Fourier Series

- Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient.
- This sum we call as Fourier Series.

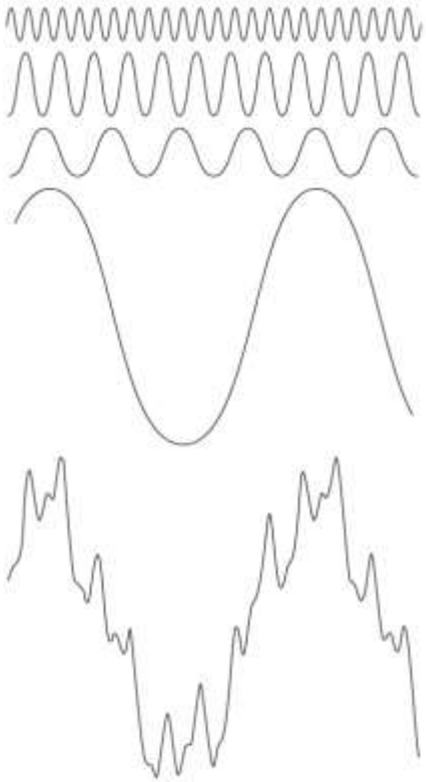


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier Series

- A function f(t) of a continuous variable t that is periodic with period T, can be expressed in Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Fourier Transform

- Even functions that are not periodic(but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighing function.
- The formulation in this case is the Fourier transform.
- A function expressed either in Fourier series or transform, can be reconstructed (recovered) completely via an inverse process, with no loss of information.
- This characteristic allows us to work in the "Fourier domain" and then return to the original domain of the function without losing any information

Fourier Transform

- The Fourier Transform of a continuous function f(t) of a continuous variable, t, is defined by the equation

$$\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt$$

Where μ is also a continuous variable.

- The Fourier transform is a function of only μ, because ‘t’ is integrated out.
- So for convenience $\Im\{f(t)\}$ can be written as

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt$$

Fourier Transform

- Using Euler's formula we can express the Fourier transform of function f(t) as

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\left[\cos(2\pi\mu t) - j\sin(2\pi\mu t)\right] dt$$

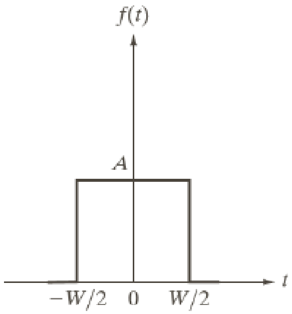
Inverse Fourier Transform

- Conversely given F(μ), we can obtain f(t) using the inverse Fourier transform

$$f(t) = \Im^{-1}\{F(\mu)\}$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} \, d\mu$$

- Compute Fourier Transform of a simple continuous function f(t) shown in the below figure



$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

where we used the trigonometric identity $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$. In this case, the complex terms of the Fourier transform combined nicely into a real sine function. The result in the last step of the preceding expression is known as the *sinc* function, which has the general form

$$\operatorname{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)} \qquad (4\text{-}23)$$

where $\operatorname{sinc}(0) = 1$ and $\operatorname{sinc}(m) = 0$ for all other *integer* values of m. Figure 4.4(b) shows a plot of $F(\mu)$.

In general, the Fourier transform contains complex terms, and it is customary for display purposes to work with the magnitude of the transform (a real quantity), which is called the *Fourier spectrum* or the *frequency spectrum*:

$$|F(\mu)| = AW\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$$

Figure 4.4(c) shows a plot of $|F(\mu)|$ as a function of frequency. The key properties to note are (1) that the locations of the zeros of both $F(\mu)$ and $|F(\mu)|$ are inversely proportional to the width, W, of the "box" function; (2) that the height of the lobes decreases as a function of distance from the origin; and (3) that the function extends to infinity for both positive and negative values of μ. As you will see later, these properties are quite helpful in interpreting the spectra of two dimensional Fourier transforms of images.

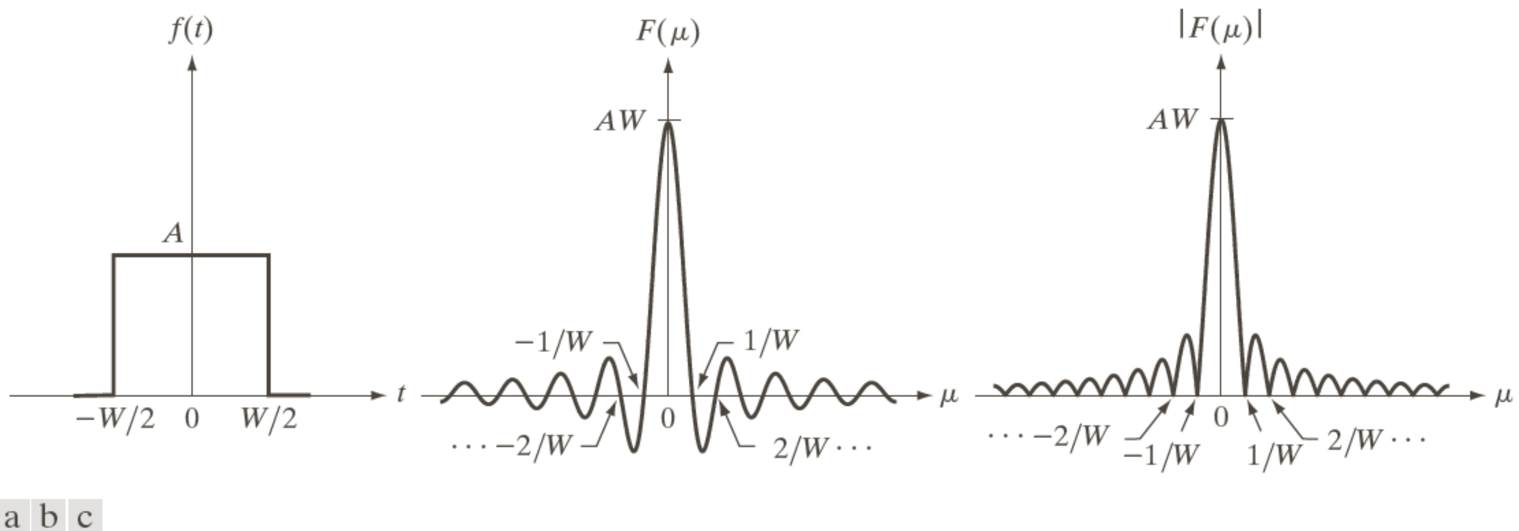
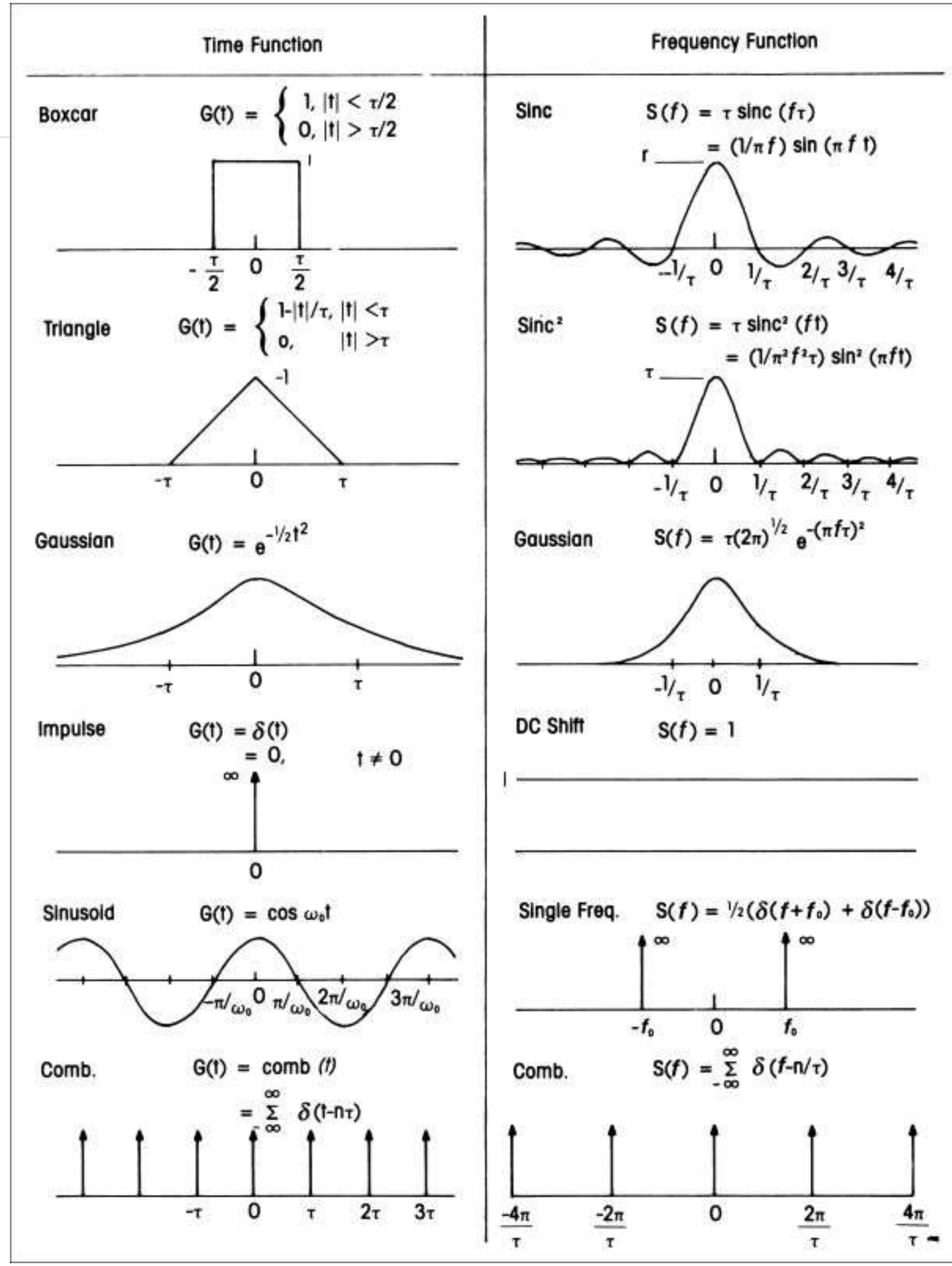


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.


Time Function
Frequency Function
Boxcar
G(t) = { 1, |t| < τ/2 ; 0, |t| > τ/2
Sinc
S(f) = τ sinc (fτ)
= (1/πf) sin (π f t)
Triangle
G(t) = { 1-|t|/τ, |t| <τ ; 0, |t| >τ
Sinc²
S(f) = τ sinc² (ft)
= (1/π² f² τ) sin² (πft)
Gaussian
G(t) = e^{-½t²}
Gaussian
S(f) = τ(2π)^{½} e^{-(πfτ)²}
Impulse
G(t) = δ(t)
= 0, t ≠ 0
DC Shift
S(f) = 1
Sinusoid
G(t) = cos ω₀t
Single Freq.
S(f) = ½(δ(f+f₀) + δ(f-f₀))
Comb.
G(t) = comb (t)
= Σ δ(t-nτ)
Comb.
S(f) = Σ δ(f-n/τ)

Impulse and it's Sifting property

- A unit impulse of a continuous variable t located at origin (t = 0), denoted as δ(t), is defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

- It satisfies the identity

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- An impulse has the sifting property with respect to integration ,

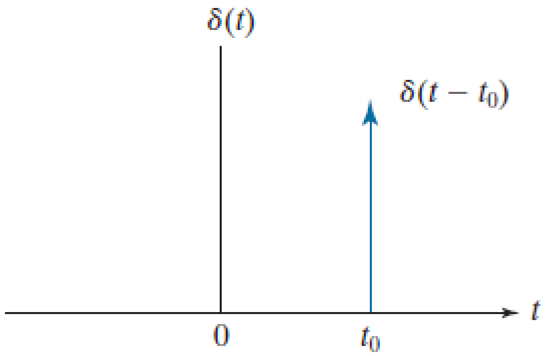
$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0)$$

Impulses and their sifting property

- A unit impulse of a continuous variable t located at t_0, denoted as $\delta(t - t_0)$, has the sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

- Which yields the value of the function at the impulse location, t_0.



- A unit discrete impulse of a variable x located at origin (x = 0), denoted as δ(x), is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

- It satisfies the identity

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

- The sifting property for discrete variables has the form

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

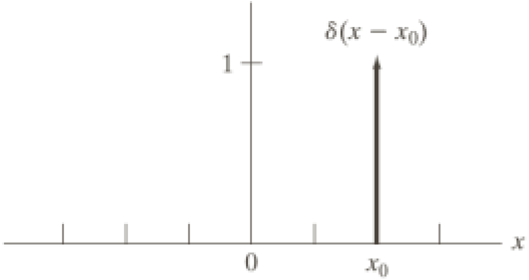
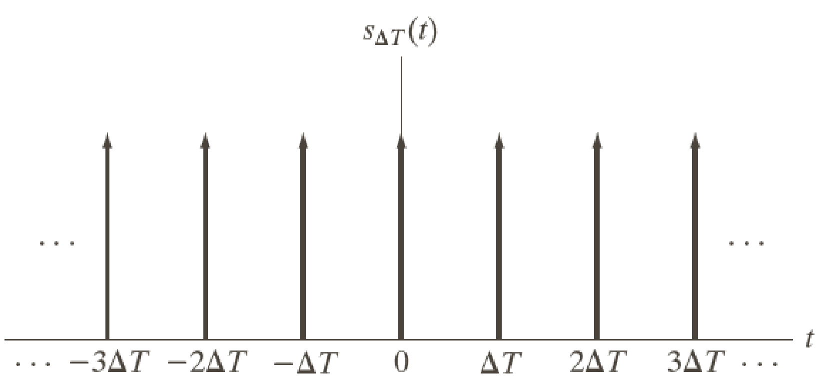


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Impulse Train

- An Impulse train, $s_{\Delta T}(t)$, defined as the sum of infinitely many periodic impulses ΔT units apart is given as

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



- Compute Fourier Transform of a unit impulse located at origin.

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t)\, dt \\
&= e^{-j2\pi\mu 0} = e^{0} \\
&= 1
\end{aligned}
$$

- Compute Fourier Transform of a unit impulse located at t_0.

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt$$

$$= e^{-j2\pi\mu t_0}$$

$$= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)$$

Fourier series of an Impulse Train function

- Impulse train is a periodic function with period ΔT, so it can expressed as a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

where

$$\begin{aligned} c_n &= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t)\, e^{-j\frac{2\pi n}{\Delta T}t}\, dt \\ &= \frac{1}{\Delta T} e^0 \\ &= \frac{1}{\Delta T} \end{aligned}$$

- The Fourier series expansion then becomes

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Fourier Transform of an Impulse Train function

- We know that

$$\Im\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

So, $S(\mu)$, the Fourier transform of the periodic impulse train $s_{\Delta T}(t)$, is

$$\begin{aligned} S(\mu) &= \Im\left\{s_{\Delta T}(t)\right\} \\ &= \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\ &= \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right) \end{aligned}$$

Convolution

- The convolution of two continuous functions, f(t) and h(t), of one continuous variable , t, is defined as

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

Fourier Transform of Convolution

- Compute the Fourier Transform of the convolution of the functions f(t) and h(t).

$$\Im\{f(t)\star h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau$$

$$= H(\mu)F(\mu)$$

Convolution theorem

- Consider the domain of ‘t’ as spatial domain and domain of ‘μ’ as the frequency domain.
- Fourier transform of the convolution of two functions in the spatial domain is equal to the product in the frequency domain of the Fourier transform of the two functions.
- Conversely , if we have the product of two transforms, we can obtain the convolution in the spatial domain by computing the inverse Fourier transform.
- *This* result is one-half of the *convolution theorem and is written as*

$$f(t) \star h(t) \Leftrightarrow H(\mu) F(\mu)$$

- The LHS and RHS *are a Fourier transform pair*
- The double arrow indicates that the expression on the right is obtained by taking the *forward Fourier transform of the expression on the left, while* the expression on the left is obtained by taking the *inverse Fourier transform of the* expression on the right.
- The other half of convolution theprem states that “ convolution in frequency domain is analogous to product in spatial domain.

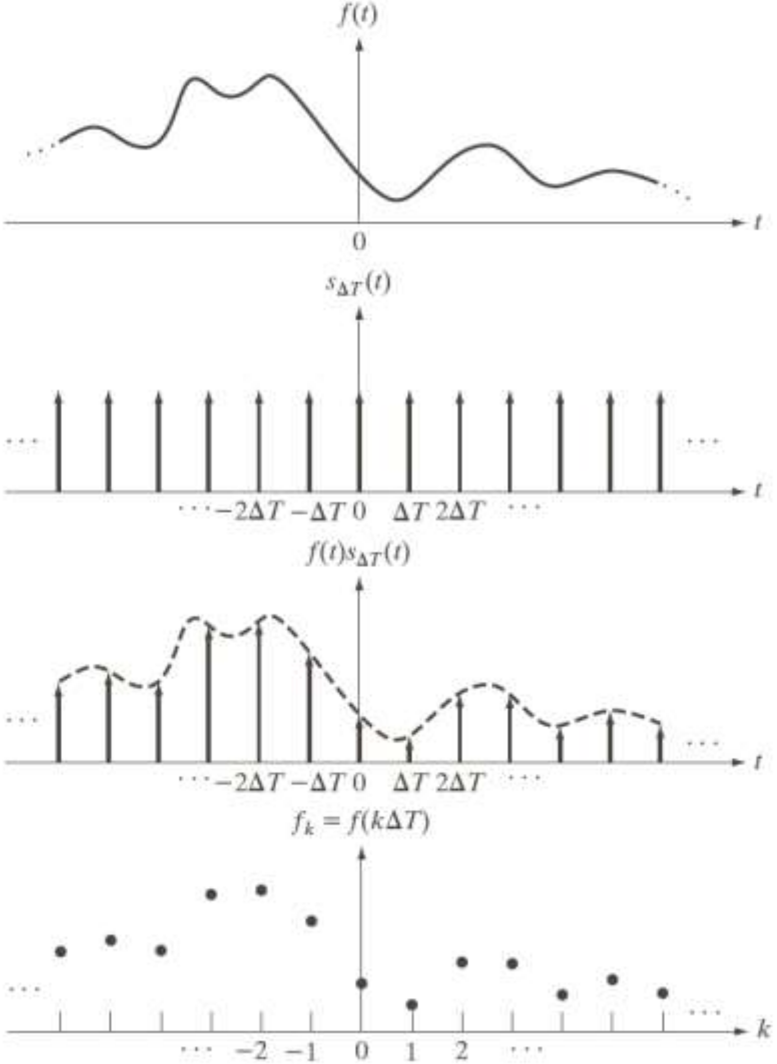
$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

Obtaining Sampled function

- Sampled function, $\tilde{f}(t)$, of a continuous function f(t) is obtained by multiplying f(t) by the sampling function (the impulse train) $s_{\Delta T}(t)$.

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\, dt$$
$$= f(k\Delta T)$$



Fourier Transform of Sampled function

$$\begin{aligned}\tilde{F}(\mu) &= \Im\{\tilde{f}(t)\} \\ &= \Im\{f(t)s_{\Delta T}(t)\} \\ &= F(\mu) \star S(\mu)\end{aligned}$$

where

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

- So we obtain convolution of $F(\mu)$ and $S(\mu)$ as shown below

Fourier Transform of Sampled function contd…

$$\begin{aligned}
\tilde{F}(\mu) &= F(\mu) \star S(\mu) \\
&= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau)\, d\tau \\
&= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

- The summation in the last line shows that
  - The Fourier transform of the sampled function is an infinite, periodic sequence of copies of *F(μ)*, the transform of the original, continuous function, *f(t)*.
  - The separation between the copies is determined by the value of $1/\Delta T$.

Graphical summary of preceding results

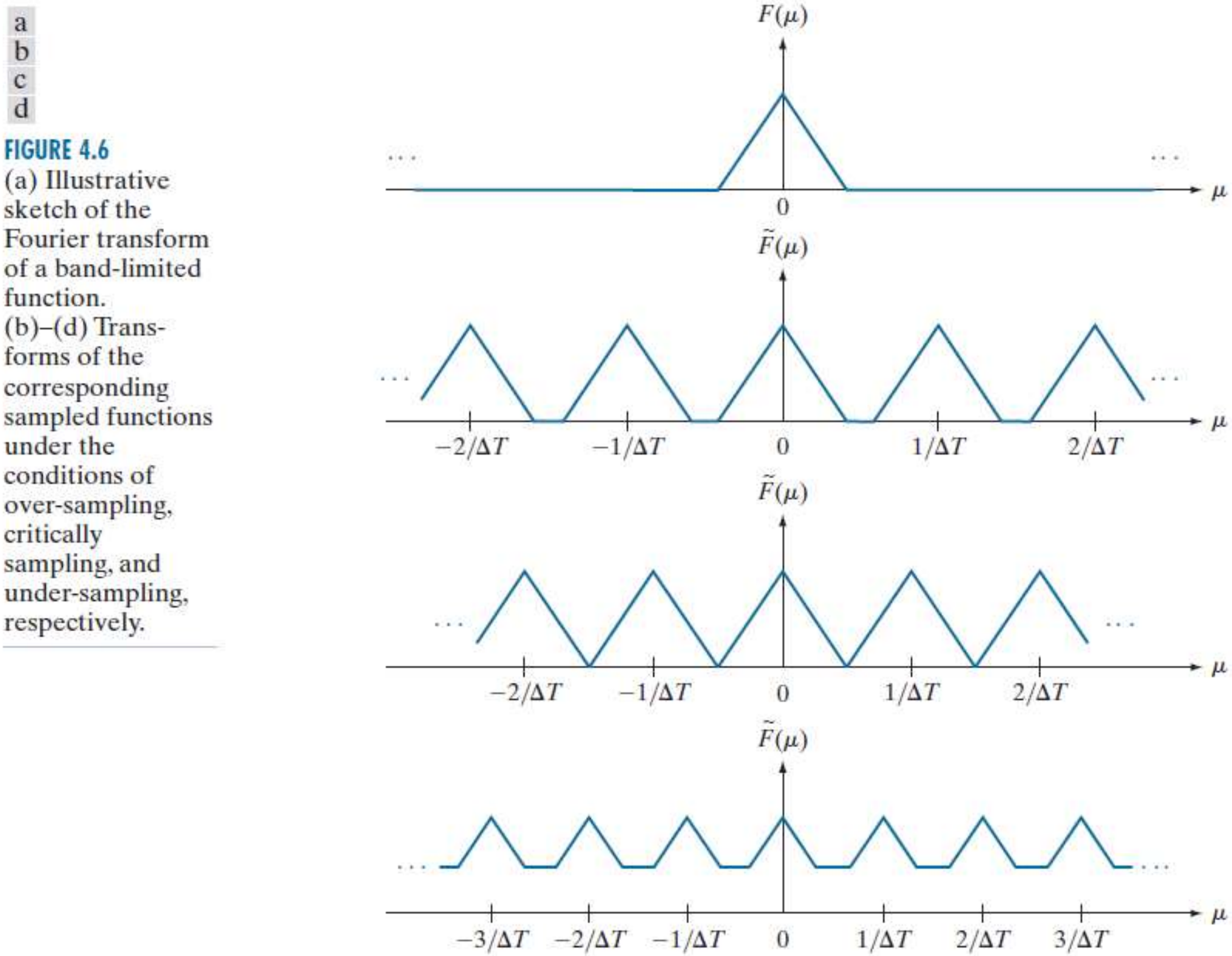


FIGURE 4.6 (a) Illustrative sketch of the Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled functions under the conditions of over-sampling, critically sampling, and under-sampling, respectively.

Sampling Theorem and Nyquist rate

- A Continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.

  i.e $1/\Delta T > 2\mu_{max}$
- This result is known as Sampling Theorem.
- A sampling rate *exactly equal to twice* the highest frequency is called the *Nyquist rate.*

  i.e $1/\Delta T = 2\mu_{max}$

Procedure for recovering f(t)

- By multiplying $\tilde{F}(\mu)$ with *H(μ)* we can recover *F(μ)* .

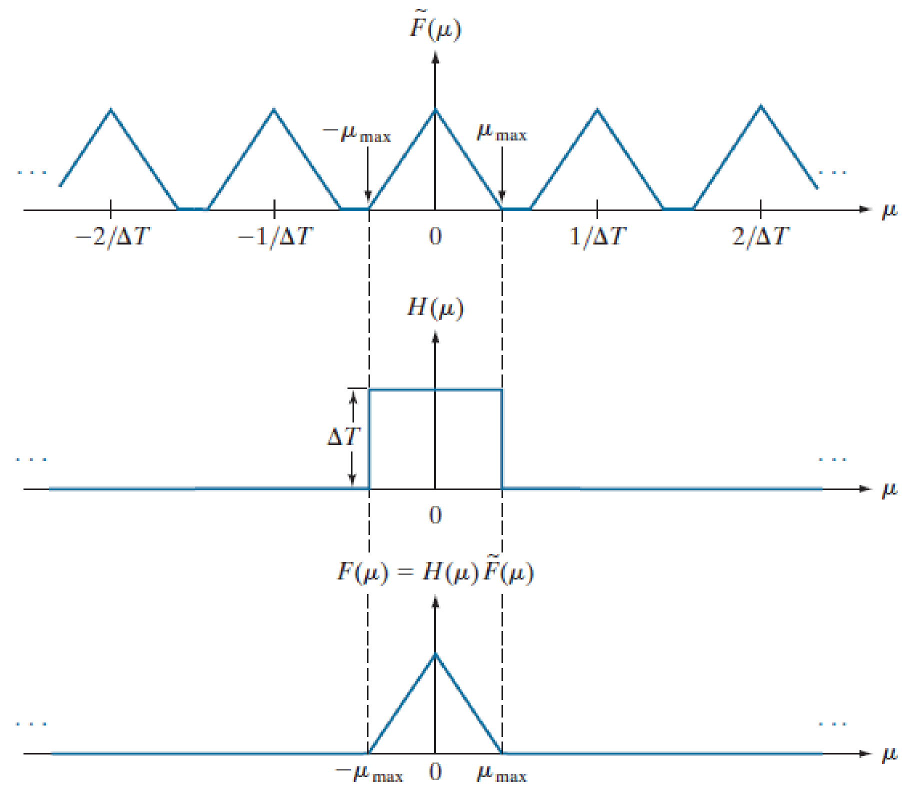
$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

Where

$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & otherwiswe \end{cases}$$

- f(t) can be recovered by using inverse Fourier transform :

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, d\mu$$



Aliasing

- If a band-limited function is sampled at a rate less than Nyquist rate then it is called as under-sampled.
- The inverse transform of F(μ) of an under-sampled function will recover a corrupted *f(t).*
- This effect caused by under-sampling a function, is known as ***frequency aliasing*** or simply ***aliasing***.

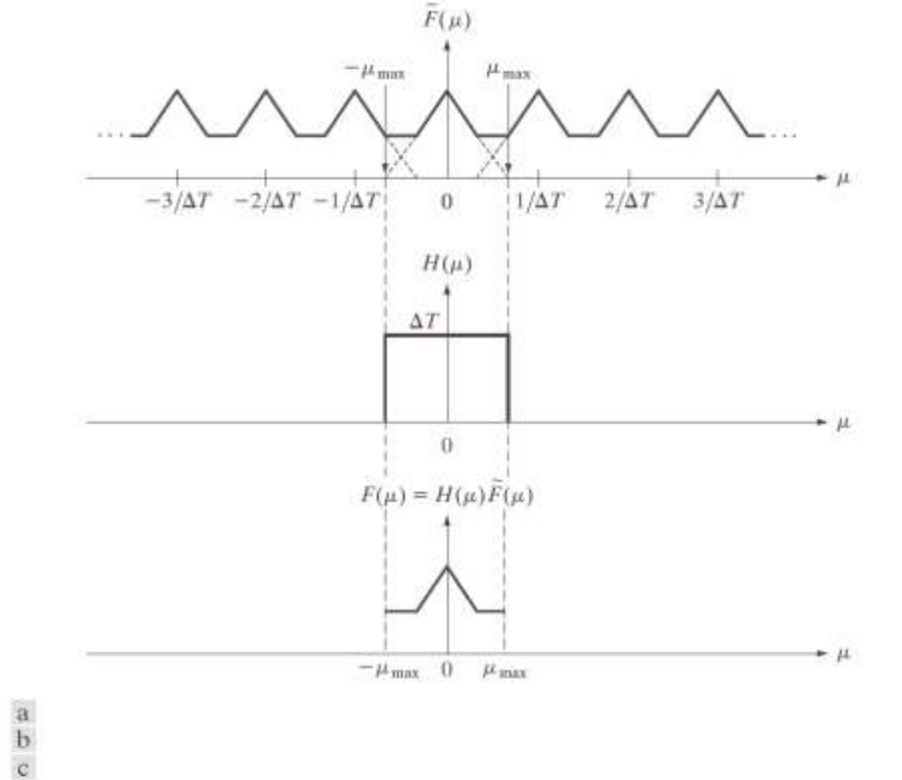


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

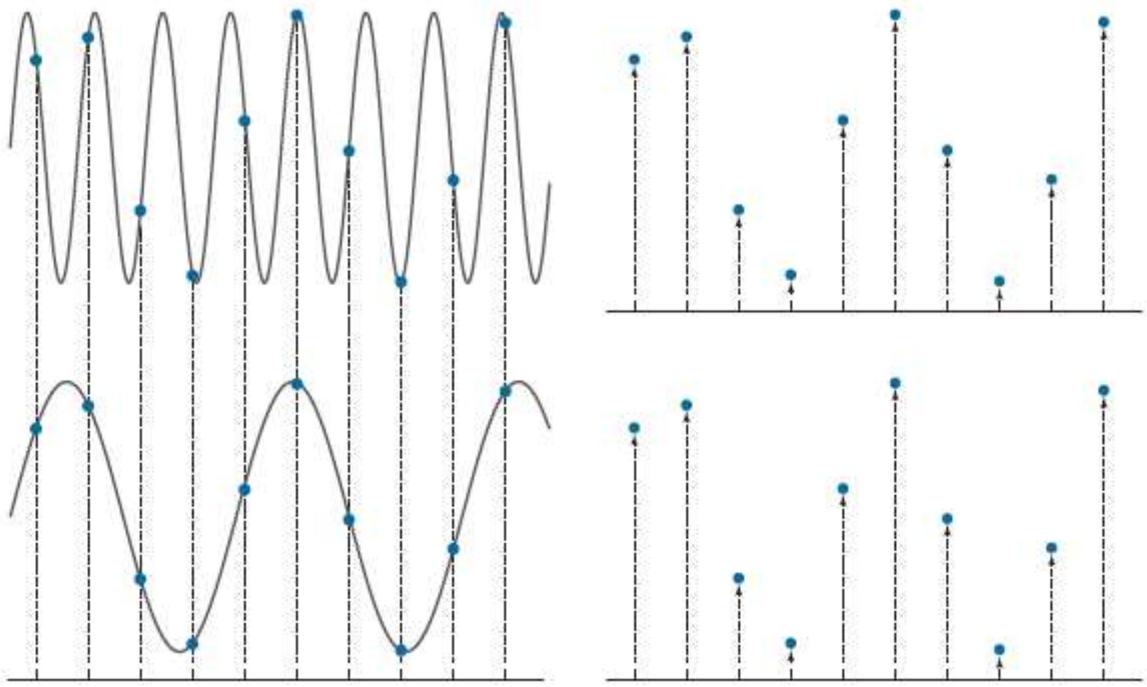


FIGURE 4.9 The functions in (a) and (c) are totally different, but their digitized versions in (b) and (d) are identical. Aliasing occurs when the samples of two or more functions coincide, but the functions are different elsewhere.

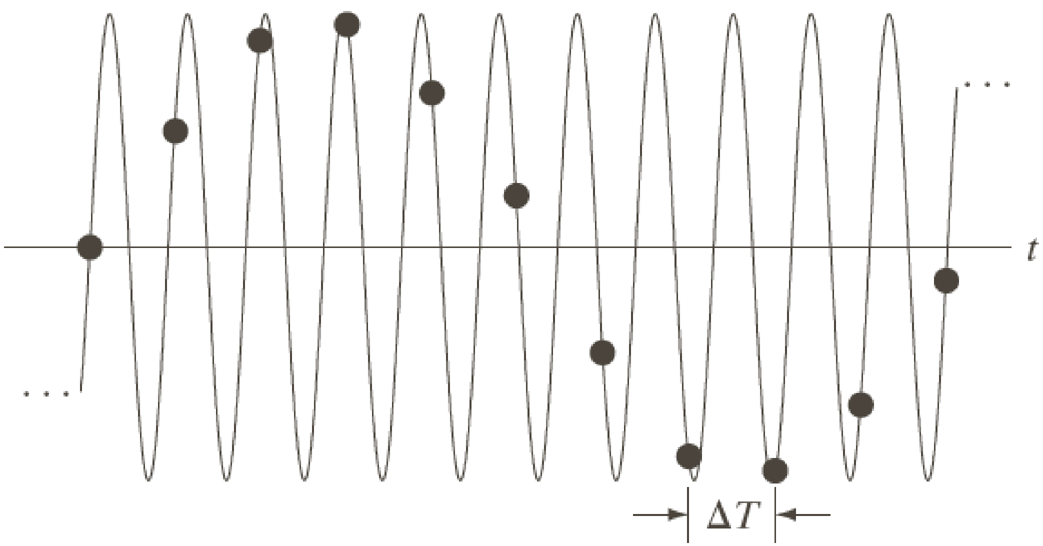


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Discrete Fourier Transform (DFT) of One Variable

- DFT is obtained from the continuous transform of a sampled function.

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

- Although f_n is a discrete function its Fourier transform is continuous and infinitely periodic with period $1/\Delta T$.
- Therefore we need to characterize $\tilde{F}(\mu)$ in one period.
- Take the samples at the following frequencies to obtain M equally spaced samples of $\tilde{F}(\mu)$ over the period $\mu = 0$ to $\mu = 1/\Delta T$.

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

- The DFT is written as

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- Similarly IDFT is written as

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

Notation of variables

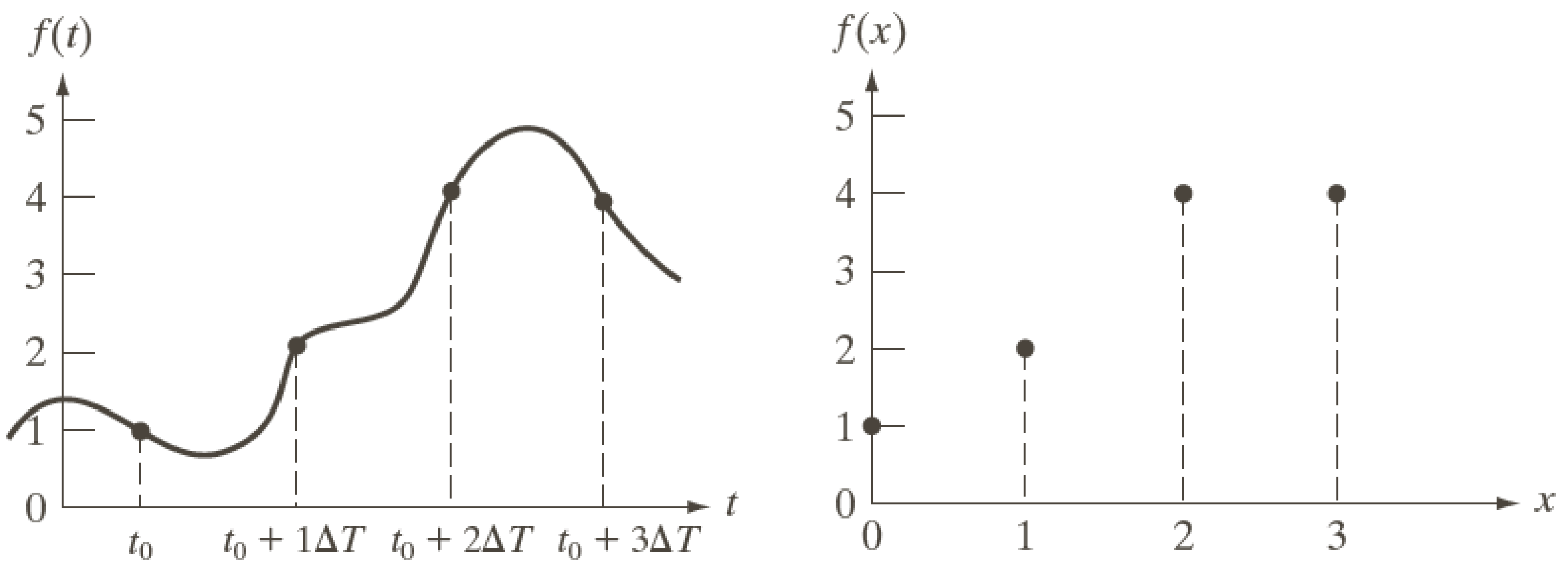
- One-dimensional continuous variables
  - t in spatial domain
  - μ in frequency domain
- One-dimensional discrete variables
  - x in spatial domain
  - u in frequency domain
- Two-dimensional continuous variables
  - (t, z) in spatial domain
  - (μ,ν) in frequency domain
- Two-dimensional discrete variables
  - (x, y)in spatial domain
  - (u,v) in frequency domain

- Finally following the above variable notation the one dimensional DFT and IDFT become

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

- Compute DFT for the discrete function f(x) = { 1, 2, 4, 4} obtained from the continuous function f(t).



EXAMPLE 4.4: The mechanics of computing the DFT.

Figure 4.12(a) shows four samples of a continuous function, $f(t)$, taken ΔT units apart. Figure 4.12(b) shows the samples in the x-domain. The values of x are 0, 1, 2, and 3, which refer to the number of the samples in sequence, counting up from 0. For example, $f(2) = f(t_0 + 2\Delta T)$, the third sample of $f(t)$.

From Eq. (4-44), the first value of $F(u)$ [i.e., $F(0)$] is

$$F(0) = \sum_{x=0}^{3} f(x) = [f(0) + f(1) + f(2) + f(3)] = 1 + 2 + 4 + 4 = 11$$

The next value of $F(u)$ is

$$F(1) = \sum_{x=0}^{3} f(x)e^{-j2\pi(1)x/4} = 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

Similarly, $F(2) = -(1 + 0j)$ and $F(3) = -(3 + 2j)$. Observe that *all* values of $f(x)$ are used in computing *each* value of $F(u)$.

If we were given $F(u)$ instead, and were asked to compute its inverse, we would proceed in the same manner, but using the inverse Fourier transform. For instance,

$$f(0) = \frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)} = \frac{1}{4}\sum_{u=0}^{3} F(u) = \frac{1}{4}[11 - 3 + 2j - 1 - 3 - 2j] = \frac{1}{4}[4] = 1$$

which agrees with Fig. 4.12(b). The other values of $f(x)$ are obtained in a similar manner.

- A two point DFT is expressed as the multiplication

$$\begin{bmatrix} F(0) \\ F(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

- A two point IDFT is expressed as the multiplication

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \end{bmatrix}$$

- A Four point DFT is expressed as the multiplication

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

- A Four point IDFT is expressed as the multiplication

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \left(\frac{1}{M}\right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix}$$

- Compute DFT for the 1-D discrete function f(x) = { 1, 2, 3, 4}

sol:

A Four point DFT is expressed as the multiplication

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -1-2j \end{bmatrix}$$

- Given a 1-D function, F(u) = {11, -3+2j, -1, -3-2j} , in frequency domain. Compute its IDFT.

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \left(\frac{1}{M}\right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \left(\frac{1}{4}\right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 11 \\ -3+2j \\ -1 \\ -3-2j \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

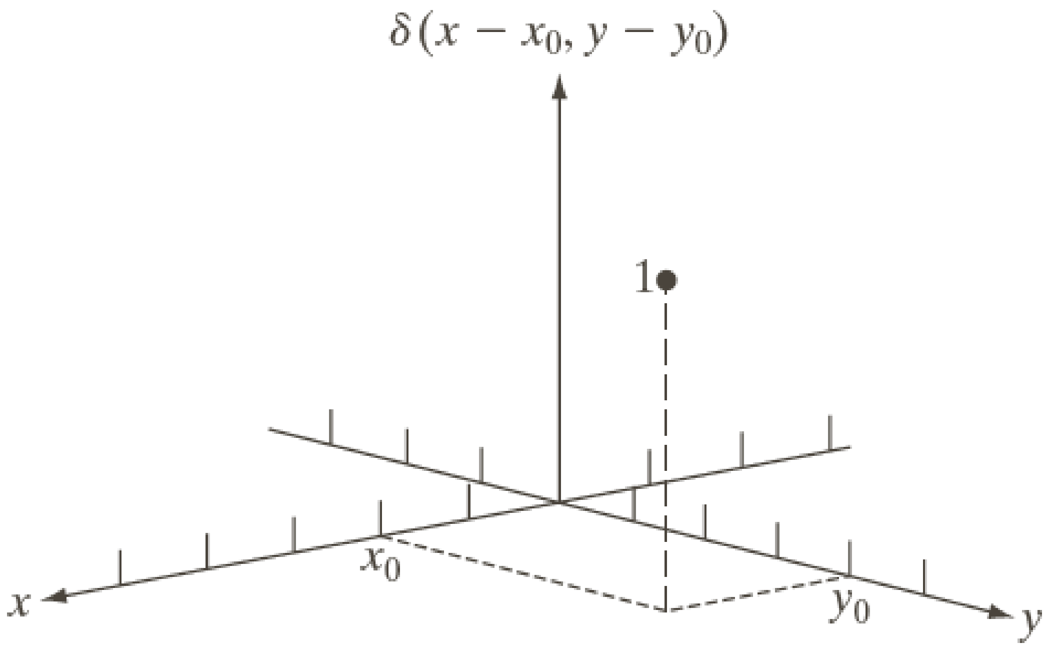


FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

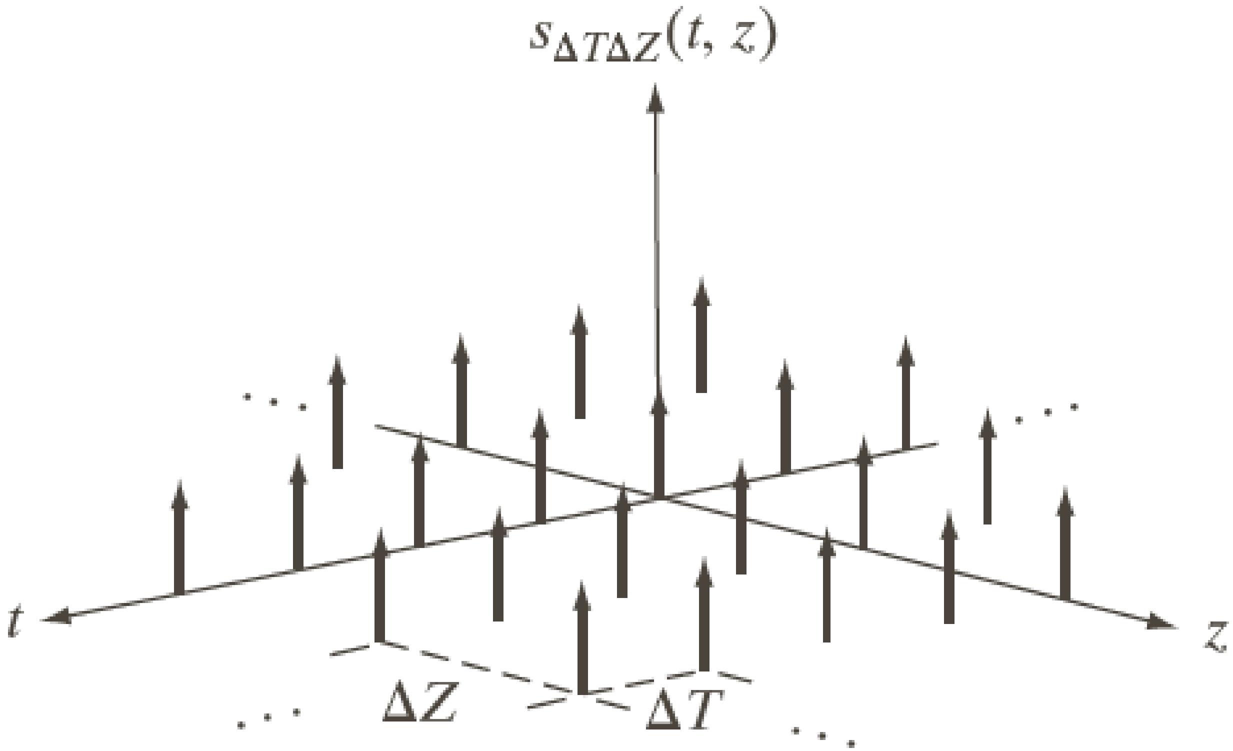
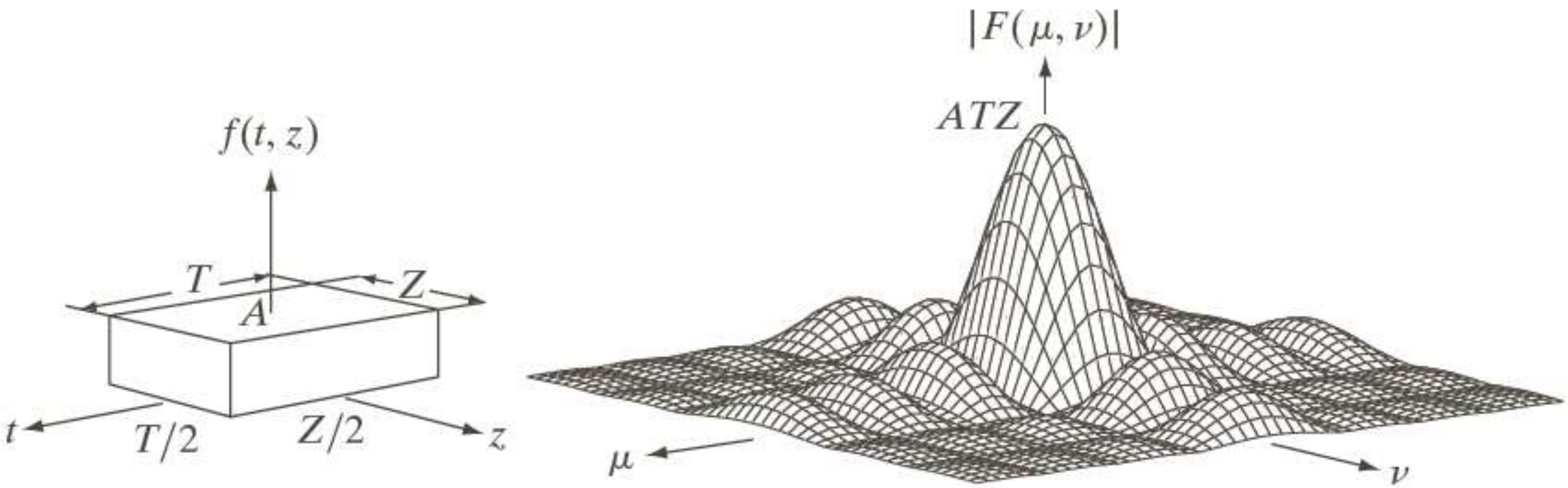


FIGURE 4.14 Two-dimensional impulse train.



a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

2-D Continuous Fourier Transform pair

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt\, dz$$

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu\, d\nu$$

2-D Discrete Fourier Tranform and its Inverse

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

- Let r(x,y,u,v) and s(x,y,u,v) are forward and inverse transformation kernels respectively.
- Forward transformation kernel r(x, y,u,v) is separable if $r(x,y,u,v) = r_1(x,u)r_2(y,v)$
- and symmetric if r1 is functionally equal to r2 so

  $$r(x,y,u,v) = r_1(x,u)\, r_1(y,v)$$
- The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result.

- Compute DFT for the image $f(x,y) = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$

Sol: The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result.

$$\begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T$$

$$\begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T$$

$$\begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix} = \begin{bmatrix} 11 & -1 \\ -1 & -5 \end{bmatrix}$$

- Given an image in spatial domain, compute its DFT.

$$f(x,y) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 2 \\ 0 & 2 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

- Given an image in frequency domain, compute its IDFT.

$$F(u,v) = \begin{bmatrix} 11 & -1 \\ -1 & -5 \end{bmatrix}$$

- Sol

$$\begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} = \left(\frac{1}{MN}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} F(0,0) & F(0,1) \\ F(1,0) & F(1,1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} = \left(\frac{1}{4}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 11 & -1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$$

Summary of DFT Definitions and Corresponding expressions

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \|F(u, v)\| e^{j\phi(u,v)}$
4) Spectrum	$\|F(u, v)\| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \|F(u, v)\|^2$
7) Average value	$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$ $= F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$ $= f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(Continued)

TABLE 4.3
Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

*

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

Basic properties of frequency domain

- Since frequency is directly related to spatial rates of change.
- Frequencies in the Fourier transform of an image correspond to patterns of intensity variations in the image.
- The slowest varying frequency component (u=v=0) corresponds to the average gray level of an image.
- As we move away from origin the lower frequencies correspond to the slowly varying components of an image.
- As we move further away from origin the higher frequencies begin to correspond to faster gray level changes in the image.
- In a DFT image of a room, lower frequencies correspond to gray level variations on the walls and floor and the higher frequencies correspond to edges of objects , noise (abrupt changes in gray level)

Frequency Domain filters

- Frequency domain filters are broadly two types
  - Low pass filters.
  - High pass filters.
- A filter that attenuates high frequencies while passing low frequencies is called a lowpass filter.
- A filter that attenuates low frequencies while passing high frequencies is called a highpass filter.
- Low pass filters are used for image smoothing.
- High pass filters are used for image sharpening.

FREQUENCY DOMAIN FILTERING FUNDAMENTALS

- Filtering in the frequency domain consists of modifying the Fourier transform of animage, then computing the inverse transform to obtain the spatial domain representation of the processed result.
- Thus, given (a padded) digital image, *f (x, y), of size P ×Q pixels, the basic filtering equation in which we are interested has the form:*

$$g(x,y) = \text{Real}\left\{\Im^{-1}\left[H(u,v)F(u,v)\right]\right\}$$

- *F(u,v) is the DFT of the input image, f (x, y),*
- *H(u,v) is a filter transfer function (which we often call just a filter or filter function),*
- *and g(x, y)* is the *filtered (output) image.*
- *Functions F, H, and g are arrays of size P ×Q, the same* as the padded input image.
- The product *H(u,v)F(u,v) is formed using elementwise* multiplication.
- The filter transfer function modifies the transform of the input image to yield the processed output, *g(x, y).*
- *The task of specifying H(u,v) is simplified considerably by using functions that are symmetric about* their center, which requires that *F(u,v) be centered also.*
- This is accomplished by multiplying the input image by $(-1)^{x+y}$ *prior to computing* its transform

SUMMARY OF STEPS FOR FILTERING IN THE FREQUENCY DOMAIN

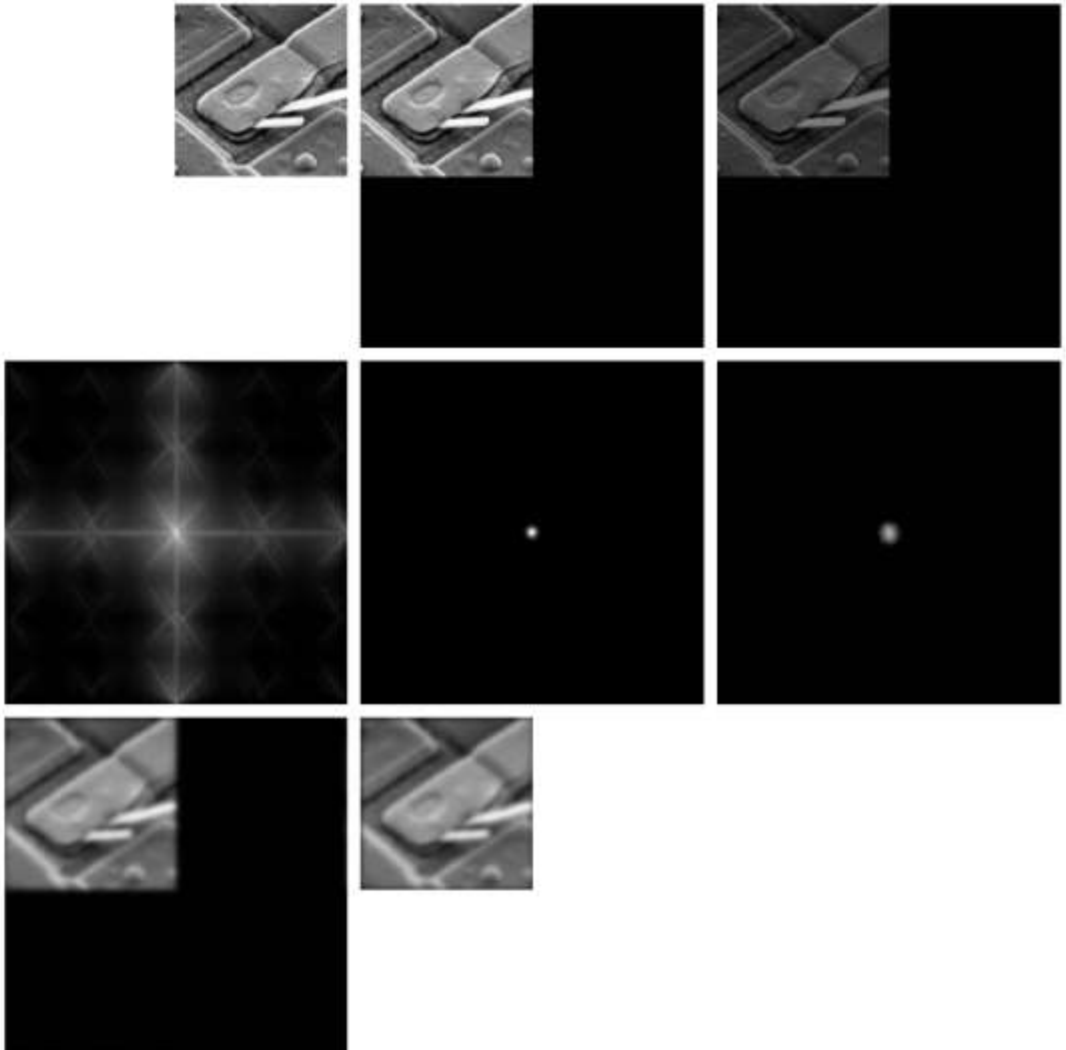
The process of filtering in the frequency domain can be summarized as follows:

1. Given an input image $f(x,y)$ of size $M \times N$, obtain the padding sizes P and Q using Eqs. (4-102) and (4-103); that is, $P = 2M$ and $Q = 2N$.
2. Form a padded† image $f_p(x,y)$ of size $P \times Q$ using zero-, mirror-, or replicate padding (see Fig. 3.39 for a comparison of padding methods).
3. Multiply $f_p(x,y)$ by $(-1)^{x+y}$ to center the Fourier transform on the $P \times Q$ frequency rectangle.
4. Compute the DFT, $F(u,v)$, of the image from Step 3.
5. Construct a real, symmetric filter transfer function, $H(u,v)$, of size $P \times Q$ with center at $(P/2, Q/2)$.
6. Form the product $G(u,v) = H(u,v)F(u,v)$ using elementwise multiplication; that is, $G(i,k) = H(i,k)F(i,k)$ for $i = 0,1,2,\ldots,M-1$ and $k = 0,1,2,\ldots,N-1$.
7. Obtain the filtered image (of size $P \times Q$) by computing the IDFT of $G(u,v)$:

$$g_p(x,y) = \left(\text{real}\left[\Im^{-1}\{G(u,v)\}\right]\right)(-1)^{x+y}$$

8. Obtain the final filtered result, $g(x,y)$, of the same size as the input image, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x,y)$.

An example



a	b	c
d	e	f
g	h	

FIGURE 4.36
(a) An $M \times N$ image, f.
(b) Padded image, f_p of size $P \times Q$.
(c) Result of multiplying f_p by $(-1)^{x+y}$.
(d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$.
(f) Spectrum of the product HF_p.
(g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p.
(h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

Low-pass filters for smoothing

TABLE 4.4
Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Ideal Lowpass filter

- A 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius D_0 from the origin and "cuts off" all frequencies outside this circle is called an *ideal lowpass filter.*
- It is specified by the function

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases} \quad (4.8\text{-}1)$$

where D_0 is a positive constant and $D(u, v)$ is the distance between a point (u, v) in the frequency domain and the center of the frequency rectangle; that is,

$$D(u, v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2} \quad (4.8\text{-}2)$$

Where P and Q are the padded sizes. Typically we select P=2M and Q = 2N.

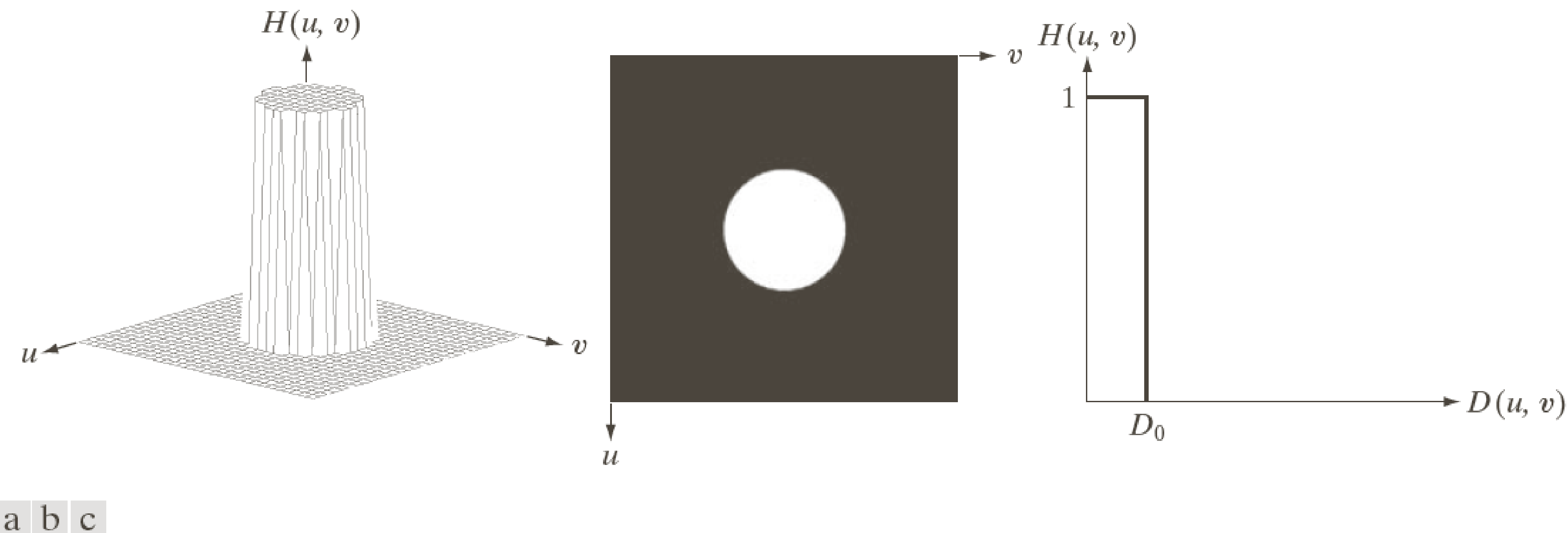
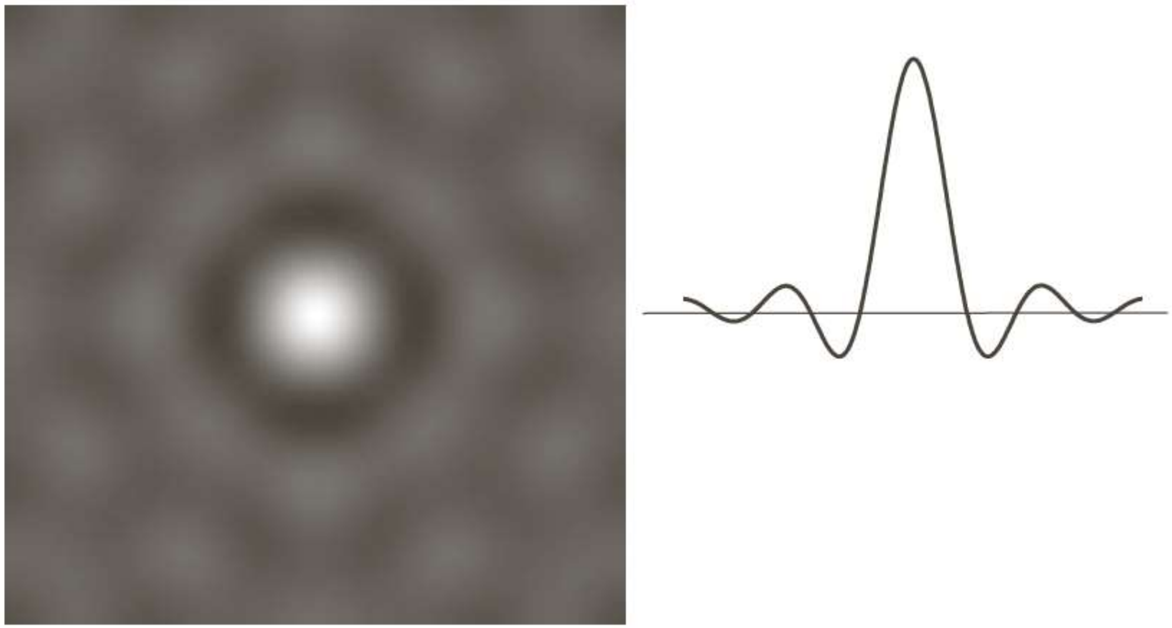
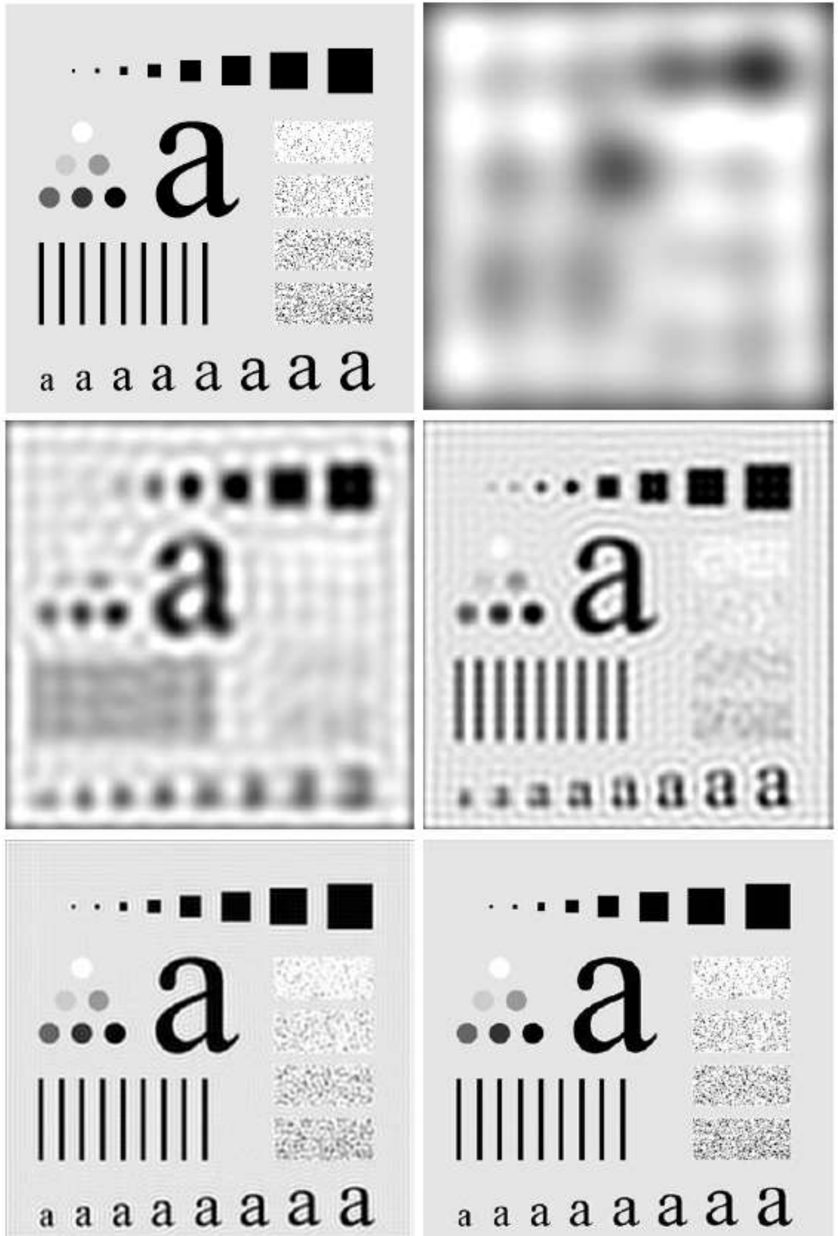


FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000. (b) Intensity profile of a horizontal line passing through the center of the image.



a b
c d
e f

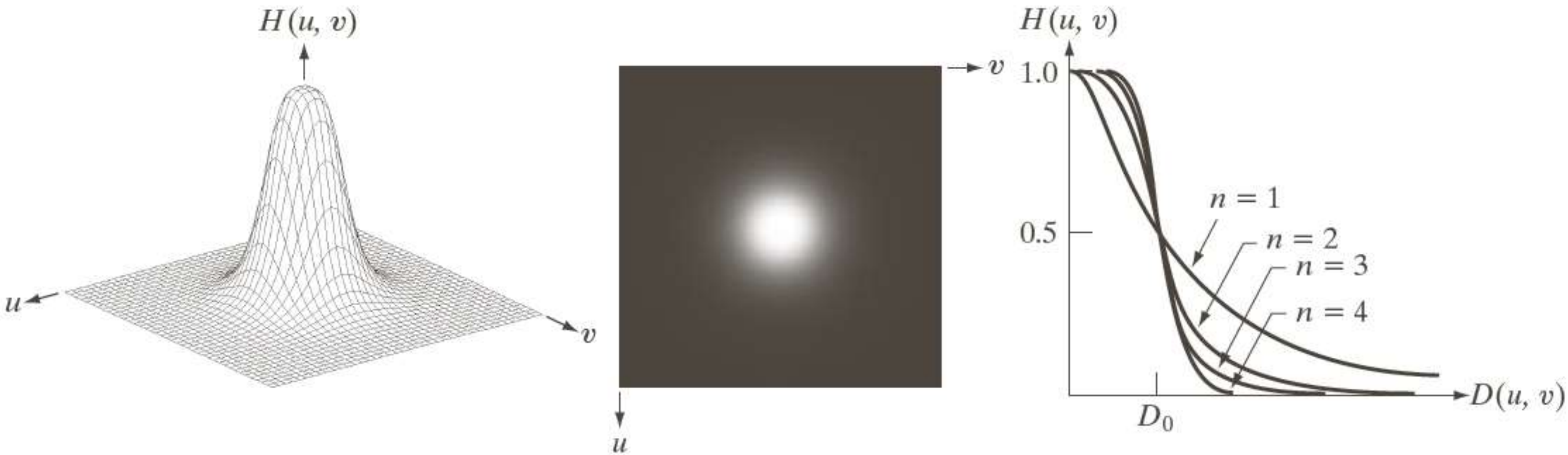
FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Butterworth Lowpass filter

The transfer function of a Butterworth lowpass filter (BLPF) of order n, and with cutoff frequency at a distance D_0 from the origin, is defined as

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}} \tag{4.8-5}$$

where $D(u, v)$ is given by Eq. (4.8-2). Figure 4.44 shows a perspective plot, image display, and radial cross sections of the BLPF function.



a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

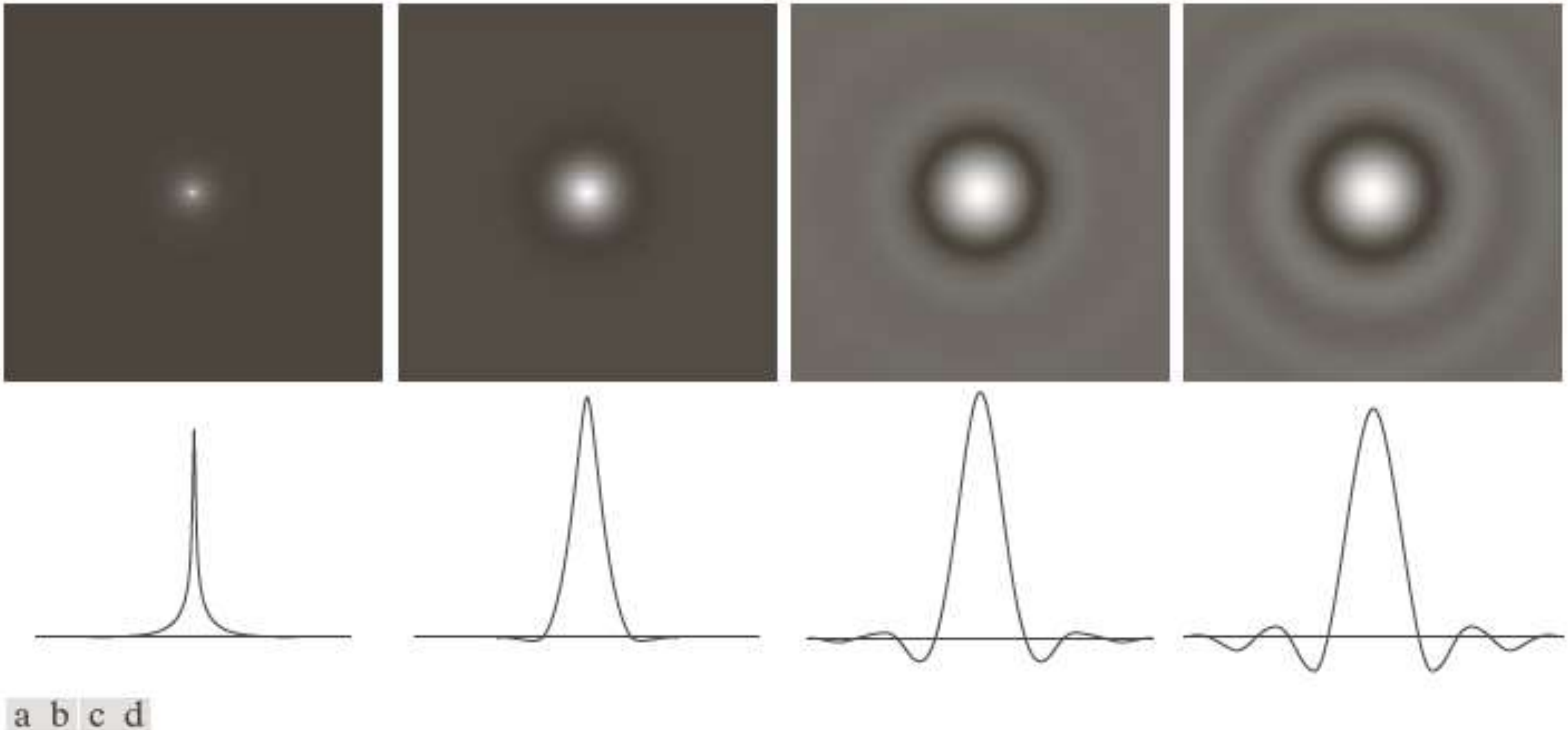
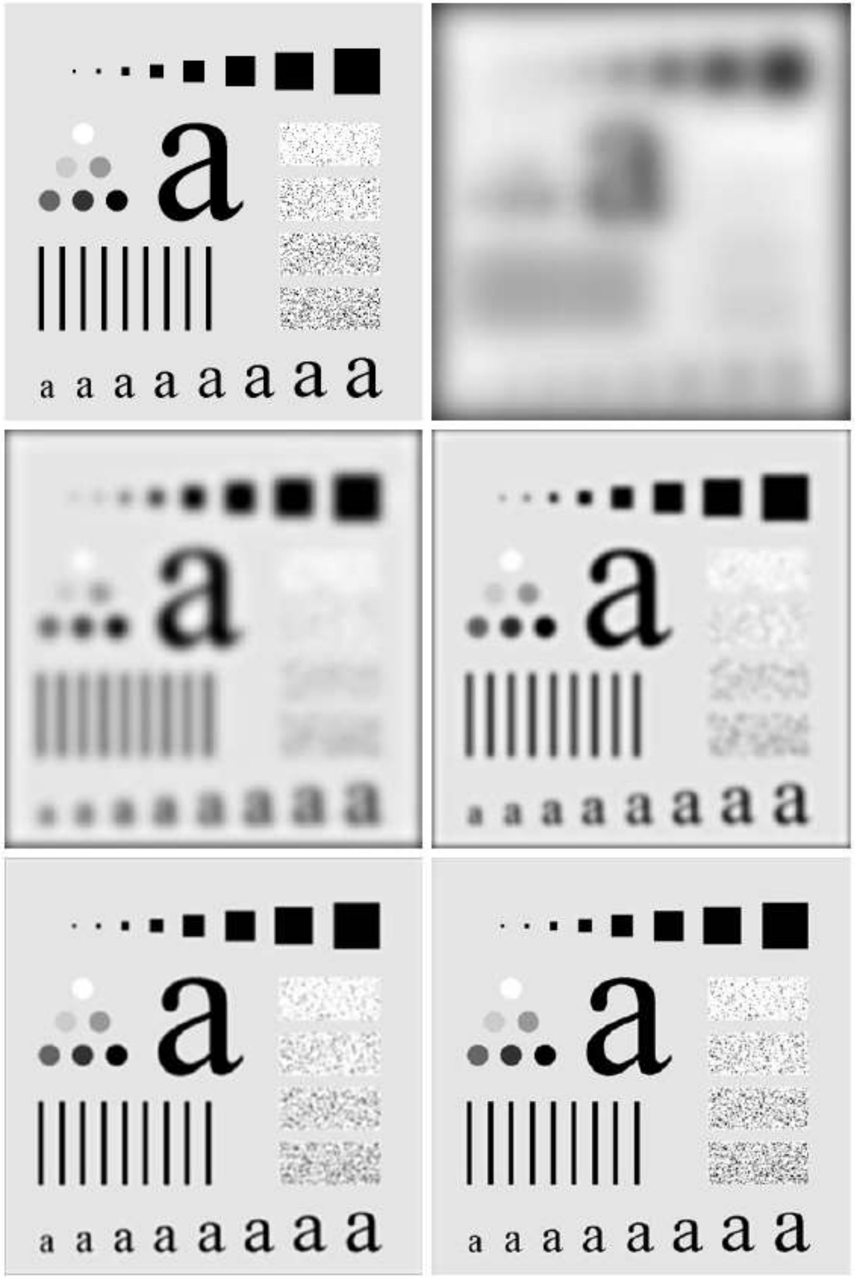


FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.



a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Gaussian Lowpass Filter

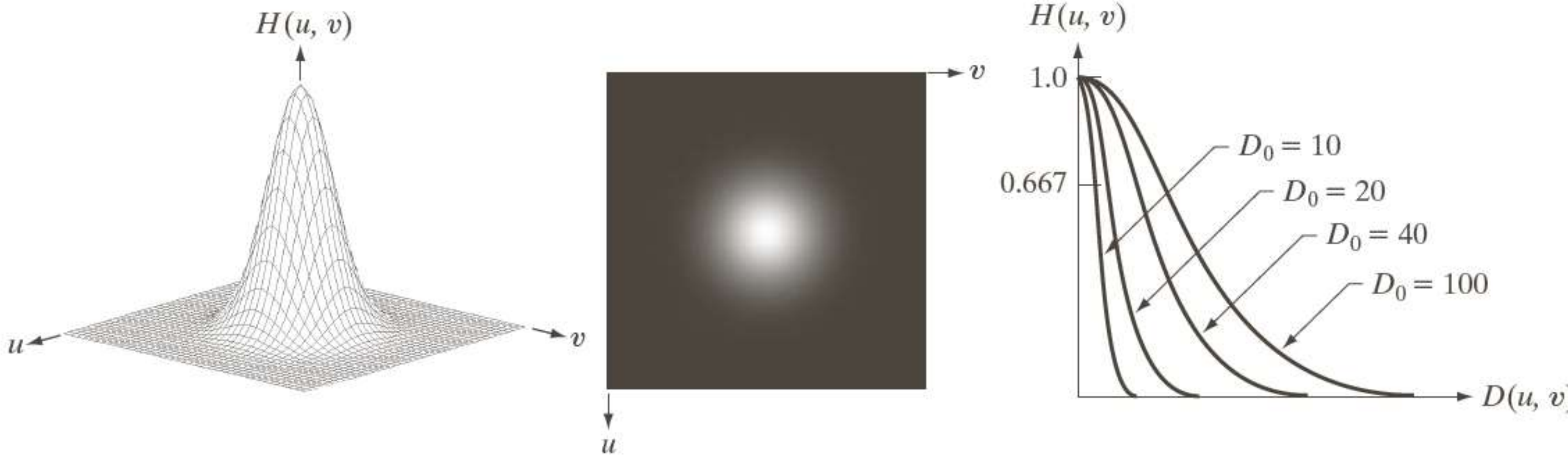
- The 2-D Gaussian filter is given by the equation

$$H(u, v) = e^{-D^2(u, v)/2\sigma^2} \tag{4.8-6}$$

where, as in Eq. (4.8-2), $D(u, v)$ is the distance from the center of the frequency rectangle. Here we do not use a multiplying constant as in Section 4.7.4 in order to be consistent with the filters discussed in the present section, whose highest value is 1. As before, σ is a measure of spread about the center. By letting $\sigma = D_0$, we can express the filter using the notation of the other filters in this section:

$$H(u, v) = e^{-D^2(u, v)/2D_0^2} \tag{4.8-7}$$

where D_0 is the cutoff frequency. When $D(u, v) = D_0$, the GLPF is down to 0.607 of its maximum value.



a b c

FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

n

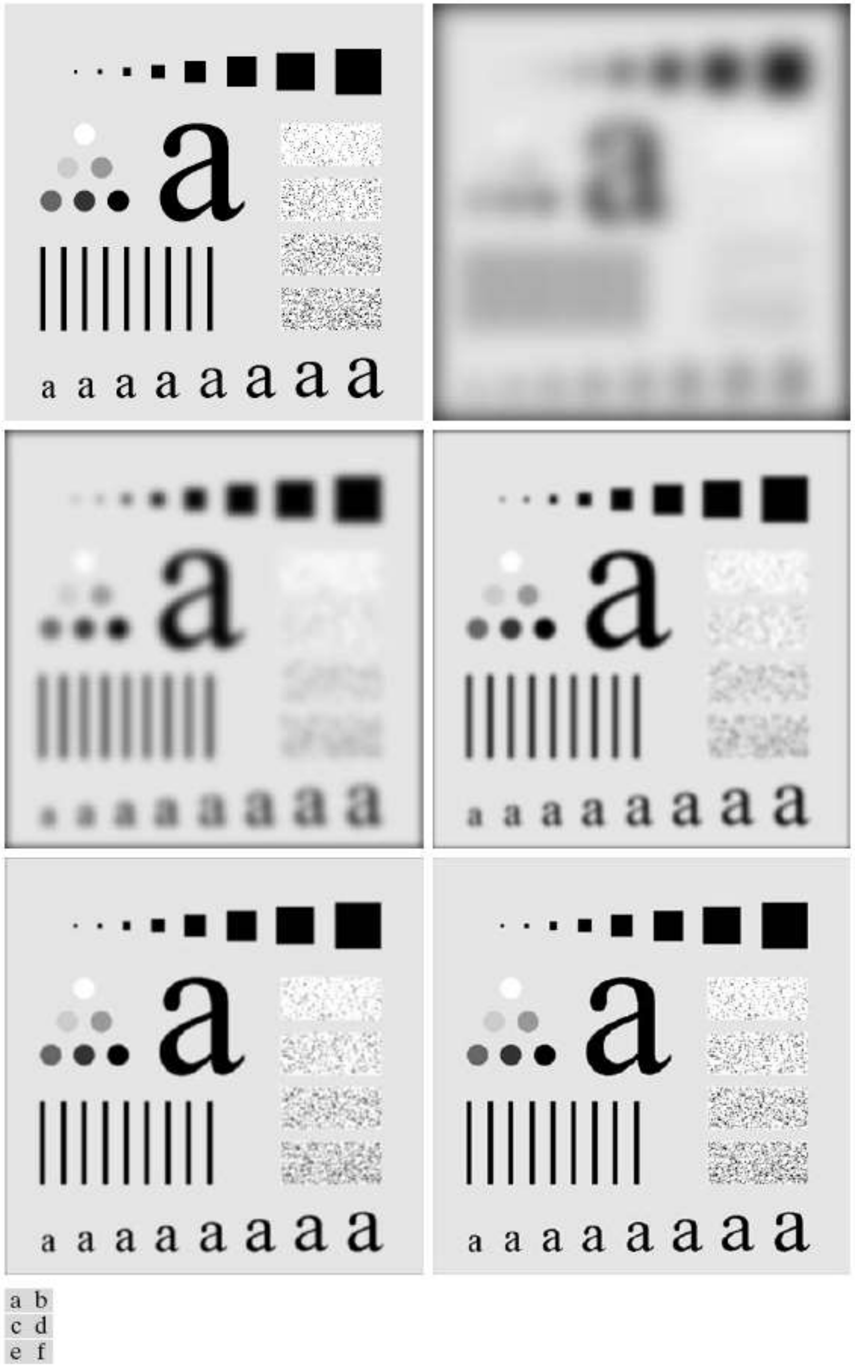
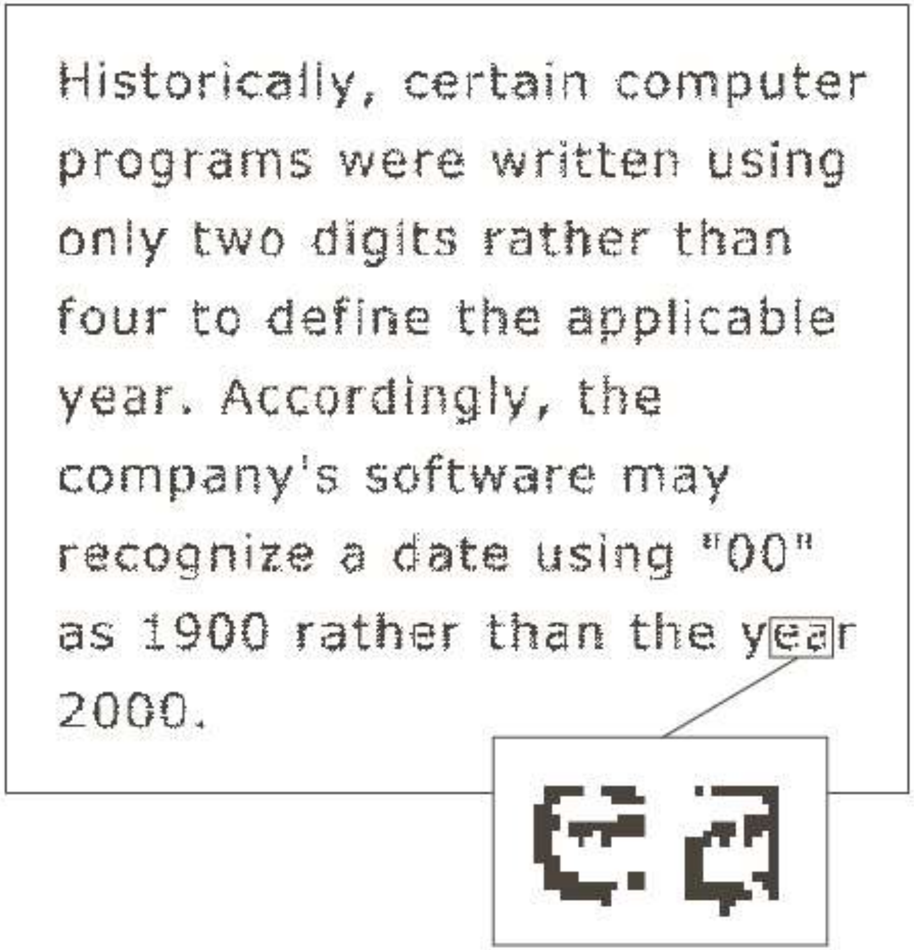
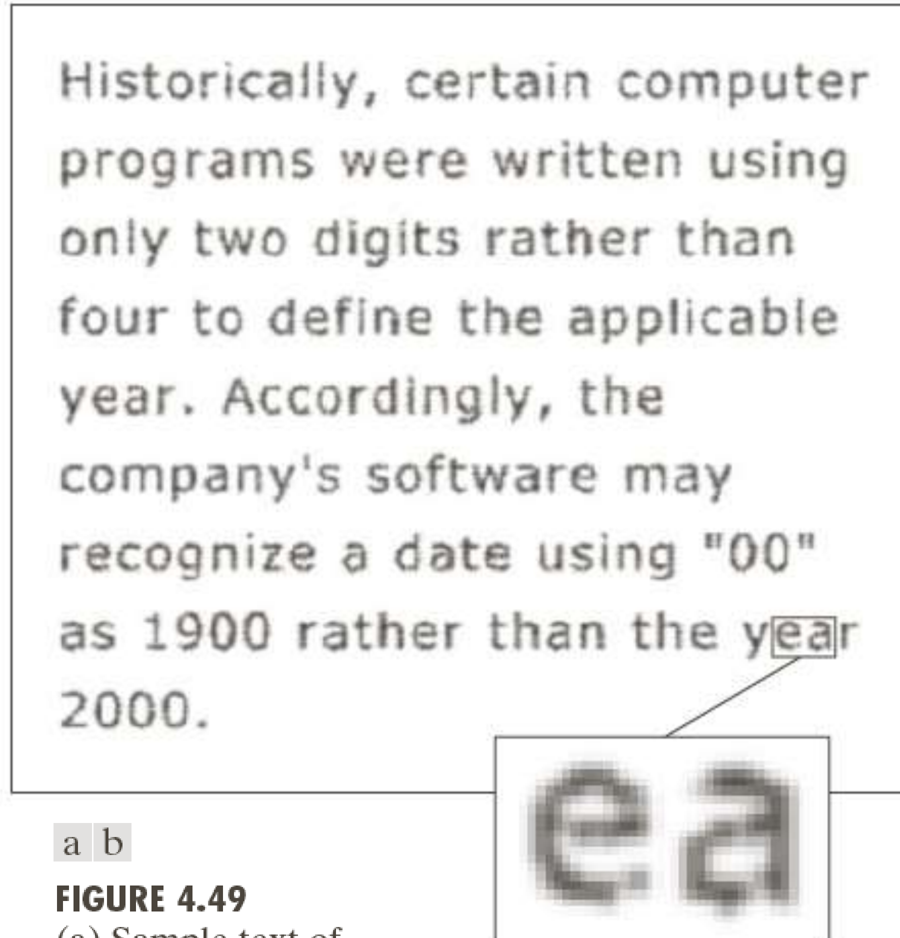


FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.





a b

FIGURE 4.49
(a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

High-pass filters for sharpening

TABLE 4.5
Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

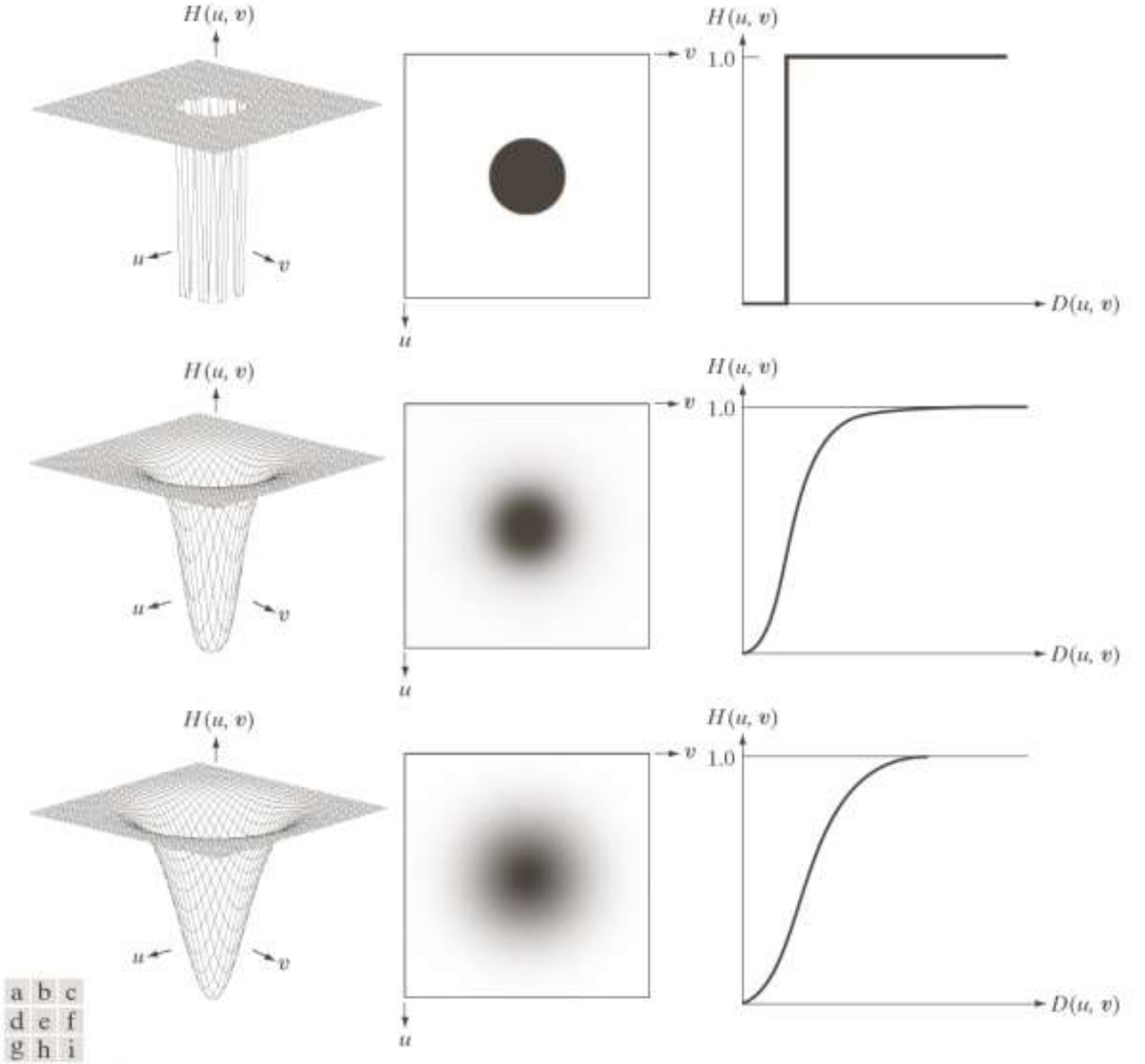
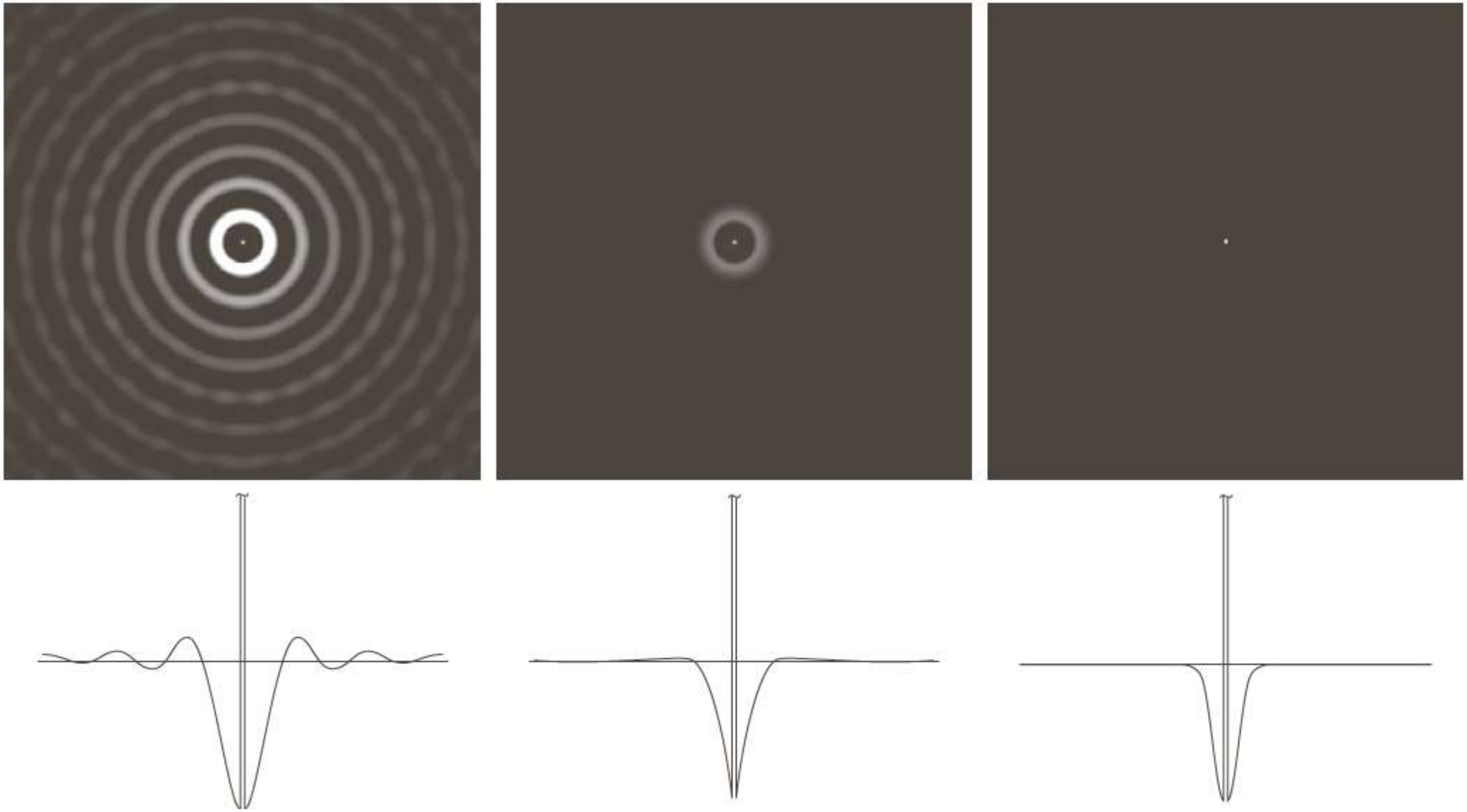


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.



a b c

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and 160.

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

a b c

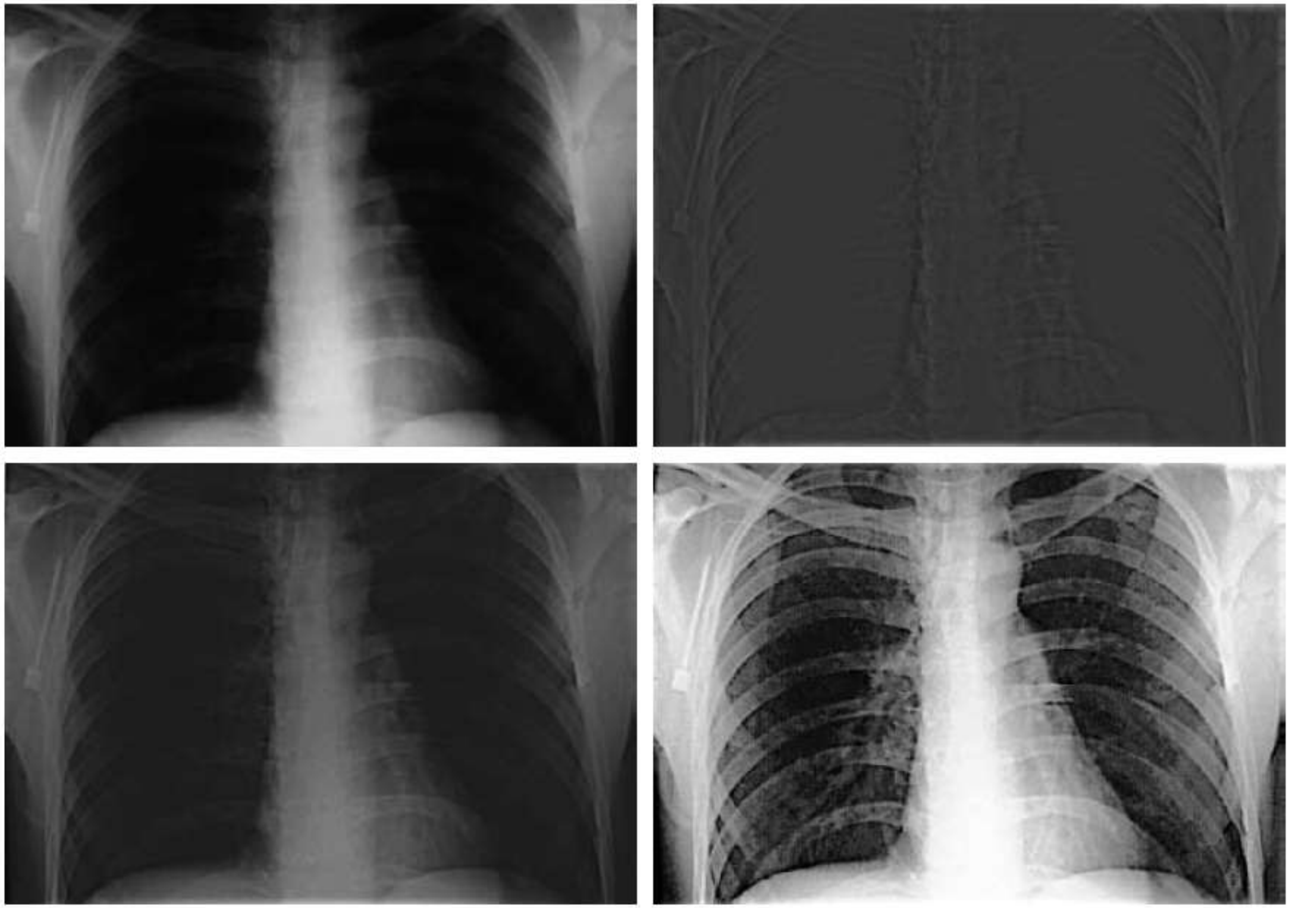
FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

a b

FIGURE 4.58 (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).



a b
c d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)