Experiment 221: The Borda Pendulum

Aim

To gain experience in the planning and execution of an experiment and to obtain a measurement which is as precise as the means available allow. Your objective in this particular experiment is to determine the local value of the gravitational acceleration g as accurately as possible using the apparatus provided.

Theory

The "simple pendulum" (a point mass suspended by a massless string) is a useful teaching model but a theoretical fantasy. In experimental physics idealized models do not exist and the pendulum mounted on the wall in the laboratory (see Figure 1) is no exception. The period of this pendulum can be calculated as

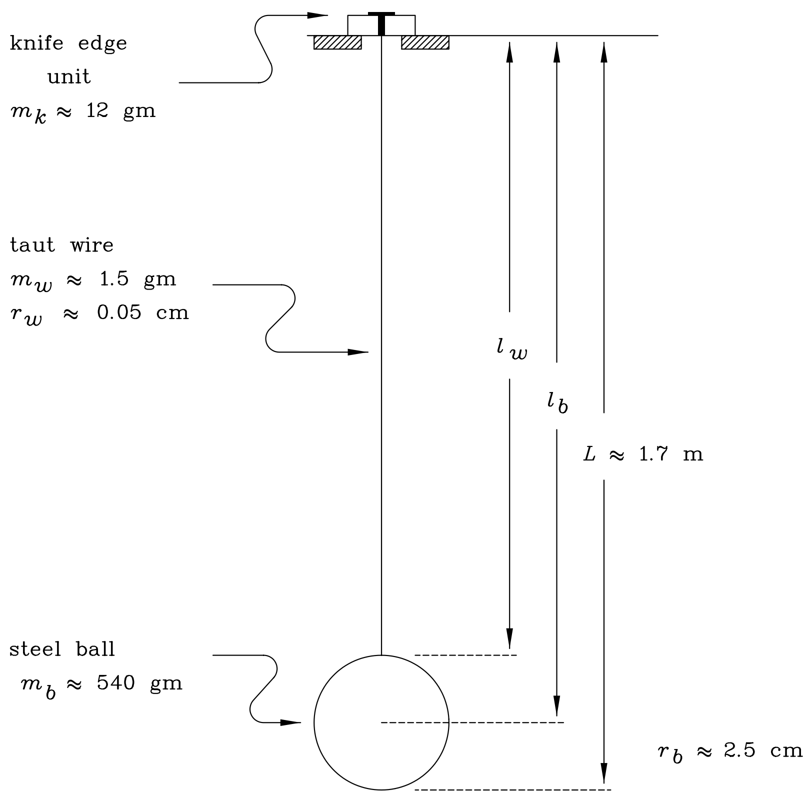

Figure 1: Schematic diagram of the Borda pendulum.

follows. Assume the pendulum is swinging and makes an instantaneous angle θ with the vertical. Newton's law for rotational motion relates the total torque τ to the moment of inertia I and the angular acceleration. Hence:

$$\tau = I\frac{d^2\theta}{dt^2}$$

But for the pendulum turning about the knife edge we have:

$$\tau = -\sum m_i g l_i \sin\theta$$

where the pendulum is regarded as made up of masses m_i with centres of mass a distance l_i from the pivot. Similarly we write $I = \sum I_i$. Substituting, we get:

$$-\sum m_i l_i g \sin\theta = \sum I_i \frac{d^2\theta}{dt^2}$$

For small angles $\sin\theta \approx \theta$, therefore:

$$\frac{d^2\theta}{dt^2} = -\frac{\sum m_i l_i}{\sum I_i}\, g\,\theta$$

The basic equation for SHM is:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Hence the pendulum motion is SHM of period T where:

$$T^2 = \frac{4\pi^2}{g}\frac{\sum I_i}{\sum m_i l_i} \tag{1}$$

Using the parallel axis theorem[1], the total moment of inertia of the pendulum about the knife edge support is given by the sum of:

$$\begin{aligned}
&\text{M.I. of steel ball} &&= m_b l_b^2 + \frac{2}{5} m_b r_b^2 \\
&\text{M.I. of taut wire} &&= \frac{1}{3} m_w l_w^2 + \frac{1}{4} m_w r_w^2 \\
&\text{M.I. of knife edge unit} &&= \text{negligible}
\end{aligned}$$

Hence, the gravitational acceleration g is given by:

$$g = \frac{4\pi^2}{T^2}\left(\frac{m_b l_b^2 + \frac{2}{5} m_b r_b^2 + \frac{1}{3} m_w l_w^2 + \frac{1}{4} m_w r_w^2}{m_b l_b + \frac{1}{2} m_w l_w}\right) \tag{2}$$

Preliminary analysis and practice measurements

The following instructions are aimed at helping you plan and carry out the necessary measurements to determine g as accurately as the apparatus provided will allow.

Equation (2) will be used to calculate g, but for the purpose of error analysis it is convenient to omit small terms. Using the approximate data given in Figure 1 make a numerical estimate of the terms in the large brackets in equation (2) and show that for error analysis equation (2) may be written as:

$$g = \frac{4\pi^2}{T^2}\,(L - r_b) \tag{3}$$

Equation (3) also demonstrates that accurate measurements are needed of T, L and r_b. The period T is measured with a stop-watch, the overall length L of the pendulum is measured using a telescopic length-transfer rod and a cathetometer (the solidly constructed apparatus with the travelling microscope) and the radius r_b of the steel ball is measured using a vernier caliper. In each case, an accurate measurement depends more on your experimental skill than on the measuring instruments.

Period

The digital stop-watch can be taken as your absolute time standard. It is quartz controlled and at temperatures encountered in the laboratory, it keeps time to better than a few parts in 10^6. The main uncertainty in measuring the period is the uncertainty in your reaction time.

(1) With the pendulum stationary, adjust the knife edge so the pendulum wire is in sync with the 0 mark on the wall scale. This is then the middle position of the pendulum.

Set the pendulum swinging and *practice your timing technique* by measuring the time for 10 swings. Do this by sitting in front of the pendulum and starting and stopping the stop-watch when the pendulum wire crosses the middle position. Always time when the pendulum is moving in the same direction. Do this 5 times and record your results.

[1] See, for example, Serway, *Physics for Scientists and Engineers,* Chap. 10.

(2) You will also need to *practice using the lap counter function of the stop watch.* Do this as follows. Use the time you have just measured to predict the times for $10, 20, 30, 40$ and 50 swings. Write these down. Now also write down the expected times for $9, 19, 29, 39$ and 49 swings. Call these trigger times.

Now we will measure the time for $10, 20, 30, 40$ and 50 swings all at once using the lap counter function of the stop watch. Use the first trigger time to alert you to the need to press the lap counter button the next time the pendulum wire crosses the centre graticule line. This should be the time for 10 swings. Record this result. Continue on to get the times for $20, 30, 40$ and 50 swings. Record the predicted and measured values.

Overall length

The cathetometer is your "absolute" standard of length. You may assume that at temperatures encountered in the laboratory, its accuracy is better than 1 part in 10^6. A telescopic rod, set to L , is used to transfer the length of the pendulum on to the cathetometer.

(3) *Practice the use of the telescopic rod and cathetometer.* Firstly examine the bearing surfaces on which the knife edge of the pendulum rests, so that you know where to put the 'hook' of the telescopic rod. Place the triangular support below the pendulum and keeping it level (use the air bubble), raise its level until the pendulum bob is just not free to move. Hang the telescopic rod and adjust it to be equal to the length from the knife edge bearing surface to the top of the triangular support. Now use the cathetometer to measure the length of the telescopic rod. You need to make sure that the telescopic rod is supported so that it is parallel to the cathetometer. The measurement needs to be done in two steps because the length L is greater than the length of the cathetometer. Record your results.

Radius r_b of the steel ball and radius r_w of the wire

(4) A vernier caliper and a micrometer are used to measure the diameter of the steel ball and the diameter of the wire respectively. Examine the vernier caliper and the micrometer and make sure you are able to use and read them.

Serious Measurements

(5) Check that the knife edges are properly positioned on their support surfaces.

Period T

(6) You need a good estimate of the random error in your timing. It is assumed that this error is independent of *how many swings* you are timing. You also need a good preliminary estimate of the period T. Measure and record the time for 20 swings. Do this 10 times ($N = 10$). This gives $T_{20,i}$ where $i = 1$ to 10. Calculate:

(i) the sample mean $T_{20} = \frac{1}{N} \sum_{i=1}^{N} T_{20,i}$

(ii) the population standard deviation $\sigma = \sqrt{\sum_{i=1}^{N} (T_{20} - T_{20,i})^2 / (N-1)}$

(iii) the standard error in the sample mean $\sigma_{20} = \sigma/\sqrt{N}$

The best preliminary estimate of the period is:

$$T \pm \sigma_T = (T_{20}/20) \pm (\sigma_{20}/20)$$

The population standard deviation σ gives an estimate of the uncertainty in your reaction time. This is independent of the total number of swings timed.

Accurate measurement of the period T using the method of refined approximations

We will now endeavour to time 1200 swings of the pendulum without counting each swing! However, if we attempt to predict the time for 1200 swings based on the results for 20 swings obtained above, the uncertainty in our prediction may be larger than the time for one swing. So we will not be sure whether we have timed exactly 1200 swings or 1201 swings (or some other, unknown number of swings). This problem can be avoided by using the method of refined approximations. The idea is to use the 20 swings data to predict the time for 99 swings which will be used as the trigger time for 100 swings. The time for 100 swings can be experimentally measured as the pendulum passes the 100$^{\text{th}}$ swing on its way towards 1200 swings by using the *lap counter* function of the stop watch. This then gives an improved estimate of the period T which can be used to calculate a trigger time for 200 swings, and so on.

(7) Before beginning timing measurements, record the amplitude of the swing in cm.

(8) The above measurements allow you to predict the time for 100 swings, i.e.

$$T_{100}^{\text{pred}} = 5T_{20} \pm 5\sigma_{20}$$

Use the lap counter to measure T_{100}. The uncertainty in this measurement is σ because we have assumed that the random error in your timing is independent of the number of swings. If

$$\left|T_{100} - T_{100}^{\text{pred}}\right| < 2\sqrt{\sigma^2 + (5\sigma_{20})^2} < T/2$$

you have made a reliable measurement and your best estimate of T is now $(T_{100} \pm \sigma)/100$. Use this to calculate T_{200}^{pred}. Measure T_{200}, T_{400}, etc and complete the following table as you proceed:

N	T_N^{pred}	T_N	$T_N^{\text{pred}} - T_N$	$T \pm \sigma_T$
20	–	–	–	
100				
200				
400				
600				
800				
1000				
1200				

Note that at each stage the best estimate for T is updated and calculated using

$$T \pm \sigma_T = (T_N \pm \sigma)/N$$

(9) Record the amplitude of the swing in cm at the end of your timing measurements. Check the following:

(i) Is $\left|T_N - T_N^{\text{pred}}\right| < T/2$?

(ii) Are the successive values of T the same within the uncertainty σ_T ?

(iii) Have you achieved an accuracy of a few parts in 10^4 ?

Your observed period T_{obs} can be taken as $(T_{1200} \pm \sigma)/1200$.

Overall length

(10) Measure L five times and record your measurements. Be sure to reset the telescopic rod and the triangular table between measurements. Check that the 5 measurements are consistent. Calculate the mean and its standard error. Your standard error should be better than 2 parts in 10^4. If it is too big, make measurements of L until the accuracy improves.

Note: it is important to realise that to repeat a measurement of L you must readjust both the triangular table and the telescopic rod.

Radius r_b of the steel ball and radius r_w of the wire

(11) Measure the diameter of the steel ball using the vernier calipers 6 times. Calculate the mean and standard error of r_b. Using the micrometer measure the diameter of the wire at several places. Calculate the mean and standard error of r_w.

(12) A correction of T for finite angles of swing (i.e. $\sin\theta \neq \theta$) may be applied using Bessel's formula:

$$T_{\text{obs}} = T_{\text{corr}}\left[1 + \frac{1}{4}\sin^2\left(\frac{\alpha}{2}\right)\right] = T_{\text{corr}}\left[1 + \frac{\alpha^2}{16}\right] \tag{4}$$

where α = amplitude of the swing **in radians**.

If α_1 and α_2 are the amplitudes of swing at the beginning and the end of the timing interval, then:

$$T_{\text{corr}} = T_{\text{obs}}\left(1 - \frac{\alpha_1\alpha_2}{16}\right) \tag{5}$$

(13) Determine g directly from equation (2) using the above correction. Prepare an error table based on equation (3). Your error table should look similar to the example given below:

Quantity	Value	Error	Fractional error
L			–
r_b			–
$L - r_b$			
T			
T^2			
g			

(14) Show your final value of g and its error to your demonstrator before writing up.

Write-up

The following points should be covered in the write-up:

1. Justify the neglect of the knife edge terms in equation (2).
2. Justify using equation (3) for the error analysis.
3. Explain the necessity for resetting the telescopic rod and the triangular table between each measurement of L.
4. Explain why the pendulum is timed as it passes through its middle position.
5. Tabulate all your measurements for finding T by refined approximations.
6. Tabulate all your measurements used to determine L.
7. An error table based on equation (3) as described in (13) above.
8. Show the final calculation of g using equation (2) and compare your result with the value given at the gravity station at the bottom of the stair-well next to the goods lift.

List of equipment

1. Borda Pendulum
2. Telescopic rod
3. Cathetometer
4. Digital Stop-watch
5. Vernier Caliper
6. Micrometer

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