TD2 - Convex Functions

Q2 - 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex, and $a, b \in \mathbf{dom} f$ with $a < b$

(a) Show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

(b) Show that

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

(c) Suppose f is differentiable. Use the result in (b) to show that

$$f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(a)$$

Note that these inequalities also follow from $f(y) \geq f(x) + \nabla f(x)^T (y-x)$:

$$f(b) \geq f(a) + f'(a)(b-a), \qquad f(a) \geq f(b) + f'(b)(a-b)$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \geq 0$ and $f''(a) \geq 0$

Q2 - 2. For each function below, determine whether it is convex, strictly convex, strongly convex or none of the above.

(a) $f(x) = (x_1 - 3x_2)^2$

(b) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2$

(c) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2 + x_1^3$

(d) $f(x) = |x| \quad (x \in \mathbb{R})$

Q2 - 3. (a) Show that every affine function $f(x) = ax+b, x \in \mathbb{R}$ is convex, but not strictly convex.

(b) Show that $f(x) = x^2, x \in \mathbb{R}$ is strictly convex.

(c) Show that the two-dimensional ball $B = \{[x, y] : x^2 + y^2 \leq 1\}$ is convex.

(d) Show that the complement of the ball $B^C = \{[x, y] : x^2 + y^2 > 1\}$ is not convex.

Q2 - 4. Show that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its value at the endpoints of the segment: for every $x, y \in \mathbb{R}^n$,

$$\int_0^1 f(x + \lambda(y-x)) dA \leq \frac{f(x)+f(y)}{2}$$

Q2 - 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex with $\mathbb{R}_+ \subseteq \mathbf{dom} f$. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \qquad \mathbf{dom} F = \mathbb{R}_{++}$$

is convex. We can assume that f is differentiable.

Q2 - 6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex with $\mathbf{dom} f = \mathbb{R}^n$, and bounded above on $\mathbb{R}^n$. Show that f is constant.

Q2 - 7. Compute the Hessian of the function $f(x, y) = x - y - x^2$, and show that f is a concave function defined on $D_f = \mathbb{R}^2$. Determine if $g(x, y) = e^{x-y-x^2}$ is a convex or a concave function on $\mathbb{R}^2$.

Q2 - 8. Let $f(x, y) = ax^2 + bxy + cy^2 + dx + ex + f$ be a general polynomial in two variables of degree two. For which values of the parameters is this function (strictly) convex and (strictly) concave?

Q2 - 9. Determine the values of the parameter α for which the function

$$f(x, y) = -6x^2 + (2\alpha + 4)xy - y^2 + 4\alpha y$$

is convex and concave on $\mathbb{R}^2$.

Q2 - 10. Prove that a twice differentiable function f is convex if and only if its domain is convex and $\nabla^2 f(x) \succeq 0$ for all $x \in$ **dom** f.

Q2 - 11. Let $F \in \mathbb{R}^{n\times m}, \hat{x} \in \mathbb{R}^n$. The *restriction* of $f : \mathbb{R}^n \to \mathbb{R}$ to the affine set $\{Fz+\hat{x} \mid z \in \mathbb{R}^m\}$ is define as function : $\mathbb{R}^m \to \mathbb{R}$ with

$$\tilde{f}(z) = f(Fz + \hat{x}), \qquad \mathbf{dom}\ \tilde{f} = \{z \mid Fz + \tilde{x} \in \mathbf{dom}\ f\}$$

Suppose f is twice differentiable with a convex domain.

(a) Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\ \tilde{f}$

$$F^T\nabla^2 f(Fz + \hat{x})F \succeq 0$$

.

(b) Suppose $A \in \mathbb{R}^{p\times n}$ is a matrix whose nullspace is equal to the range of F, *i.e.*, $AF = 0$ and $\mathbf{rank}A = n - \mathbf{rank}F$. Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\ \tilde{f}$ there exists a $\lambda \in \mathbb{R}$ such that

$$\nabla^2 f(Fz + \hat{x}) + \lambda A^T A \succeq 0$$

.

Q2 - 12. For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$

(d) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$

Q2 - 13. Suppose $p < 1, p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$$

with dom $f = \mathbb{R}^n_{++}$ is concave. This include as special cases $f(x) = \left(\sum_{i=1}^n \sqrt{x_i}\right)^2$ and the harmonic mean $f(x) = \left(\sum_{i=1}^n 1/x_i\right)^{-1}$.

Q2 - 14. Adopting the proof of concavity of the log-determinant function, show the following:

(a) $f(X) = \mathbf{tr}(X^{-1})$ is convex on $\mathbf{dom} f = \mathbb{S}^n_{++}$.

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom} f = \mathbb{S}^n_{++}$.

Q2 - 15. Show that the following function $f : \mathbb{R}^n \to \mathbb{R}$ are convex.

(a) $f(x) = ||Ax - b||$, where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, and $|| \cdot ||$ is norm on $\mathbb{R}^m$.

(b) $f(x) = -(\det(A_0 + x_1A_1 + \cdots + x_nA_n))^{1/m}$, on $\{x \mid A_0 + x_1A_1 + \cdots + x_nA_n \succ 0\}$ where $A_i \in \mathbb{S}^m$

(c) $F(X) = \mathbf{tr}(A_0 + x_1A_1 + \cdots + x_nA_n)^{-1}$, on $\{x \mid A_0 + x_1A_1 + \cdots + x_nA_n \succ 0\}$, where $A_i \in \mathbb{S}^m$.

Q2 - 16. *Perspective of a function.*

(a) Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{||x||_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$.

(b) Show that

$$f(x) = \frac{||Ax + b||_2^2}{c^T x + d}$$

is convex on $\{x \mid c^T x + x > 0\}$, where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

Q2 - 17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ as the pointwise supremum all affine functions that are global understimators of f:

$$(x) = \sup\{g(x) \mid g \text{ affine}, g(z) \leq f(z) \text{ for all } z\}$$

(a) Show that $f(x) = \tilde{f}(x)$ for $x \in$ **int dom**f.

(b) Show that $f =$ if f is closed (*i.e.,* **epi** f is a closed set).

Q2 - 18. Give a direct proof that the perspective function g, of a convex function f is convex. Show that **dom**g is a convex set, and that for $(x, t), (y, s) \in$ **dom**g, and $0 \leq \theta \leq 1$, we have

$$g(\theta x + (1-\theta)y, \theta t + (1-\theta)s) \leq \theta g(x, t) + (1-\theta)g(y, s)$$

.

Q2 - 19. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and twice continuously differentiable. Suppose $\bar{y}$ and $\bar{x}$ are related by $\bar{y} = \nabla f(\bar{x})$, and that $\nabla^2 f(\bar{x}) \succ 0$.

(a) Show that $\nabla f^*(\bar{y}) = \bar{x}$.

(b) Show that $\nabla^2 f^*(\bar{y}) = \nabla^2 f(\bar{x})^{-1}$

Q2 - 20. Show that the following functions are log-concave.

(a) *Logistic function:* $f(x) = e^x/(1+e^x)$ with $\mathbf{dom} f = \mathbb{R}$.

(b) *Harmonic mean:*

$$f(x) = \frac{1}{1/x_1 + \cdots + 1/x_n}, \qquad \mathbf{dom} f = \mathbb{R}^n_{++}$$

.

(c) *Determinant over trace:*

$$f(X) = \frac{\mathbf{det} X}{\mathbf{tr} X}, \qquad \mathbf{dom} f = \mathbb{S}^n_{++}$$

Q2 - 21. Show that the function $f(X) = X^{-1}$ is matrix convex on $\mathbb{S}^n_{++}$.

Q2 - 22. Suppose $X \in \mathbb{S}^n$ partitioned as

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where $A \in \mathbb{S}^k$. The *Schur complement* of X (with respect to A) is $S = C - B^T A^{-1} B$. Show that the Schur complement, viewed as a function from $\mathbb{S}^n$ into $\mathbb{S}^{n-k}$ is matrix concave on $\mathbb{S}^n_{++}$.