22/07/2022 UNIT-3:

VECTOR DIFFERENTIATION:

→ Scalar & vector point functions
→ Vector differential operator
→ Gradient - defn, problems, physical interpretation
→ Divergence - defn, physical interpretation. problems.
→ Curl - defn, physical interpretation, problems (geometrical)
→ Normal to a level surface.
* Unit normal vector.
→ Directional derivative
→ Angle b/w the surfaces.
→ Scalar potential function.

* Scalar Point Function:

Let R be a region of space ~~depends~~ at ~~upon~~ each point of which a scalar $\phi = \phi(x,y,z)$ is given; then ϕ is called a scalar function and R is called a scalar field.

* Ex: temperature distribution in a medium, distribution of atmospheric pressure

** Vector Point Function:

Let R be a region of space at each point of which a vector $V = \bar{v}(x,y,z)$ is given then v is called a vector point function and R is called vector field.

* Each vector $\bar{v}$ of the field is regarded as a localised vector attached to the corresponding point (x,y,z).

* Gradient of a scalar field:

Let $\phi(x,y,z)$ be a function defining a scalar field, then the vector $i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$ is the gradient of the scalar field ϕ is denoted by grad ϕ.

$$\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

The gradient of scalar field ϕ is obtained by operating on ϕ by the vector operator

$$\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

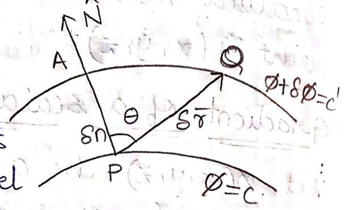
This operator is denoted by the symbol ∇, read as del (also called nabla).

$$\boxed{\text{grad}\,\phi = \nabla\phi}$$

* Geometrical interpretation of Gradient:

If a surface $\phi(x,y,z)=c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a LEVEL SURFACE through P.

* For example, if $\phi(x,y,z)$ represents potential at the point (x,y,z); the equipotential surface $\phi(x,y,z)=C$ is a level surface.



Through any point passes one and only one level surface.

Moreover, no 2 level surfaces can intersect.

Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi+\delta\phi$.

⇒ Let $\bar{r}$ and $\bar{r}+\delta\bar{r}$ be the position vectors of P and Q respectively, then $\overline{PQ}=\delta\bar{r}$

* $$\nabla\phi\cdot\delta\bar{r}=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\delta x+\hat{j}\delta y+\hat{k}\delta z\right)$$

① — $$\boxed{\nabla\phi\cdot\delta\bar{r}=\frac{\partial\phi}{\partial x}\delta x+\frac{\partial\phi}{\partial y}\delta y+\frac{\partial\phi}{\partial z}\delta z}=\delta\phi$$

If Q lies on the same level surface as P then $\delta\phi=0$

∴ ① reduces to $\boxed{\nabla\phi\cdot\delta\bar{r}=0}$

. Thus, $\nabla\phi$ is perpendicular to every $\delta\bar{r}$ lying in the surface.

. Hence $\nabla\phi$ is normal to the surface
$$\phi(x,y,z)=C$$

Grad ϕ → vector pt. function
div ϕ → scalar pt. function

*DIVERGENCE:

Let $f(x,y,z)$ be a continuously differentiable vector pt. function i.e. $\bar{f}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ then

$\nabla(f)$ with a dot product is called as divergence of a vector point function.

$$\nabla f=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(f_1\hat{i}+f_2\hat{j}+f_3\hat{k})$$

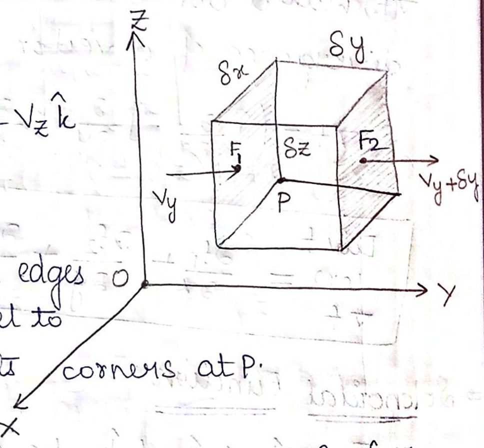
$$\boxed{\text{Div } f \text{ (or) } \nabla f=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}}$$

⇒ Solenoidal Function:

If $\bar{F}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ be a continuously differentiable vector point function and if $\boxed{\text{div } \bar{F}=0}$, then $\bar{F}$ is said to be Solenoidal vector function.

* Geometrical Interpretation of Divergence:

Consider a fluid having density $\rho = \rho(x, y, z, t)$ and velocity $\bar{v} = \bar{v}(x, y, z, t)$ at a point (x, y, z) at time t. Let $\bar{V} = \rho\bar{v}$, then $\bar{V}$ is a vector having the same direction as $\bar{v}$ and magnitude $\rho|\bar{v}|$. It is known as flux. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed $\perp$ar to the direction of flow.



Velocity: $\bar{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

at point $P(x, y, z)$

∴ Parallelopiped with edges $\delta x, \delta y, \delta z$ parallel to axes with one of its corners at P.

Mass of fluid entering face $F_1 = V_y\,\delta x\,\delta z$

Mass of fluid flowing out of face $F_2 = V_{y+\delta y}\,\delta x\,\delta z$

$$= \left(V_y + \frac{\partial V_y}{\partial y}\cdot\delta y\right)\delta x\,\delta z \quad \text{(Taylor's series)}.$$

* net decrease in the mass of fluid flowing across these 2 faces.

$$= \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z - V_y\,\delta x\,\delta z = \frac{\partial V_y}{\partial y}\delta x\,\delta y\,\delta z$$

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\,\delta y\,\delta z$$

Divide by Volume of parallelopiped $= \delta x\,\delta y\,\delta z$

Rate of loss fluid per unit time $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$\Rightarrow \operatorname{div}\bar{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

∴ div $\bar{V}$ gives rate of out flow per unit volume at a point of fluid.

* CURL:

If $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a continuously differentiable vector point function; then ∇ of vector point function, then ∇ of vector point function with the cross product is said to be the curl of the vector point function $\bar{F}$.

$$\nabla \times \bar{F} = \operatorname{curl}\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

* Irrotational Function:

If $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a continuosly different vector point function; then if curl $\bar{F}=\bar{0}$; then $\bar{F}$ is said to be irrotational.

* **Geometrical Interpretation of Curl:**

Consider a rigid body rotating about a fixed axis through O with uniform angular velocity

$$\bar{\Omega}=w_1\hat{i}+w_2\hat{j}+w_3\hat{k}$$

The velocity $\bar{V}$ of any point $P(x,y,z)$ on the body is given by $\bar{V}=\bar{\Omega}\times\bar{r}$ where

$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k} \Rightarrow$ position vector of P.

$$\bar{V}=\bar{\Omega}\times\bar{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ w_1 & w_2 & w_3\\ x & y & z\end{vmatrix}$$

$$\bar{V}=(w_2z-w_3y)\hat{i}+(w_3x-w_1z)\hat{j}+(w_1y-w_2x)\hat{k}$$

$$\text{curl}\,\bar{V}=\nabla\times\bar{V}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ w_2z-w_3y & w_3x-w_1z & w_1y-w_2x\end{vmatrix}$$

$$\text{curl}\,\bar{V}=(w_1+w_1)\hat{i}+(w_2+w_2)\hat{j}+(w_3+w_3)\hat{k}$$

$$=2(w_1\hat{i}+w_2\hat{j}+w_3\hat{k})=2\bar{\Omega}$$

$$\boxed{\bar{\Omega}=\frac{1}{2}\,\text{curl}\,\bar{V}}$$

*→ The angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

* **Level Surface:** If a surface $\phi(x,y,z)=c$ be drawn through any point $P(x,y,z)$ such that at each point; the given function has same value, then such surface is called level surface of ϕ at P.

Ex: equipotential, isothermal system.

* **Unit normal:**

Consider a vector pt. function $\bar{F}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ then unit normal is given by

$$\hat{n}=\frac{\bar{F}}{|\bar{F}|}=\frac{f_1\hat{i}+f_2\hat{j}+f_3\hat{k}}{\sqrt{f_1^2+f_2^2+f_3^2}}$$

(OR)

If a scalar point function is given then

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}\quad\text{since }\nabla\phi\text{ is a vector.}$$

* **Magnitude: (Greatest Value/Maximum Value).**

If $\phi(x,y,z)$ be a scalar pt. fn, then $|\nabla\phi|$ (or) $|\text{grad}\,\phi|$ is said to be magnitude of a given function.

* Directional Derivative:

The directional derivative of ϕ in the direction of $\bar{f}$ is given by $\boxed{\nabla\phi \cdot \frac{\bar{f}}{|\bar{f}|}}$

→ Find the D.D of $\phi(x, y, z)$ in the direction of $\bar{f}$.

$$\boxed{D.D = (\nabla\phi) \cdot \frac{\hat{f}}{|f|}}$$

D.D of 1^{st} fn
↓
vector
in the direction of II^{nd} fn

$$\hat{n} = \frac{\bar{f}}{|\bar{f}|}$$

→ Find the D.D of $\bar{f}$ in the direction of ϕ

$$\boxed{D.D = \bar{f} \cdot \frac{\nabla\phi}{|\nabla\phi|}}$$

→ Find the D.D of ϕ in the direction of ψ.

$$\boxed{D.D = \nabla\phi \cdot \frac{\nabla\psi}{|\nabla\psi|}}$$

→ Find the D.D of $\bar{f}$ in the direction of $\bar{g}$.

$$\boxed{D.D = \bar{f} \cdot \frac{\bar{g}}{|\bar{g}|}}$$

→ Find the D.D of ϕ at $P(x_1, y_1, z_1)$ in the direction of normal at $Q(x_2, y_2, z_2)$

$$\boxed{D.D = (\nabla\phi) \frac{\overline{PQ}}{|\overline{PQ}|}}$$

Note: $\boxed{\text{normal} = \text{vector} = \text{grad}\,\phi}$

normal $\neq$ unit normal.

* Find the normal of $\log(x^2+y^2+z^2)$

Let:
$\log(x^2+y^2+z^2) = \phi$

$$\boxed{\begin{aligned} \because \bar{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ |\bar{r}| &= \sqrt{x^2+y^2+z^2} \end{aligned}}$$

normal = grad ϕ

$$\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{2x}{x^2+y^2+z^2} + \hat{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \hat{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$\boxed{\text{grad}\,\phi = \frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2+y^2+z^2}}$$

$$\boxed{\text{grad}\,\phi = \frac{2\bar{r}}{r^2}}$$

* $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}; \quad \frac{\partial r}{\partial x} = \frac{x}{r}}$$

* Find grad ϕ where $\phi = x^2y + y^2z + z^2x$ at $(1, 0, -1)$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\boxed{\nabla\phi = \hat{i}(2xy + z^2) + \hat{j}(x^2 + 2yz) + \hat{k}(y^2 + 2zx)}$$

$$(\text{grad}\,\phi)_{(1,0,-1)} = \hat{i} + \hat{j} - 2\hat{k}$$

$$|\nabla\phi| = \sqrt{1+1+4} = \sqrt{6}$$

$$\boxed{\text{unit normal} = \frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}}$$

* grad ϕ where $\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\cancel{\phi^2 = y^2z^2 + z^2x^2 + x^2y^2}$$

$$\cancel{\frac{2\partial\phi}{\partial x}}$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(z+y) + \hat{j}(z+x) + \hat{k}(x+y).$$

* Find div ϕ where $\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\text{div}\,\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)$$

$$\text{div}\,\phi = 0$$

$\therefore$ ϕ is irrotational

$(\text{grad}\,\phi)_{(1,0,-1)} = \hat{i}+\hat{j}-2\hat{k}$

$|\nabla\phi| = \sqrt{1+1+4} = \sqrt{6}$

$$\boxed{\text{unit normal} = \frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}}$$

* grad ϕ where $\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$\cancel{\phi^2 = y^2z^2 + z^2x^2 + x^2y^2}$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(z+y) + \hat{j}(z+x) + \hat{k}(x+y).$$

* Find div ϕ where $\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\text{div}\,\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)$$

$\text{div}\,\phi = 0$

$\therefore \phi$ is irrotational

* Find the D·D· of ϕ in the direction of tangent planes to the curves.

$x = f(t)$; $y = g(t)$; $z = h(t)$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{r} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

$$D\cdot D\cdot = \nabla\phi\cdot\frac{\bar{f}}{|\bar{f}|}$$

$$\boxed{\bar{f} = \frac{d\bar{r}}{dt} = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}}$$

* Angle between the surfaces:

27/07/2022

(i) If $\phi(x,y,z)$ and $\psi(x,y,z)$ be 2 scalar point functions at $p(x,y,z)$:

Then the angle b/w the surfaces is given by

$$\boxed{\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}}$$

where $\bar{n}_1, \bar{n}_2$ are the normals of the given surfaces

$\bar{n}_1 = (\nabla\phi)_p$; $|\bar{n}_1| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}$

$\bar{n}_2 = (\nabla\psi)_p$; $|\bar{n}_2| = \sqrt{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2}$

$$\boxed{\theta = \cos^{-1}\left(\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}\right)}$$

(ii) If $\phi(x,y,z)$ be a scalar point function at $p(x,y,z)$ and $Q(x,y,z)$:

Then the angle b/w the surfaces is given by:

$$\boxed{\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}}$$

where $\bar{n}_1 = (\nabla\phi)_{p(x,y,z)}$

$\bar{n}_2 = (\nabla\phi)_{Q(x,y,z)}$.

$$\theta = \cos^{-1}\left(\frac{\bar{n_1}\cdot\bar{n_2}}{|\bar{n_1}||\bar{n_2}|}\right)$$

* If $r = |\bar{r}|$ where $\bar{r}$ is the position vector; find ∇r^n

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$r = \sqrt{x^2+y^2+z^2}$

$$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \hat{i}\, n r^{n-1}\cdot\frac{\partial r}{\partial x} + \hat{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \hat{k} n r^{n-1}\frac{\partial r}{\partial z}$$

$$\nabla r^n \left(\hat{i}\cdot\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) n r^{n-1}$$

$$= (\hat{i}x + y\hat{j} + z\hat{k})\, n r^{n-2}$$

$$\boxed{\nabla r^n = n r^{n-2}\cdot\bar{r}}$$

* $\nabla\left(\frac{1}{r}\right)$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \frac{-1}{r^2}\frac{\partial r}{\partial x}\hat{i} - \frac{1}{r^2}\frac{\partial r}{\partial y}\hat{j} - \frac{1}{r^2}\frac{\partial r}{\partial z}$$

$$= \frac{-1}{r^2}\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right]$$

$$= \boxed{\frac{-\bar{r}}{r^3} = \nabla\left(\frac{1}{r}\right)}$$

* ∇r:

$$\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \frac{\bar{r}}{r}$$

* $\nabla(\log \bar{r})$

$$\hat{i}\frac{1}{r}\frac{\partial r}{\partial x} + \hat{j}\frac{1}{r}\frac{\partial r}{\partial y} + \hat{k}\frac{1}{r}\frac{\partial r}{\partial z}$$

$$\frac{1}{r}\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$

$$= \frac{\bar{r}}{r^2}$$

* $\nabla(f(r))$

$$\hat{i} f'(r)\frac{x}{r} + \hat{j} f'(r)\frac{y}{r} + \hat{k} f'(r)\frac{z}{r}$$

$$= \frac{f'(r)\cdot\bar{r}}{r}$$

* ∇e^{r^2}

$$\hat{i}e^{r^2}\cdot 2r\frac{\partial r}{\partial x} + \hat{j}e^{r^2}\cdot 2r\cdot\frac{\partial r}{\partial y} + \hat{k}e^{r^2} 2r\frac{\partial r}{\partial z}$$

$$= 2re^{r^2}\cdot\frac{\bar{r}}{r}$$

$$= 2e^{r^2}\cdot\bar{r}$$

* Find a unit normal vector to the surface $x^3+y^3+3xyz=3$ at the p. $(1,2,-1)$

$\phi(x,y,z) = x^3+y^3+3xyz-3$

$\frac{\partial\phi}{\partial x} = 3x^2+3yz$ $\qquad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\frac{\partial\phi}{\partial y} = 3y^2+3xz$

$\frac{\partial\phi}{\partial z} = 3xy$

$$= \frac{\hat{i}(3x^2+3yz) + \hat{j}(3y^2+3xy) + \hat{k}(3xy)}{|\nabla\phi|}$$

at $P(1,2,-1)$

$$\hat{n} = \frac{\hat{i}(-3) + \hat{j}9 + 6\hat{k}}{\sqrt{9+81+36}}$$

$$= \frac{3(-\hat{i} + 3\hat{j} + 2\hat{k})}{3\sqrt{14}}$$

$$\boxed{\hat{n} = \frac{-\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{14}}}$$

90
36
126
6×21
$6\times 7\times 3$
3×2

* Find the unit normal vector to the surface $xy^3z^2 = 4$ at point $(-1,-1,2)$

$\phi = xy^3z^2 - 4$

$\frac{\partial\phi}{\partial x} = y^3z^2 \; ; \; \frac{\partial\phi}{\partial y} = 3xy^2z^2 \; ; \; \frac{\partial\phi}{\partial z} = 2xy^3z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{4\sqrt{1+9+1}}$$

$$= \frac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$$

* Find the greatest value of D.D of the function $f = x^2yz^3$ at $(2,1,-1)$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

$$(\nabla f)_P = -4\hat{i} + (-4\hat{j}) + 12\hat{k}$$

$$= -4(\hat{i} + \hat{j} - 3\hat{k})$$

$$\frac{(\nabla f)_P}{|\nabla f|} = \frac{\cancel{+4}(\hat{i} + \hat{j} - 3\hat{k})}{\cancel{4}\sqrt{1+1+9}}$$

$$= \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$$

$\therefore$ The greatest value of D.D of $f = 4\sqrt{11}$

* In what direction from $(3,1,-2)$ in the D.D. of $\phi = x^2y^2z^4$ is maximum and what is its magnitude.

* What is the greatest rate of increase of $u = x^2 + yz^2$ at the point $(1,-1,3)$?

* The temp. at a point $P(x,y,z)$ in space is given by $T(x,y,z) = x^2 + y^2 - z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that; it will get warm as soon as possible. In what direction it should fly?

* Find the angle b/w the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at $P(2,-1,2)$

$\phi(x,y,z) = x^2+y^2+z^2-9$.

$\psi(x,y,z) = x^2+y^2-z-3$.

$$\hat{n}_1 = (\nabla\phi)_P$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\boxed{\bar{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}}$$

$|\bar{n}_1| = \sqrt{16+4+16}$

$= 6$

$$\bar{n}_2 = (\nabla\psi)_P$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\boxed{\bar{n}_2 = 4\hat{i} - 2\hat{j} - \hat{k}}$$

$|\bar{n}_2| = \sqrt{16+4+1}$

$= \sqrt{21}$

$$\cos\theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{16+4-4}{6\sqrt{21}}$$

$$\boxed{\cos\theta = \frac{16}{6\sqrt{21}}} \qquad \boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$

1) $\phi = x^2y^2z^4$ $\quad P(3,1,-2)$

$\nabla\phi = 2xy^2z^4\hat{i} + 2x^2yz^4\hat{j} + 4x^2y^2z^3\hat{k}$

$18 \times 16 = 108$ $\quad 36 \times 8 = 288$

$= 96\hat{i} + 288\hat{j} - 288\hat{k}$

$= 96(\hat{i} + 3\hat{j} - 3\hat{k})$

$|\nabla\phi| = 96\sqrt{1+9+9}$

$= \underline{96\sqrt{19}}$

2) $U = x^2 + yz^2$ $\quad P(1,-1,3)$

$\nabla U = 2x\hat{i} + z^2\hat{j} + 2zy\hat{k}$

$= 2\hat{i} + 9\hat{j} - 6\hat{k}$

$|\nabla U| = \sqrt{4+81+36} = \sqrt{121}$

$= \underline{11}$

3) $T(x,y,z) = x^2 + y^2 - z$ $\quad P(1,1,2)$

$\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$= 2\hat{i} + 2\hat{j} - \hat{k}$

$|\nabla T| = \sqrt{4+4+1} = \underline{3}$

$\hat{n} = \dfrac{\nabla T}{|\nabla T|} = \underline{\dfrac{2\hat{i} + 2\hat{j} - \hat{k}}{3}}$

28/07/2022

* Find DD of $xy + yz + zx$ in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$ at $(1,2,0)$.

$\phi = xy + yz + zx$ $\quad \bar{f} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\nabla\phi = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

$\hat{n} = \dfrac{\bar{f}}{|\bar{f}|} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

$D \cdot D = \nabla\phi \cdot \hat{n}$

$= \left[(y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}\right]\left[\dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right]$

$(DD)_{P(1,2,0)} = \dfrac{[2\hat{i} + \hat{j} + 3\hat{k}] \cdot [\hat{i} + 2\hat{j} + 2\hat{k}]}{3}$

$= \dfrac{2+2+6}{3} = \underline{\dfrac{10}{3}}$

* $\phi = xy + yz + zx$ $\quad \bar{v} = \hat{i} + 2\hat{j} + 2\hat{k}$

$D \cdot D = \dfrac{\bar{v} \cdot \nabla\phi}{|\nabla\phi|}$

$= \dfrac{(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (2\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{14}}$

$\underline{D \cdot D} = \underline{\dfrac{10}{\sqrt{14}}}$

* Find the D·D of $xyz^2 + xz$ at $(1,1,1)$ in the direction of normal to the surface $3xy^2 + y - z = 0$ at $(0,1,1)$

$\phi = xyz^2 + xz$ $\quad \psi = 3xy^2 + y - z$

$\nabla\phi = (yz^2 + z)\hat{i} + xz^2\hat{j} + (2xyz + x)\hat{k}$

$= 2\hat{i} + \hat{j} + 3\hat{k}$

$\nabla\psi = 3y^2\hat{i} + (6xy+1)\hat{j} + (-\hat{k})$

$= 3\hat{i} + \hat{j} - \hat{k}$

$$D\cdot D = \nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

* Find D.D of $x^2-y^2+2z^2$ at $(1,2,3)$ in the direction of line $\overline{PQ}$ where $Q(5,0,4)$

$$\nabla\phi = 2x\hat{i}-2y\hat{j}+4z\hat{k} = 2\hat{i}-4\hat{j}+12\hat{k}$$

$$\overline{PQ} = \overline{OQ}-\overline{OP} = (4,-2,1)$$

$$\bar{f} = \overline{PQ} = 4\hat{i}-2\hat{j}+\hat{k}$$

$$|\bar{f}| = \sqrt{16+4+1} = \sqrt{21}$$

$$D\cdot D = \nabla\phi\cdot\frac{\bar{f}}{|\bar{f}|} = \frac{2(\hat{i}-2\hat{j}+6\hat{k})\cdot(4\hat{i}-2\hat{j}+\hat{k})}{\sqrt{21}}$$

$$= \frac{2(4+4+6)}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

* Find the D.D of $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t, y=t^2, z=t^3$ at the point $(1,1,1)$

$$\bar{f} = \frac{d\bar{r}}{dt} = \hat{i}+2t\hat{j}+3t^2\hat{k} = \hat{i}+2\hat{j}+3\hat{k}$$

$x=1$
$y=1$
$z=1$

$$\phi = xy^2+yz^2+zx^2$$

$$\nabla\phi = (y^2+2xz)\hat{i}+(2xy+z^2)\hat{j}+(2yz+x^2)\hat{k}$$

$$= 3\hat{i}+3\hat{j}+3\hat{k}$$

$x=t=1$
$y=t^2=1$
$t=\pm1$
$z=t^3=1$
$t=1,\omega,\omega^2$

$$D\cdot D = \frac{3(\hat{i}+\hat{j}+\hat{k})(\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{1+4+9}}$$

$$= \frac{3(1+2+3)}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

* Evaluate the angle b/w the normals to the surface $xy=z^2$ at $P(4,1,2)$ & $Q(3,3,-3)$.

Let $\phi = xy-z^2$

$$\nabla\phi = y\hat{i}+x\hat{j}-2z\hat{k}$$

$$\hat{n}_1 = (\nabla\phi)_{P(4,1,2)} = \hat{i}+4\hat{j}-4\hat{k}$$

$$\hat{n}_2 = (\nabla\phi)_{Q(3,3,-3)} = 3\hat{i}+3\hat{j}+6\hat{k}$$

$\bar{n}_1, \bar{n}_2$ be the normals to the surface at the 2 points P, Q.

$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{3+12-24}{\sqrt{1+16+16}\cdot\sqrt{9+9+36}}$$

$$= \frac{-9}{\sqrt{33}\cdot\sqrt{54}} = \cancel{\frac{-9}{3\sqrt{102}}}$$

$17 \times 6 = 102$

$$\boxed{\cancel{\cos\theta = \frac{-3}{\sqrt{102}}}}$$

$$\theta = \cos^{-1}\cancel{\left(\frac{-3}{\sqrt{102}}\right)}$$

$$\cos\theta = \frac{-9}{4\sqrt{33}\cdot 3\sqrt{6}}$$

$$= \frac{-3}{\sqrt{3\times11\times3\times2}} = \frac{-1}{\sqrt{22}}$$

* Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z+47=0$ at the point $(4,-3,2)$.

$\phi = x^2+y^2+z^2-29$

$\nabla\phi = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$(\nabla\phi)_{(4,-3,2)} = 8\hat{i}-6\hat{j}+4\hat{k} = 2(4\hat{i}-3\hat{j}+2\hat{k})$

$\psi = x^2+y^2+z^2+4x-6y-8z+47$

$\nabla\psi = (2x+4)\hat{i}+(2y-6)\hat{j}+(2z-8)\hat{k}$

$(\nabla\psi)_{(4,-3,2)} = 12\hat{i}-12\hat{j}-4\hat{k} = 4(3\hat{i}-3\hat{j}-\hat{k})$

$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{\cancel{8}(12+9-2)}{2\times4\cancel{8}\sqrt{29\times19}}$$

$$= \frac{19}{\sqrt{29\times19}} = \sqrt{\frac{19}{29}}$$

* Find the values of a & b the surfaces $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$ intersect orthogonally at the point $(1,-2,2)$

$\cos\theta = 0$

$\hat{n}_1\cdot\hat{n}_2 = 0$

$\hat{n}_1 = \nabla\phi_1 = (2xa-a-2)\hat{i}-bz\hat{j}-by\hat{k}$

$= (a-2)\hat{i}-2b\hat{j}+2b\hat{k}$

$\hat{n}_2 = 8xy\hat{i}+4x^2\hat{j}+3z^2\hat{k}$

$= -16\hat{i}+4\hat{j}+12\hat{k}$

$\hat{n}_2 = 4(-4\hat{i}+\hat{j}+3\hat{k})$

$\hat{n}_1\cdot\hat{n}_2 = 4[(-4a+8)-2b+6b] = 0$

$-4a+8+4b=0$

$4(-a+2+b)=0$

$\boxed{a-b=2}$

$\cancel{a}+4b=\cancel{a}+2$

$\boxed{b=\frac{1}{2}}$ $\boxed{a=\frac{5}{2}}$

* Find the normal vector to the surface $z=\sqrt{x^2+y^2}$ at the point $(3,4,5)$

normal $=\nabla\phi$

$\phi=\sqrt{x^2+y^2}-z$

$\nabla\phi=\frac{2x}{2\sqrt{x^2+y^2}}\hat{i}+\frac{2y}{2\sqrt{x^2+y^2}}\hat{j}-1\hat{k}$

$=\cancel{\frac{x+y}{\sqrt{x^2+y^2}}}=\cancel{\frac{7}{5}-1}$

$=\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}-\hat{k}=\cancel{\frac{2}{5}}$

$\phi=x^2+y^2-z^2$

$=6\hat{i}+8\hat{j}-10\hat{k}$

* Find angle b/w the surfaces $x\log z=y^2-1$ and $x^2y=2-z$ at the point $(1,1,1)$

$\phi=x\log z-y^2+1$

$\nabla\phi=\log z\hat{i}-2y\hat{j}+\frac{x}{z}\hat{k}$

$=-2\hat{j}+\hat{k}$ $\quad|\nabla\phi|=\sqrt{5}$

$\psi=x^2y-2+z$

$\nabla\psi=2xy\hat{i}+x^2\hat{j}+\hat{k}$

$=2\hat{i}+\hat{j}+\hat{k}$ $\quad|\nabla\psi|=\sqrt{4+1+1}=\sqrt{6}$

$\cos\theta=\frac{-2+1}{\sqrt{30}}=\frac{-1}{\sqrt{30}}$

$\theta=\cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$

* Find the DD of $f(x,y)=x^2y^3+xy$ at $(2,1)$ in the direction of unit vector which makes an angle of $\pi/3$ with x-axis.

$\cancel{\phi=x^2y^3+xy}$

$\cancel{\nabla\phi=2xy^3\hat{i}+x\hat{j}=4\hat{i}+2\hat{j}}$

$\cancel{f=\cos\theta\hat{i}+\sin\theta\hat{j}}$

$\phi=x^2y^3+xy$

$\nabla\phi=(2xy^3+y)\hat{i}+(3y^2x^2+x)\hat{j}$

$=5\hat{i}+14\hat{j}$

$\bar{f}=\cos\theta\hat{i}+\sin\theta\hat{j}=\frac{\hat{i}}{2}+\frac{\sqrt{3}}{2}\hat{j}$

$D.D=\frac{(5\hat{i}+14\hat{j})\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right)}{\sqrt{\frac{1}{4}+\frac{3}{4}}}=\frac{\frac{5}{2}\hat{i}+7\sqrt{3}\hat{j}}{\cancel{1}}$

* Find a scalar function f; $V=\nabla f$ where $V=xy(2yz\hat{i}+2xz\hat{j}+xy\hat{k})$.

$\nabla f=V=2xy^2z\hat{i}+2x^2zy\hat{j}+x^2y^2\hat{k}$

$f=x^2y^2z+x^2y^2z+x^2y^2z$

$\boxed{f=x^2y^2z}$

* Find the divergence of vector field $(x^2y^2 - z^3)\hat{i} + (2xyz)\hat{j} + e^{xyz}\hat{k}$

$$\text{div}\,\bar{\phi} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\bar{\phi} = (x^2y^2 - z^3)\hat{i} + (2xyz)\hat{j} + e^{xyz}\hat{k}$$

$$\boxed{\text{div}\,\bar{\phi} = 2xy^2 + 2xz + xye^{xyz}}$$

at $P(1,0,2)$

$$\text{div}\,\bar{\phi} = 0 + 4 + 0 = 4$$

* Find divergence of $\frac{\bar{r}}{r^3}$.

$$\bar{f} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$r^3 = (x^2+y^2+z^2)^{3/2}$$

$$\cancel{\text{div}\,\bar{f} = \frac{1}{x^2+y^2+z}}$$

$$\cancel{\bar{f} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \frac{y}{(x^2+y^2+z^2)}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}}$$

$$\cancel{\text{div}\,\bar{f} = \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}\, 2y}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)\,2z}{(x^2+y^2+z^2)^3}}$$

$$\bar{f} = \frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}$$

$$\bar{f} = xr^{-3}\hat{i} + yr^{-3}\hat{j} + zr^{-3}\hat{k}$$

$$\bar{f} = r^{-3} + -3r^{-4}\frac{x^2}{r} + r^{-3} - 3r^{-4}\frac{y^2}{r} + r^{-3} - 3r^{-4}\frac{z^2}{r}$$

$$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2)$$

$$= 3r^{-3} - 3r^{-3}$$

$$= 0$$

$f = \frac{\bar{r}}{r^3} \Rightarrow$ solenoidal

$\text{div}(\text{curl}\,\bar{v}) = 2y - 2y + 0 = 0$

Show that the vector field is irrotational and find the scalar potential function $f(x, y, z)$ such that $\bar{v} = \text{grad}\, f$.

To satisfy scalar potential function

$\text{curl}\,\bar{v} = 0$ $\qquad$ $\bar{v} = \text{grad}\, f$

$\text{curl}\, F = 0$ $\qquad$ $\bar{F} = \text{grad}\, \phi$

$\Rightarrow V = (3x^2y^2z^4)\hat{i} + (2x^3yz^4)\hat{j} + (4x^3y^2z^3)\hat{k}$

Prove that $\text{curl}\,\bar{v} = 0$

$$\text{curl}\,\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2z^4 & 2x^3yz^4 & 4x^3y^2z^3 \end{vmatrix}$$

$$= (8x^3yz^3 - 8x^3yz^3)\hat{i} - \hat{j}(12x^2y^2z^3 - 12x^2y^2z^3) + \hat{k}(6x^2yz^4 - 6x^2yz^4)$$

$\bar{v} = \text{grad}\,\phi$

$\phi = x^3y^2z^4$

$$= \underline{0}$$

* $(e^{xy} \cdot y)\hat{i} + e^{xy} x\hat{j} + 2e^{z}\hat{k}$

$$\text{curl}\,\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xy}y & xe^{xy} & 2e^{z} \end{vmatrix}$$

$= 0\hat{i} - 0\hat{j} + \hat{k}(xye^{xy} + e^{xy} - xye^{xy} - e^{xy})$

$= \underline{0}$

$\bar{V} = ye^{xy}\hat{i} + xe^{xy}\hat{j} + ze^{z}\hat{k}$

$\phi = e^{xy} + ze^{z}$

* Find the constant a, b, c

$\bar{v} = (3x + ay + z)\hat{i} + (2x - y + bz)\hat{j} + (x + cy + z)\hat{k}$

is irrotational

Hence find scalar potential function

$$\text{curl}\,\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x+ay+z & 2x-y+bz & x+cy+z \end{vmatrix}$$

$= \hat{i}\left(\frac{\partial}{\partial y}(x+cy+z) - \frac{\partial}{\partial z}(2x-y+bz)\right)$

$-\hat{j}\left[\frac{\partial}{\partial x}(x+cy+z) - \frac{\partial}{\partial z}(3x+ay+z)\right]$

$+\hat{k}\left[\frac{\partial}{\partial x}(2x-y+bz) - \frac{\partial}{\partial y}(3x+ay+z)\right]$

$= (c-b)\hat{i} + (1-1)\hat{j} + \hat{k}(2-a) = 0$

$c = b \Rightarrow \boxed{\begin{matrix} b = c = k \\ a = 2 \end{matrix}}$

Let $k = 1 \Rightarrow b = c = 1$

$\bar{v} = \text{grad}\,\phi$

$(3x + 2y + z)\hat{i} + (2x - y + bz)\hat{j} + (x + y + z)\hat{k}$

$= \hat{i}\frac{\partial f_1}{\partial x} + \hat{j}\frac{\partial f_2}{\partial y} + \hat{k}\frac{\partial f_3}{\partial z}$

$\frac{\partial f_1}{\partial x} = 3x + 2y + z = \frac{3x^2}{2}$

$\frac{\partial f_2}{\partial y} = 2x - y + bz = \frac{-y^2}{2}$

$\frac{\partial f_3}{\partial z} = x + y + z = \frac{z^2}{2}$

$\phi = \frac{3x^2 - y^2 + z^2}{2}$

* Find the constant a, b, c so that $(x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. Hence find scalar potential function $\phi(x, y, z)$.

→ S.T. $\text{curl}(\text{curl}(\text{curl}(\text{curl} f))) = \nabla^4 f$

→ P.T $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

→ If $\bar{a} = 3xz^2\hat{i} - yz\hat{j} + (x + 2z)\hat{k}$

find $\text{curl}(\text{curl}\,\bar{a})$

→ Show that vector field defined by $xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$ is conservative (irrotational $\text{curl}\,\bar{v} = 0$)

→ If $\bar{a}$ is constant vector $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Find $\text{curl}(\bar{a} \times \bar{r})$.

$\bar{v} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

curl $\bar{v} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$= \hat{i}(c-1) - (4+a)\hat{j} + (b+2)\hat{k} = \bar{0}$

$c = 1,\ a = -4\ \ b = -2$

$\bar{v} = \text{grad } f$

$\bar{v} = (x+2y-4z)\hat{i} + (-2x-3y-z)\hat{j}$

$+ (4x+y+2z)\hat{k}$

$\frac{\partial f_1}{\partial x} = x+2y-4z$

$= \frac{x^2}{2}$

$\frac{\partial f_2}{\partial y} = \frac{-3y^2}{2}$

$\frac{\partial f_3}{\partial z} = z^2$

$$\boxed{f = \frac{x^2-3y^2+z^2}{2}}$$

$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

$\nabla f(r) = \frac{\partial}{\partial f}(f(r)) = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{f'(r) \cdot \cancel{x}}{\cancel{x}}$

$\nabla^2 f(r) = \frac{1}{r^2}\left[r\left(f''(r) \cdot \frac{x^2}{r} + f'(r)\right) - x f'(r) \cdot \frac{x}{r} \right]$

$= \frac{x}{r^2} f''(r) \cdot \frac{x^2}{r} + \frac{f'(r)}{r} - \frac{x^2 f'(r)}{r^3}$ — ①

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2 + y^2 + z^2$

$y \quad = \frac{y^2}{r^2} f''(r) + \frac{f'(r)}{r} - \frac{y^2 f'(r)}{r^3}$ — ②

$z \quad = \frac{z^2}{r^2} f''(r) + \frac{f'(r)}{r} - \frac{z^2 f'(r)}{r^3}$ — ③

① + ② + ③

$\frac{3}{r} f'(r) - \frac{x^2+y^2+z^2}{r^3} f'(r) + \frac{(x^2+y^2+z^2) f''(r)}{r^3}$

$= \frac{3f'(r)}{r} - \frac{f'(r)}{r} + \frac{f''(r)}{r}$

$= \frac{f''(r)}{r} + \frac{2f'(r)}{r}$

$\Rightarrow \bar{a} = 3xz^2\hat{i} - 3y\hat{j} + (x+2z)\hat{k}$

$$\text{curl } \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xz^2 & -3y & x+2z \end{vmatrix}$$

curl $\bar{a} = y\hat{i} - (1-6xz)\hat{j}$

$$curl(curl(\bar{a})) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & (1-6xz) & 0 \end{vmatrix}$$

$$= 6x\hat{i} - (-6z-1)\hat{j}$$

• $xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$

$$\phi = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$$

$$curl\ \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \bar{0}$$ irrotational

* $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$curl(\bar{a} \times \bar{r})$

$$\bar{a} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

$$curl(\bar{a} \times \bar{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2z - a_3y & a_1z - a_3x & a_1y - a_2x \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \bar{0}$$ ⇒ irrotational

* Max rate & value:

1) $\phi = \tan(x^2 + y^2)$

$$\nabla\phi = \frac{2x}{(x^2+y^2)^2+1}\hat{i} + \frac{2y\,\hat{j}}{1+(x^2+y^2)^2}$$ $x = y = \sqrt{\frac{\pi}{3}}$

$$= 2\left[\sqrt{\frac{\pi}{3}}\right]\left[\frac{\hat{i}+\hat{j}}{1+\frac{4\pi^2}{9}}\right] = \frac{2\sqrt{\frac{\pi}{3}}}{1+\frac{4\pi^2}{9}}[\hat{i}+\hat{j}]$$

$$|\nabla\phi| = \frac{2\sqrt{\frac{2\pi}{3}}}{1+4\pi^2/9}$$

2) $\sqrt{xy}\,e^z = \phi$ $P(4,4,1)$

$$\nabla\phi = \frac{e^z y\,\hat{i}}{2\sqrt{xy}} + \frac{e^z x\,\hat{j}}{2\sqrt{xy}} + \sqrt{xy}\,e^z\hat{k}$$

$$= \frac{4e}{8}\hat{i} + \frac{4e}{8}\hat{j} + 4e\hat{k}$$

$$= \frac{e}{2}\hat{i} + \frac{e}{2}\hat{j} + 4e\hat{k}$$

$$\text{rate} = e\left[\frac{\hat{i}}{2} + \frac{\hat{j}}{2} + 4\hat{k}\right]$$

$$\text{value} = e\sqrt{\frac{1}{4}+\frac{1}{4}+16} = e\sqrt{\frac{1}{2}+16} = e\sqrt{\frac{33}{2}}$$