13/03/2023

TRANSFORM TECHNIQUES, PROBABILITY AND STATISTICS

→ Fourier series: periodic function

Eg: Heartbeat; ECG, musical instruments

→ Fourier Transforms: sine & cos transforms

Eg: used in water coolers to check the voltage in accordance with cooling nature.

→ Writing algorithms by checking for any errors is given by 4A series.

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UNIT-1 : FOURIER SERIES

→ Defn of periodic fn - Examples

~~→ Examp~~

→ Defn of Fourier series - expansion - Euler's formulae

→ Dirichlet's conditions for expansion of Fourier series

→ change of interval - problems

→ Expansion of Fourier series for even and odd functions - Problems.

→ Expansion of Fourier Series for half

range sine and cosine functions

* Periodic Function

Periodic function occur frequently in various engineering problems.

~~Their representation in terms of sine and cosine.~~

Their representation in terms of simple functions in terms of sine and cosine which leads to Fourier series.

Fourier series is a powerful tool for solving ordinary and partial differential equations.

Applications of Fourier Series:

→ Study of periodic function in conduction of heat, electrodynamics and ecostics, magneto motive force, flux density, current and voltage etc.

→ It is also useful for non-periodic functions

→ Definition of Periodic Function:

A function $f(x)$ is said to be a periodic function with period $T>0$ if $\forall x$, $f(x+T) = f(x)$; $\boxed{f(x+nT) = f(x)}$ where $n = 1, 2, 3, \ldots$

*→ $f(x) = f(x+T) + f(x+2T) + \cdots + f(x+nT)$

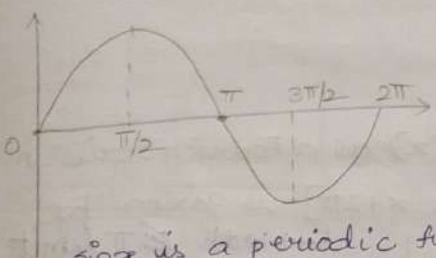
and T is the least of such values.

and T is the period of $f(x)$.

Ex:

$\sin x = \sin(x+2\pi) = \cdots = \sin(x+2n\pi)$

where $n = 1, 2, 3, \ldots$



$\sin x$ is a periodic function of period 2π.

*Note:

→ $\sin nx$ & $\operatorname{cosec} nx$ is a periodic function with period $\frac{2\pi}{n}$; $n = 1, 2, \ldots$

→ $\cos nx$ & $\sec nx$ is a periodic function with period $\frac{2\pi}{n}$; $n = 1, 2, \ldots$

→ $\tan nx$ & $\cot nx$ is a periodic function with period $\frac{\pi}{n}$; $n = 1, 2, \ldots$

→ Definition of Fourier Series:

Fourier Series is an infinite series representation of periodic function in terms of sines and cosines.

* Expansion of f(x) as a Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

↳ $\boxed{\text{with } \pi \text{ as limits}}$

where a_0, a_n & b_n are Euler constants (or) Fourier coefficients.

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→ Expansion of f(x) as a Fourier Series in the interval $[\alpha, \alpha+2\pi]$ is given by

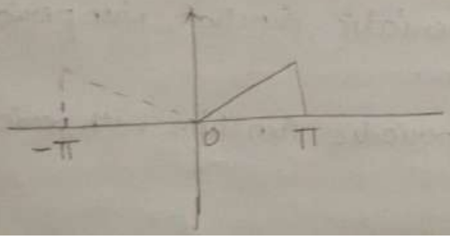
↳ with π limits

put $\alpha = 0$ in $[\alpha, \alpha+2\pi]$

$[0, 2\pi]$

put $\alpha = -\pi$ in $[\cancel{\alpha}, \cancel{\alpha}+2\pi]$

$[-\pi, \pi]$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where a_0, a_n & b_n are Euler constants (or) Fourier coefficients.

$\boxed{\frac{a_0}{2} \rightarrow \text{value is used to check \& correct the error}}$

↳ error will be reduced to large extent if divided by even no.

If interval is $[\alpha, \alpha+2c]$

$\alpha = 0$: $[0, 2c]$

$\alpha = -c$: $[-c, c]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

→ Euler's Formulae:

$$\boxed{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$$

* → Case(i): put $\alpha = 0$ in $[\alpha, \alpha+2\pi]$ ~~as~~

we get $[0, 2\pi]$

∴ Expansion of f(x) as a Fourier series in the interval $[0, 2\pi]$ given as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

case(ii)

put $\alpha = -\pi$ in $[\alpha, \alpha+2\pi]$ we get $[-\pi, \pi]$

$\therefore$ Expansion of $f(x)$ as a finite series in the interval $[\pi, 2\pi]$ given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

* Useful Integrals:

1) $$\int_c^{c+2\pi} \sin nx\, dx = \left(-\frac{\cos nx}{n}\right)_c^{c+2\pi} = 0$$

2) $$\int_c^{c+2\pi} \cos nx\, dx = \left(\frac{\sin nx}{n}\right)_c^{c+2\pi} = 0$$

3) $$\int_c^{c+2\pi} \sin^2 nx\, dx = \int_c \frac{1-\cos 2nx}{2}\, dx$$

$$= \left(\frac{x}{2} - \frac{\sin 2nx}{4n}\right)_c^{c+2\pi} = \pi$$

4) $$\int_c^{c+2\pi} \cos^2 nx\, dx = \int_c^{c+2\pi} \frac{1+\cos 2nx}{2}\, dx$$

$$= \left(\frac{x}{2} + \frac{\sin 2nx}{4n}\right)_c^{c+2\pi} = \pi$$

5) $$\int_c^{c+2\pi} \sin mx \cdot \cos nx\, dx = \frac{1}{2}\left[\int_c^{c+2\pi} (\sin(m+n)x + \sin(m-n)x)\, dx\right]$$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi}$$

$$= 0$$

6) $\int_{c}^{c+2\pi} \cos mx \cdot \cos nx \, dx$

$= \frac{1}{2}\int_{c}^{c+2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$

$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right]_{c}^{c+2\pi}$

$= 0$

7) $\int_{c}^{c+2\pi} \sin mx \cdot \sin nx \, dx = \frac{1}{2}\int_{c}^{c+2\pi} [\cos(m-n)x - \cos(m+n)x] \, dx$

$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{c}^{c+2\pi}$

$= 0$

8) $\int_{c}^{c+2\pi} \sin nx \cdot \cos nx \, dx = 0$

9) $\int u \cdot v \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

* Results:

1) $\sin n\pi = \sin 2n\pi = 0$

2) $\cos n\pi = (-1)^n$; $\cos 2n\pi = 1$

3) $\sin(2n+1)\pi/2 = (-1)^n$

4) $\cos(2n+1)\pi/2 = 0$

5) $\sin \frac{n\pi}{2} = \begin{cases} (-1)^{n-1/2} & ; n \to \text{odd} \\ 0 & ; n \to \text{even} \end{cases}$

6) $\cos \frac{n\pi}{2} = \begin{cases} 0 & ; n \to \text{odd} \\ (-1)^{n/2} & ; n \to \text{even} \end{cases}$

$n \in Z$; $Z \to$ set of all integers.

* Derivation of Euler Formulae in the interval $[\alpha, \alpha + 2\pi]$

Given $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —(1)

Case(i): to get a_0:

Integrate (1) w.r.to 'x' with limits α to $\alpha + 2\pi$

$\int_{\alpha}^{\alpha+2\pi} f(x) \, dx = \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2} \, dx + \sum_{n=1}^{\infty}\left[a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx\right]$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}(\alpha+2\pi-\alpha) + 0 + 0$$

$$\boxed{\because \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = 0}$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{\cancel{2}} \times \cancel{2}\pi$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$$

case(ii) to get a_n:

Multiply ① with $\cos nx$

$$f(x)\cos nx = \frac{a_0 \cos nx}{2} + \sum_{n=1}^{\infty}\left(a_n \cos^2 nx + b_n \sin nx \cos nx\right) \quad -②$$

Integrate ② wrt x with limits α to $\alpha+2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx\right]$$

$$\boxed{\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \pi}$$

$$\boxed{\because \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = 0}$$

$$= \frac{a_0}{2}(0) + \sum_{n=1}^{\infty}(a_n \pi + 0)$$

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = 0 + a_n\pi + 0$$

$$\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$$

case(iii) to get b_n

Multiply ① with $\sin nx$

$$f(x)\sin nx = \frac{a_0 \sin nx}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin nx + b_n \sin^2 nx\right) \quad -③$$

Integrate ③ wrt x with limits α to $\alpha+2\pi$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx + \sum_{n=1}^{\infty}\left(a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \sin nx\,dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx\right)$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = 0 + b_n\pi$$

$$\boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$$

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* DIRICHLET'S CONDITIONS FOR A FOURIER EXPANSION:

→ Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions possess valid Fourier expansions

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n; provided in the interval $c \leq x \leq c+2\pi$

(i) $f(x)$ is well defined and single valued, except possibly at a finite no. of points

(ii) $f(x)$ has only a finite no. of discontinuities

(iii) $f(x)$ has only a finite no. of maxima & minima

These conditions are known as Dirichlet conditions.

→ The above conditions are sufficient but not necessary.

* Obtain the Fourier series for the fn $f(x) = x^3$ in the interval $[0, 2\pi]$

∵ The given interval is $[0, 2\pi]$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

is the Fourier Series expansion.

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^3\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{4}(2\pi)^4$$

$$= \frac{16\pi^4}{4\pi}$$

$$\boxed{a_0 = 4\pi^3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^3 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^3 \frac{\sin nx}{n} - \int_0^{2\pi} \frac{3x^2}{n}\sin nx\,dx\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{x^3 \sin nx}{n} - \frac{3}{n}\int_0^{2\pi} x^2 \sin nx\,dx\right]_0^{2\pi}$$

$\Rightarrow a_n$

$$= \frac{1}{\pi}\left[\frac{x^3 \sin nx}{n}\right]_0^{2\pi} - \frac{1}{\pi}\cdot\frac{3}{n}\left[x^2\left(-\frac{\cos nx}{n}\right) - \int_0^{2\pi} 2x\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3 \sin 2n\pi - 0}{n}\right] + \frac{3}{n\pi}\left[\frac{x^2 \cos nx}{n}\right]_0^{2\pi} - \frac{6}{n\pi}\left[x\frac{\sin nx}{n} - \int_0^{2\pi}\frac{\sin nx}{n}dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{3}{n\pi}\left[\frac{4\pi^2 \cos 2n\pi}{n} - 0\right]\right] - \frac{6}{n^3\pi}\left[\frac{x \sin nx}{1}\right]_0^{2\pi} + \frac{6}{n^3\pi}\left[\frac{-\cos nx}{n}\right]_0^{2\pi}$$

$$= \left[\frac{4\pi^2}{n}\times\frac{3}{n\pi}\right] - \frac{6}{n^3\pi}\left[\frac{2\pi \sin 2n\pi - 0}{1}\right] + \frac{6}{n^3\pi}\left[\frac{-(\cos 2n\pi - 1)}{n}\right]$$

$$= \frac{12\pi}{n^2} - \frac{3}{n^3\pi}\cdot(0-0) + \frac{3}{n^3\pi}\left[-(0)\right.$$

$$= \frac{12\pi}{n^2}$$

$$\boxed{a_n = \frac{12\pi}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^3 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{-\sin nx}{n^4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[8\pi^3\left(\frac{-\cos 2n\pi}{n}\right) - 0 + \frac{3}{n^2}\left(4\pi^2(0) - 0\right) + 6\left(\frac{2\pi - 0}{n^3}\right) + \frac{6}{n^4}\left(0 - 0\right)\right.$$

$$= \frac{1}{\pi}\left[\frac{-8\pi^3(1)}{n} + \frac{12\pi}{n^3}\right]$$

$$= \frac{-8\pi^2}{n} + \frac{12}{n^3}$$

$$\boxed{b_n = \frac{12 - 8n^2\pi^2}{n^3}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{a_0}{2} + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \cdots$$

$$= 2\pi^3 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \cdots$$

$\Rightarrow a_n = \frac{12\pi}{n^2}$ | $b_n = \frac{12 - 8n^2\pi^2}{n^3}$

$a_1 = 12\pi$ $\qquad b_1 = 12 - 8\pi^2$

$a_2 = 3\pi$ $\qquad b_2 = \frac{12 - 32\pi^2}{8}$

$a_3 = \frac{4}{3}\pi$ $\qquad b_3 = \frac{12 - 72\pi^2}{27}$

$$f(x) = 2\pi^3 + 12\pi\cos x + (12 - 8\pi^2)\sin x + 3\pi\cos 2x + \frac{(12 - 32\pi^2)}{8}\sin 2x + \frac{4\pi}{3}\cos 3x + \frac{(12 - 72\pi^2)}{27}\sin 3x + \cdots$$

* Write the Fourier Series expansion of function $f(x) = e^{-x}$ in the range $[0, 2\pi]$

Given $f(x) = e^{-x}$ in the range $[0, 2\pi]$

$\therefore$ Fourier Series Expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$= \frac{-1}{\pi}\left[e^{-2\pi} - 1\right]$$

$$\boxed{a_0 = \frac{1 - e^{-2\pi}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cos nx$$

$$a_n = \frac{1}{\pi}\left[e^{-x}\cdot\frac{\sin nx}{n} + \frac{(-1)}{n^2}\int_0 e^{-x} \cos nx\, dx\right]$$

$$\cancel{a_n = \frac{-1}{n^3}\frac{1}{\pi}\left[\frac{e^{-2\pi}(0) - 1(0)}{n} - \frac{1}{n^2}\right]}$$

$$\cancel{a_n = \left[\frac{e^{-x}\sin nx}{n\pi}\right]_0^{2\pi} - \frac{1}{n^2}\cdot\frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx}$$

$$\cancel{a_n = \frac{e^{-2\pi}\sin 2n\pi - e^0\sin(0)}{n} - \frac{1}{n^2}a_n}$$

$$\cancel{a_n + \frac{1}{n^2}a_n = 0}$$

$$\cancel{a_n\left[\frac{n^2+1}{n^2}\right] = 0}$$

$\cancel{a_n =}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$\boxed{\int_0^{2\pi} e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$\cancel{= \frac{1}{\pi}\left[\frac{e^{-x}}{4\pi^2}\left[2\pi\cos\right.\right.}$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{(n^2+1)\pi}\left[\left(-\cos 2n\pi + n\sin 2n\pi\right)e^{-2\pi} - e^0\left(\cos(0) - n\sin(0)\right)\right]$$

$$= \frac{1}{(n^2+1)\pi}\left[-e^{-2\pi} + 0 + 1 - 0\right]$$

$$\boxed{a_n = \frac{1-e^{-2\pi}}{(n^2+1)\pi}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$\boxed{\int_0^{2\pi} e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-\sin nx - n\cos nx\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\left(0 - n(1)\right) - \left(0 - n(1)\right)\right]$$

$$= \frac{1}{(n^2+1)\pi}\left(-ne^{-2\pi} + n\right)$$

$$b_n = \frac{n(-e^{-2\pi}+1)}{(n^2+1)\pi}$$

$$\boxed{b_n = \frac{n(1-e^{-2\pi})}{(n^2+1)\pi}}$$

$$a_n = \frac{(1-e^{-2\pi})}{(n^2+1)\pi} \qquad b_n = \frac{n(1-e^{-2\pi})}{(n^2+1)\pi}$$

$$a_1 = \frac{1-e^{-2\pi}}{2\pi} \qquad b_1 = \frac{(1-e^{-2\pi})}{2\pi}$$

$$a_2 = \frac{1-e^{-2\pi}}{5\pi} \qquad b_2 = \frac{2(1-e^{-2\pi})}{5\pi}$$

$$a_3 = \frac{1-e^{-2\pi}}{10\pi} \qquad b_3 = \frac{3(1-e^{-2\pi})}{10\pi}$$

$$a_0 = \frac{(1-e^{-2\pi})}{\pi}$$

$$\cancel{f(x) = \frac{(1-e^{-2\pi})}{\pi}\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{5}\right.}$$

$$f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \frac{1}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x\right.$$

$$\left. + \frac{1}{10}\cos 3x + \frac{3}{10}\sin 3x + \dots\right]$$

* Find the coefficient of $\cos 3x$ in Fourier series expansion of the function $x\cos x$ in the interval $[0, 2\pi]$.

Given function $f(x) = x\cos x$

$\therefore$ Fourier Series Expansion of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} x\cos x \cos 3x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(\cos(4x) + \cos 2x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos 4x\, dx + \frac{1}{\pi}\int_0^{2\pi} x\cos 2x\, dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin 4x}{4} + \frac{\cos 4x}{16}\right]_0^{2\pi} + \frac{1}{\pi}\left[x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$a_3 = \frac{1}{\pi}\left[0 + \frac{1}{16} - 0 - \frac{1}{16}\right]$$

$$+ \frac{1}{\pi}\left[0 + \frac{1}{4} - 0 - \frac{1}{4}\right] = 0$$

$$\boxed{a_3 = 0}$$

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* Find the Fourier Series expansion of $f(x) = x - x^2$ in the interval $[-\pi, \pi]$

Given $f(x) = x - x^2$ in the interval $[-\pi, \pi]$

$\therefore$ Fourier Series Expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\underset{(odd)}{\int_{-\pi}^{\pi} x\,dx} - \underset{(even)}{\int_{-\pi}^{\pi} x^2\,dx}\right]$$

** if $f(x) = f(-x)$; $f(x)$ is even

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

** if $f(x) = -f(-x)$; $f(x)$ is odd

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$a_0 = \frac{1}{\pi}\left[0 - 2\int_{0}^{\pi} x^2\,dx\right]$$

$$= \frac{-2}{\pi}\left(\frac{\pi^2}{\pi}\right) \Rightarrow \boxed{a_0 = -2\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\underset{(odd)}{\int_{-\pi}^{\pi} x\cos nx\,dx} - \underset{(even)}{\int_{-\pi}^{\pi} x^2\cos nx\,dx}\right]$$

$$= \frac{1}{\pi}\left[0 - 2\int_{0}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{-2}{\pi}\left[x^2\frac{\sin nx}{n} - 2\int_{0}^{\pi} x\frac{\sin nx}{n}\,dx\right]$$

$$\cancel{= \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2}{n^2}\left[\frac{x(-\sin nx)}{n} + \frac{\cos nx}{n^3}\right]\right]}$$

$$= \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} - 2\int_0^{\pi} x \frac{\sin nx}{n}\, dx\right]$$

$$= \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2}{n}\left[\frac{x(-\cos nx)}{n} + \int_0 \frac{\cos nx}{n}dx\right]\right]$$

$$= \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2}{n^3}\frac{\sin nx}{1}\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\left[\frac{\pi^2(0)}{n} - 0 + \frac{2\pi(-1)^n - 0}{n^2} - 2(0)\right]$$

$$= \frac{-2}{\pi} \cdot \frac{2\pi(-1)^n}{n^2}$$

$$= \frac{4(-1)^{n+1}}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^{n+1}}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{+\pi} (x - x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$\Downarrow$ even $\qquad \Downarrow$ odd

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx - 0$$

$$= \frac{2}{\pi}\left[x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi\cos n\pi - 0}{n} + \frac{\sin n\pi - 0}{n^2}\right]$$

$$= \frac{2}{n\pi}\frac{\pi}{n}(-1)^n$$

$$\boxed{b_n = \frac{2(-1)^n}{n^2}}$$

$\therefore$ Sum of (a_3, b_2, a_4, b_6)

$$a_3 = \frac{4(-1)^4}{9} = \frac{4}{9}$$

$$a_4 = \frac{4(-1)^5}{16} = \frac{-4}{16} = \frac{-1}{4}$$

$$b_2 = \frac{2(-1)^2}{4} = \frac{1}{2}$$

$$b_6 = \frac{2(-1)^6}{36} = \frac{1}{18}$$

$$\text{Sum} = \frac{4}{9} - \frac{1}{4} + \frac{1}{2} + \frac{1}{18} = \frac{9}{18} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

* $f(x) = |x| - x$ in the interval $[-\pi, \pi]$

$$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$$f(x) = \begin{cases} -x - x & -\pi \leq x < 0 \\ x - x & 0 \leq x \leq \pi \end{cases}$$

$$\boxed{f(x) = \begin{cases} -2x & -\pi \leq x < 0 \\ 0 & 0 \leq x \leq \pi. \end{cases}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -2x \, dx + \int_{0}^{\pi} 0 \, dx\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{-2x^2}{2}\right]_{-\pi}^{0} + 0$$

$$= \frac{-2}{2\pi}(0 - \pi^2) = \frac{2\pi^2}{2\pi} = \pi$$

$a_0 = \pi$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-2x) \cos nx \, dx + \int_{0}^{\pi} (0) \cos nx \, dx\right]$$

$$= \frac{-2}{\pi}\int_{-\pi}^{0} x \cos nx \, dx$$

$$= \frac{-2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{-2}{\pi}\left[0 + \frac{1}{n^2} - 0 - \frac{(-1)^n}{n^2}\right]$$

$$\boxed{a_n = \frac{-2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-2x) \sin nx \, dx + \int_{0}^{\pi} (0) \sin nx \, dx\right]$$

$$= \frac{-2}{\pi}\int_{-\pi}^{0} x \sin nx \, dx = \frac{-2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{-2}{\pi}\left[0 + 0 - \frac{\pi(-1)^n}{n} - 0\right]$$

$$\boxed{b_n = \frac{2(-1)^n}{n}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

→ Fourier Series

$$f(x) = \frac{\pi}{2} + \left(a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \dots \right.$$

$$a_n = \frac{-2}{n^2 \pi} \left(1 - (-1)^n\right) \qquad b_n = \frac{2(-1)^n}{n}$$

$$a_1 = \frac{-4}{\pi} \qquad b_1 = \frac{-2}{1}$$

$$a_2 = \frac{-2}{4\pi}(0) = 0 \qquad b_2 = \frac{2(-1)^2}{2} = 1$$

$$a_3 = \frac{-2}{9\pi}(2) = \frac{-4}{9\pi} \qquad b_3 = \frac{2(-1)^3}{3} = \frac{-2}{3}$$

$$f(x) = \frac{\pi}{2} + \left(\frac{-4}{\pi} \cos x - 2 \sin + \cos 2x + \left(\frac{-4}{9\pi}\right) \cos 3x - \frac{2}{3} \sin 3x + \dots \right.$$

20/03/2023

* $f(x) = x - x^2 \; [-\pi, \pi]$ - hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}$$

$$x - x^2 = -\frac{\pi^2}{3} + \frac{4}{1^2}\cos x - \frac{4}{2^2}\cos 2x + \frac{4}{3^2}\cos 3x - \cdots\cdots \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x + \cdots\cdots$$

put $x = 0$

$$0 = \frac{-\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}$$

* Expand $f(x)$ as a Fourier Series in the interval $[0, 2\pi]$; $f(x) = \sqrt{1 - \cos x}$.

Hence evaluate $\frac{1}{1*3} + \frac{1}{3*5} + \frac{1}{5*7} + \cdots$

$$f(x) = \sqrt{1 - \cos x} = \sqrt{2}\sin x/2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sqrt{2} \sin x/2 \, dx$$

$$= \frac{\sqrt{2}}{\pi} \left[\frac{-\cos x/2}{1/2} \right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi} \cdot \frac{-2}{1} \left[-1 - 1 \right]$$

$$= \frac{2\sqrt{2}}{\pi} \cdot \cancel{2} = \underline{\underline{\frac{4\sqrt{2}}{\pi}}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sqrt{2} \sin x/2 \cos nx \, dx$$

$$= \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} 2 \sin x/2 \cos nx \, dx$$

$$= \frac{1}{\sqrt{2}\pi} \left\{ \int_0^{2\pi} \left[\sin\left(\frac{x}{2} + nx\right) + \sin\left(\frac{x}{2} - nx\right) \right] dx \right.$$

$$= \frac{1}{\sqrt{2}\pi} \left[\int_0^{2\pi} \sin\left(\frac{x}{2} + nx\right) dx + \int_0^{2\pi} \sin\left(\frac{x}{2} - nx\right) dx \right.$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\left[-\frac{\cos\left(\frac{1}{2}+n\right)x}{\left(\frac{1}{2}+n\right)}\right]_0^{2\pi} + \left[-\frac{\cos\left(\frac{1}{2}-n\right)x}{\left(\frac{1}{2}-n\right)}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{2}{1+2n}\left[-\cos\left(\frac{1}{2}+n\right)x\right]_0^{2\pi} + \frac{2}{1-2n}\left[-\cos\left(\frac{1}{2}-n\right)x\right]_0^{2\pi}\right\}$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{2}{1+2n}\left[-\cos\left(\frac{2n+1}{\cancel{2}}\right)\cancel{2}\pi + \cos(0)\right] + \frac{2}{1-2n}\left[-\cos\left(\frac{1-2n}{\cancel{2}}\right)\cancel{2}\pi + \cos(0)\right]\right\}$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{2}{1+2n}\left[+1+1\right] + \frac{2}{1-2n}\left[+1+1\right]\right\}$$

$$= \frac{2}{\sqrt{2}\pi}\left[\frac{2}{(2n+1)} - \frac{2}{(2n-1)}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{(2n+1)(2n-1)}\right]$$

$$= -\frac{4\sqrt{2}}{\pi}\cdot\frac{1}{(4n^2-1)}$$

$$= \frac{-8}{\pi\sqrt{2}(4n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\,\sin x/2 \cdot \sin nx \, dx$$

$$= \frac{-1}{\sqrt{2}\pi}\int_0^{2\pi} -2\sin x/2 \cdot \sin nx \, dx$$

$$= \frac{-1}{\sqrt{2}\pi}\left\{\int_0^{2\pi}\left[\cos\left(\frac{x}{2}+nx\right) - \cos\left(\frac{x}{2}-nx\right)\right]dx\right.$$

$$= \frac{-1}{\sqrt{2}\pi}\left\{\int_0^{2\pi}\cos\left(\frac{x}{2}+nx\right)dx - \int_0^{2\pi}\cos\left(\frac{x}{2}-nx\right)dx\right\}$$

$$= \frac{-1}{\sqrt{2}\pi}\left\{\left[\frac{\sin\left(\frac{1}{2}+n\right)x}{\left(\frac{1}{2}+n\right)}\right]_0^{2\pi} - \left[\frac{\sin\left(\frac{1}{2}-n\right)x}{\left(\frac{1}{2}-n\right)}\right]_0^{2\pi}\right.$$

$$= \frac{-1}{\sqrt{2}\pi}\left\{\frac{2}{2n+1}\left[\sin\left(\frac{1+2n}{\cancel{2}}\right)\cancel{2}\pi - \sin(0)\right] - \frac{2}{1-2n}\left[\sin\left(\frac{1-2n}{2}\right)2\pi - \sin(0)\right]\right\}$$

$$= \frac{-1}{\sqrt{2}\pi}\left\{ 0 - 0 \right\}$$

$$= \underline{0.}$$

$b_1 = 0$

$$f(x) = \frac{2\sqrt{2}}{\pi} + \left[\frac{-8}{\pi\sqrt{2}(3)}\cos x - \frac{8}{\pi\sqrt{2}\,3.5}\cos 2x - \frac{8}{\pi\sqrt{2}(5.7)}\cos 3x - \cdots \right]$$

$$= \frac{2\sqrt{2}}{\pi} - \frac{8}{\pi\sqrt{2}}\left[\frac{1}{3}\cos x + \frac{1}{3.5}\cos 2x + \cdots \right]$$

put $x = 0$

$$\sqrt{2}\sin\frac{\pi}{2} = \frac{2\sqrt{2}}{\pi} - \frac{8}{\pi\sqrt{2}}\left[\frac{1}{3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \right]$$

$$\sqrt{2} - \frac{2\sqrt{2}}{\pi} = -\frac{8}{\pi\sqrt{2}}\left[\frac{1}{3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \right]$$

$$\frac{\cancel{\sqrt{2}}(\pi - 2)}{\cancel{\pi}} = -\frac{\overset{4}{\cancel{8}}}{\cancel{\pi}\cancel{\sqrt{2}}}\left[\frac{1}{3} + \frac{1}{3.5} + \cdots \right]$$

$$\frac{2-\pi}{4} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots$$

→ P.T. $f(x) = x^2$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}, \quad [-\pi, \pi]$$

hence show that

(i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

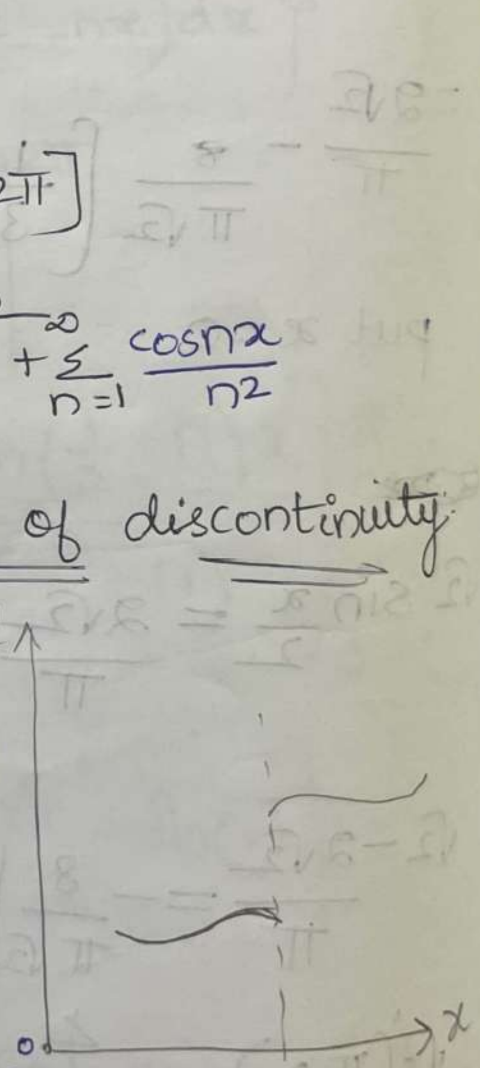
~~(iii)~~

(iii) $f(x) = \left(\frac{\pi - x}{2}\right)^2, \quad [0, 2\pi]$

S.T. $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

23/03/2023

* Functions having points of discontinuity



For instance; if in the interval $(\alpha, \alpha + 2\pi)$ $f(x)$ is defined by

$$f(x) = \phi(x) \quad \alpha < x < c$$
$$= \psi(x), \quad c < x < \alpha + 2\pi$$

i.e. c is the point of discontinuity, then

(In deriving the Euler's formulae for a_0, a_n, b_n; it was assumed that $f(x)$ was continuous. Instead a fn may have a finite no. of points of finite discontinuity i.e. its graph may consist of a finite no. of different curves given by different equations. Even then such a function is expressible as a Fourier Series)

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\cos nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\sin nx\,dx + \int_{c}^{\alpha+2\pi} \phi(x)\sin nx\,dx\right]$$

* **Note:**

$$f(x) = \frac{a_0}{2} + c_1\cos x + b_1\sin x + a_2\cos 2x + b_2\cos 2x + \ldots\ldots$$

$$\frac{1}{2}\left[f(c-0) + f(c+0)\right] = ''$$

*→ Find the Fourier Expansion for $f(x)$

if $f(x)$ is ~~f(x) = -1~~

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

*→ Fourier Series Expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$\frac{1}{2}\left[-\pi + x\right] = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$
$$+ a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$\therefore a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \, dx + \int_{0}^{\pi} x \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(0+\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi} \cdot \frac{-\pi^2}{2} = \frac{-\pi}{2}$$

$$a_0 = -\pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[-\pi\frac{(\sin nx)}{n}\right]_{-\pi}^{0} + \left[\frac{x(\sin nx)}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[0 - 0\right] + \left(\frac{\pi(0)}{n} + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right)\right\}$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$\boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$$

$$a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{-2}{n^2\pi} & n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[+\pi\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{x(-\cos nx)}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi - (-1)^n\pi}{n} + \frac{-\pi(-1)^n}{n} + 0 - 0 - 0\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{\pi}(1-(-1)^n)}{n} - \frac{\cancel{\pi}}{n}(-1)^n\right]$$

$$b_n = \frac{(1-(-1)^n) - (-1)^n}{n}$$

$$\boxed{b_n = \frac{1-2(-1)^n}{n}}$$

$b_1 = 3$

$b_2 = \frac{-1}{2}$

$b_3 = 1.$

$$= \frac{-\pi}{4} - \frac{2}{\pi}\cos x - \frac{2}{9\pi}\cos 3x - \frac{2}{25\pi}\cos 5x - \ldots$$

$$+ 3\sin x - \frac{1}{2}\sin 2x + \sin 3x + \ldots$$

put $x = 0$ / $x = \pi$

$$\cancel{\frac{+1}{2}[\pi + x] = \frac{a_0}{2} +}$$

$$\frac{+1}{2}[-\pi + x] = \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\frac{-\pi}{2} + \frac{\pi}{4} = \frac{-2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$-\frac{\pi}{4} \times \frac{\pi}{2} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

* $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$

P.T $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$

Hence deduce that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots = \frac{1}{4}(\pi-2)$$

$$f(x) = \frac{1}{2}[0+\sin x] = \frac{\sin x}{2}$$

* Fourier Series Expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\,dx\right]$$

$$= -\frac{[\cos x]_0^{\pi}}{\pi}$$

$$= \frac{-1}{\pi}(-1-1) = \frac{2}{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos nx \, dx + \frac{1}{2}\int_{0}^{\pi} 2\sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\int_{0}^{\pi}\left(\sin(x+nx) + \sin(x-nx)\right)dx$$

$$= \frac{1}{2\pi}\left[\int_{0}^{\pi}\sin x(n+1)\,dx + \int_{0}^{\pi}\sin x(1-n)\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-\cos x(n+1)}{n+1}\right]_{0}^{\pi} + \left[\frac{-\cos x(1-n)}{1-n}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-1}{n+1}\left((-1)^n - 1\right) + \left(\frac{-1}{1-n}\right)\left((-1)^n - 1\right)\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-1}{(n+1)}\left((-1)^n - 1\right) + \left(\frac{-1}{1-n}\right)\left((-1)^n - 1\right)\right\}$$

$$a_n = \frac{\left((-1)^n - 1\right)}{2\pi}\left[-1\left[\frac{1-\cancel{n}+1+\cancel{n}}{(1+n)(1-n)}\right]\right]$$

$$= \frac{1-(-1)^n}{2\pi}\left[\frac{2}{1-n^2}\right]$$

$$=\frac{1}{2\pi}\left\{((-1)^n-1)\left[(-1)\left[\frac{1-n+1+n}{(1+n)(1-n)}\right]\right]\right.$$

$$=\frac{1-(-1)^n}{\cancel{2}\pi}\left[\frac{\cancel{2}}{1-n^2}\right]$$

$$=\frac{(1-(-1)^n)}{\pi(1-n^2)} = \frac{(1+(-1)^{n+1})}{\pi(1-n^2)}$$

$n=2n$

$$\frac{1+(-1)^{2n+1}}{(-\pi)(4n^2-1)}$$

$$=\frac{-1+(-1)^{2(n+1)}}{\pi(4n^2-1)}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\,dx+\frac{1}{2}\int_{0}^{\pi}2\sin x\sin nx\,dx\right]$$

$\therefore b_n=0$

$n=1$

$$=\frac{1}{\pi}\cdot\frac{1}{2}\int_0^{\pi}2\sin^2 x\,dx$$

$$=\frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$=\frac{1}{2\pi}\left[(\pi-0)-\frac{\sin 2x}{2}\right]$$

$$b_1=\frac{1}{2}$$

$$\boxed{f(x)=\frac{1}{\pi}+\frac{\sin x}{2}+\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{\pi(1-n^2)}}$$

* Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx .$$

$$f(x) = \frac{1}{\pi} + b_1 \sin x + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{\cos 2nx}{n^2-1}\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \left(\frac{-2}{\pi}\right)\left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right]$$

$$f(x) = \frac{0+\sin x}{2} = \frac{\sin x}{2} \quad \text{at } x = \pi/2$$

$$f(\pi/2) = \frac{1}{2}$$

$$\frac{\sin x}{2} =$$

$$x = \pi/2$$

$$\cancel{\frac{1}{2} = \frac{1}{\pi} + \frac{1}{2}(1) - \frac{2}{\pi}}$$

$$\frac{1}{2} = \frac{1}{\pi} + \frac{2}{\pi}\left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right]$$

$$\frac{\pi - 2}{2\times 2} = \left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots\right]$$

$$\frac{1}{4}(\pi - 2) = \left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots\right]$$

* Given $f(x) = x^2 \quad [-\pi, \pi]$

Fourier Series Expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

By Euler's formulae.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^3 + \pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n\,x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \Rightarrow \text{even fn}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - \int 2x \frac{\sin nx}{n}\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{x^2 \sin nx}{n} - 2\left[x\left(-\frac{\cos nx}{n^2}\right) + \frac{\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

odd fn

$$b_n = 0$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$* \quad f(x) = \left(\frac{\pi - x}{2}\right)^2 \quad [0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{3(-1)}\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[(\pi - x)^3\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left((-\pi)^3 - \pi^3\right) = \underline{\underline{\frac{\pi^2}{6}}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 + x^2 - 2\pi x)\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\frac{\pi^2}{n}\sin nx + x^2\frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}\right.$$

$$\left. - 2\pi\left[x\frac{\sin nx}{n} - \int\frac{\sin nx}{n}\,dx\right]\right\}$$

$$= \frac{1}{4\pi}\left[\frac{\pi^2\sin nx}{n} + \frac{x^2\sin nx}{n} - 2\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right.$$

$$\left. - 2\pi\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[0 + 0 - 2\left(0 + \frac{1}{n^2} - 1\right) - 2\pi\left(0 + 0\right] - 0\right.$$

*Change of Interval: 24/03/2023

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha+2c)$.

To change the problem to period 2π.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \text{—(1)}$$

put $\alpha = 0$ in (1); we get the results for the interval $(0, 2c)$ & put $\alpha = -c$ in (1); we get the results for the interval $(-c, c)$.

* $f(x) = 2x - x^2$ in the interval $[0, 3]$.

deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\cdots = \frac{\pi^2}{12}$

$[0, 2c] = [0, 3]$

$c = 3/2$ Fourier Series Expansion in $[0, c]$

$\cancel{a_0 = \frac{2}{3}\int^{3}}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n x\pi}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx \qquad a_0 = \frac{1}{c}\int_0^{2c} f(x)\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[(9-0) - \frac{1}{3}(27-0)\right]$$

$$= \frac{2}{3}(9-9) = 0$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n x\pi}{c}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{n x\pi\, 2}{3}\,dx$$

$$a_n = \frac{2}{3}\left[2\int_0^3 x\cos\frac{2n\pi x}{3}\,dx - \int_0^3 x^2\cos\frac{2n\pi x}{3}\,dx\right]$$

$$a_n = \frac{2}{3}\left[2\left[\frac{x\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} + \frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right]_0^3\right.$$

$$\left. - \left[\frac{x^2\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} - \frac{2\times 3}{2n\pi}\int x\sin\frac{2n\pi}{3}x\,dx\right]_0^3\right]$$

$$= \quad a_n = \frac{2}{3}\left[2\left(0 + \frac{9}{4n^2\pi^2}(0)\right)\right]$$

$$+\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{2}\times\cancel{3}}{\cancel{2}n\pi}\left[x\left(\frac{-\cos 2n\pi x/3}{2n\pi/3}\right) + \frac{3}{2n\pi}\,\frac{\sin 2n\pi x/3}{2n\pi/3}\right]_0^3$$

$$= \frac{\cancel{2}}{n\pi}\left[\frac{-3\times 3}{\cancel{2}n\pi}(1) - 0 + 0\right]$$

$$\boxed{a_n = \frac{-9}{n^2\pi^2}}$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[2\int_0^3 x\sin\frac{2n\pi x}{3}\,dx - \int_0^3 x^2\sin\frac{2n\pi x}{3}\,dx\right]$$

$$= \frac{4}{3}\left[x\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) + \frac{\sin\frac{2n\pi x}{3}\cdot 9}{4n^2\pi^2}\right]_0^3$$

$f(x) = e^{-x}$

$$-\frac{2}{3}\left[x^2\frac{\left(-\cos\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} + 2\int x\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\,dx\right]_0^3$$

$$= \frac{4}{3}\left[3(-1)\frac{3}{2n\pi} + 0 - 0 - 0\right]$$

$$-\frac{2}{3}\left[\frac{9(-1)\cancel{3}}{2n\pi} - 0\right]$$

$f(x) = e^{-x}$

$[-l, l]$

$$-\frac{4}{3}\left[\frac{3}{2n\pi}\right]\left[x\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} + \frac{3}{2n\pi}\cdot\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right]_0^3$$

$$= \frac{\cancel{4}^{2}}{\cancel{3}}\left(\frac{-\cancel{9}^{3}}{\cancel{2}n\pi}\right) + \frac{9}{n\pi} - \frac{2}{n\pi}\left[0 + \frac{\cancel{9}}{\cancel{4}n^2\pi^2} - 0 - \frac{\cancel{9}}{\cancel{4}n^2\pi^2}\right]$$

$$= \frac{-6}{n\pi} + \frac{9}{n\pi} = \frac{3}{n\pi}$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$$\cancel{f(x) = \frac{-9}{\pi^2}}$$

Beauty itself

$$f(x) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{-9}{n^2\pi^2} \cos\frac{n\pi x}{c} + \frac{3}{n\pi} \sin\frac{n\pi x}{c}\right)$$

$$c = \frac{3}{2}$$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{n\pi}\left[\frac{-3}{n\pi} \cos\frac{2n\pi x}{3} + \sin\frac{2n\pi x}{3}\right]\right)$$

$$= \frac{3}{n\pi} \sum_{n=1}^{\infty} \left(\frac{-3}{n\pi} \cos\frac{2n\pi x}{3} + \sin\frac{2n\pi x}{3}\right)$$

$$= \frac{3}{n\pi}\left[\frac{-3}{\pi} \cos\frac{2\pi x}{3} - \frac{3}{2\pi} \cos\frac{4\pi x}{3} - \cdots \right.$$

$$\left. + \sin\frac{2\pi x}{3} + \sin\frac{4\pi x}{3} + \cdots\right]$$

$$2x - x^2 = \frac{3}{n\pi}\left[\frac{-3}{\pi} \cos 2\pi x - \frac{3}{2\pi}\right.$$

$$3 - \frac{9}{4} = \frac{3}{4} \qquad \frac{9}{\pi^2}$$

$$2x - x^2 = \frac{-9}{\pi^2} \cos\frac{2\pi x}{3} + \frac{3}{\pi} \sin\frac{2\pi x}{3} - \frac{9}{4\pi^2} \cos\frac{4\pi x}{3}$$

$x = 3$

$$+ \frac{3}{2\pi} \sin\frac{4\pi x}{3} + \cdots$$

$$\frac{3}{4} = \frac{-9}{\pi^2} - \frac{9}{4\pi^2} - \cdots$$

$$\frac{3}{4} = \frac{9}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$* f(x) = e^{-x} \quad [-l, l]$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$\boxed{a_0 = \frac{e^{l} - e^{-l}}{l}}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx$$

$$= \frac{-1}{l}\left(e^{-l} - e^{l}\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[e^{-x}\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \int e^{-x}\frac{\sin\frac{\pi x}{l}}{n\pi/l}\,dx\right]_{-l}^{l}$$

$$= \frac{1}{\cancel{l}}\cdot\frac{\cancel{l}}{n\pi}\left[e^{-x}\sin\frac{n\pi x}{l}\right]_{-l}^{l}$$

$$+ \frac{1}{l}\cdot\frac{l}{n\pi}\left[e^{-x}\frac{\left(-\cos\frac{n\pi x}{l}\right)}{n\pi/l} - \frac{l}{n\pi}\int e^{-x}\cos\frac{n\pi x}{l}\,dx\right]_{-l}^{l}$$

$$= \frac{1}{n\pi} \cdot \frac{l}{n\pi} \left[e^{-x} \left(\cos \frac{n\pi x}{l} \right) \right]_{-l}^{l} - \frac{l^2}{n^2\pi^2} \frac{1}{l} \int e^{-x} \cos$$

$$b = \frac{l}{n^2\pi^2} \left[e^{-l} (-1)^{n+1} - e^{l} (-1)^{n+1} \right]$$

$$- \frac{l^2}{n^2\pi^2} a_n$$

$$a_n \left[\frac{n^2\pi^2 + l^2}{\cancel{n^2\pi^2}} \right] = \frac{l}{\cancel{n^2\pi^2}} (-1)^{n+1} \left[e^{-l} - e^{l} \right]$$

$$\boxed{a_n = \frac{l(-1)^n}{(n^2\pi^2 + l^2)} \left(e^{l} - e^{-l} \right)}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \int_{-l}^{l} e^{-x} \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \cdot \frac{e^{-x}}{1 + \frac{n^2\pi^2}{l^2}} \left[-\sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l} \right]$$

$$= \left\{ \frac{-1}{l} \frac{l^2 e^{-x}}{n^2\pi^2 + l^2} \left[\sin\frac{n\pi x}{l} + \frac{n\pi}{l} \cos\frac{n\pi x}{l} \right] \right\}_{-l}^{l}$$

$$= \left\{ \frac{-l e^{-x}}{n^2\pi^2 + l^2} \left(\sin\frac{n\pi x}{l} + \frac{n\pi}{l} \cos\frac{n\pi x}{l} \right) \right\}_{-l}^{l}$$

$$= \frac{-l e^{-l}}{n^2\pi^2 + l^2} \left(0 + \frac{n\pi}{l} (-1)^n \right)$$

$$+ \frac{l e^{l}}{n^2\pi^2 + l^2} \left(0 + \frac{n\pi}{l} (-1)^n \right)$$

$$= \frac{-l}{n^2\pi^2 + l^2} \cdot \frac{n\pi}{\cancel{l}} (-1)^n \frac{\left(e^{-l} - e^{l} \right)}{1}$$

$$= \frac{n\pi (-1)^n}{n^2\pi^2 + l^2} \left(e^{l} - e^{-l} \right)$$

25/03/2023

* $f(x) = \begin{cases} +\pi x & 0 < x \leq 1 \\ \pi(2-x) & 1 \leq x \leq 2 \end{cases}$

Hence ~~deda~~ evaluate $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Given interval $= [0, 2]$

$$2l = 2$$
$$l = 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{1} \left[\int_0^1 \pi x\, dx + \int_1^2 (2\pi - \pi x)\, dx \right]$$

$$= \frac{\pi}{2} + 2\pi(2-1) - \frac{\pi}{2}(4-1)$$

$$= \frac{\pi}{2} + 2\pi - \frac{3\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$a_0 = \pi$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 (2\pi - \pi x)\cos n\pi x\,dx$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} - \int \frac{\sin n\pi x}{n\pi}\,dx\right]_0^1$$

$$+ \left[2\pi\frac{\sin n\pi x}{n\pi}\right]_1^2 - \pi\left[x\frac{\sin n\pi x}{n\pi} - \int\frac{\sin n\pi x}{n\pi^2}\,dx\right]_1^2$$

$$= \pi\left[0 + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$+ 2\pi(0) - \pi\left[0 + \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\frac{(-1)^n - 1}{n^2\pi^2}\right] - \pi\left[\frac{1-(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{\cancel{\pi}}{n^2\pi^{\cancel{2}}}\left[(-1)^n - 1 - 1 + (-1)^n\right]$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{2}{n^2\pi}\left((-1)^n - 1\right)}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\, \sin\frac{n\pi x}{l}\, dx$$

$$= \int_0^2 f(x)\, \sin n\pi x\, dx$$

$$= \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 (2\pi - \pi x) \sin n\pi x$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$+\left[2\pi\left(\frac{-\cos n\pi x}{n\pi}\right)\right]_1^2 - \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\frac{-(-1)^n}{n\pi} + 0 + 0 + 0\right] + 2\pi\left[(-1)\left[\frac{1 - (-1)^n}{n\pi}\right]\right]$$

$$-\pi\left[\frac{(-2)(1)}{n\pi}+0-\frac{(-1)(-1)^n}{n\pi}-0\right]$$

$$=\frac{\cancel{\pi}(-1)^{n+1}}{n\cancel{\pi}}+2\cancel{\pi}\left(\frac{(-1)^n-1}{n\cancel{\pi}}\right)-\cancel{\pi}\left(\frac{-2+(-1)^n}{n\cancel{\pi}}\right)$$

$$=\frac{(-1)^{n+1}}{n}+\frac{2(-1)^n-2}{n}+\frac{2-(-1)^n}{n}$$

$$=\frac{-\cancel{(-1)^n}+2(-1)^n-\cancel{2}+\cancel{2}-\cancel{(-1)^n}}{n}$$

$$b_n=0$$

$$b_1=\int_0^1 \pi x\sin\pi x\,dx+\int_1^2(2\pi-\pi x)\sin\pi x\,dx$$

Fourier Series:

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\left((-1)^n-1\right)\cos nx+0\right)$$

$$\cancel{=\frac{\pi}{2}}$$

$$f(x) = \frac{\cancel{\pi x} + 2\pi - \cancel{\pi x}}{2}$$

$$= \pi$$

$$\pi = \frac{\pi}{2} + \frac{2}{\pi}(-2)\cos\frac{x\pi}{l}$$

$$+ \frac{2}{4\pi}(0)\cos 2\frac{x\pi}{l}$$

$$+ \frac{2}{9\pi}(-2)\cos 3\frac{x\pi}{l} + \cdots$$

$x = x$

$$\frac{\pi}{2} = \frac{-4}{\pi}\left[(-1)\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]\right]$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots}$$

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

$$f(x) = 1 \qquad [-l, l] = [-2, 2]$$

$$l = 2$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 1\,dx\right]$$

$$= \frac{1}{2}\left[0 + (2-0)\right] =$$

$$a_0 = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad a_1 = 0$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2}\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right]_0^2$$

$$= \frac{1}{n\pi}(0-0)$$

$$a_n = 0.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2}\, dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0 \sin\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_{0}^{2} 1 \sin\frac{n\pi x}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{b_n = \frac{1-(-1)^n}{n\pi}}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1+(-1)^{n+1}}{n\pi}\right)$$

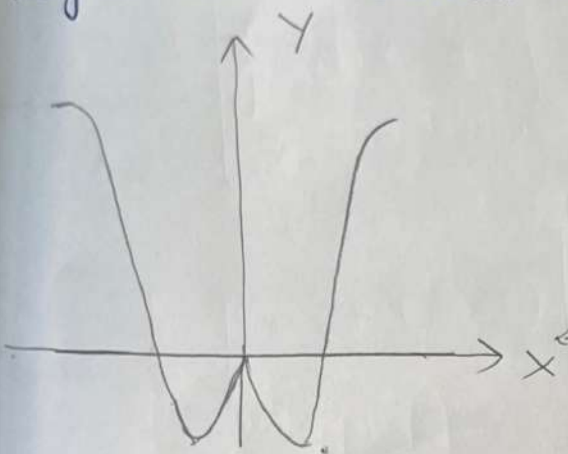
* EVEN AND ODD FUNCTIONS: 27/03/2023

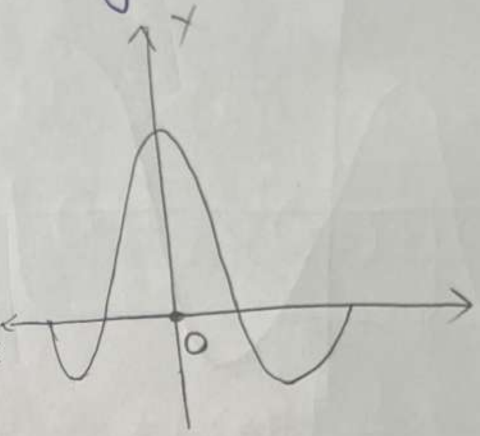
→ Even function:

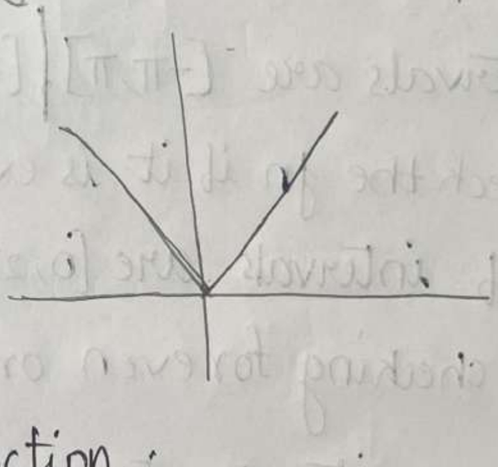
A function $f(x)$ is said to be even if $\boxed{f(-x) = f(x)}$

Ex: $\cos x, \sec x, x^2$

** Graphically even function is symmetrical about the y-axis.







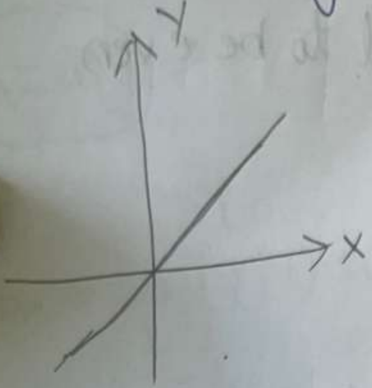
→ Odd function:

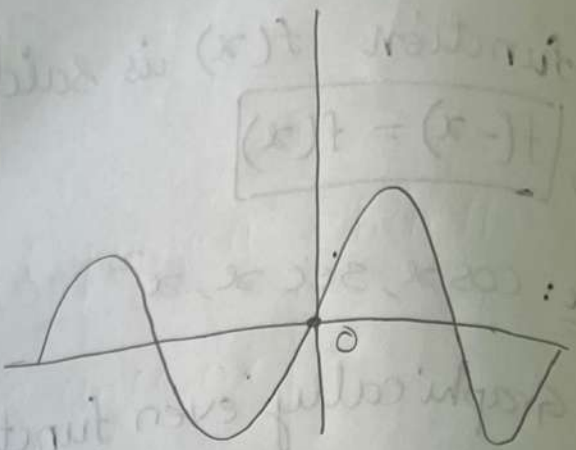
A function $f(x)$ is said to be odd if

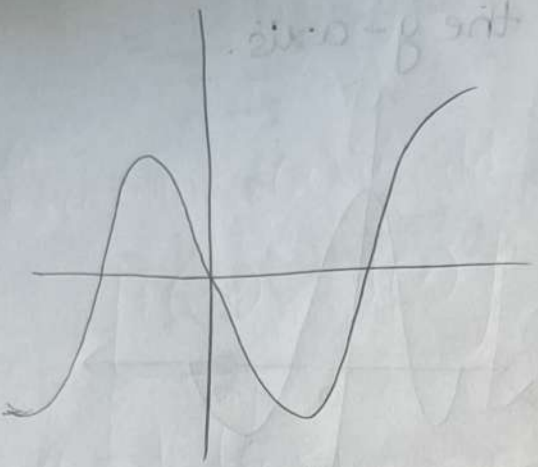
$$f(-x) = -f(x)$$

Ex: $\sin x, \tan x, x^3$.

Graphically an odd fn is symmetrical about the origin







→ If the intervals are $[-\pi, \pi]$ / $[-l, l]$ then we must check the fn if it is even or odd and if intervals are $[0, 2\pi]$ / $[0, 2l]$ no need of checking for even or odd fn.

- then we can write the Fourier series and find the Euler's constants.

* $+ \times + = + = - \times - =$ Even * Even = Odd * Odd.
* $- * + = - =$ Even * Odd = Odd * Even.

* **Property of definite integral :**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx & f(x) \text{ is even.} \end{cases}$$

→ We have an expansion of $f(x)$ as a Fourier Series in the interval $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

*→ **Case (i):** $f(x)$ is an even function.

$$\boxed{f(-x) = f(x)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)}^{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\boxed{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}} \underbrace{\cos nx}_{\text{even}}\,dx$$

$$\boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx}$$

$$b_n = \int_{-\pi}^{\pi} \underset{\text{even}}{f(x)} \cdot \underset{\text{odd}}{\sin nx}\, dx$$

$$\Rightarrow \boxed{b_n = 0}$$

$\therefore$ Fourier Series Expansion: when $f(x)$ is even

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

* case(ii): $f(x)$ is odd function

$$\boxed{f(-x) = -f(x)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$ (Property of definite integral)

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{f(x)}\, \underset{\text{even}}{\cos nx}\, dx$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{f(x)} \cdot \underset{\text{odd}}{\sin nx}\, dx$$

$$\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

Fourier Series Expansion - when $f(x)$ is odd

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

→ We have an expansion of $f(x)$ as a Fourier Series in the interval $[-l, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Case(i): If $f(x)$ is an even function:

$$\boxed{f(-x) = f(x)}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \Rightarrow$$

$$\boxed{a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{even}} \underbrace{\cos\frac{n\pi x}{l}}_{\text{even}}\,dx$$

$$\boxed{a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx}$$

$$b_n = \frac{1}{\pi}\int_{-l}^{l} \underbrace{f(x)}_{\text{even}} \underbrace{\sin\frac{n\pi x}{l}}_{\text{odd}} dx$$

$$\boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

Case(ii): $f(x)$ is odd function.

$$\boxed{f(-x) = -f(x)}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd}} dx$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd}} \underbrace{\cos\frac{n x\pi}{l}}_{\text{even}} dx$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin\frac{n x\pi}{l}}_{\text{odd}} dx$$

$$\boxed{b_n = \frac{2}{l}\int_{0}^{l} f(x) \sin\frac{n x\pi}{l} dx}$$

~~bn=~~

$\cancel{f(x) = \frac{a_0}{2} +}$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

* Expand Fourier Series $f(x) = x^2$ in the interval $[-l, l]$.

$f(x) = x^2 \quad ; \quad f(-x) = x^2$

$\Rightarrow f(x)$ is even fn.

$\therefore$ Fourier Series Expansion of $f(x) = x^2$ in the interval $[-l, l]$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left(\frac{l^3}{3}\right) =$$

$$\boxed{a_0 = 2l^2/3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}$$

$$= \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2 \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \int 2x \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l x^2}{n\pi}\sin\frac{n\pi x}{l} + \frac{2x\,l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} - 2\sin\frac{n\pi x}{l} \times \frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{n\pi}(0) + \frac{2l^3}{n^2\pi^2}(-1)^n - 0 - 0 - 0 - 0\right]$$

$$= \frac{2}{\cancel{l}} \cdot \frac{2l^{\cancel{3}\,2}}{n^2\pi^2}(-1)^n$$

$$= \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$\boxed{b_n = 0}$$

$$\boxed{a_n = \frac{4l^2(-1)^n}{n^2\pi^2}}$$

$$\boxed{f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l(-1)^n}{n^2\pi^2} \cos nx}$$

* $|x| = f(x)$ $[-\pi, \pi]$ hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$$

* $f(x) = \sin ax$ $[-\pi, \pi]$

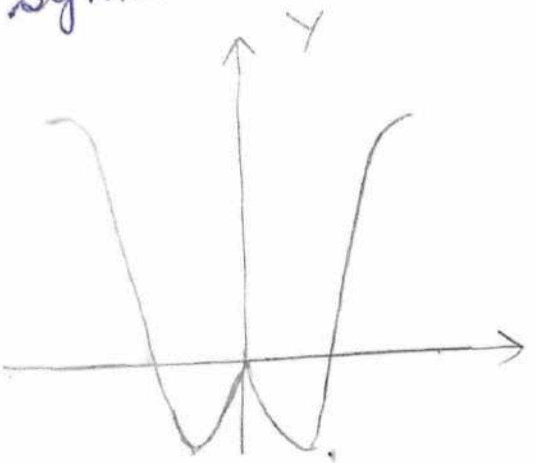
* EVEN AND ODD FUNCTIONS: 27/03/2023

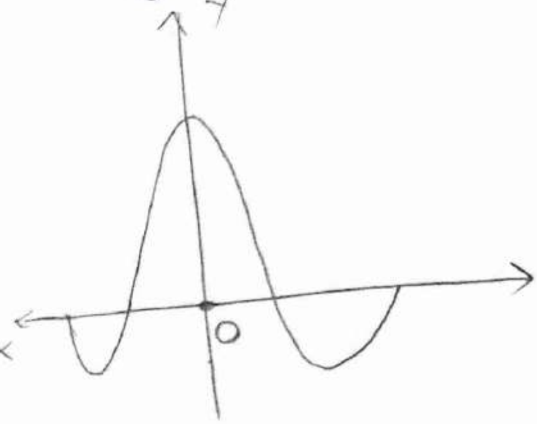
→ Even function:

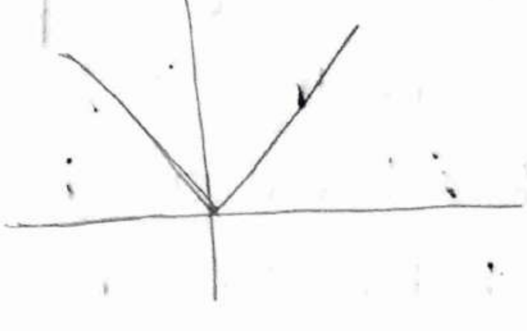
A function $f(x)$ is said to be even if $\boxed{f(-x) = f(x)}$

Ex: $\cos x$, $\sec x$, x^2

** Graphically even function is symmetrical about the y-axis.







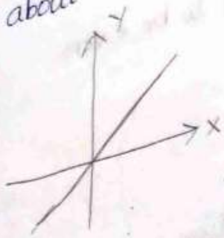
→ Odd function:

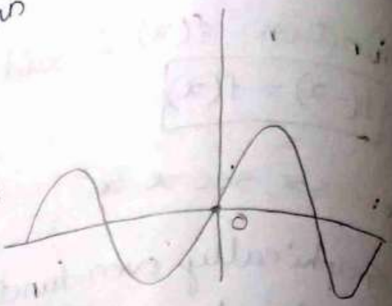
A function $f(x)$ is said to be odd if

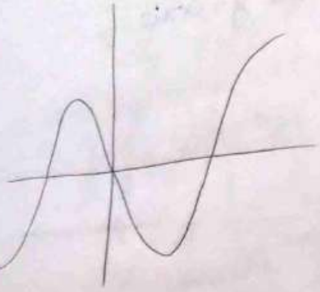
$$f(-x) = -f(x)$$

Ex: $\sin x$, $\tan x$, x^3.

Graphically an odd fn is symmetrical about the origin







→ If the intervals are $[-\pi,\pi]$ / $[-l,l]$ then we must check the fn if it is even or odd and if intervals are $[0,2\pi]$ / $[0,2l]$ no need of checking for even or odd fn.

then we can write the Fourier series and find the Euler's constants.

* $+ \times + = + = - \times -$ = Even * Even = Odd * Odd

* $- * + = -$ = Even * Odd = Odd * Even.

* **Property of definite integral:**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx & f(x) \text{ is even.} \end{cases}$$

→ We have an expansion of $f(x)$ as a Fourier Series in the interval $[-\pi,\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

*→ Case (i): $f(x)$ is an even function.

$$\boxed{f(-x) = f(x)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\to \text{even}}{f(x)}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\boxed{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}} \underbrace{\cos nx}_{\text{even}}\,dx$$

$$\boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx}$$

$$b_n = \int_{-\pi}^{\pi} \underset{\text{even}}{f(x)} \cdot \underset{\text{odd}}{\sin nx}\, dx$$

$$\Rightarrow \boxed{b_n = 0}$$

$\therefore$ Fourier Series Expansion: when $f(x)$ is even

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

→ case(ii): $f(x)$ is odd function

$$\boxed{f(-x) = -f(x)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \quad \text{(Property of definite integral)}$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{f(x)}\ \underset{\text{even}}{\cos nx}\,dx$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{f(x)}\ \underset{\text{odd}}{\sin nx}\,dx$$

$$\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

Fourier Series Expansion - when $f(x)$ is odd

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

→ We have an expansion of $f(x)$ as a Fourier Series in the interval $[-l, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Case(i): If $f(x)$ is an even function:

$$\boxed{f(-x) = f(x)}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$\boxed{a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underset{\text{even}}{f(x)}\ \underset{\text{even}}{\cos\frac{n\pi x}{l}}\,dx$$

$$\boxed{a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx}$$

$$b_n = \frac{1}{\pi}\int_{-l}^{l} \underset{\text{even}}{f(x)}\ \underset{\text{odd}}{\sin\frac{n\pi x}{l}}\,dx$$

$$\boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

case(ii): $f(x)$ is odd function.

$$\boxed{f(-x) = -f(x)}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} \underset{\text{odd}}{f(x)}\,dx$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underset{\text{odd}}{f(x)}\ \underset{\text{even}}{\cos\frac{nx\pi}{l}}\,dx$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underset{\text{odd}}{f(x)}\ \underset{\text{odd}}{\sin\frac{nx\pi}{l}}\,dx$$

$$\boxed{b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{nx\pi}{l}\,dx}$$

~~b_n~~

$\cancel{f(x) = \frac{a_0}{2}} +$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

* Expand Fourier Series $f(x) = x^2$ in the interval $[-l, l]$.

$f(x) = x^2$, $f(-x) = x^2$

$\Rightarrow f(x)$ is even fn.

$\therefore$ Fourier Series Expansion of $f(x) = x^2$ in the interval $[-l, l]$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_{0}^{l} f(x)\,dx = \frac{2}{l}\left(\frac{l^3}{3}\right) =$$

$$\boxed{a_0 = 2l^2/3}$$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}$$

$$= \frac{2}{l}\int_{0}^{l} x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \int 2x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right]_0^l$$

$$= \frac{2}{l}\left[\frac{lx^2}{n\pi}\sin\frac{n\pi x}{l} + \frac{2x\,l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} - 2\sin\frac{n\pi x}{l}\times\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{n\pi}(0) + \frac{2l^3}{n^2\pi^2}(-1)^n - 0 - 0 - 0 - 0\right]$$

$$= \frac{2}{l} \cdot \frac{2l^{3}}{n^2\pi^2}(-1)^n$$

$$= \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$\boxed{b_n = 0}$$

$$\boxed{a_n = \frac{4l^2(-1)^n}{n^2\pi^2}}$$

$$\boxed{f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l(-1)^n}{n^2\pi^2}\cos nx}$$

* $|x| = f(x)$ $[-\pi, \pi]$ hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$$

* $f(x) = \sin ax$ $[-\pi, \pi]$

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* $f(x) = \begin{cases} x & [0, \pi] \\ -x & [-\pi, 0] \end{cases}$ $[-\pi, \pi]$

$[-\pi, 0]$, $[0, \pi]$
$\pi^2/2$ $\pi^2/2$

Fourier Series Expansion in the interval $[-\pi, \pi]$

$\because |x| \Rightarrow$ even fn

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left(\frac{-1}{2}(0-\pi^2) + \frac{1}{2}(\pi^2 - 0)\right)$$

$$= \frac{1}{\pi}(\pi^2) = \pi$$

$$\boxed{\frac{a_0}{2} = \pi/2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{-1}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{-1}{\pi}\left[0 + \frac{1}{n^2} - 0 - \frac{(-1)^n}{n^2}\right]$$

$$+ \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{(-1)^n - 1}{n^2\pi} + \frac{(-1)^n - 1}{n^2\pi}$$

$$= \frac{2\left[(-1)^n - 1\right]}{n^2\pi}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi}\cos nx$$

$$|x| = \frac{\pi}{2} + \frac{-4}{\pi}\cos x + 0 + \frac{-4}{9\pi}\cos 3x + \cdots$$

$x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right] = \pi/2$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}}$$

* $f(x) = \sin ax \quad [-\pi, \pi]$

$f(x)$ is a odd fn.

$\Rightarrow$ Fourier series expansion for $f(x)$ which is a odd fn in the interval $[-\pi, \pi]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx$$

$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$$= \frac{2}{\pi}\int_0^{\pi} \sin ax \cdot \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin ax \cdot \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(a-n)x\, dx$$

$$- \frac{1}{\pi}\int_0^{\pi} \cos(a+n)x\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{(a-n)}\right]_0^{\pi} - \frac{1}{\pi}\left[\frac{\sin(a+n)x}{(a+n)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{(a-n)} - \frac{\sin(a+n)x}{(a+n)}\right]_0^{\pi}$$

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$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & -\pi \leq x \leq 0 \\ 1-\frac{2x}{\pi} & 0 \leq x \leq \pi \end{cases}$$

$f(-x) = 1-\frac{2x}{\pi}$ in $(-\pi, 0)$ $= f(x)$ in $(0,\pi)$

$f(-x) = 1+\frac{2x}{\pi}$ in $(0,\pi)$ $= f(x)$ in $(-\pi, 0)$ =f(x)

Given fn is an even fn.

Its Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[(\pi-0) - \frac{2}{\pi}\cdot\frac{1}{2}(\pi^2-0)\right]$$

$$= \frac{2}{\pi}\left[\pi - \pi\right] = 0$$

$a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n} - \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - 2(0) - \frac{2}{\pi}\frac{\cos n\pi}{n^2}\right] - \frac{2}{\pi}\left[0 - \frac{2}{\pi}(0) - \frac{2}{\pi}\frac{1}{n^2}\right]$$

$$= -\frac{2}{\pi}\frac{(-1)^n}{n^2} + \frac{2}{\pi}\cdot\frac{2}{\pi}\cdot\frac{1}{n^2}$$

$$= \frac{-4}{\pi n^2}\left[\frac{1-(-1)^n}{1}\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{\pi^2 n^2}\left(1-(-1)^n\right).$$

* Half Range Series:

It is often required to obtain Fourier series of a function $f(x)$ in the interval $(0, \pi)$

→ Sine Series:

If it be required to expand $f(x)$ as a sine series in $(0, \pi)$, ~~we are~~ defined an odd function $f_1(x)$ in $(-\pi, \pi)$ identical with $f(x)$ in $(0, \pi)$. i.e. we extend the function reflecting it w.r.to the origin so that $f(-x) = -f(x)$

→ Hence the half-range sine series in $(0, \pi)$ is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \end{aligned}}$$

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \\ b_n &= \frac{2}{\pi}\int_0^{\cancel{l}} f(x) \sin \frac{n\pi x}{l} \, dx \end{aligned}}$$

→ Cosine Series:

If it be required to expand $f(x)$ as a cosine series in $(0, \pi)$ we define an even function $f_2(x)$ in $(-\pi, \pi)$ identical with $f(x)$ in $(0, \pi)$; i.e. we extend the function reflecting it w.r.to the y-axis so that $f(-x) = f(x)$

Hence the half-range cosine series in $(0, \pi)$ is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n \cos nx + \frac{a_0}{2} \\ a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x) \, dx \\ a_n &= \frac{2}{\pi}\int_0^{\pi} f_1(x) \cos nx \, dx \end{aligned}}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \frac{a_0}{2}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

* $f(x) = e^x$ in $(0,\pi)$

Expansion of $f(x)$ as a half range sine series in $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} e^x \sin nx\,dx\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{e^{\pi}}{n^2+1}(0 - \pi\cos n\pi) - \frac{1}{1+n^2}(0-n)\right]$$

Sivani

$$= \frac{2}{\pi(n^2+1)}\left[-n\cos n\pi\, e^{\pi} + n\right]$$

$$= \frac{2n}{\pi(n^2+1)}\left[-e^{\pi}\cos n\pi + 1\right]$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2n}{\pi(n^2+1)}\left(1 - e^{\pi}(-1)^n\right)\sin nx}$$

* $f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & l/2 \le x \le l \end{cases}$

→ Half range cosine series

Expansion of $f(x)$ as a half range cosine series in $(0,l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\frac{a_0}{2} = \frac{2}{l}\cdot\frac{1}{2}\int_0^l f(x)\,dx$$

$$\frac{a_0}{2} = \frac{1}{l}\left[k\int_0^{l/2} x\,dx + k\int_{l/2}^{l}(l-x)\,dx\right]$$

$$= \frac{1}{l}\left[\frac{k}{2}\cdot\frac{l^2}{4} + kl\left(\frac{l}{2}\right) - \frac{k}{2}\left(\frac{3l^2}{4}\right)\right]$$

$$= \frac{k}{2l}l^2\left[\frac{1}{4} + 1 - \frac{3}{4}\right] = \frac{kl}{2}\left[\frac{1}{2}\right] = \frac{kl}{4}$$

$$\boxed{\frac{a_0}{2} = \frac{kl}{4}}$$

$$a_n = \frac{2}{\pi}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{\pi}\left[k\int_0^{l/2} x\cos\frac{n\pi x}{l}\,dx + k\int_{l/2}^{l}(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{\pi}\left[kx\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} + k\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^{l/2}$$

$$+ \frac{2}{\pi}\left[kl\int_{l/2}^{l}\cos\frac{n\pi x}{l}\,dx - k\int_{l/2}^{l} x\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{kl}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{l}\cdot\frac{l}{2} + \frac{kl^2}{n^2\pi^2}(0) - 0 - \frac{kl^2}{n^2\pi^2}(0)\right]$$

$$+ \frac{2}{\pi}\left[\frac{kl.l}{n\pi}\sin\frac{n\pi x}{l}\right]_{l/2}^{l} - \frac{2k}{\pi}\left[\frac{xl}{n\pi}\sin\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]$$

$$= \frac{2}{\pi}\left[\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2}\right]$$

$$+ \frac{2}{\pi}\left[\frac{kl^2}{n\pi}\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$- \frac{2k}{\pi}\left\{\left[\frac{l^2}{n\pi}(0) - \frac{l^2}{n^2\pi^2}(-1)^n\right] - \left[\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}(0)\right]\right\}$$

$$= \frac{kl^2}{n\pi^2}\sin\frac{n\pi}{2} - \frac{2kl^2}{n\pi^2}\sin\frac{n\pi}{2}$$

$$+ \frac{2kl^2}{n^2\pi^3}(-1)^n + \frac{l^2}{2n\pi}\sin n\pi/2$$

$$a_n = \left(\frac{kl^2}{n\pi^2} - \frac{2kl^2}{n\pi^2} + \frac{l^2}{2n\pi}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{2kl^2}{n^2\pi^3}(-1)^n$$

$$f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{-kl^2}{n\pi^2} + \frac{l^2}{2n\pi}\right)\sin\frac{n\pi}{2} + \frac{(-1)^n 2kl^2}{n^2\pi^3}\right]$$

$$= \frac{kl}{4} + \left(\left(\frac{-kl^2}{\pi^2} + \frac{l^2}{2n\pi}\right) - \frac{2kl^2}{\pi^3}\right)\cos x$$