Worksheet 2

1. Product of eigenvalues = determinant of matrix .

Sum of eigenvalues = trace of matrix .

if rank $(A) < n$, then det $(A) = 0$

$\Rightarrow \lambda = 0$ will be an eigenvalue .

Here A, B are 2×2 matrices with rank ≤ 1 (< 2)

$\Rightarrow$ $\lambda_1 = 0$ will be an eigenvalue for both.

$\lambda_1 + \lambda_2 = tr(A)$

~~$tr(A) = \lambda_2$~~ $\lambda_2 = tr(A)$.

$\therefore$ $p_A(t) = (t - 0)(t - tr(A)$ $= t(t - tr(A))$.

$p_B(t) = t(t - tr(B))$.

$$\therefore \lim_{t \to 0} \frac{p_A(t)}{p_B(t)} = \lim_{t \to 0} \frac{t - tr(A)}{t - tr(B)} = \underline{\frac{tr(A)}{tr(B)}}$$

2. $p_A(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_{n-1} t + a_n$

(a) if $t = 1$ is a root, then.

$1 + a_1 + a_2 + \dots + a_{n-1} + a_n = 0$.

$$\Rightarrow \sum_{i=1}^{n} a_i = -1.$$

(b) if $t = 0$ is a root, then,

$0 + a_1 . 0 + a_2 . 0 + \dots + a_{n-1} . 0 + a_n = 0$

$\Rightarrow$ $a_n = 0$.

3. $A : n \times n$

$P_A(\cdot)$: char poly of A.

$f(t)$ ~~is~~ : has degree $m > n$.

$P_A(t) \mid f(t)$.

$\Rightarrow$ $f(t) = P_A(t)\, q(t)$.

$f(A) = P_A(A).\, q(A)$

$= [0]$

$\therefore$ it has only one eiginvalue, namely, 0.

4. $X = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

$$\det(X - \lambda I) = \begin{vmatrix} 1-\lambda & a \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$$

~~det~~ will be diagonalzale only if min poly, $m_A(\lambda) = 1 - \lambda$.

~~But $m_A(A) \neq 0 \Rightarrow$ not diagonalizable~~.

But $m_A(A) = 0$ iff $A = I$ iff $a = 0$.

~~5.~~

5. $X = \begin{bmatrix} 0 & 0 & 1 \\ a & 1 & b \\ 1 & 0 & 0 \end{bmatrix}$

$$\det(X - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ a & 1-\lambda & b \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda[\lambda(\lambda-1)] + \lambda - 1$$

$$= -\lambda^2(\lambda-1) + (\lambda-1)$$

$$= (\lambda-1)(1-\lambda^2) = \underline{-(\lambda-1)^2(1+\lambda)}$$

diagonalizable iff min poly is $(\lambda-1)(\lambda+1)$.

$$\Leftrightarrow (A-I)(A+I)=0.$$

$$\Leftrightarrow \begin{bmatrix} -1 & 0 & 1 \\ a & 0 & b \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ a & 2 & b \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ a+b & 0 & a+b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\Leftrightarrow a+b=0$$

$$\Leftrightarrow a=-b$$

$Xv=v \qquad (X-I)v=0.$

$$\begin{bmatrix} -1 & 0 & 1 \\ a & 0 & -a \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_3 \to R_3+R_1$

$R_2 \to R_2+aR_1$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-x+z=0 \qquad x=z.$

$$\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

6. $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

$\det(A - \lambda I) = 0$

$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(2-\lambda)(3-\lambda) - 2 = 0$

$\Rightarrow 6 - 5\lambda + \lambda^2 - 2 = 0.$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 1) = 0$

$\lambda = 1, 4.$

$\underline{\underline{\lambda = 1}}$ $\quad Ax = x.$

$(A - I)x = 0$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x + 2y = 0 \Rightarrow x = -2y.$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\underline{\underline{\lambda = 4}}$ $\quad Ax = 4x$

$(A - 4I)x = 0$

$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to 2R_2 + R_1$

$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x + 2y = 0 \Rightarrow x = y.$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad P^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}.$

$A = PDP^{-1}$

(a) $e^A = Pe^D P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2e & e^4 \\ e & e^4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3}\underline{\underline{\begin{bmatrix} 2e+e^4 & -2e+2e^4 \\ -e+e^4 & e+2e^4 \end{bmatrix}}}.$$

(b) $\sqrt{A} = P\sqrt{D}\,P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3}\underline{\underline{\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}}}.$$

7) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ Find Q such that

(a) $D = Q^{-1}AQ$ where Q is non-orthogonal

(b) $D = Q^{T}AQ$ where Q is orthogonal

A: <u>Find eigenvalues and eigenvectors</u>

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)^2 - 4\right] - 2\left[2(3-\lambda) - 4\right] + 2\left[4 - 2(3-\lambda)\right] = 0$$

$$(3-\lambda)\left[(3-\lambda)^2 - 4\right] - 4\left[(3-\lambda) - 2\right] + 4\left[2 - (3-\lambda)\right] = 0$$

$$\left[(3-\lambda) - 2\right]\left[(3-\lambda)(3-\lambda+2) - 4 - 4\right] = 0$$

$$(1-\lambda)\left[(3-\lambda)(5-\lambda) - 8\right] = 0$$

$$(1-\lambda)\left[15 - 8\lambda + \lambda^2 - 8\right] = 0$$

$$(1-\lambda)(\lambda^2-8\lambda+7)=0$$

$$(1-\lambda)(\lambda-1)(\lambda-7)=0$$

$$(\lambda-1)^2(\lambda-7)=0$$

$$\lambda = 1, 1, 7.$$

eigenvectors

$\underline{\lambda = 1}$

$Ax = x$

$(A-I)x = 0$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + 2y + 2z = 0$

$\Rightarrow \quad x = -y - z.$

eigenvector : $\begin{bmatrix} -y-z \\ y \\ z \end{bmatrix}$

$$= y\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} + z\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$\underline{\lambda = 7}$

$Ax = 7x$

$(A - 7I)x = 0$

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + 4R_1$
$R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-6y + 6z = 0 \Rightarrow y = z$

$-4x + 2y + 2z = 0$

$-2x + z + z = 0$

$-2x + 2z = 0$

$\Rightarrow x = z \quad \therefore z\underline{\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}$

~~Let~~ $Q = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

will satisfy $Q^{-1}AQ = D$.

b) $$Q = \left[\begin{array}{ccc} \multicolumn{2}{c}{\overbrace{\hspace{4em}}^{\lambda=1}} & \lambda=7. \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ \underline{0} & \underline{1} & \underline{1} \\ u & v & w \end{array}\right]$$ → have to make this orthogonal.

$u.v = 2 \neq 0$

$u.w = -1+1=0$ ✓

$v.w = -1+1=0$ ✓.

have to replace u or v with another column that makes $u.v=0$.

Let us keep u as such and change v.

choose a new 'v' from the set of eigenvectors of $\lambda=1$.

Let $v = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix}$

$u.v=0$

$$\Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \quad y + z + y = 0$

$\Rightarrow \quad 2y + z = 0$

choose $y = 1$ and $z = -2$.

Then $v = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad u \cdot v = 0.$

now. make magnitude of u, v and w 1.

$|u| = \sqrt{-1^2 + 1^2} = \sqrt{2}$

$|v| = \sqrt{1^2 + 1^2 + -2^2} = \sqrt{6}$

$|w| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$\therefore \quad Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \text{satisfies } Q^T A Q = D.$$

8) $f(x,y) = 2x^2 + 4xy + 5y^2$

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$

$f(x,y) = u^T A u$

Find orthogonal diagonalization of A.

$|A - \lambda I| = 0$ (to find eigenvalues)

$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \quad \Rightarrow (2-\lambda)(5-\lambda) - 4 = 0$

$\Rightarrow 10 - 7\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \underline{\underline{\lambda = 1, 6}}$

eigenvector :

$\lambda = 1$

$(A-I)x = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + 2y = 0$

$\Rightarrow x = -2y$

$$\underline{\underline{y\begin{bmatrix} -2 \\ 1 \end{bmatrix}}}$$

$\lambda = 6$

$(A-6I)x = 0$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x + 2y = 0$

$\Rightarrow y = 2x$

~~x_2~~ $x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\therefore \quad P = \begin{array}{c} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \\ \begin{array}{cc} \underline{v} & \underline{w} \end{array} \end{array} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

$v.w = -2+2 = 0$ ✓

$|v| = \sqrt{-2^2+1^2} = \sqrt{5}.$

$|w| = \sqrt{1^2+2^2} = \sqrt{5}.$

$$\therefore \quad P = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$ will satisfy $P^T A P = D$

$\Rightarrow A = P D P^T$

we have $f(x,y) = u^T A u$

$= u^T P D P^T u$

$= \underline{u^T P} \; D \; \underline{P^T u}$

Let $v = P^T u$ and $v = \begin{bmatrix} r \\ s \end{bmatrix}$. ⇒

⇒ $v^T = u^T P$

$\therefore f(r,s) = v^T D v$

$$= \begin{bmatrix} r & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \underline{\underline{r^2 + 6s^2}}.$$

now relation between r, s, x, y.

given $x = \phi(r,s)$ and $y = \psi(r,s)$.

we have $v = P^T u$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -2x + y \\ x + 2y \end{bmatrix}$$

$\therefore \frac{1}{\sqrt{5}}(-2x+y) = r \Rightarrow -2x + y = \sqrt{5}\, r$ — ①

$x + 2y = \sqrt{5}\, s$ — ②

$\frac{1}{\sqrt{5}}(x+2y) = s$

① + 2×② → $5y = \cancel{3\sqrt{5}s} \Rightarrow \sqrt{5}\, r + 2\sqrt{5}\, s$

$= \sqrt{5}(r + 2s)$

$y = \frac{1}{\sqrt{5}}(r+2s)$ ✓

② − 2×① → $5x = \sqrt{5}\, s - 2\sqrt{5}\, r$

$\underline{x = \frac{1}{\sqrt{5}}(s - 2r)}$ ✓

9. $f(x,y,z) = 3x^2 - 4xy + 6y^2 + 2xz - 4yz + 3z^2$

A: Let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

$f(x,y,z) = u^T A u.$

<u>Diagonalyzation of A</u> (orthogonal)

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -2 & 1 \\ -2 & 6-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(6-\lambda)(3-\lambda)-4\right] + 2\left[-2(3-\lambda)+2\right] + \left[4-6+\lambda\right] = 0$$

$$(3-\lambda)\left[18-9\lambda+\lambda^2-4\right] + 4\left[1-3+\lambda\right] + \left[\lambda-2\right] = 0.$$

$$(3-\lambda)\left[\lambda^2-9\lambda+14\right] + 4\left[\lambda-2\right] + \left[\lambda-2\right] = 0.$$

$$3\lambda^2 - \underline{27\lambda} + \underline{42} - \lambda^3 + \underline{9\lambda^2} - \underline{14\lambda} + 5\lambda - \underline{10} = 0.$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0.$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$(\lambda-2)^2(\lambda-8) = 0 \Rightarrow \lambda = 2, 2, 8.$$

$Ax = 2x$

$(A-2I)x = 0$

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & -8 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\xrightarrow{} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 2y + z = 0$

$x = 2y - z$ $\therefore$ eigenvector . $\begin{bmatrix} 2y-z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$Ax = 8x$

$(A-8I)x = 0$

$$\begin{bmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 - 2R_1$

$R_3 \to 5R_3 + R_1$

$$\begin{bmatrix} -5 & -2 & 1 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 1R_3 - 2R_2$

$$\begin{bmatrix} -5 & -2 & 1 \\ 0 & -6 & -12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = -2z$

$-5x + 4z + z = 0 \Rightarrow x = z$

∴ eiginvector $= \begin{bmatrix}1\\-2\\1\end{bmatrix}$

$$P=\begin{array}{c}\begin{array}{cc}\overbrace{\lambda=2\qquad\quad} & \lambda=8\end{array}\\ \begin{bmatrix}2&-1&1\\1&0&-2\\0&1&1\end{bmatrix}\\ \begin{array}{ccc}u&v&w\end{array}\end{array}\qquad D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&8\end{bmatrix}$$

satisfies $P^{-1}AP=D$

Have to make P orthogonal

$u.v \neq 0$

$u.w=0, \quad v.w=0.$

Have to change v.

let $v=\begin{bmatrix}2y-z\\y\\z\end{bmatrix}$ ~~⇒~~ $u.v=0 \Rightarrow \begin{bmatrix}2\\1\\0\end{bmatrix}\cdot\begin{bmatrix}2y-z\\y\\z\end{bmatrix}=0$

$\Rightarrow 4y-2z+y=0$

$5y-2z=0.$

choose $y=2, z=5$

$\therefore v=\begin{bmatrix}-1\\2\\5\end{bmatrix}$

$$P=\begin{bmatrix}2&-1&1\\1&2&-2\\0&5&1\end{bmatrix}$$

$|u|=\sqrt{2^2+1^2}=\sqrt{5}$

$|v|=\sqrt{-1^2+2^2+5^2}=\sqrt{30}$

$|w|=\sqrt{1^2+-2^2+1^2}=\sqrt{6}$

$$P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}$$ satisfies $P^TAP = D$

$\Rightarrow A = PDP^T.$

we have $f(x,y,z) = u^TAu$

$= \underline{u^TP}\, D\, \underline{P^Tu}$

Let $P^Tu = v = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ $\Rightarrow$ $v^T = u^TP$

$$\therefore \ f(r,s,t) = v^TDv$$

$$= \begin{bmatrix} r & s & t \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$= \underline{\underline{2r^2 + 2s^2 + 8t^2}},$$

where $$\cancel{\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

where $v = P^Tu$

$(\because P^T = P^{-1})$

$\Rightarrow u = Pv$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$\Rightarrow x = \frac{2}{\sqrt{5}}r - \frac{1}{\sqrt{30}}s + \frac{1}{\sqrt{6}}t$

$y = \frac{1}{\sqrt{5}}r + \frac{2}{\sqrt{30}}s - \frac{2}{\sqrt{6}}t$

$z = \frac{5}{\sqrt{30}}s + \frac{1}{\sqrt{6}}t.$

10) $M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Find a, b, c such that $6M^{-1} = aM^2 + bM + cI$.

multiply both sides with M,

$6I = aM^3 + bM^2 + cM$. —①

<u>characteristic eqn of M:</u>

$$|M - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2+\lambda)(3+\lambda)\right] \cancel{+2(} = 0$$

$$(1-\lambda)\left[6 + 5\lambda + \lambda^2\right] = 0$$

$$6 + 5\lambda + \lambda^2 - 6\lambda - 5\lambda^2 - \lambda^3 = 0.$$

$$-\lambda^3 - 4\lambda^2 - \lambda + 6 = 0.$$

$$\lambda^3 + 4\lambda^2 + \lambda - 6 = 0.$$

From Cayley Hamilton theorem, M also satisfies its characteristic eqn.

$\rightarrow \cancel{M^3 + 4\lambda^2}$

$$\Rightarrow M^3 + 4M^2 + M - 6I = 0 \quad \text{—} ②$$

From eqn ①,

$$aM^3 + bM^2 + cM - 6I = 0 \quad \text{—} ③$$

equating ② and ③,

$$\underline{\underline{a = 1, b = 4, c = 1}}$$