Experiment 318: Flow studies using the electric analogue plotter

Aim

In this experiment the flow around a cylinder and a flat plate in conditions with and without circulation will be measured and compared with theory. From such results lift forces (but not drag) may be calculated. The method can also be applied to complicated shapes not amenable to analysis.

References

1. E.L. Houghton and associates, "Aerodynamics for Engineering Students", 4th edition (Edward Arnold Ltd.).

   This is available in the Advanced Laboratory, however almost any other fluid dynamics text would provide suitable background reading.

2. G.I. Taylor and C.F. Sharman "A Mechanical Method for Solving Problems of Flow in Compressible Fluids", Proc. Roy. Soc. Lond. 121, Series A, 194 (1928).

   A copy of this paper, which is the basis of the method used in this experiment, is in the folder with the apparatus.

Introduction

The application of the electric analogue plotter to fluid flow problems rests on an analogy between the equations of two dimensional irrotational fluid flow and the equations of the flow of electric current in a conducting sheet. This analogy is explained well in Ref. 2.

In this experiment we make use of analogy B of Taylor and Sharman (Ref. 2) so that we can represent the effects of circulation. We also deal only with incompressible flow and treat the thickness t and conductivity σ of the paper as constant.

The equations for hydrodynamic flow are:

$$\mathbf{v} = -\nabla\phi$$

or in two dimensions:

$$\frac{\partial\phi}{\partial x} = -v_x \qquad \frac{\partial\phi}{\partial y} = -v_y \tag{1}$$

$$\frac{\partial\psi}{\partial x} = \rho v_y \qquad \frac{\partial\psi}{\partial y} = -\rho v_x \tag{2}$$

where ϕ is the velocity potential, ψ is the stream function (see Ref. 1 for definition) and ρ is the specific density, which is a dimensionless constant for incompressible flow)

The equations of electric flow may be written in an analogous way:

$$\mathbf{j} = -\sigma\nabla V$$

or in two dimensions:

$$\frac{\partial V}{\partial x} = -\frac{j_x}{\sigma} \qquad \frac{\partial V}{\partial y} = -\frac{j_y}{\sigma} \tag{3}$$

$$\frac{\partial W}{\partial x} = t j_y \qquad \frac{\partial W}{\partial y} = -t j_x \tag{4}$$

where $\mathbf{j}$ is the current density, V is the electric potential, W is the current potential, σ is the conductivity of the sheet and $t = t\,(x, y)$ is the thickness of the conducting sheet.

Analogy B of Taylor and Sharman (Ref. 2) is obtained by associating the hydrodynamic and electric flow quantities in the following way:

$$W \to \phi \quad \text{so that} \quad v_x = -tj_y \quad v_y = -tj_x$$
$$V \to -\psi \quad \text{so that} \quad \frac{j_x}{\sigma} = \rho v_y \quad \frac{j_y}{\sigma} = -\rho v_x$$

These are consistent if $t = 1/(\sigma\rho)$.

For the conducting paper which is reasonably homogeneous, σ is constant. Since t is also constant for the conducting paper, only incompressible flow can be modelled.

It is clear from the above that in this analogy streamlines are represented by equipotential lines on the conducting paper.

The lift force for a body in a fluid flow is defined as the force perpendicular to the undisturbed flow direction. (Drag is the force parallel to the flow direction). The lift may be calculated immediately if a quantity called "circulation" is first determined.

The circulation Γ is defined by:

$$\Gamma = \oint \mathbf{v}\cdot\mathrm{d}\mathbf{s} \tag{5}$$

$\mathbf{v}$ is the velocity vector at the point where the line increment $\mathrm{d}\mathbf{s}$ is taken. The integral is over any closed path surrounding the object. Sections 4.1.2 and 4.1.3 of Ref. 1 (pages 112, 137, and 142 in the 3rd edition) explain circulation and its relation to lift in greater detail. Fig. 3.41 of Ref. 1 (Figs. 5.41 and 5.55 in the 3rd edition) show the typical flow around a cylinder with and without circulation. In this case circulation can be induced by rotating the cylinder resulting in a lift force as is well known from the flight of spinning balls. The flow around certain airfoil shapes may also have circulation even though no physical rotation is present.

The following considerations show that flow with circulation may be modelled by adjusting the current flow out of the model.

Analogy B may be written:

$$\phi = mW \quad \psi = -nV$$

where m and n are constants.

The equations are then consistent if:

$$t = \frac{n}{m\rho\sigma}$$

i.e.,

$$\phi = \frac{nW}{t\rho\sigma}$$

For the two-dimensional case:

$$\mathbf{v} = -\nabla\phi = -\left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y}\right)$$
$$d\mathbf{s} = \hat{\imath}\,\mathrm{d}x + \hat{\jmath}\,\mathrm{d}y$$

So

$$\mathbf{v}\cdot\mathrm{d}\mathbf{s} = -\frac{\partial\phi}{\partial x}\,\mathrm{d}x - \frac{\partial\phi}{\partial y}\,\mathrm{d}y = -\mathrm{d}\phi$$

The circulation Γ then is:

$$\Gamma = \oint \mathbf{v}\cdot\mathrm{d}\mathbf{s} = -\oint \mathrm{d}\phi = -\frac{n}{t\rho\sigma}\oint \mathrm{d}W$$

From equation (4) $\mathrm{d}W$ is given by:

$$\mathrm{d}W = \frac{\partial W}{\partial x}\,\mathrm{d}x + \frac{\partial W}{\partial y}\,\mathrm{d}y = tj_y\,\mathrm{d}x - tj_x\,\mathrm{d}y$$

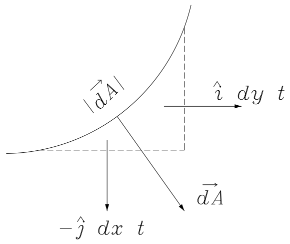

Figure 1: Line element along boundary through which current density $\mathbf{j}$ flows

Now it can be seen from Fig. 1 that:

$$\mathbf{j}\cdot\mathrm{d}\mathbf{A} = (\hat{\imath}\, j_x + \hat{\jmath}\, j_y)\cdot(\hat{\imath}\,\mathrm{d}y - \hat{\jmath}\,\mathrm{d}x)\, t = j_x \mathrm{d}y\, t - j_y \mathrm{d}x\, t = -\mathrm{d}W$$

Thus

$$\Gamma = -\frac{n}{t\rho\sigma}\oint \mathrm{d}W = \frac{n}{t\rho\sigma}\oint \mathbf{j}\cdot\mathrm{d}\mathbf{A} = \frac{n}{t\rho\sigma} I \tag{6}$$

where is the total current flow out of the model.

A uniform "field of flow" is established by two parallel plates of length l and separation L, with a potential difference V_0 (see Fig. 2).

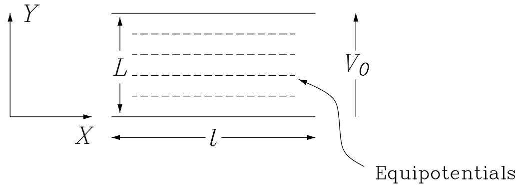

Figure 2: Generating a uniform electric field

The equipotentials are equivalent to streamlines and the current flow is everywhere perpendicular to the equipotentials.

Let U = undisturbed flow velocity. Then for uniform flow the definition of the stream function — see page 155 of Ref. 1 (page 98 in the 3rd edition) — gives:

$$\psi = Uy$$

But $\psi = -nV$ so

$$n = -\frac{Uy}{V}$$

For uniform flow

$$V = \frac{V_0 y}{L}$$

i.e.,

$$n = -\frac{UL}{V_0}$$

Using equation (6) the circulation is:

$$\Gamma = -\frac{UL}{V_0}\,\frac{1}{t\rho\sigma}\, I \tag{7}$$

If the current flow between the parallel electrodes is I_p, then:

$$j_y = \frac{I_p}{lt} = \sigma E_y = \sigma\frac{V_0}{L}$$

i.e.,

$$\sigma = \frac{I_p}{t} \frac{L}{l} \frac{1}{V_0}$$

For incompressible flow $\rho = 1$ and substituting this value for σ into equation (7) gives:

$$\frac{\Gamma}{U} = -\frac{lI}{I_p} \tag{8}$$

where l is the length of the parallel plates, I is the current out of the model and I_p is the current between the parallel plates in the absence of the model.

The current may be induced by biasing the model so that its potential is different from that which it would acquire if "floating" electrically.

Experiment

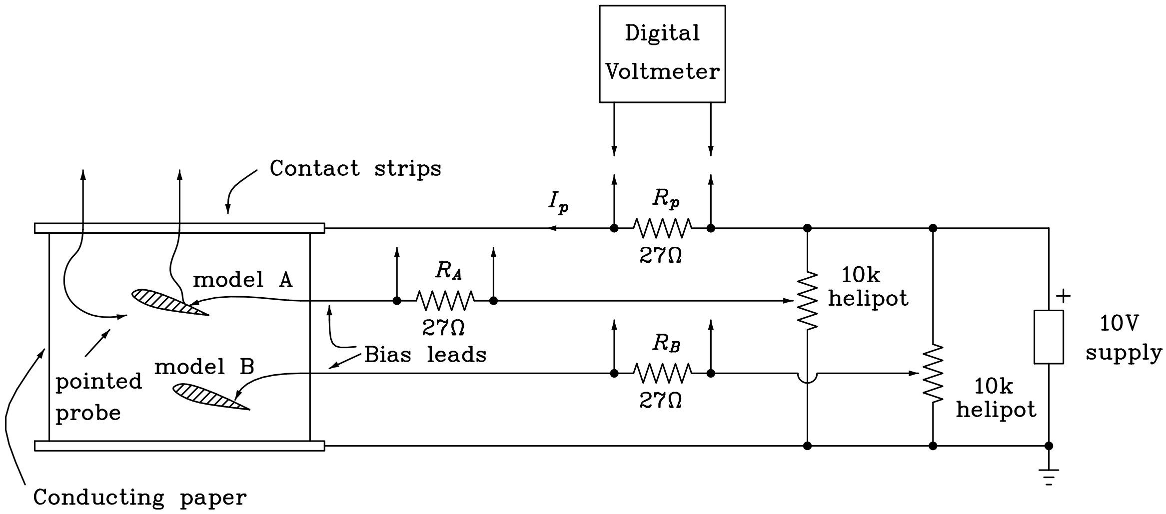

Figure 3: Experimental setup

The experimental set-up is shown in Fig. 3. The control box contains the biasing and monitoring circuits. The position of the rotary switch on top of the control box determines which voltage the digtal voltmeter is reading:

Rotary Switch Position	**Digital Voltmeter Reading**
Probe	between probe and ground
I_p	across R_p
Bias A	across R_A
Bias B	across R_B
Probe A	between probe and model A
Probe B	between probe and model B

The following cases are to be studied experimentally:

A. Flow around a cylinder without circulation.

B. Flow around a cylinder with circulation.

C. Determination of the circulation from line integral evaluation.

D. Flow around a flat plate with circulation as a function of angle of attack.

E. Circulation as a function of angle of incidence for symmetric and cambered air foil sections.

F. Interaction between lifting surfaces ("optional extra").

A. Flow around a cylinder without circulation

In this part of the experiment streamlines (or equipotentials) are to be plotted around a cylinder.

(1) Place a sheet of conducting paper under the edge strips and screw them down firmly to ensure good electrical contact.

(2) Apply 10 V across the paper and check that the 5 V equipotential is parallel to the edge strips within 1 mm over its full length. If not, tighten the edge strips.

(3) Place the cylinder in the centre of the board and apply a weight to it to ensure good electrical contact to the paper. Measure the potential of the paper somewhere near the cylinder while simultaneously pushing down on the cylinder to ensure there is no substantial change and hence a good electrical contact between model and paper.

(4) Plot a set of equipotential lines which are equally spaced in potential. The equipotentials may be plotted by pushing the pointed probe through the paper leaving a hole. It is best to measure the equipotentials with respect to the cylinder. The zero volt line is then the "stagnation line" which separates the flow each side of the cylinder. It always meets the model normal to the surface.

B. Flow around a cylinder with circulation

When there is no circulation the stagnation points are diametrically opposite in the line of the undisturbed flow — see Fig. 3.41 of Ref. 1 (Fig. 5.55 in the 3rd edition).

As the circulation is increased the stagnation points move around the circumference of the cylinder toward each other. When $\Gamma = 4\pi a U$ where a is the radius of the cylinder, the stagnation points coincide.

In this part of the experiment you are required to set the bias to correspond to this condition and compare it with the value calculated from the geometry of the apparatus.

(5) With the model in place but not connected, determine I_p by measuring the voltage drop across R_p.

(6) Connect one of the bias leads to the cylinder. With one terminal of the voltmeter connected to the cylinder, trace around the cylinder close to the surface and locate the stagnation points. Turn the appropriate helipot until the two stagnation points coincide. At this point the circulation is given by $\Gamma = 4\pi a U$ (see Fig. 3.41 of Ref. 1).

(7) Check this condition by comparing the theoretical and measured circulations (actually Γ/U). Measure the voltage across the appropriate 27 Ω resistor to determine I and hence the circulation.

(8) Plot a set of streamlines for this condition.

C. Determination of the circulation from line integral evaluation

If the circulation is known the lift force (which is perpendicular to the drag and the undisturbed flow) can be calculated from the Kutta-Zhukovsky theorem — see Section 4.1.3 of Ref. 1 (page 142 in the 3rd edition) — which gives the lift as:

$$F = \rho' U \Gamma \tag{9}$$

where ρ' is the fluid density.

If the flow field is known the circulation can be determined from a numerical evaluation of the integral:

$$\Gamma = \oint \mathbf{v} \cdot \mathbf{dl}$$

(9) Plot the flow lines in the vicinity of the cylinder with circulation, being sure that they are equally spaced in potential. Adjust the bias so that the condition $\Gamma < 4\pi a U$ results, i.e. there are two stagnation points on the cylinder. Remember that symmetry requires that only half the diagram need be plotted.

We require to determine the quantity:

$$\frac{\Gamma}{U} = \frac{1}{U} \oint \mathbf{v} \cdot \mathbf{dl}$$

and compare it with the result obtained by measuring I and I_p. The form of the flow lines indicates whether Γ/U is greater or less than $4\pi a$. Check this from the cylinder size.

The line integral is estimated from the sum (see Fig. 4):

$$\frac{\Gamma}{U} \approx \sum_{i=1}^{n} \frac{v_i}{U} \cos\theta_i \, \Delta l_i$$

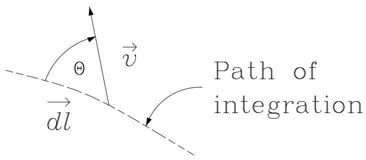

Figure 4: Geometry involved in line integral

Since any path of integration may be chosen it is simplest to arrange that for each element $\theta = 0°$, $90°$ or $180°$, so that $\cos\theta = \pm 1$ or zero.

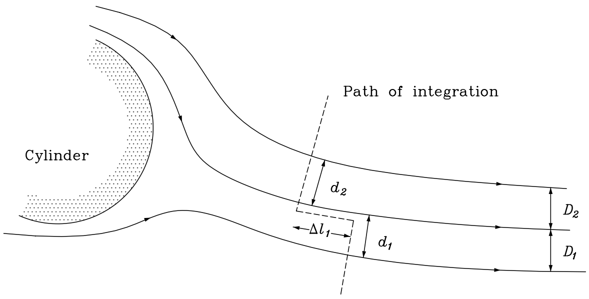

Figure 5: Choosing the path of integration to find Γ

Using the fact that the closer the streamlines, the higher the velocity (see Fig. 5), one can determine the velocity ratio v_i/U as the average of the ratio of the distances between streamlines near the path of integration and streamlines in the region of the undisturbed flow, on either side of the streamline, i.e.

$$\frac{v_i}{U} \approx \frac{1}{2}\left(\frac{D_1}{d_1} + \frac{D_2}{d_2}\right)$$

Note that if equipotentials (streamlines) are plotted at equal voltage increments, then $D_1 = D_2 =$ etc.

D. Flat plate with circulation

The object of this section is to confirm the theoretical result that the circulation and hence the lift is proportional to the angle of attack — see Section 4.4.1 of Ref. 1 (Section 6.4.1 in the 3rd edition).

Flow without circulation around a flat plate at an angle of attack is shown in Fig. 6.

This is not a realistic model of fluid flow since all fluids are viscous and have mass. The large accelerations and viscous forces required for the fluid to follow around the sharp trailing edge to the rear stagnation point cannot be sustained. Instead the rear stagnation point moves to the trailing edge producing a trailing vortex together with equal and opposite circulation (the "bound vortex") around the plate. This condition, with the stagnation flow line coming tangentially from the trailing edge, is known as the Kutta condition and is representative of realistic flow with circulation from all lifting surfaces as in Fig. 7.

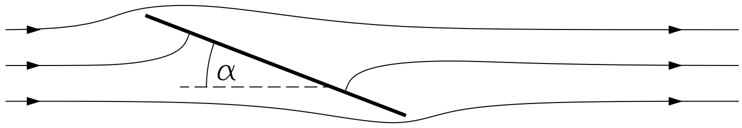

Figure 6: Flow around a flat plate with no circulation

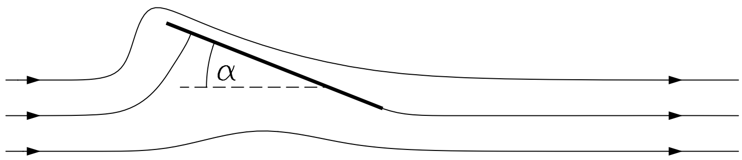

Figure 7: Flow around a flat plate with circulation

Since the stagnation flow line has the same potential as the plate the Kutta condition may be verified by checking that there is no potential difference between the plate and the probe placed just behind it. For a given angle of attack the circulation is adjusted until the Kutta condition is satisfied.

(10) Current is led into the flat plate through one of the bias leads.

Set the plate to a number of angles of attack between 2° and 20°. At each angle adjust the helipot so that the voltage between the plate and a point on the paper a few millimetres behind the trailing edge is zero (the Kutta condition). Measurements at negative angles of attack over the same range should also be made.

(11) Determine I for each value of α by measuring the voltage across the appropriate resistor. I_p should be found from the voltage across the $27\,\Omega$ resistor with the flat plate disconnected and $\alpha = 0$. All these measurements should be made with a high input impedance digital voltmeter whose input is isolated from ground.

For a flat plate at an angle of incidence α, where α is small, the lift per unit span is given by — see pages 234–5 of Ref. 1 (page 215 in the 3rd edition):

$$F = \pi\alpha\rho' U^2 c \tag{10}$$

where c is the length of the flat plate. Combining this with equations (8) and (9) gives:

$$I = -\frac{\pi c I_p}{l}\alpha \tag{11}$$

(12) From your measurements and the dimensions of the apparatus verify that this equation is obeyed.

(13) Plot a set of streamlines for one angle of incidence with the Kutta condition filled.

E. Symmetric and asymmetric airfoil sections with circulation

The theory of thin airfoils is given in Section 4.4.2 of Ref. 1 (page 221 in the 3rd edition) where it is seen that the lift is proportional to Fourier components describing the camber.

(14) Measure the circulation as a function of the angle of incidence for both airfoils and comment on the results. Compare with the data given in the folder with the apparatus. The symmetric section is specified by the number NACA 0012 and the asymmetric by NACA 2412.

F. Interaction between lifting surfaces

(15) This is an "optional extra". Provision is made for imposing the Kutta condition on two models simultaneously. In this way the interaction between wing and tailplane of an aircraft can be studied, or that between keel and rudder or mainsail and foresail of a sailing boat.

Questions

These must be answered as part of the write-up.

1. Prove equation (11).
2. In what important way does this model fail to represent real fluid flow especially for large values of α?
3. Briefly discuss the uses and limitations of this method of studying fluid flow.

List of equipment

1. Board with roll of conducting paper
2. Digital voltmeter
3. Control Box
4. Models: cylinder, airfoils (2), curves plate (2), flat plate and cylinder weight
5. Weighted probe

R. Garrett

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