Algorithms for solving the Algebraic Riccati Equation

Several algorithms from Petkov *et al.*[1] were presented in lecture. Here's are basic versions of some of the algorithms.

Notation

The Continuous-time Algebraic Riccati Equation (CARE) will be written as

$$A^T P + PA + Q - PSP = 0, \tag{1}$$

where $S = BR^{-1}B^T$.

The condition number of the CARE is given by

$$c_R = \frac{2\|A\|_F + \|Q\|_F/\|P\|_F + \|BR^{-1}B^T\|_F\|P\|_F}{\text{sep }\left[(A - BK^T), -(A - BK)\right]} \tag{2}$$

where

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}} \tag{3}$$

and

$$\text{sep }\left[(A, -A)\right] = \min_i \sigma_i \tag{4}$$

where σ_i is a singular value of the matrix

$$I_n \otimes A^T + A^T \otimes I_n,$$

and $\otimes$ represents the Kronecker product.

Under the conditions that

- (A, B) is stabilizable,
- (C, A) is detectable, and
- $Q = C^T C$,

the unique positive semidefinite solution P may be obtained by one of the methods described below.

Newton's Method

After checking to make sure that A, B, Q, R satisfy the above conditions, proceed as follows. First, select a matrix K_0 such that all eigenvalues of $A - BK_0$ lie in the open left half plane.

[1] Petkov, P., N Christov, and M. Konstantinov, *Computational Methods for Linear Control Systems*, New York: Prentice, 1991.

1. Set $A_k = A - BK_k$.
2. Set $Q_k = Q + K_k^T R K_k$.
3. Solve $A_k^T P_k + P_k A_k + Q_k = 0$ for P_k.
4. Set $K_{k+1} = R^{-1} B^T P_k$.
5. Compute the condition number c_R of the Riccati equation

$$c_R = \frac{2\|A\|_F + \|Q\|_F/\|P_k\|_F + \|BR^{-1}B^T\|_F\|P_k\|_F}{\text{sep }\left[\left(A - BK_{k+1}^T\right), -\left(A - BK_{k+1}\right)\right]} \tag{5}$$

6. Compute

$$\frac{\|P_k - P_{k-1}\|_F}{\|P_{k-1}\|_F}. \tag{6}$$

7. If the desired accuracy has not been reached, iterate.

It should be noted that the stopping criteria should be no smaller than ϵc_R where ϵ is the larger of the machine precision and

$$\max\left\{\frac{\|\Delta A\|}{\|A\|}, \frac{\|\Delta Q\|}{\|Q\|}, \frac{\|\Delta S\|}{\|S\|}\right\}, \tag{7}$$

where $\|\Delta M\|$ is the uncertainty in the matrix M.

Matrix Sign Function Method

After checking to make sure that A, B, Q, R satisfy the indicated conditions, proceed as follows. The matrix sign function is defined as follows. Let

$$J = X^{-1}AX =: D + N, \tag{8}$$

where D is diagonal and N is nilpotent, be the Jordan form of the matrix A. Then, when all eigenvalues of A have nonzero real part, we define

$$\text{Sign}\,(A) = XYX^{-1} \tag{9}$$

where

$$y_{ii} = \text{Sign}\,(d_{ii}) = \begin{cases} 1 & \text{if } \text{Re}(d_{ii}) > 0 \\ -1 & \text{if } \text{Re}(d_{ii}) < 0 \end{cases} \tag{10}$$

and $y_{ij} = 0 \;\; \forall j \neq i$. Let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \text{Sign}\,(H) \tag{11}$$

where

$$H = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}, \tag{12}$$

and let

$$M = \begin{bmatrix} W_{12} \\ W_{22} + I \end{bmatrix} \tag{13}$$

$$N = \begin{bmatrix} W_{11} + I \\ W_{21} \end{bmatrix} \tag{14}$$

Let $W_0 = H$. Proceed as follows:

1. Iterate using $W_{k+1} = W_k - \frac{1}{2}\left(W_k - W_k^{-1}\right)$ to obtain Sign(H).
2. Solve $MP = -N$ for P.

The Schur Method

After checking to make sure that A, B, Q, R satisfy the conditions indicated above, proceed as follows[2]. As Laub notes, "[t]he Schur vector approach is obviously not well-suited to hand computation." Examples can be found starting on page 26 of the report (p. 28 of the pdf file).

1. Find an orthogonal transformation $\tilde{U}$ that reduce H to real Schur form $\tilde{T} = \tilde{U}^T H \tilde{U}$ where $\tilde{T}$ is block upper triangular, *i.e.* the block $\tilde{T}_{21} = 0$.
2. Use additional orthogonal transformations to reorder the Schur form so that all all negative eigenvalues precede all non-negative eigenvalues on the diagonal of the upper triangular matrices.
3. Let U be the composition of all of the orthogonal transformations. Partition $T = U^T HU$ so that the square submatrix with the negative eigenvalues is T_{11}.
4. Partition U compatibly with T so $[U_{11}^T,\, U_{21}^T]^T$ are the Schur vectors corresponding to T_{11}.
5. Solve $U_{11}^T P = U_{21}^T$.

[2] Laub, A. J. "A Schur Method for Solving Algebraic Riccati Equations", Massachusetts Institute of Technology, Laboratory for Information and Decision Systems, LIDS Report number 859, 1978. Available modulo missing pages at `http://dspace.mit.edu/bitstream/handle/1721.1/1301/R-0859-05666488.pdf?sequence=1`