Carter (1968)

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0.1 Tetrads

Carter researched metrics of spacetime that the Hamilton–Jacobi equation of a particle is separable in terms of two abelian groups. The metrics have six functions $(P_\lambda, P_\mu, Q_\lambda, Q_\mu, \Delta_\lambda, \Delta_\mu)$ of coordinates λ, μ. These are subscripts and are called non-ignorable because parameters of two abelian groups are not λ, μ. These are similar to the radius coordinate r of the spherical coordinates. The remaining coordinates are denoted by ψ and χ, which is called ignorable. Using the coordinate basis $(d\lambda, d\mu, d\psi, d\chi)$ of the dual tangent space, the metric is restricted to certain symmetric forms

$$\begin{aligned} ds^2 &= \frac{Z}{\Delta_\lambda} d\lambda^2 + \frac{Z}{\Delta_\mu} d\mu^2 + \frac{\Delta_\mu}{Z} \left[P_\lambda d\psi - Q_\lambda d\chi\right]^2 - \frac{\Delta_\lambda}{Z} \left[P_\mu d\psi - Q_\mu d\chi\right]^2, \\ Z &= P_\lambda Q_\mu - P_\mu Q_\lambda. \end{aligned} \tag{0.1.1}$$

Carter discussed the separability of variables of Hamilton–Jacobi equations with electromagnetic fields. Electromagnetic fields without source fields are described by four-potential in spacetime. A classical four-potential is now one-form

$$A = \frac{P_\lambda X_\mu + P_\mu X_\lambda}{Z} d\psi - \frac{Q_\lambda X_\mu + Q_\mu X_\lambda}{Z} d\chi, \tag{0.1.2}$$

with functions X_λ and X_μ and the corresponding electromagnetic field (strength) tensor is $F = dA$ ($\times 2$ on the Carter's paper).

Carter classifies metrics of

$$\text{type } A \begin{cases} P_\lambda = \lambda^2, P_\mu = -\mu^2, Q_\lambda = 1, Q_\mu = 1, \\ \Delta_\lambda = \dfrac{1}{3}\Lambda\lambda^4 + h\lambda^2 - 2m_+\lambda + n + e^2,\ \Delta_\mu = \dfrac{1}{3}\Lambda\mu^4 - \left(h\mu^2 - 2m_-\mu - n\right), \end{cases} \tag{0.1.3}$$

$$\text{type } B(+) \begin{cases} P_\lambda = \lambda^2 + 1, P_\mu = 2\mu, Q_\lambda = 0, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\left(\dfrac{1}{3}\lambda^4 + 2\lambda^2 - 1\right) + \left(h\left(\lambda^2 - 1\right) - 2m_+\lambda + e^2\right),\ \Delta_\mu = -\left(h\mu^2 - 2m_-\mu + n\right), \end{cases} \tag{0.1.4}$$

$$\text{type } C(+) \begin{cases} P_\lambda = \lambda^2, P_\mu = 0, Q_\lambda = 0, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\lambda^4 + \left(h\lambda^2 - 2m_+\lambda + e^2\right),\ \Delta_\mu = -\left(h\mu^2 - 2m_-\mu + n\right), \end{cases} \tag{0.1.5}$$

$$\text{type } D \begin{cases} P_\lambda = \dfrac{1}{2}, P_\mu = \dfrac{1}{2}, Q_\lambda = 1, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\lambda^2 + \left(e^2\lambda^2 - 2m_+\lambda + n_+\right),\ \Delta_\mu = \Lambda\mu^2 - \left(e^2\mu^2 - 2m_-\mu + n_-\right). \end{cases} \tag{0.1.6}$$

We can deduce ortho-normal tetrads whose superscripts are numbered as non-ignorable and ignorable using a condition $U = Z$,

$$\omega^i = \left(\frac{U}{\Delta_i}\right)^{1/2} d\lambda^i,\ \omega^{2i} = \frac{\left(U\Delta_{-i}\right)^{1/2}}{Z} \left((r)\, Z_i^r d\varphi^{2r}\right) \tag{0.1.7}$$

for an integer index $i \in \{+1, -1\}$ and summed indices $r = +1, -1$. The non-ignorable coordinates correspond to two ones that the metric component depends on, in other words, $\left(\lambda^{+1}, \lambda^{-1}\right) \equiv (\lambda, \mu)$. On the other hand, the ignorable ones $\left(\varphi^{+2}, \varphi^{-2}\right)$ are equivalent to (ψ, χ). A round bracket (r) means summed signs, namely $(r)\, Z_i^r d\varphi^{2r} = +Z_i^{+1} d\varphi^{+2} - Z_i^{-1} d\varphi^{-2} = P_i d\psi - Q_i d\chi$. Then non-ignorable-numbered tetrads are represented by those greek letters as

$$\omega^{+1} = \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} d\lambda,\ \omega^{-1} = \left(\frac{Z}{\Delta_\mu}\right)^{1/2} d\mu \tag{0.1.8}$$

using $U = Z$ and the remainder are

$$\omega^{+2} = \frac{(Z\Delta_\mu)^{1/2}}{Z}\left(+P_\lambda d\psi - Q_\lambda d\chi\right),\ \omega^{-2} = \frac{(Z\Delta_\lambda)^{1/2}}{Z}\left(+P_\mu d\psi - Q_\mu d\chi\right). \tag{0.1.9}$$

These relations are summarized by a matrix

$$\hat{E} = \begin{bmatrix} \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \left(\frac{Z}{\Delta_\mu}\right)^{1/2} & 0 & 0 \\ 0 & 0 & \frac{(Z\Delta_\mu)^{1/2}}{Z} P_\lambda & -\frac{(Z\Delta_\mu)^{1/2}}{Z} Q_\lambda \\ 0 & 0 & \frac{(Z\Delta_\lambda)^{1/2}}{Z} P_\mu & -\frac{(Z\Delta_\lambda)^{1/2}}{Z} Q_\mu \end{bmatrix}, \tag{0.1.10}$$

which combines the ortho-normal tetrads ω^js ($j \in \{\pm 1, \pm 2\}$)

$$\begin{bmatrix} \omega^{+1} \\ \omega^{-1} \\ \omega^{+2} \\ \omega^{-2} \end{bmatrix} = \hat{E} \begin{bmatrix} d\lambda \\ d\mu \\ d\psi \\ d\chi \end{bmatrix}. \tag{0.1.11}$$

The determinant of the matrix $\hat{E}$ equals $-Z$:

$$\begin{aligned} \det \hat{E} &= \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} \left(\frac{Z}{\Delta_\mu}\right)^{1/2} \frac{(Z\Delta_\lambda)^{1/2}}{Z} \frac{(Z\Delta_\mu)^{1/2}}{Z} \left(-P_\lambda Q_\mu + P_\mu Q_\lambda\right) \\ &= \left(\frac{Z^2}{\cancel{Z\Delta_\lambda}}\right)^{1/2} \left(\frac{Z^2}{\cancel{Z\Delta_\mu}}\right)^{1/2} \frac{\cancel{(Z\Delta_\lambda)^{1/2}(Z\Delta_\mu)^{1/2}}}{Z^{\cancel{2}\,1}} (-\cancel{Z}) \\ &= |Z|^2/Z \times (-1) = -Z \end{aligned} \tag{0.1.12}$$

Moreover, we can deduce the ortho-normal basis. Let $\{\partial_\lambda, \partial_\mu, \partial_\psi, \partial_\chi\}$ be coordinate basis vectors and $\{e_{+1}, e_{-1}, e_{+2}, e_{-2}\}$ be ortho-normal basis vectors corresponding to the tetrads ω^js. These ortho-normal vectors satisfy

$$\begin{bmatrix} e_{+1} & e_{-1} & e_{+2} & e_{-2} \end{bmatrix} = \begin{bmatrix} \partial_\lambda & \partial_\mu & \partial_\psi & \partial_\chi \end{bmatrix} \hat{E}^{-1}, \tag{0.1.13}$$

where the inverse matrix $\hat{E}^{-1}$ of the transformation matrix Eq. (0.1.10) is

$$\hat{E}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{Z/\Delta_\lambda}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{Z/\Delta_\mu}} & 0 & 0 \\ 0 & 0 & \frac{Q_\mu}{\sqrt{Z\Delta_\mu}} & -\frac{Q_\lambda}{\sqrt{Z\Delta_\lambda}} \\ 0 & 0 & \frac{P_\mu}{\sqrt{Z\Delta_\mu}} & -\frac{P_\lambda}{\sqrt{Z\Delta_\lambda}} \end{bmatrix} \tag{0.1.14}$$

If $\mathbf{E}^{-1}$ is expand concretely, each ortho-normal vector in $\{e_{+1}, e_{-1}, e_{+2}, e_{-2}\}$ is represented by

$$\begin{aligned} e_{+1} &= \left(\frac{\Delta_\lambda}{Z}\right)^{1/2} \partial_\lambda, e_{-1} = \left(\frac{\Delta_\mu}{Z}\right)^{1/2} \partial_\mu, \\ e_{+2} &= \frac{1}{(Z\Delta_\mu)^{1/2}} \left(Q_\mu \partial_\psi + P_\mu \partial_\chi\right), e_{-2} = \frac{1}{(Z\Delta_\lambda)^{1/2}} \left(-Q_\lambda \partial_\psi - P_\lambda \partial_\chi\right). \end{aligned} \tag{0.1.15}$$

using the coordinate basis $\{\partial_\lambda, \partial_\mu, \partial_\psi, \partial_\chi\}$. The tetrads construct the metric tensor

$$g = \omega^{+1} \otimes \omega^{+1} + \omega^{-1} \otimes \omega^{-1} + \omega^{+2} \otimes \omega^{+2} - \omega^{-2} \otimes \omega^{-2}. \tag{0.1.16}$$

We can also check the tetrads are the dual basis of the ortho-normal basis:

$$\begin{bmatrix} \omega^{+1} \\ \omega^{-1} \\ \omega^{+2} \\ \omega^{-2} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} e_{+1} & e_{-1} & e_{+2} & e_{-2} \end{bmatrix} = \hat{E} \begin{bmatrix} d\lambda \\ d\mu \\ d\psi \\ d\chi \end{bmatrix} \begin{bmatrix} \partial_\lambda & \partial_\mu & \partial_\psi & \partial_\chi \end{bmatrix} \hat{E}^{-1} = 1. \tag{0.1.17}$$

That is why we can confirm the ortho-normality

$$g\left(e_j, e_j\right) = +1 \text{ for } j \in \{+1, -1, +2\},\ g\left(e_{-2}, e_{-2}\right) = -1,\ \text{ otherwise vanish.} \tag{0.1.18}$$

- Lorentzian Manifold M
- An open set U
- coordinates: $(\lambda, \mu, \psi, \chi)$
- Independent functions: $\Delta_\lambda, \Delta_\mu, P_\lambda, P_\mu, Q_\lambda, Q_\mu$
- $Z = P_\lambda Q_\mu - P_\mu Q_\lambda$

$$A = \sum_i \frac{X_{-i}}{Z} \left((r)\, Z_i^r d\varphi^{2r}\right),\ A = \sum_i \frac{X_{-i}}{\left(U\Delta_{-i}\right)^{1/2}} \omega^{2i} \tag{0.1.19}$$

$$A = \frac{P_\lambda X_\mu + P_\mu X_\lambda}{Z} d\psi - \frac{Q_\lambda X_\mu + Q_\mu X_\lambda}{Z} d\chi$$

Cartan's first structure formula $d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = \tau^\mu$ with torsion free, i.e., $\tau^\mu = 0$ gives conditions of the connection form

$$\omega^\mu_\nu \wedge \theta^\nu = -d\theta^\mu. \tag{0.1.20}$$

We use the interior product ι to solve ω^μ_νs. The interior product