TD1 - Convex Set

Exercise 1. Let C be a nonempty convex subset of $\mathbb{R}^n$

(a) Let $f : C \to \mathbb{R}$ be a convex function, and $g : \mathbb{R} \to \mathbb{R}$ be a function that is convex and montonically nondecreasing over a convex set that contains the set of values that f can take, $\{f(x) \mid x \in C\}$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. In addition, if g is montonically increasing and f is strictly convex, then h is strictly convex.

(b) Let $f = (f_1, ..., f_m)$, where $f_i : C \to \mathbb{R}$ is convex function, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a function that is convex and monotonically nodecreasing over a convex set that contains the set $\{f(x) \mid x \in C\}$, in the sense that for all u, u_1 in this set such that $u \leq u_1$, we have $g(u) \leq g(u_1)$. Show that function h defined by $h(x) = g(f(x))$ is convex over $C \times \cdots \times C$.

Exercise 2. What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^\top x = b_1\}$ and $\{x \in \mathbb{R}^n : a^\top x = b_2\}$.

Exercise 3. Which of the following sets are convex? Justify your answer.

(a) A *slab, i.e.,* a set of the form $\{x \in \mathbb{R}^\times : \alpha \leq a^\top x \leq \beta\}$

(b) A *rectangle, i.e.,* a set of the form $\{x \in \mathbb{R}^n : \alpha_i \leq x_i \leq \beta_i, \forall i = 1, ..., n\}$

(c) A *wedge, i.e.,* a set of the form $\{x \in \mathbb{R}^n : a_1 x \leq b_1, a_2^\top x \leq b_2\}$

(d) The set of points closer to a given point than to a given (arbitrary) set $S \subset \mathbb{R}^n$, *i.e.,*

$$\{x \in \mathbb{R}^n : ||x - x_0||_2 \leq ||x - y||_2 \text{ for all } y \in S\}$$

(e) The set of points closer to one set than another, *i.e.,*

$$\{x : \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{||x - z||_2 : z \in S\}$$

(f) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.,* the set $\{x : ||x - a|_2 \leq \theta ||x - b||_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

(g) The set $\{x : x + S_2 \subset S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Exercise 4. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n : x^\top A x + b^\top x + c \leq 0\},$$

with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^\top x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^\top \succeq 0$ for some $\lambda \in \mathbb{R}$

Exercise 5. *Hyperbolic sets.* Show that the *hyperbolic set* $\{x \in \mathbb{R}^2_+ : x_1x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}^n_+ : \prod_{i=1}^{n} x_i \geq 1\}$ is convex. *Hint.* If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$.

Exercise 6. Show that if S_1 and S_2 are convex set in $\mathbb{R}^{m\times n}$, then so is their

$$S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

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Exercise 7. Let study the image of hyperplanes, halfspaces, and polyhedra under the perspective function $P(x, t) = x/t$, with $\mathbf{dom}P = \mathbb{R}^n \times R_{++}$. For each of the following sets C, give a simple description of

$$P(C) = \{v/t \mid (v, t) \in C, t > 0\}$$

(a) The polyhedron $C = \mathbf{conv}\{(v_1, t_1), ..., (v_k, t_k)\}$ where $v_k \in \mathbb{R}^n$ and $t_i > 0$.

(b) The hyperplane $C = \{(v, t) \mid f^T v + gt = h\}$, with f and g not both zero.

(c) The halfspace $C = \{(v, t) \mid f^T v + gt \leq h\}$, with f and g not both zero.

Exercise 8. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad \mathbf{dom}f = \{x \mid c^T x + d > 0\}$$

We want to study the inverse image of a convex set C under f, *i.e.,*

$$f^{-1}(C) = \{x \in \mathbf{dom}f \mid f(x) \in C\}$$

For each of the following sets $C \subseteq \mathbb{R}^n$, given a simple description of $f^{-1}(C)$.

(a) The halfspace $C = \{y \mid g^T y \leq h\}$ with $g \neq 0$.

(b) The polyhedron $C = \{y \mid Gy \succeq h\}$.

(c) The ellipsoid $\{y \mid y^T P^{-1} y \leq 1\}$, where $P \in \mathbb{S}^n_{++}$.

Exercise 9. Given an example of two closed convex sets that are disjoint but cannot be strictly separated.

Exercise 10. Supporting hyperplanes.

(a) Express the closed convex set $\{x \in \mathbb{R}^2_+ \mid x_1x_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{x \in \mathbb{R}^n \mid ||x||_\infty \leq 1\}$, the l_∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{a}$ be a point in the boundary of C. Identity the supporting hyperplanes of C at $\hat{x}$ explicitly.

Exercise 11. *Positive semidefinite cone for* $n = 1, 2, 3$. Given an explicit description of the positive semidefinite cone $\mathbb{S}^n_+$ in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element on $\mathbb{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \quad \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix}$$

Exercise 12. *Cones in* $\mathbb{R}^2$. Suppose that $K \subseteq \mathbb{R}^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements ($x = r(\cos\theta, \sin\theta)$) with $r \geq 0$).

(b) Give a simple description of K^*, and draw a plot illustrating the relation between K and K^*.

(c) Where is K pointed?

(d) Where is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $x \preceq_K y$ means when K is proper.

Exercise 13. *Properties of dual cones.* Let K^* be the dual cone of a convex cone K, as defined as

$$K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$$

Prove the following.

(a) K^* is indeed a convex cone.

(b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

(c) K^* is closed.

(d) The interior of K^* is given by $\mathbf{int}K^* = \{y \mid y^T x > 0 \text{ for all } x \in \mathbf{cl}K\}$

Exercise 14. Find the dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in \mathbb{R}^{m\times n}$

Exercise 15. Let K_{pol} be the set of (coefficients of) non-negative polynomials of degree $2k$ on $\mathbb{R}$

$$K_{\text{pol}}\{x \in \mathbb{R}^{2k+1} \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k} \geq 0 \text{ for all } t \in \mathbb{R}\}$$

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(a) Show that K_{pol} is a proper cone.

(b) A basis result state that a polynomial of degree $2k$ is nonnegative on $\mathbb{R}$ if and only if it can be expressed as the sum of squares of two polynomials of degree k or less. In other words, $x \in K_{\text{pol}}$ if and only if the polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}$$

can be expressed as

$$p(t) = r(t)^2 + s(t)^2$$

Use this result to show that

$$K_{\text{pol}}\left\{x \in \mathbb{R}^{2k+1} \;\middle|\; x_i = \sum_{m+n=i+1} Y_{mn} \text{ for some } Y \in \mathbb{S}_+^{k+1}\right\}$$

In other word, $p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}$ is nonnegative if and only if there exists a matrix $Y \in \mathbb{S}_+^{k+1}$ such that

$$\begin{aligned} x_1 &= Y_{11} \\ x_2 &= Y_{12} + Y_{21} \\ x_3 &= Y_{13} + Y_{22} + Y_{31} \\ &\vdots \\ x_{2k+1} &= Y_{k+1,k+1} \end{aligned}$$

(c) Show that $K_{\text{pol}}^* = K_{\text{han}}$ where

$$K_{\text{han}} = \{z \in \mathbb{R}^{2k+1} \mid H(z) \succeq 0\}$$

and

$$H(z) = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_k & z_{k+1} \\ z_2 & z_3 & z_4 & \cdots & z_{k+1} & z_{k+2} \\ z_3 & z_4 & z_5 & \cdots & z_{k+2} & z_{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z_k & z_{k+1} & z_{k+2} & \cdots & z_{2k-1} & z_{2k} \\ z_{k+1} & z_{k+2} & z_{k+3} & \cdots & z_{2k} & z_{2k+1} \end{pmatrix}$$

This is the *Hankel matrix* with coefficient $z_1, ..., z_{2k+1}$.