23/01/2023

UNIT-3

THE PRINCIPLE OF INCLUSION - EXCLUSION

Recall the principles

① $|A \cup B| = |A| + |B| - |A \cap B|$

② $|\bar{A}| = S - |A|$ where S is the universal set.

③ $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$ and $|\overline{A \cup B}| = S - |A \cup B|$

$$\boxed{|\overline{A \cup B}| = S - |A| - |B| + |A \cap B|} \text{ — } (*)$$

(*) is called the principle of inclusion - exclusion for 2 sets.

→ **Principle of Inclusion - Exclusion for n sets:**

Let 'S' be a finite set and $A_1, A_2, \ldots A_n$ be subset of S then the principle of inclusion - exclusion for $A_1, A_2, \ldots, A_n$ states that

$$\boxed{|\overline{A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n}| = S - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_i \cap A_j \cap \cdots \cap A_n|.}$$

— (*2)

(*2) is called the principle of inclusion;

exclusion for n sets.

*Generalisation: If $S_0 = |S|$, $S_i = \sum |A_i|$,

$S_2 = \sum |A_i \cap A_j|$; $S_3 = \sum |A_i \cap A_j \cap A_k|$ ....

$S_n = \sum |A_i \cap A_j \cap \cdots \cap A_n|$. The principle of inclusion – exclusion as given by expression

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = \bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n.$$

Note:

① The no. of elements in S that satisfy exactly m of the n conditions ($0 \le m \le n$) is given by $E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2}$

$$\mp \cdots (-1)^{n-m} \binom{n}{m-m}$$

② The no. of elements in S that satisfy atleast 'm' of the n conditions ($1 \le m \le n$) is given by.

$$l_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \mp \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

PROBLEMS:

1) Out of 30 students in a hostel; 15 study history, 8 study economics and 6 study geography. It is known that 3 students study all these subjects. Show that 7 (or) more students study none of these subjects.

Let S = set of all students in the hostel = 30

A_1 = set of students who study history = 15

A_2 = set of students who study economics = 8

A_3 = set of students who study geography = 6

$S_1 = \sum |A_i| = A_1 + A_2 + A_3 = 15 + 8 + 6 = 29$

$S_3 = |A_1 \cap A_2 \cap A_3| = 3.$

The no. of students who do not study any of three subjects is $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}|$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| \text{ (or) } |\overline{A_1 \cup A_2 \cup A_3}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k|$$

$$= |S| - S_1 + S_2 - S_3 = 30 - 29 + S_2 - 3 = S_2 - 2$$

where $S_2 = \sum |A_i \cap A_j|$

We know that $(A_1 \cap A_2 \cap A_3)$ is a subset of

$(A_i \cap A_j)$ for $i \neq j = 1, 2, 3$.

$\because$ Each of $|A_i \cap A_j|$ which are 3 in number is greater than (or) equal to $|A_1 \cap A_2 \cap A_3|$

$$S_2 = \sum |A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9$$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| \geq 9 - 2 = 7.$$

② How many integers b/w 1 & 300 (inclusive) are?

(i) divisible by atleast one of 5, 6, 8?

(ii) divisible by none of 5, 6, 8?

Sol: Let $S = \{1, 2, \dots, 300\}$

so that $|S| = 300$

Let A_1, A_2, A_3 be subset of whose elements are divisible by 5, 6, 8 resp.

(i) the no. of elements of S that are divisible by atleast one of 5, 6, 8 is $|A_1 \cup A_2 \cup A_3|$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

We know that

$|A_1| = 60$; $|A_2| = 50$; $|A_3| = 37$

$|A_1 \cap A_2| = 10$; $|A_1 \cap A_3| = 7$; $|A_2 \cap A_3| = 12$;

$|A_1 \cap A_2 \cap A_3| = 2$

$$|A_1 \cup A_2 \cup A_3| = (60 + 50 + 37) - (10 + 7 + 12) + 2 = 120.$$

Thus 120 elements of S are divisible by atleast 5, 6, 8.

(ii) The no. of elements of S that are divisible by none of 5, 6, 8 $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - |A_1 \cup A_2 \cup A_3|$

$$= 300 - 120 = 18.$$

3) In how many ways 5 number of a's, 4 no of b's and 3 no. of c's can be arranged so that all the identical letters are not in a single block?

The given letters are $5 + 4 + 3 = 12$.

Out of 12 letters 5 are a's; 4 are b's & 3 are c's. If S is the set of all arrangements (permutations) of these letters. $\boxed{|S| = \frac{12!}{5!4!3!}}$

Let A_1 be the set of arrangements of letters where the 5 a's are in a single block.

The no. of such arrangements is

$$|A_1| = \frac{8!}{4!3!}$$

|||ly A_2 is the set of arrangements of the letters where the 4 b's are in a single block.

$$|A_2| = \frac{9!}{5!\,3!}$$

A_3 is the set of arrangements of the letters where the 3 c's are in a single block.

$$|A_3| = \frac{10!}{5!\,4!}$$

$$|A_1 \cap A_2| = \frac{5!}{3!} \; ; |A_1 \cap A_3| = \frac{6!}{4!} \; ; |A_2 \cap A_3| = \frac{7!}{5!}$$

$$|A_1 \cap A_2 \cap A_3| = 3!$$

The required no. of arrangements is

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - |A_1 \cap A_2 \cap A_3| - \{|A_1| + |A_2| + |A_3|$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\}$$

$$= \frac{12!}{5!\,4!\,3!} - \left\{\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!}\right\} + \frac{5!}{3!} + \frac{6!}{4!} + \frac{7!}{5!}$$

$$= 27720 - (280 + 504 + 1260) + (20 + 30 + 42) - 3!$$

$$- 6 = 25\cancel{7}\cancel{6}2$$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| \geq 7$$

④ In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns (CAR, POG, PUN (OR) BYTE occurs)?

Let S denote the set of all permutations in which ~~car appears~~ of the 26 letters.

Then $|S| = 26!$

Let A_1 be the set of all permutations in which car appears. This word car consists of 3 letters which form a single block.

* Set A_1 therefore consists of all permutations which contains this single block and the 23 remaining letters $|A_1| = 24!$

|||ly A_2, A_3, A_4 are the set of all permutations which contain DOG, PUN and BYTE respectively

$$|A_2| = 24! \quad |A_3| = 24! \; ; |A_4| = 23!$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26-6+2)!$$

$$= 22!$$

$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26-7+2)! = 21!$$

$$|A_1 \cap A_2 \cap A_3| = (26-9+3)! = 20!$$

$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4|$$

$$= (26-10+3)!$$

$$= 19!$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26-13+4)! = 17!$$

The required no. of permutations is given by $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4|$

$$= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!) - (20! + 3 \times 19!) + 17!$$

⑤ In how many ways can one arrange the letters in the word CORRESPONDENT so that.

(i) There is no pair of connective identical letters.

(ii) There are exactly two pairs of consecutive identical letters?

(iii) There are atleast 3 pairs of consecutive identical letters?

In the word CORRESPONDENT; there occurs one each of C, P, D, T and two each of O, R, E, S, N

If S is the set of all permutations of these 14 letters. $|S| = \frac{14!}{(2!)^5}$

Let A_1, A_2, A_3, A_4, A_5 be the set of the permutations in which O's, R's, E's, N's, S's appear in pairs resp.

There $|A_i| = \frac{13!}{(2!)^4}$ for $i = 1, 2, 3, 4, 5$

Also $|A_i \cap A_j| = \frac{12!}{(2!)^3}$ $|A_i \cap A_j \cap A_k| = \frac{11!}{(2!)^2}$

$|A_i \cap A_j \cap A_k \cap A_p| = \frac{10!}{2!}$, $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 9!$

So, $N = |S| = \frac{14!}{(2!)^5}$, $S_1 = C(5,1) \times \frac{13!}{(2!)^4}$.

$S_2 = C(5,2) \times \frac{12!}{(2!)^3}$, $S_3 = C(5,3) \times \frac{11!}{(2!)^2}$

$S_4 = C(5,4) \times \frac{10!}{(2!)^1}$

$S_5 = C(5,5) \times 9!$

Accordingly the no. of permutations where there is no pair of consecutive identical letter is

$$E_0 = S_0 - \binom{1}{1} S_1 + \binom{2}{2} S_2 - \binom{3}{3} S_3 + \binom{4}{4} S_4 - \binom{5}{5} S_5$$

$$= \frac{14!}{(2!)^5} - \binom{5}{\cancel{4}5} \times \frac{13!}{(2!)^4} \cdots\cdots - \binom{5}{5} \times 9!$$

The no. of permutations where there are exactly 2 pairs of consecutive identical letters;

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$$

$$= \binom{5}{2} \times \frac{12!}{(2!)^3} - \binom{3}{1}\binom{5}{3} \times \frac{11!}{(2!)^2} + \binom{4}{2}\binom{5}{4} \times \frac{10!}{(2!)}$$

The no. of permutations where there are atleast three pairs of consecutive identical letters is

$$E_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{3} S_5$$

$$= \binom{5}{3} \times \frac{11!}{(2!)^2} - \binom{3}{2}\binom{5}{4} \frac{10!}{(2!)} + \binom{4}{2}\binom{5}{5} \times 9!$$

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* Find the no. of non-negative integer solutions of the equation.

$$x_1 + x_2 + x_3 + x_4 = 18$$

Under the conditions $x_i \leq 7$ for $i = 1,2,3,4$.

Let S denote the set of all non-negative integer solutions of the given equation.

The no. of such solutions is $C(4+18-1, 18)$
$= C(21, 18)$

$|S| = C(21, 18)$

Let A be the subset of S that contains the non-negative integer solutions of the given equation under the conditions $x_i > 7, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.

$A_1 = \{(x_1, x_2, x_3, x_4) \in S / x_1 > 7\}$

Similarly $A_2 = \{(x_1, x_2, x_3, x_4) \in S / x_2 > 7\}$

$A_3 = \{(x_1, x_2, x_3, x_4) \in S / x_3 > 7\}$

$A_4 = \{(x_1, x_2, x_3, x_4) \in S / x_4 > 7\}$

$\therefore$ the required sol^n $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}|$

Let us set $y_1 = x_1 - 8$; then $x_1 > 7 (x \geq 8)$ Corresponds to $y_1 \geq 0$ when written in terms of y_1, $y_1 + x_1 + x_2 + x_3 + x_4 = 10$

The no. of non-negative integer sol's of the equation is $C(4+10-1, 10) = C(13, 10)$

$|A_1| = C(13, 10)$

$|A_2| = |A_3| = |A_4| = C(13, 10)$

Let $y_1 = x_1 - 8$; $y_2 = x_2 - 8$ then $x_1 > 7$ and $x_2 > 7$ correspond to $y_1 \geq 0$ and $y_2 \geq 0$

When written in terms of Y_1 & Y_2.

$Y_1 + Y_2 + X_3 + X_4 = 2$

The no. of non-negative integer sol^n's of this equation is $C(4+2-1, 2) = C(5,2)$

$|A_1 \cap A_2| \therefore |A_1 \cap A_2| = C(5,2)$

$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4|$
$= |A_3 \cap A_4| = C(5,2)$

The given equation; more than 2 x's cannot be greater than 7 simultaneously

$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| = \Sigma|A_i| + \Sigma|A_i \cap A_j|$
$- \Sigma|A_i \cap A_j \cap A_k|$
$+ |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= C(21,18) - 4C_1 \times C(13,10) + 4C_2\, C(5,2)$
$- 0 + 0$

$= 1330 - (4 \times 286) + 6 \times 30 = 366.$

*** DERANGEMENTS:**

A permutation of n distinct objects in which none of the objects is in its natural place is called a derangement

Formula for d_n

The following is the formula for d_n for $n \geq 1$

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= n! \sum_{k=0} \frac{(-1)^k}{k!}$$

For example

$$D_2 = 2!\left[1 - \frac{1}{1!} + \frac{1}{2!}\right] = 1$$

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 1\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2$$

$D_4 = 9$, $D_5 = 44$, $D_6 = 256$; $D_7 = 1854$

Problems:

(1) Evaluate d_5, d_6, d_7, d_8.

Sol:

$$d_5 = 5!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$$= 120\left\{\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right\} = 44$$

$$d_6 = 6!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right\}$$

$$= 720\left\{\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right\} = 256.$$

$d_7 \approx [7! \times e^{-1}] \approx [5040 \times 0.3679] \approx 1854$

$d_8 \approx [8! \times e^{-1}] \approx [40320 \times 0.3679] \approx 14833$

② From the set of all permutations of n distinct objects; one permutation is chosen at random. What is the probability that is not a derangement?

Sol^n

The no. of permutations of n distinct objects is $n!$

The no. of derangements of these objects is d_n.

The probability that a permutation chosen is not a derangement.

$$P = 1 - \frac{d_n}{n!} = 1 - \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!}$$

3) In how many ways can the integers 1, 2, ..., 10 be arranged in a line so that no even integer is in its natural place?

Let A_1 be the set of all permutations of the given integer where 2 is in its natural place.

A_2 be the set of all permutations in which 4 is in its natural place and so on. The no. of permutations where no even integer is in its natural place is $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4} \cap \bar{A_5}|$.

This is given by

$$|\bar{A_1} \cap \bar{A_2} \cdots \cap \bar{A_5}| = |S| - S_1 + S_2 - S_3 + S_4 - S_5.$$

$|S| = 10!$

Now, the permutations in A_1 are all of the form $b_1, b_3 \cdots b_{10}$ where $b_1 b_3 b_4 \cdots b_{10}$ is a permutation of 1, 3, 4, 5, ...10 as such

$|A_1| = 9!$

$|||^{ly}$ $|A_2| = |A_3| = |A_4| = |A_5| = 9!$

$S_1 = \Sigma |A_i| = 5 \times 9! = C(5,1) \times 9!$

The permutations in $A_1 \cap A_2$ are all of the form $b_1\, 2\, b_3\, 4\, b_5\, b_6 \cdots b_{10}$ where $b_1 b_3 b_5 b_6 \cdots b_{10}$ is a permutations of 1, 3, 5, 6, ...10.

$|A_1 \cap A_2| = 8!$

$|||^{ly}$ each of $|A_i \cap A_j| = 8!$ Are there are $C(10,2)$ such terms $S_2 = \Sigma |A_i \cap A_j| = C(5,2) \times 8!$

$S_3 = C(5,3) \times 7!$ $S_4 = C(5,4) \times 6!$

$S_5 = C(5,5) \times 5!$

$|\bar{A}_1 \cap \bar{A}_2 \cdots \cap \bar{A}_5|$

$= 10! - C(5,1) \times 9! + C(5,2) \times 8!$
$- C(5,3) \times 7! + C(5,4) \times 6! - C(5,5) \times 5!$

$= 2170680$

*

4) Prove that, for any +ve integer n,

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_k.$$

For any +ve integer n; the total no. of permutations of 1, 2, 3 ... N is n!
In each such permutations there exists k elements which are in their natural positions called fixed elements and n-k elements which are not in their original positions. The k element can be chosen in $\binom{n}{k}$ ways and the remaining n-k elements can then be chosen in d_{n-k} ways.

Hence there are $\binom{n}{k} d_{n-k}$ permutations of 1, 2, 3, ... n with k fixed elements and n-k deranged elements. As k varies from 0 to n; we count all of the permutations of 1, 2, 3, ... n

Thus
$$n! = \sum_{k=0}^{n} nC_k\, d_{n-1}$$
$$= nC_0\, d_n + nC_1\, d_{n-1} + nC_2\, d_{n-2} + \cdots + nC_n\, d_0$$

$$\boxed{= \sum_{k=0}^{n} nC_{n-k}\, d_k = \sum_{k=0}^{n} {}^nC_k\, d_k}$$

30/01/2023

* ROOK POLYNOMIAL:

Consider a board that represents a full chess board (or) a part of it. Let 'n' be the no. of squares present in the board. Two pawns (or) rooks placed on a board are said to be non-capturing position if they are not in same row (or) same column.

For $2 \leq k \leq n$. Let r_k denotes the no. of ways in which 'k' rooks can be placed on a board such that no 2 rooks capture each other.

If the board is denoted by C then the polynomial is denoted by $r(C, x)$

$r(C,x) = 1 + r_1 x + r_2 x^2 + \dots + r_n x^n$

is called rook polynomial.

*Note:

1) Rook polynomial is defined for $n \geq 2$

if $n=1$; then board contain only one square so that $r_2 = r_3 = \dots = r_n = 0$

($r_n = 1$ = no. of squares in the board)

$\boxed{r(C,x) = 1+x}$

2) If $n=2$

$r_3 = r_4 = \dots = r_n = 0$ ($r_n = 2$ = no. of squares.

$r(C,x) = 1 + x + x^2 + \dots$

3) Rook polynomial for $n \times n$ board is given by

$$r(C_{n\times n}, x) = 1 + (nC_1)^2 x + 2!(nC_2)^2 x^2 + \dots + n!(nC_n)^2 x^n$$

* Expansion Formula:

If a given board C, suppose we choose a particular square and mark it as (*)

Let D denotes the board obtained from C by deleting the row and column containing the square (*) and E be the board obtained from C by deleting only the square (*). Then the rook polynomial for the board C is given by

$$\boxed{r(C,x) = x \cdot r(D,x) + r(E,x)}$$

This is known as expansion formula for $r(C,x)$.

* Product Formula:

Suppose a board is made up of 2 parts C_1 & C_2 where C_1 and C_2 have no squares in the same row (or) same column of C; such parts are called disjoint subboard of C. Then the rook polynomial for the board C is given by

$$r(C,x) = r_1(C_1,x) \times r_2(C_2,x)$$

* Arrangements with forbidden positions:

Suppose m objects are to be placed in n places where $n \geq m$. Suppose there are constrains under which some objects cannot occupy certain places; such places are called forbidden positions for the said objects.

The no. of ways of carrying out this task is given by

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

where $S_n = n!$; $S_K = (n-K)! \times r_k$

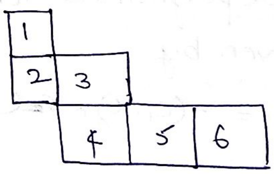
for $K = 1, 2, 3, \ldots n$

r_k is the coeff of x^k in the rook polynomial of the board of 'm' rows and 'n' columns whose squares represent the forbidden places.

*PROBLEMS:

1) Find the rook polynomial of the following board:

C:



Sol^n: Here $r_1 = 6$ ($\because$ 6 squares are in board C)

positions of 2 non-capturing rooks are

(1, 3); (1, 4); (1, 5); (1, 6), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6)

$r_2 = 9$

positions of 3 non-capturing rooks are

(1, 3, 5) (1, 3, 6) $\quad r_3 = 2$

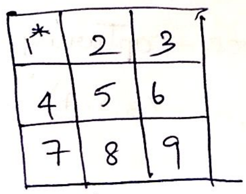
Rook polynomial is given by

$$r(C, x) = 1 + r_1 x + r_2 x^2 + r_3 x^3$$

$$= 1 + 6x + 9x^2 + 2x^3$$

2) Suppose the board containing more no. of squares. Find the rook polynomial for 3×3 board using expansion formula.

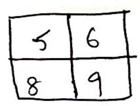
C:



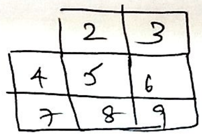
We shall now split the board C into the board D & E using the procedure of expansion formula.

denoting square ① as (*)

D



E:



For the board D:

$r_1 = 4$ ($\because$ 4 squares are in board D)

positions of 2 non-capturing rooks are

(5, 9), (6, 8) thus $r_2 = 2$

$r(D, x) = 1 + 4x + 2x^2$

For Board E:

$r_1 = 8$ ($\because$ 8 squares are in the board E)

positions of 2 non-capturing rooks are:

(2, 4), (2, 6), (2, 7), (2, 9), (3, 4), (3, 5), (3, 7), (3, 8), (4, 8), (4, 9), (5, 7), (5, 9), (6, 7), (6, 8)

$r_2 = 14$

positions of 3 non-capturing rooks.

(2, 4, 6), (2, 6, 7), (3, 5, 7), (3, 4, 8).

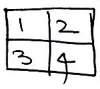
$r_3 = 4$

$r(E, x) = 1 + 8x + 14x^2 + 4x^3$

$r(C, x) = x\, r(D, x) + r(E, x)$

$= x(1 + 4x + 2x^2) + (1 + 8x + 14x^2 + 4x^3)$

$= 1 + 9x + 18x^2 + 6x^3.$

* Consider the board  find the rook polynomial.

①

Soln: Here $r_1 = 4$ ($\because r_1 = n = 4 =$ no. of squares)

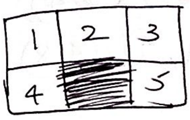
The positions of 2 non-capturing rooks are (1, 4), (2, 3) $\therefore r_2 = 2$

The board has no positions for more than 2 non-capturing rooks

$\therefore r_3 = r_4 = 0.$

$\therefore$ Rook polynomial is

$$\boxed{r(C, x) = 1 + 4x + 2x^2}$$

② 

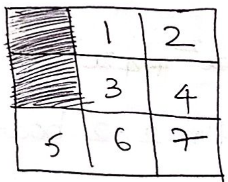
Here $n = r_1 = 5$

positions of 2 non-capturing rooks are (1, 5), (2, 4), (2, 5), (3, 4) $r_2 = 4$

$\therefore r_3 = r_4 = r_5 = 0$

$\therefore$ Rook polynomial is

$$\boxed{r(C, x) = 1 + 5x + 4x^2}$$

③ 

Here $r_1 = n = 7$

positions of 2 non-capturing are (1, 7), (1, 4), (1, 5), (2, 3), (2, 6), (2, 7), (3, 5), (3, 5), (4, 5), (4, 6)

$r_2 = 10$

positions of 3 non capturing rooks are:

(1, 4, 5), (2, 3, 5)

$\therefore r_3 = 2$

$\therefore r_4 = \cdots r_7 = 0.$

$\therefore r(C,x) = 1+7x+10x^2+2x^3$.

④

1	2	3
4		5
6	7	8

$r_1 = n = 8$

positions of 2 non-capturing rooks are:

(1,5), (1,8), (1,7), (2,4)
(2,5), (2,6), (2,8), (3,4),
(3,6), (3,7); (4,7), (4,8)
(5,7) (5,6)

$r_2 = 14$

positions of 3 non-capturing rooks:

(1,5,7), (3,7,4), (5,2,6), (4,2,8)

$r_3 = 4$

$\therefore$ Rook polynomial is

$r(C,x) = 1+8x+14x^2+4x^3$

⑤ Find the rook polynomial for the 2x2 board using expansion formula.

Sol^n:

C

1*	2
3	4

D

4

($\because$ deleting consecutive row & column of *)

$\therefore r_1 = 1$

$\boxed{r(D,x) = 1+x}$

E

	2
3	4

($\because$ deleting the * box)

$r_1 = 3$

$r_2 = (2,3) = 1$

$r(E,x) = 1+3x+x^2$

$\therefore$ rook polynomial by using expansion formula:

$r(C,x) = x \cdot r(D,x) + r(E,x)$

$= x(1+x) + 1+3x+x^2$

$\boxed{r(C,x) = 1+4x+2x^2}$

⑥ Find the rook polynomial for 3x3 board using Expansion formula:

C

1	2	3
4	5*	6
7	8	9

D

1	3
7	9

E

1	2	3
4		6
7	8	9

D:

$r_1 = 4$

(1,9), (3,7)

$r_2 = 2$

$\boxed{r(D,x) = \cancel{\phi}\ 1+4x+2x^2}$

E:

$r_1 = 8$

$r_2 \Rightarrow 14$

(1,8), (1,6), (1,9) (2,4) (2,6), (2,7) (2,9) (3,4) (3,7), (3,8) (4,8) (4,9) (6,7), (6,8)

$\cancel{r(E,x) = 1+8x+14x^2}$

$r_3 = (1,8,6), (4,2,9), (7,2,6), (3,4,8)$
$= 4$

$\boxed{r(E,x) = 1+8x+14x^2+4x^3}$

$\therefore r(C,x) = x\, r(D,x) + r(E,x)$

$= x(1+4x+2x^2) + 1+8x+14x^2+4x^3$

$\boxed{r(C,x) = 1+6x^3+18x^2+9x}$

Verification :

We have $n \times n$ board, Expansion formula is

$r(C_{n\times n}, x) = 1 + ({}^nC_1)^2 x + 2!\,({}^nC_2)^2 x^2$

$+ 3!\,({}^nC_3)^2 x^3$

$= 1 + ({}^3C_1)^2 x + 2!\,({}^3C_2)^2 x^2$

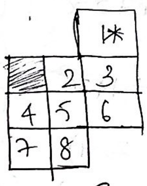
$+ 3!\,({}^3C_3)^2 x^3$

$= 1 + 9x + 2 \times 9x^2 + 6 \times x^3$

$= 1 + 9x + 18x^2 + 6x^3$

⑦ Using the expansion formula, obtain the rook polynomial for the board as shown below :

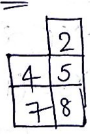
Given :



C

Sol^n

D:



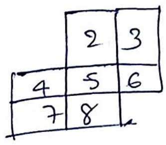
$r_1 = 5$

$r_2 \Rightarrow$ (2,4), (2,7), (4,8), (5,7), = 4

$r_3 = 0 = r_4 = \dots$

$\therefore \boxed{r(D,x) = 1+5x+4x^2}$

E:



$r_1 = 7$

$r_2 \Rightarrow$ (2,6), (2,4), (2,7), (3,4), (3,5), (3,7), (3,8), (4,8), (5,7), (6,7), (6,8), = 11

$r_3 \Rightarrow$ (6,2,7), (3,5,7), (4,8,3) = 3

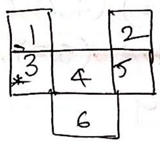
$\boxed{r(E,x) = 1+7x+11x^2+3x^3}$

$r(C,x) = x\,r(D,x) + r(E,x)$

$= x(1+5x+4x^2) + 1+7x+11x^2+3x^3$

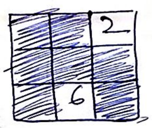
$= 1 + 8x + 16x^2 + 7x^3$

⑧ Using the expansion formula find the rook polynomial for the board C as shown below:

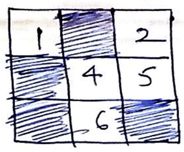


C

D:



E:



$r_1 = 2$

$r_2 = 1\ (2,6)$

$r(D,x) = 1 + 2x + x^2$

$r_1 = 5$

$r_3 = (1, \cancel{6}, \cancel{5}) = 1$

$r_2 \Rightarrow$ (1,4) (1,6), (1,5) (2,4), (2,6), ~~5~~ (5,6) = 6

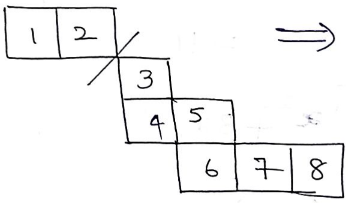
$r(E,x) = 1+5x+6x^2+x^3$

$r(C,x) = x\,r(D,x) + r(E,x)$

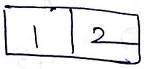
$= x(1+2x+x^2) + 1+5x+6x^2+x^3$

$= 1+6x+8x^2+2x^3$

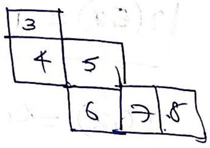
9. Obtain the rook polynomial for the chess board.



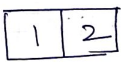
$\Longrightarrow C_1 \Rightarrow$



$C_2 \Rightarrow$



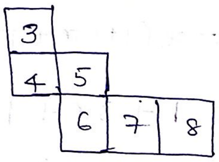
C_1



$r_1 = 2$

$r_2 = r_3 = 0 \ldots$

$\therefore \boxed{r(C_1,x) = 1+2x}$



$r_1 = 6$

$r_2 \Rightarrow (3,5),(3,6),(3,7),(3,8),(4,6),(4,7),(4,8),(5,7),(5,8)$

$= 9$

$r_3 \Rightarrow (3,5,7)\,(3,5,8)$

$= 2$

$\boxed{r(C_2,x) = 1+6x+9x^2+2x^3}$

$\therefore r(C,x) = r(C_1,x) * r(C_2,x)$

$= (1+2x) * 1+6x+9x^2+2x^3$

$= \cancel{1+7x+10x^2+2x^3}$

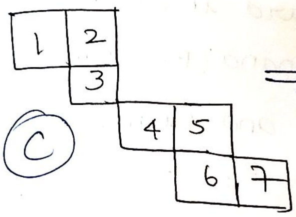
$= (1+2x)(1+6x+9x^2+2x^3)$

$= 1+6x+9x^2+2x^3+2x+12x^2+18x^3+4x^4$

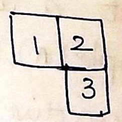
$\boxed{r(C,x) = 1+8x+21x^2+20x^3+4x^4}$

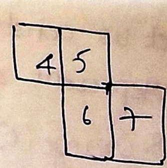
10. Find the rook polynomial for the board C.

C

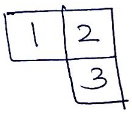


$\Longrightarrow$





C_1

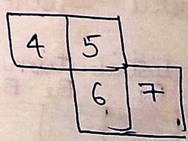


$r_1 = 3$

$r_2 \Rightarrow (1,3) = 1$

$r(C_1,x) = 1+3x+x^2$

C_2



$r_1 = 4$

$r_2 \Rightarrow (4,6);(4,7)$

$(5,7) = 3$

$r(C_2,x) = 1+4x+3x^2$

$r(C,x) = r(C_1,x) * r(C_2,x)$

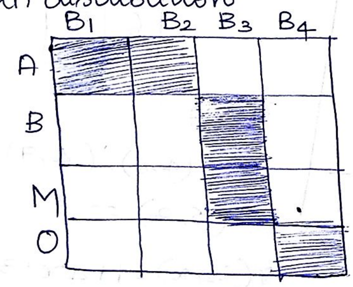
$= (1+3x+x^2) * (1+4x+3x^2)$

$= 1+4x+3x^2+3x+12x^2+9x^3+x^2+4x^3+3x^4$

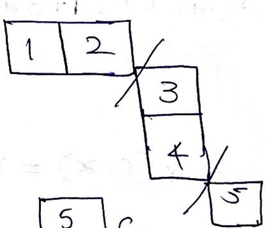
$= 1+7x+16x^2+13x^3+3x^4$

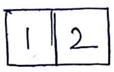
(11). An apple, a banana, a mango and an orange are to be distributed to 4 boys B_1, B_2, B_3, B_4. The boys B_1 & B_2 do not want to have an apple; the boy B_3 does not want banana & mango and B_4 refuses oranges. In how many ways can the distribution is made so that no boy is displeased.

Let us construct a board in which rows represent apple (A), banana (B), Mango (M), & Orange (O) respectively and columns represents boys B_1, B_2, B_3, B_4 respectively; shaded square represents forbidden positions in distribution.



C: the board 'C' consisting of shaded squares.



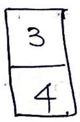


C_1

$r_1 = 2$

$r_2 = 0$

$r(C_1, x) = 1 + 2x$



C_2

$r_1 = 2$

$r_2 = 0$

$r(C_2, x) = 1 + 2x$

5

C_3

$r_1 = 1$

$r(C_3, x) = 1 + x$

$$r(C,x) = r(C_1,x) * r(C_2,x) * r(C_3,x)$$

$$= (1+2x)^2 \times (1+x) = (1+4x+4x^2)(1+x)$$

$$= 1 + x + 4x + 4x^2 + 4x^2 + 4x^3$$

$$\boxed{r(C,x) = 1 + 5x + 8x^2 + 4x^3}$$

Thus for board C

$r_1 = 5, \; r_2 = 8, \; r_3 = 4$

We have $S_0 = 4! = 24$

$S_1 = (4-1)! \, r_1 = 6 \times 5 = 30$

$S_2 = (4-2)! \, r_2 = 2 \times 8 = 16$

$S_3 = (4-3)! \, r_3 = 1 \times 4 = 4$

$\overline{N} = S_0 - S_1 + S_2 - S_3$

$= 24 - 30 + 16 - 4$

$= \underline{\underline{6}}$.

$\therefore$ There are 6 ways of distributing fruits so that no boy is displeased.

*RELATIONS – POSETS – HASSE DIAGRAMS:

*→ Cartesian Product of Sets:

Let A and B be 2 non-empty sets then the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$ is called cartesian product (or) cross product (or) product of A & B and is denoted by $A \times B$.

Thus: $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

Note:

① $A \times B$ is not same as $B \times A$

$\because B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}$

$\therefore (a,b) \neq (b,a)$ in general.

② Suppose (a,b) & (c,d) are ordered pairs with $(a,b) = (c,d)$ iff $a = c$ & $b = d$.

③ If A & B are finite sets with $|A| = m$, $|B| = n$ then $|A \times B| = mn = |A| \cdot |B|$.

④ No. of subsets of $A \times B$: $|(A \times B)| = 2^{mn}$.

where $|A| = m$, $|B| = n$.

*Relations:

Let A and B be 2 non-empty sets then a subset of $A \times B$ is called a binary relation (or) Relation from A to B [i.e. $R \subseteq A \times B$]

→ If $(a,b) \in R$; we say that "a is related to b" by R. This is denoted by $a R b$

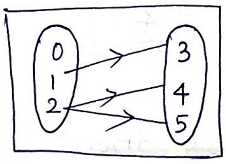
eg: $A = \{0,1,2\}$; $B = \{3,4,5\}$

$A \times B = \{(0,3),(0,4),(0,5),(1,3),(1,4),$
$(1,5),(2,3),(2,4),(2,5)\}$

Let $R = \{(1,3),(2,4),(2,5)\} \subseteq A \times B$

Here R is a subset of $A \times B$.

R is a relation from A to B.



⇒ figure(a)

This relation 'R' is clearly represented in the figure(a); we call this diagram as arrow diagram.

Note: If A has 'm' elements and B has 'n' elements, then no. of relations from A to B is 2^{mn}.

Problems:

1) Let A and B be finite sets with $|B|=3$. If there are 4096 relations from A to B; then find $|A|=?$

We know that if $|A|=m$, $|B|=n$; then

No. of relations from A to B $=2^{mn}$

Here $|B|=3$; $2^{mn}=4096$; $|A|=?$

$4096=2^{m(3)}$

$$3|A| = \frac{\log_e 4096}{\log_e 2} = \frac{1}{3}\cdot\frac{12\log_e 2}{\log_e 2}$$

$$= \underline{4}$$

2) Let $A=\{1,2,3\}$ & $B=\{2,4,5\}$ then determine the following:

(i) $|A\times B|$ (ii) No. of relations from A to B

(iii) No. of binary relations on A

(iv) No. of relations from A to B that contains $(1,2)$ & $(1,5)$.

(v) No. of relations from A to B that contains exactly 5 ordered pairs.

(vi) No. of relations from A to B that contains atleast 7 ordered pairs.

* Given $|A|=m=3$; $|B|=n=3$

(i) $|A\times B| = mn = 3\times 3 = 9$

(ii) No. of binary relations from A to B

$$=2^{mn}=2^9=512$$

(iii) No. of binary relations on A $=2^{m^2}=2^9$

$$=512$$

(iv) Let $R_1=\{(1,2),(1,5)\}$; every relation from A to B that contains the elements $(1,2),(1,5)$ is of the form $R_1\cup R_2$ where R_2 is a subset of $\bar{R_1}$ in $A\times B$.

$\therefore$ No. of such relations = No. of subsets of $\bar{R_1}$

$$=2^7=128 \; (\because |\bar{R_1}|=9-2=7)$$

(v) No. of relations from A to B that contains exactly 5 ordered pairs $= {}^9C_5 = {}^9C_4$

$$= \frac{9\times 8\times 7\times \cancel{6}}{\cancel{24}}$$

$$= \underline{126}$$

(vi) No. of relations from A to B that contains atleast 7 ordered pairs

$$= {}^9C_7 + {}^9C_8 + {}^9C_9 = 46$$

*MATRIX OF A RELATION:

Consider the sets $A=\{a_1,a_2,\ldots,a_m\}$; $B=\{b_1,b_2,\ldots,b_n\}$ of orders m,n

respectively; then 'A × B' contains all order pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$ which are mn in number.

Let R be a relation from A to B so that R is a subset of A × B. Let $m_{ij} = (a_i, b_j)$ where

$$M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases} \text{ zero-one matrix.}$$

Here the matrix formed by m_{ij} of size $m \times n$ is called the relation matrix for R, or the adjacency matrix (or) zero-one matrix for R and is denoted by M_R (or) $M(R)$.

Note:
Rows of M_R correspond to elements of 'A' and columns correspond to elements of 'B'

Eg: ① Let $A = \{1, 2, 3, 4\}$, $B = \{4, 5\}$ and $R = \{(1,4), (2,5)\}$ then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

② Let $A = \{p, q, r\}$; $R = \{(p,p), (p,q), (r,r), (r,q)\}$

then

$$M_R = \begin{array}{c} \\ p \\ q \\ r \end{array} \begin{array}{c} \begin{array}{ccc} p & q & r \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

* Properties on Relations:

1) **Reflexive relation:** A relation R on a set 'A' is said to be reflexive if $(a,a) \in R$, $\forall a \in A$, i.e. aRa; $\forall a \in A$.

2) **Non-Reflexive relation:** R is said to be non-reflexive if $\exists$ some $a \in A$ such that $(a,a) \notin R$.

Eg: ① $\le, =, \ge$ are reflexive relations on the set of all real numbers.

② $<, >$ are non-reflexive on R (real nos).

③ If $A = \{1,2,3,4\}$, then $R = \{(1,1), (2,2), (3,3)\}$, is not a reflexive relation since $(4,4) \notin R$.

3) **Irreflexive relation:** A relation R on a set 'A' is said to be irreflexive if $(a,a) \notin R$, $\forall a \in A$.

Eg: "<", ">" are irreflexive on the set of all real numbers.

Note:

(i) A non-reflexive relation need not be irreflexive.

(ii) A relation can be neither reflexive nor irreflexive

4) Symmetric relation: A relation 'R' on set 'A' is said to be symmetric whenever $(a,b) \in R$, then $(b,a) \in$; $\forall a,b \in A$.

5) Assymmetric Relation: A relation 'R' on a set 'A' is said to be assymmetric whenever $(a,b) \in R$, then $(b,a) \notin R$ $\forall a,b \in A$.

6) Anti-Symmetric Relation: A relation 'R' on a set 'A' is said to be anti-symmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$. (OR)

$(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$.

Eg: '$\leq$' is antisymmetric on the set of all real numbers ($\because$ if $a \leq b$ & $b \leq a$, then $a = b$).

Note: ① Asymmetric & antisymmetric relations are not same.

② A relation can be both symmetric and anti-symmetric. It can be neither symmetric nor anti-symmetric.

7) Transitive Relation: A relation R on a set 'A' is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, $\forall a,b,c \in A$.

Eg: '$\leq$', '$\geq$' are transitive on the set of all real numbers ($\because$ if $a \leq b$ & $b \leq c \Rightarrow a \leq c$ &
if $a \geq b$ & $b \geq c \Rightarrow b \geq c$)

8) Equivalence relation: A relation R on a set A is said to be an equivalence relation on 'A' if R is reflexive, symmetric & transitive on A.

Eg: '=' is an equivalence relation on the set of all real numbers.

9) Partial order relation: A relation 'R' on a set 'A' is said to be partial order relation, if R is reflexive, antisymmetric & transitive

Eg: '$\leq$' is a partial order relation on the set of all real numbers.

* PROBLEMS:

1) Let $A = \{1,2,3\}$. Determine the nature of the following relations defined on A.

(i) $R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$

Irreflexive, symmetric & non-transitive.

(ii) $R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$

Reflexive, asymmetric & transitive.

(iii) $R_3 = \{(1,1),(2,2),(3,3)\}$

Reflexive, anti-symmetric, transitive.

(iv) $R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$\underbrace{\text{Reflexive, Symmetric, transitive.}}$

Equivalence Relation.

(v) $R_5 = \{(2,3),(3,4),(2,4)\}$

Irreflexive, Assymmetric, transitive.

2) If $A = \{1,2,3,4\}$ and R be a relation on A then give an example of a relation for each of the following.

(i) Reflexive, symmetric but not transitive

(ii) Reflexive, transitive but not symmetric

(iii) Symmetric, transitive but not reflexive.

A: (i) $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)\}$

(ii) $R = \{(1,1),(2,2),(3,3),(4,4),(1,2)\}$

(iii) $R = \{(1,1),(2,2),(1,2),(2,1)\}$

3) Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ be a relation on A. Verify that R is an equivalence relation.

To verify 'R' is an equivalence relation; we have to show that R is reflexive, symmetric & transitive.

(i) As $(1,1),(2,2),(3,3),(4,4) \in R$
i.e. $(a,a) \in R \ \forall a \in A$ } $\therefore$ R is reflexive.

(ii) As $(1,2),(2,1) \in R$ & $(4,3),(3,4) \in R$
i.e. if $(a,b) \in R \Rightarrow (b,a) \in R$.
$\therefore$ R is symmetric.

(iii) As $(1,2),(2,1) \in R \Rightarrow (1,1) \in R$.
$(1,2),(2,2) \in R \Rightarrow (1,2) \in R$.
$(1,1),(1,2) \in R \Rightarrow (1,2) \in R$
$(4,3),(3,4) \in R \Rightarrow (4,4) \in R$.

i.e. if $(a,b), (b,c) \in R \Rightarrow (a,c) \in R$

$\forall a,b,c \in A$.

$\therefore$ R is transitive.

Hence R is an equivalence relation.

4) Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ on A define a relation R on 'A' such that $(x,y) \in R$ iff $(x-y)$ is a multiple of 5; then verify that 'R' is an equivalence relation.

* Here $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

R is defined as $(x,y) \in R$ iff $(x-y)$ is a multiple of 5

(i) For any $x \in A$, we have $x - x = 0$

$\Rightarrow x - x = 5(0)$ [is a multiple of 5]

$\Rightarrow (x,x) \in R$. (i.e. $x R x$)

$\therefore$ R is reflexive.

(ii) For any $x,y \in A$ such that

$(x,y) \in R \Rightarrow (x-y)$ is a multiple of 5

$\Rightarrow (x-y) = 5k \ (k \in \mathbb{Z})$

$(y-x) = -5k = 5(-k)$; $(-k \in \mathbb{Z})$

$\Rightarrow (y,x) \in R$. multiple of 5.

$\therefore$ R is symmetric.

(iii) For any $x,y,z \in A$ such that $(x,y) \in R$ & $(y,z) \in R$;

if $(x,y) \in R \Rightarrow (x-y) = 5k_1 \ (k_1 \in \mathbb{Z})$

$(y,z) \in R \Rightarrow (y-z) = 5k_2 \ (k_2 \in \mathbb{Z})$.

Now: $(x-y)+(y-z) = 5(k_1+k_2) = 5k$.

$(k = k_1+k_2 \in \mathbb{Z})$

$(x-z) =$ multiple of 5

$\therefore (x,z) \in R \Rightarrow$ R is transitive.

$\therefore$ Hence 'R' is equivalence relation.

* For a fixed integer $n>1$; prove that the relation "congruent modulo m" is an equivalence relation or the set of all integers $\mathbb{Z}$. (OR)

~~Equivalence class~~:

P.T. the relation R defined on set of all integers such that $x R y$ iff $x \equiv y \pmod{m}$ is an equivalence relation for all $x,y \in \mathbb{Z}$.

Equivalence ~~Rela~~ class:

Let R be an equivalence relation on a set A and $a \in A$; then the set of all those elements of A which are related to 'a' by R is called the equivalence class of 'a' with respect to R. This is denoted by [a] (or) R(a)

thus, $[a] = R(a) = \{ x \in A \mid (x,a) \in R \}$

Eg: Let $R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ defined on a set $A = \{1,2,3\}$

clearly R is a equivalence relation

$[1] = \{1,2\}$

$[2] = \{1,2\}$ & $[3] = \{3\}$

* Partition of a set:

Let 'A' be a non-empty set. Suppose there exists non-empty subsets $A_1, A_2, \ldots, A_K$ of A such that the following 2 conditions holes.

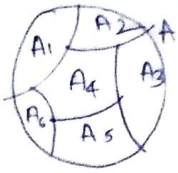
(i) $A = \bigcup_{i=1}^{K} A_i$ (ii) $A_i \cap A_j = \phi$ (for $i \neq j$)

then $P = \{A_1, A_2, \ldots, A_K\}$ is called a partition of A. Also $A_1, A_2, \ldots, A_K$ called the blocks (or) cells of the partition.

A partition of a set with 6 blocks is shown here

Note:

Equivalence class can act as partition blocks.



* Problems:

1) For the equivalence relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ defined on the set $A = \{1,2,3,4\}$. Determine the partitions induced.

* Given equivalence relation is

$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$

Equivalence classes:

$[1] = \{1,2\}$; $[2] = \{1,2\}$; $[3] = \{3,4\}$

$[4] = \{4,3\}$.

Among above [1], [3] are distinct

and $[2] \cup [3] = \{1,2\} \cup \{3,4\}$

$= \{1,2,3,4\} = A$

$\therefore P = \{[2],[3]\}$ is partition of A.

2) Find the partition of A induced by R, given $A=\{1,2,3,4,5,6,7,8,9,10,11,12\}$ where $R=\{(x,y) \mid (x,y)$ iff $(x-y)$ is multiple of $5\}$

Here R is clearly equivalence relation (we already proved in previous problems)

<u>Equivalence classes</u>:

$[1]=\{1,6,11\}=[6]$

$[2]=\{2,7,12\}=[7]$

$[3]=\{3,8\}=[8]$

$[4]=\{4,9\}=[9]$

$[5]=\{5,10\}=[10]$

$[6]=\{1,6,11\}=[12]$

Distinct Equivalence classes are $[1]$, $[2],[3],[4],[5]$. and

$[1]\cup[2]\cup[3]\cup[4]\cup[5]=A$.

$P=\{[1],[2],[3],[4],[5]\}$ is the partition of A.

3) Let $A=\{1,2,3,4,5\}$. Define a relation R on $A\times A$ by $(x_1,y_1)\,R\,(x_2,y_2)$ iff $x_1+y_1=x_2+y_2$

(i) Verify 'R' is an equivalence relation on $A\times A$.

(ii) Determine the equivalence classes $[(1,3)],[(2,4)]$

(iii) Determine the partition of $A\times A$ induced by R.

* POSET (Partial Order Set):

Let R is a partial order relation defined on a set 'A' then (A, R) is the partial order set (poset)

Eg: The relation "$\leq$" on the set all integers 'Z' is a partial order relation.

$\therefore$ $(Z, \leq)$ is a poset.

NOTE:

A poset (A, R) is said to be "Total order set". if the partial order relation 'R' satisfies the property, $x, y \in A$ either $x R y$ (or) $y R x$.

Eg:

The relation "$\leq$" defined on set of all real numbers R then $(R, \leq)$ is a totally ordered set.

* Every totally ordered set is a poset.