System of Linear Equations

General form:

$$(A)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Variables $\rightarrow x_1, x_2, \ldots x_n$

Coefficients $\rightarrow a_{11}, a_{12}, \ldots a_{mn}$

Constants $\rightarrow b_1, b_2, \ldots b_m$

(A) is a system of m-linear equations in 'n' variables. Any n-tuple $x = (x_1, x_2, \ldots x_n)$ which satisfies each of the equations in (A) is called a solution.

A system of linear equation has,

1) No solution (or) $\longrightarrow$ Inconsistent
2) Exactly one solution, (or)
3) Infinitely many solutions.

$\left.\begin{matrix}\text{2)}\\ \text{3)}\end{matrix}\right\} \longrightarrow$ Consistent.

Matrix Notation

The system (A) can be written as,

$$AX = B, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If $B = 0$, then the system is Homogeneous,

otherwise, i.e. if $B \neq 0$, then the system is Non-homogeneous.

Elementary Row operations

1) Interchange two rows

Eg:- $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$$R_1 \longleftrightarrow R_2$$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

2) Multiply all entries in a row by a non-zero constant (c)

Eg:- $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$R_1 \longrightarrow 2R_1 \qquad A = \begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$$

3) Replace one row by the sum of itself and a multiple of another row.

Eg :-
$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$R_1 \longrightarrow R_1 + 2R_2$$

$$A = \begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix}$$

Elimination method

i)
$$x_1 + 2x_2 = 4$$
$$2x_1 - x_2 = 3$$

$$Ax = B \Rightarrow A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Convert A into an upper triangular matrix.

$$R_2 \longrightarrow R_2 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$$

Perform the same row operation on B.

$$\begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$x_1 + 2x_2 = 4$$

$$-5x_2 = -5 \Rightarrow x_2 = 1$$

$$x_1 + 2 \times 1 = 4 \Rightarrow x_1 = 2$$

Solution $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Echelon form

A rectangular matrix is in Echelon form if it has the following properties:

1) All ~~non~~ zero rows are below the non-zero rows.
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3) All Entries in a column below a leading entries are zeros.

Eg:- $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rank of a matrix

The order of the largest submatrix with non-zero determinant.

The no. of non-zero rows in the row-Echelon form.

Eg :-

1) $$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

A is not in Echelon form.

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Echelon form of } A.$$

No. of non-zero rows $= 3 = \text{rank}(A)$
$= \rho(A)$

2) $$A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & -2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

A is not in Echelon form.

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - R_1$.

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & -10 & -4 \\ 0 & -4 & -1 \end{bmatrix}$$

$-4 - \frac{-4}{10}$

$-1 + \frac{4 \cdot 4}{10}$

$-1 + \frac{16}{10}$

$R_3 \longrightarrow R_3 - \frac{4}{10} R_2$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & -10 & -4 \\ 0 & 0 & 6/10 \end{bmatrix} \Rightarrow \rho(A) = 3$$

5)
$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 4 & 8 & 3 \\ 2 & 6 & 0 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 2R_1$
$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -2 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{r(A) = 2}}$$

Solution of <u>Homogeneous and Non-homogeneous</u> System

1) <u>Homogeneous</u> $AX = 0$:-

Let A be an $m \times n$ matrix.

→ Always consistent ($X = 0$ is a solution)

1) Unique solution if and only if $r(A) = n$ = no. of variables or no. of columns in A.

2) Infinite no. of solution if ~~r(A) < n~~ ($r(A) \neq n$)

2) <u>Non-homogeneous</u> $AX = B$:-

1) If $r(A) = r(A|B) = n$, unique solution
2) If $r(A) = r(A|B) \neq n$, infinite no. of solution } Consistent
3) If $r(A) \neq r(A|B)$, no solution (Inconsistent)

→ If A is an $m \times n$ matrix, then,

$$\rho(A) \leq \min\{m, n\}.$$

→ If A is a square matrix ($n \times n$ matrix), then $\rho(A) = n$ if and only if A is invertible (i.e., $\det(A) \neq 0$).

$$\det(A) = 0 \Longleftrightarrow \rho(A) \neq n.$$

1) Solve

$$x + 3y = 1$$
$$2x + 4y + 8z = 3$$
$$2x + 6y = 7$$

Matrix form:-

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 8 \\ 2 & 6 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

$B \neq 0$, Non-homogeneous.

We have to check the $\rho(A)$ and $\rho(A|B)$.

$$A|B = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 4 & 8 & 3 \\ 2 & 6 & 0 & 7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -2 & 8 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$\rho(A) = 2$

$\rho(A|B) = 3$ $\quad\quad \rho(A) \neq \rho(A|B)$

No solution.

2) $x + 3y = 1$

$2x + 4y + 8z = 3$

$2x + 6y = 2$

$$A|B = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 4 & 8 & 3 \\ 2 & 6 & 0 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -2 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = \rho(A|B) = 2.$

Infinite no. of solutions.

$x + 3y = 1 \quad \Rightarrow x = 1 - 3y$

$-2y + 8z = 1 \quad \Rightarrow 8z = 1 + 2y$

$z = \frac{1+2y}{8}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-3y \\ y \\ \frac{1+2y}{8} \end{bmatrix}$$

$y = 1 \quad X = \begin{bmatrix} -2 \\ 1 \\ 3/8 \end{bmatrix}$ is a solution.