# Experiment 222: The Mechanical Oscillator

## Aim

To study (A) damped harmonic motion and (B) forced oscillations and resonance.

## Experimental arrangement

The oscillatory mechanical system consists of a mass suspended by two springs (see Fig. 1). The upper spring hangs from a fixed support. The bottom end of the lower spring is attached by a wire and pulley to a variable speed motor. The bottom end of the spring provides a fixed support if the motor is turned off. The lower support moves in approximate vertical simple harmonic motion if the motor is turned on. A strain gauge attached to the bar on which the upper spring is suspended measures a strain which is proportional to the tension in the upper spring. Since the tension is proportional to the extension of the spring the signal from the strain gauge indicates the displacement of the mass. The signal is sampled every 10 ms using an analogue to digital converter (ADC) under the control of a microcontroller. The data is sent to a computer which records each strain as a number in the range 0 to 1023 using the program. The data recorded in the graph seen on the computer can be saved to disk by clicking on the `Save As...` button under the `File` menu.

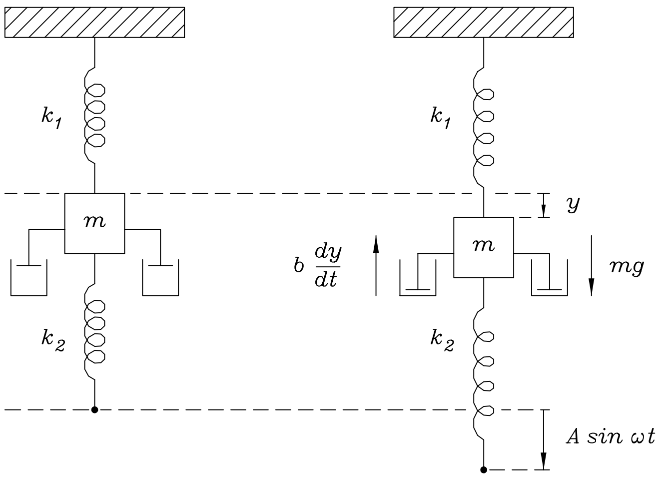


Figure 1: The mechanical oscillator

Damping is provided by a ‘dash-pot’, i.e. loose-fitting piston inside a cylinder of fluid. Damping may be removed if the cylinder of fluid is lowered clear of the moving piston. This case will be referred to as *low damping* as there is still some damping, in this case due to air friction and flexing of the springs. *Medium damping* and *high damping* are obtained with the piston in the fluid and with the holes in the piston either open or closed.

The springs have a combined spring constant $k$ so they exert a force $ky$ upwards when the mass is displaced downwards a distance $y$ from equilibrium, as shown in Fig. 1. The damping may be assumed to be proportional to the speed of the mass and so the damping force is given by $b\left(\frac{dy}{dt}\right)$ and is in an opposite direction to the velocity (see Fig.1).

# Note:

In this experiment the natural angular frequency ($\omega_0$) and quality factor ($Q$) of the mechanical oscillator will be determined. Two methods are used (hence part A and part B). One method may be more useful when the damping is low and the other when the damping is high. Think about this as you do the experiment; don't invest excessive amounts of time trying to obtain perfect data if you think the method inappropriate for the level of damping.

**WARNING! When the damping is low, be careful when driving the oscillator near its natural frequency. The amplitude of motion may become so large as to cause the apparatus to self-destruct, sending the oscillator flying around the room.**

# A. Damped Harmonic Motion

The differential equation governing vertical oscillations may be written:

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \tag{1}$$

This equation has solution:

$$y\left(t\right) = A\exp\left(-\frac{t}{\tau}\right)\cos\left(\omega' t + \delta\right) \tag{2}$$

where

$$\tau = \frac{2m}{b} \tag{3}$$

and

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{4}$$

The parameter $\tau$ is the time constant and is the time required for the amplitude of oscillation to decay by a factor of 1/e. The parameter $\omega'$ is the natural angular frequency of the damped oscillations. In the experiment we will measure $\tau$ and $\omega'$ for various amounts of damping. We also define:

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{5}$$

which is the undamped angular frequency. The differential equation can then be written in the alternative form:

$$\frac{d^2y}{dt^2} + \frac{2}{\tau}\frac{dy}{dt} + \omega_0^2 y = 0 \tag{6}$$

We also have:

$$\left(\omega'\right)^2 = \omega_0^2 - \left(1/\tau\right)^2 \tag{7}$$

The damped and undamped periods of oscillation are given by:

$$T' = \frac{2\pi}{\omega'} \tag{8}$$

and

$$T_0 = \frac{2\pi}{\omega_0} \tag{9}$$

In studying oscillatory systems it is also convenient to define a parameter $Q$ (called the '$Q$ factor' or 'Quality factor'). There are several equivalent definitions of $Q$ including:

$$Q = \frac{\omega_0 \tau}{2} \tag{10}$$

If $Q > 5$ then $\omega' = \omega_0$ to within 1 %.

# Procedure

**CAUTION: keep hands and loose clothing clear of moving parts**

(1) Ensure the computer is turned on and run the data acquisition program by navigating to the menu item `Start->All Programs->Physics->AdvLab->Expt222-Mechanical Oscillator` and clicking on `Mechanical Oscillator`. The screen shows (from top to bottom) a main menu, an area for selecting settings, a graph area, a text area for data and a status bar. Click on the button labelled `Find Devices`. If everything is connected properly, the `Select Device` combo box and the `Open Device` button will be enabled. If not, see a demonstrator. Under `Select Device` you should see `exp222` and the words `USBXpress Device` should appear next to it. Click on `Open Device`. On the Status Bar you should start to see data constantly changing. Pull the mass of the mechanical oscillator down by a few centimetres, and let go. Click on `Start`. A graph should appear as 1 second worth of data is collected and a record of the data will be displayed in the read-only text area. Press `Clear`. Familiarize yourself with the program by performing the following steps:

(a) Set the mass into motion again if it has come to rest. Click on the `Clear` button. This clears the graph and the read-only data area. Data is collected at a rate determined by the `Sample Period` selector. The default is 10 ms or 100 points per second. Notice that as points are collected, the horizontal axis is automatically rescaled so that all the data are visible on the screen.

(b) After a time set by the `Time to sample` selector, the `Stop` button is automatically pressed. Move the mouse pointer into the region containing the graph. Notice how the coordinates of the pointer position appear in the Status Bar region.

(c) In order to zoom into the graph for more accurate measurements, use the left mouse button to drag a "zoom rectangle" around the portion of the graph which is to be examined. The zoom rectangle must be drawn starting at its upper left corner and finishing at its lower right corner. In order to undo the zoom function and display all the available data again, click on the graph again and this time drag a rectangle from top right to bottom left.

(d) When measuring the period of the oscillation, it is preferable to measure the time for several oscillations, and to divide the interval by the number of cycles. Use the zoom feature to allow the "time" to be measured to sufficient accuracy from the Status Bar, and the pan feature to scroll through the required number of cycles. Panning is enabled by dragging the graph horizontally while holding down the right mouse button. Once the appropriate portion of the graph is visible, the mouse pointer coordinates may be read out again from the Status Bar.

Please **ask a demonstrator** for assistance if you have any problems using the program.

(2) Set up the pendulum for low damping. Start the oscillator by pulling the mass down $\sim$1cm (try and keep the mass and springs in a straight line) and then releasing it. Shortly after the oscillations start, depress the `Start` button and wait for the oscillations to drop away by a factor of approximately 1/e (select an appropriate `Time to sample`). This may take several minutes. Once enough data has been collected, press the `Stop` button or wait for it to stop automatically.

(a) Using the zoom function if necessary, measure the "period" $T'$ using the Status Bar. You will need to estimate the equilibrium position of the mass from the values of amplitude on the Status bar. Ensure that you also record the uncertainty in your value of $T'$

(b) Measure the factor by which the oscillation amplitude has been reduced during the time of your data collection. Calculate the value and the uncertainty in $\tau$ using

$$\tau = \frac{t_{\text{final}} - t_{\text{initial}}}{\ln\left(\dfrac{A_{\text{initial}}}{A_{\text{final}}}\right)},$$

where $A_{\text{initial}}$ and $A_{\text{final}}$ are the amplitudes of oscillation at times $t_{\text{initial}}$ and $t_{\text{final}}$ respectively.

(c) Calculate $T_0, \omega_0$ and $Q$ using equations (7) through (10).

(3) Repeat the above procedure for medium and high damping. Note that the equilibrium position will be different for the different cases. Why?

(4) Tabulate your results for $T', \tau, T_0, \omega_0$ and $Q$ with their errors for low, medium and high damping.

(5) Comment on whether this has been a good method of measuring $\omega_0$ and $Q$.

## B. Forced Oscillations and Resonance

With the motor turned on, the lower support is moved approximately in vertical SHM of angular frequency $\omega$. This is equivalent to applying a periodic force $F \sin \omega t$ directly to the mass and the differential equation becomes:

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F}{m}\sin \omega t \tag{11}$$

The general solution of this equation has two parts. The transient solution decays exponentially with time constant $\tau$ the same as for damped harmonic motion. Since $(1/\mathrm{e})^5 \approx 0.007$ the transient solution will have decayed to negligible amplitude after a time of about $5\tau$.

The steady state solution is the steady oscillatory motion which remains when the transient solution has decayed. For the rest of the experiment we will be concerned only with the steady state solution.

The steady state solution is:

$$y\left(t\right) = A \sin\left(\omega t + \phi\right) \tag{12}$$

where

$$A = \frac{F}{\left[m^2\left(\omega_0^2 - \omega^2\right)^2 + b^2\omega^2\right]^{1/2}} \tag{13}$$

and

$$\tan \phi = -\frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)} \tag{14}$$

The quality factor may be written:

$$Q = \frac{m\omega_0}{b} \tag{15}$$

$$A = \frac{F/\left(m\omega\omega_0\right)}{\left[\left(\omega_0/\omega - \omega/\omega_0\right)^2 + 1/Q^2\right]^{1/2}} \tag{16}$$

We shall measure amplitude as a function of frequency and obtain a plot corresponding to equation (16). For $Q > 1$ the amplitude becomes large when $\omega \approx \omega_0$. Thus it shows resonant behaviour when the angular frequency of the driving force is close to the natural angular frequency $\omega'$ of the system. [Note that we have already shown that $\omega \approx \omega_0$ above.]

The amplitude is maximum at angular frequency given by:

$$\omega_A = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \tag{17}$$

It is convenient to consider also the kinetic energy because the kinetic energy amplitude is resonant exactly at $\omega = \omega_0$. This can be shown as follows:

The velocity of the mass is given by:

$$\frac{dy}{dt} = \omega A \cos\left(\omega t + \phi\right) \tag{18}$$

The kinetic energy of the mass is given by:

$$K = E\cos^2(\omega t + \phi) \tag{19}$$

where the kinetic energy amplitude is given by:

$$E = \frac{1}{2} m\,\omega^2 A^2 \tag{20}$$

which can be written

$$E = \frac{(F/m\omega_0)^2/2}{\left[(\omega_0/\omega - \omega/\omega_0)^2 + 1/Q^2\right]} \tag{21}$$

The kinetic energy amplitude is clearly maximum when $\omega = \omega_0$. This corresponds to maximum velocity of the mass and is often taken as the condition for resonance. It is not the condition for maximum amplitude of displacement as noted above.

The $Q$ factor is readily measured from a plot of equation (21) as follows. If $\omega_1$ and $\omega_1 + \triangle\omega$ are the angular frequencies for which $E$ is half its value at resonance then it can be shown that $Q$ is given by:

$$Q = \frac{\omega_0}{\triangle\omega} \tag{22}$$

# Procedure

(6) Damping is achieved by filling the container with water, and choosing the holes in the plunger to be completely open (low damping), half closed (medium damping) or all the way closed (high damping).

(7) With the medium damping system adjust the frequency of the motor to obtain maximum amplitude of oscillation. This is to determine the scale of your graphs to be plotted in the following. Prepare a graph with amplitude on the vertical scale and angular frequency up to about $2\omega_0$ (using the result from damped harmonic motion) on the horizontal scale.

When making amplitude measurements do not assume that the centre of the oscillation will remain in the same place for different damping systems. Why?

(8) We wish to make measurements of steady state amplitude $S$ at various frequencies. Note that each time the frequency of the driving motor is changed, transients are introduced, and you must wait for the steady state to be reached (about $5\tau$) before making measurements. When the steady state is reached press the `Start` button to start collecting data and wait for a reasonable amount of data to be collected before analyzing for $T$ and $S$.

(9) Start the system with high damping. Measure $T$ and $S$ for a range of frequencies around the resonance and tabulate your results in columns to give $T$, $\omega$, $S$ and $\omega^2 S^2$. [This last column is proportional to the amplitude of the kinetic energy].

Plot the steady state amplitude $S$ as a function of $\omega$. On a second graph plot $\omega^2 S^2$ as a function of $\omega$. This can be regarded as a plot of the kinetic energy amplitude as a function of $\omega$.

Plot your graphs **as the data is collected.** Start near resonance and move to higher frequencies. Then start again at resonance and move to lower frequencies. By plotting the graph as the data are collected, you should be able to obtain values which clearly define the steeper parts of the resonance curves. Use equation (22) to determine the value of $Q$.

(10) Determine $\omega_0$ from your graph of kinetic energy amplitude as a function of $\omega$.

(11) Determine $\omega_A$ from your graph of amplitude as a function of $\omega$.

(12) Repeat procedures (9) – (11) for medium damping.

Plot your results on the same graph paper as for the high damping case.

(13) Tabulate values of $Q, \omega_0, \omega_A$ (with error estimates) and the theoretical value of $\omega_A$ calculated using equation (17).

(14) Comment on the usefulness of this as a method of measuring $\omega_0$ and $Q$.

(15) Comment on the results for $\omega_0$ and $Q$ obtained in parts A and B of the experiment.

## List of Equipment

1. Mechanical oscillatory system
2. Strain gauge and converter system
3. USB interface to computer
4. Personal Computer with `Mechanical Oscillator` programme
5. Variable speed motor.

W.J. Simpson

S.M. Tan

C.T. Tindle

M.D. Hoogerland

This version: June 25, 2014