MATH221
Mathematics for Computer Science

Unit 7
Mathematical Induction

OBJECTIVES

- Understand and apply Mathematical Induction.
- Understand and apply the Generalized Principle of Mathematical Induction.
- Understand the Sigma Notation and Recursive Definitions.
- Understand and apply the Principle of Strong Mathematical Induction.

The Basic Principle of Mathematical Induction: Example

Prove $n^3 > 2n - 2$ for all natural numbers, n.

Can we use the exhaustive method to prove this statement ? No.

Let CLAIM(n) be "$n^3 > 2n – 2$".
For $n = 1$: CLAIM(1) is $1^3 > 2 \times 1 – 2$
LHS = 1; RHS = 0 $\Rightarrow$ LHS > RHS.
For $n = 2$: CLAIM(2) is $2^3 > 2 \times 2 – 2$
LHS = 8; RHS = 2 $\Rightarrow$ LHS > RHS.
For $n = 3$: CLAIM(3) is …

The process above will NEVER prove the CLAIM for **ALL** natural number n.

The Principle of Mathematical Induction

For all natural numbers n, let CLAIM(n) be a statement. If
1. CLAIM(1) is true, and
2. the truth of CLAIM(k) implies the truth of CLAIM(k + 1) for all natural numbers k,

then CLAIM(n) is true for all natural numbers n.

The Principle of Mathematical Induction: Proof

Let $T = \{t \in \mathbb{N}$: CLAIM(t) is false$\}$.

If we can show that T is empty, then CLAIM(n) is true for $n \in \mathbb{N}$.

Note that T is a subset of $\mathbb{N}$. Also, we know the following:

1. CLAIM(1) is true, and
2. the truth of CLAIM(k) implies the truth of CLAIM(k + 1) for all natural numbers k.

Assume T is not empty.

Then T has a least element.

[A result of well-ordering property on $\mathbb{N}$]

The Principle of Mathematical Induction: Proof

Let this least element be t_0, so

CLAIM(t_0) is false. (*)

By statement 1 above $t_0 \neq 1$, so t_0 - 1 is a natural number.

Also, $t_0 - 1$ is not in T. Therefore, CLAIM($t_0 - 1$) is true. However, by statement 2 above,

CLAIM($t_0 - 1$) true implies that

CLAIM($t_0 - 1 + 1$) = CLAIM(t_0) is true.

This is a contradiction with statement (*).

So, our assumption that T is not empty must be **false**.

Hence, T must be empty.

Therefore, CLAIM(n) is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction: Example

Prove $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$, for $n \in \mathbb{N}$.

Let CLAIM(n) be "$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$".

Does the claim satisfy the conditions of the Principle of Mathematical Induction?

1. Is CLAIM(1) true?

   CLAIM(1) is $1 = \frac{1(1+1)}{2}$. LHS = 1; RHS = $\frac{2}{2} = 1$.

   Therefore, LHS = RHS and so CLAIM(1) is true.

The Principle of Mathematical Induction: Example

2. We must show that for $k \geq 1$,

   CLAIM(k) is true $\Rightarrow$ CLAIM(k + 1) is true.

   Assume CLAIM(k) is true, that is,

   $$1 + 2 + \ldots + k = \frac{k(k+1)}{2} \quad (*)$$

   Prove CLAIM(k + 1) is true, that is,

   $$1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+1+1)}{2}$$
   $$= \frac{(k+1)(k+2)}{2}$$

The Principle of Mathematical Induction: Example

LHS = 1 + 2 + … + k + (k + 1)

$$= \frac{k(k+1)}{2} + (K+1) \quad \text{(Using *)}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Therefore, LHS = RHS.
Therefore, CLAIM(k + 1) is true.
So, CLAIM(n) satisfies both the conditions of the Principle of Mathematical Induction.
Therefore, by induction, CLAIM(n) is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction: Summary

To establish an infinite family of claims, (CLAIM(1), CLAIM(2), CLAIM(3), …) using the Principle of Mathematical Induction, it is sufficient to carry out the following two steps:

Step 1: Prove that the first claim, CLAIM(1) is true.

Step 2: Give a ***general proof*** that if CLAIM(k) is true then CLAIM(k + 1) is also true, whatever the value of $k \in \mathbb{N}$.

The first step is called the **basis step** and the second step is called the **inductive step**.

Aside: Note on Formula for Algebraic Expansion

$(x + y)^2 = x^2 + y^2 + 2xy$

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

The Principle of Mathematical Induction: Example

Prove $n^3 > 2n - 2$ for all natural numbers, n.

Let CLAIM(n) be "$n^3 > 2n - 2$".

Step 1: CLAIM(1) is $1^3 > 2 \times 1 - 2$.
LHS = 1; RHS = 0.
Therefore, LHS > RHS and so CLAIM(1) is true.

Step 2: Assume CLAIM(k) is true for some $k \in \mathbb{N}$, that is,
$k^3 > 2k - 2$. (*)
Prove CLAIM(k + 1) is true, that is, prove that
$(k + 1)^3 > 2(k + 1) - 2$.
that is, to prove that $(k + 1)^3 > 2k$.

The Principle of Mathematical Induction: Example

LHS $= (k + 1)^3$
$= k^3 + 3k^2 + 3k + 1$ (refer to the Appendix for the formula)
$> (2k - 2) + 3k^2 + 3k + 1$ (using *)
$\geq 2k - 2 + 3 + 3 + 1$ (since $k \geq 1$)
$= 2k + 5$
$> 2k$
$=$ RHS

Therefore, $(k + 1)^3 > 2(k + 1) - 2$ and so CLAIM($k + 1$) is true.

Therefore, by induction, CLAIM(n) is true for all $n \in \mathbb{N}$.

The Generalized Principle of Induction: Example

Prove that $2^n > 2n + 1$ by induction.

Let CLAIM(n) be $2^n > 2n + 1$ for $n \in \mathbb{N}$.
Step 1: CLAIM(1) is $2^1 > 2(1) + 1$.
LHS = 2; RHS = 3.
LHS < RHS!!!
The fact is that $2^n > 2n + 1$ is true for $n \geq 3$, $n \in \mathbb{N}$.

Sometimes we would like to prove a CLAIM that only works for values greater than 4, say.

The Principle of Mathematical Induction can be generalized to "accommodate" this type of problem.

The Generalized Principle of Induction

Suppose that q is some integer. For all integers $n \geq q$, let CLAIM(n) be a statement. If

1. CLAIM(q) is true, and

2. the truth of CLAIM(k) implies the truth of CLAIM($k + 1$) for all integers $k \geq q$,

then CLAIM(n) is true for all integers $n \geq q$.

Note that the proof is almost identical to the proof to that for the Principle of Mathematical Induction.

The Generalized Principle of Induction: Note

This generalized principle is applied in the same way as before, except that at

Step 1: we prove *CLAIM(q)* instead of CLAIM(1), and

Step 2: we may need the information that $k \geq q$ instead of $k \geq 1$.

Note that q can be any integer: positive, negative or zero.

Exercise:

Prove that $2^n > 2n + 1$ for $n \geq 3$, $n \in \mathbb{N}$.

Let CLAIM(n) be $2^n > 2n + 1$.

Step 1: CLAIM(3) is $2^3 > 2(3) + 1$.

LHS = 8; RHS = 7.

Therefore, LHS > RHS and so CLAIM(3) is true.

Step 2: Assume CLAIM(k) is true for some $k \geq 3$, $k \in \mathbb{N}$, that is,

$$2^k > 2k + 1. \qquad (*)$$

Prove CLAIM($k + 1$) is true, that is, prove that

$$2^{k+1} > 2(k + 1) + 1 = 2k + 3.$$

Exercise:

$$\begin{aligned} \text{LHS} &= 2^{k+1} \\ &= 2^k(2) \\ &> (2k + 1)(2) && \text{(using *)} \\ &= 4k + 2 \\ &= 2k + 2k + 2 \\ &> 2k + 1 + 2 && (\text{since } k \geq 3,\ 2k > 1) \\ &= 2k + 3 \\ &= \text{RHS} \end{aligned}$$

Therefore, $2^{k+1} > 2k + 3$ and so CLAIM($k + 1$) is true.

Therefore, by induction, CLAIM(n) is true for all natural numbers $n \geq 3$.

Sigma Notation

The notation $\sum_{i=1}^{n} a_i$ denotes the **sum** of the numbers

$a_1, a_2, a_3, \ldots, a_n$; that is

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

Sigma Notation: Examples

(i) $\sum_{i=1}^{4} 2i = (2 \times 1) + (2 \times 2) + (2 \times 3) + (2 \times 4)$

$= 2 + 4 + 6 + 8$

$= 20$

(ii) $\sum_{i=4}^{12} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

$= 9$

Sigma Notation: Equalities

(i) $\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{n} a_i$

(ii) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n$

(iii) $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$

(iv) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i, \quad n \geq m$

(v) $\sum_{i=m}^{n} k a_i = k \sum_{i=m}^{n} a_i, \quad n \geq m$

Discussion:

True or False? $\sum_{i=1}^{3} 2i - 1 = 9$

$$\sum_{i=1}^{3} 2i - 1 = 2\sum_{i=1}^{3} i - 1$$

$$= 2(1 + 2 + 3) - 1$$

$$= 11 \quad (\neq 9)$$

Recursive Definitions

Sequences of numbers are defined ***recursively*** if each new number depends on previous values.

Recursive Definitions: Example

Let $u_1, u_2, \ldots, u_n, \ldots$ be real numbers defined by

$u_1 = 2$ and $u_2 = 4$

and

$u_n = 5u_{n-1} - 6u_{n-2}$, for $n \geq 3$.

Find u_4:

$u_3 = 5u_2 - 6u_1$ $(n = 3)$
$= 5 \times 4 - 6 \times 2$
$= 8$

$u_4 = 5u_3 - 6u_2$ $(n = 4)$
$= 5 \times 8 - 6 \times 4$
$= 16.$

Recursive Definitions: Note

The numbers can often be defined directly.

For example, $u_n = 2^n$ in the previous example.

The Principle of Strong Mathematical Induction

For integer n, let CLAIM(n) be a statement and let a and b be fixed integers with $a \leq b$. If

1. CLAIM(a), …, CLAIM(b) are true, and

2. the truth of CLAIM(i) for all integers i with $a \leq i < k$ implies the truth of CLAIM(k) for all integers $k > b$,

then CLAIM(n) is true for all integers $n \geq a$.

Aside: Notes on Divisibility

The expression "$a \mid b$" reads "a divides b", or, "a is a factor of b", or, "a is a divisor of b".

It means "b is divisible by a", or, "b is a multiple of a", or more formally, there is an $l \in \mathbb{Z}$ so that $b = al$.

Examples.

(i) $6 \mid 12$

(ii) $7 \mid (a^2 - 2)$ $\Leftrightarrow$ there is $l \in \mathbb{Z}$, $a^2 - 2 = 7l$.

(iii) $6 \nmid 25$ reads "6 does not divide 25".

When a proof involves showing $a \mid b$, you need to express b as a multiple of a, that is, ***find*** $l \in \mathbb{Z}$ so that $b = al$.

The Principle of Strong Mathematical Induction: Example

Suppose that $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$b_1 = 4$, $b_2 = 12$,

$b_k = b_{k-2} + b_{k-1}$ for all integers $k \geq 3$.

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

Let CLAIM(n) be b_n is divisible by 4.

Step 1: CLAIM(1) is $b_1 = 4$ is divisible by 4.

CLAIM(2) is $b_2 = 12$ is divisible by 4.

Since 4 and 12 are divisible by 4, CLAIM(1) and CLAIM(2) are true.

The Principle of Strong Mathematical Induction: Example

Step 2: Let $k > 2$ be an integer, and suppose Claim(i) is true for all integers i *with* $1 \leq i < k$. That is, b_i is divisible by 4 for all integers i *with* $1 \leq i < k$. That is, (*)

Prove that CLAIM(k) is true, that is, prove that b_k is divisible by 4.

Now, $b_k = b_{k-2} + b_{k-1}$.

Since $k > 2$, $1 \leq k - 2 < k$ and $1 < k - 1 < k$, and so, by inductive hypothesis, both b_{k-2} and b_{k-1} are divisible by 4.

From definition of divisibility, this implies that $b_{k-2} = 4x$ and $b_{k-1} = 4y$, where x, y $\in$ Z.

Thus, $b_k = b_{k-2} + b_{k-1} = 4(x + y)$, where $x + y \in Z$.

Hence, b_k *is divisible by 4.*

Therefore, CLAIM(k) is true.

Therefore, by induction, CLAIM(n) is true for all integers $n \geq 1$.

End of Unit 7