UNIT-3- VECTOR SPACES

* Definition of vector space - axioms - problems.
* Def - Vector sub space - axioms - problems.
* Linearly dependence and Linearly independence of vectors.
* *** Basis and dimension of vector space.
* Linear transformation - def - properties - problems.
* Inverse linear transformation - def - properties - problem
* Range and kernel of linear map - def - properties
* Dimension of range and kernal - def - problems.
* Rank and nullity - def - properties - problems.
* Rank. nullity theorem - without proof.

Binary operation:- when we use addition and multiplicative signs with any function is said to be a binary operation.

Binary Operation:- Given a non-empty set A, any function from $A \times A$ to A is called a binary operation on A.

In otherwords, to each ordered pair (x, y) of elements x and y of A, the binary operation * on A associates a unique element z of A, we write $z = x * y$.

Ex: Addition is a binary operation on R or N or Q or the set E of even integers, It is not a binary operation on the set of odd integers (∵ sum of two odd integers is not an odd integer.

Some important properties:-

CLOSURE:- let $B \subset A$, let $*$ be a binary operation on A. If for each pair of elements (x, y) in B, $x * y$ belongs to B, we say that B is closed under '$*$'

Ex:- 1.) The set of natural number (N) is not closed under division, but it is closer under addition.

2.) The set of ~~all~~ integers is not closed under the usual addition of numbers, but it is closed under multiplication.

ASSOCIATIVE:- A binary operative $*$ on a set A is said to be associative if $a*(b*c) = (a*b)*c \quad \forall\, a,b,c \in A$

COMMUTATIVE:- (Abelian property):-

A binary operation $*$ on a set A is said to be commutative if $a*b = b*a \quad \forall\, a,b \in A$.

Algebraic Structures.

Fields:- The standard properties of real numbers w.r.t addition and multiplication as so fundamental in mathematics that whenever an set x with two binary operations '+' and '$\cdot$' (called addition and multiplication). satisfies these properties called field.

* The following properties holds for a field.

* let R be the set of real numbers and the operation $+$ and $\cdot$ in R.

1.) Addition is associative

2.) There exists an element $0 \in R$ with the property

$$0+a = a = a+0 \text{ for all } a \in R$$

3.) For each $a \in R$, there exists $-a \in R$ such that

$$a+(-a) = 0 = (-a)+a$$

4.) Addition is commutative.

5.) multiplication is associative.

6.) There exists an element $1 \in R$ such that $1 \cdot a = a \cdot 1$ for all $a \in R$

7.) for each $a (\neq 0)$ in R, there exists an element such that

$$a \cdot \frac{1}{a} = 1 = a \cdot \frac{1}{a}$$

8.) multiplication is commutative.

9.) multiplication is distributive over addition that is for all $a, b, c \in R$ such that

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

and $(a+b) \cdot c = a \cdot c + b \cdot c$

Ex: 1.) The set Q of rational numbers under '+' and '·' is a field

2.) The set of R (real numbers), C (complex numbers) and field. Z (integer) is not a field under addition and multiplication ($\because$ prop 7 is not satisfied)

4) N is not a field.

Now we take up another structure in algebra, which is simpler than a field.

Group:- Let G be a non-empty set on which a binary operation '$*$' is defined. Then G is said to be a group, under the operation '$*$', if the following axioms are satisfied:

1.) $\boxed{* \text{ is closure:-}}$

For each $a, b \in G$ such that $a * b \in G$

2.) $\boxed{* \text{ is associative.}}$

For each $a, b, c \in G$ such that.

$$(a*b)*c = a*(b*c)$$

3.) There exists an element $e \in G$ such that.

$a * e = a = e * a$ for all $a \in G$

$\therefore$ e is called an identity of $*$

4.) For each $a \in G$, $\exists$ an element $x \in G$. Such that $a * x = e = x * a$

x is said to be an inverse of a for the operation $*$

In addition to the above axioms.

The operation $*$ is commutative.

i.e. There exists $a, b \in G$ such that

$$a * b = b * a$$

Then the group is said to be commutative (or) abelian.

Ex:- 1.) R, Q and Z are groups under the usual addition operation.

2.) N under addition is not a group because axioms 3, 4 are not satisfied.

Theorem 1 :- In a group G there is only one identity.
i.e the identity is unique.

Theorem 2 :- In a group G there is only on inverse
i.e the inverse is unique.

Semi group :- Let S be a non-empty set and $S \subseteq G$, $*$ be a binary operation on S. Then $(S, *)$ is called a semi group if the binary operation $*$ satisfies the following condition.

(i) $*$ is closure i.e for each $a, b \in G$ such that $a * b \in G$

(ii) $*$ is associative i.e for each $a, b, c \in G$ such that

$$(a * b) * c = a * (b * c)$$

Ring :- A non-empty set R equipped with two binary operations called (+) and multiplication (·) is said to be ring if it satisfies the following properties.

1.) $(R, +)$ is an abelian group
- closure with '+'
- associative with '+' 2.)
- identity with '+' 3.)
- Inverse with '+' 4.)
- commutative with '+' 5.)

2.) $(R, \cdot)$ is a semi group
- closure with '·' 6.)
- associative with '·' 7.) $a(u+v) = au + av$

3.) Distributive laws :-

$a \cdot (b+c) = a \cdot b + a \cdot c$ left distributive law hold.

$(b+c) \cdot a = b \cdot a + c \cdot a$ right distributive law hold.

8) $(a+b)u = au + bu$.
9) $(ab)u = a(bu)$
10) $1 \cdot (u) = u(1) = u$.

The ring is denoted by $(R, +, \cdot)$

Commutative Ring:- A ring R is said to be commutative ring if $a \cdot b = b \cdot a \ \forall \ a, b \in R$.

Eg:- The set of R, Q, Z and C are rings usual operation of addition and multiplication.

Field:- A ring R with atleast two elements is said to be field if satisfies 1.) commutative 2.) has unity $1 \in R$ such that for each non-zero element posses multiplicative inverse.

Ex:- The ring $(R, +, \cdot)$, $(C, +, \cdot)$, $(Q, +, \cdot)$ are fields $(Z, +, \cdot)$ is not a field because all non-zero elements of Z are not units.

Internal composition:- Let V be a non-empty set. The elements of V are called vectors.

A map $f : V \times V \to V$ is called an internal composition in V

External composition:- Let V & F be two non-empty sets. The elements of 'V' are called vectors where as the elements of F are called the scalars.

$\Rightarrow$ A map $f : F \times V \to V$ is called an external composition in V over F

Eg:- * If $a \circ \alpha \in V \ \forall \ a \in F$ & $\alpha \in V$ then '$\circ$' is called external composition in V over F

* If $\alpha + \beta \in V \ \forall \ \alpha, \beta \in V$ then $+$ is called an internal composition in V over F.

Axioms, preferences.

⇒ start from distributive law to check vector space i.e $a(u+v) = au+bv$ (I)

⇒ commutative (II)

⇒ Associative (III)

⇒ $1 \cdot u = u = u \cdot 1$ (IV)

⇒ closure (V)

⇒ Identity (VI)

⇒ Inverse (VII)

Q.) Let $V = R^2$ defined vector addition & scalar multiplication.

$$\boxed{a \oplus b = a^b}, \quad \boxed{k \odot a = ka.}$$

vector addition Scalar addition.

Step 1:- Verify whether it is vector space (OR) not.

Check By commutative law,

we have to verify → $a \oplus b = b \oplus a$ — ①

By the definition. $a \oplus b = a^b$ LHS of ①

$b \oplus a = b^a$ RHS of ①

$a^b \neq b^a$.

here commutative law fails.

∴ V is not a vector space.

Q.) Let $V = R$ $a \oplus b = 2a + 2b$. $k \odot a = ka$.

verify that V is a vector space.

⇒ By commutative.

$a \oplus b = b \oplus a$.

$2a + 2b = 2b + 2a$.

Satisfies.

$\Rightarrow$ Associative.

we have $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

LHS $\quad (2a + 2b) \oplus c$

$2(2a+2b) + 2c = 4a + 4b + 2c$

RHS $\quad a \oplus (b \oplus c)$

$a \oplus (2b + 2c)$

$2(a) + 2(2b + 2c)$

$= 2a + 4b + 4c.$

$(a \oplus b) \oplus c \neq a \oplus (b \oplus c)$

here associative fails

So, V is not a vector Space.

Q.) Show that $V = R^2$ is a vector space

1.) $\boxed{\text{Closure}}$:- for each $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in R^2$, $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2$

Such that $\quad u + v := \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in R^2.$

2.) $\boxed{\text{Associative}}$:- for each $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in R^2$, $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2$.

$w = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \in R^2$

$u + (v + w) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 + x_3 \\ y_2 + y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \end{bmatrix}$

$$(u+v)+w = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3 \\ y_1+y_2+y_3 \end{bmatrix} \in R^2.$$

So, $u+(v+w) = (u+v)+w$.

3.) $\boxed{\text{Identity}}$:-

For each $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in R^2 \ni$

$$u+\mathbf{0} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = u = \mathbf{0}+u \in R^2$$

$\forall u \in R^2$

4.) $\boxed{\text{Inverse}}$:-

$$-u = \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} \in R^2$$

$$u+(-u) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-u)+u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad u+(-u) = 0 = (-u)+u$$

5.) $\boxed{\text{Commutative}}$.

$$u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in R^2, \quad v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2.$$

$$u+v = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}, \quad v+u = \begin{bmatrix} x_2+x_1 \\ y_2+y_1 \end{bmatrix}$$

$$u+v = v+u \in R^2 \ \forall \ u, v \in R^2$$

Vector Subspace:- A subspace W of a vector space V is a non-empty subset that is itself a vector space w.r.to the inherited operations of vector addition and scalar multiplication on V ie

$u, v \in V$, $a, b \in F$ (i) $au \in V$ (ii) $bv \in V$ (iii) $au + bv \in V$.

Theorem:- Let W be a non-empty subset of the vector space V iff W is closed under addition and scalar multiplication $au + bv \in W$

Q.) Let $V = R$. Define addition and scalar multiplication by $a \oplus b = 2a + 2b$ and $k \odot a = ka$.

Show that addition is commutative but not associative.

Ans:- Since the usual addition of real numbers (on right-hand side) is commutative.

$$a \oplus b = 2a + 2b = 2b + a = b \oplus a.$$

Thus the operation $\oplus$ is commutative.

To determine whether addition is associative,

$$(a \oplus b) \oplus c = (2a + 2b) \oplus c$$
$$= 2(2a + 2b) + 2c$$
$$= 4a + 4b + 2c$$

and

$$a \oplus (b \oplus c) = a \oplus (2b + 2c)$$
$$= 2a + 2(2b + 2c)$$
$$= 2a + 4b + 4c.$$

$$(a \oplus b) \oplus c \neq a \oplus (b \oplus c)$$

$\therefore$ The associative property is not satisfied so, V is not a vector space.

Q.) Let $V = R$. Define addition and scalar multiplication by $a \oplus b = a^b$ and $k \odot a = ka$ show that V is not a vector space

In this case $a \oplus b = a^b$ and $b \oplus a = b^a$

Since $a^b \neq b^a$ for all choices of a & b, the commutative

property of addition is not up held and V is not a vector space

$\therefore$ V is not a vector space.

NOTE:-

* The set $V = R^n$ with the standard operations of addition and scalar multiplication is a vector space.

* The set $V = M_{m \times n}$ of all $m \times n$ matrices is a vector space over the scalar field R with $\oplus$ and $\odot$ defined component wise.

* The set $V = P_N$ of all polynomials of degree n is a vector space.

* The set V = set of real-valued functions defined on a common domain given by the interval $[a,b]$ is a vector space.

* The set $V = C$ of complex numbers is a vector space.

Theorem:- In a vector space V, additive inverses are unique

Proof:- let u be an element of V, ~~additive~~ suppose that v and u are elements of V and both are additive inverse of u.

we show that $v = w$ since

$$u \oplus v = 0 \quad \text{and} \quad u \oplus w = 0$$

$$v = v \oplus 0 = v \oplus (u \oplus w) = (v \oplus u) \oplus w = 0 \oplus w = w$$

Q) let $V = R^3$ show that V with the given operations for $\oplus$ and $\odot$ is not a vector space.

1.) $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}$, $c \odot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$

By commutative property $u \oplus v = v \oplus u$ — ①.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \neq \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \oplus \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Since, $\begin{bmatrix} x_1 - x_2 \\ y_1 + y_2 \\ z_1 - z_2 \end{bmatrix} \neq \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$

2.) $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 - 1 \\ z_1 + z_2 - 1 \end{bmatrix}$, $c \odot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$

⇒ <u>closure</u> property satisfies. $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in R^3$

⇒ <u>commutative</u> property satisfies $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \oplus \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

⇒ <u>Associative</u> property satisfies $\begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 - 1 \\ z_1 + z_2 - 1 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 - 2 \\ y_1 + y_2 + y_3 - 2 \\ z_1 + z_2 + z_3 - 2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 + x_3 - 1 \\ y_2 + y_3 - 1 \\ z_2 + z_3 - 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 + x_2 - 2 \\ y_1 + y_2 + y_3 - 2 \\ z_1 + z_2 + z_3 - 2 \end{bmatrix}$$

⇒ <u>Distributive</u> we have to prove that $a(u+v) = au + av$

L.H.S $= a\left[\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right] = a\begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 - 1 \\ z_1 + z_2 - 1 \end{bmatrix} = \begin{bmatrix} ax_1 + ax_2 - a \\ ay_1 + ay_2 - a \\ az_1 + az_2 - a \end{bmatrix}$

R.H.S ⇒ $au = a\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ay_1 \\ az_1 \end{bmatrix}$

$$av = a\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} ax_2 \\ ay_2 \\ az_2 \end{bmatrix}$$

$$au + av = \begin{bmatrix} ax_1 + ax_1 - 1 \\ ay_1 + ay_2 - 1 \\ az_1 + az_2 - 1 \end{bmatrix}$$

$$a(u+v) \neq au + av$$

$\therefore$ V is not a vector space.

$\because$ (External composition fails).

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 2x_2 \\ 2y_1 + 2y_2 \\ 2z_1 + 2z_2 \end{bmatrix}, \quad c \odot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$$

Associative law fails.

$$\therefore u \oplus (v \oplus w) \neq (u \oplus v) \oplus w$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} 2x_2 + 2x_3 \\ 2y_2 + 2y_3 \\ 2z_2 + 2z_3 \end{bmatrix} \neq \begin{bmatrix} 2x_1 + 2x_2 \\ 2y_2 + 2y_2 \\ 2z_1 + 2z_2 \end{bmatrix} \oplus \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 4x_2 + 4x_3 \\ 2y_1 + 4y_2 + 4y_3 \\ 2z_1 + 4z_2 + 4z_3 \end{bmatrix} \neq \begin{bmatrix} 4x_1 + 4x_2 + 2x_3 \\ 4y_1 + 4y_2 + 2y_3 \\ 4z_1 + 4z_2 + 2z_3 \end{bmatrix}$$

$\therefore$ It is not a vector space.

4) $$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}, \quad c \odot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} c + x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

we have to verify that external compositions.

$$a(u+v) = au + av$$

$$a\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} a + x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$au = a\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a+x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $av = a\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} a+x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$au + av = \begin{bmatrix} 2a + x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

$\therefore a(u+v) \neq au + av$

$\therefore$ It is not a vector space.

5) write out all 10 vector space axioms to show R^2 with the standard component wise operations is a vector space.

6) write out all 10 vector space axioms to show that $M_{2\times2}$ with the standard component wise operations is a vector space. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

7) let $V = R^2$ and defined addition as the standard component wise addition and defino scalar multiplication by $c \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+c \\ y \end{bmatrix}$ show that V is not a vector space.

By 10th axiom, $1 \odot u = 1 \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ y \end{bmatrix}$

fails

So, $V = R^2$ is not a vector space.

8) let $V = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \middle| a, b \in R \right\}$ with the standard component wise operations. Show that V is not a vector space.

(b) If addition and scalar multiplication are defined component wise only on the first two components and third is always 1 show that V is a vector space.

If any element in a given definition is const. mostly closure over addition fails.

(a) let $u = \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix}$; $v = \begin{bmatrix} a_2 \\ b_2 \\ 1 \end{bmatrix}$

(i) $u + v = v + u$ (commutative law satisfies)

(ii) $(u+v) + w = u + (v+w) \Rightarrow$ Associative law satisfies

(iii) $k(u+v) = k\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ 2 \end{bmatrix} = \begin{bmatrix} ka_1+ka_2 \\ kb_1+kb_2 \\ 2k \end{bmatrix}$

$ku = k\begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} = \begin{bmatrix} ka_1 \\ kb_1 \\ k \end{bmatrix}$, $kv = \begin{bmatrix} ka_2 \\ kb_2 \\ k \end{bmatrix}$

$ku + kv = \begin{bmatrix} ka_1+ka_2 \\ kb_1+kb_2 \\ 2k \end{bmatrix}$ Distributive law satisfies

(iv) $1 \cdot u = 1 \cdot \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix}$ satisfies

(v) $u + v \in R$ ~~does closure~~ satisfies.

$\hookrightarrow \begin{bmatrix} a_1+a_2 \\ b_2+b_2 \end{bmatrix} \longrightarrow$ So, closure fails | here $2 \in R$ but it is fixed that 3rd element is 1

S.T $V = R^3$ is not a vector space.

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}$.

$u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in R^3$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in R^3$, $w = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \in R^3$

we have to verify associative axiom.

$(u \oplus v) \oplus (w) = u \oplus (v \oplus w)$ — ①.

L.H.S of ①

$[u \oplus v] \oplus w = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \overset{I}{\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}} \oplus \overset{II}{\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}} = \begin{bmatrix} x_1 - x_2 - x_3 \\ y_1 - y_2 - y_3 \\ z_1 - z_2 - z_3 \end{bmatrix}$

$$au = a\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a+x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad av = a\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} a+x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$au + av = \begin{bmatrix} 2a + x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$\therefore \; a(u+v) \neq au + av$

It is not a vector space.

5) write out all 10 vector space axioms to show R^2 with the standard component wise operations is a vector space.

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By 10th axiom, $1 \odot u = 1 \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ y \end{bmatrix}$

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(ii) $(u+v) + w = u + (v+w) \Rightarrow$ Associative law satisfies

(iii) $k(u+v) = k\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ 2 \end{bmatrix} = \begin{bmatrix} ka_1+ka_2 \\ kb_1+kb_2 \\ 2k \end{bmatrix}$

$ku = k\begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} = \begin{bmatrix} ka_1 \\ kb_1 \\ k \end{bmatrix}$, $kv = \begin{bmatrix} ka_2 \\ kb_2 \\ k \end{bmatrix}$

$ku + kv = \begin{bmatrix} ka_1+ka_2 \\ kb_1+kb_2 \\ 2k \end{bmatrix}$ Distributive law satisfies

(iv) $1 \cdot u = 1 \cdot \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix}$ satisfies

(v) $u + v \in R$ ~~does~~ closure satisfies.

$\hookrightarrow \begin{bmatrix} a_1+a_2 \\ b_2+b_2 \end{bmatrix} \longrightarrow$ So, closure fails | here $2 \in R$ but it is fixed that 3rd element is 1

S.T $V = R^3$ is not a vector space.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}.$$

$u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in R^3$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in R^3$, $w = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \in R^3$

we have to verify associative axiom.

$(u \oplus v) \oplus (w) = u \oplus (v \oplus w)$ — ①.

L.H.S of ①

$$[u \oplus v] \oplus w = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \overset{I}{\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}} \oplus \overset{II}{\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}} = \begin{bmatrix} x_1 - x_2 - x_3 \\ y_1 - y_2 - y_3 \\ z_1 - z_2 - z_3 \end{bmatrix}$$

R.H.S $= u \oplus [v \oplus w] = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \end{bmatrix}$

$= \begin{bmatrix} x_1 - x_2 + x_3 \\ y_1 - y_2 + y_3 \\ z_1 - z_2 + z_3 \end{bmatrix}$ Associative law doesn't hold.

Additive identity:-

$u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in R^3$, $e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$

$u \oplus e = u = e \oplus u$.

L.H.S $\begin{bmatrix} x_1 - e_1 \\ y_1 - e_2 \\ z_1 - e_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$x_1 - e_1 = x_1$ | $y_1 - e_2 = y_1$ | $z_1 - e_3 = z_1$

$\boxed{e_1 = 0}$ | $\boxed{e_2 = 0}$ | $\boxed{e_3 = 0}$

$e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

R.H.S

$u = e \oplus u$.

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} e_1 - x_1 \\ e_2 - y_1 \\ e_3 - z_1 \end{bmatrix}$

i.e $x_1 = e_1 - x_1$ | $y_1 = e_2 - y_1$ | $z_1 = e_3 - z_1$

$2x_1 = e_1$ | $e_2 = 2y_1$ | $e_3 = 2z_1$

$e = \begin{bmatrix} 2x_1 \\ 2y_1 \\ 2z_1 \end{bmatrix}$

L.H.S $\neq$ R.H.S

So, additive identity fails.

Additive inverse:-

$u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in e^3$, $k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \rightarrow$ additive inverse.

LHS

$u + k = e = k + u$.

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \oplus \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in L.H.S.

$x_1 - k_1 = 0$

$x_1 = k_1$, $y_1 = k_2$, $z_1 = k_3$. $\Rightarrow k = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$k \oplus u = \begin{bmatrix} k_1 - x_1 \\ k_2 - k_2 \\ k_3 - k_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 2x_1 \\ 2y_1 \\ 2z_1 \end{bmatrix}$.

$k_1 = 3x_1$, $k_2 = 3y_1$, $k_3 = 3z_1$.

$k = \begin{bmatrix} 3x_1 \\ 3y_1 \\ 3z_1 \end{bmatrix}$

L.H.S $\neq$ R.H.S So, additive inverse fails

* If identity is checked, no need to check inverse.

Axiom 10:-

$1 \cdot u = 1 \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$

Distributive : $a(u+v) = au + av$

$$a\cdot(u+v) = a\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}$$

$$au = \begin{bmatrix} ax_1 \\ ay_1 \\ az_2 \end{bmatrix}, \quad av = \begin{bmatrix} ax_2 \\ ay_2 \\ az_2 \end{bmatrix}$$

$$+av \quad \begin{bmatrix} ax_1 - ax_2 \\ ay_1 - ay_3 \\ az_1 - az_2 \end{bmatrix}$$

So, L.H.S = R.H.S.
Distributive holds good.

$$(a+b)u = au \oplus bu$$

$$\begin{bmatrix} (a+b)x_1 \\ (a+b)y_1 \\ (a+b)z_1 \end{bmatrix} = \begin{bmatrix} ax_1 - bx_1 \\ ay_1 - by_1 \\ az_1 - bz_2 \end{bmatrix}$$

Ans → 4) $1 \odot u = 1\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1+x_1 \\ y_1 \\ z_1 \end{bmatrix} \neq \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $\quad 1 \odot u \neq u$.

So, multiplicative identity fails.

let $V = \left\{ \begin{bmatrix} t \\ 1+t \end{bmatrix} \mid t \in R \right\}$ define.

$$\begin{bmatrix} t_1 \\ 1+t_1 \end{bmatrix} \oplus \begin{bmatrix} t_2 \\ 1+t_2 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ 1 + t_1 + t_2 \end{bmatrix}$$

$$c \odot \begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} ct \\ 1+ct \end{bmatrix}$$

(a) find additive identity

Associative $u \oplus (v + w) = \begin{bmatrix} t_1 \\ 1+t_1 \end{bmatrix} \oplus \begin{bmatrix} t_2 + t_3 \\ 1 + t_2 + t_3 \end{bmatrix}$

$$= \begin{bmatrix} t_1 + t_2 + t_3 \\ 2 + t_2 + t_3 + t_1 \end{bmatrix}$$

$$(u \oplus v) \oplus w = \begin{bmatrix} t_1 + t_2 \\ 1 + t_1 + t_2 \end{bmatrix} \oplus \begin{bmatrix} t_3 \\ 1 + t_3 \end{bmatrix}$$

$$= \begin{bmatrix} t_1 + t_2 + t_3 \\ t_1 + t_2 + t_3 + 2 \end{bmatrix}$$

∴ Associative satisfies.

Distributive :-

$a(u \oplus v)$	$au \oplus av$
$a\begin{bmatrix} t_1 + t_2 \\ 1 + t_1 + t_2 \end{bmatrix}$	$\begin{bmatrix} at_1 \\ 1 + at_1 \end{bmatrix} \oplus \begin{bmatrix} at_2 \\ 1 + at_2 \end{bmatrix}$
$\begin{bmatrix} at_1 + at_2 \\ 1 + at_1 + at_2 \end{bmatrix}$	$\begin{bmatrix} at_1 + at_2 \\ 1 + at_1 + at_2 \end{bmatrix}$

∴ distributive satisfy.

Additive identity :- $v + e = v$ $\quad e = \begin{bmatrix} e \\ 1+e \end{bmatrix}$

$$\begin{bmatrix} t \\ 1+t \end{bmatrix} + \begin{bmatrix} e \\ 1+e \end{bmatrix} = \begin{bmatrix} t+e \\ 1+t+e \end{bmatrix} = \begin{bmatrix} t \\ 1+t \end{bmatrix}$$

$\boxed{e = 0}$

Additive inverse

So, $k = \begin{bmatrix} (-1)t \\ 1 + (-1)t \end{bmatrix} = (-1)\begin{bmatrix} t \\ 1+t \end{bmatrix}$

$u \oplus k = e = k \oplus u$.

$$\begin{bmatrix} k_1 + t \\ 1 + k_1 + t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{k_1 = -t}$$

additive inverse satisfies

(b) Show that V is a vector space in above problem.

(c) verify $0 \cdot v = 0 = v \cdot 0$ in above problem.

$$v = \begin{bmatrix} t_2 \\ 1+t_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 0 \cdot \begin{bmatrix} t_2 \\ 1+t_2 \end{bmatrix} = \begin{bmatrix} 0 \cdot t_2 \\ 1+0 \cdot t_2 \end{bmatrix}$$

---

let $$V = \left\{ \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} \middle| \, t \in R \right\}.$$

Define $$\begin{bmatrix} 1+t_1 \\ 2-t_1 \\ 3+2t_1 \end{bmatrix} \oplus \begin{bmatrix} 1+t_2 \\ 2-t_2 \\ 3+2t_2 \end{bmatrix} = \begin{bmatrix} 1+(t_1+t_2) \\ 2-(t_1+t_2) \\ 3+(2t_1+2t_2) \end{bmatrix}$$

$$c \odot \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} = \begin{bmatrix} 1+ct \\ 2-ct \\ 3+2ct \end{bmatrix}.$$

a.) Find the additive identity and inverse

b.) show that V is a vector space

c.) verify $0 \odot v = 0$ for all v.

let $f(x) = x^3$ defined on R and let $V = \{ f(x+t) / t \in R \}$

Define $f(x+t_1) \oplus f(x+t_2) = f(x+t_1+t_2)$

$c \odot f(x+t) = f(x+ct)$

Additive identity:-

$$V + e = V$$

$$f(x+t) + f(x+e) = f(x+t)$$

$$f(x+t+e) = f(x+t)$$

$$e = 0$$

The zero vector is given by $f(x+0) = x^3$

Inverse:-

$$-f(x+t) = f(x-t)$$

(b) each of the 10 vector space axioms are satisfied

vector space.

Vector subspace:- A subspace W of a vector space V is a non-empty subset that is itself a vector space w.r.t vector addition and scalar multiplication on V.

(or) A non-empty subset W of a v.s $V(F)$ is said to be subspace of V if W itself forms a v.s over the same field F w.r.t vector addition and scalar multiplication in V.

NOTE:- ① If W is a subspace of $V(F)$ then W is a subgroup of V w.r.t addition.

∴ let $V(F)$ be a v.s then $\{\bar{0}\}$ and $V(F)$ are always of subspace of $V(F)$.

Trivial (or Improper) subspaces and Non-trivial (or proper) Subspaces:-

∵ Every vector space $V(F)$ ($V \neq \{\bar{0}\}$) has atleast two subspaces $\{\bar{0}\}$ and V itself. These two subspaces are called Trivial subspaces of $V(F)$. If $V(F)$ has any subspace other than these two then they are called proper subspaces of $V(F)$.

Properties:- ① A non-empty subset W of a vector space $V(F)$ to be a subspace of $V(F)$ $\Leftrightarrow$ (i) $\alpha+\beta \in W \ \forall \alpha, \beta \in W$
(ii) $a\alpha \in W \ \forall \alpha \in W, \forall a \in F$

② A non-empty subset W of a vector space $V(F)$ to be a subspace of $V(F)$ $\Leftrightarrow$ $a\alpha + b\beta \in W \ \forall \alpha, \beta \in W \ \forall a, b \in F$

③ A non-empty subset W of a vector space $V(F)$ to be a subspace of $V(F)$ $\Leftrightarrow$ (i) $\alpha - \beta \in W \ \forall \alpha, \beta \in W$
(ii) $a\alpha \in W \ \forall \alpha \in W \ \forall a \in F$

④ A non-empty subset W of a vector space $V(F)$ to be a subspace of $V(F)$ $\Leftrightarrow$ $a\alpha + \beta \in W \ \forall \alpha, \beta \in W \ \forall a \in F$.

Theorem:- let W be a non-empty subset of the vector space V. Then W is a subspace of V iff W is closed under addition and scalar multiplication.

Theorem:- The intersection of any collection of subspaces of vector space is a subspace of the vector space.

Theorem:- The union of two subspaces of a vector space need not be a subspace unless one is contained in the other.

Ex:- let $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} / x \in R \right\}$ and $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} / y \in R \right\}$

$$W_1 \cup W_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} / x = 0 \text{ (or) } y = 0 \right\}$$

This set not closed under addition since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is not in $W_1 \cup W_2$

Linear combination:- let $S = \{v_1, v_2 \ldots v_k\}$ be a set of vectors in vector space V. and let $c_1, c_2 \ldots c_k$ be scalars. A linear combination of the vectors of S is an expression of the form

$$(c_1 \odot v_1) \oplus (c_2 \odot v_2) \oplus \cdots \oplus (c_k \odot v_k)$$

(or) $$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \sum_{i=1}^{k} c_i v_i$$

Span of a set of vectors:- let V be a vector space and let $S = \{v_1, v_2 \ldots v_n\}$ be a (finite) set of vectors in V. The span of S denoted by span (S) is the set

$$\text{span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n / c_1, c_2 \ldots c_n \in R\}$$

Theorem:- If $S = \{v_1, v_2 \ldots v_n\}$ is a set of vectors in a vector space V, then span (S) is a subspace.

Linear independence and linear dependence:- The set of vectors $S = \{v_1, v_2 \ldots v_n\}$ in a vector space V is called linearly independent provided that the only solution to the equation

$c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0$ is a trivial solution $c_1 = c_2 \ldots c_n = 0$

If the equation has a non-trivial solution, then the set S (atleast one of the value is non-zero) is called linearly dependent.

Basis for a vector space :- A subset B of a vector space V is a basis for V provided that

1.) B is a linearly independent set of vectors in V.

2.) Span (B) = V

Dimension of a vector space :- The dimension of the vector space V denoted by dim V, is the number of vectors in any basis of V.

Co-ordinates :- Let $B = \{v_1, v_2 --- v_n\}$ be an ordered basis for the vector space V. Let v be a vector in V, and let $c_1, c_2 --- c_n$ be the unique scalar such that

$v = c_1v_1 + c_2v_2 + --- c_nv_n$. Then $c_1, c_2 --- c_n$ are called the co-ordinates of v relative to B.

$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ and refer to the vector $[v]_B$ as the co-ordinate vector of v relative to B.

Problems: Let S be the subset of the vector space R^3 defined by $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ Show that $v = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$ is in span (S)

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$$

$$2c_1 + c_2 + c_3 = -4$$
$$-c_1 + 3c_2 + c_3 = 4$$
$$-2c_2 + 4c_3 = -6$$

Solving this linear system, we obtain

$c_1 = -2$, $c_2 = 1$ & $c_3 = -1$.

This shows that v is a linear combinations of the vectors in S and is thus in span (S)

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ show that the span of S is the subspace of $m_{2\times2}$ of all symmetric matrices

Consider a 2x2 symmetric matrix of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Since any matrix in span (S) has the form

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

span (S) is the collection of all 2x2 symmetric matrices

NOTE:- If the values satisfies all the eqn ①, then it is in linear combination ← linear span.
It should satisfy all eqn.

consistant → unique
→ infinite.

span → non-homogenous.

To find LD and LI consider always homogenous system.

To find span consider non homogenous system.

Q:) $S = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ verify $(\text{span}(S)) = R^3$

Method 1:-

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\2\end{bmatrix} + c_3\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$c_1 + c_2 + c_3 = a$

$c_1 + c_3 = b$

$c_1 + 2c_2 = c.$

$c_2 = a - b.$

$c_1 = c - 2(a-b) = 2b - 2a + c$

$c_3 = 2a - c - b.$

$\therefore \text{span}(S) = R^3.$

method-2:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 2 & 0 & c \end{array}\right]$$ Row reduction.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & b-a \\ 0 & 1 & -1 & c-a \end{array}\right]$$

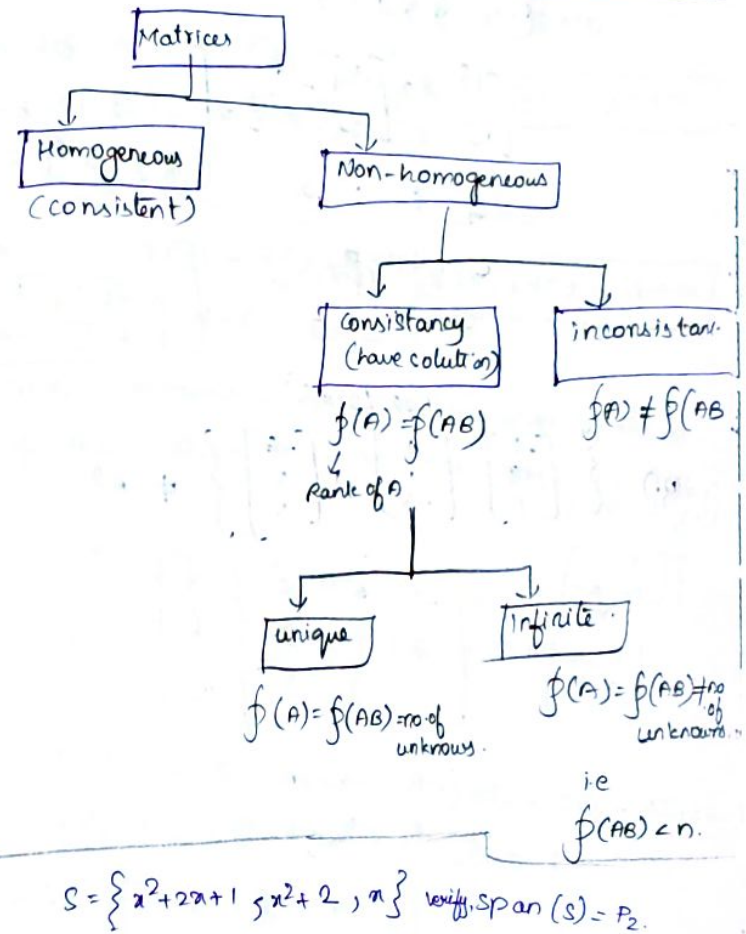
$R_3 \to R_2 + R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 0 & b-a \\ 0 & 0 & -1 & b+c-2a \end{array}\right]$$

$R_1 \to R_1 + R_2 + R_3.$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2a+2b+c \\ 0 & 1 & 0 & a-b \\ 0 & 0 & 1 & 2a-b-c \end{array}\right]$$

The system is consistant and it is $\text{span}(S) = R^3$.



Q) Determine $S = \{x^2+2x+1, x^2+2, x\}$ verify, $\text{span}(S) = P_2$.

(method 1)

$$c_1(x^2+2x+1) + c_2(x^2+2) + c_3 x = ax^2 + bx + c.$$

$c_1 + c_2 = a$

$2c_1 + c_3 = b$

$c_1 + 2c_2 = c.$

$c_2 = c - a.$

$c_1 = 2a - c$

$c_3 = 2c + b - 4a.$

Method-2

$n=3$ (no. of vectors)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 2 & -4 & c \\ -1 & 1 & -3 & d \end{array}\right]$$

$R_4 \to R_4 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 2 & -4 & c \\ -1 & -1 & +1 & d-c \end{array}\right]$$

$R_4 \to R_2 + R_4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 2 & -4 & c \\ 0 & 0 & 0 & b+d-c \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 2 & -4 & c \\ 0 & 0 & 0 & b+d-c \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 0 & c-2b+2a \\ 0 & 0 & 0 & b+d-c \end{array}\right]$$

So, it is not consistant.

$\rho(AB) = 4$, $\rho(A) = 2$

$\rho(A) \neq \rho(AB)$

$\text{span}(S) \neq m(2\times 1)$

Basis (both condns must be satisfied) → must satisfy span. ($\text{span}(S) = B$)
→ linearly independent.

$\rho(A) = 2 < 3$ → less than unknowns

So linearly dependent

I

Linearly Dependent :- It is a simple sq. matrix → Determinant
$|A| = 0$

Linearly independent :- $|A| \neq 0$ → $|A|_{3\times 3} \neq 0$
$\rho(A) = 3$.

II

It is not a simple square matrix (or) Rectangular. → Echlon form

Linearly Dependent (L.D) :- $\rho(A) < n$

Linearly independent (L.I) :- $\rho(A) = n$.

rank (no. of non-zero rows)

III

Linear combination → L.D

Eg :- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

no linear combination → L.I

Q.) $S = \left\{ \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ find span(S)

Is S L.I ?

$$A = \left[\begin{array}{ccc|c} 2 & 1 & -3 & a \\ -3 & 1 & 1 & b \\ 0 & 1 & 1 & c \\ 0 & 0 & 0 & d \end{array}\right]$$

$R_2 \to 2R_2 + 3R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & a \\ 0 & 5 & -7 & 2b+3a \\ 0 & 1 & 1 & c \\ 0 & 0 & 0 & d \end{array}\right]$$

$R_3 \to 5R_3 - R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & a \\ 0 & 5 & -7 & 2b+3a \\ 0 & 0 & 2 & 5c-2b-3a \\ 0 & 0 & 0 & d \end{array}\right]$$

Inconsistent

$\rho(A) = 3 \neq \rho(AB)(4)$

It doesn't span

$\rho(A) = 3$, no of vectors $= 3$.

$\rho(A) = n$.

So, Linearly independent.

43) Q.) $S = \{ \underset{v_1}{1}, \underset{v_2}{x-3}, \underset{v_3}{x^2+2x}, \underset{v_4}{2x^2+3x+5} \}$ is S linearly independent?

$n=4$

$$\begin{array}{c} \\ x^2 \\ x \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} \text{eqn1} & \text{eqn2} & \text{eqn3} & \text{eqn4} \end{array} \\ \left[\begin{array}{cccc|c} 0 & 0 & 1 & 2 & a \\ 0 & 1 & 2 & 3 & b \\ 1 & -3 & 0 & 5 & c \end{array}\right] \end{array}$$

$ax^2 + bx + c$

span(S)

$\rho(A) = 3 < 4$ L.D

35) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $n=3$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$c_1 - c_2 - c_3 = 1$ — ①

$c_1 - c_2 + 2c_3 = -1$ — ②

$c_2 = 1$ ① − ②

$-3c_3 = 2$

$\boxed{c_3 = \frac{-2}{3}}$

$c_1 - 1 + \frac{2}{3} = 1 \Rightarrow c_1 = 2 - \frac{2}{3}$

$= \boxed{\frac{6-2}{3} = \frac{4}{3}}$

Span.

34) $$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & -6 \\ 1 & -2 & 0 & -11 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & -6 \\ 0 & -2 & -3 & -5 \end{array}\right]$$

Span

26) $$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 1 & -1 & 2 & 7 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -2 \\ 0 & 0 & 3 & 9 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

$\boxed{c_2 = 3}$

$c_1 - c_2 - c_3 = -2$ — ①

$c_1 - c_2 + 2c_3 = 7$ — ②

$-3c_3 = -9$

$\boxed{c_3 = 3}$ $c_1 = 4$

Span.

28) $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 2 \\ -1 & 1 & -3 & -3 \end{array}\right]$$

$R_4 \to R_4 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 2 \\ 0 & 1 & -2 & -2 \end{array}\right]$$

$R_4 \to 2R_4 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

$\rho(A) \neq \rho(AB)$

3 4

Not a span.

30)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & -1 \\ 1 & -2 & 0 & -4 \end{array}\right]$$

$C_2 = 3$

$C_2 = 3$

$C_1 + 3C_3 = -1$

$C_1 - 2C_2 = -4$

$+ \quad + \quad +$

$3C_3 + 6 = 6$

$3C_3 = -$

$C_3 = -1/3$

$C_1 - 1 = -1$

$C_1 = 0$

span

31.)
$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ -1 & 3 & 0 & b \\ 2 & -1 & 0 & c \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ -1 & 3 & 0 & b \\ 0 & 0 & 0 & c-a \end{array}\right]$$

Not a span

32.)
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 3 & 2 & b \\ 2 & 1 & -1 & c \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 3 & 2 & b \\ 1 & -2 & -3 & c-b \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 3 & 2 & b \\ 0 & -4 & -4 & c-b-a \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & -4 & -4 & c-b-a \end{array}\right]$$

$R_3 \to R_3 + 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a+4b-4a \end{array}\right]$$

$\rho(A) \neq \rho(AB)$

not in span

33)
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 2 & -1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 & c \\ 0 & 1 & 0 & 0 & d \end{array}\right]$$

$R_3 \to R_3 - R_1$

$\rho(A) \neq \rho(AB)$ not in span

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 2 & -1 & 0 & 0 & b \\ 0 & -1 & 0 & 0 & c-a \\ 0 & 1 & 0 & 0 & d \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 2 & -1 & 0 & 0 & b \\ 0 & -1 & 0 & 0 & c-a \\ 0 & 0 & 0 & 0 & c-a+d \end{array}\right]$$

34.)
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & b \\ 1 & -1 & 1 & c \\ 0 & -1 & 1 & d \end{array}\right]$$

$R_4 \to R_4 + R_1 - R_2 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & b \\ 1 & -1 & 1 & c \\ 0 & 0 & 1 & d+a-b-c \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & b \\ 0 & -2 & 1 & c-a \\ 0 & 0 & 1 & d+a-b-c \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & -1 & c-a+2b \\ 0 & 0 & 1 & d+a-b-c \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & -1 & c-a+2b \\ 0 & 0 & 0 & d+b \end{array}\right]$$

$\rho(A) \neq \rho(AB)$

Not in span

35.)
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 0 & 1 & 1 & c \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 0 & 0 & 0 & c-b \end{array}\right]$$

$\rho(A) \neq \rho(AB)$

not in span

36.)
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -1 & 1 & b \\ -4 & 2 & 2 & c \end{array}\right]$$

$R_3 \to R_3 + 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -1 & 1 & b \\ 0 & 2 & 6 & c+4a \end{array}\right]$$

L.I

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -1 & 1 & b \\ 0 & 0 & 8 & c+4a-b \end{array}\right]$$

span

37)
$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & a \\ 1 & 0 & 0 & b \\ -1 & -2 & 0 & c \end{array}\right]$$

$n = 3$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & -2 & 0 & c+b \end{array}\right]$$

span

L.D

38)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & -1 & 5 & b \\ 2 & 1 & 1 & c \end{array}\right]$$

$R_3 \to R_3 - R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & -1 & 5 & b \\ 0 & 2 & -6 & c-a-b \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & -1 & 3 & b-a \\ 0 & 2 & -6 & c-a-b \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & -1 & 3 & b-a \\ 0 & 0 & 0 & c-a-b+2b-2a \end{array}\right]$$

Not a span

L.D

39)
$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & a \\ 3 & 1 & -1 & b \\ 2 & 0 & -1 & c \end{array}\right]$$

$R_3 \to 3R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & a \\ 3 & 1 & -1 & b \\ 0 & 0 & -5 & 3c-2a \end{array}\right]$$

span

L.I

40.) $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 2 & 0 & 1 & 1 & b \\ 1 & 3 & 1 & 1 & c \end{array}\right]$

LD

$R_3 \to R_3 - R_1$

$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 2 & 0 & 1 & 1 & b \\ 0 & 4 & 1 & -1 & c-a \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 0 & 2 & 1 & -3 & b-2a \\ 0 & 4 & 1 & -1 & c-a \end{array}\right]$

$R_3 \to R_3 - 2R_2$

$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & a \\ 0 & 2 & 1 & -3 & b-2a \\ 0 & 0 & -1 & 5 & c-a-b+4a \end{array}\right]$

Span

BASIS:- If a non-empty set B said to be basis of a vector space. It must satisfy two conditions.

① Span(S) = V

② S must have linearly independent vectors in vectorspace V

Dimensions:- [must be unique]

No. of vectors in a basis is said to be dimension of vector space.

NOTE :- Basis is not unique.

Problems on basis.

Verify the set $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

To check Linearly Independent, we do det of matrix

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1(-1-1) - 1(-1) \Rightarrow -2+1 = -1 \neq 0$$

$\boxed{LI}$

$n = 3$

Span:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 1 & -1 & c \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 0 & 1 & b-a \\ 0 & 1 & -1 & c \end{array}\right]$$

Swap $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & c \\ 0 & 0 & 1 & b-a \end{array}\right]$$

$n = \rho(A) = \rho(AB) = 3$ consistent.

B is a basis

Dimensions = (No. of vectors) = 3.

Verify the set $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ -1 & 0 & 0 & d \end{array}\right] \quad R_4 \to R_4 + R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & d+a \end{array}\right]$$

$= 3$, $\rho(AB) = 4$

$\rho(A) \neq \rho(AB)$ not a span

so, not a basis & L.D

$\rho(A) = n$

$\{x+1, x-1, x^2\} = S$ given set is polynomial second degree

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & -1 & 0 & c \end{array}\right] \quad R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 0 & b \\ 0 & -2 & 0 & c-b \end{array}\right] \quad \text{Swap } R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & b \\ 0 & -2 & 0 & c-b \\ 0 & 0 & 1 & a \end{array}\right] \leftarrow \text{Swap } R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & -2 & 0 & c-b \end{array}\right]$$

$= 3 = \rho(A) = \rho(AB)$ L.I It is basis.

Dimensions $= 3$.

Find basis for the span of S.

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \left[\begin{array}{ccc|c} 2 & 2 & 1 & a \\ 2 & 0 & 2 & b \\ -1 & -2 & 1 & c \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & a \\ 0 & -2 & 1 & b-a \\ 0 & 0 & 2 & 2c-b+2a \end{array}\right] \leftarrow \left[\begin{array}{ccc|c} 2 & 2 & 1 & a \\ 0 & -2 & 1 & b-a \\ 0 & -2 & 1 & 2c+a \end{array}\right] \leftarrow \left[\begin{array}{ccc|c} 2 & 2 & 1 & a \\ 0 & -2 & 1 & b-a \\ -1 & -2 & 1 & c \end{array}\right]$$

$R_3 \to R_3 - R_2$ $\quad R_3 \to 2R_3 + R_1$

$\rho(A) \neq \rho(AB)$

not a basis.

$$S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 & 0 & 2 & 4 \\ -3 & 2 & -1 & 0 \\ 0 & 2 & 2 & 4 \end{bmatrix} \quad C_3 \to 2C_3 - C_4 \quad \begin{bmatrix} 2 & 0 & 0 & 4 \\ -3 & 2 & -2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$C_2 \to C_2 + C_3$

$$\begin{bmatrix} 2 & 0 & 0 & 4 \\ -3 & 0 & -2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$C_1 \to 2C_1 - C_4$

$$\begin{bmatrix} 0 & 0 & 0 & 4 \\ -6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad C_3 \to 3C_2 - C_1 \quad \begin{bmatrix} 0 & 0 & 0 & 4 \\ -6 & 0 & -2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

It is basis.

as this vector is zero

$S = \{V_1, V_2, V_4\}$

Dimensions $= 3$.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

find basis and dimensions.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \quad C_3 \to C_3 + C_5 \quad \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$C_4 \to C_4 + C_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & 1 & 0 & -1 & -2 \end{bmatrix}$$

$C_5 \to C_5 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & 1 & 0 & -1 & -1 \end{bmatrix}$$

$C_4 \to C_4 - C_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$C_3 \to C_3 - C_4$

Basis Dimensions $= \{C_1, C_2, C_5\}$

3

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - C_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad C_5 \to C_5 + C_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Find a basis for R^4 that contains the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$R^4 = \{ e1, e2, e3, e4 \}$ standard vectors.

$$R^4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$S = \{ v_1, v_2, e1, e2, e4 \}$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad C_2 \to C_2 + C_1 \quad \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$C_4 \to C_4 - C_2$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_1 - C_3$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_5 \to C_5 - C_1$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In a vector space V_2 show that $(3,7) \in [(1,2),(0,1)]$

↳ linear combination

we have to prove that $(3,7)$ is a linear combination of $(1,2), (0,1)$

$$(3,7) = a(1,2) + b(0,1)$$

$\boxed{a=3}$, $2a + b = 7$

$\boxed{b=1}$

$(3,7) = 3(1,2) + 1(0,1)$

$3 = 3 + 0$

$7 = 6 + 1$

$\in$ indicates linear combination.

Q) $(3,7) \notin [(1,2),(2,4)]$

$(3,7) = a(1,2) + b(2,4)$

$3 = a + 2b \times 2$

$7 = 2a + 4b$

Q $(1+i, 1-i) \in [(1+i,1),(1,1-i)]$

$(1+i, 1-i) = a(1+i,1) + b(1,1-i)$

$1+i = a + ai + b \rightarrow ①$

$1-i = a + b - bi \rightarrow ②$

$1+i = a + ai + b.$

$2 = a(1+i) + 2b \rightarrow ② \times (1+i)$

$1+i-2 = -b.$

$\boxed{1-i = b}$

$1+i = a(1+i) + (1-i)$

$a(1+i) = 2i$

$a = \dfrac{2i(1-i)}{(1+i)(1-i)}$

$\boxed{a = 1+i}$

which of following sets are linearly independent & also find their linear span.

$S_1(1, x, x^2)$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 1 & 0 & 0 & c \end{array}\right]$$

Span

L.I

$\rho(A) = \rho(AB) = n = 3$

$S_2(1, x, x^2, 1+x^2)$

L.D

$\rho(A) \neq n$

3 4

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 0 & b \\ 1 & 0 & 0 & 1 & c \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & c \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & a \end{array}\right]$$

Span

NOLL matrix rank and dimension = 0

Dimension :-

Standard basis.

which of the following sets are basis. where $V = R^4$ find dimentions.

$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \right\}$$

$[1\ -1\ 0\ 1]$ $[0\ 0\ 0\ 1]$ $[2\ -1\ 0\ 1]$ $[3\ 2\ 1\ 0]$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & b \\ 2 & -1 & 0 & 1 & c \\ 3 & 2 & 1 & 0 & d \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 3 & 2 & 1 & 0 & d \\ 2 & -1 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & b \end{array}\right]$$

$R_3 \to R_3 + R_1 - R_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 3 & 2 & 1 & 0 & d \\ 0 & -4 & -1 & 2 & c+a-d \\ 0 & 0 & 0 & 1 & b \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 0 & 5 & 1 & -3 & d-3a \\ 0 & -4 & -1 & 2 & c+a-d \\ 0 & 0 & 0 & 1 & b \end{array}\right]$$

$n = 4$

$\rho(A) = \rho(AB) = n$

basis.

$R_3 \to 5R_3 + 4R_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & a \\ 0 & 5 & 1 & -3 & d-3a \\ 0 & 0 & -1 & -2 & 5c+5a-5d+4d-12a \\ 0 & 0 & 0 & 1 & b \end{array}\right]$$

$V = R^0$; $\beta(A) = 1$

Q.) $S = \{[1, 2, 3, 4]\}$ → Not a basis ∵ dimensions → 4
∵ no. of vectors → 1

NOTE: $\boxed{\text{No. of vectors} = \beta(A) = \text{dimension of } V}$

$S = \{[1\ -1\ 2\ 0],\ [1\ 1\ 2\ 1],\ [2\ 3\ 5\ -1],\ [2\ 2\ 3\ 2]$
not a basis $[2\ 0\ 4\ 0]\}$. $V \in R^4$

no. of vectors ⇒ 5

$\beta(A) = 4$

dimensions = 5.

Find the dimension of V. where $V = \{y / (D^2+3D+2)y = 0\}$

$D^2 + 3D + 2 = 0$.

$D^2 + 2D + D + 2 = 0$

$D(D+2) + 1(D+2) = 0$

$(D+2)(D+1) = 0$

$D = -1, -2$.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Dimension = 2.

Span.

$V = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in F \right\}$ find dimension of V.

$\left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b \in F \right\}$.

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

vectors = 2 ; span.

$\boxed{\text{Dimensions} = 2.}$

standard basis of $M_{2\times2}$ matrix.

$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \middle/ a, b, c \in F \right\}$ → 3 dimensions.

$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ → 6 dimensions.

$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + h \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$+ g \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

* Coordinate Vector:- Let $B = \{v_1, v_2 \cdots v_n\}$ be an ordered basis for vector space 'V'. let 'v' be a vector in a vector space 'V' and let $c_1, c_2 \cdots c_n$ be the scalars corresponding to vectors in 'V' such that 'v' is combination of all vectors in ordered basis. i.e

$$\boxed{v = c_1 v_1 + c_2 v_2 + \cdots c_n v_n}$$

* then values of $c_1, c_2 \cdots c_n$ are called co-ordinates of 'v' relative to the ordered basis and is denoted by $[v]_B$

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

* Standard basis → having identity.

$P_2(x) = \{1, x, x^2\}$ standard basis. ($a_0 + a_1 x + a_2 x^2$)

* Ordered basis:- $P_2(x) = \{1, 1+x, x^2-1\}$ ordered basis.

Q:) Find the coordinates of vector relative to the ordered basis B

$$B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}, \quad v = \begin{bmatrix} 8 \\ 0 \end{bmatrix}.$$

$v = c_1 v_1 + c_2 v_2$

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$8 = 3c_1 - 2c_2$

$0 = c_1 + 2c_2$

$\boxed{c_1 = 2}$

$8 = 6 - 2c_2$

$\boxed{c_2 = -1}$

so $[v]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Q:) $B = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad v = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$0 = 2c_1 + c_2$

$1 = 2c_1$

$1/2 = 1c_1 + 2c_2 + c_3$

$c_1 = 1/2$

$c_2 = -1$

$\boxed{c_3 = 2}$

$$\boxed{[v]_B = \begin{bmatrix} 1/2 \\ -1 \\ 2 \end{bmatrix}}$$

Q:) $B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$

$$v = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$1 = c_1 + c_3 + c_4$

$3 = -c_1 + c_2$

$-2 = c_2 - c_4$

$2 = -c_3$

$c_1 + c_4 = 3$

$c_4 - c_1 = 5$

$\boxed{c_4 = 4}$

$3 = 1 + c_2$

$c_2 = 2$

$c_3 = -2$

$c_1 = -1$

$$[v]_B = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}.$$

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \right\}$$

$$V = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$2 = c_1 + c_3 + c_4$

$-2 = -c_1 + c_2 - c_3 + c_4$

$1 = c_2 + c_3$

$3 = c_1 + 2c_2 + 3c_4.$

$\boxed{c_2 + c_4 = 0}$

$1 = c_1 + c_4 - c_2.$

$$\boxed{\begin{bmatrix} 3/2 & 1 \\ 0 & 1/2 \end{bmatrix}}$$

$B = \{x^2 + 2x + 2,\ 2x + 3,\ -x^2 + x + 1\}$ $\quad v = p(x) = -3x^2 + 6x + 8$

$$\begin{bmatrix} -3 \\ 6 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$-3 = c_1 - c_3$

$6 = 2c_1 + 2c_2 + c_3$

$8 = 2c_1 + 3c_2 + c_3.$

$6 = 2c_1 + 4 + c_3$

$2 = 2c_1 + c_3$

$-6 = 2c_1 - 2c_1$

$-2 = -c_2.$

$\boxed{c_2 = 2}$

$8 = 3c_3$

$\boxed{c_3 = 8/3}$

$-3 = x - \frac{8}{3}$

$3x - 8 = -9.$

$3x = -1$

$x = -1/3$

$$[v]_B = \begin{bmatrix} -1/3 \\ 2 \\ 8/3 \end{bmatrix}$$

Q. Find the cordinates of $\{2, 3, 4, -1\}$ of V_4 relative to the standard basis of V_4. $\{2, 3, 4, -1\}$

2 3
4 -1

* Vector itself is a coordinate vector for standard basis. *

$S = \{[1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]\}$ is an ordered basis of a vector space (C^3) of c then find coordinate vector of $(3+4i, 6i, 3+7i)$

$c_1 + c_2 + c_3 = 3 + 4i$

$c_2 + c_3 = 6i$

$c_3 = 3 + 7i$

$c_2 = -3 - i$

$c_1 = 3 + 4i - 6i$

$c_1 = -2i + 3.$

$$\Rightarrow \begin{bmatrix} -2i + 3 \\ -3 - i \\ 3 + 7i \end{bmatrix}.$$

LINEAR TRANSFORMATION :- let $U(F)$ and $V(F)$ be two vectors over the same field 'F'. A mapping $T: U(F) \to V(F)$ is said to be linear transformation such that

(1) $\boxed{T(u+v) = T(u) + T(v)}$ → internal composition

(2) $\boxed{T(au) = a\,T(u)}$ → external composition $\forall\ a \in F$, $u \in U(F)$

(3) $\boxed{T(au + bv) = a\,T(u) + b\,T(v)}$ $\forall\ a, b \in F$, $u \in U(F)$, $v \in V(F)$

↳ internal & external composition.

NOTE :- $\boxed{\text{Linear Transformation (LT)} \Rightarrow \text{Homomorphism but converse need not be true.}}$

Operator :- Let $V(F)$ be a vector space. A linear transformation $T: V(F) \to V(F)$ is called linear operator (T) on V to V.

Let $V(F)$ be a vector space in field (F) if $T: V(F) \to F$ is a linear transformation. Then T is called linear function on 'V'.

Q.) $T: R^3 \to R^2$ defined by $T(x,y,z) = (x+y, y+z)$ then T is a L.T

S.T

$u_1 = (x_1, y_1, z_1) \Rightarrow T(u_1) = (x_1+y_1, y_1+z_1)$

$u_2 = (x_2, y_2, z_2) \Rightarrow T(u_2) = (x_2+y_2, y_2+z_2)$

we have to P.T

$T(au_1+bu_2) = aT(u_1) + bT(u_2)$

L.H.S.

$au_1+bu_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= [ax_1+bx_2, ay_1+by_2, az_1+bz_2]$

Taking L.T on both sides

$T(au_1+bu_2) = T[ax_1+bx_2, ay_1+by_2, az_1+bz_2]$

$= (ax_1+bx_2+ay_1+by_2, ay_1+by_2+az_1+bz_2)$

$= (a(x_1+y_1) + b(x_2+y_2), a(y_1+z_1) + b(y_2+z_2))$

$= a[x_1+y_1, y_1+z_1] + b[x_2+y_2, y_2+z_2]$

$= aT(u_1) + bT(u_2) = R.H.S.$

* To be in L.T range elements must have same degree

Q.) NOTE:- $T: R^2 \to R^3$ defined by $T(x,y) = (x+y, xy, y-x)$.

(1) here T is not a linear transformation

Reason :-

$x+y$ = degree 1

xy = degree 2

$y-x$ = degree 1

→ All degrees not same so not a L.T

Q.) $P: R \to R^2$ defined by $P(x) = x, x^2$ then T is

(2) not a L.T since x^2 is not a 1st degree term.

i.e x & x^2 has different degree.

Q.) If $T(x,y,z) = (x+k, y, z)$ then $\boxed{T \text{ is}}$ where k is any constant func

(3)

* if $k=0$ → $\boxed{L.T}$

* if $k \neq 0$
  - → 1st degree → $\boxed{L.T}$
  - → Not a 1st degree → $\boxed{\text{not a L.T}}$

Q.) $T: R^3 \to R^2$ defined by $T(x,y,z) = (|x|, 0)$ verify Linear Transformation.

$u_1 = (x_1, y_1, z_1) \Rightarrow T(u_1) = (|x_1|, 0)$

$u_2 = (x_2, y_2, z_2) \Rightarrow T(u_2) = (|x_2|, 0)$

We have to prove:

$$T(au_1 + bu_2) = aT(u_1) + bT(u_2)$$

L.H.S

$$au_1 + bu_2 = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$

$$= (ax_1 + x_2, ay_1 + y_2, az_1 + bz_2)$$

$$T(u_1 + u_2) = (|ax_1 + x_2|, 0) \quad -(1)$$

$$T(u_1) = (|x_1|, 0) \quad -(2)$$

$$T(u_2) = (|x_2|, 0) \quad -(3)$$

$$|x_1 + x_2| \neq |x_1| + |x_2|$$

By triangular inequality.

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

$T(u_1 + u_2) = T(u_1) + T(u_2)$ is not always true.

So T is not a L.T.

$T: R^3 \to R^2$ defined by $T(1,2,3) = (5,0)$ and $T(2,3,7) = (0,11)$ the find the value of $T(5,8,17) = ?$

$$(x,y,z) = p(1,2,3) + q(2,3,7) \quad -(1)$$

$$x = p + 2q \quad -(2)$$

$$y = 2p + 3q \quad -(3)$$

$$z = 3p + 7q \quad -(4)$$

$2 \times$ eqn (2) $-$ (3).

$$2p + 4q = 2x.$$

$$2p + 3q = y$$

$$2x - y = q.$$

$$\boxed{q = 2x - y.}$$

$$\boxed{p = 2y - 3x.}$$

$$p + 4x - 2y = x.$$

$$p = 2y - 3x.$$

Substitute p, q in (1).

$$(x,y,z) = (2y - 3x)(1,2,3) + (2x - y)(2,3,7)$$

Taking L.T on both sides

$$T(x,y,z) = (2y - 3x)T(1,2,3) + (2x - y)T(2,3,7)$$

$$T(5,8,17) = (2y - 3x)(5,0) + (2x - y)(0,11)$$

$$= (16 - 15)(5,0) + (10 - 8)(0,11)$$

$$= (5,0) + (0,22)$$

$$\boxed{T(5,8,17) = (5,22)}$$

Q) Describe the explicity of L.T $T: R^2 \to R^2$ such that $T(2,3) = (0,0)$

$T(1,0) = (0,0)$

$$(x,y) = a(2,3) + b(1,0)$$

$x = 2a + b$

$y = 3a$

$x = \frac{2y}{3} + b$

$a = \frac{y}{3}$, $\frac{3x - 2y}{3} = b$

$$(x,y) = \frac{y}{3}(2,3) + \left(\frac{3x-2y}{3}\right)(1,0)$$

apply L.T

$$T(x,y) = \frac{y}{3}T(2,3) + \left(\frac{3x-2y}{3}\right)T(1,0)$$

$$T(x,y) = \frac{y}{3}(4,5) + \left(\frac{3x-2y}{3}\right)(0,0)$$

$$\boxed{T(x,y) = \left(\frac{4y}{3}, \frac{5y}{3}\right)}$$

Q) Describe the explicity of L.T $T: F^2 \to F^2$ $T(e_1) = (a,b)$

$T(e_2) = (c,d)$ $e_1 = (1,0)$, $e_2 (0,1)$

$$(x,y) = p(1,0) + q(0,1)$$

$x = p$

$y = q$

Apply L.T.

$$T(x,y) = x\,T(1,0) + y\,T(0,1)$$

$$= x(a,b) + y(c,d)$$

$$\boxed{T(x,y) = (ax + cy, bx + dy)}$$

NOTE- Range must be Linearly dependent then only we can apply L.T.
signifies basis

$T: V^3 \to V^3$ such that $T(0,1,2) = (3,1,2)$ and $T(1,1,1) = (2,2,2)$ then determine explicit of L.T.

$$S = \{(0,1,2), (1,1,1)\}$$

but it is not a basis since ($L(S) \to$ span) does n't exist.

$$|A| = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 2 & 1 & 0 & c \end{array}\right]$$

$$|A| = +1(1-2) = -1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & -1 & 0 & c \end{array}\right]$$

we observed that $(1,0,0)$ is span(S) and Linearly independent of S.

$$S = \{(0,1,2), (1,1,1), (1,0,0)\}.$$

Q) let $T: P_2^{\mathbb{R}} \to \mathbb{R}$ so that $T(1+x) = x+x^2$, $T(1-x) = 1-x^2$

$T(x^2) = \frac{x^3}{3}$ then find L.T?

$S = \{1+x, 1-x, x^2\}$. ordered basis, span

Linearly dependent

$$ax^2+bx+c = p(1+x) + q(1-x) + r(x^2)$$

$$ax^2+bx+c = rx^2 + x(p-q) + p+q.$$

$$a = r; \quad b = p-q; \quad c = p+q.$$

$$2p = b+c \qquad q = c - \frac{b+c}{2}.$$

$$\boxed{p = \frac{b+c}{2}} \qquad \boxed{q = \frac{c-b}{2}}$$

$$ax^2+bx+c = \left(\frac{b+c}{2}\right)(1+x) + \left(\frac{c-b}{2}\right)(1-x) + a(x^2)$$

$$T(ax^2+bx+c) = \left(\frac{b+c}{2}\right)T(1+x) + \left(\frac{c-b}{2}\right)T(1-x) + aT(x^2)$$

$$= \left(\frac{b+c}{2}\right)(x+x^2) + \left(\frac{c-b}{2}\right)(1-x^2) + a\left(\frac{1+x^3}{3}\right)$$

$$T(ax^2+bx+c) = \frac{ax^3}{3} + \frac{cx^2+bx^2}{2} + \frac{bx}{2} + \left(\frac{c}{2}+a\right)$$

$$\boxed{T(ax^2+bx+c) = \frac{ax^3}{3} + \left(\frac{c+b}{2}\right)x^2 + \frac{bx}{2} + \left(\frac{c+2a}{2}\right)}$$

* Null space :- (or) kernal of a L.T

let $T: U(F) \to V(F)$ be a L.T. Then the nullspace of T is denoted by $N(T)$ (or) $\ker(T)$ and defined as

$N(T) = \{u \in U(F) \mid T(u) = 0_V\}$ where 0_V is a zero vector in V

Property:-

* If $T: U(F) \to V(F)$ be a L.T then Nullspace of T is a subspace of U(F)

& Range is always subspace of V(F)

Range of L.T (or) Image of L.T :- let $T: U(F) \to V(F)$ be a L.T. Then the range of T is denoted by $R(t)$ ~~(or) T(u)~~ and is defined as $R(T) = \{T(u) \mid u \in U(F)\}$.

If $T: U(F) \to V(F)$ be a L.T then $R(T)$ is a subspace of $V(F)$.

Subspace
↓
$au + bv$ must satisfy.

Nullity of L.T :-

let $T: U(F) \to V(F)$ be a L.T and U(F) (or) N(T) is finite dimensional vector space (F.D.V.S) then the dim N(T) is known as Nullity of T denoted by.

$$\boxed{\dim N(T) = n(T)}$$

Rank of a L.T :-

let $T: U(F) \to V(F)$ be a L.T and ~~V(F) (or) R(T)~~ U be a F.D.V.S then the dim R(T) is said to be Rank of T

and it is denoted by.

$$\boxed{\rho(T) = \dim R(T)}$$

* dim of nullspace = nullity
* dim of range = rank

Q.) $T: R^2 \to R^3$ be a L.T defined by $T(x,y) = (x, y, x+y)$ then find
1.) $N(T)$
2.) $\eta(T)$
3.) $R(T)$
4.) $\rho(T)$.

(i) $N(T) \Rightarrow$ null space (or) $\ker(T)$.

$$N(T) = \{u \in U(F) \mid T(U) = 0_V\}$$

$$= \{(x,y) \in R^2 \mid T(x,y) = (0,0,0)\}$$

$$= \{(x,y) \in R^2 \mid (x, y, x+y) = (0,0,0)\}$$

$$= \{(x,y) \in R^2 \mid x=0, y=0, x+y=0\}$$

$$N(T) = \{(0,0) \in R^2\}$$

(ii) $\eta(T) \to$ nullity $\dim N(T)$.

$$N(T) = \{(0,0) \in R^2\}$$

$$a(0) + b(0) = L.S \text{ (span)}$$

$(0,0) \to$ L.D (linearly dependent.

$$\boxed{\eta(T) = 0 = \dim(N(T)) = \text{Nullity}}$$

* Dimension of null vector / zero vector is zero.
* Dimension of singleton non-zero vector is one.
  ↳ Linearly independent

(iii) Range of T:-

$$R(T) = \{u \in (U(F)) \mid T(u)\}$$

$$= \{(x,y) \in R^2 \mid T(x,y)\}$$

$$= \{(x,y) \in R^2 \mid (x, y, x+y)\}$$

$$R(T) = \{(x,y) \in R^2 \mid x(1,0,1) + y(0,1,1)\}$$

it is in combination means it in linear span.

$$S = \{(1,0,1), (0,1,1)\} \text{ is L.I.}$$

$$\boxed{\rho(T) = \dim R(T) = \text{rank} = 2.}$$

$T: R^3 \to R^2$, $\dim(U(F)) = 3$, $\dim(V(F)) = 2$ is a L.T defined by $T(x,y,z) = (x+y, y+z)$ then find (i) $N(T)$ (ii) $\eta(T)$ (iii) $R(T)$ (iv) $\rho(T)$

(i) $N(T) = \{u \in U(F) \mid T(U) = 0_V\}$.

$$= \{(x,y,z) \in R^3 \mid T(x,y,z) = (0,0)\}$$

$$= \{(x,y,z) \in R^3 \mid (x+y, y+z) = (0,0)\}$$

$\{(x,y,z) \in R^3 \mid x = -y,\ z = -y\}$

$\{(-y, y, -y) \in R^3 \mid y \in R\}$

Q.) $N(T) = \{y(-1, 1, -1) \in R^3 \mid 1 \in R\}$.

(ii) $\eta(T) = \dim N(T) = 1$

Since L.S $= N(T) = \{y(-1, 1, -1) \in R^3 \mid 1 \in R\}$

(iii) $R(T) = \{(x, y, z) \in R^3 \mid T(x, y, z)\}$.

$= \{(x, y, z) \in R^3 \mid (x+y, y+z)\}$.

$R(T) = \{(x, y, z) \in R^3 \mid x(1,0) + y(1,1) + z(0,1)\}$

(iv) $\rho(T)$:-

$\{(x, y, z) \in R^3 \mid x(1,0) + y(1,1) + z(0,1)\} = L(S)$

$\Rightarrow$ But $S = \{(1,0), (1,1), (0,1)\}$ is not L.I. ($|S|$ can be four)

$\Rightarrow S' = \{(1,0), (0,1)\}$ is L.I (or) $\{(1,0), (1,1)\}$ is L.I. $\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq$

S' is a basis

$$\boxed{\rho(T) = \dim(R(T)) = 2}$$

Rank - Nullity Theorem:- Let $T: U(F) \to V(F)$ be a L.T and U is a ~~fin~~ (F.D.V.S) then $\dim(R(T)) + \dim(N(T)) = \dim(U)$

$$\boxed{\dim(R(T)) + \dim(N(T)) = \dim(U)}$$

$$\boxed{\rho(T) + \eta(T) = \dim(U)}$$

Q.) $T: R^3 \to R^3$ be a L.T defined by

$T(x, y, z) = (2x, y+z, x+y+z)$.

find $\rho(T)$ & $\eta(T)$

$\boxed{|A| = 0}$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\rho(T) = 2$

Given $\dim(U) = 3$

By rank - Nullity theorem.

$\rho(T) + \eta(T) = \dim(U)$

$2 + \eta(T) = 3$

$$\boxed{\eta(T) = 1}$$

$T: V_2 \to V_3$ be L.T $T(x, y) = (x, x+y, y)$

$$\boxed{\begin{aligned} \rho(T) &= 2 \\ \eta(T) &= 0 \end{aligned}}$$

$A' = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Q. $T: V_3 \to V_3$

$T(x_1, x_2, x_3) = (x_1, x_1+x_2, x_1+x_2+x_3, x_3)$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\rho(T) = 3$, $\eta(T) = 0$.

Q.) $T: V_4 \to V_3$ be a L.T

$T(e_1) = (1,1,1)$, $T(e_2) = (1,-1,1)$, $T(e_3) = (1,0,0)$

$T(e_4) = (1,0,1)$

Then find $N(T)$, $\eta(T)$, $R(T)$, $\rho(T)$.

$$T(e_1, e_2, e_3, e_4) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$N(T) = \{ (e_1, e_2, e_3, e_4) \in V_4 \mid$

Q.) $V = M_{2\times 2}(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ a,b,c,d \in R \right\}$ is a V.S over the field of real no and $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ be the matrix in V and $T: V \to V$ defined by $T(A) = PA \ \forall A \in V$.

Then find the rank and nullity of a L.T.

$$N(T) = \{ A \in V / T(A) = 0_V \} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ PA = 0_V \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ \begin{matrix} a = c \\ b = d \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} c & d \\ c & d \end{bmatrix} \middle/ c,d \in R \right\}$$

$$= \left\{ c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \middle/ c,d \in R \right\}$$

$\therefore N(T) = L(S)$

where $S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \to$ L.I

$\therefore$ S is a basis for $N(T)$

$$\boxed{\eta(T) = \dim(N(T)) = 2}$$

$\rho(T) + \eta(T) = \dim(U)$ $\Rightarrow$ $\boxed{\rho(T) = 2}$

$\rho(T) + 2 = 4$