Program	**B. Tech. (SoCS)**	**Semester**	**IV**
Course	**Linear Algebra**	**Course Code**	**MATH 2059**
Session	**Jan-May 2025**	**Topic**	**Linear mapping, Inner product space, Orthogonalization and SVD**

1. Determine whether the following mappings are linear or not; clearly mentioning the properties of a linear mapping that hold or DO NOT hold.

(a) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as $T(x, y) = (ax + by, cx + dy)$ for some real numbers a, b, c and d.

(b) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined as $T(x, y, z) = (x + y + 1, x - y)$.

(c) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined as $T(x, y) = (x + y, x - y, xy)$.

(d) $T: \mathcal{M}(2, \mathbb{R}) \longrightarrow \mathbb{R}$ defined as $T(A) = trace\,(A)$.

(e) $T: \mathcal{M}(2, \mathbb{R}) \longrightarrow \mathbb{R}$ defined as $T(A) = \det A$.

(f) $T: \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ defined as $T(p(x)) = p(1)$.

2. Find the matrix representation of the following linear mappings with respect to the given basis.

(a) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as

$$T(x, y, z) = (2x + 3y - z, x + y - 2z, 3x + 4y - 3z)$$

with respect to the standard basis $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$.

(b) $T: \mathcal{M}(2, \mathbb{R}) \longrightarrow \mathcal{M}(2, \mathbb{R})$ defined as

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -a & 2b \\ d & -2c \end{bmatrix}$$

with respect to an ordered basis $\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ of $\mathcal{M}(2, \mathbb{R})$.

(c) $T: \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ defined as

$$T\big(p(x)\big) = p'(x) + p(0)$$

with respect to the standard basis $\mathcal{B} = \{1, x, x^2, x^3\}$ of $\mathcal{P}_3(\mathbb{R})$.

(d) $T: \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ defined as

$$T\big(p(x)\big) = p'(x) + \int_0^x p(t)\, dt$$

with respect to the standard bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, x, x^2, x^3\}$ of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$ respectively.

3. Find the dimensions of $rangespace$ (T) and $nullspace\,(T)$ where T is a linear map $T: \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$ defined as $T\big(p(x)\big) = p(x+1) - p(x-1)$. Hence, verify Rank-Nullity theorem. Is T bijective on $\mathcal{P}_2(\mathbb{R})$?

4. Suppose α, β and γ are in arithmetic progression such that $\alpha < \beta < \gamma$. Find the dimension of vector space $V = \{v \in \mathbb{R}^4 |\, Tv = 0\}$ where the matrix representation of linear map $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is

$$\begin{bmatrix} \alpha & 1 & 0 & -1 \\ \beta & 0 & 1 & 1 \\ \gamma & -1 & 2 & 3 \end{bmatrix}$$

Is $T(\mathbb{R}^4) = \mathbb{R}^3$? Justify.

5. Which of the following define an inner product on the given vector space?

(a) $< u, v > = x_1x_2 - x_1y_2 - x_2y_1 + 3y_1y_2$ on $V = \mathbb{R}^2$ where $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

(b) $< u, v > = x_1y_2x_3 + x_2y_1y_3$ on $V = \mathbb{R}^3$ where $u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $v = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

6. For what values of a and b, the function

$$f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = a^2x_1x_2 + abx_1y_2 + abx_2y_1 + b^2y_1y_2$$

represents an inner product on $\mathbb{R}^2$? Justify.

7. Find the matrix associated with the inner product function on vector space $\mathcal{P}_2(\mathbb{R})$ defined as

$$< p(t), q(t) > = \int_{-1}^{1} p(t)q(t)dt$$

with respect to the standard basis $\{1, t, t^2\}$ of $\mathcal{P}_2(\mathbb{R})$.

8. Use Gram-Schmidt process to obtain an orthonormal basis from the given basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ for a two-dimensional subspace of $V = \mathcal{M}(2, \mathbb{R})$ equipped with the inner product defined by

$$< A, B > = trace\ (B^T A)$$

where B^T is the transposed matrix B.

9. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree $n \leq 2$ having standard basis $\mathcal{B} = \{1, t, t^2\}$. Normalize this basis using Gram-Schmidt orthonormalizing process.

10. Find the Singular value decomposition $A = U\sum V^T$ for the matrix

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

where matrices U and V both are orthogonal and matrix $\sum$ is a diagonal.

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