ASSIGNMENT-3

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Ans 1. $\frac{d^4y}{dx^4}=0$; $y(0)=y'(0)=y''(1)=y'''(1)=0$

G.S. $\rightarrow$ $y(x)=ax^3+bx^2+cx+d$

$y(0)=0 \Rightarrow d=0$
$y'(0)=0 \Rightarrow c=0$
$y''(1)=0 \Rightarrow 6ax+2b \Rightarrow 2b+6a=0$
$y'''(1)=0 \Rightarrow 6a=0 \quad \Rightarrow a=0$
$\qquad\qquad\qquad\qquad\qquad b=0$

As $a=b=c=d=0$, so only trivial soln. possible.

$$G(x,t)=\begin{cases} A_1x^3+A_2x^2+A_3x+A_4 & 0\le x<t \\ b_1x^3+b_2x^2+b_3x+b_4 & t<x\le 1\end{cases}$$

① Continuous at $x=t$

i.e. $A_1t^3+A_2t^2+A_3t+A_4=b_1t^3+b_2t^2+b_3t+b_4$

$(b_1-A_1)t^3+(b_2-A_2)t^2+(b_3-A_3)t+(b_4-A_4)=0$

Putting $b_K-A_K=C_K$

$c_1t^3+c_2t^2+c_3t+c_4=0$ —— ①

② $\left(\frac{\partial^3 G}{\partial x^3}\right)_{t+0}-\left(\frac{\partial^3 G}{\partial x^3}\right)_{t-0}=-1$

$6b_1-6A_1=-1$

$6c_1=-1 \qquad c_1=\frac{-1}{6}$ —— ②

③ Satisfying boundary Cond.

$y(0) = A_4 = 0$ — ③

$y'(0) = A_3 = 0$ — ④

$y''(1) = 0 \Rightarrow b_2 = 0$ — ⑤

$y'''(1) = 0 \Rightarrow b_1 = 0$ — ⑥

④ G' is Cont. at $x = t$

$3c_1t^2 + 2c_2t + c_3 = 0$

$6c_1t + 2c_2 = 0$

$c_2 = -3c_1t = \frac{t}{2}$

$c_3 = -3\left(\frac{-1}{6}\right)t^2 - 2\left(\frac{t}{2}\right)t$

$= -\frac{t^2}{2}$

$c_4 = \frac{1}{6}t^3 - \frac{t^3}{2} + \frac{t^3}{2} = \frac{t^3}{6}$

$c_2 = \frac{t}{2}$

$c_3 = \frac{-t^2}{2}$

$c_4 = \frac{t^3}{6}$

$b_1 = 0,\ A_1 = \frac{1}{6}$

$b_2 = 0\ A_2 = -t/2$

$A_3 = 0\ b_3 = -t^2/2$

$A_4 = 0\ b_4 = \frac{t^3}{6}$

$$G(x,t) = \begin{cases} \frac{x^3}{6} - \frac{tx^2}{2} + 0 & 0 \le x < t \\ 0 + \left(-\frac{t^2x}{2}\right) + \frac{t^3}{6} & t < x \le t \end{cases}$$

(ii) $\frac{d^3y}{dx^3} = 0 \qquad y(0) = y'(1) = 0,\ y'(0) = y(1)$

$y(x) = Ax^2 + Bx + C$

$y(0) = C = 0$

~~$y(x) = Bx$~~

$y'(1) = 0 \qquad 2A + B = 0$

$y'(0) = B = A + B + 0$

$A = 0,\ B = 0,\ C = 0$ Trivial Soln. exist.

$$G(x,t) = \begin{cases} A_1x^2 + A_2x + A_3 & 0 \le x < t \\ b_1x^2 + b_2x + b_3 & t < x \le 1 \end{cases}$$

① G, G' are continuous at $x = t$

$$C_1 t^2 + C_2 t + C_3 = 0 \quad \text{— ①}$$
$$2C_1 t + C_2 = 0 \quad \text{— ②}$$

② $$\left(\frac{\partial^2 G}{\partial x^2}\right)_{t+0} - \left(\frac{\partial^2 G}{\partial x^2}\right)_{t-0} = -1$$

$$2C_1 = -1 \qquad C_1 = -\frac{1}{2} \quad \text{— ③}$$

③ Boundary Cond. ~~at~~

$y(0) = A_3 = 0$

$y'(1) = 2B_1 + B_2 = 0$

$y'(0) = y(1) \Rightarrow A_2 = B_1 + B_2 + B_3$

$A_2 = -B_1 + B_3$

$C_2 = -2C_1 t = t$

$C_3 = -C_1 t^2 - C_2 t = -\frac{t^2}{2}$

$B_3 = -\frac{t^2}{2}$

$t = B_2 - A_2 = B_1 + B_2 - B_3$

$B_1 + B_2 = t + B_3 = t - \frac{t^2}{2}$

$2B_1 + B_2 = 0$

$B_1 = -\frac{t^2}{2} - t$

$A_1 = \frac{t^2}{2} - t + \frac{1}{2}$

$B_2 = -2B_1 = 2t - t^2$

$A_2 = 2t - t^2 - t = t - t^2$

$A_3 = 0 \quad B_3 = -\frac{t^2}{2}$

$$G(x,t) = \begin{cases} \left(\frac{t^2}{2} - t + \frac{1}{2}\right)x^2 + (t - t^2)x + 0 & 0 \le x < t \\ \left(\frac{t^2}{2} - t\right)x^2 + (2t - t^2)x - \frac{t^2}{2} & t < x \le 1 \end{cases}$$

(iii) $y''' = 0 \qquad y(0) = y(1) = 0, \quad y'(0) = y'(1)$

$y(x) = Ax^2 + Bx + C$

$y(0) = 0 \Rightarrow C = 0, \quad y(1) = 0 \Rightarrow A + B = 0$

$y'(0) = B = 2A + B \qquad A = B = C = 0$ Trivial Soln exist.

$$G(x,y) = \begin{cases} A_1x^2 + A_2x + A_3 & 0 \le x < t \\ b_1x^2 + b_2x + b_3 & t \le x < 1 \end{cases}$$

① G, G' are cont. at $x = t$

$$c_1t^2 + c_2t + c_3 = 0$$

$$2c_1t + c_2 = 0$$

② $\left(\frac{\partial^2 G}{\partial x^2}\right)_{t+0} - \left(\frac{\partial^2 G}{\partial x^2}\right)_{t-0} = -1$

$$2c_1 = -1 \quad \Rightarrow c_1 = \frac{-1}{2} = b_1 - A_1$$

$G(0,t) = 0 \quad A_3 = 0$

$G(1,t) = 0 \quad b_1 + b_2 + b_3 = 0$

$G'(0,t) = G'(1,t) = A_2 - b_2 + 2b_1 \quad \Rightarrow -2b_1 = b_2 - A_2 = t$

$$b_1 = \frac{-t}{2}$$

$$c_2 = -2c_1t = t$$

$\cancel{}\ b_2 - A_2 = t$

$$c_3 = -c_1t^2 - c_2t = \frac{1}{2}t^2 - t^2 = \frac{-t^2}{2}$$

$b_3 = \frac{-t^2}{2}, \quad A_3 = 0$

$b_2 = -b_3 - b_1 = \frac{t^2}{2} + \frac{t}{2}, \quad A_2 = \frac{t^2}{2} - \frac{t}{2}, \quad A_1 = -\frac{t}{2} + \frac{1}{2}, \quad b_1 = \frac{-t}{2}$

$$G(x,t) = \begin{cases} \left(-\frac{t}{2} + \frac{1}{2}\right)x^2 + \left(\frac{t^2}{2} - \frac{t}{2}\right)x + 0 & 0 \le x < t \\ -\frac{t}{2}x^2 + \left(\frac{t^2+t}{2}\right)x + \left(\frac{-t^2}{2}\right) & t \le x < 1 \end{cases}$$

Ans 2. (i) $y'' + \pi^2 y = \cos \pi x$, $y(0) = y(1), y'(0) = y'(1)$

$m^2 + \pi^2 = 0 \qquad m = \pm \pi i$

$y(x) = A \cos \pi x + B \sin \pi x$

$A = -A \Rightarrow A = 0$ Trivial Soln. so G is unique

$B\pi = -B\pi \Rightarrow B = 0$

$$G(x,t) = \begin{cases} a_1 \cos \pi x + a_2 \sin \pi x & 0 \le x < t \\ b_1 \cos \pi x + b_2 \sin \pi x & t \le x < 1 \end{cases}$$

$c_1 \cos \pi t + c_2 \cos \pi t = 0$

~~$-\pi c_1 \sin \pi t$~~

$-\pi c_1 \cos \pi t + c_2 \pi \cos \pi t = -1$

$c_1 \cos \pi t + c_2 \sin \pi t = 0$

$c_1 \sin \pi t - c_2 \cos \pi t = -\frac{1}{\pi}$

$c_1 = \frac{-\sin \pi t}{\pi}$, $c_2 = \frac{\cos \pi t}{\pi}$

$a_1 = -b_1$

$\pi a_2 = -b_2 \pi$

$a_2 = -b_2$

$b_1 - a_1 = 2b_1 = \frac{-\sin \pi t}{\pi}$, $b_1 = \frac{-1}{2} \frac{\sin \pi t}{\pi}$, $a_1 = \frac{1}{2} \frac{\sin \pi t}{\pi}$, $b_2 = \frac{1}{2} \frac{\cos \pi t}{\pi}$

$a_2 = \frac{-1}{2} \frac{\cos \pi t}{\pi}$

$$G(x,t) = \begin{cases} \frac{1}{2} \frac{\sin \pi t}{\pi} \cos \pi x + \left(-\frac{1}{2} \frac{\cos \pi t}{\pi} \sin \pi x\right) & 0 \le x < t \\ \frac{1}{2\pi} \left[-\sin \pi t \cos \pi x + \cos \pi t \sin \pi x\right] & t \le x < 1 \end{cases}$$

$y(x) = \int_0^1 G(x,t)\, \phi(t)\, dt$

$$y(x) = \frac{-1}{2\pi}\int_0^x (\sin\pi x \cos^2\pi t - \cos\pi x \sin\pi t \cos\pi t)\, dt$$

$$- \frac{1}{2\pi}\int_x^1 (\cos\pi x \sin\pi t \cos\pi t - \sin\pi x \cos^2\pi t)\, dt$$

$$= \frac{-1}{2\pi}\left[\frac{\sin\pi x\,(-\pi + 2\pi x + \sin 2\pi x)}{2\pi} + \frac{\cos\pi x\,(\cos 2\pi x - 1)}{2\pi}\right]$$

$$= \frac{-1}{4\pi^2}\left[\sin\pi x\,(-\pi + 2\pi x + \sin 2\pi x) + \frac{\cos\pi x\,(\cos 2\pi x - 1)}{2\pi}\right]$$

~~(iii) $y'' - y = 2\sin$~~

(ii) $y'' + y = x^2$, $y(0) = y(\frac{\pi}{2}) = 0$

$m^2 + 1 = 0$

GS $\rightarrow$ $y = A\cos x + B\sin x$

$y(0) = 0$ $A = 0$ $\phi(x) = -x^2$

$y(\pi/2) = 0$ $B = 0$

$$G(x,y) = \begin{cases} A_1\cos x + A_2\sin x & 0 \le x < t \\ b_1\cos x + b_2\sin x & t \le x < 1 \end{cases}$$

$c_1\cos t + c_2\sin t = 0$

$-(b_1 - A_1)\sin t + (b_2 - A_2)\cos t = -1$

$-c_1\sin t + c_2\cos t = -1$

$c_1 = \sin t$ $y(0) = 0$ $A_1 = 0$ $b_1 = \sin t$

$c_2 = -\cos t$ $b_2 = 0$ $A_2 = \cos t$

$$G(x,t) = \begin{cases} \cos t \sin x & 0 \le x < t \\ \sin t \cos x & t < x \le \frac{\pi}{2} \end{cases}$$

$$y(x) = \int_0^x -\cos x \sin t (t^2)\, dt + \int_x^{\pi/2} -\sin x \cos t\, (t^2)\, dt$$

$$= -\cos x \int_0^x t^2 \sin t\, dt - \sin x \int_x^{\pi/2} t^2 \cos t\, dt$$

$$= -\cos x \left[\left[-t^2 \cos t \right]_0^x + \int_0^x 2t \cos t\, dt \right] - \sin x \left[\left[t^2 \sin t \right]_x^{\pi/2} - \int_x^{\pi/2} 2t \sin t\, dt \right]$$

$$= -\cos x \left[-x^2 \cos x + 2x \sin x + 2\cos x - 2 \right]$$

$$- \sin x \left[\frac{\pi^2}{4} - x^2 \sin x - 2x \cos x - 2(1 - \sin x) \right]$$

$$= x^2 + 2(\sin x + \cos x) - 2 - \frac{\pi^2}{4} \sin x$$

(iii) $y'' - y = 2 \sinh$, $y(0) = y(1) = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$y(x) = c_1 e^x + c_2 e^{-x}$

$c_1 + c_2 = 0$

$c_1 e + c_2 e^{-1} = 0$

$c_1 e + c_2 e = 0$

$c_2 (e^2 - 1) = 0 \Rightarrow c_2 = 0$, $c_1 = 0$ Trivial soln. $\rightarrow$ G is unique

$$G(x,t) = \begin{cases} A_1 e^x + A_2 e^{-x} & 0 \le x < t \\ b_1 e^x + b_2 e^{-x} & t \le x < 1 \end{cases}$$

$c_1 e^t + c_2 e^{-t} = 0$

$c_1 e^t - c_2 e^{-t} = -1$

$c_2 = \frac{1}{2} e^t$, $c_1 = -\frac{1}{2} e^{-t}$

$A_1 + A_2 = 0$, $B_1 e + B_2 e^{-1} = 0$

$B_1 - C_1 + B_2 - C_2 = 0$

$B_1 + \frac{1}{2}e^{-t} + B_2 - \frac{1}{2}e^{t} = 0$

$B_1 + B_2 = \dfrac{e^t - e^{-t}}{2}$ $\qquad$ $B_2 = \dfrac{e^2(e^t - e^{-t})}{2(e^2-1)}$

$(1-e^2)B_1 = \dfrac{e^t - e^{-t}}{2}$ $\qquad$ $B_1 = \dfrac{e^t - e^{-t}}{2(1-e^2)}$

$$A_1 = \frac{e^t - e^{-t}}{2(1-e^2)} + \frac{1}{2}e^{-t} = \frac{e^t - e^2 . e^{-t}}{2(1-e^2)}$$

$$A_2 = \frac{e^2(e^t - e^{-t})}{2(e^2-1)} - \frac{e^t}{2} = \frac{e^t + e^2 . e^{-t}}{2(1-e^2)}$$

$$G(x,t) = \begin{cases} A_1 e^x + A_2 e^{-x} & 0 \le x \le t \\ B_1 e^x + B_2 e^{-x} & t \le x \le 1 \end{cases}$$

$$y(x) = -\sin h \int_0^x (A_1 e^t + A_2 e^{-t})\,dt - \sin h \int_x^1 (B_1 e^t + B_2 e^{-t})\,dt$$

$$= -\sin h\left[A_1(e^x - 1) - A_2(e^{-x} - 1)\right] - \sin h\left[B_1(e - e^x) - B_2(e - e^{-x})\right]$$

Ans