10/01/2021

UNIT 2

MULTI-VARIABLE CALCULUS

(FUNCTIONS OF SEVERAL VARIABLES)

TOPICS:

1) Limits and Continuity definition problems
2) Partial derivatives $\begin{cases} \to 1^{st} \text{ order} \\ \to \text{higher order.} \end{cases}$
3) Taylor's Series and Maclaurin's Series (with 2 variables)
   — Statements & Problems.
4) Maxima & Minima in 2 variables
   - without constraints
   - with constraints (Lagrange's Method)

*LIMITS:

A function $f(x,y)$ is said to tend to limit 'l', if corresponding to a +ve number 'ϵ' then there exists another +ve number 'δ' such that

$$\boxed{|f(x,y) - l| < \epsilon} \quad \text{when } |x-a|<\delta \ \& \ |x-b|<\delta$$

(or) $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = l \text{ (finite \& unique)}}$

*CONTINUITY:

A function $f(x,y)$ is said to be continuous at a,b if corresponding to a +ve number ϵ $\exists$ another +ve number 'δ' such that

$$\boxed{|f(x,y) - f(a,b)| < \epsilon} \quad \text{when } |x-a|<\delta \ \& \ |y-b|<\delta$$

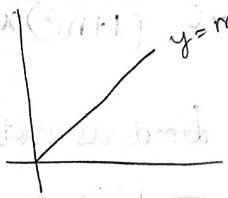
(or) $\boxed{\lim_{\substack{x\to a \\ y\to b}} f(x,y) = f(a,b)}$

~~*PARTIAL DIFFERENTIAL~~

*PROBLEMS

• Note: whenever given point is (0,0) (along path)

* we can use the path $y=mx$ / $y=m^2x^2$ / $y=mx^2$



$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = 0$: Existence of limit / continuity

$= \infty$: Limit / continuity does not exist.

in terms of $= m$: limit is not unique so limit / continuity does not exist

* Problems on ...

Q: $\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2+y^2}\right)$

$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{xy}{x^2+y^2}\right)\right) = 0$

$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{xy}{x^2+y^2}\right)\right) = 0$

$\cancel{\lim_{x\to 0}\left[\lim_{y\to mx} x(mx)\,y\right]}$

$\lim_{x\to 0}\left(\lim_{y\to mx} \frac{x\,\cancel{(mx)}\,y}{x^2+y^2}\right)$

$\lim_{x\to 0} \frac{x(mx)}{(1+m^2)x^2} = \frac{m}{1+m^2}$

limit is not unique as 'm' changes

⇒ limit does not exist.

$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = l_1$

$\lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right] = l_2$

① = ②

limit exists

Q: Determine $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x-y}$ if it exists

$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x^2+y^2}{x-y}\right)\right) = 0$

$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{x^2+y^2}{x-y}\right)\right) = 0$

$\lim_{x\to 0}\left(\lim_{y\to mx}\left(\frac{x^2+m^2x^2}{x-mx}\right)\right)$

$\lim_{x\to 0}\left(\frac{x^2(1+m^2)}{x(1-m)}\right) = 0$

Hence Limit exists.

Q: Does $\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y^2}$ exists?

$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x+\sqrt{y}}{x^2+y^2}\right)\right) = \infty$

$\lim_{y\to 0}\left(\frac{1}{y^{3/2}}\right) = \infty$

$\lim_{x\to 0}\left(\lim_{y\to m^2x^2}\left(\frac{x+\sqrt{y}}{x^2+y^2}\right)\right) =$

$= \lim_{x\to 0} \frac{x+mx}{x^2+m^4x^4} = \lim_{x\to 0} \frac{x(1+m)}{x^2(1+m^4x^2)} = \infty$

Hence, limit does not exist.

Q: Show that $\lim_{(x,y)\to(0,0)} \left(\frac{x^3y}{x^6+y^2}\right)$ does not exist

$\cancel{\lim_{x\to 0}\left(\lim_{y\to 0} \frac{1}{x^3}\right) = \infty}$

$\lim_{y\to 0}\left(\frac{1}{y}\right) = \infty$

$\cancel{\lim_{x\to 0}\left(\lim_{y\to mx^3}\left(\frac{x^3(mx)}{x^6+m^2x^2}\right)\right) = \lim_{x\to 0} \frac{mx^4}{x^2(m^2x^4)}}$

$= \lim_{x\to 0}$

* $$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^6+y^2}\right)=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3y}{x^6+y^2}\right)=0$$

$$\lim_{x\to 0}\left(\lim_{y\to mx^3}\frac{x^3 y}{x^6+y^2}\right)$$

$$\lim_{x\to 0}\frac{x^3 . mx^3}{x^6+m^2x^6} = \frac{m}{1+m^2}$$

Q: $$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x-y}{2x+y}$$

$$\left.\begin{aligned}&\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{2x+y}\right)=\frac{1}{2}\\ &\lim_{y\to 0}\left(\cancel{\lim_{x\to 0}}(-1)\right)=-1\end{aligned}\right\}\Rightarrow \text{limit does not exist}$$

$$\lim_{x\to 0}\left(\lim_{y\to mx}\left(\frac{x-y}{2x+y}\right)\right)$$

$$\lim_{x\to 0}\left(\frac{x(1-m)}{(2x+mx)}\right)=\frac{1-m}{2+m}$$ as 'm' is arbitrary ∴ so limit is not unique.

limit does not exist.

Q: $$\lim_{(x,y)\to(1,1)}\frac{x(y-1)}{y(x-1)}$$

$$\lim_{x\to 1}\left\{\lim_{y\to 1}\frac{x(y-1)}{(x-1)y}\right\}=\lim_{x\to 1}\frac{x(0)}{x-1}=0$$

$$\lim_{y\to 1}\left(\lim_{x\to 1}\frac{x(y-1)}{y(x-1)}\right)=\infty$$

limit does not exist

11/01/2021

* Investigate the continuity of the function $f(x,y)$ at $(0,0)$ where

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Let us consider the limit of the function for testing the continuity along the path $y=mx$

$$\lim_{\substack{x\to 0\\ y\to mx}} f(x,y)=\lim_{\substack{y\to mx\to 0\\ x\to 0}}\frac{xy}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x^2 m}{x^2(1+m^2)}=\frac{m}{1+m^2}$$

limit is not unique (∵ m is arbitrary).

$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist along path

* Examine for continuity at origin of the function defined by.

$$f(x,y)=\frac{x^2}{\sqrt{x^2+y^2}} \quad \text{for } x\neq 0;\ y\neq 0$$

$$=0 \quad \text{for } x=0,\ y=0$$

$$\lim_{x\to 0}\frac{x^2}{\sqrt{x^2(1+m^2)}}=\lim_{x\to 0}\frac{x}{\sqrt{1+m^2}}=0 \Rightarrow \text{limit exists}$$ Continuity exists.

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{\sqrt{x^2+y^2}}\right)=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+y^2}}\right)=0$$

Limit exists ; continuous at (0,0) / origin.

* Show that the function $f(x)=\begin{cases}\frac{x^2+y^2}{x-y} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

is continuous at (0,0).

$$\lim_{x\to 0}\frac{x^2(1+m^2)}{x(1-m)}=0 \rightarrow \text{limit exists}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2+y^2}{x-y}\right)=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2+y^2}{x-y}\right)=0$$

$\Rightarrow$ limit exists function is continuous

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* Discuss the continuity of the function

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\lim_{x\to 0}\frac{2x^2m}{x^2(1+m^2)}=\frac{2m}{1+m^2}$$

$\Rightarrow$ function is not continuous as limit does not exist

* $$\lim_{\substack{x\to 1\\ y\to 2}}\frac{2x^2y}{x^2+y^2+1}$$

$$\lim_{x\to 1}\left(\lim_{y\to 2}\frac{2x^2y}{x^2+y^2+1}\right)=\lim_{x\to 1}\frac{4x^2}{x^2+5}=\frac{4}{6}=\frac{2}{3}$$

$$\lim_{y\to 2}\left(\lim_{x\to 1}\frac{2x^2y}{x^2+y^2+1}\right)=\lim_{y\to 2}\frac{2y}{y^2+2}=\frac{4}{6}=\frac{2}{3}$$

$\Rightarrow$ limit exists

function is continuous.

PARTIAL DIFFERENTIATION: (∂)

Differentiation of dependant variable w.r.to more than one independent variables is said to be ~~denoted~~ partial differentiation

Let $z=f(x,y)$

z : D.V x, y : I.V

$(I.V)^1 = 1^{st}$ order

~~$(\partial)^1=1$~~

$(I.V)^2 = 2^{nd}$ order

$(I.V)^n = n^{th}$ order

Consider $z = f(x,y)$ be a function of 2 independent variables of x & y.

$$z = f(x, y) \quad \text{— (1)}$$

1st order partial derivatives:

Diff. (1) partially wrt 'x' (treating 'y' as const) on both sides

$$p = \frac{\partial z}{\partial x} (z_x) = \frac{\partial f}{\partial x} (f_x)$$

Diff (2) partially wrt 'y' (treating 'x' as const) on both sides

$$q = \frac{\partial z}{\partial y} (z_y) = \frac{\partial f}{\partial y} (f_y$$

* Higher order partial derivatives: $\Rightarrow z = f(x,y)$

$$p = \frac{\partial z}{\partial x} \quad \text{— (2)}$$

$$q = \frac{\partial z}{\partial y} \quad \text{— (3)}$$

2nd order partial derivatives:

Diff. (2) partially wrt 'x' on both side ($y \to$ const)

$$r = \frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2} (z_{xx})$$

$$s = \frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} (z_{yx}) = \frac{\partial}{\partial y}(f_x) \quad (x \to \text{const})$$

Diff (3) partially wrt 'x' on both sides ($y \to$ const)

$$s = \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} (z_{xy}) = \frac{\partial}{\partial y}(f_y)$$

$$\boxed{z_{yx} = z_{xy}}$$

Diff (3) partially wrt 'y' on both sides ($x \to$ const)

$$t = \frac{\partial q}{\partial y} = \frac{\partial^2 z}{\partial y^2} (z_{yy})$$

$w = f(u, v)$.

$$p = \frac{\partial w}{\partial u} \Big/ \frac{\partial f}{\partial u}$$

$$q = \frac{\partial w}{\partial v} \Big| \frac{\partial f}{\partial v}$$

$$r = \frac{\partial^2 w}{\partial u^2} \Big| \frac{\partial^2 f}{\partial u^2}$$

$$s = \frac{\partial^2 w}{\partial u \partial v} = \frac{\partial q}{\partial u} = \frac{\partial p}{\partial v}$$

$$t = \frac{\partial^2 w}{\partial v^2} = \frac{\partial^2 f}{\partial v^2}$$

12/01/2022

Using first principle formula of PDs:

Note: $f(x,y) = \begin{cases} g(x,y) ; (x,y) \neq (0,0) \\ 0 \quad (x,y) = (0,0) \end{cases}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(or)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We always use 1st principle formula

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Formulae of 1^{st} & 2^{nd} order PD by using 1^{st} principle:

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = p$$

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial x} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = q$$

II^{nd} order P.D's of $z = f(x,y)$.

1) $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial (f_x)}{\partial x} = \lim_{\Delta x \to 0} \frac{f_x(x+\Delta x, y) - f_x(x,y)}{\Delta x} = r$$

2) $$\frac{\partial^2 z}{\partial x \partial y} = \lim_{\Delta x \to 0} \frac{f_y(x+\Delta x, y) - f_y(x,y)}{\Delta x} = s$$
$$= \frac{\partial}{\partial x}(f_y)$$

3) $$\frac{\partial^2 z}{\partial y \partial x} = \lim_{\Delta y \to 0} \frac{f_x(x, y+\Delta y) - f_x(x,y)}{\Delta y} = s.$$
$$= \frac{\partial}{\partial y}(f_x)$$

4) $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = \lim_{\Delta y \to 0} \frac{f_y(x, y+\Delta y) - f_y(x,y)}{\Delta y} = t$$

* Find all the first order partial derivatives of the following:

(i) $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$p = \frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{\cancel{(x+y)^2}}{(x+y)^2+(x^2+y^2)^2} \cdot \frac{(x+y)(2x)-(x^2+y^2)}{\cancel{(x+y)^2}}$$

$$= \frac{2x^2+2xy-x^2-y^2}{x^2+y^2+2xy+x^4+y^4+2x^2y^2}$$

$$\boxed{p = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}}$$

$$q = \frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2+(x^2+y^2)^2} \cdot \frac{(x+y)2y - x^2 - y^2}{(x+y)^2}$$

$$\boxed{q = \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}}$$

(ii) $u = y^x$

$p = \frac{\partial u}{\partial x} = \boxed{y^x \cdot \log y = p}$

$q = \frac{\partial u}{\partial y} = \boxed{x y^{x-1} = q}$

HW

(i) $u = \log(x^2+y^2)$ (iii) $u = \frac{x}{y} \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $u = x^2 \sin\left(\frac{y}{x}\right)$

* $u = x^2+y^2+z^2$ $x u_x + y u_y + z u_z = 2u$

$\frac{\partial u}{\partial x} = 2x = u_x$ $2x^2+2y^2+2z^2 = 2(x^2+y^2+z^2)$

$= \underline{\underline{2u}}$

$\frac{\partial u}{\partial y} = 2y = u_y$

$\frac{\partial u}{\partial z} = 2z = u_z$

* $z = \log(x^2+xy+y^2)$ P.T. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ?$

$\frac{\partial z}{\partial x} = \frac{2x+y}{x^2+xy+y^2}$ $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial y} = \frac{2y+x}{x^2+y^2+xy}$ $= \frac{2x^2+xy+2y^2+xy}{x^2+y^2+xy}$

$=2$

* $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ $x u_x + y u_y + z u_z = 2 = 0$

$u_x = \cancel{\frac{1}{z}} - \frac{z}{x^2} + \frac{1}{y}$ $\cancel{\frac{-z}{x^2} + \frac{x}{y} \cdot \frac{1}{y}}$

$u_y = \frac{-x}{y^2} + \frac{1}{z}$ $\frac{-z}{x} + \frac{x}{y} - \frac{x}{y} + \frac{y}{z}$

$u_z = \frac{-y}{z^2} + \frac{1}{x}$ $-\frac{y}{z} + \frac{z}{x} = \underline{0}$

* $u = \log(\tan x + \tan y)$

$u_x \sin 2x + u_y \sin 2y = ?$

$u_x = \frac{\sec^2 x}{\tan x + \tan y}$

$u_y = \frac{\sec^2 y}{\tan x + \tan y}$

$\frac{2\tan x + 2\tan y}{\tan x + \tan y} = \underline{\underline{2}}$

* Verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

1) $u = ax^2 + 2hxy + by^2$

$$\frac{\partial u}{\partial x} = 2ax + 2hy$$

$$\frac{\partial u}{\partial y} = 2hx + 2by.$$

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2h \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2h.\end{aligned}\right\} \text{Equal}$$

2) $u = \tan^{-1}\left(\frac{x}{y}\right)$ 3) $u = e^{ax}\sin by$

*If $u = \log(x^3+y^3+z^3-3xyz)$ S.T.

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2)-3(xy+yz+zx)}{x^3+y^3+z^3-3xyz}.$$

$\frac{1}{2}(x+y+z)$

$$= \frac{3\left(x^2+y^2+z^2-(xy+yz+zx)\right)}{x^3+y^3+z^3-3xyz}.$$

$$= \frac{3\cancel{(x^2+y^2+z^2-(xy+yz+zx))}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{x+y+z}$$

$$LHS = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

* $u=\sqrt{x^2+y^2+z^2}$; $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=?$

$$\frac{\partial u}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{2\sqrt{x^2+y^2+z^2}}$$

$$=\frac{4(x^2+y^2+z^2)}{4(x^2+y^2+z^2)}=\underline{\underline{1}}$$

* $u=e^{x-at}\cos(x-at)$

S.T. $\boxed{\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}}$

* $x^x y^y z^z=c$ S.T. $x=y=z$. $\frac{\partial^2 z}{\partial x\partial y}=-(x\log_e x)^{-1}$

$x\log x+y\log y+z\log z=\log c$ — 1

$\cancel{\frac{x}{x}}+\cancel{\log x}$ $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$$0+\frac{y}{y}+\log y+\frac{z}{z}\frac{\partial z}{\partial y}+\frac{\partial z}{\partial y}\cdot\log z=0$$

$$\frac{\partial z}{\partial y}=\frac{-1-\log y}{1+\log z} \quad \text{— 2}$$

$$\frac{\partial z}{\partial x}=\frac{-(1+\log x)}{1+\log z} \quad \text{— 3}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=$$

Diff 2 wrt 'x' on both sides.

$\cancel{\frac{\partial^2 z}{\partial x\partial y}=-\left[(1+\log z)\left(\frac{1}{y}\right)\frac{\partial y}{\partial x}-(1+\log\right.}$

$$\frac{\partial z}{\partial y}(1+\log z)=-(1+\log y)$$

$$\frac{1}{z}\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}+(1+\log z)\frac{\partial^2 z}{\partial x\partial y}=0$$

$$\frac{1}{z}\cdot\frac{(1+\log x)(1+\log y)}{(1+\log z)^3}=-\frac{\partial^2 z}{\partial x\partial y}$$

$(x=y=z)$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{z}\,\frac{1}{(1+\log z)}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-1}{x(\log e+\log x)}$$

$$\frac{\partial^2 z}{\partial x\partial y}=-(x(\log ex))^{-1}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -(x \log(ex))^{-1}}$$

Q. $u = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\cancel{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)}$$

$$\cancel{= \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2}}$$

$$\cancel{= \frac{-2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{y^2}{x^2+y^2} \cdot \frac{-x}{y^2} = \frac{-x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1) + x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - 2y(y)}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$\Rightarrow u = e^{ax} \sin by$

$$\frac{\partial u}{\partial x} = e^{ax} \cdot a \sin by$$

$$\frac{\partial u}{\partial y} = e^{ax}\, b \cos by$$

$$\cancel{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = a\frac{\partial}{\partial x}\left(a e^{ax} \sin by\right)}$$

$$\cancel{= a\left(e^{ax}(0) + \sin by\, a \cdot e^{ax}\right)}$$

$$\cancel{= a^2 e^{ax} \sin by}$$

$$\cancel{\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial}\right.}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = b\left(\frac{\partial}{\partial x}\left(e^{ax} \cos by\right)\right)$$

$$= b\left[e^{ax}(0) - e^{ax} \cos by\, a\right.$$

$$= ab\, e^{ax} \cos by$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = a\frac{\partial}{\partial y}\left(e^{ax}\sin by\right)$$

$$= a\left(be^{ax}\cos by + 0\right)$$

$$= abe^{ax}\cos by$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

* $u = \log(x^2+y^2)$

$$p = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \qquad q = \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

* $u = \frac{x}{y}\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{x}{y} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{x(0) - y}{x^2} + \frac{\tan^{-1}(y/x)}{y}$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{-x}{x^2+y^2}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{-x}{x^2+y^2} + \frac{\tan^{-1}(y/x)}{y}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{y} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{x - y(0)}{x^2} + \frac{-1}{y^2}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{x^2}{y(x^2+y^2)} - \frac{1}{y^2}\tan^{-1}\left(\frac{y}{x}\right)}$$

$$u = \frac{x}{y}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\cancel{\frac{\partial u}{\partial y} = \frac{x}{y} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{x - 0}{x^2}}$$

* $u = x^2\sin\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = x^2\cos\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2} + 2x\sin\left(\frac{y}{x}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = 2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}$$

$$\frac{\partial u}{\partial y} = x^2\cos\left(\frac{y}{x}\right) \cdot \frac{x}{x^2} + 0$$

$$\boxed{\frac{\partial u}{\partial y} = x\cos\left(\frac{y}{x}\right)}$$

* $u = e^{x-at}\cos(x-at)$. $\qquad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial x} = e^{x-at}\cos(x-at) - e^{x-at}\sin(x-at)$$

$$\frac{\partial u}{\partial x} = e^{x-at}\left[\cos(x-at) - \sin(x-at)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = e^{x-at}\left[-\sin(x-at) - \cos(x-at)\right] + e^{x-at}\left[\cos(x-at) - \sin(x-at)\right]$$

$$= e^{x-at}\left[-2\sin(x-at)\right]$$

$$= -2e^{x-at}\left(\sin(x-at)\right)$$

$$\frac{\partial^2 u}{\partial t^2} = e^{x-at}(-a)\cos(x-at) + ae^{x-at}\sin(x-at)$$

$$= ae^{x-at}\left(\sin(x-at) - \cos(x-at)\right)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left[a e^{x-at}\left(\sin(x-at) - \cos(x-at)\right)\right]$$

$$= a\left[e^{x-at}\left(\cos(x-at)(-a) - a\sin(x-at)\right) - a\, e^{x-at}\left(\sin(x-at) - \cos(x-at)\right)\right]$$

$$= a^2 e^{x-at}\left[-2\sin(x-at)\right]$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

17/01/2022

* **Partial derivatives at origin:**

$$f_x = \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$f_y = \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\Delta y \to 0} \frac{f(0, y+\Delta y) - f(0,0)}{\Delta y}$$

Q: S.T $f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is

not continuous at $(0,0)$ but its partial derivatives f_x & f_y exist at $(0,0)$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy}{x^2+2y^2}\right) = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy}{x^2+2y^2}\right) = 0$$

$$\lim_{x\to 0}\left(\frac{x(mx)}{x^2+2m^2x^2}\right) = \lim_{x\to 0} \frac{\cancel{x^2}\, m}{(1+2m^2)\cancel{x^2}} = \frac{m}{1+2m^2}$$

$\Rightarrow$ m is arbitrary

$\Rightarrow$ limit does not exist

$\Rightarrow$ $f(x)$ is discontinuous at $(0,0)$

Existence of PDs

$$f_x = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - 0}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\dfrac{(\Delta x)(0)}{(\Delta x)^2 + 0}}{\Delta x} = 0$$

$$f_y = \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - 0}{\Delta y}$$

$$= \lim_{\Delta y\to 0} \frac{\dfrac{(0)(\Delta y)}{0 + 2(\Delta y)^2}}{\Delta y} = 0$$

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = 0 \qquad \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 0}$$

$\downarrow$

$\boxed{\text{Derivative should be finite}}$

$\Rightarrow$ f_x & f_y exists at $(0,0)$

Q: S.T. the function $f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right) & x+y \neq 0 \\ 0 & x+y = 0 \end{cases}$

is continuous at $(0,0)$ but its partial derivatives f_x & f_y does not exist

$$\cancel{\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x+y}{\sin(x+y)}\right) = \lim_{x\to 0} \frac{x}{\sin x} = 0}$$

$$\cancel{\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x+y}{\sin(x+y)}\right) = \lim_{y\to 0}}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}(x+y)\sin\left(\frac{1}{x+y}\right)\right) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0}\frac{\sin(1/x)}{1/x}$$

$$= 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}(x+y)\sin\left(\frac{1}{x+y}\right)\right) = \lim_{y\to 0} y\sin\left(\frac{1}{y}\right) = \lim_{y\to 0}\frac{\sin(1/y)}{1/y}$$

$$= 1$$

$\Rightarrow$ ~~f(x) is cont~~

$$\lim_{x\to 0}(x+mx)\sin\left(\frac{1}{(1+m)x}\right) = 1$$

$\Rightarrow$ ~~f(x) is continuous at (0,0)~~

$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-0}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\Delta x\cdot\sin\left(\frac{1}{\Delta x}\right)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\sin\left(\frac{1}{\Delta x}\right) = \infty \text{ (undefined)}$$

$$\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-0}{\Delta y}$$

$$= \lim_{\Delta y\to 0}\sin\left(\frac{1}{\Delta y}\right) = \infty \text{ (undefined)}$$

$\Rightarrow$ P.Ds f_x & f_y does not exist at $(0,0)$.

* δ-ϵ method:

for continuity

$$|f(x,y)-f(0,0)|<\epsilon \qquad |x-a|\le\delta \ \ \& \ \ |y-b|\le\delta.$$

$$\left|(x+y)\sin\left(\frac{1}{x+y}\right)-0\right|<\epsilon \qquad |x-0|<\delta,\ |y-0|<\delta,\ x<\delta,\ y<\delta$$

$$\left|(x+y)\sin\left(\frac{1}{x+y}\right)\right|<\epsilon$$

$$\le |x+y|$$

$$\le |x|+|y|$$

$$\le 2\sqrt{x^2+y^2}<\epsilon$$

$$2\sqrt{2\delta^2}<\epsilon$$

$$\boxed{\delta<\frac{\epsilon}{2\sqrt{2}}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

H.W

S.T. the function $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{|x|+|y|} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

is continuous at $(0,0)$ but its P.Ds f_x & f_y do not exist at $(0,0)$

$$f_x = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{(\Delta x)^2}{|\Delta x|} \Rightarrow \begin{cases}1 & \text{if } \Delta x>0\\ -1 & \Delta x<0\end{cases}$$

$$f_y = \lim_{\Delta y \to 0} \frac{(\Delta y)^2}{|\Delta y|} \qquad \begin{cases} +1 & \Delta y > 0 \\ -1 & \Delta y < 0 \end{cases}$$

limit is not unique.

$\delta - \epsilon$ method

$|f(x,y) - f(0,0)| < \epsilon \qquad |x| < \delta$

$\left|\frac{x^2+y^2}{|x|+|y|}\right| < \epsilon \qquad |y| < \delta$

$\leq \frac{(|x|+|y|)^2}{|x|+|y|}$

$\leq |x|+|y| \qquad \lim_{(x,y)\to(0,0)} f(x,y) = 0$

$\leq 2\sqrt{x^2+y^2} < \epsilon \qquad = f(x,y)$

$2\sqrt{2}\,\delta < \epsilon$

$\delta < \frac{\epsilon}{2\sqrt{2}}$

18/01/2022

CHAIN RULE – TOTAL DERIVATIVE – IMPLICIT FUNCTION OF PARTIAL DERIVATIVES

ODE: Diff. of dependent variable wrt only one independent variable is called ODE

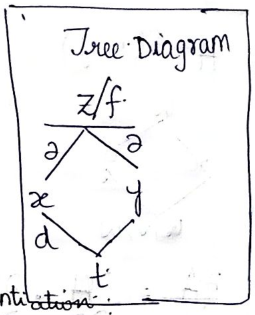
$y = f(x) \qquad \frac{dy}{dx} \qquad$ Ex: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \sin x$

PDE: Diff. of dependent variable wrt more than one independent variable is called PDE

$\partial z/\partial x \;/\; \partial z/\partial y$

$$\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$



The chain Rule of Partial Differentiation:

If $z = f(x,y)$ where $x = \phi(t)$; $y = \psi(t)$ then z is called a composite function of a variable t.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

$z = f(x,y) \qquad x = \phi(u,v) \qquad y = \psi(u,v)$ then

z is called a composite function of a variable u, v

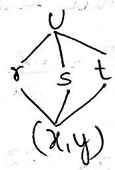
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \qquad \text{'v' is constant.}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

* $u = f(r,s,t) \qquad r = \phi(x,y)$

$s = \psi(x,y)$

$t = \tau(x,y)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

Q: $A = f(x,y)$ $\quad x = \phi(u,v,w)$
$y = \psi(u,v,w)$

$$\frac{\partial A}{\partial u} = \frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial u} + \cancel{\frac{\partial w}{\partial z}\cdot\frac{\partial z}{}}$$

$$\frac{\partial A}{\partial v} = \frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial A}{\partial w} = \frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial w}$$

H.W

$\boxed{\begin{array}{l} z = f(u,v) \\ u = \phi(x,y) \\ v = \psi(x,y) \end{array}}$

TOTAL DIFFERENTIAL COEFFICIENT:

Let $z = f(x,y)$ where $x = \phi(t)$; $y = \psi(t)$

Substituting x & y in $z = f(x,y)$

z becomes a function of a single variable t called total derivative of z.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

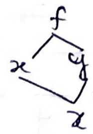
$$dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy$$

$$\boxed{dz = p\,dx + q\,dy}$$

IMPLICIT FUNCTION:

If $f(x,y) = 0$ (or) const

Then the derivative of y w.r.t 'x' i.e. $\frac{dy}{dx}$ is called an implicit function.



$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$0 = f_x + f_y\cdot\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

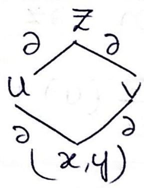
$\rightarrow z(u,v) = 2 \longrightarrow$ implicit

Implicit function $\dfrac{dv}{du} = -\dfrac{z_u}{z_v} = -\dfrac{\frac{\partial z}{\partial u}}{\frac{\partial z}{\partial v}}$

$z_u = \dfrac{\partial z}{\partial u} \qquad z_v = \dfrac{\partial z}{\partial v}$

$z = f(u,v)$

$u = \phi(x,y) \qquad v = \psi(x,y)$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

19/01/2022

1) If $u = f(r,s,t)$ where $r = \dfrac{x}{y}$ $\quad s = \dfrac{y}{z}$ $\quad t = \dfrac{z}{x}$

Show that $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} + z\cdot\dfrac{\partial u}{\partial z} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y} + \frac{\partial u}{\partial s}\cdot(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{y}\cdot\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\cdot\frac{z}{x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{-x}{y^2} + \frac{\partial u}{\partial s}\cdot\frac{1}{z} + \frac{\partial u}{\partial t}(0)$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{-x}{y^2}\cdot\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\frac{-y}{z^2} + \frac{\partial u}{\partial t}\cdot\frac{1}{x}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{1}{x}\frac{\partial u}{\partial t} - \frac{y}{z^2}\frac{\partial u}{\partial s}}$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z}$$

$$= \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}\cdot\frac{z}{x}} - \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} + \cancel{\frac{y}{z}\frac{\partial u}{\partial s}}$$
$$+ \cancel{\frac{z}{x}\frac{\partial u}{\partial t}} - \cancel{\frac{y}{z}\frac{\partial u}{\partial s}}$$

$= \underline{0}$.

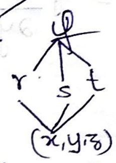
$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0}$$

2) If $u = f(2x-3y, 3y-4z, 4z-2x)$

P.T. $\frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{3}\frac{\partial f}{\partial y} + \frac{1}{4}\frac{\partial f}{\partial z} = 0$

$\cancel{\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x-3y)}$

$\cancel{\frac{\partial f}{\partial x} = 2}$



consider

$r = 2x-3y$

$s = 3y-4z$

$t = 4z-2x$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial f}{\partial r}\cdot(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}\cdot(-2)$$

$$\frac{1}{2}\frac{\partial f}{\partial x} = \frac{2\partial f}{\partial r} - \frac{\partial f}{\partial t}(2) \Rightarrow$$

$$\boxed{\frac{1}{2}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}} \quad \text{— (1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + 0$$

$$\boxed{\frac{1}{3}\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}} \quad \text{— (2)}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= 0 + (-4)\frac{\partial f}{\partial s} + (4)\frac{\partial f}{\partial t}$$

$$\boxed{\frac{1}{4}\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}} \quad \text{— (3)}$$

① + ② + ③

$$\boxed{\frac{1}{2}\frac{\partial U}{\partial x}+\frac{1}{3}\frac{\partial U}{\partial y}+\frac{1}{4}\frac{\partial U}{\partial z}=0}$$

3) $z=f(x,y)$ where $x=e^{u}+e^{-v}$
$y=e^{-u}-e^{v}$.

S.T. $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial u}$$

$$=\frac{\partial z}{\partial x}\cdot e^{u}+\frac{\partial z}{\partial y}\cdot(-e^{-u})$$

$$\frac{\partial z}{\partial u}=e^{u}\left(\frac{\partial z}{\partial x}\right)-\left(\frac{\partial z}{\partial y}\right)e^{-u} \quad \text{— 1}$$

$$\frac{\partial z}{\partial v}=-e^{v}\left(\frac{\partial z}{\partial y}\right)-\left(\frac{\partial z}{\partial x}\right)e^{-v}$$

$$\cancel{e^{u}\frac{\partial z}{\partial x}-e^{u}\frac{\partial z}{\partial y}+e^{v}\frac{\partial z}{\partial y}}$$

$$\frac{\partial z}{\partial v}=-\left(e^{v}\frac{\partial z}{\partial y}+e^{-v}\frac{\partial z}{\partial x}\right) \rightarrow 2$$

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=$$

$$e^{u}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\cdot e^{-u}+e^{v}\frac{\partial z}{\partial y}+e^{-v}\frac{\partial z}{\partial x}$$

$$=\frac{\partial z}{\partial x}\cdot(e^{u}+e^{-v})-\frac{\partial z}{\partial y}(e^{-u}-e^{v})$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y} \quad =\text{RHS}$$

H.W

$u=f(x^2+2yz,\ y^2+2zx)$

P.T $(y^2-zx)u_x+(x^2-yz)u_y+(z^2-xy)u_z=0$

4) Find $\frac{du}{dt}$ if $u=x^2y^3$ $x=\log t$
$y=e^{t}$

$\cancel{\frac{dz}{dt}=\partial z}$ By total derivative

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{du}{dt}=2xy^3\frac{1}{t}+3y^2x^2\cdot e^{t}$$

$$\frac{du}{dt}=xy^2\left[\frac{2y}{t}+3xe^{t}\right]$$

$$=\log t\, e^{2t}\left(\frac{2e^{t}}{t}+3\log t\, e^{t}\right)$$

$$\boxed{\frac{du}{dt}=\log t\cdot e^{3t}\left(\frac{2}{t}+3\log t\right)}$$

5) If $u = \log(x+y+z)$

$x = e^{-t}$ $\quad y = \sin t$; $z = \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{1}{x+y+z}\cdot e^{-t} + \frac{1}{x+y+z}\cos t - \frac{1}{x+y+z}\sin t$$

$$= \frac{-e^{-t} + \cos t - \sin t}{e^{-t} + \sin t + \cos t}$$

$$\boxed{\frac{du}{dt} = \frac{-e^{-t} + \cos t - \sin t}{e^{-t} + \sin t + \cos t}}$$

6) If $u = \sin(x^2+y^2)$ where $a^2x^2+b^2y^2=c^2$ find $\frac{du}{dx}$. $\quad \cancel{y^2 = c^2 - a^2x^2}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \cos(x^2+y^2)\cdot 2x + \cos(x^2+y^2)\cdot 2y\cdot\frac{dy}{dx}$$

$$a^2 2x + 2y\frac{dy}{dx}\cdot b^2 = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{a^2x}{b^2y}}$$

$$\frac{du}{dx} = \left[x\cos(x^2+y^2) + y\cos(x^2+y^2)\cdot -\frac{a^2x}{b^2y}\right]2$$

$$\frac{du}{dx} = 2\left[x\right]\left[\cos(x^2+y^2)\right]\left(1 - \frac{a^2}{b^2}\right)$$

$$\boxed{\frac{du}{dx} = 2x\cos(x^2+y^2)\left(1-\frac{a^2}{b^2}\right)}$$

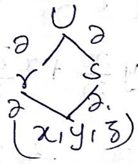
H.W

$u = f(x^2+2yz, y^2+2zx)$.

$\Rightarrow$ Let us take $r = x^2+2yz$

$s = y^2+2zx$

$u = f(r,s)$.



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot 2x + \frac{\partial u}{\partial s}\cdot 2z.$$

$$u_x = 2\left[x\frac{\partial u}{\partial r} + z\frac{\partial u}{\partial s}\right] \quad —(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot 2z + \frac{\partial u}{\partial s}\cdot 2y.$$

$$u_y = 2\left[z\frac{\partial u}{\partial r} + y\frac{\partial u}{\partial s}\right] \quad —(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot 2y + \frac{\partial u}{\partial s}\cdot 2x$$

$$U_z = 2\left[y\frac{\partial U}{\partial r} + x\frac{\partial U}{\partial s}\right] \quad (3)$$

$$(y^2 - zx)U_x + (z^2 - xy)U_y + (x^2 - yz)U_z = 0$$

LHS.

$$(y^2 - zx)\cdot 2\left[x\frac{\partial U}{\partial r} + z\frac{\partial U}{\partial s}\right]$$

$$= 2\left[xy^2\frac{\partial U}{\partial r} + y^2 z\frac{\partial U}{\partial s} - zx^2\frac{\partial U}{\partial r} - z^2 x\frac{\partial U}{\partial s}\right] \quad (4)$$

$$(z^2 - xy)\cdot 2\left[y\frac{\partial U}{\partial r} + x\frac{\partial U}{\partial s}\right]$$

$$= 2\left[z^2 y\frac{\partial U}{\partial r} + z^2 x\frac{\partial U}{\partial s} - xy^2\frac{\partial U}{\partial r} - x^2 y\frac{\partial U}{\partial s}\right] \quad (5)$$

$$(x^2 - yz)\cdot 2\left[z\frac{\partial U}{\partial r} + y\frac{\partial U}{\partial s}\right]$$

$$= 2\left[x^2 z\frac{\partial U}{\partial r} + x^2 y\frac{\partial U}{\partial s} - yz^2\frac{\partial U}{\partial r} - y^2 z\frac{\partial U}{\partial s}\right] \quad (6)$$

(4) + (5) + (6)

$$= 2\Big[\cancel{xy^2\frac{\partial U}{\partial r}} + \cancel{y^2 z\frac{\partial U}{\partial s}} - \cancel{zx^2\frac{\partial U}{\partial r}} - \cancel{z^2 x\frac{\partial U}{\partial s}}$$

$$+ \cancel{z^2 y\frac{\partial U}{\partial r}} + \cancel{z^2 x\frac{\partial U}{\partial s}} - \cancel{xy^2\frac{\partial U}{\partial r}} - \cancel{x^2 y\frac{\partial U}{\partial s}}$$

$$+ \cancel{x^2 z\frac{\partial U}{\partial r}} + \cancel{x^2 y\frac{\partial U}{\partial s}} - \cancel{yz^2\frac{\partial U}{\partial r}} - \cancel{y^2 z\frac{\partial U}{\partial s}}\Big]$$

$= 0$ (RHS)

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Q: $x^y + y^x = c$ find $\frac{dy}{dx}$

By implicit function.

$x^y + y^x = c$; $f = x^y + y^x - c$

diff. w.r.to 'x'

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}}$$

$\frac{d}{dx}(\text{const})$ Variable

$= (\text{const})^{\text{Variable}}$ $\log(\text{const})$

$$\frac{\partial f}{\partial x} = y(x^{y-1}) + y^x \log y$$

$$\frac{\partial f}{\partial y} = x^y \log x + x y^{x-1}$$

$$\boxed{\frac{dy}{dx} = \frac{-[y x^{y-1} + y^x \log y]}{x^y \log x + x y^{x-1}}}$$

Q: If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$

find $\frac{du}{dx}$. $\quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

$$\frac{\partial v}{\partial x} = \frac{x\cdot y}{xy} + \log xy$$
$$= 1 + \log xy$$
$$\frac{\partial v}{\partial y} = \frac{x\cdot x}{xy} = \frac{x}{y}$$

$$\cancel{3}x^2 + \cancel{3}y^2\,dy/dx + \cancel{3}\left[x\frac{dy}{dx} + y\right] = 0$$
$$x^2 + y^2\frac{dy}{dx} + y = -x\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{(x^2+y)}{(y^2+x)}$$

$$\cancel{\frac{du}{dx} = 1 + \log xy + \left(\frac{x}{y}\right)\left(-(x^2+y^2+y)\right)}$$
$$\boxed{\frac{dy}{dx} = -\frac{(x^2+y)}{x+y^2}}$$
$$\cancel{\frac{du}{dx} = 1 + \log xy - \frac{x^3}{y} - xy - x}$$
$$\boxed{\frac{du}{dx} = 1 + \log xy - \frac{x}{y}\left(\frac{x^2+y}{x+y^2}\right)}$$

Q: If $z = f(x,y)$; $u = lx + my$
$v = ly - mx$

P.T. $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

$u = lx + my$
$v = ly - mx$

$$\boxed{\frac{v+mx}{l} = y} \qquad u = lx + \frac{mv + m^2x}{l}$$
$$\boxed{\frac{ul - mv}{m^2+l^2} = x}$$

$$y = \frac{v}{l} + \frac{m}{l}\left(\frac{ul - mv}{m^2+l^2}\right)$$
$$= \frac{v}{l} + \frac{mul - m^2v}{l(m^2+l^2)} = \frac{m^2v + l^2v + mul - m^2v}{l(l^2+m^2)}$$
$$y = \frac{l(lv+mu)}{l(l^2+m^2)} \quad \boxed{\frac{lv+mu}{l^2+m^2} = y}$$

Q: If $z = \sin^{-1}(x-y)$
$x = 3t \qquad y = 4t^3$.
$$\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}.$$

By total derivative
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 - \frac{12t^2}{\sqrt{1-(x-y)^2}}$$
$$= \frac{3 - 12t^2}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-t^2(3-4t^2)^2}}$$
$$= \frac{3(1-4t^2)}{\sqrt{1-t^2(9+16t^4-24t^2)}}$$
$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$(1-4t^2)^2$
$1+16t^4-8t^2$

$$= \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-16t^6+8t^4-t^2}}$$
$$= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2 - t^2(1-4t^2)^2}} = \frac{3\cancel{(1-4t^2)}}{\cancel{(1-4t^2)}\sqrt{1-t^2}}$$
$$\boxed{\frac{dz}{dt} = \frac{3}{\sqrt{1-t^2}}}$$

Q: $u = ax + by$ $\quad v = bx - ay$.

find $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

Q: If $y \log(\cos x) = x \log(\sin y)$ find $\frac{dy}{dx}$

Q: If $x^3 + y^3 - 3axy = 0$ at $(1,2)$

find $\frac{dy}{dx}$.

Q: $z = \tan^{-1}\left(\frac{x}{y}\right)$ where $x = 2t$

$y = 1 - t^2$

prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$

$$x = \frac{lu - mv}{l^2+m^2} \qquad y = \frac{lv + mu}{m^2+l^2}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$\frac{\partial x}{\partial u} = \frac{l}{l^2+m^2} \qquad \frac{\partial y}{\partial u} = \frac{m}{m^2+l^2}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{l}{m^2+l^2}\right) + \frac{\partial z}{\partial y} \cdot \frac{m}{l^2+m^2}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{l}{m^2+l^2}\right)^2 + \frac{\partial^2 z}{\partial y^2}\left(\frac{m}{l^2+m^2}\right)^2 \quad -(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

$$\frac{\partial x}{\partial v} = \frac{-m}{l^2+m^2} \qquad \frac{\partial y}{\partial v} = \frac{l}{m^2+l^2}$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} \frac{m}{l^2+m^2} + \frac{\partial z}{\partial y} \frac{l}{m^2+l^2}.$$

$$\frac{\partial^2 z}{\partial v^2} = +\frac{\partial^2 z}{\partial x^2} \cdot \left(\frac{m}{l^2+m^2}\right)^2 + \frac{\partial^2 z}{\partial y^2}\left(\frac{l}{m^2+l^2}\right)^2$$

RHS

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{l^2+m^2}{l^2+m^2}\right) + \frac{\partial^2 z}{\partial y^2}\left(\frac{l^2+m^2}{l^2+m^2}\right)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

---

* $u = ax + by$ $\quad v = bx - ay$.

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_y = a} \qquad \left(\frac{\partial x}{\partial u}\right)_v =$$

$$\cancel{\frac{\partial v}{\partial x}} \qquad \frac{v + ay}{b} = x. \qquad \boxed{\frac{bx - v}{a} = y}$$

$$\cancel{u = \frac{av + a^2 y}{b} + by} \qquad u = ax + \frac{b^2 x - bv}{a}$$

$$\cancel{u = \frac{av + y(a^2+b^2)}{b}} \qquad \frac{(a^2+b^2)x - bv}{a}$$

$$\cancel{u = \frac{av}{b} + \frac{y(a^2+b^2)}{b}}$$

$$\frac{au + bv}{a^2+b^2} = x$$

$$\boxed{\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2+b^2}}$$

$$\boxed{\frac{u - by}{a} = x}$$

$$v = \frac{bu - b^2 y - a^2 y}{a}$$

$$av - bu = -y(b^2 + a^2)$$

$$\boxed{y = \frac{bu - av}{a^2+b^2}}$$

$\left(\frac{\partial y}{\partial v}\right)$

$$y(a^2+b^2) = b(ax+by) - av$$

$$y(a^2+b^2) = abx + b^2y - av$$

$$a^2 y = abx - av$$

$$y = \frac{a(bx - v)}{a^2} = \frac{bx - v}{a}$$

$$\boxed{\left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{a}}$$

$\left(\frac{\partial v}{\partial y}\right)_u$

$$y(a^2+b^2) - bu = -av$$

$$\frac{bu - y(a^2+b^2)}{a} = v$$

$$\boxed{\left(\frac{\partial v}{\partial y}\right)_u = \frac{-(a^2+b^2)}{a}}$$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u =$$

$$a \times \frac{a}{a^2+b^2} \times \frac{-1}{a} \times \frac{-(a^2+b^2)}{a}$$

$$= 1$$

TAYLOR'S AND MACLAURIN'S SERIES FOR A FUNCTION OF TWO VARIABLES:

TAYLOR'S statement: If $f(x, y)$ possess continuous partial derivatives of n^{th} order in any neighbourhood of a point (x, y) and $(x+h, y+k)$ is any point of this neighbourhood.

Then

$$f(x+h, y+k) = f(x, y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \frac{1}{3!}[h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{yyx} + k^3 f_{yyy}] + \dots \quad \text{--- (1)}$$

Note:

1) Put $x = a$; $y = b$

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy} + 3hk^2 f_{yyx} + k^3 f_{yyy}] + \dots$$

2) Put $a+h = x$; $b+k = y$ in eq (1)

$$f(2x-a, 2y-b) = f(x, y)$$

$$= f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(y-b)^2(x-a) f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

Put $a=0$; $b=0$ in above.

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$+ \cdots$$

* $y \log(\cos x) = x \log(\sin y)$

$$\frac{dy}{dx} = ?$$

$$u = y \log(\cos x) - x \log(\sin y)$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} = \frac{y - \sin x}{\cos x} - \log(\sin y)$$

$$\frac{\partial u}{\partial x} = -y\tan x - \log(\sin y) = -(y \tan x + \log(\sin y))$$

$$\frac{\partial u}{\partial y} = \log(\cos x) - \frac{x \cdot \cos y}{\sin y}$$

$$= \log(\cos x) - (x \cot y)$$

$$\frac{dy}{dx} = \frac{y \tan x + \log(\sin y)}{\log(\cos x) - x \cot y}$$

* $x^3 + y^3 - 3axy = 0 = u$ at $(1,2)$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{-(3x^2 - 3ay)}{(3y^2 - 3ax)}$$

$$= -\frac{(x^2 - ay)}{(y^2 - ax)}$$

$$\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{-(1-2a)}{(4-a)}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{(1-2a)}{a-4}}$$

* $z = \tan^{-1}\left(\frac{x}{y}\right)$ $\quad x = 2t$, $y = 1-t^2$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1(1/y)}{\frac{x^2+y^2}{y^2}}\cdot 2 + \frac{-x/y^2}{\frac{x^2+y^2}{y^2}}\cdot(-2t)$$

$$= \frac{2y}{x^2+y^2} + \frac{2xt}{x^2+y^2}$$

$$= \frac{2(xt+y)}{x^2+y^2} = \frac{2(2t^2+1-t^2)}{4t^2+1+t^4-2t^2}$$

$$\cancel{= \frac{2(1+t^2+2t-2t^2)}{(1+t^2)^2}}$$

$$\cancel{= \frac{2[(1+t)^2-2t^2]}{(1+t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2}{1+t^2}$$

$$\boxed{\frac{dz}{dt} = \frac{2}{1+t^2}}$$

31/01/2022

PROBLEMS ON TAYLOR'S & MACLAURIN'S SERIES WITH THE FUNCTION OF 2 VARIABLES:

→ Taylor's Series with the function of 2 variables:

Expansion of $f(x,y)$ about $x=a$ & $y=b$ is given by

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$
$$+ \cdots$$

Maclaurin's Series with the function of 2 variables i.e. in powers of x & y:

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+ \cdots$$

* ~~Expand~~ Problems on power series:

1) Expand the function $f(x,y) = e^x \log(1+y)$ in terms of x and y upto 3rd degree.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \cdots$$
$$+ \frac{1}{2!}[x^2 f$$

By Maclaurin's Series.

$f(x,y) = e^x \log(1+y)$ | $f(0,0)=0$

$f_x = e^x \log(1+y)$ | $f_x(0,0)=0$

$f_y = \dfrac{e^x}{1+y}$ | $f_y(0,0)=1$

$f_{xx} = \dfrac{\partial f_x}{\partial x} = e^x \log(1+y)$ | $f_{xx}(0,0)=0$

$f_{xy} = \dfrac{\partial f_y}{\partial x} = \dfrac{e^x \cancel{(1+y)}}{(1+y)}$ | $f_{xy}(0,0)=1$

$f_{yy} = \dfrac{\partial f_y}{\partial y} = \dfrac{-e^x}{(1+y)^2}$ | $f_{yy}(0,0)=-1$

$f_{xxx} = \dfrac{\partial (f_{xx})}{\partial x} = e^x \log(1+y)$ | $f_{xxx}(0,0)=0$

$f_{xxy} = \dfrac{\partial f_{xy}}{\partial x} = \dfrac{e^x}{1+y}$ | $f_{xxy}=1$

$f_{xyy} = \dfrac{\partial f_{yy}}{\partial x} = \dfrac{-e^x}{(1+y)^2}$ | $f_{xyy}=-1$

$f_{yyy} = \dfrac{\partial (f_{yy})}{\partial y} = \dfrac{+2e^x}{(1+y)^3}$ | $f_{yyy}=2$

$$f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right] + \cdots$$

$$\boxed{f(x,y) = e^x\log(1+y) = y + \frac{2xy - y^2}{2} + \frac{3x^2y - 3xy^2 + 2y^3}{6} + \cdots}$$

2) Expand e^{xy} in neighbourhood of $(1,1)$.

$f(x,y) = e^{xy}$

By Taylor's Series:

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

$f(x,y) = e^{xy}$ | $f(1,1) = e$

$f_x = e^{xy}.y$ | $f_x(1,1) = e$

$f_y = e^{xy}.x$ | $f_y(1,1) = e$

$f_{xx} = \dfrac{\partial f_x}{\partial x} = y^2 e^{xy}$ | $f_{xx} = e$

$f_{xy} = \dfrac{\partial f_y}{\partial x} = e^{xy}.xy + e^{xy}.1$ | $f_{xy} = 2e$

$f_{yy} = \dfrac{\partial f_y}{\partial y} = e^{xy}.x^2$ | $f_{yy} = e$

$$e^{xy} = e + [(x-1)e + (y-1)e] + \frac{1}{2!}\left[(x-1)^2 e + 2\overset{xy-x-y+1}{(x-1)(y-1)}(2e) + (y-1)^2 e\right]$$

$$= e\left[1 + x + y - 2 + \frac{1}{2!}\left(x^2 + y^2 - 2x - 2y + 2 + 4xy - 4x + 4 - 4y\right) + \cdots\right]$$

$$= e\left[1 + x + y - 2 + \frac{x^2+y^2}{2} - x - xy + x + 2xy - 2x + 2 - 2y + \cdots\right]$$

$$= e\left[\frac{x^2+y^2}{2} + 2xy - 2x - 2y + 2\right] + \cdots$$

$$e^{xy} = e\left[1 + (x-1) + (y-1)\right] + e\left[\frac{(x-1)^2}{2} + \cancel{2}(x-1)(y-1) + \frac{(y-1)^2}{2}\right] + \cdots$$

3) Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$ upto 3rd degree.

$f(x,y) = x^2y + 3y - 2$ $\quad a = 1,\ b = -2$

$f(x,y) = x^2y + 3y - 2$ $\quad f(1,-2) = -2-6-2 = \underline{\underline{-10}}$

$f_x(x,y) = 2xy$ $\quad f_x(1,-2) = -4$

$f_y(x,y) = x^2 + 3$ $\quad f_y(1,-2) = 4$

$f_{xx} = \frac{\partial f_x}{\partial x} = 2y$ $\quad f_{xx}(1,-2) = -4$

$f_{xy} = \frac{\partial f_y}{\partial x} = 2x$ $\quad f_{xy}(1,-2) = 2$

$f_{yy} = \frac{\partial f_y}{\partial y} = 0$ $\quad f_{yy}(1,-2) = 0$

$f_{xxx} = \frac{\partial f_{xx}}{\partial x} = 0$ $\quad f_{xxx}(1,-2) = 0$

$f_{xxy} = \frac{\partial f_{xy}}{\partial x} = 2$ $\quad f_{xxy}(1,-2) = 2$

$f_{xyy} = \frac{\partial f_{yy}}{\partial x} = 0$ $\quad f_{xyy}(1,-2) = 0$

$f_{yyy} = \frac{\partial f_{yy}}{\partial y} = 0$ $\quad f_{yyy}(1,-2) = 0$

$$x^2y + 3y - 2 = \cancel{10} + \left[(x-1)(-4) + (y+2)(4)\right] + \frac{1}{2!}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y-2)^2(0)\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)\right] + \cdots$$

$$x^2y + 3y - 2 = -10 + \left[(1-x)4 + 4(y+2)\right] + \frac{1}{2!}\left[-4(x-1)^2 + 4(x-1)(y+2)\right] + \frac{1}{3!}\left[6(x-1)^2(y+2)\right] + \cdots$$

4) Expand $\sin x \sin y$ at $(0,0)$

5) Expand $e^x \cos y$ at $(1, \pi/4)$

4) $f(x,y) = \sin x \cdot \sin y$ at $(0,0)$

$f(x,y) = \sin x \sin y$ $\quad f(0,0) = 0$

$f_x(x,y) = \cos x \cdot \sin y$ $\quad f_x(0,0) = 0$

$f_y(x,y) = \sin x \cos y$ $\quad f_y(0,0) = 0$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(\cos x \sin y) = -\sin x \sin y$ $\quad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \frac{\partial}{\partial x}(\sin x \cos y) = \cos x \cos y$ $\quad f_{xy}(0,0) = \underline{1}$

$f_{yy}(x,y) = \frac{\partial f_y}{\partial y} = -\sin x \sin y$ $\quad f_{yy}(0,0) = 0$

By Taylor's Series at $x = 0$; $y = 0$ $\Rightarrow$ By Maclaurin's Series

$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$
$+ \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$
$+ \frac{1}{3!} + \ldots$

$\sin x \sin y = \cancel{0}\ xy + \ldots$

5) $f(x,y) = e^x \cos y$ at $a = 1$, $b = \pi/4$

$f(x,y) = e^x \cos y$ $\quad f(1, \frac{\pi}{4}) = \frac{e}{\sqrt{2}}$

$f_x(x,y) = e^x \cos y$ $\quad f_x(1, \frac{\pi}{4}) = \frac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \sin y$ $\quad f_y(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{xx}(x,y) = e^x \cos y$ $\quad f_{xx}(1, \frac{\pi}{4}) = \frac{e}{\sqrt{2}}$

$f_{xy}(x,y) = -e^x \sin y$ $\quad f_{xy}(1, \frac{\pi}{4}) = -\frac{e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \cos y$ $\quad f_{yy}(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

By Taylor's Series

$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)]$
$+ [(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)$
$+ (y-b)^2 f_{yy}(a,b)]$
$+ \ldots$

$\Rightarrow e^x \cos y = \frac{e}{\sqrt{2}} + [(x-1)\frac{e}{\sqrt{2}} - (y - \frac{\pi}{4})\frac{e}{\sqrt{2}}]$
$+ \frac{1}{2}[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y-\frac{\pi}{4})(-\frac{e}{\sqrt{2}})$
$- (y - \frac{\pi}{4})^2 \frac{e}{\sqrt{2}}] + \ldots$

$\Rightarrow e^x \cos y = \frac{e}{\sqrt{2}}[1 + (x-1) - (y - \frac{\pi}{4}) + \frac{(x-1)^2}{2} - (x-1)(y-\frac{\pi}{4})$
$- (y - \frac{\pi}{4})^2] + \ldots$

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6) $z = y + f(u)$ $\quad u = \frac{x}{y}$; find the value of $u z_x + z_y = ?$

$z_x = 0 + \frac{1}{y} \cdot f'(u)$

$z_x = \frac{f'(u)}{y}$

$z_y = 1 + \frac{f'(u)(-x)}{y^2}$

$z_y = 1 - \frac{x}{y^2} f'(u)$

$= u z_x + z_y$

$= \cancel{\frac{x}{y} \cdot \frac{f'(u)}{y}} + 1 - \cancel{\frac{x}{y^2} f'(u)}$

$= \underline{\underline{1}}$

7) $z = f(x,y)$ $\quad x = e^{2u} + e^{-2v}$ $\quad y = e^{-2u} + e^{2v}$

find $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}$

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$

$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v}$

f
$x \quad y$
u, v

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot e^{2u}\cdot 2 + \frac{\partial f}{\partial y} - 2e^{-2u}$$

$$= +2e^{2u}\left[e^{2u}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}e^{-2u}\right] = 2\left[e^{2u}\frac{\partial f}{\partial x} - e^{-2u}\right.$$

$$\frac{\partial f}{\partial v} = 2e^{2v}\left[e^{2v}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}e^{-2v}\right] = 2\left[\frac{\partial f}{\partial y}\cdot e^{2v} - \frac{\partial f}{\partial x}\cdot e\right.$$

5 $$\cancel{\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[(e^{2u} - e^{2v})\frac{\partial f}{\partial x} + (e^{2u}+\right.}$$

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[\frac{\partial f}{\partial x}(e^{2u} + e^{-2v})\right.$$

$$\left. - \frac{\partial f}{\partial y}(+e^{2v} + e^{-2u})\right]$$

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$$

8) Obtain the total differential of $z = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$= \cancel{\frac{\partial z}{\partial x}}\ \frac{1/y}{1+\frac{x^2}{y^2}} + \frac{y^2\cdot x/-y^2}{x^2+y^2}\cdot\frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y} - \frac{x}{x^2+y^2}\cdot\frac{dy}{dx}$$

$$\boxed{\frac{dz}{dx} = \frac{y - x(dy/dx)}{x^2+y^2}}$$

9) Find the total differential of $u = \left(xz + \frac{x}{z}\right)^y,\ z \neq 0$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$u = \left(xz + \frac{x}{z}\right)^y$$

$$\frac{\partial u}{\partial x} = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$\frac{\partial u}{\partial y} = \left(xz + \frac{x}{z}\right)^y\cdot\log\left(xz + \frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = y\left(xz + \frac{x}{z}\right)^{y-1}\cdot\left(x - \frac{x}{z^2}\right)$$

$$\frac{du}{dx} = \left(xz + \frac{x}{z}\right)^y\left[\frac{y\left(z+\frac{1}{z}\right)}{\left(xz+\frac{x}{z}\right)} + \log\left(xz + \frac{x}{z}\right)\frac{dy}{dx} + \frac{xy\left(1-\frac{1}{z^2}\right)}{xz+\frac{x}{z}}\cdot\frac{dz}{dx}\right]$$

$$\frac{du}{dx} = \left(xz + \frac{x}{z}\right)^y\left[\frac{zy + \frac{y}{z}}{xz + \frac{x}{z}} + \log\left(xz + \frac{x}{z}\right)\frac{dy}{dx} + \frac{\cancel{x}y(z^2-1)}{z^2 1}\cdot\frac{\cancel{z}}{\cancel{x}(z^2+1)}\frac{dz}{dx}\right]$$

$$= \left(xz + \frac{x}{z}\right)^y\left[\frac{(z^2+1)\,y}{(z^2+1)\,x} + \log\left(xz + \frac{x}{z}\right)\frac{dy}{dx} + \frac{y}{z}\left(\frac{z^2-1}{z^2+1}\right)\frac{dz}{dx}\right]$$

$$\frac{du}{dx} = \left(xz + \frac{x}{z}\right)^y\left[\frac{y}{x} + \log\left(xz + \frac{x}{z}\right)\frac{dy}{dx} + \frac{y}{z}\left(\frac{z^2-1}{z^2+1}\right)\frac{dz}{dx}\right]$$

10) Expand $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ & $(y-1)$ upto 3rd degree.
Hence compute $f(1.1, 0.9)$ approx.

$$f(x,y) = \tan^{-1}(y/x) \qquad f(1,1) = \frac{\pi}{4}$$

$$f_x = \frac{\frac{-y}{x^2}}{\frac{x^2+y^2}{x^2}} = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = -\frac{1}{2}$$

$$f_y = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}} = \frac{x}{x^2+y^2} \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{+y\cdot 2x}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-x\,(2y)}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{-1}{2}$$

$$f_{xxx} = 2y\left[\frac{(x^2+y^2)^2 - 2(x^2+y^2)\cdot 2x^2}{(x^2+y^2)^4}\right]$$

$$f_{xxx}(1,1) = 2\left[\frac{4-4(2)}{16}\right] = \frac{-1}{2}$$

$$f_{xxy} = \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)\,2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{-8-0}{16} = \frac{-1}{2}$$

$$f_{xyy} = -2y\left[\frac{(x^2+y^2)^2 - x2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$$

$$= -2\left[\frac{4-4(2)}{16}\right] = \frac{1}{2}$$

$f_{xxy} = \frac{-1}{2}$

$f_{xyy} = \frac{1}{2}$

$f_{yyy} = \frac{1}{2}$

$$f_{yyy} = -2x\left[\frac{(x^2+y^2)^2 - y(2(x^2+y^2)\cdot 2y)}{(x^2+y^2)^4}\right]$$

$$= \frac{1}{2}$$

Taylor's Series expansion

$$f(x,y) = f(a,b) + [(x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \cdots$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$
$$+\frac{1}{2}\left[(x-1)^2\left(\frac{1}{2}\right) + (y-1)^2\left(\frac{-1}{2}\right)\right]$$
$$+\frac{1}{6}\left[\cancel{(x-1)^3\left(\frac{-1}{2}\right)} + \frac{3(x-1)^2(y-1)}{2} + \frac{3}{2}(x-1)(y-1)^2 + \frac{1}{\cancel{2}}\cancel{(y-1)^3}\right] + \cdots$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left[\frac{-0.1}{2} - \frac{0.1}{2}\right]$$
$$+\frac{1}{2}\left[\frac{0.01}{2} - \frac{0.01}{2}\right] + \cancel{\frac{1}{6}\left[\frac{0.001}{2}\right.}$$

$= \frac{\pi}{4} - 0.1 + \frac{1}{2}(0) + \frac{1}{6}[0]$

$= \frac{3.14}{4} - 0.1$

$= 0.785 - 0.1 = 0.685$

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5 MAXIMA AND MINIMA FOR A FUNCTION IN 2 VARIABLES :

Step - 1 : Find 1st order Partial derivatives

$p = \frac{\partial z}{\partial x} = 0$ — (1)
$q = \frac{\partial z}{\partial y} = 0$ — (2)
} Simultaneous Equations.

we get the values of (x_1, y_1) & (x_2, y_2)

Step - 2 : Find the 2nd order partial derivatives

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y} \Big/ \frac{\partial^2 z}{\partial y \partial x}$.

$t = \frac{\partial^2 z}{\partial y^2}$.

Step - 3 : At (x_1, y_1)

$rt - s^2 > 0$ & if $r < 0$. f has max at (x_1, y_1)

$rt - s^2 > 0$, $r > 0$ f has min at (x_1, y_1)

$rt - s^2 = 0$; → Not specific.

$rt - s^2 < 0$ → f has neither max & min.

Same process to be continued at other points.

$rt - s^2 > 0$, $r > 0$ f has min at (x_1, y_1).

$rt - s^2 = 0$; $\rightarrow$ Not specific.

$rt - s^2 < 0 \longrightarrow$ f has neither max & min.

Same process to be continued at other points.

$\rightarrow$ A function $f(x, y)$ is said to have a maximum value at $x = a$; $y = b$ if $f(a, b) > f(a+h, b+k)$; for small & independent values of h and k; positive or negative.

$\rightarrow$ A function $f(x, y)$ is said to have a minimum value at $x = a$, $y = b$ if $f(a, b) < f(a+h, b+k)$ for small & independent values of h & k; +ve (or) -ve.

* Rules to find extreme values of a function $f(x, y)$:

1) Find all the 1^{st} order partial derivatives.

2) Equating $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Solve the simultaneous equations:

we get (x_1, y_1); (x_2, y_2) be the stationary points of these functions.

3) For each stationary point in step-2; find

$$r = \frac{\partial^2 f}{\partial x^2} \; ; \; s = \frac{\partial^2 f}{\partial x \partial y} \text{ (or) } \frac{\partial^2 f}{\partial y \partial x} \; ; \; t = \frac{\partial^2 f}{\partial y^2}.$$

4) (i) If $rt - s^2 > 0$; $r < 0$ for a particular solution (x_1, y_1) of step-2; then f has a maximum value at (x_1, y_1).

f_{max} = substituting (x_1, y_1) in the given eqn.

(ii) If $rt-s^2>0$ and $r>0$ for a particular solution (x_1, y_1) of step-(2) then f has minimum value at (x_1, y_1)

f_{min} = substituting (x_1, y_1) in given fr

(iii) If $rt-s^2<0$ for a particular solution $(x_1, y$ of step-2; then f has no extreme values at (x_1, y_1) ; f has neither max nor min at (x_1, y_1) ; (x_1, y_1) is called saddle point

(iv) If $rt-s^2=0$; the case is doubtful & requires further simplification.

Similarly ~~do the~~ check the above ~~points~~ cond for diff. stationary points.

<u>Note</u>: The above procedure is applicable to find max & min of $f(x, y)$ without having constraints.

* PROBLEMS

1) Examine the function x^3+y^3-3axy for maxima & minima.

$$f(x, y) = x^3+y^3-3axy.$$

1) $$\frac{\partial f}{\partial x} = 3x^2-3ay = 0$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax = 0$$

$$x^2 = ay \qquad y^2 = ax$$

$$\frac{y^4}{a^2} = ay \qquad \boxed{x = \frac{y^2}{a}}$$

$y^3 = a^3$ $\quad$ $x = \frac{a^2}{a}$

$\boxed{y = a}$ $\quad$ $\boxed{x = a}$ $\quad$ $\boxed{x = 0}$

$y=0$

$P(a, a)$. $\quad Q(0,0)$

$\boxed{\text{at } Q(0,0)}$

$\frac{\partial^2 f}{\partial x^2} = 6x = r$
$6(a) = 0$

$\frac{\partial^2 f}{\partial y^2} = 6y. = t$
$6(0) = 0.$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = 0 - 3a = -3a. = s.$

$rt - s^2 = 36xy - 9a^2$
$= 36(0) - 9a^2 = -9a^2 < 0$

$\Rightarrow$ function has neither max nor min at $(0,0)$.
$Q(0,0)$ is a saddle point.

$\Rightarrow$ $\boxed{\text{at } (a, a)}$

$r = 6x$ $\quad$ $t = 6y$ $\quad$ $s = -3a.$

$r = 6a$ $\quad$ $t = 6a.$ $\quad$ $s = -3a.$

$rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\boxed{r > 0}$

f_{min} = Substitute (a, a) in $f(x, y)$.

$= a^3 + a^3 - 3a^3$

$\boxed{f_{min} = -a^3}$

* The point where the function has neither ~~the~~ maximum (or) min. at that point is called a saddle point

2) $f(x,y) = x^3 y^2 (1-x-y)$

$$\frac{\partial f}{\partial x} = y^2\left[3x^2(1-x-y) + x^3(-1)\right] = 0$$

$$\frac{\partial f}{\partial y} = x^3\left[2y(1-x-y) + y^2(-1)\right] = 0$$

$$3x^2y^2 - 3x^3y^2 - 3x^2y^3 - x^3y^2 = 0$$

$$2yx^3 - 2yx^4 - 2x^3y^2 - x^3y^2 = 0.$$

$$\boxed{y=0;\ x=0}$$

$$y^2\left[x^2(3-3x-3y-x)\right] = 0$$

$$\boxed{3 = 4x + 3y}$$

$$x^3\left[y(2-2x-2y-y)\right] = 0$$

$$\boxed{2 = 2x+3y}$$

$$3 = 4x+3y$$
$$\underline{-\ 2 = 2x + 3y}$$
$$1 = 2x$$

$x = \frac{1}{2}$, $y = \frac{1}{3}$

~~P(0,0)~~

P(0,0) & Q$\left(\frac{1}{2}, \frac{1}{3}\right)$ are the stationary points

$$\therefore f = x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2yx^3 - 2yx^4 - 3x^3y^2$$

$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\cancel{f_{yy} = 6x^2y - 8x^3y^2}$

$f_{yy} = 2x^3 - 2x^4 - 6yx^3.$

$f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

at $(0,0)$

$f_{xx} = 0 \quad f_{xy} = 0 \quad rt - s^2 = 0$

$f_{yy} = 0 \quad \Rightarrow$ Further process to be ~~d~~ determined.

at $\left(\frac{1}{2}, \frac{1}{3}\right)$

$f_{xx} = \left(6 \times \frac{1}{2} \times \frac{1}{9}\right)_3 - 12 \times \frac{1}{4} \times \frac{1}{9}_3 - 6 \times \frac{1}{2} \times \frac{1}{27}_9.$

$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = \frac{-1}{9} < 0 = r$

$f_{xy} = 6\left(\frac{1}{4} \times \frac{1}{3}\right) - 8 \times \frac{1}{8} \times \frac{1}{3} - 9 \times \frac{1}{4} \times \frac{1}{9}$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{6-4-3}{12} = \frac{-1}{12} = s$

$f_{yy} = 2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6 \times \frac{1}{8} \times \frac{1}{3} \quad \cancel{rt - s^2}.$

$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8} = 0 = t.$

H.W 1) $f(x,y) = x^4 + 2x^2y - x^2 + 3y^2$

2) $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$rt - s^2 = 0 - \frac{1}{144} < 0 \quad r < 0$

$f_{min} = f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8} \times \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right)$

$= \frac{1}{72} \times \frac{1}{6}.$

$= \frac{1}{432}$

5) A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction

Let $x \longrightarrow$ length

$y \longrightarrow$ breadth

$z \longrightarrow$ height are the dimensions

$V = xyz$ $\boxed{z = \frac{V}{xy}}$

~~$S = 2xy(x+y)$~~

$S \rightarrow$ surface area of rectangular box

$S = xy + 2yz + 2zx$ (since open at the top)

$= xy + \frac{2Vy}{xy} + 2x\frac{V}{xy}$

$$\boxed{S(x,y) = xy + \frac{2V}{x} + \frac{2V}{y}}$$

$$\frac{\partial S}{\partial x} = y - \frac{2V}{x^2} = 0$$

$$\frac{\partial S}{\partial y} = x - \frac{2V}{y^2} = 0$$

$2V = x^2 y$ $\quad xy^2 = 2V$

$y = \frac{2V}{4V^2} \cdot \times y^4$ $\quad x = \frac{2V}{y^2}$

$\frac{1}{2V} y^4 - y = 0$

$y\left[\frac{y^3}{2V} - 1\right] = 0$

$y = 0;\ y^3 = 2V$

~~$y = 2\sqrt{2V}$~~

$$\boxed{y = \sqrt[3]{2V}}$$

$$x = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3} = y.$$

$$z = \frac{V}{xy} = \frac{V}{(4V^2)^{1/3}} = \left(\frac{V}{4}\right)^{1/3} = \frac{V}{(2V)^{2/3}}$$

$$= \frac{V^{1/3}}{2^{2/3}}$$

$$x = (2V)^{1/3} \qquad y = (2V)^{1/3} \qquad z = \frac{(2V)^{1/3}}{2}$$

$$\boxed{x = y = 2z}$$

$$S_{min} = (2V)^{2/3} + \frac{2V}{(2V)^{1/3}} + \frac{2V}{(2V)^{1/3}}$$

$$= 3 \times (2V)^{2/3}$$

$$= \underline{\underline{3x^2}}$$

08/02/2022

Q: Find the extreme points of

$\sin x + \sin y + \sin(x+y) = f(x,y)$

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0$ —(1)

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0$ —(2)

$\cos x = -\cos(x+y)$

$\cos y = -\cos(x+y)$.

$\cos x - \cos y = 0$

~~$\cos x + \cos x \cos y - \sin x \sin y = 0$~~

~~$\cos x(1+\cos y) = \sin x \sin y$.~~

~~$\tan x = \frac{2\cos^2 y/2}{2\sin\frac{y}{2}\cos y/2}$~~

~~$\tan x = \tan\frac{y}{2}$~~

~~$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} =$~~ $\boxed{x=y}$ —(3)

Put (3) in (1)

$\cos x + \cos 2x = 0$

$2\cos^2 x + \cos x - 1 = 0$

$2\cos x(\cos x + 1) - (\cos x + 1) = 0$

$\cos x = \frac{1}{2}$, $\cos x = -1$

$\boxed{x = \frac{\pi}{3}, \pi}$

$\boxed{y = \frac{\pi}{3}, \pm\pi}$

$P\left(\frac{\pm\pi}{3}, \frac{\pm\pi}{3}\right)$. $Q(\pm\pi, \pm\pi)$. $R\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$.

$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$ $r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$ $s = -\frac{\sqrt{3}}{2}$

$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$ $t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

~~$rt - s^2 = \left(\frac{3}{4} + \frac{1}{4} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}$~~

~~$\boxed{rt - s^2 = \frac{3}{4} - \frac{\sqrt{3}}{2}}$ $= 0.75 - 0.866 < 0$~~

~~f has neither maxima nor minima.~~

$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$ $r < 0$

$(f_{max}) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.

$R\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$ $rt - s^2 > 0$ $r > 0$

$f_{min} = \frac{-3\sqrt{3}}{2}$

H.W 1) $f(x,y) = \sin x \sin y \sin(x+y)$

* Find the min value of $x^2+y^2+z^2$ given

$xyz = a^3$.

$$f(x,y) = x^2 + y^2 + \frac{a^6}{x^2y^2}$$

$$\frac{\partial f}{\partial x} = 2x - \frac{2a^6}{y^2}\cdot\frac{1}{x^3}$$

$$\boxed{\frac{\partial f}{\partial x} = 2x - \frac{2a^6}{x^3y^2}}$$

$$x^4y^2 - a^6 = 0$$

$$(x^2y)^2 = (a^3)^2$$

$$\boxed{x^2y = a^3}$$

$$y = \frac{a^3}{x^2}$$

$$\boxed{\frac{\partial f}{\partial y} = 2y - \frac{2a^6}{y^3x^2}}$$

$$y^4x^2 = a^6$$

$$y^2x = a^3$$

$$\frac{a^6}{x^3} = a^3$$

$$x = a$$

$$y = a$$

* LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:

(FUNCTION WITH CONDITION:)

Sometimes it is required to find the stationary points of a function of several variables which are not all independent but are connected by some given relations.

It is required to find the extreme values of a function subject to some conditions involving the variables, such type of problems can be solved by using the method of Lagrange's undetermined multipliers.

Working Rule:

Suppose it is required to find the extremum for the function $f(x,y,z)$ Subject to condition

$$\phi(x,y,z) = 0$$

① $\boxed{F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)}$

$\lambda \to$ Lagrange's Multiplier;

which is determined by following condition.

② Obtain the equations

$f_x = 0 \Rightarrow f_x + \lambda\phi_x = 0$

$f_y = 0 \Rightarrow f_y + \lambda\phi_y = 0$

$f_z = 0 \Rightarrow f_z + \lambda\phi_z = 0$

③ Solve the equations: Values of (x,y,z) are obtained will give the stationary points of $f(x,y,z)$

We find Putting stationary points

$f_{max} =$ / $f_{min} =$ / Both.

* $f(x,y,z) = x^2+y^2+z^2$. $\phi(x,y,z) = xyz - a^3$

Consider Lagrange's function

$$F(x,y,z) = x^2+y^2+z^2 + \lambda(xyz - a^3)$$

$$\frac{\partial F}{\partial x} = 2x + yz\lambda = 0$$

$$\boxed{\lambda = \frac{-2x}{yz}} \quad -(1)$$

$$\frac{\partial F}{\partial y} = 2y + xz\lambda = 0$$

$$\boxed{\lambda = \frac{-2y}{xz}} \quad -(2)$$

$$\frac{\partial F}{\partial z} = 2z + xy\lambda = 0$$

$$\boxed{\lambda = \frac{-2z}{xy}} \quad -(3)$$

$$x^2 z = y^2 z$$

$$z(x^2 - y^2) = 0$$

$z = 0$; $x^2 = y^2$

$$\boxed{x = y}$$

(2) = (3)

$$y^2 x = z^2 x$$

$$\boxed{y = z}$$

$$\boxed{x = y = z}$$

$\cancel{\lambda = \frac{-2x}{x^2}} = \cancel{\frac{-2}{x}}$

$\cancel{\boxed{\lambda = -\frac{2}{x}}}$

$xyz = a^3$ $\quad x^3 = a^3$

$$\boxed{a = x = y = z}$$

$f(x,y,z) = x^2+y^2+z^2$ $\quad$ (a,a,a) is a stationary point.

$$\boxed{f_{min} = 3a^2}$$

2) Find the min of $x^2+y^2+z^2$ given $x+y+z = 3a$.

* Find the max. and min value of $x+y+z$. Subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$

$$f(x,y,z) = x+y+z$$

$$\phi(x,y,z) = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} - 1$$

$$F(x,y,z) = x+y+z+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$$\frac{\partial F}{\partial x} = 1 - \frac{\lambda}{x^2} = 0$$

$$\boxed{\lambda = x^2} \quad -(1)$$

$$\frac{\partial F}{\partial y} = 1 - \frac{\lambda}{y^2} = 0$$

$$\boxed{\lambda = y^2} \quad -(2)$$

$$\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2} = 0$$

$$\boxed{\lambda = z^2} \quad (3)$$

$$\boxed{x = y = z}$$

$$\frac{1}{\sqrt{\cancel{x}}} + \frac{1}{\sqrt{\cancel{x}}} + \frac{1}{\sqrt{\cancel{x}}} = 1$$

$$\frac{9}{x} = 1$$

$$\cancel{\boxed{x = \pm 3}} \quad \boxed{\lambda = 9}$$

$$\boxed{x = y = z = \pm 3}$$

$(3, 3, 3)$ $\quad (-3, -3, -3)$

$f(3,3,3) = 9 \longrightarrow$ max

$f(-3,-3,-3) = -9 \longrightarrow$ min

* Find a point on the plane $3x + 2y + z - 12 = 0$ which is nearest to the origin.

$\phi(x, y, z) = 3x + 2y + z - 12 = 0$

$P(x, y, z)$

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$\cancel{F(x, y, z) = 3x + 2y + z - 12 + \lambda\sqrt{x^2 + y^2 + z^2}}$

$\cancel{f_x = 3 + \frac{\lambda \cdot 2x}{2\sqrt{x^2 + y^2 + z^2}}}$

$\cancel{\boxed{\lambda = \frac{-3\sqrt{x^2 + y^2 + z^2}}{x}}}$

* $\phi(x, y, z) = 3x + 2y + z - 12$

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow x^2 + y^2 + z^2$

$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$F_x = 2x + 3\lambda = 0$

$F_y = 2y + 2\lambda = 0$

$F_z = 2z + \lambda = 0$

$$\boxed{\begin{aligned} x &= -\frac{3\lambda}{2} \\ y &= -\lambda \\ z &= -\frac{\lambda}{2} \end{aligned}}$$

$3x + 2y + z = 12$

$-\frac{9\lambda}{2} - 2\lambda - \frac{\lambda}{2} = 12$

$-\frac{14\lambda}{2} = 12$

$$\boxed{\lambda = -\frac{12}{7}}$$

$x = \frac{3}{\cancel{2}} \times \frac{\cancel{12}^{6}}{7}$

$$\boxed{x = \frac{18}{7}} \quad \boxed{y = \frac{12}{7}} \quad \boxed{z = \frac{6}{7}}$$

* Find max value of $x^2 y^3 z^4$ if $2x + 3y + 4z = a$.
* Find the shortest distance from the origin to the surface; $xyz^2 = 2$
* Given that $x + y + z = a$; find the max value of $x^m y^n z^p$.
* $f(x, y) = \sin x \cdot \sin y \sin(x+y)$

$\frac{\partial f}{\partial x} = \cos x \sin y \sin(x+y) + \sin y \sin x \cos(x+y)$

$\frac{\partial f}{\partial x} = \sin y\left[\cos x \sin(x+y) + \sin x \cos(x+y)\right]$

$\frac{\partial f}{\partial x} = \sin y\left[\sin(2x+y)\right]$

$$\frac{\partial f}{\partial y} = \sin x\left(\sin(x+2y)\right).$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$\sin y \sin(2x+y) = 0$; $\sin x \cdot \sin(x+2y) = 0$

$y = 0$; $\boxed{2x = -y}$; $x = 0$; $\boxed{x = -2y}$

$\cancel{x = -\frac{y}{2}}$

$\cancel{\frac{\partial f}{\partial y} = \sin x(\sin(x+2y))}$

$P(0,0)$

$$\frac{\partial f}{\partial x} = \sin y \sin(2x+y)$$

$$\frac{\partial f}{\partial y} = \sin x \cdot \sin(x+2y)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\sin y \cos(2x+y) \qquad r = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2\sin x \cos(x+2y) \qquad t = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cos x \sin(x+2y) + \sin x \cos(x+2y)$$

$s = 0 + 0 = 0$

$rt - s^2 = 0 \Rightarrow$ further simplification is required.

* $x^2+y^2+z^2 = f(x,y,z)$ $\qquad x+y+z = 3a$

$\phi(x,y,z) = x+y+z-3a$

Lagrange's function:

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$F(x,y,z) = x^2+y^2+z^2+\lambda(x+y+z-3a)$.

$$\frac{\partial F}{\partial x} = 2x+\lambda = 0 \qquad x = -\frac{\lambda}{2}$$

$$\frac{\partial F}{\partial y} = 2y+\lambda = 0 \qquad y = -\frac{\lambda}{2}$$

$$\frac{\partial F}{\partial z} = 2z+\lambda = 0 \qquad z = -\frac{\lambda}{2}$$

$\boxed{x = y = z}$

$3x = 3a$ $\qquad \boxed{x = a = y = z}$

$P(a,a,a)$ is the ~~point~~ stationary point.

$f(a,a,a) = \underline{3a^2}$

* $f(x,y,z) = x^2y^3z^4$ $\qquad \phi(x,y,z) = 2x+3y+4z-a$

$F(x,y,z) = x^2y^3z^4 + \lambda(2x+3y+4z-a)$

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0$$

$\lambda = -xy^3z^4$ — (1)

$$\frac{\partial F}{\partial y} = 3y^2x^2z^4 + 3\lambda = 0$$

$\lambda = -x^2y^2z^4$ — (2)

$$\frac{\partial F}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0 \qquad \lambda = -x^2y^3z^3$$

$\lambda = -xy^3z^4$ — (1) $\qquad 2x+3y+4z-a=0$

$\lambda = -x^2y^2z^4$ — (2)

$\lambda = -x^2y^3z^3$ — (3)

(1) = (2) $\qquad$ (2) = (3)

$\boxed{y = x} \qquad \boxed{z = y}$

$\Rightarrow \boxed{x = y = z}$

$2x+3x+4x-a=0$

$9x = a$

$\boxed{x = \frac{a}{9}} \qquad P\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right)$

$f\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right) = x^2y^3z^4$

$= \frac{a^2}{(9)^2} \cdot \frac{a^3}{(9)^3} \cdot \frac{a^4}{9^4}$

$= \left(\frac{a}{9}\right)^9$

* $f(x,y,z) = \sqrt{x^2+y^2+z^2} \Rightarrow x^2+y^2+z^2$

$\phi(x,y,z) = xyz^2 - 2$

Lagrange's Function :

$F(x,y,z) = f(x,y,z) + \lambda(xyz^2-2)$

$= x^2+y^2+z^2+\lambda(xyz^2-2)$

$\frac{\partial F}{\partial x} = 2x + yz^2\lambda = 0$

$\lambda = \frac{-2x}{yz^2} \qquad \boxed{\frac{x}{yz^2} = \frac{-\lambda}{2}}$ — (1)

$\frac{\partial F}{\partial y} = 2y + xz^2\lambda = 0 \qquad \boxed{\frac{y}{xz^2} = \frac{-\lambda}{2}}$ — (2)

$\frac{\partial F}{\partial z} = 2z + 2xyz\lambda = 0 \qquad \boxed{\frac{1}{xy} = -\lambda}$ — (3)

$x^2z^2 = y^2z^2$

$\boxed{x = y}$

$\frac{1}{2xy} = \frac{y}{xz^2}$

$xz^2 = 2xy^2$

$\boxed{z = \sqrt{2}\,y}$

$xyz^2 = 2 \Rightarrow x^2 \cdot 2x^2 = 2$

$x^4 = 1$

$x = \pm 1$

$P(1, 1, \sqrt{2}) \qquad Q(-1, -1, \sqrt{2})$

$f(1,1,\sqrt{2}) = \sqrt{1+1+2}$

$= 2$

$f(-1,-1,\sqrt{2}) = 2$

$\therefore$ Shortest distance from origin to surface $= 2$

* $x+y+z = a \Rightarrow \phi(x,y,z) = x+y+z-a$

$f(x,y,z) = x^m y^n z^p$

Lagrange's function:

$$F(x,y,z) = f(x,y,z) + \lambda(\phi(x,y,z))$$

$$F(x,y,z) = x^m y^n z^p + \lambda(x+y+z-a)$$

$$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda = 0$$

$$\boxed{\lambda = -m x^{m-1} y^n z^p} \quad (1)$$

$$\frac{\partial F}{\partial y} = n y^{n-1} x^m z^p + \lambda$$

$$\boxed{\lambda = -n y^{n-1} x^m z^p} \quad (2)$$

$$\frac{\partial F}{\partial z} = p z^{p-1} x^m y^n$$

$$\boxed{\lambda = -p z^{p-1} x^m y^n} \quad (3)$$

$\cancel{\lambda x}$

$(1) = (2)$

$$\frac{m}{x} \cdot x^m y^n z^p = \frac{n}{y} x^m y^n z^p$$

$$\boxed{my = nx} \quad (4)$$

$(2) = (3)$

$$\boxed{\frac{n}{y} = \frac{p}{z}} \qquad \boxed{nz = py} \quad (5)$$

$$\boxed{x = \frac{my}{n} \qquad z = \frac{py}{n}}$$

$$x + y + z = a$$

$$\left(\frac{m}{n} + 1 + \frac{p}{n}\right) \cancel{x}y = a$$

$$\boxed{\cancel{x}y = \frac{an}{m+n+p}}$$

$$\boxed{x = \frac{am}{m+n+p}}$$

$$\boxed{z = \frac{ap}{m+n+p}}$$

$$f(x,y,z).$$

$$f_{max} = \frac{a^{m+n+p}}{(m+n+p)^{m+n+p}} \cdot m^m n^n p^p$$

09/02/2022

1) Find the max & min distances of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=1$.

$P(x,y,z)$ be any point on sphere.

$Q(3,4,12)$

$$PQ = f(x,y,z) = \left(\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}\right)^2$$

$$f(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$$

$$\phi(x,y,z) = x^2+y^2+z^2-1$$

$$F(x,y,z) = f(x,y,z) + \lambda(\phi(x,y,z))$$

$$F(x,y,z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$$

$$\frac{\partial F}{\partial x} = 2(x-3) + 2x\lambda = 0$$

$$x(1+\lambda) = 3$$

$$\boxed{x = \frac{3}{1+\lambda}} \quad \text{— (1)}$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2y\lambda = 0$$

$$y(1+\lambda) = 4$$

$$\boxed{y = \frac{4}{1+\lambda}} \quad \text{— (2)}$$

$$\frac{\partial F}{\partial z} = 2(z-12) + 2z\lambda = 0$$

$$\boxed{z = \frac{12}{1+\lambda}} \quad \text{— (3)}$$

$$\frac{9+16+144}{(1+\lambda)^2} = 1$$

$$1+\lambda = \pm 13$$

$1+\lambda = 13$ | $1+\lambda = -13$

$\lambda = 12$ | $\lambda = -14$

$x = \frac{3}{13}$, $y = \frac{4}{13}$, $z = 12/13$ | $x = \frac{-3}{13}$, $y = \frac{-4}{13}$, $z = -\frac{12}{13}$

$$f\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) = \sqrt{\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2}$$

$$= \sqrt{\frac{9\times 144}{169} + \frac{16\times 144}{169} + \frac{144\times 144}{13}}$$

$$= \frac{12}{13} \times 13 = \underline{\underline{12}}$$

$$f\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right) = \sqrt{\left(\frac{14}{13}\right)^2 (9+16+144)}$$

$$= \frac{14}{13} \times 13 = \underline{\underline{14}}$$

* $f(x,y,z) = xy + 2xz + 2yz$

$$\phi(x,y,z) = xyz - 32$$

$$F(x,y,z) = xy + 2xz + 2yz + \lambda(xyz - 32)$$

$$\frac{\partial F}{\partial x} = y + 2z + yz\lambda = 0 \quad \text{— (1)}$$

$$\frac{\partial F}{\partial y} = x + 2z + xz\lambda = 0 \quad \text{— (2)}$$

$$\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy = 0 \quad \text{— (3)}$$

(1) $\times x$ − (2) $\times y$.

$$\cancel{xy} + 2zx + \cancel{xyz\lambda} = 0$$

$$-\cancel{xy} + 2zy + \cancel{xyz\lambda} = 0$$

$$\boxed{x = y}$$

(2) $\times y$ − (3) $\times z$

$$\cancel{xy} + \cancel{2yz} + \cancel{\lambda xyz} - 2zy - 2xz - \cancel{\lambda xyz} = 0$$

$$y = 2z$$

$$xyz = 32$$

$$x^2 \cdot \frac{x}{2} = 32$$

$$x^3 = 64$$

$$x = 4$$

$$\boxed{4, 4, 2}$$

$f_{min} = 16 + 16 + 16 = 48$ sq. units

→ S.T. the rectangular solid of max volume that can be inscribed in a sphere is a cube.

Let $2x, 2y, 2z$ be l, b, h of rectangular solid.

$$V = 8xyz = f(x, y, z)$$

$r \rightarrow$ radius of sphere.

$$\phi(x, y, z) = x^2 + y^2 + z^2 - r^2$$

$$F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - r^2)$$

$$\frac{\partial F}{\partial x} = 8yz + 2x\lambda = 0$$

$$\boxed{\lambda = \frac{-4yz}{x}}$$

$$\frac{\partial F}{\partial y} = 8xz + 2y\lambda$$

$$\boxed{\lambda = -\frac{4xz}{y}}$$

$$\frac{\partial F}{\partial z} = 8xy + 2z\lambda$$

$$\lambda = -\frac{4xy}{z}$$

$$\frac{yz}{x} = \frac{xz}{y}$$

$$\boxed{x = y} \qquad \boxed{y = z}$$

$$3x^2 = r^2$$

$$\boxed{x = \frac{r}{\sqrt{3}}}$$

$$\boxed{V = \frac{8r^3}{3\sqrt{3}}}$$

The rectangular solid is a cube.

→ Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Let $2x, 2y, 2z$ be l, b, h of rectangular parallelopiped.

$$V = 8xyz$$

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\frac{\partial F}{\partial x} = 8yz + \frac{2x\lambda}{a^2} = 0$$

$$\frac{\lambda}{4a^2} = \frac{-yz}{x}$$

$$\frac{\lambda}{4b^2} = -\frac{x\cancel{z}}{\cancel{z}y}$$

$$\frac{\lambda}{4c^2} = -\frac{xy}{z}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{3x^2}{a^2} = 1$$

$$x = \frac{a}{\sqrt{3}} \quad y = \frac{b}{\sqrt{3}} \; ; \; z = \frac{c}{\sqrt{3}}.$$

$$V = \frac{8abc}{3\sqrt{3}}$$

Q: If $u = x^2+y^2+z^2$; given $ax^2+by^2+cz^2=1$ & $lx+my+nz=0$; find extreme values of u.

Lagrange's formula

$$F(x,y,z) = f(x,y,z) + \lambda(\phi_1(x,y,z)) + \mu(\phi_2(x,y,z))$$

$$F(x,y,z) = x^2+y^2+z^2 + \lambda(ax^2+by^2+cz^2) + \mu(lx+my+nz).$$

$$F_x = 2x + 2ax\lambda + \mu l = 0 \quad \text{—}(1) \times x$$

$$F_y = 2y + 2ay\lambda + \mu m = 0 \quad \text{—}(2) \times y$$

$$F_z = 2z + 2az\lambda + \mu n = 0 \quad \text{—}(3) \times z.$$

$$2(x^2+y^2+z^2) + 2\lambda(ax^2+by^2+cz^2) + \mu(lx+my+nz) = 0$$

$$2u + 2\lambda = 0$$

$$\lambda = -u$$

$$2x - 2aux + \mu l = 0$$

$$x = \frac{\mu l}{2(au-1)} \qquad y = \frac{\mu m}{2(bu-1)}$$

$$z = \frac{\mu n}{2(cu-1)}$$

Sub x, y, z in

$$lx + my + nz = 0$$

$$\frac{\mu l^2}{2(au-1)} + \frac{\mu m^2}{(bu-1)} + \frac{\mu n^2}{(cu-1)} = 0$$

$$\mu\left[l^2(bcu^2 - u(b+c)+1) + m^2(acu^2 - u(a+c)+1) + \cancel{n^2}\right.$$

* Find the extreme values of $f(x,y,z) = 2x+3y+z$ such $x^2+y^2=5$ and $x+z=1$

$$F(x,y,z) = 2x+3y+z) + \lambda_1(x^2+y^2-5) + \lambda_2(x+z-1)$$

$$\frac{\partial F}{\partial x} = 2 + 2x\lambda_1 + \lambda_2 = 0$$

$$\frac{\partial F}{\partial y} = 3 + 2y\lambda_1 = 0$$

$$\frac{\partial F}{\partial z} = 1 + \lambda_2 = 0.$$

$$\boxed{\lambda_2 = -1}$$

$$\cancel{2(1+x\lambda_1) = +1}$$

$$\cancel{1+x\lambda_1 = \frac{+1}{2}}$$

$$\lambda_1 = \frac{-3}{2y} \qquad x = \frac{-1}{2\lambda_1}$$

$$y = \frac{-3}{2\lambda_1}$$

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$$

$$10 = 20\lambda_1^2 \qquad \lambda_1^2 = \frac{1}{2}$$

$$\cancel{\lambda_1^2 = \frac{5}{2}}$$

$$\cancel{\lambda_1 = \pm\frac{\sqrt{5}}{\sqrt{2}}}$$

$$\boxed{\lambda_1 = \pm\frac{1}{\sqrt{2}}}$$

$$x = \frac{-1}{2\times 1} \times \sqrt{2}$$

$$\boxed{x = \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}$$

$$y = \frac{-3}{2} \times \sqrt{2}$$

$$\boxed{y = \frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}}$$

$$z = 1 - x$$

$$= 1 + \frac{1}{\sqrt{2}}$$

$$\boxed{z = \frac{\sqrt{2}+1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}}$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{\sqrt{2}+1}{\sqrt{2}}\right)$$

$$\frac{-2}{\sqrt{2}} - \frac{9}{\sqrt{2}} + \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$= \frac{-11+\sqrt{2}+1}{\sqrt{2}}$$

$$= \frac{-10+\sqrt{2}}{\sqrt{2}} = \frac{-5\sqrt{2}+1}{} = \boxed{1-5\sqrt{2}}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{9}{\sqrt{2}} + \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$= \frac{10+\sqrt{2}}{\sqrt{2}} = 1+5\sqrt{2}$$

* Shortest distance b/w the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Let (x, y) be a point on ellipse
(u, v) be a point on line.

$$\phi(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$$

$$\psi(x, y) = 2u + v - 10$$

$$f(x, y, z) = (x-u)^2 + (y-v)^2$$

$$F(x, y, z) = (x-u)^2 + (y-v)^2 + \lambda\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \mu(2u + v - 10)$$

$$F_x = 2(x-u) + \frac{2x\lambda}{4} = 0$$

$$F_y = 2(y-v) + \frac{2\lambda y}{9} = 0$$

$$F_u = -2(x-u) + 2\mu = 0$$

$$F_v = -2(y-v) + 2\mu = 0$$

$$\cancel{x = \frac{8u}{8+\lambda}}$$

$$2x\left(1 + \frac{\lambda}{4}\right) = 2u$$

$$x = \frac{u \times 4}{4+\lambda}$$

$$\boxed{x = \frac{4u}{4+\lambda}}$$