A New Structural-Differential Property of 5-Round AES

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Introduction (1/2)

Secret-Key Distinguisher: one of the weakest cryptographic attack.

Setting: *Two Oracles*:

- one simulates the block cipher for which the cryptography key has been chosen at random;
- the other simulates a truly random permutation.

Goal: distinguish the two oracles, i.e. decide which oracle is the cipher.

Introduction (2/2)

AES is probably the most widely studied and used block cipher.

So far, non-random properties which are independent of the secret key are known for up to 4 rounds of AES.

We propose a new structural property for up to 5 rounds of AES which is independent of the secret key.

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AES Subspace Trail

AES

High-level description of AES:

- block cipher based on a design principle known as *substitution-permutation network*;
- block size of 128 bits = 16 bytes, organized in a 4×4 matrix;
- key size of 128/192/256 bits;
- 10/12/14 rounds:

$$R^i(x) = k^i \oplus MC \circ SR \circ \text{S-Box}(x).$$

Subspace Trail

Recently introduced at FSE 2017.

Definition

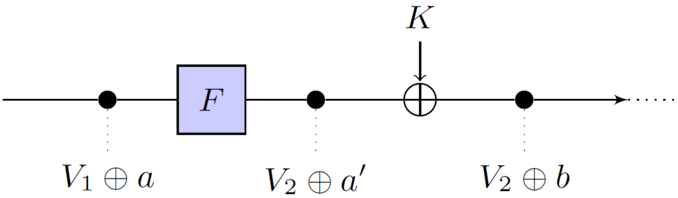
Let $(V_0, V_1, ..., V_r)$ denote a set of $r+1$ subspaces with $\dim(V_i) \leq \dim(V_{i+1})$. If for each $i = 0, ..., r-1$ and for each $a_i \in V_i^{\perp}$, there exists (unique) $a_{i+1} \in V_{i+1}^{\perp}$ such that

$$F(V_i \oplus a_i) \subseteq V_{i+1} \oplus a_{i+1},$$

then $(V_0, V_1, ..., V_r)$ is a subspace trail of length r for the function F.

It allows to describe key-recovery attacks and secret-key distinguisher in an *easier and more formal way* than “classical notation”.

Subspace Trail - Example



Example of Subspace Trail: $\forall a \in V_1^{\perp}$ there exists $b \in V_2^{\perp}$ s.t.

$$F_k(V_1 \oplus a) \subseteq V_2 \oplus b.$$

Subspaces for AES

We define the following subspaces:

- *column space* $\mathcal{C}_I$;
- *diagonal space* $\mathcal{D}_I$;
- *inverse-diagonal space* $\mathcal{ID}_I$;
- *mixed space* $\mathcal{M}_I$.

The Column Space

Definition

The *column spaces* $\mathcal{C}_i$ for $i \in \{0, 1, 2, 3\}$ are defined as

$$\mathcal{C}_i = \langle e_{0,i}, e_{1,i}, e_{2,i}, e_{3,i} \rangle.$$

E.g. $\mathcal{C}_0$ corresponds to the symbolic matrix

$$\mathcal{C}_0 = \left\{ \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{bmatrix} \middle| \forall x_1, x_2, x_3, x_4 \in \mathbb{F}_{2^8} \right\} \equiv \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{bmatrix}$$

The Diagonal Space

Definition

The *diagonal spaces* $\mathcal{D}_i$ for $i \in \{0, 1, 2, 3\}$ are defined as

$$\mathcal{D}_i = SR^{-1}(\mathcal{C}_i) = \langle e_{0,i}, e_{1,(i+1)}, e_{2,(i+2)}, e_{3,(i+3)} \rangle.$$

E.g. $\mathcal{D}_0$ corresponds to symbolic matrix

$$\mathcal{D}_0 \equiv \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}$$

for all $x_1, x_2, x_3, x_4 \in \mathbb{F}_{2^8}$.

The Inverse-Diagonal Space

Definition

The *inverse-diagonal spaces* $\mathcal{ID}_i$ for $i \in \{0, 1, 2, 3\}$ are defined as

$$\mathcal{ID}_i = SR(\mathcal{C}_i) = \langle e_{0,i}, e_{1,(i-1)}, e_{2,(i-2)}, e_{3,(i-3)} \rangle.$$

E.g. $\mathcal{ID}_0$ corresponds to symbolic matrix

$$\mathcal{ID}_0 \equiv \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & x_3 & 0 \\ 0 & x_4 & 0 & 0 \end{bmatrix}$$

for all $x_1, x_2, x_3, x_4 \in \mathbb{F}_{2^8}$.

The Mixed Space

Definition

The *i-th mixed spaces* $\mathcal{M}_i$ for $i \in \{0, 1, 2, 3\}$ are defined as

$$\mathcal{M}_i = MC(\mathcal{ID}_i).$$

E.g. $\mathcal{M}_0$ corresponds to symbolic matrix

$$\mathcal{M}_0 \equiv \begin{bmatrix} 0x02 \cdot x_1 & x_4 & x_3 & 0x03 \cdot x_2 \\ x_1 & x_4 & 0x03 \cdot x_3 & 0x02 \cdot x_2 \\ x_1 & 0x03 \cdot x_4 & 0x02 \cdot x_3 & x_2 \\ 0x03 \cdot x_1 & 0x02 \cdot x_4 & x_3 & x_2 \end{bmatrix}$$

for all $x_1, x_2, x_3, x_4 \in \mathbb{F}_{2^8}$.

Subspace Trail for AES (1/2)

Definition

Let $I \subseteq \{0,1,2,3\}$. The subspaces $\mathcal{C}_I$, $\mathcal{D}_I$, $\mathcal{ID}_I$ and $\mathcal{M}_I$ are defined as:

$$\mathcal{C}_I = \bigoplus_{i \in I} \mathcal{C}_i, \quad \mathcal{D}_I = \bigoplus_{i \in I} \mathcal{D}_i, \quad \mathcal{ID}_I = \bigoplus_{i \in I} \mathcal{ID}_i, \quad \mathcal{M}_I = \bigoplus_{i \in I} \mathcal{M}_i.$$

For each $a \in \mathcal{D}_I^{\perp}$, there exists unique $b \in \mathcal{C}_I^{\perp}$ s.t.

$$R(\mathcal{D}_I \oplus a) = \mathcal{C}_I \oplus b.$$

For each $b \in \mathcal{C}_I^{\perp}$, there exists unique $c \in \mathcal{M}_I^{\perp}$ s.t.

$$R(\mathcal{C}_I \oplus b) = \mathcal{M}_I \oplus c.$$

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For each $b \in \mathcal{C}_I^{\perp}$, there exists unique $c \in \mathcal{M}_I^{\perp}$ s.t.

$$R(\mathcal{C}_I \oplus b) = \mathcal{M}_I \oplus c.$$

Subspace Trail for AES (2/2)

Theorem

For each $a \in \mathcal{D}_I^{\perp}$, there exists unique $b \in \mathcal{M}_I^{\perp}$ s.t.

$$R^2(\mathcal{D}_I \oplus a) = \mathcal{M}_I \oplus b.$$

Lemma

For each x, y:

$$Prob(R^2(x) \oplus R^2(y) \in \mathcal{M}_I \,|\, x \oplus y \in \mathcal{D}_I) = 1.$$

Subspace Trail for AES (2/2)

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For each x, y:

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Example: the diagonal space $\mathcal{D}_i$

Plaintexts p^1 and p^2 satisfy $p^1 \oplus p^2 \in \mathcal{D}_i$ *if and only if* p^1 and p^2 are equal in all bytes expect for ones in the i-th diagonal.

E.g. $p^1 \oplus p^2 \in \mathcal{D}_0$ iff

$$p^1 \oplus p^2 \equiv \begin{bmatrix} ? & 0 & 0 & 0 \\ 0 & ? & 0 & 0 \\ 0 & 0 & ? & 0 \\ 0 & 0 & 0 & ? \end{bmatrix}$$

Example: the mixed space $\mathcal{M}_I$

Assume final MixColumns is omitted. Ciphertexts c^1 and c^2 satisfy $c^1 \oplus c^2 \in \mathcal{ID}_{\{0,1,2,3\}\setminus i}$ *if and only if* c^1 and c^2 are equal in the bytes in the i-th anti-diagonal.

E.g. $c^1 \oplus c^2 \in \mathcal{ID}_{\{0,1,2,3\}\setminus 3} \equiv \mathcal{ID}_{0,1,2}$ iff

$$c^1 \oplus c^2 \equiv \begin{bmatrix} ? & ? & ? & 0 \\ ? & ? & 0 & ? \\ ? & 0 & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

If the final MixColumns is not omitted, then $c^1 \oplus c^2 \in \mathcal{M}_{\{0,1,2,3\}\setminus i}$ *iff* $MC^{-1}(c^1 \oplus c^2) \equiv$ $\equiv MC^{-1}(c^1) \oplus MC^{-1}(c^2) \in \mathcal{ID}_{\{0,1,2,3\}\setminus i}$.

Part II

Secret-Key Distinguisher on 4 Rounds of AES

Secret Key Distinguisher on to 4 Rounds

Let $I, J \subseteq \{0,1,2,3\}$. Consider $2^{32\cdot|I|}$ plaintexts in the same coset of $\mathcal{D}_I$ - i.e. $p^0, p^1, ..., p^{32\cdot|I|-1} \in \mathcal{D}_I \oplus a$ - an the corresponding ciphertexts $c^0, c^1, ..., c^{32\cdot|I|-1}$ - i.e. $c^i = R^4(p^i)$.

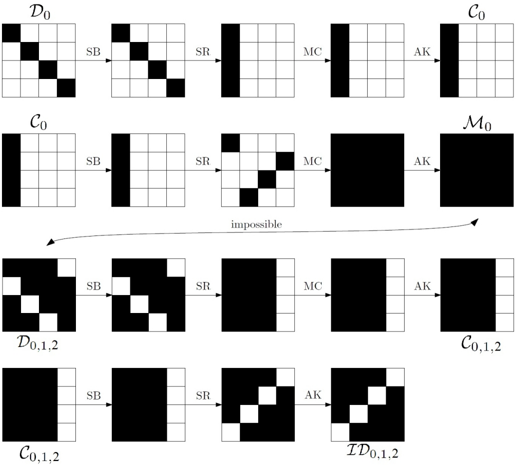
- Integral Property

$$\bigoplus_i p^i_{j,k} = \bigoplus_i c^i_{j,k} = 0 \qquad \forall j, k = 0, ..., 3;$$

- Impossible Differential Property

$$c^j \oplus c^k \notin \mathcal{M}_J \qquad \forall |J| + |I| \leq 4.$$

Impossible Differential Distinguisher - 4 Rounds



$$Prob(R^4(p^1) \oplus R^4(p^2) \in \mathcal{M}_{0,1,2} \,|\, p^1 \oplus p^2 \in \mathcal{D}_0) = 0.$$

Balance Property - 4-round AES

Given 2^{32} plaintexts in the same coset of a diagonal space $\mathcal{D}_0$:

$$\mathcal{D}_0 \oplus a \equiv \begin{bmatrix} A & C & C & C \\ C & A & C & C \\ C & C & A & C \\ C & C & C & A \end{bmatrix} \xrightarrow{R^4(\cdot)} \begin{bmatrix} B & B & B & B \\ B & B & B & B \\ B & B & B & B \\ B & B & B & B \end{bmatrix} \xrightarrow{R(\cdot)} ?$$

Given the same set of plaintexts $\mathcal{D}_0$, is there any property which is independent of the secret key after 5-round AES?

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Part III

Structural Property for up to 5 Rounds of AES

Structural Property for 5 Rounds of AES

Given $\mathcal{D}_I \oplus a$ (i.e. an arbitrary coset of $\mathcal{D}_I$), consider all the $2^{32\cdot|I|}$ plaintexts and the corresponding ciphertexts after 5 rounds, i.e. $(p^i, c^i \equiv R^5(p^i))$ for $i = 0, ..., 2^{32\cdot|I|} - 1$ where $p^i \in \mathcal{D}_I \oplus a$.

Theorem

For a fixed $J \subseteq \{0, 1, 2, 3\}$, let n the number of different pairs of ciphertexts (c^i, c^j) for $i \neq j$ such that $c^i \oplus c^j \in \mathcal{M}_J$ (i.e. c^i and c^j belong to the same coset of $\mathcal{M}_J$)

$$n := |\{(p^i, c^i), (p^j, c^j) \,|\, \forall p^i, p^j \in \mathcal{D}_I \oplus a,\ p^i < p^j \text{ and } c^i \oplus c^j \in \mathcal{M}_J\}|.$$

The number n is a multiple of 8, i.e. $\exists\, n' \in \mathbb{N}$ s.t. $n = 8 \cdot n'$.

Partial Order of the Plaintexts

Definition

Given two different texts t^1 and t^2, we say that $t^1 \leq t^2$ if $t^1 = t^2$ or if there exists $i, j \in \{0, 1, 2, 3\}$ such that

1. $t^1_{k,l} = t^2_{k,l}$ for all $k, l \in \{0, 1, 2, 3\}$ with $k + 4 \cdot l < i + 4 \cdot j$
2. $t^1_{i,j} < t^2_{i,j}$.

If $t^1 \leq t^2$ and $t^1 \neq t^2$, then $t^1 < t^2$.

Distinguisher on 5-round of AES (1/2)

Goal: Distinguish 5-round of AES from random permutation.

Consider 2^{32} plaintexts in the same coset of $\mathcal{D}_I$ for $|I| = 1$.

Count the number n of pairs of ciphertexts (after 5 rounds) that belong to the same coset of $\mathcal{M}_J$ for fixed $|J| = 3$.

If $n \bmod 8 \neq 0$, then the permutation is a random one.

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Distinguisher on 5-round of AES (2/2)

Using an initial coset of $\mathcal{D}_I$ for $|I| = 1$, the probability of success is higher than 99.5%:

- data cost: 2^{32} chosen plaintexts/ciphertexts;
- computational cost: $2^{35.6}$ table look-ups on table of size 2^{36} bytes.

Practically verified on a small-scale AES
https://github.com/Krypto-iaik/AES_5round_SKdistinguisher

It works also in the decryption direction (i.e. using chosen ciphertexts instead of plaintexts).

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Part IV

Sketch of the Proof

Reduction to a Single Round (1/2)

Remember:

$$R^2(\mathcal{D}_I \oplus a) = \mathcal{M}_I \oplus b.$$

Given a coset of $\mathcal{D}_I$, count the number of collisions among the ciphertexts after 5 rounds in the same coset of $\mathcal{M}_J$.

Since

$$\mathcal{D}_I \oplus a \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_I \oplus b \xrightarrow{R(\cdot)} \mathcal{D}_J \oplus a' \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_J \oplus b',$$

we can focus only on the middle round!

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we can focus only on the middle round!

Reduction to a Single Round (2/2)

Given $\mathcal{M}_I \oplus a$ (i.e. an arbitrary coset of $\mathcal{M}_I$), consider all the $2^{32\cdot|I|}$ plaintexts and the corresponding ciphertexts after 1 round, i.e. $(p^i, c^i \equiv R(p^i))$ for $i = 0, ..., 2^{32\cdot|I|} - 1$ where $p^i \in \mathcal{M}_I \oplus a$.

Lemma

Let n the number of different pairs of ciphertexts (c^i, c^j) for $i \neq j$ such that $c^i \oplus c^j \in \mathcal{D}_J$ (i.e. c^i and c^j belong to the same coset of $\mathcal{D}_J$)

$$n := |\{(p^i, c^i), (p^j, c^j) \,|\, \forall p^i, p^j \in \mathcal{M}_I \oplus a,\ p^i < p^j \text{ and } c^i \oplus c^j \in \mathcal{D}_J\}|.$$

The number n is a multiple of 8, i.e. $\exists\, n' \in \mathbb{N}$ s.t. $n = 8 \cdot n'$.

Sketch of the Proof

W.l.o.g. $I = \{0\}$.

Given $p^1, p^2 \in \mathcal{M}_0 \oplus a$, there exist $x^1, y^1, z^1, w^1 \in \mathbb{F}_{2^8}$ and $x^2, y^2, z^2, w^2 \in \mathbb{F}_{2^8}$ s.t.:

$$p^i = a \oplus \begin{bmatrix} 2 \cdot x^i & y^i & z^i & 3 \cdot w^i \\ x^i & y^i & 3 \cdot z^i & 2 \cdot w^i \\ x^i & 3 \cdot y^i & 2 \cdot z^i & w^i \\ 3 \cdot x^i & 2 \cdot y^i & z^i & w^i \end{bmatrix},$$

for $i = 1, 2$ and where $2 \equiv$ 0x02 and $3 \equiv$ 0x03.

For the following: p^1 "$\equiv$" $\langle x^1, y^1, z^1, w^1 \rangle$ and p^2 "$\equiv$" $\langle x^2, y^2, z^2, w^2 \rangle$.

Sketch of the Proof

Study the following cases:

- 3 variables are equal, e.g. $x^1 \neq x^2$ and $y^1 = y^2$, $z^1 = z^2$, $w^1 = w^2$;
- 2 variables are equal, e.g. $x^1 \neq x^2$,$y^1 \neq y^2$ and $z^1 = z^2$, $w^1 = w^2$;
- 1 variable is equal, e.g. $x^1 \neq x^2$, $y^1 \neq y^2$, $z^1 \neq z^2$ and $w^1 = w^2$;
- all variables are different, e.g. $x^1 \neq x^2$, $y^1 \neq y^2$, $z^1 \neq z^2$, $w^1 \neq w^2$.

If 3 variables are equal, then $R(p^1) \oplus R(p^2) = c^1 \oplus c^2 \notin \mathcal{D}_J$ with prob. 1.

Sketch of the Proof

Study the following cases:

- 3 variables are equal, e.g. $x^1 \neq x^2$ and $y^1 = y^2$, $z^1 = z^2$, $w^1 = w^2$;
- 2 variables are equal, e.g. $x^1 \neq x^2$,$y^1 \neq y^2$ and $z^1 = z^2$, $w^1 = w^2$;
- 1 variable is equal, e.g. $x^1 \neq x^2$, $y^1 \neq y^2$, $z^1 \neq z^2$ and $w^1 = w^2$;
- all variables are different, e.g. $x^1 \neq x^2$, $y^1 \neq y^2$, $z^1 \neq z^2$, $w^1 \neq w^2$.

If 3 variables are equal, then $R(p^1) \oplus R(p^2) = c^1 \oplus c^2 \notin \mathcal{D}_J$ with prob. 1.

Sketch of the Proof (2 variables are different)

W.l.o.g. consider $p^1 \equiv \langle x^1, y^1, z, w\rangle$ and $p^2 \equiv \langle x^2, y^2, z, w\rangle$.
$R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ if and only if

$$R(\hat{p}^1) \oplus R(\hat{p}^2) \in \mathcal{D}_J$$

where

$$\hat{p}^1 \equiv \langle x^1, y^2, z, w\rangle, \qquad \hat{p}^2 \equiv \langle x^2, y^1, z, w\rangle.$$

It is sufficient to prove that $R(p^1) \oplus R(p^2) = R(\hat{p}^1) \oplus R(\hat{p}^2)$.

$$\begin{aligned}
&(R(p^1) \oplus R(p^2))_{0,0} = \\
=&2 \cdot [\text{S-Box}(2 \cdot x^1 \oplus a_{0,0}) \oplus \text{S-Box}(2 \cdot x^2 \oplus a_{0,0})] \oplus \\
&\oplus 3 \cdot [\text{S-Box}(y^1 \oplus a_{1,1}) \oplus \text{S-Box}(y^2 \oplus a_{1,1})] = \\
=&(R(\hat{p}^1) \oplus R(\hat{p}^2))_{0,0}.
\end{aligned}$$

Sketch of the Proof (2 variables are different)

Given $p^1 \equiv \langle x^1, y^1, z, w\rangle$ and $p^2 \equiv \langle x^2, y^2, z, w\rangle$ s.t. $R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ then

$$R(\hat{p}^1) \oplus R(\hat{p}^2) \in \mathcal{D}_J$$

where

$$\hat{p}^1 \equiv \langle x^1, y^1, z, w\rangle, \qquad \hat{p}^2 \equiv \langle x^2, y^2, z, w\rangle$$

or

$$\hat{p}^1 \equiv \langle x^1, y^2, z, w\rangle, \qquad \hat{p}^2 \equiv \langle x^2, y^1, z, w\rangle$$

for all $z, w \in \mathbb{F}_{2^8}$.

It is sufficient to prove that $R(p^1) \oplus R(p^2) = R(\hat{p}^1) \oplus R(\hat{p}^2)$ doesn't depend on z and w.

Sketch of the Proof (2 variables are different)

Is it possible that $p^1 \equiv \langle x^1, y^1, 0, 0\rangle$ and $p^2 \equiv \langle x^2, y^2, 0, 0\rangle$ such that $R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ exist?
Answer: *Yes, if* $|J| = 3$ *since the branch number of the MixColumns matrix is 5*.

Indeed, consider the first column of:

$$(SR \circ \text{ S-Box}(p^1) \oplus SR \circ \text{ S-Box}(p^2))_{\cdot,0} \equiv \begin{bmatrix} \text{S-Box}(2 \cdot x^1 \oplus a_{0,0}) \oplus \text{S-Box}(2 \cdot x^2 \oplus a_{0,0}) \\ \text{S-Box}(y^1 \oplus a_{1,1}) \oplus \text{S-Box}(y^2 \oplus a_{1,1}) \\ 0 \\ 0 \end{bmatrix}.$$

Since...

Sketch of the Proof (2 variables are different)

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Sketch of the Proof (3 variables are different)

W.l.o.g. consider $p^1 \equiv \langle x^1, y^1, z^1, w\rangle$ and $p^2 \equiv \langle x^2, y^2, z^2, w\rangle$.
$R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ if and only if $R(\hat{p}^1) \oplus R(\hat{p}^2) \in \mathcal{D}_J$ where

$$\begin{aligned}
\hat{p}^1 &\equiv \langle x^1, y^1, z^1, w\rangle, & \hat{p}^2 &\equiv \langle x^2, y^2, z^2, w\rangle \\
\hat{p}^1 &\equiv \langle x^2, y^1, z^1, w\rangle, & \hat{p}^2 &\equiv \langle x^1, y^2, z^2, w\rangle \\
\hat{p}^1 &\equiv \langle x^1, y^2, z^1, w\rangle, & \hat{p}^2 &\equiv \langle x^2, y^1, z^2, w\rangle \\
\hat{p}^1 &\equiv \langle x^1, y^1, z^2, w\rangle, & \hat{p}^2 &\equiv \langle x^2, y^2, z^1, w\rangle
\end{aligned}$$

for each $w \in \mathbb{F}_{2^8}$.

Note: $p^1 \equiv \langle x^1, y^1, z^1, 0\rangle$ and $p^2 \equiv \langle x^2, y^2, z^2, 0\rangle$ such that
$R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ can exist if and only if $|J| \geq 2$.

Sketch of the Proof (4 variables are different)

W.l.o.g. consider $p^1 \equiv \langle x^1, y^1, z^1, w^1 \rangle$ and $p^2 \equiv \langle x^2, y^2, z^2, w^2 \rangle$. $R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ if and only if $R(\hat{p}^1) \oplus R(\hat{p}^2) \in \mathcal{D}_J$ where

$$\begin{aligned}
\hat{p}^1 &\equiv \langle x^2, y^1, z^1, w^1 \rangle, & \hat{p}^2 &\equiv \langle x^1, y^2, z^2, w^2 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^2, z^1, w^1 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^1, z^2, w^2 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^1, z^2, w^1 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^2, z^1, w^2 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^1, z^1, w^2 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^2, z^2, w^1 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^1, z^2, w^2 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^2, z^1, w^1 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^2, z^1, w^2 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^1, z^2, w^1 \rangle; \\
\hat{p}^1 &\equiv \langle x^1, y^2, z^2, w^1 \rangle, & \hat{p}^2 &\equiv \langle x^2, y^1, z^1, w^2 \rangle.
\end{aligned}$$

Note: $p^1 \equiv \langle x^1, y^1, z^1, w^1 \rangle$ and $p^2 \equiv \langle x^2, y^2, z^2, w^2 \rangle$ such that $R(p^1) \oplus R(p^2) \in \mathcal{D}_J$ can exist if and only if $|J| \geq 1$.

Sketch of the Proof

$$n := |\{(p^i, c^i), (p^j, c^j) \,|\, \forall p^i, p^j \in \mathcal{M}_I \oplus a,\ p^i < p^j \text{ and } c^i \oplus c^j \in \mathcal{D}_J\}|.$$

- If $|J| = 1$, then $n = 8 \cdot n'$;
- If $|J| = 2$, then $n = 8 \cdot n' + 4 \cdot 2^8 \cdot n''$;
- If $|J| = 3$, then $n = 8 \cdot n' + 4 \cdot 2^8 \cdot n'' + 2 \cdot 2^{16} \cdot n'''$.

The number of collisions n is a multiple of 8 independently of I, J, the secret key, the details of the S-Box and the MixColumns operation (expect for the branch number equal to 5).

Part V

Conclusion and Open Problems

Conclusion and Open Problems

- First 5-round Secret-Key Distinguisher for AES independent of the secret key.
- Open Problems:
  - Set up a *6-round Secret-Key Distinguisher for AES* independent of the secret key;
  - Set up a *key recovery attack that exploits this 5-round secret key distinguisher* (or a modified version of it);
  - Apply "similar" distinguisher to other constructions.

Thanks for your attention!

Questions?

Comments?