Program	**B. Tech. (SoCS)**	**Semester**	**IV**
Course	**Linear Algebra**	**Course Code**	**MATH 2059**
Session	**Jan-May 2025**	**Topic(s)**	**Vector Spaces, Subspaces, Basis, Dimension**

1. Determine whether the following subsets V_i under the given vector addition and scalar multiplication operations forms a vector space over the given field F or not. Justify your answer with giving supporting examples.

a) $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$ with vector addition and scalar multiplication defined as:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$\alpha(x_1, y_1, z_1) = (x_1, \alpha y_1, z_1)\ \forall\ \alpha \in \mathbb{R}.$$

b) $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ with vector addition and scalar multiplication defined as:

$$(x_1, y_1) + (x_2, y_2) = (0, y_1 + y_2)$$
$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)\ \forall\ \alpha \in \mathbb{R}.$$

c) $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ with vector addition and scalar multiplication defined as:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\alpha(x_1, y_1) = (\alpha x_1, 0)\ \forall\ \alpha \in \mathbb{R}.$$

2. Which of the following is/are subspace of $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$ of real numbers.

a) $W = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

b) $W = \{(x, y) \in \mathbb{R}^2 \mid 3x - 4y = 0\}$

c) $W = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{16} = 1\}$

d) $W = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$

e) $W = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$

f) $W = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$

g) $W = \{(x, y) \in \mathbb{R}^2 \mid \sin^2 x + \sin^2 y = 0\}$

h) $W = \{(x, y) \in \mathbb{R}^2 \mid \sin^2 x + \cos^2 x = 1\}$

3. Which of the following subsets is/are the subspace(s) of $V = \mathbb{R}^3$ over the field $F = \mathbb{R}$ of real numbers.

a) $W = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

b) $W = \{(x, x, x) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$

c) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$
d) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$
e) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9\}$
f) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 1\}$
g) $W = \{(x,y,z) \in \mathbb{R}^3 \mid z = 1\}$
h) $W = \{(x,y,z) \in \mathbb{R}^3 \mid y = 2x, z = -x\}$
i) $W = \{(x,2x,3x) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$
j) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$
k) $W = \{(x + y, x - y, x) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$
l) $W = \{(x + 2y, x + 1, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$
m) $W = \{(x, x + 5, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

4. Find the real values of a for which the set $\{(1,1,1 + a), (2, 2 + a, 2 + a), (3 + a, 3 + a, 3 + a)\}$ is linearly independent.

5. Show that the set of all 3×3 real symmetric matrices is a subspace of $M(3,\mathbb{R})$, the vector space of all 3×3 real square matrices over the field $F = \mathbb{R}$ of real numbers. Also, find its basis and dimension.

6. Show that the set of all 3×3 real skew-symmetric matrices is a subspace of $M(3,\mathbb{R})$, the vector space of all 3×3 real square matrices over the field $F = \mathbb{R}$ of real numbers. Also, find its basis and dimension.

7. Consider the subspaces W_1 and W_2 of vector space $\mathbb{R}^4$ as follows:

$$W_1 = \{(x_1, x_2, x_3, x_4) \in V \mid x_2 + x_3 + x_4 = 0\},$$
$$W_2 = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 = 0, x_3 = 2x_4\}.$$

a) Express explicitly the subspace $W_1 \cap W_2$.
b) Find a basis and dimension of W_1, W_2 and $W_1 \cap W_2$. Also, find the dimension of $W_1 + W_2$.

8. Find the dimension of the following subspaces of the vector space $V = \mathbb{R}^4$.

a) $W_1 = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_3 + x_4 = 0, x_2 + x_3 + x_4 = 0\}$
b) $W_2 = \{(x_1, x_2, x_3, x_4) \in V \mid x_1 + x_2 - x_3 + x_4 = 0, x_1 + x_2 + 2x_3 = 0, x_1 + 3x_2 = 0\}$

9. Consider the subspaces W_1 and W_2 of vector space $V = M(2,\mathbb{R})$ as follows:

$$W_1 = \left\{ \begin{pmatrix} a & -a \\ c & d \end{pmatrix} \middle| \ a, c, d \in \mathbb{R} \right\},$$
$$W_2 = \left\{ \begin{pmatrix} a & -b \\ -a & d \end{pmatrix} \middle| \ a, b, d \in \mathbb{R} \right\}.$$

a) Express explicitly the subspace $W_1 \cap W_2$.
b) Find a basis and dimension of W_1, W_2 and $W_1 \cap W_2$. Also, find the dimension of $W_1 + W_2$.

10. Find the dimension of the subspace $W = \left\{[a_{ij}]_{10\times 10} \,\middle|\, a_{ij} = 0 \text{ if } i \text{ is even}\right\}$ of the vector space $V = M(10, \mathbb{R})$ over the field $F = \mathbb{R}$ of real numbers.

11. Show that the subset $W = \left\{[a_{ij}]_{3\times 3} \,\middle|\, a_{ij} = a_{ji},\ a_{11} = 0,\ trace(A) = 0\right\}$ is a subspace of the vector space $V = M(3, \mathbb{R})$ over the field $F = \mathbb{R}$ of real numbers. Also, find the dimension of W.

12. Show that the subset $W = \{p(x) \in P_5(\mathbb{R}) \mid p(1) = p'(2) = 0\}$ is a subspace of the vector space $V = P_5(\mathbb{R})$, the vector space of all real polynomials of degree ≤ 5, over the field $F = \mathbb{R}$ of real numbers. Also, find the dimension of W.

13. Let $V = P(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V, where

a) W consists of all polynomials with integral coefficients.
b) W consists of all polynomials with degree ≥ 6, and the zero polynomial.
c) W consists of all polynomials with only even powers of t.

14. Let W be a subspace of $\mathbb{R}^5$ spanned by $u_1 = (1,2,-1,3,4)$, $u_2 = (2,4,-2,6,8)$, $u_3 = (1,3,2,2,6)$, $u_4 = (1,4,5,1,8), u_5 = (2,7,3,3,9)$. Find a basis and dimension of W.