Experiment 218: Gravity Waves and Dispersion

Aim

The object of this experiment is to use gravity waves on water to examine four fundamental properties of waves: phase velocity; group velocity; dispersion; and the time–frequency uncertainty.

NOTE: It will be helpful for you to have some experience with Python before doing this experiment. You may find it useful to first do Expt 213: Data Analysis with Python.

Introduction

In this experiment, we examine waves which occur on the interface of two fluids of different densities, air and water. If the interface is not horizontal and plane, restoring forces due to gravity and surface tension act to make the interface horizontal again, resulting in wave propagation. For waves of wavelength longer than a few centimetres, the effects of gravity dominate over surface tension.

To a very good approximation, the speed of sound in air is independent of frequency over the audio frequency range. Speech and music would almost certainly be impossible if different tones travelled at different speeds. For the great majority of waves, however, the speed depends on the frequency, and the propagation is described as being **dispersive**. In this experiment, we measure the variation of wave speed with frequency for interface waves and show how this dispersion affects the propagation of a wave packet.

Background Information

A development of the linearised theory of gravity waves is given after the experimental procedure. To get the most benefit from the experiment, you should work through that section after you obtaining your results.

Wave propagation in a medium is characterized by a **dispersion equation**, which relates the angular frequency $\omega = 2\pi\nu$ to the wavenumber $k = 2\pi/\lambda$ of sinusoidal travelling waves, where ν denotes the frequency and λ denotes the wavelength. For non-dispersive waves which all travel at the same speed c, the relation $c = \nu\lambda$ turns into the nondispersive dispersion relation

$$\omega = ck.$$

When the medium is dispersive, the ratio ω/k is called the **phase velocity** (more correctly, the phase speed) and is denoted c_p. If we send a sinusoidal wave of constant angular frequency ω, the crests (and troughs) of the waves will travel at the phase velocity. Unlike the situation which holds in a non-dispersive medium, the phase velocity is now dependent on frequency, so ω is not directly proportional to k.

If we send a short train of oscillations by starting and then stopping the source (rather than by allowing the source to oscillate continuously), an interesting effect occurs. The **envelope** (or profile) of the short train of waves generated (also known as a **wave packet**) travels at a speed called the **group velocity** which is usually different from the phase velocity of the waves which comprise the packet. The group velocity can also be calculated from the dispersion relation using

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

For the nondispersive dispersion relation, the group and phase velocities coincide, but this does not happen in general if the medium is dispersive. Note that the group velocity is only well defined for wavetrains which are **narrowband**, which means that the frequency (or wavenumber) components are limited to a small range around some central value. The derivative in the definition of c_g needs to be evaluated at this central value of k.

As you will see in the theory section, the dispersion relation for water waves in which gravity provides the restoring force is

$$\omega = \sqrt{gk \tanh kh} \tag{1}$$

where h denotes the height of the water and g is the acceleration due to gravity.

Note: If you have not previously met the hyperbolic functions, we define the hyperbolic sine, cosine and tangent via the relations:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}, \qquad \cosh x = \frac{\exp(x) + \exp(-x)}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(x)}.$$

These are analogous to the definitions of the usual sine, cosine and tangent in terms of exponentials, except that there are no imaginary quantities anywhere. You can check that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x,$$

which differs from the trigonometric functions in that the signs are always positive.

Note also that if x is small, $\tanh x \approx \sinh x \approx x$, whereas if x is large, $\tanh x \approx 1$.

Question 1: Show that for gravity waves the phase velocity is

$$c_p = \sqrt{\frac{g}{k}\tanh kh} \tag{2}$$

and that the group velocity is

$$c_g = \frac{c_p}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{3}$$

Question 2: What do these expressions tend to in the limits:

(a) the water is deep compared to a wavelength (e.g., $\lambda < 20h$)?

(b) the water is shallow compared to a wavelength ($\lambda > 3h$)?

Question 3: Assume the Pacific Ocean is circular, with a diameter of 12000 km and a uniform depth of 5 km. Calculate the time taken to traverse the Pacific Ocean by:

(a) a Tsunami (tidal wave) with dominant period of 10 minutes.

(b) the onset of storm-generated ocean swells with dominant period of 10 seconds.

Procedure

The gravity waves are observed in a 2.5 metre long tank, which should be filled with water to a depth of about 16 cm. The waves are generated by a paddle which is driven in vertical motion by a variable speed motor. The speed of this motor is altered by turning the speed knob. Two different paddles are provided, and they may be interchanged by unhooking the drive cord from the paddle, removing the pulley over which the cord passes and replacing the paddle. Start off with the paddle with the more acute pitch (the "small" paddle).

At the other end of the tank there is a baffle which acts to absorb the waves and minimize reflections. A series of vertical marks are spaced at 25 cm intervals along the tank to assist in timing the passage of waves. The pressure at a point within the tank may be measured by immersing a pipe which is connected to a solid-state pressure transducer via a plastic hose. The transducer converts the pressure into a voltage which

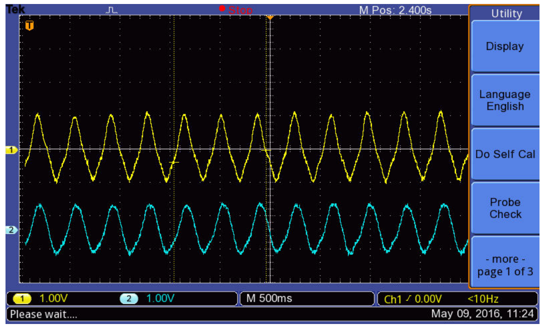

Figure 1: Measuring the phase velocity.

is displayed on an oscilloscope. Two such transducers are provided, and for best results the pipes should be immersed just below the water surface, so that the pressure fluctuations are largest when a wave passes.

Before doing the experiment, it is necessary to adjust the knobs on the front panel of the transducer box so that the output voltage is zero when no wave is passing. The pipes leading to transducers A and B should be positioned two metres apart. Select a horizontal sweep rate of 0.5 or 1 s/div. In order to "freeze" the trace on the screen, press the **RUN/STOP** button.

You can save the data using a USB flash drive: use the **SAVE/RECALL** button. Set the *Action* to "Save All" and the option "Print Button saves all to files": then you are able to save both channels by pressing the 'Print' button (the one in the shape of a floppy disk). It is helpful for you to use a small capacity flash drive (e.g. < 4 GB), as the oscilloscope takes a long time to verify large flash drives.

The oscilloscope allows time intervals to be measured using a pair of cursors. Ask a demonstrator for help about this feature, if you do not know how to use it. A stopwatch is also required for timing purposes.

(1) Measure the height of the water in the tank, h.

(2) In this section we will measure the phase velocity of gravity waves for a range of frequencies. The oscilloscope can be used for both an accurate measurement of the frequency, and for a measurement of the phase velocity. A sample oscilloscope trace, showing the pressure variation at both Pipes A and B, is shown in Figure 1.

The phase velocity is the rate at which the phase propagates in a continuous wave. Thus the phase velocity can be measured by measuring the time taken for a crest of the wave to travel from Pipe A to Pipe B. Set the paddle to oscillate continuously and measure the frequency. To get an estimate for the phase velocity, measure the time (using a stopwatch) for the crest of a wave to propagate from the first to the second pipe. The oscilloscope can be used to refine your measurement: set the cursor spacing to your stopwatch reading, and find the nearest time between a crest at Pipe A and Pipe B. Repeat for a range of frequencies.

Some notes: When changing the driving frequency, wait for steady-state conditions before making any measurements.

The "small" paddle should be used for as wide a frequency range as possible, but at low frequencies you may find it desirable to use the "large" paddle, since this generates waves with a larger amplitude. At frequencies lower than about 1 Hz, partial reflection of waves from the baffle may cause standing wave effects which can disguise the passage of the travelling waves. Avoid using such regions for timing purposes.

From your time-of-passage values calculate the phase velocity c_p and then the wavelength λ using $c_p = \nu\lambda$, where ν is the frequency. Plot your data points of c_p against λ superimposed on the theoretical curve given by equation (2) and the definition of k. Estimate the errors in c_p and λ and show error bars on your graph.

(3) In this section you will investigate the propagation of wave packets, and measure the group velocity for packets with various centre frequencies. The goal is to create a short wavepacket — only one or

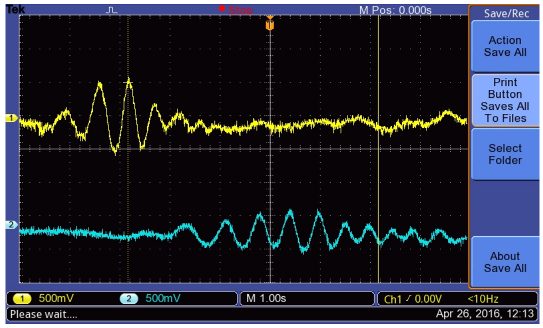

Figure 2: Measuring the group velocity.

two wavelengths wide — and investigate how this wavepacket behaves differently from the continuous wave case. A sample oscilloscope trace is shown in Figure 2.

With the “small” paddle attached, select a frequency of about 2 Hz and accurately measure the frequency for a continuous wave. Turn the paddle off, and wait for all oscillations to die down completely. Now create a wavepacket by turning the paddle on for only one or two oscillations.

You should find that your oscilloscope trace is similar to that in Fig 2. Measure the group velocity: this is the speed of the wave *envelope*. You have some freedom in how you do this: you may use the cursors to estimate the centre of the envelope, or save the data to a flash drive and analyse it in Python. You **must** save the data for both channels at this stage for later Fourier analysis anyway.

Measure the group velocity for a variety of centre frequencies. Plot the dispersion relation and compare this with the phase velocity dispersion, and with the theoretical values. In your discussion you should list your observations and comment on the shape of the final waveform.

Spectrograms, Fourier Analysis and the Time–Frequency Uncertainty Principle

One of the most useful developments in the history of physics and mathematics is the concept of the Fourier transform. Any signal can be decomposed and written in terms of an integral over sine waves (you may have met its cousin, the Fourier series, which relates to any *periodic* signal being written as a discrete *sum* of sine waves). This section will show you how useful this concept is, particularly applied to waves.

We will use Python to analyse the waveforms from the group velocity experiments. We will give you code snippets, and then you should put them together to do the analysis. If you are not familiar with **Expt 213: Data Analysis with Python**, then you should find a copy of this pamphlet for reference.

(4) Load the data into python using the read_csv function in the pandas library:

```
import pandas as pd

#There are 5 columns in the csv file, the first three of which are meaningless.
    This code lets you name the 4th and 5th columns 'time' and 'volt', and
    extract them as arrays.
df = pd.read_csv('F0000CH1.CSV',header=None,index_col=False,names=['a','b','c',
    'time','volt'])

t = df['time']
y = df['volt']
```

The first thing we will do is look at the spectrum of the wavepacket, which we will do with a Fast Fourier transform (FFT). Use the data from the first sensor (the short wavepacket). You can use numpy's FFT library:

```
import matplotlib.pyplot as plt
import numpy as np
```

```
#Because the scan function on the oscilloscope leaves some data values empty,
    you will find it useful to remove these values. Look at the data and find
    which data points to remove.

#Have a look at data:
plt.plot(t,y)

#Remove bad values
y = y[? : ??]
t = t[? : ??]

#Subtract off DC component:
y = y - np.mean(y)

#Time interval:
dt = t[2] - t[1]

#The FFT. The n=10000 performs a zero-padding:
myfft = np.fft.fft(y,n=10000)
#If you're uncertain about the effect of zero padding, try the same code
    without the n=10000 argument

 #The Fourier frequencies:
myFs = np.fft.fftfreq(np.size(myfft),dt)

#Look at power spectrum:
plt.figure()
plt.plot(myFs,abs(myfft)**2)
plt.xlim((0,10))
```

You should find that you have a broad peak. Find the half-width-at-half-maximum (HWHM): this is your uncertainty in frequency. Look at your waveform in the first figure and estimate the HWHM of the envelope: this is your uncertainty in time. Now check how well your data corresponds to the time–frequency uncertainty principle:

$$\Delta f \Delta t \geq \frac{1}{4\pi} \tag{4}$$

Note that this is the *same* uncertainty principle that Heisenberg so famously postulated for position and momentum in quantum mechanics, because time and frequency are a *Fourier transform pair.* This uncertainty relation is completely general for all wave phenomena.

(5) Finally, we will look at the effect of dispersion. We will use a tool known as a 'spectrogram'. A spectrogram is very simple - it takes a FFT over a subset of data, and then shifts the window along to take another FFT. Plotted as a contour or surface plot, this shows how the frequency of a signal changes in time.

Load the second waveform into Python. We will use the spectrogram tool from scipy's signal library.

```
from scipy import signal

df = pd.read_csv('F0000CH2.CSV',header=None,index_col=False,names=['a','b','c','time','
    volt'])
t2 = df['time']
y2 = df['volt']

#Remove bad values
t2 = t2[? : ??]
y2 = y2[? : ??]

dt2 = t2[2] - t2[1]

#Sampling frequency:
fs = 1.0/dt2

theF, theT, myspec = signal.spectrogram(y2,fs,nperseg=???,noverlap=???,nfft=????)
#The spectrogrm parameters are: nperseg (the window size in datapoints of the FFT),
    noverlap (the amount of overlap between neighbouring FFTs), and nfft (the total
    size of the zero-padded fft).
```

```
#Now that you have some knowledge of FFTs, choose sensible values for the spectrogram
    parameters to give you the best looking spectrogram.

#this plots the data
plt.pcolormesh(theT,theF,myspec)
plt.ylim((0,10))
plt.xlim((0,9))
```

Question 4: What do you notice about the spectrogram (in particular its 'chirp'?) Explain this, and why the wavepacket at the second sensor is so much more spread out than at the first sensor.

You may find it interesting to compare the frequency at the beginning of the waveform compared with the end of the waveform. How well does this agree with your FFT from the channel 1 data? You may also find it interesting to compare the width of the FFT power spectrum for a wavepacket and for a continuous wave.

Theory

Consider a tank partly filled with water to a height h. With no wave present, the surface of the water is the plane $y = h$. For simplicity, we shall only consider waves travelling in the direction of the positive x axis, so that we may ignore motion in the z direction. If the atmospheric pressure is P_0, the pressure in the water when there is no wave is

$$P(x,y) = P_0 - \rho g(y-h)$$

where ρ is the density of water and g is the acceleration of gravity.

When a wave is present, the particles of water are in motion. As discussed in the appendix, at time t, the velocity vector $\mathbf{v}$ of a water particle at (x,y) has components v_x and v_y where

$$\begin{aligned} v_x &= Ak\cosh ky \sin(kx-\omega t) \\ v_y &= -Ak\sinh ky\cos(kx-\omega t) \end{aligned}$$

The amplitude of the wave is proportional to A, the angular frequency is ω and the wavenumber is k. As usual, $\omega = 2\pi\nu$ and $k = 2\pi/\lambda$ where ν is the frequency and λ is the wavelength. The phase velocity (speed) of the wave is $c_p = \nu\lambda = \omega/k$.

Question 5: Recall that a flow is incompressible iff

$$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

Evaluate the partial derivatives for the flow given above, and verify that the condition for incompressibility is satisfied.

If we assume that the waves are small so that the height is much less than a wavelength, and terms involving A^2 may be neglected compared to those which are linear in A, the pressure in the water is

$$P(x,y,t) = P_0 - \rho g(y-h) + A\omega\rho\cosh ky\sin(kx-\omega t)\,. \tag{5}$$

In order to visualize the motion of the water particles, we can find approximate expressions for the displacement vector of the particle which was originally at (x,y). When the wave passes through, this particle is displaced to $(x+d_x, y+d_y)$ where

$$d_x = \frac{kA}{\omega}\cosh ky\cos(kx-\omega t) \tag{6}$$

$$d_y = \frac{kA}{\omega}\sinh ky\sin(kx-\omega t)\,. \tag{7}$$

These equations could be used for a computer based simulation.

Question 6 (Optional): A particle originally on the surface at (x_0, h) is moved to

$$x = x_0 + \frac{kA}{\omega} \cosh kh \cos(kx_0 - \omega t)$$
$$y = h + \frac{kA}{\omega} \sinh kh \sin(kx_0 - \omega t).$$

These define the new free surface of the water. On this surface, the pressure must be equal to P_0. Use (5) to evaluate

$$P\left(x_0 + \frac{kA}{\omega} \cosh kh \cos(kx_0 - \omega t), h + \frac{kA}{\omega} \sinh kh \sin(kx_0 - \omega t), t\right)$$

ignoring any terms higher than first order in A. Show that for this to be equal to P_0, we recover

$$\omega = \sqrt{gk \tanh kh}$$

which is the dispersion relation for gravity waves.

List of Equipment

1. Large glass tank.
2. Variable speed, motor-driven wave generator.
3. 2 x P.V.C. plastic paddles
4. Baffle.
5. Metre rule.
6. Two solid state pressure transducers and sensor amplifiers.
7. Tektronix digital storage oscilloscope

S.M. Tan, March 2001

Revised: R. Au-Yeung, February 2014

Revised: D.H. White, May 2016