Carter (1968)

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§1 metrics

Carter (1968) は時空の対称性を仮定して以下のような計量を研究した．

$$
\begin{aligned}
ds^2 &= \frac{Z}{\Delta_\lambda}d\lambda^2 + \frac{Z}{\Delta_\mu}d\mu^2 + \frac{\Delta_\mu}{Z}\left[P_\lambda d\psi - Q_\lambda d\chi\right]^2 - \frac{\Delta_\lambda}{Z}\left[P_\mu d\psi - Q_\mu d\chi\right]^2 \\
Z &= P_\lambda Q_\mu - P_\mu Q_\lambda
\end{aligned} \tag{1.0.1}
$$

Carter は目的である Hamilton–Jacobi 方程式の (変数) 分離可能性を，電磁場を入れて議論した．

古典的 4 元ポテンシャルは 1 形式

$$
A = \frac{P_\lambda X_\mu + P_\mu X_\lambda}{Z}d\psi - \frac{Q_\lambda X_\mu + Q_\mu X_\lambda}{Z}d\chi \tag{1.0.2}
$$

のような形をしており，電磁場テンソルはその外微分 $F = dA$ (論文では $\times 2$). この電磁場テンソルと エネルギー・運動量テンソル の関係は $4\pi T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$

(ref. Gravitation p. 141, p. 471)

行列計算するには ($F_{\alpha\beta}F^{\alpha\beta} = F_{\alpha\beta}g^{\alpha\alpha'}F_{\alpha'\beta'}g^{\beta'\beta}$ なので)

$4\pi(T_{\mu\nu}) = (F_{\mu\alpha})(g^{\alpha\beta})(F_{\beta\nu})^T - \frac{1}{4}(g_{\mu\nu})\cdot \mathrm{tr}[(F_{\alpha\beta})^T(g^{\alpha\alpha'})(F_{\alpha'\beta'})(g^{\beta'\beta})]$

あるいは

$4\pi T = ...$

を用いる．

A

$ds^2 = \left(\lambda^2+\mu^2\right)\left[\frac{d\lambda^2}{\Delta_\lambda} + \frac{d\mu^2}{\Delta_\mu}\right] + \frac{\Delta_\mu\left[d\chi - \lambda^2 d\psi\right]^2 - \Delta_\lambda\left[d\chi + \mu^2 d\psi\right]^2}{\lambda^2+\mu^2}$

- $Z = \lambda^2 + \mu^2$
- $\Delta_\lambda = \frac{1}{3}\Lambda\lambda^4 + h\lambda^2 - 2m\lambda + p + e^2$
- $\Delta_\mu = \frac{1}{3}\Lambda\mu^4 - h\mu^2 + 2q\mu + p$
- $P\left(\lambda\right) = -\lambda^2$
- $Q\left(\lambda\right) = -1$
- potential $A = e\left\{\frac{\lambda\mu(\mu\cos\alpha + \lambda\sin\alpha)}{\lambda^2+\mu^2}d\psi + \frac{\lambda\cos\alpha - \mu\sin\alpha}{\lambda^2+\mu^2}d\chi\right\}$
-

Field strength:

$F = dA$

$\tilde{B}(+)$

$$
\begin{aligned}
ds^2 &= \left(\lambda^2+l^2\right)\left\{\frac{d\lambda^2}{\Delta_\lambda} + \frac{d\mu^2}{\Delta_\mu} + \Delta_\mu d\psi^2\right\} - \frac{\Delta_\lambda[d\chi + 2l\mu d\psi]^2}{\lambda^2+l^2} \\
&= Z\left\{\frac{d\lambda^2}{\Delta_\lambda} + \frac{d\mu^2}{\Delta_\mu} + \Delta_\mu d\psi^2\right\} - \frac{\Delta_\lambda[d\chi + 2l\mu d\psi]^2}{Z}
\end{aligned} \tag{1.0.3}
$$

- $Z=\lambda^2+l^2$
- $\Delta_\lambda=\Lambda\left(\frac{1}{3}\lambda^4+2l^2\lambda^2-l^4\right)+h\left(\lambda^2-l^2\right)-2m\lambda+e^2$
- $\Delta_\mu=-h\mu^2+2q\mu+p$
- $A=e\left\{\frac{\mu\left[2l\lambda\cos\alpha+\left(\lambda^2+l^2\right)\sin\alpha\right]}{\lambda^2+l^2}d\psi+\frac{\lambda\cos\alpha}{\lambda^2+l^2}d\chi\right\}$
- $g_{\lambda\lambda}=Z/\Delta_\lambda$, $g_{\mu\mu}=Z/\Delta_\mu$
- $g_{\chi\chi}=-\Delta_\lambda/Z$, $g_{\psi\psi}=(+Z-4l^2\mu^2\Delta_\lambda)\Delta_\mu$
- $g_{\chi\psi}=g_{\psi\chi}=-2l\mu\Delta_\lambda/Z$

$\tilde{B}(-)$
$ds^2=\left(\mu^2+k^2\right)\left\{\frac{d\lambda^2}{\Delta_\lambda}+\frac{d\mu^2}{\Delta_\mu}-\Delta_\lambda d\psi^2\right\}+\frac{\Delta_\lambda[d\chi-2k\lambda d\psi]^2}{\mu^2+k^2}$

- $Z=\mu^2+k^2$
- $\Delta_\lambda=h\lambda^2-2m\lambda+n$
- $\Delta_\mu=\Lambda\left(\frac{1}{3}\mu^4+2k^2\mu^2-k^4\right)-h\left(\mu^2-k^2\right)+2q\mu-e^2$
- $A=e\left\{\frac{\lambda\left[\left(\mu^2+k^2\right)\cos\alpha+2k\mu\sin\alpha\right]}{\mu^2+k^2}d\psi+\frac{\mu\sin\alpha}{\mu^2+k^2}d\chi\right\}$
-

D
$ds^2=\frac{d\lambda^2}{\Delta_\lambda}+\frac{d\mu^2}{\Delta_\mu}+\Delta_\mu d\chi^2-\Delta_\lambda d\psi^2$

- $Z=1$
- $\Delta_\lambda=\left(\Lambda+e^2\right)\lambda^2-2m\lambda+n$
- $\Delta_\mu=\left(\Lambda-e^2\right)\mu^2+2q\mu+p$
- $P\left(\lambda\right)=0$
- $Q\left(\lambda\right)=-1$
- $A=e\{\lambda\cos\alpha d\psi-\mu\sin\alpha d\chi\}$
-

電磁場テンソルの計算
$F/2=dA=e\{\cos\alpha d\lambda\wedge d\psi-\sin\alpha d\mu\wedge d\chi\}$
だから $F_{\lambda\psi}=e\cos\alpha, F_{\mu\chi}=-e\sin\alpha$
(λ,μ,χ,ψ) の順番で書くと
$\tilde{A}$
$ds^2=\left[\left(c\lambda\cos\gamma+k\right)^2+\left(c\mu\sin\gamma+l\right)^2\right]\left\{\frac{d\lambda^2}{\Delta_\lambda}+\frac{d\mu^2}{\Delta_\mu}\right\}$
$+\frac{\Delta\left\{\sin\gamma d\chi-\left[\left(c^2\lambda^2+k^2+\lambda^2\right)\cos\gamma+2ck\lambda\right]d\psi\right\}^2}{\left(c\lambda\cos\gamma+k\right)^2+\left(c\mu\sin\gamma+l\right)^2}-\frac{\Delta\left\{\cos\gamma d\chi+\left[\left(c^2\mu^2+k^2+\lambda^2\right)\sin\gamma+2cl\mu\right]d\psi\right\}^2}{\left(c\lambda\cos\gamma+k\right)^2+\left(c\mu\sin\gamma+l\right)^2}$
where
$\Delta_\lambda=\frac{1}{3}\Lambda c^2\lambda^4\left(\cos\gamma\right)^2+\frac{4}{3}\Lambda ck\lambda^3\cos\gamma+\left(2\Lambda l^2+h\right)\lambda^2-2m\lambda+n$
$\Delta_\mu=\frac{1}{3}\Lambda c^2\mu^4\left(\sin\gamma\right)^2+\frac{4}{3}\Lambda cl\mu^3\sin\gamma+\left(2\Lambda k^2-h\right)\mu^2+2q\mu+p$
$e^2=\Lambda\left(k^4-l^4\right)+h\left(k^2+l^2\right)+2c\left(km\cos\gamma+lq\sin\gamma\right)+c^2\left(n\left(\cos\gamma\right)^2-p\left(\sin\gamma\right)^2\right)$