Elements of Probability Summer 2019

Discrete probability spaces — 07.03, 2019

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1 Overview

In the last lecture we discussed probability spaces and σ fields.

In this lecture we discussed probability spaces and geometric probability.

2 Introduction

Let's start with some vocabulary first:

- $\omega \in \Omega$ is called elementary event
- $A \subseteq \Omega$ is called an event

If (A_n) is a sequence of sets from $\mathcal{S}$ such that
$(\forall n, m)(n \neq m \to A_n \cup A_m = \emptyset)$
Then

$$P(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} P(A_n) \tag{1}$$

So: $A, B \in \mathcal{S}, A, B = \emptyset \implies P(A \cup B) = P(A) + P(B)$
This works for disjoint sets. Let's assume $A, B \in \mathcal{S}$
$P(A \cup B) = P((A \cap B') \cup P(A \cap B) \cup P(A' \cap B)) = P(A' \cap B) + P(A \cap B) + P(A \cap B')$

$P(A) + P(B) + P(A \cap B) = P((A \cap B') \cup (A \cap B)) + P((A \cap B) \cup (A' \cap B)) - P(A \cap B) =$
$= P(A' \cap B) + P(A \cap B) + P(A \cap B')$
which works for intersecting sets too.

Theorem $P(A \cup B) = P(A' \cap B) + P(A \cap B) + P(A \cap B')$

3 Discrete uniform probability spaces

We have a finite set Ω. Let $|\Omega| = n$.
We put $\mathcal{S} = P(\Omega)$
We put $P(A) = \frac{|A|}{|\Omega|}$.

$(\Omega, \mathcal{P}(\Omega), P)$ is a discrete (also known as combinatorial) probability space.

Example 1 Let $\Omega = \{1, ..., 1000\}$ $\quad A = \{k \in \Omega : 2|k \vee 3|k\}$ $\quad$ What is P(A)?
Denote $A = \{k \in \Omega : 2|k\} \cup \{k \in \Omega : 3|k\}$ or otherwise $A = B \cup C$
$P(A) = P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$B \cap C = \{k \in \Omega : 2|k \wedge 3|k\} = \{k \in \Omega : 6|k\}$

$P(B) = \frac{\lfloor \frac{1000}{2} \rfloor}{1000} = \frac{500}{1000}$

$P(C) = \frac{\lfloor \frac{1000}{3} \rfloor}{1000} = \frac{333}{1000}$

$P(B \cap C) = \frac{\lfloor \frac{1000}{6} \rfloor}{1000} = \frac{166}{1000}$

$P(A) = \frac{500+333-166}{1000} \sim 0.667$

Example 2 $\Omega = \{1, ..., n\}^2$ $\quad A = \{(x, y) \in \Omega : x < y\}$ $\quad |\Omega| = n^2$ $\quad P(A) = \frac{|A|}{n^2}$

Let's prepare a square of size x by x. Let's divide it into small squares of size 1. As we can now see $|A| = \frac{n^2-n}{2}$ and $P(A) = \frac{n^2-n}{2n^2} = \frac{1}{2}(1 - \frac{1}{n})$

3.1 Inclusion/exclusion formula

This formula is used to count probability of events that may be joint.

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{T \in [n]^k} P(\bigcap_{i \in T} A_i) \tag{2}$$

Example 3 Sometimes lazy teachers will mix students exams randomly and give them out to students so they would check exams of theirs peers. What is the probability of failure (ie. student gets his own test), given sheets were randomly mixed?
Fix n. Let Ω be set of all permutations of $\{1, ..., n\}$. Then $|\Omega| = n!$. $F_n = \{\pi \in \Omega : (\exists i)(\pi(i) = i)\}$.
For two students $P(F_2) = 1/2$. Let $B_i = \{\pi \in \Omega : \pi(i) = i\}$ (ie. B_i is a particular situation in which student got his own test).
Then $F = \bigcup_{i=1}^{n} B_i = \sum_{k=1}^{n} (-1)^{k+1} \sum_{T \in [n]^k} P(\bigcap_{i \in T} B_i)$.
Suppose we have set $T \subseteq \{1, ..., n\}$ and $B_i = \{\pi \in \Omega : (\forall i \in T)(\pi(i) = i)\}$, thus
$|\bigcap_{i \in T} B_i| = (n - |T|)!$.
If we substitute to equation (2) we get

$$P(F) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{T \in [n]^k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \tag{3}$$

Which should look familiar, as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{4}$$

so

$$P(F_n) \sim 1 - \frac{1}{e} \tag{5}$$

4 Discrete Finite Probability Space

Let $\Omega = \{\omega_1, ..., \omega_n\}, \mathcal{S} = \mathcal{P}(\Omega)$
$(p_1, ..., p_n) \in [0,1]^n$ Let remember that

1. $0 \leqslant p_i \leqslant 1$
2. $\sum_{k=1}^{n} p_i = 1$

For $A \subseteq \Omega, P(A) = \sum_{k,\omega_k} p_i$; special case - when $p_i = \frac{1}{|\Omega|}$

5 Geometric Probability

Imagine $\mathcal{S} = Borel([0,1]^2)$ (ie. a square of side 1) and A is $\mathcal{S} = Borel(\left[0, \frac{1}{2}\right]^2)$ (ie. square of side $\frac{1}{2}$). What is P(A)? Instinctively we can say it's $P(A) = \frac{1}{4}$.

Let's think about another example. $\Omega = \{(x,y) : x^2 + y^2 \leqslant r^2\}$. In such situation $P(A) = \frac{areaofA}{\pi r^2}$.
Say
$A = \{(x,y) : x^2 + y^2 \leqslant (\frac{1}{3})^2\}$
then
$P(A) = \frac{\pi(\frac{1}{3})^2}{\pi 1^2} = \frac{1}{9}$.