Advanced Topics in Algebra Summer 2019

Transition Matrices — 20.03, 2019

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1 Overview

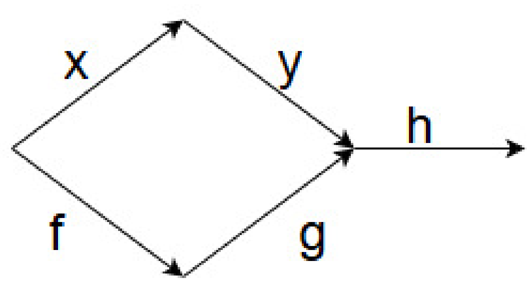
In the last lecture we covered:

- **kernel** and **image of function**,
- **injection** (as a type of function),
- **matrix of a function**,
- **rank of matrix**.

In this lecture we talked about diagrams and transition matrices.

2 Diagram

In this lecture we will use diagrams. In diagram points are sets and arrows are functions. In linear algebra arrows are linear maps. Exemplary diagram is shown on below. We say diagram commutes if $h \oplus g \oplus f = h \oplus y \oplus x$.



3 Reminder

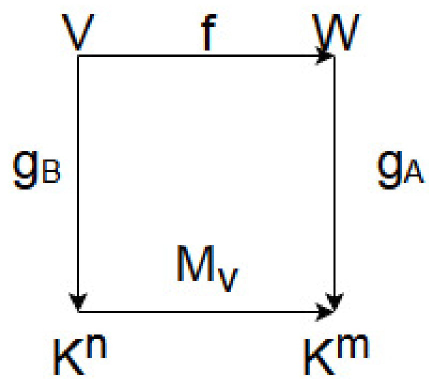
Let V be a vector space over K, $B = \{b_1, b_2, ..., b_n\}$ be a base of V and v∈V. Then $(k_1, ..., k_n)$ is a coordinate of v in base B, if $v = k_1 b_1 + ... + k_n b_n$.

Observation

1. $\{(k_1, ..., k_n) : k \in K\} = K^n$ - vector space
2. $\phi : V \to K^n, \phi_B(v) = v_B$ is a linear map and bijection
3. $\phi_B^{-1} : K^n \to V$ is a linear bijection

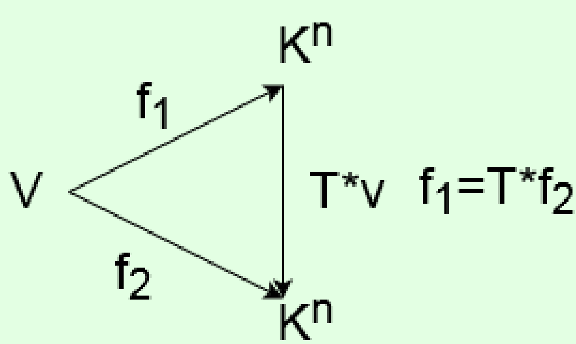
Proof is by definition of linear map, bijection and surjection and is left for the reader to prove.

Let V,W be vector spaces over K. B is a base of V and A is a base of W.
$f : V \to W$ is a linear map.
A matrix of f in basis A,B is matrix M such that the following diagram commutes.



4 Transition matrix

Definition 4.1. *Let V be a vector space over K, B_1, B_2 - basis of V. A transition matrix from B_1 to B_2 such that the following diagram commutes. Where $f1$ is f_{B_1} and $f2$ analogically.*



Fact A is a transition matrix

Proof By definition $f_{B_1}^{-1}$ and f_{B_1} are linear maps. $f_{B_2} \oplus f_{B_1}^{-1}$ is a linear map $f_{B_2} \oplus f_{B_1}^{-1} : K^n \to K^n$.
T is a matrix of map $f_{B_2} \oplus f_{B_1}^{-1}$ in standard basis.

Observation Let V be a vector space over K, B_1, B_2 basis of V, T - transition matrix B_2 to B_1.
$\forall v \in VT * V_{B_1} = V_{B_2}$.

Example Let $V = \mathbb{R}^2$, $E = \{(1,0),(0,1)\}$, $B = \{(1,2),(3,4)\}$
$T \in \mathbb{R}^{2x2} : \forall v \in \mathbb{R}^2 : T_{V_B} = V_E$
T = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Note: it is enough to check the above equation on a vector from base.