Introuction to Complex Systems	Summer 2019

Non-linear dynamics — 07.03, 2019

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1 Overview

In the last lecture we talked about the definition of complex systems, basic rules of modelling and population dynamics.

In this lecture we discussed non-linear dynamics.

2 Logistic model

Before we started the lecture prof. Weron recommended two books: ”Chaos and Fractals” by Petingen, Jürgens andSaupe, as well as ”Nonlinear Dynamics and Chaos” by Strogatz.

As a reminder, last week we talked about logistic equation by Verhulst, in form 1

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{1}$$

which for discrete measurements, simplifies to form 2

$$c_{t+1} = ac_t(1 - c_t) \tag{2}$$

This equation behaves dependant on a, in a way shown in picture 1 (full credit to prof. Weron - picture taken from her presentation).

Fixed point is a point in which we will obtain the same result in the next step. Mathematically it can be written as $c_{t+1} = f(c_t) = c_t \equiv c^*$. The fixed point we will denote as c^*, and thus by solving the logistic equation we obtain two points: $c^* = 0$ or $c^* = \frac{a-1}{a}$. By analogy to rolling a ball on uneven surface, imagine we have two fixed points (ie. such points in which with flow of time ball will not move) - the pit and the top. Again, by analogy only one of those points is stable. We know, that if we push the ball on top it will go down, and the ball in the pit will stay in place. This is stability of a fixed point.
Mathematically we can denote it as:

$$x_t = x^* + \epsilon_t$$
$$x_{t+1} = x* + \epsilon_{t+1}$$
$$f(x^*) = x^*$$

Let ϵ_{t+1} be small, then

$$x_{t+1} = f(x_t) = f(x^* + \epsilon_t) \sim f(x^*) + f'(x^*) * \epsilon_t = x^* + \lambda\epsilon_t$$

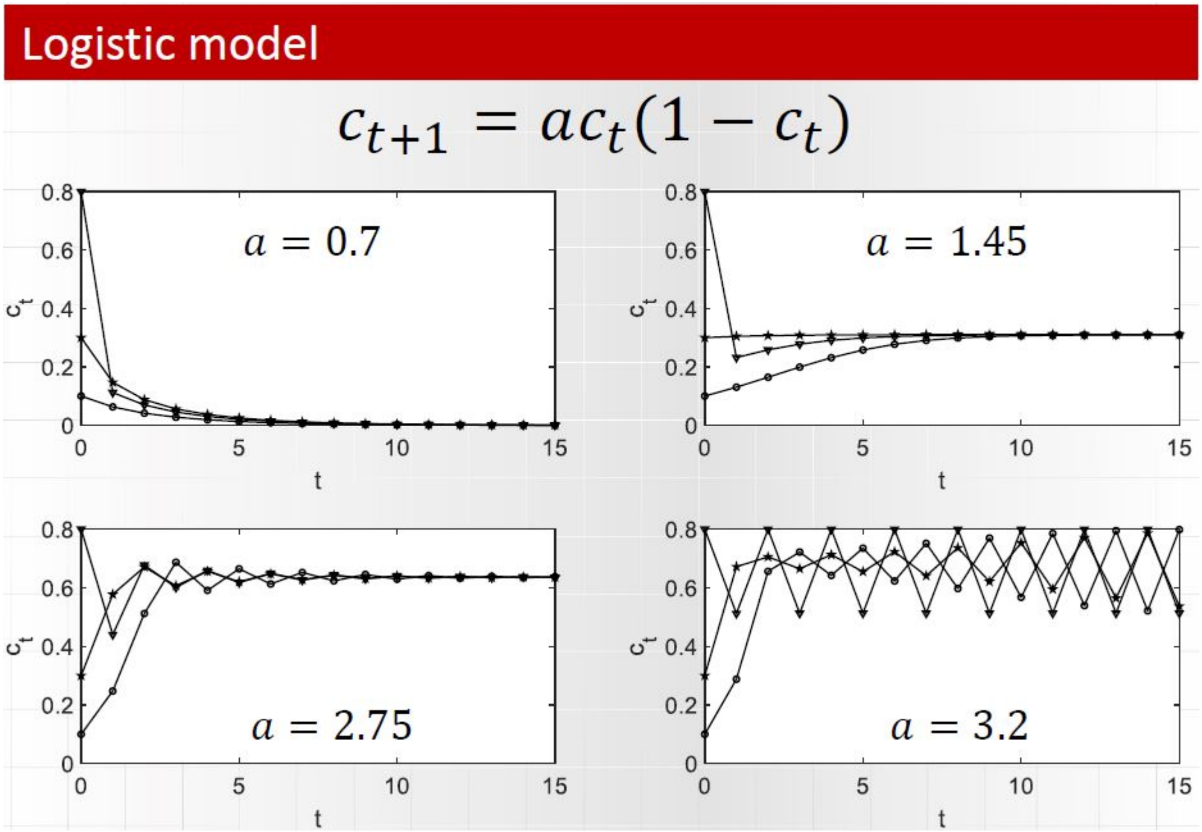

Figure 1: Different behaviour of logistic models, depending on parameter a [prof. Weron Lecture 2 notes]

Thus

$$x_{t+1} \sim x^* + \lambda\epsilon_t \rightarrow \epsilon_{t+1} \sim \lambda\epsilon_t$$
$$\lambda = f'(x*)$$

System is unstable when $|\lambda| > 1$ and stable when $|\lambda| < 1$

3 Deterministic chaos

After that we calculated the stability criteria for logistic model and using graphical iteration method for different values of a. When a=4 we encounter deterministic chaos. At this point we are sensitive to initial conditions. This means we cannot predict the future. This can be showed using The Period Doubling Tree. When we zoom this tree in, we can see "windows" for some values of a. This is called intermittency, ie. irregular alterations of phases of apparently periodic and chaotic dynamics.

Doing same thing for complex numbers can generate so called Julia sets (strictly speaking mapping $z_{n+1} = z_n^2 + c$ can). For any choice of c we can end up with either one piece, or dust (disconnected set). In 1979 Mandelbrot suggested way to picture dichotomy. It is the most famous Mandelbrot fractal (just look up Mandelbrot - it comes up). If we plot them against, we can also see the correlation between Mandelbrot's set and Doubling Period Tree.

3.1 Feigenbaum constant

When we measure the distance between the bifurcation points and calculate ratio, we can see that they tend to one value - 4.669...