09/05/2022

DIFFERENTIAL CALCULUS:

DIFFERENTIAL EQUATION

→ Ordinary D.E

→ Partial DE

* **Derivative:**

Differentiation (or) derivative w.r.t only one independent variable is denoted by $\frac{dy}{dx}$ | y' | y_1.

LINEAR →
- → Degree 1
- → Dependent variable should not multiply with ~~vara~~ derivatives.

⇒ ~~O~~. ORDER : highest derivative.

⇒ DEGREE: power of highest derivative

→ Free from radicals and fractional powers.

* $$\frac{d^2y}{dx^2} = 0$$

Integrate wrt 'x' on both sides.

$$\frac{dy}{dx} = c_1$$

Integrate wrt 'x' on both sides

$\boxed{y(x) = C_1 x + C_2}$ ⇒ General Solⁿ of DE

Primitive (Integration)

$\boxed{\text{no. of arbitrary const} = \text{order of DE}}$

SOLUTIONS

General
- no. of ac = order of DE

Particular
- Particular values of arbitrary const. from general solⁿ using initial value conditions.

Singular
- Cartesian form
- Not found from other solⁿ.

Use of diff. equations

Initial value conditions : Value of 'x' is same.

Boundary value conditions : Value of 'x' is diff.

09/05/2022

UNIT-1 :

ORDINARY DIFFERENTIAL EQUATIONS OF 1st ORDER :

1) Exact D.E. of 1st order - Definition, Derivation, Problems.

2) Non-Exact DE (Integrating factor) of 1st order - Definition, Problems, 5 standard types.
Direct, Homo, (5)

3) Linear DE of 1st order - Derivations, Problem.

4) Clairaut's (Singular Solⁿ)
100% SAQ.

5) Applications of ordinary DE :-
- ** Orthogonal Trajectory (cartesian) — SAQ, LAQ
- LR circuits
- RC circuits

* **Uses of ordinary differential equations for CSE students :**

It is essential to describe the nature of the physical universe using differential equations and it is also essential to model computer graphics and vision using differential equations

In scientific computing ; differential equatio and its numerical solution are usually used since the world around us governed by differential equations.

In science and engineering many practical problems are formulated by one quantity is related to one or more other quantities defined in the problems. The study of this relationship gives rise to differential equations.

* Differential equations:

A differential equation is an equation involving differentials (or) differential coefficients

$$\frac{dy}{dx} = x^2 - 1 \quad -(1)$$

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \quad -(2)$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2} \quad -(3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad -(4)$$

There are 2 types of differential equations:

1) Ordinary DE: A differential equation involving derivatives wrt a single independent variable is called ordinary differential equation.

Ex: 1, 2, 3, are ODE.

2) Partial DE: A differential equation involving partial derivatives wrt more than one independent variable is called partial differential equation.

Ex: 4 is a PDE.

*** Order of DE:** The order of the differential equation is the order of the highest order derivative occuring in the differential equations.

Ex: (1): order = 1

(2): order = 2

(3): order = 3

Ex: $\frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0$

*** Degree of DE:** The degree of differential equation is the degree of the highest order derivative which occurs in it provided the equations has been made free of radicals signs and fractional power as far as derivatives are concerned.

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}.$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\boxed{\text{degree} = 2}$$

* Solution of DE:

A solution (integral/primitive) of a differential equation is a relation free from derivatives between the variables.

Ex:

$\boxed{y = C_1\cos x + C_2\sin x}$ is the solution of the DE $\frac{d^2y}{dx^2} + y = 0$

$$\frac{dy}{dx} = -C_1\sin x + C_2\cos x.$$

$$\frac{d^2y}{dx^2} = -C_1\cos x - C_2\sin x$$

$$= -y$$

→ **General or Complete solution:**

The differential equation is that in which the number of independent arbitrary constants is equal to the order of DE.

→ **Particular Solution:**

A particular solution of a differential equation is that which is obtained from the general solutions giving particular values to the arbitrary constants.

* The solution of a DE of n^{th} order is the particular sol^n if it contains less than 'n' arbitrary constants.

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* $(5x^4+3x^2y^2-2xy^3)\,dx + (2x^3y-3x^2y^2-5y^4)\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = 5x^4+3x^2y^2-2xy^3$$

$$N = 2x^3y-3x^2y^2-5y^4$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 0+6x^2y-6xy^2 \\ \frac{\partial N}{\partial x} &= 6x^2y-6xy^2\end{aligned}\right\} \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Necessary condition is satisfied.

$\therefore$ Given equation is an exact differential equation.

$\therefore$ General Solution:

$$\int M dx + \int N dy = \text{const} = C$$

y is const: terms ~~of x~~ independent of x

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\, dx + \int (-5y^4)\, dy = C$$

$$x^5 + x^3 y^2 - x^2y^3 - y^5 = C$$

$$\boxed{x^5 + x^3y^2 - x^2y^3 - y^5 = C}$$ is the general solution.

* EXACT DIFFERENTIAL EQUATIONS:

A differential equation obtained from its primitive directly by differentiation, without any operation of (multiplication, elimination or reduction etc.) is said to be an exact differential equation.

Given function: $\boxed{U(x,y) = C}$

Differential equation: $\boxed{M dx + N dy = du}$

By Total derivative:

$$\frac{du}{dx} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dx}$$

$$\boxed{0 = M dx + N dy}$$

**

THEOREM:

The necessary and sufficient condition for the DE $M dx + N dy = 0$ to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}}$$

Proof:

$$du = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} \cdot dy$$

($\because u$ is const)

$$M = \frac{\partial U}{\partial x}; \quad N = \frac{\partial U}{\partial y}$$

$$\boxed{M dx + N dy = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} \cdot dy}$$

**

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

continuously homogeneous functions

$$\because \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} \quad \text{(Suppose)}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ necessary condition of exactness.

Sufficient Condition:

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x}$

integrate wrto 'y'

$M = \frac{\partial U}{\partial x}$

We have to prove

$Mdx + Ndy = 0$

$\Rightarrow \boxed{M = \frac{\partial U}{\partial x}}$ — (1)

$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$

$\frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y} \Rightarrow$ integrate wrt o x $\Rightarrow \boxed{N = \frac{\partial U}{\partial y} + f(y)}$

$\boxed{N = \frac{\partial U}{\partial y} + f(y)}$ — (2)

$$Mdx + Ndy = \frac{\partial U}{\partial x} \cdot dx + \left(\frac{\partial U}{\partial y} + f(y)\right) dy$$

$$Mdx + Ndy = \left(\frac{\partial U}{\partial x} \cdot dx + \frac{\partial U}{\partial y} \cdot dy\right) + f(y)\, dy$$

$$= du + \int f(y)\, dy$$

$Mdx + Ndy = 0$ is exact

Method of Solution:

$u + \int f(y)\, dy = 0$ $\quad u = \int M dx$

Solution of $Mdx + Ndy = 0$ $\quad f(y) =$ terms ind. of x in N

is $\int M dx + \int N dy = \text{const}$

$y \to$ const $\quad$ terms ind. of x

*PROBLEMS:

1) Solve:

$$\left(xe^{xy} + 2y\right)\frac{dy}{dx} + ye^{xy} = 0$$

$$\left(xe^{xy} + 2y\right)dy + ye^{xy}dx = 0$$

$$\Rightarrow \boxed{Mdx + Ndy = 0}$$

$M = ye^{xy}$; $N = xe^{xy} + 2y$

Condition: $\frac{\partial M}{\partial y} = e^{xy} + ye^{xy} \cdot x$

$= e^{xy}(1 + xy)$

$\frac{\partial N}{\partial x} = e^{xy} + xye^{xy} + 0$

$= e^{xy}(1 + xy)$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\because$ Condition is satisfied; $\therefore$ The given DE is exact

$\therefore$ General Solution:

$\int M dx + \int N dy = C$

(y const) $\quad$ terms ind. of x

$$\int e^{xy} y\, dx + \int 2y\, dy = C$$

$$y\frac{e^{xy}}{y} + y^2 = C$$

$$\boxed{e^{xy} + y^2 = C}$$

2) $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$

$$\boxed{Mdx+Ndy=0}$$

$M=3x^2+6xy^2$

$N=6x^2y+4y^3$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ Condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=12xy.\\ \frac{\partial N}{\partial x}&=12xy\end{aligned}\right\} \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

$\Rightarrow$ Given differential equation is exact.

$\therefore$ General solution

$$\boxed{\underset{y\to\text{const}}{\int Mdx}+\underset{\text{terms independent of }x}{\int Ndy}=c}$$

$$\int(3x^2+6xy^2)\,dx+\int 4y^3\,dy=c$$

$$\boxed{x^3+3x^2y^2+y^4=c}$$

3) $(x^2+y^2-a^2)x\,dx+(x^2-y^2-b^2)\,y\,dy=0$

$$\boxed{Mdx+Ndy=0}$$

$M=(x^2+y^2-a^2)x$

$N=(x^2-y^2-b^2)y$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ necessary condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=2yx\\ \frac{\partial N}{\partial x}&=2xy\end{aligned}\right\} \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$\Rightarrow$ Given differential equation is exact.

$\therefore$ General Solution

$$\boxed{\underset{y\to\text{const}}{\int Mdx}+\underset{\text{terms indep. of }x}{\int Ndy}=c}$$

$$\int(x^3+xy^2-a^2x)\,dx+\int(-y^3-b^2y)\,dy=c$$

$$\frac{x^4}{4}+\frac{x^2y^2-a^2x^2}{2}-\frac{y^4}{4}-\frac{b^2y^2}{2}=c.$$

$$\boxed{\frac{x^4}{4}+\frac{x^2(y^2-a^2)}{2}-\frac{y^4}{4}-\frac{b^2y^2}{2}=c}$$

4) $(y\cos x+1)\,dx+\sin x\,dy=0$

$$\boxed{Mdx+Ndy=0}$$

$M=y\cos x+1 \qquad N=\sin x$

$\frac{\partial M}{\partial y}=\cos x \qquad \frac{\partial N}{\partial x}=\cos x$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ Given differential equation is exact.

$\therefore$ General Solution:

$$\int Mdx+\int Ndy=c$$

$$\int (y\cos x + 1)\,dx + \int 0\,dy = c$$

$$\boxed{+y\sin x + x = c}$$

5) $(1+e^{x/y})dx + \left(1-\frac{x}{y}\right)e^{x/y}dy = 0$

$$\boxed{M\,dx + N\,dy = 0}$$

$M = 1 + e^{x/y}$

$N = \left(1-\frac{x}{y}\right)e^{x/y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ necessary condition for exactness.

$\frac{\partial M}{\partial y} = e^{x/y} \cdot \frac{d}{dy}\left(\frac{x}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$

$\cancel{\frac{\partial N}{\partial x} = \frac{1}{y} e^{x/y}\left(\frac{-x}{y^2}\right) + \frac{x^2}{y^3}e^{x/y} + e^{x/y}\frac{x}{y^2}}$

$\frac{\partial N}{\partial x} = \cancel{\frac{e^{x/y}}{y}} - \frac{x}{y}\frac{e^{x/y}}{y} - \cancel{\frac{e^{x/y}}{y}}$

$= -\frac{x e^{x/y}}{y^2}$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \Rightarrow$ Given differential equation is exact.

∴ Given General Solution:

$$\boxed{\int M\,dx + \int N\,dy = c}$$

$$\int (1+e^{x/y})\,dx + \int 0\,dy = c$$

$$\boxed{x + e^{x/y}y = c}$$

$1 + e^{x/y}\cdot\frac{dy}{dx} + y e^{x/y}\left(\frac{-x\frac{dy}{dx} + y}{y^2}\right) = 0$

$1 + e^{x/y}\frac{dy}{dx} - e^{x/y}\frac{x}{y}\frac{dy}{dx} + e^{x/y} = 0$

6) $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y\,dy = 0$

$$\boxed{M\,dx + N\,dy = 0}$$

$M = \sec x \tan x \tan y - e^x$

$N = \sec x \sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ necessary condition for exactness

$\left.\begin{aligned}\frac{\partial M}{\partial y} &= \sec x \tan x \sec^2 y \\ \frac{\partial N}{\partial x} &= \sec x \tan x \sec^2 y\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Given differential equation is exact

∴ General Solution:

$$\boxed{\int M\,dx + \int N\,dy = C}$$

$$\int (\sec x \tan x \tan y - e^x)\,dx + \int 0\,dy = C$$

$$\boxed{\tan y \sec x - e^x = C}$$

7) $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)\,dy = 0$

$$\boxed{Mdx + Ndy = 0}$$

$M = y^2 e^{xy^2} + 4x^3$

$N = 2xye^{xy^2} - 3y^2$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2ye^{xy^2} + 2y^3 x e^{xy^2} + 0\\ \frac{\partial N}{\partial x} &= 2ye^{xy^2} + 2xy \cdot e^{xy^2} \cdot y^2\\ &= 2ye^{xy^2} + 2xy^3 e^{xy^2}\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given differential equation is exact.

General Solution: $\boxed{\int M dx + \int N dy = c}$

$$\int (y^2 e^{xy^2} + 4x^3)\,dx + \int -3y^2\,dy = c$$

$$\frac{y^2 e^{xy^2}}{y^2} + x^4 - y^3 = c$$

$$\boxed{e^{xy^2} + x^4 - y^3 = c}$$

8) $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0$

$$\boxed{Mdx + Ndy = 0}$$

$M = 2xy\cos x^2 - 2xy + 1$

$N = \sin x^2 - x^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ necessary condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2x\cos x^2 - 2x\\ \frac{\partial N}{\partial x} &= 2x\cos x^2 - 2x\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given differential equation is exact.

$\therefore$ General Solution

$$\boxed{\int M dx + \int N dy = c}$$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$\boxed{y\sin x^2 - yx^2 + x = c.}$$

9) $(\sin x\cos y + e^{2x})\,dx + (\cos x\sin y + \tan y)\,dy = 0$

$$\boxed{Mdx + Ndy = 0}$$

$M = \sin x\cos y + e^{2x}$

$N = \cos x\sin y + \tan y$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -\sin x \cdot \sin y\\ \frac{\partial N}{\partial x} &= -\sin x\sin y\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Given differential equation is exact.

$\therefore$ General Solution

$$\boxed{\int M dx + \int N dy = c}$$

$$\int(\sin x\cos y+e^{2x})dx+\int\tan y\,dy=c$$

$$-\cos x\cos y+\frac{e^{2x}}{2}+\ln|\sec x|\pm c$$

$$\boxed{e^{2x}-2\cos x\cos y+2\ln|\sec x|=c}$$

10) $x\,dy+y\,dx+\dfrac{x\,dy-y\,dx}{x^2+y^2}=0$

$$\int d(xy)+\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=0$$

Integrating on both sides.

$$\cancel{xy+\frac{xy}{x}\tan^{-1}\left(\frac{xy}{x}\right)-\frac{1}{2}\ln\left|1+\frac{xy^2}{x^2}\right|=c}$$

$$\cancel{\boxed{xy+\frac{y}{x}\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right|=c}}$$

$$\boxed{xy+\tan^{-1}\left(\frac{y}{x}\right)=c}$$

16/05/2022

* $\dfrac{dy}{dx}+\dfrac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

$\sin x\,dy+x\cos y\,dy+x\,dy+(y\cos x+\sin y+y)dx=0$

$M\,dx+N\,dy=0$

$M=y\cos x+\sin y+y$

$N=\sin x+x\cos y+x$

$\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}\Rightarrow$ condition for exactness

$\dfrac{\partial M}{\partial y}=\cos x+\cos y+1$

$\dfrac{\partial N}{\partial x}=\cos x+\cos y+1$

$\therefore\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial y}\Rightarrow$ D.E. is exact.

General Solution:

$$\underset{y\to\text{const}}{\int M\,dx}+\underset{\text{terms independent of } x}{\int N\,dy}=c$$

$$\int(y\cos x+\sin y+y)dx+\int 0\,dy=c$$

$$\boxed{y\sin x+x(\sin y+y)=c}$$

* Solve $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+(x+\log x-x\sin y)dy=0$

$M\,dx+N\,dy=0$

Comparing:

$M=y+\dfrac{y}{x}+\cos y$

$N=x+\log x-x\sin y$

$\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}\Rightarrow$ necessary condition

$\left.\begin{aligned}\frac{\partial M}{\partial y}&=1+\frac{1}{x}-\sin y\\ \frac{\partial N}{\partial x}&=1+\frac{1}{x}-\sin y\end{aligned}\right\}\ \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

$\Rightarrow$ DE is exact equation

* General Solution :

$$\int M\,dx + \int N\,dy = C$$
$y \to$ const terms indep. of x

$$\int \left(y + \frac{y}{x} + \cos y\right)dx + \int 0\,dy = C$$

$$xy + y\log x + x\cos y = C$$

$$\boxed{x(y+\cos y) + y\log x = C}$$

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1) $(x^2 - ay)dx = (ax - y^2)dy$

$(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0$

$\boxed{M\,dx + N\,dy = 0}$ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\Rightarrow$ necessary condition for exactness

$M = x^2 - ay$

$N = y^2 - ax$

$\frac{\partial M}{\partial y} = -a$ $\frac{\partial N}{\partial x} = -a$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ D.E is exact.

General Solution :

$$\int M\,dx + \int N\,dy = \text{const}$$
$y \to$ const terms indp of x

$$\int (x^2 - ay)\,dx + \int y^2\,dy = \text{const} = C$$

$$\frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

$$\boxed{\frac{x^3 + y^3}{3} - axy = C}$$

2) $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$

$M\,dx + N\,dy = 0$

$M = (x^2 + y^2 - a^2)x$

$N = (x^2 - y^2 - b^2)y$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ necessary condition for exactness.

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^3 + xy^2 - a^2x) = 2xy$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2y - y^3 - b^2y) = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ DE is exact

$\Rightarrow$ General Solution

$$\int M\,dx + \int N\,dy = \text{const} = C$$
$y \to$ const terms independent of x

$$\int (x^3 + xy^2 - a^2x)\,dx + \int (-y^3 - b^2y)\,dy = C$$

$$\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{b^2y^2}{2} = C$$

$$\boxed{x^4 - y^4 + 2x^2y^2 - 2(a^2x^2 + b^2y^2) = C}$$

3) $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

$Mdx+Ndy=0$

$M=x^2-4xy-2y^2$

$N=y^2-4xy-2x^2$

$\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$ → condition neccessary for exactness

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-4x-4y\\ \frac{\partial N}{\partial x}&=-4y-4x\end{aligned}\right\}\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

DE is exact.

General Solution:

$\underset{y\to\text{const}}{\int M dx}+\underset{\text{terms ind of } x}{\int N dy}=\text{const}$

$\int(x^2-4xy-2y^2)\,dx+\int y^2dy=c$

$$\boxed{\frac{x^3}{3}-2x^2y-2y^2x+\frac{y^3}{3}=C}$$

4) $(x^4-2xy^2+y^4)\,dx-(2x^2y-4xy^3+\sin y)dy=0$

$(x^4-2xy^2+y^4)dx+(4xy^3-2x^2y-\sin y)dy=0$

$\boxed{Mdx+Ndy=0}$

$M=x^4-2xy^2+y^4$ $\quad N=4xy^3-2x^2y-\sin y$

$\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$ → condition necessary for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=-4xy+4y^3\\ \frac{\partial N}{\partial x}&=4y^3-4xy\end{aligned}\right\}\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

⇒ DE is exact.

⇒ General Solution:

$\underset{y\to\text{const}}{\int M dx}+\underset{\text{terms ind of } x}{\int N dy}=\text{Const.}$

$\int(x^4-2xy^2+y^4)\,dx+\int(-\sin y)\,dy=\text{const}$

$\dfrac{x^5}{5}-x^2y^2+y^4x+\cos y=c.$

$$\boxed{x^5-5x^2y^2+5\cos y+5xy^4=C}$$

5) $ye^{xy}dx+(xe^{xy}+2y)\,dy=0$

$\boxed{Mdx+Ndy=0}$

$M=ye^{xy}dx$

$N=xe^{xy}+2y$

$\dfrac{\partial M}{\partial y}=ye^{xy}x+e^{xy}$
$=e^{xy}(1+xy)$

$\dfrac{\partial N}{\partial x}=xe^{xy}\cdot y+e^{xy}$
$=e^{xy}(1+xy)$

$\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$ ⇒ condition is satisfied

DE is exact

⇒ General Solution:

$\underset{y\to\text{const}}{\int M dx}+\underset{\text{terms ind of } x}{\int N dy}=\text{const}$

$$\int y e^{xy} dx + \int 2y\, dy = c$$

$$\frac{ye^{xy}}{y} + y^2 = c$$

$$\boxed{e^{xy} + y^2 = c}$$

6) $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

$$\boxed{M dx + N dy = 0}$$

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$
$$N = 2x^3y - 3x^2y^2 - 5y^4$$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\Rightarrow$ necessary condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 0 + 6x^2y - 6xy^2 \\ \frac{\partial N}{\partial x} &= 6x^2y - 6xy^2\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

DE is exact.

$\Rightarrow$ General Solution

$$\underset{y \to \text{const}}{\int M dx} + \underset{\text{terms ind of } x}{\int N dy} = C$$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\, dy = c$$

$$\boxed{x^5 + x^3y^2 - x^2y^3 - y^5 = c}$$

7) $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$

$$\boxed{M dx + N dy = 0}$$

$$M = 3x^2 + 6xy^2$$
$$N = 6x^2y + 4y^3$$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\Rightarrow$ necessary condition for exactness.

$$\frac{\partial M}{\partial y} = 12xy. \qquad \frac{\partial N}{\partial x} = 12xy.$$

$\Rightarrow$ DE is exact.

$\Rightarrow$ General Solution:

$$\underset{y \to \text{const}}{\int M dx} + \underset{\text{terms ind of } x}{\int N dy} = C$$

$$\int (3x^2 + 6xy^2)\,dx + \int 4y^3 dy = c$$

$$\boxed{x^3 + 3x^2y^2 + y^4 = c}.$$

8) $\frac{2x}{y^3}\,dx + \frac{y^2 - 3x^2}{y^4}\,dy = 0$

$$\boxed{M dx + N dy = 0}$$

$$M = \frac{2x}{y^3} \qquad N = \frac{y^2 - 3x^2}{y^4}$$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\Rightarrow$ necessary condition for exactness.

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2x\left(\frac{-3}{y^4}\right) = -\frac{6x}{y^4} \\ \frac{\partial N}{\partial x} &= 0 - \frac{6x}{y^4} = -\frac{6x}{y^4}\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ DE is exact

⇒ General Solution :

$$\int M\,dx + \int N\,dy = C$$

$y \to$ const   terms ind of x

$$\int \frac{2x}{y^3}\,dx + \int \frac{1}{y^2}\,dy = C$$

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

$$\boxed{\frac{x^2}{y^3} - \frac{1}{y} = C}$$

9) $y \sin 2x\,dx - (1+y^2+\cos^2 x)\,dy = 0$

$$\boxed{M\,dx + N\,dy = 0}$$

$M = y\sin 2x$

$N = -(1+y^2+\cos^2 x)$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ⇒ necessary condition for exactness.

$$\frac{\partial M}{\partial y} = \sin 2x$$

$$\frac{\partial N}{\partial x} = +2\cos x \sin x = \sin 2x$$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ⇒ DE is exact

⇒ General Solution

$$\int M\,dx + \int N\,dy = C$$

$y \to$ const   terms ind of x

$$\int y\sin 2x\,dx + \int -(1+y^2)\,dy = C$$

$$\frac{y(-\cos 2x)}{2} - y - \frac{y^3}{3} = C$$

$$\boxed{-\frac{y\cos 2x}{2} - y - \frac{y^3}{3} = C}$$

10) $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

$$\boxed{M\,dx + N\,dy = 0}$$

$M = 2xy + y - \tan y$

$N = x^2 - x\tan^2 y + \sec^2 y$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ⇒ necessary condition for exactness.

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y$$

DE is exact

⇒ General Solution :

$$\int M\,dx + \int N\,dy = C$$

$y \to$ const   terms indep of x

$$\int (2xy + y - \tan y)\,dx + \int \sec^2 y\,dy = C$$

$$\boxed{x^2y + x(y-\tan y) + \tan y = C}$$

30/05/2022

1) Determine for what values of a and b the following DE is exact. Hence find the general solution of exact equation.

$$(y+x^3)dx + (ax+by^3)dy = 0$$

$\Rightarrow M\,dx + N\,dy = 0$

$M = y+x^3$

$N = ax+by^3$

$\because$ DE is exact

$\frac{\partial M}{\partial y} = 1$ $\quad\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\frac{\partial N}{\partial x} = a$ $\quad\boxed{a=1}$

$\therefore$ $\quad\boxed{\therefore b \in R}$

General Solution:

$\int M\,dx + \int N\,dy = C$

$y \to$ const $\quad$ terms no x

$\int (y+x^3)dx + \int by^3 dy = C$

$$\boxed{xy + \frac{x^4}{4} + \frac{by^4}{4} = C}$$

Solve:

2) $(3x^2+2e^y)dx + (2xe^y+3y^2)dy = 0$

$\boxed{M\,dx + N\,dy = 0}$

Comparing $M = 3x^2+2e^y$

$N = 2xe^y+3y^2$

$\left.\begin{array}{l}\frac{\partial M}{\partial y} = 2e^y \\ \frac{\partial N}{\partial x} = 2e^y\end{array}\right\}$ $\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\therefore$ DE is exact

General Solution:

$\int M\,dx + \int N\,dy = C$

$y \to$ const $\quad$ terms no x

$\int (3x^2+2e^y)dx + \int 3y^2 dy = C$

$$\boxed{x^3 + 2xe^y + y^3 = C}$$

3) Solve the initial value condition/problem:

$e^x(\cos y\,dx - \sin y\,dy) = 0\ ;\ y(0) = 0$

$e^x\cos y\,dx + (-\sin y\,e^x)dy = 0$

Comparing $\boxed{M\,dx + N\,dy = 0}$

$M = e^x\cos y$

$N = -\sin y\,e^x$

$\frac{\partial M}{\partial y} = -\sin y\,e^x \qquad \frac{\partial N}{\partial x} = -\sin y\,e^x$

$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\therefore$ DE is exact

General Solution:

$$\int M dx + \int N dy = C$$

$y \to$ const terms no x

$$\int e^x \cos y \, dx + \int 0 \, dy = C$$

$$e^x \cos y = C$$

$x = 0; \; y = 0$

$$e^0 \cos 0 = C$$

$$\boxed{C = 1}$$

$\boxed{\therefore e^x \cos y = 1}$ is required particular solution

4) Under what conditions ; the following DEs are exact?

a) $xy^3 dx + ax^2y^2 dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2y^2 \qquad 3\cancel{x^2}\cancel{y^2} = 2\cancel{x}\cancel{y^2}a$$

$$\frac{\partial N}{\partial x} = 2xay^2 \qquad \boxed{a = \frac{3}{2}}$$

b) $(ax+y)dx + (Kx+by)dy = 0$

$$\frac{\partial M}{\partial y} = 1 \qquad \boxed{K = 1}$$

$$\frac{\partial N}{\partial x} = K \qquad \boxed{a, b \in R}$$

c) $(a \sinh x \cos y + b \cosh x \sin y)dx +$

$(c \sinh x \cos y + d \cosh x \sin y)dy = 0$

$N = c \sinh x \cos y + d \cosh x \sin y$

$M = a \sinh x \cos y + b \cosh x \sin y$

$$\frac{\partial M}{\partial y} = -a \sinh x \sin y + b \cosh x \cos y$$

$$\frac{\partial N}{\partial x} = c \cosh x \cos y - d \sinh x \sin y$$

$$\boxed{a = d; \quad b = c}$$

$a, b, c, d \in R$

5) For following DE check whether the equations are exact and find its G.S.

a) $(1+e^x)dx + y\,dy = 0$.

$$\boxed{M dx + N dy = 0}$$

$M = 1 + e^x$

$N = y$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 \\ \frac{\partial N}{\partial x} &= 0 \end{aligned}\right\} \quad \text{DE is exact} \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

General Solution:

$$\int M dx + \int N dy = C$$

$y \to$ const terms ~~of~~ no x.

$$\int (1+e^x)dx + \int y\,dy = C$$

$$\boxed{x + e^x + \frac{y^2}{2} = C}$$

2) $2\cosh x \, dx + \sinh x \, dy = 0$

$\boxed{M dx + N dy = 0}$

$M = 2\cosh x \qquad N = \sinh x$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = \cosh x$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ DE is not exact

~~General Solution :-~~

$\cancel{\int M dx + \int N dy = c}$

$\cancel{\int 2\cosh x \, dx + \int 0 \, dy = c}$

$\cancel{\boxed{2\sinh x = c}}$

3) $\left(3x^2y + \frac{y}{x}\right)dx + (x^3 + \ln x)\,dy = 0$

$M dx + N dy = 0$

$M = 3x^2y + \frac{y}{x} \qquad N = x^3 + \ln x$

$\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x} \qquad \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ∴ DE is exact

General Solution:

$\int M dx + \int N dy = c$

y → const terms no

$\int\left(3x^2y + \frac{y}{x}\right)dx + \int 0 \, dy = c$

$\boxed{x^3y + y\ln x = c}$

4) $x\,dy + 2y\,dx - xy\,dy = 0$

$2y\,dx + (x - xy)\,dy = 0$

$M dx + N dy = 0$

$M = 2y \qquad \frac{\partial M}{\partial y} = 2$

$N = x - xy \qquad \frac{\partial N}{\partial x} = 1 - y$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ DE is not exact.

5) $x\,dx + y\,dy = 2yx^2\,dy + 2y^3\,dy$

$x\,dx + (y - 2yx^2 - 2y^3)\,dy = 0$

$M dx + N dy = 0$

$M = x \qquad N = y - 2yx^2 - 2y^3$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0 - 4x - 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

DE is not exact.

6) $y(1 + 6xy)\,dx + (4y - x)\,dy = 0$

$M dx + N dy = 0$

$M = y + 6xy^2 \qquad N = 4y - x$

$$\frac{\partial M}{\partial y} = 1 + 12xy \qquad \frac{\partial N}{\partial x} = 0 - 1 = -1.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{DE is not exact.}$$

7) $(1+x^2)dy + 2xy\,dx = 0$

$$M\,dx + N\,dy = 0$$

$$M = 2xy \qquad N = 1+x^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{DE is exact}$$

General Solution:

$$\int M\,dx + \int N\,dy = c$$

$$\int 2xy\,dx + \int 1\,dy = c$$

$$\boxed{x^2y + y = c}$$

8) $(e^{2y}+1)\cos x\,dx + 2e^{2y}\sin x\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = (e^{2y}+1)\cos x \qquad N = 2e^{2y}\sin x$$

$$\frac{\partial M}{\partial y} = e^{2y}\cdot 2\cos x \qquad \frac{\partial N}{\partial x} = 2e^{2y}\cos x$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{DE is exact}$$

General Solution:

$$\int M\,dx + \int N\,dy = c$$

$$\int (e^{2y}+1)\cos x\,dx + \int 0\,dy = c$$

$$\boxed{e^{2y}\sin x + (+\sin x) = c}$$

9) $y\,dx + x\,dy + xy\,dy = 0$

$$d(xy) + xy\,dy = 0$$

$$y\,dx + (x+xy)\,dy = 0$$

$$M = y \quad ; N = x + xy$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1 + y$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{DE is not exact.}$$

10) $\sinh x \cos y\,dx - \cosh x \sin y\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = \sinh x \cos y$$

$$N = -\cosh x \sin y$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= -\sinh x \sin y \\ \frac{\partial N}{\partial x} &= -\sinh x \sin y\end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ DE is exact

$\therefore$ General Solution:

$$\int M\,dx + \int N\,dy = C$$

$$\int \sinh x \cos y\, dx + \int 0\, dy = C$$

$$\boxed{\cosh x \cos y = C}$$

11) $(xe^{xy} + 2y)\,dy + ye^{xy}\,dx = 0$

$$M\,dx + N\,dy = 0$$

$$M = ye^{xy}$$
$$N = (xe^{xy} + 2y)$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= xye^{xy} + e^{xy}\\ \frac{\partial N}{\partial x} &= xye^{xy}\end{aligned}\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ DE is not exact

12) $x\,dy - y\,dx = e^{y}(x^2+y^2)\,dy$

$$[x - e^{y}(x^2+y^2)]\,dy - y\,dx = 0$$

$$M = -y$$
$$N = x - e^{y}(x^2+y^2)$$

$$\frac{\partial M}{\partial y} = -1 \qquad \frac{\partial N}{\partial x} = 1 - 2xe^{y} - 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{DE is not exact}$$

13) $x\,dy - y\,dx + y^2\,dx = 0$

$$x\,dy + (y^2 - y)\,dx = 0$$

$$N\,dy + M\,dx = 0 \qquad M = y^2 - y$$
$$N = x$$

$$\frac{\partial M}{\partial y} = 2y - 1$$

$$\frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{DE is not exact}$$

14) $(2x + e^{y})\,dx + xe^{y}\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = 2x + e^{y}$$
$$N = xe^{y}$$

$$\frac{\partial M}{\partial y} = e^{y} \qquad \frac{\partial N}{\partial x} = e^{y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{DE is exact}$$

~~15) 2xy~~ General Solution

$$\int M\,dx + \int N\,dy = C$$

$$\int (2x + e^{y})\,dx + 0 = C$$

$$\boxed{x^2 + xe^{y} = C}$$

15) $2xy\,dx + (x^2+1)\,dy = 0$

$$Mdx + Ndy = 0$$

$$M = 2xy \quad N = x^2+1$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{DE is exact}$$

General Solution:

$$\int M dx + \int N dy = C$$

$$\int 2xy\,dx + \int 1\,dy = C$$

$$\boxed{x^2y + y = C}$$

01/06/2022

*** Integrating factor: (IF)**

Non-exact differential equation can be made into exact differential equation by multiplying a suitable non-zero factor (in terms of x&y (or) only in terms of x (or) y) is said to be an integrating factor.

- Integrating factor is not unique for a problem; a problem can have more than 1 I.F.

Ex: $y\,dx - x\,dy = 0$ — (1)

$$Mdx + Ndy = 0$$

$$M = y\,;\ N = -x$$

$$\frac{\partial M}{\partial y} = 1\,;\ \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

eqn (1) is non-exact

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{x}$$

$$I.F = e^{\int -2/x\,dx} = e^{-2\log x} = \cancel{e^{\log x^{-2}}} = x^{-2} = \frac{1}{x^2}$$

(1) × I.F

$$\frac{y}{x^2}dx - \frac{1}{x}dx = 0 \quad \text{— (2)}$$

$$M_1 = \frac{y}{x^2} \quad \Big| \quad N_1 = \frac{-1}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^2} \quad \Big| \quad \frac{\partial N_1}{\partial x} = \frac{1}{x^2}$$

eqn (2) is exact DE

→ Multiplying eq (1) becomes exac

1) with $\frac{1}{x^2}$; IF $= \frac{1}{x^2}$

2) with $\frac{1}{y^2}$; I.F $= \frac{1}{y^2}$

3) $\frac{1}{xy}$; IF $= \frac{1}{xy}$.

General Solution:

$$\int M dx + \int N dy = c$$

$y \to$ const terms indp of x

① $\int \frac{y}{x^2} dx + \int 0\, dy = c$

$$\boxed{\frac{-y}{x} = c}$$

② $\int \frac{1}{y} dx + \int 0\, dy$

$$\boxed{\frac{x}{y} = c}$$

③ $\int \frac{1}{x} dx + \int \frac{-1}{y} dy = c$

$\log x - \log y = \log c$

$$\boxed{\frac{x}{y} = c}$$

* Note: For each different integrating factor; we get a same general solution for a differential equation.

There are 5 standard types to find an integrating factor:

1) Integrating factor found by direct method:

In a no. of problems a little analysis is helpful to find I.F. The following are the ~~…~~ list of differentials will be helpful to solve the problem

1) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$ 2) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

3) $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

4) $\frac{x\,dy - y\,dx}{xy} = d\left(\log\left(\frac{y}{x}\right)\right)$

5) $\frac{y\,dx + x\,dy}{xy} = d\left(\log(xy)\right)$

6) $\frac{x\,dx + y\,dy}{x^2+y^2} = d\left(\frac{1}{2}\log\left(x^2+y^2\right)\right)$

7) $\frac{x\,dy - y\,dx}{x^2-y^2} = d\left(\frac{1}{2}\left(\log\left(\frac{x+y}{x-y}\right)\right)\right)$

Ex: $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$ — ①

Compare $M dx + N dy$

$M = y + 3x^2y^2e^{x^3}$

$N = -x$

$\frac{\partial M}{\partial y} = 1 + 6x^2ye^{x^3}$ $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Given equation is non-exact DE.

Divide ① by y^2 on both sides

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2 e^{x^3} dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int d\left(e^{x^3}\right) = 0$$

$$\boxed{\frac{x}{y} + e^{x^3} = C}$$

$*$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2 + 6x^2 y e^{x^3}$

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{y(1+3x^2ye^{x^3})}(2)\left(1+3x^2ye^{x^3}\right)$$

$$= \frac{-2}{y}$$

$$I.F = e^{\int \frac{-2}{y} dy} = \frac{1}{y^2}$$

Multiply eqn ① with I.F.

$$\frac{dx}{y} - \frac{x\,dy}{y^2} + 3x^2 e^{x^3} dx = 0 \rightarrow \text{Exact}$$

$$\int M\,dx + \int N\,dy = C$$

$$\int \left(\frac{1}{y} + 3x^2 e^{x^3}\right) dx + \int 0\,dy = C$$

$$\underline{\frac{x}{y} + e^{x^3} = C}$$

2) If the given eq. is non exact and it is a homogeneous differential equation (degree of x & y are same); then the integrating factor is $\boxed{I.F = \frac{1}{Mx+Ny}}$ and multiplying this I.F with the given equation gives an exact differential equation.

3) If the given eq. is non-exact and it is of the form $\boxed{y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0}$ then integrating factor $\boxed{I.F = \frac{1}{Mx - Ny}}$ & multiplying this factor with given equation gives an exact DE.

4) If the given eq. is non-exact & if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is divisible by N i.e.

$\boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)}$; then I.F is

$\boxed{IF = e^{\int f(x)\,dx}}$ and multiply this with ①; we will get exact DE.

5) If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is divisible by M.

i.e. $\boxed{\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)}$ then

the I.F is $\boxed{I.F = e^{\int g(y)\,dy}}$ multiply this factor with given equation ; we get exact DE.

* $2\cosh x\,dx + \sinh x\,dy = 0$ — (1)

$M dx + N dy = 0$

$M = 2\cosh x \qquad N = \sinh x$

$\frac{\partial M}{\partial y} = 2\sinh x \qquad \frac{\partial N}{\partial x} = \cosh x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2\sinh x - \cosh x$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2\sinh x - \cosh x}{\sinh x}$

$= 2 - \coth x$

$I.F = e^{\int f(x)\,dx} = e^{\int (2 - \coth x)\,dx}$

$= e^{2x - \ln(\sinh x)}$

$\boxed{I.F = \frac{e^{2x}}{\sinh x}}$

Multiply eq (1) with I.F.

$2e^{2x}\coth x\,dx + e^{2x}\,dy = 0$

$M_1 = 2e^{2x}\coth x \qquad N_1 = e^{2x}$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N_1}{\partial x} =$

* $x\,dy + 2y\,dx - xy\,dy = 0$ — (1)

$M dx + N dy = 0$

$M = 2y \qquad N = x - xy$

$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 1 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + y.$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1+y}{1-y}$

$\because$ Eq (1) is of the form $y(2)\,dx + x(1-y)\,d$

$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy - xy + x}$

$= \frac{1}{xy + xy^2}$

$I \cdot F = \frac{1}{xy(1+y)}$

Multiply eq ① with I.F

$\frac{dy}{y(1+y)} + \frac{2dx}{x(1+y)} - \frac{dy}{1+y} = 0$

$M_1 = \frac{2}{x(1+y)}$ $\quad N_1 = \frac{1}{y(1+y)} - \frac{1}{1+y}$

$= \frac{1-y}{(1+y)y}$

$\frac{\partial M_1}{\partial y} = \frac{2}{x} \frac{1}{(1+y)^2}$

02/06/2022

* $2\cosh x \, dx + \sinh x \, dy = 0$ — ①

③ $M = 2\cosh x$ $\quad N = \sinh x$

$\frac{\partial M}{\partial y} = 0$ $\quad \frac{\partial N}{\partial x} = \cosh x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\cosh x$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -1$

$IF = e^{\int f(x)dx} = e^{\int -1 dx} = e^{-x}$

Multiply IF with ①

$2e^{-x}\cosh x \, dx + e^{-x}\sinh x \, dy = 0$

$M_1 = 2e^{-x}\cosh x$ $\quad N_1 = e^{-x}\sinh x$

$2\coth x \, dx + dy = 0$

$\boxed{2\ln \sinh x + y = c}$

* $x\,dx + y\,dy = 2yx^2 dy + 2y^3 dy$

$x\,dx + (y - 2yx^2 - 2y^3)dy$

$M = x$ $\quad N = y - 2yx^2 - 2y^3$

$\frac{\partial M}{\partial y} = 0$ $\quad \frac{\partial N}{\partial x} = -4xy$

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -1$

$\cancel{IF = e^{\int -1 dx} = e^{-x}}$

$\cancel{xe^{-x}dx + e^{-x}(y - 2yx^2 - 2y^3}$

$x\,dx + y\,dy - 2(y)(x^2 dy + y^2 dy) = 0$

9) $\dfrac{x\,dx + y\,dy}{x^2+y^2} - 2y\,dy = 0$

$$\int d\left(\frac{1}{2}\log(x^2+y^2)\right) - \int 2y\,dy = 0$$

$$\boxed{\frac{1}{2}\log(x^2+y^2) - y^2 = C}$$

7) $\dfrac{x\,dy + 2y\,dx}{xy} = xy\,dy$

$$\frac{dy}{y} + \frac{2dx}{x} = dy$$

$$\log y + 2\log x = y$$

$$\log y + 2\log x - y = C$$

2) $y\,dx + x\,dy + xy\,dy = 0$

$$d(xy) + xy\,dy = 0$$

$$y\,dx + (x + xy)\,dy = 0$$

$M = y \qquad N = x(1+y)$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1 + y.$$

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{+y}{\cancel{1+y}\,y} = 1$$

$$e^{\int 1\,dy} = e^y$$

$$ye^y\,dx + xe^y(1+y)\,dy = 0$$

$M_1 = ye^y \qquad N_1 = xe^y(1+y)$

$$\frac{\partial M_1}{\partial y} = ye^y + e^y \qquad \frac{\partial N_1}{\partial x} = (1+y)e^y$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\therefore \int M\,dx + \int N\,dy = C$$

$$ye^y x + 0 = C$$

$$\boxed{xye^y = C}$$

8) $\dfrac{x\,dy - y\,dx}{x^2+y^2} - e^y\,dy = 0$

$$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) - \int e^y\,dy = 0$$

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - e^y = C}$$

10) $\dfrac{x\,dy - y\,dx}{y^2} + dx = 0$

$$-\int d\left(\frac{x}{y}\right) + \int dx = 0$$

$$\boxed{-\frac{x}{y} + x = C}$$

11) $(y+6xy^2)dx + (4y-x)dy = 0$

$y\,dx - x\,dy + 6xy^2 dx + 4y\,dy = 0$

divide with y^2

$$\int \frac{y\,dx - x\,dy}{y^2} + \int 6x\,dx + \int \frac{4}{y}\,dy = 0$$

$$\boxed{\frac{x}{y} + 3x^2 + 4\log y = C}$$

1) $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

Divide equation with y^2

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int d(e^{x^3}) = C$$

$$\boxed{\frac{x}{y} + e^{x^3} = C}$$

2) $x^2y\,dx - 2xy^2dx - x^3dy + 3x^2y\,dy = 0$ (1)

$x^2(y\,dx - x\,dy) + xy(3x\,dy - 2y\,dx) = 0$

$(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

$M = x^2y - 2xy^2$ $\quad N = 3x^2y - x^3$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ $\quad \frac{\partial N}{\partial x} = 6xy - 3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ DE is non-exact

By standard type : 2 $\quad IF = \frac{1}{Mx+Ny}$

$$IF = \frac{1}{\cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}}$$

$$\boxed{IF = \frac{1}{x^2y^2}}$$

Multiply eq 1 with IF.

$$\frac{dx}{y} - \frac{2}{x}dx - \frac{x}{y^2}dy + \frac{3}{y}dy = 0$$

11) $(y+6xy^2)dx+(4y-x)dy=0$

$y\,dx - x\,dy + 6xy^2dx + 4y\,dy = 0$

divide with y^2

$$\int \frac{y\,dx - x\,dy}{y^2} + \int 6x\,dx + \int \frac{4}{y}\,dy = 0$$

$$\boxed{\frac{x}{y} + 3x^2 + 4\log y = C}$$

1) $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

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$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0$$

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By standard type : 2 $IF = \frac{1}{Mx + Ny}$

$$IF = \frac{1}{\cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}}$$

$$\boxed{IF = \frac{1}{x^2y^2}}$$

Multiply eq 1 with IF.

$$\frac{dx}{y} - \frac{2}{x}dx - \frac{x}{y^2}dy + \frac{3}{y}dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$$

$$M_1 = \frac{1}{y} - \frac{2}{x} \qquad N_1 = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

DE is exact.

General solution:

$$\int M dx + \int N dy = C$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = C$$

$$\boxed{\frac{x}{y} - 2\log x + 3\log y = C}$$

3) $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ —(1)

~~xy^2~~

$Mdx + Ndy = 0$

$M = xy^2 + 2x^2y^3 \quad N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

DE is not exact.

$IF = \frac{1}{Mx - Ny}$

$= \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3}$

$$\boxed{I.F = \frac{1}{3x^3y^3}}$$

Multiply I.F with (1)

$\frac{1}{3x^2y}dx + \frac{2}{3x}dx + \frac{1}{3xy^2}dy - \frac{1}{3y}dy$

$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$

$M_1 = \frac{1}{3x^2y} + \frac{2}{3x} \quad N_1 = \frac{1}{3xy^2} - \frac{1}{3y}$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{3x^2y^2}\log y \qquad \frac{\partial N_1}{\partial x} = \frac{-1}{3y^2x^2}\log x$$

$$xy(y\,dx + x\,dy) + 2x^2y^3\,dx - x^3y^2\,dy = 0$$

$$y\,dx + x\,dy + 2xy^2\,dx - x^2y\,dy = 0$$

$$(y + 2xy^2)\,dx + (x - x^2y)\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + 4xy \qquad \frac{\partial N}{\partial x} = 1 - 2xy$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2xy$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{xy + 2x^2y^2 - xy + x^2y^2}$$

$$= \frac{1}{3x^2y^2}$$

$$\cancel{\frac{1}{3x^2y}} + \cancel{\frac{2}{3x}}$$

$$\int M\,dx + \int N\,dy = C$$

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int \frac{-1}{3y} = C$$

$$\frac{1}{3y}\cdot\frac{x^{-1}}{-1} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$$

$$\boxed{\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C}$$

4) $(xy^2 - e^{x^{1/3}})dx - x^2y\,dy = 0$ — (1)

$$M = xy^2 - e^{x^{1/3}} \qquad N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = -2xy$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{4}{x}$$

$$I.F = e^{\int -4/x\,dx} = e^{-4\log x}$$
$$= x^{-4} = \frac{1}{x^4}$$

Mutliply ~~IF with~~ (1) with I.F

$$\frac{y^2}{x^3}dx - \frac{e^{x^{1/3}}}{x^4}dx - \frac{y}{x^2}dy = 0$$

$$M_1 = \frac{y^2}{x^3} - \frac{e^{x^{1/3}}}{x^4} \qquad N_1 = \frac{-y}{x^2}$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{2y}{x^3}} \qquad \boxed{\frac{\partial N_1}{\partial x} = \frac{+2y}{x^3}}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow \text{DE is exact}$$

<u>General Solution</u>:

$$\int \frac{y^2}{x^3}dx - \int \frac{e^{x^{1/3}}}{x^4}dx - 0 = c$$

$\cancel{\frac{-y^2}{x^2}}$ $\cancel{e^{x^{1/3}}}$

$I.F = \frac{1}{x^2y^2 - e^{x^{1/3}} \cdot x -}$

5) $(xy^3 + y)\,dx + (2)(x^2y^2 + x + y^4)\,dy = 0$.

$M\,dx + N\,dy = 0$

$M = xy^3 + y \qquad N = 2(x^2y^2 + x + y^4)$

$\frac{\partial M}{\partial y} = 3xy^2 + 1 \qquad \frac{\partial N}{\partial x} = 2(2xy^2 + 1)$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ DE is not exact.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2 + 1 - 4xy^2 - 2$

$= -1 - xy^2 \qquad \frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y}$

$\cancel{xy^3dx + ydx + 2x^2y^2dy + 2xdy + 2y^4dy = 0}$

~~divide with xy~~

$\cancel{y^2dx + \frac{dx}{x} + 2xy\,dy + 2\frac{dy}{y} + \frac{2y^3}{x}dy = 0}$

$d(xy^2) \qquad I.F = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

$(xy^4 + y^2)\,dx + 2y(x^2y^2 + x + y^4) = 0$

$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \qquad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y$

$\int M\,dx + \int N\,dy = c \Rightarrow \int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$

$\Rightarrow \boxed{\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c}$

6) $\frac{y\,dx - x\,dy}{\cancel{y^2}} + \frac{e^{y/x}\,dx}{\cancel{y^2}} = 0$ —— (1)

$(y + e^{y/x})\,dx - x\,dy = 0$

$M\,dx + N\,dy = 0$

$M = y + e^{1/x} \qquad N = -x$

$\frac{\partial M}{\partial y} = +1 \qquad \frac{\partial N}{\partial x} = -1$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ DE is not exact.

$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{-x} = \frac{-2}{x}$

$I.F = e^{\int \frac{-2}{x}dx} = e^{-2\log x}$

$I.F = \frac{1}{x^2}$

Multiply eq (1) with I.F

$e^{1/x} \cdot 1$

$\frac{y\,dx - x\,dy}{x^2} + \frac{e^{1/x}}{x^2}\,dx = 0$

$-\int d\left(\frac{xy}{x}\right) + \int -d\left(e^{1/x}\right) = 0$

$\boxed{\frac{y}{x} + e^{1/x} = c}$

7) $(5x^3+12x^2+6y^2)dx + 6xy\,dy = 0$ —(1)

$M\,dx + N\,dy = 0$

$M = 5x^3+12x^2+6y^2$ $\quad N = 6xy$

$\frac{\partial M}{\partial y} = 12y$ $\quad \frac{\partial N}{\partial x} = 6y$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ DE is not exact

$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{1}{6xy}(6y) = \frac{1}{x}$

$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Multiply eq (1) with I.F

$5x^4dx + 12x^3dx + 6y^2x\,dx + 6x^2y\,dy = 0$

$M_1 = 5x^4 + 12x^3 + 6y^2x$ $\quad N_1 = 6x^2y$

$\frac{\partial M_1}{\partial y} = 12xy$ $\quad \frac{\partial N_1}{\partial x} = 12xy$

$\because \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ DE is exact

$\Rightarrow$ General Solution:

$\int M\,dx + \int N\,dy = C$

$\int (5x^4+12x^3+6xy^2)dx + 0 = C$

$\boxed{x^5 + 3x^4 + 3x^2y^2 = C}$

8). $x\,dy + xy^2dy + y\,dx + x^2y\,dx = 0$

$(x\,dy + y\,dx) + xy(y\,dy + x\,dx) = 0$

~~dx~~

$$\Rightarrow (y+x^2y)dx + (x+xy^2)dy = 0$$

$$M\,dx + N\,dy = 0$$

$$M = y+x^2y \qquad N = x+xy^2$$

$$\frac{\partial M}{\partial y} = 1+x^2 \qquad \frac{\partial N}{\partial x} = 1+y^2$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{DE is not exact}$$

$$\Rightarrow (y\,dx + x\,dy) + xy(x\,dx + y\,dy) = 0$$

Divide with xy

$$\frac{dx}{x} + \frac{dy}{y} + x\,dx + y\,dy = 0 \quad \text{— (2)}$$

$$M = \frac{1}{x} + x \qquad N = \frac{1}{y} + y$$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \text{DE is exact}$$

Integrate eq (2)

$$\log x + \log y + \frac{x^2}{2} + \frac{y^2}{2} = C$$

$$\boxed{2\log(xy) + x^2 + y^2 = C}$$

9) $(2xy+x^2)\,dy = (3y^2+2xy)\,dx$ — ①

$$(3y^2+2xy)\,dx - (2yx+x^2)\,dy = 0$$

$$M = 3y^2+2xy \qquad N = -(2xy+x^2)$$

$$\frac{\partial M}{\partial y} = 6y+2x \qquad \frac{\partial N}{\partial x} = -(2y+2x)$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ DE is not exact.

$$\cancel{\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{-1}{2(x+y)}(6y+2x+2y+2x)}$$

$$\cancel{= \frac{-1}{2(x+y)}(8y+4x)}$$

$=$

~~$\because$ Eq ① is homogeneous $\Rightarrow$ IF $= \frac{1}{Mx+Ny}$~~

$$\cancel{IF = \frac{1}{3xy^2}}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{-1}{x(2y+x)}(8y+4x)$$

$$= \frac{-4}{x}\frac{(2y+x)}{2y+x} = \frac{-4}{x}$$

$$I.F. = e^{\int \frac{-4}{x}dx}$$

$$= e^{\log x^{-4}} = x^{-4}$$

$$\left(\frac{3y^2}{x^4}+\frac{2y}{x^3}\right)dx - \left(\frac{2y}{x^3}+\frac{1}{x^2}\right)dy = 0$$

$M_1 = \frac{3y^2}{x^4} + \frac{2y}{x^3}$ $\quad N_1 = \left(\frac{2y}{x^3} + \frac{1}{x^2}\right)(-1)$

$\frac{\partial M_1}{\partial y} = \frac{6xy}{x^4} + \frac{2}{x^3}$, $\quad \frac{\partial N_1}{\partial x} = \frac{6y}{x4} + \frac{2}{x^3}$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

General Solution:

$\int M dx + \int N dy = C$

$\int \frac{3y^2}{x4} dx + \int \frac{2y}{x^3} dx = C$

$3y^2 \cdot \frac{x^{-3}}{-3} + 2y \cdot \frac{x^{-2}}{-2} = C$

$\boxed{-\frac{y^2}{x^3} - \frac{\cancel{2}y}{x^2} = C}$

$\boxed{+\frac{y^2}{x^3} + \frac{y}{x^2} = C}$

10) $(ye^{xy} + 4y^3)\, dx + (xe^{xy} + 12xy^2 - 2y)dy =$

$M dx + N dy = 0$

$M = ye^{xy} + 4y^3$

$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + 12y^2$

$N = xe^{xy} + 12xy^2 - 2y$

$\frac{\partial N}{\partial x} = e^{xy} + 12y^2 + xye^{xy}$

$\cancel{\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow DE \text{ is not exact}}$

$\cancel{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{xy} + xye^{xy} + 12y^2 - e^{xy} - 12y^2}$

10) $(ye^{xy} + 4y^3)dx + (xe^{xy} + 12xy^2 \cancel{xy} - 2y)dy = 0$

$y(0) = 2$

$M = ye^{xy} + 4y^3$

$\frac{\partial M}{\partial y} = e^{xy} + ye^{xy}(x) + 12y^2$

$= e^{xy} + xye^{xy} + 12y^2$

$N = xe^{xy} + 12xy^2 - 2y$

$\frac{\partial N}{\partial x} = xe^{xy}(y) + e^{xy} + 12y^2$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ DE is ~~not~~ exact

G.S:

$\int M dx + \int N dy = c$

$\int (ye^{xy} + 4y^3) dx + \int -2y\, dy = c$

$\frac{ye^{xy}}{y} + 4xy^3 - y^2 = c$

$$\boxed{e^{xy} + 4xy^3 - y^2 = c}$$

$x = 0 \quad y = 2$

$e^0 + 0 - 4 = c$

$$\boxed{c = 3}$$

11) $(\cos x + y\sin x)dx - \cos x\, dy = 0$ $\boxed{y(\pi) = 0}$

$M\,dx + N\,dy = 0$

$M = \cos x + y\sin x$ $N = -\cos x$

$\frac{\partial M}{\partial y} = 0 + \sin x$ $\frac{\partial N}{\partial x} = \sin x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ DE is exact

$\Rightarrow$ General Solution :

$\int M\,dx + \int N\,dy = c$

$\int (\cos x + y\sin x)\,dx = c$

$\boxed{\sin x - y\cos x = c}$

$x = \pi ;\ y = 0$

$c = 0$

$\Rightarrow$ Particular Solution of given DE is

$\boxed{\sin x = y\cos x}$

07/06/2022

Definition of Linear DE of first order:
(Leibnitz Linear differential equation)

Linear $\Rightarrow$ order = anything
degree = 1.

A DE is said to be linear if the dependent variable and its derivatives (differential coefficients) occur only in 1^{st} degree and not multiplied together.

Standard equation of Linear first order equation:

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \quad -(1)$$

$$\frac{dx}{dy} + P(y)\cdot x = Q(y) \quad -(2)$$

$$I.F = e^{\int(\text{Function of I.V}) \text{ wrt I.V.}}$$

(1) $(I.F) = e^{\int p(x)dx}$

(2) $I.F = e^{\int P(y)dy}$

General Solution:

$$\boxed{(D.V)(IF) = \int Q(I.F) \text{ wrt I.V.} + C}$$

(1) $y(I.F) = \int Q(x)(I.F)dx + C$

(2) $x(I.F) = \int Q(y)(I.F)dy + C.$

* DERIVATION OF I·F and G·S of 1st order:

Consider standard equation of LDE:

$$\boxed{\frac{dy}{dx} + P(x)\cdot y = Q(x)}$$

$$[P(x)\cdot y - Q(x)]\,dx + dy = 0 \quad -① $$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = P(x) \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Eq ① is non-exact}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = P(x)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x)$$

$$\boxed{IF = e^{\int P(x)\,dx}}$$

Multiply I·F with ①

$$\left[P(x)\cdot y\cdot e^{\int P(x)dx} - Q(x)\cdot e^{\int P(x)dx}\right] + dy\, e^{\int P(x)dx}$$

General Solution:

$$\int M\,dx + \int N\,dy = C \quad -② $$

$$\int \left[P(x)\cdot y\cdot e^{\int P(x)\cdot dx} - Q(x)\cdot e^{\int P(x)\cdot dx}\right] dx = C$$

$$\int P(x)\cdot y\cdot e^{\int P(x)\cdot dx}\,dx - \int Q(x)\cdot e^{\int P(x)\cdot dx} =$$

$$\Rightarrow \boxed{y\, e^{\int P(x)\cdot dx} = \int Q(x)\cdot e^{\int P(x)\cdot dx} + C}$$

III^{ly}: if $\frac{dx}{dy} + P(y)\cdot x = Q(y)$

we get $\boxed{I\cdot F = e^{\int P(y)dy}}$

$$\boxed{x\, e^{\int P(y)dy} = \int Q(y)\cdot e^{\int P(y)dy} + C}$$

* Non-linear DE:

A DE is said to be non-linear if D.V and its derivatives occur other than 1^{st} degree (or) multiplied together.

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$$

$$\frac{dx}{dy} + P(y)\,x = Q(y)$$

$$P(y) = \frac{1}{y\log y} \qquad Q(y) = \frac{1}{y}$$

$$I\cdot F = e^{\int P(y)\,dy} = e^{\int \frac{1}{y\log y}} = e^{\log(\log y)}$$

$$\boxed{I\cdot F = \log y}$$

$\cancel{x\cdot \log y = \int}$

$$x(I\cdot F) = \int Q(y)\cdot I\cdot F\cdot dy + C$$

$$x\log y = \int \frac{\log y}{y}\,dy + C$$

$$\boxed{x \log y = \frac{(\log y)^2}{2} + c}$$

04/06/2022

1) $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2 x\, y = \frac{\sin x}{\cos x} \times \frac{1}{\cos^2 x}$$

$$\frac{dy}{dx} + (\sec^2 x) y = \frac{\sin x}{\cos^3 x} = \tan x \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + P(x) y = Q(x)$$

$P(x) = \sec^2 x \qquad Q(x) = \tan x \sec^2 x$

$$I \cdot F = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

General solution:

$$y(I \cdot F) = \int Q(x) \cdot I \cdot F \cdot dx + c$$

$$y(e^{\tan x}) = \int e^{\tan x} \tan x \cdot \sec^2 x + c$$

$$= \tan x \int d(e^{\tan x}) - \int \sec^2 x \cdot e^{\tan x} + c$$

$$y e^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + c$$

$$\boxed{y = \tan x - 1 + c e^{-\tan x}}$$

2) $x \log x \dfrac{dy}{dx} + y = \log x^2$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

$$\frac{dy}{dx} + y(P(x)) = Q(x)$$

$$P(x) = \frac{1}{x\log x} \qquad Q(x) = \frac{2}{x}$$

$$I \cdot F = e^{\int P(x) \cdot dx} = e^{\int \frac{1}{x\log x} dx}$$

$$= e^{\log(\log x)} = \log x$$

General Solution:

$$y(I \cdot F) = \int Q(x) \cdot (I \cdot F)\, dx + C$$

$\cancel{y(e^{\tan x})}$

$$y \log x = \int \frac{2\log x}{x} dx + C$$

$$y \log x = \frac{(\log x)^2}{2} \times 2 + C$$

$$\boxed{y \log x = (\log x)^2 + C}$$

3) $2\cos x \dfrac{dy}{dx} + 4y\sin x = \sin 2x$ $y = 0$, $x = \pi/3$

$$\frac{dy}{dx} + 2y\tan x = \sin x$$

$$P(x) = 2\tan x \qquad Q(x) = \sin x$$

$$I \cdot F = e^{\int 2\tan x\, dx} = e^{2\log|\sec x|} = \sec^2 x$$

General Solution:

$y\sec^2 x = \int \sin x \sec^2 x$

$y\sec^2 x = \int \tan x \sec x \, dx + c$

$y\sec^2 x = \sec$

$y\sec^2 x = \sin x \tan x - \int \cos x \tan x \, dx +$

$$\boxed{y\sec^2 x = \sin x \tan x + \cos x + c}$$

$$\boxed{\int f(x)\cdot g(x)\cdot dx = f(x)\int g(x)\,dx - \int \frac{df(x)}{dx}\cdot \int g(x)\,dx}$$

4) $\cosh x \frac{dy}{dx} + y\sinh x = 2\cosh^2 x \sinh x$

$$\frac{dy}{dx} + y\tanh x = 2\cosh x \sinh x$$

$$\frac{dy}{dx} + y\tanh x = \sinh 2x$$

$P(x) = \tanh x \quad Q(x) = \sinh 2x$

$$I.F = e^{\int \tanh x \, dx} = e^{\operatorname{sech}^2 x}$$

General Solution:

$$y\,e^{\operatorname{sech}^2 x} = \int \sinh(2x)\cdot e^{\operatorname{sech}^2 x}\,dx + C$$

$$y\,e^{\operatorname{sech}^2 x} = \int \sinh(2x)\cdot e^{\operatorname{sech}^2 x}\,dx + C$$

5) $(1-x^2)\dfrac{dy}{dx} - xy = 1$

$$\frac{dy}{dx} + \left(\frac{-x}{1-x^2}\right)y = \frac{1}{1-x^2}$$

$$P(x) = \frac{-x}{1-x^2} \qquad Q(x) = \frac{1}{1-x^2}$$

$$I.F = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$I.F = \sqrt{1-x^2}$$

General Solution :

$$y\sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}}\,dx + C$$

$$\cancel{y\sqrt{1-x^2} = \sin^{-1}x + C}$$

$$\boxed{y\sqrt{1-x^2} = \sin^{-1}x + C}$$

6) $(1-x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}$.

$$\frac{dy}{dx} + \frac{2xy}{1-x^2} = \frac{x}{\sqrt{1-x^2}}$$

$$P(x) = \frac{2x}{1-x^2} \qquad Q(x) = \frac{x}{\sqrt{1-x^2}}$$

$$I.F = e^{-\int \frac{-2x}{1-x^2}} = e^{-\log(1-x^2)}$$

$$= \frac{1}{1-x^2}$$

G.S:

$$\frac{y}{1-x^2} = \frac{-1}{2}\int \frac{-2x}{(1-x^2)^{3/2}}\, dx + C$$

$$= \frac{-1}{\cancel{2}} \frac{(1-x^2)^{-1/2}}{\cancel{-1/2}} + C$$

$$\boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C}$$

7) $$\frac{dy}{dx} = \frac{-x - y\cos x}{1+\sin x}$$

$$\frac{dy}{dx} = \frac{-x}{1+\sin x} - y\left(\frac{\cos x}{1+\sin x}\right)$$

$$\frac{dy}{dx} + \left(\frac{\cos x}{1+\sin x}\right) y = \frac{-x}{1+\sin x}$$

$$P(x) = \frac{\cos x}{1+\sin x} \qquad Q(x) = \frac{-x}{1+\sin x}$$

$$I.F = e^{\int \frac{\cos x}{1+\sin x} dx} = e^{\log(1+\sin x)}$$

$$= 1+\sin x$$

$$y(1+\sin x) = \int -x \, dx + C$$

$$\boxed{y(1+\sin x) = -\frac{x^2}{2} + C}$$

8) $\frac{dr}{d\theta} + 2r\cot\theta + \sin 2\theta = 0.$

$$\frac{dr}{d\theta} + (2\cot\theta)\, r = -\sin 2\theta$$

$P(\theta) = 2\cot\theta \qquad Q(\theta) = -\sin 2\theta$

$$I.F = e^{\int 2\cot\theta\, d\theta} = e^{2\log|\sin\theta|} = \sin^2\theta$$

$$\Rightarrow r\sin^2\theta = \int -2\sin^3\theta\cos\theta\, d\theta + C$$

$$r\sin^2\theta = -\int \sin^2\theta \cdot \sin 2\theta\, d\theta + C,$$

$$\boxed{r\sin^2\theta = -\frac{\sin^3\theta}{3} + C}$$

9) $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

$P(x) = 2x \qquad Q(x) = 2e^{-x^2}$

$$I.F = e^{\int 2x\, dx} = e^{x^2}$$

$$\Rightarrow ye^{x^2} = \int 2\, dx + C$$

$$\boxed{ye^{x^2} = x + C}$$

10) $(x + 2y^3)\frac{dy}{dx} = y.$

$$\frac{dy}{dx} = \frac{y}{x + 2y^3}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$$I.F = e^{\int -\frac{1}{y} dy}$$

$$= e^{-\log y}$$

$$= \frac{1}{y}$$

General Solution:

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + C$$

$$\cancel{\frac{x}{y}} \quad \boxed{\frac{x}{y} = y^2 + C}$$

11) $\sqrt{1-y^2}\, dx = (\sin^{-1}y - x)\, dy$

$$\frac{dx}{dy} = \frac{\sin^{-1}y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}$$

$$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$$

$$I.F = e^{\int \frac{1}{\sqrt{1-y^2}}} = e^{\sin^{-1}y}$$

General Solution:

$$x e^{\sin^{-1}y} = \int \frac{e^{\sin^{-1}y} \cdot \sin^{-1}y}{\sqrt{1-y^2}} + C$$

$$xe^{\sin^{-1}y} = \sin^{-1}y \cdot e^{\sin^{-1}y} - e^{\sin^{-1}y} + c$$

$$\boxed{x = \sin^{-1}y - 1 + ce^{-\sin^{-1}y}}$$

12) $ye^y dx = (y^3 + 2xe^y) dy$

$$\frac{dx}{dy} = \frac{y^2}{e^y} + \frac{2x}{y}$$

$$\frac{dx}{dy} - \frac{2x}{y} = \frac{y^2}{e^y}$$

$$I.F = e^{-\int \frac{2}{y} dy} = e^{-2\log y} = \frac{1}{y^2}$$

General Solution:

$$\frac{x}{y^2} = \int \frac{1}{e^y} dy + c$$

$$\boxed{\frac{x}{y^2} = -e^{-y} + c}$$

13) $(1+y^2) dx + (x - e^{-\tan^{-1}y}) dy = 0$

$$(1+y^2) dx = (e^{-\tan^{-1}y} - x) dy$$

$$\frac{dx}{dy} = \frac{e^{-\tan^{-1}y}}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y} = e^{\tan^{-1}y}$$

$$xe^{\tan^{-1}y} = \int \frac{1}{1+y^2} dy + c$$

$$\boxed{x e^{\tan^{-1} y} = \tan^{-1} y + C}$$

14) $e^{-y} \sec^2 y \, dy = dx + x \, dy$

$$e^{-y} \sec^2 y = \frac{dx}{dy} + x$$

$$\frac{dx}{dy} + x = e^{-y} \sec^2 y$$

$$I.F = e^{\int 1 dy} = e^{y}$$

General Solution:

$$x e^{y} = \int \sec^2 y \, dy + c$$

$$\boxed{x e^{y} = \tan y + C}$$

06/06/2022

NON-LINEAR DIFFERENTIAL EQUATION OF 1^{st} ORDER: BERNOULLI'S EQUATION:

Standard Equation of non-linear DE:

$$\boxed{\frac{dy}{dx} + P(x) y = Q(x) \cdot y^n}$$

Working Rule to solve non-linear DE make into Linear DE by using substitution method.

Note:

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

Divide by y^n.

1) $$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

consider the substituting term as always coefficient of $P(x)$

$\Rightarrow$ put $y^{1-n} = t$

$$(1-n)\frac{dy}{dx} \cdot y^{-n} = \frac{dt}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{dt}{dx} \cdot \frac{1}{1-n}$$

2) $$\frac{1}{1-n}\frac{dt}{dx} + P(x) \cdot t = Q(x)$$

$$\boxed{\frac{dt}{dx} + [(1-n)P(x)] \cdot t = (1-n)\, Q(x)}$$

$$e^{1-n\int P(x)\,dx} = e^{\int (1-n) P(x)\,dx}$$

$$\boxed{I \cdot F = e^{\int (1-n) P(x)\,dx}}$$

General Solution

$$\boxed{t \cdot e^{(1-n)\int P(x)\,dx} = \int Q(x) \cdot e^{(1-n)\int P(x)\,dx}\, dx + C}$$

$$\boxed{y^{1-n} \cdot e^{(1-n)\int P(x)\,dx} = \int Q(x) \cdot e^{(1-n)\int P(x)\,dx}\, dx + C}$$

Solve:

$$* \frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

$$\sec^2 y\frac{dy}{dx} + x(2\tan y) = x^3$$

$$2\tan y = t$$

$$2\sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{2}\frac{dt}{dx} + x\cdot t = x^3$$

$$\frac{dt}{dx} + 2x\cdot t = 2x^3$$

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

G.S:

$$t\cdot e^{x^2} = \int 2x^3\cdot e^{x^2}dx + C$$

$$t\cdot e^{x^2} = \int x^2(2xe^{x^2})\,dx + C$$

$$t\cdot e^{x^2} = x^2\cdot e^{x^2} - \int 2x\cdot e^{x^2} + C$$

$$\boxed{t\cdot e^{x^2} = x^2\cdot e^{x^2} - e^{x^2} + C}$$

$$\boxed{2\tan y = x^2 - 1 + Ce^{-x^2}}$$

* $x\frac{dy}{dx} + y = x^3 y^6$

$\rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2 y^6$ (Divide by x)

$\rightarrow y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$ (Divide by y^6)

$$y^{-5} = t$$

$$-5y^{-6} \cdot \frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}$$

$$\left(\frac{-1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2\right) \times -5$$

$$\boxed{\frac{dt}{dx} - \frac{5}{x}t = -5x^2}$$

$$I \cdot F = e^{\int -5/x\, dx} = e^{-5\log x} = x^{-5} = \frac{1}{x^5}$$

$$\boxed{I \cdot F = \frac{1}{x^5}}$$

G.S:

$$t \cdot \frac{1}{x^5} = \int \frac{1}{x^5}(-5x^2)\, dx + C$$

$$\frac{t}{x^5} = -5\int \frac{1}{x^3}\, dx + C$$

$$\frac{y^{-5}}{x^5} = -5\frac{x^{-2}}{-2} + C$$

$$\boxed{\frac{1}{x^5 y^5} = \frac{5}{2x^2} + C}$$

* $x^3\sec^2 y \frac{dy}{dx} + 3x^2\tan y = \cos x$

$\tan y = t$

$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$x^3 \frac{dt}{dx} + 3x^2 t = \cos x$

$$\boxed{\frac{dt}{dx} + \frac{3t}{x} = \frac{\cos x}{x^3}}$$

$I.F = e^{\int \frac{3}{x} dx} = e^{3\int \frac{1}{x} dx} = e^{3\log x}$

$$\boxed{I.F = x^3}$$

G.S:

$t \cdot x^3 = \int \cos x \, dx + C$

$x^3 t = \sin x + C$

$$\boxed{x^3 \tan y = \sin x + C}$$

W:

$x\frac{dy}{dx} + y = x^2y^2\log x$

$xy(1+xy^2)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{xy(1+xy^2)}$

$y\,dy = \frac{dx}{x} \cdot \frac{1}{1+xy^2}$

$$y\,dy + y^3 x\,dy = \frac{dx}{x}$$

$$\frac{dx}{x} - dy\left(y + y^3 x\right) = 0$$

$$\frac{dx}{dy} - x\left(x y^3 + y\right) = 0$$

$$\frac{dx}{dy} - xy = x^2 y^3$$

$$\cancel{y^{-3}\frac{dx}{dy} - xy^{-2} = x^2}$$

$$x^{-2}\frac{dx}{dy} - \frac{y}{x} = y^3$$

$$\boxed{\begin{aligned} &\frac{-1}{x} = t \\ &\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy} \end{aligned}}$$

$$\frac{dt}{dy} + yt = y^3$$

$$IF = e^{\int y\,dy} = e^{y^2/2}$$

$$x e^{y^2/2} = \int e^{y^2/2} \cdot y^3\,dy + C$$

$$x e^{y^2/2} = \int e^{y^2/2} \cdot y^2\,dy + C$$

$$\cancel{x e^{y^2/2} = 2\left[y^2 e^{y^2/2} - \int y e^{y^2/2}\,dy\right] + C}$$

$$x e^{y^2/2} = \int \left(y e^{y^2/2}\right) \cdot y^2\,dy + C$$

$$= y^2 e^{y^2/2} - \int 2y \cdot e^{y^2/2}\,dy + C$$

$$x e^{y^2/2} = y^2 e^{y^2/2} - 2 e^{y^2/2} + C$$

$$\boxed{x = y^2 - 2 + c e^{-y^2/2}}$$

* $x \frac{dy}{dx} + y = x^2 y^2 \log x$

$$\frac{dy}{dx} + \frac{y}{x} = x y^2 \log x$$

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{xy} = x \log x$$

$$y^{-1} = t$$

$$-y^{-2} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{-dt}{dx} + \frac{1}{x} t = x \log x$$

$$\frac{dt}{dx} - \frac{1}{x} t = -x \log x \qquad I.F = e^{\int -\frac{1}{x} dx} = \frac{1}{x}$$

$$\frac{t}{x} = \int \log x \, dx + c$$

$$\frac{1}{xy} = x \log x - \int \frac{1}{x} \cdot x \, dx + c$$

$$\boxed{\frac{1}{xy} = x \log x - x + c}$$

* CLAIRAUT'S EQUATION:

Standard equation of Clairaut's is given by:

$$\boxed{y = \frac{dy}{dx}x + f(p)}$$ $$\boxed{p = y'}$$

$$\boxed{y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)}$$

Note: There are 2 solutions exists in Clairaut's

- → General solution
- → Singular solution $(f(x,y))$

$$\boxed{y = px + f(p)} \quad \text{— (1)}$$

Diff wrt 'x' on both sides.

$$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0$$

case(i) $\frac{dp}{dx} = 0$

$$\boxed{p = c}$$

$$\boxed{y = cx + f(c)}$$

is the general solution

case(ii): $x + f'(p) = 0$

$y = cx + f(c) \Rightarrow x + f'(c) = 0$

Diff. wrt 'c' on both sides [x, y → const] and eliminate 'c' we get singular solution

case(ii) $x + f'(p) = 0$

$x = -f'(p)$

Substitute 'x' in $y = px + f(p)$

$$\boxed{y = -pf'(p) + f(p)}$$

We get the y value

then substitute back and we will get eliminate 'p' from the eqn $\Rightarrow \boxed{y = px + f(p)}$

08/06/2022

→ Ordinary DE have only general soln. but Clairaut's equation have both general and singular solution.

Find the general and singular solution of $y = xy' - (y')^3$

$p = y'$

$y = xp - p^3$ — (1)

diff. wrto x

$$\frac{dy}{dx} = x\frac{dp}{dx} + p - 3p^2\frac{dp}{dx}$$

$$\cancel{p} = (x - 3p^2)\frac{dp}{dx} + \cancel{p}$$

i) $(x - 3p^2) = 0$

$x = 3p^2$

Sub in (1)

$y = 3p^3 - p^3$

$y = 2p^3$

$$y = 2\left(\sqrt{\frac{x}{3}}\right)^3$$

$$y = 2\left(\frac{x}{3}\right)^{3/2}$$

$$y^2 = 4\left(\frac{x}{3}\right)^3$$

$\boxed{27y^2 = 4x^3}$ is the singular sol^n.

(ii) $\int \frac{dp}{dx} = \int 0$

$p = c$

$\boxed{y = xc - c^3}$ is the general sol^n

* $y = xc - c^3$

diff. wrto c

$0 = x - 3c^2$

$x = 3c^2$

$$c = \sqrt{\frac{x}{3}}$$

$$y = \frac{x^{3/2}}{\sqrt{3}} - \left(\frac{x}{3}\right)^{3/2}$$

$$= \frac{x^{3/2}}{\sqrt{3}} - \frac{x\sqrt{x}}{3\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left(\sqrt{\frac{x}{1}}\right)^3$$

$$\boxed{y = \frac{2}{\sqrt{3}}x^{3/2}}$$

$* \; y = (x-a)p - p^2 \quad \text{--- (1)}$

$y = xp - ap - p^2$

$y = xp - p(a+p)$ $\quad y' = p$

diff. wrto 'x'

$$\frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} - a\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$-\frac{dp}{dx}(-x+a+2p) = 0$$

(i) $\frac{dp}{dx} = 0$

$\boxed{p = c}$

$\boxed{y = xc - ac - c^2}$

(ii) $-x+a+2p = 0$

$x = +a+2p \quad p = \frac{(a+x)}{2}$

Sub in (1)

$y = (+a+2p)p - p^2$

$\cancel{y = -ap - 2p^2 - p^2}$

$\cancel{y = -ap - 3p^2}$

$\cancel{y = +a\frac{(a-x)}{2} - 3\left(\frac{a-x}{2}\right)^2}$

$y = ap + 2p^2 - p^2 - ap$

$y = \cancel{ap} + p^2 - \cancel{ap}$

$y = \cancel{\frac{a^2+ax}{2}} + \frac{a^2+2ax+x^2}{4}$ $\cancel{\frac{a^2-ax}{2} - \frac{3}{4}(a^2-2ax}$

$\cancel{4y = 2a^2 - 2ax - 3a^2 + 6ax -}$

$\cancel{= -a^2 + 4ax - 3x^2}$

$\cancel{4y = 2a^2 + 2ax}$

$\cancel{+a^2 + 2ax + x^2}$

$\boxed{4y = (x-a)^2}$

$p = \sin(y - xp) \rightarrow (1)$

$\sin^{-1} p = y - xp$ $\quad$ $y' = p$

$y = xp + \sin^{-1} p \quad - (1)$

diff. w.r.to 'x'

$$\cancel{\frac{dy}{dx}} = x\frac{dp}{dx} + \cancel{p} + \frac{1}{\sqrt{1-p^2}}\cdot\frac{dp}{dx}$$

$$\left(x + \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$$

(i) $\frac{dp}{dx} = 0$

$p = C$

$\boxed{y = xc + \sin^{-1} c}$

is the general sol^n.

(ii) $x = \frac{-1}{\sqrt{1-p^2}}$

Sub in (1)

$$y = \frac{-p}{\sqrt{1-p^2}} + \sin^{-1} p$$

$$y = \frac{-p}{\sqrt{1-p^2}} + \sin^{-1} p$$

$\cancel{y\sqrt{1-p^2} = -1 + \sqrt{1-p^2}\sin^{-1} p}$

$\cancel{px = \frac{-1}{\sqrt{1-p^2}}}$

$\cancel{x^2p^2 = \frac{1}{1-p^2}}$

$\cancel{x^2p^2(1-p^2) = 1}$

$\cancel{p^2(1-p^2) = \frac{1}{x^2}}$

$1 - p^2 = \frac{1}{x^2}$

$p^2 = 1 - \frac{1}{x^2}$

$$\boxed{p = \sqrt{1 - \frac{1}{x^2}}}$$

$$y = \frac{\sqrt{x^2-1}/x}{-1/x} + \sin^{-1}\left(\sqrt{1-\frac{1}{x^2}}\right)$$

$\boxed{y + \sqrt{x^2-1} = \sin^{-1}\sqrt{1-\frac{1}{x^2}}}$

*$P = \log(px - y)$

*$(y - px)(p-1) = p$

*$yp - y - p^2x + px = p$

diff w.r.to 'x'

$y\frac{dp}{dx} + p - p - 2p\frac{dp}{dx}\cdot x - p^2 + p^2 + x\frac{dp}{dx} = \frac{dp}{dx}$

$(y - 2px)\frac{dp}{dx} + x\frac{dp}{dx} = \frac{dp}{dx}$

$(y - 2px + x - 1)\frac{dp}{dx} = 0$

$y(p-1) - x(p^2 - p) = p$

$y(p-1) = xp^2 - xp + p$

$y(p-1) = xp(p-1) + p$

$$\boxed{y = px + \frac{p}{p-1}}$$

diff w.r.to 'x'

$$\cancel{\frac{dy}{dx}} = \cancel{p} + x\frac{dp}{dx} + \frac{(p-1)\frac{dp}{dx} - p\frac{dp}{dx}}{(p-1)^2}$$

$$\left(x(p-1)^2 + \cancel{(p-1)} - \cancel{p}\right)\frac{dp}{dx} = 0$$

$$\left(x(p^2+1-2p) - 1\right)\frac{dp}{dx} = 0$$

(i) $\frac{dp}{dx} = 0$

$p = c$

$$\boxed{y = cx + \frac{c}{c-1}}$$

is the general solution

(ii) $\boxed{x = \frac{1}{(p-1)^2}}$

$$y = \frac{p}{(p-1)^2} + \frac{p}{p-1}$$

$$y = p\left[\frac{1 + p - 1}{(p-1)^2}\right]$$

$$y = \frac{p^2}{(p-1)^2}$$

$(p-1)^2 = x$

$p - 1 = \sqrt{x}$

$$\boxed{p = \sqrt{x} + 1}$$

$$\sqrt{y} = \frac{\sqrt{x}+1}{\sqrt{x}}$$

$$\boxed{xy = (\sqrt{x}+1)^2}$$

* $p = \log(px - y)$

$10^p = px - y$

$$\boxed{y = px - 10^p} \quad -(i)$$

diff. wrto 'x'

$$\frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} - 10^p\frac{dp}{dx}$$

$$(x - 10^p)\frac{dp}{dx} = 0$$

(i) $\frac{dp}{dx} = 0$

$p = c$

$\boxed{y = cx - 10^c}$

↳ General Solution

(ii) $x - e^p = 0$

$x = 10^p$ $\quad \log x = p$.

in (1)

$y = p(10^p - 1)$.

$\boxed{y = \log x\,(x-1)}$

is the ~~particular~~ singular s

ORTHOGONAL TRAJECTORIES:

Working procedure:

$f(x, y, c) = 0$ — ①
family of curve

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ only after eliminating the arbitrary constant

diff ① w.r.to 'x' and eliminate 'c'

$f\left(x, y, \frac{dy}{dx}\right) = 0$ — ②
DE of family of curves

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$f\left(x, y, -\frac{dx}{dy}\right) = 0$ — ③

Solve ③ using suitable method (variable separable).

$g(x, y, t) = 0$ is orthogonal trajectory of given family of curves.

Trajectory: A curve which cuts every member of a given family of curves according to some definite law is called trajectory of the family.

Orthogonal trajectory: A curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family

09/06/22

* Find the orthogonal trajectory of x^2+y^2

- $x^2+y^2=a^2$ — ① is given family of curve

$a \to$ const

Diff ① w.r.to 'x' on both sides

$2x+2yy_1=0$

$x+y\frac{dy}{dx}=0$ — ② is DE of given family of curve

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ②

$x-y\frac{dx}{dy}=0$ — ③ is ⊥ar DE of family of curves.

$\frac{dx}{x}=\frac{dy}{y}$ [variable separable].

Integrate

$\log x=\log y+\log c$

$\boxed{\frac{x}{y}=c}$ is O.T. of given family of curve.

* Find the O.T. of parabola $y^2=4ax$

$y^2=4ax$ — ① is given family of parabola

$2yy_1=4a$

$2y\frac{dy}{dx}=4a$ — ③

from ① $4a = \frac{y^2}{x}$

$2y\frac{dy}{dx} = \frac{y^2}{x}$

$\boxed{\frac{dy}{dx} = \frac{y}{2x}}$ — ⑤ is DE of given family of curve.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ②

$-\frac{dx}{dy} = \frac{y}{2x}$

$\boxed{-2x\,dx = y\,dy}$ → Variable Seperable.

Integrate:

$-x^2 = \frac{y^2}{2} + c.$

$\boxed{2x^2 + y^2 = C}$ is O.T. of given family of curve.

* Find O.T. of $xy = c$.

$xy = c$ — ① is given family of curve.

$c \to$ const

diff ① w.r.to 'x'

$x\frac{dy}{dx} + y = 0$ — ② is DE of given family of curve.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$-x\frac{dx}{dy} + y = 0$ — ③ is $\perp$ar DE of family of curves

$x\,dx = y\,dy$

Integrate.

$\frac{x^2}{2} = \frac{y^2}{2} + K$

$\boxed{x^2 - y^2 = K}$ is the O.T. of given family of curve.

* Semi cubical parabola $ay^2 = x^3$

$ay^2 = x^3$ — ① is the given family of curve.

Diff w.r.to 'x'

$a = \frac{x^3}{y^2}$

$a\,2y\frac{dy}{dx} = 3x^2$

$\frac{2x^3}{y}\frac{dy}{dx} = 3x^2$

$\boxed{\frac{dy}{dx} = \frac{3y}{2x}}$ — ② is the DE of family of curve.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$\boxed{-\frac{dx}{dy} = \frac{3y}{2x}}$ — ③ is the ⊥ar DE of family of curve.

$-2x\,dx = 3y\,dy$

$-x^2 = \frac{3y^2}{2} + c$

$\boxed{2x^2 + 3y^2 = K}$ is the O.T. of given family of curve.

Find O.T of $ax^2=y$

* $ax^2=y$ — ① is the given family of curve.

Diff w.r.to 'x'

$$2ax=\frac{dy}{dx} \qquad a=\frac{y}{x^2}$$

$$\boxed{\frac{2y}{x}=\frac{dy}{dx}}$$ — ② is the DE of family of curve.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\boxed{-\frac{dx}{dy}=\frac{2y}{x}}$$ — ③ is the $\perp$ar DE of family of curve.

$$-x\,dx=2y\,dy$$

$$-\frac{x^2}{2}=y^2+c$$

$$\boxed{x^2+2y^2=c}$$ → ④ is the O.T of given family of curve.

* $\left(\frac{dy}{dx}\right)^2=\left(\sqrt{\frac{a}{x}}\right)$

$\frac{dy}{dx}=\sqrt{\frac{a}{x}}$ is the DE of family of curve.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$-\frac{dx}{dy}=\sqrt{\frac{a}{x}}$$ → ② is the $\perp$ar DE of family of curve.

$$-dx\sqrt{x}=\sqrt{a}\,dy$$

Integrate

$$-\frac{x^{3/2}}{3/2}=\sqrt{a}\,y+C$$

$$-\frac{2}{3}x^{3/2} = \sqrt{a}\,y + c$$

$$\boxed{\frac{2}{3}x^{3/2} + \sqrt{a}\,y = c}$$

* $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{—(1)}$$ is the given family of curve.

Diff. w.r.t. 'x' on both sides.

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x(b^2+\lambda)}{a^2 y} \quad \text{—(2)}$$

Eliminate λ from (1) & (2)

(1)

$$\frac{y^2}{b^2+\lambda} = 1 - \frac{x^2}{a^2}$$

$$b^2+\lambda = \frac{a^2y^2}{a^2-x^2}$$

$$\frac{dy}{dx} = -\frac{x}{\cancel{a^2}\cancel{y}} \cdot \frac{\cancel{a^2}y^{\cancel{2}}}{a^2-x^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{xy}{a^2-x^2}} \quad \text{—(3)}$$ is the DE of given family of curve

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\boxed{\cancel{-}\frac{dx}{dy} = \frac{\cancel{-}xy}{a^2-x^2}}$$ — ③ is the ⊥ar DE of family of curve.

$$\frac{dx}{x}(a^2-x^2) = y\,dy.$$

$$\frac{a^2}{x}dx - x\,dx = y\,dy.$$

$$a^2\log x - \frac{x^2}{2} = \frac{y^2}{2} + c$$

$$2a^2\log x - x^2 - y^2 = C$$

$$\boxed{x^2+y^2-2a^2\log x = c}$$

* Show that the system con-focal and co-axial parabolas is $y^2 = 4a(x+a)$ is self orthogonal.

$$y^2 = 4a(x+a) \rightarrow ①$$ is the given family of curve.

Diff w.r.to 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{y}{2}\frac{dy}{dx} = a.$$

$$4ax + 4a^2 = y^2$$

$$4a(x+a) = y^2$$

$$\frac{4y}{2}\frac{dy}{dx}\left(x+\frac{y}{2}\frac{dy}{dx}\right) = y^2$$

$$2yy'\left(x + \frac{y}{2}y'\right) = y^2$$

$$2xyy' + y^2(y')^2 = y^2$$

$$2xyy' = y^2\left(1-(y')^2\right) \quad \text{— (2)}$$

$$\frac{dy}{dx} = \frac{y\left(1-(y')^2\right)}{2xy'}$$

→ DE. of given family of curves

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ in (2)

$$\cancel{-\frac{dx}{dy} = \frac{y\left(1-\left(\frac{1}{y'}\right)^2\right)}{-2x\frac{1}{y'}}} \qquad \cancel{\frac{-1}{y'} = \frac{y\left(1-\frac{1}{(y')^2}\right)}{2x\left(\frac{-1}{y'}\right)}}$$

$$\cancel{\frac{-1}{y'} = \frac{y\left((y')^2-1\right)}{-2xy'}} \qquad \cancel{\frac{-1}{y'} = \frac{y\left((y')^2-1\right)}{+2xy'}}$$

$$\cancel{2x = y\left((y')^2-1\right)}$$

$$2xy\left(\frac{-1}{y'}\right) = y^2\left(1-\left(\frac{1}{y'}\right)^2\right)$$

$$\frac{-2xy}{y'} = y^2 - \frac{y^2}{(y')^2}$$

$$-2xyy' = (y')^2y^2 - y^2$$

$$\boxed{y^2 = (y')^2y^2 + 2xyy'} \quad \text{— (3)}$$

∴ (2) is equal (3), This is the O.T. of family of curves

* P.T. the family of con-focal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ are self-orthogonal.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad -(1)$$ is the given family of con-focal conics.

Diff. w.r.t. 'x'

$$\frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0$$

$$\frac{x}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}=0$$

$$x(b^2+\lambda)=-(a^2+\lambda)yy'$$

$$b^2x+\lambda x=-a^2yy'-\lambda yy'$$

$$\lambda(x+yy')=-(a^2yy'+b^2x)$$

$$\lambda=\frac{-(a^2yy'+b^2x)}{x+yy'}$$

$$a^2+\lambda=\frac{a^2x+a^2yy'-a^2yy'-b^2x}{x+yy'}$$

$$=\frac{(a^2-b^2)x}{x+yy'}$$

$$b^2+\lambda=\frac{b^2x+b^2yy'-a^2yy'-b^2x}{x+yy'}=\frac{-(a^2-b^2)yy'}{x+yy'}$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \leftarrow (1)$$

$$\frac{x(x+yy')}{a^2-b^2}-\frac{y(x+yy')}{(a^2-b^2)y'}=1$$

$x^2y' + xy(y')^2 - xy - y^2y' = (a^2-b^2)y'$

$(x^2-y^2)y' + xy[(y')^2-1] = (a^2-b^2)y'$

Replace y' with $-1/y'$ DE of given → ②

$(x^2-y^2)\left(\frac{-1}{y'}\right) + \frac{xy}{(y')^2} - xy = (a^2-b^2)\left(\frac{-1}{y'}\right)$

$\frac{y^2-x^2}{y'} + \frac{xy}{(y')^2} - xy = \frac{b^2-a^2}{y'}$

$y^2y' - x^2y' + xy - xy(y')^2 = (b^2-a^2)y'$

$x^2y' - y^2y' - xy + xy(y')^2 = (a^2-b^2)y$ → ③

$\because$ ② = ③

$\Rightarrow$ ② is the O.T. of family of curves.

$\Rightarrow$ Given family of curves is self-orthogonal.

* Find the O.T. of asteroid: $\boxed{x^{2/3} + y^{2/3} = a^{2/3}}$

$x^{2/3} + y^{2/3} = a^{2/3}$ is the given family of curves

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = \frac{-x^{-1/3}}{y^{-1/3}}$

$\boxed{y' = -\left(\frac{y}{x}\right)^{1/3}}$ → ② is the DE of given family of curves

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\boxed{\frac{-1}{y'} = -\left(\frac{y}{x}\right)^{1/3}}$$ — (3) is the ⊥ or DE of given family of curves.

$$-\frac{dx}{dy} = + \frac{y^{1/3}}{x^{1/3}}$$

$$x^{1/3}dx = y^{1/3}dy$$

$$\frac{x^{4/3}}{4/3} = \frac{y^{4/3}}{4/3} + C$$

$$\boxed{x^{4/3} = y^{4/3} + K}$$ → is the O.T. of family of curves

* Find the equation of family of all O.T. of family of circles which passes through (2,0) & (−2,0)

$x^2+y^2+2gx+2fy+c=0$ — (1) is the given family of curves.

∵ (2,0) & (−2,0) passes through (1)

$$4+\cancel{4g}+c=0$$
$$4-\cancel{4g}+c=0$$
$$8+2c=0$$

$c=-4$

$g=0$.

$x^2+y^2+2fy-4=0$ — (1) is the given family of curve.

Diff. w.r.to 'x' on both sides.

$$\cancel{2}x+\cancel{2}yy'+\cancel{2}fy'=0$$
$$y'(y+f)=-x$$

$$\boxed{f = \frac{-x-yy'}{y'}}$$

$$x^2+y^2-4+2y\left(\frac{-x-yy'}{y'}\right)=0$$

$$x^2+y^2-4-\frac{2xy}{y'}-2y^2=0$$

$$x^2-y^2-4=\frac{2xy}{y'}$$

$$\boxed{\frac{dy}{dx}=\frac{2xy}{x^2-y^2-4}}$$ —(2) is the DE of given family of curves.

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$\boxed{-\frac{dx}{dy}=\frac{2xy}{x^2-y^2-4}}$$ —(2) is the DE of family of curves.

$$(x^2-y^2-4)\,dx=-2xy\,dy$$

$$x^2dx-y^2dx-4dx+2xy\,dy$$

$$x^2dy-y^2dy-4dy=2xy\,dx$$

$$x^2dy-2xy\,dx=y^2dy\mp 4dy$$

* Self Orthogonal Trajectory:

A family of curve is said to be self orthogonal if 2 curves of the family intersect orthogonally.

∴ The D.E. ... remains unchanged if we replace $\frac{dy}{dx}$ with $-dx/dy$

$$(x^2-y^2-4)dx + 2xydy = 0$$

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Non-exact}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}\left(-2y-2y\right) = \frac{-2}{x}$$

$$I.F = e^{\int \frac{-2}{x}dx} = e^{\log x^{-2}} = \frac{1}{x^2}$$

$$\left(1-\frac{y^2}{x^2}-\frac{4}{x^2}\right)dx + \frac{2y}{x}dy = 0$$

$\Rightarrow$ Exact

$$\int M dx + \int N dy = C$$

$$\int\left(1-\frac{y^2}{x^2}-\frac{4}{x^2}\right)dx + \int 0\, dy = C$$

$$\boxed{x + \frac{y^2}{x} + \frac{4}{x} = C} \rightarrow \text{O.T. of the given family of curve.}$$

13/06/2022.

ELECTRIC CIRCUITS:

The formation of DE for an electric circuit depends upon following law:

Notations (LR circuit)

Let 'i' be the current (A) and 'v' be the voltage (V)

'L' be the inductance (H)

'R' be the resistance (Ω)

'E' be the Electromotive force (emf) (V)

Note:

- Voltage drop across resistance (R) $= iR$

Voltage drop across inductance (L) $= L\frac{di}{dt}$

Kirchoff's Law:

Voltage Law: The algebraic sum of the voltage drops in each part of any electrical circuit is equal to resultant EMF of the circuit

* LR CIRCUITS:

Let 'i' be the current flowing in the circuit containing resistance 'R' and inductance 'L' connecting in series with battery / voltage source (emf) at any time 't'.

$$\boxed{L\frac{di}{dt} + iR = E}$$

$$\boxed{\frac{di}{dt} + \frac{iR}{L} = \frac{E}{L}}$$ linear DE in terms of i

comparing $$\boxed{\frac{dy}{dx} + P(x)\cdot y = q(x)}$$

$$I\cdot F = e^{\int P(x)\,dx} = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

$$\boxed{I\cdot F = e^{Rt/L}}$$

General Solution:

$$i(I\cdot F) = \int Q(I\cdot F)\,dt + K$$

$$ie^{Rt/L} = \int \frac{E}{L}\cdot e^{Rt/L} dt + K$$

$$ie^{Rt/L} = \frac{E}{L}\cdot\frac{L}{R} e^{Rt/L} + K$$

$$\boxed{i = \frac{E}{R} + Ke^{-Rt/L}}$$ is the required current of the circuit.

at $t=0$; minimum $\Rightarrow$ $i = 0$ $\boxed{K = -\frac{E}{R}}$ $\boxed{i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)}$

at $t=\infty$; maximum $\Rightarrow$ $\boxed{i = \frac{E}{R}}$

S.T. the DE for the current "i" is an electrical circuit containing a resistance and inductance (L) and acted on by an cell of emf. $E\sin\omega t$;

By voltage Law

LR circuit

$$L\frac{di}{dt} + iR = E\sin\omega t$$

$$\frac{di}{dt} + \frac{iR}{L} = \frac{E}{L}\sin\omega t$$

$$I.F = e^{\int\frac{R}{L}dt} = e^{Rt/L}$$

S: $i \cdot e^{Rt/L} = \int \frac{E}{L} \cdot e^{Rt/L}\sin\omega t\, dt + K$

$$\cancel{i \cdot e^{Rt/L} = \frac{E}{R} e^{Rt/L} + K}$$

$$i \cdot e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\sin\omega t\, dt + K$$

$$i \cdot e^{Rt/L} = \frac{E}{L}\left[\sin\omega t \frac{L}{R} \cdot e^{Rt/L} - \int \cos\omega t \cdot \frac{R}{L} e^{Rt/L} dt\right] + K$$

$$i \cdot e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\sin\omega t\, dt + K.$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$a = \frac{R}{L}$ $b = \omega$

$\cancel{i \cdot e^{Rt/L} = e^{Rt/L}}$

$$i \cdot e^{Rt/L} = \frac{Le^{Rt/L}}{R^2 + \omega L^2}\left[\frac{R}{L}\sin\omega t - \omega\cos\omega t\right] + K$$

$$\boxed{i = \frac{EL^2}{(L)R^2 + \omega L^2}\left[\frac{R}{L}\sin\omega t - \omega\cos\omega t\right] + Ke^{-Rt/L}}$$

$$\boxed{i = \frac{E}{R^2 + \omega L^2}\left[R\sin\omega t - \omega L\cos\omega t\right] + Ke^{-Rt/L}}$$

15/06/2022

* emf = 40V R = 10Ω, L = 0.2H.

By Kirchoff's law:

$$L\frac{di}{dt} + iR = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

$$\frac{di}{dt} + \frac{10}{0.2}i = \frac{40}{0.2}$$

$$\frac{di}{dt} + 50i = 200$$

$P = 50$ | $Q = 200$

$$I.F = e^{\int 50dt} = e^{50t}$$

Required Current:

$$i(I \cdot F) = \int Q(I \cdot F)\, dt + K;$$

$$ie^{50t} = \int 200\, e^{50t}\, dt + K$$

$$ie^{50t} = \frac{200}{50} \cdot e^{50t} + K$$

$$\boxed{i = 4\cancel{0} + Ke^{-50t}}$$

$i = 0 \,;\, t = 0$

$$K = -4\cancel{0}$$

$$\boxed{i = 4\left(1 - e^{-50t}\right)}$$

* EMF = $750 \cos \omega t$ $\quad R = 10\Omega \quad L = 0.2H$

By Kirchoff's Law

$$L\frac{di}{dt} + iR = E$$

$$\frac{di}{dt} + i\frac{10}{0.2} = \frac{750 \cos \omega t}{0.2}$$

$$\frac{di}{dt} + 50i = 3250 \cos \omega t$$

$P = 50 \quad Q = 3250 \cos \omega t$

$$I \cdot F = \int e^{P dt} = e^{50t}$$

Current: $i\left(e^{50t}\right) = \int 3250 \cos \omega t\, e^{50t}\, dt + K$

$$ie^{50t} = 3250 \int e^{50t} \cos \omega t\, dt + k$$

$$= 3250 \frac{e^{50t}}{(50)^2 + \omega^2}\left(50 \cos \omega t + \omega \sin \omega\right.$$

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$$i = \frac{3750}{2500+\omega^2}(50\cos\omega t + \omega\sin\omega t) + Ke^{-50t}$$

$\omega = 200$

$$\begin{array}{r} 2500 \\ 40000 \\ \hline 42500 \end{array}$$

$$i = \frac{3750}{42500}\left[50\cos 200t + 200\sin\omega t\right] + Ke^{-50t}$$

$i = 0 \quad t = 0$

$$\begin{array}{r} 25 \\ \times \end{array}$$

$$\ldots +0) + K = 0$$

$$K = -\frac{\cancel{375}^{15} \times 5\cancel{0}}{\cancel{4250}_{17\cancel{0}}} = -\frac{750}{17}$$

$$i = \frac{3750}{42500}\left[50\cos 200t + 200\sin 200t\right] - \frac{75}{17}e^{-50t}$$

* emf $= E$ V $R = R\,\Omega$ in series and a constant inductance L H. If initial current is zero; the current builds half its theoretical maximum $\frac{L\log 2}{R}$ sec;

By Kirchoff's law:

$$L\frac{di}{dt} + iR = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

$$I\cdot F = e^{\int \frac{R}{L}dt} = e^{Rt/L}$$

$$ie^{Rt/L} = \int \frac{E}{L}\cdot e^{Rt/L}\,dt + K$$

$$\Rightarrow i e^{Rt/L} = \left[\frac{E}{L} \times \frac{L}{R} e^{Rt/L} \right] + K$$

$$i = \frac{E}{R} + K e^{-Rt/L}$$

at $t=0$

$i=0$

$K = -\frac{E}{R}$

at $t=\infty$

$$i = \frac{E}{R}$$

* at $t = \frac{L \log 2}{R}$

$$i = \frac{E}{R} + \left(\frac{-E}{R}\right) e^{-\frac{\cancel{R}}{\cancel{L}} \cdot \frac{\cancel{L} \log 2}{\cancel{R}}}$$

$$i = \frac{E}{R} - \frac{E}{2R} = \frac{E}{2R} = \text{half of maximum}$$

R-C Circuits:

Let 'i' be the current (A); 'R' be the resistance (Ω); q be the charge (C); E be the emf (V) and C be the capacitance represented (F)

By Kirchoff's law:

$$\boxed{\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}}$$

$$\boxed{Ri + \frac{q}{C} = E}$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{RC} \qquad Q = \frac{E}{R}$$

$$I.F = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

Required charge:

$$q e^{t/RC} = \int \frac{E}{R} \cdot e^{t/RC} dt + K$$

$$q e^{t/RC} = \frac{E}{\cancel{R}} \times \cancel{R}C\, e^{t/RC} + K$$

$$q e^{t/RC} = EC\, e^{t/RC} + K.$$

$$\boxed{q = EC + Ke^{-t/RC}}$$

To get current

$$i = Ke^{-t/RC} \cdot \frac{1}{RC}$$

$$\boxed{i = \frac{Ke^{-t/RC}}{RC}} \qquad \boxed{i = -\frac{E}{R} e^{-t/RC}}$$

at $t=0$; $q=0$

$$0 = EC + K$$

$$\boxed{K = -EC}$$

: $C = 0.01F$ in a series with a resistance $R = 20\Omega$ is charged from a battery $E = 10V$. Assuming that initially the capacitor is uncharged; determine the charge $q(t)$; voltage and current i.

$C = 0.01F$; $R = 20\Omega$ $E = 10V$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

$$\frac{dq}{dt} + \frac{q}{0.2} = \frac{1\cancel{0}}{2\cancel{0}} = \frac{1}{2}$$

$$\frac{dq}{dt} + 5q = \frac{1}{2}$$

$P = 5$ $Q = \frac{1}{2}$

$I.F. = e^{\int 5dt} = e^{5t}$

$$qe^{5t} = \int \frac{1}{2} e^{5t} dt + K$$

$$qe^{5t} = \frac{1}{10} e^{5t} + K$$

$$\boxed{q = \frac{1}{10} + Ke^{-5t}}$$

at $q = 0$ $t = 0$

$$\frac{1}{10} + K = 0$$

$$\boxed{K = -\frac{1}{10}}$$

$$q = \frac{1}{10}\left(1 - e^{-50t}\right)$$

$$i = \frac{1}{10}\left(0 + e^{-5t}(-5)\right)$$

$$i = \frac{5}{10} e^{-5t}$$

$$\boxed{i = 0.5e^{-5t}}$$

$$V = \frac{q}{C}$$

$$= \frac{0.1}{0.01}\left(1 - e^{-5t}\right)$$

$$\boxed{V = 10\left(1 - e^{-5t}\right)}$$

* Find current 'i' at any time t of the equation $E = Ri + \int \frac{i}{C}\, dt$ $\quad E = E_0 \sin wt$

$$E = iR + \int \frac{i}{C}\, dt$$

$$iR + \int \frac{i}{C} dt = E_0 \sin wt$$

diff wrt 't'

$$R\frac{di}{dt} + \frac{i}{C} = E_0 w \cos wt$$

$$\frac{di}{dt} + \frac{i}{RC} = \frac{E_0 w}{R} \cos wt$$

$$P = \frac{1}{RC} \qquad Q = \frac{E_0 w}{R} \cos wt$$

$$e^{\int \frac{1}{RC} dt} = \underline{\underline{e^{t/RC}}}$$

$$ie^{t/RC} = \int e^{t/RC} \cdot \frac{E_0 w}{R} \cos wt \, dt + K$$

$$ie^{t/RC} = \frac{E_0 w}{R} \int e^{t/RC} \cos wt \, dt + K$$

$$ie^{t/RC} = \frac{E_0\omega}{R}\,\frac{e^{t/RC}}{\frac{1}{R^2C^2}+\omega^2}\left[\frac{1}{RC}\cos\omega t + \omega\sin\omega t\right]$$

$$i = \frac{E_0\omega}{\cancel{R}}\cdot\frac{\cancel{R^2}C^{\cancel{2}}}{1+\omega^2R^2C^2}\left[\frac{\cos\omega t + \omega RC\sin\omega t}{\cancel{RC}}\right]$$

$$i = \frac{E_0\omega C}{1+\omega^2R^2C^2}\left[\cos\omega t + \omega RC\sin\omega t\right] + k$$

$$t = 0,\ i = 0$$

$$\frac{E_0\omega C}{1+\omega^2R^2C^2}\left[1+0\right] + K = 0$$

$$K = \frac{-E_0\omega C}{1+\omega^2R^2C^2}$$

$$i = \frac{E_0\omega C}{1+\omega^2R^2C^2}\left[\cos\omega t + \omega RC\sin t - 1\right]$$

16/06/2022 UNIT-2.

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER:

→ Homogeneous linear differential equations with constant coefficients.
- → Complementary function.
- → Problems.

→ Non-Homogeneous linear DE with constant coefficients
- → Complementary function
- → Particular Integral
- → Problems

→ Method of variation of parameters - problems

→ Applications of second order equations
- LCR circuits.

$$\boxed{D = \frac{d}{dx}}$$

$$* \quad \frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + K_n y = 0 \quad — (1)$$

$$\left[D^n + K_1 D^{n-1} + \cdots + K_n\right] y = 0 \quad — (2)$$

(2) is the operator | standard form.

Let $y = e^{mx}$

$Dy = me^{mx}$

$D^2 y = m^2 e^{mx}$

$D^n y = m^n e^{mx}$

Auxiliary equation:

$$e^{mx}\left[m^n + K_1 m^{n-1} + K_2 m^{n-2} + \cdots + K_n\right] = 0 \quad — (3)$$

~~f(D)~~ $f(m) = 0 \qquad e^{mx} \neq 0$

$$\left[(D-m_1)(D-m_2) \cdots (D-m_n)\right] y = 0$$

$$\left[(D-m_1)(D-m_2)\right]y = 0.$$

O.E:

$$f(D)\cdot y = 0$$

A.E:

$$f(m) = 0$$

(i) $(D-m_1)\,y = 0$

$$\frac{dy}{dx} - m_1 y = 0$$

$$\frac{dy}{dx} = m_1 y$$

$$\frac{dy}{y} = m_1 dx$$

$$\log y = m_1 x + \log c$$

$$y = c_1 e^{m_1 x}$$

$$y = c_1 e^{m_1 x}$$

(ii) $(D-m_2)\,y = 0$

$$y = c_2 e^{m_2 x}$$

$$y = c_3 e^{m_3 x} \cdots\cdots$$

$$\boxed{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots\cdots}$$

General solution of linear Homogeneous Differential Equation is a complementary function only.

Roots:
- → Real and distinct
- → Real and equal
- → Imaginary roots

* If $m_1 = m_2$

$$\left[(D-m_1)(D-m_1)\right]y = 0$$

* Assume

$$(D-m_1)\,y = z$$

$$(D-m_1)\,z = 0$$

$$\frac{dz}{dx} - m_1 z = 0$$
$$z = c_1 e^{m_1 x}$$

$$(D - m_1)y = z$$
$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$
$$I.F = e^{-m_1 x}$$
$$y e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x} dx + c_2$$
$$y e^{-m_1 x} = c_1 x + c_2$$
$$\boxed{y = (c_1 x + c_2) e^{m_1 x}}$$

If roots are 3, 3, 3

$$y = (c_1 x^2 + c_2 x + c_3) e^{3x}$$

If roots are −1, −1, 2, 4

$$y = (c_1 x + c_2) e^{-x} + c_3 e^{2x} + c_4 e^{4x}$$

***Imaginary Roots:** $a \pm ib$

$a+ib,\ a-ib$

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$
$$= c_1 e^{ax} \cdot e^{ibx} + c_2 e^{ax} \cdot e^{-ibx}$$
$$y = c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\cos bx - i \sin bx)$$
$$y = e^{ax}[\cos bx (c_1 + c_2) + \sin bx (ic_1 - ic_2)]$$
$$\boxed{y = e^{ax}[c_1 \cos bx + c_2 \sin bx]}$$

Nature of Roots	Complementary Fun...
1) m_1, m_1, m_2, m_2, m_5.	$(c_1x+c_2)e^{m_1x} + (c_3x+c_4)...$
2) m_1, m_1, m_1, m_2	$(c_1x^2+c_2x+c_3)e^{m_1x} + c_4e...$
3) $m_1, m_2, m_3, m_3 \cdots$	$c_1e^{m_1x} + c_2e^{m_2x} + (c_3x+c_4)...$
4) $(a \pm ib)^2$	~~$e^{ax}(c_1\cos bx + c_2\sin bx) + e^{ax}(c$~~ $e^{ax}\left[(c_1x+c_2)\cos bx + (c_3x+c_4)\sin b...\right.$

* 2^{nd} order DE: (or) higher order DE:

A differential equation is said to be linear differential equation if its dependent variable and its derivatives occurs in the first degree and are not multiplied together.

$\Rightarrow$ Operator(D):

$$\frac{d}{dx} \to D ; \frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} \to D^2y$$

$$\frac{d^ny}{dx^n} \to D^ny$$

where

$D \to$ differential operator.

18/06/2022

* Solve the following DE:

1) $y'' + 36y = 0$

$$\frac{d^2y}{dx^2} + 36y = 0 \quad -(1)$$

Operator form of given DE is

$$(D^2 + 36)y = 0 \quad -(2)$$

Auxiliary equation of (2) is

$$(m^2+36)e^{mx} = 0 \qquad e^{mx} \neq 0$$

$$m^2+36 = 0$$

$$m^2 = -36$$

$$m = \pm 6i$$

∴ Here the roots are imaginary roots.

$$(D-6i)(D+6i) = 0$$

Complementary Function:

Roots are imaginary.

$$C.F(y_c) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$= e^{0(x)}(c_1 \cos 6x + c_2 \sin 6x)$$

$$\boxed{C.F = c_1 \cos 6x + c_2 \sin 6x}$$

2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0 \quad -(1)$

Operator form of given DE is

$$(D^2 + 2D + 4)y = 0 \quad -(2)$$

Auxiliary equation of (2) is

$$(D^2+2D+4)\,y=0$$

$$m^2+2m+4=0. \qquad e^{mx}\neq 0$$

$$m=\frac{-2\pm\sqrt{4-16}}{2}$$

$$=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm 2\sqrt{3}i}{2}$$

$m=-1\pm\sqrt{3}i$ Roots are imaginary

$$C.F=\boxed{y_c=e^{-x}\left(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x\right)}$$

3) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

O.F: $D^2+2D+1=0$

A.F: $m^2+2m+1=0$

$$(m+1)^2=0$$
$$m=-1,-1$$

Roots are real and equal

$$\boxed{y_c=(c_1x+c_2)\,e^{-x}}$$

4) $(D^2+3D+2)y=0$

$O.F=(D^2+3D+2)\,y$

$A.F=\boxed{m^2+3m+2=0}$

$$m=-2,-1$$

Required General Solution:

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$$

$4y'' - 8y' + 3y = 0 \quad y(0) = 1,\ y'(0) = 3$

$$4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 3y = 0$$

O.F. of given DE

$(4D^2 - 8D + 3)y = 0$

A.F of given DE:

$4m^2 - 8m + 3 = 0$

$4m^2 - 6m - 2m + 3 = 0$

$2m(2m-3) - (2m-3) = 0$

$(2m-3)(2m-1) = 0$

$m_1 = \frac{3}{2} \quad m_2 = \frac{1}{2}$

$\boxed{C.F = c_1 e^{3/2 x} + c_2 e^{1/2 x}} \Rightarrow$ General Sol^n.

$\cancel{4y'' + 4y' + y = 0}$

$\frac{dy}{dx} = c_1 \cdot \frac{3}{2} e^{3/2 x} + \frac{c_2}{2} e^{1/2 x}$

$y(0) = 1$

$x = 0;\ y = 1$

$\boxed{c_1 + c_2 = 1}$

$3 = \frac{3}{2} c_1 + \frac{c_2}{2}$

$3c_1 + c_2 = 6$

$c_1 + c_2 = 1$

$2c_1 = 5$

$c_1 = 5/2$

$c_2 = -\frac{3}{2}$

$$\boxed{y(x) = \frac{5}{2} e^{3/2 x} - \frac{3}{2} e^{1/2 x}}$$

2) $4y'' + 4y' + y = 0$

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

$(4D^2 + 4D + 1)y = 0 \Rightarrow$ O.F of given DE

A.F $\Rightarrow 4m^2 + 4m + 1 = 0$

$(2m+1)^2 = 0$ Roots are real & equal.

$2m + 1 = 0$

$m = -1/2$

$m_1 = m_2 = -1/2$

$$\boxed{y_c = (c_1 x + c_2)e^{-x/2}}$$

3) $y'' + 4y' + 13y = 0$ $y(0) = 0; y'(0) = 1$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

O.F $= (D^2 + 4D + 13)y = 0$

A.F $= m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 52}}{2}$$ Roots are imaginary.

$$= \frac{-4 \pm \sqrt{36}}{2}$$

$$= \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

$$y_c = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$$

$x = 0 ; y = 0$

$$\boxed{c_1 = 0}$$

$$y'(x) = e^{-2x}[c_1(-\sin 3x)3 + c_2(3)(\cos 3x)] + e^{-2x}(-2)[c_1 \cos 3x + c_2 \sin 3x]$$

$y'(0) = 1$

$x = 0 ; y = 1$

$$1 = [0 + 3c_2] - 2(0 + 0)$$

$$\boxed{c_2 = +\frac{1}{3}}$$

$$y_c = e^{-2x}\left[+\frac{1}{3} \sin 3x\right]$$

4) $(D^2 + 9D)y = 0$

$O.F \Rightarrow (D^2 + 9D)y = 0$

$A.F \Rightarrow m^2 + 9m = 0$

$$m(m+9) = 0$$

$$m = 0, -9$$

$$y_c = c_1 e^{(0)x} + c_2 e^{-9x}$$

$$\boxed{y_c = c_1 + c_2 e^{-9x}} \Rightarrow CF.$$

5) $y'' + 2y' + 2y = 0 \quad y(0) = 1 \, ; \, y(\pi/2) = e^{-\pi/2}$

$O \cdot F = (D^2 + 2D + 2)y = 0$

$A \cdot F = m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i$$

$C \cdot F = e^{-x}[c_1 \cos x + c_2 \sin x]$

$y(0) = 1$ $\qquad$ $y(\pi/2) = e^{-\pi/2}$

$x = 0 \, ; \, y = 1$ $\qquad$ $C_2 = 1$

$1 = c_1 + 0 \quad \boxed{c_1 = 1}$

$y'(c) = e^{-x}[-c_1 \sin x + c_2 \cos x]$

$\quad - e^{-x}[c_1 \cos x + c_2 \sin x]$

$x = \pi/2 \quad y = e^{-\pi/2}$

$e^{-\pi/2} = e^{-\pi/2}[-1 + c_2(0)]$

$\quad - e^{-\pi/2}[0 + c_2]$

$1 = -1 - c_2$

$\boxed{c_2 = -2}$

$$\boxed{y_c = e^{-x}[\cos x + \sin x]}$$

6) $(D^2+4)y=0$

$O \cdot F \Rightarrow (D^2+4)\,y=0$

$A \cdot F \Rightarrow m^2+4=0$ $\boxed{m^2=\pm 2i}$

$m=\pm 2i$

$y_C = e^{0(x)}\left[c_1\cos 2x + c_2 \sin 2x\right]$

$y_C = c_1\cos 2x + c_2\sin 2x$

$m^2=-2i$ ×

$m^2 = 2i$

$m = \pm\sqrt{2i}$

$y_C = e^{0x}\left[c_1\cos\sqrt{2i}\,x + c_2\sin\sqrt{2i}\,x\right]$

6) $(D^4+4)y=0$

$y(D^4+4+4D^2-4D^2)=0$

$y\left[(D^2+2)^2-(2D)^2\right]=0$

$A \cdot F \Rightarrow y\left[(m^2+2)^2-(2m)^2\right]=0$

$(m^2+2-2m)(m^2+2+2m)=0$

$m^2-2m+2=0$ $\qquad$ $m^2+2m+2=0$

$m=\dfrac{2\pm\sqrt{4-8}}{2}$ $\qquad$ $\boxed{m=-1\pm i}$

$=\dfrac{2\pm\sqrt{-4}}{2}$

$=\dfrac{2\pm 2i}{2}=1\pm i$

$$y_c = e^{x}[c_1\cos x + c_2\sin x] + e^{-x}[c_1\cos x + c_2\sin x]$$

$$\boxed{y_c = (e^{x}+e^{-x})c_1\cos x + (e^{x}+e^{-x})c_2\sin x}$$

20/06/2022

1) $(4D^2+12D+9)y=0$

$O.F = (4D^2+12D+9)y=0$

$A.E = 4m^2+12m+9=0$

$4m^2+6m+6m+9=0$

$2m(2m+3)+3(2m+3)=0$

$(2m+3)(2m+3)=0$

$m_1=m_2=-3/2$ [Nature of roots are real & equal]

C.F: General sol^n.

$$\boxed{y_c = (c_1x+c_2)e^{-3/2x}}$$

2) $y''' - 6y'' + 11y' - 6y = 0$

$y(0)=0$, $y'(0)=-4$, $y''(0)=-18$

$O.F = (D^3-6D^2+11D-6)y=0$

$A.E = m^3-6m^2+11m-6=0$

$m_1=1,\ m_2=2,\ m_3=3.$

Nature of root are real and unequal.

$\Rightarrow$ C.F : General Sol^n.

$$\boxed{y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$$

$x = 0 ; y = 0$

$$\boxed{0 = c_1 + c_2 + c_3} \qquad c_1 = -c_2 - c_3$$

$x = 0 ; y = -4$

$$\boxed{-4 = c_1 + 2c_2 + 3c_3}$$

$$\boxed{-18 = c_1 + 4c_2 + 9c_3}$$

$-4 = c_2 + 2c_3 \times 3 \Rightarrow -12 = \cancel{3c_2} + 6c_3$

$-18 = 3c_2 + 8c_3 \qquad \mp 18 = \cancel{3c_2} \pm 8c_3$

$\qquad\qquad\qquad\qquad 6 = -2c_3$

$c_2 = -4 + 6$

$c_2 = 2 \qquad\qquad c_3 = -3$

$c_1 = 1$

3) $y^{iv} - 5y'' + 4y = 0$

O.F $= (D^4 - 5D^2 + 4)\, y = 0$

A.F $= m^4 - 5m^2 + 4 = 0$

$m^4 - 4m^2 - m^2 + 4 = 0$

$m^2(m^2 - 4) - 1(m^2 - 4) = 0$

$m^2 = 4 ; m^2 = 1$

$m = \pm 2 \quad m = \pm 1$

Roots are real & distinct

C·F: General Solⁿ

$$\boxed{y_c = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}}$$

4.) Solve the boundary value problem.

$y'' + \omega^2 y = 0 \qquad y(0) = 0 \; ; \; y(l) = 0$

$(D^2 + \omega^2) y = 0 \Rightarrow$ Standard Form.

A·E: $(m^2 + \omega^2) = 0$

$m^2 = -\omega^2$

$m = \pm \omega i$

Roots are imaginary

C·F: General Sol.ⁿ

C·F $= y_c(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$.

$y_c(x) = e^{0(x)}(c_1 \cos \omega x + c_2 \sin \omega x)$

$y_c(x) = c_1 \cos \omega x + c_2 \sin \omega x$

$\rightarrow x = 0 ; y = 0$

$0 = c_1 + c_2(0)$

$c_1 = 0$

$\rightarrow x = l \; ; y = 0$

$0 = c_1 \cos \omega l + c_2 \sin \omega l$.

$c_2 (\sin \omega l) = 0$

(i) if $c_2 = 0 \Rightarrow y_c(x) = 0$ × (Trivial Sol(n))

∵ Every system has Non-trivial

Sol^n

(ii) $\Rightarrow \sin \omega l = 0$ $\quad n = 1, 2, 3, \ldots$

$\omega . l = n\pi$

$\boxed{l = \pm \frac{n\pi}{\omega}}$

$\boxed{\omega = \pm \frac{n\pi}{l}}$

$\boxed{y_c(x) = C_2 \sin \frac{n\pi x}{l}}$

By super-position principle.

$\boxed{y(x) = \sum_{n=1}^{\infty} C_K \sin \frac{n\pi x}{l}}$

5) Find the non-trivial solution of boundary value problems (BVP) $y^{IV} - \omega^4 y = 0$.

$y(0) = 0$; $y''(0) = 0$ $\quad y(l) = 0 \quad y''(l) = 0$

Operator Form:

$(D^4 - \omega^4) y = 0$

A.E: $m^4 - \omega^4 = 0$

$(m^2 + \omega^2)(m^2 - \omega^2) = 0$

$m^2 = -\omega^2 \qquad m^2 = \omega^2$

$m = \pm \omega i \qquad m = \pm \omega$

$\cancel{y_c(x) \Rightarrow C.F}$ 2 Roots are real

$\cancel{y_c(x) = C_1 e^{\omega i x} + C_2}$ and 2 roots imaginary

$y_c x =$

C.F $\Longrightarrow$ General Solution

$$y_c = c_1 e^{\omega x} + c_2 e^{-\omega x} + e^{(0x)}[c_3 \cos\omega x + c_4 \sin\omega x]$$

$$\boxed{y_c = c_1 e^{\omega x} + c_2 e^{-\omega x} + c_3 \cos\omega x + c_4 \sin\omega x}$$

$x = 0 ;\ y = 0$

$0 = c_1 + c_2 + c_3$

$\boxed{c_1 + c_2 + c_3 = 0}$

$\cancel{c_2 = 0}$

$\boxed{c_1 + c_2 + c_3 = 0}$ — (1)

$x = 0\ y'(0) = 0$

$\cancel{c_1\omega + c_2(-\omega) - \omega c_1(0) + \omega c_2 = 0}$

$c_1\omega = 0$

$\cancel{c_1 = 0}$

$y'(x) = c_1\omega e^{\omega x} - c_2 e^{-\omega x}\omega - c_3\omega\sin\omega x + c_4\cos\omega x$

$y''(x) = c_1\omega^2 e^{\omega x} + c_2\omega^2 e^{-\omega x} - c_3\omega^2\cos\omega x - c_4\omega^2\sin\omega x$

$x = 0.\ y''(0)$

$\boxed{0 = c_1\omega^2 + c_2\omega^2 - c_3\omega^2}$ — (3)

$x = l ;\ y = 0$

$\boxed{c_1 e^{\omega l} + c_2 e^{-\omega l} + c_3\cos\omega l + c_4\sin\omega l}$ — (4)

$x = l ;\ y''(l)$

$\boxed{c_1\omega^2 e^{\omega l} + c_2\omega^2 e^{-\omega l} - c_3\omega^2\cos\omega l - c_4\omega^2\sin\omega l}$

— (3/2)

$$\omega^2(c_1 + c_2 - c_3) = 0 \quad - (1)$$

$$c_1 + c_2 + c_3 = 0 \quad - (3)$$

$$\Rightarrow \boxed{c_1 + c_2 = 0} \quad - (5) \quad \times e^{\omega l}$$

(2) & (4)

$$\boxed{c_1 e^{\omega l} + c_2 e^{-\omega l} = 0} \quad - (6)$$

$$\cancel{c_1 e^{\omega l}} + c_2 e^{\omega l} = 0$$

$$\cancel{c_1 e^{\omega l}} + c_2 e^{-\omega l} = 0$$

$$\boxed{c_2(e^{\omega l} - e^{-\omega l}) = 0}$$

if $c_2 = 0$; $c_1 = 0$ $\quad c_3 = 0$.

$$y_c(x) = c_4 \sin \omega x \quad - (7)$$

$$y'(x) = c_4 \cos \omega x \cdot \omega.$$

$$y''(x) = -c_4 \omega^2 \sin \omega x$$

$x = 0$; $y = 0$ in (7)

$x = l$; $y = 0$.

$$c_4 \sin \omega l = 0$$

$c_4 \neq 0 \quad \omega l = n\pi$

$$\boxed{\omega = \frac{n\pi}{l}}$$

$$\boxed{y_c(x) = c_4 \sin \frac{n\pi x}{l}}$$

**

NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS:

N.H.L.D.E are always having complete (general) solution i.e. [complementary function and particular integral].

$$\boxed{\text{Complete Solution} = \text{Complementary Function} + \text{Particular Integral}}$$

General form of NHLDE:

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + K_n y = Q(x) \quad \text{—(1)}$$

$$Q(x) \neq 0$$

Operator Form:

$$\left(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_n\right) y = Q(x) \quad \text{—(2)}$$

To find C.F:

H.L.DE:

$$\left(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_n\right) y = 0$$

$$f(m) = 0$$

Find the roots and follow the previous method

To find Particular Integral:

$\boxed{f(D)y = Q(x)}$

- $Q(x) = e^{ax}$
- $Q(x) = \sin ax / \cos ax$
- $Q(x) = x^k$
- $Q(x) = e^{ax} \cdot V(x)$
  - $\sin ax$
  - $\cos ax$
  - x^k
- $Q(x) = x^k \, V(x)$
  - $\sin ax$
  - $\cos ax$