Thm: (Laplace Expansion). Consider a matrix $A \in \mathbb{R}^{n \times n}$. Then, for all $j = \overline{1, n}$:

1. Expansion along column j

$$|A| = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} |A_{k,j}| \quad — ①$$

2. Expansion along row j

$$|A| = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} |A_{j,k}| \quad — ②$$

e.g: compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol^n

Using (2), we obtain

$$|A| = (-1)^{1+1} \cdot 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{1+2} \cdot 2 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$$

$$+ (-1)^{1+3} \cdot 3 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= \underline{-5}$$

Def: Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A and $v \in \mathbb{R}^n \setminus \{0\}$ is the corresponding eigenvector of A if $Av = \lambda v$.

e.g: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Sol^{n}:

- Characteristic polynomial

$$Av = \lambda v \quad , \quad v \neq 0 \quad (\text{From def})$$

$$\Leftrightarrow (A - \lambda I) v = 0$$

OR $$|A - \lambda I| = 0$$

$$\Leftrightarrow \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda)(5-\lambda) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} \lambda = 2 \\ \lambda = 5 \end{array} \right. \quad (\text{Eigenvalue})$$

Consider
$$\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} v = 0$$

For $\lambda = 5$, we obtain

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -x_1 + 2x_2 = 0 \\ x_1 - 2x_2 = 0 \end{cases}$$

$$\Leftrightarrow x_1 - 2x_2 = 0$$

$$\Leftrightarrow x_1 = 2x_2$$

$$E_5 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$ (Eigenspace)

For $\lambda = 2$, we obtain

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} v = 0, \quad v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = -x_1$$

$$E_2 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ (Eigenspace)

e.g.: compute the eigendecomposition of

$$A = \frac{1}{2} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

- Compute eigenvalues and eigenvectors

The characteristic polynomial of A is

$$|A - \lambda I| = 0 \iff (\lambda - \tfrac{7}{2})(\lambda - \tfrac{3}{2}) = 0$$

$\lambda_1 = 7/2, \quad \lambda_2 = \frac{3}{2}$

The associate eigenvector

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} P_1 = \frac{7}{2} P_1 \quad \& \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} P_2 = \frac{3}{2} P_2$$

That is $P_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $P_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- check for existence: the eigenvectors P_1 & P_2 form a basis of $\mathbb{R}^2$

Thus, A can be diagonalized.

- Construct the matrix P to diagonalize A.

$$P = [P_1, P_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1} A P = \begin{bmatrix} 7/2 & 0 \\ 0 & 3/2 \end{bmatrix} = D$$

$$\underbrace{\frac{1}{2}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}}_{A} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 7/2 & 0 \\ 0 & 3/2 \end{bmatrix}}_{D} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{P^{-1}}$$