Experiment 326: Gaussian Beam Optics (The ABCD law)

Aim

To investigate quantitatively the propagation of Gaussian beams through simple optical systems, and compare the results to the theory.

Note: This experiment is appropriate for students who are doing PHYSICS 326, and it helps (but it is not essential) if you have done the PYTHON experiment (213). This experiment deals with the fact that laser beams do not have plane wavefronts but a Gaussian intensity distribution. You will learn more about this in PHYSICS 326 but this hand-out contains all the important information.

References

1. A. Yariv, "Optical Electronics" 4th Edition, Saunders College Publ.
2. B.E.A. Saleh, M.C. Teich: "Fundamentals of Photonics" Wiley-Interscience

Introduction

In geometrical optics, a paraxial beam at a certain position can be described by a vector $\mathbf{x}$ where the first element is given by x the distance of the beam from the optical axis while the second element is given by the slope x' of the beam in respect to the optical axis. The propagation of the beam from an interface (1) to an interface (2) is governed by the equation:

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

where the elements of the $ABCD$-matrix depend on the optical elements in between the two interfaces. Propagating through n optical elements one can use matrix multiplication to determine the matrix of the total system:

$$\begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \times \cdots \times \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \times \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

The matrices for the three simplest optical elements are given below:

straight section of length d and constant refractive index n: $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

dielectric interface ($n_1 \to n_2$): $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$

thin lens of focal length f: $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$

Geometrical optics is only a good approximation if you neglect diffraction and assume infinite intensity-independent wavefronts, e.g. plane waves or spherical waves. In the PHYSICS 326 course, you will learn that the intensity of a laser beam has a Gaussian shape (some of the theory is included in the appendix). At any optical interface along the z-axis, a Gaussian beam can be characterised by ω and R where ω is the half width of the beam profile and is defined as the radius where the intensity has dropped to $1/\mathrm{e}^2$ of its value on the axis, and R is the radius of curvature of the circle formed by the wavefront of the beam. The wavefront is always perpendicular to the direction of propagation. Fig. 1 shows the form of a propagating Gaussian beam.

It is emphsized that:

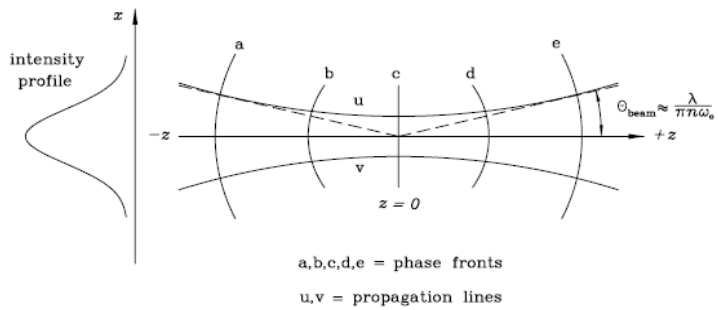

Figure 1: A propagating Gaussian beam

- Both ω and R vary with z (even for a beam propagating through empty space).
- ω is the physical width of the beam and is often referred to as "the beam radius" or "spot size".
- ω is the quantity that is measureable in this experiment.
- R is also a radius (don't get confused!) but is not directly measureable in this experiment.

Make sure you understand ω and R before going further.

ω and R are real numbers. They are combined into one complex parameter called the q-parameter:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \mathrm{j}\frac{\lambda}{\pi n \omega^2(z)}$$

where λ is the wavelength of the laser ($\lambda_{\mathrm{HeNe}} = 632.8\,\mathrm{nm}$) and n is the refractive index of the medium.

The propagation of a Gaussian beam can now be described by the evolution of the q-parameter. Propagating from an interface (1) to an interface (2), the equation for q is given by:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

where A, B, C and D are the same elements as given in the ray matrices. This equation is also called the $ABCD$ law. One can even use the same matrix formalism to determine the q-parameter if propagating through more than one optical element, e.g. for two elements:

$$q_3 = \frac{A_T q_1 + B_T}{C_T q_1 + D_T}$$

where the matrix elements for the total system are given by:

$$\begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \times \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

On the other hand, one can determine the curvature of the wavefront by measuring the spot size at several different positions of a straight section and using the $ABCD$ law. Knowing the q-parameter in front and behind a thin lens allows us to calculate the focal length of this lens. Contrary to geometrical optics the spot size of a Gaussian beam will always be finite. The minimum value of the spot size is called the beam waist, and is denoted by w_0. At this position ($z = 0$ in Fig. 1) the wavefront will be rectilinear, i.e. $R = \infty$.

If the position and the radius ω_0 of a beam waist are known one can calculate the radius of curvature $R(z)$ and the beam radius $w(z)$ at any distance from the beam waist (if there are no optical elements in between) using:

$$R(z) = z\left(1 + \frac{z_0^2}{z^2}\right)$$

$$\omega^2(z) = \omega_0^2\left(1 + \frac{z^2}{z_0^2}\right)$$

where z_0 is called the Rayleigh range and is defined by:

$$z_0 \equiv \frac{\pi n \omega_0^2}{\lambda}$$

In the far-field approximation ($z \gg z_0$) these equations reduce to:

$$R = z$$
$$\omega = \text{constant} \times z$$

The second equation allows finding the position of the beam waist by extrapolation using two beam radii which have been measured at distances $z \gg z_0$. With this result the radius of curvature R can be calculated for the positions where the beam radius was measured. Once ω and R are known for a certain position it is easy to calculate the q-parameter.

You will find some more about the theory in the appendix or in Chapter 2 of Ref 1. This book also contains some worked examples, and is the textbook for PHYSICS 326.

Equipment

The Gaussian beam source is a helium neon laser. The beam height and angle are adjusted using two mirrors, so that the beam is at the height of the detector and parallel to the optical table. The detector is a photo-diode which produces a current proportional to the intensity of light falling on it. The photo-diode is mounted behind a pinhole aperture, on a translation stage. For the different measurements you might have to use different pinholes. The whole detector head (including the translation stage) can be rotated by 90° to enable you to take horizontal and vertical measurements.

An HP PC computer is used to both move the translation stage and read the detector's photo-current level, enabling the beam's intensity profile to be scanned and stored in the computer for later analysis. The program doing this is called '**Stepper**'. Here is a summary of using the program — skip over this now but refer back to it throughout the experiment:

- The two outputs '**Mean voltage**' and '**Standard deviation**' provide means to measure the average background and the background noise. To get the noise to an acceptable level you have to average the signal by choosing the right number for the box '**Number of averages**'.
- To be able to see the wings of the Gaussian shape over several orders of magnitude you have to subtract the average background from the reading of the detector. This is done by putting the correct background level in the box named '**Subtract background**'.
- The step size of the stepping motor is fixed. You will have to calculate this number and put it in the box named '**μm per step**'. This number will be used by the computer to calculate how many steps it has travelled for each measurement.
- The distance between two measurement points you can choose by setting the box called '**Sample spacing (# steps)**'. Choose this spacing according to the aperture you are using. The sample spacing in terms of μm is calculated automatically based on the value entered for '**μm per step**' and is shown in the box named '**Sample spacing (μm)**'.
- Finally you have to put a number in the box '**Number of points**'. The sample spacing (μm) multiplied by this number gives you the total distance the detector will be moved by the stepping motor.

(1) **Getting started**

Turn on the laser, tilt the detector head so that the translation stage moves horizontally, and ensure that the laser beam is parallel to the table at the height of the photo-detector by adjusting both mirrors.

Turn on the computer and start the data acquisition program entitled '**Stepper**'.

Turn on the stepper motor power supply, located near the bottom of the magnetic base, and ensure that the data transfer cables are connected.

Question: Given that the drive screw on the translation stage has 40 turns per inch, and the stepping motor turns 1.8 degrees per step, what is the distance travelled by the translation stage for 1 step?

Enter this value into the box entitled '**μm per step**' in the data acquisition program.

(2) **Ensuring low noise photo-current measurements:**

The photo-diode is subject to several random processes which introduce noise on the photo-current, limiting the minimum light intensity which can be measured. In this initial part of the experiment you will improve the counting statistics by averaging the photo-current several times at each position.

Set the '**Number of points**' to 100 and the '**Number of averages**' to 1. This means the detector will move to 100 positions and, at each position, measure the intensity just once (if '**Number of averages**' was 5 the detector would measure the intensity at each position five times and return for that position the average of the five measurements). **Without** any laser light falling on the detector, set the acquisition program running by clicking '**Run**'.

Record the mean voltage, and the standard deviation of the noise waveform (you might want to do this measurement with and without the room lights turned on to see whether you are able to do the experiment with the lights on).

Make a table in your laboratory book, and record these statistics for increasing the number of averages in steps of 4.

Important: Make sure you change the scan direction after each scan has finished.

Determine, by plotting a graph of number of averages versus standard deviation in voltage, the number of averages required to reduce the noise to below 100 μvolts.

Do one more scan, with that number of averages, and record the mean voltage level. Enter this value into the box labelled '**Subtract background**'. If this scan takes more than 5 minutes discuss it with your demonstrator. Be aware that you might have to repeat this procedure for each aperture!

(3) **Saturation and dynamic range:**

The AD-converter will saturate at a signal level of about 8 volts. Therefore make sure that your signal level always stays below this value (you might have to change the aperture in front of the detector to achieve this). On the other hand, you get the highest dynamic range if your signal is large compared with the noise. Therefore you cannot always use the smallest aperture! Adjust the '**Sample spacing**' in the program accordingly!

(4) **Recording beam profiles:**

With the beam falling on the pinhole, maximise the photo-current level using the '**Power meter**' button by making minor adjustments to the mirrors. Make sure that the translation stage is in the middle of its scan range when the beam falls onto the aperture (otherwise the translation stage might hit one of the micro-switches giving you strange beam profiles). Record the beam profile using the '**Stepper**' acquisition program. Use the graph to determine the beam radii (on both sides!) at $1/e^2$ of the maximum intensity, and write it down in your lab book. You may zoom in the graph by highlighting the area (from top left to bottom right) you wish to view. To zoom out, reverse the zoom in process, i.e. highlight any area on the graph from bottom right to top left.

Hint: You might want to switch from a linear to a logarithmic graph to determine the beam radius. Ask your demonstrator how to do it.

(5) **Moving the motor manually**

The motor can be manually moved using the '**Move Motor**' button. Once the button is clicked, use the keyboard **UP** and **DOWN** keys to move the motor continuously, and **Esc** to return to the main program. Alternatively, you can perform a dummy scan with both '**Number of points**' and '**Number of averages**' set to 1, and '**Sample spacing (# steps)**' set to the number of steps you wish the motor to move.

(6) **Getting consistent measurements:**

To see whether the measurements you have made are consistent, you have to measure the beam radius at each point **horizontally** and **vertically**! The detector head is so designed that you can rotate it by 90°. If your measurements are consistent, you are expected to do the calculations only once!

Experiment

(7) Determine the position of the beam waist and its radius ω_0 by measuring ω at at least two different positions (approximately 0.1 m and 0.5 m behind the last mirror). To work out the position of the beam waist use the far-field divergence angle of the laser which is given by (see Fig. 1):

$$\theta = \frac{\lambda}{\pi n \omega_0}$$

Using two of your measurements of the beam radius, you can now calculate θ and hence the location and the size of the beam waist. Furthermore, the far-field approximation allows you to set R equal to z for $z \gg z_0$.

(8) Put the concave lens into the beam (about 0.1 m away from the last mirror) and measure the beam radius at 4 different positions after the lens (choose at least 2 of these positions so you can use the far-field approximation). From these measurements calculate the q-parameter at the second interface of this lens and the focal length of this lens (using the q-parameter at the first interface of the lens).

(9) Now place the convex lens approximately 10 cm behind the concave lens so that there is a beam waist about 50 cm behind the lens system. Write down the distance between the 2 lenses!

Question: What is meant by 'the distance between the 2 lenses'?

Measure ω at at least 10 positions behind the lens system. Make sure that you take measurements in front of, at, and behind the focal point. To do this properly you might have to change the aperture in front of the detector at some stage. Save some of these measurements on a disk (choose '**Save**' or '**Save As**' from the '**File**' menu), and include some print-outs in your write-up (You can use PYTHON to process the data). Measure the distance between the lens system and the beam waist, judging the beam waist with your eyes.

(10) **Important:** When you have finished the experiment, make sure to switch off the stepper motor power supply and the '**Stepper**' program. Doing so ensures that the stepper motor is turned off to avoid overheating.

Other things you have to include in your write-up:

(11) Draw a graph which shows the beam radius as a function of the distance to the lens system. Use the data on the graph to determine the location of the beam waist and spot size at the beam waist. Compare your result for the location of the beam waist with that judged by eye in (8). Compare the calculated spot size at the beam waist with the spot size at the beam waist on your graph.

(12) From your results, calculate the q parameter on either side of the convex lens and use the ABCD law to work out the focal length of the lens.

(13) Work out the matrix for the lens system. Calculate the 'back focal length' (distance between the last interface of the lens system and the focal point as defined by the thin lens approximation), and compare it with the measured distance between the lens system and the beam waist.

Question: Do these two figures match? If not explain why!

Questions

1. What is the effect of the finite size of the aperture in front of the photo-detector? Does it have an effect on your results?
2. If you were to point this laser at the moon, what would be the beam radius on the lunar surface?
3. How could you achieve a smaller beam radius on the moon (it has to be a practical solution!)?

Appendix

Theory (following Chapter 3 of Ref. 2)

Waves with wavefront normals making small angles with the z-axis are called paraxial waves. They must satisfy the paraxial Helmholtz equation. An important solution of this equation that exhibits the characteristics of an optical beam is a wave called the Gaussian beam. The intensity distribution in any transverse plane is a circularly symmetric Gaussian function centred about the beam axis. The width of this function is minimum at the beam waist and grows gradually in both directions. The wavefronts are approximately planar near the beam waist, but they gradually curve and become approximately spherical far from the waist. The angular divergence of the wavefront normals is the minimum permitted by the wave equation. Under ideal conditions, the light from a laser takes the form of a Gaussian beam. A paraxial wave is a plane wave $\exp(-\mathrm{j}kz)$ (with wave number $k = 2\pi n/\lambda$ and vacuum wavelength λ) modulated by a complex envelope $A(\mathbf{r})$ that is a slowly varying function in the direction of propagation (i.e. of z). The complex amplitude is:

$$U(\mathbf{r}) = A(\mathbf{r}) \exp(-\mathrm{j}kz)$$

For the complex amplitude to satisfy the Helmholtz equation, $\nabla^2 U + k^2 U = 0$, the complex envelope $A(\mathbf{r})$ must satisfy the paraxial Helmholtz equation:

$$\nabla_T^2 A - \mathrm{j}2k\frac{\partial A}{\partial z} = 0$$

where $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse part of the Laplacian operator. One simple solution to the paraxial Helmholtz equation provides the paraboloidal wave for which:

$$A(\mathbf{r}) = \frac{A_1}{z}\exp\left(-\mathrm{j}k\frac{\rho^2}{2z}\right)$$
$$\rho^2 = x^2 + y^2$$

where A_1 is a constant. The paraboloidal wave is the paraxial approximation of the spherical wave when x and y are much smaller than z. Another solution of the paraxial Helmholtz equation provides the Gaussian beam. It is obtained from the paraboloidal wave by a simple transformation. Since the complex envelope of the paraboloidal wave is a solution of the paraxial Helmholtz equation, a shifted version of it, with $z - \zeta$ replacing z, where ζ is a constant, is also a solution. This provides a paraboloidal wave centred about the point $z = \zeta$ instead of $z = 0$. When ζ is purely imaginary, say $\zeta = -\mathrm{j}z_0$, where z_0 is real, this gives rise to the complex envelope of the Gaussian beam:

$$A(\mathbf{r}) = \frac{A_1}{q(z)}\exp\left(-\mathrm{j}k\frac{\rho^2}{2q(z)}\right)$$
$$q(z) = z - \mathrm{j}z_0$$

To separate the amplitude and phase of this complex envelope, we write the complex function $1/q(z) = 1/(z - \mathrm{j}z_0)$ in terms of its real and imaginary parts by defining two new real functions $R(z)$ and $\omega(z)$, such that:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \mathrm{j}\frac{\lambda}{\pi n \omega^2(z)}$$

where $w(z)$ and $R(z)$ are measures of the beam width and wavefront curvature, respectively. They are given by:

$$\omega(z) = \omega_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{\lambda z_0}{\pi}n}$$
$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

List of Equipment

1. HeNe laser ($\lambda_{\mathrm{HeNe}} = 632.8$ nm)
2. 2 Beam-steering mounts with mirrors
3. Concave lens and Convex lens
4. Photo-detector with translation stage, amplifier, and controller
5. 3 interchangeable apertures (diameters of 100μm, 25μm, and 5μm)
6. HP PC computer

R. Leonhardt
D. A. Wardle
Y. H. Lo

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