TP1: Roots of Equations

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1 Introduction to Python

1. Explain and develop a computer program to find roots of the equation,

$$bx + c = 0$$

2. Explain and develop a computer program to find roots of the equation,

$$ax^2 + bx + c = 0$$

3. Develop computer programs to compute the following sums with loops and then with mathematical formula for $n = 10, 20, 50$.

   (a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

   (b) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = 1 - \dfrac{1}{n+1}$

4. Develop computer programs to compute the sum $\displaystyle\sum_{k=1}^{n} \frac{1}{k!}$.

5. Develop a computer program to determine the exponential function which can be written in infinite series form

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

   then compute the value of e^x for $x = 0.1, 0.5, 1$ with polynomial approximation of degree 5.

6. Develop a computer program to compute the value of polynomial $p(x) = \displaystyle\sum_{k=0}^{n} a_k x^k$ at $x = x_0$. Use your program to compute $p(0.1)$ if $p(x) = 1 - 3x^2 + 2x^3 - x^5 + 4x^6$.

2 Solutions of Equations in One Variable

1. Let $f(x) = (x-1)^{10}, p = 1$ and $p_n = 1 + \dfrac{1}{n}$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

2. Let (p_n) be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that (p_n) diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

3. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.

   (a) $[0, 1]$
   (b) $[1, 3.2]$
   (c) $[3.2, 4]$

5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.

   (a) $x - 2^{-x} = 0$ for $0 \le x \le 1$
   (b) $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$
   (c) $2x\cos(2x) - (x+1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$
   (d) $x\cos(x) - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$

6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

7. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

8. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

   (a) $2 + \sin x - x = 0$ use $[2, 3]$
   (b) $x^3 - 2x - 5 = 0$ use $[2, 3]$
   (c) $3x^2 - e^x = 0$
   (d) $x - \cos x = 0$

9. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

10. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find P_2.

11. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.

    (a) Use the secant Method.
    (b) Use the method of False Position.
    (c) Which of a) or $(b$) is closer to $\sqrt{6}$?

12. Let $f(x) = -x^3 - \cos x$. With $p_0 = 0$ and $p_1 = 0$, find p_3.

    (a) Use the Secant method.
    (b) Use the method of False Position.

13. Find solutions accurate to within 10^{-5} for the problem $e^x - 3x^2 = 0$ for $0 \le x \le 1$ and $3 \le x \le 5$ using

    (a) Newton's method.

    (b) Secant method.

    (c) The method of False Position.

14. The four-degree polynomial $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$ has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^6 using the

    (a) Method of False Position

    (b) Secant method

    (c) Newton'w method

    Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

15. The accumulated value of a savings account based on regular periodic payments can be determined from the annuity due equation,

$$A = \frac{P}{i}[(1 + i)^n - 1].$$

    In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at $\$\ 750,000$ upon retirement in 20 years and can afford to put $\$\ 1500$ per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

16. Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^{-4} for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

    (a) Bisection method

    (b) Newton's method

    (c) Secant method

    (d) method of False Position

    (e) Müller's method.

17. Find approximations to within 10^{-5} to all the zeros of each of the following polynomials

    (a) $x^4 + 5x^3 - 9x^2 - 85x - 136$

    (b) $x^5 + 11x^4 - 21x^3 - 10x^2 - 21x - 5$

    using Müller's method and Lagurre's method.