Experiment 363: Poles and Zeros (Elvis Edition)

Aim

To study the relationships between Bode plots, Nyquist plots and the poles and zeros of a network.

Introduction

The transfer function of a linear circuit consisting of a finite number of lumped-parameter components is expressible as the ratio of two polynomials in the complex frequency s, i.e.,

$$T(s) = \frac{P(s)}{Q(s)}$$

The values of s for which $P(s)$ is zero (and hence for which $T(s)$ is zero) are called the zeros of the transfer function. The values of s for which $Q(s)$ is zero (and hence for which $T(s)$ is infinite) are called the poles of the transfer function.

When a sinusoidal input signal of angular frequency ω is applied to the circuit, the ratio of output amplitude to input amplitude is given by $|T(\mathrm{j}\omega)|$ while the phase shift is given by $\arg[T(\mathrm{j}\omega)]$. By measuring these quantities over a range of frequencies, we may plot graphs of $|T(\mathrm{j}\omega)|$ in decibels and of $\arg[T(\mathrm{j}\omega)]$ as a function of the frequency (plotted on a logarithmic scale). These are known as the amplitude and phase Bode plots of the transfer function respectively. The locus of $T(\mathrm{j}\omega)$ in the complex plane as ω is varied is known as the Nyquist plot. In this experiment we shall obtain the Bode and Nyquist plots of a variety of circuits and relate them to the positions of the poles and zeros.

1 National Instruments Elvis II

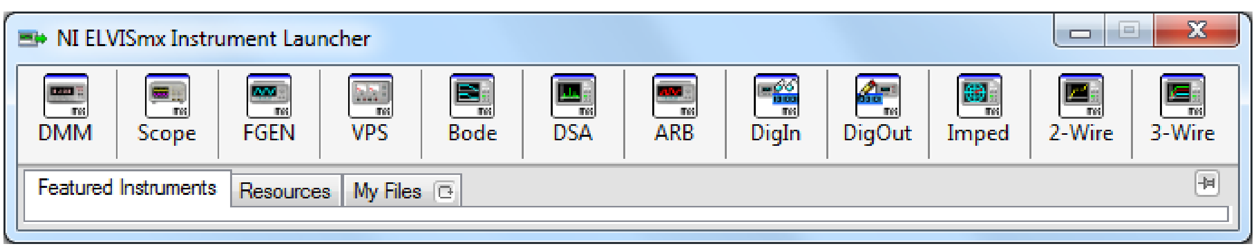

Figure 1: The instrument launcher for the NI ELVIS II, containing such useful devices as the digital multimeter (DMM), an oscilloscope (Scope), a function generator (FGEN) and a Bode analyser (Bode)

This experiment designed to make use of the Elvis II platform from National Instruments. **All the circuits that you will study in this lab need to be built and tested on the Elvis. The electronic components and wires you need to complete this experiment can be found in the components cabinet next to the LCR meter in the corner of the lab.** The layout of the Elvis II allows circuits to be built without having to use solder to connect components, and also contains devices such as an oscilloscope and a function generator, all accessible via a USB connection to a PC with the appropriate software installed.

To access these components, ensure that the USB plug is connected to a PC and the Elvis II is powered on using the switch on the side of the device. The LED indicator for the USB should display 'READY'. You can now launch the 'NI ELVISmx Instrument Launcher', shown in Fig. 1. It is important to understand the layout of the NI Elvis to ensure that the circuits that you build are wired correctly. An overview is shown in Fig. 2, and some important features are numbered as follows:

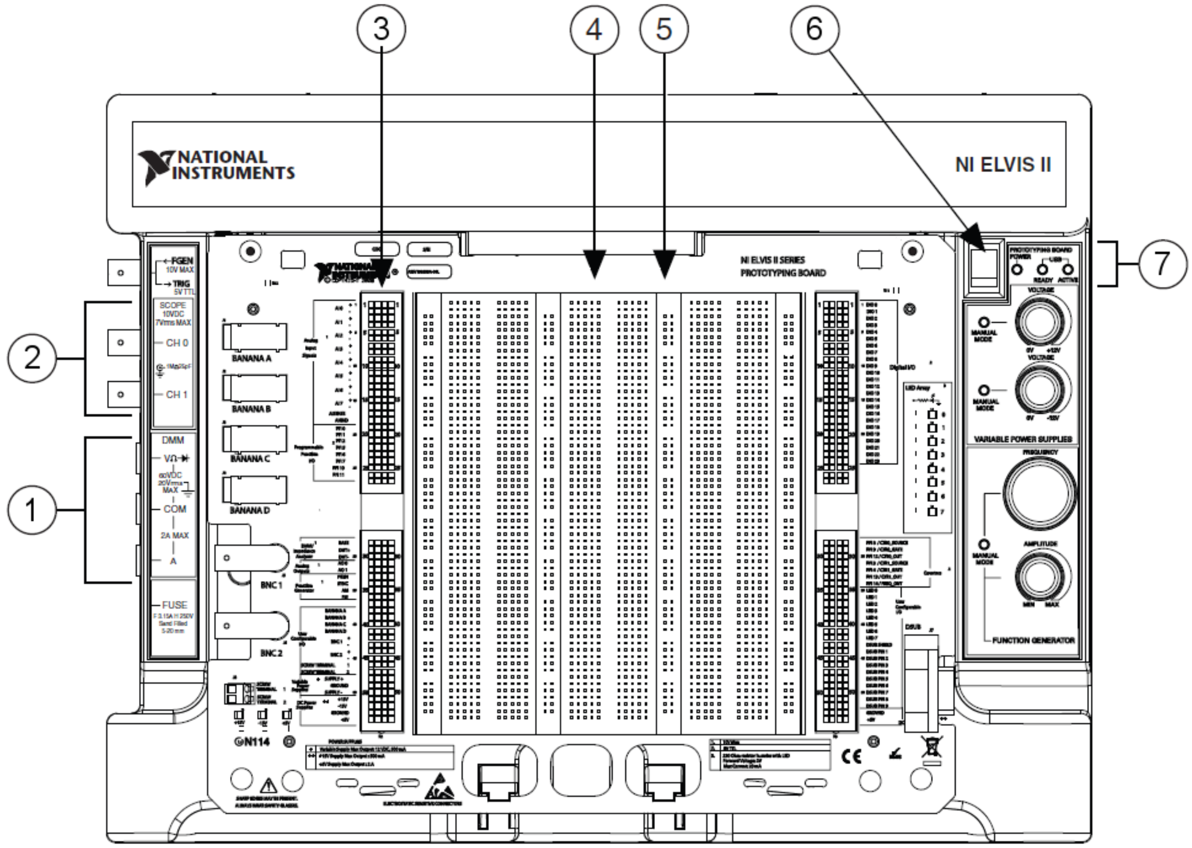

Figure 2: The Elvis II from National Instruments.

1. Connections for the digital multimeter (DMM) leads. To use this instrument, select DMM from the instrument launcher, select the appropriate function and set it to 'Run'.
2. Connections for two oscilloscope (Scope) channels, labeled CH 0 and CH 1. The oscilloscope can be accessed from the instrument launcher.
3. Various connections that can be wired to circuits on the prototyping board. In this experiment we shall use the DC power supplies (pins 51-54) and the function generator (FGEN, pin 33).
4. The prototyping board, where circuits can be constructed. Ensure you understand how the pinholes are connected before you begin to wire up your circuits. An IC chip can be inserted so that its legs occupy columns E and F of the prototyping board, and wires can be taken from neighbouring pinholes to connect to other parts of your circuit.
5. Two columns of pinholes denoted by a red + and a blue −. It is good practise to take a wire from a power supply and connect it to the appropriate rail. Subsequent wires can then be used to power the components of your circuit.
6. Power switch for the prototyping board. This must be active for the power supplies and function generator to operate.
7. LED status indicators to show USB activity, as well as the prototyping board's power status.

In this experiment we will primarily use the Elvis' built-in Bode plot analyzer. This requires three connections to be made between the Elvis and the circuit board under test.

1. Connect a BNC cable with an alligator clip end between oscilloscope channel 0 (top left hand side of the Elvis) and the input of the circuit you wish to test.

2. Connect a BNC cable with an alligator clip end between oscilloscope channel 1 (top left hand side of the Elvis) and the output of the circuit you wish to test.
3. Connect wires from pin 33 (FGEN, left hand plug board of ELVIS) and pin 49 (GND, left hand plug board of ELVIS) to the input of the circuit you wish to test.
4. Launch the Bode function analyzer program and measure Bode amplitude and phase plots over suitable frequency ranges.

Note: Be sure to include oscilloscope plots and Bode plots in your report where appropriate. You should do this by importing your data into Python.

Data Processing

For each of the circuits from which data are collected, you should draw the experimental Nyquist and Bode plots together with theoretical plots computed from the component values and the transfer functions given. For example, suppose the transfer function of a network (after substituting the component values) is given by:

$$T(s) = \frac{10^7}{(s+10^3)(s+10^4)}$$

We can calculate a vector of values of T at the frequencies specified by vector `f` as follows:

```
import numpy as np
import matplotlib.pyplot as plt

def xfer(f):
    s = 1j * 2*np.pi*f
    T = 1e7 / ((s+1e3)*(s+1e4))
    return T

f = np.logspace(np.log10(20),np.log10(2000000),500)
T = xfer(f)

plt.figure(0)
plt.semilogx(f, 20*np.log10(abs(T)))
plt.title('Bode amplitude plot')
plt.grid()

plt.figure(1)
plt.semilogx(f, (180/np.pi)*np.unwrap(np.angle(T)))
plt.title('Bode phase plot')
plt.grid()

plt.figure(2)
plt.plot(np.real(T), np.imag(T))
plt.title('Nyquist plot')
plt.grid()

plt.show()
```

it is easy to obtain the theoretical plots for this transfer function. You will need to add the code to read in your experimental data and plot them on the same axes as the theoretical plots. On your graphs you should then mark in the significant points or lines as discussed below in connection with each circuit.

Instead of using the parameters for the transfer function calculated from the component values, you may prefer to use the "model" function `xfer` together with a suitably written misfit function and `scipy.optimize.fmin` in PYTHON to find the best-fitting values of the parameters. The theoretical plots may be plotted using these best-fitting parameters together with the experimental data.

2 One-Pole Circuits

2.1 One-pole low-pass circuit

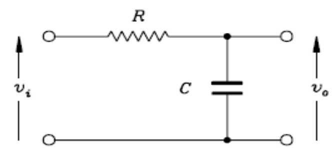

Figure 3: Low-pass circuit

The transfer function for the low-pass circuit in Fig. 3 is:

$$T_L(s) = \frac{(1/RC)}{s + (1/RC)} = \frac{\omega_0}{s + \omega_0} \text{ where } \omega_0 = \frac{1}{RC}$$

The transfer function involves only a single parameter ω_0, and the position of the pole is at $s = -\omega_0$.

(1) Obtain the gain and phase response of the first-order low-pass filter in which $R = 56\,\text{k}\Omega$ and $C = 4.7\,\text{nF}$ and plot the experimental and theoretical Nyquist and Bode plots.

(2) Calculate the corner frequency, and check that the gain and phase at this frequency are as expected.

(3) Sketch the Bode amplitude plot asymptotes and the phase guidelines for the network.

(4) Prove that the Bode phase plot possesses rotational symmetry about the corner frequency $\nu_0 = \omega_0/(2\pi)$. This means that if we consider the phase responses at angular frequencies ω_0/k and $k\omega_0$ which are at equal distances from ω_0 on a logarithmic scale, we should find that:

$$|\arg T_L(\mathrm{j}\omega_0/k) - \arg T_L(\mathrm{j}\omega_0)| = |\arg T_L(\mathrm{j}k\omega_0) - \arg T_L(\mathrm{j}\omega_0)|$$

Verify that this is so for your experimental data, choosing, for example the frequencies corresponding to $k = 3$.

2.2 One-pole and one-zero high-pass circuit

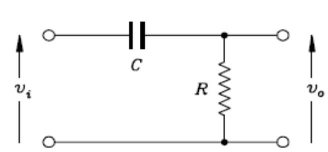

Figure 4: High-pass circuit

The transfer function for the high-pass circuit in Fig. 4 is:

$$T_H(s) = \frac{s}{s + (1/RC)} = \frac{s}{s + \omega_0}$$

The transfer function involves only a single parameter ω_0, the position of the pole is at $s = -\omega_0$ and the zero is at the origin.

(5) Obtain the gain and phase response of the first-order high-pass filter in which $R = 56\,\mathrm{k}\Omega$ and $C = 4.7\,\mathrm{nF}$ and plot the experimental and theoretical Nyquist and Bode plots.

(6) Calculate the corner frequency, and check that the gain and phase at this frequency are as expected.

(7) Sketch the Bode amplitude plot asymptotes and the phase guidelines for the network.

(8) By noting that $T_H(s)$ may be written as:

$$T_H(s) = s\left(\frac{1}{s+\omega_0}\right)$$

explain how the amplitude and phase Bode plots for may be derived by combining the separate amplitude and phase Bode plots for the factors s and $1/(s+\omega_0)$.

(9) By noting that $T_H(s)$ may also be written as:

$$T_H(s) = 1 - \left(\frac{\omega_0}{s+\omega_0}\right)$$

explain how the Nyquist plot for may be obtained from that of the single-pole low-pass filter $T_L(s)$.

(10) [Optional] Show that

$$T_H\left(\mathrm{j}\omega_0^2/\omega\right) = \overline{T_L(\mathrm{j}\omega)}$$

where the line above the expression on the right-hand side denotes the complex conjugate. Hence find expressions for $\left|T_H\left(\mathrm{j}\omega_0^2/\omega\right)\right|$ and $\arg T_H\left(\mathrm{j}\omega_0^2/\omega\right)$ in terms of $|T_L(\mathrm{j}\omega)|$ and $\arg T_L(\mathrm{j}\omega)$. Comment on the implications of these expressions to the relationship between the Bode plots of the high-pass and low-pass transfer functions.

2.3 Circuit with Zero at $-\omega_1$ and Pole at $-\omega_2$ $(\omega_2 > \omega_1)$

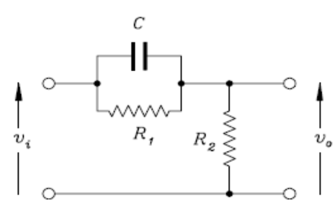

Figure 5: Modified high-pass circuit

The transfer function for the modified high-pass circuit in Fig.5 is:

$$T(s) = \frac{s + 1/(R_1C)}{s + 1/(R_\mathrm{p}C)} = \frac{s+\omega_1}{s+\omega_2}$$

where $R_\mathrm{P} = R_1R_2/(R_1+R_2)$, $\omega_1 = 1/(R_1C)$ and $\omega_2 = 1/(R_\mathrm{p}C)$. This transfer function involves two parameters ω_1 and ω_2.

(11) Obtain the gain and phase response of the modified high-pass circuit in which $R_1 = 220\,\mathrm{k}\Omega$, $R_2 = 18\,\mathrm{k}\Omega$ and $C = 4.7\,\mathrm{nF}$ and plot the experimental and theoretical Nyquist and Bode plots.

(12) By expressing $T(s)$ as:

$$T(s) = (s+\omega_1)\left(\frac{1}{s+\omega_2}\right)$$

explain how the amplitude and phase Bode plots for $T(s)$ may be derived by combining the separate amplitude and phase Bode plots for the factors $(s+\omega_1)$ and $1/(s+\omega_2)$. Hence determine and sketch the Bode amplitude plot asymptotes and phase guidelines.

(13) Prove that the Bode phase plot is symmetrical about the angular frequency $\omega_0 = \sqrt{\omega_1 \omega_2}$ on a logarithmic scale. i.e., show that for any k,

$$\arg T\left(\mathrm{j}\omega_0/k\right) = \arg T\left(\mathrm{j}k\omega_0\right)$$

Confirm this relationship holds for your experimental data for any value of k of your choice.

(14) Show that we may write:

$$T\left(s\right) = 1 - \left(\frac{\omega_2 - \omega_1}{s + \omega_2}\right)$$

and describe how this may be used to draw the Nyquist plot of this network starting from the Nyquist plot of $\omega_0 / \left(s + \omega_0\right)$. (Hint: consider scaling and translating the plot appropriately).

(15) From your Nyquist plot, determine the value of the maximum phase shift ϕ_M using elementary trigonometry, and compare the result with your measurements.

2.4 Circuit with Zero at $-\omega_1$ and Pole at $-\omega_2$ $\left(\omega_1 > \omega_2\right)$

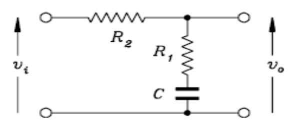

Figure 6: Modified low-pass circuit

The transfer function for the modified low-pass circuit in Fig. 6 is:

$$T\left(s\right) = \frac{R_1}{R_1 + R_2}\left[\frac{s + 1/\left(R_1 C\right)}{s + 1/\left[\left(R_1 + R_2\right) C\right]}\right] = \frac{\omega_2}{\omega_1}\left(\frac{s + \omega_1}{s + \omega_2}\right)$$

where $\omega_1 = 1/\left(R_1 C\right)$ and $\omega_2 = 1/\left[\left(R_1 + R_2\right) C\right]$.

(16) Obtain the gain and phase response of the modified high-pass circuit in which $R_1 = 18\,\mathrm{k\Omega}$, $R_2 = 220\,\mathrm{k\Omega}$ and $C = 4.7\,\mathrm{nF}$ and plot the experimental and theoretical Nyquist and Bode plots.

(17) By factorizing $T\left(s\right)$ appropriately, determine and sketch the Bode amplitude plot asymptotes and phase guidelines.

(18) Express $T\left(s\right)$ in a form that allows its Nyquist plot to be derived from earlier results.

(19) From your Nyquist plot, determine the minimum (most negative) phase shift $-\phi_M$ using elementary trigonometry, and compare the result with your measurements.

(20) [Optional] Consider $T\left(\mathrm{j}\omega_0^2/\omega\right)$ where $\omega_0 = \sqrt{\omega_1 \omega_2}$, and comment on the relationships between the Bode plots of this circuit and the previous one.

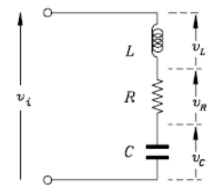

Figure 7: Series LCR circuit

3 Two-Pole Circuits

3.1 Series LCR Circuit

Consider the series LCR circuit in Fig. 7 with the output taken across each of the three components in turn. The corresponding transfer functions are:

$$\frac{V_C(s)}{V_i(s)} = \frac{\omega_0^2}{s^2 + (\omega_0/Q)\,s + \omega_0^2}$$
$$\frac{V_R(s)}{V_i(s)} = \frac{(\omega_0/Q)\,s}{s^2 + (\omega_0/Q)\,s + \omega_0^2}$$
$$\frac{V_L(s)}{V_i(s)} = \frac{s^2}{s^2 + (\omega_0/Q)\,s + \omega_0^2}$$

where

$$\omega_0^2 = \frac{1}{LC} \quad \text{and} \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

All three transfer functions have the same poles. They differ only in the number of zeros at the origin.

For $Q \leq \frac{1}{2}$, poles lie on the real axis and are inverse points with respect to the circle of radius ω_0.

For $Q > \frac{1}{2}$, poles are complex conjugates and lie on this circle.

The transfer functions are classified as low-pass, band-pass or high-pass according to whether they possess none, one or two zeros respectively at the origin. In the range of frequencies of interest in this experiment, it is difficult to obtain inductors which act as ideal elements in the series LCR circuit. Instead, an active circuit which possesses transfer functions similar (but not identical) to those of the series LCR circuit is used (see Fig. 8). **You are required to build this circuit on the Elvis.** The op-amp we will use is the LF356, which is a modern, high-gain, low-noise, op-amp. The pin-out for this chip is shown in Fig. 9. The Balance pins can be left unconnected. The V_+ and V_- pins require $+15V$ and $-15V$ respectively.

Note: This circuit is reasonably complicated so some initial planning of the layout of your circuit **before you start building** will make your life considerably simpler.

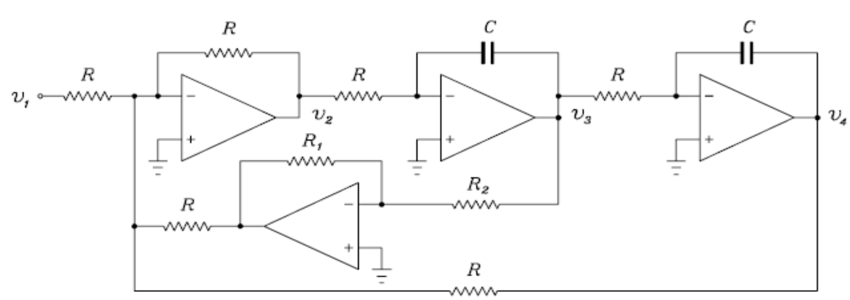

Figure 8: Active filter circuit

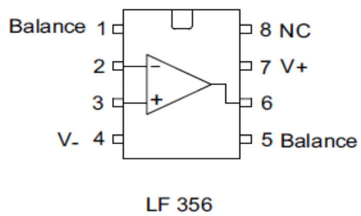

Figure 9: Pin-out of the LF356 operational amplifier

For the active filter circuit the following transfer functions can be deduced:

$$\frac{V_2(s)}{V_1(s)} = T_H(s) = -\left[\frac{s^2}{s^2 + R_1 s/(R_2 RC) + 1/(RC)^2}\right] = -\left[\frac{s^2}{s^2 + (\omega_0/Q)s + \omega_0^2}\right]$$

$$\frac{V_3(s)}{V_1(s)} = T_B(s) = +\left[\frac{s/(RC)}{s^2 + R_1 s/(R_2 RC) + 1/(RC)^2}\right] = +\left[\frac{\omega_0 s}{s^2 + (\omega_0/Q)s + \omega_0^2}\right]$$

$$\frac{V_4(s)}{V_1(s)} = T_L(s) = -\left[\frac{1/(RC)^2}{s^2 + R_1 s/(R_2 RC) + 1/(RC)^2}\right] = -\left[\frac{\omega_0^2}{s^2 + (\omega_0/Q)s + \omega_0^2}\right]$$

where $\omega_0 = 1/(RC)$ and $Q = R_2/R_1$.

3.2 The Low-Pass Transfer Function

Consider the low-pass transfer function of the active filter circuit, viz:

$$T_L(s) = -\left[\frac{\omega_0^2}{s^2 + (\omega_0/Q)s + \omega_0^2}\right]$$

(21) Obtain the gain and phase responses of the two pole low-pass filter implemented on circuit board 2, where $R = 56\,\mathrm{k}\Omega$, $C = 4.7\,\mathrm{nF}$, $R_1 = 47\,\mathrm{k}\Omega$ and R_2 can be selected from among 10 kΩ, 33 kΩ and 150 kΩ, giving three possible values of Q. Plot the experimental and theoretical Nyquist and Bode plots for each value of Q. If you choose to use PYTHON to find the parameters of this transfer function, it is convenient to use ω_0, Q and an overall gain (which should ideally be unity) as the parameters.

(22) By factorizing appropriately, show that for $Q \leq \frac{1}{2}$, the "corners" of the amplitude Bode plot occur at angular frequencies ω_1 and ω_2 where:

$$\omega_1, \omega_2 = \omega_0 \left[\frac{1}{2Q} \mp \sqrt{\frac{1}{4Q^2} - 1}\right]$$

(23) Show that the poles of the transfer function become complex for $Q > \frac{1}{2}$ and that the angle θ that the poles make with the negative real axis is given by $\theta = \cos^{-1}[1/(2Q)]$.

(24) Show that the amplitude Bode plot possesses a maximum (at $\omega \neq 0$) only if $Q > 1/\sqrt{2}$. Show that the angular frequency ω_3 at which this maximum occurs is:

$$\omega_3 = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

(25) Show that the angular frequencies ω_4 and ω_5 at which the phase shift is 135° and 45° respectively are given by:

$$\omega_4, \omega_5 = \omega_0 \left[\sqrt{1 + \frac{1}{4Q^2}} \mp \frac{1}{2Q}\right]$$

(Hint: Write X for $\omega/\omega_0 - \omega_0/\omega$ and show that $|QX| = 1$ when $\omega = \omega_4, \omega_5$.

3.3 The Band-Pass Transfer Function

Consider the band-pass transfer function of the active filter circuit, viz:

$$T_B(s) = \left[\frac{\omega_0 s}{s^2 + (\omega_0/Q)\, s + \omega_0^2}\right]$$

(26) Obtain the gain and phase responses of the two pole band-pass filter for each of the three values of and plot the experimental and theoretical Nyquist and Bode plots.

(27) By considering the limits of low frequency and high frequency respectively, show that the low frequency Bode amplitude asymptote is given by $20\log_{10}(\omega/\omega_0)$ and that the high frequency Bode amplitude asymptote is given by $20\log_{10}(\omega_0/\omega)$.

(28) Show that at ω_0, the response of the bandpass filter is Q.

(29) Show that at frequencies ω_4 and ω_5, the gain of the bandpass filter is 3 dB lower than the gain at the centre frequency ω_0. The difference $\Delta\omega = \omega_5 - \omega_4$ is called the (angular) bandwidth of the filter. Verify that $\omega_0/\Delta\omega = Q$.

(30) Show that the bandpass transfer function may be rewritten as:

$$T_B(j\omega) = \frac{Q}{1 + \mathrm{j}QX}$$

where $X = \omega/\omega_0 - \omega_0/\omega$. Hence describe why the Nyquist plot for the bandpass transfer function is a circle.

3.4 The High-Pass Transfer Function

Consider the high-pass transfer function of the active filter circuit, viz:

$$T_H(s) = -\left[\frac{s^2}{s^2 + (\omega_0/Q)\, s + \omega_0^2}\right]$$

(31) Obtain the gain and phase responses of the two pole high-pass filter for each of the three values of Q and plot the experimental and theoretical Nyquist and Bode plots.

(32) [Optional] Show that $T_H\left(\mathrm{j}\omega_0^2/\omega\right) = \overline{T_L(\mathrm{j}\omega)}$ for the two-pole filters and comment on the relationships between the Bode plots for T_L and T_H.

S.M. Tan

February 1997

Revised by S. Coen and H. Oudenhoven, March 2011.

Revised by SGM, August 2014.

Revised for Python: 24 November 2014