DEPARTMENT OF MATHEMATICS WORKSHEET 3

UPES UNIVERSITY OF TOMORROW

Program	B. Tech. (SoCS)	Semester	IV
Course	Linear Algebra	Course Code	MATH 2059
Session	Jan-May 2025	Topic	Vector spaces, Subspaces, Basis, Dimension

1. Each of the following subsets V_i fail to be a vector space over the given field $\mathcal{F}$. Determine which axiom(s) of the vector space is (are) violated by giving supporting example(s).

(a) $V_1 = \mathbb{R}^n$ over the field $\mathcal{F} = \mathbb{C}$ of complex numbers. $i(1,0,0,\dots,0) = (i,0,0,\dots,0) \notin \mathbb{R}^n$

(b) $V_2 = \{(x_1, x_2, \dots \dots, x_n) | \, x_i \in \mathbb{Q}\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers., $\pi(1,0,0,\dots) = (\pi,0,\dots) \notin \mathbb{Q}^n$

(c) $V_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin V_3$

(d) $V_4 = \left\{ A = [a_{ij}]_{n \times n} \middle| \, a_{ij} \in \mathbb{R}, \det A = 0 \right\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin V_4$

(e) $V_5 = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n | a_i \in \mathbb{R} \; \forall \; 0 \leq i \leq n, a_0 \neq 0\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers. $0 \notin V_5$.

2. Find the constants α, β and γ so that the arbitrary vector $v = (a, b, c) \in \mathbb{R}^3$ can be expressed as the linear span of vectors $e_1 = (1,0,0), e_2 = (1,1,0)$ and $e_3 = (1,1,1)$ i.e.

$$v = \alpha e_1 + \beta e_2 + \gamma e_3$$

Can the set $\mathcal{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ serve as a basis for the vector space $\mathbb{R}^3$?

3. Prove that each of the following subsets W_i is a vector subspace of the given vector space V. Also, find the dimension in each case.

(a) $W_1 = \{(x_1, x_2, \dots \dots, x_n) | \, x_i = 0 \text{ if } i \text{ is even}\}$ of the vector space $V = \mathbb{R}^n$ for $n \geq 2$.

(b) $W_2 = \left\{ A = [a_{ij}]_{3 \times 3} \middle| \, a_{ij} \in \mathbb{R}, trace(A) = 0 \right\}$ of the vector space $V = \mathcal{M}(2, \mathbb{R})$ i.e. space of all 2×2 real matrices.

(c) $W_3 = \left\{ A = [a_{ij}]_{4 \times 4} \middle| \, a_{ij} \in \mathbb{R}, a_{ij} = a_{ji}, trace(A) = 0 \right\}$ of the vector space $V = \mathcal{M}(4, \mathbb{R})$ i.e. space of all 4×4 real matrices.

(d) $W_4 = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n | a_i = \; 0 \text{ if } i \text{ is even}\}$ vector space $V = \mathcal{P}_n(\mathbb{R})$ i.e. space of all real polynomials of degree $\leq n$ having real coefficients.

$p(x), q(x) \in W_3$ $\Rightarrow g(x) = p(x) + \alpha q(x)$, Now $g(1) = p(1) + \alpha q(1) = 0$

(e) $W_3 = \{p(x) \in V \mid p(1) = 0\}$ vector space $V = \mathcal{P}_n(\mathbb{R})$ i.e. space of all real polynomials of degree $\leq n$ having real coefficients.

4. Consider the set V consisting of all polynomials $p(x) \in \mathcal{P}_n(\mathbb{R})$ that satisfy $p(-x) = p(x)$ i.e. $p(x)$ is an even function. Prove that V is a vector space in itself.

† **Hint:** *V must be a subspace of $\mathcal{P}_n(\mathbb{R})$.*

$\Rightarrow$ $p(x), q(x) \in V$
$g(x) = p(x) + \alpha q(x)$
$g(-x) = p(-x) + \alpha q(-x) = p(x) + \alpha q(x)$ ✓

5. Consider the vector space V defined as follows:

$$V = \{a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2024}x^{2024} \mid p(1) = 0, p(-1) = 0\}$$

Find the dimension of V.

† **Hint:** *The answer has three distinct prime divisors.*

6. Consider the subspaces W_1 and W_2 of vector space $\mathcal{M}(2, \mathbb{R})$ as follows:

$$W_1 = \left\{\begin{pmatrix} a & b \\ -a & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\right\}$$

$$W_2 = \left\{\begin{pmatrix} a & b \\ c & 2a \end{pmatrix} \mid a, b, c \in \mathbb{R}\right\}$$

(a) Express explicitly the subspace $W_1 \cap W_2$.

(b) Also find $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$.

$\Rightarrow c = 2a$

$\Rightarrow \begin{pmatrix} a & b \\ -a & 2a \end{pmatrix}$

$W_1 \cap W_2 = \left\{\begin{pmatrix} a & b \\ -a & 2a \end{pmatrix}\right\}$.

$a\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
$= 3 + 3 - 2$
$= 4$

7. Consider the subspaces W_1 and W_2 of vector space $\mathcal{P}_7(\mathbb{R})$ as follows:

$$W_1 = \{p(x) \in \mathcal{P}_7(\mathbb{R}) \mid p(1) = 0, p\left(\frac{1}{2}\right) = 0, p(4) = 0\}$$

$$W_2 = \{p(x) \in \mathcal{P}_7(\mathbb{R}) \mid p(-1) = 0, p\left(\frac{1}{2}\right) = 0, p(-4) = 0\}$$

(c) Express explicitly the subspace $W_1 \cap W_2$.

(d) Also find $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$.

8. Suppose W_1 and W_2 are two subspaces of a $7-$dimensional vector space V such that $\dim W_1 = 4$ and $\dim W_2 = 5$. What are the possible values of $\dim(W_1 \cap W_2)$?

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Definition of a Vector Space over a field $\mathbb{F}$

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every **u**, **v**, and **w** in V and every scalar [redacted] c and d, then V is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
4. V has a **zero vector 0** such that for every **u** in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	Additive identity
5. For every **u** in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V, $c \in \mathbb{F}$	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity

1. Each of the following subsets V_i fail to be a vector space over the given field $\mathcal{F}$. Determine which axiom(s) of the vector space is (are) violated by giving supporting example(s).

(a) $V_1 = \mathbb{R}^n$ over the field $\mathcal{F} = \mathbb{C}$ of complex numbers.

(b) $V_2 = \{(x_1, x_2, \dots\dots, x_n) \mid x_i \in \mathbb{Q}\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers.

(c) $V_3 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc \neq 0\right\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers.

(d) $V_4 = \left\{A = [a_{ij}]_{n\times n} \middle| \, a_{ij} \in \mathbb{R}, \det A = 0\right\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers.

(e) $V_5 = \{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \mid a_i \in \mathbb{R} \, \forall \, 0 \leq i \leq n, a_0 \neq 0\}$ over the field $\mathcal{F} = \mathbb{R}$ of real numbers.

Solⁿ: (a) $V_1 = \mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$

$\mathbb{F} = \mathbb{C}$.

Let $v = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$

$i \in \mathbb{F}$

Then $iv = (i, 0, 0, \dots, 0) \notin \mathbb{R}^n$, since $i \notin \mathbb{R}$

$\therefore iv \notin V_1$ ((6) property is not satisfied)

$\therefore V_1$ is not a vector space.

(b) $V_2 = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{Q}\}$. $\mathbb{F} = \mathbb{R}$.

Let $v = (1, 0, 0, \dots, 0) \in V_2$. $\sqrt{2} \in \mathbb{F}$

$\sqrt{2}v = (\sqrt{2}, 0, 0, \dots, 0) \notin V_2$, $\because \sqrt{2} \notin \mathbb{Q}$. ($\sqrt{2}$ is irrational)

$\therefore \sqrt{2}v \notin V_2 \Rightarrow V_2$ is not a vector space (6 property is not satisfied)

(c) $V_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \ ad - bc \neq 0 \right\}, \ \mathbb{F} = \mathbb{R}$

Note that $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in V_3$ Since $1.1 - 0.0 = 1 \neq 0$

$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in V_3$ Since $(-1).(-1) - 0 = 1 \neq 0$

Now, $A + B = \begin{pmatrix} \overset{a}{0} & \overset{b}{0} \\ \underset{c}{0} & \underset{d}{0} \end{pmatrix} \notin V_3$ Since $ad - bc = 0.0 - 0.0 = 0$

$\therefore$ V_3 is not a vector Space (property (1) is not satisfied)

(d) $V_4 = \left\{ A = [a_{ij}]_{n \times n} : a_{ij} \in \mathbb{R}, \ \det(A) = 0 \right\}, \ \mathbb{F} = \mathbb{R}.$

Take $n = 2$

$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V_4$ Since

$\det(A) = 0, \ \det B = 0.$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$ $\det(A + B) = 1 \neq 0$

$\therefore$ $A + B \notin V_4$. That is, V_4 is not a vector Space

(e) $V_5 = \left\{ \underset{\text{constant term}}{\underbrace{a_0}} + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \ 0 \leq i \leq n, \ a_0 \neq 0 \right\}, \ \mathbb{F} = \mathbb{R}.$

Here, $v = 1 + x \in V_5$, $a_0 = 1 \neq 0$.

Take $c = 0 \in \mathbb{F}$ (which is a Scalar)

$\therefore$ $cv = 0(1 + x) = 0$. Here constant term $a_0 = 0$. $\therefore$

$cv = 0 \notin V_5$

So, V_5 is not a vector Sp

2. Find the constants α, β and γ so that the arbitrary vector $v = (a,b,c) \in \mathbb{R}^3$ can be expressed as the linear span of vectors $e_1 = (1,0,0), e_2 = (1,1,0)$ and $e_3 = (1,1,1)$ i.e.

$$v = \alpha e_1 + \beta e_2 + \gamma e_3$$

Can the set $\mathcal{B} = \{(1,0,0),(1,1,0),(1,1,1)\}$ serve as a basis for the vector space $\mathbb{R}^3$?

Sol^n: $v = \alpha e_1 + \beta e_2 + \gamma e_3$

$(a,b,c) = \alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1)$

$(a,b,c) = (\alpha,0,0) + (\beta,\beta,0) + (\gamma,\gamma,\gamma)$

$(a,b,c) = (\alpha+\beta+\gamma, \beta+\gamma, \gamma)$

$\Rightarrow$ $\alpha+\beta+\gamma = a$ --- (1)

$\beta+\gamma = b$ --- (2)

$\boxed{\gamma = c}$ --- (3)

$\beta + \gamma = b$ (from (2))

$\Rightarrow \beta = b - \gamma = b - c \Rightarrow \boxed{\beta = b-c}$

Now $\alpha+\beta+\gamma = a$ (from (1))

$\alpha + (b-c) + c = a$

$\boxed{\alpha = b-a}$

Now $\mathcal{B} = \{(1,0,0),(1,1,0),(1,1,1)\}$.

$\dim(\mathbb{R}^3) = 3$. Since $\mathcal{B}$ has 3 elements, if $\mathcal{B}$ is is linearly independent, then $\mathcal{B}$ becomes a basis.

So, let $\alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1) = (0,0,0)$

$\Rightarrow (\alpha+\beta+\gamma, \beta+\gamma, \gamma) = (0,0,0)$

$\Rightarrow$ $\alpha+\beta+\gamma = 0$

$\beta+\gamma = 0$

$\gamma = 0$

So, $\beta+\gamma=0 \Rightarrow \beta = 0$. & $\alpha+\beta+\gamma = 0 \Rightarrow \alpha+0+0=0$

$\alpha = 0$.

$\therefore$ $\alpha=\beta=\gamma=0$. $\Rightarrow \mathcal{B}$ is linearly independent. Hence $\mathcal{B}$ is a basis

Def^n: Let V be a vector Space over $\mathbb{F}$. Then $W \subseteq V$ is a subspace iff $\omega_1 + \alpha \omega_2 \in W$, $\alpha \in \mathbb{F}$, $\omega_1, \omega_2 \in W$

3. Prove that each of the following subsets W_i is a vector subspace of the given vector space V. Also, find the dimension in each case.

(a) $W_1 = \{(x_1, x_2, \dots \dots, x_n) |\ x_i = 0 \text{ if } i \text{ is even}\}$ of the vector space $V = \mathbb{R}^n$ for $n \geq 2$.

(b) $W_2 = \left\{A = [a_{ij}]_{3\times 3} \middle| a_{ij} \in \mathbb{R}, trace(A) = 0\right\}$ of the vector space $V = \mathcal{M}(2, \mathbb{R})$ i.e. space of all 2×2 real matrices.

(c) $W_3 = \left\{A = [a_{ij}]_{4\times 4} \middle| a_{ij} \in \mathbb{R}, a_{ij} = a_{ji}, trace(A) = 0\right\}$ of the vector space $V = \mathcal{M}(4, \mathbb{R})$ i.e. space of all 4×4 real matrices.

(d) $W_4 = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n | a_i = 0 \text{ if } i \text{ is even}\}$ vector space $V = \mathcal{P}_n(\mathbb{R})$ i.e. space of all real polynomials of degree $\leq n$ having real coefficients.

Sol^n: (a) $W_1 = \{(x_1, x_2, \dots, x_n) \mid x_i = 0 \text{ if } i \text{ is even}\} \subseteq \mathbb{R}^n$

Case I: $n =$ even.

Let $\omega_1 = (x_1, 0, x_3, 0, \dots, x_{n-1}, 0) \in W_1$
$\omega_2 = (y_1, 0, y_3, 0, \dots, y_{n-1}, 0) \in W_2$

Let $\alpha \in \mathbb{F} = \mathbb{R}$

If $\omega_1 + \alpha \omega_2 \in W_1$, then W_1 is a subspace.

So, $\omega_1 + \alpha \omega_2 = (x_1, 0, x_3, 0, \dots, x_{n-1}, 0) + (\alpha y_1, 0, \alpha y_3, 0, \dots, \alpha y_{n-1}, 0)$
$= (x_1 + \alpha y_1, 0, x_3 + \alpha y_3, 0, \dots, x_{n-1} + \alpha y_{n-1}, 0)$
$\in W_1$, since coordinates at even places is 0.

$\therefore \omega_1 + \alpha \omega_2 \in W_1$. Hence W_1 is a subspace.

Let $\omega = (x_1, 0, x_3, 0, \dots, x_{n-1}, 0) \in W_n$

$\omega = x_1(1, 0, 0, \dots, 0) + x_3(0, 0, 1, 0, \dots, 0) + \dots + x_{n-1}(0, 0, \dots, 1, 0)$

i.e. $B = \{(1, 0, 0, \dots, 0), (0, 0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 1, 0)\}$ spans W_1

& B is linearly independent. Therefore B is a basis of W_1.

No. of elements in $B = \frac{n}{2} = \dim W_1$

Case II : $n =$ odd

$$W_1 = \{(x_1, 0, x_2, 0, \cdots, 0, x_n)\}$$

Similarly, as above W_1 is a subspace.

Let $\omega = (x_1, 0, x_2, 0, \cdots, 0, x_n) \in W_1$

Then $(x_1, 0, x_3, 0, \cdots, 0, x_n) = x_1(1, 0, 0, \cdots, 0) + x_3(0, 0, 1, 0, \cdots, 0)$
$+ x_n(0, 0, \cdots, 1)$.

Here $B = \{(1, 0, \cdots, 0), (0, 0, 1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$

Spans W_1 & is linearly independent. Hence

B is a basis.

No. of elements in $B = \frac{n+1}{2} = \dim W_1$

(b) $W_2 = \{A = [a_{ij}]_{3\times 3} : a_{ij} \in \mathbb{R} \text{ \& trace}(A) = 0\} \subseteq M(2, \mathbb{R})$

Let $\omega_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\omega_2 \in \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in W_2$

Here $\text{trace}(\omega_1) = 0$ & $\text{trace}(\omega_2) = 0$

$a_{11} + a_{22} + a_{33} = 0$ & $b_{11} + b_{22} + b_{33} = 0$

Now $\omega_1 + \alpha\omega_2 = \begin{pmatrix} a_{11} + \alpha b_{11} & a_{12} + \alpha b_{12} & a_{13} + \alpha b_{13} \\ a_{21} + \alpha b_{21} & a_{22} + \alpha b_{22} & a_{23} + \alpha b_{23} \\ a_{31} + \alpha b_{31} & a_{32} + \alpha b_{32} & a_{33} + \alpha b_{33} \end{pmatrix}$

Now $\text{trace}(\omega_1+\alpha\omega_2) = (a_{11}+\alpha b_{11}) + (a_{22}+\alpha b_{22}) + (a_{33}+\alpha b_{33})$

$= (a_{11}+a_{22}+a_{33}) + \alpha(b_{11}+b_{22}+b_{33})$

$= 0+0 = 0$

$\therefore$ $\omega_1+\alpha\omega_2 \in W_2$. So, W_2 is a subspace.

(c) $W_3 = \{A=[a_{ij}]_{4\times 4} \mid a_{ij}\in\mathbb{R},\ \text{trace}(A)=0\} \subseteq M(4,\mathbb{R})$

Let $\omega_1 = [a_{ij}]_{4\times 4}$, $\omega_2 = [b_{ij}]_{4\times 4} \in W_3$.

$tr(\omega_1)=0$ & $a_{ij}=a_{ji}$, $tr(\omega_2)=0$ & $b_{ij}=b_{ji}$

To show $\omega_1+\alpha\omega_2 \in W_3$.

$\omega_1+\alpha\omega_2 = [a_{ij}+\alpha b_{ij}]_{4\times 4} = [c_{ij}]_{4\times 4}$, $c_{ij} = a_{ij}+\alpha b_{ij}$

We need to show: (a) $tr(\omega_1+\alpha\omega_2)=0$ & $c_{ij}=c_{ji}$

Now, $\text{trace}(\omega_1+\alpha\omega_2) = c_{11}+c_{22}+c_{33}+c_{44}$

$= a_{11}+\alpha b_{11} + a_{22}+\alpha b_{22} + a_{33}+\alpha b_{33} + a_{44}+\alpha b_{44}$

$= (a_{11}+a_{22}+a_{33}) + \alpha(b_{11}+b_{22}+b_{33}+b_{44})$

$= tr(\omega_1) + \alpha\, tr(\omega_2)$

$= 0+\alpha 0 = 0$

$= 0.$

Now To show: $c_{ij} = c_{ji}$.

Now $c_{ij} = a_{ij}+\alpha b_{ij}$

$= a_{ji}+\alpha b_{ji}$

$c_{ij} = c_{ji}$

$\therefore$ $\omega_1+\alpha\omega_2 \in W_3$

$W_4 = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i = 0, \text{ when } i = \text{even}\} \subseteq P_n(\mathbb{R})$.

Case I: n = even.

Then $W_4 = \{a_1 x + a_3 x^3 + a x^5 + \cdots + a_{n-1} x^{n-1}\}$ | n = even

Let $\omega_1 = a_1 x + a_3 x^3 + \cdots + a_{n-1} x^{n-1} \in W_4$

$\omega_2 = b_1 x + b_3 x^3 + \cdots + b_{n-1} x^{n-1} \in W_4$

& $\alpha \in \mathbb{F} = \mathbb{R}$

Now $\omega_1 + \alpha\omega_2 = (a_1 x + a_3 x^3 + \cdots + a_{n-1} x^{n-1}) + \alpha b_1 x + \alpha b_3 x^3 + \cdots + \alpha b_{n-1} x^{n-1}$

$= (a_1 + \alpha b_1) x + (a_3 + \alpha b_3) x^3 + \cdots + (a_{n-1} + \alpha b_{n-1}) x^{n-1}$

$\omega_1 + \alpha\omega_2 \in W_4$ ($\because$ only odd powers are present).

$\therefore W_4$ is a subspace

Case II: n = odd

$W_4 = \{a_1 x + a_3 x^3 + \cdots + a_n x^n\}$

Similarly as above, one can show that W_4 is a subspace.

(e) $W_3 = \{p(x) \in V \mid p(1) = 0\}$ vector space $V = \mathcal{P}_n(\mathbb{R})$ i.e. space of all real polynomials of degree $\leq n$ having real coefficients.

4. Consider the set V consisting of all polynomials $p(x) \in \mathcal{P}_n(\mathbb{R})$ that satisfy $p(-x) = p(x)$ i.e. $p(x)$ is an even function. Prove that V is a vector space in itself.

† **Hint:** *V must be a subspace of $\mathcal{P}_n(\mathbb{R})$.*

$p(x), q(x) \in V$

$g(x) = p(x) + \alpha q(x)$

$g(-x) = p(-x) + \alpha q(-x) = p(x) + \alpha q(x)$

5. Consider the vector space V defined as follows:

$$V = \{a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{2024} x^{2024} \mid p(1) = 0, p(-1) = 0\}$$

Find the dimension of V.

† **Hint:** *The answer has three distinct prime divisors.*

(e) $W_3 = \{p(x) \in V \mid p(1) = 0\} \subseteq V = P_n(\mathbb{R})$.

Let $p(x), q(x) \in W_3$. Then $p(1) = 0$, $q(1) = 0$. Let $\alpha \in \mathbb{F}$.

To show $r(x) = p(x) + \alpha q(x) \in W_3$.

Now $r(1) = p(1) + \alpha q(1) = 0 + \alpha 0 = 0$

$\therefore r(x) \in W_3$. $\therefore W_3$ is a subspace.

(4) Let $p(x)$ & $q(x) \in V$. Then

$p(x) = p(-x)$, $q(x) = q(-x)$. Let $\alpha \in \mathbb{F}$.

To show, $r(x) = p(x) + \alpha q(x) \in V$.

Now $r(-x) = p(-x) + \alpha q(-x)$

$= p(x) + \alpha q(x)$ $\left(\because \begin{array}{l} p(x) = p(-x) \\ q(x) = q(-x) \end{array} \right)$

$r(-x) = r(x)$

$\therefore r(x) \in V$.

Q.5 $V = \{ p(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2024} x^{2024} \mid p(1) = p(-1) = 0 \}$

Let $p(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2024} x^{2024}$

Then, $p(1) = a_0 + a_2 (1)^2 + \dots + a_{2024} (1)^{2024}$

$p(1) = a_0 + a_2 + \dots + a_{2024}$ --- (1)

$p(-1) = a_0 + a_2 (-1)^2 + \dots + a_{2024} (-1)^{2024}$

$p(-1) = a_0 + a_2 + \dots + a_{2024}$ --- (2)

From (1) & (2) the condition $p(1) = 0$ & $p(-1)$ are exactly the same.

$\text{Dim}(V) =$ Total no. of terms $-$ Total no. of conditions

$= 1013 - 1$

$= 1012$

Total no. of terms in $\underbrace{a_0 + a_2 x^2 + \dots + a_{2024} x^{2024}}$

$1 + \frac{2024}{2}$

$1 + 1012 = 1013$

6. Consider the subspaces W_1 and W_2 of vector space $\mathcal{M}(2,\mathbb{R})$ as follows:

$$W_1 = \left\{\begin{pmatrix} a & b \\ -a & c \end{pmatrix} \mid a,b,c \in \mathbb{R}\right\}$$

$$W_2 = \left\{\begin{pmatrix} a & b \\ c & 2a \end{pmatrix} \mid a,b,c \in \mathbb{R}\right\}$$

(a) Express explicitly the subspace $W_1 \cap W_2$.

(b) Also find $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$.

(a) Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in W_1 \cap W_2$

$\therefore A \in W_1$ & $A \in W_2$

$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in W_1 \Rightarrow z = -x$

$\therefore A = \begin{pmatrix} x & y \\ -x & w \end{pmatrix}$. Also, $A = \begin{pmatrix} x & y \\ -x & w \end{pmatrix} \in W_2$

$\Rightarrow w = 2x$

$\therefore \boxed{A = \begin{pmatrix} x & y \\ -x & 2x \end{pmatrix}}$

$\therefore W_1 \cap W_2 = \left\{\begin{pmatrix} x & y \\ -x & 2x \end{pmatrix} \mid x,y \in \mathbb{R}\right\}$.

(b) Let $A = \begin{pmatrix} x & y \\ -x & 2x \end{pmatrix} \in W_1 \cap W_2$

then, $A = \begin{pmatrix} x & y \\ -x & 2x \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Basis of $W_1 \cap W_2 = \left\{\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

$\therefore \dim(W_1 \cap W_2) = 2$

We know that

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Let $\begin{pmatrix} a & b \\ -a & c \end{pmatrix} \in W_1$

$$\Rightarrow \begin{pmatrix} a & b \\ -a & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Basis of $W_1 = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\dim W_1 = 3$

Let $\begin{pmatrix} a & b \\ c & 2a \end{pmatrix} \in W_2$

$$\Rightarrow \begin{pmatrix} a & b \\ c & 2a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Basis of $W_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$.

$\therefore \dim W_2 = 3$

$$\therefore \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
$$= 3 + 3 - 2$$
$$\dim(W_1 + W_2) = 4$$

7. Consider the subspaces W_1 and W_2 of vector space $\mathcal{P}_7(\mathbb{R})$ as follows:

$$W_1 = \{p(x) \in \mathcal{P}_7(\mathbb{R}) |\, p(1) = 0, p\left(\frac{1}{2}\right) = 0, p(4) = 0\}$$

$$W_2 = \{p(x) \in \mathcal{P}_7(\mathbb{R}) |\, p(-1) = 0, p\left(\frac{1}{2}\right) = 0, p(-4) = 0\}$$

(c) Express explicitly the subspace $W_1 \cap W_2$.

(d) Also find $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$.

Sol^{n}: Do as in 6.

8. Suppose W_1 and W_2 are two subspaces of a 7 $-$dimensional vector space V such that $\dim W_1 = 4$ and $\dim W_2 = 5$. What are the possible values of $\dim(W_1 \cap W_2)$?

Theorem: If W is a subspace of V, then $\dim W \leq \dim V$.

Corrollary: $W_1 + W_2$ is a subspace of V. So $\dim(W_1 + W_2) \leq \dim V$

Sol^n:

$$\dim(W_1 \cap W_2) \le \dim W_1 = 4 \quad \cdots ①$$

$$\dim(W_1 \cap W_2) \le \dim W_2 = 5 \quad \cdots ②$$

$$\dim(W_1 + W_2) \le 7$$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$7 \geqslant 5 + 4 - \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad 7 \geqslant 9 - \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad \dim(W_1 \cap W_2) \geq 9 - 7$$

$$\dim(W_1 \cap W_2) \geq 2$$

$$2 \le \dim(W_1 \cap W_2) \le 4 \quad (\text{see eq}^n ②)$$