Chapter 4: Optimization for Data Science
Duality

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Lagrangian Duality

$$P : \begin{cases} \min_{x \in \mathbb{R}^n} & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \forall i = 1, ..., m \\ & h_i(x) = 0, \forall i = 1, ..., p \end{cases}$$

- $x \in \mathbb{R}$ decision variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ objective function
- $f_i : \mathbb{R}^n \to \mathbb{R}$ inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}$ equality constraint functions

Assume for simplicity that function values cannot be ∞

Lagrangian

Definition: The Lagrangian L of the problem P is defined as the function $L : \mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^p \to \mathbb{R}$

$$L(x, \lambda, y) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

- Weighted sum of objective and constraint functions.
- λ_i is the Lagrange multiplier corresponding to $f_i(x) \leq 0$.
- μ_i is the Lagrange multiplier corresponding to $h_i(x) = 0$.

The Lagrangian is concave (affine) in (λ, μ) for any fixed x.
If P is convex optimization problem, the Lagrangian is convex in x for any fixed (λ, μ), i.e., L is saddle function.

Lagrangian Cont'd

The Lagrangian allows us to reexpress the optimization problem P as a min-max problem. Indeed, define

$$f(x) = \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p}$$

and note that

$$f(x) = \begin{cases} f_0(x), & \text{if } f_i(x) \leq 0 \forall i \text{ and } h_i(x) = 0 \forall i \\ \infty & \text{otherwise.} \end{cases}$$

Thus, we obtain

Lagrangian Reformulation:

$$\inf P = \inf_{x \in \mathbb{R}^n} f(x) = \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p} L(x, \lambda, \mu)$$

The Dual Problem

Below we refer to P as the primal problem. Using the Lagrangian we can introduced a dual problem with objective function.

$$g(\lambda, \mu) = \infty_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

Dual Problem:

$$D : \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p} g(\lambda, \mu)$$

By construction, D is equivalent to a min-max problem.

$$\sup D = \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p} g(\lambda, \mu) = \sup_{\lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p} \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

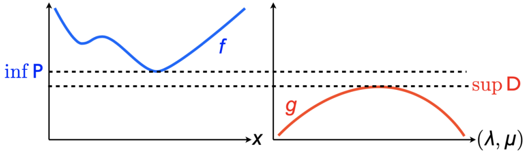
Weak Duality

Proposition: $g(\lambda, \mu) \geq f(x), \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p$
Proof: By definition we have

$$g(\lambda, \mu) = \inf_{\bar{x} \in \mathbb{R}^n} L(\bar{x}, \lambda, \mu) \leq L(x, \lambda, \mu) \leq \sup_{\bar{\lambda}, \bar{\mu}} L(x, \bar{\lambda}, \bar{\mu}) = f(x)$$

Corollary (Weak Duality):

$$\sup D \leq \inf P$$



Thus if D is unbounded ($\sup D = \infty$), then P must be infeasible. If P is unbounded ($\inf P = -\infty$), then D must be infeasible.

Significance of Dual Solutions

Note: Every feasible solution of P (D) provides an upper (lower) bound on both $\inf \mathsf{P}$ and $\sup \mathsf{D}$.

Assume $\hat{x}$ is a (feasible) candidate solution for P. Its quality is quantified by $f_0(\hat{x}) - \inf \mathsf{P}$. However, $\inf \mathsf{P}$ is unknown!

A dual feasible solution (λ, μ) provides a proof or certificate that

$$\inf \mathsf{P} \geq g(\lambda, \mu) \Rightarrow f_0(\hat{x}) - \inf \mathsf{P} \leq f_0(\hat{x}) - g(\lambda, \mu)$$

Thus, if $f_0(\hat{x}) - g(\lambda, \mu) \leq \varepsilon$, then $\hat{x}$ is an ε-optimal solution.

Strong Duality

Definition: $\Delta = \inf P - \sup P$ is called the duality gap. By weak duality, we have that $\Delta \geq 0$. If $\Delta = 0$, we say that strong duality holds.

$$P : \begin{cases} \min_{x \in \mathbb{R}^n} & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \forall i = 1, \dots, m \\ & h_i(x) = 0, \forall i = 1, \dots, p \end{cases}$$

Strong Duality

- does not hold in general.
- always holds if P is a convex problem satisfying a constraint qualification.

Slater's constraint qualification holds if there exist x_S with

- $f_i(x_S) < 0, \forall i = 1, \dots, m$
- $h_i(x_S) = 0, \forall i = 1, \dots, p$

Least squares problem

Primal Problem

$$\begin{cases} \min_{x \in \mathbb{R}^n} & x^T x \\ \text{s.t.} & Ax = b \end{cases}$$

Lagrangian

$$L(x, \mu) = x^T x + \mu^T (Ax - b)$$

Dual Objective

$$\begin{aligned} &(\mu) = \min_{x \in \mathbb{R}^n} L(x, \mu) \\ &\Rightarrow \nabla_x L(x, \mu) = 2x + A^T \mu = 0 \\ &\Rightarrow x = -\frac{1}{2} A^T \mu \\ &\Rightarrow g(\mu) = -\frac{1}{4} \mu^T A A^T \mu - b^T \mu \end{aligned}$$

Dual Problem

$$\max_{\mu \in \mathbb{R}^m} -\frac{1}{4} \mu^T A A^T \mu - b^T \mu$$

Standard form Linear Program

Primal problem
$$\begin{cases} \min_{x \geq 0} & c^T x \\ \text{s.t.} & Ax = b \end{cases}$$

Lagrangian
$$L(x, \lambda, \mu) = c^T x - \lambda^T x + \mu^T (Ax - b)$$

Dual objective
$$g(\lambda, \mu) = \min_x L(x, \lambda, \mu)$$

$$= \begin{cases} -b^T \mu, & c - \lambda + A^T \mu = 0 \\ -\infty & \text{else} \end{cases}$$

Dual Problem
$$\begin{cases} \max_{\mu, \lambda \geq 0} & -b^T \mu \\ \text{s.t.} & c - \lambda + A^T \mu = 0 \end{cases}$$

$$\iff \begin{cases} \max_{\mu} & b^T \mu \\ \text{s.t.} & A^T \mu \leq c \end{cases}$$

Quadratic Program

Primal Problem $(P \succ 0)$
$$\begin{cases} \min_x & x^T P x \\ \text{s.t.} & Ax \le b \end{cases}$$

Lagrangian
$$L(x, \lambda) = x^T P x + \lambda^T (Ax - b)$$

Dual Objective
$$\begin{aligned} g(\lambda, \mu) &= \min_x L(x, \lambda, \mu) \\ &= -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda \end{aligned}$$

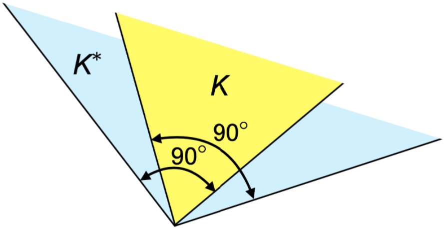
Dual Problem
$$\max_{\lambda \ge 0} -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

Dual Cones

Definition: If K is a cone, then the set

$$K^* = \{y \in \mathbb{R}^n : x^T y \geq 0, \forall x \in K\}$$

is the dual cone of K.

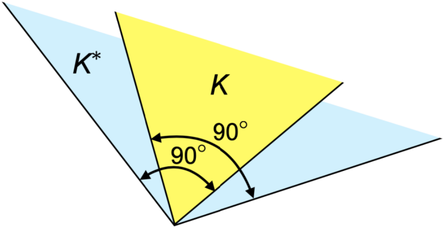


Note that K^* is a cone by construction.
Example: $(\mathbb{S}^n_+) = \{Y \in \mathbb{S}^n : \mathrm{tr}(X^T Y) \geq 0, \forall X \in \mathbb{S}^n_+\}$

Dual Cones Cont'd

Properties of the dual cone:

- K^* is closed and convex.
- $K_2 \subset K_1 \implies K_1^* \subseteq K_2^*$ (the smaller K, the larger K^*).
- $K^{**} = \mathrm{cl}(\mathrm{conv}(K))$, the smallest convex closed superset of K.
- If a convex cone K is proper, then K^* is proper and $K^{**} = K$



Definition: A cone K is called self-dual, if $K^* = K$.
Example of self-dual cones: $\mathbb{R}^n_+$, the second-order cone $\mathbb{S}^n_+$.

Problems with Generalized Inequality

$$P : \begin{cases} \min_{x \in \mathbb{R}^n} & f_0(x) \\ \text{s.t.} & f_i(x) \preceq_{K_i} 0, \forall i = 1, \dots, m \\ & h_i(x) = 0, \forall i = 1, \dots, p \end{cases}$$

Where $K_i \subset \mathbb{R}^{r_i}$ is a proper convex cone.

- Assign a Lagrange multiplier $\lambda_i \in K_i^*$ to $f_i(x) \preceq_{K_i} 0$.
- Assign a Lagrange multiplier $\mu_i \in \mathbb{R}$ to $h_i(x) = 0$.

The Lagrangian $L : \mathbb{R}^n \times K_1^* \times K_m^* \times \mathbb{R}^p \to \mathbb{R}$ is defined as

$$L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

Problems with Generalized Inequality Cont'd

$$P : \inf_{x \in \mathbb{R}^n} \qquad f(x) = \sup_{\lambda \in K^*, \mu \in \mathbb{R}^p} L(x, \lambda, \mu)$$

$$D : \sup_{\lambda \in K^*, \mu \in \mathbb{R}^p} g(\lambda, \mu) \qquad g(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

- Weal duality holds because
  - $f_i(x) \in K_i \Rightarrow \forall \lambda_i \in K_i^* : \lambda_i^T f_i(x) \leq 0$
  - $f_i(x) \notin K_i \Rightarrow \forall \lambda_i \in K_i^* : \lambda_i^T f_i(x) > 0$
- Strong duality holds for covex problems satisfy a constraint qualification.
- Slater's constraint qualification holds if there exist x_S with
  - $f_i(x_S) \prec 0, \forall i = 1, ..., m$
  - $h_i(x_S) = 0, \forall i = 1, ..., p$

SOCP Duality

Primal Program:

$$P : \begin{cases} \min_{x \in \mathbb{R}^n} & f^T x \\ \text{s.t.} & ||A_i x - b_i||_2 \leq c_i^T x + d_i, i = 1, ..., m \end{cases}$$

Dual Program:

$$D : \begin{cases} \max_{V_i, \mu \in \mathbb{R}^m} & -\sum_{i=1}^m (b_i^T v_i + d_i \mu_i) \\ \text{s.t.} & \sum_{i=1}^m (A_i^T v_i + c_i \mu_i) = f \\ \text{s.t.} & ||V_i|| \leq \mu_i, i = 1, ..., m \end{cases}$$

Derivation: Keep as Homework.

SPD Duality

Primal Program:

$$\begin{cases} \min_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & F_1x_1 + \cdots + F_nx_x \preceq G \end{cases}$$

Dual Program:

$$\begin{cases} \min_{\wedge\succeq 0} & -\mathrm{tr}(\wedge^T G) \\ \text{s.t.} & \mathrm{tr}(\wedge^T F_i) = -c_i, \forall i = 1, ..., n \end{cases}$$

Derivation: Homework to proof.

Summary

- **Lagrangian:** weighted sum of objective and constraints; saddle function for convex problems; used to construct primal objective (partial maximum w.r.t. Lagrange multipliers) and dual objective (partial minimum w.r.t. decision variables).
- **Duality:** $\sup \mathsf{D}$ never exceeds $\inf \mathsf{P}$ (weak duality); $\sup \mathsf{D}$ equals $\inf \mathsf{P}$ for convex problems satisfying a constraint qualification (strong duality); all primal (dual) feasible solutions offer upper (lower) bounds on both $\inf \mathsf{P}$ and $\sup \mathsf{D}$.
- **Explicit dual problems:** The dual of an LP/QP/SOCP/SDP is also an LP/QP/SOCP/SDP, respectively.
- **Dual cone:** a proper convex cone coincides with the dual of its dual (bidual) cone; the nonnegative orthant, the second-order cone and the positive semidefinite cone are self-dual.