Institute of Technology of Cambodia
Midterm Exam

Class: Engineering | **Year: III**
Subject: Optimization | **Duration: 2 hours**

Name:................................. **ID:..................................**

AMS-Group:....................... **Date:................................**

(25 pts) Part I: In this section, you have to select one correct answer.

(1) A **Convex set** is defined as $C \subseteq \mathbb{R}^n$ such that $x, y \in C \implies tx + (1-t)y \in C$, $\forall 0 \leq t \leq 1$.

- ☐ True. correct.
- ☐ False.

(2) A norm ball is given as $||\cdot||$ is convex.

- ☐ True. correct.
- ☐ False.

(3) Define halfspace general form:

- ☐ $\{x \mid a^T x = b\}$.
- ☐ $\{x \mid a^T x \leq b\}$. correct.
- ☐ $\{x \mid Ax = b\}$.

(4) Any subspace is not convex.

- ☐ True.
- ☐ False. correct.

(5) Let $C = \{x \in \mathbb{R}^n \mid x^T Ax + b^T x + c \leq 0\}$ with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. It is convex depends on

- ☐ $A = 0$.
- ☐ $A \succeq 0$. correct.
- ☐ $A \succ 0$.

(6) A wedge is convex which its general form is

- ☐ $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$. correct.

- ☐ $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \geq b_2\}$.
- ☐ $\{x \in \mathbb{R}^n \mid a_1^T x \geq b_1, a_2^T x \geq b_2\}$.

(7) The set of symmetric positive semidefinite matrices denotes as

- ☐ $\mathbb{S}_+^n = \{X \in \mathbb{R}^{n\times n} \mid X \succeq 0\}$. correct.
- ☐ $\mathbb{S}_+^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^T\}$.
- ☐ $\mathbb{S}_+^n = \{X \in \mathbb{R}^{n\times n} \mid X \succ 0\}$.

(8) The intersection of convex sets is convex.

- ☐ True. correct.
- ☐ False.

(9) Let K be a cone. The set of $K^* = \{y \mid x^T y \geq 0, \forall x \in K\}$ is called the dual cone of K.

- ☐ True. correct.
- ☐ False.

(10) The definition of convex function stated that $f : \mathbb{R}^n \to \mathbb{R}$, such that $\mathbf{dom} f \subseteq \mathbb{R}^n$ is convex, and $t \in (0, 1)$, then $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$.

- ☐ True.
- ☐ False. correct.

(11) Strictly convexity implies strong convexity.

- ☐ True.
- ☐ False. correct.

(12) Quadratic function of $f(x) = \dfrac{1}{2}x^T Ax$ is strictly convex if and only if

- ☐ $X \geq 0$.
- ☐ $A \succeq 0$
- ☐ $A \succ 0$ correct.
- ☐ $X \succ 0$

(13) Any norm is convex function except l_p, which defines as $||x||_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}, \forall p \geq 1$.

- ☐ True.
- ☐ False. correct.

(14) Affine function, $f(x) = a^T x + b$ is neither convex nor concave.

- ☐ True.
- ☐ False. correct.

(15) Let $f(x, y) = 5x^2 + 2xy + y^2 - x + 2y + 3$, then

- ☐ $H(f)$ is positive definite and f is strictly convex. correct.
- ☐ $H(f)$ is positive semidefinite and f is convex.
- ☐ $H(f)$ is negative semidefinite and f is concave.

(16) Let $f(u,v) = (u+2v+1)^2 - \ln(uv)^2$, where domain is $u > 1, v > 1$.

- ☐ $H(f)$ is positive definite and f is strictly convex. correct.
- ☐ $H(f)$ is positive semidefinite and f is convex.
- ☐ $H(f)$ is negative semidefinite and f is concave.

(17) The conjugate function of $f(x) = 3x^2 + 4x$ is

- ☐ $f^*(y) = (y-4)^4$
- ☐ $f^*(y) = \dfrac{(y-4)^4}{4}$
- ☐ $f^*(y) = \dfrac{(y-4)^4}{8}$
- ☐ $f^*(y) = \dfrac{(y-4)^4}{12}$ error-free mark. The answer is $f^*(y) = \dfrac{(y-4)^2}{12}$

(18) The optimal point is

- ☐ an x that achieves the optimal value. correct.
- ☐ an x that has the optimal set.
- ☐ an x that has the optimal function.

(19) The optimal point is defined as

- ☐ $p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) = 0, i = 1, ..., p\}$ correct.
- ☐ $p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) \geq 0, i = 1, ..., p\}$
- ☐ $p^* = \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) \leq 0, i = 1, ..., p\}$

(20) The domain of optimization problem is defined as

- ☐ $D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{i=0}^{p} \mathrm{dom} h_i$
- ☐ $D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{i=1}^{p} \mathrm{dom} h_i$ correct.
- ☐ $D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{i=2}^{p} \mathrm{dom} h_i$

(21) $\varepsilon-$suboptimal is defined as

- ☐ $f_0(x) \leq p^* + \varepsilon$, where $\varepsilon > 0$ correct.
- ☐ $f_0(x) + f_0(y) \leq p^* + \varepsilon$, where $\varepsilon > 0$
- ☐ $f(x) \geq p^* + \varepsilon$, where $\varepsilon > 0$

(22) The optimality criterion for for differentiable f_0 where $x, y \in \textbf{dom} f_0$ is denoted as:

- ☐ $f_0(y) = f_0(x) + \nabla f_0(x)^T(y - x)$
- ☐ $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y - x)$ correct.
- ☐ $f_0(y) \leq f_0(x) + \nabla f_0(x)^T(y - x)$

(23) The optimality criterion for differentiable f_0 where $x, y \in \textbf{dom} f_0$ and it feasible set is

- ☐ $X = \{x \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) = 0, i = 1, ..., p\}$ correct.
- ☐ $X = \{x \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) \geq 0, i = 1, ..., p\}$
- ☐ $X = \{x \mid f_i(x) \leq 0, i = 1, ..., m, h_i(x) \leq 0, i = 1, ..., p\}$

(24) Check the convex optimization below and known as.

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq 0 \\ & Ax = b \end{array}$$

where $G \in \mathbb{R}^{m \times n}, A \in \mathbb{R}^{p \times n}$

- ☐ Affine Programming
- ☐ Linear Programming correct.
- ☐ Quadratic Programming

(25) Are the two convex optimization problem equivalent?.

$$\begin{array}{llll} \textbf{Left} & & \textbf{Right} & \\ \text{minimize} & f_0(x) = x_1^2 + x_2^2 & \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 & \text{subject to} & f_1(x) = x_1 \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 & & h_1(x) = x_1 + x_2 = 0 \end{array}$$

- ☐ They are different.
- ☐ They are equivalence. correct.
- ☐ They are not convex optimization problem.

(15 pts) Part II: Solve the following questions:

(1) Prove that $f(x) = \frac{1}{2}x^TQx + b^Tx + c$ is convex function using its definition.

(2) Let $f(x, y, z) = \ln(xyz)$ is defined on $\mathbf{dom} f = \{(x, y, z) \mid x > 0, y > 0, z > 0\}$. Determine if this function is convex or concave.

(3) Find the conjugate function of $f(x) = \frac{1}{2}x^TAx + b^Tx + c$, where $A \succeq 0$.

(4) *(fill in the blank)*: The general terminology of convex optimization problem can we written as:

$$\begin{array}{ll} \text{..............} & f_0(x) \\ \text{..............} & f_i(x) \leq 0, \;\; i = 1, ..., m \\ & h_i(x) \leq 0, \;\; i = 1, ..., m \end{array}$$

Where

$$\begin{array}{ll} f_i(x) \leq 0 & \text{is called ....} \\ h_i(x) \leq 0 & \text{is called ....} \\ f_i(x) = 0 & \text{is called ....} \\ h_i(x) = 0 & \text{is called ....} \end{array}$$

(5) A publisher has orders from Manchester for 600 copies of an optimization textbook and orders from Oxford of 400 copies. The company has 700 copies in a warehouse in Liverpool and 800 copies in a warehouse in London. It costs \$5 to ship a book from Liverpool to Manchester, but it costs \$10 to ship it to Oxford. It costs \$15 to ship a text from London to Manchester, but it costs \$4 to ship it from London to Oxford. The problem is to determine how many copies should be shipped from each warehouse to Manchester and Oxford to fill the order at the least cost. Formulate a linear programming problem of the form

$$\begin{array}{ll} \text{minimize} & x^Tcx \\ \text{subject to} & Ax \leq b \end{array}$$

Then, we can summary as following by let x_1 and x_2 denote the number of texts shipped to Manchester from Liverpool and London, respectively, and x_3 and x_4 the units shipped to Oxford from Liverpool and London. The objective total cost of the shippings,

$$\begin{array}{ll} \text{minimize} & 5x_1 + 15x_2 + 10x_3 + 4x_4x_4 \\ \text{subject to} & x_1 + x_2 = 600 \\ & x_3 + x_4 = 400 \\ & x_1 + x_3 \leq 700 \\ & x_2 + x_4 \leq 800 \\ & x_i \geq 0, 1 \leq i \leq 4 \end{array}$$

Find the optimal solution of the LP above. *(hints: you have to solve for x_1, x_2, x_3, x_4 by consider all the constraint conditions).*

Solution of Part II

(1) Prove that $f(x) = \frac{1}{2}x^T Qx + b^T x + c$ is convex function using its definition.

Using definition of convex function, let $x, y \in \textbf{dom}f$, and $\theta \in [0,1]$ such that $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$

Then

$$\begin{aligned}
f(\theta x + (1-\theta)y) &= \frac{1}{2}(\theta x + (1-\theta)y)^T Q(\theta x + (1-\theta)y) + b^T(\theta x + (1-\theta)y) + c \\
&= \frac{1}{2}(\theta x^T + (1-\theta)y^T)Q(\theta x + (1-\theta)y) + \theta b^T x + (1-\theta)b^T x + \theta c + (1-\theta)c \\
&= \frac{1}{2}(\theta x^T Q - (1-\theta)y^T Q)(\theta x + (1-\theta)y) + \theta b^T x + (1-\theta)b^T x + \theta c + (1-\theta)c \\
&\leq \frac{1}{2}(\theta x^T Ax + \theta(1-\theta)y^T Qy) + \theta b^T x + (1-\theta)b^T y + \theta c + (1-\theta)c \\
&\leq \theta\left[\frac{1}{2}x^T Qx + b^T x + c\right] + (1-\theta)\left[\frac{1}{2}y^T Qy + b^T y + c\right] \\
&\leq \theta f(x) + (1-\theta)f(y) \qquad \blacksquare
\end{aligned}$$

Noted: the inequality happen when $(y^T - x^T)(y - x) < 0$

(2) Let $f(x,y,z) = \ln(xyz)$ is defined on $\textbf{dom}f = \{(x,y,z) \mid x > 0, y > 0, z > 0\}$. Determine if this function is convex or concave.

$$\begin{aligned}
\nabla f &= \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \\
H(f) &= \begin{pmatrix} -1/x^2 & 0 & 0 \\ 0 & -1/y^2 & 0 \\ 0 & 0 & -1/z^2 \end{pmatrix}
\end{aligned}$$

We have $D_1 = -\frac{1}{x^2} < 0, D_2 = \frac{1}{x^2y^2} > 0, D_3 = -\frac{1}{x^2y^2z^2} < 0$ since $x, y, z > 0$.

$\implies H(f)$ is negative definite, which mean f is concave. $\blacksquare$

Noted I ask to find out whether function is convex or concave, strictly or not is not important. :)

(3) Find the conjugate function of $f(x) = \frac{1}{2}x^T Ax + b^T x + c$, where $A \succeq 0$.

We have to find $f^*(y) = \sup_{x \in \textbf{dom}f}(y^T x - f(x))$, by knowing that $A \succeq 0$

$$
\begin{aligned}
f^*(y) &= y^T x - \left(\frac{1}{2}x^T Ax + b^T x + c\right) \\
&= y^T x - \frac{1}{2}x^T Qx - b^T x - c \\
\frac{\partial f^*(y)}{\partial x} &= y - xQ - b = 0 \implies x = Q^{-1}(y-b) \\
\implies f^*(y) &= y^T Q^{-1}(y-b) - \frac{1}{2}\left[Q^{-1}(y-b)\right]^T Q \left[Q^{-1}(y-b)\right] - b^T(y-b)Q^{-1} - c \\
\implies f^*(y) &= y^T Q^{-1}(y-b) - \frac{1}{2}(y-b)^T Q^{-1}(y-b) - b^T(y-b)Q^{-1} - c \\
&= (y^T - b)Q^{-1}(y-b) - \frac{1}{2}(y-b)^T Q^{-1}(y-b) - c \\
&= (y-b)^T Q^{-1}(y-b) - \frac{1}{2}(y-b)^T Q^{-1}(y-b) - c \\
&= \frac{1}{2}(y-b)^T Q^{-1}(y-b) - c
\end{aligned}
$$

Therefore, the conjugate function is $f^*(y) = \frac{1}{2}(y-b)^T Q^{-1}(y-b) - c$ ■

(4) *(fill in the blank)*: The general terminology of convex optimization problem can we written as:

$$
\begin{array}{ll}
\textit{minimize} & f_0(x) \\
\textit{subject to} & f_i(x) \le 0, \;\; i = 1, ..., m \\
 & h_i(x) \le 0, \;\; i = 1, ..., m
\end{array}
$$

Where

$$
\begin{array}{ll}
f_i(x) \le 0 & \text{is called } \textit{inequality constraints} \\
h_i(x) \le 0 & \text{is called } \textit{inequality constraints} \\
f_i(x) = 0 & \text{is called } \textit{equality constraints} \\
h_i(x) = 0 & \text{is called } \textit{equality constraints}
\end{array}
$$

(5) You can get all of shipments to Machester from Liverpool, and all the ones to Oxford from London, by givening the optimal solution is $x_1 = 600, x_2 = 0$, and $x_3 = 400, x_4 = 0$. Actually you don't need to solve anything, since you can give minimize the shipments by using only two equality constraint while it still holds for inequality constraints. What a weir day, right!.