MM-ASSIGN-1 $\underline{DB}$

19MA20039
RAHUL SAINI

Ans 1: $xy'' + 2y' + \frac{xy}{2} = 0$ $\quad z = y\sqrt{x}$

$$y = \frac{z}{\sqrt{x}} \quad , y' = \frac{1}{\sqrt{x}} \times \frac{dz}{dx} - \frac{z}{2x^{3/2}}$$

$$= \frac{z'}{\sqrt{x}} - \frac{z}{2x\sqrt{x}}$$

Substituting y, y' & y'', we get

$$\sqrt{x}\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}}\frac{dy}{dx} + \frac{3y}{4x\sqrt{x}} + \frac{2}{\sqrt{x}}\frac{dy}{dx} - \frac{y}{x\sqrt{x}} + \frac{\sqrt{x}\,y}{2} = 0$$

Multiplying both sides by $x\sqrt{x}$,

$$x^2y'' - xy' + \frac{3y}{4} + 2xy' - y + \frac{x^2y}{2} = 0$$

$$\underbrace{x^2y'' + xy' + \left(\frac{x^2}{2} - \frac{1}{4}\right)y = 0}_{\text{Bessel eq.}} \longrightarrow y = AJ_n(\lambda x) + BJ_{-n}(\lambda x)$$

$$\lambda = \frac{1}{\sqrt{2}}, \ n = \frac{1}{2} \Rightarrow y = AJ_{1/2}\left(\frac{x}{\sqrt{2}}\right) + BJ_{-1/2}\left(\frac{x}{\sqrt{2}}\right)$$

$$y = \frac{z}{\sqrt{x}} = \frac{1}{\sqrt{x}}\left[AJ_{1/2}\left(\frac{x}{\sqrt{2}}\right) + BJ_{-1/2}\left(\frac{x}{\sqrt{2}}\right)\right]$$

Ans 2: $\displaystyle\int_0^1 \frac{\mu J_0(x\mu)\,d\mu}{\sqrt{1-\mu^2}} = \frac{\sin x}{x}$

$$J_n(x\mu) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2\,\Gamma(r+n+1)} \times \left(\frac{x\mu}{2}\right)^{2r+n}$$

$$J_0(x\mu) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{x\mu}{2}\right)^{2r}$$

$$\text{Integral } (I) = \int_0^1 \frac{\mu}{\sqrt{1-\mu^2}} \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{x\mu}{2}\right)^{2r} d\mu$$

$$= \underbrace{\sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{x}{2}\right)^{2r}}_{K} \int_0^1 \frac{\mu^{2r+1}}{\sqrt{1-\mu^2}}\, d\mu$$

Substituting $\mu = \sin\theta$

$d\mu = \cos\theta \, d\theta$

$$I = K \int_0^{\pi/2} \frac{(\sin\theta)^{2r+1} \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= K \int_0^{\pi/2} (\sin\theta)^{2r+1} (\cos\theta)^{2\left(\frac{1}{2}\right)-1}\, d\theta$$

$$= K \, \frac{\Gamma(r+1)\,\Gamma(1/2)}{2\,\Gamma(r+3/2)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(\underline{|r})^2} \left(\frac{x}{2}\right)^{2r} \frac{\Gamma(r+1)\,\Gamma(1/2)}{2\,\Gamma(r+3/2)}$$

$$I = \left[1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \frac{x^6}{\lfloor 7} + \cdots\cdots\right] = \frac{\sin x}{x}$$

<u>Ans 3.</u> $$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x}{2}\right)^{2r}$$

$$J_0(\sin\theta\, x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x \sin\theta}{2}\right)^{2r}$$

$$I = \frac{x^n}{2^{n-1}\Gamma(n)} \int_0^{\pi/2} \sin\theta \cos^{2n-1}\theta \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x \sin\theta}{2}\right)^{2r} d\theta$$

r is integer $\Rightarrow$ $\Gamma(r+1) = \lfloor r$

$$\int_0^{\pi/2} (\sin\theta)^{2r+1} (\cos\theta)^{2n-1}\, d\theta = \frac{\Gamma(r+1)\,\Gamma(n)}{2\,\Gamma(n+r+1)}$$

$$I = \frac{x^n}{2^{n-1}\Gamma(n)} \sum_{r=0}^{\infty} \frac{\Gamma(r+1)\Gamma(n)}{2\Gamma(n+r+1)} \times \frac{(-1)^r}{(\lfloor r)^2} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(\lfloor r)\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} = J_n(x)$$

<u>Hence Proved</u>

Ans 4. $\int_0^x t J_n^2 \, dt = J_n^2 \left[\frac{t^2}{2}\right]_0^x - \int_0^x 2 J_n J_n' \frac{t^2}{2} \, dt$

$$= \frac{x^2 J_n^2}{2} - \int_0^x t^2 J_n J_n' \, dt$$

As we know, $J_{n+1}' = J_n - \frac{(n+1)}{n} J_{n+1}$

$$J_{n-1}' = -J_n + \frac{(n-1)}{n} J_{n-1}$$

$$x J_{n-1} J_{n+1} - \frac{x^2}{2} J_n J_{n+1} + \frac{(n-1)x \, J_{n-1} J_{n+1}}{2} + \frac{x^2}{2} J_{n-1} J_n$$

$$- \frac{(n+1) x \, J_{n-1} J_{n+1}}{2} = x^2 J_n J_n'$$

$$2J_n' = J_{n-1} - J_{n+1}$$

$$I = \frac{x^2}{2} J_n^2 - \frac{x^2}{2} J_{n-1} J_{n+1} = RHS$$ Hence Proved

Ans 5. We know $2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

By putting $n = 1, 2, 3$

$$1 \cdot 2 \cdot J_1 = x(J_0 + J_2)$$
$$2 \cdot 2 \cdot J_2 = x(J_1 + J_3)$$
$$3 \cdot 2 \cdot J_3 = x(J_2 + J_4)$$

$$J_2 = \frac{2J_1}{x} - J_0 \quad , \quad J_3 = \frac{4J_2}{x} - J_1$$

$$= \frac{8J_1}{x^2} - \frac{4J_0}{x} - J_1$$

$$J_4 = \frac{6J_3}{x} - J_2 = \frac{48J_1}{x^3} - \frac{24J_0}{x^2} - \frac{6J_1}{x} - \frac{2J_1}{x} + J_0$$

$$J_4 = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1 + \left(1 - \frac{24}{x^2}\right) J_0$$ Ans