisfies $s'(0) = s'(2) = 1$.

20. Construct the natural cubic spline for the following data.

(a)

x	$f(x)$
0.8	0.22363362
1.0	0.65809197

(b)

x	$f(x)$
0.1	-0.62049958
0.2	-0.28398668
0.3	0.00660095
0.4	0.24842440

21. A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x \le 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find b, c and d.

22. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x \le 2, \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

23. A car traveling along a straight road is clocked at a number of points. The data from the observationsare given in the following table, where the time is in seconds, the distance is in feet, and the speed isin feet per second.

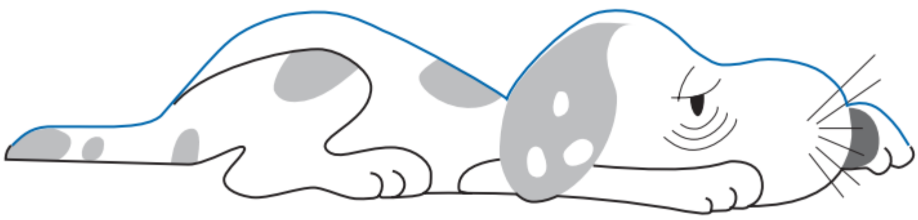
Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

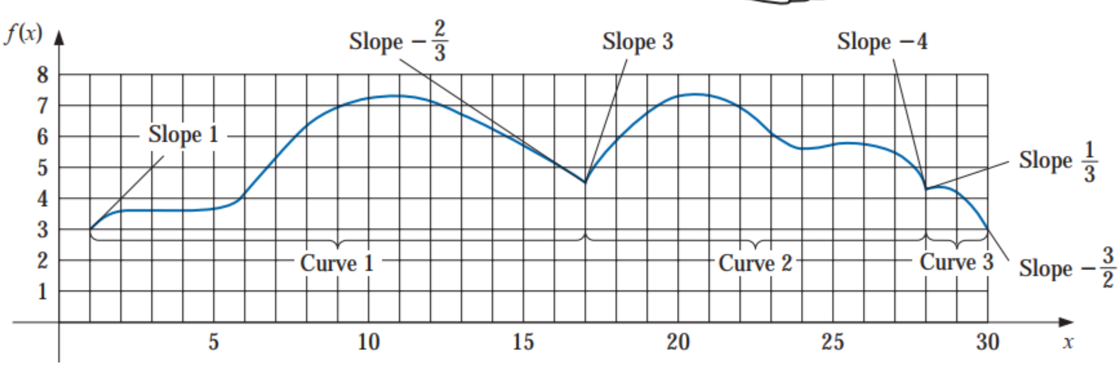
(a) Use a clamped cubic spline to predict the position of the car and its speed when $t = 10s$.

(b) Use the derivative of the spline to determine whether the car ever exceeds a 55-mi/h speed limit on the road; if so, what is the first time the car exceeds this speed?

(c) What is the predicted maximum speed for the car?

24. The upper portion of this noble beast is to be approximated using clamped cubic spline interpolants.The curve is drawn on a grid from which the table is constructed. Use Algorithm 3.5 to construct thethree clamped cubic splines.





Curve 1

i	x_i	$f(x_i)$	$f'(x_i)$
0	1	3.0	1.0
1	2	3.7	
2	5	3.9	
3	6	4.2	
4	7	5.7	
5	8	6.6	
6	10	7.1	
7	13	6.7	
8	17	4.5	-0.67

Curve 2

i	x_i	$f(x_i)$	$f'(x_i)$
0	17	4.5	3.0
1	20	7.0	
2	23	6.1	
3	24	5.6	
4	25	5.8	
5	27	5.2	
6	27.7	4.1	-4.0

Curve 3

i	x_i	$f(x_i)$	$f'(x_i)$
0	27.7	4.1	0.33
1	28	4.3	
2	29	4.1	
3	30	3.0	-1.5