Summary Note: Optimization Week 2

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Exercise 1. Let C be a nonempty convex subset of $\mathbb{R}^n$

(a) Let $f : C \to \mathbb{R}$ be a convex function, and $g : \mathbb{R} \to \mathbb{R}$ be a function that is convex and montonically nondecreasing over a convex set that contains the set of values that f can take, $\{f(x) \mid x \in C\}$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. In addition, if g is montonically increasing and f is strictly convex, then h is strictly convex.

(b) Let $f = (f_1, ..., f_m)$, where $f_i : C \to \mathbb{R}$ is convex function, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a function that is convex and monotonically nodecreasing over a convex set that contains the set $\{f(x) \mid x \in C\}$, in the sense that for all u, u_1 in this set such that $u \leq u_1$, we have $g(u) \leq g(u_1)$. Show that function h defined by $h(x) = g(f(x))$ is convex over $C \times \cdots \times C$.

Answer:

(a) We have

- $f : C \to \mathbb{R}$ a convex function.
- $g : \mathbb{R} \to \mathbb{R}$ a convex function with monotonically non-decreasing over a convex set.
- A function $\{f(x) \mid x \in C\}$
- We have to show that $h(x) = g(f(x))$ is convex over C

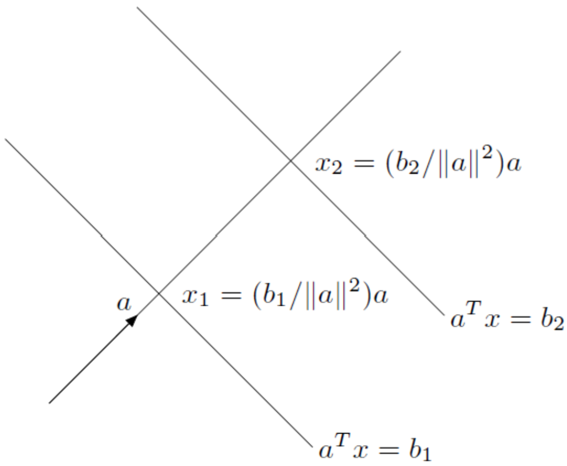
Recall: In order to prove anything is a convex or in convex set, we have to find out that two points that insider of the convex set is following the following identity $\theta x_1 + (1-\theta)x_2 \in C$, where $\theta \in [0, 1]$

Then, Let $x, y \in C$ and $\theta \in [0, 1]$ We have

$$\begin{aligned}
h[\theta x + (1-\theta)y] &= g\left[f(\theta x + (1-\theta)y)\right] \\
&= g[f(\theta x) + f[(1-\theta)y)] \\
&\leq g[\theta f(x) + (1-\theta)f(y)] && \text{convexity of } f \text{ and the mononicity of } g \\
&\leq \theta g[f(x)] + (1-\theta)g[f(y)] && \text{convexity of } g \\
&= \theta h(x) + (1-\theta)h(y)
\end{aligned}$$

Exercise 2. What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^\top x = b_1\}$ and $\{x \in \mathbb{R}^n : a^\top x = b_2\}$.

Answer:



The distance distance between the two hyperplanes is also the distance between the two points x_1 and x_2 where the hyperplane intersects the line through the origin and parallel to the normal vector a.

- Let $H_1 : a^T x = b_1$ and $H_2 : a^T x = b_2$
- x_1 lie on H_1 which satisfy the $a^T x = b_1 \implies x_1 = c_1 a$
- c_1 is the adjustment for x_1 to be lie on the intersection of H_1 and vector a.
- We can find the c_1 by substituting into H_1, then

$$\implies a^T(c_1 a) = b_1 \implies c_1 = \frac{b_1}{a^T a} = \frac{b_1}{||a||_2^2} \implies x_1 = \frac{b_1}{||a||_2^2}a$$

- We can do similarly for x_2 on H_2, then

$$\implies x_2 = \frac{b_2}{||a||_2^2}a$$

- Then, the distance between two hyperplances is

$$||x_2 - x_1|| = \left|\left|\frac{b_2}{||a||_2^2}a - \frac{b_1}{||a||_2^2}a\right|\right| = \frac{||a||}{||a||_2^2}||b_2 - b_1|| = \frac{|b_2 - b_1|}{||a||_2}$$

Therefore, the distance between the two hyperplanes is $||x_2 - x_1|| = \dfrac{|b_2 - b_1|}{||a||_2}$ ■

- **Note:** I will explain what is the meaning $||a||_2^2$ in the next class.

Exercise 3. Which of the following sets are convex? Justify your answer.

(a) A *slab, i.e.,* a set of the form $\{x \in \mathbb{R}^n : \alpha \leq a^\top x \leq \beta\}$

(b) A *rectangle, i.e.,* a set of the form $\{x \in \mathbb{R}^n : \alpha_i \leq x_i \leq \beta_i, \forall i = 1, ..., n\}$

(c) A *wedge, i.e.,* a set of the form $\{x \in \mathbb{R}^n : a_1^T x \leq b_1, a_2^\top x \leq b_2\}$

(d) The set of points closer to a given point than to a given (arbitrary) set $S \subset \mathbb{R}^n$, *i.e.,*

$$\{x \in \mathbb{R}^n : ||x - x_0||_2 \leq ||x - y||_2 \text{ for all } y \in S\}$$

(e) The set of points closer to one set than another, *i.e.,*

$$\{x : \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{||x - z||_2 : z \in S\}$$

(f) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.,* the set $\{x : ||x - a|_2 \leq \theta||x - b||_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

(g) The set $\{x : x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Answer:

(a) A *slab*, a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$

- Let $S = \{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$

$\implies$ We notice that S is an intersection of two halfspaces where by let

$\implies$ $H_1 = \{x \in \mathbb{R}^n \mid a^T x \geq \alpha\}$ and $H_2 = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$

- H_1 and H_2 is the form of halfspaces, then a slab is

$\implies$ $S = H_1 \cap H_2$, is also a convex.

(b) A *rectangle, i.e.,* a set of the form $\{x \in \mathbb{R}^n : \alpha_i \leq x_i \leq \beta_i, \forall i = 1, ..., n\}$

(c) A *wedge, i.e.,* a set of the form $\{x \in \mathbb{R}^n : a_1^T x \leq b_1, a_2^\top x \leq b_2\}$

- A wedge can be expressed as below

$$\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^\top x \leq b_2\} = \{x \in \mathbb{R}^n \mid a^T x \leq b_1\} \bigcap \{x \in \mathbb{R}^n \mid a_2^T x \leq b_2\}$$

which is the intersection of two halfspaces, therefore, a *wedge* is convex. ■

(d) The set of points closer to a given point than to a given (arbitrary) set $S \subseteq \mathbb{R}^n$, *i.e.,*

$$\{x \in \mathbb{R}^n : ||x - x_0||_2 \leq ||x - y||_2 \text{ for all } y \in S\}$$

- We have to prove that $\{x \in \mathbb{R}^n \mid ||x-a||_2 \leq ||x-b||_2\}$ is a halfspace, where $a, b \in \mathbb{R}^n$

$$\begin{aligned}
||x-a||_2 \leq ||x-b||_2 &\iff \left(||x-a||_2^2\right) \geq \left(||x-b||_2^2\right) \\
&\iff (x-a)^T(x-a) \leq (x-b)^T(x-b) \\
&\iff x^Tx - x^Ta - a^Tx + a^Ta \leq x^Tx - x^Tb - b^Tx + b^Tb \\
&\iff (x^Tb + b^Tx) - (x^Ta - a^Tx) \leq b^Tb - a^Ta \\
&\iff 2(b^T - a^T)x \leq b^Tb - a^Ta
\end{aligned}$$

Hence, we have

$$\{x \in \mathbb{R}^n \mid ||x-a||_2 \leq ||x-b||_2\} = \{x \in \mathbb{R}^n \mid 2\left(b^T - a^T\right)x \leq b^Tb - a^Ta\}$$

is a halfspaces.

- $x^Tb + b^Tx = 2b^Tx = 2x^Tb$ why? Keep as your homework.

(e) The set of points closer to one set than another, *i.e.,*

$$\{x : \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{||x-z||_2 : z \in S\}$$

(f) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.,* the set $\{x : ||x-a|_2 \leq \theta||x-b||_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

We have

$$\begin{aligned}
||x-a||_2 \leq \theta||x-b||_2 &\iff \left(||x-a||_2^2\right) \leq \theta^2\left(||x-b||_2^2\right) \\
&\iff (x-a)^T(x-a) \leq \theta^2(x-b)^T(x-b) \\
&\iff x^Tx - x^Ta - a^Tx + a^Ta \leq \theta^2(x^Tx - x^Tb - b^Tx + b^Tb) \\
&\iff (1-\theta^2)x^Tx - 2(a-\theta^2 b)^Tx + (a^Ta - b^Tb) \leq 0
\end{aligned}$$

- If $\theta = 1 \iff 2(a-\theta^2 b)^Tx \geq b^Tb - a^Ta$

$\implies$ $\{x : ||x-a|_2 \leq \theta||x-b||_2\}$ is a halfspace and then it is convex. ■

- If $0 \leq \theta \leq 1$, we can show that $\{x : ||x-a|_2 \leq \theta||x-b||_2\}$ is a closed ball centered at x_0 with radius r.

Since $1-\theta^2 > 0$ when $0 \leq \theta < 1$, then above equation become

$$x^Tx - 2\frac{(a-\theta^2 b)Tx}{1-\theta^2} + \frac{1}{1-\theta^2}(a^Ta - b^Tb) \leq 0$$

Then, let $x_0 = 2\dfrac{a - \theta^2 b}{1 - \theta^2}$ and

$$r = \sqrt{||x_0||_2^2 - \frac{1}{1 - \theta^2}\left(||a||_2^2 - \theta^2||b||_2^2\right)}$$

Then, the equation become $(x - x_0)^T(x - x_0) \leq r^2$.
$\implies \{x : ||x - a|_2 \leq \theta||x - b||_2\}$ is a closed ball and it is convex $\blacksquare$

(g) The set $\{x : x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.