TP7 Boundary-Value Problems for Ordinary Differential Equations

1 The boundary-value problem

$$y'' = 4(y - t),\ 0 \le t \le 1,\ y(0) = 0,\ y(1) = 2,$$

has the solution $y(t) = e^2(e^4 - 1)^{-1}(e^{2x} - e^{-2x}) + x$. Use the Linear Shooting method to approximatethe solution, and compare the results to the actual solution.

a With $h = 1/2$;

b With $h = 1/4$.

2 Use the Linear Shooting method to approximate the solution to the following boundary-value problems.

a $y'' = -3y' + 2y + 2t + 3,\ 0 \le t \le 1,\ y(0) = 2,\ y(1) = 1$; use $h = 0.1$.

b $y'' = -(t + 1)y' + 2y + (1 - t^2)e^{-t},\ 0 \le t \le 1,\ y(0) = -1,\ y(1) = 0$; use $h = 0.1$.

3 Although $q(x) < 0$ in the following boundary-value problems, unique solutions exist and are given. Use the Linear Shooting Algorithm to approximate the solutions to the following problems, and compare the results to the actual solutions.

a $y'' + y = 0,\ 0 \le t \le \pi/4,\ y(0) = 1,\ y(\pi/4) = 1$; use $h = \pi/20$;
actual solution $y(t) = \cos t + (\sqrt{2} - 1)\sin t$.

b $y'' = -4t^{-1}y' - 2t^{-2}y + 2t^{-2}\ln t,\ 1 \le t \le 2,\ y(1) = 1/2,\ y(2) = \ln 2$; use $h = 0.05$;
actual solution $y(t) = 4t^{-1} - 2t^{-2} + \ln t - 3/2$.

4 Let u represent the electrostatic potential between two concentric metal spheres of radii R_1 and R_2 $(R_1 < R_2)$. The potential of the inner sphere is kept constant at V_1 volts, and the potential of the outer sphere is 0 volts. The potential in the region between the two spheres is governed by Laplace's equation, which, in this particular application, reduces to

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0,\ R_1 \le r \le R_2,\ u(R_1) = V_1,\ u(R_2) = 0.$$

Suppose $R_1 = 2$ in., $R_2 = 4$ in., and $V1 = 110$ volts.

a Approximate $u(3)$ using the Linear Shooting Algorithm.

b Compare the results of part **a** with the actual potential $u(3)$, where

$$u(r) = \frac{V_1R_1}{r}\left(\frac{R_2 - r}{R_2 - R_1}\right).$$

5 Use the Nonlinear Shooting Algorithm with $h = 0.5$ to approximate the solution to the boundary-value problem

$$y'' = -(y')^2 - y + \ln t,\ 1 \le t \le 2,\ y(1) = 0,\ y(2) = \ln 2.$$

Compare your results to the actual solution $y = lnx$.

6 Use the Nonlinear Shooting method with $TOL = 10^{-4}$ to approximate the solution to the following boundary-value problems. The actual solution is given for comparison to your results.

- **a** $y'' = y' + 2(y - \ln t)^3 - t^{-1},\ 2 \le t \le 3,\ y(2) = 1/2 + \ln 2,\ y(3) = 1/3 + \ln 3;$ use $h = 0.1$; actual solution $y(t) = t^{-1} + \ln t$.
- **b** $y'' = 2(y')^2 t^{-3} - 9y^2 t^{-5} + 4t,\ 1 \le t \le 2,\ y(1) = 0,\ y(2) = \ln 256;$ use $h = 0.05$; actual solution $y(t) = t^3 \ln t$.

7 The boundary-value problem

$$y'' = y' + 2y + \cos t,\ 0 \le t \le \pi/2,\ y(0) = -0.3,\ y(\pi/2) = -0.1$$

has the solution $y(t) = -\frac{1}{10}(\sin t + 3\cos t)$. Use the Linear Finite-Difference method to approximatethe solution, and compare the results to the actual solution.

- **a** With $h = \pi/4$;
- **b** With $h = \pi/8$.

8 Use the Linear Finite-Difference Algorithm to approximate the solution to the following boundary-value problems.

- **a** $y'' = -(t+1)y' + 2y + (1-t^2)e^{-t},\ 0 \le t \le 1,\ y(0) = -1,\ y(1) = 0$; use $h = 0.1$.
- **b** $y'' = t^{-1}y' + 3t^{-2}y + t^{-1}\ln t - 1,\ 1 \le t \le 2,\ y(1) = y(2) = 0$; use $h = 0.1$.

9 Although $q(x) < 0$ in the following boundary-value problems, unique solutions exist and are given. Use the Linear Finite-Difference Algorithm to approximate the solutions, and compare the results tothe actual solutions.

- **a** $y'' + 4y = \cos t,\ 0 \le t \le \pi/4,\ y(0) = 0,\ y(\pi/4) = 0$; use $h = \pi/20$; actual solution $y(t) = -\frac{1}{3}\cos 2t - \frac{\sqrt{2}}{6}\sin 2t + \frac{1}{3}\cos t$.
- **b** $y'' = 2y' - y + te^t - t,\ 0 \le t \le 2,\ y(0) = 0,\ y(2) = -4$; use $h = 0.2$; actual solution $y(t) = \frac{1}{6}t^3e^t - \frac{5}{3}te^t + 2e^t - t - 2$.

10 The deflection of a uniformly loaded, long rectangular plate under an axial tension force is governed by a second-order differential equation. Let S represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by

$$W''(t) - \frac{S}{D}W(t) = \frac{-ql}{2D}t + \frac{q}{2D}t^2,\ 0 \le t \le l,\ W(0) = W(l) = 0,$$

where l is the length of the plate and D is the flexual rigidity of the plate. Let $q = 200$ lb/in.^{2}, $S = 100$ lb/in., $D = 8.8 \times 107$ lb/in., and $l = 50$ in. Approximate the deflection at 1-in. intervals.

11 Use Newton's method to find a solution to the following nonlinear systems in the given domain. Iterate until $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_2 < 10^{-6}$.

- **a** $\begin{cases} 3x_1^2 - x_2^2 &= 0, \\ 3x_1x_2^2 - x_1^3 &= 1. \end{cases}$ Use $\mathbf{x}^{(0)} = (1,1)^t$.

b $\begin{cases} \ln(x_1^2 + x_2^2) - \sin(x_1x_2) &= \ln 2 + \ln \pi, \\ e^{x_1 - x_2} + \cos(x_1x_2) &= 0. \end{cases}$ Use $\mathbf{x}^{(0)} = (2, 2)^t$.

12 Find a solution of

$$\begin{cases} \sin x + y^2 + \ln z - 7 &= 0, \\ 3x + 2^y - z^3 + 1 &= 0, \\ x + y + z - 5 &= 0. \end{cases}$$

using Newton's method with $\mathbf{x}^{(0)} = (1, 1, 1)^t$.

13 Use the Nonlinear Finite-Difference method with $h = 0.5$ to approximate the solution to the boundaryvalue problem

$$y'' = -(y')^2 - y + \ln t, \ 1 \le t \le 2, \ y(1) = 0, \ y(2) = \ln 2.$$

Compare your results to the actual solution $y = \ln t$.

14 Use the Nonlinear Finite-Difference Algorithm with $TOL = 10^{-4}$ to approximate the solution to the following boundary-value problems. The actual solution is given for comparison to your results.

a $y'' = y' \cos t - y \ln y, \ 0 \le t \le \pi/2, \ y(0) = 1, \ y(\pi/2) = e$; use $N = 9$; actual solution $y(t) = e^{\sin t}$.

b $y'' = (1 - (y')^2 - y \sin t)/2, \ 0 \le t \le \pi, \ y(0) = 2, \ y(\pi) = 2$; use $N = 19$; actual solution $y(t) = 2 + \sin t$.