1. $A = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}$

$z^2 - x^2 - (y+x)(z-x)$

$(z-x)(z+x-y-x)$

$(z-x)(z-y)$

$R_2 \to R_2 - xR_1$
$R_3 \to R_3 - x^2 R_1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y^2-x^2 & z^2-x^2 \end{bmatrix}$

$R_3 \to R_3 - (y+x) R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & 0 & (z-x)(z-y) \end{bmatrix}$

(a) when $x \neq y \neq z$, $(z-x)(z-y) \neq 0 \Rightarrow$ Rank$(A) = 3$

(b) $x = y \neq z$, $(z-x)(z-y) \neq 0 \Rightarrow$ ~~Rank(A)~~ $=3$.

by $y - x = 0$. $\therefore$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & z-x \\ 0 & 0 & (z-x)(z-y) \end{bmatrix}$

not in echelon form. $\therefore$ $R_3 \to R_3 - (z-y) R_2$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & z-x \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Rank$(A) = 2$.

(c) $x = y = z$, then echelon form of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Rank$(A) = 1$.

2 (a) Rank $(A+B) = \text{Rank}(A) + \text{Rank}(B)$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Rank $(A) = 2$, Rank $(B) = 2$.

$\Rightarrow$ Rank (A) + Rank $(B) = 4$.

But Rank $(A+B) = 0$.

(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Rank $(A) = 1$ Rank $(B) = 1$

Rank (A) + Rank $(B) = 2$.

But $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ Rank $(A+B) = 2$

$=$ Rank (A) + Rank (B)

~~Let~~ $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

Rank $(A+B) \leq$ Rank (A) + Rank (B).

3) A is skew symmetric $\Rightarrow A = -A^T$

$$\det(A) = \det(-A^T)$$
$$= \det(-A)$$
$$= (-1)^n \det(A)$$

if n is odd, $(-1)^n = -1$

$$\Rightarrow \det(A) = -\det(A)$$
$$2\det(A) = 0 \Rightarrow \det(A) = 0$$

$\text{Rank}(A) \neq 5$

4). $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(a) $VV^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} [x \; y \; z]_{1\times3} = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = xyz \begin{bmatrix} x & y & z \\ x & y & z \\ x & y & z \end{bmatrix}$

$\text{Rank}(VV^T) = 1.$

(b) $V^TV = [x \; y \; z]_{1\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = x^2+y^2+z^2$ $\quad \text{Rank}(V^TV) = 1$.

5) $x - y - \lambda^2 z = 0$ common line of intersection: infinite soln.
$x - y + z = 0$ common point of intersection: unique soln.
$x + y - z = 0$

$A = \begin{bmatrix} 1 & -1 & -\lambda^2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$\begin{bmatrix} 1 & -1 & -\lambda^2 \\ 0 & 0 & 1+\lambda^2 \\ 0 & 2 & -(1+\lambda^2) \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -1 & -\lambda^2 \\ 0 & 2 & -(1+\lambda^2) \\ 0 & 0 & 1+\lambda^2 \end{bmatrix}$

Rank (A) = 3 only if $1 \cancel{+} \lambda^2 \neq 0$

since $\lambda^2 > 0$ $1+\lambda^2 \neq 0$.

$\Rightarrow$ Rank (A) = 3 $\Rightarrow$ unique soln.

~~the planes intersect~~

The planes intersect by a line only if Rank (A) < 3.

which is never possible for any $\lambda \in \mathbb{R}$.

now, to find P:

$$\cancel{\begin{bmatrix} 1 & -1 & \lambda^2 \\ 0 & 2 & -(1+\lambda^2) \\ 0 & 0 & 1+\lambda^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

~~$1+\lambda$~~ Since this is a homogeneous system, (0,0,0) will be the unique soln, i.e, P = (0,0,0).

6). $2x + y + z = 1$
$3x - y + \lambda z = 2$
$x + \mu y + z = 3$

$$A|B = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & -1 & \lambda & 2 \\ 1 & \mu & 1 & 3 \end{array}\right]$$

R_2

$R_2 \to 2R_2 - 3R_1$
$R_3 \to 2R_3 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -5 & 2\lambda-3 & 1 \\ 0 & 2\mu-1 & 1 & 5 \end{array}\right]$$

$R_3 \to 5R_3 + (2\mu-1)R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -5 & 2\lambda-3 & 1 \\ 0 & 0 & 5+(2\mu-1)(2\lambda-3) & 24+2\mu \end{array}\right]$$

$5 + (2\mu-1)(2\lambda-3)$
$25 + 2\mu - 1$

will have a common line of intersection if infinite solutions,

i.e, Rank (A) = Rank (A|B) < 3.

only if $5+(2\mu-1)(2\lambda-3) = 0$
$24 + 2\mu = 0$
$\Downarrow$
$\boxed{\mu = -12}$

$\to 5 + (\cancel{3}24-1)(2\lambda-3) = 0$
$\Rightarrow 5 + -25(2\lambda-3) = 0$
$\Rightarrow 2\lambda - 3 = \frac{5}{25} \Rightarrow 2\lambda = \frac{1}{5} + 3 = \frac{16}{5}$
$\Rightarrow \boxed{\lambda = 8/5}$

$\therefore \boxed{\lambda = 8/5,\ \mu = -12}$ will give a common line of intersection L.

$2 \times \frac{8}{5} - 3 = \frac{16}{5} - 3$
$(-24)(\frac{16}{5} - 3)$
$\cancel{(-25)}(\frac{1}{5})$

To find equation of L:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -5 & 1/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow z \in \mathbb{R}$.
$-5y + z/5 = 1$

$$\Rightarrow y = \left(\frac{1-\frac{z}{5}}{-5}\right) = \frac{5-z}{-25} = \frac{z-5}{25}$$

$$2x+y+z=1 \Rightarrow x = \frac{1-y-z}{2} = \frac{1-\frac{z-5}{25}-z}{2}$$

$$= \frac{25-z+5-25z}{50}$$

$$= \frac{30-26z}{50} = \frac{15-13z}{25}$$

Let $z=t$

$$y = \frac{t-5}{25} \qquad x = \frac{3}{5} - \frac{13}{25}t$$

$$x = \frac{15-13t}{25} \qquad y = -\frac{1}{5} + \frac{1}{25}t$$

~~$y = \frac{1}{25}$~~

7 Let the two planes be

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$

They are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

not parallel $\Rightarrow$ ~~at~~ atleast two of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ will be different.

Matrix form of the system will be

$$\underset{2\times 3}{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$A|B = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{array}\right]$$

This will be solvable if Rank (A) = Rank (A|B)

$R_2 \to a_1 R_2 - a_2 R_1$
$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & a_1 b_2 - a_2 b_1 & a_1 c_2 - a_2 c_1 & a_1 d_2 - a_2 d_1 \end{array}\right].$$

Rank (A|B) = Rank(A) if atleast one of the following holds: $a_1 b_2 - a_2 b_1 \neq 0$ or

$a_1 c_2 - a_2 c_1 \neq 0$.

i.e if $a_1 b_2 \neq a_2 b_1$ or $a_1 c_2 \neq a_2 c_1$

i.e, $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ or $\dfrac{a_1}{a_2} \neq \dfrac{c_1}{c_2}$.

This holds since the planes are not parallel.

$\therefore$ The system is solvable.

Now, since A|B and A are 2×4 and 2×3 matrices respectively, their rank will be equal to 2, which is less than the no of variables.

$\therefore$ The solution will be a line (infinite solutions)

8) $P_1 : -2x - 3y + z = 0$

$P_2 : bz = 5$

$P_3 : ay + 2z = 5.$

$$A|B = \left[\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 0 & 0 & b & 5 \\ 0 & a & 2 & 5 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 0 & a & 2 & 5 \\ 0 & 0 & b & 5 \end{array}\right].$$

(a) P_3 intersects the line joining P_1 and P_2 at a unique point is equivalent to the system having a unique solution.

This happens if Rank (A) = Rank (A|B) = ~~B~~ 3

~~This~~ i.e, if ~~$a \in \mathbb{R}$~~ $b \neq 0$, $a \neq 0$. (if $a = 0$, then $b = 2$ will not give unique soln)

(b) never ~~to~~ intersects L if the system dont have a solution.

If $b = 0$, then Rank (A) = 2 $\neq$ Rank (A|B) = 3.

$a \in \mathbb{R}$.

~~(c) contains L if P_3 intersects P_1 and P_2 with a the~~

(c) contains L if the intersection of P_1, P_2 and P_3 is the line L. This will happen if

$$\text{Rank (A)} = \text{Rank (A|B)} < 3.$$

This will happen if $a = 0$, $b = 2$.

Then the last row can be made 0.

Hence Rank (A) = 2 = Rank (A|B) < 3.