Probability Theory

Events and Sample Space

- Sample Space:
  - for a procedure Sample Space consists of all possible simple events; that is, the sample space consists of all outcomes that cannot be broken down any further
- Event
  - any collection of results or outcomes of a procedure
- Simple Event
  - an outcome or an event that cannot be further broken down into simpler components

- Sample space Ω - set of all possible outcomes of a random experiment
  – Dice roll: {1, 2, 3, 4, 5, 6}
  – Coin toss: {Tails, Heads}
- Event space $\mathcal{F}$ - subsets of elements in a sample space
  – Dice roll: {1, 2, 3} or {2, 4, 6}
  – Coin toss: {Tails}

Events and Sample Space

- A pair of dice are rolled. The sample space has 36 simple events:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

where the pairs represent the numbers rolled on each dice.

- Which elements of the sample space correspond to the event that the sum of each dice is 4?

Probability

- The word 'Probability' means the chance of occurring of a particular event.
- It is generally possible to predict the future of an event quantitatively with a certain probability of being correct.
- The probability is used in such cases where the outcome of the trial is uncertain.

$$P(A) = \frac{\text{number of cases favourable to A}}{\text{number of possible outcomes}}$$

- P - denotes a probability.
- A, B, and C - denote specific events.
- P(A) - denotes the probability of event A occurring.

Probability

- Probability of an Event Defined over (Ω, F) s.t.
  - 0 < P(a) <1 for all a in F
  - P(Ω) = 1
- Probability of an event which is certain to occur is **one**.
- Probability of an event which is impossible to **zero**.
- If the probability of happening of an event P(A) and that of not happening is P(A'), then P(A) + P(A') = 1,

  where, $0 \leq P(A) \leq 1$, $0 \leq P(A') \leq 1$.

Event Relations

- **Equally Likely Events:** Events are said to be equally likely if one of them cannot be expected to occur in preference to others. In other words, it means each outcome is as likely to occur as any other outcome.
  - *Example:* When a die is thrown, all the six faces, i.e., 1, 2, 3, 4, 5 and 6 are equally likely to occur.
- **Mutually Exclusive or Disjoint Events:** Events are called mutually exclusive if they cannot occur simultaneously.
  - *Example:* Suppose a card is drawn from a pack of cards, then the events getting a jack and getting a king are mutually exclusive because they cannot occur simultaneously.

Event Relations

- **Exhaustive Events:** The total number of all possible outcomes of an experiment is called exhaustive events.
  - Example: In the tossing of a coin, either head or tail may turn up. Therefore, there are two possible outcomes. Hence, there are two exhaustive events in tossing a coin.
- **Dependent Event:** Events are said to be dependent if occurrence of one affect the occurrence of other events.
- **Independent Events:** Events A and B are said to be independent if the occurrence of any one event does not affect the occurrence of any other event.

$$P\ (A \cap B) = P\ (A)\ P\ (B).$$

Event Relations

Example: A coin is tossed thrice, and all 8 outcomes are equally likely

A: "The first throw results in heads."

B: "The last throw results in Tails."

Prove that event A and B are independent.

Solution:

Sample Space: [HHH, HHT, HTH, THH, TTT, TTH, THT, HTT]

A: [HHH, HHT, HTH, HTT]

B: [HHT, TTT, THT, HTT]

A∩B: [HHT, HTT]

$P(A) = \frac{4}{8} = \frac{1}{2}$ $P(B) = \frac{4}{8} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{2} \text{x} \frac{1}{2} = \frac{1}{4}$

Event Relations

Theorem1: If A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Proof: Let the n=total number of exhaustive cases

n_1 = number of cases favorable to A.

n_2 = number of cases favorable to B.

Now, we have A and B two mutually exclusive events. Therefore, $n_1 + n_2$ is the number of cases favorable to A or B.

$$P(A \cup B) = \frac{\text{favorable cases}}{\text{Total number of exhaustive cases}} = \frac{n_1 + n_2}{n} = \frac{n_1}{n} + \frac{n_2}{n}$$

But we have, $P(A) = \frac{n_1}{n}$ and $P(B) = \frac{n_2}{n}$

Hence, $P(A \cup B) = P(A) + P(B)$.

Event Relations

Theorem2: If A and B are two events that are not mutually exclusive, then

$P(A \cup B)=P(A)+P(B)- P(A\cap B)$.

Proof: Let n = total number of exhaustive cases

n_1=number of cases favorable to A

n_2= number of cases favorable to B

n_3= number of cases favorable to both A and B

But A and B are not mutually exclusive. Therefore, A and B can occur simultaneously. So,$n_1+n_2-n_3$ is the number of cases favorable to A or B.

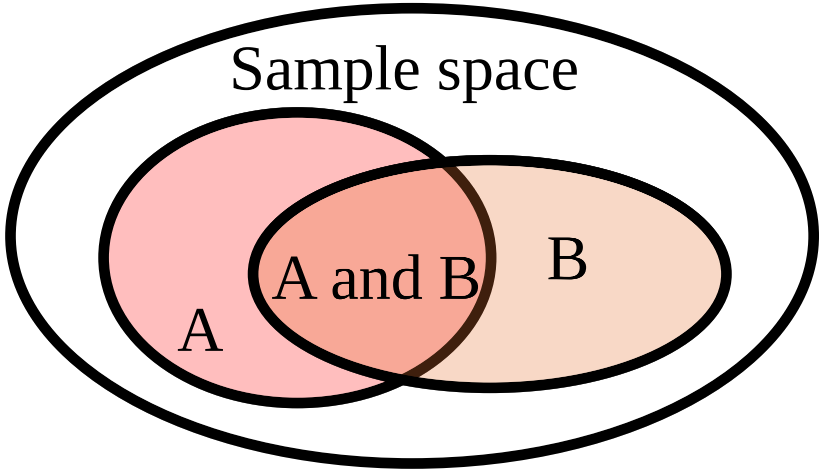
Therefore, $P(A \cup B)=\frac{n_1+n_2-n_3}{n}=\frac{n_1}{n}+\frac{n_2}{n}-\frac{n_3}{n}$

But we have, $P(A)=\frac{n_1}{n}$, $P(B)=\frac{n_2}{n}$ and $P(A\cap B)=\frac{n_3}{n}$

Hence, $P(A \cup B)=P(A)+P(B)- P(A\cap B)$.

Conditional Probability

- The probability of an event A based on the occurrence of another event B is termed conditional Probability. It is denoted as **P(A|B)** and represents the probability of A when event B has already happened.
- $P(A \mid B) = P(A \cap B) / P(B)$
- If the two events are independent:
  - $P(A \cap B) = P(A) * P(B)$
  - $P(A/B) = P(A)$



Joint Probability:

The probability of two more events occurring together and at the same time is measured it is termed as Joint Probability.
Joint probability for two events A and B is denoted as, $P(A \cap B)$.

Bayes Theorem

- Bayes theorem, also known as the Bayes Rule, is used to determine the conditional probability of event A when event B has already happened.
- "The conditional probability of an event A, given the occurrence of another event B, is equal to the product of the probability event of B given A and the probability of A divided by the probability of event B." i.e.

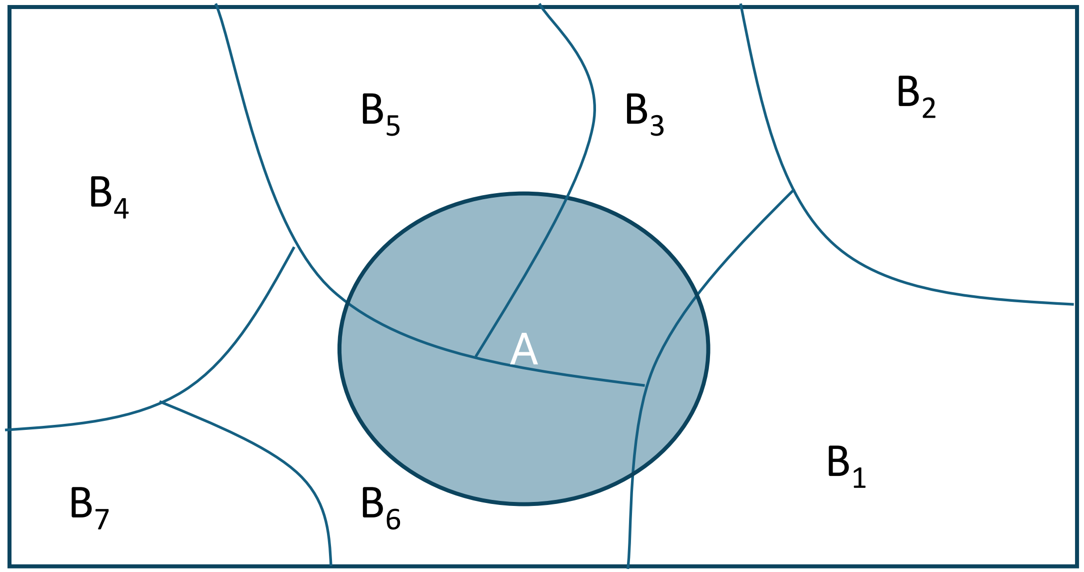
$$P(A|B) = P(B|A)P(A) / P(B) \quad \text{given} \quad P(B) \neq 0$$

- where,
  - P(A) and P(B) are the probabilities of events A and B
  - P(A|B) is the probability of event A when event B happens
  - P(B|A) is the probability of event B when event A happens

Theorem of Total Probability

- Let E1, E2,.............En are mutually exclusive and exhaustive events associated with a random experiment and lets E be an event that occurs with some Ei.
- Then,

$$P(E) = {}^{n}\sum_{i=1} P(E/E_i) \,.\, P(E_i)$$



$$p(A) = \sum P(B_i)P(A|B_i)$$

Questions

Q1. Two dice are thrown. The events A, B, C, D, E, F

A = getting even number on first die.

B= getting an odd number on the first die.

C = getting a sum of the number on dice ≤ 5

D = getting a sum of the number on dice > 5 but less than 10.

Show that:

1. A, B are a mutually exclusive event and Exhaustive Event.
2. A, C are not mutually exclusive.
3. C, D are a mutually exclusive event but not Exhaustive Event.
4. $A' \cap B'$ are a mutually exclusive and exhaustive event.

Questions

Q2. A bag contains 5 green and 7 red balls. Two balls are drawn. Find the probability that one is green and the other is red.

Q3. Find the probability of drawing a heart on each of two consecutive draws from well shuffled-packs of cards if the card is not replaced after the draw.

Q4. "$X+Y=6$ or $X+Y=7$" – given this (and only this), what is the probability of $Y=5$?

Q5. There are three urns containing 3 white and 2 black balls; 2 white and 3 black balls; 1 black and 4 white balls respectively. There is an equal probability of each urn being chosen. One ball is equal probability chosen at random. what is the probability that a white ball is drawn?