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SHORT NOTE

THE APPARENT RESISTIVITY TENSOR

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INTRODUCTION

The scalar apparent resistivity was originally defined for simple linear electrode resistivity arrays such as Wenner and Schlumberger arrays. With the extension of resistivity surveying into bipole-dipole arrays, several different and not obviously related definitions of apparent resistivity have been used by different authors despite similar field procedure. In this note we show that when a pair of current sources (or quadripole source) is used, and the corresponding electric field vectors are measured at each field station, the most comprehensive expression of the reduced data is as an apparent resistivity tensor. Other definitions of apparent resistivity can be simply related to this tensor. For example, the quadripole-quadripole apparent resistivity of Doicin (1976) is one of the tensor invariants; the maximum and minimum resistivities defined by the rotating dipole method of Furgerson and Keller (1975) can be simply derived, and their geometric mean is shown to be the quadripole-quadripole value.

DEFINITIONS OF APPARENT RESISTIVITY

For bipole (or multipole) mapping, it has become common practice to measure the total electric field vector at each survey point (Risk et al, 1970; Keller et al, 1975). There are many ways that an apparent resistivity can be defined for such a system but all make use of the current density vector $\mathbf{J}$ for a uniform half-space, determined from the location and strength of the source bipole. For a single point source of current I, the current density vector $\mathbf{J}_A$ for a uniform half-space is

$$\mathbf{J}_A = I\,\mathbf{R}_A/2\pi R_A^3, \tag{1}$$

where $\mathbf{R}_A$ is the position vector of the point of interest P relative to the current source (see Figure 1) and R_A is its magnitude. For a bipole source AB, the current density $\mathbf{J}_{AB}$ is

$$\mathbf{J}_{AB} = \mathbf{J}_A - \mathbf{J}_B = I(\mathbf{R}_A/R_A^3 - \mathbf{R}_B/R_B^3)/2\pi. \tag{2}$$

Bibby and Risk (1973) defined the apparent resistivity as ratio of the magnitudes of the measured electric field vector $\mathbf{E}_{AB}$ and the assumed current density vector:

$$\rho_{AB} = |\mathbf{E}_{AB}|/|\mathbf{J}_{AB}|. \tag{3}$$

In addition, the deviation δ of the electric field from that in a uniform half-space was also determined:

$$\delta = \cos^{-1}(\mathbf{E}_{AB}\cdot\mathbf{J}_{AB}/|\mathbf{E}_{AB}||\mathbf{J}_{AB}|). \tag{4}$$

When a second current bipole (CD, say, in Figure 1) is added with the center near the first bipole, but with differing orientation, two electric fields are measured (i.e., four independent quantities). In terms of equations (3) and (4), these can be expressed as two apparent resistivities, and two deviations of the electric field. Doicin (1976) suggested an alternative definition of apparent resistivity which would combine these to form a single scalar apparent resistivity ρ_{qq}. This is defined by taking the vector cross product of the measured electric field, so that for bipole sources AB, CD (see Figure 1)

$$\rho_{qq}^2 = |\mathbf{E}_{AB}\times\mathbf{E}_{CD}|/|\mathbf{J}_{AB}\times\mathbf{J}_{CD}|. \tag{5}$$

This scalar is much less dependent upon the orientation of the current bipoles used, and Doicin considered it to be a more useful parameter than either of the apparent resistivities defined for each bipole source

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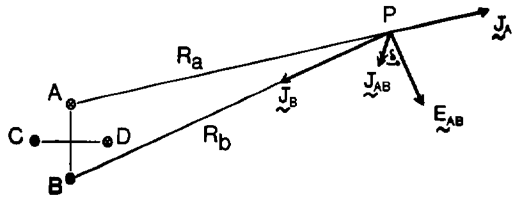

FIG. 1. Current density vector $\mathbf{J}_{AB}$ for a current bipole AB. The measured electric field $\mathbf{E}_{AB}$ is deviated from that in a uniform half-space by the amount δ.

independently. In deriving a single scalar quantity, however, four parameters have been replaced by a single one.

TENSOR APPARENT RESISTIVITY

Essentially, measurements are made of the electric field vector for each of two current sources, and for each source the corresponding current density vector for a uniform half-space is known. Sufficient information is therefore available to define a two-dimensional apparent resistivity tensor, consisting of four independent quantities. This tensor contains all the information available from the measurements.

The apparent resistivity tensor can be defined in an analogous manner to Ohm's law, i.e.,

$$\mathbf{E} = \rho_{ij}\mathbf{J}, \tag{6}$$

where ρ_{ij} consists of four elements. The values of the elements ρ_{ij} will, in general, depend upon the orientation of the coordinate system used, but certain invariants can be extracted once ρ_{ij} have been calculated.

For a chosen set of rectangular coordinates, let the uniform field current density vectors $\mathbf{J}_{AB}$ and $\mathbf{J}_{CD}$ have components in the direction of the coordinate axes (J_{11}, J_{12}) and (J_{21}, J_{22}), respectively (see Figure 2). Similarly, the corresponding components of the measured electric field vectors $\mathbf{E}_{AB}$ and $\mathbf{E}_{CD}$ will be (E_{11}, E_{12}) and (E_{21}, E_{22}). It is not, in general, necessary for the bipole sources to be perpendicular. For each bipole source, the components will be related by equation (6):

$$\begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} J_{11} \\ J_{12} \end{bmatrix}$$

and (7)

$$\begin{bmatrix} E_{21} \\ E_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} J_{21} \\ J_{22} \end{bmatrix}.$$

Solving for ρ_{ij} gives

$$\rho_{ij} = \frac{1}{|\mathbf{J}_{AB} \times \mathbf{J}_{CD}|} \begin{bmatrix} E_{11}J_{22} - E_{21}J_{12} & E_{21}J_{11} - E_{11}J_{21} \\ E_{12}J_{22} - E_{22}J_{12} & E_{22}J_{11} - E_{12}J_{21} \end{bmatrix}. \tag{8}$$

A third measurement for a current bipole with common center, but of different orientation, will not necessarily give a result that is compatible with the measurements made with the first pair of sources unless the resistivity structure under study is two-dimensional, or if the current electrode spacing is sufficiently small so as to be regarded as a dipole source.

It is of interest to note that the method used by Risk et al (1970), where three current electrodes were used and measurements were made for each of the three possible bipoles, does not give a third independent determination of the electric field. With currents I_1, I_2, and I_3, the measured electric fields $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ must satisfy the condition

$$\sum_{i=1}^{3} \mathbf{E}_i/I_i = 0.$$

Thus, provided the measurements were error free, the third measurement will always give a result compatible with the other two. The advantage of the third measurement lies in providing a check on the accuracy of the others.

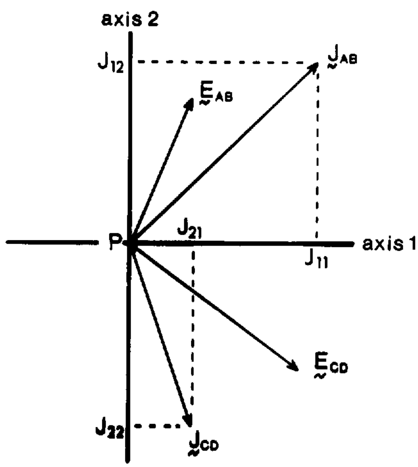

FIG. 2. Components of the current density vectors as used in the definition of the apparent resistivity tensor.

Properties of the apparent resistivity tensor

In certain simple cases of subsurface resistivity distribution, several terms of the apparent resistivity tensor will vanish. For a uniform isotropic half-space of resistivity ρ^*, $\rho_{12} = \rho_{21} = 0$, and the diagonal terms are each equal to the resistivity of the medium $\rho_{11} = \rho_{22} = \rho^*$. For a half-space in which the center of the current electrodes and the field station P lie along a vertical plane of symmetry of the resistivity structure, the nondiagonal terms will be zero. However, for a complex medium, with horizontal and lateral changes of resistivity, all terms will in general be different and nonzero.

Although the true resistivity tensor of a real medium is necessarily symmetric (Nye, 1957), the apparent resistivity tensor, which is in some manner a summation of all the subsurface material properties, is not necessarily symmetric. The condition of symmetry ($\rho_{12} = \rho_{21}$) can be derived directly from equation (8):

$$\mathbf{E}_{AB} \cdot \mathbf{J}_{CD} = \mathbf{E}_{CD} \cdot \mathbf{J}_{AB}. \tag{9}$$

The tensor ρ_{ij} has two independent invariants, which are unchanged by the choice of coordinate axes. These can be chosen as

$$\mathrm{P}_1 = \tfrac{1}{2}(\rho_{11} + \rho_{22}) = \tfrac{1}{2}\,\mathrm{trace}(\rho_{ij}), \tag{10}$$

and

$$\mathrm{P}_2^2 = \rho_{11}\rho_{22} - \rho_{12}\rho_{21} = \text{determinant } (\rho_{ij}). \tag{11}$$

These can be defined in terms of the original measured quantities from equation (8),

$$\mathrm{P}_1 = \tfrac{1}{2}|\mathbf{E}_{AB} \times \mathbf{J}_{CD} - \mathbf{E}_{CD} \times \mathbf{J}_{AB}| / |\mathbf{J}_{AB} \times \mathbf{J}_{CD}|,$$

and (12)

$$\mathrm{P}_2^2 = |\mathbf{E}_{AB} \times \mathbf{E}_{CD}| / |\mathbf{J}_{AB} \times \mathbf{J}_{CD}| = \rho_{qq}^2.$$

Thus, the definition of quadripole-quadripole apparent resistivity given by Doicin (1976) [equation (5)] is simply an invariant of the apparent resistivity tensor.

The tensor can be written as the sum of a symmetric and a residual tensor:

$$\rho_{ij} = \tfrac{1}{2}\begin{bmatrix} \rho_{11} - \rho_{22} & \rho_{12} + \rho_{21} \\ \rho_{12} + \rho_{21} & \rho_{22} - \rho_{11} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \rho_{11} + \rho_{22} & \rho_{12} - \rho_{21} \\ \rho_{21} - \rho_{12} & \rho_{11} + \rho_{22} \end{bmatrix}. \tag{13}$$

A convenient alternative form of the tensor can be obtained by introducing characteristic magnitudes and directions for each part.

$$\rho_{ij} = \Pi_1 \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} + \Pi_2 \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta \end{bmatrix}, \tag{14}$$

where

$$\Pi_1 = \tfrac{1}{2}[(\rho_{11} - \rho_{22})^2 + (\rho_{12} + \rho_{21})^2]^{1/2},$$
$$\Pi_2 = \tfrac{1}{2}[(\rho_{11} + \rho_{22})^2 + (\rho_{12} - \rho_{21})^2]^{1/2},$$
$$\tan 2\alpha = (\rho_{12} + \rho_{21})/(\rho_{11} - \rho_{22}),$$

and

$$\tan 2\beta = (\rho_{12} - \rho_{21})/(\rho_{11} + \rho_{22}).$$

Two directions, α and β, are associated with the tensor. The first α is the direction of the principal axis of the symmetric part of the tensor. This axis would define the principal axis of apparent anisotropy in the case of a symmetric tensor. The second angle β is a measure of the departure of the tensor from symmetry. When the tensor is symmetric, $\beta = 0$.

In this form, the tensor invariants are given by

$$\mathrm{P}_1 = \Pi_2 \cos 2\beta,$$

and

$$\mathrm{P}_2^2 = \Pi_2^2 - \Pi_1^2.$$

The relation between the current density vector and the electric field can be simply derived. If the current density vector is $\{J \cos\theta,\ J \sin\theta\}$, relative to the original axes, substituting in equation (6) gives the corresponding electric field vector as

$$\mathbf{E} = J \begin{bmatrix} \Pi_1 \cos(2\alpha - \theta) + \Pi_2 \cos(2\beta - \theta) \\ \Pi_1 \sin(2\alpha - \theta) - \Pi_2 \sin(2\beta - \theta) \end{bmatrix}. \tag{15}$$

Relation to other apparent resistivities

As shown above, the quadripole-quadripole apparent resistivity is one of the invariants of the apparent resistivity tensor:

$$\rho_{qq} = \mathrm{P}_2.$$

Doicin (1976) argued that this parameter was less variable than each of the apparent resistivities for a single bipole source and thus is of great use for mapping purposes.

The apparent resistivity for each of the current bipolar sources was defined by Risk et al (1970) to be the ratio of the magnitude of the electric field to that of current density vector [equation (3)]. This concept was generalized to the Rotating Dipole Method (RDM) by Furgerson and Keller (1975) in an attempt to eliminate the dependence of the resulting apparent resistivity on the orientation of the current source. The RDM, as applied by Tasci (1975), for example, uses a pair of bipole current sources, approximately at right angles, and at each field station two electric field vectors are measured. Instead of deriving two values of apparent resistivity, however, they define a complete set of values by linearly combining the measured electric fields corresponding to all directions of the current density. The apparent resistivity corresponding to a direction of current density given by $a\mathbf{J}_{AB} + b\mathbf{J}_{CD}$ is

$$\rho(a, b) = |a\mathbf{E}_{AB} + b\mathbf{E}_{CD}| / |a\mathbf{J}_{AB} + b\mathbf{J}_{CD}|, \quad (16)$$

where a and b are scalars: $a^2 + b^2 = 1$, say. By varying a and b, a value of resistivity can be defined for all directions of the current density (or electric field). As each of $\mathbf{E}_{AB}$, $\mathbf{J}_{AB}$ and $\mathbf{E}_{CD}$, $\mathbf{J}_{CD}$ are related by ρ_{ij}, then so is their linear combination $a\mathbf{E}_{AB} + b\mathbf{E}_{CD}$, $a\mathbf{J}_{AB} + b\mathbf{J}_{CD}$.

For a current density vector in the direction θ and of magnitude J, the magnitude of the corresponding electric field can be found from (15).

$$|\mathbf{E}| = J[(\Pi_1^2 + \Pi_2^2 + 2\Pi_1\Pi_2 \cos(2\theta - 2\alpha - 2\beta)]^{1/2}$$

or (17)

$$\rho_\theta = [\Pi_1^2 + \Pi_2^2 + 2\Pi_1\Pi_2 \cos(2\theta - 2\alpha - 2\beta)]^{1/2}.$$

The extreme values, which were determined by Tasci (1975) by computing ρ_θ for 40 directions using equation (16), can be found directly from equation (17):

$$\rho_{\min} = |\Pi_2 - \Pi_1|,$$

when $\theta = \alpha + \beta \pm \pi/2$. Similarly,

$$\rho_{\max} = \Pi_1 + \Pi_2,$$

when $\theta = \alpha + \beta$. It is interesting to note that the geometric mean of these resistivities is

$$(\rho_{\max}\rho_{\min})^{1/2} = |\Pi_2^2 - \Pi_1^2|^{1/2} = P_2 = \rho_{qq},$$

which is simply the quadripole-quadripole value of Doicin (1976).

CONCLUSIONS

Various attempts have been made to combine the results of resistivity measurements from two similarly located current sources of different orientation. The essential point that needs to be recognized is that there are two parameters determined from each current source—a total of four parameters in all. Although it may be diagnostically convenient to combine these in some manner to derive a single parameter, this process essentially discards valuable information. In this note we have shown that these four parameters together define a tensor apparent resistivity at each field point. This tensor contains all the information from both current sources. Other definitions of apparent resistivity may be extracted as special properties of the apparent resistivity tensor.

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