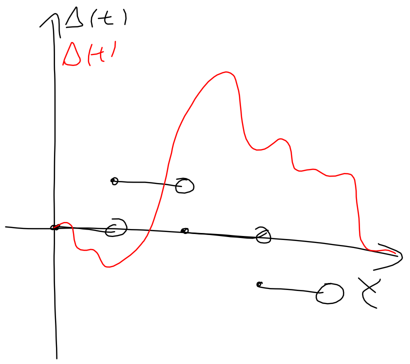
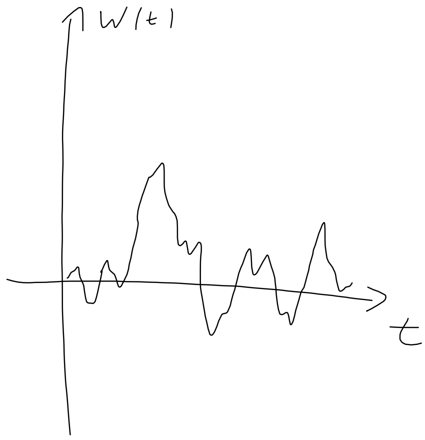
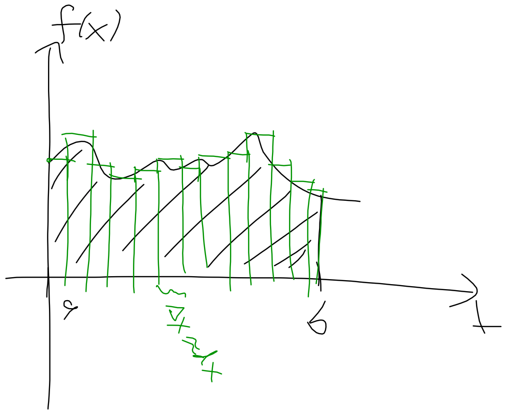






$$\int_a^b f(x)\,dx$$

$$\sum_{i=0}^{n-1} f(x_i)\underbrace{(x_{i+1}-x_i)}_{d_t}$$

Let $I(t)$, $\Delta(t)$ will be investment strategy

$W(t)$ be the value of asset

t_i be the beginning of each day

for $0 \le t < t_1$:

$$I(t) = \Delta(t_0)\left(W(t) - W(t_0)\right)$$
$$= \Delta(t_0)W(t)$$

for $t_1 \le t < t_2$:

$$I(t) = \Delta(t_0)\left(W(t_1) - W(t_0)\right) + \Delta(t_1)\left(W(t) - W(t_1)\right)$$

for $t_2 \le t < t_3$

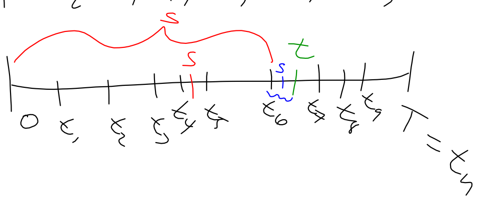
$$I(t) = \Delta(t_0)\left(W(t_1) - W(t_0)\right) + \Delta(t_1)\left(W(t_2) - W(t_1)\right) + \Delta(t_2)\left(W(t) - W(t_2)\right)$$

for $t_n \le t < t_{n+1}$

$$I(t) = \sum_{i=0}^{n-1} \Delta(t_i)\left(W(t_{i+1}) - W(t_i)\right) + \Delta(t_n)\left(W(t) - W(t_n)\right)$$

$\Delta(t)$ be simple processes

$\Pi = \{t_0, t_1, t_2, \dots, t_n\}$



<u>NTS:</u> $E[I(t) \mid \mathcal{F}(s)] = I(s)$; $s \le t$

Case 1: $t \in [t_k, t_{k+1})$ $\quad s \in [t_\ell, t_{\ell+1})$

$\ell < k$

$$I(t) = \underbrace{\sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))}_{A} + \underbrace{\Delta(t_\ell)(W(t_{\ell+1}) - W(t_\ell))}_{B}$$

$$+ \underbrace{\sum_{j=\ell+1}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))}_{C} + \underbrace{\Delta(t_k)(W(t) - W(t_k))}_{D}$$

$$E[I(t) \mid \mathcal{F}(s)] = E[A + B + C + D \mid \mathcal{F}(s)]$$

$$= E[A \mid \mathcal{F}(s)] + E[B \mid \mathcal{F}(s)] + \dots$$

A: $E\Big[\underbrace{\sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))}_{\text{Known}\ /\ \mathcal{F}(s)\ \text{meas.}} \Big| \mathcal{F}(s)\Big]$

$$= \left(\sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))\right) E[1 \mid \mathcal{F}(s)]$$

$$= \sum_{j=0}^{\ell-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))$$

B: $E[\Delta(t_\ell)(W(t_{\ell+1}) - W(t_\ell)) \mid \mathcal{F}(s)]$

$$\overset{*}{=} \Delta(t_\ell)\, E[W(t_{\ell+1}) - W(t_\ell) \mid \mathcal{F}(s)]$$

$$= \Delta(t_\ell)\, E[\underbrace{W(t_{\ell+1}) - W(s)}_{\text{ind}} + \underbrace{W(s) - W(t_\ell)}_{\text{meas.}} \mid \mathcal{F}(s)]$$

$$= \Delta(t_\ell)\Big\{\underbrace{E[W(t_{\ell+1}) - W(s) \mid \cancel{\mathcal{F}(s)}]}_{=0} + \underbrace{E[W(s) - W(t_\ell) \mid \mathcal{F}(s)]}_{W(s) - W(t_\ell)}\Big\}$$

$$= \Delta(t_\ell)(W(s) - W(t_\ell))$$

* con't.

$$= \Delta(t_\ell)\Big\{\underbrace{E[W(t_{\ell+1}) \mid \mathcal{F}(s)]}_{W(s)} - W(t_\ell)\Big\}$$

C: for $l+1 \le j \le k-1$, $j \in \mathbb{Z}^+$

$$E\{\Delta(t_j)(W(t_{j+1})-W(t_j)) \mid \mathcal{F}(s)\}$$

for $\mathcal{H} \subseteq \mathcal{G}$ $E\{E\{X \mid \mathcal{G}\} \mid \mathcal{H}\} = E\{X \mid \mathcal{H}\}$

$$= E\{E\{\Delta(t_j)(W(t_{j+1})-W(t_j)) \mid \mathcal{F}(t_j)\} \mid \mathcal{F}(s)\}$$

$$= E[\Delta(t_j)\underbrace{E\{W(t_{j+1})-W(t_j) \mid \cancel{\mathcal{F}(t_j)}\}}_{=0} \mid \mathcal{F}(s)]$$

$$= 0$$

D: $E\{\Delta(t_k)(W(t)-W(t_k)) \mid \mathcal{F}(s)\}$

$$= E\{E[\Delta(t_k)(W(t)-W(t_k)) \mid \mathcal{F}(t_k)] \mid \mathcal{F}(s)\}$$

$$= 0$$

$$E\{I(t) \mid \mathcal{F}(s)\} = \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1})-W(t_j)) + \Delta(t_l)(W(s)-W(t_l)) = I(s)$$

Case 2: $s,t \in [t_k, t_{k+1})$, $s \le t$

$$E\{I(t) \mid \mathcal{F}(s)\} = E\left\{\sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1})-W(t_j)) + \Delta(t_k)(W(t)-W(t_k)) \mid \mathcal{F}(s)\right\}$$

$$= \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1})-W(t_j)) + \underbrace{E\{\Delta(t_k)(W(t)-W(t_k)) \mid \mathcal{F}(s)\}}_{\underbrace{\Delta(t_k)\underbrace{E\{W(t)-W(t_k) \mid \mathcal{F}(s)\}}_{E\{W(t) \mid \mathcal{F}(s)\}-W(t_k)}}_{\Delta(t_k)(W(s)-W(t_k))}}$$

$$= I(s)$$

$$E\{I(t)\} = E\{I(t) \mid \mathcal{F}(0)\}$$
$$= I(0) = 0$$
$$Var(I(t)) = E[I^2(t)] - E[I(t)]^2$$
$$= E[I^2(t)]$$

$$D_\ell = W(t_{\ell+1}) - W(t_\ell) \quad \text{for } \ell < k$$
$$D_k = W(t) - W(t_k)$$

$$I(t) = \sum_{j=0}^{k} \Delta(t_j) D_j$$

$$I^2(t) = \underline{\sum_{j=0}^{k} \Delta^2(t_j) D_j^2} + \underline{2 \sum_{0 \le i < j \le k} \Delta(t_i)\Delta(t_j) D_i D_j}$$

$$(a+b+c+d)^2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & a^2 & ab & ac & ad \\ b & ba & b^2 & bc & bd \\ c & ca & cb & c^2 & cd \\ d & da & db & dc & d^2 \end{array}$$

$\Delta(t_j)$: $\mathcal{F}(t_j)$-measurable

D_j: $(W(t_{j+1}) - W(t_j))$: ind of $\mathcal{F}(t_j)$

$$E\{\Delta^2(t_j) D_j^2\} = E\{\Delta^2(t_j)\} E[D_j^2]$$

$\Delta(t_i)\Delta(t_j)D_i$: $\mathcal{F}(t_j)$-measurable

$$E[\Delta(t_i)\Delta(t_j) D_i D_j] = E[\Delta(t_i)\Delta(t_j) D_i]\, E(D_j)$$
$$= 0$$

$$E\{I^2(t)\} = \sum_{j=0}^{k} E\{\Delta^2(t_j)\} \underbrace{E\{(W(t_{j+1}) - W(t_j))^2\}}_{t_{j+1} - t_j}$$

$$= \sum_{j=0}^{k} E[\Delta^2(t_j)(t_{j+1} - t_j)]$$

$$= E\Big[\underbrace{\sum_{j=0}^{k} \Delta^2(t_j)(t_{j+1} - t_j)}_{\int_0^t \Delta^2(u)\,du}\Big]$$

$$= E\Big[\int_0^t \Delta^2(u)\,du\Big]$$

$$\Pi_j = \{t_j = s_0, s_1, s_2, \dots, s_n = t_{j+1}\}$$

$$[I,I](t) = \sum_{j=0}^{k}\left(\lim_{\|\Pi_j\|\to 0}\sum_{i=0}^{n-1}\left(I(s_{i+1})-I(s_i)\right)^2\right)$$

for $0 \le j \le k$

$$\lim_{\|\Pi_j\|\to 0}\sum_{i=0}^{n-1}\left(I(s_{i+1})-I(s_i)\right)^2$$

$$= \lim_{\|\Pi_j\|\to 0}\sum_{i=0}^{n-1}\Big(\sum_{l=0}^{j-1}\Delta(t_l)\left(W(t_{l+1})-W(t_l)\right) + \Delta(t_j)\left(W(s_{i+1})-W(t_j)\right) - \sum_{l=0}^{j-1}\Delta(t_l)\left(W(t_{l+1})-W(t_l)\right) - \Delta(t_j)\left(W(s_i)-W(t_j)\right)\Big)^2$$

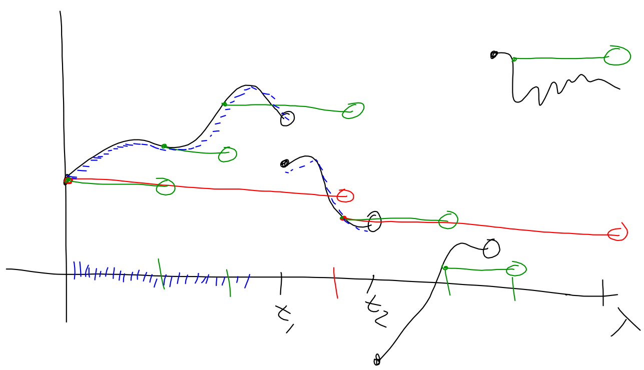
$$= \lim_{\|\Pi_j\|\to 0}\sum_{i=0}^{n-1}\Delta^2(t_j)\left(W(s_{i+1})-W(s_i)\right)^2$$

$$= \Delta^2(t_j)\underbrace{\lim_{\|\Pi_j\|\to 0}\sum_{i=0}^{n-1}\left(W(s_{i+1})-W(s_i)\right)^2}_{t_{j+1}-t_j}$$

$$= \Delta^2(t_j)(t_{j+1}-t_j)$$

$$[I,I](t) = \sum_{j=0}^{k}\Delta^2(t_j)(t_{j+1}-t_j)$$

$$= \int_0^t \Delta^2(u)\,du$$



$t \in [t_1, t_2)$

$$I(t) = \int_0^t \Delta(u)\,dW(u) = \int_0^{t_1}\Delta(u)\,dW(u) + \int_{t_1}^{t}\Delta(u)\,dW(u)$$

$$\lim_{t\searrow t_1} I(t) = \int_0^{t_1}\Delta(u)\,dW(u) + \underbrace{\lim_{t\searrow t_1}\int_{t_1}^{t}\Delta(u)\,dW(u)}_{\int_{t_1}^{t_1+dt}\Delta(u)\,dW(u)}$$

$$\int_0^T W(t)\,dW(t) \stackrel{?}{=} \frac{W^2(T)}{2} \qquad \int_0^T x\,dx = \frac{T^2}{2}$$

$\Delta(t) = W(t)$

$\Delta_n(t) \to W(t)$ divide $[0,T]$ into n pieces

$t_0 = 0,\ t_1 = T/n,\ t_2 = \frac{2T}{n},\ t_3 = \frac{3T}{n}, \ldots$

$$\Delta_n(t) = \begin{cases} W(0) & \text{for } 0 \le t < T/n \\ W(T/n) & \text{for } T/n \le t < 2T/n \\ W(2T/n) & \text{for } 2T/n \le t < 3T/n \\ \vdots & \\ W\left(\frac{(n-1)T}{n}\right) & \text{for } \frac{(n-1)T}{n} \le t < T \end{cases}$$

$$\int_0^T W(t)\,dW(t) = \lim_{n\to\infty} \int_0^T \Delta_n(t)\,dW(t)$$

$$= \lim_{n\to\infty} \left(\underline{\sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left(W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right)} \right)$$

$W_j = W\left(\frac{jT}{n}\right)$

$$\frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 = \frac{1}{2}\underbrace{\sum_{j=0}^{n-1} W_{j+1}^2}_{} - \sum_{j=0}^{n-1} W_{j+1}W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2$$

$$\sum_{k=1}^{n} W_k^2 = \sum_{k=0}^{n} W_k^2 = \sum_{k=0}^{n-1} W_k^2 + W_n^2$$

$$= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}W_j$$

$$\sum_{j=0}^{n-1} W_j(W_{j+1} - W_j) = \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2$$

$$\int_0^T W(t)\,dW(t) = \lim_{n\to\infty}\left(\frac{1}{2}W^2\left(\frac{nT}{n}\right) - \frac{1}{2}\sum_{j=0}^{n-1}\left(W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right)^2\right)$$

$$= \frac{W^2(T)}{2} - \frac{1}{2}\underbrace{\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right)^2}_{T}$$

$$= \frac{W^2(T)}{2} - T/2$$