- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a matrix A, then eigenvalues of any polynomial of A, i.e, $p(A)$ will be $p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)$.

eg: if eigenvalues of A are $1, 2, 3$,

eigenvalues of A^2+2A-I will be ~~$1^2+2.1-1$~~

$1^2+2.1-1,\ 2^2+2.2-1,\ 3^2+2.3-1$.

i.e, $\underline{\underline{2, 7, 14}}$.

- eigenvalues of a real <u>symmetric</u> matrix are always real.
- If A is diagonalizable, then rank (A) = number of non zero eigenvalues of A.
- <u>Functions of matrices</u>

If A is diagonalizable, $A = PDP^{-1}$.

Then $A^n = PD^nP^{-1}$ (any number n)

eg: $A^3 = PD^3P^{-1}$

$A^{1/2} = PD^{1/2}P^{-1}$

$e^A = \cancel{A e}\ P e^D P^{-1}$

$\sin(A) = P \sin(D) P^{-1}$

Practice set 2

(7) $\det(A) = 18$, $tr(A) = -2$, where A is 3×3 matrix

Let eigenvalues of A be $\lambda_1, \lambda_2, \lambda_3$.

then $\lambda_1 . \lambda_2 . \lambda_3 = 18$

$\lambda_1 + \lambda_2 + \lambda_3 = -2$

$\det(A + 3I) = 0 \Rightarrow -3$ is an eigenvalue.

$\therefore \lambda_3 = -3$

$\lambda_1 . \lambda_2 = -6$

$\lambda_1 + \lambda_2 = 1 \Rightarrow \lambda_1 = -2, \lambda_2 = 3$

$\therefore \lambda_1 = -2, \lambda_2 = 3, \lambda_3 = -3$.

eig values of $A^2 - 2A = (-2)^2 - 2(-2), 3^2 - 2(3), (-3)^2 - 2(-3)$

$= 4 + 4, 9 - 6, 9 + 6$

$= \underline{8, 3, 15}$.

$tr(A^2 - 2A) = 8 + 3 + 15 = \underline{26}$.

(8) char poly of $A = (x-2)(x+1)^2(x+3)^2$

eig values of $A = 2, -1, -1, -3, -3$

eig values of $(A+3I) = 5, 2, 2, 0, 0$

rank $(A+3I) = 3$ (no of non zero eigenvalues).

(9) $B = I + A + A^2 + \cdots + A^{10}$ $\quad (A : 2\times 2)$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D$$

$$\Rightarrow \quad P^{-1}BP = P^{-1}P + P^{-1}AP + (P^{-1}AP)^2 + \cdots + (P^{-1}AP)^{10}$$

$$= I + D + D^2 + \cdots + D^{10}$$

$$= \begin{bmatrix} 1 + \underbrace{1 + 1 + \cdots + 1}_{10 \text{ times}} & 0 \\ 0 & \underbrace{1 + 2 + 2^2 + \cdots + 2^{10}}_{GP} \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 0 & \dfrac{1.(2^{11}-1)}{2-1} \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 0 & 2^{11}-1 \end{bmatrix}$$

$\therefore$ $\operatorname{tr}(P^{-1}BP) = 11 + 2^{11} - 1 = \underline{10 + 2^{11}}$

given $\operatorname{tr}(P^{-1}BP) = \alpha + \beta 2^{11}$

$$\Rightarrow \alpha = 10, \; \beta = 1$$

12) A: 3x3 symmetric matrix.

eig values : 2, 3, 6

eig vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

make this orthogonal (since A is symmetric)

dot product is zero.

make magnitude of columns 1.

$$\therefore \quad P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$P^T A P = D \Rightarrow A = \underbrace{P D P^T}$$

calculate this to get A.

(14) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(2-\lambda)\right) + 1\,(\lambda - 1)$$

$$= \cancel{2}\,(1-\lambda)\left[(2-\lambda)^2 - 1\right]$$

$$= (1-\lambda)\left(\cancel{2}\,4 - 4\lambda + \lambda^2 - 1\right)$$

$$= (1-\lambda)(\lambda^2 - 4\lambda + 3)$$

$$= \lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0.$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

From Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (1)}$$

now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 7A^2 + A^2 - 3A + A + I$$

$$= A^5(0) + A(\underbrace{A^3 - 5A^2 + 7A - 3I}_{=0}) + A^2 + A + I$$

$$= \underline{A^2 + A + I} \quad \text{(calculate this using } A\text{)}.$$

15) $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$|A - \lambda I| = 0.$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow 2 - 3\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 2 = 0$ —— (1).

$\lambda = \dfrac{3 \pm \sqrt{9+8}}{2} = \dfrac{3 \pm \sqrt{17}}{2}$ are the roots.

Cayley Hamilton theorem:

verify $A^2 - 3A - 2I = 0.$

now, $2A^4 - 5A^3 - 7A + 6I$

$= 2A^4 - 6A^3 + A^3 - 4A^2 - 4A^2 - 7A + 6I.$

$= 2A^2(A^2 - 3A - 2I) + A^3 - 4A^2 - 7A + 6I.$

$= \overset{\shortparallel}{0} + A^3 - 3A^2 + A^2 - 2A - 5A + 6I.$

$= A(A^2 - 3A - 2I) + A^2 - 5A + 6I.$

$= 0 + A^2 - 3A - 2A - 2I + 8I.$

$= A^2 - 3A - 2I - 2A + 8I.$

$= 0 - 2A + 8I$ (~~take~~ calculate this).

13) The given matrices are 3x3.

Find char poly and verify Cayley Hamilton.

You will get a polynomial of the form,

$$aA^3 + bA^2 + cA + dI = 0$$

To find A^{-1}, multiply throughout by A^{-1},

$$aA^2 + bA + cI + dA^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{d}\left[\underbrace{-aA^2 - bA - cI}_{\text{calculate this.}}\right]$$

to find A^4, multiply throughout by A,

$$aA^4 + bA^3 + cA^2 + dA = 0.$$

$$A^4 = \frac{1}{a}\left[\underbrace{-bA^3 - cA^2 - dA}_{\text{calculate this.}}\right]$$