HW#3 513

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1 a

Assume v is an eigenvector of H, i.e

$$Hv = \lambda v \tag{1}$$

Since we know H is symmetric, i.e

$$H = H' \tag{2}$$

and H is orthogonal, i.e

$$H' = H^{-1} \tag{3}$$

We can conclude that

$$H = H^{-1} \tag{4}$$

So we can multiply H to both side of equation (1), get

$$\begin{aligned} HHv &= H\lambda v \\ v &= \lambda Hv \\ v &= \lambda^2 v \\ \lambda &= \pm 1 \end{aligned} \tag{5}$$

Then we have to prove H should have both 1 and -1 as eigenvalues

Assume not: H has only 1 or -1 as eigenvalue

Case 1: when H has only 1 eigenvalue, then

$$\begin{aligned} H &= QDQ^{-1} \\ H &= QIQ^{-1} \\ H &= QQ^{-1} \\ H &= I \end{aligned} \tag{6}$$

which contradicts our assumption that H is not I

Case 2: when H has only -1 eigenvalue, then by applying the same calculation, we can conclude that

$$H = -I$$

which also contradicts our assumption. Therefore H should have both 1 and -1 as its eigenvalue

2 b

Claim: Let H be symmetric, orthogonal with Schur decomposition H = QDQ'. Then H is Householder if and only if D contains exactly one -1 eigenvalue

3 c

In one direction, we want to prove that if H is householder, then D only contains exactly one -1 eigenvalue.

Assume there are two eigenvector v, w such that

$$Hv = -v, \quad Hw = -w$$

Doing some transformations (adding both side by their vector), we get

$$Hv + v = 0$$

$$Hw + w = 0$$

Moreover, those two equations are equate to

$$(H + \lambda I)v = 0$$

$$(H + \lambda I)w = 0$$

Since they are all equal to 0, we can put them together

$$(H + \lambda I)v = (H + \lambda I)w \tag{1}$$

Canceling terms, we conclude that

$$v = w$$

As a result, there is only one eigenvector such that its eigenvalue is -1.

On another direction, we want to verify that $QDQ' = I - 2ww'$ in order to prove H is a householder matrix.

we can write Q as

$$[v_1|v_2|v_3|...|w]$$

and D is a diagonal matrix except the last entry is -1.

Then

$$\begin{aligned} QDQ' &= [v1|v2|...| - w] * Q' \\ &= v1 \cdot v1' + v2 \cdot v2' + ... - w \cdot w' \end{aligned} \tag{2}$$

Moreover, we know that the identify matrix is

$$v_1 \cdot v_1' + v_2 \cdot v_2' + ... = I$$

so that

$$v_1 \cdot v_1' + v_2 \cdot v_2' + ... + w \cdot w' - w \cdot w' = I - w \cdot w'$$

$$v_1 \cdot v_1' + v_2 \cdot v_2' + ... + = I - w \cdot w'$$

substituting the equation back to equation (2), we can get

$$QDQ' = (I - w \cdot w') - w \cdot w' = I - 2ww' \tag{3}$$

As a result, we proved the claim that H is Householder if and only if D contains exactly one -1 eigenvalue