Diagonalizability of symmetric matrices

- Symmetric matrix : $A = A^T$.
- Symmetric matrix is always diagonalizable.
  (This means we will always get n eigenvectors for a symmetric matrix of order $n \times n$)
- If A is symmetric, we can find a P such that

$$P^T A P = D$$

$\rightarrow$ diagonal. $\quad$ P : orthogonal.

P will be orthogonal if and only if

- columns of P have dot product between them 0.
- magnitude of columns of P will be 1.

Q) Find a matrix Q such that $Q^T A Q = D$, is diagonal.

where $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

A: Char. eqn: $-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = 0$.

$-(\lambda-8)(\lambda-6)(\lambda-3) = 0$

$\Rightarrow \underbrace{\lambda = 8, 6, 3}$ → eigenvalues.

eigenvectors, $\lambda = 8$ ↓ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\lambda = 6$ ↓ $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ $\lambda = 3$ ↓ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

These matrices satisfy $P^{-1}AP = D$.

We need to make P into an orthogonal matrix Q so that we will have $Q^T A Q = D$.

• Check dot product between columns

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\-1\\2\end{bmatrix} = 1-1+0=0 \checkmark$$

$$\begin{bmatrix}-1\\-1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = -1-1+2=0 \checkmark$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix} = -1+1+0=0 \checkmark$$

• check if magnitude of each column is 1

$$\left\|\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\| = \sqrt{-1^2+1^2+0} = \sqrt{2}$$ ($\therefore$ divide this column by $\sqrt{2}$)

$$\left\|\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\| = \sqrt{-1^2+-1^2+2^2} = \sqrt{6}$$ ($\therefore$ divide this column by $\sqrt{6}$)

$$\left\|\begin{bmatrix}1\\1\\1\end{bmatrix}\right\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$ ($\therefore$ divid this column by $\sqrt{3}$)

matrix $Q = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$ and this satisfies

$Q^TAQ=D$.

Q. Example where the matrix has repeated eigenvalues.

Find Q such that $Q^T A Q = D$,

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

ch eqn: $\lambda^3 - 12\lambda^2 + 21\lambda + 98 = 0$

$$(\lambda-7)^2(\lambda+2) = 0$$

$$\lambda = 7, 7, -2.$$

eigenvectors: when $\lambda = -2$, $\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

when $\lambda = 7$, $Ax = 7x$

$$(A - 7I)x = 0$$

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -4 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x - 2y + 4z = 0$$

$$-2x - y + 2z = 0$$

$$y = -2x + 2z$$

$\therefore$ eigenvector $\begin{bmatrix} x \\ -2x+2z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Choose $P = \begin{bmatrix} -1 & 1 & 0 \\ -1/2 & -2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

this will satisfy $P^{-1}AP = D$.

To find Q.

- make the columns having dot product between them zero

$$\begin{array}{cccc} & \lambda=-2 & \lambda=7 & \\ P & \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 0 & 1 \end{bmatrix} & \\ & u & v \quad w & \end{array}$$

$u.v = -1+1+0 = 0$ ✓

$u.w = 0-1+1 = 0$ ✓

$v.w = 0-4+0 = -4$ ✗

dot product between eigenvectors of different eigenvalues will be 0, but dot product between eigenvectors of same eigenvalue might not be zero. Hence, we have to make it zero.

Here $v.w \neq 0$.

Let us keep v and change w.

w belongs to the set of eigenvectors corresponding to $\lambda=7$.

Choose $w = \begin{bmatrix} x \\ -2x+2z \\ z \end{bmatrix}$

we need $v \cdot w = 0$

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ -2x+2z \\ z \end{bmatrix} = 0$$

i.e, $x - 2(-2x+2z) + 0 = 0$

$\Rightarrow x + 4x - 4z = 0$

$5x - 4z = 0.$

choose $x = 4, z = 5.$

then $w = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$. which make $v \cdot w = 0$.

- make magnitude 1 :

$|u| = \sqrt{-1^2 + -\frac{1}{2}^2 + 1^2} = \frac{3}{2}.$

$|v| = \sqrt{1^2 + -2^2 + 0} = \sqrt{5}$

$|w| = \sqrt{4^2 + 2^2 + 5^2} = \sqrt{45} = \underline{\underline{3\sqrt{5}}}.$

$$\therefore \quad Q = \begin{bmatrix} -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ -1/3 & -2/\sqrt{5} & 2/3\sqrt{5} \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix}$$ satisfies $Q^T A Q = D$.

Quadratic Forms

a polynomial in which every term has power two.

In one variable, $q(x) = ax^2$

2 variables, $q(x,y) = ax^2 + bxy + cy^2$

3 variables, $q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fxz$.

- Any quadratic form can be uniquely expressed using a symmetric matrix

eg: Let $q(x,y) = 2x^2 + 4xy + y^2$.

then ~~the~~ let $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$.

Then $q(x,y) = u^T A u$.

eg: $q(x,y,z) = x^2 + 2y^2 + 3z^2 - 6xy + 8yz - 10xz$.

$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $A = \begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 1 & -3 & -5 \\ -3 & 2 & 4 \\ -5 & 4 & 3 \end{bmatrix} \end{array}$

Then $q(x,y,z) = u^T A u$.

- If you have any symmetric matrix, you can write its polynomial as well

eg: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Then, if $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $u^T A u$ will be a quadratic form.

$$u^T A u = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 2xy + 2yx + 3y^2$$
$$= x^2 + 4xy + 3y^2$$

↳ quadratic form.

Change of variables

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$ be the variables used in a function.

if we use new variables $v = \begin{bmatrix} r \\ s \end{bmatrix}$ instead of x, y such that the function $f(x,y)$ is changed into $g(r,s)$, ~~these~~ this is called change of variables.

$\begin{bmatrix} x \\ y \end{bmatrix}$ is changed to $\begin{bmatrix} r \\ s \end{bmatrix}$ with the use of some 2×2 matrix P.

~~$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} r \\ s \end{bmatrix}$ or $u = Pv$~~

$v = Pu$

$$\begin{bmatrix} r \\ s \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix}$$

P gives relation between r, s, x, y.

Note: In a quadratic form, the terms with ~~to~~ product of variables is called cross product terms.

eg: if $q(x,y) = 2x^2 + \underbrace{6xy}_{\text{cross product term.}} + 3y^2$

Q) Convert the following quadratic form in x and y into another quadratic form with variables r and s so that the new ~~fo~~ form has no ~~......~~ cross product terms.

~~$q(x,y) = 2 + 3$~~ ~~$q(x,y) = 2x^2 + 2xy + 3y^2$~~

$$q(x,y) = 2x^2 + 6xy + 2y^2.$$

A: Here $6xy$ is the product term.

we need to change x, y to r, s so that

$g(r,s) = q(x,y)$

has no product terms.

$q(x,y) = 2x^2 + 6xy + 2y^2.$

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

Then $q(x,y) = u^T A u$ —— (1).

Since A is symmetric, we can find an orthogonal matrix Q such that $Q^T A Q = D$

or $A = Q D Q^T$.

Finding Q

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(2-\lambda) - 9 = 0$

$4 - 3\lambda + \lambda^2 - 9 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = 5, -1.$

eigenvectors

$\lambda = 5$

$Ax = 5x$

$(A - 5I)x = 0$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x + 3y = 0$

$x = y, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

$Ax = -x$

$(A + I)x = 0$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x + 3y = 0$

$x = -y \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ choose $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

then $P^{-1}AP = D.$

- ~~dot~~ making Q from P:

dot product between columns, $\begin{bmatrix}1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\end{bmatrix} = -1+1=0$ ✓

magnitude of $\begin{bmatrix}1\\1\end{bmatrix} = \sqrt{1^2+1^2} = \sqrt{2}$.

magnitude of $\begin{bmatrix}-1\\1\end{bmatrix} = \sqrt{-1^2+1^2} = \sqrt{2}$.

$\therefore \quad Q = \dfrac{1}{\sqrt{2}} \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

This satisfies $Q^T A Q = D$.

$\Rightarrow A = Q D Q^T$.

Substitute in (1) to get

$q(x,y) = \underline{u^T Q}\, D\, \underline{Q^T u}$

Let $Q^T u = v \Rightarrow u^T Q = v^T$.

$\therefore \quad q(x,y) = v^T D v$.

Let $v = \begin{bmatrix} r\\ s\end{bmatrix}$.

Then $q(x,y) = \begin{bmatrix} r & s\end{bmatrix} \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix} \begin{bmatrix} r\\ s\end{bmatrix}$

$g(r,s) = \underline{\underline{5r^2 - s^2}}$.

Relation between r, s, x, y: $\quad v = Q^T u$

$\begin{bmatrix} r\\ s\end{bmatrix} = \dfrac{1}{\sqrt{2}} \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \begin{bmatrix} x\\ y\end{bmatrix} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} x+y\\ -x+y\end{bmatrix}$.