Program	B. Tech. (SoCS)	Semester	IV
Course	Linear Algebra	Course Code	MATH 2059
Session	Jan-May 2025	Topic(s)	Linear mapping, Inner product space, Orthogonalization and SVD

1. Check whether the following mappings are linear or not. Justify your answer.

a) $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by
$$T(a + bx + cx^2 + dx^3) = a + b(x+1) + c(x+1)^2 + d(x+1)^3$$

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by
$$T(x, y, z) = (|x|, 0)$$

c) $T: \mathbb{R}^2 \to \mathbb{R}$ defined by
$$T(x, y) = xy$$

d) $T: M(2, \mathbb{R}) \to M(2, \mathbb{R})$ defined by
$$T(A) = 2A + 3A^T$$

e) $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T\big(p(x)\big) = xp(x) + p(1) - 2$

2. Find the matrix representation of the following linear mappings with respect to the given ordered basis.

a) $T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by
$$T(a + bx) = a + bx + ax^2$$
with respect to an ordered basis $B = \{1, x\}$ and $B' = \{1 + x, 1 - x, x^2\}$ of $P_1(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively.

b) $T: P_2(\mathbb{R}) \to P_4(\mathbb{R})$ be defined by
$$T(a + bx + cx^2) = x^2(a + bx + cx^2)$$
with respect to the standard bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3, x^4\}$ of $P_2(\mathbb{R})$ and $P_4(\mathbb{R})$ respectively.

c) $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as
$$T(x, y, z) = (x + y - z, x + y + z, y - z)$$
with respect to an ordered basis $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$.

d) $T: \mathbb{R}^2 \to M(2, \mathbb{R})$ defined by

$$T(x,y) = \begin{pmatrix} x+y & x-y \\ 2x-y & x+2y \end{pmatrix}$$

with respect to the standard bases $B = \{(1,0),(0,1)\}$ and $B' = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ of $\mathbb{R}^2$ and $M(2, \mathbb{R})$ respectively.

3. Find the dimensions of $rangespace(T)$ and $nullspace(T)$ for the following linear transformations.

a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x,y,z) = (x-y,\ x-y,\ 0)$

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined as $T(x,y,z) = (x+y,\ x-z)$

c) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x,y,z) = (x+y,\ y+z,\ z+x)$

d) $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined as $T(p(x)) = p''(x) + p'(x)$

e) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(x,y) = (x,\ x+y,\ y)$

f) $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined as

$$T(p(x)) = \int_1^x p'(t)dt$$

g) $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined as

$$T(p(x)) = \int_0^x p(t)dt + p'(x) + p(2)$$

h) $T: M(3, \mathbb{R}) \to M(3, \mathbb{R})$ defined as

$$T(A) = \frac{A - A^T}{2}$$

i) $T: \mathbb{R}^3 \to M(2, \mathbb{R})$ defined as

$$T(x,y,z) = \begin{pmatrix} -x+z & 2x-3y \\ 3x+4y & 2y+z \end{pmatrix}$$

4. For each part in Question No. 3, check whether the given linear transformation on respective vector space is bijective or not. Justify your answer in each case.

5. Which of the following defines an inner product on the given vector space?

a) $< u, v > = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$ on $V = \mathbb{R}^2$ where $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

b) $< u, v > = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$ on $V = \mathbb{R}^2$ where $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

c) $< u, v > = 3x_1y_1 + 4x_1y_2 + 4x_2y_1 + 5x_2y_2$ on $V = \mathbb{R}^2$ where $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

6. For what value(s) of k, the following function represents an inner product on $\mathbb{R}^2$?

a) $f(x,y) = 4x_1y_1 + kx_1y_2 + kx_2y_1 + 9x_2y_2$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

b) $f(x,y) = kx_1y_1 + 5x_1y_2 + 5x_2y_1 - 2x_2y_2$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

7. Find the matrix A that represents the usual inner product on $\mathbb{R}^2$ relative to each of the following bases of $\mathbb{R}^2$:

a) $\{v_1 = (1,4), v_2 = (2,-3)\}$
b) $\{u_1 = (1,-3), u_2 = (6,2)\}$

8. Let $V = P_3(\mathbb{R})$ be the vector space of all real polynomials of degree ≤ 3 with inner product

$$< p(t), q(t) > = \int_{-1}^{1} p(t)q(t)dt$$

Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis from the standard basis $B = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$.

9. Consider the subspace U of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1,1,1,1)$, $v_2 = (1,1,2,4)$ and $v_3 = (1,2,-4,-3)$. Using the Gram-Schmidt orthogonalization process, find an orthonormal basis of U.

10. Find a singular value decomposition for the following matrices:

a) $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$
b) $B = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$