**Advanced Topics in Algebra** Summer 2019

# Eigenvalues and eigenvectors — 27.03, 2019

*dr Krzysztof Majcher* *Scribe: Krzysztof Agieńczuk*

## 1 Overview

In the last lecture we talked about **transition matrices**

In this lecture we covered **eigenvalues**, **eigenvectors** and **eigendecomposition**

## 2 Determinant

We will define determinant using the Laplace formula.

**Definition 2.1.** *Determinant of square matrix*
*For* $n \in \mathcal{N}$*, let:* $K^{nxn} \to K$*; where* $K$ *is a field*
*For* $n = 1 \to det[a] = a$
*For* $n > 1$*:* $A \in K^{n \times n}$*; fix* $j \in \{1, ..., n\}$
$det[a] = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} * det[A_{ij}]$

For $n = 2$ we can compute te determinant directly using

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \tag{1}$$

Now, let us see that the determinant tells us how much the space is "stretched" or "squished" by a given transformation. If you take field created by basis vectors in $\mathbb{R}^2$, apply a linear map to them and count the determinant, you will get a factor by which this field increased or decreased (in $\mathbb{R}^3$ it will be volume, and accordingly in higher dimensions). Determinant equal to 0 will mean, that (in $\mathbb{R}^2$) the field is now equal to 0, which means we collapsed the space ("lost" one dimension). Given that, below facts are obvious.

**Fact** For $A, B \in K^{n \times n}$

1. $det(A * B) = det(A) * det(B)$
2. $det(A^n) = (det(A))^n$
3. $det(A^{-1}) = \frac{1}{det(A)}$
4. $rank(A) = n \iff det(A) \neq 0$

**Reminder** Let $A \in K^{n \times n}$, $f_A(v) = A * v^T$. Then, $dim(im(f_A)) = rank(A)$

**Reminder** If $M_B$ is a matrix of a linear map $f : K^n \to K^n$ in base $B = \{b_1, ..., b_n\}$, then

$$M_B = P * M_B * P^-1 \tag{2}$$

where P is matrix of basis vectors.

# 3 Eigenvalue

**Definition 3.1.** *Let $K$ be a field and $M \in K^{n \times n}$ then a number $\lambda \in K$ is an eigenvalue for eigenvector $v \in K^n$ of a matrix $M$ if*

$$Mv^T = \lambda v^T \tag{3}$$

This may come as a surprise as we have multiplication Matrix times vector = scalar times vector. However you may write $\lambda$ as matrix transforming each basis vector of the space by $\lambda$. This will result in matrix with lambdas on diagonal and zeros everywhere else. This we can write as $\lambda \times$ *identity matrix*, which we will use in 3.1.

**Note** Eigenvalues must be non-zero. However they can be negative. This means our space gets "flipped".

**Example** For $K \in \mathbb{R}$ and $n = 2$, matrix $M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalue $\lambda = 3$ for eigenvector $v = (1, 0)$ and eigenvalue $\lambda = 2$ for $v = (0, 1)$.

**Definition 3.2.** *Let $K$ be a field. $f : K^n \to K^n$, then number $\lambda \in K$ is an eigenvalue for eigenvector $v$ of $f$* $\iff$

$$f(v) = \lambda * v \tag{4}$$

**Fact**

1. Let $M \in K^{n \times n}$, $\lambda$ - eigenvalue of M. Then $E_\lambda = \{v \in K : Mv^T = \lambda v^T\} \leq K^n$
2. Let $\{\lambda_1, ..., \lambda_n\}$ be a set of eigenvalues and $\{v_1, ..., v_n\}$ set of eigenvectors such that $\lambda_i$ is eigenvalue for $v_i$, then $\{v_1, ..., v_n\}$ is linearly independent.

## 3.1 Polynomials of eigenvalues

If $M \in K^{n \times n}$, then $det[M - \lambda * Id]$ is a polynomial. (Id is unit matrix).

**Definition 3.3.** *Let $M \in K^{n \times n}$. Then a polynomial $det[M\lambda * Id]$ is a characteristic polynomial of M.*

**Theorem** Let $M \in K^{n \times n}$, $\lambda$ is eigenvalue of M, then $det[M - \lambda * Id] = 0$

This means eigenvalues are square roots of characteristic polynomial.