Elements of Probability	Summer 2019
Lecture Introduction — 28.02, 2019	
Prof. dr hab. Jacek Cichoń	*Scribe: Krzysztof Agieńczuk*

1 Overview

In this lecture we discussed the probability space, σ fields and open/closed sets.

2 Probability space

σ-field on a set X is a collection Σ of subsets of X that includes X itself, is closed under complement, and is closed under countable unions.

By probability space we denote a triple (Ω, $\mathcal{S}$,P), fulfilling the following conditions:

- $\Omega \neq \emptyset$
- $\mathcal{S}$ is a σ field of subsets of Ω
- P: $\mathcal{S} \to [0, 1]$

With such conditions two conclusions appear.

- P($\emptyset$)=0, P(Ω)=1
- If $(A_n)_{n\in\mathbb{N}}$ are $\mathcal{S}$ and $(\forall n, m)(n \neq m \to A_n \cap A_m = \emptyset)$ then P($\bigcup A_n$)=$\sum_{n=0}^{\infty} P(A_n)$

2.1 Simple model

Let's use simple model. We define $\Omega = [0, 1]^2$ and $\mathcal{S}$ as a family of subsets of Ω such that they have an "area". Let P(A) be this area.

Let's now imagine two disjoint sets in this area. The rest of Ω are infinitely many empty sets. Now, we sum it all. P(0)=0, and so we can write that

$P(A \cup B) = \sum_{n=0}^{\infty} P(A_n) = P(A) + P(B) + P(0) + ... = P(A) + P(B)$

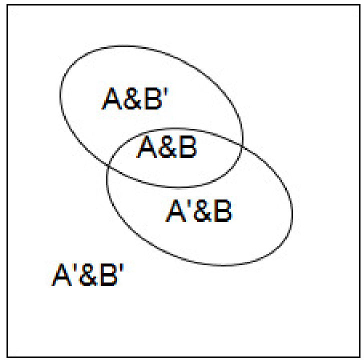
Quick Reminder DeMorgan Laws[1]

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

[1] I differ here from professor Cichoń's notation, as I'm using ' instead of ^{c}. This is just a preference. Both mean compliment and in the rest of the notes I'll be using ' sign

Question 1 Fix Ω; A,B$\subseteq$ Ω. What operations can be defined from A,B and $\cup, \cap, '$ where ' is defined as $(X' = \Omega \setminus X)$

Let's imagine situation like shown on 2.1[2]



Now let's name $A \cap B = C_0$ and so on. $C_0, ..., C_3$ are components of the family of A,B. Now, if $i \neq j$, then $C_i \cap C_j = \emptyset$. Additionally $\sum_{i=0}^{3} C_i = \Omega$

This is called partition of a set.

Now, let's say we have $\mathcal{S} = \{C_T : T \subseteq \{0, 1, 2, 3\}\}$.
If $X, Y \in \mathcal{S} \implies X', Y' \in \mathcal{S} \implies X' \cup Y' \in \mathcal{S} \implies (X' \cup Y') \in \mathcal{S} = X \cap Y$

Note During the lecture prof also showed example where A was included in B. In such case, $A \cap B' = \emptyset$.

2.2 General Case

We have $A_1, A_2, ..., A_n \subseteq \Omega$. Let's start by building all components.
For $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap ... \cap A_n^{\epsilon_n}$ and $\epsilon_i \in \{0, 1\}$ there is 2^n component each for $\epsilon \in \{0, 1\}^n$, and $A_\epsilon = \bigcap_{k=1}^{n} A_k^{\epsilon_k}$.

Assume $\mathbb{T} = \{\epsilon : \{1, ..., n\} \to \{0, 1\} : A_\epsilon \neq \emptyset\}$. For such $\mathbb{T}$, we define k as $k = |\mathbb{T}|$. There is regularity that $1 \leqslant k \leqslant 2^n$.

For $T \subseteq \mathbb{T}$ we put $A_T = \bigcup_{\epsilon \in T} A_\epsilon$.
Let's put $\mathcal{S} = \{A_T : T \subseteq \mathbb{T}\}$. Then $\mathcal{S}$ is closed under $\cap, \cup$ and ' (negation).
This way $|\mathcal{S}| = 2^k$, so $|\mathcal{S}| = 2^{(2^n)}$.

Example 1 For n=10, let's calculate $|\mathcal{S}|$
$2^{2^{10}} \approx 2^{1000} \approx (2^3)^{\frac{1000}{3}} \approx 10^3 00$

[2]Imagine & character is $\cup$. My default diagram editor doesn't support LaTeX inline :P

Definition 1 $\mathcal{S}$ is a σ-field of subsets of Ω if

- $(\forall x \in \mathcal{S})(X \subseteq \Omega)$
- $\emptyset \in \mathcal{S}$
- $(\forall x \in \mathcal{S})(X' \in \mathcal{S})$
- if $(A_n)_{n\in\mathbb{N}}$ are from $\mathcal{S}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$

Remark: $(\bigcup_{n\in\mathbb{N}} A_n)' = \bigcap_{n\in\mathbb{N}} A_n'$

Definition 2 Let $\mathcal{A}$ be a family of subsets Ω. Then $\sigma(\mathcal{A})$ is the smallest field of subsets of Ω which contains $\mathcal{A}$.

Example 2 $\mathcal{A} = \{A_1, ..., A_n\}$
$\sigma(\mathcal{A}) = \{A_T : T \subseteq \{0,1\}^n$

2.3 Theorem

$\sigma(\mathcal{A})$ exists!

Proof 1 Take $\mathcal{A}$, a family of subsets of Ω
$P(\Omega) = \{X : X \subseteq \Omega\}$ is a σ-field.[3].

Definition 3 A set $\mathcal{U} \subseteq \mathbb{R}$ is open if
$(\forall x \in \mathcal{U})(\exists \epsilon > 0)((x-\epsilon),(x+\epsilon) \subseteq \mathcal{U})$

Example 3

- (0,1) is open
- $$0,1$$
  is not open
- (-∞, 0) ∪ (1,+∞) is open

Union of two open sets is open.

[3]P(Ω) means powerset

Definition 4 Analogically, $D \subseteq \mathcal{R}$ is closed if and only if D' is open.

Example 4 [0,1) is neither open, nor closed. We will say that
OPEN($\mathbb{R}$)=$\{\mathcal{U} \subseteq \mathbb{R} : \mathcal{U} is open\}$

Definition 5 Borel($\mathbb{R}$)=σ(OPEN($\mathbb{R}$))

Definition 3 generalization A set $\mathcal{U} \subseteq \mathbb{R}^n$ is open if $(\forall x \subseteq \mathcal{U})(\exists \epsilon > 0)(B(x,\epsilon) \subseteq \mathcal{U})$[4]

Quoting prof. Cichoń "All reasonable subsets of $\mathbb{R}$ are Borel sets."

[4]B probably means Borel set, as in Definition 5, but I am not sure. Somebody can confirm.