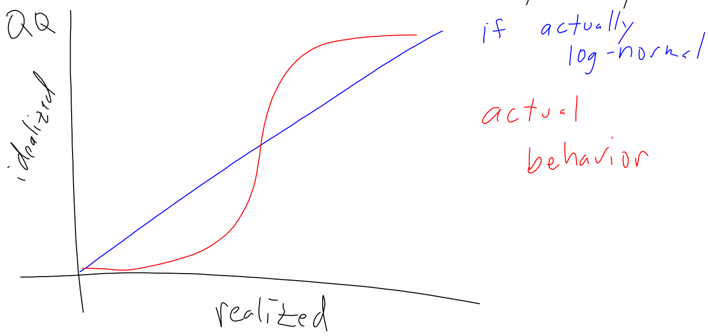
$$S(t) = S(0)\, e^{(\alpha - \sigma^2/2)t + \sigma W(t)}$$

$$\log\left(\frac{S(t+\Delta t)}{S(t)}\right) = \underbrace{\left(\alpha - \frac{\sigma^2}{2}\right)\Delta t + \sigma\left(W(t+\Delta t) - W(t)\right)}_{\sim N\left(\left(\alpha - \frac{\sigma^2}{2}\right)\Delta t,\ \sigma^2 \Delta t\right)}$$



Stochastic Vol:

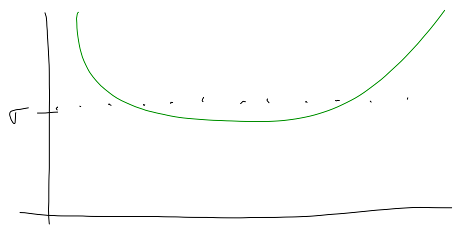
$$dS(t) = \alpha(t) S(t)\, dt + \sigma(t) S(t)\, dW(t)$$

$$d\sigma(t) = \gamma\, dt + \mathfrak{z}\, dB(t)$$

$$dW(t)\, dB(t) = \rho\, dt$$

Implied Vol.

$$C_{BS}(x, \tau, K, r, \sigma) = \text{price}$$

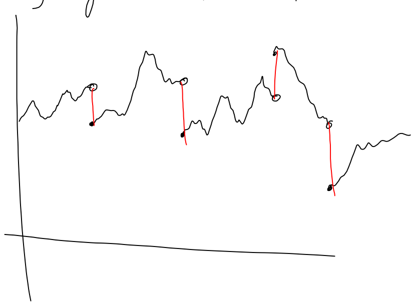


Regime Switching;

$$\{(\alpha_i, \sigma_i)\}_d \qquad d \text{ regimes}$$

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1d} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2d} \\ P_{31} & & \ddots & & \\ \vdots & & & \ddots & \\ P_{d1} & & & & P_{dd} \end{pmatrix}$$

Jump Diffusion



$$E[N(t)-N(s)] = \sum_{k=0}^{\infty} k \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

$$= \sum_{k=1}^{\infty} k \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^k (t-s)^k}{(k-1)!} e^{-\lambda(t-s)}$$

$$= e^{-\lambda(t-s)} \lambda(t-s) \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!}$$

$$= e^{-\lambda(t-s)} \lambda(t-s) \sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^k}{k!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= e^{-\lambda(t-s)} \lambda(t-s) e^{\lambda(t-s)}$$

$$= \lambda(t-s)$$

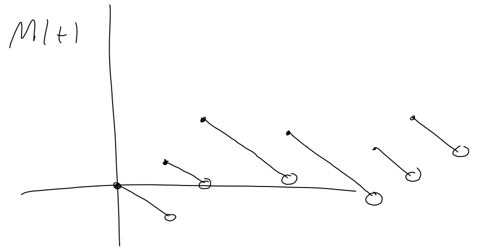
$$\mathrm{Var}(N(t)-N(s)) = E[(N(t)-N(s))^2] - E[N(t)-N(s)]^2$$

$$E[(N(t)-N(s))^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

$$= \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{(k+1)\lambda^k (t-s)^k}{k!}$$

$$= \lambda(t-s) e^{-\lambda(t-s)} \left(\underbrace{\sum_{k=0}^{\infty} \frac{k\lambda^k (t-s)^k}{k!}}_{\lambda(t-s)e^{\lambda(t-s)}} + \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^k}{k!}}_{e^{\lambda(t-s)}} \right)$$

$$= \lambda^2 (t-s)^2 + \lambda(t-s)$$



$$E\left[e^{uN(t)}\right] = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} e^{uk}$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(e^u \lambda t)^k}{k!}$$

$$= e^{-\lambda t} e^{e^u \lambda t}$$

$$= e^{\lambda t (e^u - 1)}$$

for u_1 + u_2 dummy variables

$$Y(t) = e^{u_1 W(t) + u_2 N(t) - \frac{1}{2} u_1^2 t - \lambda (e^{u_2} - 1) t}$$

Define $X(s) = u_1 W(s) + u_2 N(s) - \frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s$

$$X^c(s) = u_1 W(s) - \frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s$$

$$dX^c(s) = u_1 dW(s) - \frac{1}{2} u_1^2 ds - \lambda (e^{u_2} - 1) ds$$

$$\left(dX^c(s)\right)^2 = u_1^2 ds$$

if Y has a jump @ S

$$Y(s) = e^{u_1 W(s) + u_2 (N(s_-) + 1) - \frac{1}{2} u_1^2 s - \lambda (e^{u_2} - 1) s}$$

$$= Y(s_-) e^{u_2}$$

$$Y(s) - Y(s_-) = Y(s_-) \left(e^{u_2} - 1\right) \Delta N(s)$$

$$Y(t) = f(X(t)) = f(X(0)) + \int_0^t f'(X(s)) dX^c(s) + \frac{1}{2} \int_0^t f''(X(s)) \left(dX^c(s)\right)^2 + \sum_{0<s\le t} \Big(f(X(s)) - f(X(s_-))\Big)$$

$$= 1 + \int_0^t Y(s) u_1 dW(s) + \int_0^t -\frac{1}{2} u_1^2 Y(s) ds + \int_0^t -\lambda (e^{u_2} - 1) Y(s) ds$$

$$+ \frac{1}{2} \int_0^t Y(s) u_1^2 ds + \sum_{0<s\le t} Y(s_-) \left(e^{u_2} - 1\right) \Delta N(s)$$

$$= 1 + u_1 \int_0^t Y(s) dW(s) + \left(e^{u_2} - 1\right) \left(\int_0^t Y(s) dN(s) - \int_0^t Y(s) \lambda ds \right)$$

$$= 1 + u_1 \int_0^t Y(s) dW(s) + \left(e^{u_2} - 1\right) \int_0^t Y(s_-) \big(\underbrace{dN(s) - \lambda ds}_{dM(s)} \big)$$

$$E\left[Y(t)\right] = 1$$

$$E\left[e^{u_1 W(t) + u_2 N(t)}\right] = e^{\frac{1}{2} u_1^2 t} e^{\lambda t (e^{u_2} - 1)}$$

$$X(t) = X^c(t) + J(t)$$

$$X^c(t) = \int_0^t \Gamma(s)\,dW(s) + \int_0^t \Theta(s)\,ds$$

Define $Y(t) = e^{\int_0^t \Gamma(s)\,dW(s) + \int_0^t \Theta(s)\,ds - \frac{1}{2}\int_0^t \Gamma^2(s)\,ds}$

$$= e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)}$$

$$dY(t) = Y(t)\left(dX^c(t) - \frac{1}{2}\,d(X^c(t))^2\right) + \frac{1}{2}Y(t)\,\Gamma^2(t)\,dt$$

$$= Y(t)\left(\Gamma(t)\,dW(t) + \Theta(t)\,dt - \frac{1}{2}\Gamma^2(t)\,dt\right) + \frac{1}{2}Y(t)\left(\Gamma^2(t)\,dt\right)$$

$$= Y(t)\,dX^c(t)$$

Define $K(t) = 1$ for $0 \le t <$ "first jump"

$K(t) = \prod_{0<s\le t}(1 + \Delta X(s))$ for $t \ge$ "first jump"

$$Z^X(t) = Y(t)K(t)$$

$$K(t) = K(t_-)(1 + \Delta X(t)) \Rightarrow \begin{array}{l} K(t) - K(t_-) \\ = K(t_-)\Delta X(t) \end{array}$$

$$Z^X(t) = Y(t)K(t) = Y(0)K(0) + \int_0^t K(s_-)\,dY(s)$$

$$+ \int_0^t Y(s_-)\,dK(s) + 0$$

$$= 1 + \int_0^t K(s_-)Y(s_-)\,dX^c(s) + \int_0^t Y(s_-)K(t_-)\Delta X(t)$$

$$= 1 + \int_0^t K(s_-)Y(s_-)\,dX(s)$$

$$= 1 + \int_0^t Z^X(s_-)\,dX(s)$$

$$X(t) = \frac{\tilde{\lambda}-\lambda}{\lambda} M(t)$$

$$= \frac{\tilde{\lambda}-\lambda}{\lambda} N(t) - (\tilde{\lambda}-\lambda) t$$

$$= \frac{\tilde{\lambda}-\lambda}{\lambda} N(t) + (\lambda-\tilde{\lambda}) t$$

$$X^c(t) = (\lambda-\tilde{\lambda}) t$$

$$\Delta X(t) = X(t) - X(t_-) = \frac{\tilde{\lambda}-\lambda}{\lambda} \quad \text{if jump occurs @ } t$$

$$1 + \Delta X(t) = \frac{\tilde{\lambda}}{\lambda}$$

$$Z(t) = e^{(\lambda-\tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$$

$$= e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0<s\le t} (1 + \Delta X(s))$$

$$Z(t) = 1 + \int_0^t Z(s_-)\, dX(s)$$

$$E[Z(t)] = \underbrace{1}_{=1} + E\left[\int_0^t \frac{\tilde{\lambda}-\lambda}{\lambda} Z(s_-)\, dM(s)\right]$$

$$E[Z(T)] = 1$$

fix $Z = Z(T)$

$$\tilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}$$

$$\tilde{E}\left[e^{uN(t)}\right] = E\left[e^{uN(t)} Z(t)\right]$$

$$= E\left[e^{uN(t)} e^{(\lambda-\tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}\right]$$

$$= e^{(\lambda-\tilde{\lambda})t} E\left[e^{uN(t) + \log\left(\frac{\tilde{\lambda}}{\lambda}\right) N(t)}\right]$$

$$= e^{(\lambda-\tilde{\lambda})t} E\left[e^{N(t)\left(u + \log\left(\frac{\tilde{\lambda}}{\lambda}\right)\right)}\right]$$

$$= e^{(\lambda-\tilde{\lambda})t} e^{\lambda t\left(e^{u + \log\left(\frac{\tilde{\lambda}}{\lambda}\right)} - 1\right)}$$

$$= e^{\lambda t - \tilde{\lambda} t + \lambda t\left(\frac{\tilde{\lambda}}{\lambda} e^u - 1\right)}$$

$$= e^{\lambda t - \tilde{\lambda} t + \tilde{\lambda} t e^u - \lambda t} = e^{\tilde{\lambda} t (e^u - 1)}$$