MATH221
Mathematics for Computer Science

Unit 4
Set Theory

OBJECTIVES

- Understand the basic terminologies and concepts in set theory.
- Understand and apply the set operations.
- Simplifying set expressions
- Proving results that involve expressions on sets.

Set

A **set** is a collection of elements (objects).For example, {0, 1, -1} is a set.

The order in which the elements are listed is irrelevant; and no repetition of the same element in a set. For example, no difference between the following two sets (there are the same - equal):

{0, 1, -1}, {-1, 0, 1}

We will usually use small letters for elements and capital letters for sets.

Basic Concept and Notation

1. $x \in A$ is read "x is an element of A".
2. $x \notin A$ is read "x is not an element of A".
3. $\{x: x \in B \wedge P(x)\}$ is read "the set of all elements of B which make the statement $P(x)$ true".
   More often, this notation is more simply written $\{x \in B: P(x)\}$.

   This is called **set builder notation**.

Examples

(i) $\{x: x \in \mathbb{R} \wedge 0 \leq x \leq 1\}$
$= \{x \in \mathbb{R}: 0 \leq x \leq 1\}$
$= [0, 1]$ (interval notation in Calculus)

(ii) The set of all rational numbers, $\mathbb{Q}$ can be written as
$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \wedge b \neq 0\}.$$

(iii) $\{x \in \mathbb{R}: x^3 = x\} = \{0, 1, -1\}$

A Special Set: Empty Set

The **empty set (or null set)**, denoted by $\varnothing$, is described as the set having no elements.

Some books express as $\varnothing = \{\}$.

Example:

Write down the elements in each of the following sets.

(i) $\{x \in \mathbb{N}: x^3 = x\}$
$= \{1\}$

(ii) $\{x \in \mathbb{R}: x^2 = 9\}$
$= \{-3, 3\}$

(iii) $\{x \in \mathbb{Z}: x^3 = 7\}$
$= \varnothing$

Subsets

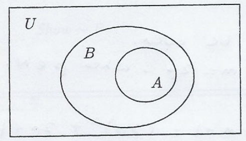
If A and B are sets, we say that ***A is a subset of B***, denoted $A \subseteq B$, if and only if, every element of A is also an element of B. That is,

$$\forall x, (x \in A \Rightarrow x \in B).$$

The phrases **A is contained in B** and **B contains A** are alternative ways of saying that A is a subset of B.

If A is a subset of B, then B is sometimes called a ***superset of A***.

A is not a subset of B is denoted as $A \not\subseteq B$.



A is a subset of B: $A \subseteq B$

Subsets: Notes

If A and B are sets, how to prove $\sim (A \subseteq B)$?

Since $\sim (\forall x, (x \in A \Rightarrow x \in B)) \equiv \exists x, \sim (x \in A \Rightarrow x \in B)$

$$\equiv \exists x, \sim (\sim (x \in A) \vee (x \in B))$$
$$\equiv \exists x, (x \in A \wedge \sim (x \in B))$$
$$\equiv \exists x, (x \in A \wedge x \notin B).$$

Hence, to to prove $\sim (A \subseteq B)$, we need to prove $\exists x, (x \in A \wedge x \notin B)$.

Discussion:

Decide whether the following are true or false.

(i) $\{1, 2\} \subseteq \{1, 2, 3\}$
True

(ii) $\{0, 2\} \subseteq \{1, 2, 3\}$
False

(iii) $\{1\}$ is the same as 1.
False

Discussion:

(iv) $1 \in \{x \in \mathbb{N}: x^2 =1\}$
True

(v) $\{1\} \in \{x \in \mathbb{N}: x^2 =1\}$
False

(vi) Let A be any set. Then $\varnothing \subseteq A$.
True

Singleton Sets

Sets having a single element are frequently called **singleton sets**.

Example: $\{1\}$ is a singleton set.

Universal Set

Most mathematical discussions are carried on within some context.

For example, in a certain situation, all sets being considered might be sets of real numbers. In such a situation, the set of real numbers would be called a **universal set** for the discussion.

Exercise: Subsets

Let A be a set. Prove that $\varnothing \subseteq A$.

Proof:

Suppose ~ ($\varnothing \subseteq A$).

Then, there exists $y \in \varnothing$ such that $y \notin A$.

This, therefore, means that $\varnothing$ is not empty, which is contradiction.

Therefore, $\varnothing \subseteq A$.

Subset: Note

We have the following relationship between sets we met earlier.

$$\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Examples:

Let S be a set. Determine whether the following are true or false.

(i) $S \in S$
False

(ii) $S \subseteq \{S\}$
False

(iii) $\varnothing \subseteq \{S\}$
True

Examples:

(iv) $\varnothing \in \{S\}$
False

(v) $\{\varnothing\} \subseteq \{S\}$
False

Proper Subset

If A and B are sets, we say that **A is a proper subset of B** if $A \subseteq B$ but $B \nsubseteq A$.

This is usually denoted by $A \subset B$.

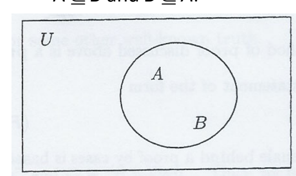
A is not a proper subset of B is denoted as $A \not\subset B$.

Proper Subset: Note

If A is a proper subset of B, there must be at least one element in B that is not in A.

Set Equality

If A and B are sets, we will say that $A = B$ if
$A \subseteq B$ and $B \subseteq A$.



$A = B$

Hence, to prove that two sets are equal from the definition, we must show two things:

(i) $A \subseteq B$ and (ii) $B \subseteq A$.

Notation: Note

Technically, the listing of elements can be done only for finite sets. For example, W = {1, 2, 3, 4}.

However, if an infinite set is defined by a "simple" rule, we sometimes write a few elements and then use "…" to mean roughly "and so on" or "by the same rule".

Notation: Example

(i) $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

(ii) If we want the set of all even integers, we have a few options.

$E = \{n \in \mathbb{Z}: n \text{ is even}\}$

$E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$E = \{n \in \mathbb{Z}: \exists p \in \mathbb{Z}, n = 2p\}$.

Note:

Can we list the elements in $\mathbb{Q}$ as we did in the last example? What about for $\mathbb{R}$? (that is, specify directly)

No

Notation: Note

If $a \in U$, then $\{x \in U: x = a\} = \{a\}$.

The set $\{a\}$ is NOT the same as a.

The former is a SET containing the ELEMENT a.
The latter is an element a.

Power Sets

Given a set A, the power set of A , denoted by $P(A)$, is the set of all subsets of A.

Power Sets: Example

Find the power set of the set $\{x, y\}$. That is find $P(\{x, y\})$.

$P(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

Operations on Sets (Set Operations)

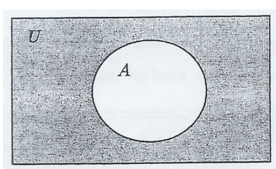
There are four main operations on sets, one corresponding to each of the logical connectives.

Set Operation	*Name*
$\overline{A}$	Complement
$\cup$	Union
$\cap$	Intersection
$-$	difference

Complements

Let U be a universal set, and let $A \subseteq U$.
Then the **complement of *A***, denoted by $\overline{A}$ or $U \setminus A$, is given by

$\overline{A} = \{x \in U: \sim(x \in A)\} = \{x \in U: x \notin A\}$.



Complement of A: $\overline{A}$ or $U \setminus A$

Complement of A is the set that contains elements in U not belonging to A.

A' and A^c are also used to denote $\overline{A}$ in some books.

Examples:

Let $U = \mathbb{Z}$. Write down $\overline{A}$ for the following sets.

(i) $A = \{x \in \mathbb{Z}: x \text{ is even}\}$
$\overline{A} = \{x \in \mathbb{Z}: \sim(x \text{ is even})\} = \{x \in \mathbb{Z}: x \text{ is odd}\}$

(ii) $A = \{x \in \mathbb{Z}: x > 0 \vee x < 0\}$

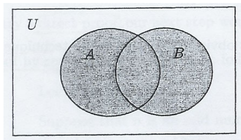
$\overline{A} = \{x \in \mathbb{Z}: \sim(x > 0 \vee x < 0)\} = \{x \in \mathbb{Z}: x = 0\} = \{0\}$

Union

Let A and B be subsets of a universal set U.

Then the **union of A and B**, denoted by $A \cup B$, is defined as:

$$A \cup B = \{x \in U: x \in A \vee x \in B\}.$$



Union: $A \cup B$

$A \cup B$ is the set that contains all the elements belonging to A or B or both

Alternatively, $A \cup B$ can also be defined as:

$$A \cup B = \{x: x \in A \vee x \in B\}$$

without referencing to U.

Example:

Let $U = \mathbb{R}$. Write down $A \cup B$ for the following sets.

(i) $A = \{1\}$ and $B = \{2\}$

$A \cup B = \{1, 2\}$

(ii) A is the set of all even integers, B is the set of all odd integers.

$A \cup B = \{x \in \mathbb{R}: x \in \mathbb{Z}\} = \mathbb{Z}$

(iii) $A = \{x \in \mathbb{R}: 0 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R}: 1 \leq x \leq 3\}$

$A \cup B = \{x \in \mathbb{R}: 0 \leq x \leq 3\}$

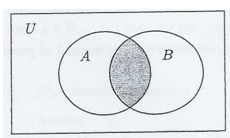
(refer to the note on analyzing intervals in the Appendix)

Intersection

Let A and B be subsets of a universe U.

Then the **intersection of A and B**, denoted by $A \cap B$, is defined as:

$$A \cap B = \{x \in U: x \in A \wedge x \in B\}.$$



Intersection: $A \cap B$

$A \cap B$ is the set that contains all the elements belonging both A and B.

Alternatively, $A \cap B$ can also be defined as:

$$A \cap B = \{x: x \in A \wedge x \in B\}$$

without referencing to U.

Example:

Let $U = \mathbb{R}$. Write down $A \cap B$ for the following sets.

(i) $A = \{1, 2, 3, 5\}$ and $B = \{1, 4, 5, 6\}$

$A \cap B = \{1, 5\}$

(ii) A is the set of all even integers, B is the set of all odd integers.

$A \cap B = \varnothing$

(iii) $A = \{x \in \mathbb{R}: 0 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R}: 1 \leq x \leq 3\}$

$A \cap B = \{x \in \mathbb{R}: 1 \leq x \leq 2\}$

(refer to the note on analyzing intervals in the Appendix)

Difference

Let A and B be subsets of a universal set U.

Then the **difference of A minus B**, denoted by $A - B$ *or* $A \backslash B$, is given by

$$A - B = \{x \in U : x \in A \wedge x \notin B\}.$$

Alternatively, $A - B$ can also be defined as:

$$A - B = \{x : x \in A \wedge x \notin B\}.$$

without referencing to U.

Difference: Notes

1. The difference of $A - B$ is sometimes called the relative complement of B in A.

2. Since $A \subseteq U$ and $B \subseteq U$, $A - B \subseteq U$.

3. If we let $A = U$, then we have

$$\begin{aligned} A - B &= \{x \in A : x \in A \wedge x \notin B\} \\ &= \{x \in U : x \notin B\} \\ &= \overline{B} \end{aligned}$$

4. Using Definitions, we can simplify the definition of – as follows:

$$\begin{aligned} A - B &= \{x \in U : x \in A \wedge x \notin B\} \\ &= \{x \in U : x \in A \wedge x \in \overline{B}\} \\ &= A \cap \overline{B} \end{aligned}$$

Disjoint Sets

Let A and B be sets.

Then A and B are said to be **disjoint** if

$$A \cap B = \varnothing.$$

That is, disjoint sets have no elements in common.

Example:

Let $U = \mathbb{R}$, $A = \{1, 2, 3\}$ and $B = \{2\}$, $C = \{2, 3, 4\}$ and $D = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

(i) Write down

(a) $A - C$

$= \{1\}$

(b) $B - C$

$= \varnothing$

(c) $D - B$

$= \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

Example:

Let $U = \mathbb{R}$, $A = \{1, 2, 3\}$ and $B = \{2\}$, $C = \{2, 3, 4\}$ and
$D = [0, 1] = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$.

(i) (a) $D - A$
$= [0, 1) = \{x \in \mathbb{R}: 0 \leq x < 1\}$
(b) $A - D$
$= \{2, 3\}$

(ii) Which pairs of sets from A, B, C, D are disjoint?
B and D, C and D.

Algebra Of Sets

A list of rules governing set theory, and the relationships between various sets and set expressions. We can prove these using the definitions we have learnt so far.

Theorem – Basic Properties in Set Theory

Let U be a universal set and let A, B, and C be elements of $P(U)$.

1. $(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$
   $A \subseteq A \cup B$
   $A \cap B \subseteq A$

2. $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$
   $A \subseteq B \Leftrightarrow A \cup B = B$
   $A \subseteq B \Leftrightarrow A \cap B = A$

Theorem – Basic Properties in Set Theory

3. $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$
   $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$

4. Commutative Laws:
   $A \cup B = B \cup A$
   $A \cap B = B \cap A$

5. Associative Laws:
   $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap B) \cap C = A \cap (B \cap C)$

Theorem – Basic Properties in Set Theory

6. Distributive Laws:
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

7. DeMorgan's Laws:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Corollary

Let U be a universal set and let A, B, and C be elements of $P(U)$.

1. Facts about Complementation:

$\overline{(\overline{A})} = A$ Double Complement Laws

$A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$

$A - B = A \cap \overline{B}$ Set Difference Law

$\overline{U} = \emptyset$

$\overline{\emptyset} = U$

Corollary

2. Properties of $\emptyset$ and U:

$A \cap U = A \quad A \cup \emptyset = A$ Identity Laws

$A \cap \emptyset = \emptyset \quad A \cup U = U$ Universal Bound Laws

$A \cap \overline{A} = \emptyset \quad A \cup \overline{A} = U$ Complement Laws

3. Subset properties of $\cup$ and $\cap$:

$(A \subseteq C \wedge B \subseteq C) \Leftrightarrow (A \cup B) \subseteq C$

$(A \subseteq B \wedge A \subseteq C) \Leftrightarrow A \subseteq (B \cap C)$

Corollary

4. Idempotent Laws:

$A \cup A = A$

$A \cap A = A$

5. Absorption Laws:

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

Proving Results on Sets

We can use any suitable method of proof discussed in Unit 3 to prove results on sets from the definitions or from the basic properties discussed earlier.

The first method is called typical "element argument" method. The framework for proving $A \subseteq B$ is as follows:

From definition, to prove that $A \subseteq B$, we must show that

$\forall x, (x \in A \Rightarrow x \in B)$.

We begin by letting $x \in A$, that is, we take x to be an arbitrary element of A.

Finally, we must prove that $x \in B$.

Proving Results on Sets

The method discussed in the previous example can be used to prove set equalities by using the definition

$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$$

then showing $A \subseteq B \wedge B \subseteq A$ using the method twice.

Using this definition to prove is sometimes called a "double containment" proof.

Proving Results on Sets: Example 1

Let U be a set and let A, B and C be elements of $P(U)$. Prove that $A \subseteq A \cup B$.

We must prove $x \in A \Rightarrow x \in A \cup B$.

Forward: Let $x \in A$. (This means we "know" $x \in A$)

Backward: By definition, $x \in A \vee x \in B \Rightarrow x \in A \cup B$.

To complete the proof, we need the step $x \in A \Rightarrow x \in A \vee x \in B$. However, in terms of logic, we note that $\boldsymbol{P \Rightarrow (P \vee Q)}$ **is a tautology**.

Proving Results on Sets: Example 1

Proof:

Let $x \in A$. Then we have

$$\begin{aligned} x \in A \quad &\Rightarrow x \in A \vee x \in B \\ &\Rightarrow x \in A \cup B. \end{aligned}$$

Therefore, $A \subseteq A \cup B$.

Proving Results on Sets: Example 2

Prove the statement $A \subseteq B \Leftrightarrow A \cup B = B$.

The proof is in two parts:

(1) $A \subseteq B \Rightarrow A \cup B = B$

(2) $A \cup B = B \Rightarrow A \subseteq B$

To prove part (1), let $A \subseteq B$. We must show that $A \cup B = B$. To do this, we must prove two things

(i) $A \cup B \subseteq B$

(ii) $B \subseteq A \cup B$

Proving Results on Sets: Example 2

To prove part (i), i.e. $A \cup B \subseteq B$:

Let $x \in A \cup B$, then by definition, $x \in A \vee x \in B$.

To complete this part of the proof, we have to prove that $x \in A \vee x \in B \Rightarrow x \in B$. This requires a proof by cases.

Case 1: If $x \in B$, there is nothing to show.

Case 2: If $x \in A$, we know that $A \subseteq B$, and so $x \in B$.

Therefore, we have established that

$x \in A \cup B \Rightarrow (x \in A \vee x \in B) \Rightarrow x \in B$

Thus $A \cup B \subseteq B$.

Proving Results on Sets: Example 2

To prove part (ii), i.e. $B \subseteq A \cup B$:

Let $x \in B$, then we have

$x \in B \quad \Rightarrow x \in A \vee x \in B$.

$\Rightarrow x \in A \cup B$

Therefore, $B \subseteq A \cup B$.

Therefore, we have proven that $A \subseteq B \Rightarrow A \cup B = B$.

Proving Results on Sets: Example 2

To prove part (2), let $A \cup B = B$. We must show that $A \subseteq B$.

Let $x \in A$. Then we have

$x \in A \quad \Rightarrow x \in A \vee x \in B$.

$\Rightarrow x \in A \cup B$ (By definition)

$\Rightarrow x \in B$ (as $A \cup B = B$)

Therefore, $A \subseteq B$.

Therefore, we have proven that $A \cup B = B \Rightarrow A \subseteq B$.

Therefore, $A \subseteq B \Leftrightarrow A \cup B = B$.

Proving Results on Sets: Example 3

Sometimes, it is faster to prove by proofing the statements defining the two sets are equivalent.

Prove that $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.

Proof:

$x \in \overline{(A \cap B)} \equiv \sim(x \in A \cap B)$ by definition of complement

$x \in \overline{A} \cup \overline{B} \equiv x \in \overline{A} \vee x \in \overline{B}$ by definition of $\cup$

$\equiv \sim(x \in A) \vee \sim(x \in B)$

$\equiv \sim(x \in A \wedge x \in B)$ by Logic (DeMorgan's law)

$\equiv \sim(x \in A \cap B)$ by definition of $\cap$

$\equiv x \in \overline{(A \cap B)}$

Therefore, $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Proving Results on Sets: Note

Some results can be proved using properties from the theorem step-by-step method.

Example.

Using (i) $A \subseteq B \Leftrightarrow A \cup B = B$

(ii) $A \subseteq B \Leftrightarrow A \cap B = A$

(iii) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Prove that $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$

Proving Results on Sets: Note

Proof:

$$
\begin{aligned}
& A \subseteq B \Leftrightarrow A \cap B = A && \text{by part (ii)} \\
& \Leftrightarrow \overline{A \cap B} = \overline{A} && \\
& \Leftrightarrow \overline{A} \cup \overline{B} = \overline{A} && \text{by part (iii)} \\
& \Leftrightarrow \overline{B} \cup \overline{A} = \overline{A} && \\
& \Leftrightarrow \overline{B} \subseteq \overline{A} && \text{by part (i)}
\end{aligned}
$$

(i) $A \subseteq B \Leftrightarrow A \cup B = B$
(ii) $A \subseteq B \Leftrightarrow A \cap B = A$
(iii) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Example:

(ii) Disprove that $A - (B - C) = (A - B) - C$.

Consider the following sets:

$A = \{1, 2, 3\}$, $B = \{3\}$, $C = \{2, 3\}$

$A - (B - C) = \{1, 2, 3\} - (\{3\} - \{2, 3\})$
$= \{1, 2, 3\}$
$(A - B) - C = (\{1, 2, 3\} - \{3\}) - \{2, 3\}$
$= \{1, 2\} - \{2, 3\}$
$= \{1\}$

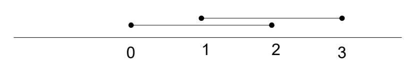
Therefore, $A - (B - C) \neq (A - B) - C$.

Appendix: Note on Analyzing Intervals

Exercise on Union:

$A = \{x \in \mathbb{R}: 0 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R}: 1 \leq x \leq 3\}$

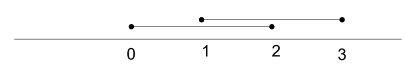
$A \cup B = \{x \in \mathbb{R}: 0 \leq x \leq 3\}$



Exercise on Intersection:

$A = \{x \in \mathbb{R}: 0 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R}: 1 \leq x \leq 3\}$

$A \cap B = \{x \in \mathbb{R}: 1 \leq x \leq 2\}$



End of Unit 4