Experiment 211: Binomial Distribution and χ^2 Test

Aim

To experimentally obtain a frequency distribution and to check that it is a sample from the expected population distribution.

Theory

When N two-sided objects (e.g. coins, tiddlywinks, etc.) are tossed n times, the number of times that x "heads" is expected to appear is given by:

$$E(x) = n \times \text{Population Probability} = n \times p(x) = n \times \frac{N!}{(N-x)!x!} p^x q^{N-x} \tag{1}$$

where $p =$ "head" probability $= 1/2$ and $q =$ "tail" probability $= (1-p) = 1/2$

$E(x)$ is known as the frequency distribution and in the case of two-sided objects, $E(x)$ is also known as a binomial distribution.

In the case of 200 tosses of 4 two-sided tiddlywinks, we have $N = 4, n = 200$ and $x = 0, 1, 2, 3, 4$. Hence

$$E(x) = 200 \times \frac{4!}{(4-x)!x!} \left(\frac{1}{2}\right)^4 = \frac{300}{(4-x)!x!} \tag{2}$$

If $f(x)$ is the actual number of times that x "heads" appear and m is the mean of the sample and s is the standard deviation of the sample, then

$$m = \sum_{x=0}^{N} \frac{f(x)\,x}{n} \qquad \text{and} \qquad s = \sqrt{\sum_{x=0}^{N} \frac{f(x)\,(x-m)^2}{n}} \tag{3}$$

If μ and σ are the mean and standard deviation of the population, the best estimate of μ is m. The best estimate of σ is not s, but is $\sqrt{n/(n-1)}s$, because μ is not known exactly. So

$$\mu = m \qquad \text{and} \qquad \sigma = \sqrt{\sum_{x=0}^{N} \frac{f(x)\,(x-m)^2}{n-1}} \tag{4}$$

Finally, σ_m, the standard error in the sample mean, gives a measure of the uncertainty in the estimation of μ. It is given by:

$$\sigma_m = \frac{\sigma}{\sqrt{n}} \tag{5}$$

Procedure

(1) Toss a set of 4 two-sided objects (coins, tiddlywinks, etc) 200 times, counting and recording the number of "heads" obtained each time.

(2) Draw a histogram (See Figure 1) of the frequency distribution you obtain. Plot this, at least roughly, as you do the experiment.

(3) Using a pen of a different colour, draw in the same diagram a histogram of the expected (theoretical) frequency distribution, $E(x)$ using equation (2).

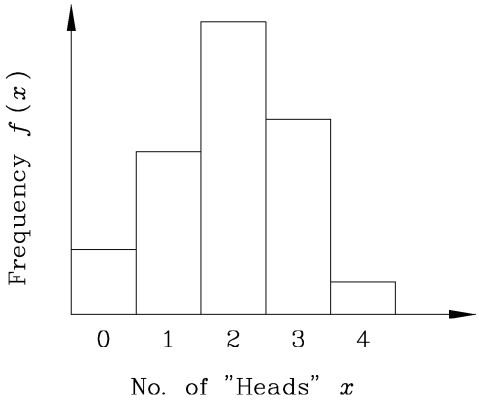

Figure 1:

(4) Calculate the mean m of the sample.

(5) Calculate σ, the best estimate of the standard deviation of the population, using equation (4).

(6) Calculate the standard error σ_m in the sample mean using equation (5).

(7) The population mean and standard deviation are, by definition:

$$\mu = \sum_{x=0}^{N} p(x)\, x \qquad \text{and} \qquad \sigma = \sqrt{\sum_{x=0}^{N} p(x)\,(x-\mu)^2} \tag{6}$$

where $p(x)$ is the probability of x occurring. Calculate these quantities.

(8) For the binomial distribution as defined in equation (1), these quantities are given by:

$$\mu = Np \qquad \text{and} \qquad \sigma = \sqrt{Np\,(1-p)} \tag{7}$$

Check that your calculation in procedure (7) above agrees with these. Prove these relations.

Hint: To prove the μ expression of equation (7) insert the expression for $p(x)$ as given in equation(1) into the μ expression of equation (6). Now try to make it mostly look like $\sum p(x)$, which of course is equal to 1. To prove the σ expression of equation (7), expand the σ expression of equation (6) and show that $\sigma^2 = \sum p(x)\, x^2 - \mu^2$. Now use the same approach again as for the proof of the μ expression of equation (7). You may find it useful to write x^2 as $x(x-1)+x$. There are more elegant approaches as given in many statistics texts.

(9) Compare the sample mean with the population mean. Are they equal within the variations expected experimentally?
Compare the estimate of the population standard deviation with the value calculated in procedures (7) and (8) above.

(10) Compare the sample distribution with the binomial distribution by using the χ^2 test. For your reference an example of a χ^2 calculation for a sample set of data is given in the Appendix.

Does the χ^2 test indicate that the experimental results support the theory? See Appendix B of Experiment 212 "Central Limit Theorem" for a full discussion.

APPENDIX

χ^2 Test Example

No. of "heads"	Observed frequency i.e. Experimental sample	Expected frequency $200 \times$ (Poulation probability)	
x	$O(x)$	$E(x)$	$\frac{\lvert O(x)-E(x)\rvert^2}{E(x)}$
0	18	$200 \times 1/16 = 12.5$	2.4
1	58	$200 \times 4/16 = 50$	1.3
2	68	$200 \times 6/16 = 75$	0.7
3	44	$200 \times 4/16 = 50$	0.7
4	12	$200 \times 1/16 = 12.5$	0.0
	$\sum_{x=0}^{4} O(x) = 200$	$\sum_{x=0}^{4} E(x) = 200$	$\chi^2 = 5.1$

Note that $O(x)$ is the same as $f(x)$ in equation (3). In this example:

$$\chi^2 = \sum_{x=0}^{4} \frac{\lvert O(x) - E(x)\rvert^2}{E(x)} = 5.1$$

There are five observed frequencies, and grouping is not necessary because all the $E(x)$ are greater than 5. One degree of freedom is lost since the total number (200) of events has been made the same for both observed and expected results. Hence χ^2 has 4 degrees of freedom.

The probability of χ^2 being greater than this value	0.3	0.71	1.06	3.36	7.78	9.49	13.28
is	99%	95%	90%	50%	10%	5%	1%

If your χ^2 value lies outside the region 0.71 to 9.49 it can mean either:

(a) Your experiment was one of the 10% which would, on statistical averages, be expected to lie outside this region or

(b) Your experiment did not satisfy the theoretical assumptions; that each toss was independent of the previous one and that "heads" and "tails" were equally likely.

Note that if $\chi^2 > 9.49$, this means that your data differ from the theory by so much that we would only expect such a large deviation (if theory were correct) 5% of the time. On the other hand if $\chi^2 < 0.71$, this means that your data fit the theory very closely indeed. This may still be disturbing, since we expect there to be some deviation, and expect to get a deviation as small as that only 5% of the time.

If your χ^2 value lies outside the region 0.30 to 13.28 it can mean either:

(a) Your experiment was one of the 2% which would, on statistical averages, be expected to lie outside this region or

(b) Your experiment did not satisfy the theoretical assumptions.

It is seen that the smaller the percentage in point (a) becomes, the more likely it is that point (b) will apply — if χ^2 is too large, perhaps the discs were not shaken well enough before throwing, or they were bent, or one side was slightly sticky. On the other hand if χ^2 is too small and the fit is too good, one might question if the experimenter was perhaps reluctant to record results which disagreed with the theory.

List of Equipment

1. 1 x Plastic Cup (shaker)
2. 4 x Plastic Tiddlywinks

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