Practice Set 3

1\. Let x be such that $x \equiv 4 \pmod{12}$

$x = 16$ is a solution since $12 \mid (16-4)$.

Remaining solutions are $16+12,\ 16+2.12,\ 16+3.12, \ldots$

$16-12,\ 16-2.12,\ 16-3.12, \ldots$

Five integers: $16, 28, 40, 52, 64$.

2\. (a) $19^2 \bmod 41 = 361 \bmod 41$

$= \underline{\underline{33}}$

$$\begin{array}{r} 8 \\ 41 \overline{)361} \\ 328 \\ \hline 33 \end{array}$$

(b) $(32^3 \bmod 13)^2$

$32^3 \bmod 13 = [(32 \bmod 13)(32 \bmod 13)(32 \bmod 13)] \bmod 13$

$= [6.6.6] \bmod 13$

$= 216 \bmod 13$

$= \underline{8}$

$$\begin{array}{r} 16 \\ 13 \overline{)216} \\ 13 \\ \hline 86 \\ 78 \\ \hline \underline{8} \end{array}$$

$(32^3 \bmod 13)^2 = 8^2 = \underline{\underline{64}}$

~~3 4, LCM $(2^3 5$~~

4\. $\text{lcm}(2^3 3^5 7^2,\ 2^4 3^3) = 2^4 . 3^5 . 7^2 = \underline{\underline{190512}}$

5\. $1000 = 2^3 5^3$

$625 = 5^4$

$\gcd(1000, 625) = 2^0 . 5^3 = 125$

$\text{lcm}(1000, 625) = 2^3 . 5^4 = 5000$.

$\gcd(1000, 625) . \text{lcm}(1000, 625) = 125 \times 5000 = 625000$

$1000 \times 625 = 625{,}000$

Hence, verified.

6) (a) $\underline{\gcd(1,5)}$

$5 = 5 \times 1 + 0$

$\gcd(1,5) = 1$

(b) $\gcd(100, 101)$

$101 = 1 \times \underline{100} + 1 \rightarrow \gcd$

$100 = 100 \times 1$

$\gcd(100, 101) = 1$

(c) $\gcd(1529, 14039)$

$14039 = 9 \times \underline{1529} + \underline{278}$

$1529 = 5 \times \underline{278} + \underline{139} \rightarrow \gcd$

$278 = 2 \times 139$.

$\gcd(1529, \cancel{139}\, 14039) = \underline{\underline{139}}$.

7) $a \equiv b \pmod m$

<u>To prove:</u> $\gcd(a,m) = \gcd(b,m)$.

$a \equiv b \pmod m \Rightarrow m \mid a-b$

$\Rightarrow a - b = mq$ for some $q \in \mathbb{Z}$.

$\Rightarrow a = b + mq$.

Let ~~if~~ $c \in \mathbb{Z}$ such that $c \mid a$ and $c \mid m$.

we have ~~then~~ $b = a - mq$

$\cancel{\Rightarrow}$ $c \mid a$ and $c \mid m$

$\Rightarrow c \mid a - mq$

$\Rightarrow c \mid b$.

$\Rightarrow$ any divisor of a and m is a divisor of b as well.

~~Thus, $\gcd(a,m) = \gcd(b,m)$.~~

8) Now, suppose $d \in \mathbb{Z}$ such that $d \mid b$ and $d \mid m$

Since $a = b + mq$ and $d \mid b$ and $d \mid m \Rightarrow d \mid b + mq$,

we have $d \mid a$.

Thus, $\gcd(a, m) = \gcd(b, m)$.

8) Inverse of 7 modulo 26 is an integer x such that

$$7x \equiv 1 \pmod{26}$$

If $x = 15$.

$$7 \times 15 - 1 = 105 - 1 = 104.$$

$$26 \mid 104 \qquad (\because 104 = 26 \times 4)$$

$$\therefore \quad 7(15) \equiv 1 \pmod{26}.$$

$\Rightarrow$ 15 is an inverse of 7 modulo 26.

10) Lights flash at interval of 12, 15, 20.

$t_1 = 12$

$t_2 = 15$

$t_3 = 20$.

They will flash together at time $t = \text{lcm}(t_1, t_2, t_3)$

$= \text{lcm}(12, 15, 20)$

$= \cancel{120\,s}$ $\underline{60 \text{ seconds}}$

11) $7^{121} \pmod{13}$.

~~121 = 9×13 + 4~~

Let $p = 13$ and $a = 7$.

$p \nmid a \Rightarrow a^{p-1} \equiv 1 \pmod p$

$7^{12} \equiv 1 \pmod{13}$

$121 = 12 \times 10 + 1$

$7^{121} = (7^{12})^{10} \cdot 7 \equiv (1)^{10} \cdot 7 \pmod{13}$

$\equiv 7 \pmod{13}$.

$\therefore 7^{121} \pmod{13} = \underline{7}$.

$23^{1002} \pmod{41}$

$p = 41$, $a = 23$

$p \nmid a \Rightarrow a^{p-1} \equiv 1 \pmod{p}$

$\Rightarrow 23^{40} \equiv 1 \pmod{41}$

$1002 = 40 \times 25 + 2$

$23^{1002} = (23^{40})^{25} \cdot 23^{2}$

$\equiv (1)^{25} \cdot 529 \pmod{41}$

$\equiv 37 \pmod{41}$ (Since $41 \times 12 = 492$)

$\therefore 23^{1002} \pmod{41} = \underline{\underline{37}}$

12) $4x \equiv 5 \pmod{9}$

Finding inverse of $4 \pmod{9}$

we have to find y such that

$4y \equiv 1 \pmod{9}$

$y = 7$ will be a solution as $9 \mid 4(7) - 1$

$\Rightarrow 4(7) \equiv 1 \pmod{9}$

multiply with 5 on both sides

$4(7)(5) \equiv 5 \pmod{9}$

$4(35) \equiv 5 \pmod{9}$

$\therefore$ Solution is 35 or $\underline{\underline{x \equiv 35 \pmod{9}}}$

13) $a = 8316$, $b = 10920$.

(a) Using Euclidean Algorithm,

$10920 = 1 \times 8316 + 2604$

$8316 = 3 \times 2604 + 504$

~~$2604 = 4 \times 504 + 588$~~

$2604 = 5 \times 504 + 84$ $\rightarrow$ gcd.

$504 = 6 \times 84$

$\gcd(8316, 10920) = 84$.

(b) $84 = 2604 - 5(504)$

$= 10920 - 8316 - 5(8316 - 3(2604))$

$= 10920 - 6(8316) + 15(2604)$

$= 10920 - 6(8316) + 15(10920 - 8316)$

$= 16(10920) - 21(8316)$

$= -21(8316) + 16(10920)$

$m = -21$, $n = 16$

14) In general, for any n,

$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$.

eg: $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$\mathbb{Z}_{10} = \{0, 1, 2, 3, \ldots, 9\}$.

Addition in $\mathbb{Z}_n$

if $a, b \in \mathbb{Z}_n$, then $a+b$ in $\mathbb{Z}_n$ is $(a+b) \pmod{m}$

eg: in $\mathbb{Z}_5$,

$3+4 = 7 \pmod 5$

i.e, $3+4 = 2$ in $\mathbb{Z}_5$

Multiplication in $\mathbb{Z}_n$

if $a, b \in \mathbb{Z}_n$ then $a.b$ is $(a.b) \pmod n$.

eg: in $\mathbb{Z}_5$,

$3.4 = 12 \pmod 5$

i.e, $3.4 = 2$ in $\mathbb{Z}_5$.

Addition table for

(a) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Multiplication table for $\mathbb{Z}_4$

$*_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

b) Addition table for

$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Multiplication table

$*_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

15) a^{-1} in $\mathbb{Z}_m$ is x such that $ax \equiv 1 \pmod{m}$

(a) $a = 37$, $m = 249$

$$37x \equiv 1 \pmod{249}$$

Find gcd (37, 249) using Euclidean Algorithm

$$249 = 6 \times \underline{37} + \underline{27}$$

$$37 = 1 \times \underline{27} + \underline{10}$$

$$27 = 2 \times \underline{10} + \underline{7}$$

$$10 = 1 \times \underline{7} + \underline{3}$$

$$7 = 2 \times \underline{3} + 1 \longrightarrow \text{gcd}$$

$$3 = 3 \times 1$$

Since gcd(37, 249) = 1, 37 has a unique inverse modulo 249.

we need to find x such that

$$37x \equiv 1 \pmod{249}$$

i.e, $249 \mid 37x - 1$

i.e, find y such that $37x - 1 = 249y$.

Find x and y $\xRightarrow{\text{such that}}$ $\underline{\underline{1 = 37x - 249y}}$ — ①

From Euclidean algorithm,

$$\begin{aligned}
1 &= 7 - 2(3) \\
&= 27 - 2(10) - 2(10 - 7) \\
&= 27 - 4(10) + 2(7) \\
&= 249 - 6(37) - 4(37 - 27) + 2(27 - 2(10)) \\
&= 249 - 10(37) + 6(27) - 4(10)
\end{aligned}$$

$$= 249 - 10(37) + 6(249 - 6(37)) - 4(37 - 27)$$

$$= 7(249) + (-10 - 36 - 4)(37) + 4(27)$$

$$= 7(249) - 50(37) + 4(249 - 6(37))$$

$$= 11(249) - 74(37)$$

$$= -74(37) + 11(249)$$

Comparing with equation 1,

$\underline{x = -74}$, $y = -11$.

$\downarrow$ inverse.

$\therefore$ ~~$x \equiv -74$ (mod~~

$-74 + 249 = 175$ is the inverse of 37 in $\mathbb{Z}_{249}$.

(b) $a = 15$, $m = 234$.

Since $\gcd(15, 234) \neq 1$, 15 does not have an inverse modulo 234.

$234 = 15 \times \underline{15} + \underline{9}$

$15 = 1 \times \underline{9} + \underline{6}$

$9 = 1 \times \underline{6} + \underline{3} \rightarrow$ gcd.

$6 = 2 \times 3$

~~Find x s.t $15x \equiv 1 \pmod{234}$~~

~~$234 \mid 15x - 1$~~

~~i.e, find x and y s.t $15x - 1 = 234y$.~~

~~$1 = 15x - 234y$ — 1.~~

Since $\gcd(15, 234) \neq 1$,
15 does not have an inverse modulo 234.

16) $7x + 5y = 1.$ — (1)

$7x - 1 = 5y.$

$7x \equiv 1 \pmod{5}$

$x = 3$ is a solution.

$x = 3 + 5n$, where $n \in \mathbb{Z}$ gives all the integer solutions.

Substitute this in equation (1). to get y.

$7(3 + 5n) + 5y = 1$

$21 + 35n + 5y = 1$

$y = \dfrac{1 - 21 - 35n}{5} = \underline{\underline{-4 - 7n}}.$

$\therefore\ x = 3 + 5n,\ y = -4 - 7n;\ n \in \mathbb{Z}$ are the solutions.

17) $3^{100} \bmod 7$.

Let $p = 7,\quad a = 3$.

$p \nmid a \implies a^{p-1} \equiv 1 \pmod p$

$3^6 \equiv 1 \pmod 7$

$100 = 6 \times 16 + 4$

$\therefore\ 3^{100} = (3^6)^{16} \cdot 3^4$

$\equiv (1)^{16} \cdot 81 \pmod 7$

$\equiv 4 \pmod 7$.

Remainder $= 4$.

18) Last digit of any number is the remainder obtained when it is divided by 10.

Here, we need to find $7^{83} \bmod 10$.

$7^4 = 2401$

$\therefore\ 7^4 \bmod 10 = 1$

$83 = 4 \times 20 + 3$.

$7^{83} \bmod 10 = (7^4)^{20} \cdot 7^3 \bmod 10$

$= [((7^4)^{20} \bmod 10)(7^3 \bmod 10)] \bmod 10$

$= [((7^4 \bmod 10)^{20} \bmod 10)(7^3 \bmod 10)] \bmod 10$

$= [(1^{20} \bmod 10)(343 \bmod 10)] \bmod 10$

$= (1 \cdot 3) \bmod 10 = 3$

19) (a) $a_n = 2a_{n-1} + 3a_{n-2}$

For a degree 2 recurrence relation,

characteristic equation is $r^2 - c_1 r - c_2 = 0$.

Here, $c_1 = 2$, $c_2 = 3$.

$\therefore$ char eqn: $r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

roots are $r_1 = 3$, $r_2 = -1$.

General solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

here $\underline{\underline{a_n = \alpha_1 3^n + \alpha_2 (-1)^n}}$

(b) $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$, $a_1 = 1$.

char eqn: $r^2 - c_1 r - c_2 = 0$.

here $c_1 = 1$, $c_2 = 1$

$\therefore$ char eqn: $r^2 - r - 1 = 0$

roots: $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$.

General solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

given $a_0 = 0 = \alpha_1 + \alpha_2$

$a_1 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$

$\Rightarrow$ $\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2$

$\alpha_1(1+\sqrt{5}) + \alpha_2(1-\sqrt{5}) = 2$

$\Rightarrow -\alpha_2(1+\sqrt{5}) + \alpha_2(1-\sqrt{5}) = 2$

$\Rightarrow \alpha_2(-1-\sqrt{5}+1-\sqrt{5}) = 2$

$\Rightarrow \alpha_2(-2\sqrt{5}) = 2$

$\Rightarrow \alpha_2 = \frac{-1}{\sqrt{5}}$

$\Rightarrow \alpha_1 = -\alpha_2 = \frac{1}{\sqrt{5}}$

$\therefore \alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = \frac{-1}{\sqrt{5}}$

$\therefore$ particular solution: $a_n = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

(c) $a_n = 6a_{n-1} - 9a_{n-2}$

char eqn: $\cancel{r^2 - 2r}$ $\quad r^2 - c_1 r - c_2 = 0$.

$c_1 = 6, \quad c_2 = -9$.

char eqn: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

roots: $r = 3, 3$ (repeated roots)

$\therefore$ general solution is $a_n = \alpha_1 r^n + \alpha_2 n r^n$

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$

(d) $a_n = 11a_{n-1} - 39a_{n-2} + 45a_{n-3}$

char eqn: $r^3 - c_1 r^2 - c_2 r - c_3 = 0$.

$c_1 = 11, \quad c_2 = -39, \quad c_3 = 45$

$\therefore$ char eqn: $r^3 - 11r^2 + 39r - 45 = 0$

$r_1 = 3$ is a solution

$(r-3)(r^2 - 8r + 15) = 0$

$(r-3)(r-3)(r-5) = 0.$

$\therefore$ 3 is a root with multiplicity 2.

$r_1 = 3$ with multiplicity 2.

$r_2 = 5.$

general solution is $a_n = \alpha_{1,0} r_1^n + \alpha_{1,1} n r_1^n + \alpha_{2,0} r_2^n$

$$a_n = \alpha_{1,0} 3^n + \alpha_{1,1} n 3^n + \alpha_{2,0} 5^n$$

(e) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ $\quad a_0 = 3,\ a_1 = 4,\ a_2 = 12$

char eqn : $r^3 - c_1 r^2 - c_2 r - c_3 = 0.$

$c_1 = 6,\ c_2 = -12,\ c_3 = 8.$

char eqn : $r^3 - 6r^2 + 12r - 8 = 0$

2 is a solution.

$$\begin{array}{r|l} & r^2 - 4r + 4 \\ \hline r-2 & r^3 - 6r^2 + 12r - 8 \\ & r^3 - 2r^2 \\ \hline & -4r^2 + 12r \\ & -4r^2 + 8r \\ \hline & 4r - 8 \\ & 4r - 8 \\ \hline & 0 \end{array}$$

$\therefore$ $(r-2)(r^2 - 4r + 4) = 0$

$(r-2)(r-2)^2 = 0.$

$(r-2)^3 = 0.$

$r_1 = 2$ is a repeated root with multiplicity 3.

gen. soln is $a_n = \alpha_{1,0} r_1^n + \alpha_{1,1} n r_1^n + \alpha_{1,2} n^2 r_1^n$

$$a_n = \alpha_{1,0} 2^n + \alpha_{1,1} n 2^n + \alpha_{1,2} n^2 2^n.$$

$a_0 = 3 = \alpha_{1,0} + 0 + 0 \quad \Rightarrow \quad \alpha_{1,0} = 3$

$a_1 = 4 = \alpha_{1,0} 2 + \alpha_{1,1} 2 + \alpha_{1,2} 2 \Rightarrow 2\alpha_{1,0} + 2\alpha_{1,1} + 2\alpha_{1,2} = 4$

$6 + 2\alpha_{1,1} + 2\alpha_{1,2} = 4$

$\alpha_{1,1} + \alpha_{1,2} = -1$ — (1)

$a_2 = 12 = \alpha_{1,0}\, 2^2 + \alpha_{1,1} . 2 . 2^2 + \alpha_{1,2} . 2^2 . 2^2$

$\Rightarrow \quad 4\alpha_{1,0} + 8\alpha_{1,1} + 16\alpha_{1,2} = 12$

$12 + 8\alpha_{1,1} + 16\alpha_{1,2} = 12$

$\cancel{2\alpha_{1,1} + 4\alpha}$

$\alpha_{1,1} + 2\alpha_{1,2} = 0 \quad — (2)$

we have (1) and (2) as

$\alpha_{1,1} + \alpha_{1,2} = -1 \quad — (1)$

$\alpha_{1,1} + 2\alpha_{1,2} = 0 \quad — (2)$

(2) − (1) $\Rightarrow \quad \alpha_{1,2} = 1$

(1) $\Rightarrow \quad \alpha_{1,1} = -1 - \alpha_{1,2} = -1-1 = -2$

$\therefore \quad \alpha_{1,0} = 3, \quad \alpha_{1,1} = -2, \quad \alpha_{1,2} = 1.$

$\therefore$ particular solution is $a_n = 3.2^n - 2n2^n + n^2 2^n.$

20) a) $a_k = 5a_{k-1} - 6a_{k-2} \qquad a_0 = 6, \; a_1 = 30$

char eqn: $r^2 - c_1 r - c_2 = 0$

$c_1 = 5, \quad c_2 = -6$

$\therefore$ char eqn: $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r_1 = 2, \quad r_2 = 3$

gen solution : $\quad a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$a_n = \alpha_1 2^n + \alpha_2 3^n$

$a_0 = 6 = \alpha_1 + \alpha_2$

$a_1 = 30 = \alpha_1 . 2 + \alpha_2 . 3$

$\Rightarrow \quad \alpha_1 + \alpha_2 = 6 \quad \Rightarrow \quad 2\alpha_1 + 2\alpha_2 = 12 \quad — (1)$

$2\alpha_1 + 3\alpha_2 = 30 \quad — (2)$

$(2) - (1) \Rightarrow \quad \alpha_2 = 18.$

$(1) \Rightarrow \quad \alpha_1 = 6 - \alpha_2 = 6 - 18 = \underline{\underline{-12}}.$

$\therefore \quad \alpha_1 = -12, \ \alpha_2 = 18.$

$\therefore$ particular solution is $a_n = -12 \cdot 2^n + 18 \cdot 3^n.$

(b) $a_n = 4a_{n-1} - 4a_{n-2} + n^2 \ ; \ a_0 = 2, \ a_1 = 5$

$a_n = a_n^{(h)} + a_n^{(p)}.$

<u>Finding $a_n^{(h)}$</u>:

$a_n = 4a_{n-1} - 4a_{n-2}$

char eqn: $r^2 - c_1 r - c_2 = 0$

$c_1 = 4, \ c_2 = -4$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0.$

$r = 2, 2.$

~~gen soln~~: $a_n^{(h)} = \alpha_1 r^n + \alpha_2 n r^n$

$a_n^{(h)} = \alpha_1 2^n + \alpha_2 n 2^n.$

<u>Finding $a_n^{(p)}$</u>:

$F(n) = n^2$

$a_n^{(p)} = A_0 + A_1 n + A_2 n^2$

we have $a_n = 4a_{n-1} - 4a_{n-2} + n^2$

$\Rightarrow \cancel{A_0 + A_1 n}$

$\Rightarrow A_0 + A_1 n + A_2 n^2 = 4(A_0 + A_1(n-1) + A_2(n-1)^2) + n^2$
$- 4(A_0 + A_1(n-2) + A_2(n-2)^2)$

$$A_0 + A_1 n + A_2 n^2 = 4(A_0 + A_1 n - A_1 + A_2 n^2 - 2A_2 n + A_2)$$
$$- 4(A_0 + A_1 n - 2A_1 + A_2 n^2 - 4A_2 n + 4A_2) + n^2$$

$$A_0 + A_1 n + A_2 n^2 = 4(\cancel{A_0} + \cancel{A_1 n} - A_1 + \cancel{A_2 n^2} - 2A_2 n + A_2$$
$$- \cancel{A_0} - \cancel{A_1 n} + 2A_1 - \cancel{A_2 n^2} + 4A_2 n - 4A_2) + n^2$$
$$= 4(A_1 + 2A_2 n - 3A_2) + n^2$$
$$= 4A_1 - 12A_2 + 8A_2 n + n^2$$

$$(A_0 - 4A_1 + 12A_2) + (A_1 + 8A_2)n + (A_2 - 1)n^2 = 0.$$

$A_2 - 1 = 0 \Rightarrow \boxed{A_2 = 1}$

$A_1 + 8A_2 = 0 \Rightarrow A_1 + 8 = 0 \Rightarrow \boxed{A_1 = -8}$

$A_0 - 4A_1 + 12A_2 = 0 \Rightarrow A_0 + 32 + 12 = 0 \Rightarrow \boxed{A_0 = -44}$

$$a_n^{(p)} = -44 + -8n + n^2$$

$\therefore$ gen soln: $a_n = a_n^{(h)} + a_n^{(p)}$
$$= \alpha_1 2^n + \alpha_2 n 2^n - 44 - 8n + n^2.$$

Finding α_1, α_2:

$a_0 = 2 \Rightarrow 2 = \alpha_1 - 44 \Rightarrow \boxed{\alpha_1 = 46}$

$a_1 = 5 \Rightarrow 5 = \alpha_1 . 2 + \alpha_2 . 2 - 44 - 8 . 1 + 1$
$$= 92 + 2\alpha_2 - 52 + 1$$
$$= 2\alpha_2 + 41 \Rightarrow \alpha_2 = \frac{5-41}{2} = -18$$

$\boxed{\alpha_2 = -18}$

$\therefore$ solution is $a_n = \underline{46 . 2^n - 18 n 2^n - 44 - 8n + n^2}$

21) Refer 19(b).

22) Let $n(J) \rightarrow$ no of people proficient in Java

$n(C) \rightarrow$ " " " in C

$n(P) \rightarrow$ " " " in Python.

Given, ~~that~~ total number of programmers, $n(U) = 100$

$n(J) = 45$

$n(C) = 30$

$n(P) = 20$

$n(C \cap J) = 6$

$n(J \cap P) = 1$

$n(C \cap P) = 5$

$n(C \cap J \cap P) = 1$.

To find : $n((C \cup J \cup P)^c)$

$$n((C \cup J \cup P)^c) = n(U) - n(C \cup J \cup P)$$

$$= n(U) - [n(C) + n(J) + n(P) - n(C \cap J) - n(J \cap P) - n(C \cap P) + n(C \cap J \cap P)]$$

$$= 100 - [45 + 30 + 20 - 6 - 1 - 5 + 1]$$

$$= 100 - [95 - 12 + 1]$$

$$= 100 - [96 - 12]$$

$$= 100 - 84$$

$$= \underline{\underline{16}}.$$

23) $n(U) = 350$

$n(B) \rightarrow$ no. of farmers who farm beetroot.

$n(Y) \rightarrow$ " " " yams

$n(R) \rightarrow$ " " " radish

$n(B) = 260$, $n(Y) = 100$, $n(R) = 70$.

$n(B \cap R) = 40$, $n(Y \cap R) = 40$, $n(B \cap Y) = 30$.

<u>To find</u>: $n(B \cap Y \cap R)$.

Since it is given 350 are farmers

$\Rightarrow \quad n(B \cup Y \cup R) = 350$.

$\cancel{n(B \cap Y \cap R) = n(B) + n(Y) + n(R) - n(B \cap Y) - n(}$

$$n(B \cap Y \cap R) = n(B \cup Y \cup R) - n(B) - n(Y) - n(R) + n(B \cap Y) + n(Y \cap R) + n(B \cap R)$$

$$= 350 - 260 - 100 - 70 + 40 + 40 + 30$$

$$= 350 - 430 + 110$$

$$= 460 - 430 = \underline{\underline{30}}$$

24) 12 → red socks
12 → blue socks.

$\frac{12}{R}$ $\frac{+2}{B}$

Has to get : atleast 2 blue socks.

If he picks 12 socks, it is possible that all 12 can be red

If he picks 2 more, that has to be blue.

Thus, he should pick a total of $12+2 = \underline{\underline{14}}$ socks.

26) Total number of people in the group = 267.

<u>case 1:</u> Let us assume every person has atleast one friend.

<u>To find</u>: atleast how many people will have the same number of friends.

<u>worst case</u>: each person has different number of friends, i.e,

person 1 → 1 friend
person 2 → 2 friends
⋮
person 266 → 266 friends. (This is the maximum no of friends a person can have since there are only 267 people).

Therefore, 267^{th} person will have number of friends equal to any one of the people above.

Therefore, atleast 2 of them will have the same no of friends.

Answer : $\underline{\underline{2}}$.

case 2: Suppose there are people with no friends. ~~as~~

worst case: each person has different no of friends

person 1 → 0 friends

person 2 → 1 friend

⋮

person 266 → 265 friends.

Since one person is not anybody's friend, the ~~total~~ ~~no~~ maximum no of friends a person can have is 265 (267 - himself - person with no friend).

∴ 267th person will have no of friends equal to one of the above.

Answer: 2

∴, atleast 2 people will have same number of friends.

27) Let a_1 be number of games played on day 1

a_2 be total no of games played till day 2

⋮

a_j be total no of games played till day j.

$a_1 < a_2 < \cdots < a_{30}$.

Suppose there is a period of consecutive days when the team plays exactly n number of games.

$a_1 + n < a_2 + n < \cdots < a_{30} + n$.

~~no~~ ~~out of all~~

Let i, j be the days ~~wh~~ during which the team Nays

n games.

i.e, $a_i - a_j = n$

$\Rightarrow a_i = a_j + n$.

This can be guaranteed if $a_1, \dots, a_{30}, a_1+n, \dots, a_{30}+n$ (60...

have not more than 59 options to choose from.

So, we have to choose n such that

$\cancel{a_{30} = 59}$, $a_{30} + n \leq 59$

a_{30} has maximum value 45.

$\therefore$ if $n \leq 59 - 45 = 14$, then we are done.

$\therefore$, answer is $\underline{14}$.

28) $P(n): 1 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

I) For $n = 1$,

$LHS = 1$

$RHS = \left(\frac{1.(1+1)}{2}\right)^2 = 1$

$LHS = RHS$

$\therefore P(1)$ is true.

II) Let $P(k)$ be true for some $k \geq 1$,

i.e, $1 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$

To prove: $P(k+1)$ is true, i.e, $1^3 + 2^3 + \dots + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

Proof: $LHS = 1 + 2^3 + \dots + k^3 + (k+1)^3$

$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4} + k+1\right]$$

$$= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 = \text{RHS}.$$

$\therefore$ $P(k+1)$ is true.

$\Rightarrow$ $P(n)$ is true ~~$\forall n \geq 1$~~ $\forall n = 1, 2, 3, \ldots$

29) Done in class

30) Let $P(n)$: $1 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

I) For $n = 1$,

LHS $= 1$

RHS $= \dfrac{1(1+1)(2+1)}{6} = 1$

LHS $=$ RHS

$\therefore$ $P(1)$ is true.

II) Let $P(k)$ be true for some $k \geq 1$.

i.e, $1 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

~~Proof: LHS =~~

To prove: $P(k+1)$ is true, i.e, $1 + 2^2 + \cdots + (k+1)^2 = \dfrac{(k+1)(k+2)(2(k+1)+1)}{6}$

Proof: LHS $= 1 + 2^2 + \cdots + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= (k+1)\left(\frac{2k^2+7k+6}{6}\right)$$

$$= (k+1)\left(\frac{2k^2+4k+3k+6}{6}\right)$$

$$= (k+1)\left(\frac{2k(k+2)+3(k+2)}{6}\right)$$

$$= (k+1)\,\frac{(2k+3)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \underline{\underline{\text{RHS}}}$$

$\therefore$ $P(k+1)$ is true

$\Rightarrow$ $P(n)$ is true for all $n = 1, 2, 3, \ldots$