# MATH221
## Mathematics for Computer Science

## Unit 5
# Relations and Functions



## OBJECTIVES

- Understand what is a relation and function.
- Determine the domain and range of relation and function.
- Evaluate the properties of relation such as reflexivity, symmetry and transitivity.
- Understand what is a function, one-to-one and onto function
- Find the existence of inverse function.



## Unit 5a
# Relations



## Cartesian Product

Given two sets $A$ and $B$, the **Cartesian product** of $A$ and $B$, denoted by $\boldsymbol{A \times B}$ (read "$A$ cross $B$"), is the set of all ordered pairs $(a, b)$, where $a$ is in $A$ and $b$ is in $B$.

Symbolically,

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$$

## Cartesian Product: Example

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, $C = \{a, b\}$. Find the followings.
a. $A \times B$ b. $(A \times B) \times C$

a. $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. $(A \times B) \times C = \{(u, v): u \in A \times B \text{ and } v \in C\}$
$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), ((y, 2), a),$
$((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), ((y, 1), b),$
$((y, 2), b), ((y, 3), b)\}$



## Relations

Let $A$ and $B$ be sets. We say that $R$ is a (binary) **relation from *A* to *B*** if $R \subseteq A \times B$.

If $R \subseteq A \times A$, we will say that $R$ is a **relation on *A***.

Further, if $(a, b) \in R$, we will frequently write $aRb$ and say that "***a* is related to b by *R***", symbolically, denoted as **aRb**.

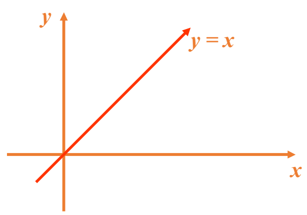


## Discussion:

Consider the relation $R$ on $\mathbb{R}$ given by

$$R = \{(x, y): x, y \in \mathbb{R} \wedge x = y\}.$$

(i) We can sketch a graph of this relation as follows:






## Discussion:

(ii) True of False? $1R1$
True

$2.1R2.2$
False

$(-3, 3) \in R$
False

(iii) If $aR100$, then $a = 100$.

## Relations: Notes

The relation in the previous discussion is usually written as

$$R = \{(x, x): x \in \mathbb{R}\}$$

and it is normally called the **identity relation on $\mathbb{R}$**.



## Examples:

1. Let $X = \{0, 1, 2, 3\}$ and let the relation $R$ on $X$ be given by $R = \{(x, y): \exists z \in \mathbb{N}, x + z = y\}$.

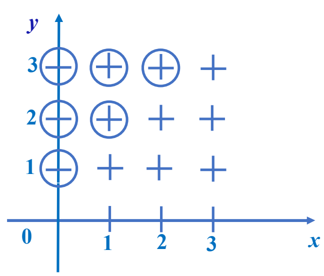
   (i) What is an easier way of expressing the relation $R$?
   $R = \{(x, y): (x, y \in X) \wedge (x < y)\}$.

   (ii) List all the elements of $R$.
   $R = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$



## Examples:

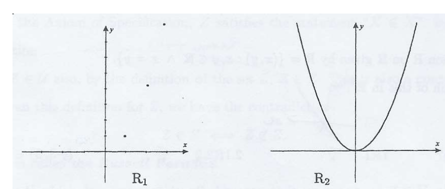
(iii) Sketch $X \times X$ and circle the elements of $R$.






## Examples

2. Let $R_1$ be the relation on $\mathbb{N}$ given by $R_1 = \{(x, y): y = x^2\}$.
   Let $R_2$ be the relation on $\mathbb{R}$ given by $R_2 = \{(x, y): y = x^2\}$. By sketching each relation, we can see how the "input set" makes a difference to the elements in the relation



## Relation: Note

We must be careful when writing relations. If a relation involves two different sets, we must make very clear the sets a relation $R$ is from and to.

Since relations are subsets of cartesian products, their unions and intersections can be calculated as for any sets.



## Example:

Let $A = \{0, 1\}$ and $B = \{-1, 0, 1\}$. Let two relations from $A$ to $B$ be given by

$$R_1 = \{(0, -1), (1, -1), (1, 0)\},$$

and $$R_2 = \{(0, 0), (1, 1), (1, -1)\}.$$

Write down $R_1 \cap R_2$ and $R_1 \cup R_2$.

$R_1 \cap R_2 = \{(1, -1)\}$

$R_1 \cup R_2 = \{(0, -1), (0, 0), (1, -1), (1, 0), (1, 1)\}$.3



## Domain and Range of Relations

Let $R$ be a relation from $A$ to $B$.

Then the **domain of $R$**, denoted by Dom $R$, is given by

$$\text{Dom } R = \{x: \exists y, xRy\}$$

and the **range of $R$**, denoted Range $R$, is given by

$$\text{Range } R = \{y: \exists x, xRy\}$$



## Examples:

1. Let $A = \{0, 1, 2, 3\}$ and let $R$ be a relation on $A$ given by
   $R_1 = \{(0, 0), (0, 1), (0, 2), (3, 0)\}$.
   Write down Dom $R_1$ and Range $R_1$.
   Dom $R_1 = \{0, 3\}$
   Range $R_1 = \{0, 1, 2\}$

2. Let $R_2$ be the relation on $\mathbb{Z}$ given by $R_2 = \{(x, y): xy \neq 0\}$.
   What are Dom $R_2$ and Range $R_2$.
   Dom $R_2 = \mathbb{Z} - \{0\}$
   Range $R_2 = \mathbb{Z} - \{0\}$

## Examples:

3. Let $R_3$ be the relation from $\mathbb{Z}$ to $\mathbb{Q}$ given by

$$R_3 = \{(x, y): x \neq 0 \wedge y = 1/x\}.$$

Find Dom $R_3$ and Range $R_3$.

Dom $R_3 = \mathbb{Z} - \{0\}$

Range $R_3 = \{1/n: n \in \mathbb{Z} \wedge n \neq 0\}$



## Domain and Range of Relations: Notes

1. Let $R$ be a relation from $A$ to $B$. Then Dom $R \subseteq A$ and Range $R \subseteq B$.

2. We could say that Dom $R$ is the set of all first elements in the ordered pairs that belong to $R$.

   Range $R$ is the set of all second elements in the ordered pairs that belong to $R$.



## Inverse Relations

Let $R$ be a relation from $A$ to $B$.

We define the **inverse relation**, $R^{-1}$, from $B$ to $A$ as

$$R^{-1} = \{(y, x): (x, y) \in R\}.$$



## Inverse Relations: Example

1. Consider the relation $R$ from $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$.

$$R = \{(a, 1), (b, 2), (c, 3), (a, 4)\}.$$

The inverse relation, $R^{-1}$, is simply each ordered pair written "in reverse".

$$R^{-1} = \{(1, a), (2, b), (3, c), (4, a)\}.$$

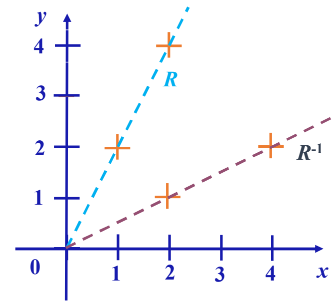
## Inverse Relations: Example

2. Define a relation $R$ on $\mathbb{N}$ as $R = \{(x, y): y = 2x\}$.
   (i) Write down three elements of $R$. Write down three elements of $R^{-1}$.
   Three elements of $R$ : (1, 2), (2, 4), (3, 6)
   Three elements of $R^{-1}$ : (2, 1), (4, 2), (6, 3)



## Inverse Relations: Example

2. (ii) Sketch a graph of $R$ and $R^{-1}$ on coordinate axes.






## Examples:

1. (iii) Write down a simple definition of $R^{-1}$.

$y = 2x$
$x = \frac{1}{2}y$
Exchanging x and y, we get the rule: $y = \frac{1}{2}x$

Hence, $R^{-1} = \{(x, y): y = ½ x\}$.

2. Let $S$ be the identity relation on the set of real numbers. What is $S^{-1}$.
   $S^{-1} = \{(x, y): y = x\}$.



## Reflexivity

Let $R$ be a relation on the set $A$.
$R$ is **reflexive** on $A$ if and only if $\forall x \in A, (x, x) \in R$.

## Reflexivity: Example

Let $R_1$ be a relation on $\mathbb{N}$ defined by

$$R_1 = \{(x, y): x \text{ is a factor of } y\}.$$

For each $x \in \mathbb{N}$, we know that $x$ is a factor of itself.
Thus, $(x, x) \in R_1$ and so $R_1$ is reflexive.

**Informally**, a relation $R$ on a set $A$ is reflexive if each element in $A$ is related to itself by R.



## Symmetry

Let $R$ be a relation on the set $A$.
$R$ is **symmetric** on $A$ if and only if

$$\forall x, y \in A, ((x, y) \in R \Rightarrow (y, x) \in R).$$



## Symmetry: Example

Let $R_2$ be the identity relation on $\mathbb{R}$.

Let $(x, y) \in R_2$. Then, $x, y \in \mathbb{R}$ and $x = y$. This implies that y = x. Hence, $(y, x) \in R_2$. So, $R_2$ is symmetric.

**Informally**, a relation $R$ on a set $A$ is symmetric if you can “swap” the ordered pairs around and still get elements of $R$.



## Transitivity

Let $R$ be a relation on the set $A$.
$R$ is **transitive** on $A$ if and only if

$$\forall x, y, z \in A, (\boxed{(x, y) \in R \wedge (y, z) \in R} \Rightarrow (x, z) \in R).$$

## Transitivity: Example

Let $R_3$ be the relation on $\mathbb{Z}$ defined by

$$R_3 = \{(x, y): x < y\}.$$

Let $(x, y) \in R_3$ and $(y, z) \in R_3$. Since $(x, y) \in R_3$, $x < y$. Since $(y, z) \in R_3$, $y < z$. This implies that $x < z$. Hence, $(x, z) \in R_3$. Therefore, $R_3$ is transitive.



## Equivalence Relations

Let $R$ be a relation on the set $A$.

$R$ is an **equivalence relation** on $A$ if and only if $R$ is *reflexive*, *symmetric* and *transitive* on $A$.



## Verifying/Proving Reflexivity, Symmetry and Transitivity

- Finite Relation:
  - Use Method of Exhaustion discussed in Unit 3 to verify all the cases.
- Infinite Relation:
  - Use a generalized method of proof discussed in Unit 3 (usually direct proof) to verify the properties based on the definition of the relation.
- Examples on both are given after this slides.



## Equivalence Relations: Examples

1. Let $R_1$ be a identity relation on $\mathbb{R}$. Then $R_1$ is an equivalence relation.

   *Proof*:

   *Reflexive*: $\forall a \in \mathbb{R}$, $a = a$, that is, $(a, a) \in R_1$.

   Thus, $R_1$ is reflexive.

   *Symmetric*: $\forall a, b \in \mathbb{R}$, if $(a, b) \in R_1$, then $a = b$.

   Thus, $b = a$, and hence, $(b, a) \in R_1$.

   Hence, $R_1$ is symmetric.

   *Transitive*:

   $\forall a, b, c \in \mathbb{R}$, if $(a, b) \in R_1$ and $(b, c) \in R_1$, then $a = b$ and $b = c$. Thus, $a = c$, and hence $(a, c) \in R_1$.

   Therefore, $R_1$ is transitive.

   Therefore, $R_1$ is an equivalence relation.

## Equivalence Relations: Examples

2. Let $R_2$ be the relation on $\mathbb{Z}$ given by

$$R_2 = \{(a, b): ab \neq 0\}.$$

*Reflexive*: We must show $\forall a \in \mathbb{Z}$, $a \times a \neq 0$.
But, we note that $0 \in \mathbb{Z}$ and $0 \times 0 = 0$.
Hence, $(0, 0) \notin R_2$.
Therefore, $R_2$ is NOT reflexive.
Therefore, $R_2$ is not an equivalence relation.



## Discussion:

Consider the relation $R_2$ on $\mathbb{Z}$ given by

$$R_2 = \{(a, b): ab \neq 0\}.$$

(i) Is $R_2$ symmetric or transitive?

*Symmetric*:
$\forall a, b \in \mathbb{Z}$, if $(a, b) \in R_2$, then $ab \neq 0$.
This implies that $ba \neq 0$.
Thus, then $(b, a) \in R_2$.
Hence, $R_2$ is symmetric.

*Transitive*:
$\forall\, a, b, c \in \mathbb{Z}$, if $(a, b) \in R_2$ and $(b, c) \in R_2$, then $ab \neq 0$ and $bc \neq 0$.
This implies that $ac \neq 0$.
Thus, $(a, c) \in R_2$.
Hence, $R_2$ is transitive.



## Discussion:

(ii) How can we adjust the relation so that it becomes and equivalence relation?

Adjust $\mathbb{Z}$ to $\mathbb{Z}$ - {0}.



## Equivalence Relations: Notes

1. To prove a relation $R$ is an equivalence relation, you must prove *all three* properties.

2. To disprove that a relation $R$ is an equivalence relation, you must show that *one* of the three properties does *not* hold, usually by **counterexample**.

## Equivalence Relations: Example

Let $A = \{0, 1, 2\}$ and let $R$ be the relation on $A$ given by

$R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$.

Prove $R$ is an equivalence relation on $A$.

*Reflexive*: For $a = 0$: $(0, 0) \in R$.

For $a = 1$: $(1, 1) \in R$.

For $a = 2$: $(2, 2) \in R$.

So, $\forall a \in A$, $(a, a) \in R$.

Thus, $R$ is reflexive.



## Equivalence Relations: Example

*Symmetric*: We must check each ordered pair $(a, b) \in R$ to see if $(b, a)$ is also in the relation.

For $(0, 0)$, $(1, 1)$, $(2, 2)$ symmetry obviously holds.

For $(0, 1)$, we need to check that $(1, 0) \in R$, which it is.

For $(1, 0)$, we need to check that $(0, 1) \in R$, which it is

Thus, $R$ is symmetric.



## Equivalence Relations: Example

*Transitive*:

We must check each pair of elements $(a, b)$, $(b, c) \in R$ that $(a, c) \in R$.

$(0, 0)$, $(0, 1) \in R$, so $(0, 1)$ should be in $R$, which it is.

$(1, 1)$, $(1, 0) \in R$, so $(1, 0)$ should be in $R$, which it is.

$(0, 1)$, $(1, 1) \in R$, so $(0, 1)$ should be in $R$, which it is.

$(0, 1)$, $(1, 0) \in R$, so $(0, 0)$ should be in $R$, which it is.

$(1, 0)$, $(0, 1) \in R$, so $(1, 1)$ should be in $R$, which it is.

$(1, 0)$, $(0, 0) \in R$, so $(1, 0)$ should be in $R$, which it is.

Thus, $R$ is transitive.

Therefore, $R$ is an equivalence relations.



## Equivalence Class

Let $R$ be an equivalence relation on the set $A$.

Then for each $a \in A$, we define the **equivalence class of *a*** as

$$\text{Class}(a) = \{b \in A: (a, b) \in R\}.$$

## Equivalence Class: Examples

1. Consider the relation $R$ in the previous example.

   $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$.

   For each element in $a$, we write the equivalence classes as follows.

   Class(0) = $\{b \in A: (0, b) \in R\} = \{0, 1\}$

   Class(1) = $\{b \in A: (1, b) \in R\} = \{1, 0\}$ = class(0)

   Class(2) = $\{b \in A: (2, b) \in R\} = \{2\}$

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## Equivalence Class: Example

2. Let $R_1$ be the identity relation on $\mathbb{R}$. Write down the following equivalence classes.

   Class(1)

   = {1}

   Class($\pi$)

   = $\{\pi\}$

   Class(1/2)

   = {1/2}

   For any $x \in \mathbb{R}$, class($x$) = $\{x\}$.

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# Unit 5b
# Functions

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## Functions

If $F$ is a relation from $A$ to $B$, then we say $F$ is a **function from *A* to *B***, if and only if

the domain of $F$ is all of $A$

and for each element $x \in A$,

there is only one value $y \in B$ such that $(x, y) \in F$.

We say that $B$ is the **codomain** of $F$.

The set $\{y: \exists x \in A, (x, y) \in F\}$ is called the **range** of F.

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## Functions: Note

As convenient, we will now change to the more familiar function notation. We write

$f: A \rightarrow B$,

to indicate that $f$ is a function from $A$ to $B$, and we write

$y = f(x)$

to mean exactly the same thing as $(x, y) \in f$.

In this case, we may say that $y$ is **the image of *x* under *f***.



## Functions: Note

A relation from $A$ to $B$ becomes a function if

the domain is all of $A$

and if

every element of the domain is related to only one element of the codomain.



## Verifying/Proving Function

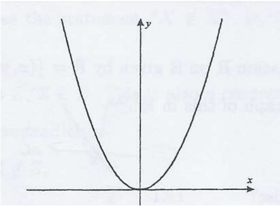
- Verifying/proofing a relation F from A to B is a function:
  - Follow an appropriate method of proof.
  - Verify the domain of F.
  - Verify that an element in the domain is related to a unique element in the range.
- Finite Case:
  - use the diagrammatical method illustrated in Appendix 1 as a working tool to verify but must present it according to the method of proof.
- Infinite Case:
  - Does the rule define an element of B for each element of A?
  - For each element $x \in A$, is there only one value $y \in B$ such that $(x, y) \in F$? Beside checking from definition, this property can be checked using the vertical line test that will be discussed in one of the examples later.



## Infinite Case: Exercise

Which of the following relations are functions? A sketch might be handy!

(i) $R_1$ on $\mathbb{R}$, $R_1 = \{(x, y): y = x^2\}$. Yes, a function.




Justification/Proof:

For each $x \in \mathbb{R}$, $y = x^2 \in \mathbb{R}$, thus, $(x, x^2) \in R_1$. Hence, $x \in$ Dom $R_1$. Hence, Dom $R_1 = \mathbb{R}$

Furthermore, for each $x \in \mathbb{R}$, there is a unique value $x^2$ value in $\mathbb{R}$ such that $(x, x^2) \in R_1$.

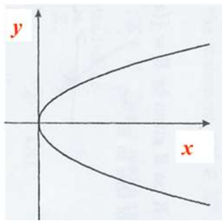
Therefore $R_1$, is a function.

## Infinite Case: Exercise

(ii) $R_2$ on $\mathbb{R}$, $R_2 = \{(x, y): x = y^2\}$.

Dom $R_2 = \{x \in \mathbb{R}: x \geq 0\}$

Since Dom $R_2 \subset \mathbb{R}$, $R_2$ is not a function.




Justification:

If x < 0, as $y^2$ is always nonnegative, hence, there is no $y \in \mathbb{R}$ such that $x = y^2 \in \mathbb{R}$, hence, $x \notin$ Dom $R_2$.

If $x \geq 0$, we take $y = \sqrt{x}$, then $x = (\sqrt{x})^2 = y^2$. Hence, $(x, y) \in R_2$.

Therefore, Dom $R_2 = \{x \in \mathbb{R}: x \geq 0\}$

Since Dom $R_2 \subset \mathbb{R}$, $R_2$ is not a function.

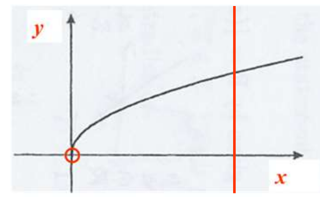


## Infinite Case: Exercise

(iii) $R_3$ on $A = \{x \in \mathbb{R}: x > 0\}$, $R_3 = \{(x, y): x = y^2\}$.

Dom $R_3 = \{x \in \mathbb{R}: x > 0\}$.

Since it satisfies the vertical line test, $R_3$ is a function.




Justification/Proof:

For each $x \in A$, as $x \geq 0$, we can take $y = \sqrt{x}$, then $x = (\sqrt{x})^2 = y^2$. Hence, $(x, y) \in R_3$.

Hence, Dom $R_3$ = A

As each vertical line in this graph intersects the curve exactly one point, hence, for each element $x \in A$, there is only one value $y \in A$, such that $(x, y) \in R_3$.

Therefore $R_3$ is a function.



## Finite Case: Exercises

Let $A = \{2, 4, 6\}$ and let $B = \{1, 3, 5\}$. Consider the following relations from $A$ to $B$.

1. $R_1 = \{(x, y): x + 1 = y\} = \{(2, 3), (4, 5)\}$.
   Dom $R_1 = \{2, 4\} \neq A$.
   Thus, $R_1$ is not a function.

2. $R_2 = \{(2, 5), (4, 1), (4, 5), (6, 5)\}$.
   Dom $R_2 = \{2, 4, 6\} = A$.
   However, 4 is related to both 1 and 5.
   Thus $R_2$ is not a function.



## Finite Case: Exercises

Let $A = \{2, 4, 6\}$ and let $B = \{1, 3, 5\}$. Consider the following relations from $A$ to $B$.

3. $R_3 = \{(2, 5), (4, 1), (6, 5)\}$.
   Dom $R_3 = A$ and each first element only appears once.
   Thus, $R_3$ is a function.

For Finite Cases, we can also use diagrammatical method shown in Appendix 1 to check. However, for justification/proof, **we should follow the method used in these exercises**.

## One-To-One Functions

Let $f: A \to B$ be a function. $f$ is **one-to-one** if and only if

$$\forall x_1, x_2 \in A, (f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$



## Verifying/Proving One-to-One Function

- Verifying/proofing a function f from A to B is a one-to-one function:
  - Follow an appropriate method of proof.
  - Verify that an element in the range is related to a unique element in the domain.
- Finite Case:
  - Can use the diagrammatical method illustrated in Appendix 1 as a working tool to verify, but for proof, we must present it according to the method of proof.
- Infinite Case:
  - For $(x_1, y_1), (x_2, y_2) \in F$, based on the rule of F, if $y_1 = y_2$, is $x_1 = x_2$?
  - The above verification/proof can be replaced by the horizontal *line test*.

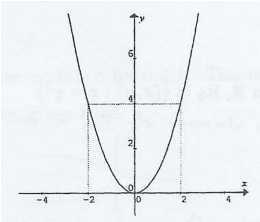


## One-To-One Functions: Example

Consider the relation $F$ on $\mathbb{R}$ given by

$$F = \{(x, y): y = x^2\}.$$

$F$ is a function, but some of the elements in the $y$ position are related to two elements in the domain. Hence, $F$ is not a one-to-one function.



Justification/Proof of Not One-to-One:

Since $F(3) = 9$ and $F(-3) = 9$, but $3 \neq -3$, hence, F is not one-to-one.

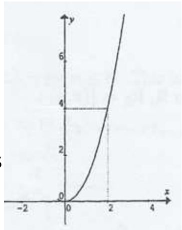


## One-To-One Functions: Example

Consider the relation $F$ on $\mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$ given by

$$F = \{(x, y): y = x^2\}.$$

Any given element from the range is the image of only one element from the domain. Therefore, $F$ is a one-to-one function, as each element in the range is the image of only one element in the domain.



Justification/Proof of One-to-One:

For each $x_1, x_2 \in \mathbb{R}^+$ such that $(f(x_1) = f(x_2)$, $x_1^2 = x_2^2$.

As both $x_1 > 0$ and $x_2 > 0$, this implies that $x_1 = x_2$.

Hence, F is one-to-one.

## One-To-One Functions: Notes

For checking a function from A to B to see whether it is one-to-one:

Infinite case: Can also be checked or proved by the horizontal line test.

Finite case: Is each element in the range always associated with a different element in the domain? (refer to Appendix 1).



## Exercises:

Which of the following are one-to-one functions?

(i) $F_1$ on $A = \{1, 2, 3\}$, $F_1 = \{(1, 2), (2, 3), (3, 1)\}$.
Yes

(ii) $F_2$ on $A = \{1, 2, 3\}$, $F_2 = \{(1, 2), (2, 1), (3, 1)\}$.
No, since the both the first element 2 and 3 have the same image 1.



## Exercise:

Which of the following functions are one-to-one functions?

(iii) $F_3$ on $\mathbb{Z}$, $F_3 = \{(x, y): y = 2x\}$.
Yes
Proof/Justify: If there are $x_1, x_2 \in \mathbb{Z}$ such that $F_3(x_1) = F_3(x_2)$
Then, $2x_1 = 2x_2$
Hence, $x_1 = x_2$
Therefore, $F_3$ is one-to-one

(iv) $F_4$ from $\mathbb{Z} - \{0\}$ to $\mathbb{R}$, $F_4 = \{(x, y): y = \sqrt{x^2 - 1}\}$.
No, since both the first element -1 and 1 have the same image 0.



## Onto Functions

Let $f: A \rightarrow B$ be a function.

$f$ is **onto** if and only if

$$\text{Range } f = B,$$

that is,

$$\forall y \in B, \exists x \in A, y = f(x).$$

## Verifying/Proving Onto Function

- Verifying/proofing a function f from A to B is a Onto:
  - ➢Follow an appropriate method of proof.
  - ➢Verify that an element in the codomain B is related to an element in the domain A.
- Finite Case:
  - ➢Can use the diagrammatical method illustrated in Appendix 1 as a working tool to verify, but for proof, we must present it according to the method of proof.
- Infinite Case:
  - ➢For y ∈ B, based on the rule of F, show that there is a solution x ∈ A satisfying the rule.



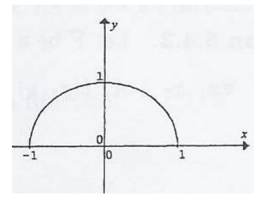
## Onto Functions: Example

Let $F$ be a function from $A = \{x \in \mathbb{R}: -1 \le x \le 1\}$ to $A$ given by

$F(x) = \sqrt{1-x^2}$ .

Dom $F = [-1, 1]$

Range $F = [0, 1]$




Justification/Proof of Not Onto:

-1 ∈ A, as $F(x) = \sqrt{1-x^2} \ge 0$, hence, there is no x ∈ A such that F(x) = -1.

Thus, -1 ∉ range F.

Hence, range $F \neq A$.

Therefore, F is not onto.

This function is clearly not onto as range $F \neq A$.



## Onto Functions: Example

$A = \{x \in \mathbb{R}: -1 \le x \le 1\}$

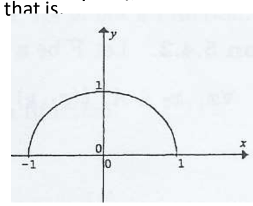
However, we can define $B = \{y \in \mathbb{R}: 0 \le y \le 1\}$, and a "new" function $f_1: A \to B$, given by the same rule as $F$, that is,

$f_1(x) = \sqrt{1-x^2}$ .

$A = \{x \in \mathbb{R}: -1 \le x \le 1\}$

Dom $F = [-1, 1]$

Range $F = [0, 1]$




Proof:

For each $y \in B$, let $x = \sqrt{1-y^2} \in A$, then $y = f_1(x)$.

Thus, Range $f_1$ = B.

Hence, $f_1$ is onto.



## Onto Functions: Notes

1. To show a function $f: A \to B$ is onto, you simply need to show that Range $f = B$, that is, for each element $y \in B$, come up with an element $x \in A$ such $y = f(x)$.

2. To prove that a function $f: A \to B$ is not onto, you just need to find one element $y \in B$ which is not the image of any $x \in A$.

## Exercise:

Determine which of the following functions are onto.

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d\}$.

(i) $f_1: A \to B$, $f_1 = \{(1, a), (2, c), (3, c), (4, d), (5, d)\}$

No, $b \in B$ but it is not an image of any element in $A$.

(ii) $f_2: A \to B$, $f_2 = \{(1, a), (2, b), (3, c), (4, d), (5, a)\}$

Yes, since range $f_2 = B$.



## Exercise:

Determine which of the following functions are onto.

(iii) $f_3: \mathbb{R} \to \mathbb{R}$ such that $f_3(x) = 4x - 1$.

For each $y \in \mathbb{R}$, let $x = \frac{y+1}{4} \in \mathbb{R}$, then $y = f_3(x)$.

Thus, Range $f_3 = \mathbb{R}$.

Hence, $f_3$ is onto

(iv) $f_4: \mathbb{Z} \to \mathbb{Z}$ such that $f_4(x) = 4x - 1$.

No, $0 \in \mathbb{Z}$ but it is not an image of any element in the domain.



## Onto Functions: Notes

If you are asked to show why a function is not onto, all you need to give is a counterexample, that is, you must give a value of
$y \in B$ for which there is no $x \in A$ to give $y = f(x)$.

For example, for $f_4$ in the previous exercise,

$f_4$ is not onto.

For $y = 1$, we need $x \in \mathbb{Z}$ so that $1 = 4x - 1$.

That is, we need $2 = 4x$ or $x = ½$.

However, $½ \notin \mathbb{Z}$, so there is no $x$ so that $y = f_4(x)$.



## Inverse Functions

For any function, there is an inverse relation, however, this inverse relation is not always a function.

The inverse relation of $F$ will also be a function when $F$ is one-to-one and onto.

## Exercise:

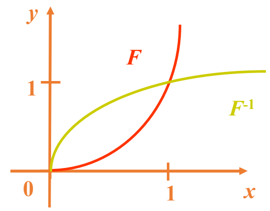
Sketch the function $F$ from $A = \{x \in \mathbb{R}: x \geq 0\}$ to A given by

$F = \{(x, y): y = x^2\}$.

Sketch $F^{-1}$. Find $F^{-1}$.

Is $F^{-1}$ a function?

$F^{-1} = \{(x, y): y = \sqrt{x}\}$




$F^{-1}$ is a function since $F$ is one-to-one and onto.

Note: Finding rule for inverse function:

$y = x^2$

$x = \sqrt{y}$ (from algebra and x $\in$ A)

Exchanging x and y, we get the rule: $y = \sqrt{x}$

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## Composition of Functions

Let f and g be two functions. Let the domains of f and g be S and T respectively. Let the codomains of f and g be T and U respectively. Then, we can define a composition function from S to U, called composition of f and g, denoted as g ∘ *f* , as follows:

$(g \circ f)(x) = g(f(x))$

Example: $f$: $\mathbb{R} \to \mathbb{R}$ defined as $f(x)= 2x$; and $g$: $\mathbb{R} \to \mathbb{R}$ defined as $g(x)= x^2$. *Then,*

$(g \circ f)(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$

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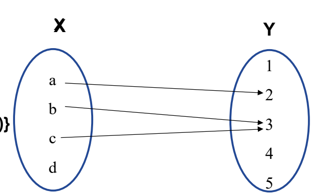
**Appendix 1: Diagrammatical Checking of Finite Relations and Functions**

**Example: Checking relations R from X = {a, b, c, d } to Y = {1, 2, 3, 4, 5}**

**xRy is shown by an arrow from x to y**

**First case:**

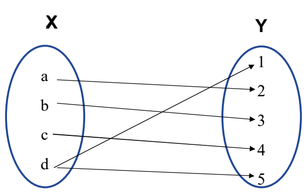
**R = {(a, 2), (b, 3), (c, 3)}**




**not a function from X to Y: domain of R ≠ X**

**Second case:**

**R = {(a, 2), (b, 3), (c, 4), (d, 5), (d, 1)}**




**not a function from X to Y: due to dR1 and dR5**

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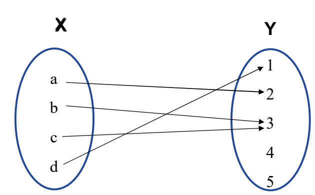
**Appendix 1: Diagrammatical Checking Finite Relations and Functions (cont'd)**

**Example: Checking relations R from X = {a, b, c, d } to Y = {1, 2, 3, 4, 5}**

**Third case:**

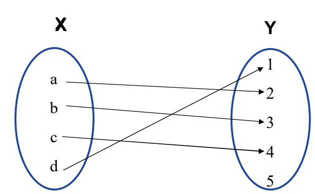
**R = {(a, 2), (b, 3), (c, 3), (d, 1)}**




**a function from X to Y but not one-to-one and also not onto**

**Fourth case:**

**R = {(a, 2), (b, 3), (c, 4), (d, 1)}**




**a one-o-one function from X to Y but not onto**

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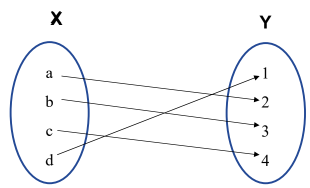
**Appendix 1: Diagrammatical Checking Finite Relations and Functions (Cont'd)**

**Example: Checking a relation R from X = {a,  b, c, d } to Y = {1, 2, 3, 4}**

**Fifth case:**

**R = {(a, 2), (b, 3), (c, 4), (d, 1)}**




**a one-o-one and onto function from X to Y**



# End of Unit 5