TD

Chapter 4: Lagrangian Duality

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Exercise 1. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbb{R}$.

(a) *Analysis of primal problem.* Given the feasible set, the optimal value, and the optimal solution.

(b) *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property $(p^* \geq \inf_x L(x,\lambda) \text{ for } \lambda \geq 0)$. Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the duel problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

(d) *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq u, \end{array}$$

as a function of the parameter u. Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

Exercise 2. Express the dual problem of

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & f(x) \leq 0, \end{array}$$

with $c \neq 0$, in term of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Exercise 3. Consider the inequality form LP

$$\begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & Ax \preceq b, \end{array}$$

with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. In this exercise we develop a simple geometric interpretation of the dual LP.

Let $w \in \mathbb{R}^m$. If x is feasible for the LP, i.e., satisfies $Ax \preceq b$, then it also satisfies the inequality

$$w^TAx \leq w^Ty$$

Geometrically, for any $w \succeq 0$, the halfspace $H_w = \{x \mid w^TAx \leq w^Ty\}$ contain the feasible set for the LP. Therefore if we minimize the objective c^Tx over the halfspace H_w we get a lower bound on p^*.

(a) Derive an expression for the minimum value of $c^T x$ over the halfspace H_w (which will depend on the choice of $w \succeq 0$).

(b) Formulate the problem of finding the best such bound, by maximizing the lower bound over $w \succeq 0$.

(c) Relate the result of *(a)* and *(b)* to the Lagrange dual of the LP,

Exercise 4. Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

Given the dual problem, and make the implicit equality constraint explicit.

Exercise 5. We consider the convex piecewise-linear minimization problem

$$\text{minimize} \quad \max_{i=1,\dots,m} (a_i^T x + b_i)$$

with variable $x \in \mathbb{R}^n$.

(a) Derive a dual problem, based on the Lagrange dual of the equivalent problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{i=1,\dots,m} y_i \\ \text{subject to} & a_i^T x + b_i = y_i, \quad i = 1, \dots, m \end{array}$$

with variables $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

(b) Formulate the piecewise-linear minimization problem as an LP, and form the dual of the LP. Relate the LP dual to the dual obtained in part (a).

(c) Suppose we approximate the objective function by the smooth function

$$f_0(x) = \log\left(\sum_{i=1}^m \exp(a_i^T x + b_i)\right)$$

and solve the unconstrained geometric program

$$\text{minimize} \quad \log\left(\sum_{i=1}^m \exp(a_i^T x + b_i)\right)$$

Let p^*_{pwl} and p^*_{gp} be the optimal values, then show that

$$0 \le p^*_{\text{gp}} - p^*_{\text{pwl}} \le \log m$$

(d) Derive similar bounds for the difference between p^*_{pwl} and the optimal value of

$$\text{minimize} \quad (1/\gamma) \log\left(\sum_{i=1}^m \exp(\gamma(a_i^T x + b_i))\right),$$

where $\gamma > 0$ is a parameter. What happens as we increase γ?

Exercise 6. Relate the two dual problems derived in example 5.9 on page 257 on main reference.

Exercise 7. Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^{N} ||A_i x + b_i||_2 + (1/2)||x - x_0||_2^2$$

The problem data are $A_i \in \mathbb{R}^{m_i \times m}, b_i \in \mathbb{R}^{m_i}, x_0 \in \mathbb{R}^n$. First introduce new variables $y_i \in \mathbb{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

Exercise 8. Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain $\{x \mid a_i^T x < b_i, i = 1, ..., m\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$.

Exercise 9. Prove that the optimal solution of the LP

$$\begin{array}{ll} \text{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{subject to} & \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} \preceq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} cv4 \end{array}$$

Exercise 10. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & ||Ax - b||_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ with $\textbf{rank} A = n$, and $G \in \mathbb{R}^{p \times n}$ with $\textbf{rank} G = p$. Give he KKT conditions, and derive expressions for the optimal solution x^* and the dual solution ν^*.

Exercise 11. Consider the l_1-norm minimization problem

$$\text{minimize} \quad ||Ax + b + \epsilon d||_1$$

with variable $x \in \mathbb{R}^3$, and

$$A = \begin{bmatrix} -2 & 7 & 1 \\ -5 & -1 & 3 \\ -7 & 3 & -5 \\ -1 & 4 & -4 \\ 1 & 5 & 5 \\ 2 & -5 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ 3 \\ 9 \\ 0 \\ -11 \\ 5 \end{bmatrix}, \quad d = \begin{bmatrix} -10 \\ -13 \\ -27 \\ -10 \\ -7 \\ 14 \end{bmatrix}$$

We denote by $p^*(\epsilon)$ the optimal value as a function of ϵ.

(a) Suppose $\epsilon = 0$. Prove that $x^* = 1$ is optimal. Are there any other optimal points?

(b) Show that $p^*(\epsilon)$ is affine on an interval that includes $\epsilon = 0$.

Exercise 12. Consider the pair of primal and dual LPs

$$\begin{array}{ll} \text{minimize} & (c+\epsilon d)^T x \\ \text{subject to} & Ax \preceq b + \epsilon f \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & -(b+\epsilon f)^T z \\ \text{subject to} & A^T z + c + \epsilon d = 0 \\ & z \succeq 0 \end{array}$$

where

$$A = \begin{bmatrix} -4 & 12 & -2 & 1 \\ -17 & 12 & 7 & 11 \\ 1 & 0 & -6 & 1 \\ 3 & 3 & 22 & -1 \\ -11 & 2 & -1 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 13 \\ -4 \\ 27 \\ -18 \end{bmatrix}, \quad f = \begin{bmatrix} 6 \\ 15 \\ -13 \\ 48 \\ 8 \end{bmatrix}$$

$c =)49, -34, -50, -5), d = (3, 8, 21, 25)$, and ϵ is a parameter.

(a) Prove that $x^* = (1, 1, 1, 1)$ is optimal when $\epsilon = 0$, by constructing a dual optimal point z^* that has the same objective value as x^*. Are there any other primal or dual optimal solutions?

(b) Give an explicit expression for the optimal value $p^*(\epsilon)$ as a function of ϵ on an interval that contains $\epsilon = 0$. Specify the interval on which your expression is valid. Also give explicit expressions for the primal solution $x^*(\epsilon)$ and the dual solution $z^*(\epsilon)$ as a function ϵ, on the same interval.

Exercise 13. Derive the Lagrange dual of the optimization problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} \phi(x_i) \\ \text{subject to} & Ax = b \end{array}$$

with variable $x \in \mathbb{R}^n$, where

$$\phi(u) = \frac{|u|}{c - |u|} = -1 + \frac{c}{c - |u|}, \quad \mathbf{dom}\phi = (-c, c)$$

c is positive parameter.

Exercise 14. Show that the dual of the SOCP

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & ||A_i x + b_i||_2 \leq c_i^T x + d_i, \quad i = 1, ..., m \end{array}$$

with variables $x \in \mathbb{R}^n$, can be expressed as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} (b_i^T u_i + d_i v_i) \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} (A_i^T u_i + c_i v_i) + f = 0 \\ & ||u_i||_2 \leq v_i, \quad i = 1, ..., m \end{array}$$

with variables $u_i \in \mathbb{R}^{n_i}, v_i \in \mathbb{R}, i = 1, ..., m$. The problem data are $f \in \mathbb{R}^n, A_i \in \mathbb{R}^{n_i \times n}, b_i \in \mathbb{R}^{n_i}, c_i \in \mathbb{R}$ and $d_i \in \mathbb{R}, i = 1, ..., m$. Derive the dual in the following two ways.

(a) Introduce new variables $y_i \in \mathbb{R}^{n_i}$ and $t_i \in \mathbb{R}$ and equalities $y_i = A_i x + b_i, t_i = c_i^T x + d_i$, and derive the Lagrange dual.

(b) Start from the conic formulation of the SOCP and use the conic dual. Use that fact that the second-order cone is self-dual.