Задача 8

Вариант 3

$$\begin{cases} \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)u(x) = -f(x), \quad x \in [0,1] \\ u(0) = 0 \quad u(1) = 1 \end{cases}$$

$$k(x) = \begin{cases} 3, & x \in (0;0,3) \\ 0,5, & x \in (0,3;\ 5/7) \\ 100, & x \in (5/7,\ 1) \end{cases} \qquad q(x) = \begin{cases} 3, & x \in (0;0,3) \\ 0, & x \in (0,3;\ 5/7) \\ 1, & x \in (5/7;\ 1) \end{cases} \qquad f(x) = \begin{cases} 0, & x \in (0;0,3) \\ 0 & x \in (0,3;\ 5/7) \\ 100 & x \in (5/7,\ 1) \end{cases}$$

① Смысл условий сопряжения:

$u_+ = u_-$ - требование непрерывности температуры в точках $\xi = 0,3;\ \xi = 5/7$

$W_+ = W_-$ - требование непрерывности теплового потока в точках $\xi = 0,3,\ \xi = 5/7$

$\Rightarrow$ температура, тепловой поток, коэф. $k(x), q(x), f(x)$ в точках $\xi = 0,3,\ \xi = 5/7$ должны быть непрерывным по x на $[0,1]$

$\Rightarrow$ гарантия существования и единств. реш. задачи

$$\begin{cases} W(x) = -3u'(x), & 0 < x < 0,3 \\ W(x) = -k_1(x)u'(x), & x = 0,3 \\ W(x) = -0,5 \cdot u'(x), & 0,3 < x < 5/7 \\ W(x) = -k_2(x)u'(x), & x = 5/7 \\ W(x) = -100\,u'(x), & 5/7 < x < 1 \end{cases}$$

1. $x \in (0;0,3)$: $3u_1''(x) - 3u_1 = 0$
   $u(0) = 0 \quad u(1) = 1$
   $u_+ = u_- \quad W_+ = W_-$
2. $x \in (0,3;\ 5/7)$: $0,5u_2''(x) = 0$
   $u(0) = 0 \quad u(1) = 1$
   $u_+ = u_- \quad W_+ = W_-$
3. $x \in (5/7,\ 1)$: $100u_3''(x) - u_3(x) = -100$
   $u_+ = u_- \quad W_+ = W_-$

2.

$$d_i = \frac{1}{h}\int_{x_i-0,5}^{x_i+0,5} q(x)\,dx \qquad i = \overline{1, n-1}$$

$$q(x) = \begin{cases} 3; & x \in (0;0,3) \\ 0; & x \in (0,3;\ 5/7) \\ 1; & x \in (5/7;\ 1) \end{cases} \qquad \frac{1}{h} = 10$$

$$d_1 = 10\int_{x_{0,5}}^{x_{1,5}} 3\,dx = 30 \cdot 0,1 = 3$$

$$d_1 = d_2 = 3$$

$$d_3 = 10\int_{0,25}^{0,3} 3\,dx + 10\int_{0,3}^{0,35} 0\,dx = 30 \cdot (0,3 - 0,25) = 30 \cdot 0,05 = 1,5$$

$$d_4 = 10\int_{0,35}^{0,45} 0\,dx = 0$$

$$d_4 = d_5 = d_6 = 0$$

$$d_7 = 10\left(\int_{0,65}^{5/7} 0\,dx + \int_{5/7}^{0,75} 1\,dx\right) = \frac{5}{14} \approx 0,357$$

$$d_8 = 10\int_{0,75}^{0,85} 1\,dx = 1$$

$$d_8 = d_9 = 1$$

3.

$$\varphi_i = \frac{1}{h}\int_{x_i-0,5}^{x_i+0,5} f(x)\,dx$$

$$f(x) = \begin{cases} 0; & (0;0,3) \\ 0; & (0,3;\ 5/7) \\ 100; & (5/7;\ 1) \end{cases} \qquad \frac{1}{h} = 10$$

$$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$$

$$\varphi_7 = 10\int_{0,65}^{5/7} 0\,dx + 10\int_{5/7}^{0,75} 100\,dx = 1000 \cdot (0,75 - 5/7) = \frac{250}{7} \approx 35,714$$

$$\varphi_8 = \varphi_9 = 100$$

② узлы $x_i = ih$ $i = \overline{0,n}$ $h = \frac{1}{n}$ – шаг сетки

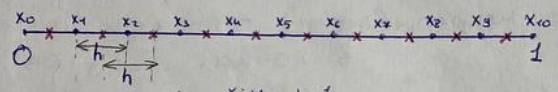
$$\begin{cases} \frac{v_{i-1} - v_i}{h^2} \cdot a_i - \frac{v_i - v_{i+1}}{h^2} a_{i+1} - d_i v_i = -\varphi_i, & i = \overline{1,n-1} \\ v_0 = \mu_1 \qquad v_n = \mu_2 \end{cases}$$

$$a_i = \left(\frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{dx}{k(x)} \right)^{-1} \quad i = \overline{1,n}$$

$$d_i = \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)\,dx \quad i = \overline{1,n-1}$$

$$\varphi_i = \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)\,dx \quad i = \overline{1,n-1}$$

③ $n = 10$ $h = 0{,}1$



1. $a_i = \left(\frac{1}{h} \int_{x_{i-1}}^{x_{i+1}} \frac{dx}{k(x)} \right)^{-1} \quad i = \overline{1,n}$

$$k(x) = \begin{cases} 3; & x \in (0; 0{,}3) \\ 0{,}5; & x \in (0{,}3; 5/7) \\ 100; & x \in (5/7; 1) \end{cases} \qquad \frac{1}{h} = 10$$

$$a_1 = \left(10 \int_{x_0}^{x_1} \frac{dx}{3} \right)^{-1} = \left(\frac{10}{3}(0{,}1 - 0) \right)^{-1} = 3$$

$$a_2 = a_3 = a_4 = 3$$

$$a_4 = \left(10 \int_{0{,}3}^{0{,}4} \frac{dx}{0{,}5} \right)^{-1} = (20 \cdot 0{,}1)^{-1} = 2$$

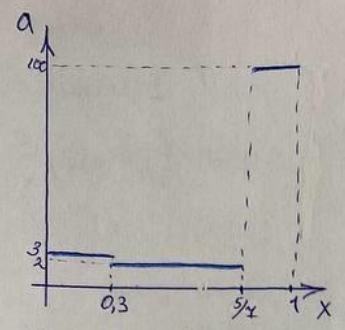
$$a_4 = a_5 = a_6 = a_7 = 2$$

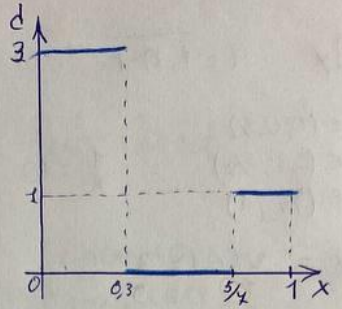
$$a_8 = \left(10 \left(\int_{0{,}7}^{5/7} \frac{dx}{0{,}5} + \int_{5/7}^{0{,}8} \frac{dx}{100} \right) \right)^{-1} = \left(10 \cdot \left(2\left(\frac{5}{7} - 0{,}7 \right) + \frac{1}{100}\left(0{,}8 - \frac{5}{7} \right) \right) \right)^{-1} =$$

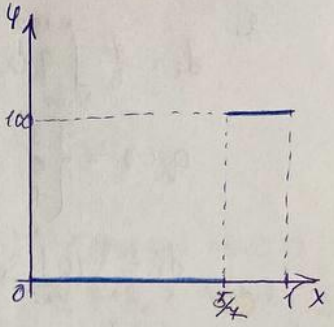
$$= \frac{350}{103} \approx 3{,}398$$

$$a_9 = \left(10 \int_{0{,}8}^{0{,}9} \frac{dx}{100} \right)^{-1} = \left(\frac{1}{10} \cdot 0{,}1 \right)^{-1} = 100$$

$$a_9 = a_{10} = 100$$







$$a=\begin{cases}3;\ x\in[0;0{,}3]\\ 2;\ x\in[0{,}3;\ 5/7]\\ \frac{350}{103};\ x=5/7\\ 100;\ x\in[5/7,\ 1]\end{cases}$$

$$d=\begin{cases}3;\ x\in[0;0{,}3]\\ 1{,}5;\ x=0{,}3\\ 0;\ x\in[0{,}3;\ 5/7]\\ \frac{5}{14};\ x=5/7\\ 1;\ x\in[5/7;\ 1]\end{cases}$$

$$\varphi=\begin{cases}0;\ x\in[0,\ 5/7]\\ \frac{250}{7};\ x=5/7\\ 100;\ x\in[5/7,\ 1]\end{cases}$$

(4) $n=10 \quad h=0{,}1$

РС в каноним. виде

$$\begin{cases}\frac{a_i}{h^2}v_{i-1}-\left(\frac{a_i+a_{i+1}}{h^2}+d_i\right)v_i+\frac{a_{i+1}}{h^2}v_{i+1}=-\varphi_i\\ v_0=0\\ v_n=1\end{cases}$$

$v_0=0$

$300v_0-(100(3+3)+3)v_1+300v_2=0 \quad i=1$

$300v_1-(100(3+3)+3)v_2+300v_3=0 \quad i=2$

$300v_2-(100(3+2)+1{,}5)v_3+200v_4=0 \quad i=3$

$200v_3-(100(2+2)+0)v_4+200v_5=0 \quad i=4$

$200v_4-(100(2+2)+0)v_5+200v_6=0 \quad i=5$

$200v_5-(100(2+2)+0)v_6+200v_7=0 \quad i=6$

$200v_6-\left(100\left(2+\frac{350}{103}\right)+\frac{5}{14}\right)v_7+\frac{350}{103}\cdot 100v_8=-\frac{250}{7} \quad i=7$

$\frac{250}{103}\cdot 100v_7-\left(100\left(\frac{350}{102}+100\right)+1\right)v_8+100\cdot 100v_9=-100 \quad i=8$

$10000v_8-(100(100+10)+1)v_9+100\cdot 100v_{10}=-100 \quad i=9$

$v_{10}=1$

Задание 9

Вариант 1

$$\begin{cases} \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)u(x) = -f(x), \quad x \in [0,1] \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

$$k(x) = \begin{cases} x+2, & x \in (0,\xi) \\ e^x, & x \in (\xi,1) \end{cases} \quad q(x) = \begin{cases} \sin(x), & x \in (0,\xi) \\ 2-x^2, & x \in (\xi,1) \end{cases} \quad f(x) = \begin{cases} \cos(2x), & x \in (0,\xi] \\ x+2, & x \in (\xi,1) \end{cases}$$

$\xi = 0{,}35$, в точке $x = \xi$ $u_+ = u_-$; $w_+ = w_-$

1) Записать схему и формулы для вычисл. её коэф.

Разн. схема имеет вид:

$$\begin{cases} \frac{a_i}{h^2}v_{i-1} + \left(\frac{a_i}{h^2} - \frac{a_{i-1}}{h^2} - d_i\right)v_i + \frac{a_{i+1}}{h^2}v_{i+1} = \varphi_i, \quad i = \overline{1, n-1} \\ v_0 = \mu_1; \quad v_n = \mu_2 \end{cases}$$

$$\varphi_i \begin{cases} \frac{1}{h}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f_1(x)dx, & x_{i+\frac{1}{2}} \le \xi \\ \frac{1}{h}\left(\int\limits_{x_{i-\frac{1}{2}}}^{\xi} f_1(x)dx + \int\limits_{\xi}^{x_{i+\frac{1}{2}}} f_2(x)dx\right), & \xi \in \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \\ \frac{1}{h}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f_2(x)\,dx, & x_{i-\frac{1}{2}} \ge \xi \end{cases}$$

$$d_i \begin{cases} \frac{1}{h}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q_1(x)dx, & x_{i+\frac{1}{2}} \le \xi \\ \frac{1}{h}\left(\int\limits_{x_{i-\frac{1}{2}}}^{\xi} q_1(x)dx + \int\limits_{\xi}^{x_{i+\frac{1}{2}}} q_2(x)dx\right), & \xi \in \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \\ \frac{1}{h}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q_2(x)dx, & x_{i-\frac{1}{2}} \ge \xi \end{cases}$$

$$a_i = \begin{cases} \left(\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \frac{dx}{k_1(x)}\right)^{-1}, & x_i \le \xi \\ \left(\frac{1}{h}\int\limits_{x_{i-1}}^{\xi} \frac{dx}{k_1(x)} + \int\limits_{x_i}^{\xi} \frac{dx}{k_2(x)}\right)^{-1}, & \xi \in (x_{i-1}, x_i) \\ \left(\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \frac{dx}{k_1(x)}\right)^{-1}, & x_{i-1} > \xi \end{cases}$$

3

$$\varphi_1 = \frac{1}{h}\int_{0,05}^{0,15} \cos 2x\,dx = 5(\sin 0{,}15 - \sin 0{,}05) \approx 0{,}97845$$

$$\varphi_2 = 5(\sin 0{,}25 - \sin 0{,}15) \approx 0{,}9795$$

$$\varphi_3 = 5(\sin 0{,}35 - \sin 0{,}25) \approx 0{,}082395$$

$$\varphi_4 = \frac{1}{h}\int_{0{,}35}^{0{,}45} (x+2)\,dx = 10\left(\frac{x^2}{2} + 2x\right)\Big|_{0{,}35}^{0{,}45} \approx 2{,}4$$

$$\varphi_5 \approx 2{,}5$$

$$\varphi_6 \approx 2{,}6$$

$$\varphi_7 \approx 2{,}7$$

$$\varphi_8 \approx 2{,}8$$

$$\varphi_9 \approx 2{,}9$$

Ур-я разн. схемы

$$v_0 = 0,\quad v_{10} = 1$$

$$\frac{a_1}{h^2}v_0 - \left(\frac{a_1+a_2}{h^2} + d_1\right)v_1 + \frac{a_2}{h^2}v_2 = -\varphi_1$$

$$\frac{a_2}{h^2}v_1 - \left(\frac{a_2+a_3}{h^2} + d_2\right)v_2 + \frac{a_3}{h^2}v_3 = -\varphi_2$$

$$\frac{a_3}{h^2}v_2 - \left(\frac{a_3+a_4}{h^2} + d_3\right)v_3 + \frac{a_4}{h^2}v_4 = -\varphi_3$$

$$\frac{a_4}{h^2}v_3 - \left(\frac{a_4+a_5}{h^2} + d_4\right)v_4 + \frac{a_5}{h^2}v_5 = -\varphi_4$$

$$\frac{a_5}{h^2}v_4 - \left(\frac{a_5+a_6}{h^2} + d_5\right)v_5 + \frac{a_6}{h^2}v_6 = -\varphi_5$$

$$\frac{a_6}{h^2}v_5 - \left(\frac{a_6+a_7}{h^2} + d_6\right)v_6 + \frac{a_7}{h^2}v_7 = -\varphi_6$$

$$\frac{a_7}{h^2}v_6 - \left(\frac{a_7+a_8}{h^2} + d_7\right)v_7 + \frac{a_8}{h^2}v_8 = -\varphi_7$$

$$\frac{a_8}{h^2}v_7 - \left(\frac{a_8+a_9}{h^2} + d_8\right)v_8 + \frac{a_9}{h^2}v_9 = -\varphi_8$$

$$\frac{a_9}{h^2}v_8 - \left(\frac{a_9+a_{10}}{h^2} + d_9\right)v_9 + \frac{a_9}{h^2}v_{10} = -\varphi_9$$

2) $n = 10 \quad \xi = 0{,}35 \quad h = \frac{1}{10}$

$x_0 = 0$

$x_1 = 0{,}1$

$x_2 = 0{,}2$

$x_3 = 0{,}3$

$x_4 = 0{,}4$

$x_5 = 0{,}5$

$x_6 = 0{,}6$

$x_7 = 0{,}7$

$x_8 = 0{,}8$

$x_9 = 0{,}9$

$x_{10} = 1$

$x_{\frac{1}{2}} = 0{,}05$

$x_{\frac{3}{2}} = 0{,}15$

$x_{\frac{5}{2}} = 0{,}25$

$x_{\frac{7}{2}} = 0{,}35$

$x_{\frac{9}{2}} = 0{,}45$

$x_{\frac{11}{2}} = 0{,}55$

$x_{\frac{13}{2}} = 0{,}65$

$x_{\frac{15}{2}} = 0{,}75$

$x_{\frac{17}{2}} = 0{,}85$

$x_{\frac{19}{2}} = 0{,}95$

$a_1 = \left(\frac{1}{h}\ln(x+2)\Big|_0^{0,1}\right)^{-1} = \left(10(\ln 2{,}1 - \ln 2)\right)^{-1} \approx 2{,}0496$

$a_2 = \left(10(\ln 2{,}2 - \ln 2{,}1)\right)^{-1} \approx 2{,}1496$

$a_3 = \left(10(\ln 2{,}3 - \ln 2{,}2)\right)^{-1} \approx 2{,}2496$

$a_4 = \left(\frac{1}{h}\int_{0,3}^{0,35}\frac{1}{x+2}dx + \int_{0,35}^{0,4}\frac{1}{e^x}dx\right)^{-1} = \left(10\left(\ln 2{,}35 - \ln 2{,}3\right) - \left(\frac{1}{e^{0,35}} - \frac{1}{e^{0,4}}\right)\right)^{-1} \approx$

$\approx 2{,}3496$

$a_5 = \left(\frac{1}{h}\int_{0,4}^{0,5}\frac{1}{e^x}dx\right)^{-1} = \left(10(-1)\left(\frac{1}{e^{0,5}} - \frac{1}{e^{0,4}}\right)\right)^{-1} \approx 1{,}5676$

$a_6 = \left(-10\left(e^{-0,6} - e^{-0,5}\right)\right)^{-1} \approx 1{,}7325$

$a_7 = \left(-10\left(e^{-0,7} - e^{-0,6}\right)\right)^{-1} \approx 1{,}9147$

$a_8 = \left(-10\left(e^{-0,8} - e^{-0,7}\right)\right)^{-1} \approx 2{,}1161$

$a_9 = \left(-10\left(e^{-0,9} - e^{-0,8}\right)\right)^{-1} \approx 2{,}3386$

$a_{10} = \left(-10\left(e^{-1} - e^{-0,9}\right)\right)^{-1} \approx 2{,}5846$

$d_1 = \frac{1}{h}\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}}\sin x\,dx = \frac{-1}{10}(\cos 0{,}15 - \cos 0{,}05) \approx 0{,}00003$

$d_2 = -\frac{1}{10}(\cos 0{,}25 - \cos 0{,}15) \approx 0{,}00006$

$d_3 = -\frac{1}{10}(\cos 0{,}35 - \cos 0{,}25) \approx 0{,}00009$

$d_4 = \frac{1}{h}\int_{0,35}^{0,45}(2 - x^2)dx = 10\left(2x - \frac{x^3}{3}\right)\Big|_{0,35}^{0,45} \approx 1{,}83916$

$d_5 \approx 1{,}74916$

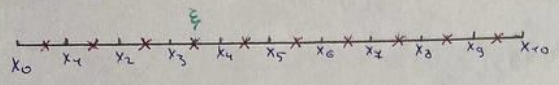
$d_6 \approx 1{,}63916$

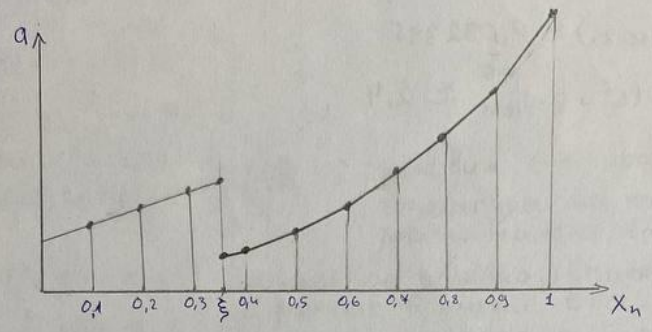
$d_7 \approx 1{,}50916$

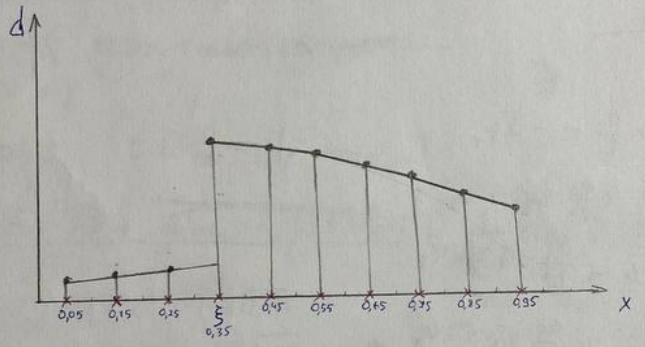
$d_8 \approx 1{,}35916$

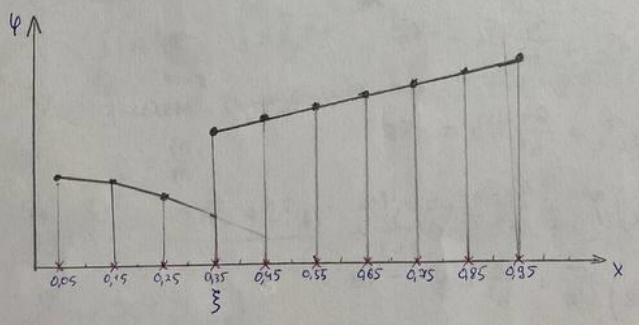
$d_9 \approx 1{,}18916$

Рисунки


ξ
x0 x1 x2 x3 x4 x5 x6 x7 x8 x9 x10


a
0,1 0,2 0,3 ξ 0,4 0,5 0,6 0,7 0,8 0,9 1
Xn


d
0,05 0,15 0,25 ξ 0,45 0,55 0,65 0,75 0,85 0,95
0,35
X


φ
0,05 0,15 0,25 0,35 0,45 0,55 0,65 0,75 0,85 0,95
ξ
X