MATH221
Mathematics for Computer Science

Unit 10
Introduction to Combinatorics and Probability

Learning Objectives

- Introduce sample space, event, possibility tree and counting rules
- Introduce basic probability
- Introduce permutation and combination
- Introduce pigeonhole principle and generalized pigeonhole principle

1: Introduction

Introduction: Tossing two coins

- 0, 1 or 2 heads?
- Does each of these events occur about 1/3 of the time?

1/4

No heads obtained

2/4

One head obtained

1/4

Two heads obtained

Introduction: Experiment on Tossing Two Coins

Table 10.1.1. Relative frequencies.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	卌 卌 |	11	22%
1 head obtained	卌 卌 卌 卌 卌 ||	27	54%
0 heads obtained	卌 卌 ||	12	24%

Formalizing the analysis, we introduce:

- random process
- sample space
- event
- probability

Introduction

To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

Definition

A **sample space** is the set of all possible outcomes of a random process or experiment.
An **event** is a subset of a sample space.

Equally Likely Probability

Notation

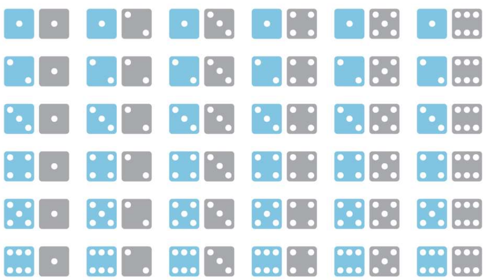
For a finite set A, $N(A)$ denotes the number of elements in A.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** of E, denoted $P(E)$, is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of outcomes in } S} = \frac{N(E)}{N(S)}$$

Equally Likely Probability: Rolling a Pair of Dice



Equally Likely Probability: Rolling a Pair of Dice

A more compact notation identifies, say, ⚁⚃ with the notation 24, ⚄⚂ with 53, and so forth.

a. Use the compact notation to write the sample space S of possible outcomes.

S = { 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 }

b. Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

E = { 15, 24, 33, 42, 51 } $P(E)$ = 5/36

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list.

For instance, how many integers are there from 5 through 12?

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

Counting the Elements of a List

More generally, if m and n are integers and $m \leq n$, how many integers are there from m through n?

Note that $n = m + (n - m)$, where $n - m \geq 0$ [since $n \geq m$].

Theorem 10.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are

$$n - m + 1$$

integers from m to n inclusive.

Counting the Elements of a List

a. How many 3-digit integers (from 100 to 999 inclusive) are divisible by 5?

100 101 102 103 104 **105** 106 107 108 109 **110** ... 994 **995** 996 997 998 999

5×**20** 5×**21** 5×**22** 5×**199**

Number of multiples of 5 from 100 to 999 = number of integers from 20 to 199 inclusive.

From Theorem 10.1.1, there are 199 – 20 + 1 = 180 such integers.
Hence, there are **180** 3-digit integers that are divisible by 5.

Counting the Elements of a List and Probability

b. What is the probability that a randomly chosen 3-digit integer is divisible by 5?

By Theorem 10.1.1, total number of integers from 100 through 999 = 999 – 100 + 1 = 900.

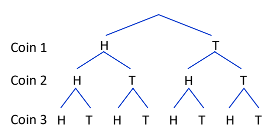
By part (a), 180 of these are divisible by 5.

Hence, answer = 180/900 = **1/5**.

2: Possibility Trees and the Multiplication Rule

Possibility Trees

A **tree structure** is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.



Example 1: Possibilities for Tournament Play

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row, or a total of three games.

One way in which this tournament can be played is for *A* to win the first game, *B* to win the second, and *A* to win the third and fourth games. Denote this by writing *A*–*B*–*A*–*A*.

Example 1: Possibilities for Tournament Play

a. How many ways can the tournament be played?

Possible ways are represented by the distinct paths from "root" (the start) to "leaf" (a terminal point) in the tree below.

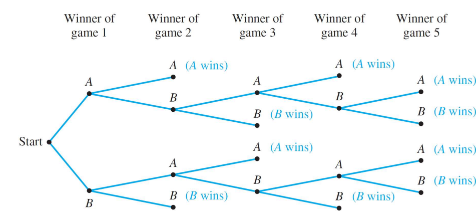


Figure 10.2.1 The Outcomes of a Tournament

Example 1: Possibilities for Tournament Play

a. How many ways can the tournament be played?

Ten paths from the root of the tree to its leaves ➔ ten possible ways for the tournament to be played.

1. A-A
2. A-B-A-A
3. A-B-A-B-A
4. A-B-A-B-B
5. A-B-B
6. B-A-A
7. B-A-B-A-A
8. B-A-B-A-B
9. B-A-B-B
10. B-B

Example 1: Possibilities for Tournament Play

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

1. A-A
2. A-B-A-A
3. A-B-A-B-A
4. A-B-A-B-B
5. A-B-B
6. B-A-A
7. B-A-B-A-A
8. B-A-B-A-B
9. B-A-B-B
10. B-B

Probability that 5 games are needed = 4/10 = **2/5**

The Multiplication Rule

Consider the following example.

Suppose a computer installation has four input/output units (*A*, *B*, *C*, and *D*) and three central processing units (*X*, *Y*, and *Z*).

Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

The Multiplication Rule

Possibility tree:

The total number of ways to pair the two types of units…

is the same as the number of branches of the tree:
$3 + 3 + 3 + 3 = 4 \times 3 =$ **12**

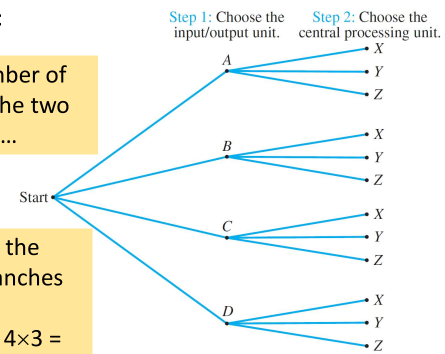


Figure 10.2.2 Pairing Objects Using a Possibility Tree

The Multiplication Rule

Theorem 10.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways (regardless of how the first step was performed),

:

the k^{th} step can be performed in n_k ways (regardless of how the preceding steps were performed),

Then the entire operation can be performed in

$$\boldsymbol{n_1 \times n_2 \times n_3 \times \ldots \times n_k} \text{ ways.}$$

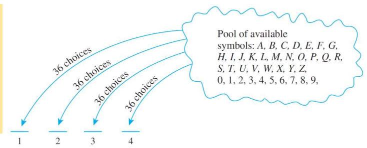
The Multiplication Rule

Example 2: No. of Personal Identification Numbers (PINs)

A typical PIN is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed. Examples: CARE, 3387, B32B, and so forth.

How many different PINs are possible?

You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.



Example 2: No. of Personal Identification Numbers (PINs)

Step 1: Choose the first symbol.
Step 2: Choose the second symbol.
Step 3: Choose the third symbol.
Step 4: Choose the fourth symbol.

There is a fixed number of ways to perform each step, namely 36, regardless how preceding steps were performed.

Hence, by the **multiplication rule**, there are:
$36 \times 36 \times 36 \times 36 = 36^4 =$ **1,679,616** PINs in all.

Example 3: No. of PINs without Repetition

Now, suppose that repetition is not allowed.

a. How many different PINs are there?

Step 1: Choose the first symbol. ← 36 ways
Step 2: Choose the second symbol. ← 35 ways
Step 3: Choose the third symbol. ← 34 ways
Step 4: Choose the fourth symbol. ← 33 ways

Hence, by the **multiplication rule**, there are:
$36 \times 35 \times 34 \times 33 =$ **1,413,720** PINs in all with no repeated symbol.

Example 3: No. of PINs without Repetition

b. If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbols?

1,679,616 PINS in all.
1,413,720 PINs with no repeated symbol.

Hence, probability that a PIN chosen at random contains no repeated symbols:

$$\frac{1{,}413{,}720}{1{,}679{,}616} \cong \mathbf{0.8417}$$

When the Multiplication Rule is Difficult/Impossible to Apply

Example 4: Consider the following problem:

Three officers – a president, a treasurer, and a secretary – are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

Is this correct?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Cyd or Dan).

Therefore, by the **multiplication rule**, $3\times3\times2$ = **18**

When the Multiplication Rule is Difficult/Impossible to Apply

It is incorrect. The number of ways to choose the secretary varies depending on who is chosen for president and treasurer.

For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan.

But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan.

When the Multiplication Rule is Difficult/Impossible to Apply

The clearest way to see all the possible choices is to construct the possibility tree, as shown below.

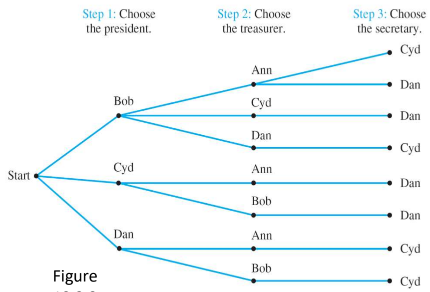


Figure 10.2.3

How many ways?

8 ways.

When the Multiplication Rule is Difficult/Impossible to Apply

Example 5: A More Subtle Use of the Multiplication Rule

Reorder the steps for choosing the officers in the previous example so that the total number of ways to choose officers can be computed using the multiplication rule.

Step 1: Choose the secretary.
Step 2: Choose the president.
Step 3: Choose the treasurer.

2 ways: Either Cyd or Dan.

2 ways: Neither Ann nor the person chosen in step 1 may be chosen, but either of the other two may.

2 ways: Either of the 2 persons not chosen in steps 1 and 2 may be chosen.

Hence, total number of ways = 2 ×2 ×2 = **8**

Example 5: A More Subtle Use of the Multiplication Rule

A possibility tree illustrating this sequence of choices:

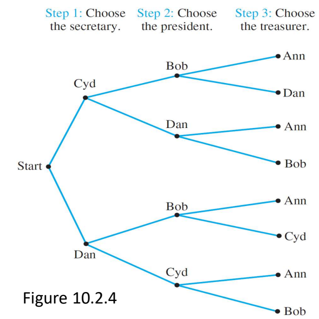


Figure 10.2.4

Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements a, b, and c has six permutations.

$$abc \quad acb \quad cba \quad bac \quad bca \quad cab$$

In general, given a set of n objects, how many permutations does the set have?

Permutations

Imagine forming a permutation as an n-step operation:

Step 1: Choose an element to write first. ← n ways
Step 2: Choose an element to write second. ← $n - 1$ ways
Step 3: Choose an element to write third. ← $n - 2$ ways
:
Step n: Choose an element to write nth. ← 1 way

By the **multiplication rule**, there are

$$n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = \mathbf{n!}$$

ways to perform the entire operation.

Permutations

In other words, there are n! permutations of a set of n elements.

Theorem 10.2.2 Permutations

The number of permutations of a set with n ($n \geq 1$) elements is ***n*!**

Permutations

Example 6 – Permutations of the Letters in a Word

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

All the eight letters in the word *COMPUTER* are distinct. Hence, **8!** = **40320**.

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

There are effectively only seven objects "CO", "M", "P", "U", "T", "E" and "R". Hence, **7!** = **5040**.

Permutations

Example 6 – Permutations of the Letters in a Word

c. If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

The probability is:

$$\frac{7!}{8!} = \frac{1}{8}$$

Permutations

Permutations of Selected Elements

Given the set {a, b, c}, there are six ways to select two letters from the set and write them in order.

ab *ac* *ba* *bc* *ca* *cb*

Each such ordering of two elements of {a, b, c} is called a ***2-permutation*** of {a, b, c}?.

Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set.
The number of r-permutations of a set of n elements is denoted ***P*(*n*, *r*)**.

Permutations of Selected Elements

Theorem 10.2.3 r-permutations from a set of n elements

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$ first version

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!}$$ second version

Permutations of Selected Elements

Example 7

a. Evaluate $P(5, 2)$.

$P(5, 2) = 5!\,/\,(5-2)! = 5 \times 4 =$ **20**

b. How many 4-permutations are there of a set of seven objects?

$P(7, 4) = 7!\,/\,(7-4)! = 7 \times 6 \times 5 \times 4 =$ **840**

c. How many 5-permutations are there of a set of five objects?

$P(5, 5) = 5!\,/\,(5-5)! = 5! =$ **120**

3: Addition and Difference Rules

Counting Elements of Disjoint Sets: The Addition Rule

The basic rule underlying the calculation of the number of elements in a union or difference or intersection is the **addition rule**.

This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Theorem 10.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k).$$

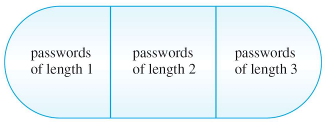
Counting Elements of Disjoint Sets: The Addition Rule

Example 8 – Counting Passwords with 3 or fewer Letters

A computer access password consists of from one to three letters chosen from the 26 letters in the alphabet with repetitions allowed. How many different passwords are possible?

The set of all passwords can be partitioned into subsets consisting of those of length 1, length 2, and length 3:

Figure 10.3.1
Set of all passwords of length ≤ 3



Counting Elements of Disjoint Sets: The Addition Rule

Example 8 – Counting Passwords with 3 or fewer Letters

By the **addition rule**, the total number of passwords equals the sum of the number of passwords of length 1, length 2, and length 3.

Number of passwords of length 1 = 26

Number of passwords of length 2 = 26^2

Number of passwords of length 3 = 26^3

Hence, total number of passwords = $26 + 26^2 + 26^3$
= **18,278**.

The Difference Rule

An important consequence of the addition rule is the fact that if the number of elements in a set *A* and the number in a subset *B* of *A* are both known, then the number of elements that are in *A* and not in *B* can be computed.

Theorem 10.3.2 The Difference Rule

If *A* is a finite set and *B* is a subset of *A*, then
$$N(A - B) = N(A) - N(B).$$

Counting Elements of Disjoint Sets: The Addition Rule

The difference rule is illustrated in Figure 10.3.2.

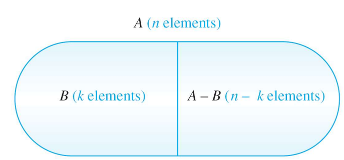


Figure 10.3.2 The Difference Rule

The Difference Rule

The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$. Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B)$$

The Difference Rule

Example 9 – Counting PINs with Repeated Symbols

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.

a. How many PINs contain repeated symbols?

There are 36^4 = 1,679,616 PINs when repetition is allowed, and there are $36 \times 35 \times 34 \times 33$ = 1,413,720 PINs when repetition is not allowed.

By the **difference rule**, there are

1,679,616 – 1,413,720 = **265,896**

PINS that contain at least one repeated symbol.

The Difference Rule

Example 9 – Counting PINs with Repeated Symbols

b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

There are 1,679,616 PINs in all, and by part (a) 265,896 of these contain at least one repeated symbol.

Thus, the probability that a randomly chosen PIN contains a repeated symbol is

$$\frac{265{,}896}{1{,}679{,}616} \cong \mathbf{0.158}$$

The Difference Rule

Example 9 – Counting PINs with Repeated Symbols

An alternative solution to part (b) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

It follows that:

$$P(S - A) = \frac{N(S - A)}{N(S)} = \frac{N(S) - N(A)}{N(S)}$$
$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$
$$= 1 - P(A)$$

$$P(A) = \frac{1{,}413{,}720}{1{,}679{,}616}$$
$$\cong 0.842$$
$$P(S - A) \cong 1 - 0.842$$
$$\cong \mathbf{0.158}$$

The Difference Rule

Probability of the Complement of an Event

This solution illustrates a more general property of probabilities: that the probability of the complement of an event is obtained by subtracting the probability of the event from the number 1.

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

The Inclusion/Exclusion Rule

The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when **some of the sets overlap**.

For simplicity, begin by looking at a union of two sets A and B, as shown in Figure 10.3.3.

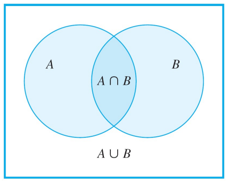


Figure 10.3.3

The Inclusion/Exclusion Rule

To get an accurate count of the elements in $A \cup B$, it is necessary to subtract the number of elements that are in both A and B. Because these are the elements in $A \cap B$,

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

Theorem 10.3.3 The Inclusion/Exclusion Rule for 2 or 3 Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

The Inclusion/Exclusion Rule

Example 10 – Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Let A = the set of all integers in [1..1000] that are multiples of 3.
Let B = the set of all integers in [1..1000] that are multiples of 5.

Then $A \cup B$ = the set of all integers in [1..1000] that are multiples of 3 or multiples of 5.

Then $A \cap B$ = the set of all integers in [1..1000] that are multiples of both 3 and 5

= the set of all integers in [1..1000] that are multiples of 15.

The Inclusion/Exclusion Rule

Example 10 – Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

As every third integer from 3 through 999 is a multiple of 3, each can be represented in the form $3k$, for some integer k in [1..333].

Hence there are 333 multiples of 3 from 1 through 1000, and so

$N(A) = 333$

Similarly, every multiple of 5 from 1 through 1000 has the form $5k$, for some integer k in [1..200].

Hence there are 200 multiples of 5 from 1 through 1000, and so

$N(B) = 200$

The Inclusion/Exclusion Rule

Example 10 – Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Finally, every multiple of 15 from 1 through 1000 has the form $15k$, for some integer k in [1..66] (since $990 = 66 \times 15$).

Hence there are 66 multiples of 15 from 1 through 1000, and so

$N(A \cap B) = 66$

It follows by the **inclusion/exclusion rule** that

$$\begin{aligned} N(A \cup B) &= N(A) + N(B) - N(A \cap B) \\ &= 333 + 200 - 66 = 467 \end{aligned}$$

Thus, **467** integers from 1 through 1000 are multiples of 3 or multiples of 5.

The Inclusion/Exclusion Rule

Example 10 – Counting Elements of a General Union

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

There are 1000 integers from 1 through 1000.

By part (a), 467 of these are multiples of 3 or multiples of 5.

Thus, by the **difference rule**, there are 1000 – 467 = **533** that are neither multiples of 3 nor multiples of 5.

The Inclusion/Exclusion Rule

Note that the solution to part (b) of Example 10 hid a use of De Morgan's law.

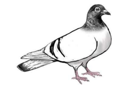
The number of elements that are neither in A nor in B is $N(A^c \cap B^c)$.
By De Morgan's law, $A^c \cap B^c = (A \cup B)^c$.

So $N((A \cup B)^c)$ was then calculated using the set difference rule: $N((A \cup B)^c) = N(U) - N(A \cup B)$, where the universe U was the set of all integers from 1 through 1,000.

4: The Pigeonhole Principle

The Pigeonhole Principle: Introduction

If n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.



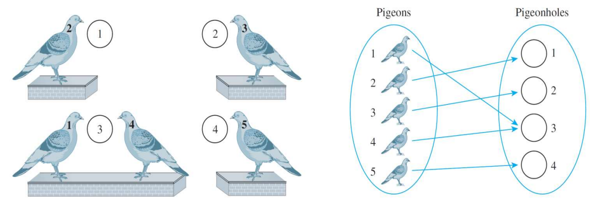
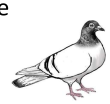


Figure 9.4.1 n = 5 and m = 4

The Pigeonhole Principle

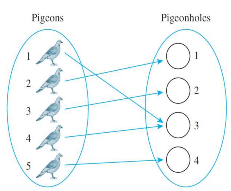
The pigeonhole principle is sometimes called the *Dirichlet box principle* because it was first stated formally by J. P. G. L. Dirichlet (1805–1859).



Mathematical formulation:

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.



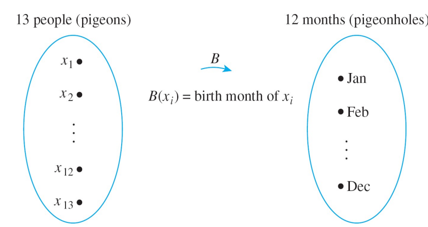
The Pigeonhole Principle

Example 11 – Applying the Pigeonhole Principle

In a group of six people, must there be at least two who were born in the same month?

No.

In a group of 13 people, must there be at least two who were born in the same month? Why?



Yes. At least 2 people must have been born in the same month.

Generalized Pigeonhole Principle

If n pigeons fly into m pigeonholes and, for some positive integer k, $k < n/m$, then at least one pigeonhole contains $k + 1$ or more pigeons.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Generalized Pigeonhole Principle

Example 12 – Applying the Generalized Pigeonhole Principle

Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial (‘A’, ‘B’, …, ‘Z’).

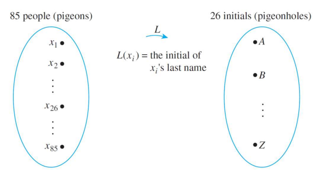
In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names. Note that

$$3 < 85/26 \cong 3.27$$

Generalized Pigeonhole Principle

Example 12 – Applying the General Pigeonhole Principle

Consider the function L from people to initials defined by the following arrow diagram.



Since $3 < 85/26$, the generalized pigeonhole principle states that some initial must be the image of at least four (3 + 1) people.

Thus, at least 4 people have the same last initial. ■

5: Combinations

Counting Subsets of a Set: Combinations

- Given a set S with n elements, how many subsets of size r can be chosen from S?
- Each subset of size r is called an *r*-combination of the set.

Definition: *r*-combination

Let n and r be non-negative integers with $r \leq n$.

An ***r*-combination** of a set of n elements is a subset of r of the n elements.

$\binom{n}{r}$, read "n choose r", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements. Other symbols used are $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$.

Counting Subsets of a Set: Combinations

Example 13-Combinations

Let S = {Ann, Bob, Cyd, Dan}. Each committee consisting of three of the four people in S is a 3-combination of S.

a. List all such 3-combinations of S.

b. What is $\binom{4}{3}$?

a. The 3-combinations are:
{Bob, Cyd, Dan}, {Ann, Cyd, Dan}, {Ann, Bob, Dan}, {Ann, Bob, Cyd}

b. $\binom{4}{3}$ = **4**

Counting Subsets of a Set: Combinations

Example 14 – Ordered and Unordered Selection

Two distinct methods that can be used to select r objects from a set of n elements:

Ordered selection
Also called *r*-permutation

Unordered selection
Also called *r*-combination

Example: S = { 1, 2, 3 }

2-permutations of S		2-combinations of S
{1, 2}	{2, 1}	{1, 2}
{1, 3}	{3, 1}	{1, 3}
{2, 3}	{3, 2}	{2, 3}

Relationship between Permutations and Combinations

Example 15 – Relationship between Permutations and Combinations

Write all 2-permutations of the set {0, 1, 2, 3}. Find an equation relating the number of 2-permutations, $P(4, 2)$, and the number of 2-combinations, $\binom{4}{2}$, and solve this equation for $\binom{4}{2}$.

{0, 1},	{1, 0},
{0, 2},	{2, 0},
{0, 3},	{3, 0},
{1, 2},	{2, 1},
{1, 3}.	{3, 1}.
{2, 3},	{3, 2}

According to Theorem 10.2.3,
$P(4, 2) = 4!/(4-2)! = 4!/2! = 12$

The construction of a 2-permutation of {0, 1, 2, 3} can be thought of comprising two steps:

Step 1: Choose a subset of 2 elements from {0, 1, 2, 3}.
Step 2: Choose an ordering for the 2-element subset.

Relationship between Permutations and Combinations

Example 15 – Relationship between Permutations and Combinations

This can be illustrated by the following possibility tree:

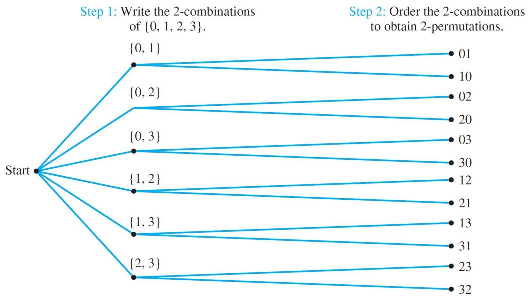


Figure 10.5.1 Relationship between Permutations and Combinations

Relationship between Permutations and Combinations

Example 15 – Relationship between Permutations and Combinations

The number of ways to perform step 1 is $\binom{4}{2}$.
The number of ways to perform step 2 is 2!

Hence,

$P(4, 2) = \binom{4}{2} \cdot 2!$

$\binom{4}{2} = P(4, 2) / 2!$
$= 12 / 2 = 6$

Relationship between Permutations and Combinations

Theorem 10.5.1 Formula for $\binom{n}{r}$

The number of subsets of size *r* (or *r*-combinations) that can be chosen from a set of *n* elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where *n* and *r* are non-negative integers with $r \leq n$.

Relationship between Permutations and Combinations

Example 16 – Teams with Members of Two Types

Suppose the group of 12 consists of 5 men and 7 women.

a. How many 5-person teams can be chosen that consist of 3 men and 2 women?

Hint: Think of it as a two-step process:
Step 1: Choose the men.
Step 2: Choose the women.

$$\binom{5}{3} \times \binom{7}{2} = \frac{5!}{3!\,2!} \times \frac{7!}{2!\,5!} = 210$$

Relationship between Permutations and Combinations

Example 16 – Teams with Members of Two Types

Suppose the group of 12 consists of 5 men and 7 women.

b. How many 5-person teams contain at least one man?

Hint: May use **difference rule** or **addition rule**.
The former is shorter.

Let A be the set of all 5-person teams,
and B be the set of 5-person teams without any men.

Then $N(A) = \binom{12}{5} = 792$, and $N(B) = \binom{7}{5} = 21$

Therefore number of 5-person teams that contain
at least one man = 792 – 21 = **771**

Relationship between Permutations and Combinations

Example 16 – Teams with Members of Two Types

Suppose the group of 12 consists of 5 men and 7 women.

c. How many 5-person teams contain at most one man?

Number of teams without any man = $\binom{5}{0} \times \binom{7}{5} = 1 \times 21$
$= 21$

Number of teams with one man = $\binom{5}{1} \times \binom{7}{4} = 5 \times 35$
$= 175$

Therefore number of 5-person teams that contain
at most one man = 21 + 175 = 196

6: The Binomial Theorem

Pascal's Formula

Pascal's formula can be derived by two entirely different arguments. One is algebraic; it uses the formula for the number of r-combinations obtained in Theorem 10.5.1

Theorem 10.6.1 Pascal's Formula

Let n and r be positive integers, $r \le n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Pascal's Formula

Pascal's triangle is a geometric version of Pascal's formula.

$n \backslash r$	0	1	2	3	4	5	$\cdots$	$r-1$	r	$\cdots$
0	1							.	.	$\cdots$
1	1	1						.	.	$\cdots$
2	1	2	1					.	.	$\cdots$
3	1	3	3	1				.	.	$\cdots$
4	1	4	6 +	4	1			.	.	$\cdots$
5	1	5	10 =	10	5	1		.	.	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\cdots$	$\binom{n}{r-1}$ +	$\binom{n}{r}$	$\cdots$
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	$\cdots$	=	$\binom{n+1}{r}$	$\cdots$
.		.	.	.	.	.		.	.	$\cdots$
.		.	.	.	.	.		.	.	$\cdots$
.		.	.	.	.	.		.	.	$\cdots$

Table 10.6.1 Pascal's Triangle

The Binomial Theorem

In algebra a sum of two terms, such as $a + b$, is called binomial.

The *binomial theorem* gives an expression for the powers of a binomial $(a + b)^n$, for each positive integer n and all real numbers a and b.

$\binom{n}{r}$ is called a binomial coefficient

Theorem 10.6.2 Binomial Theorem

Given any real numbers a and b and any non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

The Binomial Theorem

Example 17 – Substituting into the Binomial Theorem

Expand the following using the binomial theorem:

a. $(a+b)^5$ b. $(x-4y)^4$

a.

$$(a+b)^5 = \sum_{k=0}^{5} \binom{5}{k} a^{5-k} b^k$$

$$= a^5 + \binom{5}{1} a^{5-1} b^1 + \binom{5}{2} a^{5-2} b^2 + \binom{5}{3} a^{5-3} b^3 + \binom{5}{4} a^{5-4} b^4 + b^5$$

$$= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$$

The Binomial Theorem

Example 17 – Substituting into the Binomial Theorem

Expand the following using the binomial theorem:

a. $(a+b)^5$ b. $(x-4y)^4$

b.

$$(x-4y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} (-4y)^k$$

$$= x^4 + \binom{4}{1} x^{4-1} (-4y)^1 + \binom{4}{2} x^{4-2} (-4y)^2 + \binom{4}{3} x^{4-3} (-4y)^3 + (-4y)^4$$

$$= x^4 - 16x^3 y + 96x^2 y^2 - 256xy^3 + 256y^4$$

End of Unit 10