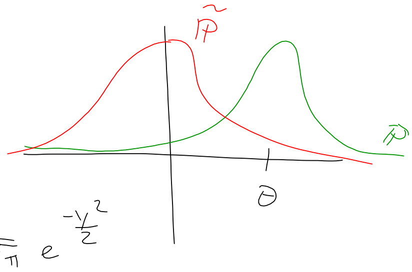
$$\mathbb{P}: \quad X \sim N(0,1)$$
$$Y = X + \theta \Rightarrow Y \sim N(\theta, 1)$$

$$\tilde{\mathbb{P}}: \quad Y \sim N(0,1)$$



$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}}}$$
$$= e^{-y\theta + \theta^2/2}$$

$$\tilde{\mathbb{P}}(Y \leq a) = \int_{Y \leq a} Z \, d\mathbb{P}$$
$$= \int_{-\infty}^{a} \underbrace{e^{-y\theta + \frac{\theta^2}{2}}}_{Z} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}} dy}_{d\mathbb{P}}$$
$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y^2 - 2y\theta + \theta^2)}{2} - y\theta + \theta^2/2} dy$$
$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \qquad x = y - \theta$$
$$= e^{-x\theta - \theta^2/2}$$

$$\tilde{\mathbb{P}}(Y \leq a) = \int_{Y \leq a} Z \, d\mathbb{P}$$
$$= \int_{x+\theta \leq a} e^{-x\theta - \theta^2/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= \int_{-\infty}^{a-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}} dx$$

$$y = x + \theta$$
$$dy = dx$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$\int_A \frac{1}{Z(s)} E[Y Z(t) \mid \mathcal{F}(s)] \, d\tilde{P}$$

$$= \int_\Omega \frac{\mathbb{1}_{\{A\}}}{Z(s)} E[Y Z(t) \mid \mathcal{F}(s)] \, d\tilde{P}$$

$$= \tilde{E}\left[\frac{1}{Z(s)} E[\mathbb{1}_{\{A\}} Y Z(t) \mid \mathcal{F}(s)] \right]$$

$$X = \frac{1}{Z(s)} E[\mathbb{1}_{\{A\}} Y Z(t) \mid \mathcal{F}(s)]$$

$$\tilde{E}[X] = E[X \, Z(s)]$$

$$= E[E[\mathbb{1}_{\{A\}} Y Z(t) \mid \mathcal{F}(s)]]$$

$$= E[\mathbb{1}_{\{A\}} Y Z(t)]$$

$$= \tilde{E}[\mathbb{1}_{\{A\}} Y]$$

$$= \int_\Omega \mathbb{1}_{\{A\}} Y \, d\tilde{P} = \int_A Y \, d\tilde{P}$$

$$Z(t) = e^{-\frac{1}{2}\int_0^t \theta^2(u)\,du - \int_0^t \theta(u)\,dW(u)}$$

$$dZ(t) = ?$$

$$X(t) = X(0) + \int_0^t \Delta(u)\,dW(u) + \int_0^t \Gamma(u)\,du$$

$$dX(t) = \Delta(t)\,dW(t) + \cancel{\Gamma(t)}\,dt$$

$$E[X(t)|\mathcal{F}(s)] = X(0) + \int_0^s \Delta(u)\,dW(u)$$
$$+ E\left[\int_0^t \Gamma(u)\,du \,\middle|\, \mathcal{F}(s)\right]$$
$$= X(s) \qquad \text{if } \Gamma(t) = 0$$

$$dY(t) = \frac{Y^2(t)}{(t+1)^2} \cos(Y(t))\,dM(t)$$

$$Y(t) - Y(0) = \int_0^t \frac{Y^2(u)}{(u+1)^2} \cos(Y(u))\,dM(u)$$

$$Y(t) = Y(0) + \int_0^t \frac{Y^2(u)}{(u+1)^2} \cos(Y(u))\,dM(u)$$

$$E\{Y(t)|\mathcal{F}(s)\} = Y(s)$$

$$Z(t) = e^{X(t)} \quad \text{where } X(t) = -\int_0^t \theta(u)\,dW(u) - \frac{1}{2}\int_0^t \theta^2(u)\,du$$

$$dX(t) = -\theta(t)\,dW(t) - \frac{1}{2}\theta^2(t)\,dt$$

$$dZ(t) = Z(t)\,dX(t) + \frac{1}{2}Z(t)(dX(t))^2$$
$$= -Z(t)\theta(t)\,dW(t) - \frac{1}{2}Z(t)\theta^2(t)\,dt$$
$$+ \frac{1}{2}Z(t)\,\theta^2(t)\,dt$$
$$= -Z(t)\theta(t)\,dW(t)$$

$$Z = Z(T)$$

$$E\{Z\} = E\{Z(T)\} = E\{Z(T)|\mathcal{F}(0)\} = Z(0) = 1$$

$\tilde{W}(0) = 0$ ✓

$[\tilde{W}, \tilde{W}](t) = t$ ✓

cont. paths. ✓

$\tilde{W}(0) = W(0) + \int_0^0 \theta(u)\,du = 0$

$\tilde{W}(t) = W(t) + \int_0^t \theta(u)\,du$

$d\tilde{W}(t) = dW(t) + \theta(t)\,dt$

$(d\tilde{W}(t))^2 = dt$

$\int_0^t (d\tilde{W}(u))^2 = t$

$\tilde{E}\{\tilde{W}(t) \mid \mathcal{F}(s)\} = \tilde{W}(s)$

$$\tilde{E}[\tilde{W}(t) \mid \mathcal{F}(s)] = \frac{1}{Z(s)} E\{\tilde{W}(t) Z(t) \mid \mathcal{F}(s)\}$$

So $\tilde{W}(t)$ is $\tilde{\mathbb{P}}$-Martingale, if $\tilde{W}(t)Z(t)$ is a $\mathbb{P}$-Mart.

$$d(\tilde{W}(t) Z(t)) = Z(t)\,d\tilde{W}(t) + \tilde{W}(t)\,dZ(t) + dZ(t)\,d\tilde{W}(t)$$

$$= Z(t)(dW(t) + \theta(t)\,dt)$$

$$- \tilde{W}(t) Z(t) \theta(t)\,dW(t)$$

$$+ (-\theta(t) Z(t)\,dW(t))(dW(t) + \theta(t)\,dt)$$

$$= Z(t)\,dW(t) + Z(t)\theta(t)\,dt$$

$$- \tilde{W}(t) Z(t) \theta(t)\,dW(t)$$

$$- \theta(t) Z(t)\,dt - 0$$

$$= (Z(t) - \tilde{W}(t) Z(t) \theta(t))\,dW(t)$$

$$D(t)S(t) = e^{-\int_0^t R(u)du}\, S(0)\, e^{\int_0^t (\alpha(u)-\frac{\sigma^2(u)}{2})du + \int_0^t \sigma(u)dW(u)}$$

$$= S(0)\, e^{\int_0^t (\alpha(u)-R(u)-\frac{\sigma^2(u)}{2})du + \int_0^t \sigma(u)dW(u)}$$

$$d(D(t)S(t)) = D(t)dS(t) + S(t)dD(t) + dD(t)dS(t)$$

$$= D(t)\alpha(t)S(t)dt + D(t)\sigma(t)S(t)dW(t)$$

$$+ -R(t)D(t)S(t)dt + 0$$

$$= D(t)S(t)(\alpha(t)-R(t))dt + D(t)S(t)\sigma(t)dW(t)$$

$$= D(t)S(t)\sigma(t)\Big(\underbrace{dW(t) + \frac{(\alpha(t)-R(t))}{\sigma(t)}dt}\Big)$$

$$\theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)} \Rightarrow \underbrace{dW(t) + \theta(t)dt}_{d\widetilde{W}(t)}$$

GBM:

$$\theta(t) = \frac{\alpha - r}{\sigma}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \frac{\alpha - r}{\sigma}du$$

$$= W(t) + \left(\frac{\alpha - r}{\sigma}\right)t$$

$$W(t) = \widetilde{W}(t) - \left(\frac{\alpha - r}{\sigma}\right)t$$

$$S(t) = S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$$= S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma(\widetilde{W}(t) - (\frac{\alpha - r}{\sigma})t)}$$

$$= S(0)\, e^{(r - \sigma^2/2)t + \sigma \widetilde{W}(t)}$$

$$\widetilde{E}\left\{e^{-rT}(S(T)-k)_+\right\}$$

$$= \widetilde{E}\left\{e^{-rT}\left(S(0)e^{(r-\sigma^2/2)T + \sigma\widetilde{W}(T)} - k\right)_+\right\}$$