Advanced Topics in Algebra	Summer 2019

Lecture Introduction — 27,02, 2019

Dr Krzysztof Majcher *Scribe: Krzysztof Agieńczuk*

1 Overview

In the this lecture we discussed the contents of the lectures, basic logical operators, complex numbers and introduction to linear algebra.

2 Contents

Topics for this lecture include:

- Least square method
- Fourier series
- Fourier transform
- Principal Component Analysis (PCA)
- SVD decomposition

Suggested literature for this lecture is Thomas Banchoff's "Linear Algebra"[1].

Well, we got dumped with symbols and definitions. I'll try to make it as easy as possible but this will be very dry. You've been warned!

3 Logic

Basic logical operators include: $\wedge, \vee, \neg, \implies, \iff$. They mean (in this order) and, or, negation, implies, if and only if. There are also quantifiers $\exists x, \forall x$ meaning there exists x, that..., and for all xs that... We can operate on sets. For sets A and B, $A \cup B, A \cap B, A \setminus B$ mean sum, intersection and set difference. AxB is product of two sets, ie. $AxB = \{(x, y) : x \in A, y \in B\}$ There is also notation for functions. For sets A,B, by f:A→B denote a function for A to B.

4 Complex numbers

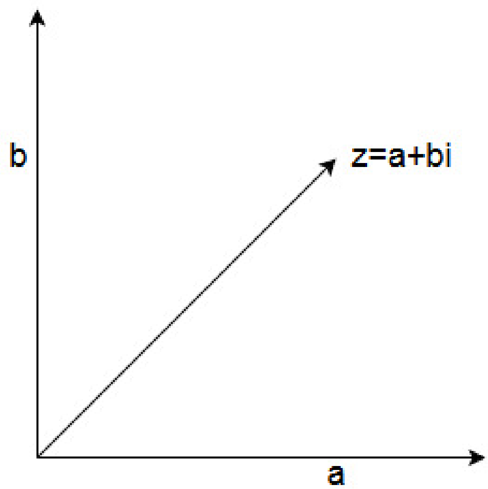
Complex number is one consisting of two parts, real and imaginary. We write it like below $c = a+bi$, where $a, b \in \mathbb{R}, i^2 = (-1)$

Example 1 $(a + bi) + (c + di) = (a + c) + (b + d)i$

Example 2 $(a + bi) * (c + di) = ac + adi + bic + bidi = (ac - bd) + (ad + bc)i$

4.1 Complex Plane

Number $z = a + bi$ can be considered as a point $(a, b) \in \mathbb{R}^3$. This is shown on a graph below.



With that in mind, we can define $|z| = \sqrt{a^2 + b^2}$. If by α we denote the angle between a and —z— vector, we can write $a = |z| * cos\alpha$ and $b = |z| * sin\alpha$. This implies $z = |z| * (cos\alpha + i * sin\alpha)$.

4.2 Euler Formula

for

$$\alpha \in \mathbb{R}, e^{\alpha * i} = cos\alpha + i * sin\alpha$$

or in other terms

$$z = |z| * e^{i * \alpha}$$

which is called exponential form of a complex number.

Fact 1 $\forall z \in \mathbb{C}, \exists r \in \mathbb{R}^+, \alpha \in \mathbb{R}$ such that $z = r * e^{\alpha * i}$ For any $k \in \mathbb{Z} re^{\alpha * i} = re^{(\alpha + 2\pi * k)i}$

Example 3 Find exponential form of (1-i).

We know $\alpha = \frac{7}{4}\pi$ and $r = \sqrt{2}$. If we substitute we get $(1-i) = \sqrt{2} * (\cos\frac{7}{4}\pi + i * \sin\frac{7}{4}\pi) = \sqrt{2} * e^{\frac{7}{4}\pi * i}$

5 Linear algebra

5.1 Linear space

Let k be $\mathbb{C}$ or $\mathbb{R}$. A linear space over K is a triple $(V, +, *)$, where

- V $\rightarrow$ set (of vectors)
- $+ \rightarrow VxV \rightarrow V$
- $* \rightarrow k * V \rightarrow V$

Conditions for linear space

- $(V, +)$ is abelian group
- $\forall k, l \in K, \forall v \in V (k + l) * v = kv + lv$
- $\forall k, l \in K, \forall v \in V (k * l) * v = k * (l * v)$
- $\forall k \in K, \forall v, w \in V k(v + w) = kv + kw$
- $\forall v \in V 1 * v = v$

Example 4 Is $(\mathbb{R}^2, +, *)$ a linear space? $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}^2\}$, $+ : (x, y)+(a, b) = (x+a, y+b)$, $* : k(x, y) = (kx, ky)$, so it is a linear space.

5.2 Linear subspace

Let $(V, +, *)$ be linear space over K, $A \subset V$. We say A is a subspace of V if

1. $\forall v, w \in A v + w \in A$
2. $\forall k \in K, v \in V k * v \in A$

Example 5 Let $V = (R^2, +, *), A = \{(0, 0)\}$. We can see that $(0, 0) + (0, 0) = (0, 0) \in A$ and that $k(0, 0) = (0, 0) \in A$. This way we have proven that A is a subspace of V.

5.3 Linear combination

Let V be a linear space over K, $v_1, v_2, ..., v_n \in V$. A linear combination of a vector $v_1, v_2, ..., v_n$ is any vector of the form $v = k_1v_1 + k_2v_2 + ... + k_nv_n$ where $k_1, k_2, ..., k_n \in K$

Example 6 Let's have a linear space $V \in \mathbb{R}$ and vectors $v1 = (1, 2, 3), v2 = (5, 3, 9)$. $v = 3(1, 2, 3) + 2(5, 3, 9)$ is a linear combination of given.

5.4 Linear closure

Let V be linear space over K, $A \in V$. Linear closure of A is set of every linear combination of vector from A. We denote it as $Span(A)$.

Example 7 Let's have a linear space $V \in \mathbb{R}^3$ and set $A = \{(1,0,0),(0,1,0)\}$. Span of A will be $Span(A) = \{x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}^2\} = \{(x, y, 0) : x, y \in \mathbb{R}^2\}$

Fact 2 Let V be linear space over K. For any $A \subset V$ Span(A) is a subspace of V.

5.5 Linear Independence

Let $A \subset V$. We say A is linearly independent if $\forall v_1, v_2, ..., v_n \in A, \forall k_1, k_2, ..., k_n \in K\ [k_1v_1 + k_2v_2 + ... + k_nv_n = 0 \to k_1 = k_2 = ... = k_n = 0]$

5.6 Base

Set $B \subset V$ linear space is a base of V if

- B is linearly independent
- Span(B)=V

For example $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is base of $\mathbb{R}^3$ as it is linearly independent and Span of B is $\mathbb{R}^3$.

Fact 3 Let V be a linear space over K. Then

1. If $B, B' \subset V$ are base of V then $|B| = |B'|$
2. If $A \subset V$ is linearly independent set, then exists B, which is base of V such that $A \subset B$
3. Let B be a base of V, for any $v \in V$ there is exactly one vector $(k_1, k_2, ..., k_n) : v = k_1b_1 + k_2b_2 + ... + k_nb_n$, where $B - \{b_1, b_2, ..., b_n\}$

6 Dimension

Let V be linear space over K. A dimension of V is a number $dim(V) = |B|$, where B is a base of V.

Example 8 Let's have $V = \mathbb{R}^3$. Then $dim(\mathbb{R}^3) = 3$, because $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a base of $\mathbb{R}^3$ and $|B| = 3$