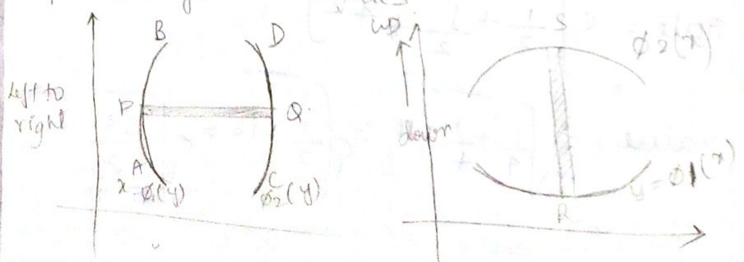
01/08/2022 UNIT – 4

* VECTOR INTEGRATION:

Multiple integrals

* double integration – cartesian form
    - with constant limits.
    - with variable limits
- Triple integration → cartesian form.
    - with constant limits.
    - with variable limits
* Change of order of integration
    - line integral
        - work done by the force.
        - surface integral: double integral
        - volume integral: triple integral
* Greece theorem in a plane – statement & problems
* Gauss divergence theorem – statement & problems.
* Stokes theorem.

strip is always for variables



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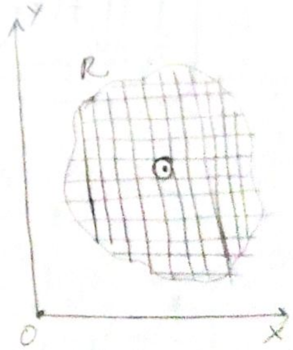
* DOUBLE INTEGRALS:

The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \dots + f(x_n)\delta x_n$ when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, \dots, \delta x_n$ are n sub-intervals into which the range $b-a$ has been divided and $x_1, x_2, \dots, x_n$ are values of x lying respectively in the first, second.....n^{th} sub interval.

*→ A double integral is its counterpart in two dimensions.

$f(x,y) \to$ single valued and bounded function.



Sum:

$$f(x_1,y_1)\delta A_1 + f(x_2,y_2)\delta A_2 + \dots + f(x_n,y_n)\delta A_n = \sum_{r=1}^{n} f(x_r,y_r)\delta A_r \quad \text{—(1)}$$

No. of sub-regions increases indefinitely, such that $\delta A \to 0$

* The limit of the sum (1), if it exists irrespective of the mode of sub-division

is called the double integral of $f(x, y)$ over the region R and is $\iint_R f(x,y)\,dA$

$$\lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r = \iint_R f(x,y)\,dA$$

$\iint_R f(x,y)\,dx\,dy$ (or) $\iint_R f(x,y)\,dy\,dx$

$$= \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy}$$

$$= \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx}$$

Evaluate: $\int_1^2\int_1^3 x y^2\,dx\,dy$

$$= \int_{y=1}^{2}\left[\int_{x=1}^{3} x\,dx\right] y^2\,dy$$

$$= \int_{y=1}^{2}\left[\left[\frac{x^2}{2}\right]_1^3 y^2\,dy\right]$$

$$= \int_1^2 4y^2\,dy$$

$$= \frac{4}{3}\left[y^3\right]_1^2 = \frac{4}{3}\times 7 = \frac{28}{3}$$

* $\int_1^a \int_1^b \frac{dy\,dx}{xy}$

$$= \int_1^a \frac{1}{x}\left[\int_1^b \frac{dy}{y}\right]dx = \int_1^a \frac{1}{x}(\log y)_1^b\,dx$$

$$= \int_1^a \frac{\log b - \log 1}{x}\,dx$$

$$= \log b\,\left[\log x\right]_1^a = \log b\cdot\log a$$

$$= \log a\cdot\log b.$$

Evaluate $\int_0^1 \int_{x=4y}^{4} x e^{x^2}\,dx\,dy$.

$$\int_0^1\left[\int_{x=4y}^{y} e^{x^2}\,dx\right]dy$$

$$=\int_0^1 \left[\frac{e^{x^2}}{2}\right]_{4y}^{4} dy$$

$$=\int_0^1 \frac{e^{16}-e^{16y^2}}{2}\, dy$$

$$=\int_0^1 \frac{e^{16}}{2}\, dy-\int_0^1 \frac{e^{16y^2}}{2}\, dy$$

$$=\frac{e^{16}}{2}-\frac{1}{2}\int_0^1 e^{16y^2}\, dy$$

* $\int_0^1\int_0^x e^{y/x}\, dx\, dy$

$$\int_0^1\left[\int_{y=0}^{x} e^{y/x}\, dy\right] dx$$

$$\int_0^1\left[\frac{e^{y/x}}{1/x}\right]_0^x dx$$

$$\int_0^1\left(ex-1(x)\right) dx$$

$$\int_0^1 (e-1)\,x\, dx=(e-1)\left[\frac{x^2}{2}\right]_0^1$$

$$=\frac{e-1}{2}$$

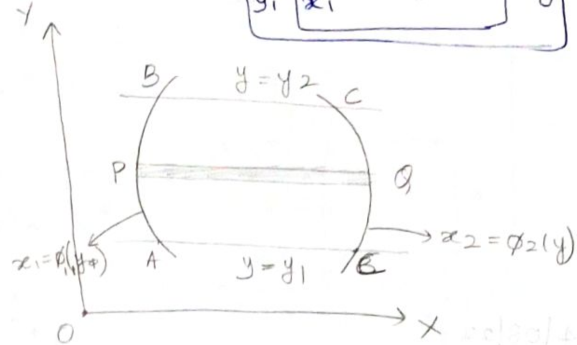
*EVALUATION OF DOUBLE INTEGRALS:

(i) When x_1, x_2 are functions of y and y_1, y_2 are constants.

Let curves $AB \rightarrow x_1=\phi_1(y)$

$CD \rightarrow x_2=\phi_2(y)$

$$\iint_R f(x,y)\,dx\,dy=\boxed{\int_{y_1}^{y_2}\boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy}$$

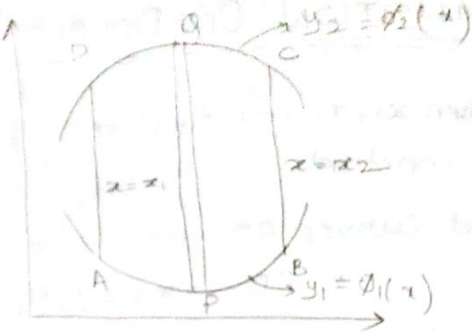


(ii) When y_1, y_2 are funtions of x and x_1, x_2 are constants

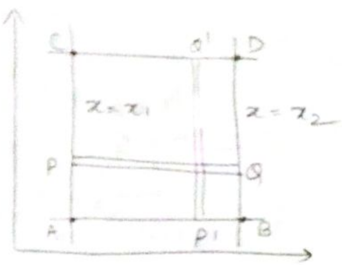
Let curves: $AB \rightarrow y_1=\phi_1(x)$

$CD \rightarrow y_2=\phi_2(x)$

$$\iint_R f(x,y)\,dx\,dy=\boxed{\int_{x_1}^{x_2}\boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx}$$



(iii) When x_1, x_2, y_1, y_2 are constants.



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* $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$\int_{y=0}^{1} \left[\int_{x=0}^{1} \frac{dx}{\sqrt{1-x^2}}\right] \frac{dy}{\sqrt{1-y^2}}$

$\int_0^1 [\sin^{-1} x]_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2} \int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$= \frac{\pi}{2} [\sin^{-1} y]_0^1$

$= \frac{\pi}{2} \left[\frac{\pi}{2} - 0\right] = \pi^2/4$

* $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

$\int_0^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$

$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2 + y^2}\right] dx$

$= \int_0^1 \frac{1}{\sqrt{x^2+1}} \left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$

$= \int_0^1 \frac{1}{\sqrt{x^2+1}} \left(\frac{\pi}{4}\right) dx$

$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{x^2+1}} = \cancel{\frac{\pi}{4} [\tan^{-1}(x)]_0^1 = \frac{\pi}{4}[\pi/4] = \pi^2/16}$

$= \frac{\pi}{4} \left[\log\left[x + \sqrt{1+x^2}\right]\right]_0^1$

$= \frac{\pi}{4} \log(1+\sqrt{2})$

* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy.$

$\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx\, dy.$

$= \cancel{\sqrt{\quad}}$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2 - x^2}\, dx \right] dy$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$= \cancel{\int_0^a \int_0^{\sqrt{a^2-y^2}}}$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-y^2}{2} \sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy.$$

$$= \int_0^a \frac{a^2-y^2}{2} \cdot \frac{\pi}{2}\, dy$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{2a^3\pi}{12} = \underline{\underline{\frac{\pi a^3}{6}}}$$

* Region of Integration:

1) $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ & $x^2 = 4y$

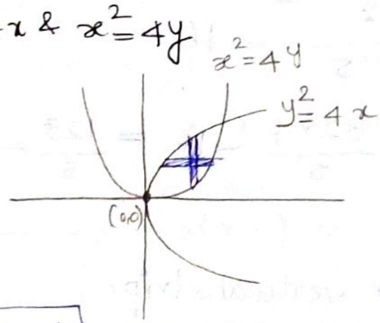
$y^2 = 4x$; $x^2 = 4y$

$y = \frac{x^2}{4}$

$\frac{x^4}{16} = 4x$

$x^3 = 64$ $\quad \boxed{y = 4}$ $\quad \boxed{x = 0;\ y = 0}$

$\boxed{x = 4}$



* Note: Strip: Variable limits

Horizontal strip: x limits (Left to Right)

Vertical strip: y limits (down to up)

* Consider the limits of x: $\frac{y^2}{4}$ to $2\sqrt{y}$

y: 0 to 4.

$$= \int_0^4 \left[\int_{y^2/4}^{2\sqrt{y}} dx \right] y\,dy$$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right) y\,dy$$

$$= \int_0^4 \left(2y\sqrt{y} - \frac{y^3}{4}\right) dy = \left[2\cdot\frac{y^{5/2}}{5/2} - \frac{y^4}{16} \right]_0^4$$

$$= \frac{4}{5}\, 4^{5/2} - \frac{4^4}{16}$$

$= \dfrac{4^{7/2}}{5} - 16$

$= \dfrac{2^{2\times 7/2}}{5} - 16$

$= \dfrac{32\times 4}{5} - 16 = \dfrac{128}{5} - 16 = \dfrac{128-80}{5}$

$= \dfrac{48}{5}$

* for vertical strip:

Consider the limits of $y : \dfrac{x^2}{4}$ to $2\sqrt{x}$

$x : 0$ to 4

$\displaystyle\int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y\,dy\right] dx = \int_0^4 \left[2x - \frac{x^4}{32}\right] dx$

$= \left[x^2 - \dfrac{x^5}{160}\right]_0^4$ $\quad \begin{array}{r} 32 \\ \times 5 \end{array}$

$= 16 - \dfrac{\cancel{256}\times 4}{\cancel{160}}$

$= 16\left[1 - \dfrac{2}{5}\right]$

$= \dfrac{48}{5}$

$= \cancel{8[16] - \dfrac{256\times 4}{160}}$

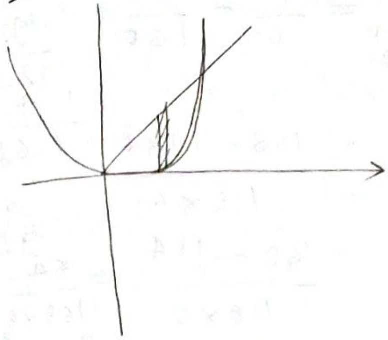
$= \cancel{32 - \dfrac{64}{5} = 16 - \dfrac{2}{5}}$

$= \cancel{32\left[1 - \dfrac{2}{3}\right] =}$

$= \cancel{32\times\dfrac{3}{5}} \to \cancel{\dfrac{78}{5}}$

$= \cancel{\dfrac{96}{5}}$

* Evaluate: $\displaystyle\iint_R xy(x+y)\,dx\,dy$ where R is the region bounded by the curves $y = x^2$ & $y = x$.



$y = x^2;\ y = x$

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$\boxed{\begin{array}{c} x = 0, 1 \\ y = 0, 1 \end{array}}$

For vertical strip:

consider the limits of $y : x^2$ to x

$x : 0$ to 1

$= \displaystyle\int_0^1 \left[\int_{x^2}^{x} xy(x+y)\,dy\right] dx$

$= \displaystyle\int_0^1 \left[\int_{x^2}^{x} (x^2y + xy^2)\,dy\right] dx = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$

$= \displaystyle\int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$

$= \left[\dfrac{x^5}{10} + \dfrac{x^5}{15} - \dfrac{x^7}{14} - \dfrac{x^8}{24}\right]_0^1$

$= \dfrac{1}{10} + \dfrac{1}{15} - \dfrac{1}{14} - \dfrac{1}{24} =$

$$= \frac{3+2}{30} - \left[\frac{24+14}{14\times 24}\right]$$

$$= \frac{1}{6} - \frac{\cancel{38}^{19}}{\cancel{14\times 24}_{12}} = \frac{1}{6} - \frac{19}{168}$$

14×12

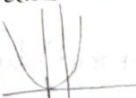
$$\begin{array}{r} 14 \\ \times 12 \\ \hline 28 \\ 14\ \\ \hline 168 \end{array}$$

$$= \frac{168 - 19\times 6}{168\times 6}$$

$$= \frac{168 - 114}{168\times 6} = \frac{\cancel{54}^{9}}{\cancel{168}\times\cancel{6}} = \frac{\cancel{9}^{3}}{\cancel{168}_{56}} = \frac{3}{56}$$

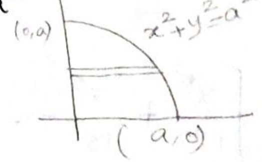
H.W

1) $\iint xy\,dx\,dy$ A : domain bounded by x-axis, ordinate x-axis and the curve $x^2 = 4ay$



$A = a^4/3$

2) $\iint xy\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$



A =

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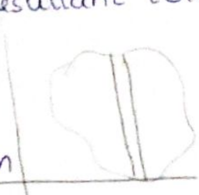
* Change of order of Integration :

1) Suppose x : constants

limits of y : variables in terms of x as shown in the figure

$$\boxed{\int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx}$$ i.e. first we integrate w.r.t 'y' then the resultant term integrate w.r.t 'x'.



⇒ By change of order integration

→ Consider the vertical strip to horizontal strip.

y : constants

x : variables in terms of y

$$\boxed{\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy}$$

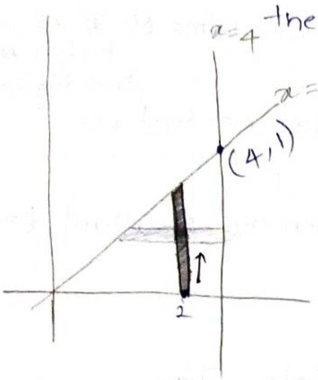
* $\displaystyle\int_0^1\int_{4y}^{4} e^{x^2}\,dx\,dy$

x : $4y$ to 4

$x = 4y$ to $x = 4$

y : 0 to 1

$\int_{y=0}^{1}\int_{x=4y}^{4} e^{x^2}\,dx\,dy$ initially we have to integrate wrt 'x' then the resultant value wrt 'y'



* By changing the order of integration taking the vertical strip instead of horizontal strip & also consider x limits & consts 'x'

$y : 0$ to $\frac{x}{4}$

$x : 0$ to 4

$$\int_{y=0}^{1}\int_{x=4y}^{4} e^{x^2}\,dx\,dy$$

$$= \int_{x=0}^{4}\int_{y=0}^{x/4} e^{x^2}\,dy\,dx$$

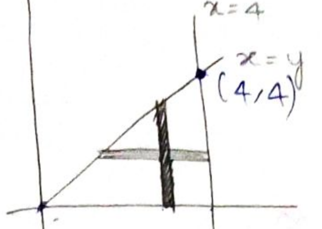
$$= \int_{0}^{4}\left[\int_{0}^{x/4} dy\right] e^{x^2}\,dx$$

$$= \int_{0}^{4}\left[\frac{x}{4} e^{x^2}\,dx\right] = \frac{1}{8}\int_{0}^{4} 2x e^{x^2}\,dx$$

$$= \frac{1}{8}\left[e^{x^2}\right]_{0}^{4}$$

$$= \frac{1}{8}\left[e^{16}-1\right] = \frac{e^{16}-1}{8}$$

* Change the order of integration $\int_{0}^{4}\int_{y}^{4} \frac{x\,dx\,dy}{x^2+y^2}$



$x=0;\ y=0$

$x : y$ to 4

$y : 0$ to 4

$y :$ ~~0~~ to ~~x~~

$x : 0$ to 4

$$= \int_{0}^{4}\int_{y=0}^{x} \frac{x\,dx\,dy}{x^2+y^2}$$

$$= \int_{x=0}^{4}\left[\int_{y=0}^{x} \frac{x\,dy}{x^2+y^2}\right] dx$$

$$= \int_{0}^{4}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]_{0}^{x} dx$$

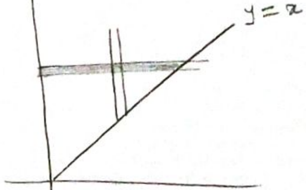
$$= \frac{\pi}{4}\int_{0}^{4} dx$$

$$= \frac{\pi}{4} \times 4 = \pi$$

* $\int_{0}^{\infty}\left[\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\right] dx$

$x : 0$ to y

$y : 0$ to ∞



$$= \int_0^\infty \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left[\int_{x=0}^{y} dx\right] \frac{e^{-y}\, dy}{y}$$

$$= \int_0^\infty y\, \frac{e^{-y}}{y}\, dy = \int_0^\infty e^{-y}\, dy$$

$$= \left[-e^{-y}\right]_0^\infty$$

$$= -e^{-\infty} + e^0$$

$$= 0 + 1 = \underline{1}$$

* $\displaystyle\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy.$

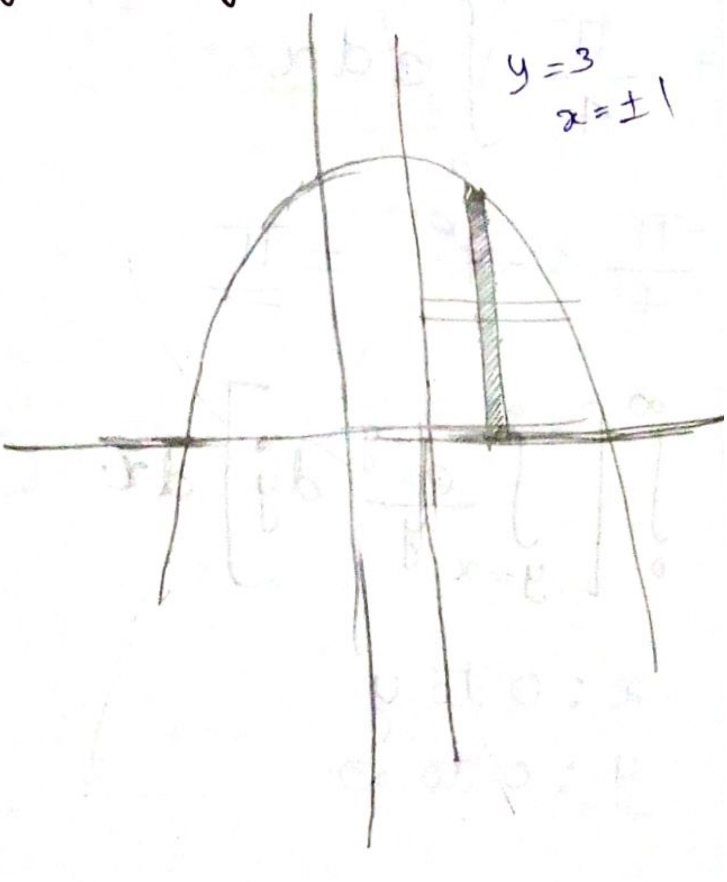
$x^2 = 4 - y$

$x^2 = -4\left(\frac{y}{4} - 1\right)$

$y = 3$

$x = \pm 1$

$$\int_{-1}^{1} \int$$



$$= \int_0^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^{\infty}\left[\int_{x=0}^{y} dx\right] \frac{e^{-y}dy}{y}$$

$$= \int_0^{\infty} y \,\frac{e^{-y}}{y}\, dy = \int_0^{\infty} e^{-y}\, dy$$

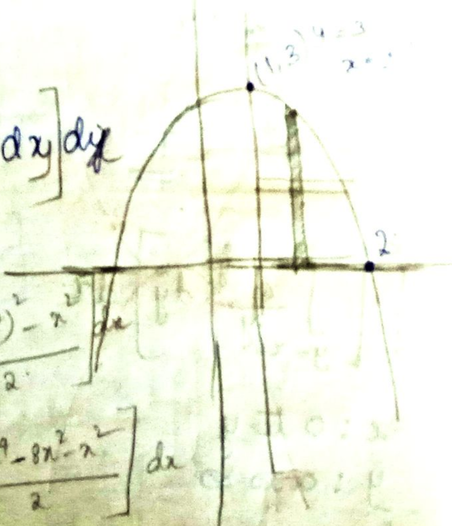
$$= \left[-e^{-y}\right]_0^{\infty}$$

$$= -e^{-\infty} + e^{0}$$

$$= 0 + 1 = 1$$

* $$\int_{y=0}^{3}\int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy$$

$$\int_1^2\left[\int_x^{4-x^2} (x+y)\, dy\right] dx$$



$$\int_1^2 \left[x(4-x^2-x) + \frac{(4-x^2)^2 - x^2}{2}\right] dx$$

$$\int_1^2 \left[4x - x^3 - x^2 + \frac{16+x^4-8x^2-x^2}{2}\right] dx$$

$$= \int_1^2\left[\int_x^{4-x^2} (x+y)\, dy\right] dx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_x^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2-x) + \frac{(4-x^2)^2}{2} - \frac{x^2}{2}\right] dx$$

$$= \int_1^2 \left[4x - x^3 - x^2 + \frac{16+x^4-8x^2}{2} - \frac{x^2}{2}\right] dx$$

$$= \int_1^2 \left[4x - x^3 - \frac{3x^2}{2} + 8 + \frac{x^4}{2} - 4x^2\right] dx$$

$$= \int_1^2 \left[4x - x^3 - \frac{11x^2}{2} + 8 + \frac{x^4}{2}\right] dx$$

$$= \left[2x^2 - \frac{x^4}{4} - \frac{11x^3}{6} + 8x + \frac{x^5}{10}\right]_1^2$$

$$= 2(3) - \frac{1}{4}(15) - \frac{11}{6}(7) + 8(1) + \frac{1}{10}(31)$$

$$= 6 - \frac{15}{4} - \frac{77}{6} + 8 + \frac{31}{10}$$

$$\begin{array}{r} 2 \\ 77 \\ \times 4 \\ \hline 308 \end{array}$$

$$= \frac{240\times 6 - 900 - 308 + 8\times 240 + 31\times 24}{\cancel{60}\, 24\times 10}$$

$$= \frac{440 - 1208 + 1920 + 744}{240}$$

$$= \frac{2896}{240} =$$

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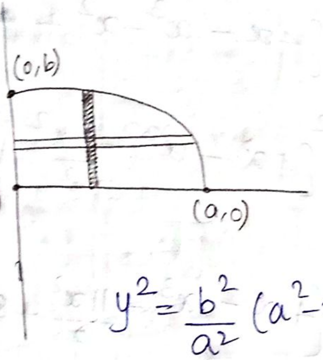
By changing the order of integration; Evaluate the following:

1) $\displaystyle\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$



$x : 0$ to $\frac{a}{b}\sqrt{b^2-y^2}$

$y : 0$ to b

By changing order of integration:

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$a^2y^2 = b^2a^2 - b^2x^2$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$x : 0$ to a

$y : 0$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} [y\,dy]\,x\,dx = \int_0^a \left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} x\,dx$$

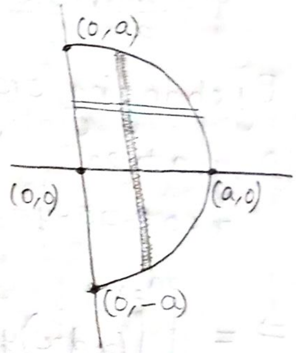
$$= \int_0^a \frac{b^2}{2a^2}(a^2-x^2)\,x\,dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{b^2}{2a^2}\left[\frac{a^2}{2}\cdot a^2 - \frac{a^4}{4}\right]$$

$$= \frac{b^2}{2a^2}\,\frac{a^4}{4} = \frac{a^2b^2}{8}$$

2) $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$



$y : -a$ to a

$x : 0$ to $\sqrt{a^2-y^2}$

$$x^2 = a^2 - y^2$$

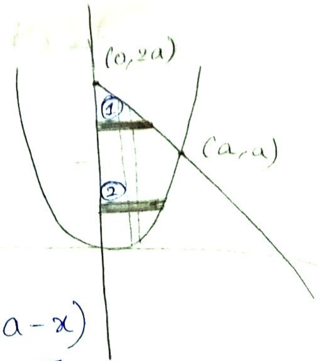
$$\boxed{x^2 + y^2 = a^2}$$

$y : -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x : 0$ to a

$$= \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy =$$

3) $\int_0^a \int_{x^2/a}^{2a-x} x y^2 \, dy \, dx$



$\frac{x^2}{a} = y$ | $x + y = 2a$

$x^2 = ay$ |

$x^2 = a(2a - x)$

$\boxed{x = -2a, a}$

$\because$ x : 0 to a

y : $\frac{x^2}{a}$ to $2a - x$

By changing order of integration :

① y : a to $2a$ | ② y : 0 to a

x : 0 to $2a - y$ | x : 0 to $\sqrt{ay}$

$$\Rightarrow = \int_a^{2a}\int_0^{2a-y} (x\,dx) y^2 dy + \int_0^a \int_0^{\sqrt{ay}} (x\,dx) y^2 dy$$

$$= \int_a^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y^2 dy + \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y^2 dy$$

$$= \frac{1}{2}\int_a^{2a} \frac{(2a-y)^2}{1} y^2 dy + \frac{1}{2}\int_0^a ay\, y^2 dy$$

$$= \frac{1}{2}\int_a^{2a} \frac{(4a^2 + y^2 - 4ay)}{1} y^2 dy + \frac{a}{2}\,\frac{a^4}{\cancel{4}}$$

$$= \frac{1}{2}\left[\int_a^{2a} (4a^2y^2 + y^4 - 4ay^3) dy\right] + \frac{a^5}{8}$$

$$= \frac{1}{2}\left[\frac{4a^2}{3}(7a^3) + \frac{31a^5}{5} - \frac{4a}{4}(15a^4)\right] + \frac{a^5}{8}$$

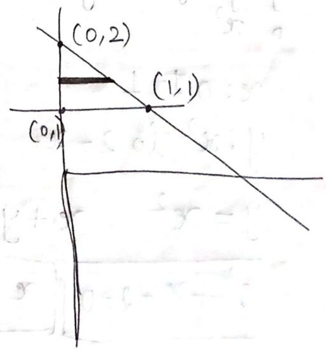
$$= \frac{14a^5}{3} + \frac{31a^5}{10} - \frac{15a^5}{2} + \frac{a^5}{8}$$

$= \cancel{\quad}$ 60 480 $= \cancel{\frac{560a^5 + 372a^5 - 900a^5 + 15a}{120}}$

$$= \frac{(2240 + 1488 - 3600 + 60)\,a^5}{480}$$

$$= \frac{188a^5}{480} = \frac{47a^5}{120}$$

4) $\int_0^1 \int_{y=1}^{2-x} xy \, dx\, dy$



$y = 1$, $x + y = 2$

$x = 1$

x : 0 to 1

y : 1 to $2 - x$

$y: 1$ to 2

$x: 0$ to $2-y$

$$=\int_1^2\int_0^{2-y}[x\,dx]\,y\,dy = \int_1^2\left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

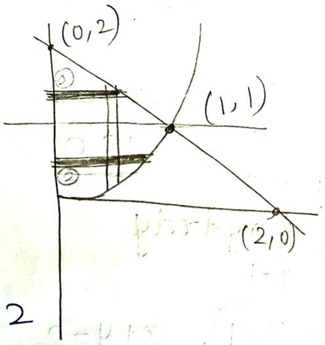
$$=\int_1^2 \frac{(2-y)^2}{2}\,y\,dy = \int_1^2\left(\frac{4+y^2-4y}{2}\right)y\,dy$$

$$=\int_1^2\left(2y+\frac{y^3}{2}-2y^2\right)dy = \frac{2}{2}(4-1)+\frac{1}{8}(15)-\frac{2}{3}(7)$$

$$=3+\frac{15}{8}-\frac{14}{3} = \frac{72+45-112}{24} = \underline{\underline{\frac{5}{24}}}$$

$\begin{array}{r}14\\ \times 8\\ \hline 112\end{array}$ $\quad \dfrac{117-112}{24}$

5) $\displaystyle\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$



$x: 0$ to 1

$y: x^2$ to $2-x$

$y = x^2 \qquad x+y=2$

$\boxed{x^2+x-2=0}$ $\boxed{x=-2,1}$

① $y: 1$ to 2, $x: 0$ to $2-y$ | ② $y: 0$ to 1, $x: 0$ to $\sqrt{y}$

$$① = \int_1^2\int_0^{2-y}(x\,dx)\,y\,dy = \int_1^2\left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$=\frac{1}{2}\int_1^2 \frac{(2-y)^2}{2}\,y\,dy$$

$$\cancel{=\frac{1}{2}\left[2\right]_1^2}$$

$$=\frac{1}{2}\int_1^2\left[4y+y^3-4y^2\right]dy$$

$$=\frac{1}{2}\left[2(y)^2+\frac{y^4}{4}-\frac{4}{3}y^3\right]_1^2$$

$\begin{array}{r}72\\ 45\\ \hline 117\end{array}$

$$=\frac{1}{2}\left[2(3)+\frac{1}{4}(15)-\frac{4}{3}(7)\right]$$

$\begin{array}{r}28\\ \times 4\\ \hline 112\end{array}$

$$=\frac{1}{2}\left[6+\frac{15}{4}-\frac{28}{3}\right] = \frac{1}{2}\left(\frac{72+45-112}{12}\right)$$

$$=\frac{1}{2}\left[\frac{5}{12}\right] = \underline{\underline{\frac{5}{24}}}$$

$$② \int_0^1\int_0^{\sqrt{y}}(x\,dx)\,y\,dy = \int_0^1\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy$$

$$=\frac{1}{2}\int_0^1 y^2\,dy = \underline{\underline{\frac{1}{6}}}$$

$$\Rightarrow \frac{5}{24}+\frac{1}{6} = \frac{9}{24} = \underline{\underline{\frac{3}{8}}}$$

06/08/2022

* TRIPLE INTEGRALS :

Let $f(x,y,z)$ be a function defined over a 3D finite region V.

Sub divisions: $\delta V_1, \delta V_2, \delta V_3 \ldots\ldots \delta V_n$

$P(x_r, y_r, z_r) \longrightarrow$ be any point in r^{th} division.

Sum $\Rightarrow \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$

limit of Sum: as $n \to \infty$ and $\delta V_r \to 0$ is called triple integral of $f(x,y,z)$ over the region V and is denoted by

$$\boxed{= \iiint f(x,y,z)\,dV}$$

* Evaluation of Triple Integrals :

~~1)~~ $\cancel{\int_a^b \int_c^d \left[\int_e^f f(x,y,z)\,dz\right] dy}$

1) $\int_a^b \left[\int_c^d \left[\int_e^f f(x,y,z)\,dz\right] dy\right] dx$

$\Rightarrow$ Depending on the integrand ; give the names to the limits

2) $\int_{y=a}^{b} \int_{z=f_1(y)}^{f_2(y)} \int_{x=g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dy\,dz\,dx$

$= \int_{y=a}^{b} \int_{z=f_1(y)}^{f_2(y)} \int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\,dz\,dy$

→ Evaluate :

$$\int_0^1 \left[\int_1^2 \left[\int_2^3 xyz\,dx\right] dy\right] dz$$

$$= \int_0^1 \left[\int_1^2 \left[\frac{x^2}{2}\right]_2^3 yz \,dy\right] dz$$

$$= \int_0^1 \frac{5}{2}\left[\frac{y^2}{2}\right]_1^2 z\,dz = \int_0^1 \frac{5}{4}(3) z\,dz$$

$$= \frac{15}{4} \cdot \frac{1}{2}(1) = \frac{15}{8}$$

* $\int_0^1 \left[\int_0^2 \left[\int_1^2 x^2 y z \, dz\right] dy\right] dx$

$= \int_0^1 \left[\int_0^2 x^2 y \left[\frac{z^2}{2}\right]_1^2 dy\right] dx$

$= \int_0^1 \left[\int_0^2 \frac{x^2 y}{2} (3) \, dy\right] dx$

$= \int_0^1 \frac{3}{\cancel{2}4} x^2 \left[y^2\right]_0^2 dx = \frac{3}{4} \times 4 \int_0^1 x^2 dx$

$= \frac{3}{4} \times 4 \times \frac{1}{3} = 1$

H.W

* $\int_0^a \int_0^{2\pi} \int_0 x^2 \sin y \, dx \, dy \, dz$

* $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2) \, dx \, dy \, dz$

* $\int_1^e \int_1^e \int_1^e (xyz)^{-1} \, dx \, dy \, dz$

* $\int_{z=-1}^{1} \left[\int_0^z \left[\int_{y=x-z}^{x+z} (x+y+z) \, dy\right] dx\right] dz$ — (1)

$= \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} = x(2z) + 4xz + z(2z)$

$= \int_{x=0}^{z} (6xz + 2z^2) \, dx = \cancel{\left[\frac{6xz^2}{2} + \frac{2z^3}{3}\right]_0^z}$

$= 3xz^2 + \frac{2z^3}{3}$

$= \left[3x^2 z + 2z^2 x\right]_0^z$

$= 3z^3 + 2z^3 = 5z^3$

$\int_{-1}^{1} 5z^3 \, dz = \frac{5}{4}(1-1) = 0$

* $\int_{x=0}^{a} \left[\int_{y=0}^{x} \left[\int_{z=0}^{x+y} e^{x+y+z} \, dz\right] dy\right] dx$

$\int_{x=0}^{a} \left[\int_{y=0}^{x} \left[\int_0^{x+y} e^x \cdot e^y \cdot e^z dz\right] dy\right] dx$

$= \int_{x=0}^{a} \left[\int_{y=0}^{x} \left[e^x e^y \left[e^z\right]_0^{x+y}\right] dy\right] dx$

$= \int_{x=0}^{a} \left[\int_{y=0}^{x} e^{2x} \cdot e^{2y} \frac{dy}{e}\right] dy \, dx$

$$\cancel{\int_{x=0}^{a} \frac{2e^{2x}}{2}\left[e^{2y}\right]_0^x dx}$$

$$\cancel{= \int_{x=0}^{a} 2e^{2x} e^{2x} dx = \left[2e^{4x} \cdot 4\right]_0^a = 8e^{4a}}$$

* $$\int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^x . e^y . e^z \, dz\, dy\, dx$$

$$\int_{x=0}^{a}\left[\int_{y=0}^{x}\left[\int_{z=0}^{x+y} e^z dz\right]e^y dy\right]e^x dx$$

$$\int_{x=0}^{a}\left[\int_{y=0}^{x}\left[e^{x+y}-1\right]e^y dy\right]e^x dx$$

$$= \int_0^a\left[\int_{y=0}^{x}\left(e^x . e^{2y} - e^y\right)dy\right]e^x dx$$

$$= \int_0^a \left[\frac{2e^{2y+x}}{2} - e^y\right]_0^x e^x dx$$

$$= \int_0^a \left[\frac{e^{3x}}{2} - e^x - 2e^x + 1\right]e^x dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - 3e^{2x} + e^x\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{2} + e^x\right]_0^a = \frac{e^{4a}}{8} - \frac{3e^{2a}}{2} + e^a - \frac{1}{8} + \frac{3}{2} - 1$$

$\frac{9}{8} - \frac{3}{2}$ $= -\frac{9+12}{8}$

Ans $= \frac{e^{4a}}{8} - \frac{3e^{2a}}{2} + e^a + \frac{3}{8}$

* $$\int_{x=1}^{e}\int_{y=1}^{\log y}\int_{z=1}^{e^x} (1 \cdot \log z)\, dz\, dy\, dx$$

$$\int_{y=1}^{e}\left[\int_{x=1}^{\log y}\left[z\log z - z\right]_1^{e^x}\right]dx$$

$$\int_{y=1}^{e}\left[\int_{x=1}^{\log y}\left(xe^x - e^x + 1\right)dx\right]dy$$

$$= \int_{y=1}^{e}\left[xe^x - e^x - e^x + x\right]_1^{\log y} dy$$

$$= \int_1^e \left[y\log y - 2y + \log y + e - 1\right]dy$$

$$= \left[\log y \cdot \frac{y^2}{2} - \frac{y^2}{4} - y^2 + y\log y - y + ey - y\right]_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - \cancel{e^2} + \cancel{e} - \cancel{e} + \cancel{e^2} - e - \left[0 - \frac{1}{4} - 1 + 0 - 1 + e - 1\right]$$

$$= \frac{e^2}{4} + \frac{1}{4} + 3 - e = \frac{1}{4}\left[e^2 + 13 - 8e\right]$$

$$= (e^2 - 8e + 13)/4$$

* $\int_0^a\int_0^{2\pi}\int_0^1 x^2 \sin y \, dx \, dy \, dz$

$= \int_0^a \left[\int_0^{2\pi}\left[\int_0^1 x^2 \, dx\right]\sin y \, dy\right] dz$

$= \int_0^a \left[\int_0^{2\pi} \frac{1}{3} \sin y \, dy\right] dz$

$= \int_0^a \frac{1}{3} \left[\cos y\right]_0^{2\pi} dz = \frac{-1}{3}\int_0^a (1-1) \, dz$

$= 0$

* $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2) \, dx \, dy \, dx$

$= \int_{-c}^{c}\left[\int_{-b}^{b}\left[\int_{-a}^{a}(x^2+y^2+z^2) \, dx\right] dy\right] dz$

$= \int_{-c}^{c}\left[\int_{-b}^{b}\left[\frac{x^3}{3}+xy^2+xz^2\right]_{-a}^{a} dy\right] dz$

$= \int_{-c}^{c}\left[\int_{-b}^{b}\left[\frac{2a^3}{3}+2y^2a+2z^2a\right] dy\right] dz$

$= \int_{-c}^{c}\left[\frac{2a^3y}{3}+\frac{2y^3}{3}a+2z^2ya\right]_{-b}^{b} dz$

$= \int_{-c}^{c}\left[\frac{2a^3}{3}(2b)+\frac{2}{3}(2b^3)a+2z^2(2b)a\right] dz$

$= \int_{-c}^{c}\left[\frac{4a^3b}{3}+\frac{a4b^3}{3}+4abz^2\right] dz$

$= \frac{4a^3b}{3}(2c)+\frac{4b^3}{3}(2c)a+\frac{4b}{3}(2c^3)a$

$= \frac{8}{3}\left[a^3bc+ab^3c+abc^3\right]$

$= \frac{8abc}{3}\left[a^2+b^2+c^2\right]$

* $\int_1^e\int_1^e\int_1^e \frac{1}{xyz} \, dx \, dy \, dz$

$= \int_1^e\left[\int_1^e\left[\int_1^e \frac{1}{x} \, dx\right]\frac{1}{y} \, dy\right]\frac{1}{z} \, dz$

$= \int_1^e\left[\int_1^e \frac{1}{y} \, dy\right]\frac{1}{z} \, dz = \int_1^e \frac{1}{z} \, dz = 1$

08/08/2022

1) Evaluate $\iiint_V (xy + yz + zx)\,dx\,dy\,dz$

where V is the region of space bounded by $x = 0$ to 1
$y = 0$ to 2
$z = 0$ to 3.

$$= \int_{z=0}^{3}\left[\int_{y=0}^{2}\left[\int_{x=0}^{1}(xy + yz + zx)\,dx\right]dy\right]dz$$

$$= \int_0^3\left[\int_0^2\left[\frac{x^2}{2}y + xyz + \frac{x^2}{2}z\right]_0^1 dy\right]dz$$

$$= \int_0^3\left[\int_0^2\left[\frac{y}{2}(1) + yz + \frac{z}{2}\right]dy\right]dz$$

$$= \int_0^3\left[\frac{y^2}{4} + \frac{y^2}{2}z + \frac{yz}{2}\right]_0^2 dz$$

$$= \int_0^3\left[1 + 2z + z\right]dz = \int_0^3[1 + 3z]\,dz$$

$$= \left[z + \frac{3}{2}z^2\right]_0^3$$

$$= 3 + \frac{3}{2}(9)$$

$= 33/2$

2) Evaluate $\iiint_V xyz\,dx\,dy\,dz$ over the +ve octant of the sphere $x^2+y^2+z^2=a^2$

$x : 0$ to a
$y : 0$ to $\sqrt{a^2-x^2}$
$z : 0$ to $\sqrt{a^2-x^2-y^2}$

$$\int_0^a x\,dx = \frac{a^2}{2}$$

$$\int_0^{\sqrt{a^2-x^2}} \frac{a^2}{2}\,y\,dy = \frac{a^2}{2}\cdot\frac{a^2-x^2}{2} = \frac{a^2(a^2-x^2)}{4}$$

$$\int_0^{\sqrt{a^2-x^2-y^2}} \frac{a^2}{4}(a^2-x^2)\,z\,dz = \frac{a^2}{8}(a^2-x^2)(a^2-x^2-y^2)$$

$$\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left(\int_0^{\sqrt{a^2-x^2-y^2}} (z\,dz)\right)y\,dy\right]x\,dx$$

$$= \int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left(\frac{a^2-x^2-y^2}{2}\right)y\,dy\right]x\,dx$$

$$= \int_0^a\left[\frac{a^2y^2}{4} - \frac{x^2y^2}{4} - \frac{y^4}{8}\right]_0^{\sqrt{a^2-x^2}} x\,dx$$

$$= \int_0^a \left[\frac{a^4 - a^2x^2}{4} - \frac{x^2}{4}(a^2-x^2) - \frac{(a^2-x^2)^2}{8}\right] x\,dx$$

$$= \cancel{\frac{a^4x}{2} - \frac{a^2x^3}{6} - \frac{a^2x^3}{12} + \frac{x^5}{20}} -$$

$\frac{a^6}{48}$

$$= \int_0^a \left[\frac{a^4x - a^2x^3}{4} - \frac{x^3a^2}{4} + \frac{x^5}{4} - \frac{a^4x}{8} - \frac{x^5}{8} + \frac{a^2x^3}{4}\right] dx$$

$$= \left[\frac{a^4x^2}{8} - \frac{a^2x^4}{16} - \frac{x^4a^2}{16} + \frac{x^6}{24} - \frac{a^4x^2}{16} - \frac{x^6}{48} + \frac{a^2x^4}{16}\right]_0^a$$

$$= \left[\frac{a^4x^2}{16} - \frac{a^2x^4}{16} + \frac{x^6}{48}\right]_0^a$$

$$= \frac{a^6}{16} - \frac{a^6}{16} + \frac{a^6}{48} = \frac{3a^6 - 3a^6 + a^6}{48}$$

$$= \frac{a^6}{48}$$

$$\cancel{\int_0^a \frac{a^2(a^2-x^2)}{2} - \frac{x^2}{4}}$$

$\iiint_V dx\,dy\,dz$ V: $x=0, y=0, z=0$ & $2x+3y+4z=12$

$x: 0$ to 6

$y: 0$ to $\frac{12-2x}{3}$

$z: 0$ to $\frac{12-2x-3y}{4}$

$$= \int_0^{3-\frac{x}{2}-\frac{3y}{4}} dz$$

$$= 3 - \frac{x}{2} - \frac{3y}{4}$$

$$\int_0^{4-\frac{2x}{3}} \left(3 - \frac{x}{2} - \frac{3y}{4}\right) dy = 3\left(4-\frac{2x}{3}\right) - \frac{x}{2}\left(4 - \frac{2x}{3}\right) - \frac{3}{8}\left(4 - \frac{2x}{3}\right)^2$$

$$= 12 - 2x - 2x + \frac{x^2}{3} - \frac{3}{8}\left(16 + \frac{4x^2}{9} - \frac{16x}{3}\right)$$

$$= 12 - 4x + \frac{x^2}{3} - 6 - \frac{x^2}{6} + 2x$$

$$\int_0^6 \left[6 - 2x + \frac{x^2}{6}\right] dx$$

$$= 6(6) - \frac{2}{2}(36) + \frac{1}{18}(216)$$

$$= 36 - 36 + 12 = \underline{12}$$

08/08/2022

VECTOR INTEGRATION:

→ **Line Integral: (Single Integral)**

* **Work done by a Force:**

If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\delta\bar{r}$ is $\bar{F}\cdot\delta\bar{r}$. Hence the total work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F}\cdot d\bar{r}$.

* If the force $\bar{F}$ is conservative i.e. $\bar{F} = \nabla\phi$ then the work done is independent of the path & vice versa. In this case $\text{curl}\,\bar{F} = \text{curl}(\text{grad}\,\phi) = \bar{0}$ ϕ is called scalar potential

→ $\bar{F}$ is conservative force field : $\nabla\times\bar{F} = 0$

→ Conservative force field is also irrotational

0/08/2022

$\bar{F} = 3xy\hat{i} - y^2\hat{j}$; Evaluate $\int_C \bar{F}\cdot d\bar{r}$ (Find work done by the force) where $C: y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$

$\bar{F}\cdot d\bar{r} = F_1dx + F_2dy + F_3dz$

where $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ $\quad d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\because$ xy-plane $\quad dz = 0$

$y = 2x^2$

$dy = 4x\,dx$

$y - 0 = 2(x-0)$

$\boxed{y = 2x}$

$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2dy$

$= 3xy\,dx - y^2dy$

$= 3x(2x^2)dx - 4x^4\cdot 4x\,dx$

$= (6x^3 - 16x^5)dx$

Work done $= \int \bar{F}\cdot d\bar{r} = \int_0^1 (6x^3 - 16x^5)dx$

$= \cancel{\frac{6}{4}(1) - \frac{16}{6}(1)}$

$= \cancel{\frac{3}{2\times3} - \frac{8}{3\times2}}$

$= \cancel{\frac{9-16}{6}} =$

$\left[\frac{6}{4}x^4 - \frac{16}{6}x^6\right]_0^1$

$= \frac{6}{4}(1) - \frac{16}{6}(1)$

$= \frac{3}{2\times3} - \frac{8}{3\times2}$

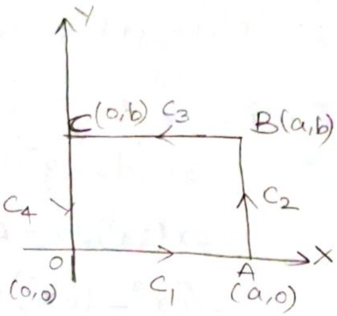
$= \frac{9-16}{6}$

$\boxed{W = -\frac{7}{6}}$

* $\bar{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$. Evaluate $\int_C \bar{F}\cdot d\bar{r}$

C: rectangle in XY-plane bounded by $y=0, y=b;\ x=0, x=a$.

$$\bar{F}\cdot d\bar{r} = \boxed{(x^2+y^2)dx - 2xydy}$$



for C_1: (0,0) to (a,0)

$\bar{F}\cdot d\bar{r} = x^2dx - 0$

$= x^2dx$

$\int_0^a \bar{F}\cdot d\bar{r} = \int_0^a x^2 dx$

$= \frac{a^3}{3}$ — (1)

for C_2: (a,0) to (a,b) $x = \text{const}; dx = 0$

$\bar{F}\cdot d\bar{r} = (a^2+y^2)\underset{0}{dx} - 2aydy$

$= -2aydy$

$\int_0^b \bar{F}\cdot d\bar{r} = \int_0^b -2ay\,dy = \left[-ay^2\right]_0^b$

$= -ab^2$ — (2)

for C_3: (a,b) to (0,b) $dy = 0$

$\bar{F}\cdot d\bar{r} = (x^2+b^2)dx$

$\int_a^0 \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_a^0$

$= \frac{-a^3}{3} + -ab^2 = -\frac{a^3}{3} - ab^2$ — (3)

for C_4: (0,b) to (0,0) $dx=0$

$\bar{F}\cdot d\bar{r} = 0$

$\int_b^0 0\,dy = 0$ — (4)

$W = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$

$= -2ab^2$

13/08/2022

* $\bar{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

$\Rightarrow \int_C \bar{F}\,d\bar{r}$ C: st. line joining (0,0,0) to (1,1,1)

St. line joining the points

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0} = t$$

$$\frac{x-0}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= t\hat{i} + t\hat{j} + t\hat{k}$

$d\bar{r} = (\hat{i}+\hat{j}+\hat{k})dt$

$\bar{F} = (3t^2+6t)\hat{i} - 14t^2\hat{j} + 20t^3\hat{k}$

$\bar{F}\cdot d\bar{r} = (3t^2+6t)dt$

$-14t^2dt + 20t^3dt$

$x=0;\ t=0$
$y=0;\ t=0$
$z=0\ t=0$
$x=y=z=1;\ t=1$

$$\int \bar{F}\cdot d\bar{r} = \int_0^1 (3t^2+6t-14t^2+20t^3)dt$$

$$= \frac{3}{3}(1-0) + \frac{6}{2}(1-0) - \frac{14}{3}(1-0) + \frac{20}{4}$$

$$= 1+3-\frac{14}{3}+5$$

$$= \frac{27-14}{3} = \frac{13}{3}$$

* $\bar{F} = x^2\hat{i} + xy\hat{j}$ $\quad (0,0)$ to $(0,1)$

(i) Along OP from ~~$x=0$ to $x=1$~~

(ii) x-axis from $x=0$ to $x=1$

(iii) the line $x=1$ from $y=0$ to $y=1$

(iv) Along the parabola $y^2=x$ from $(0,0)$ to $(1,1)$

$\bar{F}\cdot d\bar{r} = x^2dx\hat{i} + xydy\hat{j}$

(i) OP: $(0,0)$ to $(0,1)$ $\quad \bar{F}\cdot d\bar{r} = 0\hat{i} + 0\hat{j}$

$x=0,\ dx=0$ $\quad \int \bar{F}\cdot d\bar{r} = 0$

$y=0$ to 1

(ii) $y=0$ $\quad x: 0$ to 1 $\quad dy=0$

$\bar{F}\cdot d\bar{r} = x^2dx\hat{i} + xydy\hat{j}$

$\bar{F}\cdot d\bar{r} = x^2dx + 0$

$\int \bar{F}\cdot d\bar{r} = \int_0^1 x^2dx = \frac{1}{3}$

(iii) $x=1;\ y: 0$ to 1

$\bar{F}\cdot d\bar{r} = x^2dx + xydy$

$\int \bar{F}\cdot d\bar{r} = 1(0) + (1)\int_0^1 y\,dy = \frac{1}{2}$

(iv) $y^2=x$ $\quad (0,0)$ to $(1,1)$ $\quad 2ydy = dx$

$\bar{F}\cdot d\bar{r} = x^2dx + xydy$

$= x^2dx + x\sqrt{x}\cdot\frac{dx}{2\sqrt{x}}$

$= \left[x^2 + \frac{x}{2}\right]dx$

$$\int_0^1 \bar{F}\cdot d\bar{r} = \int_0^1 \left(x^2 + \frac{x}{2}\right)dx$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$W = 0 + \frac{1}{3} + \frac{1}{2} + \frac{7}{12}$$

$$= \frac{4+3+7}{12} = \frac{14}{12} = \frac{7}{6}.$$

* $(x^2 - y^2 + x)\hat{i} + (2xy + y)\hat{j}$ moves a particle in the xy plane $(0,0)$ to $(1,1)$

(i) along the parabola $y^2 = x$.

(ii) along the st. line $\therefore y = x$

~~(iii)~~

* $\bar{F} = y\hat{i} - x\hat{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0)$ to $(1,1)$

(i) along the parabola $y = x^2$

(ii) along the st. line from $(0,0)$ to $(1,0)$ to $(1,1)$

(iii) the st. line joining from $(0,0)$ to $(0,1)$

(i) along $y = x^2$ $\quad d\bar{r} = dx\hat{i} + dy\hat{j}$

$dy = 2x\,dx$

$\bar{F}\cdot d\bar{r} = y\,dx - x\,dy$

$= x^2 dx - 2x^2 dx$

$= -x^2 dx$

$\int \bar{F}\cdot d\bar{r} = \int_0^1 -x^2 dx = -\frac{1}{3}(1) = -\frac{1}{3}$

(ii) $\bar{F}\cdot d\bar{r} = y\,dx - x\,dy$ $\quad dy = dx$.

$(0,0)$ to $(1,0)$

$x = 0$ to 1

$y = 0$ $\quad dy = 0$

$\int_0^1 (0 - x(0))\,dx = 0$

$(1,0)$ to $(1,1)$

$x = 1$ $\quad dx = 0$

$y = 0$ to 1

$\int_0^1 (y(0) - 1\,dy)$

$= -1$

~~$y = 0$~~ $(0,0)$ to $(0,1)$

$x = 0$, $y: 0$ to 1

$\bar{F}\cdot d\bar{r} = y\,dx - x\,dy$

$\int_0^1 \bar{F}\cdot d\bar{r} = \int y(0) - 0(0) = 0$.

$W = -\frac{1}{3} - 1 + 0 = -\frac{4}{3}$

* $\bar{F} = (y^2\cos x + z^3)\hat{i} + (2xy\sin x - 4)\hat{j} + (3xz^2 + ~~x~~)\hat{k}$ is a conservative field. Find the work done in moving a particle in this field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$

$\frac{x-0}{\pi/2} = \frac{y-1}{-2} = \frac{z+1}{3} = t$

$x = t\frac{\pi}{2}$, $\quad y = -2t + 1$

$z = 3t - 1$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \frac{t\pi}{2}\hat{i} + (1-2t)\hat{j} + (3t-1)\hat{k}$

18/08/2022

VECTOR INTEGRATION:

* GAUSS'S DIVERGENCE THEOREM:

(Transformation b/w surface integral & volume integral)

Let S be a closed surface enclosing a volume V. If F is continuously differentiable vector point function; then

$$\boxed{\int_S \bar{F}\cdot\hat{n}\, ds = \int_V \text{div}\,\bar{F}\cdot dv}$$

when $\hat{n}$ is the outward normal vector drawn at any point of S.

$$\boxed{\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\,\bar{F}\cdot dv}$$

* Verify Gauss's theorem for $\bar{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ over surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\,\bar{F}\cdot dv \Rightarrow$$ Gauss's Divergence Theorem

RHS: $\iiint_V \text{div}\,\bar{F}\; dx\,dy\,dz$

1st octant:

XYZ

$\cancel{z}$: 0 to a

y: 0 to $a - x$

$\cancel{z}$: 0 to $a-x-y$

$\text{div}\,\bar{F} = 2x+2y+2z$

$= 2(x+y+z)$

$$= 2\int_0^a \left[\int_0^{a-x}\left[\int_0^{a-x-y}(x+y+z)\,dz\right]dy\right]dx$$

$$= 2\int_0^a \left[\int_0^{a-x} \left[xz+yz+\frac{z^2}{2}\right]_0^{a-x-y} dy\right] dx$$

$$= 2\int_0^a \left[\int_0^{a-x} \cancel{ax} - x^2 - xy + \cancel{ay} - xy - y^2 + \frac{1}{2}\left(a^2+x^2+y^2-2a\cancel{x}-2a\cancel{y}+2xy\right) dy\right]$$

$$= 2\int_0^a \left[\int_0^{a-x} \frac{a^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} - xy\right] dx$$

$$= 2\int_0^a \left[\frac{a^2 y}{2} - \frac{x^2 y}{2} - \frac{y^3}{6} - \frac{x y^2}{2}\right]_0^{a-x} dx$$

$$= 2\int_0^a \frac{a^2(a-x)}{2} - \frac{(a-x)x^2}{2} - \frac{(a-x)^3}{6} - \frac{x(a-x)^2}{2} \, dx$$

$$= 2\int_0^a \frac{a-x}{2}\left[\frac{a^2}{1} - \frac{x^2}{1} - \frac{(a-x)^2}{3} - \frac{x(a-x)}{1}\right] dx$$

$$= 2\int_0^a \frac{a-x}{2}\left[a^2 - \cancel{x^2} - ax + \cancel{x^2} - \frac{a^2 - x^2 + 2ax}{3}\right] dx$$

$$= 2\int_0^a \frac{a-x}{2}\left[\frac{2a^2}{3} - \frac{x^2}{3} - \frac{ax}{3}\right] dx$$

$$= 2\int_0^a \left[\frac{2a^3}{6} - \cancel{\frac{ax^2}{6}} - \frac{a^2x}{3} - \frac{2a^2x}{6} + \frac{x^3}{6} + \cancel{\frac{ax^2}{6}}\right] d$$

$$= 2\int_0^a \left[\frac{a^3}{3} - \frac{2a^2x}{3} + \frac{x^3}{6}\right] dx$$

$$= 2\left[\frac{a^3x}{3} - \frac{a^2x^2}{3} + \frac{x^4}{24}\right]_0^a dx$$

$$= 2\left[\frac{a^4}{3} - \frac{a^4}{3} + \frac{a^4}{24}\right] = \underline{\underline{\frac{a^4}{12}}}$$

$\underline{\underline{LHS}}$: $\iint_S \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$\because$ XY plane

$z = 0$.

$x + y = a$.

$x : 0$ to a

$y : 0$ to $a - x$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}\cdot\hat{n} = \frac{x^2+y^2+z^2}{\sqrt{3}}$$

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}$$

$$\underline{\underline{z = a - x - y}}$$

$$\int_0^a\int_0^{a-x} \frac{x^2+y^2+z^2}{\cancel{\sqrt{3}}}\cdot\frac{dx\,dy}{1/\cancel{\sqrt{3}}}$$

$$= \int_0^a \left[\int_0^{a-x} (x^2+y^2+ a^2+x^2+y^2-2ax-2ay+2xy)\, dy \right] dx$$

$$= \int_0^a \int_0^{a-x} \left[2x^2+2y^2+a^2-2ax-2ay+2xy \right] dy\, dx$$

$$= \int_0^a \left[2x^2y+\frac{2y^3}{3}+\frac{a^2y}{1}-2axy-\frac{2ay^2}{2}+\frac{2xy^2}{2} \right]_0^{a-x} dx$$

$$= \int_0^a 2x^2(a-x)+\frac{2}{3}(a-x)^3+a^2(a-x)-2ax(a-x) - a(a-x)^2+x(a-x)^2\, dx$$

$$= \int_0^a (a-x)\left[2x^2+\frac{2}{3}(a^2+x^2-2ax)+\cancel{a^2}-\cancel{2ax} -\cancel{a^2}+\cancel{ax}+\cancel{xa}-x^2 \right.$$

$$= \int_0^a (a-x)\left[x^2+\frac{2}{3}a^2+\frac{2}{3}x^2-\frac{4ax}{3} \right] dx$$

$$= \int_0^a (a-x)\left[\frac{5}{3}x^2+\frac{2a^2}{3}-\frac{4ax}{3} \right] dx$$

$$\left[\frac{5}{3}ax^2+\frac{2a^3}{3}-\frac{4a^2x}{3}-\frac{5}{3}x^3-\frac{2a^2x}{3}+\frac{4ax^2}{3} \right]_0^a$$

$$\frac{5}{\cancel{3}\,9}\cancel{a^4}+\frac{2a^4}{3}-\frac{4a^4}{6}-\frac{5}{\cancel{3}\,9}\cancel{a^4}-\frac{2a^4}{6}+\frac{4a^4}{9}$$

$$= \frac{\cancel{2}a^4}{\cancel{3}} - \frac{\cancel{2}a^4}{\cancel{3}} - \frac{a^4}{3} + \frac{4a^4}{9}$$

$$= \frac{a^4}{9}.$$

* Evaluate divergence theorem for $\bar{F}$ $4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ & $z=3$.

→ By Gauss's Divergence theorem:

$$\iiint_V \operatorname{div} \bar{F} \cdot dV$$

$z: 0$ to 3

$x^2 = 4$

$x = -2$ to 2

$y = -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

$\operatorname{div} F = 4 - 4y + 2z$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3} (4 - 4y + 2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4(3) - 4y(3) + 9\right) dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dy\, dx$$

$$\int_{-2} \left[21y - 6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 21\left(2\sqrt{4-x^2}\right) - 6\left(4 - \cancel{x^2} - 4 + \cancel{x^2}\right) dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2} - 6(0) \; dx$$

$x = 2\sin\theta$
$dx = 2\cos\theta d\theta$

$$= 42 \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta \, d\theta$$

$$= 42 \times 2 \times 2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{84}{\cancel{2}} \times \cancel{2} \int_{-\pi/2}^{\pi/2} (1 - \cos 2\theta) d\theta$$

$$= 84\left[\left(\frac{2\pi}{2}\right) - \frac{1}{2}(0 - 0)\right]$$

$$= \underline{\underline{84\pi}}$$

* Verify Gauss divergence theorem $(x^3 - y^2)\hat{i} - 2x^2y\hat{j} + z\hat{k}$ taken over the surface of the cube $x = y = z = a$ and the coordinate planes.

By Gauss's Divergence theorem

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{V}\cdot dv$$

$x: 0$ to a

$y: 0$ to a

$z: 0$ to a

$\operatorname{div}\bar{F} =$

$3x^2 - 2x^2 + 1$

$= x^2 + 1$

RHS $\iiint_V \operatorname{div}\bar{V}\cdot dv$

$$\int_0^a \int_0^a \int_0^a (x^2+1)\,dz\,dy\,dx$$

$$= \int_0^a \int_0^a (x^2 a + a)\,dy\,dx$$

$$= \int_0^a (x^2 a^2 + a^2)\,dx$$

$$= \frac{a^3\cdot a^2}{3} + a^3 = \underline{\underline{\frac{a^5}{3} + a^3}}$$

* Evaluate $\iint_S \bar{F}\cdot\hat{n}\,ds$ for $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and s is the surface $x^2+y^2=16$ included in the first octant $z=0$ to $z=5$.

xz (or) yz plane.

$\therefore$ Consider yz plane $\therefore x = 0$ $z: 0$ to 5

$y^2 = 16$ $y = \pm 4$ $\therefore y: 0$ to 4

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{\cancel{2}(x\hat{i}+y\hat{j})}{\cancel{2}\times 4} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$\bar{F}\cdot\hat{n} = \frac{xz+xy}{4}$$

$$|\hat{n}\cdot\hat{i}| = \frac{x}{4}$$

$$\int_0^5\int_0^4 \frac{xz+xy}{\cancel{4}} \cdot \frac{dy\,dz}{x} \times \cancel{4}$$

$$\int_{z=0}^{5}\left[\int_{y=0}^{4} (z+y)\,dy\right]dz = \int_0^5\left[z(4)+\frac{1}{2}(16)\right]dz$$

$$= 4\int_0^5 (z+2)\,dz$$

$$= 4\left[\frac{1}{2}(25)+2(5)\right]$$

$$= 4\left[\frac{25}{2}+10\right]$$

$$= \underline{\underline{90}}.$$

* Evaluate $\int_S \bar{F}\cdot\hat{n}\,ds$ where $\bar{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of surface of the plane $2x+3y+6z=12$ located in the first octant.

Consider yzplane $\cancel{x}=0$.

$\cancel{2x+3y=12}$. $3y+6z=12$

$\cancel{z}$: 0 to 4 ; $\cancel{y}$: 0 to $\frac{12-3y}{6}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+16}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{29}}$$

$$\bar{F}\cdot\hat{n} = 36z - 36 + 18y \qquad |\hat{n}\cdot\hat{i}| = \frac{2}{\sqrt{29}}$$

$$\int \bar{F}\cdot\hat{n}\,ds = \int_0^4\left[\int_0^{\frac{12-3y}{6}} (36z-36+18y)\,\frac{dy\,dz}{2/\sqrt{29}}\right]$$

$$= \int_0^4\left[\frac{18\sqrt{29}}{2}\int_0^{\frac{12-3y}{6}}(2z-2+y)\,dz\right]dy$$

$$= 9\sqrt{29}\int_0^4\left[\left(2-\frac{y}{2}\right)^2 - 2\left(2-\frac{y}{2}\right) + y\left(2-\frac{y}{2}\right)\right]dy$$

$$= 9\sqrt{29}\int_0^4 \cancel{4} + \frac{y^2}{4} - \cancel{2y} - \cancel{4} + y + \cancel{2y} - \frac{y^2}{2}\, dy$$

$$= 9\sqrt{29}\int_0^4 \left(\frac{-y^2}{4} + y\right) dy$$

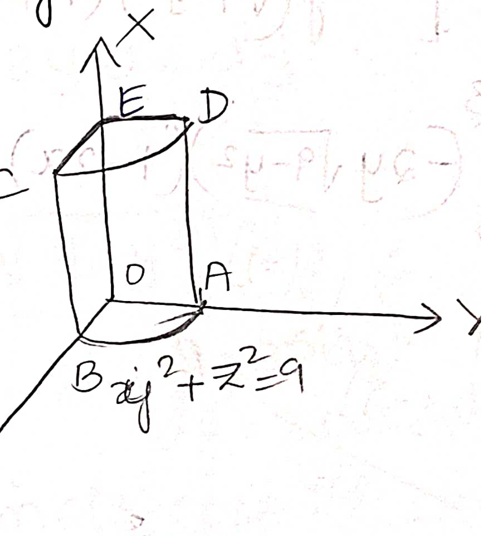
$$= 9\sqrt{29}\left[\frac{-1}{\cancel{12}_{3}}\left(\cancel{64}^{16}\right) + \frac{1}{2}(16)\right]$$

$$= 9\sqrt{29}\left[\frac{-16}{3} + 8\right]$$

$$= 9\sqrt{29} \times \frac{8}{3} = \frac{72\sqrt{29}}{3} = \underline{24\sqrt{29}}$$

22/08/2022

* Verify divergence theorem for $\bar{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ & $x=2$.



By Gauss's divergence theorem:

$$\iint_S F\cdot\hat{n}\, ds = \iiint_V \text{div}\bar{F}\cdot dV$$

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iint_{OAB} \bar{F}\cdot\hat{n}\, ds + \iint_{OADE} \bar{F}\cdot\hat{n}\, ds + \iint_{OBCE} F\cdot\hat{n}\, ds + \iint_{ECD} \bar{F}\cdot\hat{n}\, ds + \iint_{ABCD} \bar{F}\cdot\hat{n}\, ds.$$

RHS:

$\operatorname{div}\bar{F} = 4xy - 2y + 8xz$

$$\iiint_V \operatorname{div}\bar{F}\cdot dv$$

$x : 0$ to 2

$y : 0$ to 3

$z : 0$ to $\sqrt{9-y^2}$

$$\int_0^2 \int_0^3 \left[\int_{z=0}^{\sqrt{9-y^2}} \left[-2y(1-2x) + 8xz\right] dz\right] dy$$

$$\int_0^{\sqrt{9-y^2}} \left[-2y(1-2x) + 8xz\right] dz \quad \cancel{\int_0^{\sqrt{9-y^2}}}$$

$$= \left[-2y(z - 2xz) + 4xz^2\right]_0^{\sqrt{9-y^2}}$$

$$= \left[-2y\left(\sqrt{9-y^2}\right)(1-2x) + 4x(9-y^2)\right]$$

$\sqrt{9-y^2} = t$

$9-y^2 = t^2$

$-2y\,dy = 2t\,dt$

$= 2\sqrt{9-y^2}$

$$\int_0^3 \left(-2y\sqrt{9-y^2}\right)(1-2x)\, dy + \int_0^3 4x(9-y^2)\, dy$$

$$\int f'(x)\,[f(x)]^{1/2}$$

$$\frac{[f(x)]^{3/2}}{3/2} + C$$

$$= \left[\frac{(9-y^2)^{3/2}}{3/2}\right]_0^3 (1-2x) + 4x\left(9y - \frac{y^3}{3}\right)_0^3$$

$$= \left[0 - \frac{\cancel{27}^{\,9}}{\cancel{3}} \times 2\right](1-2x) + 4x(27-9)$$

$$= -\frac{54}{3}(1-2x) + 4x(18)$$

$$\int_0^2 \left(-\frac{54}{3} + \frac{108x}{\cancel{2}} + 72x\right)$$

$$= \int_0^2 -18 + 108x$$

$$= -18(2) + 54(4)$$

$$= 18(12-2) = 18 \times 10 = \underline{\underline{180}}.$$

$$\begin{array}{r} 54 \\ \times 4 \\ \hline 216 \\ 36 \\ \hline 170 \\ \hline \end{array}$$

$\ast$ **LHS**

OAB: yz plane $x=0$

$\hat{n} = -\hat{i}$

$F.\hat{n} = -2x^2y = 0$

$\iint_S F.\hat{n}\, ds = 0$

CDE: yz plane $x=2$

$\hat{n} = +\hat{i}$

$F.\hat{n} = 2x^2y = 8y.$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} 8y\, dz\, dy.$$

$$= \int_0^3 8y\sqrt{9-y^2}\, dy$$

$y = 3\sin\theta$

$dy = 3\cos\theta\, d\theta$

$$= \int_0^{\pi/2} 24\sin\theta(3)\cos\theta \cdot 3\cos\theta$$

$$= 108\int_0^{\pi/2} \sin 2\theta \cos\theta \cdot d\theta$$

$$\cancel{= 108\int_0^{\pi/2} 2\sin\theta(1-\sin^2\theta)}$$

$$\cancel{= 108\left[(2\cos\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 2\sin^3\theta]}$$

$108(-2)$

$$\cancel{= \overset{54}{\cancel{108}}(-\cancel{2})\left(\frac{3\sin\theta - \sin 3\theta}{\cancel{4}\,2}\right)}$$

$\ast$ **OBCE:**

xz plane $y=0$.

$\hat{n} = -\hat{j}$

$F.\hat{n} = -y^2 = 0$

OADE: XY plane $z=0$.

$\hat{n} = -\hat{k}$

$F.\hat{n} = -4xz^2 = 0$

$\iint_S F.\hat{n}\, ds = 0$

$$\cancel{54\left[3(0)\right.}$$

$$= -108 \times 2 - 54 \int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{4}\, d\theta$$

$$\cancel{= -216 - \frac{54}{4}\left[3(0-1) - \frac{1}{4\times 3}\left[\cos 3\theta\right]_0^{\pi/2}\right.}$$

$$\cancel{= -216 - \frac{27}{2}\left(-3 - \frac{1}{12}(0-1)\right]}$$

$$\cancel{= -216 + \frac{\overset{9}{\cancel{27}}}{2}\left(\frac{36+1}{\underset{4}{\cancel{12}}}\right)}$$

$$\begin{array}{r} 3 \\ 24 \\ \times 9 \\ \hline 216 \end{array}$$

$$= 108 \int_0^{\pi/2} 2\sin\theta \cos^2\theta \, d\theta$$

$$= 216 \int_0^{\pi/2} \left(\sin\theta - \sin^3\theta\right) d\theta$$

$$= 216\left(-\cos\theta\right)_0^{\pi/2} - \frac{216}{4}\int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{\cancel{4}}\, d\theta$$

$$= 216(-0+1) - 54\left[-3(0-1) + \left(\frac{\cos 3\theta}{3}\right)_0^{\pi/2}\right]$$

$$= 216 - 54\times 3 - 18(0-1)$$

$$= 216 - 162 + 18$$

$$= 72$$

$$\begin{array}{r} 216 \\ 18 \\ \hline 254 \\ 16 \end{array}$$

$$\begin{array}{r} 216 \\ 18 \\ \hline 198 \\ 162 \\ \hline 36 \end{array}$$

* ABCD xy plane

$\phi = y^2 + z^2 = 9$

$\nabla\phi = 2y\hat{j} + 2z\hat{k}$

$\nabla\phi = \sqrt{4(y^2+z^2)} = 2\times 3 = 6$

$\hat{n} = \frac{2(y\hat{j} + z\hat{k})}{2\times 3}$

$\hat{n}\cdot\hat{k} = \frac{(y\hat{j} + z\hat{k})\hat{k}}{3}$

$= \frac{z}{3}$

$\bar{F}\cdot\hat{n} = \frac{-y^3 + 4xz^3}{3}$

$$\int_0^2 \int_0^3 \frac{-y^3 + 4xz^3}{z}\, dy\, dx$$

$$\int_0^2 \left[\int_0^3 \frac{-y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\, dy\right] dx$$

$= 108$

* VOLUME INTEGRAL:

→ **Cartesian Form:**

Let $F(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where F_1, F_2, F_3 are functions of x, y, z, we know that $dV = dx\,dy\,dz$.

The volume integral is given by:

$$\int_V F \cdot dV = \iiint (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\, dx\,dy\,dz$$

$$= \hat{i}\iiint F_1\, dx\,dy\,dz + \hat{j}\iiint F_2\, dx\,dy\,dz$$

$$+ \hat{k}\iiint F_3\, dx\,dy\,dz.$$

Q: If $F = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ evaluate $\int_V F \cdot dV$ where V is the region bounded by the surfaces $x = 0$; $x = 2$; $y = 0$; $y = 6$; $z = x^2$, $z = 4$.

A: $128\hat{i} - 24\hat{j} + 384\hat{k}$

$$\int_V F \cdot dV = \hat{i}\int_0^6\int_0^2\int_{x^2}^{4} 2xz\, dz\, dx\, dy$$

$$+ \hat{j}\int_0^6\int_0^2\int_{x^2}^{4} (-x)\, dz\, dx\, dy$$

$$+ \hat{k}\int_0^6\int_0^2\int_{x^2}^{4} (y^2)\, dz\, dx\, dy$$

$$= \hat{i}\int_0^6\int_0^2 x(16 - x^4)\, dx\, dy$$

$$+\hat{j}\int_0^6\int_0^2 -x(4-x^2)\,dx\,dy$$

$$+\hat{k}\int_0^6\int_0^2 y^2(4-x^2)\,dx\,dy.$$

$$=\hat{i}\int_0^6 8(4)-\frac{1}{6}(64)\,dy$$

$$+\hat{j}\int_0^6 -2(4)+\frac{1}{4}(16)\,dy$$

$$+\hat{k}\int_0^6 4y^2(2)-\frac{y^2}{3}(8)\,dy$$

$$=\hat{i}\left[32(6)-\frac{32}{\cancel{3}}(\cancel{6})^{2}\right]+\hat{j}\left[-8(6)+4(6)\right]$$

$$+\hat{k}\left(\frac{8}{\cancel{3}}(\cancel{216})^{72}-\frac{8}{\cancel{9}3}(\cancel{216})^{\cancel{72}\,24}\right)$$

$$\begin{array}{r} 3 \\ 24 \\ \times 8 \\ \hline 2 \end{array}$$

$$=\hat{i}\left(32(6-2)\right)+\hat{j}\left(6(-4)\right)+\hat{k}\left(5\cancel{7}6-192\right)$$

$$\begin{array}{r} 16 \\ 5\cancel{7}6 \\ 192 \\ \hline \cancel{8}4 \end{array}$$

$$=128\hat{i}-24\hat{j}+3\cancel{8}4\hat{k}$$

$$\begin{array}{r} 1 \\ 72 \\ \times 8 \\ \hline 576 \end{array}$$

$$*\ \vec{F}=(2x^2-3z)\hat{i}-2xy\hat{j}-4x\hat{k}\ \text{then}\int \nabla\cdot\vec{F}\,dv$$

V: $x=0;\ y=0;\ z=0\qquad 2x+2y+z=4$ $\qquad \frac{A}{=}\cdot\frac{8}{3}$

$$\nabla\cdot F=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x^2-3z & -2xy & 4x\end{vmatrix}=\hat{i}(0)-\hat{j}(4+3)$$

$$+\hat{k}(-2\ 0)$$

$$=\cancel{-7}\hat{j}-2\hat{k}$$

$2x+2y+z=4$

$x: 0 \text{ to } 2$

$y: 0 \text{ to } \frac{4-2x}{2}$

$y: 0 \text{ to } 2-x$

$z: 0 \text{ to } 4-2x-2y$

*

$= \nabla \cdot F = 4x - 2x = 2x.$

$= \cancel{-(4)\hat{j} - 2\hat{k}}$

(3)

$\int \nabla F \cdot dv =$

(1) $\cancel{\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 0 \, dx\, dy\, dz = 0}$

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(2) $\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (-7)\, dz\, dy\, dx$

$= \int_0^2 \int_0^{2-x} -7(4-2x-2y)\, dy\, dx$

$= \int_0^2 \left[-28(2-x) + 14x(2-x) + 7(2-x)^2\right] dx$

$= -56(2) + 14(4) + 14(4) - \frac{14}{3}(8) +$

$\int_0^2 7(4 + x^2 - 4x)\, dx$

$= -\frac{14}{3}(8) + 28(2) + \frac{7}{3}(8) - 14(4)$

$= \frac{8}{3}(-7) + 48 = -\frac{56}{3}$

$= -\frac{56}{3} + 48 = 8\left[-\frac{7}{3} + 6\right] = \frac{88}{3}$

* **Stoke's Theorem:**

Let S be a open surface bounded by a closed, non intersecting curve C

If $\bar{F}$ is any differentiable vector point function

then $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl } \bar{F}\cdot\hat{n}\, ds$

where C is transversed in the +ve direction and $\hat{n}$ is unit outdrawn normal at any point of the surface.

(3) $\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} -2 \, dz \, dy \, dx$

$= \int_0^2 \int_0^{2-x} -2(4-2x-2y)\, dy\, dx$

$= \int_0^2 \int \left[-8(2-x) + 4(2-x)x + 2(2-x)^2\right] dx$

$= -16(2) + 4(4) + \cancel{8(4)} - \frac{4}{3}(8) + \int_0^2 2(4+x^2-4x)\, dx$

$= -\frac{32}{3} + 8(2) + \frac{2}{3}(8) - 4(4)$

$= -\frac{16}{3}$ $\quad -\frac{56}{3} - \frac{16}{3} = -\frac{72}{3}$

$= -18$

7/08/2022

* $\iiint_V \phi \, dv$ where $\phi = 45x^2y$; V: closed region bounded by the planes $4x+2y+z=8$, $x=0$; $y=0$; $z=0$

~~$x: 0$ to 2~~

$x: 0$ to 2

$y: 0$ to $4-2x$

$z: 0$ to $8-4x-2y$.

$$= 45 \int_0^2 \left[\int_0^{4-2x} \left[\int_0^{8-4x-2y} x^2 y \, dz \right] dy \right] dx$$

$$= 45 \int_0^2 \left[\int_0^{4-2x} \left[x^2 y (8-4x-2y) \, dy \right] \right] dx$$

$$= 45 \int_0^2 \left[\int_0^{4-2x} (8x^2y - 4x^3y - 2x^2y^2) \, dy \right] dx$$

$$= 45 \int_0^2 \left[4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2}{3}x^2(4-2x \ldots \right.$$

$$= 45 \left[\int_0^2 (4-2x)^2 \left[4x^2 - 2x^3 - \frac{2}{3}x^2(4-2x) \right] \right] dx.$$

$$= 45 \int_0^2 (16+4x^2-16x) \left[4x^2 - 2x^3 - \frac{8x^2}{3} + \frac{4}{3}x^3 \right] d\ldots$$

$$= 45\int_0^2 (16+4x^2-16x)\left[\frac{9x^2}{3}-\frac{9x^3}{3}\right] dx$$

$$= \cancel{45}^{15} \times \frac{9}{\cancel{3}_2} \times 4 \int_0^2 (4+x^2-4x)(2x^2-x^3)\, dx$$

$$= \cancel{1}\cancel{2}0 \int_0^2 (\cancel{8}x^2 - 4x^3 + 2x^4 - x^5 - 8x^3 + 4x^4)\, dx$$

$$= \cancel{1}\cancel{2}0 \int_0^2 (\cancel{8}x^2 - \cancel{12}x^3 + 6x^4 - x^5)\, dx \qquad \frac{4\,48\,5}{240}$$

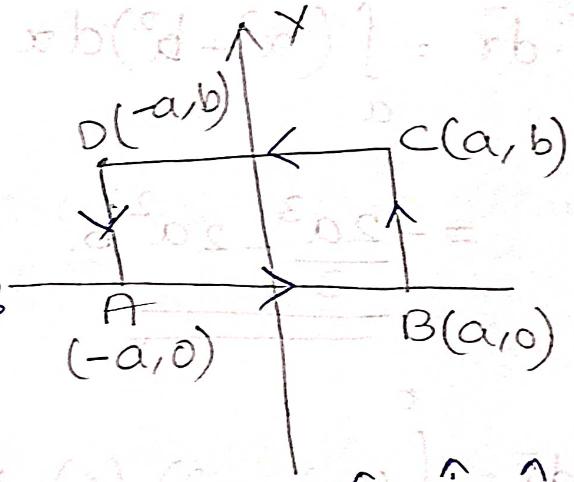
$$= \cancel{1}\cancel{2}0 \left[\frac{\cancel{8}}{3}(8) - \frac{\cancel{12}^3}{\cancel{4}}(16) + \frac{\cancel{6}}{5}(32) - \frac{1}{6}(64)\right]$$

$$= \cancel{1}\cancel{2}0 \left[\frac{\cancel{64}}{3} - \cancel{48} + \frac{32}{5} - \frac{32}{3}\right] = 120\left[\frac{32}{3} \; \frac{-48+\frac{32}{5}}{\frac{32}{\times 6}}\right]$$

$$= 120\left[\frac{32}{3} - \frac{48\times 5+32}{5}\right] = \cancel{\frac{120\times 32}{2}} - \cancel{\frac{120}{35}(-240+32)}$$

$$= 1280 -$$

* Verify $\bar{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by $x=-a$ to $x=+a$ and $y=0$ to $y=b$.



By Stokes Theorem

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\bar{F}\cdot\hat{n}\, ds$$

RHS: $\iint_S (\text{curl}\bar{F})\,\hat{n}\, ds$

$\hat{n} = \hat{k}$

$$\text{curl}\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0-0) + \hat{k}(-2y-2y)$$

$$= -4y\hat{k}$$

$\text{curl}\bar{F}\cdot\hat{n} = -4y$

$$2\int_0^a\int_0^b -4y\, dy\, dx$$

$$= -\frac{8}{2}\int_0^a b^2\, dx$$

$$= \underline{-4ab^2}$$

LHS:

$$\bar{F}\cdot d\bar{r} = (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

$y=0$; $x : -a$ to a. $dy = 0$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{-a}^{a} x^2\,dx = 2\int_0^a x^2\,dx = \frac{2a^3}{3}$$

$x=a$; $y : 0$ to b.

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int (a^2+y^2)(0) - 2ay\,dy \Rightarrow -2\int_0^b ay\,dy$$

$$= -\frac{2a}{2}b^2 = -ab^2$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_a^{-a} (x^2+b^2)\,dx = \frac{-a^3-a^3}{3} + a^2(-2a)$$

$$= \cancel{-\frac{2a^3}{3} - 2a^3}$$

$$\cancel{= -\frac{8a^3}{3}}$$

$$= -\frac{2a^3}{3} - 2a^2b.$$

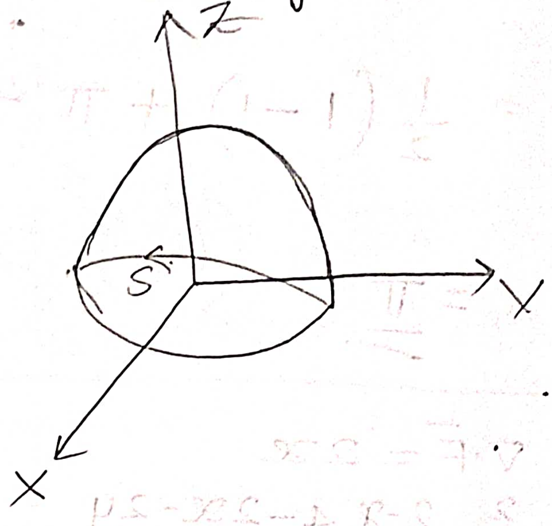
$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_b^0 (a^2+y^2)(0) + 2ay\,dy$$

$$= -ab^2.$$

$\therefore$

$$* \quad \frac{2a^3}{3} - \frac{2a^3}{3} - 4a^2b = -4a^2b$$

* Verify Stokes theorem $\bar{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ S is the upper half of the surface of the sphere $x^2+y^2+z^2=1$ and c is its boundary



Consider projection of R on XY plane.

$\hat{n} = \hat{k}$

$$\text{curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}(-2yz+2zy) - \hat{j}(0-0) + \hat{k}(0+1)$$

$$= \hat{k}$$

$$\text{curl}\bar{F} \cdot \hat{n} = 1$$

$$RHS = \iint_S \text{curl}\bar{F}\cdot\hat{n}\, ds = \iint_S ds = \underline{\underline{\pi}}$$ $(\because \text{area} = \pi r^2)$

* LHS:

The boundary c of the surface S is a circle.

$x^2+y^2=1$ $r=1$

$x = r\cos\theta$; $y = r\sin\theta$

$x = \cos\theta$ $y = \sin\theta$ θ: 0 to 2π

$dx = -\sin\theta d\theta$ ($\because$ complete revolution of circle)

$$LHS: \int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi}(-\sin 2\theta + \sin^2\theta)d\theta$$ $\bar{F}\cdot d\bar{r} = (2x-y)\,dx$

$$= \int_0^{2\pi}\left(-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right)d\theta$$ $= (2\cos\theta - \sin\theta) - \sin\theta d\theta$

$$= \left[+\frac{1}{2}\cos 2\theta\right]_0^{2\pi} + \frac{1}{2}(2\pi) - \frac{1}{4}\left[\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(1-1) + \pi - \frac{1}{4}(0-0)$$

$$= \underline{\underline{\pi}}.$$

$$\nabla\cdot\bar{F} = 2x$$

$$\int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x \; dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} (8x - 4x^2 - 4xy)\,dy\,dx$$

$$= 4\int_0^2 x\int_0^{2-x} (2-x-y)\,dy\,dx$$

$$= 4\int_0^2 x\left[2(2-x) - x(2-x) - \frac{1}{2}(2-x)^2\right]dx$$

$$= 4\int_0^2 x(2-x)\left[2-x-\frac{(2-x)}{2}\right]dx$$

$$= 4\int_0^2 (2x-x^2)\left(2-x-1+\frac{x}{2}\right)dx$$

$$= 4\int_0^2 (2x-x^2)\left(1-\frac{x}{2}\right)dx$$

$$= 4\int_0^2 \left(2x - x^2 - x^2 + \frac{x^3}{2}\right) dx$$

$$= 4\int_0^2 \left(2x - 2x^2 + \frac{x^3}{2}\right) dx$$

$$= 4\left[(4) - \frac{2}{3}(8) + \frac{1}{\cancel{8}}\left(\overset{2}{\cancel{16}}\right)\right]$$

$$= 4\left[6 - \frac{16}{3}\right]$$

$$= 4\left(\frac{2}{3}\right) = \frac{8}{3}$$

25/08/2022 UNIT-5:

BETA & GAMMA FUNCTIONS:

Special Functions:

- Improper Integrals – introduction
- Beta function – Definition – properties with proofs
  - problems.
- Gamma function – Definition – properties with proofs
  - problems.
- Relation b/w beta and gamma function – derivation with proof.
- Error function : Definition
- Complementary error function – Definition.

* IMPROPER INTEGRALS:

Consider the integral $\int_a^b f(x)dx$.

Such an integral for which

(i) either the interval of integration is not finite i.e. $a = -\infty$ (or) $b = \infty$ (or) both

(ii) or the function $f(x)$ is unbounded at one or more points in $[a, b]$ is called an improper integral.

→ Definition of Beta function:

The definite integral $\int_{x=0}^{1} x^{m-1}(1-x)^{n-1}\,dx$ is said to be Beta function, denoted by $\beta(m,n)$.

i.e. $\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx;}$

(valid) converges for $m>0;\ n>0$

It is also known as Eulerian integral of the first kind

$$\boxed{\beta(m+2,n+3) = \int_0^1 x^{(m+2)-1}(1-x)^{(n+3)-1}\,dx}$$

* Properties of Beta function:

~~By~~

1) Property 1: Symmetric property.

** $\beta(m,n) = \beta(n,m)$.

Proof: By the def. of Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

put $x = 1-y$.

$y = 1-x$

$dx = dy$

when $x=0;\ y=1$

$x=1\ \ y=0$

$$\beta(m,n) = \int_1 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$y = x$

$dy = dx$

$$= \int_0^1 x^{m-1} (1-x)^{m-1} \, dx$$

$$\boxed{\beta(m,n) = \beta(n,m)}$$

2) Property 2:

P.T. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.

RHS: $\beta(m+1,n) + \beta(m,n+1)$.

By def of Beta function

$$= \int_0^1 x^{(m+1)-1} (1-x)^{n-1} dx + \int_0^1 x^{m-1} (1-x)^{(n+1)-1} dx$$

$$= \int_0^1 \left[x^m (1-x)^{n-1} + x^{m-1} (1-x)^n\right] dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \left[\cancel{x} + 1 - \cancel{x}\right] dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \beta(m,n) = \text{LHS}.$$

*

Property 3:

P.T $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(or) $$\boxed{\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)}$$

By def of Beta function $\beta(m,n)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta$

$dx = \sin 2\theta \, d\theta$

$x = 0 ; \theta = 0$

$x = 1 ; \theta = \pi/2$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \sin 2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1} \cos^{2n-2+1} d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \underline{RHS}$$

4) Property 4:

Show that $\beta(m,n) = \boxed{\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+m}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx}$

By the definition of β function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \frac{1}{1+y}$

$dx = \frac{-1}{(1+y)^2}dy$

$x = 0 ; y = \infty$

$x = 1 ; y = 0$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-dy}{(1+y)^2}\right)$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\left(\frac{y}{y+1}\right)^{n-1} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m-1+n-1+2}}$$

$$= \int_0^{\infty} \frac{y^{n-1}\,dy}{(1+y)^{m+n}} \Rightarrow y = x \quad = \int_0^{\infty} \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$$

By symmetric property

$$= \int_0^{\infty} \frac{x^{m-1}\,dx}{(1+x)^{m+n}}.$$

5) Property 5:

To show $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$

By property $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$

$$= \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

put $x = \frac{1}{y}$

$dx = -\frac{1}{y^2}\,dy.$

$x = 1;\ y = 1$

$x = \infty;\ y = 0$

$$= \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_1^0 \frac{\frac{1}{y^{m-1}}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \frac{-1}{y^2}\,dy$$

$$= \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}} \cdot \frac{1}{y^{m+1}}\,dy.$$

$$= \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

6) **Property 6:**

To show $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}$

RHS: $a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(\frac{ax}{b}+1\right)^{m+n}} dx$.

$\boxed{\frac{ax}{b} = y \quad x = \frac{by}{a} \quad dx = \frac{b}{a}dy}$

$= \frac{a^m \cancel{b^n}}{b^m \cdot \cancel{b^n}} \int_0^\infty \frac{\left(\frac{b}{a}\right)^{m-1} \cdot y^{m-1}}{(y+1)^{m+n}} \cdot \frac{b}{a} dy$

$= \frac{a^m}{b^m} \cdot \frac{\cancel{b^m}}{a^m} \cdot \frac{\cancel{a}}{\cancel{b}} \cdot \frac{\cancel{b}}{\cancel{a}} \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}} dy$

$= \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}} dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$= \beta(m,n)$

By: $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

7) **Property 7:**

S.T. $\int_b^a (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} \beta(m,n)$

By defn: ~~$\beta(m,n) = \int_b^a (x-b)^{m-1}(a-$~~

$\boxed{\text{put } x = \frac{t-b}{a-b}}$

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$x = 0$, $t = b$.

$x = 1$, $t = a$.

$dx = \frac{dt}{a-b}$.

$= \int_b^a \left(\frac{t-b}{a-b}\right)^{m-1} \left(1 - \frac{t-b}{a-b}\right)^{n-1} \frac{dt}{(a-b)}$

$= \int_b^a \left(\frac{t-b}{a-b}\right)^{m-1} \left(\frac{a-\cancel{b}-t+\cancel{b}}{a-b}\right)^{n-1} \frac{dt}{(a-b)}$

$= \int_b^a \frac{1}{(a-b)^{m+n-2+1}} (t-b)^{m-1}(a-t)^{n-1} dt.$

$$\beta(m,n) = \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx$$

---

* Express the following in terms of β function : $\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$

put $x^2 = y$.

$2x\,dx = dy$.

$$\int_0^1 \frac{1}{2} \frac{dy}{\sqrt{1-y}} = \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{+\frac{1}{2}-1}\,dy$$

$$= \frac{1}{2}\beta\left(1,\frac{1}{2}\right)$$

* $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}}$.

$x^5 = y$

$dx(5x^4) = dy$

$$\int_0^1 \frac{y^{2/5}\frac{dy}{5y^{4/5}}}{\sqrt{1-y}}.$$

$x=0 ; y=0$

$x=1 ; y=1$

$$= \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2}\,dy$$

$$m-1 = -\frac{2}{5} \qquad n-1 = -\frac{1}{2}$$

$$\boxed{m = \frac{3}{5}} \qquad n = \frac{1}{2}$$

$$\frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2}\,dy = \frac{1}{5}\,\beta\left(\frac{3}{5},\frac{1}{2}\right)$$

8) Property 8:

Show that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$ $\qquad \frac{x}{1+a} = \frac{t}{t+a}$

By def^n of Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{put } x = \frac{(1+a)\,t}{(t+a)}$$

$$t = \frac{x(t+a)}{1+a}$$

$$\int_0^1 \frac{(1+a)^{m-1}}{(t+a)^{m-1}}\,t^{m-1}\left(1-\frac{t+at}{t+a}\right)^{n-1} \frac{x}{1+a}\,dt \qquad dt = \frac{x}{1+a}\,dt$$

$$= \int_0^1 \frac{(1+a)^{m-1}}{(t+a)^{m-1}}\,t^{m-1}\left(\frac{a-at}{t+a}\right)^{n-1} \frac{x}{1+a}\,dt$$

$x = 0$, $t = 0$
$x = 1$:
$\cancel{t} + a = \cancel{t} + at$
$t = 1$

$$= \int_0^1 \frac{(1+a)^{m-1}}{(t+a)^{m+n-2}}\,t^{m-1}\,\frac{a^{n-1}(1-t)^{n-1}\;at}{(t+a)}\,dt$$

$$= \int_0^1 \frac{(1+a)^{m-1}}{(t+a)^{m+n-1}} \cdot \frac{t^{m-1}\,a^{n-1}(1-t)^{n-1}}{1}\,dt$$

$$= \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n-1}}$$

$$= \int_0^1 \frac{(1+a)^{m-1}}{(t+a)^{m-1}} t^{m-1} \frac{a^{n-1}(1-t)^{n-1}}{(t+a)^{n-1}} \cdot \frac{t}{(t+a)} dt$$

$$= \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}} \cdot \frac{a^n \cdot (a+1)^m}{(a+1)\, a} \cdot \frac{t(t+a)}{1} dt$$

$$* \int_0^3 \frac{dx}{\sqrt{9-x^2}}$$

put $x^2 = 9y$

$x = 3\sqrt{y}$

$dx = \frac{3}{2\sqrt{y}} dy$

$x = 0 ; y = 0$

$x = 3 ; y = 1$

$$\int_0^1 \frac{\frac{3}{2\sqrt{y}} dy}{3\sqrt{1-y}}$$

$$= \int_0^1 \frac{1}{2} (y)^{-1/2} (1-y)^{-1/2} dy$$

$$= \frac{1}{2} \int_0^1 (y)^{1-1/2} (1-y)^{1-\frac{1}{2}} dy$$

$$= \frac{1}{2} \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

H.W

1) $\int_0^1 x^m (1-x^n)^p \, dx \qquad x^n = y$

2) $\int_0^1 x^3 (1-\sqrt{x})^5 \, dx$

* $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta.$

By the property $\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$2m-1 = 5$ | $2n-1 = 7/2$

$2m = 6$ | $2n = \frac{9}{2}$

$m = 3$ | $n = 9/4$

$\frac{1}{2}\beta(m,n) = \frac{1}{2}\left(3, \frac{9}{4}\right)$

* $\int_0^1 x^m (1-x^n)^p \, dx \qquad$ Let $x^n = y \qquad x = y^{1/n}$

$n x^{n-1} dx = dy$

$dx = \frac{dy}{n}$

$\int_0^1 y^{m/n} (1-y)^p \frac{dy}{n \cancel{x} y^{\frac{n-1}{n}}}$

$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p \, dy \qquad \frac{m-n+1}{n} = x-1$

$x = \frac{m-n+1}{n} + 1$

$= \boxed{\frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)} \qquad = \frac{m+1}{n}$

* $\int_0^1 x^3 (1-\sqrt{x})^5 \, dx. \qquad \sqrt{x} = y$

$x = y^2$

$dx = 2y\,dy$

$\int_0^1 y^6 (1-y)^5 \, \cancel{dx}\, 2y\,dy = 2\int_0^1 y^7 (1-y)^5$

$$\ast\rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx = \frac{\beta(m,n)}{a^n(1+a)^m}.$$

By defn of 'β' function

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad x = \frac{(1+a)t}{(t+a)}$$

$$= \int_0^1 \frac{(1+a)^{m-1} \cdot t^{m-1}}{(t+a)^{m-1}}\left(1 - \frac{\cancel{t}+at}{t+a}\right)^{n-1} \frac{a(1+a)}{(t+a)^2}dt. \qquad dx = \frac{a(1+a)}{(t+a)^2}$$

$$= \int_0^1 \frac{(1+a)^m \cdot t^{m-1}}{(t+a)^{m+1}}\left(\frac{\cancel{t}+a-\cancel{t}-at}{t+a}\right)^{n-1} a\,dt$$

$$= \int_0^1 \frac{(1+a)^m \cdot t^{m-1} \cdot a^{n-1+1} \cdot (1-t)^{n-1}}{(t+a)^{m+1}(t+a)^{n-1}}dt.$$

$$= \int_0^1 \frac{(1+a)^m a^n t^{m-1} \cdot (1-t)^{n-1}}{(t+a)^{m+n}}.$$

$$= (a+1)^m a^n \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}dt$$

$$\frac{\beta(m,n)}{(a+1)^m a^n} = \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}dt$$

$$\frac{\beta(m,n)}{(a+1)^m a^n} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx.$$

26/08/2022

P.T. $\int_5^7 (x-5)^6 (7-x)^3 dx = 2^{10} \beta(7,4)$

By property

$$\int_b^a (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} \beta(m,n)$$

on comparison:

$a = 7 \; ; \; b = 5$ $\quad m-1 = 6 \Rightarrow m = 7$

$n-1 = 3 \Rightarrow n = 4$

$$= 2^{7+4-1} \beta(7,4)$$

$$= 2^{10} \beta(7,4).$$

* Gamma Function :

~~The~~ The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is said to be a gamma function and it is denoted by $\Gamma(n)$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \; n > 0}$$

The integral $\int_0^\infty e^{-x} x^{n-1} dx$ does not converge for $n < 0$

The other name of Gamma function is Eulerian integral of 2nd kind.

* Properties of Gamma function :

Property 1 :

Show that $\Gamma(1) = 1$

By def^n of gamma function

$$\int_0^\infty e^{-x} x^{n-1} \, dx \quad n > 0.$$

put $n = 1$

$$= \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx$$

$$= \cancel{(-e^{-x}(\infty - 0))}$$

$$= -\left[e^{-x}\right]_0^\infty$$

$$= -\left(e^{-\infty} - e^{-0}\right)$$

$$= -(-1) = 1$$

$$\boxed{\Gamma(1) = 1}$$

Property 2 :

$\Gamma(n) = (n-1)\Gamma(n-1)$ where $n > 1$

By def^n of gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$= \left[x^{n-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (n-1)(x^{n-2})(-e^{-x}) \, dx$$

$$\cancel{= x^{n-1}(1) - (n-1)\int_0^\infty x^{n-2}(1) \, dx}$$

$= \cancel{x^{n-1}} \cancel{-(n-1)}$

$$= (0-0) + (n-1)\int_0^\infty e^{-x} x^{(n-1)-1}\, dx$$

$$= (n-1)\Gamma(n-1)$$

* **Note:**

A: $\boxed{\Gamma(n+1) = n\,\Gamma(n)}$

put $n = n+1$ in the above property

* **Note:**

1) $\Gamma(n)$ is defined when $n > 0$ & +ve
2) $\Gamma(n)$ is defined when n is -ve fraction.
3) But $\Gamma(n)$ is not defined when $n = 0$ and n is -ve integer.
4) If n is a +ve integer; $\boxed{\Gamma(n) = (n-1)!}$
5) If n is a +ve fraction; $\boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$
6) If n is a -ve fraction; $\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$

* Compute $\Gamma(7) = \underline{6!}$

$$\Gamma(9/2) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(-1/2) = \frac{\Gamma +1/2}{-1/2} = -2\,\Gamma 1/2 = \underline{-2\sqrt{\pi}}$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-1/2\right)}{-3/2} = \frac{-2}{3}\left(-2\sqrt{\pi}\right) = \frac{4\sqrt{\pi}}{3}$$

$$\Gamma\left(-7/2\right) = \frac{\Gamma\left(-5/2\right)}{-7/2} = \frac{-2}{7}\,\frac{\Gamma\left(-3/2\right)}{-5/2} = \frac{4}{35}\left(\frac{4\sqrt{\pi}}{3}\right) = \frac{16}{105}\sqrt{\pi}$$

$= \cancel{x^{n-1}} \cancel{-(n-1)}$

$$= (0-0) + (n-1) \int_0^\infty e^{-x} x^{(n-1)-1} dx$$

$$= (n-1)\Gamma(n-1) //$$

* Note:

$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$ put $n = n+1$ in the above property

* Note:

1) $\Gamma(n)$ is defined when $n > 0$ & +ve
2) $\Gamma(n)$ is defined when n is -ve fraction.
3) But $\Gamma(n)$ is not defined when $n = 0$ and n is -ve integer.
4) If n is a +ve integer; $\boxed{\Gamma(n) = (n-1)!}$
5) If n is a +ve fraction; $\boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$
6) If n is a -ve fraction; $\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$

* Compute $\Gamma(7) = 6!$

$$\Gamma(9/2) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}$$

$$\Gamma(-1/2) = \frac{\Gamma +1/2}{-1/2} = -2\,\Gamma 1/2 = -2\sqrt{\pi}$$

$$\Gamma\tfrac{-3}{2} = \frac{\Gamma -1/2}{-3/2} = \frac{-2}{3}(-2\sqrt{\pi})$$

$$= \frac{4\sqrt{\pi}}{3}$$

$$\Gamma -7/2 = \frac{\Gamma -5/2}{-7/2} = \frac{-2}{7} \frac{\Gamma -3/2}{-5/2}$$

$$= \frac{4}{35}\left(\frac{4\sqrt{\pi}}{3}\right) = \frac{16}{105}\sqrt{\pi}$$

27/08/2022

* Relation between β and Γ functions:

Show that $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

Proof: By the defn of Gamma function we have $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$

put $x = yt$, $dx = y\,dt$

$x = 0; t = 0$

$x = \infty; t = \infty$

$$\Gamma m = \int_0^\infty e^{-yt} y^{m-1} t^{m-1} y\,dt$$

$$\Gamma m = y^m \int_0^\infty e^{-yt} t^{m-1} dt$$

$$\boxed{\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yt} t^{m-1} dt} \quad -①$$

Multiplying ① by $e^{-y} y^{m+n-1}$ on both sides

$$\frac{\Gamma m}{y^m} \cdot e^{-y} \cdot y^{m+n-1} = \int_{t=0}^\infty e^{-yt} \cdot t^{m-1} \cdot e^{-y} \cdot y^{m+n-1} dt$$

$$\Gamma m \, e^{-y} y^{n-1} = \int_0^\infty e^{-y(1+t)} t^{m-1} y^{m+n-1} dt$$

Applying integration wrt 'y' taking limits from 0 to ∞.

$$\Gamma m \int_{y=0}^{\infty} e^{-y} y^{n-1} dy = \int_{t=0}^{\infty} \underbrace{e^{-y(1+t)} \cdot y^{m+n-1} dy}_{} \int_{t=0}^{\infty} t^{m-1} dt$$

$$\Gamma m \, \Gamma n = \int_{t=0}^{\infty} \frac{\Gamma(m+n)}{(1+t)^{m+n}} t^{m-1} dt$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\boxed{\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(m,n)}$$

*Show that $\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$

We have

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{—} (1)$$

put $m = n = \frac{1}{2}$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma \tfrac{1}{2} \cdot \Gamma \tfrac{1}{2}}{\Gamma \tfrac{1}{2} + \Gamma \tfrac{1}{2}}$$

$\Gamma(2) = (n-1)\Gamma(n-1)$

$\Gamma(2) = 1\,\Gamma(1)$

$\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$\boxed{\Gamma(n+1) = n!}$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{(\Gamma \tfrac{1}{2})^2}{\Gamma 1} = \frac{(\Gamma \tfrac{1}{2})^2}{1}$$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = (\Gamma \tfrac{1}{2})^2 \quad \text{—} (2)$$

LHS:

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \, \cos^0\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2\left(\tfrac{\pi}{2} - 0\right) = \pi$$

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \pi$$

$$(\Gamma \tfrac{1}{2})^2 = \pi$$

$$\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$$

$$\Gamma -\tfrac{1}{2} = \frac{\Gamma n+1}{n} = \frac{\Gamma \tfrac{1}{2}}{-\tfrac{1}{2}} = -2\sqrt{\pi}$$

$$\Gamma -\tfrac{3}{2} = \frac{\Gamma -\tfrac{1}{2}}{-\tfrac{3}{2}} = \frac{-2}{3}(-2\sqrt{\pi}) = \frac{4\sqrt{\pi}}{3}$$

29/08/2022

$$\boxed{\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}} \quad n>0$$

By relation b/w β & Γ

$$\boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

* $\int_0^1 x^5(1-x)^3\,dx$.

Comparing with defn of beta function.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$m-1=5 \Rightarrow m=6$

$n-1=3 \Rightarrow n=4$

$= \beta(6,4)$

$$= \frac{\Gamma 6\,\Gamma 4}{\Gamma 10} = \frac{5!\,3!}{9!} = \frac{1}{504}$$

* $\int_0^2 x(8-x^3)^{1/3}\,dx$ put $x^3=8y$

$x = 2y^{1/3}$

$dx = \frac{2}{3}y^{-2/3}dy$.

$x=0$ $y=0$; $x=2$ $y=1$

$\int_0^1 2y^{1/3}(8)^{1/3}(1-y)^{1/3}dx$

$= \int_0^1 4\,y^{1/3}(1-y)^{1/3}\cdot\frac{2}{3}y^{-2/3}dy$

$= \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\,dy$

$\boxed{\because \Gamma n = (n-1)\Gamma(n-1)}$

$m-1 = -1/3$

$m = \frac{2}{3}$ $\quad n-1=\frac{1}{3}$

$n = \frac{4}{3}$

$$\frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right) = \left(\frac{\Gamma 2/3\cdot\Gamma 4/3}{\Gamma 2}\right)\times\frac{8}{3} \qquad \Gamma 2 = 1!$$

$$= \frac{8}{3}\cdot\frac{\Gamma 2/3\cdot\frac{1}{3}\,\Gamma 1/3}{1}$$

$\boxed{\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$

$$= \frac{8}{9}\,\Gamma\tfrac{1}{3}\,\Gamma 1-\tfrac{1}{3}$$

$$= \frac{8}{9}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{8\pi}{9}\cdot\frac{2}{\sqrt3}$$

$$= \frac{16\pi}{9\sqrt3}$$

* Evaluate $\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx$

$x^2 = y$

$2x\,dx = dy$

$x=0;\ y=0$

$x=1;\ y=1$

$\int_0^1 y^{5/4}(1-y)^{3/2}\cdot\frac{dy}{2y^{1/2}}$

$= \frac{1}{2}\int_0^1 y^{3/4}(1-y)^{3/2}\,dy$.

$m-1 = \frac{3}{4}$ $\quad n-1 = \frac{3}{2}$

$m = \frac{7}{4}$ $\quad n = \frac{5}{2}$

$$= \beta\left(\frac{7}{4}, \frac{5}{2}\right) = \frac{1}{2} \frac{\Gamma 7/4 \cdot \Gamma 5/2}{\Gamma 17/4}$$

$$= \frac{\frac{3}{\cancel{4}} \Gamma 3/4 \cdot \frac{3}{2} \Gamma \frac{3}{2}}{\frac{13}{\cancel{4}} \Gamma \frac{13}{4}}$$

$$= \frac{9}{26}\left[\frac{\Gamma 3/4 \quad \frac{1}{2}\Gamma 1/2}{\frac{9}{4} \Gamma 9/4}\right] \times \frac{1}{2}$$

$$= \frac{\cancel{9}}{\cancel{52}\,13} \times \frac{4}{\cancel{9}} \sqrt{\pi} \left[\frac{\Gamma 3/4}{\frac{5}{4} \Gamma 5/4}\right] \frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{13} \times \frac{4}{5} \frac{\Gamma 3/4}{\frac{1}{4} \Gamma 1/4} \times \frac{1}{2}$$

$$= \cancel{\frac{16\sqrt{\pi}}{65}} \quad \cancel{\frac{-1/4\ \Gamma -1/4}{-3/4\ \Gamma -3/4}}$$

$$= \frac{16\sqrt{\pi}}{65} \frac{\Gamma 3/4}{\Gamma 1/4} \times \frac{1}{2}$$

$$\boxed{= \frac{8\sqrt{\pi}}{65} \frac{\Gamma 3/4}{\Gamma 1/4}.}$$

* RESULTS:

Result 1: Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

By defn of Gamma function

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad -(1)$$

put $n = \frac{1}{2}$ in eqn (1)

$$\Gamma 1/2 = \int_0^\infty e^{-x} x^{-1/2} dx \quad -(2)$$

$$= \int_0^\infty e^{-t^2} (t^2)^{-1/2} 2t\,dt$$

$$\Gamma 1/2 = 2\int_0^\infty e^{-t^2} dt.$$

$$\sqrt{\pi} = 2 \int_0^\infty e^{-t^2} dt.$$

$$\therefore \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

treating 't' as dummy variable.

$$\boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

Result 2: $\int_{-\infty}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Let $x = -t$ so that $dx = -dt$.

$\therefore \int_{-\infty}^{0} e^{-x^2} dx = \int_{\infty}^{0} e^{-t^2}(-dt)$. $x=0 \; t=0$, $x=-\infty \; t=\infty$

$= \int_0^\infty e^{-t^2} dt = \boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$

Result : 3: S.T. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

$= 2\int_0^\infty e^{-x^2} dx$ $\because f(x)$ is even, $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$= \cancel{2} \frac{\sqrt{\pi}}{\cancel{2}}$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$

* Show that $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$ $n>0$.

~~$x = \frac{1}{\log y}$~~   put $x = \log\left(\frac{1}{y}\right)$

$x = -\log y$

$dx = \frac{-1}{y} dy$

$x=0 ; y=1$

$x=\infty ; y=0$

By def^n $\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx.}$

$\Gamma n = \int_1^0 e^{+\log y}(-\log y)^{n-1} \frac{-1}{y} dy$

$\Rightarrow \Gamma n = \int_0^1 (-\log y)^{n-1} dy$

$$\boxed{\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx}$$

* $\int_0^\infty x^6 e^{-2x} dx$ $2x = t$, $dx = \frac{dt}{2}$

$\int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t} \frac{dt}{2}$

$= \frac{1}{2^7}\int_0^\infty t^6 e^{-t} dt = \frac{1}{2^7}\int_0^\infty e^{-t} \cdot t^{7-1} dt$

$= \frac{1}{2^7} \Gamma 7 = \frac{6!}{2^7}$.

* $\int_0^\infty e^{-4x} x^{3/2} dx$. $4x = y$, $dx = \frac{dy}{4}$

$\int_0^\infty e^{-y} \left(\frac{y}{4}\right)^{3/2} \frac{dy}{4}$

$= \int_0^\infty e^{-y} \frac{y^{3/2} dy}{4^{5/2}} = \frac{1}{4^{5/2}} \int_0^\infty e^{-y} y^{5/2-1} dy$

$= \frac{\Gamma 5/2}{4^{5/2}} = \frac{\frac{3}{4}\sqrt{\pi}}{2^5} = \frac{3\sqrt{\pi}}{128}$

* $\int_0^\infty x^2 \cdot e^{-x^2} dx$. $x^2 = t$, $2x\,dx = dt$, $dx = \frac{dt}{2\sqrt{t}}$.

$\int_0^\infty t \cdot e^{-t} \frac{dt}{2\sqrt{t}}$

$=\frac{1}{2}\int_0^\infty e^{-t}\, t^{1/2}\, dt$

$=\frac{1}{2}\,\Gamma 3/2 = \boxed{\frac{1}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}}{4}}$

* $\int_0^\infty e^{-a^2x^2}dx$

$a^2x^2 = y$

$x^2 = \frac{y}{a^2}$

$2x\,dx = \frac{dy}{a^2}$

$dx = \frac{dy\; a}{2a^2\sqrt{y}} = \frac{dy}{2a\sqrt{y}}$

$\int_0^\infty e^{-y}\,\frac{dy}{2a\sqrt{y}}$

$\frac{1}{2a}\int_0^\infty e^{-y}\,y^{-1/2}\,dy$

$=\frac{1}{2a}\,\Gamma 1/2 = \frac{\sqrt{\pi}}{2a}$

* $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$

$\int_0^{\pi/2}\cos^{1/2}\theta\,\sin^{-1/2}\theta\,d\theta$

$$\boxed{\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{1}{2}\beta(m,n)}$$

$=\int_0^{\pi/2} = \frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)$

$2m-1 = -\frac{1}{2}$, $m = \frac{1}{4}$ | $2n-1 = \frac{1}{2}$, $n = \frac{3}{4}$

$=\frac{1}{2}\,\frac{\Gamma 1/4\;\Gamma 3/4}{\Gamma 1}$

$=\frac{1}{2}\,\Gamma 1/4\;\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\,\frac{\pi}{\sin \pi/4} = \frac{\pi}{\sqrt{2}}$

*Evaluate: $\int_0^\infty 3^{-4x^2}dx$

$3 = e^{\log 3}$

$2x\sqrt{\log 3} = t$

$\int_0^\infty e^{-4x^2\log 3}\,dx$

$x = \frac{t}{2\sqrt{\log 3}}$

$\int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{\log 3}}$

$dx = \frac{dt}{2\sqrt{\log 3}}$

$=\frac{1}{2\sqrt{\log 3}}\int_0^\infty e^{-t}\,dt = \frac{1}{2\sqrt{\log 3}}\,\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

* $\int_0^\infty a^{-bx^2}dx$

$t = (b\log a)\,x^2$

$dt = 2xb\log a\cdot dx$

$dx = \frac{dt}{2b\log a}\sqrt{\frac{b\log a}{t}}$

$=\frac{1}{2\sqrt{b\log a}}\cdot\int_0^\infty e^{-t}\,t^{-1/2}\,dt$

$=\frac{1}{2\sqrt{b\log a}}\int_0^\infty e^{-t}\,t^{1/2-1}\,dt$

$=\frac{1}{2\sqrt{b\log a}}\,\Gamma 1/2 = \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$

*Show that $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx = \frac{6}{625}$

$\log\frac{1}{x} = y$

$\frac{1}{x} = e^{y}$

$x = e^{-y} \qquad dx = -e^{-y}dy$

$= \int_{\infty}^{0} e^{-4y}\, y^{3} - e^{-y} dy$

$= \int_{0}^{\infty} e^{-5y}\, y^{3} dy \qquad 5y = t, \quad dy = \frac{dt}{5}$

$= \int_{0}^{\infty} e^{-t} \left(\frac{t}{5}\right)^{3} \frac{dt}{5} \qquad n-1 = 3, \quad 4$

$= \frac{1}{625} \int_{0}^{\infty} e^{-t} t^{3} dt = \frac{\Gamma 4}{625}$

$= \frac{3!}{625} = \frac{6}{625}$

* $\int_{0}^{\infty} e^{-y^{1/m}} dy \qquad y^{1/m} = t$

$\int_{0}^{\infty} e^{-t} m t^{m-1} dt \qquad \log y = m \log t$

$= m \int_{0}^{\infty} e^{-t} t^{m-1} dt \qquad \frac{dy}{y} = \frac{m\, dt}{t}, \quad dy = m t^{m-1} dt$

$= m \Gamma m$

* $\int_{0}^{\pi/2} \sin^{2}\theta \cos^{4}\theta \, d\theta = \frac{1}{2} \beta\left(\frac{3}{2}, \frac{5}{2}\right) \qquad 2m-1 = 2, \quad 2n-1 = 4$

$= \frac{\Gamma 3/2\, \Gamma 5/2}{\Gamma 4} = \frac{1}{2} \, \frac{\frac{3}{2}\cdot\frac{1}{2}\pi}{3!}$

$= \frac{1}{2}\, \frac{3}{8 \times 6} \pi = \frac{\pi}{32}$

* H.W

$\int_{0}^{\infty} e^{-x^{2}} x^{7/2} dx$ using Γ function

$\int_{0}^{\infty} \frac{x\, dx}{(1+x^{6})}$ using β-Γ fn.

3) $\int_{0}^{\infty} \frac{x^{c}}{c^{x}} dx \ (c>1)$ in terms of Γ.

$\because c = e^{\log c}$

4) $\int_{0}^{\infty} \frac{x^{c}}{c^{x}} dx = \int_{0}^{\infty} \frac{x^{c}}{e^{x \log c}} = \int_{0}^{\infty} e^{-x \log c} x^{c} dx$

$= \int_{0}^{\infty} e^{-\log c \cdot x} \cdot x^{(c+1)-1} dx$

$= \frac{\Gamma(c+1)}{(\log c)^{c+1}} \qquad \left(\because \int_{0}^{\infty} x^{n-1} e^{-kx} dx = \frac{\Gamma(n)}{k^{n}}\right)$

* $\int_0^\infty e^{-x^2} x^{7/2}\, dx$

$x^2 = y \Rightarrow 2x\,dx = dy$

$dx = \frac{dy}{2\sqrt{y}}$

$\int_0^\infty e^{-y}\, y^{7/4}\, dx$ $\quad n-1 = \frac{7}{4}$

$= \Gamma 11/4 = \frac{7}{4}\Gamma 7/4$

$= \frac{3}{4}\Gamma 3/4$

* $\int_0^\infty \frac{x\,dx}{(1+x^6)} = \cancel{\int_0^\infty x(1+x^6)^{-1}\, dx}$

$x^6 = t$

$6x^5 dx = dt$

$= \int_0^\infty \frac{t^{1/6}\, dt}{(1+t)\, 6\, t^{5/6}}$ $\quad m-1 = -\frac{2}{3}$

$m+n = 1$

$= \frac{1}{6}\int_0^\infty \frac{t^{-2/3}\, dt}{(1+t)} = \beta\left(\frac{1}{3}, \frac{2}{3}\right)$

$= \frac{\Gamma 1/3\ \Gamma 2/3}{\Gamma 1} = \frac{\Gamma 1/3\ \Gamma 1-\frac{1}{3}}{1}$

$= \frac{\pi}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

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* $\int_0^\infty c^{-x} x^c\, dx$ $\quad c = e^{\log c}$

$= \int_0^\infty e^{-x\log c}\, x^c\, dx$ $\quad x\log c = t$

$\log c\, dx = dt$

$= \int_0^\infty e^{-t}\, \frac{t^c}{(\log c)^c} \cdot \frac{dt}{\log c}$

$= \frac{\int_0^\infty e^{-t} t^c\, dt}{(\log c)^{c+1}} = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

01/09/2022

INTRODUCTION:

Error function: Error function arise in the theory of probability and solution of certain type of partial differential equations.

DEVC K.SIVANI

* Error function: $erf(x)$: The error function is also called ~~error~~ integral function. It is defined by

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt}$$

Let $t^2 = u \Rightarrow 2t\,dt = du$.

$$dt = \frac{1}{2\sqrt{u}}\,du.$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} e^{-u} \cdot \frac{1}{2\sqrt{u}}\,du.$$

$$\boxed{erf(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} u^{-1/2}\, e^{-u}}$$

$$erf(\infty) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \cdot u^{-1/2}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = 1$$

* Complementary Error Function: $(erfc(x))$:

Using the definition of error function; we write $erf(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-u} u^{-1/2} du$.

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty u^{-1/2} e^{-u} du - \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty u^{-1/2} e^{-u} du$$

$$= 1 - \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty u^{-1/2} e^{-u} du.$$

$$\boxed{erf(x) = 1 - erfc(x)}$$

where we define

$$\boxed{erfc(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty u^{-1/2} e^{-u} du}$$

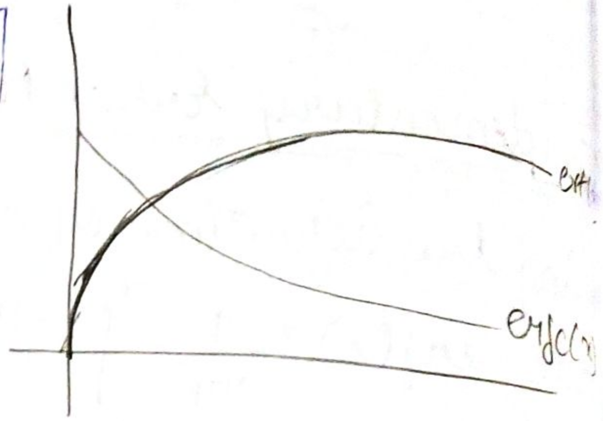
This function $erfc(x)$ is called complementary error function.

$$erf\,c(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

$$\boxed{erfc = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.}$$

* **Properties:**

1) $\boxed{erf(-x) = -erf(x)}$



using definition

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt$$

Let $t = -u$

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}(-du) = -erf(x)$$

2) $\boxed{erfc(-x) = 1 + erf(x) = 2 - erfc(x).}$

$$erfc(-x) = 1 - erf(-x)$$

$$= 1 + (1 - erfc(x))$$

$$\boxed{erfc(-x) = 2 - erfc(x)}$$