LU-decomposition

Let A be $n \times n$ matrix. Then

$$A = L \cdot U$$

where L is a $n \times n$ lower triangular matrix

U is a $n \times n$ upper triangular matrix

We use elementary row operations:

- $R_i \leftrightarrow R_j$
- $R_i \to \lambda R_i \ , \lambda \neq 0$
- $R_i \to R_i + \lambda R_j$

$A \rightsquigarrow$ row echelon form

$IA = IA = A$

Ex 1: $A = \begin{pmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{pmatrix}$

$9 - 15 = -6$

we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ \boxed{0} & 0 & 1 & 0 \\ \boxed{0} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boxed{2} & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 = R_2 - (\boxed{-2})\, R_1 \\ R_3 \to R_3 - \boxed{3} R_1 \\ R_4 \to R_4 - \boxed{2} R_1 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & \boxed{0} & 1 & 0 \\ 2 & \boxed{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & -4 & -7 & -6 \\ 0 & 1 & -8 & 4 \end{pmatrix} \quad \begin{array}{l} R_3 \to R_3 - (\boxed{-4})\, R_2 \\ R_4 \to R_4 - \boxed{1} R_2 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & \boxed{0} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \boxed{-3} & 2 \\ 0 & 0 & -9 & 2 \end{pmatrix} \quad R_4 \to R_4 - \boxed{3} R_3$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}}_{U}$$

Ex 2: Decompose A into $L.U$ where $A = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$.

PLU-decomposition

Let A be $m \times n$ matrix. Then

$$A = P \cdot L \cdot U$$

where P is $m \times m$ permutation matrix
L is $m \times m$ lower triangular matrix
U is $m \times n$ row echelon form

ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P_{1\leftrightarrow 3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$P_{1\leftrightarrow 2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P_{1\leftrightarrow 2}^2 = P_{1\leftrightarrow 2} \cdot P_{1\leftrightarrow 2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$= I$

ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, we have

$$P_{1\leftrightarrow 2} \cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix}$$

$$A \cdot P_{1\leftrightarrow 2} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$A \rightsquigarrow$ row echelon form

Ex1: Decompose $A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}_{4\times 5}$ into PLU-decomposition.

we have

$$A = P_{1\leftrightarrow 2}^2 \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix} = P_{1\leftrightarrow 2} \cdot P_{1\leftrightarrow 2} \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ \boxed{0} & 0 & 1 & 0 \\ \boxed{0} & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}$$

$R_2 \to R_2 - \boxed{0}(R_1)$
$R_3 \to R_3 - \boxed{2}R_1$
$R_4 \to R_4 - \boxed{(1)}R_1$

$$= P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & \boxed{0} & 1 & 0 \\ 1 & \boxed{0} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 2 & 4 & 3 & 5 \end{pmatrix}$$

$R_3 \to R_3 - \boxed{(1)}R_2$
$R_4 \to R_4 - \boxed{2}R_2$

$$A = P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & \boxed{0} & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} \cdot P_{3\leftrightarrow 4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \underbrace{P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4}}_{P} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{U}$$

$$P = P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

<u>Ex2</u>: Decompose A into PLU-decomposition where

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 1 \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}.$$

<u>Ans</u>: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

QR-decomposition

Let A be a square matrix and invertible. Then

$$A = Q \cdot R$$

where . Q is an orthogonal matrix, i.e, $Q \cdot Q^T = Q^T \cdot Q = I$

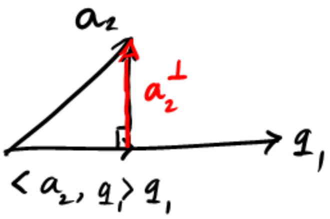
$$\Leftrightarrow Q^T = Q^{-1}$$

. $R = \begin{pmatrix} \diagdown & \\ 0 & \end{pmatrix}$ (upper triangular)

+ Suppose that $A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} \in \mathbb{R}^{3\times 3}$, then

$$A \rightsquigarrow Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}$$

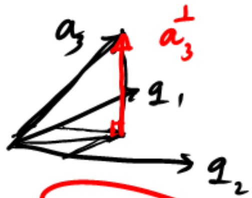
we use Gram-Schmid process



① $q_1 = \dfrac{a_1}{\boxed{\|a_1\|}}$ $\quad (\|q_1\| = 1)$

② $a_2^\perp = a_2 - \boxed{\langle a_2, q_1\rangle} q_1$

$q_2 = \dfrac{a_2^\perp}{\boxed{\|a_2^\perp\|}}$



③ $a_3^\perp = a_3 - \boxed{\langle a_3, q_1\rangle} q_1 - \boxed{\langle a_3, q_2\rangle} q_2$

$q_3 = \dfrac{a_3^\perp}{\boxed{\|a_3^\perp\|}}$

$$R = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & 0 & \square \end{pmatrix}$$

Ex1: Decompose A into QR-decomposition, where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

$a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

① $q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\boxed{\sqrt{2}}}\begin{pmatrix}1\\1\end{pmatrix}$

② $a_2^{\perp} = a_2 - \langle a_2, q_1\rangle \cdot q_1 = \begin{pmatrix}3\\-1\end{pmatrix} - \left\langle \begin{pmatrix}3\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$= \begin{pmatrix}3\\-1\end{pmatrix} - \boxed{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$

$$q_2 = \frac{a_2^{\perp}}{\|a_2^{\perp}\|} = \frac{1}{\boxed{2\sqrt{2}}}\begin{pmatrix}2\\-2\end{pmatrix}$$

So $Q = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$, $R = \begin{pmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 2\sqrt{2}\end{pmatrix}$.

where $A = QR$.

<u>Ex2</u>: Decompose A into QR-decomposition where

$$A = \begin{pmatrix}2 & -2 & -12\\ 4 & 2 & -18\\ -4 & -8 & 30\end{pmatrix}.$$

(Homework)

<u>Next week</u>: we study
- Eigendecomposition $A = PDP^{-1}$
- Singular Value Decomposition (SVD)

  $A = U\Sigma V^T$