16/09/2022

Ordinary differential equations of 1^{st} order:

* Differential equation: A equation involving differentials or differential coefficients.

→ Ordinary DE: A DE involving derivatives wrt a single independent variable.

→ Order of DE: Order of DE is the order of highest order derivative occuring in the DE.

→ Degree of DE: Degree of DE is the degree of highest order derivative occuring in the DE which occurs in it provided the equations has been made free of radical signs and fractional power.

* General/Complete solution:

no. of independent arbitrary constants = order of DE.

* Particular solution:

Obtained from general solution with particular value of the arbitrary constants.

* Exact Differential equation:

A DE is obtained from its primitively directly by differentiation without any operation is said to be an exact differential equation.

$$U(x,y) = C \Rightarrow \text{const}.$$

DE: $\boxed{Mdx + Ndy = du}$

Total derivative

$$\frac{du}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{dy} \cdot \frac{dy}{dx}.$$

$$\boxed{Mdx + N\,dy = 0}$$

Necessary & sufficient condition for DE to be exact

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$du = \frac{\partial U}{\partial x} \cdot dx + \frac{\partial U}{\partial y}\, dy$$

$$M = \frac{\partial U}{\partial x} \;;\; N = \frac{\partial U}{\partial y}$$

$$Mdx + Ndy = \frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} \;;\; \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}.$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

General Solution:

$$\underset{y \to \text{const}}{\int M\, dx} + \underset{\substack{\text{terms} \\ \text{independent of } x}}{\int N\, dy} = C$$

*Integrating factor: A non-exact DE can be made into exact DE by multiplying a suitable non-zero factor is said to be an IF.

TYPES TO FIND AN IF:

1) Integrating factor found:

1) Direct Method:

(i) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$ (ii) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iii) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$

(iv) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\left(\dfrac{y}{x}\right)\right)$

(v) $\dfrac{y\,dx + x\,dy}{xy} = d(\log(xy))$

(vi) $\frac{xdx+ydy}{x^2+y^2} = d\left(\frac{1}{2}\log(x^2+y^2)\right)$

(vii) $\frac{xdy-ydx}{x^2-y^2} = d\left(\frac{1}{2}\left(\log\left(\frac{x+y}{x-y}\right)\right)\right)$

2) Non-exact and a homogeneous f.

$$\boxed{I.F = \frac{1}{Mx+Ny}}$$

3) Non-exact and it is of the form

$yf(x,y)dx + xg(x,y)dy = 0$

$$\boxed{I.F = \frac{1}{Mx-Ny}}$$

4) Non-exact & $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ divisible by

$$\boxed{f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$I.F = e^{\int f(x)dx}$$

5) Non-exact & $\cancel{\frac{\partial N}{\partial x}}$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ divisible by $-M$

$$\boxed{g(y) = \frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$I.F = e^{\int g(y)dy}$$

* Definition of Linear DE of first order:

A DE is said to be linear if the dependent variable & its derivative occur only in 1^{st} degree & not multiplied together.

LEIBNITZ EQUATION

→ Standard Equation:

* $\frac{dy}{dx} + P(x)\cdot y = Q(x)$

$$I.F = e^{\int P(x)dx}$$

$$y(I.F) = \int Q(x)\cdot(I.F)dx + C$$

* $\frac{dx}{dy} + P(y)\cdot x = Q(y)$

$$I.F = e^{\int P(y)dy}$$

$$x(I.F) = \int Q(y)\cdot(I.F)dy + C$$

* BERNOULLI'S EQUATION:

Standard Equation of NLDE:

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

$$y^{-n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = Q(x)$$

$$y^{1-n} = t$$

$$(1-n)\frac{dy}{dx}\cdot y^{-n} = \frac{dt}{dx}$$

*CLAIRAUT'S EQUATION:

Standard equation: $y = \frac{dy}{dx} x + f(p)$

$$\boxed{y = xp + f(p)}$$

2 solutions → General solⁿ / Singular solⁿ

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\boxed{\frac{dp}{dx}(x + f'(p)) = 0}$$

$$\int \frac{dp}{dx} = \int 0$$

$$\boxed{p = c}$$

$$\boxed{y = cx + f(c)}$$

$$x + f'(c) = 0$$

$$x + f'(p) = 0$$

$$x = -f'(p)$$

$$y = px + f(p)$$

$$y = -pf'(p) + f(p)$$

$$\boxed{y = px + f(p)}$$

* ORTHOGONAL TRAJECTORIES:

$f(x, y, c) = 0$ — ①

family of curve.

diff ① wrt 'x' & eliminate 'c'

$f(x, y, \frac{dy}{dx}) = 0$ — ② DE of family of curves.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$f(x, y, -\frac{dx}{dy}) = 0$ — ③

Solve ③ using suitable method.

$g(x, y, t)$

*Trajectory: A curve which cuts every member of given family of curves according to some definite law.

→ A curve which cuts every member of given family of curves at right angles is called orthogonal trajectory.

→ A family of curves is said to be self-orthogonal if 2 curves of the family intersect orthogonally.

*ELECTRIC CIRCUITS:

I → current

V → voltage

L → Inductance

R → Resistance

E → electromotive force

C → Capacitance

Voltage drop across

Resistance (R) = IR

Inductance (L) = $L\frac{di}{dt}$

Capacitance (C) = $\frac{q}{C} = \cancel{\frac{1}{C}\frac{dv}{dt}}$

LR Circuits:

$$E = iR + L\frac{di}{dt}$$

$$\boxed{\frac{di}{dt} + \frac{iR}{L} = \frac{E}{L}}$$

$$\boxed{I.F = e^{Rt/L}}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)}$$

RC Circuits:

$$E = iR + \frac{q}{C}$$

$$\boxed{\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}}$$

$$\boxed{I.F = e^{t/RC}}$$

$$\boxed{q = EC + Ke^{-t/RC}}$$

$$\boxed{q = EC(1 - e^{-t/RC})}$$

$$\boxed{i = \frac{E}{R}e^{-t/RC}}$$

Linear Differential Equations of Second & Higher order:

Homogeneous linear Differential Equations:

$$\frac{d^ny}{dx^n} + K_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + K_ny = 0 \quad -(1)$$

$$[D^n + K_1D^{n-1} + \cdots\cdots + K_n]y = 0 \quad -(2)$$

↳ Operator form.

Auxiliary Equation:

$y = e^{mx}$

$Dy = me^{mx}$

$D^2y = m^2e^{mx}$

$\vdots$

$D^ny = m^ne^{mx}$

$$e^{mx}[m^n + K_1m^{n-1} + K_2m^{n-2} + \cdots + K_n] = 0$$

$f(m) = 0$; $e^{mx} \neq 0$

$$y[(D-m_1)(D-m_2)] = 0$$

$(D-m_1)y = 0$

$\frac{dy}{dx} - m_1y = 0$

$\boxed{y = c_1e^{m_1x}}$

$(D-m_2)y = 0$

$\frac{dy}{dx} - m_2y = 0$

$\boxed{y = C_2e^{m_2x}}$

∵ Roots are real & distinct:

$$y_c = C_1e^{m_1x} + C_2e^{m_2x} + \cdots\cdots + C_ne^{m_nx}$$

If $m_1 = m_2$

$$[(D-m_1)(D-m_1)]y = 0$$

$$(D-m_1)z = 0$$

$$\frac{dz}{dx} - m_1 z = 0$$

$$\boxed{z = c_1 e^{m_1 x}}$$

$(D-m_1)\, y = z$

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

$$I.F = e^{-m_1 x}$$

$$y e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x} dx + c_2$$

$$\boxed{y e^{-mx} = c_1 x + c_2}$$

$$\boxed{y = (c_1 x + c_2) e^{m_1 x}}$$

*** Imaginary Roots:**

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$

$$= c_1 e^{ax} \cdot e^{ibx} + c_2 e^{ax} \cdot e^{-ibx}$$

$$y = c_1 e^{ax}[(\cos bx + i \sin bx)] + c_2 e^{ax}[\cos bx - i \sin bx]$$

$$y = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right]$$

HLDE have only a general sol^n i.e. complementary function

*** Non Homogeneous Linear Differential Equations of higher order with constant coefficients**

NHLDE are always have complete solution i.e. Complementary function + Particular Integral.

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = Q(x)$$

to find CF:

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n)\, y = 0$$

$$f(m) = 0$$

to find PI:

$$\boxed{f(D) \cdot y = Q(x)}$$

- $Q(x) = e^{ax}$
- $Q(x) = \sin ax / \cos ax$
- $Q(x) = x^k$
- $Q(x) = e^{ax} \cdot V(x)$ — $\sin ax$, $\cos ax$, x^k
- $Q(x) = x\, V(x)$ — $\sin ax$, $\cos ax$

$$y_P = \frac{1}{f(D)} \cdot Q(x)$$

① $Q(x) = e^{ax}$

$$y_P = \frac{1}{f(D)} e^{ax} \qquad D = a$$

$$\boxed{y_P = \frac{e^{ax}}{f(a)}} \Rightarrow \text{if } f(a) \neq 0$$

if $f(a) = 0$

$$\boxed{y_P = \frac{x e^{ax}}{f'(a)}}$$

② $Q(x) = \sin ax / \cos ax$

$$y_P = \frac{1}{f(D)} \sin ax \qquad D^2 = -a^2$$

$$y_P = \frac{\sin ax}{f(-a^2)} \cdot \quad \boxed{f(-a^2) \neq 0}$$

if $f(-a^2) = 0$

$$y_P = \frac{x \sin ax}{f'(-a^2)}$$

③ $Q(x) = x^k \quad k>0$

$$y_p = \frac{1}{f(D)} x^k$$

$$y_p = [f(D)]^{-1} x^k$$

↳ Binomial expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

④ ~~f(D)~~ $Q(x) = e^{ax} V(x)$

$\sin bx \qquad \cos bx \qquad x^k$

$$y = \frac{1}{f(D)} e^{ax} \sin bx$$

put $D = D + a$

$$y = e^{ax} \frac{\sin bx}{\phi(D)} \qquad \text{put } D^2 = -b^2$$

$$= \underbrace{e^{ax}\left[\frac{\sin bx}{\phi(D)}\right] / e^{ax}\left[\frac{\cos bx}{\phi(D)}\right]}$$

type: 2

$$y = e^{ax} \frac{x^k}{\phi(D)}$$

type: 3

⑤ $\phi(x) = x\, V(x)$

$\sin ax \qquad \cos ax$

$$y_p = \left[x - \frac{f'(D)}{f(D)}\right] \frac{V(x)}{f(D)}$$

$V(x) = \sin ax$.

$$y_p = \underbrace{\left[x - \frac{f'(D)}{f(D)}\right]}_{\text{Simplify}} \frac{\sin ax}{f(D)} \longrightarrow \text{put } D^2 = -a^2$$

*** Direct Method to find PI:**

$Q \rightarrow$ function of x $\quad \alpha \rightarrow$ const

$$\text{PI of } \frac{1}{D-\alpha} Q = e^{\alpha x} \int Q e^{-\alpha x}\, dx$$

$$\frac{1}{D+\alpha} Q = e^{-\alpha x} \int Q e^{\alpha x}\, dx$$

$\frac{1}{D-\beta}, \frac{1}{D-\alpha}$ are 2 inverse operators

$$\frac{1}{(D-\beta)(D-\alpha)} Q = \frac{1}{(D-\beta)} \left[\frac{Q}{(D-\alpha)}\right]$$

$$= e^{\beta x}\int e^{-\beta x}\left\{e^{\alpha x}\int Q e^{-\alpha x}dx\right\}dx.$$

* Method of Variation of Parameters:

Find y_c → Real & equal
→ Real & distinct
→ Imaginary.

$$y = c_1 u(x) + c_2 v(x).$$

$u(x) =$ | $v(x) =$

$u'(x) =$ | $v'(x)$.

→ Wronskain function

$$W(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$$

$$W(x) = u(x)\cdot v'(x) - v(x)\cdot u'(x) \neq 0.$$

$$y_P = A(x)\cdot u(x) + B(x)\cdot v(x)$$

$$A(x) = -\int \frac{v(x)\cdot R(x)}{W(x)}dx$$

$$B(x) = \int \frac{u(x)\cdot R(x)}{W(x)}dx$$

$$R(x) = \frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y.$$

* LCR circuits:

$$E = \frac{dq}{dt}R + L\frac{d^2q}{dt^2} + \frac{1}{C}q$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E.$$

* VECTOR DIFFERENTIATION:

→ Scalar Point function:

R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given; then ϕ is called scalar point function and R is a scalar field.

→ Vector Point function:

R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given; then v is called vector point function and R is a vector field.

→ Gradient of a scalar field:

Let $\phi(x, y, z)$ be a function defining a scalar field; then vector $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is the gradient of the scalar field ϕ is denoted by grad ϕ.

$$\boxed{\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}}$$

$$\boxed{\text{grad}\,\phi = \nabla\phi}$$

* Geometrical Interpretation of Curl:

angular velocity
$$\bar{\Omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

velocity $\bar{v}$ at any point $P(x, y, z)$

$$\bar{v} = \bar{\Omega} \times \bar{r}$$

where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\bar{v} = (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\text{curl } \bar{v} = \nabla \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$\text{curl } \bar{v} = 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) = 2\bar{\Omega}$$

$$\boxed{\bar{\Omega} = \frac{1}{2} \text{ curl } \vec{v}}$$

* Level Surface:

If a surface $\phi(x, y, z) = C$ be drawn through any point $P(x, y, z)$ such that at each point; the given function has same value; then such surface is called level surface of ϕ at P.

* Unit normal:

vector pt. function $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

then unit normal is given by:

$$\hat{n} = \frac{\bar{f}}{|\bar{f}|} = \frac{f_1\hat{i} + f_2\hat{j} + f_3\hat{k}}{\sqrt{f_1^2 + f_2^2 + f_3^2}}$$

$\because$ scalar point fn $\phi = \phi(x, y, z)$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

* Magnitude:

$\phi(x, y, z) \longrightarrow$ Scalar pt. fn.

$|\nabla\phi|$ (or) $|\text{grad}\,\phi| \longrightarrow$ magnitude of a given fn.

* Directional Derivative:

(i) D.D. of ϕ in the direction of $\bar{f}$

$$D.D = \nabla\phi \cdot \frac{\bar{f}}{|\bar{f}|}$$

(ii) D.D of $\bar{f}$ in the direction of ϕ

$$D.D = \bar{f} \cdot \frac{\nabla\phi}{|\nabla\phi|}$$

(iii) D.D. of ϕ in the direction of ψ

$$D.D = \nabla\phi \cdot \frac{\nabla\psi}{|\nabla\psi|}$$

(iv) D·D· of $\bar{f}$ in the direction of $\bar{g}$

$$D\cdot D = \bar{f}\cdot \frac{\bar{g}}{|\bar{g}|}$$

(v) $P(x_1, y_1, z_1)$ in the direction of normal at $Q(x_2, y_2, z_2)$

$$D\cdot D = (\nabla\phi)\ \frac{\overline{PQ}}{|\overline{PQ}|}$$

*Angle between the surfaces:

1) $\phi(x,y,z)$ & $\psi(x,y,z) \longrightarrow$ 2 scalar point functions at $P(x,y,z)$

angle b/w the surfaces

$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$\bar{n}_1 = (\nabla\phi)_P$$

$$\bar{n}_2 = (\nabla\psi)_P$$

2) $\phi(x,y,z) \longrightarrow P(x,y,z)$ & $Q(x,y,z)$

$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$\bar{n}_1 = (\nabla\phi)_P$$

$$\bar{n}_2 = (\nabla\phi)_Q$$

* VECTOR INTEGRATION:

$$= \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right] dy$$

$$= \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right] dx$$

*→ Evaluation of double integrals:

1) $x_1, x_2 \longrightarrow \phi_1(y), \phi_2(y)$

$y_1, y_2 \longrightarrow$ constants

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right] dy$$

2) $y_1, y_2 \longrightarrow \phi_1(x), \phi_2(x)$

$x_1, x_2 \longrightarrow$ constants

$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right] dx$$

3) $x_1, x_2, y_1, y_2 \longrightarrow$ constants

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Horizontal strip: x limits (Left to Right)

Vertical strip: y limits (Down to up)

** Change of order of integration:

Limits of
- $x \rightarrow$ constants
  $y \rightarrow$ variables / functions in terms of x.

$$\int_{x=a}^{b}\left[\int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\right] dx$$ Vertical Strip

By change of order of integration:

$$\int_{y=c}^{d}\left[\int_{f_1(y)}^{f_2(y)} f(x,y)\, dx\right] dy$$ Horizontal Strip

** Triple Integrals:

$$\boxed{\iiint f(x,y,z)\, dv}$$

$$= \int_{z=a}^{b}\int_{y=f_1(z)}^{f_2(z)}\int_{x=g_1(y,z)}^{g_2(y,z)} f(x,y,z)\, dx\, dy\, dz$$

** Single Integral / Line Integral:

Work done by a force acting on a particle moving ~~from A~~ along the arc AB then the work done during the displacement $\delta \bar{r}$ is $F \cdot \delta \bar{r}$

$\therefore$ The total work done during displacement from A to B is given by line integral $\boxed{\int_A^B \bar{F} \cdot d\bar{r}}$

If $\bar{F}$ is conservative: $\text{curl}\, \bar{F} = \text{curl}(\text{grad}\, \phi) = \bar{0}$

** Green's Theorem: Line Integral to Double Integral

Let $M(x,y)$ and $N(x,y)$ be continuous function having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a closed region R bounded by the curve C.

then

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int M dx + N dy}$$

** Surface Integral:

An integral which is to be evaluated over a surface is called surface integral

$\iint_S \bar{F} \cdot \hat{n}\, ds \longrightarrow$ flux of F over S.

$\hat{n} \longrightarrow$ unit normal vector

$\bar{F}(x,y,z) \longrightarrow$ vector function

1) R be the projection in the xy plane

$$\iint_R \bar{F}\cdot\hat{n}\,ds = \iint_S \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

2) R be the projection in the yz plane

$$\iint_R \bar{F}\cdot\hat{n}\,ds = \iint_S \bar{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

3) R be the projection in the zx plane

$$\iint_R \bar{F}\cdot\hat{n}\,ds = \iint_S \bar{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

*→ Gauss's Divergence Theorem:

Transformation b/w. surface integral & triple integral.

$S \rightarrow$ closed surface enclosing a volume V.

$\bar{F} \rightarrow$ continuously differentiable vector point function

$\hat{n} \rightarrow$ outward normal vector drawn at any point in the surface S.

$$\boxed{\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dV}$$

*→ Volume Integral:

$\bar{F}(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where F_1, F_2, F_3 are functions of x, y, z; $dv = dx\,dy\,dz$.

$$\int_V \bar{F}\cdot dv = \iiint (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$$

*→ Stoke's Theorem:

$S \rightarrow$ open surface bounded by a closed non-intersecting curve C

$\bar{F} \rightarrow$ differentiable vector point fn.

$$\oint_C \bar{F}\cdot ds = \int_S \operatorname{curl}\bar{F}\cdot\hat{n}\,ds$$

$\hat{n} \rightarrow$ unit outward normal drawn.

* BETA & GAMMA FUNCTIONS:

1) Definition of Beta function:

Eulerian Integral of first kind:

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

Properties:

1) $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$$

put $x = 1-y$; $dx = -dy$

$x=1, y=0$; $x=0, y=1$

$$\int_1^0 (1-y)^{m-1}(y)^{n-1} - dy$$

$$= \int_0^1 (y)^{n-1}(1-y)^{m-1}$$

$$= \underline{\beta(n,m)}$$

2) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

RHS:

By def^n

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)dx$$

$$= \underline{\int_0^1 x^{m-1}(1-x)^{n-1}dx} = \beta(m,n)$$

3) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

By def^n

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta$

$x=0$; $\theta = 0$

$x=1$; $\theta = \pi/2$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(\cos^{2n-2}\theta)\,d\theta \cdot 2\sin\theta\cos\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

$$\boxed{\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta}$$

4) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

By def^n:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$x = \frac{1}{1+y}$

$dx = \frac{-1}{(1+y)^2}dy$

$x=0$; $y=\infty$

$x=1$; $y=0$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1} \cdot \frac{-1}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n-2+2}} dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$= \text{By } \beta(m,n) = \beta(n,m)$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

5) $$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

By : $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$$

$x = \frac{1}{y}$

$dx = -\frac{dy}{y^2}$

$x = \infty, y = 0$

$x = 1 \; y = 1$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^0 \frac{y^{1-m} \cdot y^{m+n}}{(y+1)^{m+n}} \cdot \frac{-dy}{y^2}$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}$$

⑥ $$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}$$

$\cancel{\frac{ax}{b} = y}$

By def^n

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

RHS: $$a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}$$

$y = \frac{ax}{b}$

$x = \frac{by}{a}$

$dx = \frac{b}{a} dy$

$x = 0; y = 0$

$x = \infty; y = \infty$

$$= \cancel{a^m} b^n \int_0^\infty \frac{\left(\frac{b}{\cancel{a}}\right)^{m-1} y^{m-1}}{(by+b)^{m+n}} \cdot \frac{b}{\cancel{a}} dy$$

$$= \frac{\cancel{b^{n+m-1+1}}}{\cancel{b^{m+n}}} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \beta(m,n)$$

⑦ $$\int_b^a (x-b)^{m-1}(a-x)^{n-1} dx = (a-b)^{m+n-1} \beta(m,n)$$

put $x = \frac{t-b}{a-b}$

$dx = \frac{dt}{a-b}$

$x = 0, t = b$

$x = 1; t = a$

By defn:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \int_b^a \frac{(t-b)^{m-1}}{(a-b)^{m-1}} \left(\frac{a-b-t+b}{a-b}\right)^{n-1} \cdot \frac{dt}{a-b}$$

$$= \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m+n-2+1}} dt$$

$$= \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1} dt}{(a-b)^{m+n-1}}$$

$$\beta(m,n) = \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx$$

$$\boxed{(a-b)^{m+n-1} \beta(m,n) = \int_b^a (x-b)^{m-1}(a-x)^{n-1} dx}$$

⑧ $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

By defⁿ

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \qquad \frac{(1+a)t}{(t+a)} = x$$

$\cancel{dx = (t+a)}$

$$x = \frac{(a+1)t}{(t+a)} \qquad \boxed{\begin{array}{l} x=0;\ t=0 \\ x=1\ \ t=1 \\ \cancel{t}+a = at+\cancel{t} \end{array}}$$

$$dx = \frac{(t+a)(a+1) - t(a+1)\cdot 1}{(t+a)^2} dt$$

$$= \frac{\cancel{at}+a^2+\cancel{t}+a-\cancel{ta}-\cancel{t}}{(t+a)^2} dt$$

$$dx = \frac{a(a+1)}{(t+a)^2} dt$$

$$\beta(m,n) = \int_0^1 \frac{t^{m-1}(1+a)^{m-1}}{(t+a)^{m-1}} \cdot \frac{(\cancel{t}+a-at-\cancel{t})^{n-1}}{(t+a)^{n-1}} dt \cdot \frac{a(a+1)}{(t+a)^2}$$

$$= \int_0^1 \frac{t^{m-1}(1+a)^m \cdot a^{n-1}(1-t)^{n-1} \cdot a}{(t+a)^{m+n}} dt$$

$$= (1+a)^m a^n \cdot \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx$$

$$\boxed{\frac{\beta(m,n)}{a^n(1+a)^m} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx}$$

2) Gamma fn: Eulerian integral of 2nd kind

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$$

Properties

1) $\Gamma(1) = 1$. $\Rightarrow$ put $n=1$ in defⁿ

2) $\Gamma(n) = (n-1)\Gamma(n-1)$

put $n = n+1 \Rightarrow \Gamma(n+1) = n\Gamma(n)$

- $n \to +ve$ integer: $\Gamma(n) = (n-1)!$
- $n \to +ve$ fraction: $\Gamma(n) = (n-1)\Gamma(n-1)$
- $n \to -ve$ fraction: $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

*** Relation between $\beta(m,n)$ & Γ**

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

By def^n of gamma fn:

$$\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$$

put $x = yt$

$x = 0$; $t = 0$

$x = \infty$; $t = \infty$

$$\Gamma m = \int_0^\infty e^{-yt} y^{m-1} t^{m-1} y \, dt$$

$$\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yt} \cdot t^{m-1} dt \quad — (1)$$

Multiply with $e^{-y} \cdot y^{m+n-1}$ on both sides

$$\Gamma m \cdot e^{-y} \cdot y^{n-1} = \int_0^\infty e^{-y(t+1)} \cdot t^{m-1} \cdot y^{m+n-1} dt$$

Apply integration wrt y taking 0 to ∞ limits

$$\Gamma m \int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty e^{-y(t+1)} \cdot y^{m+n-1} dy \int t^{m-1} dt$$

$$\Gamma m \, \Gamma n = \int_0^\infty \frac{\Gamma m+n}{(t+1)^{m+n}} \cdot t^{m-1} dt$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{t^{m-1}}{(t+1)^{m+n}} dt = \beta(m,n)$$

*** $\Gamma \frac{1}{2} = \sqrt{\pi}$**

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$m = n = \frac{1}{2}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma \frac{1}{2} \, \Gamma \frac{1}{2}}{\Gamma 1} = \frac{\left(\Gamma \frac{1}{2}\right)^2}{1}$$

$$\because \beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\frac{1}{2}\beta\left(m, n = \tfrac{1}{2}, \tfrac{1}{2}\right) \int_0^{\pi/2} 1 \cdot 1 \cdot d\theta$$

$$= \frac{\pi}{2}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$\boxed{\Gamma n \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

$$* \cancel{\int_0^{\pi/2} e^{-x^2} dx =}$$

$$* \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{0} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\boxed{\operatorname{erf}(-x) = -\operatorname{erf}(x)}$$

$$\boxed{\operatorname{erfc}(-x) = 2-\operatorname{erfc}(x)}$$

*** Error function:**

$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt}$$

$$\boxed{\operatorname{erf}(\infty) = 1}$$

Complementary Error fn:

$$\boxed{\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt}$$

$$\boxed{\operatorname{erf}(x) = 1-\operatorname{erfc}(x)}$$