

Covariance

Covariance:

Definition:

Covariance

For two jointly distributed real-valued random

variables X and Y with finite second moments

i.e., $E(X^2) < \infty$, the covariance is defined as

$$Cov(X,Y)=E[(X-\overline{X})(Y-\overline{Y})]$$

The covariance is also sometimes

denoted by σ_{XY} or $\sigma(X,Y)$.

2nd form:

$$Cov(X,Y)$$

$$= E[(X - \bar{X})(Y - \bar{Y})]$$

$$= E[XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}]$$

$$= E(XY) - E(X\bar{Y}) - E(\bar{X}Y) + E(\bar{X}\bar{Y})$$

$$= E(XY) - \bar{Y}E(X) - \bar{X}E(Y) + \bar{X}\bar{Y}E(1)$$

$$= E(XY) - \bar{Y}\bar{X} - \bar{X}\bar{Y} + \bar{X}\bar{Y}$$

$$= E(XY) - E(X)E(Y)$$

Properties:

Property 1: Covariance with itself.

For a random variable X;

$$Cov(X,X) = Var(X)$$

Proof: By definition,

$$Cov(X,X) = E(XX) - E(X)E(X)$$
$$= E(X^2) - [E(X)]^2$$
$$= Var(X)$$

i.e., the variance is a special case of the covariance in which the two variables are identical

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Property 2: For real-valued random variable X and a is real-valued constant then Cov(X,a) = 0

Proof: By definition,

$$Cov(X,a) = E(Xa) - E(X)E(a)$$
$$= aE(X) - aE(X)$$
$$= 0$$

Property 3: Covariance is symmetric

If X, Y are real-valued random variables then

$$Cov(X,Y) = Cov(Y,X)$$

Proof: By definition,

$$Cov(X,Y)$$

$$= E(XY) - E(X)E(Y)$$

$$= E(YX) - E(Y)EX(X)$$

$$= Cov(Y,X)$$

Property 4: If X, Y, Z are real-valued random variables then Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)**Proof:** By definition Cov(X+Y,Z)=E[(X+Y)Z]-E(X+Y)E(Z)= E(XZ + YZ) -[E(X) + E(Y)]E(Z)= E(XZ) - E(X)E(Z)+E(YZ)-E(Y)E(Z)

= Cov(X,Z) + Cov(Y,Z)

= ab[E(XY) - E(X)E(Y)]

= abCov(X,Y)

Property 5: If X, Y are real-valued random variables and a, b are real-valued constants then

$$Cov(aX,bY) = abCov(X,Y) : Cov(X + a,Y + b) = Cov(X,Y)$$

$$Proof: By definition,$$

$$Cov(aX,bY)$$

$$= E[(aX)(bY)] - E(aX)E(bY)$$

$$= E(abXY) - aE(X)bE(Y)$$

$$= E(xY + bX + aY + ab) - [E(xY + a)]E(YY + bX + aY + ab] - [E(xY + a)]E(YY + bX + aY + ab] - [E(xY + a)]E(YY + aBXY + a$$

Property 6: For real-valued random variables X, Y and real-valued constants a, b, c, d we have Cov(aX + bY, cX + dY) = acVar(X) + bdVar(Y) + (ad + bc)Cov(X, Y)Proof: Cov(aX + bY, cX + dY)= E[(aX + bY)(cX + dY)] - E(aX + bY)E(cX + dY) $= E[acX^{2} + bdY^{2} + (ad + bc)XY] - [aE(X) + bE(Y)][cE(X) + dE(Y)]$ $= ac \left[E(X^2) - \left(E(X) \right)^2 \right] + bd \left[E(Y^2) - \left(E(Y) \right)^2 \right]$ +(ad+bc)[E(XY)-E(X)E(Y)]= acVar(X) + bdVar(Y) + (ad + bc)Cov(X,Y)

Property 8: Covariance Relationship to inner products

Many properties of the covariance are similar to the inner product

1) Symmetric:

$$Cov(X,Y) = Cov(Y,X)$$

2) Positive semi-definite:

$$Cov(X,X) = Var(X) \ge 0$$
 and $Cov(X,X) = 0$ implies that X is constant almost surely.

3) Bilinear:

For constants a, b and random variables X, Y, Z, Cov(aX + bY, Z) = aCov(X, Z) + bCov(Y, Z)

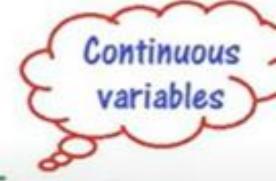
Covariance Questions:

Let X and Y be two random variables

X Y	-1	0	1
-8	0	1,0	1.0
0	0.2	0.2	0.2
1	0	1.0	1.0

Prove that they are uncorrelated.





$$f(x,y) = \begin{cases} 8xy & ; 0 \le y \le x \le 1 \\ 0 & ; elsewhere \end{cases}$$

Find the covariance of X and Y.

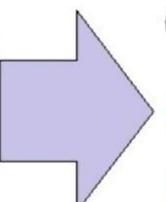
Covariance Questions:

Two Steps Approach:

Cov(X,Y) = E(XY) - E(X)E(Y)

Step 1:

Find Marginal density of X and Y



For discrete variables

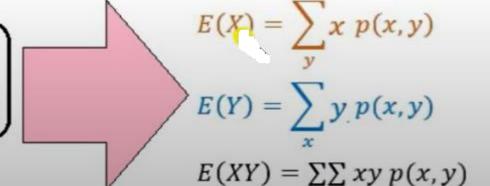
$$p(x) = \sum_{y} p(x, y)$$

$$p(y) = \sum_{x} p(x, y)$$

Step 2:

Find expectation of

X, Y and XY



For continuous variables

$$f(x) = \int f(x, y) dy$$

$$f(y) = \int f(x, y) dx$$

$$E(X) = \int x f(x) dx$$

$$E(Y) = \int y f(y) dy$$

$$E(XY) = \int \int xy f(x,y) dy dx$$

Example: Let X and Y be two random variables each taking three values -1,0,1 and having the joint

probability distribution is

Prove that X and Y have different expectations

and they are uncorrelated.

Solution:

CON(X) = 0

X	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

(i)
$$E(X) + E(Y)$$

Solution:

Marginal of X is x - 1 = 0

Marginal of Y is

$$\begin{array}{c|ccccc} y & -t & 0 & 1 \\ \hline p(y) & & & & \end{array}$$

XY	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

Marginal of X is

\boldsymbol{x}	-1	0	1
p(x)	0.2	0.6	0.2

$$E(X) = \sum xp(x)$$

$$= (-1)(0.2) + (0)(0.6)$$

$$+ (1)(0.2)$$

$$= 0$$

Marginal of Y is

$$E(Y) = \sum yp(y)$$

$$= (-1)(0.2) + (0)(0.4)$$

$$+ (1)(0.4)$$

$$= 0.2$$

Target is to prove
$$Cov(X,Y) = 0$$

i.e., $E(XY) - E(X)E(Y) = 0$
$$E(XY) = \sum \sum xyp(x,y)$$
$$= (-1)(-1)(0)$$

XY	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

$$E(XY) = \sum \sum xyp(x,y)$$

$$= (-1)(-1)(0) + (-1)(0)(0.1) + (-1)(1)(0.1)$$

$$+ \cdots \dots + (1)(0)(0.1) + (1)(1)(0.1)$$

$$= 0$$

Thus,

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0 - 0(0.2)$$

$$= 0$$

Hence, X and Y are uncorrelated.

Example...

Example: The joint probability mass function of X and Y	X	-1	1
is given below: Find the <mark>covariance</mark> of (X, Y).	0	1/8	3/8
	1	2/8	2/8

Solution:

The marginal of X is

$$\begin{array}{c|cccc} x & 0 & 1 \\ \hline p(x) & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$E(X) = \sum xp(x)$$

$$= (0)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right)$$

$$= 1/2$$

The marginal of Y is

$$\begin{array}{c|cccc} y & -1 & 1 \\ \hline p(y) & 3/8 & 5/8 \end{array}$$

$$E(Y) = \sum yp(y)$$

$$= (-1)\left(\frac{3}{8}\right) + (1)\left(\frac{5}{8}\right)$$

$$= 1/4$$

Solution....

$$E(XY) = \sum xyp(x,y)$$

$$= (0)(-1)\left(\frac{1}{8}\right) + (0)(1)\left(\frac{3}{8}\right)$$

$$+(1)(-1)\left(\frac{2}{8}\right) + (1)(1)\left(\frac{2}{8}\right)$$

$$= 0$$

Hence,
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

$$= -\frac{1}{8}$$

Example: Let X and Y be two random variables each taking three values

- 1, 0, 1 and having the joint probability distribution

Prove that X and Y are uncorrelated.

XY	-1	0	1
-1	2/16	1/16	2/16
0	2/16	2/16	2/16
1	2/16	1/16	2/16

Solution: The marginal of X is

$$E(X) = \sum xp(x) = -\frac{5}{16} + 0 + \frac{5}{16} = 0$$

The marginal of Y is

$$E(Y) = \sum yp(y) = -\frac{6}{16} + 0 + \frac{6}{16} = 0$$

Solutions

$$E(XY) = \sum xyp(x, y)$$

$$= (-1)(1)\left(\frac{2}{16}\right) + (-1)(0)\left(\frac{1}{16}\right)$$

$$+ \cdots \cdots + (1)(1)\left(\frac{2}{16}\right)$$

$$= \frac{2}{16} - \frac{2}{16} - \frac{2}{16} + \frac{2}{16}$$

$$= 0$$

$$\therefore Cov(X, Y) = E(XY) - E(X)E(Y) = 0$$
Thus, X and Y are uncorrelated.

Example: If X and Y are two independent random variables with means 5 and 10 and standard

deviations 2 and 3 respectively. Find the covariance between 3X + 4Y and 3X - Y.

Solution: Given that E(X) = 5; E(Y) = 10; Var(X) = 4; Var(Y) = 9

Let U = 3X + 4Y and V = 3X - Y

E(U) = 3E(X) + 4E(Y) = 15 + 40 = 55

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Target
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$$Cov(U,V) = E(UV) - E(U)E(V)$$

$$E(XY) = E(X)E(Y)$$

= 5X10
= 50

$$E(V) = 3E(X) - E(Y) = 15 - 10 = 5$$

$$E(UV) = E(9X^{2} + 9XY - 4Y^{2})$$

$$E(UV) = E(Y) + 9E(XY) - 4E(Y^{2})$$

$$E(X) = E(X) - E(X) + 9E(XY) - 4E(Y^{2})$$

$$E(X) = E(X) - E(X) + 9E(XY) - 4E(Y^{2})$$

$$E(X) = 247 - (55)(5) = 247$$

$$E(X) = 247 - (55)(5) = 247$$

Example: If X, Y and Z are uncorrelated random variables with zero means and Standard deviations 5, 12 and 9 respectively; U = X + Y and V = Y + Z. Find the contract between U and V.

E(UV)

Solution: Given that X, Y and Z are uncorrelated random variable

$$Cov(X,Y) = 0$$
; $Cov(X,Z) = 0$; $Cov(Y,Z) = 0$
 $E(X) = E(Y) = E(Z) = 0$;
 $Var(X) = 25$; $Var(Y) = 144$; $Var(Z) = 81$

$$E(U) = E(X + Y)$$

$$= E(X) + E(Y)$$

$$= 0$$

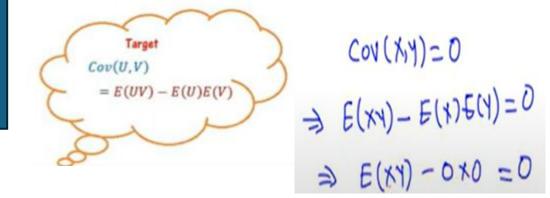
$$E(V) = E(Y + Z)$$

$$= E(Y) + E(Z)$$

$$= 0$$

$$Aos(A) = E(A_5) - D$$

 $Aos(A) = E(A_5) - E[AD]_5$



$$= E[XY + XZ + Y^2 + YZ]$$

= $E(XY) + E(XZ) + E(Y^2) + E(YZ)$

$$= 0 + 0 + E(Y^{2}) + 0$$
because $Cov(X,Y) = 0$ implies $E(XY) = E(X)E(Y)$

$$= 0 \times 0$$

$$= 0$$

$$Cov(U,V) = E(UV) - E(U)E(V)$$

$$= 144 - 0$$

$$= 144$$

Example: Two independent random variables X and Y have p.d.f's defined by

$$f(x) = \begin{cases} 4ax \ ; \ 0 \le x \le 1 \\ 0 \ ; elsewhere \end{cases} \quad \text{and} \quad f(y) = \begin{cases} 4by \ ; \ 0 \le y \le 1 \\ 0 \ ; elsewhere \end{cases}.$$

Show that X + Y and X - Y are uncorrelated.

Solution: Take U = X + Y; V = X - Y

For p.d.f. X, we have

$$\int_0^1 f(x)dx = 1$$

$$\Rightarrow 4a \int_0^1 x dx = 1$$

$$\Rightarrow 2a = 1$$

$$\Rightarrow a = \frac{1}{2}$$

For p.d.f. Y, we have

$$\int_0^1 f(y)dy = 1$$

$$\Rightarrow 4b \int_0^1 y dy = 1$$

$$\Rightarrow 2b = 1$$

$$\Rightarrow b = \frac{1}{2}$$

Target

To show U and V are uncorrelated,

it means
$$Cov(U,V)=0$$

$$Cov(U,V) = E(UV) - E(U)E(V)$$

$$f(x) = 2x ; 0 \le x \le 1$$

$$f(x) = 2x ; 0 \le x \le 1$$

$$f(y) = 2y ; 0 \le y \le 1$$

$$=E(X+Y)$$

$$= E(X) + E(Y)$$

$$= \int_0^1 x f(x) dx + \int_0^1 y f(y) dy$$

$$=2\int_{0}^{1}x^{2}dx+2\int_{0}^{1}y^{2}dy$$

$$=\frac{2}{3}+\frac{2}{3}$$

$$=\frac{4}{3}$$

$$= E(X - Y)$$

$$= E(X) - E(Y)$$

$$= \int_0^1 x f(x) dx - \int_0^1 y f(y) dy$$

$$=2\int_{0}^{1}x^{2}dx-2\int_{0}^{1}y^{2}dy$$

$$=\frac{2}{3}-\frac{2}{3}$$

$$= 0$$

E(UV)

$$=E(X^2-Y^2)$$

$$= E(X^2) - E(Y^2)$$

$$= \int_0^1 x^2 f(x) dx - \int_0^1 y^2 f(y) dy$$

$$=2\int_{0}^{1}x^{3}dx-2\int_{0}^{1}y^{3}dy$$

$$=\frac{2}{4}-\frac{2}{4}$$

$$= ($$

$$Cov(U,V) = E(UV) - E(U)E(V)$$
$$= 0 - \frac{4}{3}(0) = 0$$

Hence, U and V are uncorrelated.