Explicit Finite-Difference Methods to Solve 1D Linear Transport Equation for a Gaussian Distribution

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Linear transport theory deals with the study of equations describing the migration of particles or energy within a host medium. This report examines the one-dimensional transport of a Gaussian distribution with a mean of 0 and a standard deviation of 1 as the initial condition. The exact solution to the transport equation was first calculated. Then, numerical solutions were evaluated using three explicit finite-difference methods – 1st order upwind, Lax-Wendroff, and 4th order Runge-Kutta. The numerical solutions were calculated at different points of time for two different Courant numbers – 0.999 (close to the stability limit) and 0.5 (below the stability limit). The comparison of the numerical solutions to the exact solutions showed that the finite-difference methods were more accurate closer to the stability limit ($\bar{c} \approx 1$). Moreover, it was found that the finite-difference solutions lagged behind the actual transport of the Gaussian distribution at a lower Courant number. Furthermore, dispersion was observed at a lower Courant number with time.

Nomenclature

ϕ	= Scalar density function
x	= Spatial domain, m
x_{min}	= Lower boundary of the spatial domain, m
x_{max}	= Upper boundary of the spatial domain, m
t	= Time domain, s
q	= Flux function, m^{-1}
c	= Characteristic velocity, m/s
$\bar{c}$	= Courant number
μ	= Mean
σ	= Standard deviation

I. Introduction

Linear transport theory deals with the study of equations describing the migration of particles or energy within a host medium. This report is focused on one-dimensional linear transport equation. Consider a scalar density ϕ of some substance that is approximated by a line. For instance, some pollutant in a river or cars on a road) with the coordinate x [1]. Thus, the problem is spatially one-dimensional. The scalar density can be obtained at a point x at time t as $\phi(x,t)$. Assuming that the total amount of ϕ stays constant, i.e. there are no sources or sinks, the change of the substance in some arbitrary interval can be calculated as

$$\int_{x_1}^{x_2} \phi(x,t)dx \tag{1}$$

Moreover, its change at time *t,* assuming that ϕ is continuous, is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} \phi(x,t)dx = \int_{x_1}^{x_2} \phi_t(x,t)dx \tag{2}$$

This change can also be obtained for a flux of the same substance as

$$q(t,x_1) - q(t,x_2) \tag{3}$$

Where $q(t,x)$ is the flux, i.e. the change of the substance per unit time at point x. By conservation law [1]

$$\int_{x_1}^{x_2} \phi_t(x,t)dx = q(t,x_1) - q(t,x_2) = \int_{x_1}^{x_2} q_x(x,t)dx \tag{4}$$

In the simplest case, for a velocity *c*, the relation between the flux and the density can be obtained as

$$q(t,x) = c\phi(t,x) \tag{5}$$

Thus, the linear one-dimensional transport equation is obtained as shown in Eqn. (6).

$$\phi_t + c\phi_x = 0 \tag{6}$$

Or,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{7}$$

Thus, a hyperbolic partial differential equation is obtained representing the one-dimensional linear transport. However, an initial condition is required for this problem. For the purposes of this report, Gaussian distribution is chosen as the initial condition as shown in Eqn. (8).

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{8}$$

The exact solution for ϕ can be obtained using Eqn. (9) [2]

$$\phi(x,t) = f(x - ct) \tag{9}$$

II. Methodology

In this project, the 1D linear transport equation was solved using various numerical approaches. Finite differencing was used for Eqn. (7), and three different numerical methods were used – 1st order upwind, Lax-Wendroff, and 4th order Runge-Kutta. Gaussian distribution function was taken as the initial condition with respect to space. The domain was taken from x = 0 to x = 100.

$$\phi(x,0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{10}$$

$$\phi(0.0,t) = 0 \tag{11}$$

$$x_{min} = 0.0\ and\ x_{max} = 100.0 \tag{12}$$

The exact solution was also obtained using Eq. (9). The behavior of the different numerical approaches was then observed and compared with the exact solution.

A. 1st-order upwind

First order upwind is a simple explicit method. Eqn. (7) was first subjected to conditionally stable one-sided differencing.

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \frac{\phi_i^n - \phi_{i-1}^n}{\Delta \mathrm{x}} = 0 \tag{13}$$

The method is first-order accurate in both time and space. It was implemented for a positive wave velocity (*c* > 0.0)

$$\phi_i^{n+1} - c\frac{\Delta t}{\Delta x}(\phi_i^n - \phi_{i-1}^n) = 0, c > 0 \tag{14}$$

When combined with the equation for c < 0, the general form of first-order upwind method for the linear transport equation can be obtained as [3]:

$$\phi_i^{n+1} = \phi_i^n - c\frac{\Delta t}{\Delta x}(\phi_{i+1}^n - \phi_{i-1}^n) + \frac{|c|}{2}\frac{\Delta t}{\Delta x}(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) \tag{15}$$

The stability requirement for the 1st order upwind method is that the Courant number is less than 1 ($\bar{c} < 1.0$), where

$$\bar{c} = c\frac{\Delta t}{\Delta x} \tag{16}$$

B. Lax-Wendroff Method

Lax-Wendroff is another simple explicit method. It uses central differencing in both space and time and is second order accurate. It is highly applicable for hyperbolic problems (like the 1D linear transport equation).

The Taylor series expansion can be obtained for ϕ_i^{n+1} as

$$\phi_i^{n+1} = \phi_i^n + \Delta t\left(\frac{\partial\phi}{\partial t}\right)_i^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2\phi}{\partial t^2}\right)_i^n + O(\Delta t^2) \tag{17}$$

The time derivative can be converted into space by differentiating the original transport equation.

$$\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial x} \tag{18}$$

And

$$\frac{\partial^2\phi}{\partial t^2} = -c^2\frac{\partial^2\phi}{\partial x\partial t} \tag{19}$$

Upon substituting Eqns. (18) and (19) in Eqn. (17) and dropping the truncated error, the modified equation can be central differenced in space to obtain the following

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + c\Delta t\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta \mathrm{x}} - \frac{\Delta t^2}{2}\frac{c^2}{\Delta x^2}\,(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) = 0 \tag{20}$$

Equation (20) can then be rearranged to obtain ϕ_i^{n+1}.

$$\phi_i^{n+1} = \phi_i^n - c\Delta t\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta \mathrm{x}} + \frac{\mathrm{c}^2}{2}\frac{\Delta t^2}{\Delta x^2}\,(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n) = 0 \tag{21}$$

The stability criterion for the Lax-Wendroff method for the one-dimensional linear transport theorem is same to that of the 1st-order upwind, i.e. Courant number is less than 1 ($\bar{c} < 1.0$).

C. 4th order Runge-Kutta

The 4th order Runge-Kutta method is 2nd order accurate in space and 4th order accurate in time. It can be derived by treating the partial differencing equation as a pseudo-ordinary differential equation.

$$\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial x} = R(\phi) \tag{22}$$

This pseudo-ODE is a time-continuous equation, and any integration scheme applicable to ODEs, like the Runge-Kutta methods, may be applied [3]. The 4th order Runge-Kutta method is given in 4 steps as

Step 1:

$$\phi^{(1)} = \phi^n + \frac{\Delta t}{2}R^n \tag{23}$$

Step 2:
$$\phi^{(2)} = \phi^n + \frac{\Delta t}{2} R^{(1)} \tag{24}$$

Step 3:
$$\phi^{(3)} = \phi^n + \frac{\Delta t}{2} R^{(2)} \tag{25}$$

Step 4:
$$\phi^{n+1} = \phi^n + \frac{\Delta t}{6}\left(R^n + 2R^{(1)} + 2R^{(2)} + R^{(3)}\right) \tag{26}$$

Where $R^{()} = -c\phi_x^{()}$ for the linear transport equation.

III. Results and Discussion

The three explicit finite differencing methods were implemented in MATLAB for various Courant's number and for different points in time. The results were than compared with the exact solution for the transport of a Gaussian distribution. The mean of the distribution was taken as 0 and the standard deviation was taken as 1.0. For the purpose of this report, the characteristic velocity was chosen to be 0.5 (c = 0.5).

A. Solutions close to the stability limit (Courant number = 0.999)

First, the numerical methods were applied for a Courant number of 0.999. This was done to test the behavior close to the stability limit of 1. The 3-dimensional representation of the solutions were then plotted. Figures 1 and 2 show the comparison of exact solution with 1st order upwind method and Lax-Wendroff method respectively.

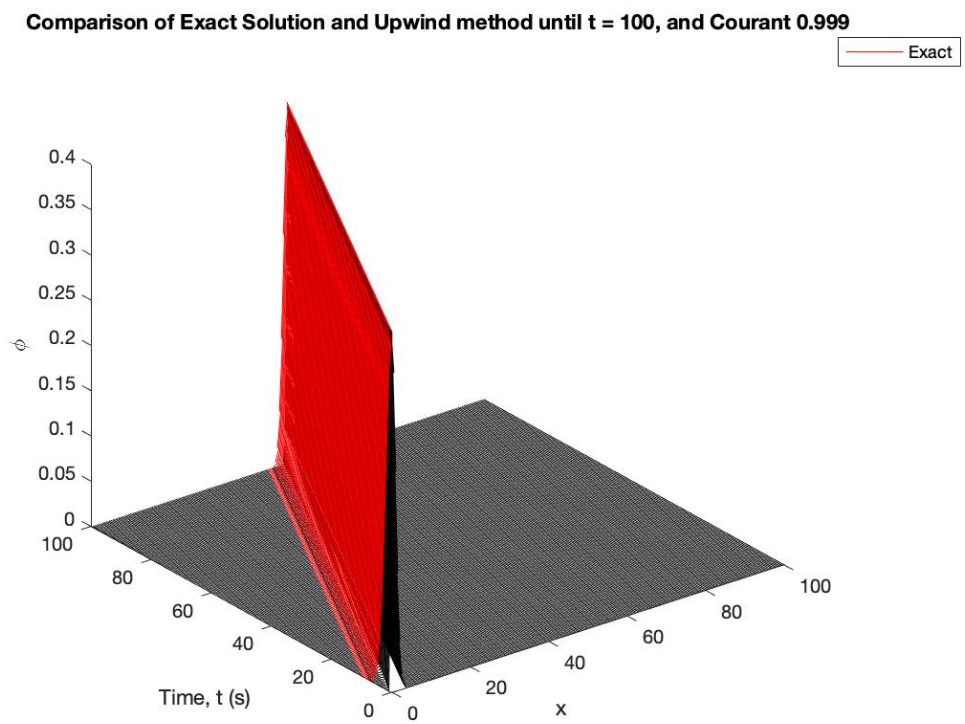

Fig. 1 Comparison of Exact solution and Upwind method close to the stability limit (Courant number = 0.999).

From the 3-dimensional plots, it was seen that the Upwind and Lax-Wendroff methods closely followed the exact solution of linear transport of the Gaussian distribution. For a better comparison, the solution of all three methods were compared to the exact solution at different points in time. These comparisons are shown in Figs. 3, 4, 5, and 6.

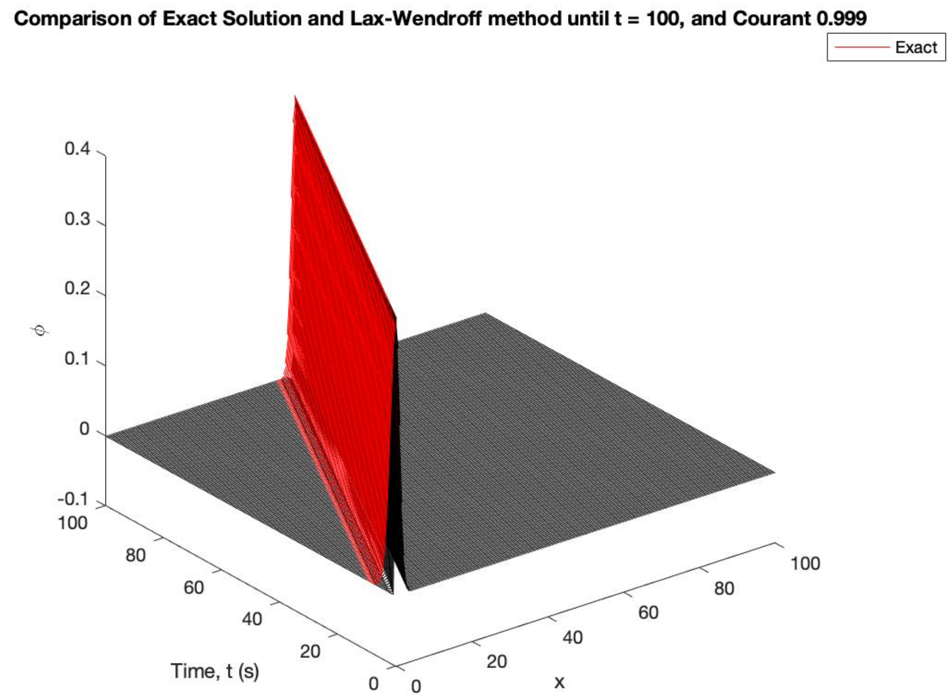

Fig. 2 Comparison of Exact solution and Lax-Wendroff method close to the stability limit (Courant number = 0.999)

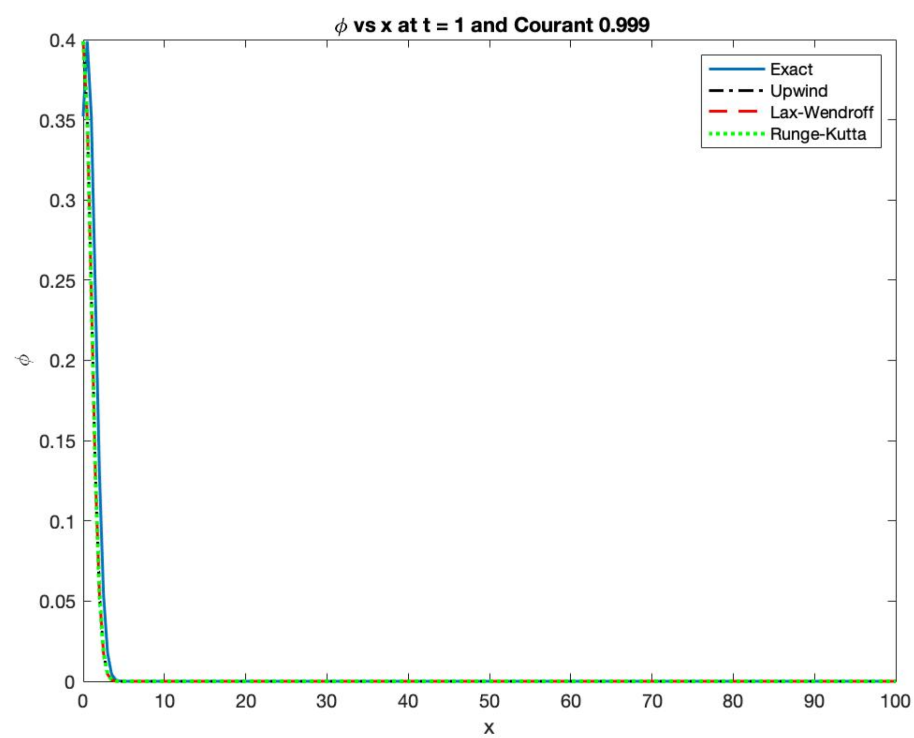

Fig. 3 Comparison of solutions at t = 1 and Courant number = 0.999.

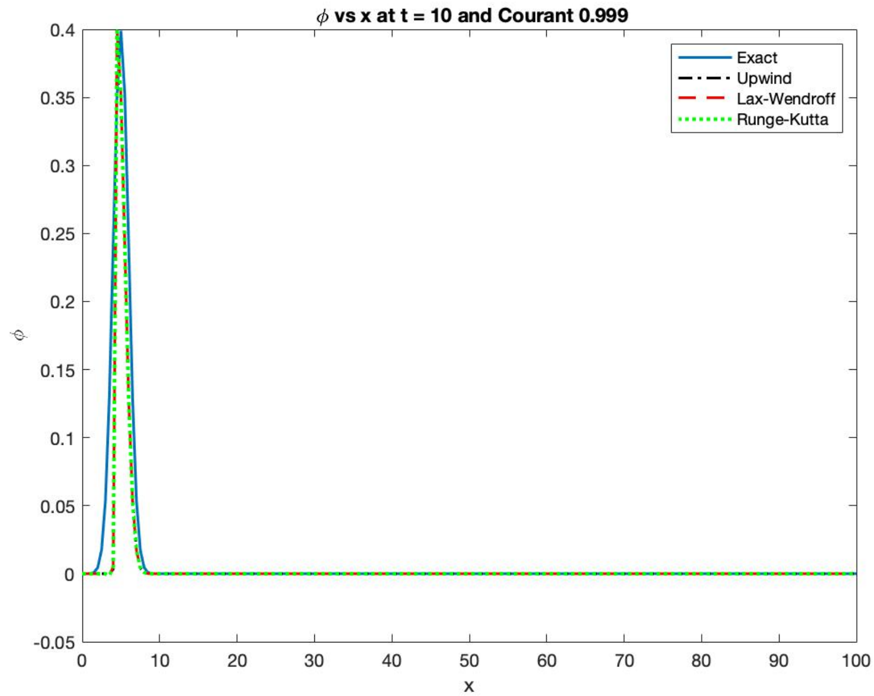

Fig. 4 Comparison of solutions at t = 10 and Courant number = 0.999.

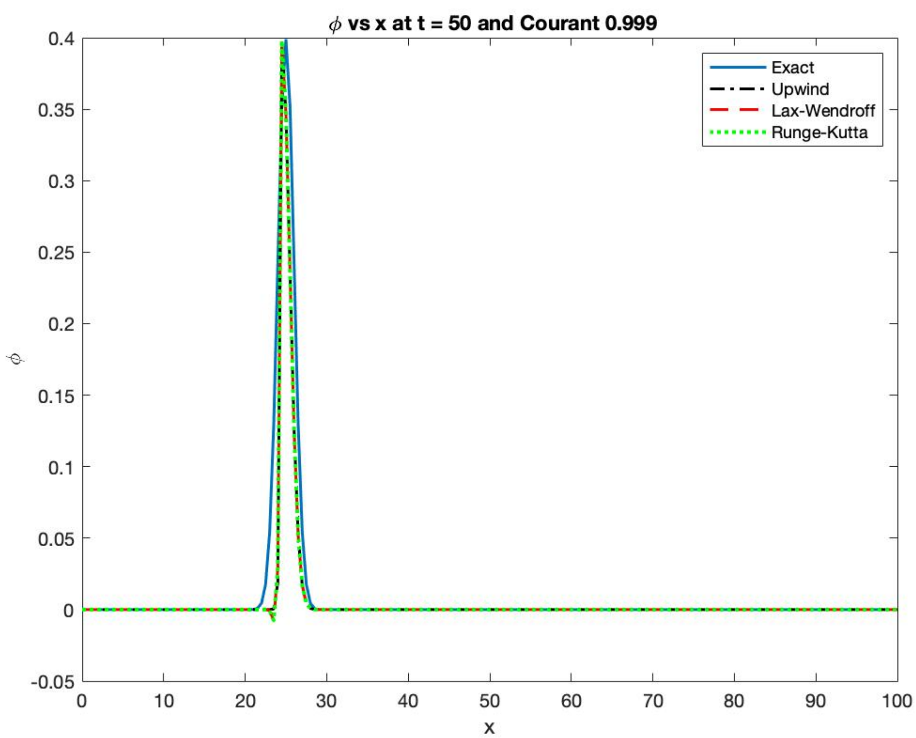

Fig. 5 Comparison of solutions at t = 50 and Courant number = 0.999.

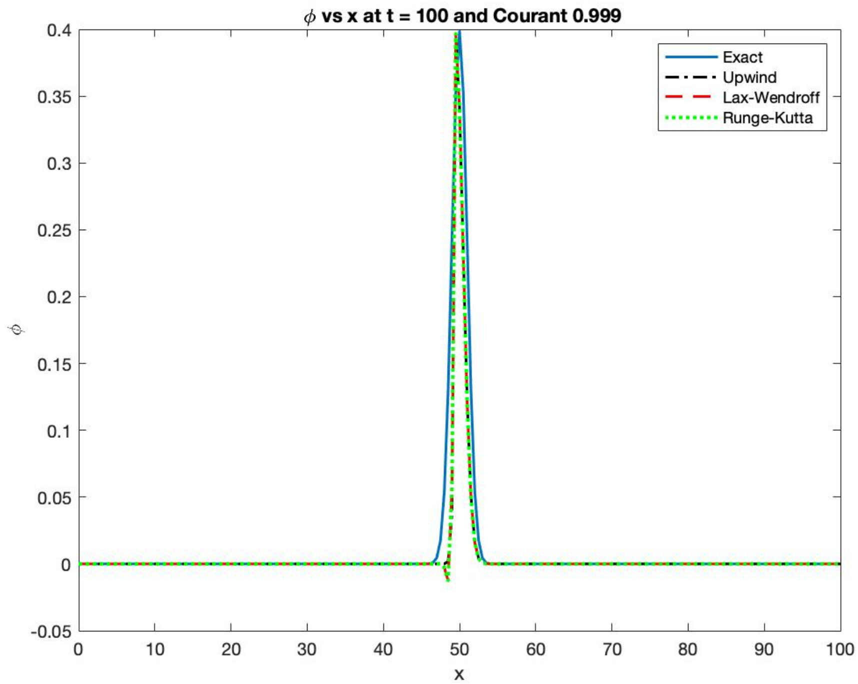

Fig. 6 Comparison of solutions at t = 100 and Courant number = 0.999.

B. Solutions for Courant number = 0.5

The numerical methods were then applied for a Courant number of 0.5. This was done to test the behavior below the stability limit of 1. The 3-dimensional representation of the solutions were then plotted. Figures 7 and 8 show the comparison of exact solution with 1st order upwind method and Lax-Wendroff method respectively.

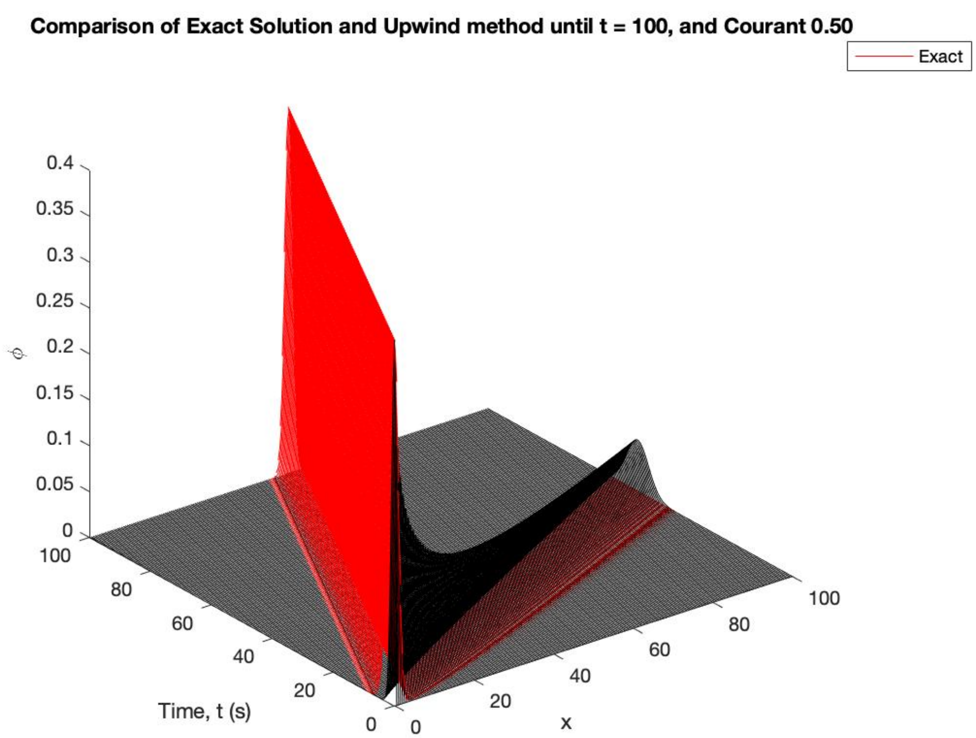

Fig. 7 Comparison of Exact solution and Upwind method below the stability limit (Courant number = 0.5).

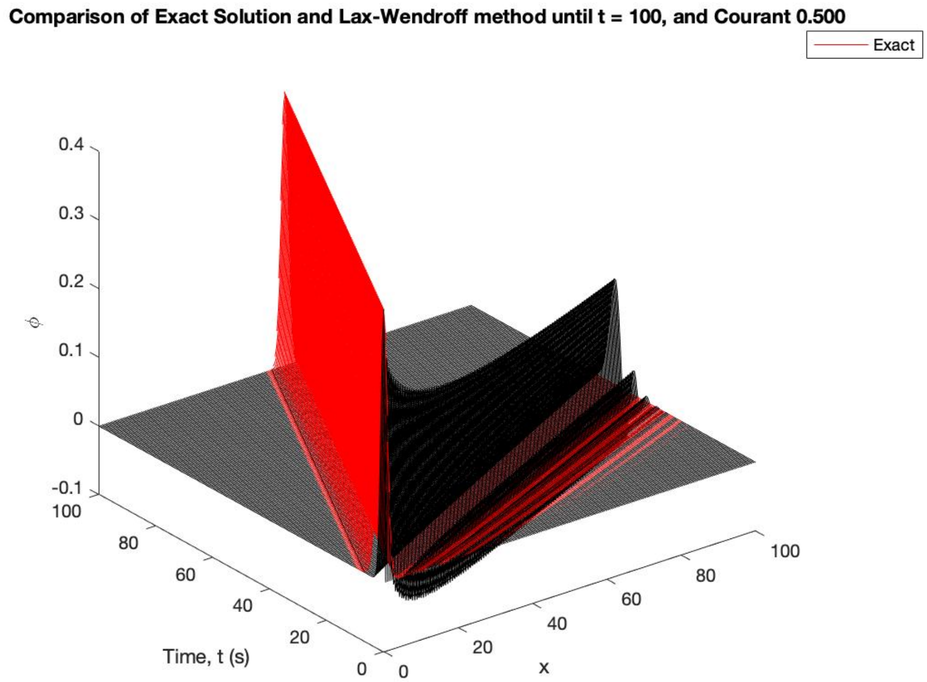

Fig. 8 Comparison of Exact solution and Lax-Wendroff method below the stability limit (Courant number = 0.5).

From the 3-dimensional plots, it was seen that the Upwind and Lax-Wendroff methods lagged behind the exact solution of linear transport of the Gaussian distribution. Although the solutions are close towards the initial time (t = 0), the numerical solution of the transport equation lags at higher points of time. Moreover, the amplitude of the distribution was observed to be reduced. For a better comparison, the solution of all three methods were compared to the exact solution at different points in time. These comparisons are shown in Figs. 9, 10, 11, and 12.

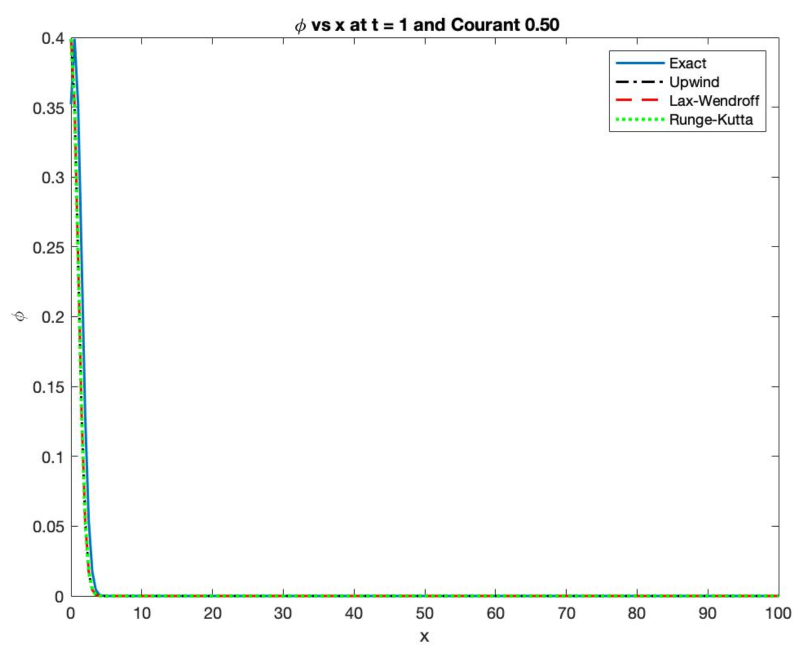

Fig. 9 Comparison of solutions at t = 1 and Courant number = 0.5.

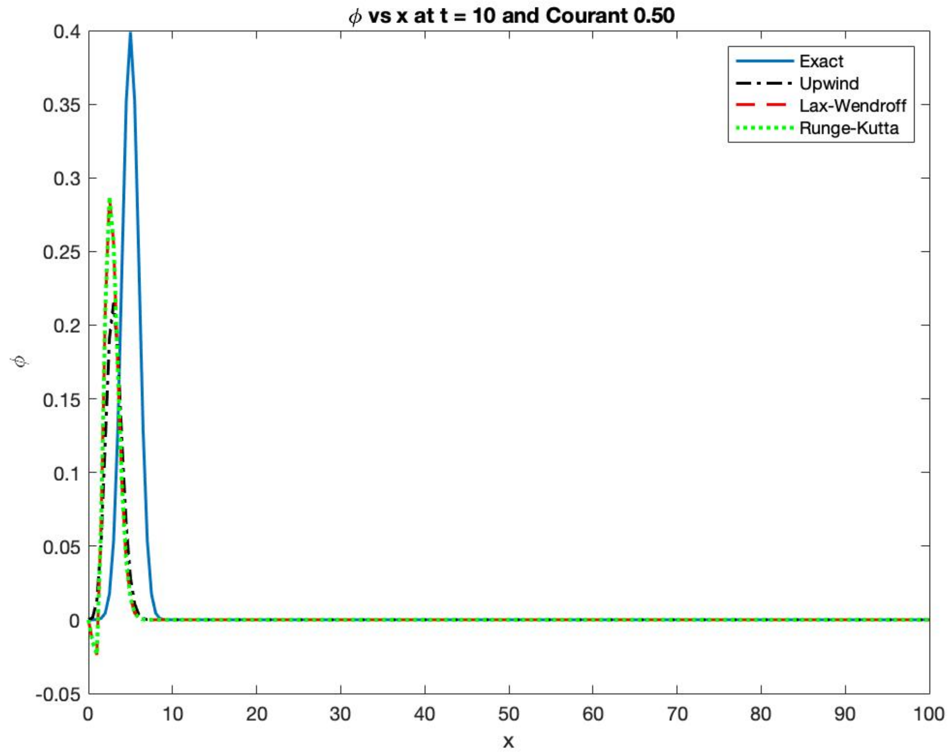

Fig. 10 Comparison of solutions at t = 10 and Courant number = 0.5.

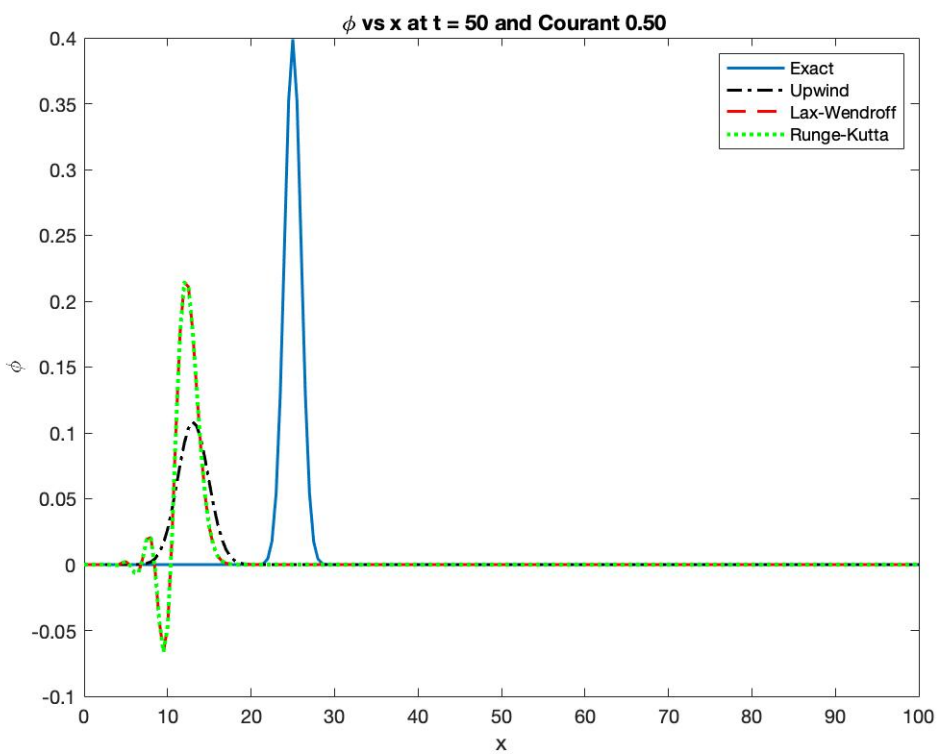

Fig. 11 Comparison of solutions at t = 50 and Courant number = 0.5.

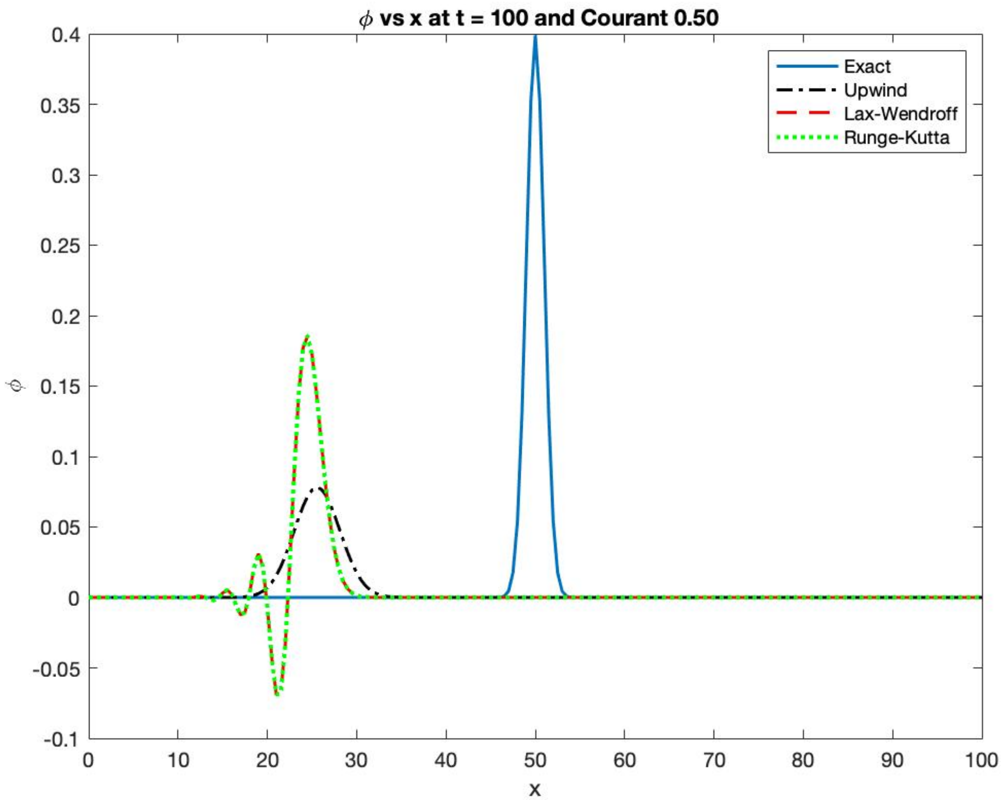

Fig. 12 Comparison of solutions at t = 100 and Courant number = 0.5.

It was observed that the numerical solutions started with the same initial conditions. However, as time passed, the numerical solutions lagged behind the actual solution. Moreover, the amplitude dropped from the actual amplitude. The greatest reduction was seen for upwind method. Furthermore, the numerical solutions exhibited dispersive property as time increased as shown in Fig. 12.

IV. Conclusions

The implementation of the three different finite difference methods for the first order linear transport of a Gaussian distribution showed that the numerical methods exhibited more accurate results closer to the stability limit (Courant number, $\bar{c} \approx 1$). However, for Courant numbers relatively below the stability limit, the amplitude of the distribution decreased in the numerical solution. Moreover, dispersion was introduced with time for lower Courant numbers. Thus, it was found that astute judgement of the stability condition is required to select a numerical approach to any partial differential equation. Otherwise, the solutions obtained may be incorrect.

References

[1] Novozhilov, A. S., "The linear transport equation," *Mechanical Methods and Measurements Laboratory Manual*, Department of Mathematics, North Dakota State University, 2016.

[2] Dennis, B., "MAE 4326 – Lecture Notes," *Computational Aerodynamics*, Department of Mechanical and Aerospace Engineering, Univeristy of Texas at Arlington, 2019.

[3] Anderson, D. A., Tannehill, J. C., and Pletcher, R. H., *Computational Fluid Mechanics and Heat Transfer*, Washington (U.A.): Hemisphere Publ. Corp. (U.A.), 1984.

Appendix

The implementation of the numerical methods in MATLAB are presented in Figs. 13, 14, 15, 16.

```
clc; clear; close all;
x_min = 0; x_max = 100; N = 200;
% vector x
dx = (x_max-x_min)/N;
x = x_min:dx:x_max;

c = .5; % wave speed
c_bar = .5; % Courant's number
t_max = 100;

dt = c_bar*dx/c;  t = 0:dt:t_max;
% Gaussian distribution values
mu = 0; sigma = 1;
% Exact solution
phi_exact = 1/(sigma * sqrt(2*pi)) * exp(-1/2*((x -c*t_max - mu)/
sigma).^2);
plot(x, phi_exact, "LineWidth", 1.5)
hold on
% Initial conditions
phi = zeros(length(x), length(t));
phi(1,:) = 0;
phi(:,1) = 1/(sigma * sqrt(2*pi)) * exp(-1/2*((x- mu)/sigma).^2);
% Upwind method
phi1 = upwind(x, t, phi, c_bar);
plot(x, phi1(:,t_max), 'k-.', "LineWidth", 1.5);
hold on
% lax wendroff method
phi2 = lax_wendroff(x,t,phi,c_bar);
plot(x, phi2(:, t_max), 'r--', "LineWidth", 1.5);
% rk4
phi3 = rk4(x, t, phi, dx, dt, c);
plot(x, phi2(:, t_max), 'g:', "LineWidth", 2);

xlabel("x"); ylabel("\phi")
head = sprintf(" \\phi vs x at t = %.0f and Courant %.3f",
 t_max,c_bar);
title(head)
dbclear all
legend("Exact", "Upwind", "Lax-Wendroff", "Runge-Kutta");

figure(2)
[xx, tt] = meshgrid(x, t);
phi5 = 1/(sigma * sqrt(2*pi)) * exp(-1/2*((xx -c*tt - mu)/sigma).^2);
plot3(xx, tt, phi5, 'r')
xlabel("x"); ylabel("Time, t (s)"); zlabel("\phi")
hold on; plot3(xx, tt, phi2, 'k'); legend("Exact")
a = sprintf("Comparison of Exact Solution and Lax-Wendroff method
 until t = %.f, and Courant %.3f", t_max, c_bar);
title(a)
```

Fig. 13 MATLAB implementation of three finite-differencing methods.

```
function y = upwind(x, t,phi, c_bar)
    y = phi;
    for n = 1:(length(t)-1)
        for i= 2:(length(x)-1)
            y(i, n+1) = y(i,n) - c_bar/2 *(y(i+1,n) - y(i-1,n)) +
 c_bar/2 * (y(i+1, n)-2*y(i,n)+y(i-1,n));
        end
    end
end
```

Fig. 14 Implementation of 1st order upwind method.

```
function y = lax_wendroff(x, t, phi, c_bar)
    y = phi;
    for n = 1:(length(t)-1)
        for i = 2:(length(x)-1)
         y(i, n+1) = y(i,n) - c_bar/2 * ( y(i+1,n)-y(i-1,n) ) +
 c_bar^2/2 * (y(i+1,n) - 2*y(i, n) + y(i-1, n) );
        end
    end
end
```

Fig. 15 Implementation of Lax-Wendroff method.

```
function y = rk4(x, t, phi, dx, dt, c)
   y = phi;
   for n = 1:(length(t)-1)
       for i = 3:(length(x)-2)
            % values from Tannehill
            k1 = (y(i+1,n)- y(i-1,n))/(2*dx);
            k2 = (y(i+1,n) + y(i-1,n) - 2*y(i,n))/dx^2;
            k3 = (y(i+2,n) - y(i-2,n) - 2*y(i+1,n) + 2*y(i-1,n) )/
(2*dx^3);
            k4 = (y(i+2,n) + y(i-2,n) - 4*y(i+1,n) -4*y(i-1,n)
 +y(i,n))/dx^4;
            % step wise solution for the pseudo-ode
            Rn = -c*k1;
            R1 = Rn + c^3*dt/3*k2;
            R2 = Rn + c^2*dt/2*k2 + c^3*dt^2/4*k3;
            R3 = Rn + c^3*dt*k2 - c^3 * dt^2*k3/2 - c^4 * dt^3 * k4/4;
            y(i, n+1) = y(i,n) + dt*(Rn + 2*R1 + 2*R2 + R3)/6;
       end
   end
```

Fig. 16 Implementation of 4th order Runge-Kutta in MATLAB.