Advanced Topics in Algebra	Summer 2019
Lecture 3 — 13.03, 2019	
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1 Overview

In the last lecture we talked about **matrices** nad **linear functions**).

In this lecture we covered:

- **kernel** and **image of function**,
- **injection** (as a type of function),
- **matrix of a function**,
- **rank of matrix**.

2 Reminder

Definition 2.1. *Linear map*

*Let V, W - lin. spaces/K. A function $f : V \to W$ is a **linear map**, if:*

1. $\forall v_1, v_2 \in V \, : \, f(v_1 + v_2) = f(v_1) + f(v_2)$
2. $\forall k \in K, \forall v \in V \, : \, f(k * v) = k * f(v)$

Example 1 $f(x, y, z) = (2x + 3y + 5z, x - y - 7z) : \mathbb{R}^3 \to \mathbb{R}^2$

3 Kernel and image

Definition 3.1. *Kernel and Image*

Let $f : V \to W$ a lin. map.

1. *A **kernel** of f is $ker(f) = \{v \in V : f(v) = \mathbb{0}\}$*
2. *An **image** of f is $im(f) = \{w \in W : (\exists v \in V : f(v) = w)\}$*

Example 2 $f(x,y,x) = (x,y,0) : \mathbb{R}^3 \to \mathbb{R}^3$

$ker(f) = \{v \in V : f(v) = \mathbb{0}\} = \{(x,y,z) \in \mathbb{R}^3 : (x,y,0) = (0,0,0)\} = \{(x,y,z) : x = y = 0\}$ "OZ"

$im(f) = \{w \in W : \exists v \in V f(v) = w\}$

$= \{(x,y,z) \in \mathbb{R}^3 : \exists (a,b,c) \in \mathbb{R}^3 : f(a,b,c) = (x,y,z)\}$

$= \{(x,y,z) \in \mathbb{R}^3 : \exists (a,b,c) \in \mathbb{R}^3 : (a,b,0) = (x,y,z)\}$

$= \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ - "plane OXY"

4 Injection

Definition 4.1. *Injection*

*A function $f : A \to B$ is **injection**, if:*

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

Fact 1. *A linear map $f : V \to W$ is an injection $\iff ker(f) = \{\mathbb{0}\}$*

Proof.

- $\longrightarrow$

  Suppouse that f is an injection.

  We know: $f(\mathbb{0}_V) = \mathbb{0}_w$

  Let $v \in ker(f)$, $f(v) = \mathbb{0}_w$

  //note here

- $\longleftarrow$ (by contradiction)

  f is not an injection:

  $\exists v_1 \neq v_2 \in V : f(v_1) = f(v_2)$

  $\exists v_1, v_2 \in V : f(v_1) - f(v_2) = \mathbb{0}_w$

  (by linearity) $f(v_1 - v_2) = \mathbb{0}_w$

  $v_1 - v_2 \neq \mathbb{0}_w$

  $v_1 - v_2 \in ker(f)$

  $ker(f) \neq \{\mathbb{0}_v\}$

$\square$

Fact 2. *Let $f : V \to W$ a lin. map.*

1. *$ker(f) \leq V$ ($ker(f)$ is a subspace of V)*
2. *$im(f) \leq W$*
3. *$dim(V) = dim(ker(f)) + dim(im(f))$*

Proof. We have to show:

- $v_1, v_2 \in ker(f)$: $v_1 + v_2 \in ker(f)$:

$$\left.\begin{array}{l} v_1 \in ker(f) \Leftrightarrow f(v_1) = \mathbb{O} \\ v_2 \in ker(f) \Leftrightarrow f(v_2) = \mathbb{O} \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} f(v_1 + v_2) \overset{\text{lin.}}{=} f(v_1) + f(v_2) = \mathbb{O} + \mathbb{O} = \mathbb{O} \\ f(v_1 + v_2) = \mathbb{O} \leftrightarrow v_1 + v_2 \in ker(f) \end{array}$$

- $k \in K, v \in ker(f)$: $k * v \in ker(f)$:

  $k \in K,\ v \in ker(f)$

  $v \in ker(f) \leftrightarrow f(v) = 0$

  $f(k * v) \overset{\text{lin.}}{=} k * f(v) = k * \mathbb{O} = \mathbb{O}$

  $f(k * v) \Leftrightarrow k * v \in ker(f)$

$\square$

Example 3 $dim(A) = dim(\{(x, y, z, t) \in \mathbb{R}^4 : x + y + z + t = 0, x + 2y + 3z + 4t = 0\})$

Consider a function $f(x, y, z, t) = (x + y + z + t, x + 2y + 3z + 4t)$ *- this is a lin map* $f : \mathbb{R}^4 \Rightarrow \mathbb{R}^2$.

$ker(f) = A$

$im(f) = \mathbb{R}^2$

$$\forall a, b \in \mathbb{R} = \begin{cases} x+y+z+t = a \\ x+2y+3z+4t = b \end{cases}$$

$f(x, y, z, t) = (a, b)$

$dim(A) = dim(ker(f)) = dim(V) - dim(im(f)) = dim(\mathbb{R}^4) - dim(\mathbb{R}^2) = 4 - 2 = 2$

5 Matrix of a function

Let:

- V, W *lin spaces over* K
- B, A *a basis of* V, W
- $f : V \to W$

(for $v \in V, v_B = (k_1, k_2, ..., k_n)$ *where* $v = k_1 b_1 + k_2 b_2 + ... + k_n b_n$*;* $b_i \in B$*)*

Definition 5.1. *Matrix of function*

A ***matrix of function*** f *with respect basis* B, A *is such matrix* M*:*

$$\forall v \in V : (f(v))_A = M * v_b$$

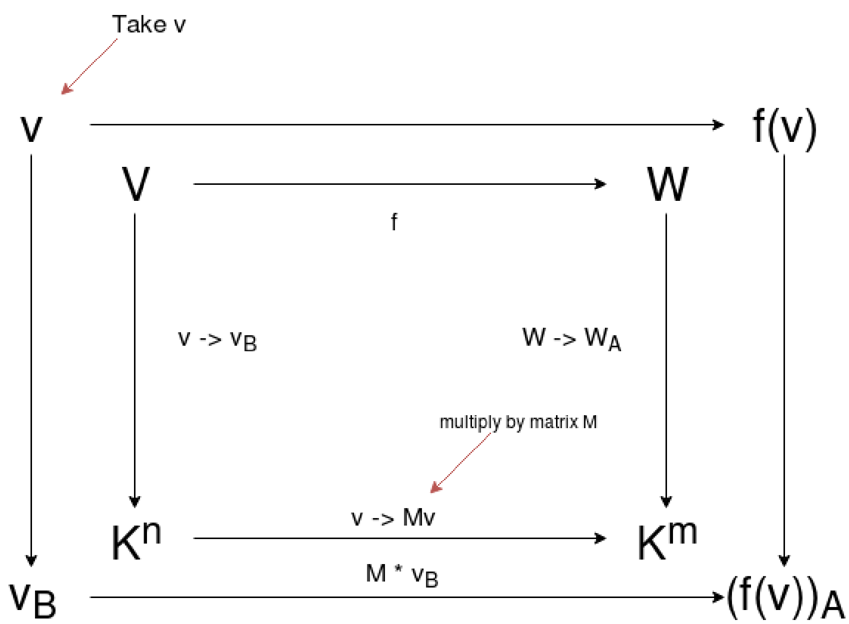

Figure 1: Definition 5.1 visualization

Example 4

$$V = W = \mathbb{R}^2, f(x, y) = (2x + 3y, 5x + 7y)$$

$$B = A = \{(1, 0), (0, 1)\}$$

$M = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ *is a matrix of* f *in* B, A

$$v \in V, (x, y) \in \mathbb{R}^2$$

$$(f(v))_B = (f(x,y))_B = (2x+3y, 5x+7y)_B = (2x+3y, 5x+7y)(*)$$

since

$$(2x+3y)*(1,0) + (5x+7y)*(0,1)$$

$$v = (x,y), v_B = (x,y) : (x,y) = x(1,0) + y(0,1)$$

$$M * V_B = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x+3y \\ 5x+7y \end{bmatrix} (= (*))$$

Example 5

$$f(x,y,z) = (3x+5y+7z, 11x+13z)$$

In standard basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subseteq \mathbb{R}^3, \{(10),(0,1)\} \subseteq \mathbb{R}^2$

$$M = \begin{bmatrix} 3 & 5 & 7 \\ 11 & 0 & 13 \end{bmatrix}$$

Example 6

$$V = W = \mathbb{R}^2[x] = \{f \in \mathbb{R}[x] : deg(f) \leq 2\}$$

$A = B = \{1, x, x^2\}$ *base of* $\mathbb{R}_2[x]$

$$L(f) = f'$$

$$M : (L(f))_A = M * f_B$$

$$f = ax^2 + bx + c, L(f) = f' = 0x^2 + 2ax + b$$

$$f_B = (c,b,a), (L(F))_A = (b, 2a, 0)$$

$$M : M * f_B = (L(f))_A$$

$$M * \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix} = \begin{pmatrix} 0c+1b+0a \\ 0c+0b+2a \\ 0c+0b+0a \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

6 Rank of matrix

Definition 6.1. *Rank of matrix*

*A **rank** of matrix $A \in K^{n \times m}$: $rk(A)$ is a maximal number of lineary independent rows (columns) of A.*

Example 7

$$rk \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} = 2$$

$$rk \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = 1$$

Fact 3. *Let $A \in K^{n \times m}$ a matrix, $R = \{\overline{r_1}, \overline{r_2}, \ldots, \overline{r_n}\}$, $\overline{r_i}$ - i=th row of A.*

Then $rk(A) = dim(Span(R))$

Fact 4. *Let $f : V \to W$ linear map, M a matrix of f in basis A, B.Then*

$$dim(im(f)) = rk(M)$$