$$W(u) - W(t) = \{ W_1(u) - W_1(t), W_2(u) - W_2(t), \dots, W_d(u) - W_d(t) \}$$

$$X_n \xrightarrow{\mathcal{L}^2} Y : \quad \lim_{n\to\infty} E[X_n] = E[Y]$$

$$\lim_{n\to\infty} Var(X_n) = Var(Y)$$

WTP $[W_i, W_j](t) = 0$

Consider $$C_\Pi = \sum_{k=0}^{n-1} \left(W_i(t_{k+1}) - W_i(t_k)\right)\left(W_j(t_{k+1}) - W_j(t_k)\right)$$

$$X_n = C_\Pi \qquad Y = 0,\ E[Y] = 0,\ Var(Y) = 0$$

$$E[C_\Pi] = E\left[\sum_{k=0}^{n-1} \left(W_i(t_{k+1}) - W_i(t_k)\right)\left(W_j(t_{k+1}) - W_j(t_k)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[W_i(t_{k+1}) - W_i(t_k)]\, E[W_j(t_{k+1}) - W_j(t_k)]$$

$$= 0$$

$$\lim_{\|\Pi\|\to 0} E[C_\Pi] = 0$$

$$Var(C_\Pi) = E[C_\Pi^2] - E[C_\Pi]^2$$

$$= E[C_\Pi^2]$$

$$E[C_\Pi^2] = E\Big[\sum_{k=0}^{n-1} \left(W_i(t_{k+1}) - W_i(t_k)\right)^2 \left(W_j(t_{k+1}) - W_j(t_k)\right)^2$$

$$+ 2 \sum_{0\le \ell < k \le n-1} \left(W_i(t_{k+1}) - W_i(t_k)\right)\left(W_i(t_{\ell+1}) - W_i(t_\ell)\right)\left(W_j(t_{k+1}) - W_j(t_k)\right) \cdot \left(W_j(t_{\ell+1}) - W_j(t_\ell)\right)\Big]$$

$$= \sum_{k=0}^{n-1} E\left[\left(W_i(t_{k+1}) - W_i(t_k)\right)^2\right] E\left[\left(W_j(t_{k+1}) - W_j(t_k)\right)^2\right]$$

$$= \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2$$

$$\lim_{\|\Pi\|\to 0} Var(C_\Pi) = \lim_{\|\Pi\|\to 0} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 = 0$$

$$\Rightarrow [W_i, W_j](t) = 0 \quad a.s.$$

If $\quad W_i(t) = W_j(t) \quad \forall t \in [0, T]$

$$X(t) = X(0) + I_1(t) + I_2(t) + R(t)$$

$$[X,X](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(X(t_{j+1}) - X(t_j)\right)^2$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(I_1(t_{j+1}) - I_1(t_j) + I_2(t_{j+1}) - I_2(t_j) + R(t_{j+1}) - R(t_j) \right)^2$$

$$= [I_1, I_1](t) + 2[I_1, I_2](t) + 2[I_1, R](t) + [I_2, I_2](t) + 2[I_2, R](t) + [R,R](t)$$

$$[I_1, I_2](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(I_1(t_{j+1}) - I_1(t_j)\right)\left(I_2(t_{j+1}) - I_2(t_j)\right)$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(\int_{t_j}^{t_{j+1}} \sigma_{1,1}(u)\,dW_1(u)\right)\left(\int_{t_j}^{t_{j+1}} \sigma_{1,2}(u)\,dW_2(u)\right)$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \sigma_{1,1}(t_j)\left(W_1(t_{j+1}) - W_1(t_j)\right) \sigma_{1,2}(t_j)\left(W_2(t_{j+1}) - W_2(t_j)\right)$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \sigma_{1,1}(t_j)\sigma_{1,2}(t_j)\left(W_1(t_{j+1}) - W_1(t_j)\right)\left(W_2(t_{j+1}) - W_2(t_j)\right)$$

$$= \int_0^t \sigma_{1,1}(u)\sigma_{1,2}(u)\underbrace{dW_1(u)\,dW_2(u)}_{=0}$$

$$= 0$$

$$[X,X](t) = [I_1, I_1](t) + [I_2, I_2](t)$$

$$dX(t)\,dX(t) = \left(\sigma_{1,1}(t)\,dW_1(t) + \sigma_{1,2}(t)\,dW_2(t) + \Theta_1(t)\,dt\right)^2$$

$$= \sigma_{1,1}^2(t)\,dt + \sigma_{1,2}^2(t)\,dt$$

$$= d[X,X](t)$$

$$\int_0^t d[X,X](u) = [X,X](t) - [X,X](0)$$

$$= \int_0^t \left(\sigma_{1,1}^2(u) + \sigma_{1,2}^2(u)\right)du \quad ?$$

$$f(t_{j+1}, X_{j+1}, Y_{j+1}) = f(t_j, X_j, Y_j) + f_t(t_j, X_j, Y_j)(t_{j+1} - t_j)$$
$$+ f_x(t_j, X_j, Y_j)(X_{j+1} - X_j) + f_y(t_j, X_j, Y_j)(Y_{j+1} - Y_j)$$
$$+ \frac{1}{2} f_{tt}(t_j, X_j, Y_j)(t_{j+1} - t_j)^2 + \dots$$
$$+ \frac{1}{6} f_{ttt}(t_j, X_j, Y_j)(t_{j+1} - t_j)^3 + \dots$$

Corollary: $f(t, x, y) = xy$

$f_t = 0$

$d[X(t)Y(t)] = ?$

$f_x = y$

$f_y = x$

$f_{xx} = 0$

$f_{xy} = 1$

$f_{yy} = 0$

$$d[X(t)Y(t)] = 0\,dt + Y(t)\,dX(t) + X(t)\,dY(t)$$
$$+ \frac{1}{2}(0)(dX(t))^2 + 1\,dX(t)\,dY(t) + \frac{1}{2}(0)(dY(t))^2$$
$$= Y(t)\,dX(t) + X(t)\,dY(t) + dX(t)\,dY(t)$$

$$d(e^{-rt}X(t)) = e^{-rt}\,dX(t) + X(t)(-re^{-rt}dt)$$
$$+ 0$$

$$d\, f(t, M(t)) = f_t(t, M(t))\,dt + f_x(t, M(t))\,dM(t) + \tfrac{1}{2} f_{xx}(t, M(t))\, \underbrace{d[M, M](t)}_{dt}$$

$$= \left(f_t(t, M(t)) + \tfrac{1}{2} f_{xx}(t, M(t))\right)dt + f_x(t, M(t))\,dM(t)$$

$$f(t, M(t)) = f(0, M(0)) + \int_0^t \left(f_t(u, M(u)) + \tfrac{1}{2} f_{xx}(u, M(u))\right)du + \int_0^t f_x(u, M(u))\,dM(u)$$

$$E[f(t, M(t))] = f(0,0) + E\left[\int_0^t \left(f_t(u, M(u)) + \tfrac{1}{2} f_{xx}(u, M(u))\right)du\right] + \underbrace{E\left[\int_0^t f_x(u, M(u))\,dM(u)\right]}_{=0}$$

$$f(t,x) = e^{sx - \frac{1}{2}s^2 t} \quad \text{for a d.u. } s$$

$$f_t = -\tfrac{1}{2} s^2 f(t,x)$$

$$f_x = s\, f(t,x)$$

$$f_{xx} = s^2 f(t,x)$$

$$E\left[e^{sM(t) - \frac{1}{2}s^2 t}\right] = 1 + E\left[\int_0^t \underbrace{\left(-\tfrac{1}{2} s^2 f(u, M(u)) + \tfrac{1}{2} s^2 f(u, M(u))\right)}_{=0} du\right]$$

$$E\left[e^{sM(t) - \frac{1}{2}s^2 t}\right] = 1$$

$$\underbrace{E\left[e^{sM(t)}\right]}_{\varphi_{M(t)}(s)} = \underbrace{e^{\frac{1}{2}s^2 t}}_{\text{MGF of } N(0,t)}$$

$$d\, f(t, M_1(t), M_2(t)) = f_t\, dt + f_x\, dM_1(t) + f_y\, dM_2(t) + \tfrac{1}{2} f_{xx} \underbrace{(dM_1(t))^2}_{dt} + f_{xy} \underbrace{dM_1(t)\,dM_2(t)}_{=0} + \tfrac{1}{2} f_{yy} \underbrace{(dM_2(t))^2}_{dt}$$

$$E[f(t, M_1(t), M_2(t))] = f(0,0,0) + E\left[\int_0^t \left(f_t + \tfrac{1}{2} f_{xx} + \tfrac{1}{2} f_{yy}\right)du\right] + 0 + 0$$

$$f(t,x,y) = e^{u_1 x + u_2 y - \frac{1}{2}(u_1^2 + u_2^2)t}$$

$$f_t = -\tfrac{1}{2}(u_1^2 + u_2^2)\, f(t,x,y)$$

$$f_x = u_1 f$$

$$f_{xx} = u_1^2 f$$

$$f_y = u_2 f$$

$$f_{yy} = u_2^2 f$$

$$E\left[e^{u_1 M_1(t) + u_2 M_2(t) - \frac{1}{2}(u_1^2 + u_2^2)t}\right] = 1$$

$$E\left[e^{u_1 M_1(t) + u_2 M_2(t)}\right] = e^{\frac{1}{2}u_1^2 t}\, e^{\frac{1}{2}u_2^2 t}$$

$$dB_1(t) = dW_1(t)$$
$$dB_2(t) = \rho\, dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)$$

$$\Rightarrow B_1(t) = W_1(t)$$
$$B_2(t) = \int_0^t \rho\, dW_1(u) + \int_0^t \sqrt{1-\rho^2}\, dW_2(u)$$
$$= \rho W_1(t) + \sqrt{1-\rho^2}\, W_2(t)$$

Is $B_2(t)$ BM?

① $B_2(0) = 0$? ✓

② $B_2(t)$ has cont. paths? ✓

③ $B_2(t)$ is Mart? ✓

$$E[B_2(t) \mid \mathcal{F}(s)]$$
$$= E[\rho W_1(t) + \sqrt{1-\rho^2}\, W_2(t) \mid \mathcal{F}(s)]$$
$$= \rho E[W_1(t) \mid \mathcal{F}(s)] + \sqrt{1-\rho^2}\, E[W_2(t) \mid \mathcal{F}(s)]$$
$$= \rho W_1(s) + \sqrt{1-\rho^2}\, W_2(s) = B_2(s)$$

④ $[B_2, B_2](t) = t$? ✓

$$[B_2, B_2](t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(B_2(t_{j+1}) - B_2(t_j)\right)^2$$
$$= \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(\rho\left(W_1(t_{j+1}) - W_1(t_j)\right) + \sqrt{1-\rho^2}\left(W_2(t_{j+1}) - W_2(t_j)\right)\right)^2$$
$$= \rho^2 [W_1, W_1](t) + 2\rho\sqrt{1-\rho^2}\,[W_1, W_2](t) + (1-\rho^2)[W_2, W_2](t)$$
$$= t$$

$$(dB_2(t))^2 = \left(\rho\, dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)\right)^2$$
$$= dt$$

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1\, dt + \sigma_1\, dB_1(t)$$
$$\frac{dS_2(t)}{S_2(t)} = \alpha_2\, dt + \sigma_2\, dB_2(t)$$

where $B_1(t)$ + $B_2(t)$ are B.M.'s

$$\mathrm{Corr}(X,Y) = \rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

$$\mathrm{Cov}(B_1(t), B_2(t)) = E[B_1(t) B_2(t)] - E[B_1(t)]E[B_2(t)]$$
$$= E[B_1(t) B_2(t)]$$

$$d(B_1(t) B_2(t)) = B_1(t)\, dB_2(t) + B_2(t)\, dB_1(t) + dB_1(t)\, dB_2(t)$$

$$B_1(t) B_2(t) = \int_0^t B_1(u)\, dB_2(u) + \int_0^t B_2(u)\, dB_1(u) + \int_0^t dW_1(u)\left(\rho\, dW_1(u) + \sqrt{1-\rho^2}\, dW_2(u)\right)$$

$$E[B_1(t) B_2(t)] = E\left[\int_0^t \rho\, du\right]$$
$$= \rho t$$

$$\mathrm{Cov}(B_1, B_2) = \rho t$$

$$\rho_{B_1(t), B_2(t)} = \frac{\rho t}{t} = \rho$$

$$dB_1(t)\, dB_2(t) = \rho\, dt$$