Advanced Topics in Algebra Summer 2019

Eigendecomposition and introduction to graphs — 01.04, 2019

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1 Overview

In the last lecture we talked about eigenvalues and eigenvectors.

In this lecture we covered: eigendecomposition, formula for n-th element of Fibonacci sequence and number of paths in a graph.

2 Reminder

Fact Let $M \in K^{n\times n}$, $\lambda_1, ..., \lambda_n$ be eigenvalues of M, $v_1, ..., v_n$ be eigenvectors of M, such that:

- $Mv_i^T = \lambda_i n_i^T$
- $\{v_1, ..., v_n\}$ is a base of M

Then

$$M = VDV^{-1} \tag{1}$$

where V is matrix of vectors $v_1, ..., v_n$ and D is diagonal matrix.

Proof As we know, if M is a matrix of a linear map F in a standard base; $F : K^{n\times n} \to k^{n\times n}$ is linear map, D is a matrix of F in a base $\{v_1, ..., v_n\}$ then $M = VDV^{-1}$ where V is matrix of vectors $v_1, ..., v_n$.

v_i is an eigenvector for eigrnvalue λ_i, so

$$(F(v_i))_v = (\lambda_i v_i)_v = \lambda_i(v_i) = \begin{bmatrix} 0 \\ ... \\ \lambda_i \\ ... \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} 0 \\ ... \\ \lambda_i \\ ... \\ 0 \end{bmatrix} = \lambda_i(v_i) \tag{2}$$

We should show that $(F(v_i))_v = (Dv_i)_v$. So

$$(F(v_i))_v = \lambda_i \begin{bmatrix} 0 \\ ... \\ 1 \\ ... \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & ... & 0 \\ & ... & \\ 0 & ... & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ ... \\ 1 \\ ... \\ 0 \end{bmatrix} = (Dv_i)_v \tag{3}$$

3 Fibonacci numbers

Fibonacci numbers is a sequence, such that

$$f(n) = \begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_{n+1} = a_n + a_{n-1} & \text{for } n \geq 1 \end{cases} \tag{4}$$

Next numbers are denoted as f_n.

Problem How to compute f_n for big n (like f_{10^9})

Let's consider different sequence, such that
$F_1 = (1, 1)$, $F_2 = (1, 2)$, $F_3(2, 3)$, etc.
So, we see that

$$F_i = (x, y) \implies F_{i+1} = (y, x + y) \tag{5}$$

Observe $F_2 = F(F_1) = F^2(1, 1)$, so by induction $F_n = F^{n-1}(1, 1)$

So now we see that F is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix (in standard base) $M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

We can now write it as

$$(f_n, f_{n+1}) = F^{n-1}(1, 1) = M^{n-1}(1, 1) = M^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

The question is how do we compute M^{n-1}?

Eigenvalues of this matrix are roots of characteristic polynomial

$$det(M - \lambda Id) = 0 \tag{7}$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \tag{8}$$

Eigenvectors of M we can compute by solving $Mv^T = \lambda v^T$. After solving we get two eigenvectors

$$\{\lambda_1 = \frac{1 + \sqrt{5}}{2}; v_1 = (1, \frac{1 + \sqrt{5}}{2})\} \tag{9}$$

$$\{\lambda_2 = \frac{1 - \sqrt{5}}{2}; v_2 = (1, \frac{1 - \sqrt{5}}{2})\} \tag{10}$$

From proof in section 2 we know that (I already plugged in the values for Fibonacci sequence)

$$M = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} \tag{11}$$

This is called eigendecomposition.

Observation Squaring matrices is hard, with one exception - squaring a diagonal matrix is simply raising elements on diagonal to 2nd power. This is true for all powers.

Also, because $P * P^{-1} = Id$, raising M^n is simply

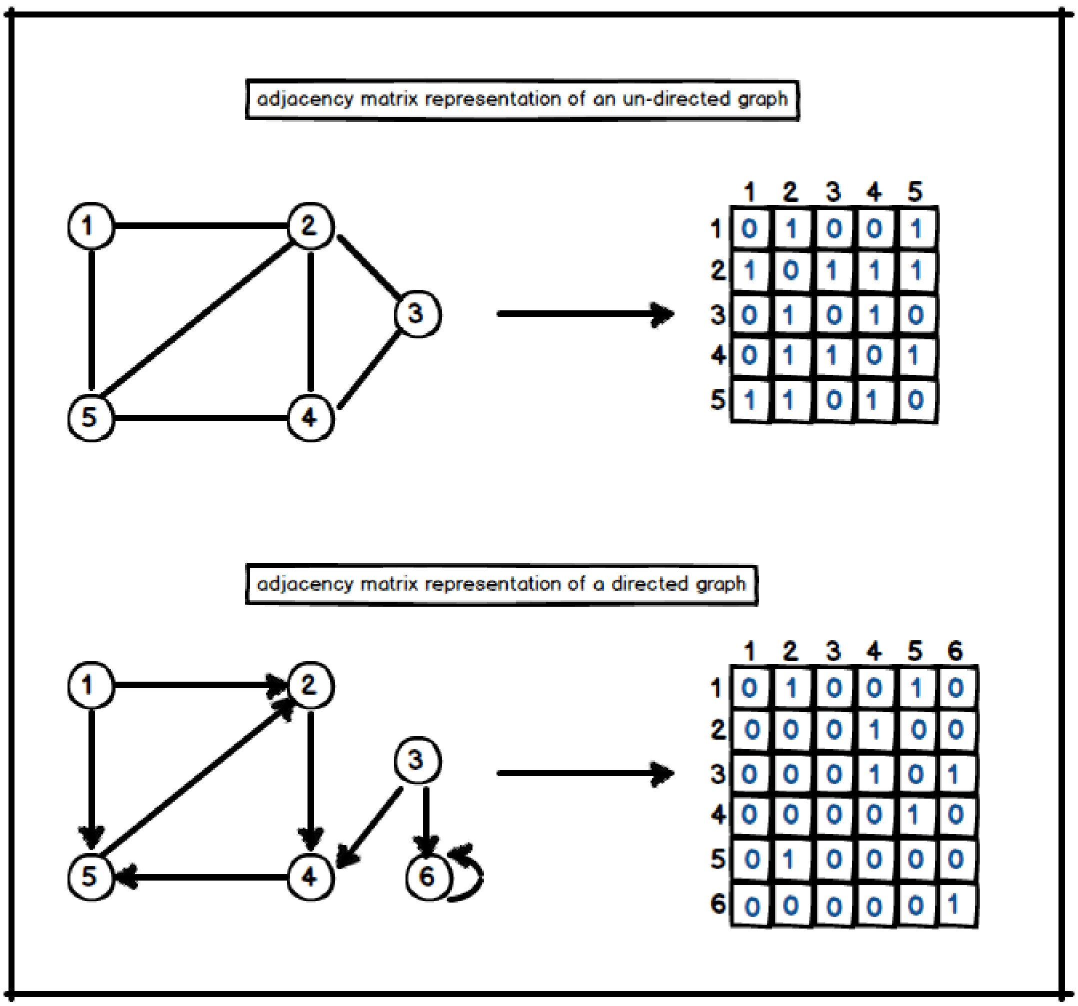
$$M^n = P * D^n * P^{-1} \tag{12}$$

As most $P * P^{-1}$ will cancel out. This enables us to calculate

$$M^{n-1} = (PDP^{-1})^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{13}$$

4 Number of paths in a graph

Adjacency matrix in a graph is matrix showing connection between vertices. It is shown on a picture below.



Theorem (Spectral Theorem) $\forall M$ - symmetric matrices
$\exists P$ - a matrix of eigenvectors of M
D - a diagonal matrix of eigenvalues of M
$M = PDP^{-1}$

In M^n value on position i,j is number of paths of length n from i-th vector to j-th vector.

4.1 Rotation

Let's take a ball. Rotation is a linear map, such that distance between two points is not changed (ie. isometry).

$R : \mathbb{R}^3 \to \mathbb{R}^3$
$\hat{R} : \mathbb{R}^3 \to \mathbb{R}^3$

Question Is it possible to find a point that after 2 rotations didn't change it's place?

Let M_1 be a matrix of R, M_2 matrix of $\hat{R}$. Then $M \leftarrow M_2 M_1$ is isometry

M has eigenvalue $(M - \lambda Id)$ of degree 3, so there is a root λ_0, so $\exists v_0 \neq 0$ such that $Mv_0 = \lambda v_0$. By calculations we get that $\lambda = 1$, so

$$\hat{R}R = Mv_0 = \lambda v_0 = 1v_0 = v_0 \tag{14}$$