17/08/2022

APPLICATIONS OF LINE INTEGRAL:

*→ GREEN'S THEOREM IN THE PLANE:

* Let $M(x,y)$ and $N(x,y)$ be continuous functions having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a closed region R bounded by the curve C; then

$$\int_C \bar{F}\cdot d\bar{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$
$$= \int M dx + N dy$$

* Verify Green's theorem in the plane $\int_C (xy+y^2)\,dx + x^2 dy$ where C: closed region of the region $y = x$ and $y = x^2$.

* By Green's theorem: $\int_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

RHS

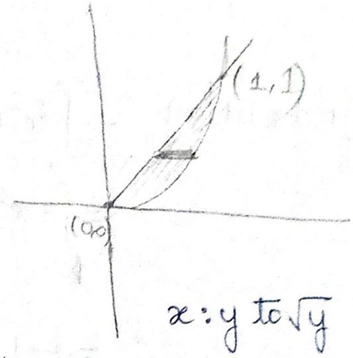
$M = xy+y^2 \quad N = x^2$

$\frac{\partial M}{\partial y} = x+2y \quad \frac{\partial N}{\partial x} = 2x.$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y = x - 2y$

$$= \iint_R (x-2y)\,dx\,dy$$

$$= \int_0^1 \left[\int_y^{\sqrt{y}} (x-2y)\,dx\right] dy$$



$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$x : y \text{ to } \sqrt{y}$

$$= \int_0^1 \left[\frac{y-y^2}{2} - 2y(\sqrt{y}-y)\right] dy$$

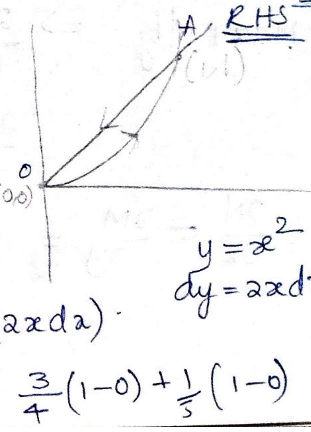
$$= \int_0^1 \left(\frac{y}{2} - \frac{y^2}{2} - 2y\sqrt{y} + 2y^2\right) dy = \left[\frac{1}{4}(1-0) - \frac{1}{6}(1-0) - \frac{2\times 2}{5}(1-0) + \frac{2}{3}(1-0)\right]$$

$$= \frac{1}{4} - \frac{1}{6} - \frac{4}{5} + \frac{2}{3} = \frac{15-10-48+40}{60}$$

$$= \frac{55-58}{60} = \frac{-3}{60} = \frac{-1}{20}$$

LHS

RHS



$$\int_C Mdx+Ndy = \int_{OA} Mdx+Ndy \;(1) + \int_{AO} Mdx+Ndy \;(2)$$

$y = x^2$
$dy = 2x\,dx$

(1) $\int_0^1 x(x^2)+x^4\,dx + x^2(2x\,dx).$

$$= \int_0^1 (3x^3+x^4)\,dx = \frac{3}{4}(1-0) + \frac{1}{5}(1-0)$$

$= \frac{19}{20}$.

② $\int_{A0} Mdx + Ndy = \int_{0} (x^2 + x^2) dx + x^2 dx$

$= \int_{1}^{0} 3x^2 dx = \frac{3}{3}(0-1)$

$= -1$

① + ② $= \frac{19}{20} - 1 = \frac{-1}{20}$.

LHS = RHS : Green's theorem is verified.

*Evaluate using Green's theorem:

$\int_C e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$ C: rectangle

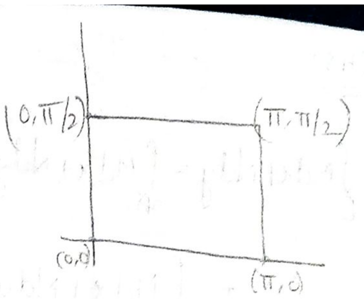
$(0,0), (\pi,0), (\pi,\pi/2), (0,\pi/2)$

$\int_C Mdx + Ndy \Rightarrow \frac{\partial M}{\partial y} = e^{-x} \cos y$

$\frac{\partial N}{\partial x} = -e^{-x} \cos y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2e^{-x} \cos y$

$\int_0^{\pi/2} \int_0^{\pi} -2e^{-x} \cos y \, dx \, dy$



$\int_0^{\pi} +2\cos y \, [e^{-x}]_0^{\pi/2} dy$

$\int_0^{\pi/2} 2 \cos y (e^{-\pi/2} - 1) dy$

$= 2(e^{-\pi/2} - 1)(1) = 2(e^{-\pi} - 1)$

* Verify Green's theorem $\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$

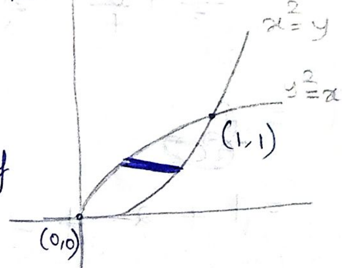
By Green's theorem

$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$ C: $y^2 = x$ & $y = x^2$

RHS

y : 0 to 1

x : y^2 to $\sqrt{y}$



$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$

$\int_0^1 \left[\int_{y^2}^{\sqrt{y}} 10y \, dx\right] dy = \int_0^1 10y(\sqrt{y} - y^2) dy$

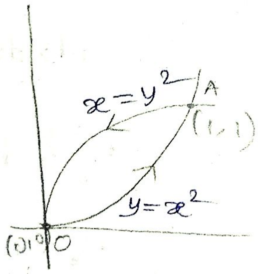
$= 10 \cdot \frac{2}{5}(1-0) - \frac{10}{4}(1-0)$

$= 4 - \frac{5}{2}$

$= \frac{3}{2}$

LHS:

$$\oint_C Mdx+Ndy = \int_{OA} Mdx+Ndy + \int_{AO} Mdx+Ndy$$



$$\int_{OA} Mdx+Ndy = \int_0^1 (3x^2-8x^4+\cancel{g}x^3-\cancel{1}2x^4)\,dx$$

$$= \int_0^1 (3x^2+8x^3-20x^4)\,dx$$

$$= \frac{3}{3}+\frac{8}{4}-\frac{20}{5} = \cancel{\frac{140-90-72}{60}} = \cancel{\frac{-22}{60}} = \cancel{-\frac{11}{30}}$$

$$= 1+2-4 = -1$$

$$\int_{AO} Mdx+Ndy = \int_1^0 (3y^4-8y^2)\,2y\,dy + (4y-6y^3)\,dy$$

$$= \int_1^0 (6y^5-22y^3+4y)\,dy$$

$$= \frac{6}{6}(-1+0)-\frac{22}{4}(-1+0)+2(-1)$$

$$= \cancel{-1+\frac{11}{4}-2} = \cancel{0}$$

$$-1-2+\frac{11}{2} = \frac{-6+11}{2} = \frac{5}{2}$$

$$(1)+(2) = -1+\frac{5}{2} = \frac{3}{2}$$

* Apply Green's theorem

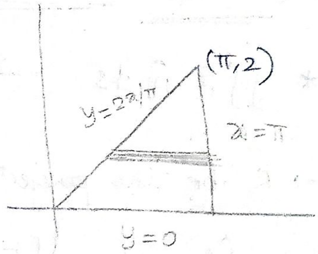
$$\oint_C (y-\sin x)\,dx+\cos x\,dy \qquad C: \text{triangle}$$

$y=0$; $x=\pi$; $\pi y=2x$

y: 0 to 2.

x: $\frac{\pi y}{2}$ to π

$$\int_0^2 \left[\int_{\pi y/2}^{\pi} \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] dx\,dy$$



$$\frac{\partial N}{\partial x} = -\sin x \qquad \frac{\partial M}{\partial y} = 1$$

$$(-\sin x-1)$$

$$\int_0^2 \left[\int_{\pi y/2}^{\pi} (-\sin x-1)\,dx\right] dy = \int_0^2 (+\cos x - x)_{\pi}^{\pi y/2}\,dy$$

$$= \int_0^2 (\cos \pi y/2 + 1) - \left(\pi\left(\frac{y}{2}-1\right)\right) dy$$

$$= \int_0^2 \left[\cos\frac{\pi y}{2}+1-\frac{\pi y}{2}+\pi\right] dy$$

$$= \left[+\frac{\sin\frac{\pi y}{2}}{} + y - \frac{\pi y^2}{4} + \pi y\right]_0^2$$

$$= -\left[\frac{2}{\pi}(0-0) + 2 - \frac{\pi}{4}(4-0) + \frac{\pi}{\cancel{2}}(\cancel{4}^{2})\right]$$

$$= -\left[+2 - \pi + 2\pi\right] = -\left[+2+\pi\right]$$

$$\frac{-\pi}{4} - \frac{2}{\pi}$$

$$= -2 - \pi$$

* SURFACE INTEGRAL: (Double Integration)

* $\iint_R F \cdot \hat{n}\, ds \Rightarrow$ flux:

→ R be the projection in the XY plane

$$\iint_R F \cdot \hat{n}\, ds = \iint_S F \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

→ R be the projection in the YZ-plane

$$\iint_R F \cdot \hat{n}\, ds = \iint_S F \cdot \hat{n} \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|}$$

→ R be the projection in the ZX plane

$$\iint_R F \cdot \hat{n}\, ds = \iint_S F \cdot \hat{n} \frac{dz\,dx}{|\hat{n} \cdot \hat{j}|}$$

Surface Integrals:

Any integral which is to be evaluated over a surface is called surface integral.

$S \rightarrow$ surface of finite area.

Suppose $f(x, y, z)$ is a single-valued function defined over surface S.

n elementary areas: $\delta S_1, \delta S_2, \delta S_3, \ldots \delta S_n$

arbitrary point $P_i(x_i, y_i, z_i)$

* Flux:

If $\overline{F}(x, y, z)$ is a vector function continuous over S and $\hat{n}$ be the unit normal vector at any point; then the integral

$\iint_S F \cdot \hat{n}\, ds$ is called flux of $\overline{F}$ over S.

$$\overline{ds} = \hat{n}\, ds.$$

$$\iint_S F \cdot \hat{n}\, ds = \iint_S F \cdot \overline{ds}$$

* Evaluate $\iint_S \bar{F}\cdot\hat{n}\, ds$ where $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

S is a part of surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

$x^2+y^2+z^2=1$

Consider XY plane put $z=0$ $\quad x^2+y^2=1$

$x: 0$ to 1

$y: 0$ to $\sqrt{1-x^2}$

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \int_0^1\int_0^{\sqrt{1-x^2}} xy\hat{k}\cdot\frac{\hat{n}}{|\hat{n}\cdot\hat{k}|}\, dx\, dy$$

$\phi = x^2+y^2+z^2-1$

$\nabla\phi = \hat{n} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \Rightarrow \boxed{\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}}$

$|\nabla\phi| = 2\sqrt{x^2+y^2+z^2} = 2$

$$\int_0^1\int_0^{\sqrt{1-x^2}} 3\frac{xyz}{z}\, dx\, dy = \int_0^1\int_0^{\sqrt{1-x^2}} 3xy\, dx\, dy$$

$$= 3\int_0^1 x\left[\int_0^{\sqrt{1-x^2}} y\, dy\right] dx$$

$$= \int_0^1 \frac{3x}{2}(1-x^2)\, dx$$

$$= 3\left[\frac{1}{4}(1-0) - \frac{1}{8}(1-0)\right]$$

$$= 3\left[\frac{1}{4} - \frac{1}{8}\right] = \cancel{3\left(\frac{6-2}{24\ 8}\right) = \frac{1}{4}}$$

$$= \frac{3}{8}$$

18/08/2022

* Evaluate $\iint_S \bar{F}\cdot\hat{n}\, ds$ where $\bar{F} = 12x^2y\hat{i} - 3yz\hat{j} + 2z\hat{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

$x+y+z=1$

$\phi = x+y+z-1$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

Consider xy plane

$z = 0$

$x+y=1$

$$\bar{F}\cdot\hat{n} = \frac{12x^2y - 3yz + 2z}{\sqrt{3}}$$

$$|\hat{n}\cdot\hat{k}| = \frac{1}{\sqrt{3}}$$

$x: 0$ to 1

$y: 0$ to $1-x$

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \int_0^1\int_0^{1-x} \frac{12x^2y - 3yz + 2z}{\sqrt{3}} \cdot \frac{dx\, dy}{\frac{1}{\sqrt{3}}}$$

$\because$ xy plane $z = 1-x-y$

$$= \int_0^1\int_0^{1-x} (12x^2y - 3 + 3x + 3y^2 + 2 - 2x - 2y)\, dx\, dy$$

$$\cancel{= \int_0^1\int_0^{1-x} (12x^2y + x + y - 1)\, dx\, dy}$$

$$= \int_0^1\left[\int_0^{1-x} (12x^2y + x + y - 1)\, dy\right] dx$$

$$= \int_0^1 \left[6x^2y^2 + xy + \frac{y^2}{2} - y\right]_0^{1-x} dx$$

$$= \int_0^1 \left[6x^2(1+x^2-2x) + x(1-x) + \frac{(1-x)^2}{2} - 1 + x\right] dx$$

$$= \int_0^1 \left[6x^2 + 6x^4 - 12x^3 + x - x^2 + \frac{1+x^2-2x}{2} - 1 + x\right] dx$$

$$= \int_0^1 6x^2 + 6x^4 - 12x^3 + \cancel{2x} - x^2 - 1 + \frac{1}{2} + \frac{x^2}{2} - \cancel{\frac{2x}{2}}$$

$$= \int_0^1 \left[\frac{11x^2}{2} + 6x^4 - 12x^3 - \frac{1}{2} + x\right] dx$$

$$= \frac{11}{6}(1-0) + \frac{6}{5}(1-0) - \frac{12}{4}(1-0) - \cancel{\frac{1}{2}(1-0)} + \cancel{\frac{1}{2}(1-0)}$$

$$= \frac{11}{6} + \frac{6}{5} - 3 = \frac{55+36-90}{30}$$

$$= \frac{91-90}{30} = \frac{1}{30}$$

$$= \int_0^1 \int_0^{1-x} (12x^2y - 1 + x + 3y^2 - 2y)\, dy\, dx$$

$$= \int_0^1 (6x^2y^2 - y + xy + y^3 - y^2)_0^{1-x} dx$$

$$= \int_0^1 6x^2(1+x^2-2x) - (1-x) + x - x^2 + 1 + x^3 - 3x + 3x^2 + 1 + x^2 - 2x)\, dx$$

$$= \int_0^1 (6x^2 + 6x^4 - 12x^3 - 1 + x + x - x^2 + 2 + x^3 + 4x^2 - 5x)\, dx$$

$$= \int_0^1 (9x^2 + 6x^4 - 11x^3 + 1 - 3x)\, dx$$

$$= 3(1-0) + \frac{6}{5}(1-0) - \frac{11}{4}(1-0) + 1 - 0 - \frac{3}{2}(1-0)$$

$$= 3 + \frac{6}{5} - \frac{11}{4} + 1 - \frac{3}{2}$$

$$= 4 + \frac{6}{5} - \frac{11}{4} - \frac{3}{2}$$

$$= \frac{80+24-55-30}{20} = 29$$

* $\bar{F} = yz\hat{i} + xz^2\hat{k} + 2y^2\hat{j}$; Find $\iint_S \bar{F}\cdot\hat{n}\, ds$

S: Surface of the cylinder $x^2+y^2=9$ contained in the I^{st} octant b/w the planes $z=0$ & $x=2$.

* Consider YZ - plane (put $x=0$)

$y^2 = 9$ $\qquad \phi = x^2+y^2-9$

$y: 0 \text{ to } 3$ $\qquad \nabla\phi = 2x\hat{i} + 2y\hat{j}$

$z: 0 \text{ to } 2$ $\qquad |\nabla\phi| = \sqrt{4x^2+4y^2}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2}{6}(x\hat{i}+y\hat{j})$ $\qquad = 2\sqrt{x^2+y^2}$

$= \frac{x\hat{i}+y\hat{j}}{3}$ $\qquad = 2\sqrt{9} = 2(3) = 6.$

$\bar{F}\cdot\hat{n} = \frac{xyz+2y^3}{3}$ $\quad = \cancel{\frac{xyz(1+z)}{3}}$

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iint_S \bar{F}\cdot\hat{n} \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \int_{y=0}^{3}\int_{z=0}^{2} \frac{xyz+2y^3}{3x} \times 3$$

$$= \int_0^3 \int_{z=0}^{2} \left(yz + \frac{2y^3}{\sqrt{9-y^2}}\right) dz\, dy$$

$$= \int_0^3 \left[y\left(\frac{4-0}{2}\right) + \frac{2y^3}{\sqrt{9-y^2}}(2-0)\right] dy$$

$$= \int_0^3 2y + \frac{4y^3}{\sqrt{9-y^2}}\, dy$$

$$= \int_0^3 \left[2y + \frac{4y^3}{\sqrt{9-9\sin^2\theta}}\right] dy \qquad y = 3\sin\theta$$

$dy = 3\cos\theta$

$$= \int_0^3 \left[2y + \frac{4\times\sin^3\theta\times 27}{3\cos\theta}\right] dy$$

$$= \frac{2}{2}(9-0) + 108\int_0^{\pi/2} \frac{\sin^3\theta}{3\cos\theta}\, 3\cos\theta\, d\theta$$

$3\sin\theta - 4\sin^3\theta = \sin 3\theta$

$$= 9 + 108\int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{4}$$

$$= 9 + 27\times 3(-0+1) + \left[\frac{\cos 3\theta}{3}\right]_0^{\pi/2} \times 27$$

$$= 9 + 81 + 9(0-1) = \underline{81}$$

* Evaluate $\iint_S \bar{F}\cdot\hat{n}\, ds$ $\quad \bar{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface $x^2+y^2=16$ included in the first octant $z=0$ to $z=5$

* Evaluate $\int_S \bar{F}\cdot\hat{n}\, ds$, where $\bar{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of surface of the plane $2x+3y+6z=12$ located in the first octant.