

Experiment 319: Viscosity of a Gas

Aim

To measure the viscosity of a gas at various temperatures.

References

1. *Thermal Physics*, Kittel, C., Kroemer, H., 2nd ed. (An excerpt can be found in Appendix B.)

Experimental Set-up

WARNING! This experiment involves potentially hazardous materials and equipment. DO NOT operate ANY of the equipment until you understand the procedure completely AND have consulted with a demonstrator.

Theory

In a gas, forces can arise within the gas if there are velocity gradients within the gas. The viscosity η (strictly the ‘coefficient of shear viscosity’) relates the forces to the velocity gradients (Equations 12 and 13 give further information, but not the complete picture). Applying $\mathbf{F} = m\mathbf{a}$ to the steady flow of an ideal gas through a capillary suggests the mass flow rate \dot{m} (mass of gas per second) entering (or leaving) the capillary is *approximately* given by

$$\dot{m} = \frac{\pi\chi}{16\eta} \frac{R^4}{L} [P(0)^2 - P(L)^2] \quad (1)$$

where χ relates density and pressure (see Equation 19), R and L are the radius and length of the capillary, and $P(0)$ and $P(L)$ are the pressures at the start and end of the capillary. There is more on this in Appendix A. Note how the flow rate scales with the different quantities in this equation (a low viscosity, for example, will give a high flow rate).

As well as being of direct relevance to any situation involving fluid flow, the study of viscosity also gives insight into the nature of matter. To give an example, in the kinetic theory of gases viscosity arises because molecules in neighbouring pieces of gas are exchanged by diffusion. Such diffusion exchanges not only molecules but also the momentum carried by the molecules — the exchanges in momentum result in the viscous forces. Using kinetic theory the viscosity is predicted to be $\bar{c}l\rho/3$ where \bar{c} and l are the mean speed and mean free path of a gas molecule and ρ is the gas density (mass per unit volume). For a Maxwellian distribution of particle velocities, $\bar{c} = \left(\frac{8kT}{\pi M}\right)^{\frac{1}{2}}$ where M is the mass of a molecule, k is Boltzmann’s constant, and T is the absolute temperature. For information about mean free path see Appendix B.

Question 1: Show that kinetic theory predicts

$$\eta = \frac{(8kT)^{\frac{1}{2}} M^{\frac{1}{2}}}{3\pi^{\frac{3}{2}} d^2} \quad (2)$$

where d is the molecular diameter.

Equation 2 predicts that the viscosity of a gas will increase with temperature (in contrast to most liquids, where the viscosity decreases with temperature) — if the predicted temperature dependence is not observed experimentally, the kinetic theory is sure to be wrong.

Method

The classical method for determining the viscosity of a gas is to observe the rate of gas flow along a capillary tube. A diagram of the experimental apparatus is shown in Figure 1.

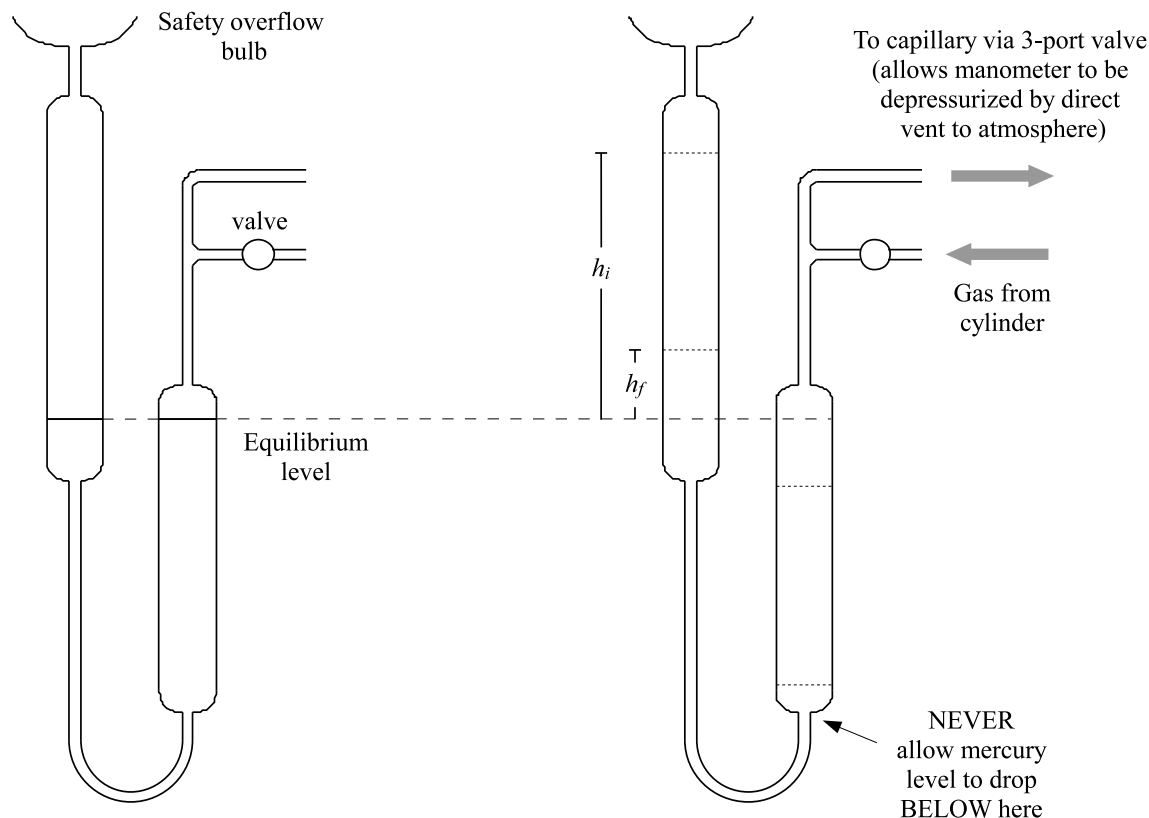


Figure 1: Schematic diagram of the wide-bore manometer before use (left). In operation (right), gas from the cylinder regulator is admitted through the valve. The mercury is displaced from the equilibrium position. The gas will slowly escape through the capillary. After filling, the time is measured for the mercury to drop from a convenient initial height h_i to a final height h_f . The dead volume V_0 is the volume of gas between the equilibrium level (on the right limb of the manometer) and the capillary.

Argon gas from the pressure cylinder is introduced into one limb of a wide-bore mercury manometer. The gas escapes to the atmosphere through the capillary. The viscosity is determined by measuring the time taken for the inlet pressure to drop from an initial to a final value. It can be shown (see Appendix C) that

$$\Delta t = \eta K \quad (3)$$

where Δt is the time taken and K is given by

$$K = \frac{4}{\pi \rho_m g} \frac{L}{R^4} \left[\left(A + \frac{2V_0}{H} \right) \ln \frac{h_i}{h_f} + \left(3A - \frac{2V_0}{H} \right) \ln \left(\frac{H + h_i}{H + h_f} \right) \right] \quad (4)$$

where

L : Length of capillary (1.0 m)
 R : radius of capillary ($125 \mu\text{m}$)
 g : gravity (9.8 ms^{-2})
 $h_{i,f}$: initial/final height of mercury

A : Cross-sectional area of manometer (18.35 mm diameter)
 ρ_m : density of mercury (13533.6 kgm^{-3})
 H : Atmospheric pressure (in height of mercury)
 V_0 : Dead volume

Procedure

Familiarisation

DO NOT operate ANY of the equipment until you understand the procedure completely AND have consulted with a demonstrator.

- (1) Flush out the manometer with a stream of gas. Follow the directions below exactly. These directions are the safest way to pressurise the manometer. **You must perform this procedure in the presence of a demonstrator initially.**
 - (a) Check that the **cylinder valve** is **closed**. The cylinder valve is closed when tightened fully **clockwise**.
 - (b) Check that the **regulator valve** is **closed**. The regulator valve is closed when the control knob has been loosened fully **counter-clockwise**.
 - (c) Check that the **manometer valve** is **open**. The manometer valve is open when the valve knob is **horizontal**.
 - (d) Re-check items (a) - (c).
 - (e) Check that the hose connecting the manometer to the capillary heat bath is secure.
 - (f) **Open** the cylinder valve.
 - (g) **Very gradually** open the regulator valve by slowly turning the regulator knob clockwise. **Always** keep your eyes on the manometer level.
NEVER ALLOW THE MERCURY TO DESCEND BELOW THE WIDE BORE SECTION.
 - (h) Once you have pressurised the manometer to the required level, **close** the **manometer valve**. Immediately **close** the **regulator valve** completely.

In order to re-pressurise the manometer, carefully open the manometer valve. Be aware that there will be some residual pressure between the manometer and regulator, and this will cause the mercury level to move slightly. Perform steps (g) and (h) above.

Always be aware of the state of the equipment, the position of valves and the pressure of gas in the manometer. Any sudden increase or decrease in pressure could result in glass breakage and mercury spillage.

When you are finished with the apparatus, ensure that the manometer is depressurised. Close the cylinder valve. Close the regulator valve.

Data acquisition

This experiment requires you to record the time required for the pressure to drop by a certain amount as the argon gas flows out of the capillary. You will make several readings at various temperatures. The temperature of the gas in the capillary is determined by the water in the heat bath. The barometer in the lab can be used to obtain the atmospheric pressure.

- (2) Fill the heat bath with water. Ensure that the water level is just below the end of the capillary.
- (3) Allow a few minutes for the heat bath and water to thermalise. Take measurements of the temperature of the water in the heat bath. Once this temperature is stable, proceed.
- (4) Pressurise the manometer to about $h_i = 14\text{cmHg}$. Record the time taken for the pressure to drop to $h_f = 8\text{cmHg}$.
- (5) Repeat the measurements several times. Record the water temperature regularly.

- (6) At this point it would be wise to perform the **dead volume measurement** as outlined below. You will then return to perform the following items.
- (7) Carefully add ice to the water in the heat bath. You will need to drain some water from the heat bath to keep the capillary end clear of the water. Monitor the water temperature and when a stable level is reached, perform items 3 to 5.
- (8) Turn on the thermostatically controlled element. Set the thermostat to medium level and allow the water to heat. Pay attention to the water level in the heat bath. Ensure that the water level does not rise above the capillary end. Monitor the water temperature. When the temperature is stable, perform items 3 to 5.
- (9) Repeat Procedure 8 for several other temperatures between 30 and 80 degrees.

Measurement of the Dead Volume

The dead volume, V_0 , of the manometer is the volume of gas bounded by the equilibrium mercury level in the right hand limb of the manometer and the capillary. The dead volume can be determined by collecting the volume of gas that leaves the capillary as the pressure in the manometer drops from an initial value h_i to a final value h_f . As the pressure drops, the gas in the dead volume will expand. It can only expand by leaving through the capillary. Hence the larger the dead volume, the more gas is expected to leave the capillary.

- (10) Add water to the heat bath such that the water covers the capillary end by about 0.5 cm. Completely fill the measuring cylinder with water. Using the glass flat, invert the cylinder and immerse the open end in the heat bath water. Just allow the cylinder to stand in the heat bath at this stage, do not clip it into the cylinder stand.

You will need to be clear on the sequence of actions that you need to perform in this part of the experiment. You will pressurise the manometer as before. You will see a stream of bubbles from the capillary end. Once the pressure reaches a predetermined level, carefully place the cylinder over the capillary end and clip into place. When the pressure drops to a predetermined level, record the volume of gas collected. In order to get your timing right, you will need to get a feel for how fast the pressure drops at various pressures.

- (11) Pressurise the manometer until the height of mercury above equilibrium is about 15 cm. As the pressure drops from $h_i \simeq 15$ cm to $h_f \simeq 10$ cm, collect the gas escaping from the capillary end. This collected volume is V_c .
- (12) Repeat Procedure 11 several times to obtain an average for V_c .

The initial pressure of the trapped gas in the right hand limb of the manometer is $P_a + 2h_i\rho_m g$ where P_a is atmospheric pressure. The volume of the trapped gas is $Ah_i + V_0$, where A is the cross-sectional area of the manometer. The final pressure in the right hand limb of the manometer is $P_a + 2h_f\rho_m g$ and the volume is $Ah_f + V_0$. The gas in the measuring cylinder at the end of the run has volume V_c and is at a pressure which depends on the height of the water column in the cylinder, h_w . Note that, while the gas in the manometer is, to good approximation, pure argon, the gas in the measuring cylinder is a mixture of argon and water vapour.

Question 2: Using the empirical results for gases in Appendix D, show that

$$V_0 = \frac{1}{2\rho_m g(h_i - h_f)} [V_c(P_a - \rho_w g h_w - P_w) - P_a A(h_i - h_f) - 2A\rho_m g(h_i^2 - h_f^2)] \quad (5)$$

where ρ_w is the density of water and P_w is the saturation vapour pressure of water at the appropriate temperature.

Analysis

- (13) Using your data and Equations 3 and 4, compute the viscosity of argon gas at the various temperatures. Tabulate and graph your results. Include an error analysis and error estimates and a comparison with accepted values.
- (14) Estimate the size of an argon atom.
- (15) Provide answers to *all* questions in your write-up.

Appendix A The origin of Equation 1

The derivation of Equation 1 involves the following assumptions

Assumption 1: There is a steady state (the flow is not changing with time) and uniform temperature.

Assumption 2: The flow at all positions is parallel to the axis of the capillary. Note that this is not the only possibility (the flow could, for example, corkscrew around the axis of the capillary).

Assumption 3: Pressure is a function of z only (refer to Figure 2 for the definition of coordinates). For the flow of a *compressible* material (such as a gas), it can be shown that this assumption and Assumption 2 are contradictory — they can't both be exactly true.

Assumption 4: The speed of the flow at position (z, r, θ) is given by

$$v(z, r) = -\frac{1}{4\eta} P'(z) (R^2 - r^2) \quad (6)$$

where $P'(z)$ is the pressure gradient (derivative of P with respect to z) evaluated at the position z .

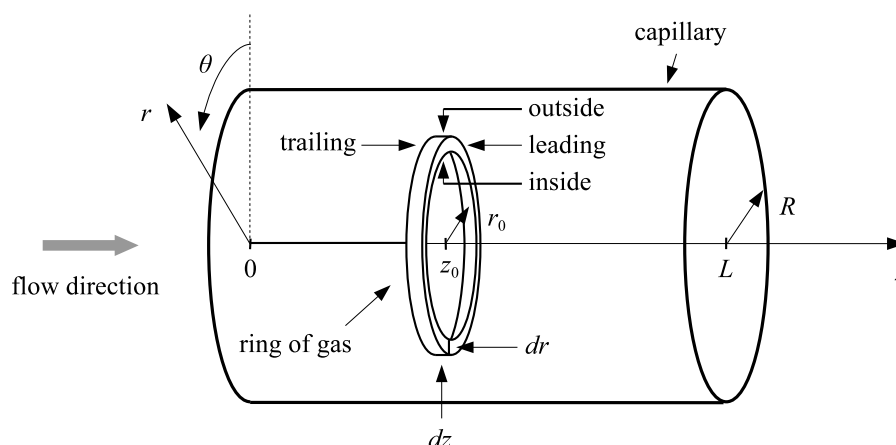


Figure 2: Capillary and coordinate frame used. The radial scale is of course drawn greatly exaggerated.

The remainder of this appendix will focus on applying $\mathbf{F} = m\mathbf{a}$ to the ring of gas in Figure 2 to show that Equation 6

- Cannot be exactly correct if the flow is compressible and Assumptions 1–3 hold.
- Is likely to be a good approximation for the flows in this experiment.

Begin with the acceleration. In the experiment, the pressure at the inlet of the capillary is greater than the pressure at the outlet of the capillary — gas moving along the capillary experiences decreasing pressure and therefore expands. Assumption 2 restricts expansion to the z direction. Such expansion requires the speed of the gas to increase as it moves down the capillary i.e. it requires that the gas be accelerating. Contrast this with steady *incompressible* flow (as studied in Experiment 217, *Viscosity and Critical Velocity of Water*) in which all parts of the fluid can move at constant speed and the acceleration be everywhere zero.

Question 3: To begin quantification of the acceleration, show that the acceleration of the ring of gas in Figure 2 is given by

$$a(z_0, r_0) = \frac{\partial v(z_0, r_0)}{\partial z} v(z_0, r_0) \quad (7)$$

where $\partial v(z_0, r_0)/\partial z$ is the partial derivative of $v(z, r)$ with respect to z evaluated at the point (z_0, r_0) . *Hint:* The only physics you need is $a = dv/dt$ and $v = dz/dt$.

To continue quantification of the acceleration, consider the amount of gas flowing past a point (z_0, r, θ) . The mass flow rate through a small area dA centred on the point and normal to the flow is

$$\rho(z_0) v(z_0, r) dA \quad (8)$$

(the gas density ρ is assumed a function of z only since $\rho \propto P$ — Equation 19 — and, by Assumption 3, the pressure is a function of z only). The mass flow rate through a similar area at a second point (z, r, θ) , up- or down- stream of the first point, is $\rho(z)v(z, r)dA$. These two mass flow rates must be equal (otherwise there would be change in the amount of mass between the two points, violating Assumption 1). Hence $\rho(z)v(z, r) = \rho(z_0)v(z_0, r)$ for all z from which it follows that $\rho(z)v(z, r)$ is a function of r only (it does *not* depend on z).

Question 4: By differentiating $\rho(z)v(z, r)$ with respect to z , show that, for the flow

$$\frac{\partial v(z, r)}{\partial z} = -\frac{v(z, r)}{\rho(z)} \rho'(z). \quad (9)$$

Question 5: Using Equations 7, 9, 19, and 6 show that, for the ring of gas,

$$ma = -\frac{\rho(z_0)}{16\eta^2} \frac{P'(z_0)^3}{P(z_0)} (R^2 - r_0^2)^2 2\pi r_0 dr dz. \quad (10)$$

Turning now to the forces acting on the ring of gas, three can be identified as follows¹:

F_1 — due to pressure: Assuming pressure is a function of z only (Assumption 3), the pressure forces on the inside and outside surfaces of the ring cancel one another. The resultant force due to pressure on the leading and trailing surfaces of the ring is

$$F_1 = -P(z_0 + dz)2\pi r_0 dr + P(z_0)2\pi r_0 dr = -P'(z_0) 2\pi r_0 dr dz \quad (11)$$

($2\pi r_0 dr$ is the area of the leading and trailing surfaces).

F_2 — due to viscosity on inside and outside surfaces: There is a viscous force on the inside surface of the ring of

$$-\eta \frac{\partial v(z_0, r_0)}{\partial r} 2\pi r_0 dz \quad (12)$$

($2\pi r_0 dz$ is the area of the inside surface of the ring) — compare with Equation 1 and Figure 1 in Experiment 217. The physical origin of this force is as follows: The gas in the region $[z_0, z_0 + dz] \times [0, r_0]$ (nearer the axis of the capillary than the ring) is moving faster than the gas in the ring. Exchange of gas by diffusion across the inside surface of the ring therefore results in a net transfer of momentum to the ring — hence the force. There is a similar viscous force on the outside surface of the ring. Let F_2 be the resultant of inside and outside forces. Using Equation 6 it is straightforward to show that $F_2 = -F_1$.

¹The capillary is assumed to be in zero-g — including gravity is easy (so can be done if required) and doesn't change any of the conclusions.

F_3 — due to viscosity on leading and trailing surfaces of the ring: There is a viscous force on the leading surface of the ring of

$$\frac{4}{3} \eta \frac{\partial v(z_0 + dz, r_0)}{\partial z} 2\pi r_0 dr. \quad (13)$$

The physical origin of this force is as follows. The gas immediately ahead of the ring is moving faster than the gas in the ring (because of the acceleration of the gas along the capillary). Exchange of gas by diffusion across the leading surface of the ring therefore results in a net transfer of momentum to the ring — hence the force. There is a similar viscous force on the trailing surface of the ring. The resultant of leading and trailing forces is

$$F_3 = \frac{4\eta}{3} \frac{\partial^2 v(z_0, r_0)}{\partial z^2} 2\pi r_0 dr dz. \quad (14)$$

Using results already established in this appendix and Equation 19, this can be converted to

$$F_3 = -\frac{P'(z_0)^3}{P(z_0)^2} (R^2 - r_0^2) 2\pi r_0 dr dz. \quad (15)$$

Question 6: Using the following values (which are order-of-magnitude estimates for those in the experiment) show that, for the experiment, $|F_1|$ ($= |F_2|$) $\gg m|a| \gg |F_3|$.

$P(z_0)$	\approx	atmospheric pressure	\approx	10^5 Pa
$P'(z_0)$	\approx	$[P(L) - P(0)]/L$	\approx	-10^4 Pa m^{-1}
$\rho(z_0)$	\approx	density of typical gas at room conditions	\approx	1 kg m^{-3}
η	\approx	viscosity of typical gas at room conditions	\approx	$10^{-5} \text{ Pa s} (= \text{kg m}^{-1} \text{ s}^{-1})$
R			\approx	10^{-4} m
r_0			\approx	$R/2$

Conclusions: For an incompressible flow, with acceleration zero everywhere, both ma and F_3 are zero. Given Assumptions 1–3, Equation 6 then results in $F = ma$ being exactly satisfied (so Equation 6 can be the correct description for the flow). For compressible flow (as is actually the case in the experiment), Equation 6 results in a failure to satisfy $F = ma$ — Equation 6 does not give a correct description of the flow. Equation 1 is therefore also not exactly correct. However, Equation 6 must be approximately correct — a small modification to the equation will introduce a small mismatch between the dominant F_1 and F_2 terms, a mismatch which could balance the remaining (small) F_3 and ma terms.

Final remarks:

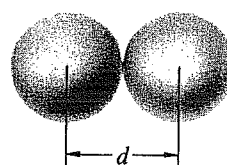
- Remaining steps to obtain Equation 1: Using Equations 6 and 19 in Equation 9 yields the differential equation $PP'' = -(P')^2$. This has solution $P(z) = \sqrt{P(0)^2 + [P(L)^2 - P(0)^2]z/L}$. The mass flow rate (Equation 1) is then straightforward to obtain by integrating $\rho(z)v(z, r)$ (using Equations 19 and 6) over any cross-section of the capillary.
- For the conditions encountered in the experiment, Equation 1 is known to be accurate to within 0.1% and more accurate approximations continue to be developed (see, for example, Prud'homme et al., *Appl. Sci. Res.* **43** (1986) 67–74 or Berg et al., *J. Fluid Mech.* **246** (1993) 1–20).

Appendix B Excerpt from Kittel & Kroemer

Collision Cross Sections and Mean Free Paths

We can estimate the collision rates of gas atoms viewed as rigid spheres. Two atoms of diameter d will collide if their centers pass within the distance d of each other. From Figure 14.4 we see that one collision will occur when an atom has traversed an average distance

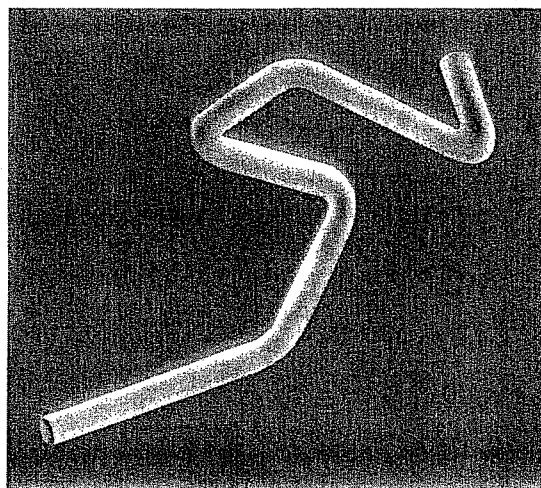
$$l = 1/n\pi d^2, \quad (13)$$



(a)

Figure 14.4 (a) Two rigid spheres will collide if their centers pass within a distance d of each other. (b) An atom of diameter d which travels a long distance L will sweep out a volume $\pi d^2 L$, in the sense that it will collide with any atom whose center lies within the volume. If n is the concentration of atoms, the average number of atoms in this volume is $n\pi d^2 L$. This is the number of collisions. The average distance between collisions is

$$l = \frac{L}{n\pi d^2 L} = \frac{1}{n\pi d^2}.$$



(b)

where n is the number of atoms per unit volume. The length l is called the **mean free path**: it is the average distance traveled by an atom between collisions. Our result neglects the velocity of the target atoms.

Appendix C The origin of Equations 3 and 4

Equation 1 assumes (among other things) a steady state (see Appendix A). The experiment is not steady — the pressure at the inlet of the capillary (the pressure in the right hand limb of the manometer), $P(0)$, decreases over time. However, over the time it takes a piece of gas to traverse the length of the capillary, the fractional change in inlet pressure is very small (a consequence of the volume of the capillary being very much smaller than the volume in the right hand limb of the manometer). The rate at which gas is removed from the right hand limb of the manometer is therefore assumed as given by Equation 1.

Question 7: Let h be the instantaneous height above equilibrium of mercury in the manometer. Using Equation 19, show that the mass of gas in the right hand limb of the manometer (including the dead volume) is given by

$$m = \chi (H + 2h) \rho_m g (Ah + V_0). \quad (16)$$

Question 8: By differentiating Equation 16 with respect to time and equating to Equation 1 show that, in a short time dt , the height of mercury in the manometer changes by dh where

$$dt = \frac{4\eta}{\pi \rho_m g} \frac{L}{R^4} \frac{-(4Ah + AH + 2V_0)}{h^2 + Hh} dh. \quad (17)$$

Integrating Equation 17 over a run in which h drops from h_i to h_f in a time Δt immediately gives

$$\Delta t = \underbrace{\eta \frac{4}{\pi \rho_m g} \frac{L}{R^4} \int_{h_i}^{h_f} \frac{-(4Ah + AH + 2V_0)}{h^2 + Hh} dh}_K. \quad (18)$$

which is Equation 3. The remaining integral can be evaluated easily using a partial fraction decomposition and this yields Equation 4.

Appendix D Some empirical results for gases

Density-pressure relationship

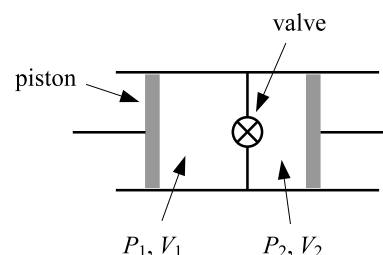
For gases at constant temperature

$$\rho = \chi P \quad (19)$$

where χ is a constant that depends on the temperature and the chemical composition of the gas (it is straightforward to check this is consistent with the ideal gas law).

Sample of gas partitioned between two enclosures

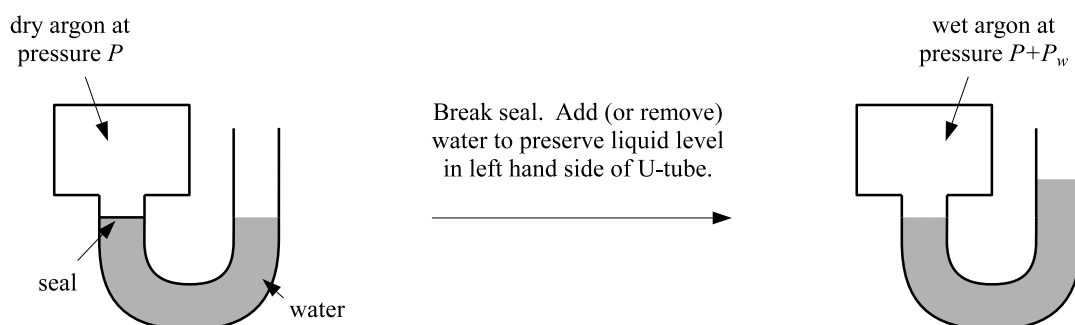
Argon trapped in two enclosures with interconnecting valve, the whole being at a single, fixed, temperature. Regardless of how the pistons and valve are manipulated, $P_1 V_1 + P_2 V_2$ does not change.



Comments: This result is true for any gas or mixture of gases (provided there are no chemical reactions, phase transitions, long-range molecular interactions, etc.). The theoretical explanation is that $P_1 V_1 + P_2 V_2$ is proportional to the total number of particles (ideal gas law) and the number of particles is conserved.

Dry gas placed in contact with water

Dry argon (pure argon gas — no water vapour) is trapped in an enclosure. The gas has pressure P . The gas is brought into contact with liquid water. Neither the volume nor the temperature of the gas is allowed to change. The pressure of the gas is observed to increase to $P + P_w$.



Comments: This result is true for any gas or mixture of gases (excluding gases which are chemically reactive with, or soluble in, water). P_w is independent of the gas, the gas volume, and P but is a function of temperature. The theoretical explanation is that water molecules leave a water surface (to the gas phase above) at a fixed rate per unit area which depends only on temperature — a steady state therefore requires a balancing return of water molecules from the gas phase (hence the gas must become argon plus water vapour and more molecules means — ideal gas law — greater pressure). P_w is referred to as the *saturation vapour pressure* of the water. As an aside, P_w is also the *partial pressure* of water vapour in the resultant gas mixture which has *total pressure* $P + P_w$ (similarly P is the partial pressure of argon in the mixture).

List of Equipment

1. Wide bore manometer
2. Heat bath and capillary
3. Stop watch
4. Volumetric flask and glass flat
5. Thermometer (0 – 100°)C
6. Argon gas cylinder and regulator
7. Thermostatically controlled element
8. Ice

Original experiment:

N. Rattenbury

Appendix D:

D.A. Wardle

Section 'Theory' and Appendices A and C:

Rattenbury & Wardle

This version: September 8, 2008.