

1 Overview

In the last lecture we talked about linear spaces, linear combinations, linear span, linear independence, bases and dimensions.

In this lecture we discussed Matrices and linear functions.

2 Matrices

Let A be a set. Then a matrix over A is a rectangular array of elements of A . Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

is matrix over \mathbb{N} . The horizontal lines we call rows and vertical - columns.

An element of i -th row and j -th column is denoted as A_{ij} .

If matrix has n rows and m columns then dimension of said matrix is $n * m$. Set of all matrices over Z of dimension $n * m$ we write as Z^{n*m} .

If number of columns is equal to number of rows, such matrix is called quadratic.

Diagonal are all elements for which $i=j$.

2.1 Operations on matrices

We talked about 4 elemental operations on matrices: Transposition, Addition, multiplication and multiplication by scalar.

Transposition For $A \in Z^{k*l}$ a transposition of A is a matrix $A^T \in Z^{l*k}$, defined as $(A^T)_{ij} = A_{ji}$. Eg.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Addition For matrices $A, B \in K^{n*n}$, $(k, +, *)$ - being a field, $(A + B) \in K^{n*n} : (A + B)_{ij} = A_{ij} + B_{ij}$. Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 6 \\ 9 & 8 & 17 \end{bmatrix}$$

Multiplication If $A \in K^{n \times m}, B \in K^{m \times k}$, then $A * B = K^{n \times k}$
 $(A * B)_{ij} = (\text{i-th row of } A) * (\text{j-th column of } B) = \sum_{t=1}^m A_{it} * B_{tj}$. Eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix} = \begin{bmatrix} 44 & 56 \\ 98 & 128 \end{bmatrix}$$

Multiplication by scalar If $k \in K, A \in K^{m \times k}$ then $(k * A)_{ij} = k * A_{ij}$. Eg.

$$7 * \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

2.2 Some facts about matrices

Fact 1

1. $A+B=B+A$
2. $A+(B+C)=(A+B)+C$
3. $(\forall k, l \in \mathbb{N})(\exists 0 \in K^{k \times l})(\forall A \in K^{k \times l})A + 0 = A$
4. $(\forall A \in K^{k \times l})(\exists B \in K^{k \times l})A + B = 0$

And $(K^{k,l}, +)$ is abelian group.

Fact 2 $A*(B*C)=(A*B)*C$

Fact 3 For $k \in \mathbb{N}$, k -field
 $(K^{k \times k}, +, *)$ is non-commutative ring.

We can use matrices for writing down permutations. Eg.

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \rightarrow A_\pi = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other uses of matrices include describing complex numbers (they can be viewed as subset of quadratic matrices) and Markov chains.

3 Linear Function

Let V, W - linear space over field K . A function $f: V \rightarrow W$ is a linear map if:

1. $\forall v_1, v_2 \in V f(v_1 + v_2) = f(v_1) + f(v_2)$

$$2. \forall k \in K, \forall v \in V f(kv) = k * f(v)$$

For example

$$f(x, y) = x + y : \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

1. Let $v_1, v_2 \in \mathbb{R}^2, v_1 = (a, b), v_2 = (p, q)$
 $f(v_1 + v_2) = f((a, b) + (p, q)) = f(a + p, b + q) = a + p + b + q$
 $f(v_1) + f(v_2) = f(a, b) + f(p, q) = a + b + p + q$
2. Let $k \in K = \mathbb{R}, V \in \mathbb{R}^2 \rightarrow k \in \mathbb{R}, v = (a, b)$
 $f(kv) = f(k(a, b)) = f(ka, kb) = ka + kb$
 $kf(v) = kf(a, b) = k(a + b) = ka + kb$

Linear Map $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

$$f(1) = a \in \mathbb{R}$$

$$\forall x \in \mathbb{R} f(x) = f(x * 1) = xf(1) = xa$$

$$f(x) = ax$$

For example: $f(x) = 2x, f(x) = 5x, g(x) = 2x + 1$. \leftarrow as seen above, $g(x)$ is not a linear map.

Linear Map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

$$\text{Let } f(1, 0) = a \in \mathbb{R}; f(0, 1) = b \in \mathbb{R}$$

$$f(x, y) = f(x(1, 0) + y(0, 1)) = f(x(1, 0)) + f(y(0, 1)) = xf(1, 0) + yf(0, 1) = xa + yb$$

$$\text{For example: } f(x, y) = ax + by = 3x + 7y$$

Linear Map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\text{Let } f(1, 0) = (a, b) \in \mathbb{R}; f(0, 1) = (p, q) \in \mathbb{R}$$

$$f(x, y) = f(x(1, 0) + y(0, 1)) = \dots (\text{using same properties as above}) = x(a + b)_y(p, q) = (ax + py, bx + qy) = f(x, y)$$

Fact 4 Let V, W - linear space over K , B -base of V . For any $f_1, f_2 : V \rightarrow W$
if $\forall b \in B f_1(b) = f_2(b)$ then $\forall v \in V f_1(v) = f_2(v)$.

Proof Base B of V

for any $v \in V$ there are vectors $b_1, b_2, \dots, b_n \in B$ and numbers k_1, k_2, \dots, k_n such that $v = k_1b_1 + k_2b_2 + \dots + k_nb_n$, so

$$f_1(v) = f_1(k_1b_1 + k_2b_2 + \dots + k_nb_n) = f_1(k_1b_1) + f_2(k_2b_2) + \dots + f_n(k_nb_n) =$$

$$k_1f_1(b_1) + k_2f_2(b_2) + \dots + k_nf_n(b_n) = k_1f_2(b_1) + \dots + k_nf_2(b_n) = f_2(k_1b_1 + k_2b_2 + \dots + k_nb_n) = f_2(v)$$