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Linear Algebra
Assignment – 2

Ans 1. To determine which two vector space axioms are broken when the addition operation is defined as function composition & scalar multiplication is kept usual, we analyze the vector space axioms.

When defining addition as function composition $f(g(x))$ with the zero vector $g(x) = x$:

1. Commutativity fails:

$$f(g(x)) \neq g(f(x)) \text{ in general.}$$

Example:

Let $f(x) = x^2$ and $g(x) = x+1$.

- $f+g = f(g(x)) = (x+1)^2$
- $g+f = g(f(x)) = x^2+1$

Clearly,

$$(x+1)^2 \neq x^2+1$$

So, $f+g \neq g+f$. Commutativity fails.

2. Distributivity fails:

$$c(f(g(x))) \neq c(f(c(g(x)))) \text{ typically.}$$

Example:

Take $c=2$, $f(x) = x^2$ and $g(x) = x+1$,

- $c(f+g) = 2 \cdot f(g(x)) = 2(x+1)^2$
- $cf + cg = (2f)((2g)(x)) = 2f(2x+2) = 2(2x+2)^2$

Clearly,

$$2x^2+4x+2 \neq 8x^2+16x+8$$

So $c(f+g) \neq cf+cg$. Distributivity fails

Ans 2. We are given a vector $x = \{x_1, x_2, x_3, x_4\} \in \mathbb{R}^4$, and we consider the 24 permutations of its components (i.e. all possible reorderings of the four entries). These 24 vectors span a subspace $S \subseteq \mathbb{R}^4$.

(a) $\dim(S) = 0$

This means that all permutations are zero vector, so:

- Choose $x = (0,0,0,0)$

✓ All 24 permutations are the zero vector → span is $\{0\}$

→ $\dim(S) = 0$

(b) $\dim(S) = 1$

This means all permutations are scalar multiples of each other, i.e., they lie along a line.

- Choose $x = (1,1,1,1)$

✓ All permutations are identical → span is just one vector

→ $\dim(S) = 1$

(c) $\dim(S) = 3$

We want the 24 permutations to span a 3D subspace.

- Choose $x = (1,1,1,0)$

Let's analyze:

- All permutations will have three 1's and one 0, like:

  $(1,1,0,1)$, $(1,0,1,1)$ etc.

- These are not all scalar multiples, but linearly dependent, because:

  → Sum of components is always 3

  → vectors lie in a 3D subspace of $\mathbb{R}^4$.

✓ 24 such vectors will span a 3D subspace → dim(S) = 3

(d) dim(S) = 4

We want the permutations to span all of $\mathbb{R}^4$.

• Choose $x = (1, 2, 3, 4)$

✓ Permutations like,

$(1,2,3,4)$, $(4,3,2,1)$, $(2,1,3,4)$ etc.

They are linearly independent enough to span $\mathbb{R}^4$.

dim(S) = 4.

Ans 3. Given: If v_1, v_2, v_3, v_4 is a basis of $\mathbb{R}^4$ and if W is a subspace, then some subset of v_i's is a basis for W.

CounterExample:

Let, • $v_1 = (1,0,0,0)$
• $v_2 = (0,1,0,0)$
• $v_3 = (0,0,1,0)$
• $v_4 = (0,0,0,1)$

So, $\{v_1, v_2, v_3, v_4\}$ is the standard basis for $\mathbb{R}^4$.

Let's define a subspace $W \subseteq \mathbb{R}^4$ as:

$$W = \text{span}\{(1,1,1,1), (1,2,3,4)\}$$

This is a 2D subspace of $\mathbb{R}^4$. Let's check:

• Suppose some subset of $\{v_1, v_2, v_3, v_4\}$ spans W. But these vectors are axis-aligned — any subset of them spans a coordinate subspace (e.g., x-y plane, or the z-axis etc.)

But:

• The vector $(1,1,1,1)$ can't be written as a linear comb^n of any proper subset of the standard basis vectors.

• Likewise, $(1,2,3,4)$ is not in the span of any 1D or
2D coordinate subspace.

Hence, no subset of $\{v_1, v_2, v_3, v_4\}$ can form a basis for W.

Ans 4. To solve the behavior of the Markov chain, we want to find the steady-state vector π such that:

$$P\pi = \pi \text{ and } \pi_1 + \pi_2 + \pi_3 = 1$$

Step 1: Write Down P.

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0.0 & 0.4 \end{bmatrix}$$

Let $\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$ be the stationary vector.

We want: $P\pi = \pi \Rightarrow (P - I)\pi = 0$

Step 2: Set Up the system of eq^n.

This yields:

$$0.5\pi_1 + 0.2\pi_2 + 0.3\pi_3 = \pi_1$$
$$0.3\pi_1 + 0.8\pi_2 + 0.3\pi_3 = \pi_2$$
$$0.2\pi_1 + 0.0\pi_2 + 0.4\pi_3 = \pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

Subtract π_i from both sides of each system of eq^n:

$$-0.5\pi_1 + 0.2\pi_2 + 0.3\pi_3 = 0 \quad (1)$$
$$0.3\pi_1 - 0.2\pi_2 + 0.3\pi_3 = 0 \quad (2)$$
$$0.2\pi_1 + 0.0\pi_2 - 0.6\pi_3 = 0 \quad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad (4)$$

Step 3: Solve the system

From eq. (3),

$$0.2\pi_1 = 0.6\pi_3 \Rightarrow \pi_1 = 3\pi_3 \quad (A)$$

Substitute into (1):

$$-0.5(3\pi_3) + 0.2\pi_2 + 0.3\pi_3 = 0$$

$$-1.5\pi_3 + 0.2\pi_2 + 0.3\pi_3 = 0$$

$$-1.2\pi_3 + 0.2\pi_2 = 0 \Rightarrow \pi_2 = 6\pi_3 \quad (B)$$

Now, substitute (A) and (B) into (4):

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$3\pi_3 + 6\pi_3 + \pi_3 = 1 \rightarrow 10\pi_3 = 1 \Rightarrow \pi_3 = \frac{1}{10}$$

Then: $\pi_3 = \frac{1}{10}$

$\pi_1 = 3/10$

$\pi_2 = 6/10$

The system converges to the steady-state vector:

$$\pi = \begin{bmatrix} 3/10 \\ 6/10 \\ 1/10 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}.$$

As time passes, $x_k \rightarrow \pi$ for any initial state x_0.

Ans 5. (a) $S = \{(1,2,-2), (1,-1,3)\} \subset \mathbb{R}^2$ w.r.t. usual dot product

We need to find:

$$S^\perp = \{v \in \mathbb{R}^3 \mid \langle v, s \rangle = 0 \text{ for all } s \in S\}$$

Let $v = (x, y, z) \in \mathbb{R}^3$. Then:

$$\begin{cases} \langle v, (1,2,-2) \rangle = x + 2y - 2z = 0 \quad (1) \\ \langle v, (1,-1,3) \rangle = x - y + 3z = 0 \quad (2) \end{cases}$$

Solve the system:

From (1): $x = -2y + 2z$

Substitute into (2):

$$(-2y + 2z) - y + 3z = 0$$

$$-3y + 5z = 0 \Rightarrow y = \frac{5}{3} z$$

Back to (1):

$$x = -2\left(\frac{5}{3} z\right) + 2z = -\frac{4}{3} z$$

So;
$$v = \left(-\frac{4}{3} z, \frac{5}{3} z, z\right) = z \cdot \left(\frac{-4}{3}, \frac{5}{3}, 1\right)$$

Therefore:

$$S^\perp = \operatorname{Span}\left\{\left(-\frac{4}{3}, \frac{5}{3}, 1\right)\right\}$$

b) $S = \{1 + x, x^2\} \subset P_2(\mathbb{R})$ with

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

We want:

$$S^\perp = \{ p(x) \in P_2(\mathbb{R}) \mid \langle p, 1+x \rangle = 0 \text{ and } \langle p, x^2 \rangle = 0 \}.$$

Let $p(x) = a + bx + cx^2$. Then:

(1) $\int_{-1}^{1} p(x)(1+x)\, dx = 0 \Rightarrow \int_{-1}^{1} (a + bx + cx^2)(1 + x^2)\, dx$

We get,

$$\int_{-1}^{1} (a+bx+cx^2)(1+x)\,dx = \int_{-1}^{1} a+(a+b)x+(b+c)x^2+cx^3$$

So,

$$2a+\frac{2}{3}(b+c)=0 \quad \rightarrow (1)$$

<u>Second</u>

$$\int_{-1}^{1} p(x).\,x^2\,dx = \int_{-1}^{1} (a+bx+cx^2)\,x^2\,dx = 0$$

$$= \int_{-1}^{1} (ax^2+bx^3+cx^4)\,dx = a.\frac{2}{3}+b.0+c.\frac{2}{5}=0$$

$$\Rightarrow \frac{2}{3}a+\frac{2}{5}c=0 \quad \rightarrow (2)$$

Solve the system:

From (1): $2a+\frac{2}{3}(b+c)=0 \Rightarrow 3a+b+c=0 \rightarrow (A)$

From (2): $\frac{2}{3}a+\frac{2}{5}c=0 \Rightarrow 5a+3c=0 \rightarrow (B)$

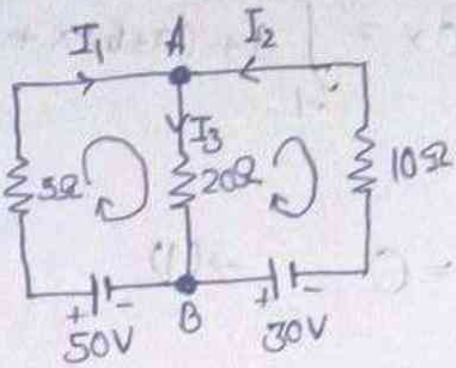
From (B): $c=\frac{-5}{3}a$

Substitute into (A): $3a+b-\frac{5}{3}a=0 \Rightarrow b=-\frac{4}{3}a$

Thus, $p(x)=a\left(1-\frac{4}{3}x+\frac{5}{3}x^2\right)$

$$S^{\perp}=\text{Span}\left\{1-\frac{4}{3}x-\frac{5}{3}x^2\right\}$$

Ans 6



Step 1: Apply Kirchoff's Current Law (KCL) at node A:

$$I_1 + I_2 = I_3 \quad -(1)$$

Apply Kirchoff's Voltage Law to the left loop

Clockwise Loop:

$$-50V + 5I_1 + 20I_3 = 0 \quad -(2)$$

Apply KVL to the right loop:

Clockwise Loop:

$$-30V + 10I_2 + 20I_3 = 0 \quad -(3)$$

Now, solving,

$$I_1 + I_2 = I_3 \Rightarrow I_1 = I_3 - I_2 \quad -(A)$$

Substitute (A) into (2):

$$5(I_3 - I_2) + 20I_3 = 50$$

$$\Rightarrow 25I_3 - 5I_2 = 50$$

$$\Rightarrow 5I_2 = 25I_3 - 50 \Rightarrow I_2 = 5I_3 - 10 \quad -(4)$$

Now, $10I_2 + 20I_3 = 30 \Rightarrow 50I_3 - 100 + 20I_3 = 30$

$$70I_3 = 130 \Rightarrow I_3 = \frac{13}{7} A$$

We get, $I_2 = 5 \cdot \frac{13}{7} - 10 = \frac{-5}{7} A$

$$I_1 = I_3 - I_2 = \frac{18}{7} A$$

$$\boxed{\begin{aligned} I_1 &= 18/7 \text{ A} \\ I_2 &= -5/7 \text{ A} \\ I_3 &= 13/7 \text{ A} \end{aligned}}$$

Solution)

→ Given A is 2×2 symmetric Matrix
→ It has orthonormal eigenvectors e_1 & e_2
→ Its eigen values are $\lambda_1 = 3$ & $\lambda_2 = -2$
we have to find U, Σ, & V^T

Step (i) Symmetric matrix & SVD

For any real symmetric matrix $A \in R^{n \times n}$ the SVD simplifies due to its symmetry.

- The left & Right singular vector are the same $U = V$
- The matrix an orthonormal eigenbasis (since it's symmetric)
- The singular values are the absolute values of the eigen values.

Step (ii) Use given data
e_1, e_2 orthonormal eigen vectors of A
$\lambda = 3, \lambda = -2$; eigen values of A.

Then
the SVD components are

$$U = [e_1 \; e_2]$$

$$\Sigma = \begin{bmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = V^T = U^T$$

2. $$\boxed{U = [e_1, e_2], \; \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \; V^T = U^T}$$

Solution 8

(a) If A is scaled by a constant then

$$4A = 4U\Sigma V^T = U(4\Sigma)V^T$$

$$\boxed{\text{SVD of } 4A: U; \Sigma' = \begin{bmatrix} 12 & 0 \\ 0 & 8 \end{bmatrix}, V^T = U^T}$$

(b) SVD of A^T?

Since A is symmetric $A^T = A$ so

SVD of A^T is the same as SVD of A

That is

$$A^T = U\Sigma V^T \text{ with } U = V, \ \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

(c) SVD of A^{-1}?

If A is invertible then:

let $A = U\Sigma V^T \Rightarrow A^{-1} = V\Sigma^{-1}U^T$

Because A is symmetric & orthogonally diagonalizable

$$A^{-1} = U\Sigma^{-1}U^T$$

The singular values of A^{-1} are $1/3$ & $1/2$

The singular vectors remains $U, V^T = U^T$

$$\therefore \text{SVD of } A^{-1}: U, \Sigma^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}, V^T = U^T$$

Ans