Eigenvalues and Eigenvectors

Polynomial of matrices

$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$P(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$, where A is sq matrix.

Characteristic polynomial

of a square matrix A is the polynomial defined by

$P_A(\lambda) = |A - \lambda I|$ or $\det(A - \lambda I)$

eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$P_A(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3$

$= 8 - 6\lambda + \lambda^2 - 3$

$= \lambda^2 - 6\lambda + 5$

eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 4 \\ 2 & 3 & 3 \end{bmatrix}$

$P_A(\lambda) = -\lambda^3 + 6\lambda^2 + 17\lambda + 13$

Characteristic equation

$\det(A - \lambda I) = 0$

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation

i.e., if $P_A(\lambda) = 0$ is the char eqn of A,

then $P_A(A) = 0$.

eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

Minimal polynomial

of a matrix A is the polynomial m_A s.t

$m_A(A) = 0$ and if $f(\lambda)$ is a polynomial with

$f_A(A) = 0$, then $m_A(\lambda) \mid f_A(\lambda)$.

eg $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Let A be any $n \times n$ square matrix. Then if ~~the~~
λ is a scalar and x is a $n \times 1$ column vector such that

$$Ax = \lambda x,$$

then, $\lambda \rightarrow$ eigenvalue of A

$x \rightarrow$ eigenvector of A corresponding to λ.

A can have maximum n eigenvalues and n eigenvectors.

To find Eigenvalue of A:

Solve the characteristic equation,

$$\det(A - \lambda I) = 0.$$

The roots λ of this equation will be the eigenvalues

- if A is a 3×3 matrix and suppose it has 3 distinct eigenvalues, then it will definitely have 3 distinct eigenvectors!

eg: $\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solve: $\det(A-\lambda I)=0$

$-\lambda^3+13\lambda-12=0$

$-(\lambda-1)(\lambda-3)(\lambda+4)=0$

$\Rightarrow$ Eigenvalues are: $\lambda=1,\ \lambda=3,\ \lambda=-4$.

Eigen vector for $\lambda=1$,

Solve $Ax=x$

i.e, $(A-I)x=0$

$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x-3y=0$
$z=0$
$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

eigenvector of $\lambda=1$.

Similarly, eigenvector of $\lambda=3$ is the solution of $(A-3I)x=0$

Solving, we get $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

eigenvector of $\lambda = -4$ is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

$\therefore$ eigenvalues : $1, 3, -4$

eigenvectors : $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

- Sometimes, there are only 2 eigenvalues (distinct) for a 3×3 matrix. In that case, it can have ~~2 or 3~~ 3 or lesser eigenvectors.

eg: $A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

$\det(A - \lambda I) = 0$

$(\lambda - 5)(\lambda - 10)^2 = 0$.

$\lambda = 5, 10$ are the eigenvalues.

when $\lambda = 5$, solve $(A - 5I)x = 0$.

~~eigenvector~~ $\begin{bmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$R_2 \leftrightarrow R_1$ $\begin{bmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + 2R_1$ $\begin{bmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 10 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$

$z \in \mathbb{R}$, $-10y - 5z = 0 \Rightarrow y = -z/2$.

$2x + 9y + 2z = 0 \Rightarrow \quad x = \dfrac{-2z - 9y}{2}$

$= \dfrac{-2z + 9z/2}{2} = \dfrac{5z}{4}$.

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z/4 \\ -z/2 \\ z \end{bmatrix} = z \begin{bmatrix} 5/4 \\ -1/2 \\ 1 \end{bmatrix} = 4z \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$.

$\therefore$ eigenvector for $\lambda = 5$ is $\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$

when $\lambda = 10$, solve $(A - 10I)x = 0$

$\begin{bmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to 5R_2 + 2R_1$
$R_3 \to 5R_3 - 4R_1$ $\begin{bmatrix} -5 & -10 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$

$-5x - 10y - 5z = 0$

$x + 2y + z = 0 \Rightarrow x = -2y - z.$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y-z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ eigenvectors of 10 are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ A has 2 distinct eigenvalues : 5, 10

multiplicity 2.

and 3 eigenvectors : $\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

- Sometimes, a matrix with order n can have less than n eigenvectors.

eg: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1, 1$ (one distinct eigenvalue)

eigenvector: $Ax = x$

$(A - I)x = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ ~~$y \in \mathbb{R}$~~ $x \in \mathbb{R}, y = 0$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Hence, even though $\lambda = 1$ is a repeated root, it has only one eigenvector.

eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -(1+\lambda) & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[-(1+\lambda)(1-\lambda)\right] = 0$

$-(1-\lambda)^2(1+\lambda) = 0$

$\Rightarrow \lambda = 1, 1, -1$ (2 distinct)

when $\lambda = -1$, we get an eigenvector

when $\lambda = 1$,

solve $(A - I)x = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$z = 0$, $x - 2y = 0 \Rightarrow x = 2y$.

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightarrow$ only one eigenvector for $\lambda = 1$.

Diagonalizability of Matrix

- A square matrix A of order n is said to be diagonalizable if there exists a $n \times n$ invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D \qquad (\text{or } A = PDP^{-1}).$$

- All matrices are not diagonalizable.

If a matrix A is diagonalizable, then its P and D will be of the following form:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & 0 & \lambda_n \\ 0 & 0 & \cdots & 0 & \end{bmatrix}, \quad P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ | & | & & | \\ | & | & & | \end{bmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A

$v_1, \ldots, v_n$ are corresponding eigenvectors.

the eigenvalues can be distinct or repeated.

- eg: $A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has eigen values $1, 3, -4$

and eigenvectors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Choose $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

verify if $P^{-1}AP = D$ (or $AP = PD$)

$$AP = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & -8 \\ 0 & 3 & 0 \end{bmatrix}$$

$$PD = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & -8 \\ 0 & 3 & 0 \end{bmatrix}.$$

Conditions for diagonalizability

An $n \times n$ matrix A is diagonalizable if and only if any ~~of~~ one of these following conditions hold:

1. A has n distinct eigenvectors (eigenvalues may or may not be repeated).
2. Minimal polynomial of A has only linear factors in it.

eg: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has only 2 eigenvectors $\therefore$, not diagonalizable.

Check the second condition for this.

char poly of A is $\cancel{P_A(A)} = p_A(\lambda) = -(1-\lambda)^2(1+\lambda)$.

$\therefore$ min poly has 2 possibilities : $(\lambda-1)(\lambda+1)$
and
$(\lambda-1)^2(\lambda+1)$

• For $(\lambda-1)(\lambda+1)$.

applying on A,

$$(A-I)(A+I) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ & & \\ & & \end{bmatrix}$$

$$\neq [0].$$

$\therefore$ min poly does not have only linear factors.

$\therefore$ not diagonalizable.