TD

Chapter 3: Optimization Problem

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Exercise 1. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 0 \\ & x \geq 0, \quad x_2 \geq 0 \end{array}$$

For each of the following objective functions, give the optimal set and optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$

(b) $f_0(x_1, x_2) = -x_1 - x_2$

(c) $f_0(x_1, x_2) = x_1$

(d) $f_0(x_1, x_2) = \max(x_1, x_2)$

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

Exercise 2. Consider the optimization proble

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain $\textbf{dom} f_0 = \{x \mid Ax \prec b\}$, where $A \in \mathbb{R}^{m \times n}$ (with row a_i^T). And assume that $\textbf{dom} f_0$ is nonempty. Prove the following facts.

(a) $\textbf{dom} f_0$ is unbounded if and only if there exists a $v \neq 0$ with $Av \preceq 0$.

(b) f_0 is unbounded below if and only if there exists a v with $Av \preceq 0, Av \neq 0$.

(c) If f_0 is bounded below then its minimum is attained., *i.e.*, there exists an x that satified the optimality condition.

Exercise 3. Some simple LPs. Give an explicit solution of each of the following LPs.

(a) *Minimizing a linear function over an affine set.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \end{array}$$

(b) *Minimize a linear function over a halfspace.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b, \end{array}$$

where $a \neq 0$

(c) *Minimizing a linear function over a rectangle.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \preceq x \preceq u, \end{array}$$

where l and u satisfy $l \preceq u$.

(d) *Minimizing a linear function over probability simplex.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^x = 1, x \succeq 0. \end{array}$$

(e) *Minimizing a linear function over a unit box with a total budget constraint.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^x = \alpha, 0 \preceq x \preceq 1. \end{array}$$

Where α is an integer between 0 and n. What happens if α is not an integer (but satisfy $0 \leq \alpha \leq n$)? What if we change the equality to an inequality $\mathbf{1}^T x \leq \alpha$?

Exercise 4. (Square LP). Consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

with A square and nonsingular. Show that the optimal value is given by

$$p* = \begin{cases} c^T A^{-1} b & A^{-1} c \preceq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Exercise 5. *Problem involving l_1 and l_∞-norms.* Formulate the following problems as LPs. Explain in detail the relation between the optimal solution for each problem and the solution of its equivalent LP.

(a) Minimize $||Ax - b||_\infty$ ($l_\infty-$norm approximation)

(b) Minimize $||Ax - b||_1$ (l_1-norm approximation)

(c) Minimize $||Ax - b||_1$ subject to $||x||_\infty \leq 1$

(c) Minimize $||x||_\infty \leq 1$ subject to $||Ax - b||_\infty$

Exercise 6. Consider the problem, with variable $x \in \mathbb{R}^n$,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \text{ for all } A \in \mathcal{A} \end{array}$$

where $\mathcal{A} \subseteq \mathbb{R}^{m \times n}$ is the set

$$\mathcal{A} = \{A \in \mathbb{R}^{m \times n} | \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij}, i = 1, ..., m, j = 1, ..., n\}$$

This problem can be interpreted as an LP where each coefficient of A is only known to lie in an interval, and we require that x must satisfy the constraints for all possible values of the coefficients. Express this problem as an LP. The LP you construct should be efficient, i.e., it should have dimensions that grow exponentially with n and m.

Exercise 7. (a) Show that the following function f is convex on $\mathbf{dom}f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{||x||_p^p}{t^{p-1}} \text{ (where } p > 0)$$

(b) Show that the following function f is convex on $\mathbf{dom}f = \{x \in \mathbb{R}^n : c^T x + d > 0\}$

$$f(x) = \frac{||Ax+b||_2^2}{c^T x + d} \text{ (where } A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R})$$

Exercise 8. Consider the problem

$$\begin{array}{ll} \text{minimize} & ||Ax-b||_1/(c^T x + d) \\ \text{subject to} & ||x||_\infty \le 1 \end{array}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R}$. We assume that $d > ||c||_1$, which implies that $c^T x + d > 0$ for all feasible x.

(a) Show that this is a quasiconvex optimization problem.

(b) Show that it is equivalent to the convex optimization problem

$$\begin{array}{ll} \text{minimize} & ||Ay-bt||_1 \\ \text{subject to} & ||y||_\infty \le t \\ & c^T y + dt = 1 \end{array}$$

where variable $y \in \mathbb{R}^n, t \in \mathbb{R}$.

Exercise 9. Given an explicit solution of each of the following QCQPs.

(a) *Minimizing a linear function over an ellipsoid centered at the origin*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x^T A x \le 1 \end{array}$$

where $A \in \mathbb{S}_{++}^n$ and $c \neq 0$. What is the solution if the problem is not convex ($A \notin \mathbb{S}_{++}^n$)?

(b) *Minimizing a linear function over an ellipsoid*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (x-x_c)^T A (x-x_c) \le 1 \end{array}$$

where $A \in \mathbb{S}_{++}^n$ and $c \neq 0$.

(c) *Minimizing a quadratic form over an ellipsoid centered at the origin*

$$\begin{array}{ll} \text{minimize} & x^T B x \\ \text{subject to} & x^T A x \le 1 \end{array}$$

where $A \in \mathbb{S}_{++}^n$ and $B \in \mathbb{S}_+^n$. Also consider the nonconvex extension with $B \notin \mathbb{S}_+^n$.

Exercise 10. Consider the QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^TPx + q^Tx + r \\ \text{subject to} & x^Tx \leq 1 \end{array}$$

where $P \in \mathbb{S}^n_{++}$. Show that $x* = -(P+\lambda I)^{-1}q$ where $\lambda = \max\{0, \bar{\lambda}\}$ and $\bar{\lambda}$ is the largest solution of the nonlinear equation

$$q^T(P+\lambda I)^{-2}q = 1$$

Exercise 11. Express the following problems as convex optimization problems.

(a) Minimize $\max\{p(x), q(x)\}$, where p and q are posynomials.

(b) Minimize $\exp(p(x)) + \exp(q(x))$, where p, and q are posynomails.

(c) Minimize $p(x)/(r(x) - q(x))$, subject to $r(x) > q(x)$, where p, q are posynomials, and r is monomial.

Exercise 12. Formulate the following optimization problems as semidefinite programs. The variable is $x \in \mathbb{R}^n$; $F(x)$ is defined as

$$F(x) = F_0 + x_1F_1 + x_2F_2 + \cdots + x_nF_n$$

with $F_i \in \mathbb{S}^m$. The domain of f in each subproblem is $\mathbf{dom} f = \{x \in \mathbb{R}^n \mid F(x) \succ 0\}$.

(a) Minimize $f(x) = c^TF(x)^{-1}c$ where $c \in \mathbb{R}^m$.

(b) Minimize $f(x) = \max\limits_{i=1,\ldots,K} c_i^TF(x)^{-1}c_i$ where $c_i \in \mathbb{R}^m, i = 1, \ldots, K$.

(c) Minimize $f(x) = \sup\limits_{||c||_2 \leq 1} c^TF(x)^{-1}c$.

Exercise 13. Consider SDP

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & x_1F_1 \mid x_2F_2 \mid \cdots \mid x_nF_n \mid G \preceq 0 \end{array}$$

where $F_i, G \in \mathbb{S}^k, c \in \mathbb{R}^n$.

(a) Suppose $R \in \mathbb{R}^{k\times k}$ is nonsingular. Show that the SDP is equivalent to the SDP

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & x_1\tilde{F}_1 + x_2\tilde{F}_2 + \cdots + x_n\tilde{F}_n + \preceq 0 \end{array}$$

where $_i = R^TF_iR, = R^TGR$.

(b) Suppose there exists a nonsingular R such that $_i$ and $\tilde{G}$ are diagonal. Show that the SDP is equivalent to an LP.

Exercise 14. *LPs, QPs, QCQPs, and SOCPs and SDPs.* Express the following problems as SDPs.

(a) The LP.

(b) The QP, the QCQP, and SOCP. *Hint.* Suppose $A \in \mathbb{S}^n_{++}, C \in \mathbb{S}^s, B \in \mathbb{R}^{r\times s}$. Then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \Longleftrightarrow C - B^T A^{-1} B \succeq 0$$

(c) The matrix fraction optimization problem

$$\text{minimize} \quad (Ax+b)^T F(x)^{-1}(Ax+b)$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$,

$$F(x) = F_0 + x_1F_1 + \cdots + x_nF_n,$$

with $F_i \in \mathbb{S}^m$, and we take the domain of the objective to be $\{x \mid F(x) \succ 0\}$. You can assume the problem is feasible (there exists at least one x with $F(x) \succ 0$).

Exercise 15. In this program, we consider a robust variation of the (convex) quadratic program

$$\begin{array}{ll} \text{minimize} & (1/2)x^TPx + q^Tx + r \\ \text{subject to} & Ax \preceq b \end{array}$$

For simplicity we assume that only the matrix P is subject to errors, and the other parameters (q, r, A, b) are exactly known. The robust quadratic program is defined as

$$\begin{array}{ll} \text{minimize} & \sup\limits_{p\in\varepsilon}((1/2)x^TPx + q^Tx + r) \\ \text{subject to} & Ax \preceq b \end{array}$$

where ε is the set of possible matrices P.

For each of the following set ε, express the robust QP as a *tractable* convex problem. Be as specific as you can. (Here, tractable means that the problem can be reduced to an LP, QP, QCQP, SOCP, or SDP. But you do not have to work out the reduction, if it is complicated; it is enough to argue that it can be reduced to one of these.)

(a) A finite set of matrices: $\varepsilon = \{P_1, ..., P_K\}$, where $P_i \in \mathbb{S}^n_+, i = 1, ..., K$.

(b) A set specified by a nominal value $P_0 \in \mathbb{S}^n_+$ plus a bound on the eigenvalues of the deviation $P - P_0$:

$$\varepsilon = \{P \in \mathbb{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$$

where $\gamma \in \mathbb{R}$ and $P_0 \in \mathbb{S}^n_+$.

(c) An ellipsoid of matrices:

$$\varepsilon = \left\{ P_0 + \sum_{i=1}^K P_i u_i \;\middle|\; ||u||_2 \le 1 \right\}$$

You can assume $P_i \in \mathbb{S}^n_+, i = 0, ..., K$.