$$dX_n = X_{n+1} - X_n$$

$$= \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

$$- \Delta_n S_n - (X_n - \Delta_n S_n)$$

$$= \Delta_n \underbrace{(S_{n+1} - S_n)}_{dS_n} + (X_n - \Delta_n S_n)\underbrace{(1+r-1)}_{dr}$$

$$dX(t) = \Delta(t)\, dS(t) + (X(t) - \Delta(t)S(t))\, r\, dt$$

$$= \Delta(t)(\alpha S(t)dt + \sigma S(t)dW(t))$$

$$+ r(X(t) - \Delta(t)S(t))dt$$

$$= (\Delta(t)\alpha S(t) - r\Delta(t)S(t) + rX(t))dt$$

$$+ \sigma \Delta(t) S(t) dW(t)$$

$$= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt$$

$$+ \Delta(t)\sigma S(t)dW(t)$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

$$d\left(e^{-rt}S(t)\right) = ?$$

$$e^{-rt}S(t) = f(t, S(t))$$

$$\boxed{\begin{aligned} f(t,x) &= e^{-rt}x \\ f_t &= -re^{-rt}x \\ f_x &= e^{-rt} \\ f_{xx} &= 0 \end{aligned}}$$

$$d(e^{-rt}S(t)) = -re^{-rt}S(t)dt + e^{-rt}dS(t) + \tfrac{1}{2}(0)\,dS(t)^2$$

$$= e^{-rt}\left(-rS(t) + \alpha S(t)\right)dt$$
$$+ e^{-rt}\sigma S(t)dW(t)$$

$$= e^{-rt}(\alpha - r)S(t)dt + e^{-rt}S(t)\sigma dW(t)$$

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

$$d\left(e^{-rt}X(t)\right) = -re^{-rt}X(t)dt + e^{-rt}dX(t) + \tfrac{1}{2}(0)\,d[X,X](t)$$

$$= e^{-rt}(\alpha - r)\Delta(t)S(t)dt + e^{-rt}\Delta(t)\sigma S(t)dW(t)$$

$$= \Delta(t)\,d\left(e^{-rt}S(t)\right)$$

---

for $c(t, S(t))$

$$d\,c(t,S(t)) = c_t(t,S(t))dt + c_x(t,S(t))dS(t)$$
$$+ \tfrac{1}{2}c_{xx}(t,S(t))\,d[S,S](t)$$

$$= c_t(t,S(t))dt + c_x(t,S(t))\left(\alpha S(t)dt + \sigma S(t)dW(t)\right)$$
$$+ \tfrac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)dt$$

$$= \left(c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right)dt$$
$$+ \sigma S(t)c_x(t,S(t))dW(t)$$

$$d\left(e^{-rt}c(t,S(t))\right) = -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t))$$

$$= e^{-rt}\left(-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right)dt$$
$$+ e^{-rt}\sigma S(t)c_x(t,S(t))dW(t)$$

Choose $\Delta(t)$ such that $X(t) = c(t, S(t))$
$\forall t \in [0, T)$

$$e^{-rt} X(t) = e^{-rt} c(t, S(t))$$

$$\int_0^T d\left(e^{-rt} X(t)\right) = \int_0^T d\left(e^{-rt} c(t, S(t))\right)$$

$$e^{-rT} X(T) - X(0) = e^{-rT} c(T, S(T)) - c(0, S(0))$$

$$\Rightarrow X(0) = c(0, S(0))$$

$$Ax + By = Cx + Dy \Rightarrow A = C, \ B = D$$

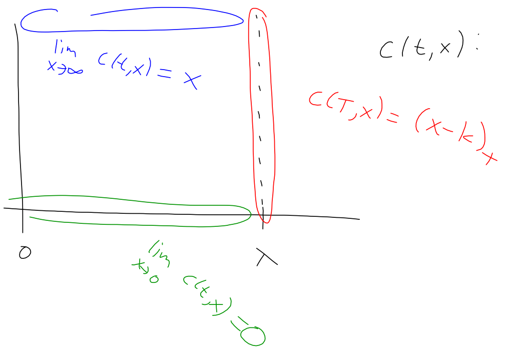
$\underline{dW(t)}$: $$e^{-rt} \Delta(t) \sigma S(t) = e^{-rt} \sigma S(t) c_x(t, S(t))$$

$$\Delta(t) = c_x(t, S(t))$$

$\underline{dt:}$ $$\cancel{e^{-rt}} (\underline{\alpha} - r) \underline{S(t) \Delta(t)} = \cancel{e^{-rt}} \Big(-r c(t, S(t)) + c_t(t, S(t)) + \underline{\alpha S(t) c_x(t, S(t))} + \tfrac{1}{2}\sigma^2 S^2(t) c_{xx}(t, S(t)) \Big)$$

$$0 = -r C(t, S(t)) + c_t(t, S(t)) + r S(t) c_x(t, S(t)) + \tfrac{1}{2}\sigma^2 S^2(t) c_{xx}(t, S(t))$$

$$0 = -r c(t, x) + c_t(t, x) + r x c_x(t, x) + \tfrac{1}{2}\sigma^2 x^2 c_{xx}(t, x)$$



$c(t,x): \quad t \in [0,T), \quad x \in [0,\infty)$

$$c(t,x) = x\,N(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

$$d_\pm(\tau,x) = \frac{\log\left(\frac{x}{K}\right) + (r \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$\lim_{t\to T} c(t,x) = \left.\begin{array}{ll} \text{if } x > K: & x-K \\ \text{if } x < K: & 0 \end{array}\right\} (x-K)_+$$

$$\lim_{x\to 0} c(t,x) = 0 \qquad \lim_{x\to\infty} c(t,x) = x - Ke^{-r(T-t)} \approx x$$

$$c_x(t,x) = N(d_+) + x N'(d_+)\frac{d}{dx}(d_+) - Ke^{-r(T-t)}N'(d_-)\frac{d}{dx}(d_-)$$

$$\frac{d}{dx}(d_\pm) = \frac{1}{\sigma\sqrt{\tau}}\left(\frac{K}{x}\right)\left(\frac{1}{K}\right) = \frac{1}{x\sigma\sqrt{\tau}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$c_x(t,x) = N(d_+) + \frac{x}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}\left(\frac{1}{x\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2}{2}}\left(\frac{1}{x\sigma\sqrt{\tau}}\right)$$

$$= N(d_+) + \frac{1}{x\sigma\sqrt{2\pi\tau}}\left(xe^{-\frac{d_+^2}{2}} - Ke^{-r\tau}e^{-\frac{d_-^2}{2}}\right)$$

$$d_+ - d_- = \frac{\log\left(\frac{x}{K}\right) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} - \frac{\log\left(\frac{x}{K}\right) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$$= \frac{\frac{\sigma^2\tau}{2} + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}} = \sigma\sqrt{\tau}$$

$$d_- = d_+ - \sigma\sqrt{\tau} \qquad (d_+ - \sigma\sqrt{\tau})^2 = d_+^2 - 2d_+\sigma\sqrt{\tau} + \sigma^2\tau$$

$$= N(d_+) + \_\_\_\left(xe^{-\frac{d_+^2}{2}} - Ke^{-r\tau}e^{-\frac{d_+^2}{2} + d_+\sigma\sqrt{\tau} - \frac{\sigma^2\tau}{2}}\right)$$

$$= N(d_+) + \_\_\_\left(x - Ke^{-r\tau - \frac{\sigma^2\tau}{2} + d_+\sigma\sqrt{\tau}}\right)$$

$$= N(d_+) + \_\_\_\left(x - Ke^{-r\tau - \frac{\sigma^2\tau}{2} + \log\left(\frac{x}{K}\right) + (r + \frac{\sigma^2}{2})\tau}\right)$$

$$= N(d_+)$$

$$c_{xx}(t,x) = \frac{N'(d_+)}{x\sigma\sqrt{\tau}}$$

$$c_t(t,x) = -rKe^{-r(T-t)}N(d_-) - \frac{x\sigma}{2\sqrt{T-t}}N'(d_+)$$

$$r\,c(t,x) = c_t(t,x) + rx\,c_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x)$$

$$r\,c(t,x) = -rKe^{-r(T-t)}N(d_-) - \cancel{\frac{x\sigma}{2\sqrt{T-t}}N'(d_+)}$$
$$+ rx\,N(d_+) + \cancel{\frac{1}{2}\sigma^2x^2\frac{N'(d_+)}{x\sigma\sqrt{T-t}}}$$
$$= -rKe^{-r(T-t)}N(d_-) + rx\,N(d_+)$$

$$c(t,S(t)) = E\left[e^{-r(T-t)}(S(T)-K)_+ \mid \mathcal{F}(t)\right]$$

$$S(t) = S(0)\, e^{(r-\sigma^2/2)t + \sigma W(t)}$$

$$c(0,S(0)) = E\left[e^{-rT}(S(T)-K)_+\right]$$

$$= E\left[e^{-rT}\left(S(0)e^{(r-\sigma^2/2)T+\sigma W(T)} - K\right)_+\right]$$

$$= \int_{-\infty}^{\infty} e^{-rT}\left(S(0)e^{(r-\sigma^2/2)T+\sigma x} - K\right)_+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx$$

$$S(0)e^{(r-\sigma^2/2)T+\sigma x} - K > 0$$

$$e^{(r-\sigma^2/2)T+\sigma x} > \frac{K}{S(0)}$$

$$\left(r-\frac{\sigma^2}{2}\right)T + \sigma x > \log\left(\frac{K}{S(0)}\right)$$

$$x > \frac{\log\left(\frac{K}{S(0)}\right) - (r-\sigma^2/2)T}{\sigma} = d^*$$

$$c(0,S(0)) = \int_{d^*}^{\infty} e^{-rT}\left(S(0)e^{(r-\sigma^2/2)T+\sigma x} - K\right)\frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx$$

$$① = \int_{d^*}^{\infty} e^{-rT} S(0)e^{(r-\sigma^2/2)T+\sigma x} \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx$$

$$② - \int_{d^*}^{\infty} e^{-rT} K \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx$$

①: $$S(0)e^{-\frac{\sigma^2 T}{2}}\int_{d^*}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{\frac{-x^2+2x\sigma T - \sigma^2T^2 + \sigma^2T^2}{2T}}\,dx$$

$$= S(0)\int_{d^*}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{\frac{-(x-\sigma T)^2}{2T}}\,dx$$

$X \sim N(\mu,\sigma^2)$ $\quad \frac{X-\mu}{\sigma} = Z \sim N(0,1)$ $\quad \frac{x-\sigma T}{\sqrt{T}} = z$ $\quad \frac{dx}{\sqrt{T}} = dz$

$$= S(0)\int_{\frac{d^*-\sigma T}{\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= S(0)\int_{-\infty}^{\frac{-d^*+\sigma T}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$\frac{-d^*+\sigma T}{\sqrt{T}} = \frac{-\frac{1}{\sigma}\left(\log\left(\frac{K}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)T\right) + \sigma T}{\sqrt{T}}$$

$$= \frac{\log\left(\frac{S(0)}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)T + \sigma^2 T}{\sigma\sqrt{T}}$$

$$= \frac{\log\left(\frac{S(0)}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_+$$

$$① = S(0)N(d_+)$$

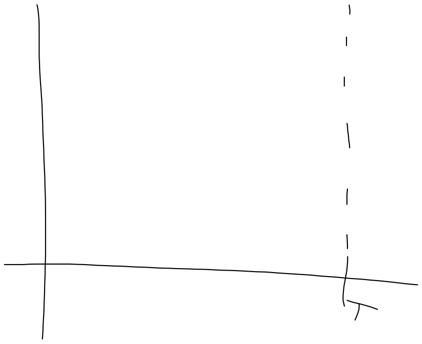
②: $$-\int_{d^*}^{\infty} e^{-rT} K \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx$$

$$= -e^{-rT}K\int_{d^*}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\,dx \qquad \frac{x}{\sqrt{T}} = z$$

$$= -e^{-rT}K\int_{\frac{d^*}{\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= -Ke^{-rT}\int_{-\infty}^{\frac{-d^*}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$\frac{-d^*}{\sqrt{T}} = \frac{\log\left(\frac{S(0)}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_-$$



put option

$$\lim_{t\to T} p(t,x) = (K-x)_+$$

$$\lim_{x\to 0} p(t,x) \approx K$$

$$\lim_{x\to\infty} p(t,x) = 0$$

Portfolio A: buy one share + sell "K" zero-coupon bonds

$$A(T) = S(T) - K$$

$$A(0) = S(0) - Ke^{-rT}$$

$$A(t) = S(t) - Ke^{-r(T-t)}$$

Portfolio B: buy one call + short one put (K)

$$B(0) = c(0,S(0)) - p(0,S(0))$$

$$B(t) = c(t,S(t)) - p(t,S(t))$$

$$B(T) = (S(T)-K)_+ - (K-S(T))_+$$

$$= \begin{cases} S(T)-K & , \text{ if } S(T)>K \\ S(T)-K & , \text{ if } K \geq S(T) \end{cases}$$

$$p(t,S(t)) = c(t,S(t)) - S(t) + Ke^{-r(T-t)}$$

$$= S(t)N(d_+) - Ke^{-r(T-t)}N(d_-) - S(t) + Ke^{-r(T-t)}$$

$$= Ke^{-r(T-t)}\left(1-N(d_-)\right) - S(t)\left(1-N(d_+)\right)$$

$$= Ke^{-r(T-t)}N(-d_-) - S(t)N(-d_+)$$

$$\lim_{t\to T} p(t,x) = \begin{cases} 0 & , x>K \\ K-x & , x<K \end{cases}$$

$$d_+ = \frac{\log\left(\frac{x}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$= (K-x)_+$$

$$\lim_{x\to 0} p(t,x) = Ke^{-r(T-t)}$$

$$\lim_{x\to\infty} p(t,x) = 0$$