16/12/2022 UNIT-4

RECURRENCE RELATIONS & GENERATING FUNCTIONS.

PART-1: RECURRENCE RELATIONS

→ Recurrence relations – Introduction, problems

→ ~~Partition of integers~~

→ Homogeneous Linear recurrence relations with constant coefficients. – Definition, Problems

→ Non-Homogeneous ~~recc~~ linear recurrence relations with constant coefficients – Definition, Problems.

PART-2: Generating Functions

→ Introduction to generating functions – Problems.

→ Partition of integers – Problems.

→ Exponential Generating Functions

→ Summation Operator

* RECURRENCE RELATIONS:

A recurrence relation for a sequence $\{a_n\}_{n=1}^{\infty}$ is an equation that expresses a_n in terms of one (or) more of the previous terms of the sequence namely $a_0, a_1, \ldots\ldots a_{n-1}$ $\forall\ n \geq 0$

→ A recurrence relation is an expression of the form

$a_n = f(a_{n-1}, a_{n-2} \ldots\ldots a_0, n)$

where f is a function of some of the variables $a_{n-1}, a_{n-2} \ldots\ldots a_0, n$.

Note that all the a_i's need not occur in the expression.

Ex: (1) $a_n = 3a_{n-1} + n$. 2) $a_n = a_{n-1} + d$

*** Solution of a recurrence relation:**

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to a solⁿ of recurrence relation if its term satisfies the recurrence relation.

Ex: 1) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are the values of a_2 and a_3?

Sol:
Recurrence relation is $a_n = a_{n-1} - a_{n-2}$

$a_0 = 3$; $a_1 = 5$

put $n = 2$

$a_2 \Rightarrow a_2 = a_1 - a_0$
$= 5 - 3$
$a_2 = 2$

$n = 3$
$a_3 = a_2 - a_1$
$= 2 - 5$
$= -3$

* Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ is the solution of recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ where $a_n = 3n \ \forall n \geq 0$

Sol:

$a_n = 2a_{n-1} - a_{n-2}$ — ①

If $a_n = 3n$ is a sol^n of ①

then it should satisfy (1)

i.e. $3n = 2(3(n-1)) - 3(n-2)$

$3n = 6n - 6 - 3n + 6$

$= 3n$

20/12/2022 Hence proved.

* Order of recurrence relation:

The order of recurrence relation defined by $a_n = f(a_{n-1}, a_{n-2}, \ldots a_{n-k}, n)$ is k where a_n depends on one or more of the previous 'k' terms and k is the smallest such integer.

The order of recurrence relation of the ~~term~~ form $a_n = f(a_{n-1}, a_{n-2}, \ldots a_0, n)$ that depends on each of its previous term is zero (or) not defined.

* Degree of Recurrence relation:

It is the degree of polynomial treated as a polynomial in its variables excluding 'n'

If f is not a polynomial in its variables, then no degree is assigned to the recurrence relation.

* Degree of recurrence relation is the difference between the greatest & least subscripts of the members of the sequence occurring in recurrence relation.

→ A recurrence relation of degree '1' is called linear recurrence relation

degree '2' → quadratic recurrence relation

degree '3' → cubic recurrence relation

degree '4' → biquadratic relation.

Ex: of order & degree:

$\rightarrow a_n - 3a_{n-1} + a_{n-2} - 4a_{n-3} = 0$

$\boxed{\text{order} = 3}$ ($\because$ least integer of suffix)

Polynomial: $t^n - 3t^{n-1} + t^{n-2} - 4t^{n-3} = 0$

Characteristic equation : $\dfrac{t^n - 3t^{n-1} + t^{n-2} - 4t^{n-3}}{t^{n-3}} = 0$

$\Rightarrow t^3 - 3t^2 + t - 4 = 0$

$\boxed{\text{degree} = 3}$

$\rightarrow a_n - \sqrt{a_{n-1}} + a_{n-2}^2 = 0.$

order = 2 (least suffix integer)
equation of RR.

~~Polynomial~~ : $t^n - t^{\frac{n-1}{2}} + t^{(n-2)2} = 0$

~~Characteric Equation~~ : $\cancel{\dfrac{t^n}{t^{2n-4}} - \dfrac{t^{n-1/2}}{t^{2n-4}} + 1 = 0}$

$\therefore$ not a polynomial.

$\therefore$ degree do not exist.

$\rightarrow a_n + 3a_{n-2} - a_0 = 0$

order = 0/not defined

degree = ?

Polynomial: $t^n + 3t^{n-2} - t^0 = 0$

Characteristic Equation : $\dfrac{t^n + 3t^{n-2} - 1}{1} = 0$

$\therefore$ degree = n

* $t^n - 9t^{n-1} + 26t^{n-2} - 24t^{n-3} = 0$
* $a^n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$

order = 3

$CE = t^3 - 9t^2 + 26t - 24 = 0$

degree = 3.

* HOMOGENEOUS LINEAR RECURRENCE RELATION WITH CONSTANT COEFFICIENTS:

If

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$

$\forall n \geq k$ where $C_0, C_1, C_2, \ldots, C_k$ are real constants. with degree k; is known as linear recurrence relation.

If f(n) is identically "zero" then the recurrence relation said to be homogeneous linear recurrence relation ...

non-homogeneous linear recurrence relation with constant coefficients.

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* **Solution of Homogeneous Linear Recurrence Relation with constant coefficients.**

*→ Given R.R is $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + \cdots + C_k a_{n-k} = 0; \ n \geq k;$ —(1)

$C_0, C_1, \cdots\cdots C_k$ are real constants of degree k.

Step-1:

Polynomial equation of (1) is

$$C_0 t^n + C_1 t^{n-1} + C_2 t^{n-2} + \cdots\cdots + C_k t^{n-k} = 0$$

Dividing by t^{n-k} on both sides.

characteristic equation of (1) is

$$C_0 t^k + C_1 t^{k-1} + C_2 t^{k-2} + \cdots\cdots + C_k = 0 \quad -(2)$$

Find the roots of eqn (2) using ~~real constants of degrees~~ synthetic division.

Step-2:

If the roots are

case(1):

→ real & distinct

Let roots are $r_1, r_2 \cdots\cdots$ (say)

The general soln of (1) is

$$\boxed{a_n = c_1 (r_1)^n + c_2 (r_2)^n}$$

case(2);

→ roots are real & equal.

Let $r_1 = r; \ r_2 = r$ (say)

The general soln of (1) is

$$\boxed{a_n = (c_1 + c_2 n) r^n}$$

* Solve $a_n = a_{n-1} + 2a_{n-2}$ $\quad a_0 = 0; a_1 = 1$

Let the HLR.R

$a_n - a_{n-1} - 2a_{n-2} = 0$ —(1)

characteristic equation of (1) order=2 degree=2

$$t^2 - t - 2 = 0$$

$$(t-2)(t+1) = 0 \qquad \boxed{t = 2, -1}$$

Roots are real & distinct

∴ General Soln of (1)

$$\boxed{a_n^H = c_1 (2)^n + c_2 (-1)^n} \quad -(3)$$

H → soln of HLR.R

$$a_0 = c_1 (2)^0 + c_2 (-1)^0$$

$$\boxed{0 = c_1 + c_2} \quad -(4)$$

$$2c_1 - \cancel{c_2} = 1$$
$$c_1 + \cancel{c_2} = 0$$
$$3c_1 = 1$$
$$\boxed{c_1 = 1/3}$$

$c_2 = -1/3$

$$a_n^H = \frac{1}{3}(2)^n + \left(\frac{-1}{3}\right)(-1)^n$$

$$\boxed{a_n^H = \frac{1}{3}\left(2^n - (-1)^n\right)}$$

* Solve the R.R.

$a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 0$ — (1)

(1) is the Recurrence Relation

Characteristic eqn of (1)

$$t^3 - 7t^2 + 16t - 12 = 0$$

1	1	-7	16	-12
	0	1	-6	10
	1	-6	10	-2

2	1	-7	16	-12
	0	2	-10	12
	1	-5	6	0

$$\Rightarrow (t-2)(t^2 - 5t + 6) = 0$$

$t = 2, 2, 3$

∵ 2 roots are equal & 1 is destinct

$$\boxed{a_n^H = (c_1 + c_2 n)2^n + c_3(3)^n}$$

* $a_n = a_{n-1} + 6a_{n-2}$; $n \geq 2$; $a_0 = 1, a_1 = 1$
* $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$; $a_0 = 0, a_3 = 3, a_5 = 10$
* $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 4$, $a_1 = 17$
* $a_n - 8a_{n-1} + 16a_{n-2} = 0$; $a_2 = 16$, $a_3 = 80$
* $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ — (1)

is the given recurrence relation ∴ characteristic equation of (1)

$$t^3 - 3t^2 + 3t - 1 = 0$$

$$\cancel{(t-1)^3 = 0}$$

$$\cancel{t^2(t-3)+1} \quad (t-1)^3 = 0$$

∴ General solution of (1)

$$\boxed{(c_1 + c_2 n + c_3 n^2)(1)^n = a_n^H}$$

$a_0 = 0$ $\quad \boxed{c_1 = 0}$

$a_3 = 3$ $\quad c_1 + 3c_2 + 9c_3 = 3$

$$3c_2 + 9c_3 = 3$$

$$\boxed{c_2 + 3c_3 = 1} \text{ — (2)}$$

$a_5 = 10 \Rightarrow \cancel{c_1 + 10c_2}$

$$(c_1 + 5c_2 + 25c_3) = 10$$

$$5c_2 + 25c_3 = 10$$

$$\boxed{c_2 + 5c_3 = 2} \text{ — (3)}$$

(2) − (3) $\quad -2c_3 = -1$

$$\boxed{c_3 = 1/2}$$

$c_2 = 2 - \frac{5}{2}$

$$\boxed{c_2 = -1/2}$$

$$\boxed{a_n^H = \left(-\frac{n}{2} + \frac{n^2}{2}\right)(1)^n}$$

$$\boxed{a_n^H = \frac{1}{2}(n^2 - n)}$$

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→ $a_n = a_{n-1} + 6a_{n-2}$ ~~$+ a_{n-3}$~~ $a_0 = \phi; a_1 = 1$

~~$a_3 = 3;$~~

~~$a_5 = 10$~~

characteristic Equation

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$t = 3, -2$

$\boxed{a_n^H = c_1(3)^n + c_2(-2)^n}$ ⇒ General Solution

$a_0 = 1 = c_1(3)^0 + c_2(-2)^0 \Rightarrow \boxed{c_1 + c_2 = 1}$

$a_1 = 1 = c_1(3) + c_2(-2)$ $\boxed{3c_1 - 2c_2 = 1}$

$3c_1 - 2(1 - c_1) = 1$ $\boxed{c_2 = \frac{2}{5}}$

$3c_1 - 2 + 2c_1 = 1$

$5c_1 = 3$

$\boxed{c_1 = 3/5}$

$$\boxed{a_n^H = \frac{3}{5}(3)^n + \frac{2}{5}(-2)^n}$$

→ $a_n = 7a_{n-1} - 10a_{n-2}$ $a_0 = 4, a_1 = 17.$

Characteristic Equation:

$t^2 - 7t + 10 = 0$

$(t-2)(t-5) = 0$

$t = 2, 5$

$$\boxed{a_n^H = c_1(2)^n + c_2(5)^n}$$

$a_0 = \boxed{4 = c_1 + c_2}$ $a_1 = 17 = 2c_1 + 5c_2$

$2c_1 + 5c_2 = 17$

$2c_1 + 5(4 - c_1) = 17$

$2c_1 + 20 - 5c_1 = 17$

$\boxed{c_2 = 3}$ $3 = 3c_1$ $\boxed{c_1 = 1}$

$$\boxed{a_n^H = (2)^n + 3(5)^n}$$

⇒ $t^2 - 8t + 16 = 0$ ⇒ Characteristic eq.

$(t-4)^2 = 0$ $t = 4, 4$

$\boxed{a_n^H = (c_1 + c_2 n)(4)^n}$ $a_2 = 16$

$a_3 = 80$

$a_2 = 16 \Rightarrow (c_1 + 2c_2)\cancel{16} = \cancel{16}$

$\boxed{c_1 + 2c_2 = 1}$ – ① ② – ①

$a_3 = 80 = (c_1 + 3c_2)64$ $\boxed{c_2 = \frac{1}{4}}$

$\boxed{c_1 + 3c_2 = \frac{5}{4}}$ – ②

$c_1 = 1 - \frac{1}{2}$

$c_1 = \frac{1}{2}$

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* Non-Homogeneous Linear Recurrence Relation with constant coefficients:

$$C_0 a_n + C_1 a_{n-1} + \dots + C_K a_{n-K} = f(n)$$

$\forall\ n \geq K$

$f(n) \neq 0$

where $C_0, C_1, C_2, \dots, C_K$ are real constants of degree K

*→ particular sol^n of ① is

$$\boxed{a_n = a_n^H + a_n^P}$$

a_n^H ↓ sol^n of HLRR

a_n^P ↓ sol^n of NHLRR (using one of the 4 types)

* Forms of particular solutions of Non-Homogeneous Linear Recurrence Relation:

S.No	f(n)	Characteristic Polynomial C(t)	Form of particular sol^n a_n^P
I 1.	Da^n	a is not a root of C(t)	$A a^n$
2.	Da^n	a is a root of C(t) with multiplicity 'm'	$A n^m a^n$.
II 3.	$Dn^s a^n$	a is not a root of C(t)	$(A_0 + A_1 n + A_2 n^2 + \dots + A_s n^s) a^n$
4.	$Dn^s a^n$	a is a root of C(t) with multiplicity 'm'	$n^m (A_0 + A_1 n + A_2 n^2 + \dots + A_s n^s) a^n$
III 5.	Dn^s	1 is not a root of C(t)	$(A_0 + A_1 n + \dots + A_s n^s)$
6.	Dn^s	1 is a root of C(t) with multiplicity 'm'	$n^m (A_0 + A_1 n + \dots + A_s n^s)$.
V 9.	constant	—	A
IV POLYNOMIAL 7.	$(P_0 + P_1 n + \dots + P_s n^s) a^n$	a is not a root of C(t)	$(A_0 + A_1 n + \dots + A_s n^s) a^n$
8.	$(P_0 + P_1 n + P_2 n^2 + \dots + P_s n^s) a^n$	a is a root of C(t) with multiplicity 'm'	$n^m (A_0 + A_1 n + \dots + A_s n^s) a^n$

* $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$ — ①

* Complete Solution of NHLRR with const. coefficients.

$$a_n = a_n^H + a_n^P \quad — ②$$

* a_n^H : HLRR

$2t^2 - 7t + 3 = 0$ is the characteristic equation.

$\rightarrow 2t^2 - 6t - t + 3 = 0$

$2t(t-3) - 1(t-3) = 0$

$t = \frac{1}{2}, 3$

roots are real & distinct

$\boxed{a_n^H = c_1\left(\frac{1}{2}\right)^n + c_2(3)^n}$ — (3)

$* \; a_n^P$ NHLRR : $2a^n - 7a_{n-1} + 3a_{n-2} = 2^n$

$f(n) = 2^n$

$\because$ 2 is not a root of the given R.R

$\therefore \; a_n^P = Aa^n$

$\boxed{a_n^P = c_3 2^n}$ — (4)

$\therefore \; a_n = a_n^H + a_n^P$

~~$\neq$ (1)~~

$\cancel{a_n = c_1\left(\frac{1}{2}\right)^n + c_2(3)^n + c_3(2)^n}$

put $a_n^P = A2^n$ in (1)

$2(A)(\cancel{2^n}) - 7(A)\frac{\cancel{2^n}}{2} + \frac{3A\cancel{2^n}}{2^2} = \cancel{2^n}$

$2A - \frac{7A}{2} + \frac{3A}{4} = \phi$

$8A - 14A + 3A = 4$

$-3A = 4$

$\boxed{A = -\frac{4}{3}}$

$\therefore \boxed{a_n = (c_1)\left(\frac{1}{2}\right)^n + c_2(3)^n - \frac{4}{3} 2^n}$

$a_0 = 0; \; a_1 = 1$

$c_1 + c_2 - \frac{4}{3} = 0$

$\boxed{c_1 + c_2 = \frac{4}{3}}$

$c_2 = \frac{4}{3} - c_1$

$1 = \frac{c_1}{2} + c_2 - \frac{8}{3}$

$\boxed{6 = 3c_1 + 6c_2 - 16}$

$\cancel{6 = 4c_1 - 3c_1^2}$

$6 = 3c_1 + 8 - 6c_1 - 16$

$* \; \underline{24|12|2022}$

$* \; a_n - 4a_{n-1} + 4a_{n-2} = 2^n$ — (1)

$\because$ Given NHLRR with const. coefficients

$a_n = a_n^H + a_n^P$

$* \; a_n^H$: ~~characteristic~~

HLRR: $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Characteristic Equation

$t^2 - 4t + 4 = 0$

$(t-2)^2 = 0$

$t = 2, 2$

roots are real & equal

$\boxed{(c_1 + c_2 n)(2)^n = a_n^H}$

a_n^P:

NHLRR $\Rightarrow a_n - 4a_{n-1} + 4a_{n-2} = 2^n$

$f(n) = 2^n$

$\because$ 2 is a root of the given NHLRR

$\therefore a_n^P = n^m a^n$ $m=2$

$= (2)$

$\boxed{a_n^P = An^2 \cdot 2^n}$ — ②

Substitute in ② in ①

$n^2 A 2^n - 4A(n-1)^2 \cdot 2^{n-1} + 4A(n-2)^2 \cdot 2^{n-2} = 1$

$n^2 - \cancel{4}^2(n^2+1-2n) \cdot \frac{1}{\cancel{2}} + \frac{\cancel{4}(n^2+4-4n)}{\cancel{4}} = 1$

$A[\cancel{n^2} - 2\cancel{n^2} - 2 + 4\cancel{n} + \cancel{n^2} + 4 - \cancel{4n}] = 1$

$+2 = \frac{1}{A}$

$\boxed{A = \frac{1}{2}}$

$\boxed{a_n^P = n^2 \cdot 2^{n-1}}$

$\therefore a_n = a_n^H + a_n^P$

$\boxed{a_n = (c_1 + c_2 n)(2)^n + n^2 \cdot 2^{n-1}}$

* Solve $a_{n+2} - a_{n+1} - 2a_n = n^2$ — ①

CE: $t^2 - t - 2 = 0 \Rightarrow (t-2)(t+1) = 0$

$t = 2, -1$

$\boxed{a_n^H = c_1(-1)^n + c_2(2)^n}$

a_n^P: $f(n) = n^2$

$= D \cdot n^s$

$D = 1, s = 2$

1 is not a root

$\boxed{a_n^P = A_0 + A_1 n + A_2 n^2}$ — ②

Sub. ② in ①

$A_0 + A_1(n+2) + A_2(n+2)^2 - A_0 - A_1(n+1) - A_2(n+1)^2 - 2[A_0 + A_1 n + A_2 n^2] = n^2$

Equating coeff. of n^2, n, const

$\cancel{A_2} - \cancel{A_2} - 2A_2 = 1$

$A_2 = -1/2$

$\cancel{A_1} + 4A_2 - \cancel{A_1} - 2A_2 - 2A_1 = 0$

$\cancel{2}A_2 - \cancel{2}A_1 = 0$

$A_1 = -1/2$

$\cancel{A_0} + 2A_1 + 4A_2 - \cancel{A_0} - A_1 - A_2 - 2A_0 = 0$

$A_1 + 3A_2 = 2A_0$

$$A_0 = \frac{A_1 + 3A_2}{2}$$

$$= \cancel{4}\left(\frac{-1}{2}\right) \cdot \frac{1}{2}$$

$$\boxed{A_0 = -1}$$

$$a_n^P = -1 + \frac{-n}{2} + \frac{-n^2}{2}$$

$$= -\frac{1}{2}\left(2 + n + n^2\right)$$

$$\boxed{a_n = c_1(-1)^n + c_2(2)^n - \frac{1}{2}\left(n^2 + n + 2\right)}$$

* Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ $\quad a_0 = 1, \; a_1 = -1$

$$t^2 - 5t + 6 = 2$$

$$a_n^H \Rightarrow (t-3)(t-2) = 0$$

$$t = 3, 2$$

$$a_n^H = c_1(2)^n + c_2(3)^n \qquad a_n^P = A$$

* we put 'A' (a_n^P) in the R.R only at the places other than a_n

$$a_{n+2} - 5a_{n+1} + 6a_n$$

$$A - 5A + 6 = 0$$

$$-4A = -4$$

$$\boxed{A = 1}$$

$$\therefore \boxed{a_n = c_1(2)^n + c_2(3)^n + 1}$$

* $a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n$ — ①

$$t^2 - 6t + 8 = 0 \qquad (t-2)(t-4) = 0 \Rightarrow t = 2, 4$$

$$\boxed{a_n^H = c_1(2)^n + c_2(4)^n}$$

$$t^2 - 6t + 8 = n \cdot 4^n \qquad s = 1, \; a = 4$$

$$\boxed{a_n^P = n\left(A_0 + A_1 n\right)(4)^n} \text{ — ②}$$

Sub ② in ①

$$n4^n(A_0 + A_1 n) - 6(n-1)4^{n-1}\left(A_0 + A_1(n-1)\right) + 8(n-2)4^{n-2}\left(A_0 + A_1(n-2)\right) = n \cdot 4^n$$

$$\cancel{A_0 + A_1 n - \frac{6}{4}}$$

$$\cancel{4^n\left[A_0 n + A_1 n^2 - \frac{6}{4}n + \frac{6}{4}\left(A_0 + A_1 n - A_1\right) + \frac{1}{2}(n-2)\left(A_0 + A_1 n - 2A_1\right)\right] = n \cdot 4^n}$$

$$A_0 n + A_1 n^2 - \frac{3}{2}A_0 n - \frac{3}{2}A_1 n^2 + \frac{3}{2}A_1 n + \frac{3}{2}A_0 + \frac{3}{2}A_1 n - \frac{3}{2}A_1 + \frac{nA_0}{2} - A_0 + \frac{n^2 A_1}{2} - A_1 n - A_1 n + 2$$

$A_0 = -1, \; A_1 = 1$

$a_n = C_1(2)^n + C_2(4)^n + n(-1+n)4^n$

$\boxed{a_n = C_1(2)^n + C_2(4)^n + n(n-1)4^n}$

* $a_n = 6a_{n-1} - 9a_{n-2} + (n^2+1)3^n$

$a_n - 6a_{n-1} + 9a_{n-2} = (n^2+1)3^n$

$\boxed{t = 3, 3}$

$\boxed{a_n^H = (c_1 + c_2 n)3^n}$

$a_n^P \Rightarrow f(n) = D(n^2+1)3^n$

$D = 1$; $(P_0 + P_1 n + P_2 n^2)a^n$

$a = 3 \Rightarrow$ root of R.R.

$a_n^P = n^m(P_0 + P_1 n + P_2 n^2)a^n$

$\boxed{a_n^P = n^2(P_0 + P_1 n + P_2 n^2)3^n}$

$n^2(P_0 + P_1 n + P_2 n^2)a^n - 6(n-1)^2[P_0 + P_1(n-1) + P_2(n-1)^2]a^{n-1}$

$- 9(n-2)^2[P_0 + P_1(n-2) + P_2(n-2)^2]a^{n-2} = (n^2+1)3^n$

$n = 0$

$-\frac{6}{3}[P_0 - P_1 + P_2] - \frac{9(4)}{9}[P_0 - 2P_1 + P_2(4)]$

$= -2(P_0 - P_1 + P_2) - 4(P_0 - 2P_1 + 4P_2) = 1$

* $a_n - 5a_{n-1} + 6a_{n-2} = 1$
* $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$.
* $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$; $a_0 = 3, a_1 = 7$.
* $a_{n+2} - 5a_{n+1} + 6a_n = n^2$.
* $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$.
* $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ & $a_1 = 4$.
* $a_n - 5a_{n-1} + 6a_{n-2} = 1$

Characteristic Equation:

$t^2 - 5t + 6 = 0$

$a_n^H \Rightarrow (t-3)(t-2) = 0 \qquad t = 3, 2$

$a_n^H = C_1(3)^n + C_2(2)^n$.

$a_n - 5a_{n-1} + 6a_{n-2} = 1$

$a_n^P = A$

$A - 5A + 6A = 1$ $\quad \boxed{A = \frac{5}{4}}$

$-4A = -5$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n + \frac{5}{4}}$$

$*$ $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = n \cdot 4^n$ —(1)

Characteristic equation

$t^3 - 7t^2 + 16t - 12 = 0$

a_n^H:

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 16 & -12 \\ & 0 & 1 & -6 & 10 \\ \hline & 1 & -6 & 10 & -2 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$(t-2)(t^2 - 5t + 6) = 0$

$t = 2, 2, 3$

$a_n^H = (c_1 + c_2 n)(2)^n + c_3(3)^n$

a_n^P: $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = n4^n$

$\boxed{a_n^P = (A_0 + A_1 n)(4)^n}$ —(2)

Sub (2) in (1)

$(A_0 + A_1 n)4^n - 7(4)^{n-1}(A_0 + A_1(n-1))$
$+ 16(4)^{n-2}(A_0 + A_1(n-2)) = n4^n$

$(A_0 + A_1 n) - \frac{7}{4}(A_0 + A_1 n - A_1) + 1(A_0 + A_1 n - 2A_1)$
$= n$

$A_1 - \frac{7}{4}A_1 + A_1 = 1$

$A_1 = 4$

$A_0 - \frac{7}{4}A_0 - \frac{7}{4}A_1 + A_0 - 2A_1 = 0$

$-\frac{3}{4}A_0 + A_0 - \frac{15}{4}A_1 = 0$

$\frac{A_0}{4} - \frac{15}{4}A_1 = 0$

$A_0 = 15A_1$

$A_0 = 60$

$$\boxed{a_n = (c_1 + c_2 n)(4)^n + (4 + 60n)(4)^n}$$

$*$ $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$ —(1) $a_0 = 3, a_1 = 7$

Characteristic Equation

$t^2 - 6t + 8 = 0$

$(t-4)(t-2) = 0$

$t = 4, 2$

$\boxed{a_n^H = c_1(4)^n + c_2(2)^n}$

$a_n^P \Rightarrow a_n - 6a_{n-1} + 8a_{n-2} = 3^n$

$\because$ 3 is not a root of RR

$\therefore a_n^P = A(3)^n$ —(2)

Sub (2) in (1)

$A(3)^n - 6A(3)^{n-1} + 8A(3)^{n-2} = 3^n$

$A - \frac{6A}{3} + \frac{8A}{9} = 1$

$\frac{9A - 18A + 8A}{9} = 1$

$-A = 9$

$A = -9$

$$\boxed{a_n = c_1(4)^n + c_2(2)^n - 9(3)^n}$$

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* GENERATING FUNCTIONS:

→ Generating function is given from a sequence.

* The generating function for the sequence $a_0, a_1, \ldots a_k$ of real number is the infinite series.

$$\boxed{G(x) = a_0 + a_1 x + \cdots + a_k x^k \cdots = \sum_{k=0}^{\infty} a_k x^k}$$

It is also called as ordinary generating function.

Ex: 1

The generating function for a sequence $\{a_k\}$ where $a_k = 2^k$ is $\sum_{k=0}^{\infty} (2x)^k$

and $a_k = 3$, we have $\sum_{k=0}^{\infty} 3x^k$

* The generating function for finite sequence: $a_0, a_1, \ldots a_n$ is given by,

$$\boxed{G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}$$

* The generating function for a finite sequence 1, 1, 1, 1, 1, 1 is $1 + x + x^2 + x^3 + x^4 + x^5$.

* The generating function for a finite sequence $a_0, a_1, \ldots, a_m$ where $a_k = C(m, k)$ & 'm' is a +ve integer and $k = 0, 1, 2, \ldots m$.

$$\boxed{a_k = C(m,k) = {}^mC_k = \frac{m!}{k!(m-k)!}}$$

is $G(x) = C(m,0) + C(m,1)x + \cdots + \cdots C(m,m)x^m$

Generating Function $\boxed{G(x) = (1+x)^m}$

* The generating function for an infinite sequence $a_0, a_1, \ldots, a_k, \ldots$

* → The generating function for a sequence $a_0, a_1, \ldots a_k, \ldots$ of real nos for infinite series.

$$\boxed{G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k}$$

Ex: $\sum_{k=0}^{\infty} (k+1) x^k$, $\sum_{k=0}^{\infty} 4^k x^k$

* Let m be a +ve integer ; let $a_k = C(m,k)$ for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence

$a_0, a_1, \ldots, a_m$.

$$G(x) = C(m,0) + C_1(m,1)\,x + \cdots + C(m,m)\,x^m.$$

$$= (1+x)^m \quad \text{(Binomial Theorem)}$$

* The function $f(x) = \frac{1}{1-ax}$ is a generating function of the sequence $1, a, a^2, \ldots$

because

$$(1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$\frac{1}{1-ax} = (1-ax)^{-1}$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

where $\boxed{|ax| < 1}$ (or) $\boxed{|x| < \frac{1}{a}}$ $a \neq 0$

* Theorem 1 :

Let $f(x) = \sum_{k=0}^{\infty} a^k x^k$ and $g(x) = \sum_{k=0}^{\infty} b^k x^k$.

then $\boxed{f(x) + g(x) = \sum_{k=0}^{\infty} (a^k + b^k)\, x^k}$ and

$$\boxed{f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} a_j b_{k-j} \right) x^k}$$

* Theorem 2 :

→ Extended binomial theorem

Let x be a real number with $|x| < 1$ and let u be a real number.

then $\boxed{(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k}$

→ Let u be a real number and k be a non-negative integer.

Then extended binomial coefficient $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \boxed{(k>0)} \\ 1 & \text{if } k = 0 \end{cases}$$

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10) $\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)(-1)^k x^k$ $\quad (-1)^k C(n+k-1, k)$

$= 1 - C(n,1)\,x + C(n+1,2)\,x^2$

11) $\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)\, a^k x^k$ $\quad C(n+k-1, k)\, a^k$

$= 1 + C(n,1)\, ax + C(n+1,2)\, a^2x^2 + \cdots$

12) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots$ $\quad 1/k!$

13) $\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ $\quad (-1)^{k+1}/k$.

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

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* TABULAR FORM OF GENERATING FUNCTIONS:

	$G(x)$	a_K
1)	$(1+x)^n = \sum_{K=0}^{n} a_K x^K$ $= \sum_{K=0}^{n} C(n,K)x^K$ $= 1 + C(n,1)x + C(n,2)x^2 + \cdots + C(n,n)x^n$	$C(n,K)$
2)	$(1+ax)^n = \sum_{K=0}^{n} C(n,K)(a^K x^K)$	$C(n,K)\,a^K$
3)	$(1+x^r)^n = \sum_{K=0}^{n} C(n,K)\,x^{rK}$ $= 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots x^{rn}$	$C(n, K/r)$ if $r \mid K$ 0 otherwise
4)	$\dfrac{1-x^{n+1}}{1-x} = \sum_{K=0}^{n} x^K = 1 + x + x^2 + \cdots x^n$	1 if $K \le n$ 0 otherwise
5)	$\dfrac{1}{1-x} = \sum_{K=0}^{\infty} x^K = 1 + x + x^2 + \cdots$	1
6)	$\dfrac{1}{1-ax} = \sum_{K=0}^{\infty} a^K x^K = 1 + ax + a^2x^2 + \cdots$	a^K
7)	$\dfrac{1}{1-x^r} = \sum_{K=0}^{\infty} x^{rK} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid K$ 0 else
8)	$\dfrac{1}{(1-x)^2} = \sum_{K=0}^{\infty} (K+1)x^K$ $= 1 + 2x + 3x^2 + \cdots$	$K+1$
9)	$\dfrac{1}{(1-x)^n} = \sum_{K=0}^{\infty} C(n+K-1, K)x^K$	$C(n+K-1, K)$

* Determine the coeff of x^{15} in

$f(x) = (x^2 + x^3 + x^4 + \cdots)^4$

$f(x) = (x^2 + x^3 + x^4 + \cdots)^4$

$= x^8(1 + x + x^2 + \cdots)^4$

$\Rightarrow$ coeff of x^{15} = coeff of $x^8 \times$ coeff of x^7 in $(1+x+x^2+\cdots)^4$

$=$ coeff of $x^8 \times ((1-x)^{-1})^4$

$= 1 \times$ coeff of $x^7 \cdot \dfrac{1}{(1-x)^4}$ $\quad n=4 \quad K=7$

$= 1 \times C(4+7-1, 7)$

$$\boxed{\dfrac{1}{(1-x)^n} = \sum_{K=0}^{\infty} C(n+K-1, K)\,x^K}$$

$= 1 \times C(10, 7) = \dfrac{10!}{7!\,3!} = \dfrac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{6}} = 120$

* Determine the coeff of x^{10} in

$f(x) = (x^3 + x^4 + \cdots)^3$

$= x^9(1 + x + \cdots)^3$

$=$ coeff of $x^9 \times$ coeff of x in $(1+x+x^2+\cdots)^3$

$= 1 \times$ coeff of x $\dfrac{1}{(1-x)^3}$

$= 1 \times C(3+1-1, 1)$

$= 1 \times C(3,1)$

$= 3$

* x^9 in $(x^2+x^3+\cdots)^3$

$x^6(1+x+\cdots)^3$

$1\times x^3 \frac{1}{(1-x)^3}$

$1\times C(3+3-1,3)$

$1\times C(5,3)$

$=\frac{5!}{3!\,2!}=\frac{5\times4}{2}=\underline{10}$

* coeff of x^9 in

$(1+x+x^2)^3$

$\boxed{\frac{1-x^{n+1}}{1-x}= \begin{cases}1 & k\le n\\ 0 & \text{otherwise}\end{cases}}$

$\underline{\underline{1+x+x^2}}$

$\boxed{n=2}$

$=\frac{1-x^{2+1}}{1-x}=\frac{1-x^3}{1-x}$

$K=9$

$K \nleq n$

$\therefore$ coeff of $x^9=\underline{0}$

*Coeff of x^{18} in $(x+x^2+\cdots+x^5)^5$

$(x+x^2+\cdots+x^5)\ (x^{10})(1+x+x^2(x^2+x^3+x^4+\cdots)^5$

$=x(1+x+x^2+\cdots+x^4)(x^{10})(1+x+x^2+\cdots)$

* $x^{11}(1+x+x^2+x^3+x^4)(1+x+x^2+\cdots)^5$

$x^{11}\left(\frac{1-x^5}{1-x}\right)(1-x)^{-5}$

$x^{11}\times$ coeff of $x^7\left(\frac{1-x^5}{1-x}\right)\frac{1}{(1-x)^5}$

$=\cancel{x^{11}\times \text{coeff of } x^7\left[\frac{1-x^5}{1-x}\cdot C(5+7-1)\right.}$

$=x^{11}\times$ coeff of $x^7\ \frac{(1-x^5)}{(1-x)^6}$

$=\cancel{\frac{x^{11}-x^{16}}{(1-x)}}\quad (x^{11}-x^{16})\left(\frac{1}{(1-x)^6}\right)$

$=x^{11}\cdot$ coeff of $x^7\frac{1}{(1-x)^6}-x^{16}\times$ coeff of $x^2\left(\frac{1}{(1-x)^6}\right)$

$=1\times C(6+7-1,7)-1\times C(6+2-1,2)$

$=C(12,7)-C(7,2)$

$=\frac{12\times11\times10\times9\times8\times\cancel{7!}}{\cancel{7!}\times5!}-\frac{7!}{2!\,5!}$

$=\frac{120\times11\times72}{120}-\frac{7\times6}{2!}$

$=11\times72-21$

$=\underline{\underline{771}}$

* Coeff of x^{20}:

$(x+x^2+\cdots+x^5)(x^2+x^3+\cdots)^5$

* coeff of x^{50}:

$(x^{10}+x^{11}+\cdots+x^{25})(x+x^2+\cdots+x^{15})$

$(x^{20}+x^{21}+\cdots+x^{45})$

*

1) $(x+x^2+\cdots+x^5)(x^2+x^3+\cdots)^5$ x^{20}

$x(1+x+x^2+x^3+x^4)\cdot x^{10}\cdot(1+x+x^2+\cdots)^5$

$\Rightarrow x^{11}\cdot\frac{1-x^5}{1-x}\cdot\frac{1}{(1-x)^5}$

$\Rightarrow x^{11}\cdot\frac{1-x^5}{(1-x)^6}$

$\Rightarrow x^{11}\left(\frac{1}{1-x^6}\right)-x^{16}\left(\frac{1}{1-x^6}\right)$

x^{11}. coeff of $x^9\left(\frac{1}{1-x^6}\right)-x^{16}$. coeff of $x^4\frac{1}{(1-x^6)}$

$= 1\times C(9+6-1,9)-1\times C(4+6-1,4)$

$= C(14,9)-C(9,4)$

$\Rightarrow x^{10}(1+x+\cdots+x^{15})\cdot x(1+x+x^2+\cdots+x^{14})$

$\cdot x^{20}(1+x+\cdots+x^{25})$

$x^{31}\frac{1-x^{16}}{1-x}\cdot\frac{1-x^{15}}{1-x}\cdot\frac{1-x^{26}}{1-x}$

$\frac{x^{31}}{(1-x)^3}\left[(1-x^{15}-x^{16}+x^{31})(1-x^{26})\right]$

$\frac{x^{31}}{(1-x)^3}\left(1-x^{26}-x^{15}+x^{41}-x^{16}+x^{42}+x^{31}-x^{57}\right)$

$x^{31}\left(\frac{1}{(1-x)^3}\right)-\frac{x^{57}}{(1-x)^3}-\frac{x^{46}}{(1-x)^3}+\frac{x^{72}}{(1-x)^3}$

$-\frac{x^{47}}{(1-x)^3}-\frac{x^{73}}{(1-x)^3}+\frac{x^{62}}{(1-x)^3}-\frac{x^{88}}{(1-x)^3}$

x^{31}. coeff of $19\left(\frac{1}{(1-x)^3}\right)$

$-x^{46}$ coeff of $x^4\left(\frac{1}{(1-x)^3}\right)$

$-x^{47}$. coeff of $x^3\left(\frac{1}{(1-x)^3}\right)$

$1\times C(3+19-1,19)-1\times C(3+4-1,4)$

$-C(3+3-1,3)$

$= C(21,19)-C(6,4)-C(5,3)$

$= \frac{21\times \cancel{20}^{10}}{\cancel{2}}-\frac{\cancel{6}^{3}\times 5}{\cancel{2}}-\frac{5\times 4}{2}$

$= 210-15-10=\underline{185}$

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* Exponential Generating Function:

If $a_0, a_1, a_2, \ldots a_n \ldots$ is an infinite sequence of real numbers then

$$f(x) = A_0 + A_1 x + \frac{A_2 x^2}{2!} + \cdots + \frac{A_n x^n}{n!} + \cdots$$

$\sum_{n=0}^{\infty} \frac{A_n x^n}{n!}$ is called exponential generating fn for the given seq.

* Summation Operator

If $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + \cdots$

then $\frac{g(x)}{1-x} =$

$$\frac{g(x)}{1-x} = g(x)\,[1-x]^{-1}$$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots + \cdots)(1-x)^{-1}$$

$$= (a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots$$

$a_0, (a_1 + a_0), (a_2 + a_1 + a_0)$ are the sequence of Summating operator.

* Find the generating function for the following equations:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

① 1, 1, 1, 1, 1 ……

$1 + x + x^2 + \cdots$

$(1-x)^{-1}$

$$\boxed{g(x) = \frac{1}{1-x}}$$

② 1, 1, 1, 1

$1 + x + x^2 + x^3$

$$\boxed{= \frac{1 - x^4}{1-x}}$$

③ 1, −1, 1, −1, ……

$1 - x + x^2 - x^3 + \cdots$

$= (1+x)^{-1}$

$$\boxed{g(x) = \frac{1}{1+x}}$$

④ 1, 0, 1, 1, 1, ……

$1 + x^2 + x^3 + x^4 + \cdots$

$+ x \ldots$

$- x$

$= (1 + x + x^2 + x^3 + \cdots) - x$

$= (1-x)^{-1} - x$

$g(x) = \frac{1}{1-x} - x$

③ 1, −2, 3, −4, ……

$= 1 - 2x + 3x^2 - 4x^3 + \cdots$

$= (1+x)^{-2}$

$g(x) = \frac{1}{(1+x)^2}$

⑥ $a_n = 5$

$g(x) = \frac{5}{1-x}$

$\sum_{n=0}^{\infty} c x^n = c(1 + x + x^2 + \cdots)$

$= c(1-x)^{-1}$

⑦ $a_n = n$

$g(x) = \sum_{n=0}^{\infty} n x^n$

$= \frac{x}{(1-x)^2}$

$g(x) = \sum_{n=0}^{\infty} n x^n$

$= x + 2x^2 + 3x^3 + \cdots$

$= x(1 + 2x + 3x^2 + \cdots$

(8) $a_n = b^n$

$$\sum_{n=0}^{\infty} b^n x^n = 1 + bx + (bx)^2 + \cdots$$

$$= \frac{1}{1-bx}$$

*

S.No	$g(x)$	a_n
1	$\frac{c}{1-x}$	$c(1)^n$
2	$\frac{1}{1-bx}$	b^n
3	$\frac{1}{1+bx}$	$(-b)^n$
4	$\frac{x}{(1-bx)^2}$	n
5	$\frac{1}{(1-bx)^2}$	$(n+1)b^n$
6	$\frac{bx}{(1-bx)^2}$	$\cancel{\frac{bx}{(1-bx)^2}}$ nb^n

* 1, 4, 16, 64, 256

$4^0, 4^1, 4^2, 4^3, 4^4$

$$= \frac{1}{(1-4x)}$$

* 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, ...

$0 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6$

$$2x\,\frac{(1-x^6)}{1-x}$$

*

* Find the coeff of x^{10} in the power series of following functions

1) $(x^3 + x^4 + x^5 + x^6 + \cdots)^3$

$x^9(1 + x + x^2 + \cdots)^3$

$= x^9 \cdot$ coeff of x in $(1 + x + x^2 + \cdots)^3$

$= x^9 \cdot$ coeff of x $\frac{1}{(1-x)^3}$

$= 1 \times C(3+1-1, 1) = C(3, 1) = \underline{3}$

2) $(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + \cdots)$

$= x^4(1 + x + x^2)\,x^3(1 + x + x^2 + x^3 + x^4) \cdot (1-x)^{-1}$

$= x^7\,\frac{1-x^3}{1-x} \cdot \frac{1-x^5}{1-x} \cdot \frac{1}{1-x}$

$= x^7\,\frac{(1-x^3)(1-x^5)}{(1-x)^3}$

$= \frac{(x^7 - x^{10})(1-x^5)}{(1-x)^3} = \frac{x^7 - x^{12} - x^{10} + x^{15}}{(1-x)^3}$

$=$

$= 1\,C(3+3-1, 3) - 1 = C(5,3) - 1$

$= 9$

$* (1+x^5+x^{10}+x^{15}+ \dots)^3$

$\frac{1}{(1-x^5)^3}$ $\quad ((x^5)^0+(x^5)^1+(x^5)^2+\dots)^3$

$1-x^5 = \text{coeff of } x^{10}$

$*$ $$\boxed{\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)\, x^k} \quad (x^5)^2$$

$$= C(n+k-1, k)$$

$k=2$

$n=3$

$$= C(3+2-1, 2)$$

$$= C(4, 2) = \frac{4!}{2!\,2!}$$

$$= \frac{24}{4} = 6$$