MM- ASSIGN- 5 DB

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Ans: Let $A^{m_1 m_2 \dots m_{K_1}}_{n_1 n_2 \dots n_{K_2}}$ be a mixed tensor of rank (K_1+K_2) & type (K_1, K_2). Given that the tensor is skew-symmetric w.r.t coordinate $(x_1, x_2 \dots x_n)$

so its given that $A^{m_1 m_2 \dots m_{K_1}}_{n_1 n_2 \dots n_{K_2}}$ is skew-symmetric.

Without loss of generality, lets take that its is a skew-symmetric w.r.t. indices m_1, m_2.

so, $A^{m_1 m_2 \dots m_{K_2}}_{n_1 n_2 \dots n_{K_2}} = -A^{m_2 m_1 \dots m_{K_1}}_{n_1 n_2 \dots n_{K_2}}$ in coordinate systen V_N $x=(x_1, x_2 \dots x_N)$

Now lets transfer coordinate system from $x \longrightarrow \bar{x}$

$\bar{x} = (\bar{x}_1, \bar{x}_2, \dots \bar{x}_N)$

Now, tranformed components will be written as - $\bar{A}^{p_1 p_2 \dots p_{K_1}}_{q_1 q_2 \dots q_{K_2}}$

We need to show that - $\bar{A}^{p_1 p_2 \dots p_{K_1}}_{q_1 q_2 \dots q_{K_2}} = -\bar{A}^{p_1 p_2 \dots p_{K_1}}_{q_1 q_2 \dots q_{K_2}}$

By law of transformation, we have -

$$LHS = \bar{A}^{p_1 p_2 \ldots p_{k_1}}_{q_1 q_2 \ldots q_{k_2}} = \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \cdot \ldots \frac{\partial \bar{x}^{p_{k_1}}}{\partial x^{m_{k_1}}} \cdot \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{n_2}}{\partial \bar{x}^{q_2}} \ldots \frac{\partial x^{n_{k_2}}}{\partial \bar{x}^{q_{k_2}}} A^{m_1 m_2 \ldots m_{k_1}}_{n_1 n_2 \ldots n_{k_2}}$$

$$= \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \ldots \frac{\partial \bar{x}^{p_{k_1}}}{\partial x^{m_{k_1}}} \cdot \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \ldots \frac{\partial x^{n_{k_2}}}{\partial \bar{x}^{q_{k_2}}} \left(-A^{m_2 m_1 \ldots m_{k_1}}_{n_1 n_2 \ldots n_{k_2}} \right)$$

$$= \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \cdot \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \ldots \frac{\partial \bar{x}^{p_{k_1}}}{\partial x^{m_{k_1}}} \cdot \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \ldots \frac{\partial x^{n_{k_2}}}{\partial \bar{x}^{q_{k_2}}} \left(-A^{m_2 m_1 \ldots m_{k_1}}_{n_1 n_2 \ldots n_{k_2}} \right)$$

$$= -\bar{A}^{p_2 p_1 \ldots p_{k_1}}_{q_1 q_2 \ldots q_{k_2}}$$

Therefore $\bar{A}$ is skew symmetric. Proved

Ans 2. Following similar steps as in Ques 1.

$A^{m_1 m_2 \ldots m_{k_1}}_{n_1 n_2 \ldots n_{k_2}}$ be a mixed tensor of rank (k_1+k_2) & type (k_1, k_2)

We know that $A^{m_1 m_2 \ldots m_{k_1}}_{n_1 n_2 \ldots n_{k_2}} = 0$

Now let coordinate transformation takes place between $x \rightarrow \bar{x}$ coordinate & the components in $\bar{x}$ system is given by -

$$\bar{A}^{p_1 p_2 \ldots p_{k_1}}_{q_1 q_2 \ldots q_{k_2}} = \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \ldots \frac{\partial \bar{x}^{p_{k_1}}}{\partial x^{m_{k_1}}} \cdot \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \ldots \frac{\partial x^{n_{k_2}}}{\partial \bar{x}^{q_{k_2}}} A^{m_1 m_2 \ldots m_{k_2}}_{n_1 n_2 \ldots n_{k_2}}$$

Since all components of A vanishes $A^{m_1 m_2 \dots m_{K_1}}_{n_1 n_2 \dots n_{K_2}} = 0 \ \forall \ m_i, n_j$

$$\Rightarrow \bar{A}^{P_1 P_2 \dots P_{K_1}}_{q_1 q_2 \dots q_{K_2}} = \left(\frac{\partial \bar{x}^{P_1}}{\partial x^{m_1}} \cdot \frac{\partial \bar{x}^{P_2}}{\partial x^{m_2}} \dots\right) \cdot \left(\frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \dots \frac{\partial x^{n_{K_2}}}{\partial \bar{x}^{q_{K_2}}}\right) \cdot 0$$

$$= 0 \quad \forall \ P_i, q_j$$

⇒ All components in all coordinate system vanishes identically.

<u>Ans3.</u> In the tensor which is mixed with rank 5 or (3,2) A^{ijK}_{lm}, if we set $K = m$ then, we get -

$A^{ijK}_{lm} = A^{ij}_{l}$, a mixed tensor of rank $(5-2=3)$

Further, by setting $j = l$, we get $A^{ij}_{l} = A^{i}$ which is tensor of rank 1.

so if one contravarient & one covarient are set equal in a tensor then, the result indicates that a summation will be taken equal ~~....~~ according to the summation convention. This resulting sum is a tensor of rank 2 less that of the original tensor. This process is called <u>Contraction</u>.

Now, we know that $a^{ij} = \frac{\text{Cofactor of } a_{ij}}{|a_{ij}|}$

where cofactor is $(-1)^{P+q} \times$ (minor of g_{pq} in $g = |g_{pq}|$)

Let us say that the coordinates are in $V_N = (x_1, x_2, \ldots x_N)$ N-dim space.

We know that summation of the product of cofactor of a_{ij} with a_{ij} results in determinant of a_{ij} into Identity Matrix.

Let a^{ij} be a contravariant tensor of rank 2 in N-dim space.
& a_{ij} " " covariant tensor of " " in same N-dim space.

Now, $a_{ij} \cdot a^{ij} = \dfrac{a_{ij} \cdot \text{Cofactor of } a_{ij}}{|a_{ij}|}$

$$= \frac{\sum_{j=1}^{N} a_{ij} \cdot \text{Cofactor of } a_{ij}}{|a_{ij}|} = \frac{|a_{ij}|}{|a_{ij}|} = I_N$$

$$a_{ij} \cdot a^{ij} = I_N$$

So $I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

So we can say $I_N = \delta^i_j$, $\Rightarrow \boxed{a_{ij} \cdot a^{ij} = \delta^i_j}$

Ans 4. If $\bar{C}_{ij}\bar{A}^i\bar{A}^j$ is invariant then $\bar{C}_{ij}\bar{A}^i\bar{A}^j = C_{lm}A^lA^m$

Since A^i, A^j are contravariant vectors, by law of transformation -

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^l}A^l \quad \& \quad \bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^m}A^m$$

Now, we have -

$$\bar{C}_{ij}\frac{\partial \bar{x}^i}{\partial x^l}A^l\frac{\partial \bar{x}^j}{\partial x^m}A^m = C_{lm}A^lA^m$$

$$\bar{C}_{ij}\frac{\partial \bar{x}^i}{\partial x^l}\cdot\frac{\partial \bar{x}^j}{\partial x^m}\cdot A^lA^m = C_{lm}A^lA^m$$

One can show for indices j & i that

$$\bar{C}_{ji}\frac{\partial \bar{x}^i}{\partial x^l}\cdot\frac{\partial \bar{x}^j}{\partial x^m}A^lA^m = C_{ml}A^lA^m$$

On adding both eq., we get -

$$A^lA^m\left[(\bar{C}_{ij}+\bar{C}_{ji})\frac{\partial \bar{x}^i}{\partial x^l}\cdot\frac{\partial \bar{x}^j}{\partial x^m} - (C_{ml}+C_{lm})\right] = 0$$

A^l & A^m are arbitrary vectors, they can't be zero

$$\Rightarrow (\bar{C}_{ji}+\bar{C}_{ij})\frac{\partial \bar{x}^i}{\partial x^l}\cdot\frac{\partial \bar{x}^j}{\partial x^m} = (C_{ml}+C_{lm})$$

$(C_{lm}+C_{ml})$

Multiply by $\frac{\partial x^l}{\partial \bar{x}^s}$, we have $(\bar{C}_{ij}+\bar{C}_{ji}).\frac{\partial \bar{x}^i}{\partial \bar{x}^s}.\frac{\partial \bar{x}^j}{\partial x^m} = \frac{\partial x^l}{\partial \bar{x}^s}.$

" " $\frac{\partial x^m}{\partial \bar{x}^t}$, " " $\delta^i_s(\bar{C}_{ij}+\bar{C}_{ji})\frac{\partial \bar{x}^j}{\partial x^m} = \frac{\partial x^l}{\partial \bar{x}^s}.\frac{\partial x^m}{\partial \bar{x}^t}(C_{ml}+C_{lm})$

$$(\bar{C}_{sm}+\bar{C}_{ms}) = \frac{\partial x^l}{\partial \bar{x}^s}.\frac{\partial x^m}{\partial \bar{x}^t}(C_{ml}+C_{lm})$$

$C_{ml}+C_{lm}$ is a contravarient tensor & since it has 2 distinct indices it is of 2^{nd} order.

Ans 5. We know that - $\left\{ {l \atop ij} \right\} = g^{lk}[ij,k]$

$$\left\{ {i \atop ij} \right\} = g^{ik}[ij,k] \quad \text{--- ①}$$

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$\left\{ {i \atop ij} \right\} = \frac{g^{ik}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k}\right) \quad \text{--- ②}$$

Now, $g^{ki}.\frac{\partial g_{ji}}{\partial x^k} = g^{ik}.\frac{\partial g_{jk}}{\partial x^i}$ --- ③

substituting ③ in ② —

$$\left\{ {i \atop i\,j} \right\} = \frac{g^{ik}}{2}\frac{\partial g_{ik}}{\partial x^j} + g^{ik}\frac{\partial g_{jk}}{\partial x^i} - g^{ik}\frac{\partial g_{ij}}{\partial x^k}$$

$$\left\{ {i \atop j\,j} \right\} = \frac{g^{ik}\,\partial g_{ik}}{2\cdot\partial x^j} + \overset{0}{\left(g^{ki}\frac{\partial g_{ji}}{\partial x^k} - \cancel{g^{ik}\frac{\partial g_{ij}}{\partial x^k}} \right)}$$

$$= \frac{g}{g} \times \frac{g^{ik}}{2} \cdot \frac{\partial g_{ik}}{\partial x^j}$$

$$= \frac{1}{2g} \frac{\partial g}{\partial x^j} \qquad \{ g = |g_{ik}| \}$$

$$\boxed{\left\{ {i \atop i\,j} \right\} = \frac{\partial \log_e \sqrt{g}}{\partial x^i}}$$

Proved