Program	**B. Tech. (SoCS)**	**Semester**	**IV**
Course	**Linear Algebra**	**Course Code**	**MATH 2059**
Session	**Jan-May 2025**	**Topic**	**Eigenvectors, Diagonalization and Quadratic forms**

1. Consider two matrices A and B both of order 2×2 such that rank (A) = rank (B) = 1 and $trace\ (B) \neq 0$. Find the value of

$$\lim_{t \to 0} \frac{p_A(t)}{p_B(t)}$$

where $p_X(t)$ denotes the characteristic polynomial of matrix X.

2. Suppose A is a real matrix of order $n \times n$ with characteristic polynomial

$$p_A(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots\dots + a_{n-1} t + a_n$$

Prove the following results:

(a) If $t = 1$ is a characteristic root of $p_A(t)$ then

$$\sum_{i=1}^{n} a_i = -1$$

(b) If $t = 0$ is a characteristic root of $p_A(t)$ then

$$a_n = 0$$

3. Suppose $p_A(t)$ is the characteristic polynomial for matrix A of order $n \times n$ and $f(t)$ is a polynomial of degree $m > n$ such that $p_A(t)$ divides $f(t)$. Prove that the set of eigenvalues of matrix $f(A)$ is a singleton set.

4. Prove that the matrix

$$X = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

is diagonalizable if and if only $a = 0$.

5. Determine the most general condition on a and b so that the real matrix

$$X = \begin{pmatrix} 0 & 0 & 1 \\ a & 1 & b \\ 1 & 0 & 0 \end{pmatrix}$$

is diagonalizable. Hence find all the linearly independent vector(s) v such that $Xv = v$.

6. Suppose

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Determine the following matrices

(a) the matrix exponential e^A

(b) the positive square root matrix $\sqrt{A}$

†**HINT: Use diagonalization of matrix A**

7. Suppose D is the diagonal matrix consisting of eigenvalues of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ in its diagonal entries. Determine a nonsingular matrix Q such that

(a) $D = Q^{-1}AQ$ where Q must not be orthogonal $i.e.$ $Q^{-1} \neq Q^T$

(b) $D = Q^TAQ$ where Q must be orthogonal $i.e.$ $Q^{-1} = Q^T$

8. For what substitutions of the form $x = \phi(r,s)$ and $y = \varphi(r,s)$ the function $f(x,y) = 2x^2 + 4xy + 5y^2$ reduces to $f(r,s) = ar^2 + by^2$ for some choice of a and b?

9. Find the orthogonal substitution expressing variables x, y, z in terms of variables r, s, t to reduce the given quadratic form to diagonal form

$$f(x,y) = 3x^2 - 4xy + 6y^2 + 2xz - 4yz + 3z^2$$

10. If the matrix

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

satisfies the condition $6M^{-1} = aM^2 + bM + cI$ where I is the identity matrix of order 3×3 then find the values of a, b and c.

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