MM ASSIGN - 2

DB

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Ans 1. To Prove → i) $J_0' = -J_1$
ii) $J_2 - J_0 = 2J_0''$
iii) $J_2 = J_0'' - \frac{1}{x} J_0'$

$$x J_n' = n J_n - x J_{n+1} \quad \text{— ①} \quad [\text{St. Bessels Recurrence}]$$

$n=0$, $\quad x J_0' = 0 - x J_1$

$$\boxed{J_0' = -J_1}$$ Proved (i)

$$2J_n' = J_{n-1} - J_{n+1} \quad \text{— ②} \quad [\text{St. B. Recurrence}]$$

$$\boxed{\begin{aligned} J_0' &= -J_1 \\ J_0'' &= -J_1' \end{aligned}}$$ (from (i)) ③

$n=1$, $\quad 2J_1' = J_0 - J_2$

$$-2J_0'' = J_0 - J_2$$

$$\boxed{J_2 - J_0 = 2J_0''}$$ Proved (ii)

Putting $n=1$ in ①, $\quad xJ_1' = J_1 - xJ_2$

$$-xJ_0'' = -J_0' - xJ_2 \quad (\text{from ③})$$

$$\boxed{J_2 = J_0'' - \frac{1}{x} J_0'}$$ Proved (iii)

Ans 2- $J_n(xy) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \gamma(n+r+1)} \left(\frac{xy}{2}\right)^{2r+n}$

RHS $\rightarrow$ $x\int_0^1 J_n(xy)\, y^{n+1} dy$

$= x \int_0^1 \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \gamma(n+r+1)} \left(\frac{xy}{2}\right)^{2r+n} y^{n+1} dy$

$= \sum_{r=0}^{\infty} \frac{(-1)^r \, x^{2r+n+1}}{\lfloor r \, \gamma(r+n+1)} \int_0^1 \frac{y^{2r+2n+1}}{2^{2r+n}} dy$

$= \sum_{r=0}^{\infty} \frac{(-1)^r . 2}{\lfloor r \, \gamma(r+n+1)} \left(\frac{x}{2}\right)^{2r+n+1} \left[\frac{y^{2r+2n+2}}{2r+2n+2}\right]_0^1$

$= \sum_{r=0}^{\infty} \frac{(-1)^r \times \cancel{2}}{\lfloor r \, \gamma(r+n+1)\, \cancel{2}(n+r+1)} \times \left(\frac{x}{2}\right)^{2r+n+1}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \gamma(r+(n+1)+1)} \left(\frac{x}{2}\right)^{2r+n+1}$

$= J_{n+1}(x) = $ LHS **Proved**

Ans 3. $J_0' = -J_1$ —— ①

$2J_n' = J_{n-1} - J_{n+1}$ —— ②

$\Rightarrow J_0'' = -J_1' = \frac{-1}{2}[J_0 - J_2]$ (from ① & ②)

$J_0''' = -\frac{1}{2}[J_0' - J_2']$

$= -\frac{1}{2}\left[J_0' - \frac{1}{2}[J_1 - J_3]\right]$

$= \frac{-J_0'}{2} - \frac{J_0'}{4} - \frac{J_3}{4}$ $(J_1 = -J_0')$

$4J_0''' + 3J_0' + J_3 = 0$

$J_3 = -4J_0''' - 3J_0'$

$\int J_3\, dx = 4J_1' - 3J_0$ $(J_0'' = -J_1')$

$\int J_3\, dx = 4J_0 - \frac{4J_1}{x} - 3J_0$ $\left(J_1' = J_0 - \frac{J_1}{x}\right)$

$$\boxed{\int J_3\, dx = J_0 - \frac{4J_1}{x}}$$

Ans 4. (i) $J_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x}{2}\right)^{2r}$

$\frac{dJ_0}{dx} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \lfloor r} \cancel{2}r \left(\frac{x}{2}\right)^{2r-1} \cdot \left(\frac{1}{x}\right)$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \lfloor r-1} \left(\frac{x}{2}\right)^{2r-1}$

$= J_{-1} = (-1)J_1 = -J_1$ **Proved**

(ii) $\int_a^b J_0(x) J_1(x) dx = \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]$

~~RHS~~ LHS $\rightarrow \int_0^b J_0(x)\,(-J_0'(x))\,dx$ $\quad (J_0' = -J_1)$

$= -\int_a^b J_0 \frac{d(J_0)}{dx} dx$

$= -\int_a^b J_0(x)\, d(J_0) = \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]$ **Proved**

$= RHS$

Ans 5. $\int_0^\infty e^{-ax} J_0(bx)\, dx$ $\qquad J_0(bx) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{bx}{2}\right)^{2r}$

$$I = \int_0^\infty e^{-ax} \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{bx}{2}\right)^{2r} dx$$

$$= \underbrace{\sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{b}{2}\right)^{2r}}_{K} \int_0^\infty e^{-ax} x^{2r}\, dx$$

put $\boxed{ax = t \qquad a\,dx = dt}$

$$= K \int_0^\infty e^{-t} \frac{t^{2r}}{a^{2r+1}}\, dt = K \frac{1}{a^{2r+1}} \int_0^\infty e^{-t}\, t^{(2r+1)-1}\, dt$$

$$= K \frac{\gamma(2r+1)}{a^{2r+1}}$$

As r = integer, $\gamma(r+1) = \underline{|2r}$

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{b}{2a}\right)^{2r} \frac{\underline{|2r}}{a}$$

$$= \frac{1}{a} \sum_{r=0}^{\infty} (-1)^r\, {}^{2r}C_r \left(\frac{b}{2a}\right)^{2r} \longrightarrow \text{LHS}$$

RHS $\longrightarrow$ $\dfrac{1}{\sqrt{a^2+b^2}} = \dfrac{1}{a}\left(1+\left(\dfrac{b}{a}\right)^2\right)^{-1/2}$

$$= \left[\frac{1}{a}\left(1+\left(\frac{-1}{2}\right)\left(\frac{b}{A}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{\lfloor 2}\left(\frac{b}{a}\right)^2+\cdots\right)\right]$$

$$= \sum_{r=0}^{\infty}\frac{1}{a}\left(\frac{b}{A}\right)^{2r}\frac{1}{\lfloor r}\cdot\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\cdots-\left(\frac{2r-1}{2}\right)$$

$$= \sum_{r=0}^{\infty}\frac{1}{a}\left(\frac{b}{a}\right)^{2r}\frac{1}{\lfloor r}(-1)^r\,\frac{1.3.5.\cdots(2r-1)}{2^r}\times\frac{(1.2.3.\ldots r)2^r}{(1.2.3.\ldots r)2^r}$$

$$= \sum_{r=0}^{\infty}\frac{1}{a}\left(\frac{b}{A}\right)^{2r}\frac{1}{\lfloor r}(-1)^r\,\frac{\lfloor 2r}{2^{2r}\lfloor r}$$

$$= \frac{1}{a}\sum_{r=0}^{\infty}(-1)^r\,{}^{2r}C_r\left(\frac{b}{a}\right)^{2r} = \text{LHS}$$ <u>Proved</u>

$$\int_0^{\infty} e^{-ax}J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$$ <u>Hence Proved</u>