$$dS(u) = \alpha S(u)\,du + \sigma S(u)\,dW(u)$$

$$d\left(\log S(t)\right) = \frac{1}{S(t)}\,dS(t) + \left(-\frac{1}{S^2(t)}\right)\frac{1}{2}\left(dS(t)\right)^2$$

$$= \alpha\,dt + \sigma\,dW(t) - \frac{1}{2}\sigma^2\,dt$$

$$= \left(\alpha - \sigma^2/2\right)dt + \sigma\,dW(t)$$

$$\int_0^t d\left(\log S(u)\right) = \int_0^t \left(\alpha - \sigma^2/2\right)du + \int_0^t \sigma\,dW(u)$$

$$\log S(t) = \log S(0) + \left(\alpha - \sigma^2/2\right)t + \sigma W(t)$$

$$S(t) = S(0)\,e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$$S(T) = S(0)\,e^{(\alpha - \sigma^2/2)T + \sigma W(T)}$$

$$S(T) = \frac{S(T)}{S(t)} \cdot S(t)$$

$$= S(t)\,e^{(\alpha - \sigma^2/2)(T-t) + \sigma\left(W(T) - W(t)\right)}$$

---

$$\int_t^T d\left(\log\left(S(u)\right)\right) = \int_t^T \left(\alpha - \sigma^2/2\right)du + \int_t^T \sigma\,dW(u)$$

$$\log S(T) = \log S(t) + \left(\alpha - \sigma^2/2\right)(T-t) + \sigma\left(W(T) - W(t)\right)$$

$$d\left(e^{b(u)}R(u)\right) = e^{b(u)}dR(u) + b'(u)e^{b(u)}R(u)du + 0$$

$$= e^{b(u)}\left(a(u) - b(u)R(u)\right)du + e^{b(u)}\sigma(u)\,d\tilde{W}(u) + b'(u)e^{b(u)}R(u)du$$

$$= e^{b(u)}\left(a(u) - b(u)R(u) + b'(u)R(u)\right)du + e^{b(u)}\sigma(u)\,d\tilde{W}(u)$$

---

$$d\left(e^{\int_0^u b(v)dv}R(u)\right) = e^{\int_0^u b(v)dv}dR(u) + b(u)e^{\int_0^u b(v)dv}R(u)du$$

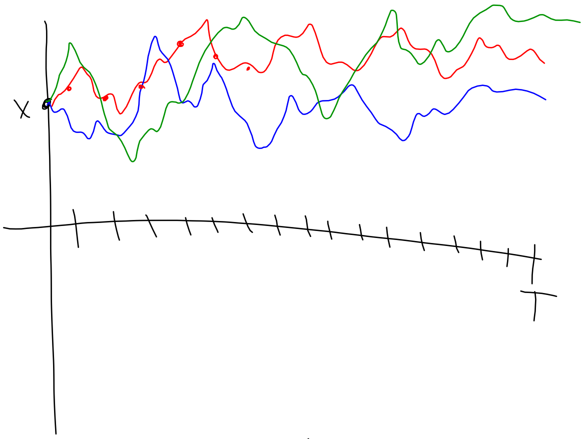
$$= e^{\int_0^u b(v)dv}\left(a(u) - R(u)b(u)\right)du + e^{\int_0^u b(v)dv}\sigma(u)\,d\tilde{W}(u) + e^{\int_0^u b(v)dv}b(u)R(u)du$$

$$= e^{\int_0^u b(v)dv}\left(a(u)du + \sigma(u)d\tilde{W}(u)\right)$$

$$e^{\int_0^T b(v)dv}R(T) = e^{\int_0^t b(v)dv}R(t) + \int_t^T e^{\int_0^u b(v)dv}a(u)du + \int_t^T e^{\int_0^u b(v)d}\sigma(u)\,d\tilde{W}(u)$$

$$R(T) = e^{-\int_t^T b(v)dv}r + \int_t^T e^{-\int_u^T b(v)dv}a(u)du + \int_t^T e^{-\int_u^T b(v)dv}\sigma(u)\,d\tilde{W}(u)$$

where $R(t) = r$




cut $T$ into $N$ pieces

$\Delta t = T/N$

$\varepsilon_i \sim N(0, \Delta t)$

$$X(0) = x$$

$$X(\Delta t) = x + \beta(0,x)\Delta t + \gamma(0,x)\varepsilon_1$$

$$X((i+1)\Delta t) = X(i\Delta t) + \beta(i\Delta t, X(i\Delta t))\Delta t + \gamma(i\Delta t, X(i\Delta t))\varepsilon_{i+1}$$

---

$$g(t, X(t)) = E[h(X(T)) \mid \mathcal{F}(t)]$$

for $s \le t$

$$E[g(t,X(t)) \mid \mathcal{F}(s)] = E\{E\{h(X(T)) \mid \mathcal{F}(t)\} \mid \mathcal{F}(s)\}$$

$$= E[h(X(T)) \mid \mathcal{F}(s)]$$

$$= g(s, X(s))$$

$$d(g(t,X(t))) = g_t(t,X(t))dt + g_x(t,X(t))dX(t) + \frac{1}{2}g_{xx}(t,X(t))(dX(t))^2$$

$$= g_t(t,X(t))dt + g_x(t,X(t))(\beta(t,X(t))dt + \gamma(t,X(t))dW(t)) + \frac{1}{2}g_{xx}(t,X(t))\gamma^2(t,X(t))dt$$

$$= (g_t(t,X(t)) + g_x(t,X(t))\beta(t,X(t)) + \frac{1}{2}g_{xx}(t,X(t))\gamma^2(t,X(t)))dt + \gamma(t,X(t))g_x(t,X(t))dW(t))$$

$$\overline{g_t(t,x) + g_x(t,x)\beta(t,x) + \frac{1}{2}g_{xx}(t,x)\gamma^2(t,x) = 0}$$

$$f(t,X(t)) = E[e^{-r(T-t)}h(X(T)) \mid \mathcal{F}(t)]$$

$s \le t$

$$E[f(t,X(t)) \mid \mathcal{F}(s)] = E[E[e^{-r(T-t)}h(X(T)) \mid \mathcal{F}(t)] \mid \mathcal{F}(s)]$$

$$= E[e^{-r(T-t)}h(X(T)) \mid \mathcal{F}(s)]$$

$$f(s,X(s)) = E[e^{-r(T-s)}h(X(T)) \mid \mathcal{F}(s)]$$

$$e^{-rt}f(t,X(t)) = E[e^{-rT}h(X(T)) \mid \mathcal{F}(t)]$$

will be m-g.

$$d(e^{-rt}f(t,X(t))) = -re^{-rt}f(t,X(t))dt$$

$$+e^{-rt}df(t,X(t)) + 0$$

$$= -re^{-rt}f(t,X(t))dt$$

$$+e^{-rt}f_t(t,X(t))dt + e^{-rt}\beta(t,X(t))f_x(t,X(t))dt$$

$$+e^{-rt}\gamma(t,X(t))f_x(t,X(t))dW(t)$$

$$+e^{-rt}\frac{1}{2}\gamma^2(t,X(t))f_{xx}(t,X(t))dt$$

$$= e^{-rt}\left(-rf + f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx}\right)dt$$

$$+ e^{-rt}\gamma f_x dW(t)$$

$$\Rightarrow -rf(t,x) + f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{xx}(t,x) = 0$$

$$\left(-rC'(t,T)-A'(t,T)\right)f(t,r)-\left(a(t)-b(t)r\right)C(t,T)f(t,r)$$

$$+\frac{1}{2}\sigma^2(t)\,C^2(t,T)f(t,r)-rf(t,r)=0$$

$$-rC'-A'-aC+bCr+\frac{1}{2}\sigma^2C^2-r=0$$

$$r(-C'+bC-1)+\left(-A'-aC+\frac{1}{2}\sigma^2C^2\right)=0$$

$$Ax+B=0$$

(1) $-C'(t,T)+b(t)C(t,T)-1=0$

(2) $-A'(t,T)-a(t)C(t,T)+\frac{1}{2}\sigma^2(t)C^2(t,T)=0$

(1). $C'(t,T)-b(t)C(t,T)=-1$

$$\underbrace{e^{-\int_0^t b(u)du}C'(t,T)-b(t)e^{-\int_0^t b(u)du}C(t,T)}_{d\left(e^{-\int_0^t b(u)du}C(t,T)\right)}=-e^{-\int_0^t b(u)du}$$

$$B(T,T)=e^{-R(T)C(T,T)-A(T,T)}=1$$

$$\Rightarrow A(T,T)=C(T,T)=0$$

$$e^{-\int_0^T b(u)du}C(T,T)-e^{-\int_0^t b(u)du}C(t,T)=\int_t^T -e^{-\int_0^s b(u)du}ds$$

$$C(t,T)=\int_t^T e^{-\int_t^s b(u)du}ds$$

$$-\left(A(T,T)-A(t,T)\right)=\int_t^T a(u)C(u,T)du-\int_t^T\frac{1}{2}\sigma^2(u)C^2(u,T)du$$

$$A(t,T)=$$

$\sigma(X)$

$$E[X|Y](\omega) = \begin{cases} E[X|Y=3] & \omega\in\{a,c\} \\ E[X|Y=4] & \omega\in\{b,d\} \end{cases}$$

$$E[W(t)\,|\,\mathcal{F}(s)] = W(s)$$

$$E[W(t)\,|\,\mathcal{F}(s),\ W(s)=4] = 4$$

---

~~$M(t_3) - M(t_2)$ is ind. of $\mathcal{F}(t_2)$~~

$$E[M(t_3) - M(t_2)\,|\,\mathcal{F}(t_2)]$$

$$= E[M(t_3)\,|\,\mathcal{F}(t_2)] - E[M(t_2)\,|\,\mathcal{F}(t_2)]$$

$$= M(t_2) - M(t_2) = 0$$

---

$$X(t) = f(t)W(t) - \int_0^t f(s)W(s)\,ds$$

$$\boxed{\begin{aligned} X(t) &= X(0) + \int_0^t \Delta(u)\,dW(u) + \int_0^t \theta(u)\,du \\ [X,X](t) &= \int_0^t \Delta^2(u)\,du \end{aligned}}$$

$$d\big(f(t)W(t)\big) = f'(t)W(t)\,dt + f(t)\,dW(t) + \tfrac{1}{2}\,0\,dt$$

$g(t,x) = f(t)x$

$$X(t) = \int_0^t f'(s)W(s)\,ds + \int_0^t f(s)\,dW(s) - \int_0^t f(s)W(s)\,ds$$

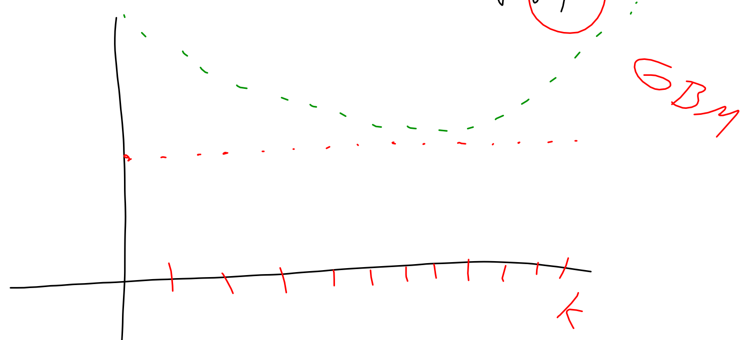
$$= \int_0^t \big(f'(s)W(s) - f(s)W(s)\big)ds + \int_0^t f(s)\,dW(s)$$

$$[X,X](t) = \int_0^t f^2(s)\,ds$$

Observables: $K$, $S(0)$, $\tau$, $r$, Option Price

$$\text{Opt. Price} = S(0)\,N(d_+) - K e^{-rT} N(d_-)$$

$$d_{\pm} = \frac{\log\left(\frac{S(0)}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$




$$Y(t,T) = -\frac{1}{T-t}\log B(t,T)$$

$$-(T-t)\,Y(t,T) = \log B(t,T)$$

$$B(t,T) = e^{-Y(t,T)(T-t)}$$

$$B(t,T) = e^{-R(t)C(t,T) - A(t,T)} = e^{-(T-t)Y(t,T)}$$

$$Y(t,T) = \frac{1}{T-t}\left(R(t)\,C(t,T) + A(t,T)\right)$$

$$B(t,T) = f(t,r) = e^{-r\,C(t,T) - A(t,T)}$$

$$f_t = \left(-r\,C'(t,T) - A'(t,T)\right) f(t,r)$$

$$f_r = -C(t,T)\,f(t,r)$$

$$f_{rr} = C^2(t,T)\,f(t,r)$$