LU-decomposition

Let A be $n \times n$ matrix. Then

$$A = L \cdot U$$

where L is a $n \times n$ lower triangular matrix

U is a $n \times n$ upper triangular matrix

We use elementary row operations:

- $R_i \leftrightarrow R_j$
- $R_i \to \lambda R_i \ , \lambda \neq 0$
- $R_i \to R_i + \lambda R_j$

$A \rightsquigarrow$ row echelon form

$IA = IA = A$

Ex1: $A = \begin{pmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{pmatrix}$

$9 - 15 = -6$

we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ \boxed{0} & 0 & 1 & 0 \\ \boxed{0} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boxed{2} & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{pmatrix} \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 = R_2 - (\boxed{-2})R_1 \\ R_3 \to R_3 - \boxed{3}R_1 \\ R_4 \to R_4 - \boxed{2}R_1 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & \boxed{0} & 1 & 0 \\ 2 & \boxed{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & -4 & -7 & -6 \\ 0 & 1 & -8 & 4 \end{pmatrix} \qquad \begin{array}{l} R_3 \to R_3 - (\boxed{-4})R_2 \\ R_4 \to R_4 - \boxed{1}R_2 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & \boxed{0} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \boxed{-3} & 2 \\ 0 & 0 & -9 & 2 \end{pmatrix} \qquad R_4 \to R_4 - \boxed{3}R_3$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}}_{U}$$

Ex2: Decompose A into $L \cdot U$ where $A = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$.

PLU - decomposition

Let A be $m\times n$ matrix. Then

$$A = P.L.U$$

where P is $m\times m$ permutation matrix

L is $m\times m$ lower triangular matrix

U is $m\times n$ row echelon form

<u>ex</u>: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P_{1\leftrightarrow 3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$P_{1\leftrightarrow 2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_{1\leftrightarrow 2}^2 = P_{1\leftrightarrow 2} \cdot P_{1\leftrightarrow 2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

<u>ex</u>: $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, we have

$$P_{1\leftrightarrow 2} \cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix}$$

$$A \cdot P_{1\leftrightarrow 2} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$A \rightsquigarrow$ row echelon form

<u>Ex1</u>: Decompose $A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}_{4\times 5}$ into PLU-decomposition.

we have

$$A = P_{1\leftrightarrow 2}^2 \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix} = P_{1\leftrightarrow 2} \cdot P_{1\leftrightarrow 2} \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ \boxed{0} & 0 & 1 & 0 \\ \boxed{0} & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - \boxed{0}(R_1) \\ R_3 \to R_3 - \boxed{2}R_1 \\ R_4 \to R_4 - \boxed{(1)}R_1 \end{matrix}$$

$$= P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & \boxed{0} & 1 & 0 \\ 1 & \boxed{0} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 2 & 4 & 3 & 5 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 - \boxed{(1)}R_2 \\ R_4 \to R_4 - \boxed{2}R_2 \end{matrix}$$

$$A = P_{1\leftrightarrow 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & \boxed{0} & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} \cdot P_{3\leftrightarrow 4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \cdot P_{3\leftrightarrow 4} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \underbrace{P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4}}_{P} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{U}$$

$$P = P_{1\leftrightarrow 2} \cdot P_{3\leftrightarrow 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot P_{3\leftrightarrow 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

<u>Ex2</u>: Decompose A into PLU-decomposition where

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 1 \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}.$$

<u>Ans</u>: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

QR-decomposition

Let A be a square matrix and invertible. Then

$$A = Q \cdot R$$

where . Q is an orthogonal matrix, i.e, $Q \cdot Q^T = Q^T \cdot Q = I$

$$\Leftrightarrow Q^T = Q^{-1}$$

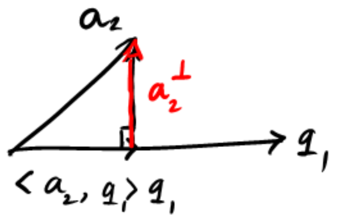
. $R = \begin{pmatrix} \diagdown & \\ 0 & \end{pmatrix}$ (upper triangular)

+ Suppose that $A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix} \in \mathbb{R}^{3\times 3}$, then

$$A \rightsquigarrow Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}$$

we use Gram-Schmid process

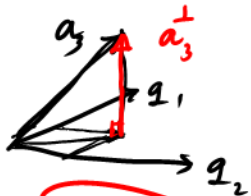
① $q_1 = \dfrac{a_1}{\boxed{\|a_1\|}}$ $\quad (\|q_1\| = 1)$



② $a_2^\perp = a_2 - \boxed{\langle a_2, q_1\rangle} q_1$

$q_2 = \dfrac{a_2^\perp}{\boxed{\|a_2^\perp\|}}$

③ $a_3^\perp = a_3 - \boxed{\langle a_3, q_1\rangle} q_1 - \boxed{\langle a_3, q_2\rangle} q_2$



$q_3 = \dfrac{a_3^\perp}{\boxed{\|a_3^\perp\|}}$

$$R = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & 0 & \square \end{pmatrix}$$

Ex1: Decompose A into QR-decomposition, where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

$$a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

① $q_1 = \dfrac{a_1}{\|a_1\|} = \dfrac{1}{\boxed{\sqrt{2}}}\begin{pmatrix}1\\1\end{pmatrix}$

② $a_2^{\perp} = a_2 - \langle a_2, q_1\rangle \cdot q_1 = \begin{pmatrix}3\\-1\end{pmatrix} - \left\langle \begin{pmatrix}3\\-1\end{pmatrix}, \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$= \begin{pmatrix}3\\-1\end{pmatrix} - \boxed{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$

$$q_2 = \frac{a_2^{\perp}}{\|a_2^{\perp}\|} = \frac{1}{\boxed{2\sqrt{2}}}\begin{pmatrix}2\\-2\end{pmatrix}$$

So $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$, $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{2} \end{pmatrix}$.

where $A = QR$.

Ex2: Decompose A into QR-decomposition where

$$A = \begin{pmatrix} 2 & -2 & -12 \\ 4 & 2 & -18 \\ -4 & -8 & 30 \end{pmatrix}.$$

(Homework)

Next week: we study
- Eigendecomposition $A = PDP^{-1}$
- Singular Value Decomposition (SVD) $A = U\Sigma V^T$

Eigendecomposition

Let A be a square matrix.

$$\boxed{A \cdot v = \lambda v}$$

where v is eigenvector of A

λ is eigenvalue corresspondiung to eigenvector v.

. If A is diagonalizable, then there exist an invertible matrix P such that $A \cdot P = P \cdot D$

$$\Leftrightarrow \quad A = P \cdot D \cdot P^{-1}$$

$$\text{or} \quad \boxed{A = Q \cdot \Lambda \cdot Q^{-1}} \qquad Q = P$$

where Q (or P) is the matrix of eigenvectors $Q = [v_1 \ v_2 \ \cdots \ v_n]$

Λ is the diagonal matrix $\Lambda = \begin{pmatrix} \lambda_1 & & (0) \\ & \lambda_2 & \\ (0) & & \ddots \ \lambda_n \end{pmatrix}$

Q^{-1} is the inverse of Q.

. If A is symmetrix, that is $A^T = A$, then there exists an orthogonal matrix Q, $(Q^T = Q^{-1})$, such that

$$\boxed{A = Q \cdot \Lambda \cdot Q^T}$$

Ex: Decompose A into eigendecomposition, where

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 4 & -1 & 4 \end{pmatrix}.$$

Solution

. Characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 4 & -1 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$p(\lambda) = (4-\lambda)\left((5-\lambda)^2 - 4\right) = (4-\lambda)(25 - 10\lambda + \lambda^2 - 4)$$
$$= (4-\lambda)(\lambda^2 - 10\lambda + 21)$$

- $p(\lambda) = 0 \Rightarrow (4-\lambda)(\lambda^2 - 10\lambda + 21) = 0$
  - $4 - \lambda = 0 \Rightarrow \lambda = 4$
  - $\lambda^2 - 10\lambda + 21 = 0$, $\Delta' = (-5)^2 - 21 = 4$, $\sqrt{\Delta'} = 2$

    $\lambda_1 = 5 - 2 = 3$, $\lambda_2 = 5 + 2 = 7$

So eigenvalues are $\lambda = 7, 4, 3$

$Av = \lambda v$

$(A - \lambda I)v = 0$

- For $\lambda_1 = 7$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_7$, then $\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 4 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$$A - 7I = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 4 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$z = t \Rightarrow y = t$, $x = t \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$

$E_7 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

- For $\lambda_2 = 4 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- For $\lambda_3 = 3 \Rightarrow v_3 = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

So $Q = (v_1 \ \ v_2 \ \ v_3) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(Q \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 6 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 6 & -1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 6 & -1 & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -4 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right)$$

$$\Rightarrow Q^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -4 & 3 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

Thus $A = Q \Lambda Q^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -4 & 3 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$

Homework: Decompose A into eigendecomposition, where

(a) $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

Singular Value Decomposition (SVD)

Let $A \in \mathbb{R}^{m \times n}$. Consider the matrix $A^T A$.

$A^T A$ is a symmetric $n \times n$ matrix which is positive semi definite.

The eigenvalues of $A^T A$ are $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq 0$.

Let $\sigma_i = \sqrt{\lambda_i}$ (σ_i is called the singular value of A)

then $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_n \geq 0$

Suppose that $\operatorname{rank}(A) = r$, so $r \leq \min(m, n)$.

- The singular value decomposition (SVD) of A is defined by

$$\boxed{A = U \cdot \Sigma \cdot V^T} = \sigma_1 u_1 v_1 + \sigma_2 u_2 v_2 + \dots + \sigma_m u_m v_m$$

where U is an $m \times m$ orthogonal matrix.

V is an $n \times n$ orthogonal matrix

Σ is an $m \times n$ matrix where diagonal elements of

$U \quad V^T$

the first 'r' rows are singular values of A and remaining elements are zeros.

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & \\ 0 & & \ddots & \sigma_r & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

★ How to find U and V:

- $U = [u_1\ u_2 \dots u_m]_{m\times m}$ where $u_1, u_2, \dots, u_m$ are eigenvectors of AA^T

- $V = [v_1\ v_2 \dots v_n]_{n\times n}$ where $v_1, v_2, \dots, v_n$ are eigenvectors of A^TA.

$U^TU = U^TU = I$

- $A = U.\Sigma.V^T \Rightarrow A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$

$$= V\Sigma U^TU\Sigma V^T$$
$$= V\Sigma^2V^T$$

So $A^TA.v_i = \sigma_i^2.v_i \Rightarrow V = [v_1\ v_2 \dots v_n]$

- Similarly $A.A^T u_i = \sigma_i^2 u_i \Rightarrow U = [u_1\ u_2 \dots u_m]$.

Ex: Find the SVD of $A = \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

$\text{rank}(A) = 2$

$AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ $\det(AA^T - \lambda I) = 0$

$\Rightarrow \lambda = 8, 2, 0$

$\Rightarrow \sigma_1 = 2\sqrt{2},\ \sigma_2 = \sqrt{2},\ \sigma_3 = 0$

$u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_1 = 8$ $\lambda_2 = 2$ $\lambda_3 = 0$

- Normalize these vectors, we have

$$u_1' = \frac{u_1}{\|u_1\|} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \quad u_2' = \frac{u_2}{\|u_2\|} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}, \quad u_3' = \frac{u_3}{\|u_3\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

So $U = [u_1' \; u_2' \; u_3'] = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

A is 3×3
U is 3×3
V is 3×3

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Homework : Find V $\rightarrow$ $A^T A = ?$

Ex2 : Find SVD of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$

Homework :

$$A = U \cdot \Sigma \cdot V^T$$

where $U = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{pmatrix}$, $V = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}_{3\times 3}$

$$\Sigma = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}.$$

Pseudoinverse

Let A be an $m \times n$ matrix.

Pseudoinverse of A, denoted by A^+, is defined by

$$\boxed{A^+ = V . \Sigma^{-1} . U^T}$$

1. If the columns of A is LI $\Rightarrow c(A) = \text{rank}(A)$, then

$$A^+ = (A^T A)^{-1} . A^T$$

2. If the rows of A is LI $\Rightarrow r(A) = \text{rank}(A)$, then

$$A^+ = A^T . (A A^T)^{-1}$$

3. If $c(A) \neq \text{rank}(A)$ and $r(A) \neq \text{rank}(A)$, then

$$A^+ = V \Sigma^{-1} U^T \quad (\text{use SVD}).$$

Ex1 : Find the pseudoinverse of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$.

$$\text{rank}(A) = 2 = c(A)$$

$$\Rightarrow A^+ = (A^T A)^{-1} . A^T = \; --- \; ?$$

Ex2 : Find the pseudoinverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

$$\text{rank}(A) = 1$$

$$A^+ = V . \Sigma^{-1} . U^T \quad , \text{where } \underbrace{A = U . \Sigma . V^T}_{\text{SVD}}$$