Unit - 5
Infinite series

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   - Divergence
   - Oscillating
     - finitely
     - infinitely

2. Series
   - Convergence
   - Divergence
   - Oscillating
     - finitely
     - infinitely

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→ **Convergence, Divergence and oscillating nature of a sequence:**

→ A sequence is said to be convergent if limit of sequence is finite.

$$\lim_{n\to\infty} u_n = l \text{ (finite)}$$

Eg: The sequence $\{u_n\} = \{1/n\}$ is a convergent sequence as its limit is zero.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

→ A sequence $\{u_n\}$ is said to be divergent if the limit of the sequence is infinite.

$$\lim_{n\to\infty} u_n = \infty$$

Eg: $\{u_n\} = \{2n-1\}$ is a divergent sequence.

$$\lim_{n\to\infty} 2n-1 = \infty$$

→ A sequence $\{u_n\}$ is said to be oscillatory if it is neither convergent nor divergent i.e it doesn't tend to a unique limit.

Eg: $\{u_n\} = \{(-n)^n\} = -1, +1, -1, +1 ----$ is an oscillatory sequence which oscillates b/w -1 & 1.

→ **Infinite series:**

If $\{u_n\} = u_1, u_2, u_3 ---- u_n$ is an infinite sequence of real numbers, then the expression $u_1 + u_2 + u_3 --- + u_n -----$ is known as an infinite series.

The sum of the terms of the sequence which are infinite in number is called infinite series.

The infinite series $u_1 + u_2 + \dots + u_n \dots$ is denoted by

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 \dots + u_n + \dots$$

→ **General properties of series:**

The convergence (or) divergence of an infinite series remains unaffected by addition (or) removal of finite no. of terms.

If a series in which all the terms are +ve and is convergent, then it remains convergent if the signs of some (or) all its terms are changed.

The convergence (or) divergence of an infinite series remains unaltered by multiplying each term by finite quantity.

Sequence:

A sequence of real numbers is a set of numbers arranged in well defined order. It is a function whose domain is a set of natural numbers and range is any subset of real numbers.

Eg: $\{u_n\} = \{1/n\} = 1, \frac{1}{2}, \frac{1}{3} \dots$

Syntactically, sequence is represented as $f: N \rightarrow S$. A sequence is generally represented as $\{u_n\}$ or $\langle u_n \rangle = u_1, u_2,$

Constant sequence: The sequence $\{u_n\}$ is denoted by $u_n = k \ (\in R)$ is called constant sequence. eg: 2, 2, 2 - - - -

Limit of a sequence: If $\{u_n\}$ be a sequence and l be any real number, then the real no. l is said to be limit of the sequence $\{u_n\}$ if to each $\epsilon > 0$, there exists a no. $m \in N$, such that $|u_n - l| < \epsilon$.

If l is the limit of the sequence u_n then if $\forall n \geq m$, is written as

$$\lim_{n\to\infty} u_n = l$$

eg: $\lim_{n\to\infty} \{u_n\} = \left\{\frac{n^2+1}{2n^2+3}\right\} = \frac{1}{2}$

Series of +ve terms:

An infinite series in which all the terms after a particular term are +ve is called a +ve term series.

eg: −7, −5, −2, +2, +7, +13, +20 - - - is a +ve term series

A series of positive terms either converges/ diverges but doesn't oscillate.

Standard limits:

(1) $\lim_{n\to\infty} (x)^{1/n} = 1 \quad (x > 0)$

(4) $\lim_{n\to\infty} x^n = \infty \quad (x \geq 1)$

(2) $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$

(5) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(3) $\lim_{n\to\infty} n^{1/n} = 1$

(6) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\forall x)$

→ Necessary condition for the convergence:

Statement -

If Σu_n converges then $\lim_{n\to\infty} u_n = 0$ (Converse it need not be true)

Pf. -

$S_n = \Sigma u_n$

$S_n = \Sigma u_n$

Given that Σu_n converges.

By the definition of convergence,

$$\lim_{n\to\infty} S_n = l \text{ (finite)} \quad \text{——(1)}$$

($\because \Sigma u_n = S_n$ series of nth partial sum)

We also know that

$$\lim_{n\to\infty} S_{n-1} = l \text{ (finite)} \quad \text{——(2)}$$

$$u_n = S_n - S_{n-1}$$

$\because$ $S_n = u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots$

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$

$S_n = u_1 + u_2 + \cdots + u_{n-1} + u_n$

Taking limits on both sides,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= l - l \quad \{\text{From (1), (2)}\}$$

$$= 0.$$

** Note: (1) Converse of the above theorem need not be true i.e "If $\lim_{n\to\infty} u_n = 0$, then Σu_n may/may not be convergent."

Ex: $1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots$

$$u_n = \frac{1}{n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

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$\sum u_n$ need not be convergence.

(ii) If $\lim_{n\to\infty} u_n \neq 0$, then $\sum u_n$ diverges.

→ Comparision test - I :

"If $\sum u_n$ & $\sum v_n$ be series of positive terms and $u_n <= v_n$ $(u_n \leq v_n)$, then the series $\sum u_n$ converges when $\sum v_n$ converges."

→ Comparision test - II :

" If $\sum u_n$ & $\sum v_n$ be series of positive terms and $u_n \geq v_n$, then the series $\sum u_n$ diverges when $\sum v_n$ diverges."

→ Comparision test - III :

" If $\sum u_n$ and $\sum v_n$ be series of positive terms, then $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite & non zero), and if

then $\sum u_n$ and $\sum v_n$ converges or diverges together. (Limit form of comparision test)

→ Geometric series test.

Statement :

The geometric series $1+r+r^2+\cdots\cdots$

(i) converges if $|r|<1$

(ii) diverges if $r \geq +1$

(iii) oscillates finitely $r=-1$

(iv) oscillates infinitely $r<-1$

Proof: given that

$$S_n = 1 + r + r^2 - - - -$$

(i) If $|r| < 1$ (i.e $-1 < r < 1$)

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{1-r} - \frac{r^n}{1-r}\right) \quad \left(\because \lim_{n\to\infty} r^n = 0 \text{ if } r<1\right)$$

$$= \frac{1}{1-r} \text{ (finite)}$$

S_n converges.

(ii) case(i) if $r = 1$

$$S_n = 1 + r + r^2 + - - -$$

$$S_n = 1 + 1 + 1 - -$$

$$= n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n$$

$$= \infty \text{ (Infinite)}$$

Diverges

case(ii) if $r > 1$

$$S_n = \frac{r^n - 1}{r - 1}$$

$$S_n = \frac{r^n}{r-1} - \frac{1}{r-1}$$

$$\lim_{n\to\infty} S_n = \infty - \text{finite}$$

$$= \infty$$

Diverges.

(iii) (i) if $r = -1$

$$S_n = 1 + r + r^2 + - - - - -$$

$$= 1 - 1 + 1 + (-1) \dots$$

$$= 1, 0, 1, 0$$

Thus, Oscillates finitely.

if $r < -1$

$$S_n = \frac{1 - r^n}{1 - r} \qquad r = -t, \ t \text{ is +ve}$$

$$= \frac{1 - (-t)^n}{1 + t}$$

$$= \infty, -\infty, \infty, -\infty$$

Oscillates infinitely.

→ p-series (Auxillary-test / p-test / Harmonic series)

Statement –

If $\sum \frac{1}{n^p} = 1 + \frac{1}{1^p} + \frac{1}{2^p} + \cdots$

(i) converges if $p > 1$

(ii) diverges if $p \leq 1$

(Proof not required)

→ Test the convergence for the foll. series.

(1) $1 + \frac{1}{2} + \frac{1}{3}$ ----------

$u_n = \frac{1}{n}$

compare with p-series i.e $\Sigma \frac{1}{n^p}$

$p = 1$

By p-series, Σu_n diverges.

(2) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$

$u_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

compare with p-series i.e $\Sigma \frac{1}{n^p}$

By p-series, $p = 1/2 < 1$

Σu_n diverges.

(3) $\Sigma \frac{2n^3+5}{4n^5+1}$

$u_n = \frac{2n^3+5}{4n^5+1}$

$u_n = \frac{n^3}{n^5}\left[\frac{2 + 5/n^3}{4 + 1/n^5}\right]$

$v_n = 1/n^2$

$\Sigma v_n = \Sigma \frac{1}{n^2}$

comparing with p-series $\Sigma 1/n^p$, $p = 2$

By p-series, $p = 2 > 1$.

$\sum V_n$ converges.

By comparision test,

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{2} \text{ (finite and non zero)}$$

Thus, $\sum U_n$ also converges.

HW

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(5) $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

$$v_n = \frac{1}{\sqrt{n}}$$

By comparing with p series $\sum \frac{1}{n^p}$

$$p = \frac{1}{2} < 1$$

$\sum v_n$ ~~converges~~ diverges

By comparision test,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \text{ (finite \& non zero)}$$

$\sum u_n$ also ~~converges~~ diverges.

(6) HW $\sum \frac{1}{2^n+3^n}$

(7) $\sum\left(\frac{n^2+1}{n^2}\right)$

$$u_n = \frac{n^2+1}{n^2}$$

$$= 1+\frac{1}{n^2}$$

$$\lim_{n\to\infty} u_n = 1 \neq 0$$

$\therefore$ Diverges.

→ Test the convergence for

→ $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \text{------}$

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$= \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$v_n = \frac{1}{n^2}$$

Comparing with $\sum \frac{1}{n^p}$

where $p=2$

By p-series $p=2>1$

then the series converges.

$\sum v_n$ converges.

By limit form of comparision test,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\cancel{n^2}}\frac{\left(2-1/n\right)}{\left(1+1/n\right)\left(1+2/n\right)}\cancel{n^2} = 2 \text{ (finite \& nonzero)}$$

Thus, Σu_n converges.

$\rightarrow \frac{2^q}{1^p} + \frac{3^q}{2^p} + \frac{4^q}{3^p} + \cdots$

$$u_n = \frac{(n+1)^q}{n^p}$$

$$u_n = \frac{n^q(1+1/n)^q}{n^p}$$

$$u_n = \frac{1}{n^{p-q}}(1+1/n)^q$$

$$v_n = \frac{1}{n^{p-q}}$$

$$\Sigma v_n = \Sigma \frac{1}{n^{p-q}} = \Sigma \frac{1}{n^p}$$

Comparing with p-series

(i) If $p>1 \Rightarrow p-q>1$

$p > 1+q$

Then, the series converges.

(ii) If $p \leq 1 \Rightarrow p-q<1$

$p < 1+q$

series diverges.

$\rightarrow \frac{1.2}{3.4.5} + \frac{2.3}{4.5.6} + \frac{3.4}{5.6.7} + \cdots$

$$u_n = \frac{n(n+1)}{(n+2)(n+3)(n+4)}$$

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$$= \frac{n^2(1+1/n)}{n^3\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+4/n\right)}$$

$$= \frac{1}{n}\frac{(1+1/n)}{(1+2/n)(1+3/n)(1+4/n)}$$

$v_n = 1/n$, $\Sigma v_n = \Sigma 1/n$

By comparing with p. series,

$\frac{1}{n^p}$, $p = 1 \geq 1$

Σv_n diverges.

By limit form of comparision test,

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (finite & non zero)

Thus, Σu_n diverges.

→ ST $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ is convergent for $p > 2$ and divergent for $p <= 2$

$$u_n = \frac{n+1}{n^p}$$

$$= \frac{n(1+1/n)}{n^p} = \frac{1}{n^{p-1}}(1+1/n)$$

$\rightarrow \sin(1/n)$

$$= \frac{1}{n} - \frac{1}{n^3 . 3!} + \frac{1}{n^5 . 5!} - - - - - -$$

$$= \frac{1}{n}\left[1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} - - - - \right]$$

$v_n = 1/n$, $\Sigma v_n = 1/n$

By comparing with p series,

$p = 1 \ngtr 1$

Σv_n diverges.

By limit form of comparision test,

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1$ (finite and non zero)

Thus, Σu_n diverges.

$\rightarrow \frac{1}{n} \sin(1/n)$

HW

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$\rightarrow 1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} - - - - - -$

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

(Neglecting first term)

$$= \frac{1}{n}\left(\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}\right)$$

$$v_n = \frac{1}{n}$$

By comparing with p-series,

$$p = 1 \leq 1$$

Σv_n diverges.

By limit form of comparision test,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{e} \text{ (finite \& non zero)}$$

$\therefore \Sigma u_n$ diverges.

$$\rightarrow \quad \sum \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$$

$$u_n = \left(\frac{2^n}{3^n}\right)^{1/2} \left[\frac{1+3/2^n}{1+1/3^n}\right]^{1/2}$$

$$v_n = \left(\frac{2}{3}\right)^{n/2}$$

Comparing with geometric series

with $r = \frac{2}{3} < 1$

$r < 1 \rightarrow$ convergent

Thus v_n converges.

But, limit form of comparision test,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \quad (\text{finite \& non zero})$$

Σu_n converges.

$\longrightarrow \sqrt[3]{n^3+1} - n$

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**

$u_n = (n^3+1)^{1/3} - n$

$(n^3)^{1/3}\left[1+\frac{1}{n^3}\right]^{1/3} - n$

$= n\left[1+\frac{1}{3.n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^6}+\cdots\right] - n$

$= \cancel{n} + \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots - \cancel{n}$

$= \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$

$v_n = \frac{1}{n^2}$

By comparing with p series,

$\Sigma \frac{1}{n^p} = \quad p>1 \Rightarrow$ converges.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ (finite & non-zero)

Σu_n converges.

→ D'Alembert's Ratio test –

If $\sum u_n$ be a series of positive terms and if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ (say), then

(i) The series converges when $l>1$
(ii) The series diverges when $l<1$
(iii) The series fails when $l=1$

Note – When the series contains factorial, repeated series, then we use D'Alembert's ratio test.

→ Test the convergence for

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+$$

$$u_n = \frac{1}{n!} \text{ (neglecting the 1st term)}$$

$$u_{n+1} = \frac{1}{(n+1)!}$$

By DART,

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = n+1$$

$$\lim_{n\to\infty} n+1 = \infty = l>1$$

Thus, the series converges.

→ Examine the convergence of

$$\sum_{n=1}^{\infty} \frac{1.3.5 \cdots (2n+1)}{2,5,8, \cdots (3n+2)}$$

$$u_n = \frac{1.3.5 \cdots (2n+1)}{2.5.8 \cdots (3n+2)}$$

$$u_{n+1} = \frac{1.3.5 \cdots (2n+1)(2n+3)}{2.5.8 \cdots (3n+2)(3n+5)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} \Rightarrow \frac{3n+5}{2n+3}$$

$$\lim_{n\to\infty} \frac{3n+5}{2n+3}$$

$$= \frac{\cancel{n}(3+5/n)}{\cancel{n}(2+3/n)}$$

$$= 3/2$$

Thus, the series converges.

→ $\sum \frac{n^2}{2^n}$

$$u_n = \frac{n^2}{2^n}$$

$$u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 2 > 1$$

Converges.

→ Cauchy's n^{th} root test:

If Σu_n be a series of +ve terms and if $\lim_{n\to\infty} u_n^{1/n} = l$ (say)

Then the series

(i) converges when $l < 1$

(ii) diverges when $l > 1$

(iii) test fails when $l = 1$

→ Σn^n

$u_n = n^n$

By using Cauchy's n^{th} root test

$$\lim_{n\to\infty} u_n^{1/n} = n^{\cancel{n} \times \frac{1}{\cancel{n}}}$$

$$= \infty > 1$$

∴ Diverges.

→ $\Sigma n^{1/n}$

$$\lim_{n\to\infty} n^{1/n} = 1 \neq 0$$

∴ Diverges.

→ $\left(1+\frac{1}{n}\right)^{-n^2}$

By using Cauchy's n^{th} root test,

$$\lim_{n\to\infty} u_n^{1/n} = \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

∴ Converges.

$$\rightarrow \sum \left(\frac{1}{1+\frac{1}{\sqrt{n}}} \right)^{-n^{3/2}} \quad (e>1) \quad \text{(Diverges)}$$

$$\rightarrow \sum \frac{1}{(\log n)^n} \quad \text{(Converges)}$$

$$\left(1/\infty = 0 \right) < 1$$

$$\rightarrow \sum \left(\frac{n+1}{n+2} \right)^n x^n$$

$$u_n^{1/n} = \frac{n+1}{n+2} \cdot x = x = l \text{ (say)}$$

(1) If $l < 1 \Rightarrow x < 1 \rightarrow$ Converges

(2) If $l > 1 \Rightarrow x > 1 \rightarrow$ Diverges

(3) If $l = 1 \Rightarrow x = 1 \rightarrow$ fails

put $x = 1$ in u_n

$$u_n = \left(\frac{n+1}{n+2} \right)^n$$

$$u_n = \frac{\cancel{n^n}\left(1 + 1/n\right)^n}{\cancel{n^n}\left(1 + 2/n\right)^n}$$

$$= \frac{1}{e} \neq 0$$

$\therefore$ Diverges.

Conclusion –

(i) If $x < 1$, then the series converges

(ii) If $x \geq 1$, then diverges.

$\rightarrow \frac{2x}{1^2} + \frac{3^2}{2^3}x^2 + \frac{4^3}{3^4}x^3 + \cdots$

$$u_n = \frac{(n+1)^n \cdot x^n}{n^{n+1}}$$

$$(u_n)^{1/n} = \lim_{n\to\infty} \frac{(n+1) \cdot x}{n \cdot n^{1/n}}$$

$$= x \text{ (say)} = l \qquad \lim_{n\to\infty} n^{1/n} = 1$$

(1) If $l < 1 \Rightarrow x < 1 \rightarrow$ converges

(2) If $l > 1 \rightarrow x > 1 \rightarrow$ Diverges

(3) If $l = 1 \rightarrow x = 1 \rightarrow$ fails

Put $x = 1$ in u_n

$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

$$= \frac{\cancel{n^n}\left(1 + \frac{1}{n}\right)^n}{\cancel{n^n} \cdot n}$$

$$u_n = 1/n$$

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$$\rightarrow \sum \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{(n+1)}{n} \right)^{-n} = u_n$$

$$(u_n)^{1/n} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n} \right]^{-1}$$

$$\lim_{n \to \infty} u_n^{1/n} = \left[\left(\frac{n+1}{n} \right)^{-1} \left[\frac{(n+1)^n}{n^n} - 1 \right]^{-1} \right]$$

$$= \left(\frac{\cancel{n^n}}{\cancel{n^n}} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^n - 1 \right]^{-1}$$

$$= \lim_{n \to \infty} \left(\left(1 + \frac{1}{n} \right)^n - 1 \right)^{-1}$$

$$= \frac{1}{e-1} < 1$$

converges.

HW

$\rightarrow \sum \frac{[(n+1)!]^2 x^{n-1}}{n}$

$\rightarrow \frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \text{------} \left(\frac{x^n}{n(n+1)}\right)$

$\rightarrow \sum \frac{x^{2n}}{(n+1)\sqrt{n}}$

$\rightarrow u_n = \frac{((n+1)!)^2 x^{n-1}}{n}$

$u_{n+1} = \frac{((n+2)!)^2 x^n}{n+1}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(n+1)!(n+1)!\cancel{x^n}.n+1}{x.\cancel{x^n}.n(n+2)!(n+2)!}$

$= \frac{n+1}{n(n+2)^2.x}$

$= \frac{\cancel{n}(1+1/n)}{n^{\cancel{2}}(1+\frac{2}{n})^2 x}$

$= 0 < 1$

Diverges. $(\forall x)$

$\rightarrow u_n = \frac{x^n}{n(n+1)}$

$u_{n+1} = \frac{x^n.x}{(n+1)(n+2)}$

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n}\right)x$$

$$= x = l \text{ (say)}$$

(i) If $l > 1 \rightarrow x > 1 \rightarrow$ converges

(ii) If $l < 1 \rightarrow x < 1 \rightarrow$ Diverges

(iii) If $l = 1 \rightarrow x = 1 \rightarrow$ test fails

Put $x = 1$ in u_n,

$$u_n = \frac{1}{n(n+1)}$$

$$u_n = \frac{1}{n^2\left(1+\frac{1}{n}\right)}$$

Let $v_n = \frac{1}{n^2}$

Comparing with p-series, $p = 2 > 1$

Thus, Σv_n converges.

By limit form of comparision test,

$\Sigma \frac{u_n}{v_n} = \frac{1}{1+1/n}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (finite & non zero)

Thus, Σu_n also converges.

(i) If $x \geq 1$, series converges.

(ii) If $x < 1$, series diverges.

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→ Alternating series :

A series whose terms are alternatively positive and negative is called alternating series.

$\{(-1)^{n+1} = (-1)^{n-1}\}$ The series is represented by

$$u_1 - u_2 + u_3 - u_4 + \text{-----}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \text{---} = \sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n}$

b) $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} \text{-----}$

Ignore 1st term.

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n}{\log n}$$

→ Leibnitz test :

If $\{u_n\}$ is a sequence of positive terms such that

(i) $u_1 \geq u_2 \geq u_3 \geq \text{--} - u_n \geq u_{n+1}$.

(ii) $\lim_{n \to \infty} u_n = 0$,

then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is convergent.

→ Absolutely Convergent:

If the given alternating series is said to be absolutely convergent, it should satisfy the foll. two conditions.

(i) By Leibnitz test, given series is convergent.

(ii) $\Sigma |u_n|$ (change Alternating → Positive) is convergent.

→ Conditionally convergent -

If the alternating series is said to be conditionally convergent, then it should follow the 2 conditions.

(i) By Leibnitz test, given series is convergent.

(ii) $\Sigma |u_n|$ is divergent.

→ $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{-------}$

The given series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$u_1 = 1,\ u_2 = 1/2^2,\ u_3 = 1/3^2,\ u_4 = \frac{1}{4^2}$

We observe that

$u_1 \geq u_2 \geq u_3 \geq u_4$ ------

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

By Leibnitz test, converges.

→ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}$

→ $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{------}$

$u_n = (-1)^{n+1} \frac{x^{2n-2}}{(2n-2)!}$

$$u_{n+1} = \cancel{(-1)} \frac{x^{2n}}{2n!}$$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x^{2n}}\ 2n!}{x^2 \cdot (2n-2)!\ \cancel{x^{2n}}}$$

$$= \frac{2n \cdot 2n-1}{x^2} = \infty > 1$$

Converges.

$\rightarrow$ ** $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2+1}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad \{\because \cos n\pi = (-1)^n\}$$

$$|u_n| = \frac{1}{n^2+1}$$

$$= \frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$$

$p = 2 > 1 \rightarrow$ Converges.

$\rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt{n+1} - \sqrt{n}\right)$

$$|u_n| = \sqrt{n+1} - \sqrt{n}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}\left(1+\sqrt{1+1/n}\right)}$$

$P = 1/2 < 1 \rightarrow$ Diverges.

$\boxed{\text{Conditionally convergent.}}$

$\rightarrow$ $$u_n = \frac{3.5.7\ldots\ldots(2n+1)}{8.10.12\ldots\ldots(2n+6)}$$

$$u_{n+1} = \frac{3.5.7\text{-----}(2n+1)(2n+3)}{8.10.12\text{-----}(2n+6)(2n+8)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+8}{2n+3} = 1$$

DART fails.

Thus, Raabie's test is applicable.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lambda \quad \begin{cases} >1 \rightarrow \text{converges} \\ <1 \rightarrow \text{diverges} \\ =1 \rightarrow \text{fails} \end{cases}$$

$$n\left(\frac{2n+8}{2n+3} - 1\right) = \lambda$$

$$\lim_{n\to\infty} \frac{5n}{2n+3} = \frac{5}{2} > 1$$

$\rightarrow$ **Integral test:**

Let $f(x)$ be a non negative decreasing function in $[1,\infty)$, then the series is

(i) Convergent when $\int f(x)dx$ = finite

(ii) Divergent when $\int f(x)dx$ = infinite

$\rightarrow$ $$\int_2^\infty \frac{1}{x\log x}dx$$

$$= \left(\log(\log(x))\right)_2^\infty$$

$= \infty$ – finite

$= \infty \rightarrow$ Diverges

(or) We observe that

$$\frac{1}{n\log n} \geq \frac{1}{n}$$

Consider $v_n = \sum \frac{1}{n}$

Diverges.

$$\rightarrow \sum_{n=2}^{\infty} \frac{\log n}{2n^3-1}$$

Consider $\frac{1}{2n^3-1} \leq \frac{1}{n^3} \quad (n \geq 2)$

$$\frac{\log n}{2n^3-1} \leq \frac{n}{n^3}$$
