Complex Systems Summer 2019

Power Laws and self organising criticality — 28.03, 2019

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1 Overview

In the last lecture we talked about **phase transitions** and **order parameter**.

In this lecture we covered **dual lattices**, **Power Laws** and **self organising criticality**.

2 Lattices

For visualisation help, especially of lattices and percolation thresholds, I send you back to prof Weron's page in hope it will not be taken down after the semester.

Imagine we have a lattice of particular shape. Eg. square. Then the dual lattice is made by putting (let's call the second-order)[1] point in the middle of those squares and connecting those second-order squares to create second lattice. Dual lattice of square lattice is also square. Of hexagonal is triangular (and th other way round).

Bond percolation is given by parameter p, meaning ability to flow/connection between the nodes. In dual lattices we have also parameter q, for second-order lattice, defined as $q = 1 - p$. If $p = q$ then the lattice is self dual.

2.1 Bethe Lattice

Bethe Lattice is a graph that is:

- infinite
- connected
- cycle-free
- in which each node is connected to z neighbours.

Number of nodes in th l-th shell is denoted as

$$N_l = z(z-1)^{l-1} \quad for \quad l > 0 \tag{1}$$

Denoting p as the probability that node is occupied, we can describe probability of going further as

$$p(z-1) \geq 1 \implies p_c = \frac{1}{z-1} \tag{2}$$

[1] I want to stress it - this is my naming, I don't know if it is scientific term

Quick explanation As we have z nodes in each shell, we have z-1 possibilities to go further, as we came from z-th. Also, we count number of nodes, not neighbours.

2.2 Mean Field Approach

Probability that the node belongs to the infinite cluster is pP_j, where p - probability there is a bond between two nodes and P_j is the probability that the j-th neighbour belongs to the infinite cluster. This is hard to calculate, so we prefer complementary probability.
Main assumption of MFA is that the system is homogenous, ie. $\forall_i P_i = P$.

For 1D: $1 - P = (1 - pP) \implies P = \frac{2p-1}{p^2} > 0 \quad for \quad p > \frac{1}{2}$. So we have an infinite cluster for 1D, which is rather obvious. It works better in higher dimensions.

3 Power Laws

Power Laws are defined as follows:

$$f(\frac{x}{a_0}) = f(\frac{x}{a_0})^b \tag{3}$$

with $a_1 = n * a_0$; where a is a scale (in other words 1m=100cm :P)

$$f(\frac{x}{a_1}) = (\frac{x}{a_1})^b = (\frac{x}{a_0 n})^b = n^{-b}(\frac{x}{a_0})^b \tag{4}$$

so they are scale free.

Simplest way to see power laws is log-log scale.

Personal income distribution in tail is a Power Law. This means there are more people earning a lot. Body size on the other hand is normal distribution. Simplest explanation of this fact is that there are or there aren't natural borders (human body cannot grow infinitely).

Using this principles, we can determine (though sometimes incorrectly) eg. sizes of dinosaurs using only parts of their skeletons. This field of study is called allometry.

3.1 Zipf's Law

Zipf's Law calculates probability of appearance. Originally Zipf calculated occurrences of words in English language. This law says that the frequency of any word is inversely proportional to its rank in the frequency table. It can be applied to other things like population or music.

As for examples in nature we can see that rainfalls, earthquakes and other natural catastrophes are Power Laws.