Advanced Topics in Algebra Summer 2019

Matrices and Linear Functions — 06.03, 2019

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1 Overview

In the last lecture we talked about linear spaces, linear combinations, linear span, linear independence, bases and dimensions.

In this lecture we discussed Matrices and linear functions.

2 Matrices

Let A be a set. Then a matrix over A is a rectangular array of elements of A. Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

is matrix over $\mathbb{N}$. The horizontal lines we call rows and vertical - columns.
An element of i-th row and j-th column is denoted as A_{ij}.
If matrix has n rows and m columns then dimension of said matrix is $n * m$. Set of all matrices over Z of dimension $n * m$ we write as Z^{n*m}.
If number of columns is equal to number of rows, such matrix is called quadratic.
Diagonal are all elements for which i=j.

2.1 Operations on matrices

We talked about 4 elemental operations on matrices: Transposition, Addition, multiplication and multiplication by scalar.

Transposition For $A \in Z^{k*l}$ a transposition of A is a matrix $A^T \in Z^{l*k}$, defined as $(A^T)_{ij} = A)ji$. Eg.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Addition For matrices $A, B \in K^{n*n}$,(k,+,*) - being a field,
$(A + B) \in K^{n*n} : (A + B)_{ij} = A_{ij} + B_{ij}$. Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 6 \\ 9 & 8 & 17 \end{bmatrix}$$

Multiplication If $A \in K^{n*m}, B \in K^{m*k}$, then $A * B = K^{n*k}$
$(A * B)_{ij}$=(i-th row of A)*(j-th column of B)=$\sum_{t=1}^{m} A_{it} * B_{tj}$. Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix} = \begin{bmatrix} 44 & 56 \\ 98 & 128 \end{bmatrix}$$

Multiplication by scalar If $k \in K, A \in K^{m*k}$ then $(k * A)_{ij} = k * A_{ij}$. Eg.

$$7 * \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

2.2 Some facts about matrices

Fact 1

1. A+B=B+A
2. A+(B+C)=(A+B)+C
3. $(\forall k, l \in \mathbb{N})(\exists 0 \in K^{k*l})(\forall A \in K^{k*l}) A + 0 = A$
4. $(\forall A \in K^{k*l})(\exists B \in K^{k*l}) A + B = 0$

And $(K^{k,l}, +)$ is abelian group.

Fact 2 A*(B*C)=(A*B)*C

Fact 3 For $k \in \mathbb{N}$, k-field
$(K^{k*k}, +, *)$ is non-commutative ring.

We can use matrices for writing down permutations. Eg.
$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \to A_\pi = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Other uses of matrices include describing complex numbers (they can be viewed as subset of quadratic matrices) and Markov chains.

3 Linear Function

Let V,W - linear space over field K. A function f:V→W is a linear map if:

1. $\forall v_1, v_2 \in V f(v_1 + v_2) = f(v_1) + f(v_2)$

2. $\forall k \in K, \forall v \in V f(kv) = k * f(v)$

For example
$f(x, y) = x + y : \mathbb{R}^2 \to \mathbb{R}^1$

1. Let $v_1, v_2 \in \mathbb{R}^2, v_1 = (a, b), v_2 = (p, q)$
   $f(v_1 + v_2) = f((a, b) + (p, q)) = f(a + p, b + q) = a + p + b + q$
   $f(v_1) + f(v_2) = f(a, b) + f(p, q) = a + b + p + q$

2. Let $k \in K = \mathbb{R}, V \in \mathbb{R}^2 \to k \in \mathbb{R}, v = (a, b)$
   $f(kv) = f(k(a, b)) = f(ka, kb) = ka + kb$
   $kf(v) = kf(a, b) = k(a + b) = ka + kb$

Linear Map $f : \mathbb{R}^1 \to \mathbb{R}^1$
$f(1) = a \in \mathbb{R}$
$\forall x \in \mathbb{R} f(x) = f(x * 1) = xf(1) = xa$
$f(x) = ax$
For example: $f(x) = 2x, f(x) = 5x, g(x) = 2x + 1$. $\leftarrow$ as seen above, g(x) is not a linear map.

Linear Map $f : \mathbb{R}^2 \to \mathbb{R}^1$
Let $f(1, 0) = a \in \mathbb{R}; f(0, 1) = b \in \mathbb{R}$
$f(x, y) = f(x(1, 0) + y(0, 1)) = f(x(1, 0)) + f(y(0, 1)) = xf(1, 0) + yf(0, 1) = xa + yb$
For example: $f(x, y) = ax + by = 3x + 7y$

Linear Map $f : \mathbb{R}^2 \to \mathbb{R}^2$
Let $f(1, 0) = (a, b) \in \mathbb{R}; f(0, 1) = (p, q) \in \mathbb{R}$
$f(x, y) = f(x(1, 0) + y(0, 1))$ =...(using same properties as above)= $x(a + b)_y(p, q) = (ax + py, bx + qy) = f(x, y)$

Fact 4 Let V,W - linear space over K, B-base of V. For any $f_1, f_2 : V \to W$
if $\forall b \in B f_1(b) = f_2(b)$ then $\forall v \in V f_1(v) = f_2(v)$.

Proof Base B of V
for any $v \in V$ there are vectors $b_1, b_2, ..., b_n \in B$ and numbers $k_1, k_2, ..., k_n$ such that $v = k_1 b_1 + k_2 b_2 + ... + k_n b_n$, so
$f_1(v) = f_1(k_1 b_1 + k_2 b_2 + ... + k_n b_n) = f_1(k_1 b_1) + f_2(k_2 b_2) + ... + f_n(k_n b_n) =$
$k_1 f_1(b_1) + k_2 f_2(b_2) + ... + k_n f_n(b_n) = k_1 f_2(b_1) + ... + k_n f_2(b_n) = f_2(k_1 b_1 + k_2 b_2 + ... + k_n b_n) = f_2(v)$