$$f_{M(t),W(t)}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{1}{2t}(2m-w)^2}$$

for $w \le m,\ m \ge 0$

$$\hat{Z}(t) = e^{-\alpha \tilde{W}(t) - \frac{1}{2}\alpha^2 t} = e^{-\alpha \hat{W}(t) + \frac{1}{2}\alpha^2 t}$$

$$\hat{\Theta}(t) = \alpha$$

$$\hat{\mathbb{P}}(A) = \int_A \hat{Z}(T)\, d\tilde{\mathbb{P}}$$

so $\hat{W}(T)$ is a BM under $\hat{\mathbb{P}}$

$$\hat{f}_{\hat{M}(T),\hat{W}(T)}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2m-w)^2}$$

$$\tilde{\mathbb{P}}(\hat{M}(T) \le m, \hat{W}(T) \le w) = \tilde{E}\left[\mathbb{1}_{\{\hat{M}(T) \le m, \hat{W}(T) \le w\}}\right]$$

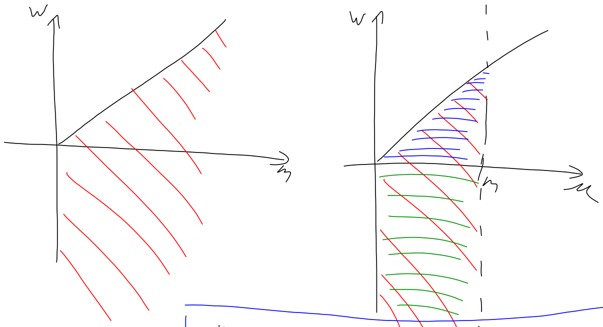
$$= \hat{E}\left[\frac{1}{\hat{Z}(T)} \mathbb{1}_{\{\hat{M}(T) \le m, \hat{W}(T) \le w\}}\right]$$

$$= \hat{E}\left[e^{\alpha \hat{W}(T) - \frac{1}{2}\alpha^2 T} \mathbb{1}_{\{\hat{M}(T) \le m, \hat{W}(T) \le w\}}\right]$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}_{\hat{M}(T),\hat{W}(T)}(x,y)\, dx\, dy$$

$$\frac{\partial^2}{\partial m\, \partial w} \tilde{\mathbb{P}}(\hat{M}(T) \le m, \hat{W}(T) \le w) = e^{\alpha w - \frac{1}{2}\alpha^2 T} \hat{f}_{\hat{M}(T),\hat{W}(T)}(m,w)$$

$$\tilde{f}_{\hat{M}(T),\hat{W}(T)}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{2m^2}{T} - \frac{2mw}{T} - \frac{w^2}{2T}}$$



$$\tilde{\mathbb{P}}(\hat m(T)\le m) = \boxed{\int_0^m \int_w^m \frac{2(2\mu-w)}{T\sqrt{2\pi T}}\, e^{\alpha w - \frac12\alpha^2 T - \frac{1}{2T}(2\mu-w)^2}\, d\mu\, dw}$$

$$+ \boxed{\int_{-\infty}^0 \int_0^m \frac{2(2\mu-w)}{T\sqrt{2\pi T}}\, e^{\alpha w - \frac12\alpha^2 T - \frac{1}{2T}(2\mu-w)^2}\, d\mu\, dw}$$

$$= -\int_0^m \left[\frac{1}{\sqrt{2\pi T}}\, e^{\alpha w - \frac12\alpha^2 T - \frac{1}{2T}(2\mu-w)^2}\Big|_{\mu=w}^{\mu=m}\right] dw$$

$$+ -\int_{-\infty}^0 \left[\frac{1}{\sqrt{2\pi T}}\, e^{\alpha w - \frac12\alpha^2 T - \frac{1}{2T}(2\mu-w)^2}\Big|_{\mu=0}^{\mu=m}\right] dw$$

$$= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac12\alpha^2 T - \frac{(2m-w)^2}{2T}}\, dw$$

$$+ \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{\alpha w - \frac12\alpha^2 T - \frac{w^2}{2T}}\, dw$$

$$-\frac{1}{2T}(w-2m-\alpha T)^2 = -\frac{1}{2T}\big(w^2 - 4mw - 2w\alpha T + 4m^2 + 4\alpha m T + \alpha^2 T^2\big)$$

$$= \frac{-(4m^2 - 4mw + w^2)}{2T} + w\alpha - 2\alpha m - \frac{\alpha^2 T}{2}$$

$$= \frac{-(2m-w)^2}{2T} + w\alpha - 2\alpha m - \frac{\alpha^2 T}{2}$$

$$-\frac{1}{2T}(w-\alpha T)^2 = -\frac{w^2}{2T} + \alpha w - \frac12 \alpha^2 T$$

$$= \frac{-e^{2\alpha m}}{\sqrt{2\pi T}} \int_{-\infty}^m e^{-\frac{1}{2T}(w-2m-\alpha T)^2}\, dw$$

$$+ \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^m e^{-\frac{1}{2T}(w-\alpha T)^2}\, dw$$

$z_1 = \frac{w-2m-\alpha T}{\sqrt T}$ $\qquad$ $z_2 = \frac{w-\alpha T}{\sqrt T}$

$dz_1 = \frac{dw}{\sqrt T}$ $\qquad$ $dz_2 = \frac{dw}{\sqrt T}$

$$= -e^{-2\alpha m} \int_{-\infty}^{\frac{-m-\alpha T}{\sqrt T}} \frac{1}{\sqrt{2\pi}}\, e^{-z_1^2/2}\, dz_1$$

$$+ \int_{-\infty}^{\frac{m-\alpha T}{\sqrt T}} \frac{1}{\sqrt{2\pi}}\, e^{-z_2^2/2}\, dz_2$$

$$= -e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt T}\right) + N\left(\frac{m-\alpha T}{\sqrt T}\right)$$

$$\frac{d}{dm}\tilde{\mathbb{P}}(\hat m(T)\le m) = \frac{2}{\sqrt{2\pi T}}\, e^{-\frac{(m-\alpha T)^2}{2T}} - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt T}\right)$$

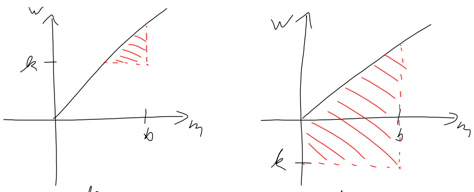
for $m \ge 0$

$$V(T) = \left(S(0)e^{\sigma \widehat{W}(T)} - K\right)_+ \mathbb{1}_{\{S(0)e^{\sigma \widehat{M}(T)} \le B\}}$$

$$= \left(S(0)e^{\sigma \widehat{W}(T)} - K\right) \mathbb{1}_{\{S(0)e^{\sigma \widehat{M}(T)} \le B,\ S(0)e^{\sigma \widehat{W}(T)} \ge K\}}$$

$$= \left(S(0)e^{\sigma \widehat{W}(T)} - K\right) \mathbb{1}_{\{\widehat{M}(T) \le b,\ \widehat{W}(T) \ge k\}}$$

where $b = \frac{1}{\sigma}\log\left(\frac{B}{S(0)}\right)$, $k = \frac{1}{\sigma}\log\left(\frac{K}{S(0)}\right)$



$k \ge 0$ $\qquad$ $k < 0$

$$V(0) = \widetilde{E}\left[e^{-rT} V(T)\right]$$

if $k \ge 0$ we integrate $k \le w \le m \le b$

if $k < 0$ over $k \le w \le m,\ 0 \le m \le b$

Combined: $\{(m,w);\ k \le w \le b,\ w_+ \le m \le b\}$

when $0 < S(0) \le B$

$$V(0) = \int_k^b \int_{w_+}^b e^{-rT}\left(S(0)e^{\sigma w} - K\right) \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\, dm\, dw$$

$$= -\int_k^b \left[e^{-rT}\left(S(0)e^{\sigma w} - K\right)\frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \Big|_{m=w_+}^{m=b} \right] dw$$

$$= -\int_k^b e^{-rT}\left(S(0)e^{\sigma w} - K\right)\frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2b-w)^2}\, dw$$

$$+ \int_k^b e^{-rT}\left(S(0)e^{\sigma w} - K\right)\frac{1}{\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{w^2}{2T}}\, dw$$

$$= S(0) I_1 - K I_2 - S(0) I_3 + K I_4$$

$$I_1 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2}\, dw$$

$$I_2 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2}\, dw$$

$$I_3 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2}{T}b^2 + \frac{2}{T}bw - \frac{1}{2T}w^2}\, dw$$

$$I_4 = \frac{1}{\sqrt{2\pi T}} \int_k^b e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2}{T}b^2 + \frac{2}{T}bw - \frac{1}{2T}w^2}\, dw$$

$$\frac{1}{\sqrt{2\pi T}} \int_k^b e^{\beta + \gamma w - \frac{1}{2T}w^2}\, dw$$

$$= \int_k^b \frac{1}{\sqrt{2\pi T}} e^{\frac{-w^2 + 2T\gamma w + 2T\beta}{2T}}\, dw$$

$$= \int_k^b \frac{1}{\sqrt{2\pi T}} e^{\frac{-(w - T\gamma)^2 + T^2\gamma^2 + 2T\beta}{2T}}\, dw$$

$$= e^{\frac{\gamma^2 T}{2} + \beta} \int_k^b \frac{1}{\sqrt{2\pi T}} e^{-\frac{(w-T\gamma)^2}{2T}}\, dw$$

$$= e^{\frac{\gamma^2 T}{2} + \beta} \int_{\frac{k - T\gamma}{\sqrt{T}}}^{\frac{b - T\gamma}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= e^{\frac{1}{2}\gamma^2 T + \beta}\left[N\left(\frac{b-\gamma T}{\sqrt{T}}\right) - N\left(\frac{k-\gamma T}{\sqrt{T}}\right)\right]$$

$$= e^{\frac{1}{2}\gamma^2 T + \beta}\left[N\left(\frac{-k+\gamma T}{\sqrt{T}}\right) - N\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right]$$

$$k = \frac{1}{\sigma}\log\left(\frac{K}{S(0)}\right) \qquad b = \frac{1}{\sigma}\log\left(\frac{B}{S(0)}\right)$$

$$= e^{\frac{1}{2}\gamma^2 T + \beta}\left[N\left(\frac{\log\left(\frac{S(0)}{K}\right) + \sigma\gamma T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{S(0)}{B}\right) + \sigma\gamma T}{\sigma\sqrt{T}}\right)\right]$$

set $\delta_{\pm}(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$

I_1: $\gamma = \alpha + \sigma \qquad \beta = -\frac{1}{2}\alpha^2 T - rT$

$$I_1 = e^{\frac{1}{2}(\sigma+\alpha)^2 T - rT - \frac{1}{2}\alpha^2 T}\left[N\left(\frac{\log\left(\frac{S(0)}{K}\right) + \sigma^2 T + \alpha T\sigma}{\sigma\sqrt{T}}\right)\right.$$

$$\left. - N\left(\frac{\log\left(\frac{S(0)}{B}\right) + \sigma^2 T + \alpha T\sigma}{\sigma\sqrt{T}}\right)\right]$$

$$\frac{1}{2}(\sigma+\alpha)^2 T - rT - \frac{1}{2}\alpha^2 T = \frac{1}{2}\sigma^2 T + \alpha\sigma T - rT$$

$$T\left(\frac{1}{2}\sigma^2 + \alpha\sigma - r\right)$$

since $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$

$$T\left(\frac{1}{2}\sigma^2 + \sigma\left(\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\right) - r\right) = 0$$

$$\sigma^2 T + \alpha\sigma T = \sigma^2 T + \left(r - \frac{1}{2}\sigma^2\right)T$$

$$= \left(r + \frac{\sigma^2}{2}\right)T$$

$$I_1 = N\left(\delta_+\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right)\right)$$

$$Y(T) = S(0)\, e^{\sigma \hat{M}(t)}\, e^{\sigma(\hat{M}(T)-\hat{M}(t))}$$
$$= Y(t)\, e^{\sigma(\hat{M}(T)-\hat{M}(t))}$$
$$\hat{M}(T)-\hat{M}(t) = \Big[\max_{t\le u\le T} \hat{W}(u) - \hat{M}(t)\Big]_+$$

$$\hat{M}(T)-\hat{M}(t) = \Big[\max_{t\le u\le T}\big(\hat{W}(u)-\hat{W}(t)\big) - \big(\hat{M}(t)-\hat{W}(t)\big)\Big]_+$$
$$\sigma\big(\hat{M}(T)-\hat{M}(t)\big) = \Big[\max_{t\le u\le T} \sigma\big(\hat{W}(u)-\hat{W}(t)\big) - \log\frac{Y(t)}{S(t)}\Big]_+$$

$$V(t) = e^{-r\tau}\,\tilde{E}\Big[Y(t)\, e^{\left[\max_{t\le u\le T}\sigma(\hat{W}(u)-\hat{W}(t)) - \log\frac{Y(t)}{S(t)}\right]_+} \Big| \mathcal{F}(t)\Big]$$
$$\underbrace{- e^{-r\tau}\,\tilde{E}\big[S(T) \,|\, \mathcal{F}(t)\big]}_{-S(t)}$$
$$e^{-r\tau}\,\tilde{E}\big[S(T) \,|\, \mathcal{F}(t)\big] = e^{rt}\,\tilde{E}\big[e^{-rT} S(T) \,|\, \mathcal{F}(t)\big], \quad \underbrace{\tilde{E}\big[e^{-rT} S(T) \,|\, \mathcal{F}(t)\big]}_{e^{-rt} S(t)}$$

$$V(t) = e^{-r\tau}\, Y(t)\, g\big(Y(t), S(t)\big) - S(t)$$

$$v(t,x,y) = e^{-r\tau}\, y\, g(y,x) - x$$
$$g(y,x) = \tilde{E}\Big[e^{[\sigma\hat{M}(\tau) - \log(\frac{y}{x})]_+}\Big]$$
$$= \tilde{E}\Big[e^{[\sigma\hat{M}(\tau) - \log(\frac{y}{x})]_+}\, \mathbb{1}_{\{\hat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\}}\Big]$$
$$+ \tilde{E}\Big[e^{[\sigma\hat{M}(\tau) - \log(\frac{y}{x})]_+}\, \mathbb{1}_{\{\hat{M}(\tau) > \frac{1}{\sigma}\log\frac{y}{x}\}}\Big]$$
$$= \tilde{\mathbb{P}}\Big[\hat{M}(\tau) \le \frac{1}{\sigma}\log\frac{y}{x}\Big]$$
$$+ \frac{x}{y}\,\tilde{E}\Big[e^{\sigma\hat{M}(\tau)}\, \mathbb{1}_{\{\hat{M}(\tau) > \frac{1}{\sigma}\log\frac{y}{x}\}}\Big]$$

$$d\left(e^{-rt}v(t,S(t),Y(t))\right)$$

$$= -re^{-rt}v(t,S(t),Y(t))dt + e^{-rt}dv(t,S(t),Y(t)) + 0$$

$$= e^{-rt}\Big(-rv\,dt + v_t\,dt + v_x\,dS(t) + v_y \underbrace{dY(t)}_{\text{no } dt}$$

$$+ \frac{1}{2}v_{xx}(dS(t))^2 + \frac{1}{2}v_{yy}\underbrace{(dY(t))^2}_{=0} + v_{xy}\underbrace{dS(t)dY(t)}_{=0}\Big)$$

$$[Y,Y](t) = \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}\left(Y(t_{j+1})-Y(t_j)\right)^2$$

$$\le \lim_{\|\Pi\|\to 0}\left(\max_{0\le k\le n-1} Y(t_{k+1})-Y(t_k)\right)\underbrace{\sum_{j=0}^{n-1}\left(Y(t_{j+1})-Y(t_j)\right)}_{Y(t)-Y(0)}$$

$$= 0$$

$$dY(t) \ne \Theta(t)dt \Rightarrow Y(t) \ne Y(0) + \int_0^t \Theta(u)\,du$$

$$dY(t)dS(t) = 0$$

$$d\left(e^{-rt}v(t,S(t),Y(t))\right) = e^{-rt}\left[-rv + v_t + rS(t)v_x + \frac{1}{2}\sigma^2 S^2(t)v_{xx}\right]dt$$

$$+ e^{-rt}\sigma S(t)v_x\,d\widetilde{W}(t)$$

$$+ \underbrace{e^{-rt}v_y\,dY(t)}_{=0}$$

$dY(t) = 0$ in "flat region"