TP5 Initial-Value Problems for Ordinary Differential Equations

1 Use Euler's method to approximate the solutions for each of the following initial-value problems.

a $y' = te^{3t} - 2y,\ 0 \le t \le 1,\ y(0) = 0$, with $h = 0.1$.

b $y' = t^{-2}(\sin 2t - 2ty),\ 1 \le t \le 2,\ y(1) = 2$, with $h = 0.1$.

2 Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t,\ 1 \le t \le 2,\ y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

a Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b Use the answers generated in part **a** and linear interpolation to approximate the following values of y, and compare them to the actual values.

i $y(1.04)$

ii $y(1.55)$

c Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using $|y(t_i) - w_i| \le \dfrac{hM}{2L}[e^{L(t_i-a)} - 1]$.

3 Use Taylor's method of order two to approximate the solution for each of the following initial-value problems.

a $y' = \sin t - e^{-t},\ 0 \le t \le 1,\ y(1) = 1$, with $h = 0.1$.

b $y' = y/t - (y/t)^2,\ 1 \le t \le 1.2,\ y(1) = 1$, with $h = 0.1$.

4 Use the indicated method to approximate the solutions to the initial-value problems

$$y' = \frac{2 - 2ty}{t^2 + 1},\ 0 \le t \le 1,\ y(0) = 1,\ h = 0.1;\ \text{actual solution } y(t) = \frac{2t + 1}{t^2 + 1},$$

and compare the results to the actual values.

a Explicit midpoint method.

b Modified Euler's/Heun's second-order method.

c Ralston's method.

5 Use the indicated method to approximate the solutions to the initial-value problems

$$y' = t^{-2}(\sin 2t - 2ty),\ 1 \le t \le 2,\ y(1) = 2,\ h = 0.1;$$

actual solution

$$y(t) = \frac{1}{2}t^{-2}(4 + \cos 2 - \cos 2t),$$

and compare the results to the actual values.

a Runge-Kutta third-order method.

b Heun's third-order method.

c Ralston's third-order method.

d Third-order Strong Stability Preserving Runge-Kutta.

6 Use the indicated method to approximate the solutions to the initial-value problems

$$y' = t^{-2}(\cos t - 2ty),\ 1 \le t \le 2,\ y(1) = 0,\ h = 0.1;$$

actual solution

$$y(t) = t^{-2}(\sin t - \sin 1),$$

and compare the results to the actual values.

a Original Runge-Kutta method.

b 3/8-rule fourth-order method.

7 Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.

a $y' = 1 + y/t,\ 1 \le t \le 2,\ y(1) = 2$ with $h = 0.1$;
and actual solution $y(t) = t\ln t + 2t$.

b $y' = -(y+1)(y+3),\ 0 \le t \le 2,\ y(0) = -2$, with $h = 0.1$;
and actual solution $y(t) = -3 + \dfrac{2}{1 + 2^{-2t}}$.

8 Use the Adams-Predictor-Corrector method to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.

a $y' = 1 + y/t,\ 1 \le t \le 2,\ y(1) = 2$ with $h = 0.1$;
and actual solution $y(t) = t\ln t + 2t$.

b $y' = -(y+1)(y+3),\ 0 \le t \le 2,\ y(0) = -2$, with $h = 0.1$;
and actual solution $y(t) = -3 + \dfrac{2}{1 + 2^{-2t}}$.

9 Use the Runge-Kutta method for systems to approximate the solutions of the following systems of first-order differential equations, and compare the results to the actual solutions.

a For $0 \le t \le 1$ with $h = 0.2$:

$$\begin{cases} u_1' = 3u_1 + 2u_2 - (2t^2+1)e^{2t}, & u_1(0) = 1, \\ u_2' = 4u_1 + u_2 + (t^2 + 2t - 4)e^{2t}, & u_2(0) = 1, \end{cases}$$

and actual solution

$$\begin{cases} u_1(t) = \dfrac{1}{3}e^{5t} - \dfrac{1}{3}e^{-t} + e^{2t}, \\ u_2(t) = \dfrac{1}{3}e^{5t} + \dfrac{2}{3}e^{-t} + t^2e^{2t}. \end{cases}$$

b for $0 \le t \le 1$ with $h = 0.1$:

$$\begin{cases} u_1' = u_1 + 2u_2 - 2u_3 + e^{-t}, & u_1(0) = 3, \\ u_2' = u_2 + u_3 - 2e^{-t}, & u_2(0) = -1, \\ u_3' = u_1 + 2u_2 + e^{-t}, & u_3(0) = 1, \end{cases}$$

and actual solution

$$\begin{cases} u_1(t) = -3e^{-t} - 3\sin t + 6\cos t, \\ u_2(t) = \frac{3}{2}e^{-t} + \frac{3}{10}\sin t - \frac{21}{10}\cos t - \frac{2}{5}e^{2t}, \\ u_3(t) = -e^{-t} + \frac{12}{5}\cos t + \frac{9}{5}\sin t - \frac{2}{5}e^{2t}. \end{cases}$$

10 Use the Runge-Kutta for Systems Algorithm to approximate the solutions of the following higher order differential equations, and compare the results to the actual solutions.

a $y'' - 2y' + y = te^t - t,\ 0 \le t \le 1,\ y(0) = y'(0) = 0,$
with actual solution $y(t) = \frac{1}{6}t^3e^t - te^t + 2e^t - t - 2.$

b $t^3y''' - t^2y'' + 3ty' - 4y = 5t^3\ln t + t^3,\ 1 \le t \le 2,\ y(1) = 0, = y'(1) = 1, y''(1) = 3,$
with actual solution $y(t) = -t^2 + t\cos(\ln t) + t\sin(\ln t) + t^3\ln t.$

11 Use the Runge-Kutta to approximate the solutions of the Lorentz equations:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(\rho - z) - y, \\ \dot{z} = xy - \beta z, \end{cases} \quad \text{for } 0 \le t \le 100,\ \text{with } x(0) = y(0) = z(0) = 10.$$

using $h = 0.01$ and plot the approximated solution in case $\sigma = 28, \rho = 10, \beta = 8/3.$