$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f(x) - f(a) = f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots$$

Let $a = x_j$, $x = x_{j+1}$

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j) + \frac{1}{6} f'''(x_j)(x_{j+1} - x_j) + \ldots$$

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$

$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} \left(f(W(t_{j+1})) - f(W(t_j)) \right)$$

Let $x_j = W(t_j)$, $x_{j+1} = W(t_{j+1})$

$$f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} f'(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))}_{dW(t)}$$
$$+ \frac{1}{2} \sum_{j=0}^{n-1} f''(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))^2}_{(dW(t))^2}$$
$$+ \frac{1}{6} \sum_{j=0}^{n-1} f'''(W(t_j)) \underbrace{(W(t_{j+1}) - W(t_j))^3}_{(dW(t))^3} + \ldots$$

$\lim_{\|\Pi\| \to 0}$

$$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))\,dW(t) + \frac{1}{2}\int_0^T f''(W(t)) \underbrace{(dW(t))^2}_{dt}$$
$$+ \frac{1}{6}\int_0^T f'''(W(t)) \underbrace{(dW(t))^3}_{=0} + \underbrace{\ldots}_{=0}$$

$$f(t_{j+1}, X_{j+1}) - f(t_j, x_j) = f_t(t_j,x_j)(t_{j+1}-t_j)$$
$$+ f_x(t_j,x_j)(x_{j+1}-x_j) + \frac{1}{2} f_{tt}(t_j,x_j)(t_{j+1}-t_j)^2$$
$$+\frac{1}{2} f_{xt}(t_j,x_j)(x_{j+1}-x_j)(t_{j+1}-t_j) + \frac{1}{2} f_{tx}(t_j,x_j)(t_{j+1}-t_j)(x_{j+1}-x_j)$$
$$+\frac{1}{2} f_{xx}(t_j,x_j)(x_{j+1}-x_j)^2 + \frac{1}{6} \ldots$$

$$f(t,W(t)) - f(0,W(0)) = \int_0^t f_t(u,W(u))\,du + \int_0^t f_x(u,W(u))\,dW(u)$$
$$+ \frac{1}{2}\int_0^t f_{tt}(u,W(u))\underbrace{(du)^2}_{=0} + \frac{1}{2}\int_0^t f_{tx}(u,W(u))\underbrace{du\,dW(u)}_{=0}$$
$$+ \frac{1}{2}\int_0^t f_{xt}(u,W(u))\underbrace{dW(u)\,du}_{=0} + \frac{1}{2}\int_0^t f_{xx}(u,W(u))\underbrace{(dW(u))^2}_{du}$$

$$\int_0^T f_x(t,W(t))\,dW(t) = \int_0^T W(t)\,dW(t)$$

$$\Rightarrow f_x(t,W(t)) = W(t)$$

$$f_x(t,x) = x$$
$$f(t,x) = \frac{1}{2}x^2 + g(t) + C$$
$$f_t(t,x) = g'(t)$$
$$f_{xx}(t,x) = 1$$

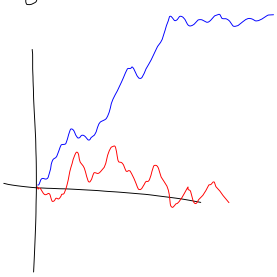
$$\frac{1}{2}W^2(T) + g(T) + C - \frac{1}{2}W^2(0) - g(0) - C$$
$$= \int_0^T g'(t)\,dt + \int_0^T W(t)\,dW(t) + \frac{1}{2}\int_0^T dt$$

$$\frac{1}{2}W^2(T)\ \cancel{+g(T) - g(0) - \int_0^T g'(t)\,dt} - \frac{1}{2}\underbrace{\int_0^T dt}_{T}$$
$$= \int_0^T W(t)\,dW(t)$$
$$= \frac{1}{2}W^2(T) - T/2$$

$$\int_0^t \cos(W(u))\,dW(u)$$

$$f_b(a,b) = \cos b$$
$$f(a,b) = \sin b$$
$$f_a(a,b) = 0$$
$$f_{bb}(a,b) = -\sin b$$

$$\int_0^t \cos(W(u))\,dW(u) = \sin(W(t)) + \frac{1}{2}\underbrace{\int_0^t \sin(W(u))\,du}$$



$$\int_0^t W(u)\,du$$

$$\int_0^\infty \sin(t)\,dt$$

$$X(t) = X(0) + \underbrace{I(t)}_{\text{Itô}} + \underbrace{R(t)}_{\text{Riemann}}$$

$$[X,X](t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(X(t_{j+1}) - X(t_j) \right)^2$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(X(0) + I(t_{j+1}) + R(t_{j+1}) - X(0) - I(t_j) - R(t_j) \right)^2$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(\left(I(t_{j+1}) - I(t_j) \right) + \left(R(t_{j+1}) - R(t_j) \right) \right)^2$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(I(t_{j+1}) - I(t_j) \right)^2$$

$$+ 2 \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(I(t_{j+1}) - I(t_j) \right)\left(R(t_{j+1}) - R(t_j) \right) \le \lim_{\|\Pi\|\to 0} \underbrace{\max_{0\le k\le n-1} \left(I(t_{k+1}) - I(t_k) \right)}_{=0} \sum_{j=0}^{n-1} \left(R(t_{j+1}) - R(t_j) \right)$$

$$+ \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(R(t_{j+1}) - R(t_j) \right)^2$$

$$= [I,I](t) + 2\underbrace{[I,R](t)}_{=0} + \underbrace{[R,R](t)}_{=0}$$

$$= [I,I](t) = \int_0^t \Delta^2(u)\,du = [X,X](t)$$

$$d[X,X](t) = \Delta^2(t)\,dt = d[I,I](t)$$

$$dX(t) = \Delta(t)\,dW(t) + \Theta(t)\,dt$$

$$\left(dX(t) \right)^2 = \Delta^2(t)\,dt$$

Is $W(t)$ an Itô process?

$X(0) = 0$, $\Theta(u) = 0$, $\Delta(u) = 1$

$$X(t) = \int_0^t dW(u) + 0 = W(t) - W(0) = W(t)$$

$$S(t) = S(0)e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$$S(t) = f(t, W(t))$$

$$f(t,x) = S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma x}$$

$$f_t = (\alpha - \sigma^2/2)\, f(t,x)$$

$$f_x = \sigma f(t,x)$$

$$f_{xx} = \sigma^2 f(t,x)$$

$$dS(t) = (\alpha - \sigma^2/2)\, S(t)dt + \sigma S(t)dW(t) + \tfrac{1}{2}\sigma^2 S(t)dt$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma dW(t)$$

$$S(t) = S(0) + \int_0^t \alpha S(u)du + \int_0^t \sigma S(u)dW(u)$$

$$S(t) = S(0)e^{\int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dW(u)}$$

$$dS(t) = ? = \cancel{f(t, W(t))}$$

$$X(t) = \log S(0) + \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dW(u)$$

$$dX(t) = \left(\alpha(t) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dW(t)$$

$$S(t) = e^{X(t)} = g(t, X(t))$$

$$d[X,X](t) = \sigma^2(t)dt$$

$$g(s,y) = e^y$$

$$g_s(s,y) = 0$$

$$g_y(s,y) = e^y$$

$$g_{yy}(s,y) = e^y$$

$$\begin{aligned} dS(t) &= S(t)dX(t) + \tfrac{1}{2} S(t)\, d[X,X](t) \\ &= S(t)\left(\alpha(t) - \frac{\sigma^2(t)}{2}\right)dt + S(t)\sigma(t)dW(t) \\ &\quad + \tfrac{1}{2} S(t)\sigma^2(t)dt \end{aligned}$$

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

$$R(t) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\, dW(s)$$

$$R(t) = f(t, X(t)) \quad ; \quad X(t) = \int_0^t e^{\beta s}\, dW(s)$$

$$f(t,x) = R(0)e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} x$$

$$f_t = -\beta R(0) e^{-\beta t} + \alpha e^{-\beta t} - \beta \sigma e^{-\beta t} x$$

$$f_x = \sigma e^{-\beta t}$$

$$f_{xx} = 0$$

$$dR(t) = \left(-\beta R(0) e^{-\beta t} + \alpha e^{-\beta t} - \beta \sigma e^{-\beta t} X(t)\right) dt + \sigma e^{-\beta t}\, dX(t)$$

$$= -\beta\left(R(0)e^{-\beta t} - \frac{\alpha}{\beta} e^{-\beta t} + \sigma e^{-\beta t} X(t)\right) dt + \sigma\, dW(t)$$

$$+\frac{\alpha}{\beta} - \frac{\alpha}{\beta}$$

$$= -\beta\left(R(t) - \frac{\alpha}{\beta}\right) dt + \sigma\, dW(t)$$

$$= \left(\alpha - \beta R(t)\right) dt + \sigma\, dW(t)$$

$$\cancel{R(t) = R(0) + \int_0^t (\alpha - \beta R(u))\,du + \int_0^t \sigma\sqrt{R(u)}\,dW(u)}$$

$$\cancel{E[R(t)] = R(0) + E\left[\int_0^t (\alpha - \beta R(u))\,du\right]}$$

$$Y(t) = e^{\beta t} R(t) \Rightarrow dY(t) = d\,f(t, R(t))$$

$$f(t,x) = x e^{\beta t}$$
$$f_t = \beta x e^{\beta t}$$
$$f_x = e^{\beta t}$$
$$f_{xx} = 0$$

$$dY(t) = d(e^{\beta t} R(t)) = \beta R(t) e^{\beta t}\,dt + e^{\beta t}\,dR(t)$$

$$= e^{\beta t}\left[\beta R(t)\,dt + (\alpha - \beta R(t))\,dt + \sigma\sqrt{R(t)}\,dW(t)\right]$$

$$= e^{\beta t}\left[\alpha\,dt + \sigma\sqrt{R(t)}\,dW(t)\right]$$

$$d(e^{\beta t} R(t)) = \alpha e^{\beta t}\,dt + e^{\beta t}\sigma\sqrt{R(t)}\,dW(t)$$

$$e^{\beta t} R(t) - R(0) = \int_0^t \alpha e^{\beta s}\,ds + \int_0^t e^{\beta s}\sigma\sqrt{R(s)}\,dW(s)$$

$$R(t) = R(0)e^{-\beta t} + e^{-\beta t}\int_0^t \alpha e^{\beta s}\,ds + e^{-\beta t}\int_0^t e^{\beta s}\sigma\sqrt{R(s)}\,dW(s)$$

$$E[R(t)] = e^{-\beta t} R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + 0$$