Numerical Analysis

Mathematical Preliminaries

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February 19, 2023

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Definition 1 (Limit)

A function f defined on a set X of real numbers has the limit L at x_0, written

$$\lim_{x \longrightarrow x_0} f(x) = L,$$

if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \text{ whenever } x \in X \text{ and } 0 < |x - x_0| < \delta.$$

Definition 2 (Continuity)

Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is continuous at x_0 if

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0).$$

The function f is continuous on the set X if it is continuous at each number in X.

Definition 3 (Limit of Sequence)

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. This sequence has the limit x (converges to x) if, for any $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$, whenever $n > N(\varepsilon)$. The notation

$$\lim_{n\to\infty} x_n, \text{ or } x_n \longrightarrow x \text{ as } n \longrightarrow \infty$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.

Theorem 4

If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

1. *f is continuous at x_0;*
2. *If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then* $\lim_{n\to\infty} f(x_n) = f(x_0)$.

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Definition 5 (Differentiability)

Let f be afunction defined in an open interval containing x_0. Thefunctionf is differentiable at x_0 if

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the derivative of f at x_0. A function that has a derivative at each number in a set X is differentiable on X.

Theorem 6

If the function f is differentiable at x_0, then f is continuous at x_0.

Theorem 7 (Rolle's Theorem)

Suppose $f \in C[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$.

Theorem 8 (Mean Value Theorem)

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 9 (Extreme Value Theorem)

If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$, for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of [a, b] or where f' is zero.

Theorem 10 (Generalized Rolle's Theorem)

Suppose $f \in C[a,b]$ is n times differentiable on (a,b). If $f(x)=0$ at the $n+1$ distinct numbers $a \le x_0 < x_1 < \cdots < x_n \le b$, then a number c in (x_0, x_n), and hence in (a,b), exists with $f^{(n)}(c)=0$.

Theorem 11 (Intermediate Value Theorem)

If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c)=K$.

Definition 12 (Integration)

The Riemann integral of the function f on the interval [a, b] is the following limit, provided it exists:

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\Delta x_i,$$

where the numbers $x_0, x_1, \dots, x_n$ satisfy $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ where $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, 2, \dots, n$, and z_i is arbitrarily chosen in the interval $[x_{i-1}, x_i]$.

Theorem 13 (Taylor's Theorem)

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

and $R_n(x) = \dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$

Definition 14 (Absolute and Relative Errors)

Suppose that p^* is an approximation to p. The absolute error is $|p - p^*|$, and the relative error is $\dfrac{|p - p^*|}{|p|}$, provided that $p \neq 0$.

Definition 15 (Significant Digits)

The number p^* is said to approximate p to t significant digits (or figures) if t is the largest nonnegative integer for which

$$\frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t}.$$

Definition 16 (Rates of Convergence)

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \leq K|\beta_n|, \text{ for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with rate, or order, of convergence $O(\beta_n)$. (This expression is read "big oh of β_n".) It is indicated by writing $\alpha_n = \alpha + O(\beta_n)$.

Definition 17

Suppose that $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$. If a positive constant K exists with

$$|F(h) - L| \le K|G(h)|, \text{ for sufficiently small } h,$$

then we write $F(h) = L + O(G(h))$.

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