Carter (1968)

K. R.

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0.1 Tetrads

Carter researched metrics of spacetime that the Hamilton–Jacobi equation of a particle is separable in terms of two abelian groups. The metrics have six functions $(P_\lambda, P_\mu, Q_\lambda, Q_\mu, \Delta_\lambda, \Delta_\mu)$ of coordinates λ, μ. These are subscripts and are called non-ignorable because parameters of two abelian groups are not λ, μ. These are similar to the radius coordinate r of the spherical coordinates. The remaining coordinates are denoted by ψ and χ, which is called ignorable. Using the coordinate basis $(d\lambda, d\mu, d\psi, d\chi)$ of the dual tangent space, the metric is restricted to certain symmetric forms

$$\begin{aligned} ds^2 &= \frac{Z}{\Delta_\lambda} d\lambda^2 + \frac{Z}{\Delta_\mu} d\mu^2 + \frac{\Delta_\mu}{Z} \left[P_\lambda d\psi - Q_\lambda d\chi\right]^2 - \frac{\Delta_\lambda}{Z} \left[P_\mu d\psi - Q_\mu d\chi\right]^2, \\ Z &= P_\lambda Q_\mu - P_\mu Q_\lambda. \end{aligned} \tag{0.1.1}$$

Carter discussed the separability of variables of Hamilton–Jacobi equations with electromagnetic fields. Electromagnetic fields without source fields are described by four-potential in spacetime. A classical four-potential is now one-form

$$A = \frac{P_\lambda X_\mu + P_\mu X_\lambda}{Z} d\psi - \frac{Q_\lambda X_\mu + Q_\mu X_\lambda}{Z} d\chi, \tag{0.1.2}$$

with functions X_λ and X_μ and the corresponding electromagnetic field (strength) tensor is $F = dA$ ($\times 2$ on the Carter's paper).

Carter classifies metrics of

$$\text{type } A \begin{cases} P_\lambda = \lambda^2, P_\mu = -\mu^2, Q_\lambda = 1, Q_\mu = 1, \\ \Delta_\lambda = \frac{1}{3}\Lambda\lambda^4 + h\lambda^2 - 2m_+\lambda + n + e^2,\ \Delta_\mu = \frac{1}{3}\Lambda\mu^4 - \left(h\mu^2 - 2m_-\mu - n\right), \end{cases} \tag{0.1.3}$$

$$\text{type } B(+) \begin{cases} P_\lambda = \lambda^2 + 1, P_\mu = 2\mu, Q_\lambda = 0, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\left(\frac{1}{3}\lambda^4 + 2\lambda^2 - 1\right) + \left(h\left(\lambda^2 - 1\right) - 2m_+\lambda + e^2\right),\ \Delta_\mu = -\left(h\mu^2 - 2m_-\mu + n\right), \end{cases} \tag{0.1.4}$$

$$\text{type } C(+) \begin{cases} P_\lambda = \lambda^2, P_\mu = 0, Q_\lambda = 0, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\lambda^4 + \left(h\lambda^2 - 2m_+\lambda + e^2\right),\ \Delta_\mu = -\left(h\mu^2 - 2m_-\mu + n\right), \end{cases} \tag{0.1.5}$$

$$\text{type } D \begin{cases} P_\lambda = \frac{1}{2}, P_\mu = \frac{1}{2}, Q_\lambda = 1, Q_\mu = 1, \\ \Delta_\lambda = \Lambda\lambda^2 + \left(e^2\lambda^2 - 2m_+\lambda + n_+\right),\ \Delta_\mu = \Lambda\mu^2 - \left(e^2\mu^2 - 2m_-\mu + n_-\right). \end{cases} \tag{0.1.6}$$

We can deduce ortho-normal tetrads whose superscripts are numbered as non-ignorable and ignorable using a condition $U = Z$,

$$\omega^i = \left(\frac{U}{\Delta_i}\right)^{1/2} d\lambda^i,\ \omega^{2i} = \frac{\left(U\Delta_{-i}\right)^{1/2}}{Z} \left((r)\, Z_i^r d\varphi^{2r}\right) \tag{0.1.7}$$

for an integer index $i \in \{+1, -1\}$ and summed indices $r = +1, -1$. The non-ignorable coordinates correspond to two ones that the metric component depends on, in other words, $\left(\lambda^{+1}, \lambda^{-1}\right) \equiv (\lambda, \mu)$. On the other hand, the ignorable ones $\left(\varphi^{+2}, \varphi^{-2}\right)$ are equivalent to (ψ, χ). A round bracket (r) means summed signs, namely $(r)\, Z_i^r d\varphi^{2r} = +Z_i^{+1} d\varphi^{+2} - Z_i^{-1} d\varphi^{-2} = P_i d\psi - Q_i d\chi$. Then non-ignorable-numbered tetrads are represented by those greek letters as

$$\omega^{+1} = \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} d\lambda,\ \omega^{-1} = \left(\frac{Z}{\Delta_\mu}\right)^{1/2} d\mu \tag{0.1.8}$$

using $U = Z$ and the remainder are

$$\omega^{+2} = \frac{(Z\Delta_\mu)^{1/2}}{Z}\left(+P_\lambda d\psi - Q_\lambda d\chi\right), \omega^{-2} = \frac{(Z\Delta_\lambda)^{1/2}}{Z}\left(+P_\mu d\psi - Q_\mu d\chi\right). \tag{0.1.9}$$

These relations are summarized by a matrix

$$\hat{E} = \begin{bmatrix} \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \left(\frac{Z}{\Delta_\mu}\right)^{1/2} & 0 & 0 \\ 0 & 0 & \frac{(Z\Delta_\mu)^{1/2}}{Z} P_\lambda & -\frac{(Z\Delta_\mu)^{1/2}}{Z} Q_\lambda \\ 0 & 0 & \frac{(Z\Delta_\lambda)^{1/2}}{Z} P_\mu & -\frac{(Z\Delta_\lambda)^{1/2}}{Z} Q_\mu \end{bmatrix}, \tag{0.1.10}$$

which combines the ortho-normal tetrads ω^js ($j \in \{\pm 1, \pm 2\}$)

$$\begin{bmatrix} \omega^{+1} \\ \omega^{-1} \\ \omega^{+2} \\ \omega^{-2} \end{bmatrix} = \hat{E} \begin{bmatrix} d\lambda \\ d\mu \\ d\psi \\ d\chi \end{bmatrix}. \tag{0.1.11}$$

The determinant of the matrix $\hat{E}$ equals $-Z$:

$$\begin{aligned} \det \hat{E} &= \left(\frac{Z}{\Delta_\lambda}\right)^{1/2} \left(\frac{Z}{\Delta_\mu}\right)^{1/2} \frac{(Z\Delta_\lambda)^{1/2}}{Z} \frac{(Z\Delta_\mu)^{1/2}}{Z} \left(-P_\lambda Q_\mu + P_\mu Q_\lambda\right) \\ &= \left(\frac{Z^2}{\cancel{Z\Delta_\lambda}}\right)^{1/2} \left(\frac{Z^2}{\cancel{Z\Delta_\mu}}\right)^{1/2} \frac{\cancel{(Z\Delta_\lambda)^{1/2}}\cancel{(Z\Delta_\mu)^{1/2}}}{Z^{\cancel{2}}} \left(-\cancel{Z}\right) \\ &= |Z|^2 / Z \times (-1) = -Z \end{aligned} \tag{0.1.12}$$

Moreover, we can deduce the ortho-normal basis. Let $\{\partial_\lambda, \partial_\mu, \partial_\psi, \partial_\chi\}$ be coordinate basis vectors and $\{e_{+1}, e_{-1}, e_{+2}, e_{-2}\}$ be ortho-normal basis vectors corresponding to the tetrads ω^js. These ortho-normal vectors satisfy

$$\begin{bmatrix} e_{+1} & e_{-1} & e_{+2} & e_{-2} \end{bmatrix} = \begin{bmatrix} \partial_\lambda & \partial_\mu & \partial_\psi & \partial_\chi \end{bmatrix} \hat{E}^{-1}, \tag{0.1.13}$$

where the inverse matrix $\hat{E}^{-1}$ of the transformation matrix Eq. (0.1.10) is

$$\hat{E}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{Z/\Delta_\lambda}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{Z/\Delta_\mu}} & 0 & 0 \\ 0 & 0 & \frac{Q_\mu}{\sqrt{Z\Delta_\mu}} & -\frac{Q_\lambda}{\sqrt{Z\Delta_\lambda}} \\ 0 & 0 & \frac{P_\mu}{\sqrt{Z\Delta_\mu}} & -\frac{P_\lambda}{\sqrt{Z\Delta_\lambda}} \end{bmatrix} \tag{0.1.14}$$

If $\hat{E}^{-1}$ is expand concretely, each ortho-normal vector in $\{e_{+1}, e_{-1}, e_{+2}, e_{-2}\}$ is represented by

$$\begin{aligned} e_{+1} &= \left(\frac{\Delta_\lambda}{Z}\right)^{1/2} \partial_\lambda, e_{-1} = \left(\frac{\Delta_\mu}{Z}\right)^{1/2} \partial_\mu, \\ e_{+2} &= \frac{1}{(Z\Delta_\mu)^{1/2}}\left(Q_\mu \partial_\psi + P_\mu \partial_\chi\right), e_{-2} = \frac{1}{(Z\Delta_\lambda)^{1/2}}\left(-Q_\lambda \partial_\psi - P_\lambda \partial_\chi\right). \end{aligned} \tag{0.1.15}$$

using the coordinate basis $\{\partial_\lambda, \partial_\mu, \partial_\psi, \partial_\chi\}$. The tetrads construct the metric tensor

$$g = \omega^{+1} \otimes \omega^{+1} + \omega^{-1} \otimes \omega^{-1} + \omega^{+2} \otimes \omega^{+2} - \omega^{-2} \otimes \omega^{-2}. \tag{0.1.16}$$

We can also check the tetrads are the dual basis of the ortho-normal basis:

$$\begin{bmatrix} \omega^{+1} \\ \omega^{-1} \\ \omega^{+2} \\ \omega^{-2} \end{bmatrix} \begin{bmatrix} e_{+1} & e_{-1} & e_{+2} & e_{-2} \end{bmatrix} = \hat{E} \begin{bmatrix} d\lambda \\ d\mu \\ d\psi \\ d\chi \end{bmatrix} \begin{bmatrix} \partial_\lambda & \partial_\mu & \partial_\psi & \partial_\chi \end{bmatrix} \hat{E}^{-1} = 1. \tag{0.1.17}$$

That is why we can confirm the ortho-normality

$$g\left(e_j, e_j\right) = +1 \text{ for } j \in \{+1, -1, +2\},\ g\left(e_{-2}, e_{-2}\right) = -1, \text{ otherwise vanish.} \tag{0.1.18}$$

The connection coefficients of the Levi-Civita connection are simplified for an orthonormal basis, namely,

$$g_{i\alpha}\gamma^{\alpha}_{lj} = \frac{1}{2}\left(g_{i\mu}c^{\mu}_{lj} - g_{j\mu}c^{\mu}_{li} + g_{l\mu}c^{\mu}_{ij}\right), \tag{0.1.19}$$

where the structure constants or coefficients can be defined by $[e_i, e_j] = \sum_\mu c^{\mu}_{ij} e_\mu$ on a finitely-generated free module of vector fields in a spacetime region having the metric. The commutator $[e_i, e_j]$

$$\operatorname{diag}\left(g_{(+1,+1)}, g_{(-1,-1)}, g_{(+2,+2)}, g_{(-2,-2)}\right) = \operatorname{diag}\left(+1, +1, +1, -1\right). \tag{0.1.20}$$

- Lorentzian Manifold M
- An open set U
- coordinates: $(\lambda, \mu, \psi, \chi)$
- Independent functions: $\Delta_\lambda, \Delta_\mu, P_\lambda, P_\mu, Q_\lambda, Q_\mu$
- $Z = P_\lambda Q_\mu - P_\mu Q_\lambda$

$$A = \sum_i \frac{X_{-i}}{Z}\left((r)\, Z^r_i d\varphi^{2r}\right),\ A = \sum_i \frac{X_{-i}}{\left(U\Delta_{-i}\right)^{1/2}}\omega^{2i} \tag{0.1.21}$$

$$A = \frac{P_\lambda X_\mu + P_\mu X_\lambda}{Z} d\psi - \frac{Q_\lambda X_\mu + Q_\mu X_\lambda}{Z} d\chi$$

Cartan's first structure formula $d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = \tau^\mu$ with torsion free, i.e., $\tau^\mu = 0$ gives conditions of the connection form

$$\omega^\mu_\nu \wedge \theta^\nu = -d\theta^\mu. \tag{0.1.22}$$

We use the interior product ι to solve ω^μ_νs. The interior product

0.2 Appendix: Einstein Cartan Theory

We review the Einstein–Cartan theory in this section. Based on general relativity and the manifold theory, the Riemannian connection, the connection one-form, and the curvature two-form can be defined. Let M be C^r-class differentiable manifold ($r \geq 2$) with dimension m, TM be the tangent bundle of M, namely,

$$TM := \coprod_{p \in M} T_p M \equiv \bigcup_{p \in M} \{p\} \times T_p M \tag{0.2.1}$$

and $\Gamma^r(TM) \equiv \Gamma(TM)$ be the set of C^r-class vector fields.

First, the tangent bundle is a trivial example of vector bundles and has an affine connection. Second, a special connection named the Riemannian (Levi-Civita) connection exists uniquely on the smooth manifold with metric. We will denote vector fields by capital X, Y, Z and so on, besides, indices by the standard notations $i, j, k, l(\ell), \mu, \nu$, etc.

Def. 0.2.1 (Affine connection). A map on vector fields $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM); (X, Y) \mapsto \nabla_X Y$ is called an affine connection on M if it satisfies the following conditions.

- ∇ is an additive group homomorphism of two arguments of vector fields.
  - $\nabla(X, Y + Z) = \nabla(X, Y) + \nabla(X, Z)$,
  - $\nabla(X + Y, Z) = \nabla(X, Z) + \nabla(Y, Z)$.
- ∇ is an action $C^r(M)$-homomorphism of the first argument: $\forall f \in C^r(M), \nabla_{fX} Y = f \nabla_X Y$, i.e. $\nabla(fX, \cdot) = f \nabla(X, \cdot)$.
- ∇ satisfies the Leibniz rule: $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

The map $\nabla_X(\cdot)$ is also called the covariant derivative to the X-direction.

Now, we can see the affine connection as a map that returns a $(1,1)$-type tensor field. That is, using a new $\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$,

$$(\nabla Y)(X) \coloneqq \nabla_X Y \Longleftrightarrow (\nabla Y)_p (X_p) \coloneqq (\nabla_X Y)_p \quad \forall p \in M. \tag{0.2.2}$$

This $(1,1)$-tensor field is called the covariant derivative of Y, which satisfies the following properties:

1. (Addictivity) $\nabla(Y + Z) = \nabla Y + \nabla Z$
2. (Leibniz rule) $\nabla(fY) = Y \otimes df + f\nabla Y$ *1

The second property, the Leibniz rule, is proven as

$$(\nabla(fY))(X) = \nabla_X(fY) = X(f)Y + f\nabla_X Y = (Y \otimes df + f\nabla Y)(X) \tag{0.2.3}$$

by definition. The last equal sign is guaranteed by $(df)_p(X_p) = X_p(f)$ for any $p \in M$.

Below, let (U, x) be some chart on M and $\{e_i\} \subsetneq \Gamma(TM|_U)$ be a moving frame, in other words, local sections such that $e_i(p)$ for any index i and for

*1 The first term $Y \otimes df$ has the second factor of the derivative of f while it is also common to define the order by $df \otimes Y$ for general connections (covariant derivatives) on vector bundles