Show that for all f that there exists a g such that

$$E[f(W(t)) \mid \mathcal{F}(s)] = g(W(s))$$

$$E[f(W(t)) \mid \mathcal{F}(s)] = E[f(W(t)-W(s)+W(s)) \mid \mathcal{F}(s)]$$

Let x be a dummy variable: i.e. $f(x) = x^2$

$$E[f(\underbrace{W(t)-W(s)}_{\text{ind. of } \mathcal{F}(s)} + x) \mid \mathcal{F}(s)]$$

$$= E[f(\underbrace{W(t)-W(s)}_{\sim N(0,t-s)} + x)]$$

$$\int_0^t x^2\,dx = \frac{t^3}{3}$$

$$= \int_{-\infty}^{\infty} f(w+x)\frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{w^2}{2(t-s)}}\,dw$$

$$= g(x)$$

$$g(W(s)) = E[f(W(t)-W(s)+W(s)) \mid \mathcal{F}(s)]$$
$$= E[f(W(t)) \mid \mathcal{F}(s)]$$

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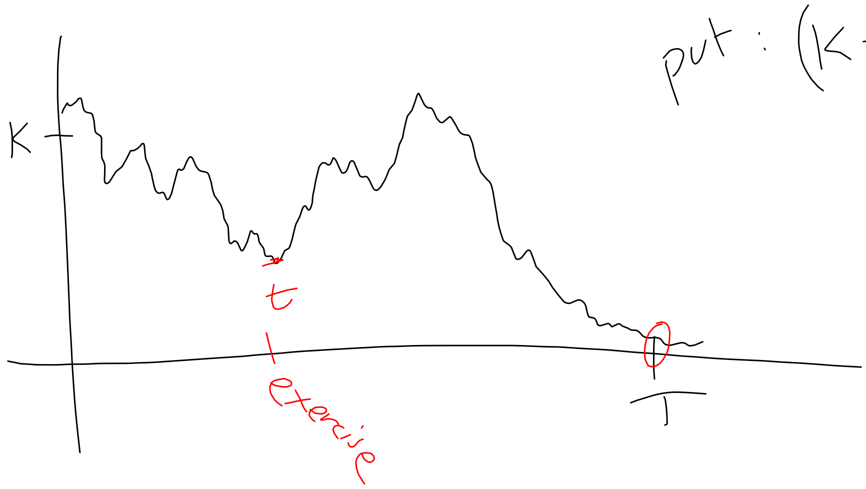
put: $(K - S(t))$



$$Z(t) = e^{\sigma W(t) - \frac{\sigma^2}{2} t}$$

$$E[Z(t) \mid \mathcal{F}(s)]$$

$$= E[Z(t) - Z(s) + Z(s) \mid \mathcal{F}(s)]$$

$$= E\Big[\underbrace{e^{\sigma W(t) - \frac{\sigma^2}{2} t} - e^{\sigma W(s) - \frac{\sigma^2}{2} s}}_{\text{not ind.}} + \underbrace{e^{\sigma W(s) - \frac{\sigma^2}{2} s}}_{\text{meas.}} \;\Big|\; \mathcal{F}(s)\Big]$$

$$= E\left[\frac{Z(t)}{Z(s)} \cdot Z(s) \;\middle|\; \mathcal{F}(s)\right]$$

$$= E\Big[\underbrace{e^{\sigma(W(t) - W(s)) - \frac{\sigma^2}{2}(t-s)}}_{\text{ind. of } \mathcal{F}(s)} \cdot \underbrace{Z(s)}_{\text{meas.}} \;\Big|\; \mathcal{F}(s)\Big]$$

$$= Z(s)\, e^{-\frac{\sigma^2}{2}(t-s)}\, E\left[e^{\sigma(W(t) - W(s))} \;\middle|\; \cancel{\mathcal{F}(s)}\right]$$

$$\varphi_Y(u) = E[e^{uY}]$$

$$Y \sim N(0, t-s)$$

$$\varphi_Y(u) = e^{\frac{1}{2} u^2 (t-s)}$$

$$E[Z(t) \mid \mathcal{F}(s)] = Z(s)\, e^{-\frac{\sigma^2}{2}(t-s)} \left(e^{\frac{1}{2}\sigma^2 (t-s)}\right)$$

$$= Z(s)$$

$$E[Z(t) \mid \mathcal{F}(s)] = E\left[e^{\sigma W(t) - \frac{\sigma^2}{2} t} \;\middle|\; \mathcal{F}(s)\right]$$

$$= e^{-\frac{\sigma^2}{2} t}\, E\left[e^{\sigma W(t)} \;\middle|\; \mathcal{F}(s)\right]$$

$$= e^{-\frac{\sigma^2}{2} t}\, E\left[e^{\sigma(W(t) - W(s))}\, e^{\sigma W(s)} \;\middle|\; \mathcal{F}(s)\right]$$

$$= e^{\sigma W(s) - \frac{\sigma^2}{2} t}\, E\left[e^{\sigma(W(t) - W(s))}\right]$$

$$= e^{\sigma W(s) - \frac{\sigma^2}{2} t}\, e^{\frac{1}{2}\sigma^2 (t-s)}$$

$$= e^{\sigma W(s) - \frac{1}{2}\sigma^2 s} = Z(s)$$

$$Z(t\wedge\tau_m) = e^{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)} \quad (x\wedge y) = \min(x,y)$$

$$E[Z(t\wedge\tau_m)] = E\{Z(t\wedge\tau_m) \mid \mathcal{F}(0)\}$$
$$= Z(0\wedge\tau_m) = Z(0) = 1$$

$$\lim_{t\to\infty} E\{Z(t\wedge\tau_m)\} = 1$$
$$\Rightarrow E\left[\lim_{t\to\infty} Z(t\wedge\tau_m)\right] = 1$$

$$\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)}$$

$$\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m)} : \begin{cases} \tau_m < \infty : & \lim_{t\to\infty}(t\wedge\tau_m) = \tau_m \\ & e^{\sigma W(\tau_m)} = e^{\sigma m} \\ \tau_m = \infty : & (t\wedge\tau_m) = t \\ & 0 < \lim_{t\to\infty} e^{\sigma W(t)} < e^{\sigma m} \end{cases}$$

$$\lim_{t\to\infty} e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} : \begin{cases} \tau_m < \infty, & \lim_{t\to\infty} e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} = e^{-\frac{1}{2}\sigma^2\tau_m} \\ \tau_m = \infty, & \lim_{t\to\infty} e^{-\frac{1}{2}\sigma^2 t} = 0 \end{cases}$$

$$\lim_{t\to\infty} e^{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)} = e^{\sigma m - \frac{1}{2}\sigma^2\tau_m} \mathbb{1}_{\{\tau_m<\infty\}}$$

$$\mathbb{1}_{\{A\}} = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases} \qquad E\{\mathbb{1}_{\{A\}}\} = \mathbb{P}(A)$$

$$\lim_{t\to\infty} E\{Z(t\wedge\tau_m)\} = E\left\{e^{\sigma m - \frac{1}{2}\sigma^2\tau_m}\mathbb{1}_{\{\tau_m<\infty\}}\right\} = 1$$

let $\sigma \searrow 0$

$$E\{\mathbb{1}_{\{\tau_m<\infty\}}\} = 1$$
$$\Rightarrow \mathbb{P}(\tau_m < \infty) = 1$$

$$\Rightarrow E\left[e^{\sigma m - \frac{1}{2}\sigma^2\tau_m}\right] = 1$$

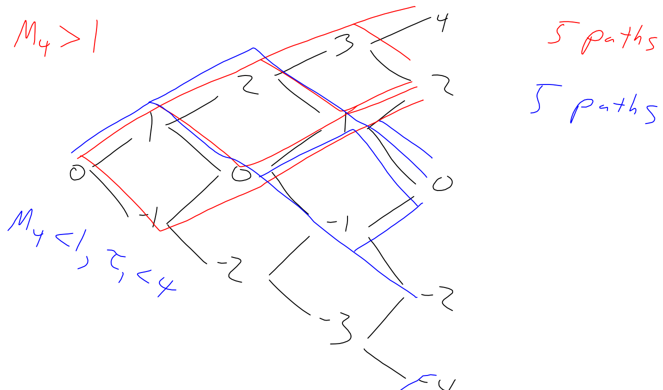
let $\alpha = \frac{1}{2}\sigma^2$, $\sigma = \sqrt{2\alpha}$

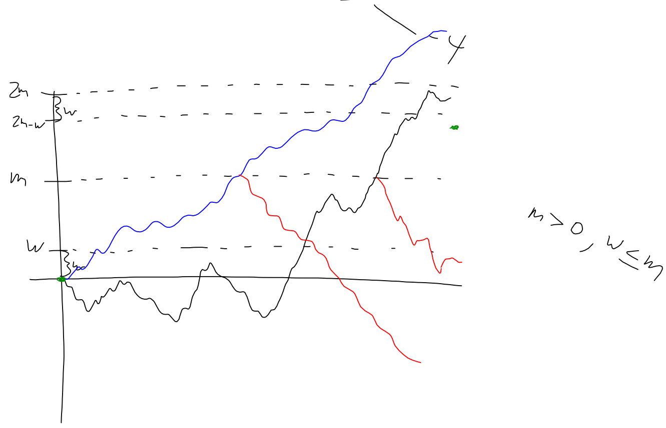
$$E\left[e^{m\sqrt{2\alpha} - \alpha\tau_m}\right] = 1$$

$$\Rightarrow E\left[e^{-\alpha\tau_m}\right] = e^{-|m|\sqrt{2\alpha}}$$

$$E\left[-\tau_m e^{-\alpha\tau_m}\right] = -\frac{|m|\sqrt{2}}{2\sqrt{\alpha}} e^{-|m|\sqrt{2\alpha}}$$

$$\lim_{\alpha\searrow 0} : \quad E[\tau_m] = \frac{|m|}{\sqrt{2}} \lim_{\alpha\searrow 0} \frac{1}{\sqrt{\alpha}} = \infty$$





let $w = m$

$$\begin{cases} \mathbb{P}(\tau_m \le t,\ W(t) \le m) = \mathbb{P}(W(t) \ge m) \\ \mathbb{P}(\tau_m \le t,\ W(t) > m) = \mathbb{P}(W(t) \ge m) \end{cases}$$

$$\{\tau_m \le t, W(t) \le m\} \cap \{\tau_m \le t, W(t) > m\} = \emptyset$$

$$\{\tau_m \le t, W(t) \le m\} \cup \{\tau_m \le t, W(t) > m\} = \{\tau_m \le t\}$$

$$\mathbb{P}(\tau_m \le t) = 2\,\mathbb{P}(W(t) \ge m)$$

$$= 2\int_{|m|}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx$$

$$\frac{d}{dx}\int_a^x g(y)\,dy = g(x)$$

$$\frac{d}{dx}\int_x^a g(y)\,dy = -g(x)$$

$$\frac{d}{dx}\int_a^{x^2} g(y)\,dy = 2x\,g(x^2)$$

$$z = \frac{x-\mu}{\sigma} = \frac{x}{\sqrt{t}}$$

$$\mathbb{P}(\tau_m \le t) = 2\int_{\frac{|m|}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

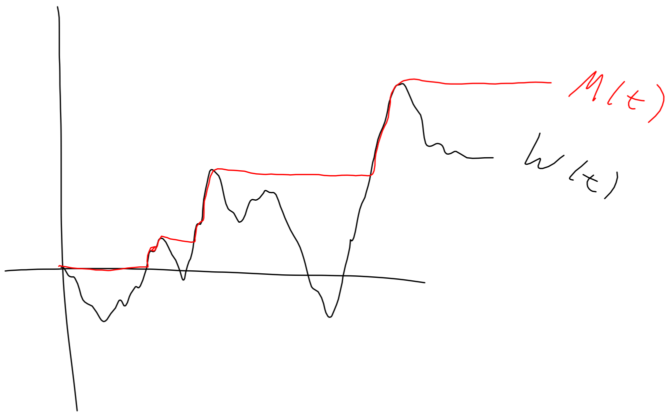
$$f_{\tau_m}(t) = -\frac{|m|\,2}{(-2)\,t^{3/2}}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{m^2}{2t}}$$

$$= \frac{|m|}{t\sqrt{2\pi t}}\,e^{-\frac{m^2}{2t}}$$

$$E\left[e^{-\alpha \tau_m}\right] = \int_0^\infty e^{-\alpha t} f_{\tau_m}(t)\, dt$$

$$= \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t} - \alpha t}\, dt$$

$$= e^{-|m|\sqrt{2\alpha}}$$



if $M(t) \geq m$, then $\tau_m \leq t$

if $\tau_m \leq t$, then $M(t) \geq m$

$$\mathbb{P}\left(M(t) \geq m,\ W(t) \leq w\right) = \mathbb{P}\left(W(t) \geq 2m - w\right)$$

for $m > 0$, $w < m$

$$\int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)\, dy\, dx = \int_{2m-w}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\, dz$$

$$\Rightarrow +\int_{-\infty}^w f_{M(t),W(t)}(m,y)\, dy = +2 \frac{1}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

$$\Rightarrow f_{M(t),W(t)}(m,w) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \left(-\frac{2(2m-w)(-1)}{2t}\right)$$

$$= \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$