Units for the Lane-Emden equation

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During one of the last classes we have derived the Lane-Emden equation for the computation of the structure of a compact star. It relies on the assumption that the equation of state of matter can be described as a so called *polytropic* function:

$$P = K\epsilon^{1+\frac{1}{n}}, \tag{1}$$

where ϵ is the *energy* density, related to the neutron number density by $\epsilon = \rho m_N c^2$. This is a somewhat simplified model, but it is still used in GR calculations. By looking at the conditions of hydrostatic equilibrium (non relativistic), after some manipulations we obtain the equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\right)\theta(\xi) = -\theta^n(\xi), \tag{2}$$

where θ is the adimensional density ρ/ρ_c, where ρ_c is the central density, and $\xi = \alpha r$, with $\alpha^2 = \left(\frac{k\rho_c^{\frac{1-n}{n}}(n+1)}{4\pi G}\right)$ is the adimensional distance from the center of the star. In order to derive the mass and the radius of the star one should solve the equation, for the adimensional variables, and then look at the various solutions that might be obtained by varying the central value of the density ρ_c. The radius is determined by the value ξ_1 such that $\theta(\xi_1) = 0$, i.e.:

$$R = \alpha\xi_1 = \left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{\frac{1}{2}} \rho_c^{\frac{1-n}{2n}}\xi_1. \tag{3}$$

At the same time, the mass is given by:

$$M = 4\pi\alpha^3\rho_c\int_0^{\xi_1}\xi^2\theta^n(\xi)d^{\xi} = \tag{4}$$

$$= 4\pi\alpha^3\rho_c\int_0^{\xi_1}\left[-\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)\right]d\xi =$$

$$-4\pi\left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\xi_1^2\frac{d\theta(\xi_1)}{d\xi}. \tag{5}$$

At this point it is possible to combine the equation for the mass and for the radius eliminating ρ_c. The result is:

$$R^{\frac{3-n}{n}}M^{\frac{1-n}{n}}=\frac{K}{GN_n}, \tag{6}$$

where N_n is an adimensional parameter that comes from the numerical solution of the equation:

$$N_n=\frac{(4\pi)^{\frac{1}{n}}}{n+1}\left(-\xi_1^2\frac{d\theta(\xi_1)}{d\xi}\right)^{\frac{1-n}{n}}\xi_1^{\frac{n-3}{n}}. \tag{7}$$

We need to determine the constant K. Let us work out the case of a non-relativistic gas of non-interacting neutrons. Introducing the adimensional Fermi momentum $x_F=p_F/mc$, the pressure, evaluated as $-\frac{\partial E}{\partial V}$, with E the total energy of the system at given density, is given by:

$$P=\frac{m_n^4c^5}{\hbar^3\pi^2}\frac{x_F^5}{15}=\frac{4}{5}\tilde{k}x_F^5. \tag{8}$$

Remembering the relation between the Fermi momentum and the number density $p_F=\hbar(r\pi^2\rho)^{\frac{1}{3}}$, one gets:

$$P=\frac{4}{5}\tilde{k}\left(\frac{3\pi^2\epsilon}{m_nc^2}\right)^{\frac{5}{3}}\frac{(\hbar c)^5}{(m_nc^2)^5}. \tag{9}$$

This expression gives the constant K in the polytropic. In fact:

$$P=\frac{\hbar c}{15\pi^2m_nc^2}\left(\frac{3\pi^2}{m_nc^2}\right)^{\frac{5}{3}}\epsilon^{\frac{5}{3}}, \tag{10}$$

which gives

$$K=\frac{\hbar c}{15\pi^2m_nc^2}\left(\frac{3\pi^2}{m_nc^2}\right)^{\frac{5}{3}}=8.79\times10^{-4}\mathrm{Mev}^{-\frac{2}{3}}\mathrm{fm}^2. \tag{11}$$

Here we made use of nuclear units, in which c=1, $\hbar c\simeq 197.32$MeV$\cdot$fmi, and the mass of the neutron is $\sim 939,6\mathrm{MeV}/c^2$. The constant in Eq. (11) can be converted in units of $M_\odot$ and km. The mass of the sun is $\sim 1.16\times 10^{60}\mathrm{MeV}/c^2$. In these units $K=10.2M_\odot^{-\frac{2}{3}}\mathrm{km}^2$. In order to express G in homogeneous units, we can exploit the definition of the Schwarzschild radius, i.e.:

$$r_s = \frac{2MG}{c^2}.$$

For the sun $r_s \sim 2.955$ km, from which we obtain $G \sim 1.48 \frac{\text{km}}{M_\odot / c^2}$. Inserting this value of see that the mass radius relation becomes:

$$RM^{\frac{1}{3}} = \frac{6.89}{N_{\frac{3}{2}}} M_\odot^{\frac{1}{3}} \text{km}. \tag{12}$$

The same procedure can be used to study the ultra-relativistic limit. In this case $K = 0.0115\ \text{MeV}^{-\frac{1}{3}} \cdot \text{fm}$ (notice that the dimensions depend on the parameter n of the polytropic!). This gives the very peculiar mass radius relation:

$$M^{\frac{2}{3}} = 247.7 M_\odot^{\frac{2}{3}}, \tag{13}$$

meaning that the (very large) mass is independent of the radius.