Experiment 362: L-C filters

Aim

The object of this experiment is to study the characteristics of some simple L-C filters.

Note: You will be measuring the voltage transfer functions of selected filters and presenting the results graphically by means of Nyquist and Bode plots. It is suggested that you plot the results on two-cycle log paper as the readings are taken. You will need a calculator to calculate the gain in decibels and the phase in degrees.

Reference

J.B. Earnshaw, "An Introduction to AC Circuit Theory," MacMillan, Chapter 8.

Theory

Voltage Transfer Function of Filter A

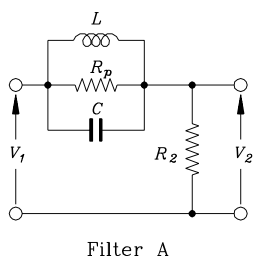

Figure 1: Circuit of L-C filter A

In the circuit of Filter A (see Fig. 1), R_p represents the power loss resistance of the inductor-capacitor parallel combination. The voltage transfer function of Filter A is given by:

$$\mathbf{T}_A = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{R_2 + z_p} \tag{1a}$$

where

$$z_p = R_p \parallel \mathrm{j}\omega L \parallel \frac{1}{\mathrm{j}\omega C} = \frac{R_p}{1 + \mathrm{j}R_p\left(\omega C - \frac{1}{\omega L}\right)} \tag{2}$$

In equation (1a) $|\mathbf{T}_A|$ is minimum when $|z_p|$ is maximum. From equation (2) it can be seen that this occurs when $\omega = \omega_0 = 1/\sqrt{LC}$, i.e. at a frequency $f_0 = 1/\left(2\pi\sqrt{LC}\right)$. This is the resonant frequency of the filter. If we define Q_p as the quality factor of the parallel inductor-capacitor combination, then by definition:

$$Q_p = \frac{R_p}{\omega_0 L} = \omega_0 R_p C \tag{3}$$

The impedance operator z_p may then be expressed in terms of Q_p as follows:

$$z_p = \frac{R_p}{1 + \mathrm{j}Q_p\beta} \tag{4}$$

where

$$\beta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{f}{f_0} - \frac{f_0}{f}$$

The voltage transfer function of Filter A is then given by:

$$\mathbf{T}_A = A\left[\frac{1 + \mathrm{j}Q_p\beta}{1 + \mathrm{j}Q_pA\beta}\right] \tag{5}$$

where

$$A = \left|\mathbf{T}_A\right|_{\mathrm{min}} = \frac{R_2}{R_2 + R_p}$$

Note that $\left|\mathbf{T}_A\right|_{\mathrm{min}}$ occurs at resonance. Hence A is the value of $\left|\mathbf{T}_A\right|$ at resonance. If we let x be the real part of $\mathbf{T}_A$ and y be the imaginary part of $\mathbf{T}_A$ then x and y can be shown to be given by:

$$x = \mathrm{Re}\left(\mathbf{T}_A\right) = A\left[\frac{1 + Q_p^2 A\beta^2}{1 + Q_p^2 A^2\beta^2}\right] \quad y = \mathrm{Im}\left(\mathbf{T}_A\right) = A\left[\frac{(1-A)\,Q_p\beta}{1 + Q_p^2 A^2\beta^2}\right]$$

The locus of $\mathbf{T}_A$ in the complex plane as the frequency is varied (i.e. y-versus-x) is called the Nyquist plot. The equation for the Nyquist plot can be derived by eliminating the frequency-dependent variable β from the equations for x and y with the result:

$$y^2 = (x - A)(1 - x) \tag{6}$$

Equation (6) can be shown to be the equation of a circle whose centre is $((1+A)/2, 0)$ and whose radius is $(1-A)/2$ [seeFig.2(a)]. The length of OP is equal to $\left|\mathbf{T}_A\right|$, the voltage gain of the filter, and ϕ is the phase angle between the output and the input voltage. It can be seen that as f increases from zero to infinity, the length of OP decreases from unity to a minimum value A at $f = f_0$ and then increases to 1 at $f = \infty$ [see Fig.2(a) and Fig. 2(c)]. Correspondingly ϕ decreases from zero to a minimum value $-\sin^{-1}\left[(1-A)/(1+A)\right]$ at $f = f_{\mathrm{min}}$. It then increases through zero at $f = f_0$ to a maximum value $\sin^{-1}\left[(1-A)/(1+A)\right]$ at $f = f_{\mathrm{max}}$ before it decreases to zero at $f = \infty$.

From equation (5) the voltage gain is given by:

$$\left|\mathbf{T}_A\right| = A\left[\frac{1 + Q_p^2\beta^2}{1 + Q_p^2 A^2\beta^2}\right]^{1/2} \tag{7}$$

The graph of -versus- can be shown to exhibit even symmetry about $f = f_0$ when a log-scale is used for frequency f [see Fig. 2(c)]. This is done by first showing that for two frequencies f_j and f_k at which $\left|\mathbf{T}_A\right|$ has the same value, $f_0/f_j = f_k/f_0$. Consequently $\log f_0 - \log f_j = \log f_k - \log f_0$ and corresponding displacements along the f-axis are equal.

The frequencies f_1 and f_2 at which $\left|\mathbf{T}_A\right| = \sqrt{2}A$ are found by solving the equation:

$$\sqrt{2}\,A = A\left[\frac{1 + Q_p^2\beta^2}{1 + Q_p^2 A^2\beta^2}\right]^{1/2}$$

so that

$$\beta = \frac{f}{f_0} - \frac{f_0}{f} = \pm\frac{1}{Q_p\sqrt{1 - 2A^2}} \tag{8}$$

The difference between the only two positive solutions of the two quadratic equations in f is:

$$f_2 - f_1 = \frac{f_0}{Q_p\sqrt{1 - 2A^2}}$$

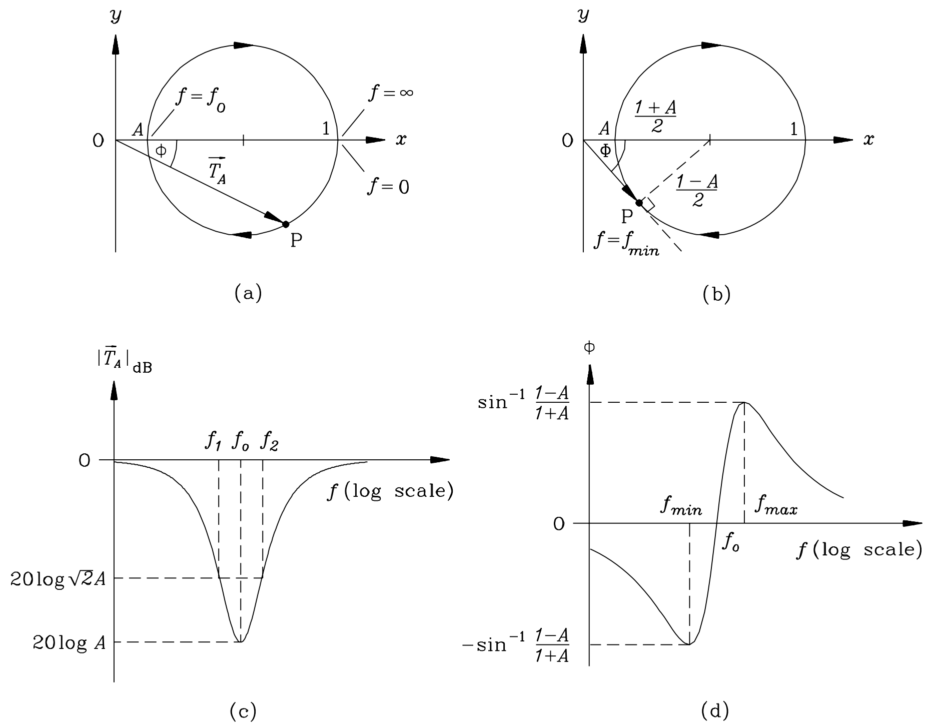

Figure 2: Nyquist and Bode plots for filter A

and hence

$$Q_p = \frac{f_0}{f_2 - f_1} \frac{1}{\sqrt{1 - 2A^2}} \tag{9}$$

We define the quality factor Q_A for the **filter** A (which is different from the quality factor Q_p of the parallel inductor-capacitor combination) to be

$$Q_A \equiv \frac{f_0}{f_2 - f_1} = Q_p \sqrt{1 - 2A^2} \tag{10}$$

Note that $Q_A \to Q_p$ as $A \to 0$, and this follows logically from the fact that R_2 effectively "monitors" the current through the parallel circuit and as $R_2 \to 0$ this current is determined solely by $|z_p|$.

The sign of ϕ is determined by the sign of β, and since the absolute value of ϕ is the same for frequencies f_j and f_k where $f_j/f_0 = f_0/f_k$, it follows that the graph of ϕ-versus-frequency [seeFig.2(d)] exhibits odd symmetry about when a log-scale is used for f. The two frequencies $f_{\min}$ and $f_{\max}$ are the frequencies at which the vector $\mathbf{T}_A$ is tangential to the Nyquist plot [see Fig. 2(b)]. From the geometry of the Nyquist plot, OP can be shown to be equal to $\sqrt{A}$ at $f = f_{\min}$ and $f = f_{\max}$. The frequencies at which $|\mathbf{T}_A| = \sqrt{A}$ are found by solving the equation:

$$\sqrt{A} = A \left[\frac{1 + Q_p^2 \beta^2}{1 + Q_p^2 A^2 \beta^2} \right]^{1/2}$$

so that

$$\beta = \frac{f}{f_0} - \frac{f_0}{f} = \pm \frac{1}{Q_p \sqrt{A}} \tag{11}$$

Equation (11) has only two positive solutions, $f_{\min}$ and $f_{\max}$ which correspond to minimum and maximum values of ϕ respectively. Solving for $f_{\min}$ and $f_{\max}$ produces the following results:

$$f_{\max} - f_{\min} = \frac{f_0}{Q_p \sqrt{A}} \quad \text{and} \quad f_{\min} f_{\max} = f_0^2 \tag{12}$$

Voltage Transfer Function of Filter B

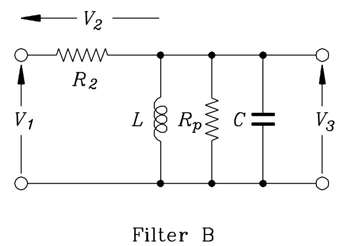

Figure 3: Circuit of L-C filter B

The voltage transfer function of Filter B (see Fig. 3) is given by:

$$\mathbf{T}_B = \frac{\mathbf{V}_3}{\mathbf{V}_1} = \frac{z_p}{R_2 + z_p} = \frac{1 - A}{1 + \mathrm{j}Q_p A\beta} \tag{13a}$$

The voltage transfer function of Filter B may be expressed in terms of that of Filter A as follows:

$$\mathbf{V}_1 = \mathbf{V}_2 + \mathbf{V}_3 \quad \text{hence} \quad \mathbf{T}_B = \frac{\mathbf{V}_3}{\mathbf{V}_1} = \frac{\mathbf{V}_1 - \mathbf{V}_2}{\mathbf{V}_1} = 1 - \frac{\mathbf{V}_2}{\mathbf{V}_1} = 1 - \mathbf{T}_A \tag{14}$$

Equation (14) suggests that the Nyquist plot for Filter A can be used to derive the Nyquist plot for FilterB. To get the amplitude vector $\mathbf{T}_B$ we add $-\mathbf{T}_A$ to $\mathbf{1}$ vectorially [see Fig. 4(a)]. Fig. 4(b) shows the amplitude vector $\mathbf{T}_B$ at the 3dB-frequencies $f = f_1$ and $f = f_2$. The Bode plots, $|\mathbf{T}_B|_{\mathrm{dB}}$-versus-$f$ (log scale) and ϕ-versus-f (log scale), shown in Fig. 4(c) and Fig. 4(d) can be deduced from the Nyquist plot for Filter B.

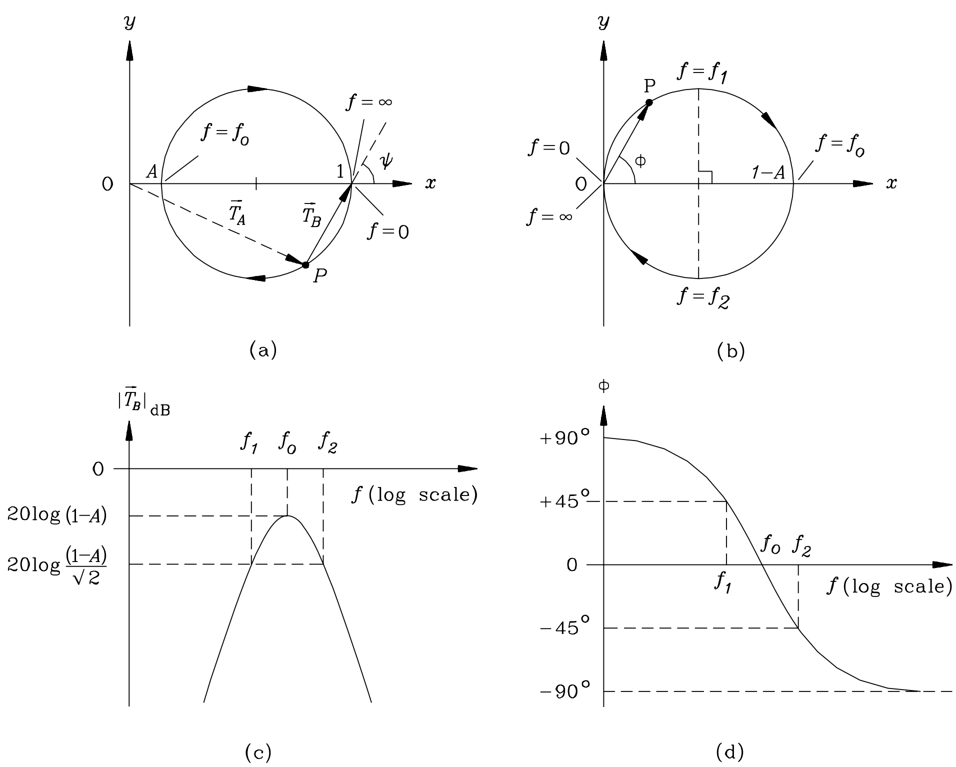

Figure 4: Nyquist and Bode plots for filter B

The graph of $|\mathbf{T}_B|_{\mathrm{dB}}$-versus-$f$ (log scale) exhibits even symmetry about f_0 when a log scale is used for f. The maximum value of $|\mathbf{T}_B|$ is $(1 - A)$ when $f = f_0$ and falls to its half-power value $(1 - A)/\sqrt{2}$ at the frequencies f_1 and f_2 where:

$$f_2 - f_1 = \frac{f_0}{Q_p A} \tag{15}$$

so the quality factor of Filter B is given by:

$$Q_B = \frac{f_0}{f_2 - f_1} = Q_p A \tag{16}$$

The graph of ϕ-versus-f decreases monotonically as f increases and exhibits odd symmetry about f_0 on a log scale for f.

For both Filter A and Filter B the quality factor of the inductor-capacitor combination sets the upper limit of selectivity. For Filter A: $Q_A \to Q_p$ as $R_2 \to 0$, wehereas for Filter B: $Q_B \to Q_p$ as $R_2 \to \infty$.

Voltage Transfer Function of Filter C

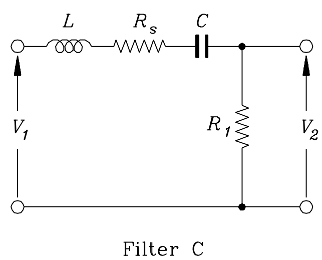

Figure 5: Circuit for L-C filter C

In the circuit of Filter C (see Fig. 5), R_s represents the power loss resistance of the inductor-capacitor series combination. The voltage transfer function of Filter C is given by:

$$\mathbf{T}_C = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_1}{R_1 + z_s} \tag{17}$$

where

$$z_s = R_s \left[1 + \mathrm{j}\left(\frac{\omega L}{R_s} - \frac{1}{\omega R_s C}\right)\right] \tag{18}$$

In equation (17) $|\mathbf{T}_C|$ is maximum when z_s is minimum. From equation (18) it can be seen that this occurs when $\omega = \omega_0 = 1/\sqrt{LC}$, i.e. at a frequency $f_0 = 1/\left(2\pi\sqrt{LC}\right)$. This is the resonant frequency of the filter. If we define Q_s to be the quality factor of the series inductor-capacitor combination then, by definition,

$$Q_s = \frac{\omega_0 L}{R_s} = \frac{1}{\omega_0 R_s C} \tag{19}$$

The voltage transfer function of Filter C in terms of Q_s is then given by:

$$\mathbf{T}_C = \frac{R_1}{R_1 + R_s\left(1 + \mathrm{j}Q_s\beta\right)} = \frac{1 - A_s}{1 + \mathrm{j}Q_s A_s \beta} \tag{20}$$

where

$$A_s = \frac{R_s}{R_1 + R_s} \tag{21}$$

Equation (20) has exactly the same form as equation (13a), so that by replacing A by A_s and Q_p by Q_s we get:

$$\mathbf{T}_C \equiv \mathbf{T}_B \tag{22}$$

The quality factor Q_C of Filter C is given by:

$$Q_C = \frac{\omega_0 L}{R_1 + R_s} = \frac{1}{\omega_0\left(R_1 + R_s\right) C} = Q_s A_s \tag{23}$$

Voltage Transfer Function of Filter D

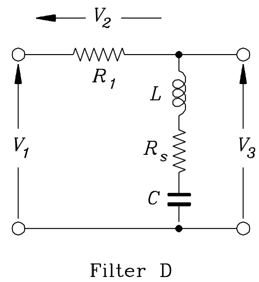

Figure 6: Circuit of L-C filter D

The voltage transfer function of Filter D (see Fig. 6) may be derived from that of Filter C as follows:

$$\mathbf{T}_D = \frac{\mathbf{V}_s}{\mathbf{V}_1} = \frac{\mathbf{V}_1 - \mathbf{V}_2}{\mathbf{V}_1} = 1 - \frac{\mathbf{V}_2}{\mathbf{V}_1} = 1 - \mathbf{T}_C = A_s \left[\frac{1 + \mathrm{j}Q_s\beta}{1 + \mathrm{j}Q_sA_s\beta} \right] \tag{24}$$

Equation (24) has exactly the same form as equation (5), so that by replacing A by A_s and Q_p by Q_s we get:

$$\mathbf{T}_D = \mathbf{T}_A \tag{25}$$

Like Filter A the quality factor Q_D for Filter D can be shown to be given by:

$$Q_D = Q_s\sqrt{1 - 2A_s^2} \tag{26}$$

Experimental Set-up

The experimental set-up is shown in Fig.7. Amplitude and phase values can be measured with the aid of the vertical and horizontal cursors of the oscilloscope. To minimise loading of the filter under study it is suggested that you use oscilloscope probes, set to 10× attenuation, for your measurements. Note that the signal from the function generator has a small DC offset (even when the DC offset button is off). Therefore use AC coupling when observing filter input and output waveforms. This is particularly important when measuring the phase of the filter. The frequency display on the function generator may not be reliable. Read the signal frequency off the oscilloscope display.

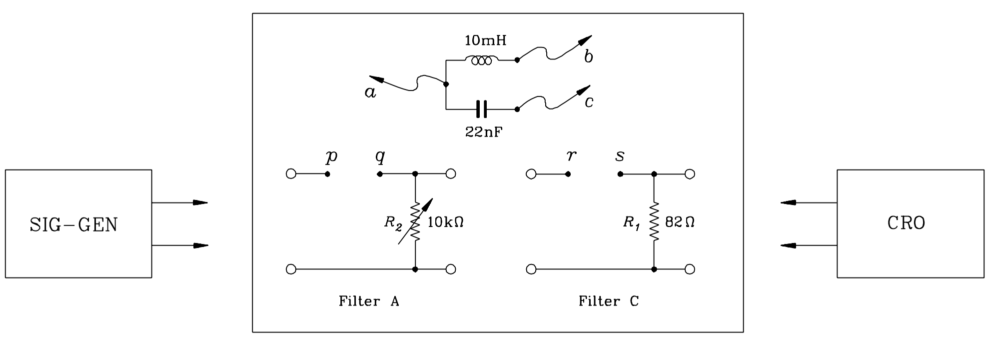

Figure 7: Experimental set-up

Measurements on Filter A

(1) Set up Filter A by connecting alligator clip a to terminal p and alligator clips b and c to terminal q (see Fig. 7). The resistance of Filter A circuit (see Fig. 1) is the parallel loss resistance of the inductor (the capacitor has negligible loss resistance).

Adjust R_2 to give a value of $A = \frac{1}{4}$ (i.e. $|A|_{\mathrm{dB}} = -12\,\mathrm{dB}$) at the resonant frequency of the filter circuit. A convenient way of doing this is to take advantage of the fact that at resonance the input and the output voltages are in phase. Using a $4\times$ sensitivity difference between the input and output waveform displays, adjust the frequency and the variable resistance until the input and output waveforms superimposed on each other. Note down the resonant frequency f_0.

(2) Measure $|\mathbf{T}_A|$ and ϕ over a range of frequencies between 1 kHz and 100 kHz, and record the results in tabular form. You will need a calculator to determine the gain in decibels and the phase in degrees. A convenient peak-to-peak amplitude for V_1 is 10 V. Then one-tenth the numerical value of the peak-to-peak voltage of is the gain of the circuit. For each frequency re-adjust V_1 to 10 V peak-to-peak. Plot $|\mathbf{T}_A|_{\mathrm{dB}}$-versus-$f$ and ϕ-versus-f on two-cycle log paper as the readings are taken so that appropriate selection of frequencies can be made.

(3) Draw the theoretical Nyquist plot for Filter A, and plot the experimental points $(|\mathbf{T}_A|, \phi)$ around it as the readings are taken. This will act as a check that the correct measurements are being made as well as ensure suitable selection of frequencies. For your report you may use MATLAB to plot all your curves from readings you have recorded.

(4) Measure R_2 using the Laboratory's RCL meter and hence find R_p. From the $|\mathbf{T}_A|_{\mathrm{dB}}$-versus-$f$ (log scale) experimental graph find f_1 and f_2 and determine.Q_p. Calculate the values of L, C and Q_A.

(5) From the ϕ-versus-f (log scale) experimental graph find $f_{\min}$ and $f_{\max}$ and compare the value of $f_{\max} - f_{\min}$ with that of $f_0/\left(Q_p\sqrt{A}\right)$.

Measurements on Filter B

(6) From the points in the Nyquist plot for Filter A, derive appropriate points for the Nyquist plot for Filter B, and draw the graphs for $|\mathbf{T}_B|_{\mathrm{dB}}$-versus-$f$ and ϕ-versus-f on two-cycle log paper. Determine f_1 and f_2 and calculate Q_B.

Measurements on Filter C

(7) Set up Filter C by connecting alligator clip b to terminal r and alligator clip c to terminal s (see Fig.7). Alligator clip a should be left free. Repeat procedures (2) to (4) for Filter C.

Measurements on Filter D

(8) From the points in the Nyquist plot for Filter C, derive appropriate points for the Nyquist plot for Filter D, and draw the graphs for $|\mathbf{T}_D|_{\mathrm{dB}}$-versus-$f$ and ϕ-versus-f on two-cycle log paper. Determine f_1 and f_2 and calculate Q_D.

List of Equipment

1. Circuit Board Housing with Circuit Board
2. Noytronix Model 300 MSTPC/02 Function Generator
3. Hitachi Model V6465 Oscilloscope with probes

J.B. Earnshaw

Z.C. Tan

May 1999