01/04/2023 UNIT-2

FOURIER TRANSFORMS:

→ Fourier Integral Theorem (without proof)
→ Fourier Sine integral.
→ Fourier Cosine integral
→ Complex form of a Fourier integral.
→ Fourier transform.
→ Inverse Fourier Transform
→ Fourier Sine Transform.
→ Inverse of Fourier Sine Transform.
→ Fourier Cosine Transform.
→ Inverse of Fourier Cosine Transform.
→ Properties of Fourier Transform.
→ Problems related to above topics.

* INTRODUCTION:

A transformation is a mathematical device which transforms one fn to another fn.

→ A periodic function $f(x)$ defined in a finite interval $(-l, l)$ can be expressed in terms of Fourier series.

→ In many Engineering applications; it is required to expand an arbitrary function like impressed force, voltage etc in a series of sinusoidal functions.

→ By extending this concept, non-periodic function defined in the interval $-\infty < x < \infty$ ∀ all x can be expressed as a Fourier Integral; since all fns are not always periodic.

* ~~So, FOURIER TRANSFORM is to reduce the no. of independent variables by one.~~

→ Fourier transforms is an integral transform to reduce the no. of independent variables by one.

→ Fourier transforms are widely used to solve various boundary value problems of engineering such as vibrations of a string, conduction of heat, transverse oscillations of an electric beam, free and forced vibrations of a membrane etc

→ **Integral Transform:**

The integral transform of a function $f(x)$ denoted by

$I[f(x)]$ (or) $\bar{f}(s)$ is defined as

$$I[f(x)] = \bar{f}(s) = \int_{x_1}^{x_2} f(x)\, k(s, x)\, dx$$

where $k(s, x)$ is the fn of s and x is the "Kernel of the transform".

→ Depending upon the kernel we obtain the following various transforms.

(i) If $k(s, x) = e^{-sx}$, we obtain the Laplace transform of $f(x)$

i.e. $L[f(x)] = \bar{f}(s) = \int_0^\infty e^{-sx} f(x)\, dx$

(ii) If $k(s, x) = e^{isx}$, we obtain the Fourier transform of $f(x)$

i.e. $F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

(iii) If $k(s, x) = x^{s-1}$, we obtain the Mellin transform of $f(x)$

i.e. $M[f(x)] = M(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$

(iv) If $K(s,x) = x J_n(sx)$, which is the Bessel's fn, then we obtain the Henkel transform of $f(x)$.

i.e. $H[f(x)] = H(s) = \int_0^\infty f(x)\, x\, J_n(sx)\, dx$

In the same process by taking various kernels we obtain various transform.

→ Before going to define the Fourier transform, first we have to know about the Fourier integral by the Fourier Integral Theorem. Before proving this theorem we must see the well known Dirichlet's conditions which play an important role in study of Fourier transforms & covers engineering problems.

* DIRICHLET'S CONDITIONS:

A function $f(x)$ is said to satisfy Dirichlet's conditions in the interval (a,b) if

(i) $f(x)$ is defined and is single-valued function except possibly at a finite no. of points in the interval (a,b)

(ii) $f(x)$ & $f'(x)$ are piecewise continuous in (a,b).

* FOURIER INTEGRAL THEOREM:

1) A function $f(x)$ which is piecewise continuous in every finite interval and if absolutely integrable i.e $\int_{-\infty}^{\infty} |f(x)|\, dx$ be convergent; then the Fourier integral representation of $f(x)$ is,

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dt\, d\lambda}$$

Proof:

→ $f(x)$ be a fn satisfying the Dirichlet conditions in every $(-l,l)$ and defined as $f(x) = \frac{1}{2}[f(x_0+0) + f(x_0-0)]$ at every point of discontinuity

→ Fourier Series of $f(x)$ in $(-l,l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad -(1)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Substitute a_0, a_n, b_n in (1)

$$f(t) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} \cos\left(\frac{n\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) f(t)\,dt$$

$$+ \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} \sin\left(\frac{n\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) f(t)\,dt$$

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A \sin B}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} \cos\left(\frac{n\pi(t-x)}{l}\right) f(t)\,dt \quad -(2)$$

Let $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges i.e. has a finite value.

→ assume limits as $l \to \infty$ in (2)

first term

$$\lim_{l\to\infty}\left[\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right] = 0$$

$$\because \left|\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right| \le \frac{1}{2l}\int_{-l}^{l} |f(t)|\,dt$$

put $\frac{\pi}{l} = \delta\lambda$ in second term of (2)

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-l}^{l} \cos(n\,\delta\lambda(t-x))\, f(t)\,dt\right]\delta\lambda \quad -(3)$$

$l \to \infty$ (or) $\delta\lambda \to 0$ eq. (3) takes the form

$$\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} \cos(\lambda(t-x))\, f(t)\,dt\right]\delta\lambda$$

$$\because \lim_{\delta\lambda\to 0}\sum_{n=1}^{\infty} \phi(n\,\delta\lambda)\,\delta\lambda = \int_{0}^{\infty} \phi(\lambda)\,d\lambda$$

Thus as $l \to \infty$ eq (2) takes the form

$$\boxed{f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} \cos\lambda(t-x)\, f(t)\,d\lambda} \quad -(4)$$

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* FOURIER SINE & COSINE INTEGRALS

By Fourier Integral Theorem

We have $f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{t=-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\right]d\lambda$ — ①

Expand $\cos\lambda(t-x)$ from eqn ① we get

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\left[\cos\lambda t\cos\lambda x + \sin\lambda t\sin\lambda x\right]dt\right]d\lambda$$

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\infty}\cos\lambda x\left[\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right]d\lambda$$

$$+ \frac{1}{\pi}\int_{\lambda=0}^{\infty}\sin\lambda x\left[\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\right]d\lambda \quad \text{— ②}$$

case(i) If $f(t)$ is an odd fn; we get Fourier Sine Integral

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin\lambda x\left[\int_{0}^{\infty} f(t)\sin\lambda t\,dt\right]d\lambda$$

[$\because$ $f(t)$ is odd, $\cos\lambda t$ is even, $f(t)\cos\lambda t$ is odd. So 1st term ~~variables~~ vanishes from ②]

case(ii) If $f(t)$ is an even fn; we get Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos\lambda x\left[\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right]d\lambda$$

[$\because$ $f(t)$ is even, $\sin\lambda t$ is odd; $f(t)\sin\lambda t$ is odd. So 2nd term vanishes from ②]

*→ Using Fourier Integral; show that

$$\int_{0}^{\infty}\frac{1-\cos\pi\lambda}{\lambda}\sin\lambda x\,d\lambda = \begin{cases}\pi/2 & 0<x<\pi \\ 0 & x>\pi\end{cases}$$

Given integrand contains $\sin\lambda x$: Fourier Sine Integral formula

*→ Fourier Sine Integral formula:

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

consider $f(t) = \begin{cases} \pi/2 & \alpha\ t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left\{\left[\int_0^\pi \frac{\pi}{2}\sin\lambda t\,dt\right] + \left[\int_\pi^\infty 0\sin\lambda t\,dt\right]\right\}d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\frac{\pi}{2}\cdot\left[\frac{-\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= -\int_0^\infty \sin\lambda x\left[\frac{\cos\lambda\pi - 1}{\lambda}\right]d\lambda$$

$$= \int_0^\infty \sin\lambda x\left(\frac{1-\cos\pi\lambda}{\lambda}\right)d\lambda$$

$$\boxed{f(x) = \int_0^\infty\left(\frac{1-\cos\pi\lambda}{\lambda}\right)\sin\lambda x\cdot d\lambda}$$

*→ Using Fourier Integral

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x\,d\lambda}{\lambda^2+a^2}.$$

$$f(x) = e^{-ax}$$

$$f(t) = e^{-at}$$

∵ Integrand is having $\cos\lambda x$

*→ Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty f(t)\cos\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty f(t)\cos\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-at}\cos\lambda t\,dt\right]d\lambda$$

$$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\frac{e^{-at}}{a^2+\lambda^2}\left[-a\cos\lambda t + \lambda\sin\lambda t\right]\right]_0^\infty d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[0 + \frac{a}{a^2+\lambda^2}\right]d\lambda$$

$$\boxed{e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{a^2+\lambda^2}\,d\lambda}$$

$e^{-\infty} = 0$

* Using F.I.

P.T. $\displaystyle\int_0^\infty \frac{\sin\lambda\pi\,\sin\lambda x}{1-\lambda^2}$

$$= \begin{cases} \frac{\pi}{2}\sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

01/04/23

* FOURIER SINE & COSINE INTEGRALS

By Fourier Integral Theorem

We have $f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{t=-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\right]d\lambda$ — ①

Expand $\cos\lambda(t-x)$ from eqn ① we get

$$f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\left[\cos\lambda t\cos\lambda x + \sin\lambda t\sin\lambda x\right]dt\right]d\lambda$$

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

$$= \frac{1}{\pi}\int_{\lambda=0}^{\infty}\cos\lambda x\left[\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right]d\lambda$$

$$+ \frac{1}{\pi}\int_{\lambda=0}^{\infty}\sin\lambda x\left[\int_{-\infty}^{\infty} f(t)\sin\lambda t\,dt\right]d\lambda \quad \text{—②}$$

Case(i) If $f(t)$ is an odd fn; we get

Fourier Sine Integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

[∵ f(t) is odd,
cos λt is even
f(t) cos λt is odd.
So 1st term vanishes from ②]

case(ii)

If f(t) is an even fn; we get
Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_{-\infty}^{\infty} f(t)\cos\lambda t\,dt\right]d\lambda$$

[∵ f(t) is even,
sin λt is odd;
f(t) sin λt is odd
So 2nd term vanishes from ②]

*→ Using Fourier Integral; show that

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin\lambda x\,d\lambda = \begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

Given integrand contains sin λx : Fourier Sine Integral formula

→ Fourier Sine Integral formula:

$$\cancel{f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\infty f(t)\sin\lambda t\, dt\right] d\lambda}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\infty f(t)\sin\lambda t\, dt\right] d\lambda.$$

Consider $f(t) = \begin{cases} \pi/2 & \text{if } t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\left[\int_0^\pi \frac{\pi}{2}\sin\lambda t\, dt\right] + \left[\int_\pi^\infty 0 \sin\lambda t\, dt\right]\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \frac{\pi}{2} \cdot \left[\frac{-\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= -\int_0^\infty \sin\lambda x \left[\frac{\cos\lambda\pi - 1}{\lambda}\right] d\lambda$$

$$= \int_0^\infty \sin\lambda x \left(\frac{1 - \cos\pi\lambda}{\lambda}\right) d\lambda$$

$$\boxed{f(x) = \int_0^\infty \left(\frac{1-\cos\pi\lambda}{\lambda}\right) \sin\lambda x \cdot d\lambda}$$

→ Using Fourier Integral

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos\lambda x \, d\lambda}{\lambda^2+a^2}.$$

$$f(x) = e^{-ax}$$

$$f(t) = e^{-at}$$

→ ∵ Integrand is having $\cos\lambda x$

→ Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos\lambda x \left[\int_0^\infty f(t) \cancel{\cos}\lambda t \, dt\right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \cos\lambda x \left[\int_0^\infty f(t) \cos\lambda t \, dt\right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \cos\lambda x \left[\int_0^\infty e^{-at} \cos\lambda t \, dt\right] d\lambda$$

$$\boxed{\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

$$= \frac{2}{\pi} \int_0^\infty \cos\lambda x \left[\frac{e^{-at}}{a^2+\lambda^2}\left[-a\cos\lambda t + \lambda\sin\lambda t\right]\right]_0^\infty d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \cos\lambda x \left[0 + \frac{a}{a^2+\lambda^2} \right] d\lambda$$

$$\boxed{e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos\lambda x}{a^2+\lambda^2}\, d\lambda}$$

$e^{-\infty}$

* Using F.I.

P.T. $\displaystyle\int_0^\infty \frac{\sin\lambda\pi \sin\lambda x}{1-\lambda^2}$

$$f(x) = \begin{cases} \pi/2 \sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

$$f(t) = \begin{cases} \pi/2 \sin t & 0 \le t \le \pi \\ 0 & x > \pi \end{cases}$$

$\because$ Eq. contains $\sin\lambda x$

$\therefore$ It is Fourier Sine Integral:

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\int_0^\infty f(t) \sin\lambda t\, dt \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\int_0^\pi \frac{\pi}{2} \sin t \sin\lambda t\, dt + \int_\pi^\infty 0 \sin\lambda t\, dt \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\frac{\pi}{4} \left(\int_0^\pi 2\sin t \sin\lambda t \, dt \right) \right] d\lambda$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4} \int_0^\infty \sin\lambda x \left[\int_0^\pi \left\{ \cos(t-\lambda t) - \cos(t+\lambda t) \right\} dt \right] d\lambda$$

$$= \frac{1}{2} \int_0^\infty \sin\lambda x \left[\frac{\sin(t-\lambda t)}{1-\lambda} - \frac{\sin(t+\lambda t)}{1+\lambda} \right]_0^\pi d\lambda$$

$$= \frac{1}{2} \int_0^\infty \sin\lambda x \left[\frac{\sin(1-\lambda)\pi}{1-\lambda} - \frac{\sin(1+\lambda)\pi}{1+\lambda} \right] d\lambda$$

$$= \frac{1}{2} \int_0^\infty \sin\lambda x \left[\frac{\sin\lambda\pi}{1-\lambda} + \frac{\sin\lambda\pi}{1+\lambda} \right] d\lambda$$

$$= \frac{1}{2} \int_0^\infty \frac{\sin\lambda x \sin\lambda\pi}{1-\lambda^2} \left(1+\cancel{\lambda}+1-\cancel{\lambda} \right) d\lambda$$

$$\boxed{= \int_0^\infty \sin\lambda x \sin\lambda\pi \, d\lambda}$$

03/04/2023

* Complex form of Fourier ~~Integral~~ Series:

→ We know that Fourier Integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(t)\cos(t-x)\lambda\, dt\right] d\lambda$$

∵ $\cos\lambda(t-x)$ is an even fn of λ, the above eqn can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\right] d\lambda \quad -①$$

Also ∵ $\sin\lambda(t-x)$ is an odd fn of λ we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cdot\sin\lambda(t-x)\,dt\,d\lambda \quad -②$$

Now multiplying ② by i & add eq ①

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(t)\left[\cos\lambda(t-x) + i\sin\lambda(t-x)\right] dt\right] d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\right] d\lambda$$

* Fourier Transform of f(x):

→ Writing the exponential function $e^{i\lambda(t-x)}$ as a product of 2 exponential fn

$e^{i\lambda(t-x)} = e^{i\lambda t}, e^{-i\lambda x}$ the Fourier integral of $f(x)$ in complex form may be in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\left[\int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt\right] d\lambda \quad \text{— (1)}$$

→ Hence the Fourier transform from eq: (1) is of the form:

$$\boxed{F[f(x)] = \int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt}$$

→ Inverse Fourier Transform:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f(x)]\, e^{-i\lambda x}\, d\lambda}$$

Fourier Sine Transform:

We know that Fourier sine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[\int_0^\infty \sin \lambda t \, f(t) \, dt \right] d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[\int_0^\infty \sin \lambda t \, f(t) \, dt \right] d\lambda \quad -(1)$$

From eq: (1) ⇒ Fourier Sine Transform

$$F_S[f(x)] = \int_0^\infty f(t) \sin \lambda t \, dt \quad -(2)$$

Sub (2) in (1) Inverse Sine Transform

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty F_S[f(x)] \sin \lambda x \, d\lambda}$$

Fourier Cosine Transform:

We know that Fourier Cosine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[\int_0^\infty \cos \lambda t \, f(t) \, dt \right] d\lambda \quad -(1)$$

from eq: (1)

Fourier Cosine Transform

$$F_c\left[f(x)\right] = \int_0^\infty f(t)\cos\lambda t\, dt \quad -②$$

Sub ② in ① Inverse Cosine Transform:

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_c\left[f(x)\right]\cos\lambda x\, d\lambda}$$

* Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$$

then evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$

$$f(x) = \begin{cases} 1 & \text{for } |x|<1 \text{ i.e. } -1<x<1 \\ 0 & \text{for } |x|>1 \text{ i.e. } 1<x<8 \end{cases}$$

Fourier Transform of $f(x)$

$$\boxed{F\left[f(x)\right] = F(S) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$= \int_{-1}^{1} 1\cdot e^{isx}\, dx$$

$$\boxed{e^{i\theta} - e^{-i\theta} = 2\sin\theta\, i}$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{1}{is}\left(e^{is} - e^{-is}\right) \qquad \underline{2\sin s\, i}$$

$$= \frac{+2\sin s\, \cancel{i}}{\cancel{i}s} = \frac{2\sin s}{s}$$

Inverse Fourier Transform:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, dx}$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} F(s)\, e^{-isx}\, ds$$

$$f(x) = \cancel{2}\,\frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin s}{s}\, e^{-isx}\, ds$$

$x=0$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin s}{s}\, ds$$

$\because$ even fn. $\frac{-\sin s}{-s} = \frac{\sin s}{s}.$

$\cancel{\frac{\pi}{2} f(0) = \int}$

$f(0) = 1$

$$\boxed{\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin s}{s}\, dx}$$

* Find the fourier transform of

$$f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

hence evaluate $\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right)$

06/04/2023

* Fourier Transform of $f(x)$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(s) = \int_{-1}^{1} e^{isx}(1-x^2)\, dx$$

$$= \int_{-1}^{1} e^{isx}\, dx - \int_{-1}^{1} x^2 e^{isx}\, dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} - \left[x^2 \frac{e^{isx}}{is} - \int 2x \frac{e^{isx}}{is}\, dx\right]_{-1}^{1}$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[\frac{x^2 e^{isx}}{is} - 2\left(x \frac{e^{isx}}{(is)^2} - \frac{e^{isx}}{(is)^3}\right)\right]_{-1}^{1}$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[\frac{x^2 e^{isx}}{is} - \frac{2x\, e^{isx}}{(is)^2} + \frac{2e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[\left(\frac{e^{is}}{is} - \frac{2e^{is}}{(is)^2} + \frac{2e^{is}}{(is)^3}\right) - \left(\frac{e^{-is}}{is} + \frac{2e^{-is}}{(is)^2} + \frac{2e^{-is}}{(is)^3}\right)\right]$$

$$= \frac{2i\sin s}{is} - \left[\frac{2(\sin s)i}{is} \, \frac{2}{i}\left(\frac{e^{is} + e^{-is}}{(is)^2} \right) + \frac{2(2\sin s)i)}{(is)^3} \right]$$

$$= \frac{2\sin s}{s} - \frac{2\sin s}{s} - \frac{2(2\cos s)}{(is)^2} + \frac{4\sin s}{i^2 s^3}$$

$$= \frac{-4\cos s}{i^2 s^2} + \frac{4\sin s}{i^2 s^3}$$

$$= \frac{4\cos s}{s^2} - \frac{4\sin s}{s^3}$$

$$= 4\left[\frac{s\cos s - \sin s}{s^3} \right]$$

Inverse Fourier Transform :

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$= \frac{4}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[\frac{s\cos s - \sin s}{s^3} \right]$$

$$\cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}}$$

$$x = \frac{1}{2}$$

$$f(x) = \frac{4}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left[\frac{s\cos s - \sin s}{s^3}\right]$$

$$1 - \frac{1}{4} = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{is/2}\left[\frac{s\cos s - \sin s}{s^3}\right] ds$$

$$\frac{1}{2} \times \frac{3}{4} \times \frac{\pi}{2} = \int_{0}^{\infty}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)\left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

$$\boxed{-\frac{3\pi}{16} = \int_{0}^{\infty} \cos\frac{s}{2}\left(\frac{s\cos s - \sin s}{s^3}\right) ds}$$

* Find the Fourier Transform of $e^{-|x|}$ and hence evaluate $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

$$e^{-|x|} = \begin{cases} e^{x} & x < 0 \\ e^{-x} & x > 0 \end{cases}$$

$$F.T = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{isx} f(x)\, dx + \int_{0}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{isx} e^{x}\, dx + \int_{0}^{\infty} e^{isx} e^{-x}\, dx$$

$$= \int_{-\infty}^{0} e^{(is+1)x}\, dx + \int_{0}^{\infty} e^{x(is-1)}\, dx$$

$$= \left[\frac{e^{(is+1)x}}{(1+is)}\right]_{-\infty}^{0} + \left[\frac{e^{x(is-1)}}{(is-1)}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{1+is} - 0\right] + \left[0 + \frac{1}{1-is}\right]$$

$$= \frac{1-is+1+is}{1-i^2s^2} = \underline{\underline{\frac{2}{1+s^2}}}$$

Inverse Fourier Transform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$= \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-isx}}{1+s^2}\, ds$$

$x = 0$

$$\frac{1}{2}\left[e^{x} + e^{-x}\right] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-isx}}{1+s^2}\, ds$$

$$\frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{\cancel{e^{is}}\,1}{1+s^2}$$

* Show that the Fourier Transform of $e^{-x^2/2}$ is $\sqrt{2\pi}\, e^{-s^2/2}$ by finding the F.T. of $e^{-a^2x^2}$

08/04/2023

Consider $f(x) = e^{-a^2x^2}$

Fourier Transform

$$= \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = \int_{-\infty}^{\infty} e^{isx} e^{-a^2x^2} dx$$

$$= \int_{-\infty}^{\infty} e^{(is-a^2x)x} dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$a^2x^2 - isx$
$= (ax)^2 - 2ax\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$
$= \left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}$

$$= \int_{-\infty}^{\infty} e^{\left(\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right)}$$

$$= \left(\int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx\right) \cdot e^{-s^2/4a^2}$$

$ax - \frac{is}{2a} = t$
$a\,dx = dt$
$dx = \frac{dt}{a}$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a} e^{-s^2/4a^2} \sqrt{\pi}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}}$$

$$F(s) = \frac{e^{-s^2/4a^2}\sqrt{\pi}}{a}$$

$$F\left[e^{-a^2x^2}\right] = e^{-s^2/4a^2} \frac{\sqrt{\pi}}{a}$$

$$a = 1/\sqrt{2}$$

$$\boxed{F\left[e^{-x^2/2}\right] = e^{-s^2/2}\sqrt{2\pi}}$$

* Find the F.T $f(x) = \begin{cases} e^{-x^2/2} & -\infty < x < \infty \end{cases}$

(OR) S.T. the F.T. of $e^{-x^2/2}$ is reciprocal.

$$F\left[e^{-x^2/2}\right] = \sqrt{2\pi}\, e^{-s^2/2}$$

$$\text{F.T of } e^{-x^2/2} = \int_{-\infty}^{\infty} e^{isx} e^{-x^2/2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx\right]} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2xis + (is)^2 - (is)^2\right]} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[x-is]^2} \cdot e^{+s^2/2}\, dx$$

$$= e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2} dx \qquad \frac{x-is}{\sqrt{2}} = t$$

$$\frac{dx}{\sqrt{2}} = dt$$

$$= \sqrt{2}\, e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt = \underline{\underline{\sqrt{2\pi}\, e^{-s^2/2}}}$$

$*$ S.T. F.T. of $f(x) = \begin{cases} a - |x| & |x| < a \\ 0 & |x| > a \end{cases}$

is $\frac{1-\cos as}{s^2}$. Hence evaluate

$$\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt.$$

$\therefore$ Fourier Transform of $f(x)$

$$f(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \begin{cases} a + x & -a < x < 0 \\ a - x & 0 < x < a. \end{cases}$$

$$F(s) = \int_{-a}^{0} e^{isx}(a+x)\, dx + \int_{0}^{a} e^{isx}(a-x)\, dx.$$

$$= \left[(a+x)\frac{e^{isx}}{is} - (1)\frac{e^{isx}}{(is)^2} \right]_{-a}^{0}$$

$$+ \left[(a-x)\frac{e^{isx}}{is} + (1)\frac{e^{isx}}{(is)^2} \right]_{0}^{a}$$

$$= \left[\left(\cancel{\frac{a(1)}{is}} + \frac{1}{s} \right) - \left(0 + \frac{e^{is(-a)}}{s^2} \right) \right]$$

$$+ \left[\left(0 - \frac{e^{isa}}{s^2} \right) - \left(\cancel{\frac{a(1)}{is}} - \frac{1}{s^2} \right) \right]$$

$$F(s) = \frac{2}{s^2} - \frac{1}{s^2}\left[e^{isa} + e^{-isa} \right]$$

$$= \frac{2}{s^2} - \frac{1}{s^2}\left[2\cos(sa) \right]$$

$$= \frac{2 - 2\cos(sa)}{s^2}$$

Inverse Fourier Transform of $f(x)$

* Find the F.T of $e^{-a|x|}$ $(a>0)$

hence S.T $\int_0^\infty \frac{\cos sx}{a^2+s^2} = \frac{\pi}{2a} e^{-a|x|}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$a = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \frac{2(1-\cos(sa))}{s^2}\, ds.$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-isx} \frac{2\sin^2\left(\frac{sa}{2}\right)}{s^2}\, ds.$$

$x = 0 ; a = 2$

$$\cancel{2} = \frac{\cancel{2}}{\pi}\int_{0}^{\infty} \frac{2\sin^2(s)}{s^2}\, ds.$$

$$\boxed{\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}$$

*Sine and Cosine Transforms:

*» Find the Fourier sine and cosine transform of e^{-ax} $(a>0)$ & deduce the integral

(i) $\displaystyle\int_0^{\infty} \frac{s \sin sx}{s^2+a^2}\, ds$ (ii) $\displaystyle\int_0^{\infty} \frac{\cos sx}{s^2+a^2}\, ds.$

*Fourier Sine Transform.

$$f_s(s) = \int_0^\infty f(x) \sin sx \, dx.$$

$$= \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)}$$

$$= \left[\frac{e^{-ax}}{a^2+s^2} \left(-a \sin sx - s \cos sx \right) \right]_0^\infty.$$

$$= \frac{e^{-\infty}}{a^2+s^2} \, \underset{x\to\infty}{Lt} \left(-a \sin sx - s \cos x \right)$$

$$- \frac{e^{0}}{s^2+a^2} \left[-a \sin s(0) - s \cos(0) \right]$$

$$= \frac{-1}{s^2+a^2} \left(s \right)$$

$$f_s(s) = \frac{s}{s^2+a^2}$$

→ Inverse Fourier Sine Transform.

$$\cancel{f(x) = \int} \quad f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+a^2}\sin sx\, ds$$

$$\frac{\pi e^{-ax}}{2} = \int_0^\infty \frac{s\sin sx}{s^2+a^2}\, ds$$

* Fourier Cosine Transform.

$$f_c(s) = \int_0^\infty f(x)\cos sx\, dx$$

$$= \int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-a)$$

$$\therefore F_c(s) = \frac{a}{s^2+a^2}$$

Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\, ds.$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds$$

$$\boxed{\frac{\pi e^{-ax}}{2a} = \int_0^\infty \frac{\cos sx}{a^2+s^2}\, ds}$$

* Find the fourier sine & cosine transform of $\frac{e^{-ax}}{x}$ and deduce that $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} dx = \tan^{-1}\left(\frac{s}{a}\right) + \tan^{-1}\left(\frac{s}{\ldots}\right)$

* Fourier Sine Transform :

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \quad -(1)$$

$$F_s(s) = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

diff. wrt 's' on both sides.

$$\frac{d}{ds} F_s(s) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx \cdot \cancel{x} \, dx$$

$$\frac{d}{ds} F_s(s) = \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{x^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+x^2}(-a) = \frac{a}{s^2+x^2}$$

$$F_s(s) = \int \frac{a}{s^2+x^2} \, ds$$

$$\boxed{F_s(s) = \tan^{-1}\left(\frac{s}{a}\right)}$$

→ ~~Inverse Fourier Sine Transform~~ :

$$\int_0^\infty \frac{e^{-bx}}{x} \sin sx \, dx = \tan^{-1}\left(\frac{s}{b}\right)$$

$$\therefore \int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right) \sin sx \, dx = \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$$

* Fourier Cosine Transform:

$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx \quad -(1)$$

diff wrt s. on both sides

$$\frac{d}{ds} F_c(s) = \int_0^\infty \frac{e^{-ax}}{x} -\sin sx \cdot x \, dx$$

$$\frac{d}{ds} F_c(x) = -\int_0^\infty e^{-ax} \sin sx \, dx$$

$$= -\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\sin sx - s\cos sx\right)\right]_0^\infty$$

$$= \left[\frac{e^{-ax}}{s^2+a^2}\left(a\sin sx + s\cos sx\right)\right]_0^\infty$$

$$\frac{d}{ds} F_c(x) = 0 - \frac{s}{s^2+a^2}$$

$$\frac{d}{ds} F_c(x) = \frac{-s}{s^2+a^2}$$

$$F_c(x) = \frac{-1}{2} \log (a^2 + s^2)$$

$$\int_0^\infty \cos sx \frac{e^{-bx}}{x} = \frac{-1}{2} \log (b^2+s^2)$$

$$\Rightarrow \int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) dx \cos sx = \frac{1}{2} \log \left(\frac{b^2+s^2}{a^2+s^2}\right)$$

$$\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) dx \cdot \cos sx = \frac{1}{2} \left(\log \left(\frac{b^2+s^2}{a^2+s^2}\right)\right)$$

* Find the Fourier sine transform of

$$\frac{x}{x^2+a^2} \qquad f_s(s) = \int_0^\infty \frac{x}{x^2+a^2} \sin sx \, dx$$

$$F_s(e^{-ax}) = \frac{s}{s^2+a^2}$$

then inverse ~~sine~~ fourier sine transform of e^{-ax} is

$$f(x) = \frac{2}{\pi} \int_0^\infty f_s(s) \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\frac{\pi e^{-ax}}{2} = \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, dx.$$

$$x = s \qquad s = x$$

$$\boxed{\frac{\pi e^{-as}}{2} = \int_0^\infty \frac{x}{x^2+a^2} \sin sx \, dx.}$$

* Find the Fourier Cosine Transform of $\frac{1}{a^2+x^2}$

$$f_c(e^{-ax}) = \frac{a}{s^2+a^2}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty f_c(s) \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2} ds$$

$$\frac{\pi e^{-ax}}{2a} = \int_0^\infty \frac{\cos sx}{s^2+a^2} ds$$

$$\boxed{\frac{\pi e^{-as}}{2a} = \int_0^\infty \frac{\cos sx}{x^2+a^2} dx.}$$

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* Find Fourier Sine Transform of $e^{-|x|}$, hence evaluate $\int_0^\infty \frac{x \sin sx}{1+x^2} dx$

$$F_s[f(x)] = F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

$$F_s[e^{-|x|}] = \int_0^\infty e^{-|x|} \sin sx \, dx$$

$|x| = x ; x > 0$

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

$a = -1 ; b = s$

$$F_s[e^{-|x|}] = \cancel{\int_0^\infty e^{-|x|}} \left[\frac{e^{-x}}{1+s^2} \{ -\sin sx - s \cos sx \} \right]_0^\infty$$

$$F_s[e^{-|x|}] = \left[0 + \frac{s}{s^2+1} \right]$$

$$\boxed{F_s[e^{-|x|}] = \frac{s}{s^2+1}}$$

Inverse Fourier Sine Transform.

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx \, F_s(s) \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx \cdot \frac{s}{s^2+1} ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+1} ds$$

$$\frac{\pi e^{-x}}{2} = \int_0^\infty \frac{s \sin sx}{s^2+1} ds$$

$x = s$ $\quad$ $s = x$

$$\boxed{\int_0^\infty \frac{x \sin sx}{x^2+1} dx = \frac{\pi e^{-s}}{2}}$$

* $F_s(e^{-ax}) = \dfrac{s}{s^2+a^2}$ $\quad$ $f\left(\dfrac{x}{x^2+a^2}\right) = \dfrac{\pi}{2} e^{-as}$

$F_c(e^{-ax}) = \dfrac{a}{s^2+a^2}$ $\quad$ $f\left(\dfrac{1}{x^2+a^2}\right) = \dfrac{\pi}{2a} e^{-as}$

* Properties of Fourier Transform:

1) Linearity property:

If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ & $g(x)$ respectively then

$$F[a f(x) \pm b g(x)] = a F[f(x)] \pm b F[g(x)]$$

$= a F(s) \pm b G(s)$

* Given that $F(s)$ & $G(s)$ are Fourier Transforms of $f(x)$ & $g(x)$ resp.

$$F(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{isx} dx \quad \text{--- (1)}$$

$$G(s) = F(g(x)) = \int_{-\infty}^{\infty} g(x) e^{isx} dx \quad \text{--- (2)}$$

LHS:

$F[a f(x) \pm b g(x)]$

(∵ by defn of F.T)

$$= \int_{-\infty}^{\infty} [a f(x) \pm b g(x)] e^{isx} dx$$

$$= \int_{-\infty}^{\infty} a f(x) e^{isx} dx \pm \int_{-\infty}^{\infty} b g(x) e^{isx} dx$$

$$= a \int_{-\infty}^{\infty} f(x) e^{isx} dx \pm b \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

by (1) & (2)

$= a F(s) \pm b G(s)$

~~2) Change of Scale Property:~~

* Note:

(i) $F_s[a f(x) \pm b g(x)] = a F_s(s) \pm b G_s(s)$

(ii) $F_c[a f(x) \pm b g(x)] = a F_c(s) \pm b G_c(s)$

2) Change of Scale Property:

If $F(s)$ is the Fourier transform of $f(x)$ then

(i) $F(f(ax)) = \frac{1}{a} F(s/a)$

(ii) $F(f(x/a)) = a F(as)$

* Given $F(s)$ is the Fourier transform of $f(x)$ i.e.

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$F(f(ax)) = \int_{-\infty}^{\infty} e^{isx} f(ax) dx$$

$ax = t$

$x = t/a$

$dx = dt/a$

$$= \int_{-\infty}^{\infty} e^{is(t/a)} f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t)\, dt$$

$= \frac{1}{a} F(s/a)$

$|||^{ly}$ we can prove $F(f(x/a)) = aF(as)$

<u>Note:</u> a) $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

b) $F_s(f(x/a)) = a F_s(as)$

$\ast\rightarrow$ $F_c(f(ax)) = \frac{1}{a} F_c(s/a)$

$F_c(f(x/a)) = a F_c(as)$

$$= \frac{1}{a} F(s/a)$$

IIIly we can prove $F(f(x/a)) = aF(as)$

<u>Note:</u> a) $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

b) $F_s(f(x/a)) = a F_s(as)$

$\ast \rightarrow$ $F_c(f(ax)) = \frac{1}{a} F_c(s/a)$

$F_c(f(x/a)) = a F_c(as)$

$$F[f(x/a)] = \int_{-\infty}^{\infty} f(x/a)\, e^{isx}\, dx$$

$$= a\int_{-\infty}^{\infty} f(t)\, e^{isat}\, dt \qquad \frac{x}{a} = t,\quad x = at,\quad dx = a\,dt$$

$$= a\int_{-\infty}^{\infty} f(t)\, e^{i(sa)t}\, dt$$

$$= aF(as)$$

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3) <u>Shifting property</u>:

If $F(s)$ is the Fourier Transform of $f(x)$

then (i) $F[f(x-a)] = e^{ias}F(s)$

(ii) $F[f(x+a)] = e^{-ias}F(s)$

* <u>Proof:</u>

$\rightarrow$ <u>Defn of F.T.</u>:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s) \quad \text{— (1)}$$

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx \qquad x - a = t,\quad x = t + a,\quad \therefore dx = dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= e^{isa}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias}F(s)$$

$$F[f(x+a)] = \int_{-\infty}^{\infty} f(x+a)\, e^{isx}\, dx \qquad x + a = t,\quad x = t - a$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{is(t-a)}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{ist}\cdot e^{-ias}\, dt$$

$$= e^{-as}F(s)$$

4) Modulation Property: 20/04/2023

If $F(s)$ is the F.T. of $f(x)$

then $F[f(x)\cos ax]$

$$= \frac{1}{2}\left[f(s-a) + f(s+a)\right]$$

→ By defn of F.T. of $f(x)$

i.e. $F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F[s]$ — (1)

LHS: $F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$

$$= \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx}\, dx$$

$$\left(\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x) e^{i(s+a)x}\, dx + \int_{-\infty}^{\infty} f(x) e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right] \quad (\because \text{by } 1)$$

$= RHS$; Hence the modulation property.

Note:

(i) $F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$

(ii) $F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(s+a) + F_s(s-a)\right]$

(iii) $F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$

(iv) $F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$.

5) Show that $\boxed{F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]}$

→ We know that,

Fourier Transform of $f(x)$

RHS: $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx = F(s)$

Consider $F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$ — (1)

diff (1) w.r.t 's' on both sides.

$$\frac{d}{ds} F(s) = \int_{-\infty}^{\infty} f(x) \frac{d}{ds}\left(e^{isx}\right) dx$$

$$\frac{d}{ds} F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, i x\, dx$$

$$= i \int_{-\infty}^{\infty} f(x) e^{isx}\, x\, dx$$

$$\frac{d}{ds} F[s] = i \int_{-\infty}^{\infty} [x f(x)]\, e^{isx}\, dx \quad -(2)$$

Again diff (2) wrt 's' on both sides

$$\frac{d^2}{ds^2} F[s] = (i)^2 \int_{-\infty}^{\infty} [x^2 f(x)\, e^{isx}\, dx$$

→ Continuing diff, we get the general form:

$$\frac{d^n}{ds^n} F(s) = (i)^n \int_{-\infty}^{\infty} [x^n f(x)]\, e^{isx}\, dx$$

$$\int_{-\infty}^{\infty} [x^n f(x)]\, e^{isx}\, dx = \frac{1}{(i)^n} \frac{d^n}{ds^n} F[s]$$

$$\boxed{F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]}$$

6) Show that $\boxed{F_c[x f(x)] = \frac{d}{ds} F_s[s]}$

RHS: F.S.T. of $f(x)$ is given by

$$F_s[f(x)] = \int_0^{\infty} f(x) \sin sx\, dx = F_s[s]$$

consider $F_s[s] = \int_0^{\infty} f(x) \sin sx\, dx \quad -(1)$

diff (1) wrt 's' on both sides

$$\frac{d}{ds} F_s(s) = \int_0^{\infty} f(x) \cos sx\, (x)\, dx$$

$$= \int_0^{\infty} (x f(x)) \cos sx\, dx$$

$$\boxed{\frac{d}{ds} F_s(s) = F_c(x f(x))}$$

7) Show that $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$

* Find the Fourier sine Transform of $\frac{1}{x}$

→ Fourier sine transform of $\frac{1}{x}$.

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty \frac{1}{x} \sin sx \, dx \quad -(1)$$

Consider $\int_0^\infty e^{-ax} \sin sx \, dx = \frac{s}{s^2+a^2} \quad -(2)$

Integrate (2) wrt 'a' taking a_1 to a_2 limits on both sides.

$$\int_0^\infty \left[\int_{a_1}^{a_2} e^{-ax} dx\right] \sin sx \, dx = \int_{a_1}^{a_2} \frac{s}{s^2+a^2} da.$$

$$\int_0^\infty \left[\frac{e^{-ax}}{-x}\right]_{a_1}^{a_2} \sin sx \, dx = \left[\tan^{-1}\left(\frac{a}{s}\right)\right]_{a_1}^{a_2}$$

$$\int_0^\infty \left[e^{-a_1 x} - e^{-a_2 x}\right] \frac{\sin sx}{x} dx = \tan^{-1}\frac{a_2}{s} - \tan^{-1}\frac{a_1}{s}$$

put $a_1 \to 0$; $a_2 \to \infty$ on both sides

$$\int_0^\infty [1-0] \frac{\sin sx}{x} dx = \tan^{-1}\infty - \tan^{-1} 0$$

$$= \frac{\pi}{2}$$

$$\boxed{\int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}}$$

* Find the F.C.T. of $\frac{1}{1+x^2}$ & hence find F.S.T. of $\frac{x}{1+x^2}$.

→ Fourier cosine transform.

$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

$$F_c\left[\frac{1}{1+x^2}\right] = \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx \quad -(1)$$

$$I = \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx \quad -(1)$$

diff. wrt 's' on both sides

$$\frac{dI}{ds} = \int_0^\infty \frac{1}{1+x^2} - \sin sx \, (x) \, dx$$

$$= -\int_0^\infty \frac{x}{1+x^2} \sin sx \, dx$$

$$\frac{dI}{ds} = -\int_0^\infty \frac{x}{1+x^2} \sin sx \, dx \quad \text{(Mul. Nr \& Dr by } x\text{)}$$

$$\frac{dI}{ds} = -\int_0^\infty \frac{x^2}{x(1+x^2)} \sin sx \, dx.$$

$$= -\int_0^\infty \left[\frac{(x^2+1)-1}{x(1+x^2)}\right] \sin sx \, dx$$

$$= -\left[\int_0^\infty \frac{\sin sx}{x} dx - \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx\right]$$

$$= -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \quad \text{---} (2)$$

diff. (2) wrt 's'

$$\frac{d^2I}{ds^2} = 0 + \int_0^\infty \frac{\cos sx \cdot x}{x(1+x^2)} dx$$

$$= \int_0^\infty \frac{\cos sx}{1+x^2} dx$$

$$\frac{d^2I}{ds^2} = I \quad (\text{from } (1))$$

$$\frac{d^2I}{ds^2} - I = 0 \qquad O.E \Rightarrow (D^2-1)I = 0$$

$$A.E \quad m^2 - 1 = 0$$

$$m = \pm 1$$

roots are real & distinct

$$\therefore I = C_1 e^s + C_2 e^{-s} \quad \text{---} (3)$$

$$\frac{dI}{ds} = C_1 e^s - C_2 e^{-s} \quad \text{---} (4)$$

put $s = 0$ in (1) & (3)

$$\text{from } (1) \quad = \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx = \int_0^\infty \frac{dx}{1+x^2}$$

$$= (\tan^{-1} x)_0^\infty = \pi/2$$

$$\boxed{I = \pi/2}$$

from (3)

$$I = C_1 + C_2 = \pi/2 \quad \text{---} (*_1)$$

put $s = 0$ in (2) & (4)

$$\text{from } (2) \quad \frac{dI}{ds} = -\pi/2$$

$$\text{from } (4) \quad \frac{dI}{ds} = C_1 - C_2 = -\pi/2 \quad \text{---} (*_2)$$

Solve $(*_1)$ & $(*_2)$

$$C_1 = 0; \; C_2 = \pi/2$$

Sub C_1, C_2 in (3)

$$I = \pi/2 \, e^{-s}$$

$$\therefore \int_0^\infty \frac{1}{1+x^2} \cos sx = \frac{\pi}{2} e^{-s}$$

$$\frac{dI}{ds} = \int_0^\infty \frac{x \sin sx}{1+x^2}\, dx = \frac{\pi}{2} e^{-s}.$$

* Find $f(x)$ if its Fourier Sine Transform is $\frac{s}{1+s^2}$.

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \sin sx\, ds.$$ (We have find $f(x)$ by Inverse F.S.T).

Multiply Nr & Dr by s.

$$= \frac{2}{\pi}\int_0^\infty \frac{s^2}{s(s^2+1)} \sin sx\, ds$$

$$\cancel{= \frac{2}{\pi}\int_0^\infty (1+s^2)}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{((1+s^2)-1)}{s(s^2+1)} . \sin sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}\, ds - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds$$

$$= 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds$$

$$f(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds \quad — (1)$$

diff w.r.to 'x' on both sides

$$\frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos sx}{\cancel{s}(1+s^2)} \cdot \cancel{s}\, ds \quad — (2)$$

$$\frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\, ds \quad — (2)$$

diff. wrt x.

$$\frac{d^2f}{dx^2} = +\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{(1+s^2)}\, s\, ds$$

$$\frac{d^2f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} \sin sx\, ds$$

$$\frac{d^2f}{dx^2} = f$$

$$f = Ae^{x} + Be^{-x} \quad — (3)$$

$$\frac{df}{dx} = Ae^{x} - Be^{-x} \quad — (4)$$

from (1) put $x = 0$; $f = 1$

from (3) put $x = 0$; $f = A + B = 1$ — (*1)

from (2) put $x = 0$

$$\frac{df}{dx} = -\frac{2}{\pi} \cdot \frac{\pi}{2} = -1$$

from (4) put $x = 0$

$A - B = -1$ — (*2)

$A = 0$; $B = 1$

$$\boxed{f(x) = e^{-x}}$$