$$M_k - M_\ell = \sum_{i=1}^{k} X_i - \sum_{i=1}^{\ell} X_i \qquad \text{for } k > \ell$$

$$= \sum_{i=\ell+1}^{k} X_i$$

$$E[M_k - M_\ell] = E\left[\sum_{i=\ell+1}^{k} X_i\right]$$

$$= \sum_{i=\ell+1}^{k} E[X_i]$$

$$E[X_i] = \sum_{\omega \in \{H,T\}} X_i(\omega)\mathbb{P}(\omega)$$

$$= X(H)\mathbb{P}(H) + X(T)\mathbb{P}(T)$$

$$= 1\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) = 0$$

$$E[M_k - M_\ell] = \sum_{i=\ell+1}^{k} 0 = 0$$

$$Var(M_k - M_\ell) = Var\left(\sum_{i=\ell+1}^{k} X_i\right)$$

$$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + 2\alpha\beta\, Cov(X,Y) + \beta^2 Var(Y)$$

$$\overset{?}{=} \alpha Var(X) + \beta Var(Y)$$

$$\alpha = \beta = 1, \quad Cov(X,Y) = 0$$

$$Var\left(\sum_{i=\ell+1}^{k} X_i\right) = \sum_{i=\ell+1}^{k} Var(X_i)$$

$$Var(X_i) = E[X_i^2] - \underbrace{E[X_i]^2}_{=0}$$

$$E[X_i^2] = \sum_{\omega \in \{H,T\}} X_i^2(\omega)\mathbb{P}(\omega)$$

$$= \tfrac{1}{2}(1)^2 + \tfrac{1}{2}(-1)^2 = 1$$

$$Var(M_k - M_\ell) = \sum_{i=\ell+1}^{k} 1 = k - \ell$$

$$E[M(\ell) \mid \mathcal{F}(k)]$$

$$= E[\underbrace{M(\ell) - M(k)}_{\text{ind of } \mathcal{F}(k)} + \underbrace{M(k)}_{\text{measurable wrt } \mathcal{F}(k)} \mid \mathcal{F}(k)]$$

$$= E[M(\ell) - M(k) \mid \mathcal{F}(k)] + E[M(k) \mid \mathcal{F}(k)]$$

$$= E[M(\ell) - M(k)] + M(k)$$

$$= M(k)$$

Markov?

$$E[f(M(\ell)) \mid \mathcal{F}(k)] \overset{?}{=} g(M(k))$$

$\forall$ function f, there exists such a g

$$E[f(M(\ell)) \mid \mathcal{F}(k)]$$

$$= E[f(\underbrace{M(\ell) - M(k)}_{\text{ind.}} + \underbrace{M(k)}_{\text{meas.}}) \mid \mathcal{F}(k)]$$

Let x be dummy variable

$$E[f(M(\ell) - M(k) + x) \mid \mathcal{F}(k)]$$

$$= E[f(M(\ell) - M(k) + x)]$$

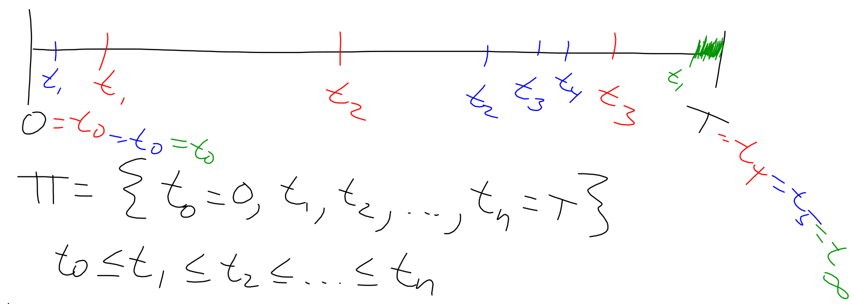
$$= \sum_{j=0}^{\ell-k} f(-(\ell-k) + 2j + x) \binom{\ell-k}{j} \left(\frac{1}{2}\right)^{\ell-k}$$

$$= g(x)$$

$$g(M(k)) = E[f(M(\ell) - M(k) + M(k)) \mid \mathcal{F}(k)]$$

$$= E[f(M(\ell)) \mid \mathcal{F}(k)]$$

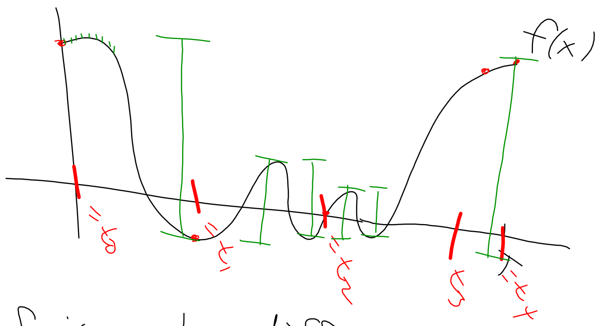
$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left| f(t_{j+1}) - f(t_j) \right|$$



$$\Pi = \{ t_0 = 0, t_1, t_2, \dots, t_n = T \}$$

$$t_0 \le t_1 \le t_2 \le \dots \le t_n$$

$$\|\Pi\| = \max_{0 \le k \le n-1} t_{k+1} - t_k$$



f is cont. + diff

then by MVT: $\exists\, t_j^* \in [t_j, t_{j+1}]$

$$f'(t_j^*) = \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j}$$

$$f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$$

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)| \underbrace{(t_{j+1} - t_j)}_{dt}$$

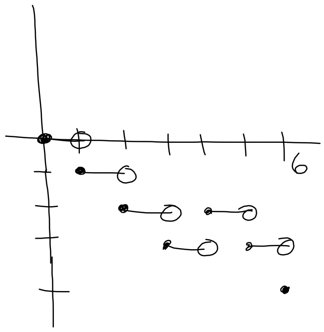
$$= \int_0^T |f'(t)|\, dt$$

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(f(t_{j+1}) - f(t_j) \right)^2$$

$$= \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} f'(t_j^*)^2 (t_{j+1} - t_j)^2$$

$$\le \lim_{\|\Pi\| \to 0} \left[\underbrace{\|\Pi\|}_{\to 0} \cdot \underbrace{\sum_{j=0}^{n-1} f'(t_j^*)^2 (t_{j+1} - t_j)}_{\int_0^T f'(t)^2 dt < \infty} \right]$$

$$= 0$$



$$[M,M](6) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(M(t_{j+1}) - M(t_j) \right)^2$$

$$= \sum_{j=0}^{5} \left(M(j+1) - M(j) \right)^2$$

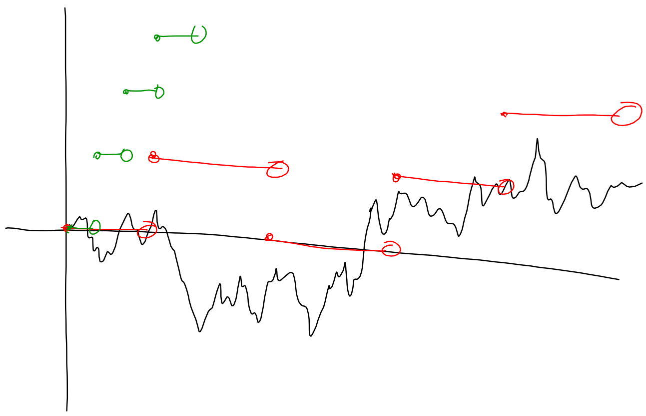
$$= \sum_{j=0}^{5} \left(X_{j+1} \right)^2 = 6$$

$$N(k) = \sum_{j=1}^{k} Y_j \ , \quad Y_j(\omega_j) = \begin{cases} 1 & \omega \in \{1,2\} \\ 0 & \omega \in \{3,4\} \\ -1 & \omega \in \{5,6\} \end{cases}$$

$$[N,N](k) = \sum_{j=0}^{k-1} \left(N(j+1) - N(j) \right)^2$$

$$= \sum_{j=0}^{k-1} Y_{j+1}^2$$

$$Var\left(N(k)\right) = \sum_{j=1}^{k} Var(Y_j) = \frac{2k}{3}$$



$$E\left[e^{Xt} \right] = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2 + 2x\mu - \mu^2 + 2x\sigma^2 t}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2 + 2x(\mu + \sigma^2 t) - \mu^2}{2\sigma^2}} dx$$

$$\text{WTP: } \lim_{n\to\infty}\left[\varphi_{W^{(n)}(t)}(u)\right] = e^{\frac{1}{2}u^2 t}$$

$$\varphi_{W^{(n)}(t)}(u) = E\left[e^{uW^{(n)}(t)}\right]$$

$$= E\left[e^{\frac{u}{\sqrt{n}}M_{nt}}\right]$$

$$= E\left[e^{\frac{u}{\sqrt{n}}\sum_{j=1}^{nt}X_j}\right]$$

$$= E\left[\prod_{j=1}^{nt} e^{\frac{u}{\sqrt{n}}X_j}\right]$$

$$= \prod_{j=1}^{nt} E\left[e^{u/\sqrt{n}\,X_j}\right]$$

$$= \prod_{j=1}^{nt}\left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)$$

$$= \left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^{nt}$$

Equivalent to show

$$\lim_{n\to\infty} \log\left(\varphi_{W^{(n)}(t)}(u)\right) = \frac{1}{2}u^2 t$$

$$\lim_{n\to\infty} \log\left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)^{nt}$$

$$= \lim_{n\to\infty}\left[nt\cdot\log\left(\frac{1}{2}e^{u/\sqrt{n}} + \frac{1}{2}e^{-u/\sqrt{n}}\right)\right]$$

Let $x = \frac{1}{\sqrt{n}} \Rightarrow n = \frac{1}{x^2}$

$$\lim_{n\to\infty} = \lim_{x\searrow 0} = \lim_{x\to 0^+}$$

$$= t\lim_{x\to 0^+}\left[\frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}\right]$$

$$\overset{L}{=} t\lim_{x\to 0^+}\left[\frac{\frac{1}{\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}}\left(\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}\right)}{2x}\right]$$

$$= \frac{ut}{2}\lim_{x\to 0^+}\left[\frac{\frac{1}{2}e^{ux} - \frac{1}{2}e^{-ux}}{x}\right]$$

$$\overset{L}{=} \frac{ut}{2}\lim_{x\to 0^+}\left[\frac{\frac{u}{2}e^{ux} + \frac{u}{2}e^{-ux}}{1}\right] = \frac{u^2 t}{2}$$

$\{X_n\} \xrightarrow{\mathcal{L}^2} Y$ (Mean - Square Convergence)

$$\lim_{n\to\infty} E[X_n] = E[Y]$$

$$\lim_{n\to\infty} E[X_n^2] = E[Y^2] \quad \therefore \lim_{n\to\infty} Var(X_n) = Var(Y)$$

$$Q_\Pi = \sum_{j=1}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

$$\lim_{\|\Pi\|\to 0} E[Q_\Pi] = T$$

$$\lim_{\|\Pi\|\to 0} Var(Q_\Pi) = 0$$

$$E[Q_\Pi] = E\left[\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2\right]$$

$$= \sum_{j=0}^{n-1} E\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right]$$

$$Var(X) = E[X^2] - E[X]^2$$

$$\underbrace{Var\left(W(t_{j+1}) - W(t_j)\right)}_{t_{j+1}-t_j} = E\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] - \underbrace{E\left[\left(W(t_{j+1}) - W(t_j)\right)\right]}_{=0}^2$$

$$E[Q_\Pi] = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = \cancel{t_1} - t_0 + \cancel{t_2} - \cancel{t_1} + \cancel{t_3} - \cancel{t_2} + \ldots + t_n - \cancel{t_{n-1}}$$

$$= T$$

$$\lim_{\|\Pi\|\to 0} T = T$$

$$Var(Q_\Pi) = Var\left(\sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2\right)$$

$$= \sum_{j=0}^{n-1} Var\left(\left(W(t_{j+1}) - W(t_j)\right)^2\right)$$

$$Var\left(\left(W(t_{j+1}) - W(t_j)\right)^2\right) = E\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right] - \underbrace{E\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right]^2}_{(t_{j+1}-t_j)^2}$$

$$E\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right]$$

Let $W(t_{j+1}) - W(t_j) = Y \sim N(0, \tau)$ where $\tau = t_{j+1} - t_j$

$$\varphi_Y(u) = e^{\frac{1}{2}u^2\tau}$$

$$\varphi_Y'(u) = u\tau e^{\frac{1}{2}u^2\tau}$$

$$\varphi_Y'(0) = E[Y] = 0$$

$$\varphi_Y''(u) = \tau e^{\frac{1}{2}u^2\tau} + u^2\tau^2 e^{\frac{1}{2}u^2\tau}$$

$$= (u^2\tau^2 + \tau) e^{\frac{1}{2}u^2\tau}$$

$$\varphi_Y''(0) = E[Y^2] = \tau$$

$$\varphi_Y'''(u) = 2u\tau^2 e^{\frac{1}{2}u^2\tau} + (u^3\tau^3 + u\tau^2) e^{\frac{1}{2}u^2\tau}$$

$$= (u^3\tau^3 + 3u\tau^2) e^{\frac{1}{2}u^2\tau}$$

$$\varphi_Y^{(iv)}(u) = 3\tau^2 e^{\frac{1}{2}u^2\tau} + u(\_\_\_\_\_)$$

$$\varphi_Y^{(iv)}(0) = 3\tau^2$$

$$Var\left(\left(W(t_{j+1}) - W(t_j)\right)^2\right) = 2(t_{j+1} - t_j)^2$$

$$Var(Q_\Pi) = 2\sum_{j=0}^{n-1} (t_{j+1} - t_j)^2$$

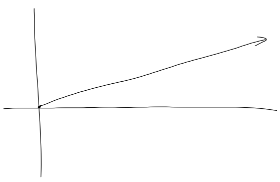
$$\lim_{\|\Pi\|\to 0} Var(Q_\Pi) = 2 \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2$$

$$\leq 2 \lim_{\|\Pi\|\to 0} \left[\|\Pi\| \cdot \underbrace{\sum_{j=0}^{n-1} (t_{j+1} - t_j)}_{T}\right]$$

$$= 0$$

$$= 2\,[f, f](T) \quad \text{where } f(t) = t$$

$$= 0$$



Cross Variation:

$$[X,Y](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(X(t_{j+1}) - X(t_j)\right)\left(Y(t_{j+1}) - Y(t_j)\right)$$

$$[W,t](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)\left(t_{j+1} - t_j\right)$$

$$\leq \lim_{\|\Pi\| \to 0} \left[\underbrace{\max_{0 \leq k \leq n-1} \left(W(t_{k+1}) - W(t_k)\right)}_{=0} \cdot \underbrace{\sum_{j=0}^{n-1} (t_{j+1} - t_j)}_{=T} \right]$$

$$= 0$$

$$d[W,W](t) = dt$$

$$\Pi = \{ t_0 = t, \ t_1 = t + dt \}$$

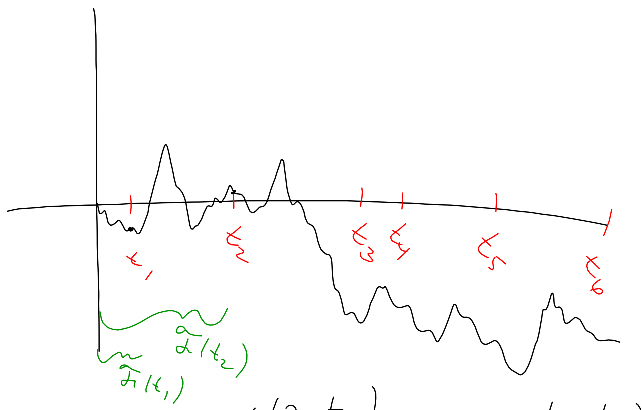
$$d[W,W](t) = [W,W](dt)$$

$$= \big(\underbrace{W(t+dt) - W(t)}_{dW(t)}\big)^2$$

$$= dW(t) \cdot dW(t) = dt$$

$$\Rightarrow dW(t) \cdot dt = 0 = dt \cdot dW(t)$$

$$\Rightarrow dt \cdot dt = 0$$



$W(t_2) \sim N(0, t_2)$ $\quad$ $Cov(W(t_2), W(t_1)) \neq 0$

$W(t_1) \sim N(0, t_1)$

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observe "often" stock price

$S(t_0), S(t_1), S(t_2), \ldots, S(t_m)$

$t_0 = T_1$ $\quad$ $t_m = T_2$

$$\log\left(\frac{S(t_{j+1})}{S(t_j)}\right) = \log\left(\frac{S(0)e^{(\alpha - \sigma^2/2)t_{j+1} + \sigma W(t_{j+1})}}{S(0)e^{(\alpha-\sigma^2/2)t_j + \sigma W(t_j)}}\right)$$

$$= (\alpha - \sigma^2/2)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))$$

$$\sum_{j=0}^{m-1}\left(\log\left(\frac{S(t_{j+1})}{S(t_j)}\right)\right)^2 = \sum_{j=0}^{m-1}\left((\alpha - \sigma^2/2)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))\right)^2$$

$$= (\alpha - \sigma^2/2)^2 \sum_{j=0}^{m-1}(t_{j+1} - t_j)^2 \quad \approx 0$$

$$+ 2(\alpha - \sigma^2/2)\sigma \sum_{j=0}^{m-1}(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) \quad \approx 0$$

$$+ \sigma^2 \sum_{j=0}^{m-1}(W(t_{j+1}) - W(t_j))^2 \quad \approx T_2 - T_1$$

$$\sum_{j=0}^{m-1}\left(\log\left(\frac{S(t_{j+1})}{S(t_j)}\right)\right)^2 \approx \sigma^2(T_2 - T_1)$$

$$\sigma \approx \sqrt{\frac{1}{T_2 - T_1}\sum_{j=0}^{m-1}\left(\log\left(\frac{S(t_{j+1})}{S(t_j)}\right)\right)^2}$$

Realized Vol.