$$\int_0^t \theta(u)\cdot dW(u) = \int_0^t \sum_{j=1}^d \theta_j(u)\,dW_j(u)$$

$$= \sum_{j=1}^d \int_0^t \theta_j(u)\,dW_j(u)$$

$$\|\theta(u)\| = \left(\sum_{j=1}^d \theta_j^2(u)\right)^{1/2}$$

$$\tilde{W}(t) = \left(\tilde{W}_1(t), \tilde{W}_2(t), \ldots, \hat{W}_d(t)\right)$$

$$\tilde{W}_j(t) = W_j(t) + \int_0^t \theta_j(u)\,du, \quad j=1,\ldots,d$$

$$\sigma_i(t) = \sqrt{\sum_{j=1}^d \sigma_{ij}^2(t)}$$

$$B_i(t) = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\,dW_j(u), \quad i=1,\ldots,m$$

$$dB_i(t) = \sum_{j=1}^d \frac{\sigma_{ij}(t)}{\sigma_i(t)}\,dW_j(t)$$

$B_i(t)$ is a cont. martingale

$$\left(dB_i(t)\right)^2 = \sum_{j=1}^d \left(\frac{\sigma_{ij}(t)}{\sigma_i(t)}\right)^2 dt$$

$$= \frac{dt}{(\sigma_i(t))^2} \sum_{j=1}^d \sigma_{ij}^2(t) = dt$$

$$[B_i, B_i](t) = t$$

$$B_i(0) = 0$$

So $B_i(t)$ is a BM

$$dS_i(t) = \alpha_i(t) S_i(t)\,dt + \sigma_i(t) S_i(t)\,dB_i(t)$$

for $i \neq k$:

$$dB_i(t)\,dB_k(t) = \left(\sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)}\,dW_j(t)\right)\left(\sum_{j=1}^{d} \frac{\sigma_{kj}(t)}{\sigma_k(t)}\,dW_j(t)\right)$$

$$= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}\,dt$$

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t)$$

$$d\left(B_i(t)B_k(t)\right) = B_k(t)\,dB_i(t) + B_i(t)\,dB_k(t) + dB_i(t)\,dB_k(t)$$

$$B_i(t)B_k(t) = \int_0^t B_k(u)\,dB_i(u) + \int_0^t B_i(u)\,dB_k(u) + \int_0^t \rho_{ik}(u)\,du$$

$$\mathrm{Cov}(B_i(t), B_k(t)) = E\left[B_i(t)B_k(t)\right]$$

$$= E\left[\int_0^t \rho_{ik}(u)\,du\right]$$

$$D(t) = e^{-\int_0^t R(u)\,du}$$

$$dD(t) = -R(t)D(t)\,dt$$

$$d\big(D(t)S_i(t)\big) = S_i(t)\,dD(t) + D(t)\,dS_i(t) + 0$$

$$= D(t)S_i(t)\Big[\big(\alpha_i(t)-R(t)\big)dt + \sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t)\Big]$$

$$= D(t)S_i(t)\Big[\big(\alpha_i(t)-R(t)\big)dt + \sigma_i(t)\,dB_i(t)\Big]$$

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\Big[\underbrace{\Theta_j(t)\,dt + dW_j(t)}_{d\widetilde{W}_j(t)}\Big]$$

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t)$$

m-equations

Solve for $\Theta(t)$

Suppose $m=2$ + $d=1$ + all coefficients constant

$$\alpha_1 - r = \sigma_1\theta$$
$$\alpha_2 - r = \sigma_2\theta$$

$$\theta = \frac{\alpha_1-r}{\sigma_1} = \frac{\alpha_2-r}{\sigma_2}$$

assume $\dfrac{\alpha_1-r}{\sigma_1} > \dfrac{\alpha_2-r}{\sigma_2}$ then $\underline{\underline{\text{no}}}$ solution

define $\mu = \dfrac{\alpha_1-r}{\sigma_1} - \dfrac{\alpha_2-r}{\sigma_2} > 0$

$$\Delta_1(t) = \frac{1}{S_1(t)\sigma_1} \qquad \Delta_2(t) = -\frac{1}{S_2(t)\sigma_2}$$

$$\Delta_1(0) = \frac{1}{S_1(0)\sigma_1} \qquad \Delta_2(0) = -\frac{1}{S_2(0)\sigma_2}$$

$$X(0) = 0 = \Delta_1(0)S_1(0) + \Delta_2(0)S_2(0) - (\quad\underline{\qquad\qquad}\quad)$$
$$= \frac{1}{\sigma_1} - \frac{1}{\sigma_2} - \left(\frac{\sigma_2-\sigma_1}{\sigma_1\sigma_2}\right)$$

$$dX(t) = \Delta_1(t)\,dS_1(t) + \Delta_2(t)\,dS_2(t) + r\big(X(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)\big)dt$$

$$= \frac{\alpha_1}{\sigma_1}dt + dW(t) - \frac{\alpha_2}{\sigma_2}dt - dW(t) + rX(t)\,dt - \frac{r}{\sigma_1}dt + \frac{r}{\sigma_2}dt$$

$$= rX(t)\,dt + \left(\frac{\alpha_1-r}{\sigma_1} - \frac{\alpha_2-r}{\sigma_2}\right)dt$$
$$= rX(t)\,dt + \mu\,dt$$

$$d\big(D(t)X(t)\big) = \mu\,dt$$

$$D(t)X(t) = \underset{\text{''}}{X(0)} + \int_0^t \mu\,du = X(0) + \mu t$$

by existence of $\widetilde{\mathbb{P}}$

$$\widetilde{E}[D(T)X(T)] = X(0)$$

Let $X(0)=0$

$$\widetilde{E}\{D(T)X(T)\}=0$$

if $\mathbb{P}(X(T)<0)=0$

then $\widetilde{\mathbb{P}}(X(T)<0)=0$

but with $\widetilde{E}[D(T)X(T)]=0$

then, $\widetilde{\mathbb{P}}(X(T)>0)=0$

so $\mathbb{P}(X(T)>0)=0$

$$V(t) = \widetilde{E}\left[e^{-\int_t^T R(u)du} V(T) \mid \mathcal{F}(t)\right], \; 0 \le t \le T$$

$$D(t)V(t) = \widetilde{E}\{D(T)V(T) \mid \mathcal{F}(t)\}$$

$$D(t)V(t) = V(0) + \sum_{j=1}^{d} \int_0^t \widetilde{\Gamma}_j(u)\, d\widetilde{W}_j(u)$$

$$d(D(t)X(t)) = \sum_{i=1}^{m} \Delta_i(t)\, d(D(t)S_i(t))$$

$$= \sum_{i=1}^{m} \Delta_i(t)\, D(t) S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\, d\widetilde{W}_j(t)$$

$$= \sum_{j=1}^{d} \sum_{i=1}^{m} \Delta_i(t) D(t) S_i(t) \sigma_{ij}(t)\, d\widetilde{W}_j(t)$$

$$D(t)X(t) = \sum_{j=1}^{d} \int_0^t \sum_{i=1}^{m} \Delta_i(u) D(u) S_i(u) \sigma_{ij}(u)\, d\widetilde{W}_j(u)$$

$$X(0) = V(0) \qquad + \qquad \frac{\widetilde{\Gamma}_j(t)}{D(t)} = \sum_{i=1}^{m} \Delta_i(t) S_i(t) \sigma_{ij}(t)$$

$\Rightarrow$ Suppose $\widetilde{\mathbb{P}}_1$ & $\widetilde{\mathbb{P}}_2$ exist

Let $A \in \mathcal{F} = \mathcal{F}(T)$

$$V_A(T) = \mathbb{1}_{\{A\}} \frac{1}{D(T)}$$

$$X_A(0) = \widetilde{E}_1\{D(T)V_A(T)\}$$

$$= \widetilde{E}_1[\mathbb{1}_{\{A\}}] = \widetilde{\mathbb{P}}_1(A)$$

$$X_A(0) = \widetilde{E}_2\{D(T)V_A(T)\}$$

$$= \widetilde{E}_2\{\mathbb{1}_{\{A\}}\} = \widetilde{\mathbb{P}}_2(A)$$

$$dX(t) = \Delta(t)\, dS(t) + R(t)\big(X(t) - \Delta(t)S(t)\big)dt$$
$$+ \Delta(t)A(t)S(t)dt$$
$$= \Delta(t)\alpha(t)S(t)dt + \Delta(t)\sigma(t)S(t)dW(t)$$
$$- \Delta(t)S(t)A(t)dt + R(t)X(t)dt - R(t)\Delta(t)S(t)dt$$
$$+ \Delta(t)A(t)S(t)dt$$
$$= R(t)X(t)\,dt + \big(\alpha(t) - R(t)\big)\Delta(t)S(t)dt$$
$$+ \Delta(t)S(t)\sigma(t)dW(t)$$
$$= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)\big(\underbrace{\Theta(t)dt + dW(t)}_{d\widetilde{W}(t)}\big)$$

$$d\big(D(t)X(t)\big) = \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W}(t)$$

$$V(t) = \widetilde{E}\left[e^{-\int_t^T R(u)du} V(T) \,\middle|\, \mathcal{F}(t)\right]$$

$$S(t) = S(0)e^{\int_0^t \left(\alpha(u) - A(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dW(u)}$$
$$S(t) = S(0)\, e^{\int_0^t \left(R(u) - A(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)d\widetilde{W}(u)}$$
$$D(t)S(t) = S(0)e^{\int_0^t \left(-A(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)d\widetilde{W}(u)}$$
$$e^{\int_0^t A(u)du} D(t)S(t) = e^{-\int_0^t \frac{\sigma^2(u)}{2}du + \int_0^t \sigma(u)d\widetilde{W}(u)}$$

let r, σ, a be constants

$$S(t) = S(0)e^{\left(r - a - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}(t)}$$
$$V(t) = \widetilde{E}\left[e^{-r(T-t)}\big(S(T) - K\big)_+ \,\middle|\, \mathcal{F}(t)\right]$$
$$= \widetilde{E}\left[e^{-r(T-t)}\Big(S(t)e^{(r-a-\sigma^2/2)(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t))}\right.$$
$$\left. - K\Big)_+ \,\middle|\, \mathcal{F}(t)\right]$$
$$c(\tau, x) = \widetilde{E}\left[e^{-r\tau}\left(xe^{(r-a-\sigma^2/2)\tau - \sigma\sqrt{\tau}Y} - K\right)_+\right]$$
$$Y = -\frac{\big(\widetilde{W}(T) - \widetilde{W}(t)\big)}{\sqrt{\tau}} \sim N(0,1)$$
$$xe^{(r-a-\sigma^2/2)\tau - \sigma\sqrt{\tau}Y} > K$$
$$e^{(r-a-\sigma^2/2)\tau - \sigma\sqrt{\tau}Y} > \frac{K}{x}$$
$$(r - a - \sigma^2/2)\tau - \sigma\sqrt{\tau}Y > \log\left(\frac{K}{x}\right)$$
$$Y < \frac{\log\left(\frac{x}{K}\right) + (r - a - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_-$$

$$c(\tau, x) = e^{-r\tau}\int_{\infty}^{d_-}\left(xe^{(r-a-\sigma^2/2)\tau - \sigma\sqrt{\tau}y} - K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$
$$= xe^{-a\tau}\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}e^{-\frac{\sigma^2\tau}{2} - \sigma\sqrt{\tau}y - \frac{y^2}{2}}dy$$
$$- Ke^{-r\tau}\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$
$$= xe^{-a\tau}\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy - Ke^{-r\tau}N(d_-)$$
$$= xe^{-a\tau}\int_{-\infty}^{d_- + \sigma\sqrt{\tau}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz - KN(d_-)e^{-r\tau}$$
$$= xe^{-a\tau}N(d_+) - Ke^{-r\tau}N(d_-)$$
$$d_\pm = \frac{\log\left(\frac{x}{K}\right) + \left(r - a \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$D(t)V(t) = \widetilde{E}\left[D(T)V(T) \mid \mathcal{F}(t)\right]$$

$$= \widetilde{E}\left[D(T) \mid \mathcal{F}(t)\right]$$

$$V(t) = \widetilde{E}\left[\frac{D(T)}{D(t)} \mid \mathcal{F}(t)\right]$$

$$B(t,T) = \widetilde{E}\left[e^{-\int_t^T R(u)du} \mid \mathcal{F}(t)\right], \quad 0 \le t \le T \le \bar{T}$$

$$V(t) = \frac{1}{D(t)}\widetilde{E}\left[D(T)\left(S(T)-K\right) \mid \mathcal{F}(t)\right]$$

$$0 = \frac{1}{D(t)}\widetilde{E}\left[D(T)S(T) \mid \mathcal{F}(t)\right] - \frac{K}{D(t)}\widetilde{E}\left[D(T) \mid \mathcal{F}(t)\right]$$

$$0 = S(t) - K\,B(t,T)$$

$$K = \frac{S(t)}{B(t,T)}$$

$$Fut_S(t,T) = \widetilde{E}\left[S(T) \mid \mathcal{F}(t)\right]$$

$$\widetilde{E}\left[Fut_S(t,T) \mid \mathcal{F}(u)\right] \quad \text{where } u < t$$

$$= \widetilde{E}\left[\widetilde{E}\left[S(T) \mid \mathcal{F}(t)\right] \mid \mathcal{F}(u)\right]$$

$$= \widetilde{E}\left[S(T) \mid \mathcal{F}(u)\right] = Fut_S(u,T)$$

$$\frac{1}{B(0,T)}\widetilde{Cov}\left(D(T), S(T)\right)$$

$$= \frac{1}{B(0,T)}\left(\widetilde{E}\left[D(T)S(T)\right] - \widetilde{E}\left[D(T)\right]\widetilde{E}\left[S(T)\right]\right)$$

$$= \frac{\widetilde{E}\left[D(T)S(T)\right]}{B(0,T)} - \widetilde{E}\left[S(T)\right]$$

$$= \frac{S(0)}{B(0,T)} - \widetilde{E}\left[S(T)\right]$$

$$= For_S(0,T) - Fut_S(0,T)$$