

MATH221 Mathematics for Computer Science

Unit 6 Number Systems

OBJECTIVES

- Know what is the natural numbers and integers.
- Understand what is binary operation and closed binary operation.
- Understand the algebraic terminologies, identity, inverse, commutative, associative and distributive.
- Know what is the well-ordered property.
- Understand the application of the definition of odd and even integers.
- Know what are prime and composite numbers.

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The Natural Numbers

The Natural Numbers, denoted by \mathbb{N} , consist of all positive “whole” numbers.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Is 0 an element of \mathbb{N} (notation $0 \in \mathbb{N}$)?

No.

Binary Operations

An **operation** on a set S is a rule for combining one or more elements of S .

A **binary operation** on S is a rule for combining pairs of elements of S to produce another element.

We denote a general operation by $*$. A binary operation $*$ on S is **closed** on S if

$$x, y \in S \Rightarrow x*y \in S.$$

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Binary Operations

On numbers we have four main binary operations: addition, subtraction, multiplication and division.

But on which number systems are these binary operations closed?

Binary Operation

We can picture such an operation as a “black box” that takes two inputs from the set and produces one output in the set:



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Binary Operation: Example

If the set is \mathbb{N} and the operation is $+$, then the inputs 1 and 2 would result in the output 3.



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Exercise:

Among addition, subtraction, multiplication and division are all binary operations on \mathbb{R} , which of them are closed on \mathbb{N} ?
Why or why not?

Addition $[4 + 2 = 6]$

Yes

Subtraction $[4 - 2 = 2]$

No. $2 - 4 = -2$

Multiplication $[4 \times 2 = 8]$

Yes

Division $[4 \div 2 = 2]$

No. $2 \div 4 = 1/2$

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Identity

Suppose $*$ is a binary operation on a set S . An element e of S is called an **identity** if for all $x \in S$ we have

$$e * x = x \text{ and } x * e = x$$

Does \mathbb{N} have an identity under the addition and multiplication operations?

Addition: no

Multiplication: yes, it is 1 as $1 \times n = n$ and $n \times 1 = n$.

Inverse

Suppose $*$ is a binary operation on a set S , and $e \in S$ is an identity. An element $x \in S$ is called **invertible** if there exists an element $y \in S$ such that

$$x * y = e \text{ and } y * x = e.$$

In this case y is called the **inverse** of x .

What are the invertible elements of \mathbb{N} under the addition and multiplication operations?

Addition: not applicable

Multiplication: 1 only

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Commutativity and Associativity

A binary operation $*$ on a set S is commutative if

$$x * y = y * x \text{ for all } x, y \in S.$$

A binary operation $*$ on a set S is associative if

$$(x * y) * z = x * (y * z) \text{ for all } x, y, z \in S.$$

Are addition and multiplication on \mathbb{N} commutative and associative operations? Yes

Example: Paper-Scissors-Rock

Let $M := \{p, s, r\}$ and consider the paper-scissors-rock-inspired binary operation $*$ on M given by

$$p * r = r * p = p \text{ (paper beats rock)}$$

$$p * s = s * p = s \text{ (scissors beats paper)}$$

$$s * r = r * s = r \text{ (rock beats scissors)}$$

$$p * p = p \text{ (paper ties with paper)}$$

$$s * s = s \text{ (scissors ties with scissors)}$$

$$r * r = r \text{ (rock ties with rock).}$$

Note that $*$ is closed on M .

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Example: Paper-Scissors-Rock (cont'd)

This binary operation is by definition commutative. But it is not associative:

$$\begin{aligned} \underline{(r * p) * s = p * s = s} \\ \underline{r * (p * s) = r * s = r} \end{aligned}$$

Distributivity

For each $a, b, c \in \mathbb{N}$ we have,

$$\begin{aligned} a \times (b + c) &= (a \times b) + (a \times c) \\ (a + b) \times c &= (a \times c) + (b \times c) \end{aligned}$$

We say multiplication distributes over addition in \mathbb{N} .

Note that addition does not distribute over multiplication. For instance?

$$2 + (3 \times 1) \neq (2 + 3) \times (2 + 1)$$

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Exercise:

Carefully write down every small step you take when simplifying the following expressions without the aid of a calculator. Can you give reasons for each step?

$$\begin{aligned} 3x + 4y + 2x + y & \\ = 3x + (4y + 2x) + y & \text{Associative law} \\ = 3x + (2x + 4y) + y & \text{Commutative law} \\ = (3x + 2x) + 4y + y & \text{Associative law} \\ = (3 + 2)x + 4y + y & \text{Distributive law} \\ = 5x + 4y + y & \text{Arithmetic} \\ = 5x + (4y + y) & \text{Associative law} \\ = 5x + (4 + 1)y & \text{Distributive law} \\ = 5x + 5y & \text{Arithmetic} \end{aligned}$$

Well-ordering property

A set S together with an order \leq is called well-ordered if every nonempty subset of S has a least element. That is, if A is a nonempty subset of S , then there is an $s_0 \in A$ such that $s_0 \leq s$ for all $s \in A$.

Axiom:

The natural numbers \mathbb{N} with the usual order \leq is well-ordered.

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The integers Z

The integers Z is the collection of all the “whole” positive and negative numbers, and 0. So

$$\underline{Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}}.$$

The order \leq on N extends to the natural order \leq on Z.

Axiom:

Any nonempty subset of integers that are greater than a fixed integer with the usual order \leq is well-ordered.

Operations, identities and inverses on Z

We have the usual binary operations $+$, $-$, \times and \div on Z. But only the operations $+$, $-$ and \times are closed on Z.

Does Z have an identity under the $+$ and \times operations?

What are the invertible elements in Z under the $+$ and \times operations?

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Commutativity and associativity

Addition and multiplication are commutative operations on Z. But subtraction is not:

$$\underline{10 - 5 \neq 5 - 10}$$

Addition and multiplication are associative operations on Z. But subtraction is not:

$$(10 - 2) - 3 \neq 10 - (2 - 3)$$

Distributivity

For each $a, b, c \in Z$ we have

$$\begin{aligned} a \times (b \pm c) &= (a \times b) \pm (a \times c) \\ (a \pm b) \times c &= (a \times c) \pm (b \times c) \end{aligned}$$

We say multiplication distributes over addition and subtraction in Z.

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Well-ordering property

Recall that a set S together with an order \leq is called well-ordered if every nonempty subset of S has a least element, that is, if A is a nonempty subset of S , then there is a $s_0 \in A$ such that $s_0 \leq s$ for all $s \in A$.

We have seen that the natural numbers N with the usual order \leq is well-ordered.

Fact: The integers Z are not well-ordered with \leq .

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Evens, odds, primes and composites

An integer $m \in Z$ is **even** if $m = 2j$ for some $j \in Z$.

An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

An integer $n > 1$ is **prime** if whenever $n = r \times s$ for positive integers r and s , we have $r = 1$ or $s = 1$. An integer $n > 1$ is **composite** if it is not prime.

Example: -2 is an even integer, 5 and -3 are odd integers, 5 is a prime integer and 4 is a composite integer.

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The rationals Q

The rationals Q is the set of numbers q that can be written in the form:

$$q = \frac{a}{b}$$

where $a, b \in Z$ and $b \neq 0$.

Each number in Q is called a rational number.

Example:

$0.4 = 2/5$ is a rational number.

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Irrational

A number is irrational if it is not a rational number; that is, if it CANNOT be written in the following form:

$$q = \frac{a}{b}$$

where $a, b \in Z$ and $b \neq 0$.

Example:

$\sqrt{2}$ is an irrational number.

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End of Unit 6