(vi) Identity $\Rightarrow \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + \begin{bmatrix} e \\ e \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{matrix} a+e=a \\ b+e=b \end{matrix} \Rightarrow e=0 \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(vii) Inverse $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow a+x=0 \Rightarrow x=-a,\ y=-b$

$x = \begin{bmatrix} -a \\ -b \\ 1 \end{bmatrix}$

(b) Let u, v be two vectors of V and c be an arbitrary constant.

$u \oplus v = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ 1 \end{bmatrix}, \quad c \odot u = c \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \begin{bmatrix} ca \\ cb \\ 1 \end{bmatrix}$

All 10 axioms of vectors space are satisfied.

$\therefore$ V is a vector space.

→ Let $V = \mathbb{R}^2$ and define $\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+2c \\ b+2d \end{bmatrix}, c \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$

Determine whether V is a vector space.

Commutative: $\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+2c \\ b+2d \end{bmatrix}$

$\begin{bmatrix} c \\ d \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c+2a \\ d+2b \end{bmatrix}$

Commutative law fails.

$V = \mathbb{R}^2$ is not a vector space.

→ Let $V = \left\{ \begin{bmatrix} t \\ -3t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$ and let addition and scalar multiplication be the standard operations on vectors. Determine V is a vector space.

$V_1 = \begin{bmatrix} t_1 \\ -3t_1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} t_2 \\ -3t_2 \end{bmatrix}, \quad V_3 = \begin{bmatrix} t_3 \\ -3t_3 \end{bmatrix}$

$(V_1+V_2)+V_3 = \begin{bmatrix} t_1+t_2 \\ -3(t_1+t_2) \end{bmatrix} + \begin{bmatrix} t_3 \\ -3t_3 \end{bmatrix} = \begin{bmatrix} t_1+t_2+t_3 \\ -3(t_1+t_2+t_3) \end{bmatrix}$

$V_1+(V_2+V_3) = (V_1+V_2)+V_3$ → Associative law satisfies.

$a(V_1+V_2) = a \begin{bmatrix} t_1+t_2 \\ -3(t_1+t_2) \end{bmatrix}$

$aV_1 = \begin{bmatrix} at_1 \\ -3t_1a \end{bmatrix}, \quad aV_2 = \begin{bmatrix} at_2 \\ -3t_2a \end{bmatrix} = aV_1 + aV_2 = \begin{bmatrix} at_1+at_2 \\ -3at_1-3at_2 \end{bmatrix}$

→ Distributive law satisfies

$1 \cdot u = 1 \cdot \begin{bmatrix} t \\ -3t \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix} = u$ P-14

Identity: $v + e = v \Rightarrow \begin{bmatrix} t \\ -3t \end{bmatrix} + \begin{bmatrix} e \\ -3e \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix}$ $t+e=t$, $e=0$ $v+e=v$ $\begin{bmatrix} t \\ -3t \end{bmatrix} + \begin{bmatrix} e \\ -3e \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix}$ $e=0$

$e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Inverse $v + x = e \Rightarrow \begin{bmatrix} t \\ -3t \end{bmatrix} + \begin{bmatrix} x \\ -3x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -t$ $t + e = t$ $x = e$ $\begin{bmatrix} -3t \\ -t \\ 3t \end{bmatrix}$

All axioms satisfies. $\therefore$ It is a vector space

→ let $V = \left\{ \begin{bmatrix} t+1 \\ 2t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$ and let addition and scalar multiplication be the standard operations on vectors. Determine whether V is a vector space.

Closure

$\boxed{\text{The zero vector is given by } 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ since this vector is not in } V, \text{ then } V \text{ is not a vector space}}$

$v + e = v$ $\begin{bmatrix} t+1 \\ 2t \end{bmatrix} + \begin{bmatrix} e+1 \\ 2e \end{bmatrix} = \begin{bmatrix} t+1 \\ 2t \end{bmatrix}$ $t+1+e+2 = t+1$ $e=-1$

$v_1 + v_2 = \begin{bmatrix} t_1+t_2+2 \\ 2t_1+2t_2 \end{bmatrix} \notin V$ Closure doesn't satisfies

$\therefore V$ is not a vector space.

→ let $V = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ and let addition and Scalar Multiplication be the standard Componentwise operations. Determine whether V is a vector space

$t+1+e+2 = t+1$ $e=-1$ 0

Soln: We observe that Closure, Associative, Distributive, 10th axiom, Commutative laws are satisfied.

$\therefore$ V is a vector space $\frac{t_1+1}{2h}$ $\frac{t_2+1}{2h}$ = t_1+t_2+4

→ let $V = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$

a) If addition and scalar multiplication are the standard componentwise operations, show that V is not a vector space.

$v_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & 1 \end{bmatrix}$

$0 \cdot u = \bar{0}$

$v_1 + v_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & 2 \end{bmatrix}$

$a \cdot \bar{0} = \bar{0}$

V is not closed under vector addition

$\therefore V$ is not a vector space.

(b) Define $\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \oplus \begin{bmatrix} d & e \\ f & 1 \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+f & 1 \end{bmatrix}$ and

$k \odot \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & 1 \end{bmatrix}$ s.t V is a vector space}.

soln. All 10 vector space axioms are satisfied.

$\therefore$ V is a vector space.

Let V be the set of 2×2 matrices with the standard (componentwise) definitions for vector addition and scalar multiplication. Determine whether V is a vector space. If V is not a vector space show that at least one of the 10 axioms does not hold.

→ Let V be the set of all Skew-Symmetric Matrices i.e the set of all matrices such that $A^T = -A$.

$A \oplus B = (A+B)^T = A^T + B^T$

$A^T = A$

$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

→ Let V be the set of all upper triangular matrices.

Let $A = \left\{ \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix} \middle| a, b, c, e, f, i \in R \right\}$

All 10 axioms of vector space satisfies.

$\therefore$ V = A is a vector space.

→ Let V be the set of all real Symmetric matrices.

that is, the set of all matrices such that $A^T = A$

Closure: $(A+B)^T = A^T + B^T = A + B$ satisfies (by $A^T = A$)

Associative $A^T + (B+C)^T = (A+B)^T + C^T$ satisfies

Identity: $A^T + e^T = A^T \Rightarrow A + e = A$ satisfies.

Inverse $A^T + x^T = e^T \Rightarrow A + x = e$ satisfies

Distributive $\alpha . (A+B)^T = a\left[A^T + B^T\right] = a(A+B)$ satisfies.

10th axiom: $1 . A^T = 1 . A = A$

$\therefore$ V = A (Set of all real Symmetric matrices) is a vector space,

→ Let V be the set of all invertible matrices. (A is invertible. $AX = B$, $X = A^{-1}B$) ($\because B = A^{-1}$) ($\because$ ~~$AB = BA$~~ $= I$)

Let $A = I$
$B = -I$ then $A+B$ is not invertible
Closure property doesn't holds

$\therefore$ V is not a vector space.

→ Let V be the set of all idempotent matrices.
Consider $A^2 = A$

Closure property fails ($\because (A+B)^2 \neq A^2 + B^2$)
V is not a vector space

→ Let B be a fixed matrix, and let V be the set of all matrices A such that $AB = 0$.
V is a vector space. ($\because$ It satisfies all the properties)

→ Let $V = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle/ a, b, c \in R \right\}$ and define addition and scalar multiplication as the standard componentwise operations. Determine whether V is a vector space.

→ Let V denote the set of 2x2 invertible matrices

Define $A \oplus B = AB$, $C \odot A = CA$

a) Determine the additive Identity and additive Inverse.

b) Show that V is not a vector space.

a) Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is an inverse of A

Both are 2x2 invertible matrices ($\because AB = BA = I$)

Additive Identity:

$\cancel{A + e} = A$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\cancel{1 + e_1 = 1} \Rightarrow \cancel{1 + e_2 = 1}$

$\cancel{e_1 = 0}$ $\cancel{e_2 = 0}$

$\cancel{1 + e_3 = 1}$ $\cancel{2 + e_4 = 2}$

$e_3 = 0$ $\cancel{e_4 = 0}$

by the def

$e_1 + e_3 = 1$

$e_2 + e_4 = 1$

$e_1 + 2e_3 = 1$

$e_2 + 2e_4 = 2$

$e_3 = 0$

$e_1 = 1$

$-e_4 = -1$

$e_4 = 1$

$e_2 = 0$

Additive Identity is $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Additive Inverse

$A + x = e$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} =$ Inverse of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore$ Additive Inverse of A is A^{-1} i.e $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

b) If $c = 0$

Then cA is not in Vector V.

$\therefore$ V is not a vector space.

→ let $V = \left\{ \begin{bmatrix} t \\ 1+t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$ define $\begin{bmatrix} t_1 \\ 1+t_1 \end{bmatrix} \oplus \begin{bmatrix} t_2 \\ 1+t_2 \end{bmatrix} = \begin{bmatrix} t_1+t_2 \\ 1+t_1+t_2 \end{bmatrix}$

$$c \odot \begin{bmatrix} t \\ 1+t \end{bmatrix} = \begin{bmatrix} ct \\ 1+ct \end{bmatrix}$$

a) Find the additive identity and inverse.

additive identity:

we have to p.T $V+e=V$ Consider $e = \begin{bmatrix} a \\ 1+a \end{bmatrix}$

$$\begin{bmatrix} t \\ 1+t \end{bmatrix} + \begin{bmatrix} a \\ 1+a \end{bmatrix} = \begin{bmatrix} t \\ 1+t \end{bmatrix}$$

$t+a=t$ | $1+t+a=1+t$ (∵ by the def.

$a=0$ | $a=0$

put $a=0$ in e

$e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the additive identity.

additive inverse

we have to PT $V+x=e$ Conside $x = \begin{bmatrix} x \\ 1+x \end{bmatrix}$

$$\begin{bmatrix} t \\ 1+t \end{bmatrix} + \begin{bmatrix} x \\ 1+x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$t+x=0 \Rightarrow x=-t$

$1+t+x=1$

$x=-t$ (by the def)

put $x=-t$ in x

$x = \begin{bmatrix} -t \\ 1-t \end{bmatrix}$ is the additive inverse.

b) show that V is a vector space.

All 10 axioms satisfies, So V is a vector space.

c) verify that $0 \odot v = 0$ for all v.

$$v = \begin{bmatrix} t \\ 1+t \end{bmatrix}$$

$$0 \odot v = \begin{bmatrix} 0t \\ 1+0t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0.$$

$0 \odot v \neq 0.$

→ let $V = \left\{ \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} \middle| \, t \in R \right\}$ Define $\begin{bmatrix} 1+t_1 \\ 2-t_1 \\ 3+2t_1 \end{bmatrix} \oplus \begin{bmatrix} 1+t_2 \\ 2-t_2 \\ 3+2t_2 \end{bmatrix}$

$$= \begin{bmatrix} 1+(t_1+t_2) \\ 2-(t_1+t_2) \\ 3+(2t_1+2t_2) \end{bmatrix}$$

$$c \odot \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} = \begin{bmatrix} 1+ct \\ 2-ct \\ 3+2ct \end{bmatrix}$$

a) Additive Identity :-

$$v + e = v \Rightarrow \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} + \begin{bmatrix} 1+e \\ 2-e \\ 3+2e \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+(t+e) \\ 2-(t+e) \\ 3+(2t+2e) \end{bmatrix} = \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix}$$

$$\Rightarrow 1+t+e = 1+t$$

$$e = 0$$

$$\therefore \quad e = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Additive Inverse

$$v + x = e \Rightarrow \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} + \begin{bmatrix} 1+x \\ 2-x \\ 3+2x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$1+t+x = 1 \quad \Rightarrow \quad x = -t$$

$$2-(t+x) = 2$$

$$3+(2t+2x) = 3$$

$$x = \begin{bmatrix} 1-t \\ 2+t \\ 3-2t \end{bmatrix}$$

b) V is a vector space ($\because$ each of the 10 vector space axioms are satisfied)

c) verify that $0 \odot V = 0$

$$0 \odot \begin{bmatrix} 1+t \\ 2-t \\ 3+2t \end{bmatrix} = \begin{bmatrix} 1+0 \\ 2-0 \\ 3+0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ let $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $S = \{ au+bv \,/\, a, b \in R \}$

Show that S with the standard componentwise operations is a V.S

→ let v be a vector in R^n and let $S = \{v\}$ Defined $\oplus$ & $\odot$ by $v \oplus v = v$, $c \odot v = v$ S.T S is a V.S.

Each of the 10 V.S axioms are Satisfied

$\therefore$ $V = R^n$ is a vector space

(Consider $u = (x_1, x_2, \ldots, x_n) \in R^n$, $v = (y_1, y_2, \ldots y_n) \in R^n$)

→ $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle/ 3x - 2y + z = 0 \right\}$ S.T S with the standard componentwise operations is a vector space. p-20

→ $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right.$ in R^3 such that $x + y - z = 0$ and $2x - 3y + 2z = 0$ S.T S with the standard Componentwise Operations is a V.S.

→ let $V = \left\{ \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \middle/ t \in R \right\}$ and define $\begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} \oplus \begin{bmatrix} \cos t_2 \\ \sin t_2 \end{bmatrix} = \begin{bmatrix} \cos(t_1+t_2) \\ \sin(t_1+t_2) \end{bmatrix}$

$c \odot \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos ct \\ \sin ct \end{bmatrix}$

a) Determine the additive identity & Inverse.

Additive Identity:-

$$v + e = v$$

$$\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \begin{bmatrix} \cos e \\ \sin e \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\cos(t+e) = \cos t \Rightarrow t + e = t \Rightarrow \boxed{e = 0}$$

$$\sin(t+e) = \sin t$$

Additive Inverse:-

$$\therefore \; e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v + x = e$$

$$\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} = \begin{bmatrix} \cos 0 \\ \sin 0 \end{bmatrix} \Rightarrow \cos(t+x) = \cos 0$$

$$\sin(t+x) = \sin 0$$

$$t + x = 0$$

$$\boxed{x = -t}$$

$$x = \begin{bmatrix} \cos(-t) \\ \sin(-t) \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

b) V is a vector space ($\because$ All 10 V.S axioms holds)

c) Show that if $\oplus$ and $\odot$ are the standard Componentwise Operations, then V is not a vector space.

→ Let V be the set of all real-valued functions defined on R with the standard operations that satisfy $f(0) = 1$. Determine whether V is a vector space

$(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$, then V is not closed under addition and hence it is not a vector space.

→ Let V be the set of all real-valued functions defined on R

Define $f \oplus g$ by $(f \oplus g)x = f(x) + g(x)$

and define $c \odot f$ by: $(c \odot f)(x) = f(x + c)$

Determine whether V is a vector space.

by 10th axiom

we have to P.T $1 \odot f = f$

$$(1 \odot f)(x) = f(x+1) \neq f(x)$$

so V is not a Vector space.

→ Let $f(x) = x^3$ defined on R and let $V = \{f(x+t) / t \in R\}$

Define $f(x+t_1) \oplus f(x+t_2) = f(x+t_1+t_2)$

$$c \odot f(x+t) = f(x+ct)$$

<u>Additive Identity</u>:

$$v + e = v$$

$$f(x+t) + f(x+e) = f(x+t)$$

$$f(x+t+e) = f(x+t)$$

$$e = 0$$

The zerovector is given by $f(x+0) = x^3$

<u>Inverse</u>

$$-f(x+t) = f(x-t)$$

b) Each of the 10 vector space axioms are satisfied

∴ V is a vector space.

<u>Vector Subspace</u>. A subspace W of a vector space V is a non-empty subset that is itself a vector space w.r.t vector addition and scalar multiplication on V.

(or) A non-empty subset W of a V.S V(F) is said to be Subspace of V if W itself forms a V.S over the same field F w.r.t vector addition and scalar multiplication in V.

<u>Note</u> ① If W is a Subspace of V(F) then W is a ~~Subspace~~ subgroup of V w.r.t addition.

② Let V(F) be a V.S then $\{\bar{0}\}$ and V(F) are always of Subspaces of V(F).

Trivial (or Improper) Subspaces and Non-trivial (or proper) Subspaces:

Every vector space $V(F)$ $(V \neq \{\bar{0}\})$ has atleast two spa. Subspaces $\{\bar{0}\}$ and V itself. These two Subspaces are Called Trivial Subspaces of $V(F)$. If $V(F)$ has any subspace other than these two then they are Called proper Subspaces of $V(F)$.

properties: ① A non-empty Subset W of a vector space $V(F)$ to be a Subspace of $V(F)$ $\Leftrightarrow$ (i) $\alpha+\beta \in W$ $\forall \alpha, \beta \in W$
(ii) $a\alpha \in W$ $\forall \alpha \in W, \forall a \in F$

② A non-empty Subset W of a vector space $V(F)$ to be a Subspace of $V(F)$ $\Leftrightarrow$ (i) $\alpha-\beta \in W$ $\forall \alpha, \beta \in W$
(ii) $a\alpha \in W$ $\forall \alpha \in W$ $\forall a \in F$.

③ A non-empty Subset W of a vector space $V(F)$ to be a Subspace of $V(F)$ $\Leftrightarrow$ $a\alpha + b\beta \in W$ $\forall \alpha, \beta \in W$ $\forall a, b \in F$

④ A non-empty Subset W of a vector space $V(F)$ to be a Subspace of $V(F)$ $\Leftrightarrow$ $a\alpha+\beta \in W$ $\forall$ $\alpha, \beta \in W$ $\forall a \in F$.

Theorem:- Let W be a non-empty Subset of the vector space V. Then W is a Subspace of V iff W is closed under addition and scalar Multiplication.

Theorem: The intersection of any collection of Subspaces of a vector space is a Subspace of the vector space.

Theorem: The union of two Subspaces of a vector space need not be a Subspace unless one is Contained in the other.

Ex: Let $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} / x \in R \right\}$ and $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} / y \in R \right\}$

$W_1 \cup W_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} / x = 0 \text{ (or) } y = 0 \right\}$

This set not closed under addition Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in $W_1 \cup W_2$.

Linear Combination: Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in a vector space V and let $c_1, c_2, \ldots, c_k$ be scalars.
A linear combination of the vectors of S is an expression of the form

$$(c_1 \odot v_1) \oplus (c_2 \odot v_2) \oplus \cdots \oplus (c_k \odot v_k)$$

(or)

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = \sum_{i=1}^{k} c_iv_i$$

Span of a set of vectors: Let V be a vector space and let $S = \{v_1, v_2, \ldots, v_n\}$ be a (finite) set of vectors in V.
The span of S denoted by $\text{span}(S)$ is the set

$$\text{Span}(S) = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n / c_1, c_2, \ldots, c_n \in \mathbb{R}\}$$

Thm: If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in a vector space V, then $\text{Span}(S)$ is a subspace.

Linear Independence and Linear dependence:
The set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called linearly independent provided that the only solution to the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$

If the equation has a non-trivial solution (at least one of the value is non-zero), then the set S is called linearly dependent

Basis for a vector space: A subset B of a vector space V is a basis for V provided that

1) B is a linearly independent set of vectors in V.
2) $\text{Span}(B) = V$.

Dimension of a vector space: The dimension of the vector space V denoted by $\dim V$, is the number of vectors in any basis of V.

Co-ordinates: let $B = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for the vector space V. Let 'v' be a vector in V, and let $c_1, c_2, \ldots, c_n$ be the unique scalars such that p-24

$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$. Then $c_1, c_2, \ldots, c_n$ are called the Co-ordinates of v relative to B

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ and refer to the vector $[v]_B$

as the Co-ordinate vector of v relative to B.

problems: Let S be the subset of the vector space R^3 defined by $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ show that

$v = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$ is in Span(S).

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix}$$

$$2c_1 + c_2 + c_3 = -4$$
$$-c_1 + 3c_2 + c_3 = 4$$
$$-2c_2 + 4c_3 = -6$$

solving this linear system, we obtain

$c_1 = -2$, $c_2 = 1$ and $c_3 = -1$

This shows that v is a linear Combination of the vectors in S and is thus in Span(S).

→ let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ show that the span of S is the Subspace of $M_{2\times 2}$ of all symmetric matrices.

sol: Consider a 2×2 ~~Square~~ symmetric matrix of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Since any matrix in Span(S) has the form

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Span(S) is the Collection of all 2×2 symmetric matrices.