ASSIGN-3

DB

19MA20039
RAHUL SAINI

Ans 1. $J_0\left[(x(t-x))^{1/2}\right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{(x(t-x))^{1/2}}{2}\right)^{2r}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \frac{x^r (t-x)^r}{2^{2r}}$$

Now, $I = \int_0^t J_0\left[(x(t-x))^{1/2}\right] dx = \int_0^t \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \frac{x^r (t-x)^r}{2^{2r}} dx$

Assuming uniform convergence of summation,

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \times \frac{1}{2^{2r}} \underbrace{\int_0^t x^r (t-x)^r dx}_{I_0}$$

$$I_0 = \int_0^t x^r (t-x)^r dx$$

Putting $x = ty \Rightarrow dx = t\,dy$

$$I_0 = \int_0^1 t^r y^r (t-ty)^r t\,dy = t^{2r+1} \underbrace{\int_0^1 (1-y)^r y^r dy}_{I_1}$$

$$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Therefore $I_{01} = \beta(r+1, r+1)$

$$\beta(r+1, r+1) = \frac{\gamma(r+1)\,\gamma(r+1)}{\gamma(2r+2)}$$

As $r \to$ integer $\longrightarrow \gamma(r) = \underline{\lfloor r-1}$

So, $$I_{01} = \frac{\underline{\lfloor r}\;\underline{\lfloor r}}{\underline{\lfloor 2r+1}}$$

$$I_0 = t^{2r+1} \frac{(\underline{\lfloor r})^2}{(\underline{\lfloor 2r+1})}$$

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{\cancel{(\underline{\lfloor r})^2}} \frac{1}{2^{2r}} t^{2r+1} \frac{\cancel{(\underline{\lfloor r})^2}}{\underline{\lfloor 2r+1}}$$

$$= \sum_{r=0}^{\infty} 2 \cdot (-1)^r \cdot \left(\frac{t}{2}\right)^{2r+1} \cdot \frac{1}{\underline{\lfloor 2r+1}}$$

$$= 2\left[\frac{t}{2} - \frac{(t/2)^3}{\underline{\lfloor 3}} + \frac{(t/2)^5}{\underline{\lfloor 5}} + \cdots\right]$$

$$= 2\sin\left(\frac{t}{2}\right)$$

Proved

Ans 2: ~~Generating~~ As we know that,

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{r=-\infty}^{\infty} z^{r} J_{r}(x)$$

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{r=-\infty}^{\infty} z^{r} J_{r}(x) \quad \text{——} \; ①$$

$$e^{\frac{y}{2}\left(z-\frac{1}{z}\right)} = \sum_{r=-\infty}^{\infty} z^{r} J_{r}(y) \quad \text{——} \; ②$$

Putting $r \to n-r$ in eq. ②, we get –

$$e^{\frac{y}{2}\left(z-\frac{1}{z}\right)} = \sum_{n-r=-\infty}^{\infty} z^{n-r} J_{n-r}(y) \quad \text{——} \; ③$$

By doing ③ × ①, we get

$$e^{\frac{x+y}{2}\left(z-\frac{1}{z}\right)} = \left(\sum_{n-r=-\infty}^{\infty} z^{n-r} J_{n-r}(y)\right)\left(\sum_{r=-\infty}^{\infty} z^{r} \left(J_{r}(x)\right)\right)$$

$$= \sum_{n-r=-\infty}^{\infty} \sum_{r=\infty}^{\infty} \left(z^{r} z^{n-r} J_{n-r}(y) J_{r}(x)\right)$$

$$= \sum_{n=-\infty+r}^{\infty} \sum_{r=-\infty}^{\infty} \left(z^{n} J_{n-r}(y) J_{r}(x)\right)$$

$$\sum_{n=-\infty}^{\infty} z^n J_n(x+y) = \sum_{n=-\infty}^{\infty} z^n \sum_{r=-\infty}^{\infty} J_{n-r}(y) J_r(x)$$

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_{n-r}(y) J_r(x) \qquad \underline{\text{Proved}}$$

Ans 3. $F(a-1; b-1; c; x) - F(a; b-1; c; x)$

$$= \sum_{r=0}^{\infty} \left[\frac{(a-1)_r (b-1)_r}{(c)_r} \cdot \frac{x^r}{\lfloor r} - \frac{(a)_r (b-1)_r}{(c)_r} \cdot \frac{x^r}{\lfloor r} \right]$$

$$(a-1)_r - (a)_r = (a-1)a(a+1)\ldots(a+r-2) - a(a+1)\ldots(a+r-1)$$

$$= a(a+1)\ldots(a+r-2)\left[(a-1)-(a+r-1)\right]$$

$$= (a)_{r-1}(-r)$$

Now exp. becomes –

$$F(a-1; b-1; c; x) - F(a, b-1; c; x) = \sum_{r=0}^{\infty} \frac{(a)_{r-1}(b-1)_r}{(c)_r} \cdot \frac{x^r}{\lfloor r}(-1)(r)$$

$$= \sum_{r=1}^{\infty} \frac{(a)_{r-1}(b-1)_r}{(c)_r} \frac{x^r}{\lfloor r-1}(-1)$$

Now $\rightarrow$ $(\alpha)_r = \alpha(\alpha+1) \ldots (\alpha+r-1) = \alpha(\alpha+1)_{r-1}$

so, $(b-1)_r = (b-1)(b)_{r-1}$

$(c)_r = c\,(c+1)_{r-1}$

Now RHS $= \sum_{r=1}^{\infty} \frac{(A)_{r-1}(b-1)(b)_{r-1}}{c\,(c+1)_{r-1}} \cdot \frac{x^{r-1}}{\lfloor r-1} \cdot (-x)$

$$= \frac{x(1-b)}{c} \sum_{r=1}^{\infty} \frac{(A)_{r-1}(b)_{r-1}}{(c+1)_{r-1}} \cdot \frac{x^{r-1}}{\lfloor r-1}$$

Substituting $r-1 = K$

$$RHS = \frac{x(1-b)}{c} \sum_{K=0}^{\infty} \frac{(A)_K (b)_K}{(c+1)_K} \frac{x^K}{\lfloor K}$$

$$= \frac{x(1-b)}{c} \cdot {}_2F_1(A; b; c+1; x)$$

Hence Proved

Ans 4. (i) As we know G. hyper-geometric eq. is -

$x(1-x)y'' + (\gamma-(\alpha+\beta+1))y' - \alpha\beta y = 0 \quad \gamma \neq 0, -1, -2 \ldots$

Soln. of above eq. -

$$y = A\left[{}_2F_1(\alpha; \beta; \gamma; x)\right] + B\left[x^{1-\gamma}\,{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)\right]$$

Now, $\gamma = \frac{3}{2}$, $\alpha+\beta = 1$ $\quad \alpha\beta = -2$

$\alpha = 2, \beta = -1$ or $\beta = 2$, $\alpha = -1$

So Soln. of $\rightarrow x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$ is $\rightarrow$

$$y = A\left[{}_2F_1\left(2, -1; \frac{3}{2}; x\right)\right] + B\left[x^{-1/2}\, {}_2F_1\left(\frac{3}{2}, \frac{-3}{2}; \frac{1}{2}; x\right)\right]$$

(ii) Here $\gamma = \frac{3}{2}$, $\alpha+\beta = 1$ $\quad \alpha\beta = 1/4$

$\Rightarrow \alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{3}{2}$

So Soln. of $\rightarrow x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' - \left(\frac{1}{4}\right) = 0$ is $\rightarrow$

$$y = A\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x\right) + x^{-1/2} B\left[{}_2F_1\left(0, 0; \frac{1}{2}; x\right)\right]\right.$$

Ans 5. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}.(2x)$$

similarly $\frac{d^2y}{dx^2} = 4x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dt}$

$$\text{Eq.} \rightarrow (1-x^2)\left[4x^2\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right] - 2x\frac{dy}{dt}\cdot 2x + n(n+1)y = 0$$

$$4(1-x^2)x^2\frac{d^2y}{dt^2} + (2-2x^2-4x^2)\frac{dy}{dt} + n(n+1)y = 0$$

By putting $x^2 = t$,

$$(1-t)t\frac{d^2y}{dt^2} + \left(\frac{1}{2} - \frac{3}{2}t\right)\left(\frac{dy}{dt}\right) + \frac{n(n+1)}{4}y = 0 \quad \text{—— (1)}$$

Hyper-geometric eq. $\rightarrow$

$$(1-x)x\frac{d^2y}{dx^2} + (\gamma - (\alpha+\beta+1)x)\frac{dy}{dx} - \alpha\beta y = 0$$

From (1), we can say – $\gamma = \frac{1}{2}$, $\alpha+\beta+1 = \frac{3}{2}$, $\alpha\beta = \frac{-n(n+1)}{4}$

$\alpha = \frac{-n}{2}$, $\beta = \frac{(n+1)}{2}$, $\gamma = \frac{1}{2}$

$\alpha+1-\gamma = \frac{(1-n)}{2}$, $\beta+1-\gamma = \frac{n+2}{2}$, $2-\gamma = \frac{3}{2}$

Soln $\rightarrow$ $y(t) = A\left[{}_2F_1(\alpha, \beta; \gamma; t)\right] + x^{1-\gamma}B\left[{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; t)\right]$

$$= A\left[{}_2F_1\left(\frac{-n}{2}, \frac{(n+1)}{2}; \frac{1}{2}; x^2\right)\right] + x^{1/2}B\left[{}_2F_1\left(\frac{1-n}{2}, \frac{n+2}{2}; \frac{3}{2}; t\right)\right]$$

Ans