

# MATH221 Mathematics for Computer Science

## Unit 6 Number Systems

### OBJECTIVES

- Know what is the natural numbers and integers.
- Understand what is binary operation and closed binary operation.
- Understand the algebraic terminologies, identity, inverse, commutative, associative and distributive.
- Know what is the well-ordered property.
- Understand the application of the definition of odd and even integers.
- Know what are prime and composite numbers.

1

2

### The Natural Numbers

The Natural Numbers, denoted by  $\mathbb{N}$ , consist of all positive “whole” numbers.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Is 0 an element of  $\mathbb{N}$  (notation  $0 \in \mathbb{N}$ )?

No.

3

### Binary Operations

An **operation** on a set  $S$  is a rule for combining one or more elements of  $S$ .

A **binary operation** on  $S$  is a rule for combining pairs of elements of  $S$  to produce another element.

We denote a general operation by  $*$ . A binary operation  $*$  on  $S$  is **closed** on  $S$  if

$$x, y \in S \Rightarrow x*y \in S.$$

4

## Binary Operations

On numbers we have four main binary operations: addition, subtraction, multiplication and division.

But on which number systems are these binary operations closed?

## Binary Operation

We can picture such an operation as a “black box” that takes two inputs from the set and produces one output in the set:



5

6

## Binary Operation: Example

If the set is  $\mathbb{N}$  and the operation is  $+$ , then the inputs 1 and 2 would result in the output 3.



7

## Exercise:

Among addition, subtraction, multiplication and division are all binary operations on  $\mathbb{R}$ , which of them are closed on  $\mathbb{N}$ ? Why or why not?

Addition [ $4 + 2 = 6$ ]

Yes

Subtraction [ $4 - 2 = 2$ ]

No.  $2 - 4 = -2$

Multiplication [ $4 \times 2 = 8$ ]

Yes

Division [ $4 \div 2 = 2$ ]

No.  $2 \div 4 = 1/2$

8

## Identity

Suppose  $*$  is a binary operation on a set  $S$ . An element  $e$  of  $S$  is called an **identity** if for all  $x \in S$  we have

$$e * x = x \text{ and } x * e = x$$

Does  $\mathbb{N}$  have an identity under the addition and multiplication operations?

Addition: no

Multiplication: yes, it is 1 as  $1 \times n = n$  and  $n \times 1 = n$ .

9

## Inverse

Suppose  $*$  is a binary operation on a set  $S$ , and  $e \in S$  is an identity. An element  $x \in S$  is called **invertible** if there exists an element  $y \in S$  such that

$$\underline{x * y = e \text{ and } y * x = e.}$$

In this case  $y$  is called the **inverse** of  $x$ .

What are the invertible elements of  $\mathbb{N}$  under the addition and multiplication operations?

Addition: not applicable

Multiplication: 1 only

10

## Commutativity and Associativity

A binary operation  $*$  on a set  $S$  is commutative if

$$\underline{x * y = y * x \text{ for all } x, y \in S.}$$

A binary operation  $*$  on a set  $S$  is associative if

$$(x * y) * z = x * (y * z) \text{ for all } x, y, z \in S.$$

Are addition and multiplication on  $\mathbb{N}$  commutative and associative operations? Yes

11

## Example: Paper-Scissors-Rock

Let  $M := \{p, s, r\}$  and consider the paper-scissors-rock-inspired binary operation  $*$  on  $M$  given by

$$\underline{p * r = r * p = p \text{ (paper beats rock)}}$$

$$\underline{p * s = s * p = s \text{ (scissors beats paper)}}$$

$$\underline{s * r = r * s = r \text{ (rock beats scissors)}}$$

$$\underline{p * p = p \text{ (paper ties with paper)}}$$

$$\underline{s * s = s \text{ (scissors ties with scissors)}}$$

$$\underline{r * r = r \text{ (rock ties with rock).}}$$

Note that  $*$  is closed on  $M$ .

12

## Example: Paper-Scissors-Rock (cont'd)

This binary operation is by definition commutative. But it is not associative:

$$\begin{aligned} (r * p) * s &= p * s = s \\ r * (p * s) &= r * s = r \end{aligned}$$

## Distributivity

For each  $a, b, c \in \mathbb{N}$  we have,

$$\begin{aligned} a \times (b + c) &= (a \times b) + (a \times c) \\ (a + b) \times c &= (a \times c) + (b \times c) \end{aligned}$$

We say multiplication distributes over addition in  $\mathbb{N}$ .

Note that addition does not distribute over multiplication. For instance?

$$2 + (3 \times 1) \neq (2 + 3) \times (2 + 1)$$

13

14

## Exercise:

Carefully write down every small step you take when simplifying the following expressions without the aid of a calculator. Can you give reasons for each step?

$$\begin{aligned} &3x + 4y + 2x + y \\ &= 3x + (4y + 2x) + y && \text{Associative law} \\ &= 3x + (2x + 4y) + y && \text{Commutative law} \\ &= (3x + 2x) + 4y + y && \text{Associative law} \\ &= (3 + 2)x + 4y + y && \text{Distributive law} \\ &= 5x + 4y + y && \text{Arithmetic} \\ &= 5x + (4y + y) && \text{Associative law} \\ &= 5x + (4 + 1)y && \text{Distributive law} \\ &= 5x + 5y && \text{Arithmetic} \end{aligned}$$

15

## Well-ordering property

A set  $S$  together with an order  $\leq$  is called well-ordered if every nonempty subset of  $S$  has a least element. That is, if  $A$  is a nonempty subset of  $S$ , then there is an  $s_0 \in A$  such that  $s_0 \leq s$  for all  $s \in A$ .

Axiom:

The natural numbers  $\mathbb{N}$  with the usual order  $\leq$  is well-ordered.

16

## The integers $\mathbb{Z}$

The integers  $\mathbb{Z}$  is the collection of all the “whole” positive and negative numbers, and 0. So

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The order  $\leq$  on  $\mathbb{N}$  extends to the natural order  $\leq$  on  $\mathbb{Z}$ .

Axiom:

Any nonempty subset of integers that are greater than a fixed integer with the usual order  $\leq$  is well-ordered.

17

## Commutativity and associativity

Addition and multiplication are commutative operations on  $\mathbb{Z}$ . But subtraction is not:

$$\underline{10 - 5 \neq 5 - 10}$$

Addition and multiplication are associative operations on  $\mathbb{Z}$ . But subtraction is not:

$$(10 - 2) - 3 \neq 10 - (2 - 3)$$

19

## Operations, identities and inverses on $\mathbb{Z}$

We have the usual binary operations  $+$ ,  $-$ ,  $\times$  and  $\div$  on  $\mathbb{Z}$ . But only the operations  $+$ ,  $-$  and  $\times$  are closed on  $\mathbb{Z}$ .

Does  $\mathbb{Z}$  have an identity under the  $+$  and  $\times$  operations?

What are the invertible elements in  $\mathbb{Z}$  under the  $+$  and  $\times$  operations?

18

## Distributivity

For each  $a, b, c \in \mathbb{Z}$  we have

$$\begin{aligned} a \times (b \pm c) &= (a \times b) \pm (a \times c) \\ (a \pm b) \times c &= (a \times c) \pm (b \times c) \end{aligned}$$

We say multiplication distributes over addition and subtraction in  $\mathbb{Z}$ .

20

## Well-ordering property

Recall that a set  $S$  together with an order  $\leq$  is called well-ordered if every nonempty subset of  $S$  has a least element, that is, if  $A$  is a nonempty subset of  $S$ , then there is a  $s_0 \in A$  such that  $s_0 \leq s$  for all  $s \in A$ .

We have seen that the natural numbers  $\mathbb{N}$  with the usual order  $\leq$  is well-ordered.

Fact: The integers  $\mathbb{Z}$  are not well-ordered with  $\leq$ .

21

## Evens, odds, primes and composites

An integer  $m \in \mathbb{Z}$  is **even** if  $m = 2j$  for some  $j \in \mathbb{Z}$ .

An integer  $n$  is **odd** if  $n = 2k + 1$  for some  $k \in \mathbb{Z}$ .

An integer  $n > 1$  is **prime** if whenever  $n = r \times s$  for positive integers  $r$  and  $s$ , we have  $r = 1$  or  $s = 1$ . An integer  $n > 1$  is **composite** if it is not prime.

Example: -2 is an even integer, 5 and -3 are odd integers, 5 is a prime integer and 4 is a composite integer.

22

## The rationals $\mathbb{Q}$

The rationals  $\mathbb{Q}$  is the set of numbers  $q$  that can be written in the form:

$$q = \frac{a}{b}.$$

where  $a, b \in \mathbb{Z}$  and  $b \neq 0$ .

Each number in  $\mathbb{Q}$  is called a rational number.

Example:

$0.4 = 2/5$  is a rational number.

23

## Irrational -

A number is irrational if it is not a rational number; that is, if it CANNOT be written in the following form:

$$q = \frac{a}{b}.$$

where  $a, b \in \mathbb{Z}$  and  $b \neq 0$ .

Example:

$\sqrt{2}$  is an irrational number.

24

**End of Unit 6**