Experiment 312: Coupled Pendulums

Aim

The aim of this experiment is to introduce the student to (i) the factors which determine the frequency of an oscillator, (ii) linear systems, (iii) the principle of superposition, (iv) normal modes of oscillation, and (v) the usefulness of models. Of these the first four ideas are very closely related.

Theory

Single Pendulum

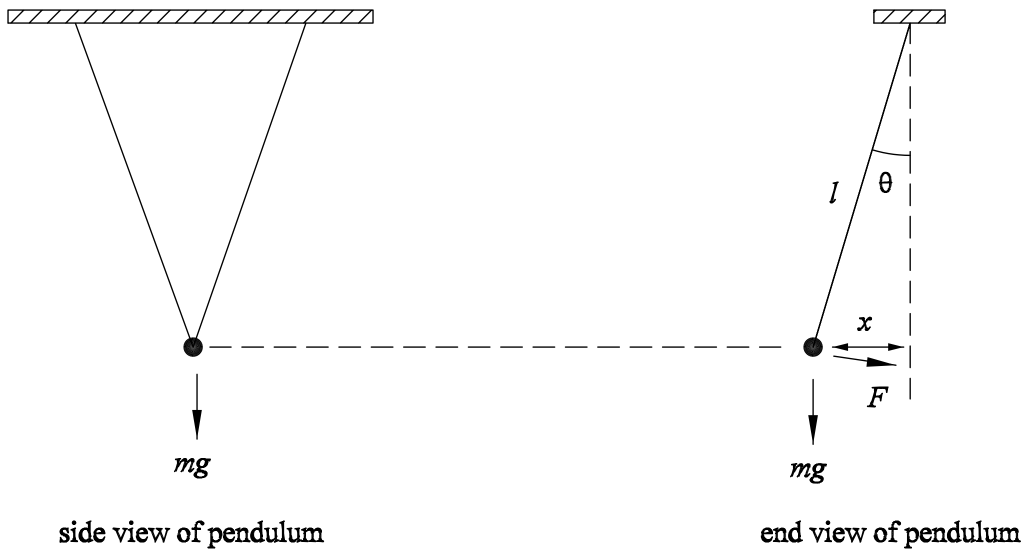

Figure 1: Single pendulum

For the single pendulum shown in Fig. 1, the restoring force is given by $F = -mg\sin\theta$. If the angle of swing is small, $\sin\theta \approx \theta \approx x/l$ then:

$$F \approx -\frac{mg}{l}x \tag{1}$$

Using Newton's second law:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{mg}{l}x \tag{2}$$

Equation (2) is the differential equation for simple harmonic motion. The angular frequency ω_0 is given by:

$$\omega_0^2 = \left[\frac{1}{m}\right]\left[\frac{mg}{l}\right] \tag{3}$$

Notice that the frequency of oscillation depends on only two terms, one is the mass of the bob, and the other is the gradient of the restoring force versus displacement graph (see Fig. 2). For the simple pendulum the mass appears in both terms, cancelling overall. The same two terms determine the frequency of any mechanical simple harmonic oscillator.

Equation (1) shows how the restoring force is linearly related to the position; if the position doubles then so does the restoring force. Because of this relationship the pendulum is described as a linear system. Linear systems occur widely in all branches of physics. This is well worth knowing as all linear systems share a

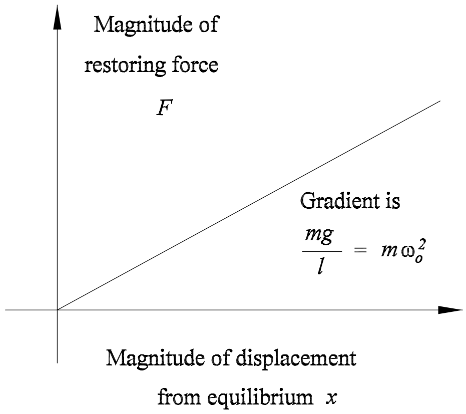

Figure 2: Force vs displacement graph for a simple harmonic oscillator

common set of properties. Among the most important of these is the principle of superposition. This states that if $x(t)$ and $y(t)$ are possible motions for the system, then so is $z(t) = \alpha x(t) + \beta y(t)$, where α and β are any real numbers.

For the single pendulum one possible motion is $x(t) = A\cos\omega_0 t$. According to the principle of superposition $z(t) = \alpha x(t) + 0y(t) = \alpha A\cos\omega_0 t$ is also a solution. This demonstrates that the amplitude of oscillation has no effect on the frequency (remember that the frequency depends only on two factors, neither of which involves amplitude).

Some aspects of non-linear systems are investigated in Experiment 314 “The Non-linear Pendulum”.

Two coupled pendulums

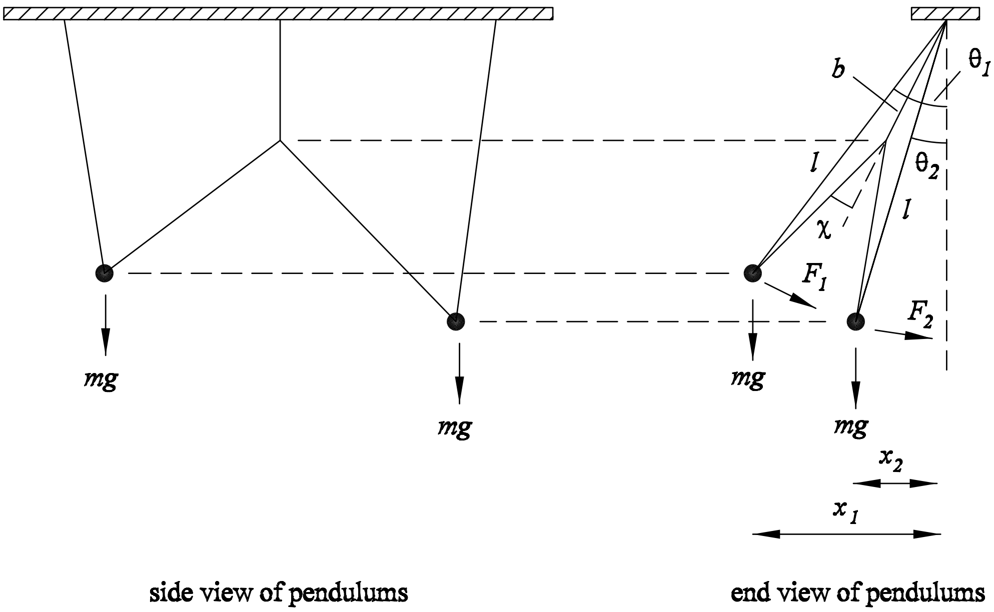

Figure 3: Two coupled pendulums

For the two coupled pendulums shown in Fig. 3, assuming the angles are small and the tensions in the strings are equal, the force F_1 is given by:

$$F_1 = -\frac{mg}{2}\theta_1 - \frac{mg}{2}\left(\frac{\theta_1 + \theta_2}{2} + \chi\right) \tag{4}$$

Using $\theta_1 \approx x_1/l$, $\theta_2 \approx x_2/l$ and $\chi \approx (x_1 - x_2)/2(l-b)$, this can be conveniently written as:

$$F_1 \approx -\frac{mg}{l}x_1 - \frac{mgb}{4l(l-b)}(x_1 - x_2) \tag{5}$$

The expression for F_2 is similar. Notice that the forces are still linearly related to the positions so that the two coupled pendulums are another example of a linear system. The resulting equation of motion is:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\omega_0^2 - k & k \\ k & -\omega_0^2 - k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6}$$

where ω_0^2 is given by equation (3) and k, another constant, is given by:

$$k = \frac{gb}{4l(l-b)} \tag{7}$$

The equation of motion can be solved by finding the eigenvalues and eigenvectors of the matrix. Following this procedure yields the two eigensolutions:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_{\mathrm{sym}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_{\mathrm{sym}}t + \phi_{\mathrm{sym}}) \tag{8}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_{\mathrm{asym}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_{\mathrm{asym}}t + \phi_{\mathrm{asym}}) \tag{9}$$

where

$$\omega_{\mathrm{sym}} = \omega_0 \tag{10}$$

$$\omega_{\mathrm{asym}} = \sqrt{\omega_0^2 + 2k} \tag{11}$$

and A_{sym}, A_{asym}, ϕ_{sym} and ϕ_{asym} are arbitrary constants.

The eigensolutions are characterized by both pendulums moving in constant amplitude sinusoidal motion with the same frequency. When all the oscillators in a set of coupled oscillators move in this way the motion is called a normal mode. Notice that in a normal mode there is no transfer of energy from one oscillator to another. Equation (8) describes a normal mode in which the two pendulums swing in phase ("symmetric mode"). Similarly, equation (9) describes a normal mode in which the two pendulums swing in antiphase ("antisymmetric mode").

Other motions can be obtained from the normal modes using the principle of superposition. For example, choosing $A_{\mathrm{sym}} = A_{\mathrm{asym}} = A$ and $\phi_{\mathrm{sym}} = \phi_{\mathrm{asym}} = 0$, then adding equations (8) and (9) gives:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_{\mathrm{sym}}t) + A\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_{\mathrm{asym}}t) \tag{12}$$

Using trigonometric identities this can be written as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2A\begin{pmatrix} \cos(\omega_{\mathrm{pend}}t)\cos\left(\frac{\omega_{\mathrm{beat}}}{2}t\right) \\ \sin(\omega_{\mathrm{pend}}t)\sin\left(\frac{\omega_{\mathrm{beat}}}{2}t\right) \end{pmatrix} \tag{13}$$

where

$$\omega_{\mathrm{pend}} = \frac{\omega_{\mathrm{asym}} + \omega_{\mathrm{sym}}}{2} \tag{14}$$

and

$$\omega_{\mathrm{beat}} = \omega_{\mathrm{asym}} - \omega_{\mathrm{sym}} \tag{15}$$

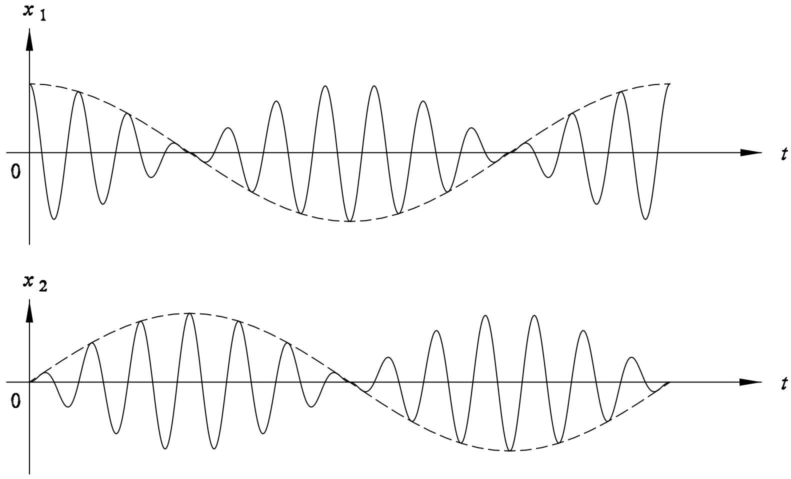

Figure 4: A possible motion of two coupled pendulums, showing beats

The motion described by equation (13) is illustrated in Fig. 4. As shown the motion will result when one (and only one) of the pendulums is displaced from its equilibrium position and released from rest. Notice how there is a periodic transfer of energy from one pendulum to the other; when one pendulum achieves maximum amplitude the other is momentarily at rest. This beating arises due to interference between the two different normal mode frequencies.

A phasor diagram can help to make this clear. Fig. 5 shows how the instantaneous amplitude of oscillation for a pendulum can be obtained by adding two phasors; one for the symmetric contribution and one for the antisymmetric contribution. As the two phasors rotate at slightly different frequencies they are sometimes aligned and the amplitude is large (Fig. 5(a)). At other times they are approximately anti-parallel and the amplitude is small (Fig. 5(b)).

A numerical simulation of the two coupled pendulums is described in the last part of Experiment 215 "Numerical Modelling".

Question 1

The antisymmetric and symmetric normal modes can be superposed to obtain new solutions. Can any solution be written as a superposition of these two modes? Justify your answer.

Hint: Is a solution completely characterized by its initial positions and velocities?

Procedure

Two coupled pendulums

(1) Equalize the lengths of the strings so that both pendulums have the same angular frequency when uncoupled (uncoupled means $b = 0$ in Fig. 3). Measure the uncoupled angular frequency by measuring the period of oscillation. Remember to choose your amplitude of swing so that the approximation $\sin\theta \approx \theta$ is valid.

(2) Set $b = l/5$. Excite the symmetric and antisymmetric normal modes and measure ω_{sym} and ω_{asym}. How do they compare with ω_0?

Use equation (15) and these measured values to predict ω_{beat}. Be warned! Equation (15) involves the subtraction of two nearly equal numbers and this can result in loss of precision. You must calculate ω_{beat} and its percentage error while you are doing the experiment. If the percentage error is too great (for example, more than 10% or 15%) you will need to obtain more precise values for ω_{sym} and ω_{asym}.

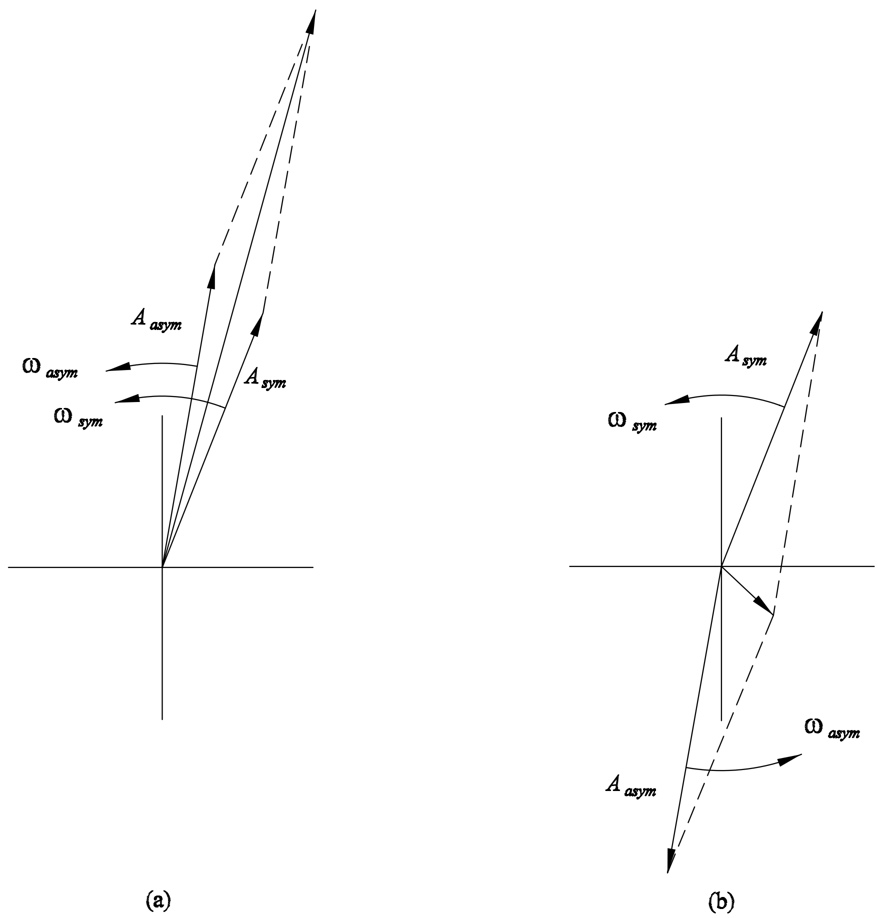

Figure 5: Origin of beats as seen in a phasor diagram

Excite the motion depicted in Fig. 4 and measure ω_{beat}. How do the predicted and measured values compare?

(3) Repeat procedure (2) for several different values of b.

(4) For one of the values of already explored, make appropriate measurements to verify equations (11) and (14).

Question 2

The two coupled pendulums can exert forces on each other through the strings coupling them. With this in mind, and with reference to Fig. 3, explain the trends in ω_{sym} and ω_{asym} as b is increased.

Three coupled pendulums

(5) How many normal modes are there for the three coupled pendulums? If the normal modes for two coupled pendulums are written $\{(1,1),(1,-1)\}$ how would the normal modes for three coupled pendulums be written? The apparatus is provided to help you answer these questions.

Spring oscillator (Wilberforce pendulum)

The spring oscillator actually consists of two oscillators (vertical and rotational) interacting because of the helical shape of the spring. Despite the fact that it looks intimidating it behaves in exactly the same way as any other pair of linear, coupled oscillators, including the two coupled pendulums used in the first part of the experiment. Herein lies the utility of models; if we encounter a problem with the spring oscillator we can solve the problem by thinking about the apparently simpler, but in abstract terms identical, system of two coupled pendulums.

(6) One of the objectives of the next procedure is to excite the symmetric and antisymmetric normal modes of the spring oscillator. This can be made easier by adjusting the oscillator so that the modes are approximately $\{(1,1),(1,-1)\}$ rather than, for example, $\{(1,10^{-3}),(10^{-3},-1)\}$. To achieve this, adjust the moment of inertia, so that when the oscillator is displaced in the vertical direction only, the motion resembles that depicted in Fig. 4, in particular so that the transfer of energy between the vertical and rotational motions occurs approximately completely (remember to use small displacements so that the spring will obey Hooke's law and hence behave like a linear system).

(7) Use trial and error to find the symmetric and antisymmetric normal modes of the spring oscillator. (You may find this confusing because the two oscillators (vertical and rotational) are different. If you are confused try thinking about the two coupled pendulums. How would you go about using trial and error to find the normal modes of this system if the two pendulums were different, for example, if one pendulum was longer than the other?) Give a single sentence, qualitative description of each motion (or maybe draw two diagrams). Measure ω_{sym} and ω_{asym}. Displace the oscillator in the vertical direction only and release from rest. Measure ω_{beat}. Use your measurements to test the validity of equation (15).

Question 3

Do you expect equation (15) to hold for the spring oscillator? Briefly explain your answer.

List of Equipment

1. Two coupled pendulums
2. Three coupled pendulums
3. Spiral pendulum
4. Stopwatch

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This version: October 14, 2008.