



Probability Theory

Expectation of Random Variable

- The expected value, or mean, of a random variable x denoted by $E(x)$ or μ is a weighted average of the values the random variable may assume.
- In the discrete case the weights are given by the probability mass function, and in the continuous case the weights are given by the probability density function.

- The expectation of a discrete random variable X is given by:

$$\mu = E(X) = \sum x f(x), \text{ where } f(x) \text{ is the PMF of } x.$$

- The expectation of a continuous random variable X is given by:

$$\mu = E(X) = \int x f(x) dx, \text{ where } f(x) \text{ is the PDF of } x.$$

Variance of Random Variable

- The **variance** of a random variable, denoted by $\text{Var}(x)$ or σ^2 , is a weighted average of the squared deviations from the mean.
 - For discrete RV: $\text{Var}(x) = \sigma^2 = \sum (x - \mu)^2 f(x)$
 - For continuous RV: $\text{Var}(x) = \sigma^2 = \int (x - \mu)^2 f(x) dx$
- The **standard deviation**, denoted σ , is the positive square root of the variance.

Median of Random Variable

- The **median** of the discrete random variable X , is the value of x for which $P(X \leq x)$ is greater than or equal to 0.5 and $P(X \geq x)$ is greater than or equal to 0.5.

Median of Random Variable

- Let X be a continuous rv with probability density function, $f(x)$. The median of X can be obtained by solving for c in the equation below:

$$\int_{-\infty}^c f(x)dx=0.5$$

- That is, it is the value for which the area under the curve from negative infinity to c is equal to 0.50.

Quantiles of Random Variable

- Let X is a real-valued random variable with Cumulative Distribution Function F .
- For $p \in (0,1)$, a value of x is called **a quantile of order p** for the distribution if

$$F(x-) = P(X < x) \leq p \text{ and } F(x) = P(X \leq x) \geq p .$$

- A quantile of order p is a value where the graph of the cumulative distribution function crosses p .
- Median is also called 50th percentile.

Questions

Q1. A service organization in a large town organizes a raffle each month. One thousand raffle tickets are sold for \$1 each. Each has an equal chance of winning. First prize is \$300, second prize is \$200, and third prize is \$100. Let, X denote the net gain from the purchase of one ticket.

- Construct the probability distribution of X
- Find the probability of winning any money in the purchase of one ticket.
- Find the expected value of X , and interpret its meaning.

Questions

a) If a ticket is selected as the first prize winner, the net gain to the purchaser is the \$300 prize less the \$1 that was paid for the ticket, hence $X=300-1=299$. There is one such ticket, so $P(299)=0.001$

Applying the same “income minus outgo” principle to the second and third prize winners and to the 997 losing tickets yields the probability distribution:

x	299	199	99	-1
P(x)	0.001	0.001	0.001	0.997

b) Let W denote the event that a ticket is selected to win one of the prizes. Using the table

$$P(W) = P(299) + P(199) + P(99) = 0.003$$

$$\begin{aligned} \text{c) } E(X) &= (299) \cdot (0.001) + (199) \cdot (0.001) + (99) \cdot (0.001) + (-1) \cdot (0.997) \\ &= -0.4 \end{aligned}$$

Questions

Q2. A discrete rv X has following probability distribution:

x	-1	0	1	4
$P(x)$	0.2	0.5	c	0.1

Find

- a) c 0.2
- b) $P(0)$ 0.5
- c) $P(X > 0)$ 0.3
- d) $P(X \geq 0)$ 0.8
- e) $P(X \leq -2)$ 0
- f) The mean μ of X 0.4
- g) The variance σ^2 of X 1.84
- h) The standard deviation σ of X 1.3565

Questions

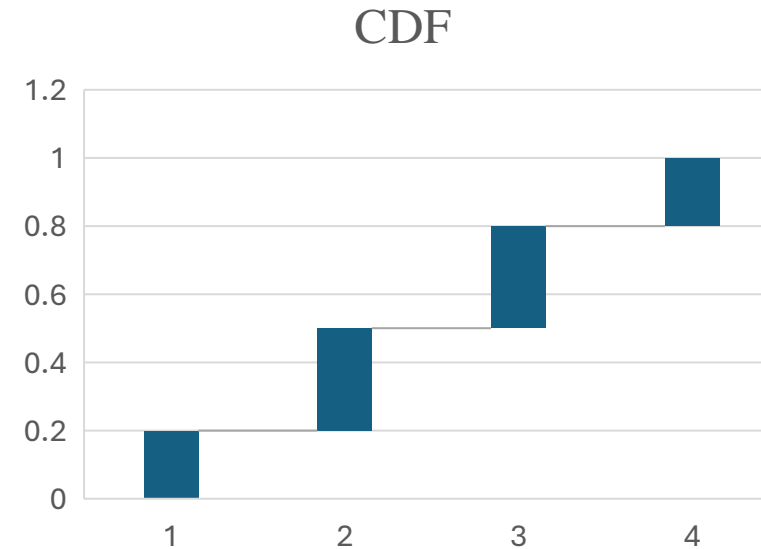
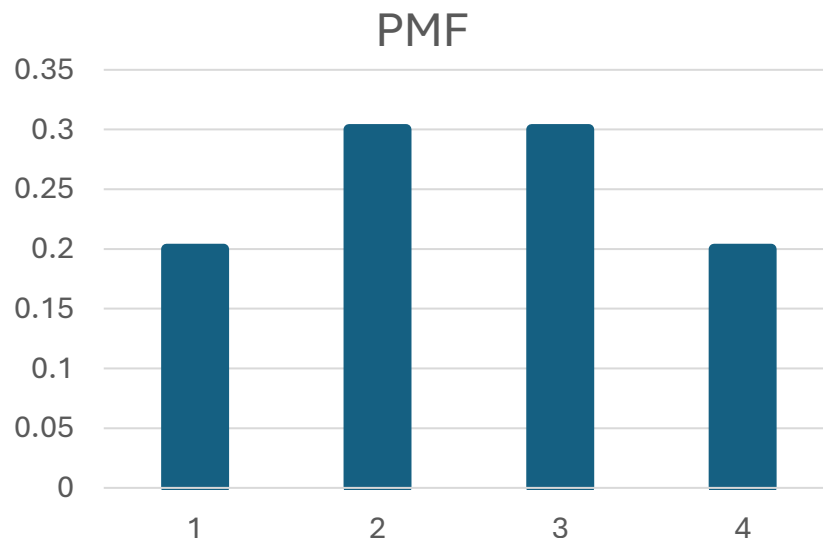
Q3. Given the following probability density function of a discrete random variable,

calculate the median of the distribution: $f(x) = \begin{cases} 0.2 & x = 1, 4 \\ 0.3 & x = 2, 3 \end{cases}$

- Solution: The median of the distribution above is 2 because;

$$P(X \leq 2) = P(X=1) + P(X=2) = 0.2 + 0.3 = 0.5 \quad \text{and,}$$

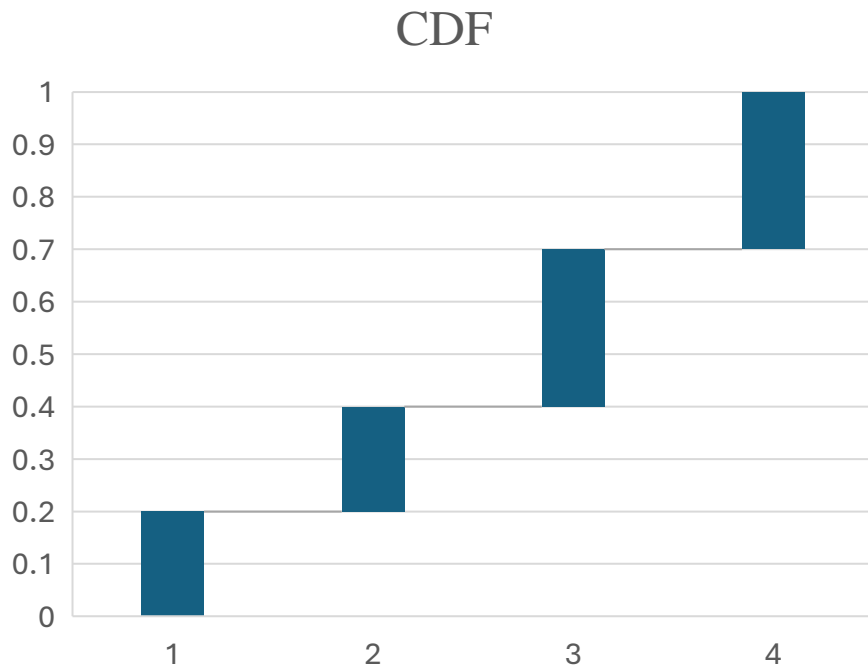
$$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) = 0.3 + 0.3 + 0.2 = 0.8$$



Questions

Q4. Given the following probability density function of a discrete random variable, calculate the 75th Percentile of the distribution:

$$f(x) = \begin{cases} 0.2 & x = 1, 4 \\ 0.3 & x = 2, 3 \end{cases}$$



Solution: The 75th percentile of the distribution is 4 because;

$$\begin{aligned} P(X < 3) &= P(X=1) + P(X=2) \\ &= 0.2 + 0.3 = 0.5 \quad \text{and,} \end{aligned}$$

$$\begin{aligned} P(X \leq 3) &= P(X=1) + P(X=2) + P(X=3) \\ &= 0.2 + 0.3 + 0.3 = 0.8 \end{aligned}$$