Experiment 314: The Chaotic Pendulum

Aim

To study the motion of a simple non-linear system, and how it can give rise to chaotic motion. Data acquired from the apparatus are imported into PYTHON and analysed.

Note: Some familiarity with PYTHON is assumed. In particular, it is useful to have completed the second year laboratory experiment "Non-linear Curve Fitting". If you have any difficulties you should consult a demonstrator.

References

1. Baker and Gollub, "Chaotic Dynamics, an Introduction", Cambridge University Press (1990)
2. Sommerfeld, "Lectures on Theoretical Physics", Vol.1, Academic Press (1952), 531. S69Y.
3. Kreyszig, "Advanced Engineering Mathematics", Wiley (1988)
4. Blackburn and Smith, "Driven Pendulum for Studying Chaos", Rev. Sci. Instr., 60, 422, (1989).

Ref. 1 is available in the laboratory. This excellent book covers most aspects of this experiment and then some.

Ref. 2 is a clear and straightforward text on classical physics. Sommerfeld lived during the transition from classical to quantum physics and is mainly remembered for his attempts to explain atomic phenomena in a "semi-classical" manner.

Ref. 4 gives a thorough introduction to the pendulum assembly.

Introduction

This pamphlet contains some theory as well as a description of the apparatus and software. The actual experiment begins at section 5. It is important to carefully read the first four sections before attempting the experiment. Your written report should demonstrate your understanding of the theory, and contain answers to all questions asked throughout the pamphlet.

1 Free, frictionless pendulum

Many of the systems we study in physics are linear. If not, we often "linearise" them to simplify our analysis. The classic example is the freely swinging frictionless pendulum. The differential equation describing its motion is:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgr\sin\theta = 0 \tag{1}$$

This is difficult to solve analytically, since the $\sin\theta$ term introduces an obvious non-linearity. To tidy up the mathematics we demand that the amplitude θ be made vanishingly small so that we can put $\sin\theta \approx \theta$ (θ in radians). The solution then gives for the oscillation frequency:

$$\omega_0 = \sqrt{\frac{mgr}{I}} \tag{2}$$

You should be familiar with the properties of the pendulum system under this condition.

It is, however, possible to solve (1) analytically (see Ref. 2). As shown in the appendix:

$$\text{period } T = \frac{4K}{\omega_0} \quad \text{or} \quad \omega = \frac{\pi\omega_0}{2K}$$

where

$$K = \int_0^{\pi/2} \frac{dv}{\sqrt{1 - k^2 \sin^2 v}}$$

The quantity $k = \sin(\theta_m/2)$ is the "modulus". As will be seen later, K may be calculated using Python's `scipy` function `special.ellipk(M)`.

Since $K = \pi/2$ as $k \to 0$ (i.e., as $\theta_m \to 0$), $\omega \to \omega_0$ as expected. In the opposite extreme, $K \to \infty$ as $k \to 1$, and this corresponds to the amplitude $\theta_m = \pi$. For this case $\omega \to 0$. Thus, there can be a large variation of ω for this apparatus.

2 Free damped pendulum

If a damping term proportional to velocity is introduced, equation (1) becomes:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgr\sin\theta = 0 \tag{3}$$

Velocity dependent damping is obtained by inducing eddy currents in a fixed copper plate – see later. The linearised solution of this for the underdamped case where $\omega_0 > b/2I$ is:

$$\theta = \theta_m \mathrm{e}^{-\alpha t}\cos(\omega_1 t) \tag{4}$$

where

$$\alpha = \frac{b}{2I} \quad \text{and} \quad \omega_1^2 = \omega_0^2 - \alpha^2 \tag{5}$$

Thus, the period of free oscillation is determined both by damping and amplitude. See Ref. 3.

3 Damped, driven pendulum

The apparatus is also capable of supplying a sinusoidally varying torque. The differential equation of motion then becomes:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgr\sin\theta = \tau_0\sin(2\pi f t) \tag{6}$$

Introducing the "quality factor":

$$Q = \frac{\omega_0}{b/I} \tag{7}$$

we express (6) in the form:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\omega_0}{Q}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2\sin\theta = \frac{\tau_0}{I}\sin(2\pi f t) \tag{8}$$

where f is the drive frequency.

3.1 Chaotic motion

The necessary conditions for chaotic motion are that:

1. the system has at least three independent dynamical variables
2. the equations of motion contain a non-linear term that couples several of the variables.

To see that equation (8) satisfies these conditions, write it as a set of first order equations:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\omega_0}{Q}\omega - \omega_0^2 \sin\theta + \frac{\tau_0}{I}\sin\phi \tag{9}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \tag{10}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2\pi f \tag{11}$$

The necessary three variables are ω, θ and ϕ. The $\sin\theta$ and $\sin\phi$ terms are clearly non-linear and the former couples the variables θ and ω. Whether the motion is chaotic depends on the parameters τ_0/I, f and Q. For some values the pendulum locks on to the driving torque, oscillating in a periodic motion whose frequency is the driving frequency with possibly some harmonics and sub-harmonics.

For other parameter values, the pendulum frequency varies constantly in a chaotic manner. The transition to chaos occurs suddenly as one of the parameters moves through some critical value. The motion of a chaotic system depends heavily on the initial conditions. For non-chaotic systems an uncertainty in the initial conditions leads to an error in prediction that grows linearly with time. For chaotic systems the error grows exponentially with time so that after a short while the state of the system is essentially unknown. Remember, however, that the system is still correctly described by deterministic equations based on Newton's second law.

3.2 Phase diagrams

A phase diagram is a useful method for representing the motion. It is formed by plotting $\mathrm{d}\theta/\mathrm{d}t$-vs-$\theta$. For an undamped linear pendulum the phase diagram is a circle about the origin, and for a damped pendulum a spiral into the origin. For a non-linear pendulum it is more like an ellipse, and for a damped non-linear pendulum a spiral into the origin. See Ref. 1 for further explanation.

4 Description of the apparatus

4.1 Mechanical parts

Fig. 1 shows the pendulum, damping and driving hardware. The micrometer (F) controls the damping by moving the orange copper plate (B) with respect to the black annular magnet (C) which rotates with the pendulum. The position of the pendulum (θ) is determined by the optical encoder wheel (A).

The data are fed to an interface board in the computer where θ and $\mathrm{d}\theta/\mathrm{d}t$ are calculated, and these variables should be sampled every 1 ms or about 1000 times per second. It is recommended that you read Ref. 4 for a more detailed description of the pendulum apparatus. Torque is delivered to the pendulum by a motor.

The driving hardware is connected to the Pendulum Driver via an eight-way ribbon cable, and to the Chaotic Pendulum Computer Interface via a four-way ribbon cable. This last piece of apparatus is connected to the Pendulum Driver via a coaxial cable, and to the computer via a USB cable.

(1) Connect the eight-way cable with the 15-pin 'D' plug to the corresponding jack on the rear panel of the Pendulum Driver at "DRIVE AMPLITUDE".

(2) Connect the four-way ribbon cable with the 9-pin 'D' plug to the corresponding jack mounted on the Chaotic Pendulum Computer Interface box.

(3) Connect the dual banana plug-to-BNC cable supplied with the driver next. The dual banana plug end connects to the rear panel of the Pendulum Driver, into the pair of jacks labelled "OUTPUT SYNC". Be sure to observe correct polarity when making this connection (the tab on the side of the plug indicates the ground). The BNC end of the cable connects to the jack on the Chaotic Pendulum Computer Interface box.

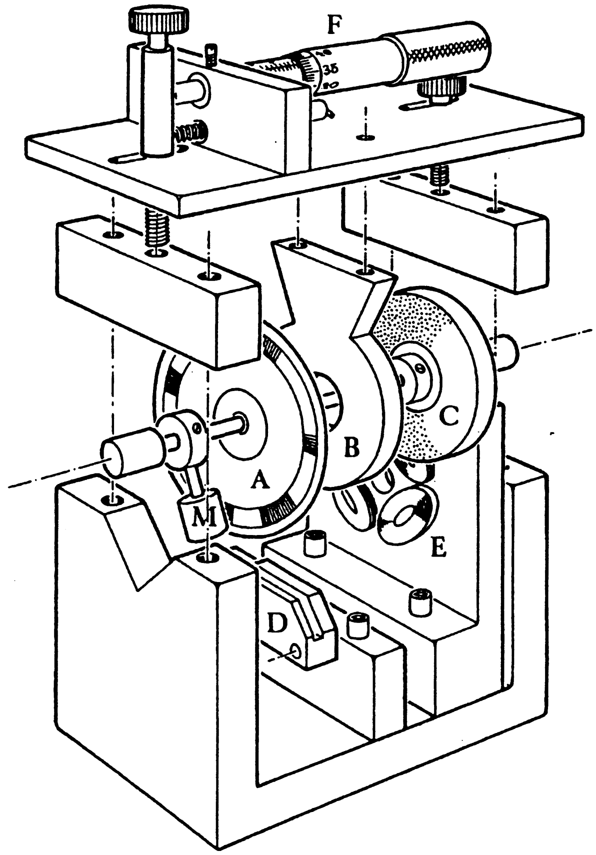


Figure 1: Exploded view of the damped, driven pendulum. A: Optical encoder wheel; B: Copper damping plate; C: Annular ring magnet; D: Encoder module; E: Drive coils; F: Micrometer; M: Pendulum bob.

(4) Connect the USB cable to the computer.

(5) Connect the BNC-to-Banana plug cable from “Drive Amplitude” on the back of the Pendulum Driver to the oscilloscope input.

This completes all connections required for Pendulum operation and computer data collection.

4.2 Expt 314-Chaotic Pendulum Software

Find Devices searches for available devices (specifically, e314) connected via USB cable.

Open Device opens a USB communications link to the selected device (e314). Conversely, **Close Device** closes the USB link.

Reset resets the Quadrature counter to zero.

Clear deletes your data.

Start starts data collection.

Stop stops collecting data collection.

Continuous plots data continuously.

Zoom All to view your entire graph.

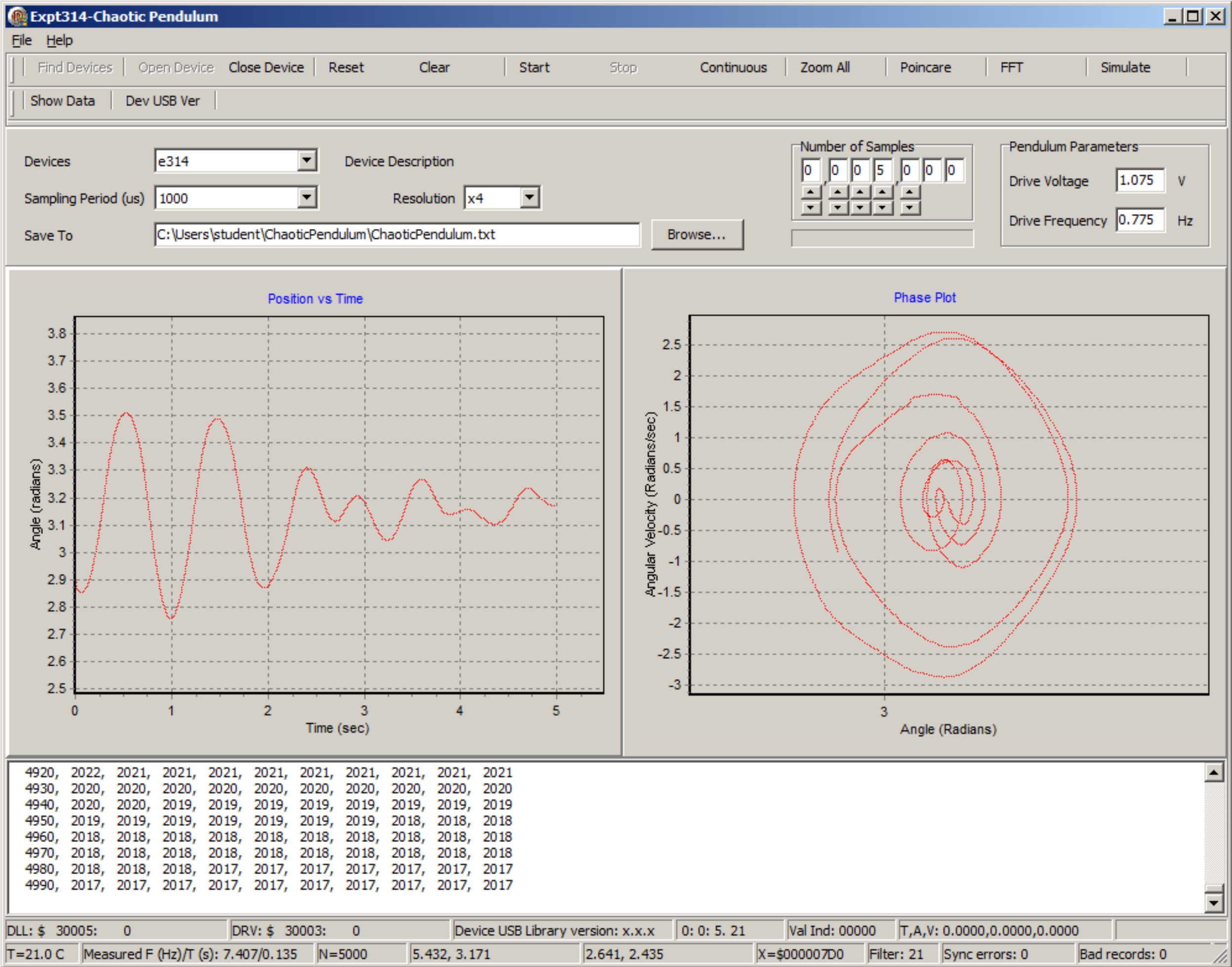


Figure 2: ChaoticPendulum Software

Poincare draws a Poincaré plot.

FFT calculates a Cooley Tukey fast Fourier transform to generate a Power spectrum from phase data.

Simulate uses a numerical fourth order Runge-Kutta routine to solve the driven pendulum equation. You enter various parameters to draw phase and Poincaré plots.

Show Data writes your data to memo. You must write the memo before saving data for the Position vs Time graph.

Sampling Period takes data at specified time intervals. For this experiment, a setting of 1000 microseconds (approximately 1000 samples per second) is sufficient.

Number of Samples effectively determines duration of data collection, e.g. taking 5000 samples for a 1000 μs sampling period gives 5 s worth of data.

Saving your data: Make sure you select **Show Data**. Then go to File–Save. Your data is saved in three separate .txt files: ChaoticPendulum (Position vs Time), ChaoticPendulum-Pha (Phase Plot) and ChaoticPendulum-Poi (Poincaré Plot). You may find it helpful to take a screenshot of your graphs by pressing Alt+PrtScn.

Start by adjusting the settings like in Fig. 2. You should use a sampling period of 1000 μs and resolution of x4 throughout the experiment. Please do not change the settings in File–Setup.

The two graphs in Fig. 2 are Position vs Time and Phase Plot. The latter displays phase-plane trajectories of the pendulum in real-time. Instantaneous pendulum angle θ in radians and angular velocity $d\theta/dt$ in radians per second are recorded by the system at intervals of 1 millisecond (approximately 1000 samples per second) and plotted as (x, y) points on the screen.

A "Poincaré plot" displays Poincaré sections in real-time. These are generated by recording $(\theta, \mathrm{d}\theta/\mathrm{d}t)$ once in each cycle of the harmonic drive torque. This forms a quasi-stroboscopic picture of the phase diagram. Non-chaotic motion produces one or more discrete spots. Chaotic motion results in a continuous distribution having the properties of a fractal - a mathematical set of non-integer dimension. Units are the same as for the phase plot.

5 Operating the pendulum

(6) Turn on the Pendulum Driver power switch and ensure that it is connected to the PC. After extended periods of operation the cover of the pendulum driver will get fairly hot. This is normal operation and should not be cause for alarm.

(7) **If the "DRIVE" switch on the Pendulum Driver is illuminated, press once to turn it off**. With the switch turned off, no torque should be applied to the pendulum. Note that this is not the same as setting the drive amplitude to zero without turning off this switch.

(8) Displace the pendulum bob slightly. If the copper damping plate (B) is close to the ring magnet (C), < 0.5 cm, the pendulum should halt its motion within a few swings. By turning the micrometer handle (F), the damping coefficient can be changed.

Be careful when adjusting the damping with the micrometer. **Do not move the copper plate so far that it touches the magnet**. If you are adjusting the damping while the pendulum is oscillating, it is generally advisable to move the copper plate towards the ring magnet. This provides a smoother, more positive operation.

(9) Click on the "Start" button. In the Phase Plot window, a spiral trace should appear when the pendulum bob is allowed to swing freely.

(10) Now, apply a torque to the pendulum. Turn on the "DRIVE" switch. If there is no motion, try turning up the "AMPLITUDE" control. To become familiar with the controls, try obtaining a response from the pendulum that is periodic. This is accomplished most easily with a high damping coefficient (< 0.5 cm) and small drive amplitudes.

Even when the pendulum is operating in a mode in which it approximates a linear system, the transient responses are slow. A settling time somewhere between several seconds and several minutes is normal.

(11) When a stable, periodic response is obtained, experiment by adjusting the "AMPLITUDE" and "FREQUENCY" controls and observing their effects.

(12) If you haven't already done so, quantify the output amplitude of the drive torque by connecting an oscilloscope to the output jacks on the Pendulum Driver. A slowly changing sine wave can be observed.

6 Determination of Q as a function of micrometer setting

The purpose of this section is to fit the data for the undriven pendulum to theory.

Note: The following functions can be found in the `phylab.py` file.

Equation 4 predicts an exponential decay in amplitude with a frequency change dependent on the damping. The theory given in the Appendix also shows that the frequency depends on the amplitude. Thus, combining the amplitude decays according to

$$\theta = \theta_m \exp\left(-\alpha t\right) \cos\left\{ \int_0^t \frac{\pi \omega_0}{2K\left(k\right)}\, dt' + \phi \right\} + \text{ const} \tag{12}$$

where $K(k)$ is the elliptic integral whose modulus k depends on the time-dependent amplitude of the oscillation, i.e. $k(t') = \sin(\theta_m \exp(-\alpha t')/2)$.

In Eqn. 12 the integral is necessary since the term $\omega = \pi\omega_0/(2K(k))$ is a function of time. So we need to add up all the $\omega t'$ values between $t' = 0$ and $t' = t$ to get the correct angular displacement and phase. This can be done neatly in PYTHON by using `expfit`, which minimises the function in `ellipmodel`. Starting values for the fit parameters, `c`, must first be loaded into `expfit`. Use `c = [1 1 1 1 1]` initially, then try new `c` values if the fit is not good.

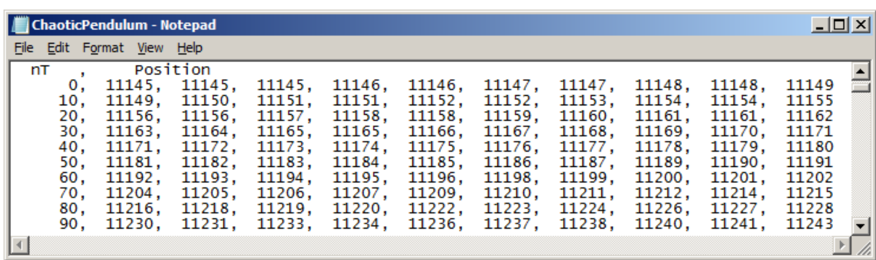
Make a fit for a number of micrometer values between 4 mm and 10 mm.

(13) Make sure the "DRIVE" switch is turned off.

By considering the sampling rate, adjust the number of samples to cover the whole decay time of the pendulum. Carefully displace the pendulum through about 180° and release it just before collecting data.

(14) Import data into PYTHON to perform further analysis. You may like to delete the header in the .txt file so that PYTHON can easily read the file. You should also delete the first few lines of data if they are constant. This can occur if you started data collection before releasing the pendulum. Such constant data interferes with fits to be made later.

Note: In the Position vs Time .txt file (Fig. 3), the first column represents the number of 1 ms intervals. Each row to the right of this column shows the pendulum position after each interval. These Position values are arbitrary numbers that represent how many of the code wheel slots have passed a reference line since you switched on the Pendulum Driver (see Ref. 4). Increasing and decreasing values represent different directions of rotation. To make sense of these values, convert Position to something meaningful, e.g. radians.




```
 nT   ,     Position
    0,  11145,  11145,  11145,  11146,  11146,  11147,  11147,  11148,  11148,  11149
   10,  11149,  11150,  11151,  11151,  11152,  11152,  11153,  11154,  11154,  11155
   20,  11156,  11156,  11157,  11158,  11158,  11159,  11160,  11161,  11161,  11162
   30,  11163,  11164,  11165,  11165,  11166,  11167,  11168,  11169,  11170,  11171
   40,  11171,  11172,  11173,  11174,  11175,  11176,  11177,  11178,  11179,  11180
   50,  11181,  11182,  11183,  11184,  11185,  11186,  11187,  11189,  11190,  11191
   60,  11192,  11193,  11194,  11195,  11196,  11198,  11199,  11200,  11201,  11202
   70,  11204,  11205,  11206,  11207,  11209,  11210,  11211,  11212,  11214,  11215
   80,  11216,  11218,  11219,  11220,  11222,  11223,  11224,  11226,  11227,  11228
   90,  11230,  11231,  11233,  11234,  11236,  11237,  11238,  11240,  11241,  11243
```

Figure 3: Position vs Time data

(15) Analyse the data for each value of the micrometer setting and plot Q vs *micrometer setting*. Determine the natural frequency of the pendulum.

You should plot some of your data with fits on the same graph. Comment on the fits.

(16) As an optional extra adapt your script to fit the velocity curve and hence plot the phase diagram with its fit.

(17) Use the following script to perform a Fourier transform of the position data. (This is not in `phylab.py`.) (You may like to also compare your results with those produced by the software's **FFT** option.) Comment on the results.

```
import numpy as np
import matplotlib.pyplot as plt

Y = np.fft.fft(pos,n=4096)       # transform the data, adjusting length of vector to 4096
Pyy = Y*Y.conj()/4096.0          # Calculate the power in each frequency component
Nyy = Pyy/sum(Pyy)               # Normalise the values of power so that their sum is one
f = 500*np.arange(0,2048)/4096.0 # Define frequency values in s^(-1)
```

```
plt.figure()
plt.plot(f[0:80],Nyy[0:80])      # Plot lowest 80 frequency components in the spectrum
plt.xlabel('Frequency (Hz)')
plt.ylabel('Power Spectrum')
```

7 Conditions for chaos

In this section you will do a set of runs in which just one parameter is varied, the others being kept constant. In principle, this could be either the damping, driving amplitude or driving frequency. A possible combination would be to fix the driving frequency at 2.0 Hz and the damping micrometer at 5 mm.

(18) Do up to 20 runs for values of the driving amplitude between about 0.7 V and 3.0 V, as measured from the slowly oscillating trace on the oscilloscope. The associated Poincaré plots will have a single spot for the non-chaotic situation and the FFT (which you will calculate later) will show just the 2 Hz driving frequency. Regions of period doubling, tripling and quadrupling should be observable and indicated by 2, 3 or 4 spots on the Poincaré diagram. Regions of chaotic behaviour will produce a continuous Poincaré plot which has the properties of a fractal. Observe the physical motion of the pendulum during these runs.

(19) For each run take about 20 seconds of data. Read the phase plot files into PYTHON to:

(a) plot the pendulum velocity and position on the y-axis vs time on the x-axis. Show only about four or five cycles.

(b) do a Fourier analysis of the position data

(c) show the phase diagram and Poincaré plot.

8 Python simulations

This section simulates the dynamics of a chaotic pendulum by using the functions in `phylab.py`. You should open up this programme and study the PYTHON code to get a general sense of how these work. You need to run each of these functions for two or three sets of parameters and discuss the physics behind your results.

For the `ginput` function, you will interactively select data points off a logistic map. This can be done by saving any changes you made to `phylab.py`, then run `ginput` by pressing F10 in Spyder. You will get a pop-up window of the logistic map.

(20) Run the `pendulum` function. This solves the dimensionless form of the damped, driven pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{1}{Q}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \sin\theta = g\,\cos\omega t$$

which is specified by the function `pendeq`. This form is used as it minimises computation time. As with the real pendulum you can experiment with variables Q, g and ω and observe the effects. Compare your results with those in Ref. 1, pp. 47 – 54.

(21) Bifurcation diagrams. Run `bifurc`. This is a plot of driving force, g vs *angular velocity* values taken from the Poincaré plot. It gives an overall picture of the dynamics of the system showing regions of chaos, frequency doubling, etc. This script takes some time to run. The progress of the calculation is shown by the value of `j` in the Spyder console. The calculation finishes when `j = 111`.

(22) Perform a series of experiment runs similar to those in section 7. Note that each point on the Poincaré plot corresponds to one cycle of the driving frequency. Roughly 100 points is sufficient. By plotting just the velocity column for each value of the driving force you can build up a crude experimental bifurcation diagram having similarities to the simulated one for a pendulum.

(23) Although the pendulum is the simplest physical system exhibiting chaotic behaviour, its simulation involves solving a difficult non-linear differential equation. However there are other mathematical models not involving differential equations which also exhibit similar chaotic properties. One of these is the difference equation. The standard example is the so called "logistic map"

$$x_{n+1} = \mu x_n \left(1 - x_n\right), \qquad x_n \in [0, 1]$$

This is described in Ref. 1, p. 77. Run `logisticmap` and you will see the diagrams shown on pp. 78 – 80. Now run `logistic` and a bifurcation diagram qualitatively similar to that for the pendulum will be obtained. Because computation is faster, this diagram is much more detailed.

(24) Period doubling. Bifurcation points are clearly evident with the logistic map. It turns out that the μ values where bifurcation occurs are related to a universal constant, the Feigenbaum number, which is independent of the nature of the map or the differential equation describing the motion. If the first bifurcation occurs at μ_1, the second at μ_2, etc. then the Feigenbaum number is defined by:

$$\lim_{k \to \infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = 4.669\ldots$$

Check this approximately by using the `ginput` function to determine the μ values of the bifurcation points.

(25) Run `logistic` over a small range of μ, say 3.52 to 3.6. This "magnified" diagram has the same form as the original in the range $2.6 < \mu < 3.9$. This is characteristic of a fractal.

List of Equipment

1. Pendulum assembly
2. Pendulum Driver box with frequency meter
3. Chaotic Pendulum Computer Interface box
4. Oscilloscope
5. PC with Pendulum software and Python installed
6. Python file: `phylab.py`, which contains the functions `expfit`, `ellipmodel`, `bifurc`, `logistic`, `logisticmap`, `pendeq`, and `pendulum`.

R. Garrett: May, 2001.

Updated by ...

M. Hoogerland

October 7, 2009.

February 16, 2011

Updated: R. Au-Yeung, H. Oudenhoven, August 8 2013

Python revision: R. Au-Yeung, November 21 2014

Appendix

Conservation of energy gives:

$$\frac{I}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + mgh = mgH \tag{13}$$

where

$$h = r\left(1-\cos\theta\right) \quad \text{and} \quad H = r\left(1-\cos\theta_m\right). \tag{14}$$

Substituting (14) into (13) gives:

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = 2\omega_0^2\left(\cos\theta - \cos\theta_m\right) \tag{15}$$

Applying the equality:

$$\cos\theta - \cos\theta_m = 2\left[\sin^2\left(\theta_m/2\right) - \sin^2\left(\theta/2\right)\right]$$

to (15) gives

$$\omega_0\,\mathrm{d}t = \frac{\mathrm{d}\theta}{2\sqrt{\sin^2\left(\theta_m/2\right) - \sin^2\left(\theta/2\right)}} \tag{16}$$

which may be transformed by making the substitution:

$$\sin\left(\theta/2\right) = \sin\left(\theta_m/2\right)\sin v \tag{17}$$

then

$$\sqrt{\sin^2\left(\theta_m/2\right) - \sin^2\left(\theta/2\right)} = \sin\left(\theta_m/2\right)\cos v \tag{18}$$

and

$$\frac{\mathrm{d}\theta}{2\sqrt{\sin^2\left(\theta_m/2\right) - \sin^2\left(\theta/2\right)}} = \frac{\mathrm{d}v}{\cos\left(\theta/2\right)} = \frac{\mathrm{d}v}{\sqrt{1-k^2\sin^2 v}} \tag{19}$$

where the quantity $k = \sin\left(\theta_m/2\right)$ is called the "modulus". Equation (16) then becomes:

$$\omega_0\,\mathrm{d}t = \frac{\mathrm{d}v}{\sqrt{1-k^2\sin^2 v}} \tag{20}$$

In order to calculate the period T, we integrate the LHS of (20) from 0 to $T/4$ and the RHS of (20) from 0 to a limit corresponding to θ_m. Putting $\theta = \theta_m$ in (18) gives this limit as $v = \pi/2$. This then yields T in terms of a "complete elliptic integral of the first kind":

$$T = \frac{4K}{\omega_0} \quad \text{or} \quad \omega = \frac{\pi}{2K}\omega_0 \tag{21}$$

where

$$K = \int_0^{\pi/2} \frac{\mathrm{d}v}{\sqrt{1-k^2\sin^2 v}} \tag{22}$$