MATH221
Mathematics for Computer Science

Unit 2
Predicate Logic

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OBJECTIVES

- Understand what is a predicate.
- Understand what are universal and existential quantifiers.
- Understand the use of universal and existential quantifiers to write statements.
- Apply the knowledge of universal and existential quantifiers to find the negation of statements that include quantifiers.

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Discussion:

In Mathematics we use variables (usually ranging over numbers) in various ways.

How does x differ in what it represents in the following statements?

(i) $x^2 = 0$

(ii) $x > 2$

(iii) $x + 0 = x$

(iv) $x^2 + 1 = 0$

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Predicate

A *predicate* is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

The *domain* of a predicate variable consists of all values that may be substituted in place of the variable.

The symbolic analysis of propositions is called *propositional calculus* and the symbolic analysis of predicates is called *predicate calculus*.

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Predicate: Examples and Notation

"*x is a student at Bedford College*" and "*x is a student at y*" are both predicates, where *x* and *y* are predicate variables that take values in appropriate sets.

We can use symbols to represent predicates. For example:

We can use P(x) to represent the predicate "*x* is a student at Bedford College" and *Q(x, y)* to represent the predicate "*x* is a student at *y*"

where *x* and *y* are predicate variables that take values in appropriate sets.

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Quantifiers

$\forall x$ read "for all *x*" or "for each *x*

$\exists x$ read "there exists an *x*" or "for some *x*

$\forall$ is called the ***Universal*** quantifier.
$\exists$ is called the ***Existential*** quantifier.

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Universal And Existential Statements

Let *Q(x)* be a predicate and *D* be the domain of *x*.

A ***universal statement*** is a statement of the form "$\forall x \in D, Q(x)$".

A ***existential statement*** is a statement of the form "$\exists x \in D, Q(x)$".

Note that *x* is the predicate variable.

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Discussion:

Returning to the statements in the Discussion, we can use the quantifier notation to restate them clearly:

(i) $x^2 = 0$

       $\exists x \in \mathbb{R}, x^2 = 0$

(ii) $x > 2$

       $\exists x \in \mathbb{R}, x > 2$

(iii) $x + 0 = x$

       $\forall x \in \mathbb{R}, x + 0 = x$

(iv) $x^2 + 1 = 0$

       $\forall x \in \mathbb{R}, \sim (x^2 + 1 = 0)$

Note:
1.   The symbol $\in$ means "belongs to".
2.   The domain of all predicate variables in the above statements is $\mathbb{R}$.

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Examples:

1. Write each of the following statements using the universal quantifier.
    (i) All dogs are animals.

    $\forall$ dog x, x is an animal.

    (ii) The square of any real number is positive or zero.

    $\forall x \in \mathbb{R}, x^2 \geq 0$.

Examples:

2. Write each of the following statements using the existential quantifier.
    (i) There exists a real number whose square is negative.

    $\exists x \in \mathbb{R}, x^2 < 0$.

    (ii) Some dogs are vegetarians.

    $\exists$ dog x, x is vegetarian.

Universal Statements: Important

A *universal* statement is defined to be true if and only if $Q(x)$ is true for *every* x in D.

A **universal** statement is defined to be false if and only if $Q(x)$ is false for *at least one* x in D.

A value of x for which $Q(x)$ is false is called a *counterexample* to the universal statement.

Existential Statements: Important

An *existential* statement is defined to be true if and only if $Q(x)$ is true for *at least one* x in D.

An e*xistential* statement is defined to be false if and only if $Q(x)$ is false for *every* x in D.

Truth Set of a Predicate

The **truth set of a predicate** is the set of elements in the domain that make the predicate a true statement.

Now try if you can determine whether the universal and existential statements in the previous discussion and examples are true or false.

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Aside

Real numbers are further classified as natural numbers, integers, rational numbers, irrational numbers. These sets of numbers are often used as the domain of the variable. The "standard" notation for these categories of numbers are as follows:

1. $\mathbb{N}$ represents the set of all natural numbers, or positive whole numbers.

   $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

2. $\mathbb{Z}$ represents the set of all integers or whole number (positive, negative and zero).

   $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

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Aside

3. $\mathbb{Q}$ represents the set of all rational number. A rational number is one that can be written as a fraction.

   Examples: $2 = 2/1$; $0.\dot{3} = 0.333\ldots = 1/3$

   An irrational number is one that cannot be written as a fraction, usually has a non-repeating, infinite decimal expansion.

   Examples: π; e

4. $\mathbb{R}$ represents the set of all real number.
   Note that $\mathbb{R}$ is formed by $\mathbb{Q}$ and the set of irrational numbers.

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Examples:

1. Write the following statements using quantifiers. Determine whether each statement is true or false.

   (i) Every integer is a rational number.

   $\forall x \in \mathbb{Z}, x \in \mathbb{Q}$.

   True

   (ii) Some real number is rational.

   $\exists x \in \mathbb{R}, x \in \mathbb{Q}$.

   True

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Examples:

1. (iii) No real numbers have squares equal to -1.

$\forall x \in \mathbb{R}, \sim (x^2 = -1)$.

True

Examples:

2. Write the following statements in words.

(i) $\forall x \in \mathbb{N}, \quad \sqrt{x} \in \mathbb{N}$.

The square root of every natural number is a natural number.

(ii) $\exists x \in \mathbb{Z}, \dfrac{1}{x} \notin \mathbb{Q}$.

The reciprocal of some integers are not rational.

(iii) $\forall$ person x, $\exists$ person y, y was the mother of x.

For every person, there was a mother of that person.

Quantified Statements: Examples

1. Every student likes at least one mathematics subject.

$\forall$ student S, $\exists$ mathematics subject y, S likes y.

2. All persons are equal.

$\forall$ person a, $\forall$ person b, a is equal to b.

3. Not all natural numbers are even.

$\sim (\forall n \in \mathbb{N}, n$ is even$)$

Discussion:

What is the definition of an even number?

An integer n is even if, and only if, n equals twice some integer.

Symbolically, if n is an integer, then

n is even $\Leftrightarrow \exists p \in \mathbb{Z}, n = 2p$.

Hence, the sentence "Not all natural numbers are even." can be expressed in one of the following two ways symbolically:

$\sim (\forall n \in \mathbb{N}, n$ is even$)$

$\sim (\forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p)$

Equivalent Forms of Universal Statements

A statement of the form
$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$
can be rewritten in the form
$$\forall x \in D, Q(x).$$

Note: $D \subseteq U$ and D is the truth set of $P(x)$.

Equivalent Forms of Universal Statements: Example

A statement of the form
$$\forall x \in \mathbb{R}, \text{ if } x \in \mathbb{Z} \text{ then } x \in \mathbb{Q}.$$
can be rewritten in the form
$$\forall x \in \mathbb{Z}, x \in \mathbb{Q}. \qquad (\text{Note: } \mathbb{Z} \subseteq \mathbb{R})$$

Examples:

Express the following laws of arithmetic using quantifiers.

(i)   Commutative law for +:
       If x and y are real numbers, then $x + y = y + x$.
       $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = y + x.$

(ii) Distributive law for $\times$ over +:
       If x, y and z are real numbers, then $x(y + z) = xy + xz$.
       $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x(y + z) = xy + xz.$

Quantifications Of Two Variables

Statement	When True?	When False?
$\forall x \, \forall y \, P(x, y)$ $\forall y \, \forall x \, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \, \exists y \, P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.

Quantifications of Two Variables

Statement	When True?	When False?
$\exists x \, \forall y \, P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \, \exists y \, P(x, y)$ $\exists y \, \exists x \, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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Discussion:

Consider the following statements.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0. \qquad (1)$$
$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0. \qquad (2)$$

(i) Are the statements the same?

No.

(ii) Determine whether each statement is true or false?

(1) is true.

(2) is false.

(iii) Is the order in which $\forall$ and $\exists$ appear important?

Yes.

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Negations Of Universal Statements

The negation of a statement of the form
$$\forall x \in D, Q(x)$$
is logically equivalent to a statement of the form
$$\exists x \in D, \sim Q(x).$$
Symbolically,
$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x).$$

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Negations Of Existential Statements

The negation of a statement of the form
$$\exists x \in D, Q(x)$$
is logically equivalent to a statement of the form
$$\forall x \in D, \sim Q(x).$$
Symbolically,
$$\sim (\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x).$$

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Negations Of Universal and Existential Statements: Example

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

$\sim (\forall$ primes p, p is odd)

$\equiv \exists$ a prime p, p is not odd.

b. $\exists$ a triangle T, the sum of the angles of T equals 200°.

$\sim (\exists$ a triangle T, the sum of the angles of T equals 200°)

$\equiv \forall$ triangles T, the sum of the angles of T does not equals 200°.

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Negations Of Universal Conditional Statements

By definition of the negation of a *for* all statement,

$\sim (\forall x \in U, P(x) \Rightarrow Q(x)) \equiv \exists x \in U, \sim (P(x) \Rightarrow Q(x))$.

But negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$\sim (P(x) \Rightarrow Q(x)) \equiv \sim (\sim P(x) \lor Q(x))$

$\equiv P(x) \land \sim Q(x)$.

Therefore,

$\sim (\forall x \in U, P(x) \Rightarrow Q(x)) \equiv \exists x \in U, P(x) \land \sim Q(x)$.

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Negations Of Quantified Statements (with two Quantifiers)

$\sim (\forall x \in D, \exists y \in E, P(x, y))$

$\equiv \exists x \in D, \forall y \in E, \sim P(x, y)$.

$\sim (\exists x \in D, \forall y \in E, P(x, y))$

$\equiv \forall x \in D, \exists y \in E, \sim P(x, y)$.

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Negations Of Quantified Statements (with two Quantifiers): Example

Write a formal negation for the following statement:

a. For all square x, there is a circle y such that x and y have the same colour.

$\forall$ square x, $\exists$ circle y, x and y have the same colour.

Negation is:

$\exists$ a square x, $\forall$ circle y, x and y do not have the same colour.

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Negations Of Quantified Statements (with two Quantifiers): Example

Write a formal negation for the following statement:

b.   There is a triangle x such that for all square y, x is to the right of y.

$\exists$ a triangle x, $\forall$ square y, x is to the right of y.

Negation is:

$\forall$ triangles x, $\exists$ a square y, x is not to the right of y.

End of Unit 2