14/03/2022 UNIT-4

MATRICES AND INNER PRODUCT SPACE:

→ Rank of a matrix – Definition & Problems.

→ Echelon form – definition & Problems (Rank using this method)

→ Linearly dependence & independence of vectors – Defn & problems

(character)

** → Eigen values – defn, properties, problems.

* eigen values are same as trace elements only in triangular matrix.

* $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A; then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ are eigen values of A^{-1},

$\lambda_1^3, \lambda_2^3, \lambda_3^3$ are eigen values of A^3.

$\lambda_1^n, \lambda_2^n, \lambda_3^n$ are eigen values of A^n.

eigen values ⇔ latent / characteristics roots

→ Eigen vectors – defn, problems.

→ Diagonalization ~~of~~ ~~~~ using – Similarity transformation, Orthogonal transformation

*

→ Inner product spaces.

→ Gram Schmidth process.

Rank; Pg: 36

*RANK OF MATRIX:

Highest order non-zero determinant (minor) of a given matrix is said to be rank of matrix that order.

$\boxed{\text{Denoted by } \rho(A).}$

→ Properties of Rank

1) Rank of null matrix is zero.
2) Identity matrix of order 'n' has a rank of 'n'.
3) Rank of A = Rank of A^T.
4) Rank of singular matrix of (m×n) is min of (m,n)
5)

*ECHELON FORM:

→ If possible make sure that first element is unity

→ No. of zeroes in preceding rows should be less than or equal to no. of zeroes in the succeeding rows.

→ (zero after element is valid) & zeroes preceds. non-zero elements.

→ no. of ~~zeroes~~ non-zero rows in a matrix ~~~~ is called rank.

*Note: Rank of the echelon form = No. of ~~zero~~ non-zero rows in column.

→ Applying only elementary row operations (or) column operations can be used in echelon form.

* Column operations are not preferrable in echelon form.

* EIGEN VALUES:

If 'A' is any square matrix of order 'n' then determinant of $\boxed{|A-\lambda I|=0}$ is said to be ~~a~~ the characteristic equation; where λ is a parameter; I is the n^{th} order unit matrix.

The roots of the characteristic equation $|A-\lambda I|=0$ ~~is said to be the~~ are called eigen values (or) characteristics/latent roots.

Note:

The characteristic equation of $n\times n$ square matrix takes

$$\boxed{(-1)^n\lambda^n + K_1(\lambda^{n-1}) + K_2(\lambda^{n-2}) + \cdots + K_n I = 0}$$

where $K_1, K_2, \ldots, K_n$ are elements of A_{ij} matrix

Note:

Square matrix of order 3

$$(-1)^3\lambda^3 + K_1\lambda^2 + K_2\lambda + K_3 = 0$$

$$\boxed{\lambda^3 - K_1\lambda^2 + K_2\lambda - K_3 = 0}$$ Characteristic equation of 3x3 matrix.

* EIGEN VECTORS:

Let A be any square matrix of order 'n' then there exists a non-singular column matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that its linear transformation $\boxed{Y=AX}$ which transforms into λX given by..

$\boxed{AX=\lambda X}$ $\boxed{[A-\lambda I]X}$, where 'X' is said to be eigen vector corresponding to eigen value.

→ Properties:

* Any square matrix 'A' and its transpose A^T have the same eigen value.
* Eigen values of diagonal matrix are the diagonal elements of the matrix.

Eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⇒ eigen values = 2, 5, 1

* Eigen values of triangular matrix (upper &

lower) are the principal diagonal elements.

* Sum of eigen values of matrix is its trace (sum of principal diagonal elements)

* Product of Eigen values of matrix is equal to its determinant.

* If λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

* If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are the eigen values of A^m.

* If λ is an eigen value of A then $\frac{\det A}{\lambda}$ is an eigen value of adjoint A.

*

* Determine the rank of the following matrices:

1) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 12 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

* Find eigen values of matrix

$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ Eigen values of given matrix A are 3, 2, 5

$\because$ A is an upper triangular matrix.

$\therefore$ Eigen values are diagonal elements.

* Find Eigen values of matrix

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $Y = AX$

$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0.$$

$$\left| \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)(5-6\lambda+\lambda^2-1) - 1(1-\lambda-3) + 3(1-15+3\lambda) = 0$$

$$5 - \cancel{6\lambda} + \lambda^2 - 1 - \cancel{5\lambda} + 6\lambda^2 - \lambda^3 + \cancel{\lambda} + 2 + \cancel{\lambda} + \cancel{9\lambda} - 42 = 0$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

$$\lambda^3 - (\text{trace of } A)\lambda^2 + \left(\begin{array}{c}\text{Sum of minors}\\ \text{of } a_{11}, a_{22}, a_{33}\end{array}\right)\lambda - (|A|) = 0$$

$\lambda^3 - 7\lambda^2 + 36 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 0 & 36 \\ & 0 & 1 & -6 & -6 \\ \hline & 1 & -6 & -6 & \times \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & -7 & 0 & 36 \\ & 0 & 3 & -12 & -36 \\ \hline & 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrrr} 6 & 1 & -7 & 0 & 36 \\ & 0 & 6 & -6 & -36 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$-2, 3, 6$ are the eigen values of A.

$-2+3+6 = 1+5+1$

* Find characteristic equation of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$|A - \lambda I| = \left|\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{bmatrix}\right|$$

$\lambda^3 - (0)\lambda^2 + (-20)\lambda + 8 = 0$

$\lambda^3 - 20\lambda + 8 = 0$

* Find sum and product of eigen values of $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

-24
2
2
$1(-12-12)$
$-1(-4-6)$
$+3(-4+6)$
$= -24+10+6$

Sum $= 2+1+1 = 4$

Product $= 2(1-4) - 1(1-2) - 1(-2+1)$

$= 2(-3) + 1 + 1$

$\boxed{\text{Product} = -4}$

* Find the eigen values of A^2 & A^{-1} of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

eigen values of $A = 3, 2, 5$

$A^{-1} = \frac{1}{3}, \frac{1}{2}, \frac{1}{5}$

$A^2 = 9, 4, 25$.

eigen value of Adjoint A.

$30(10)$

$\frac{30}{3}, \frac{30}{2}, \frac{30}{5}$

$= 10, 15, 6$

* 2 eigen values of matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are 1, 1 ; find eigen values of A^{-1}

$1 + 1 + x = 7$

$x = 5$

Eigen values of A^{-1} = $1, 1, \frac{1}{5}$

1) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 12 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$R_2 \to R_2 - 2R_1$

$\approx \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & -3 \end{bmatrix}$

$\approx -2 \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & -3 \end{bmatrix} \Rightarrow$ rank $= 3$

2) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix} \approx 3 \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\approx \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - R_1$

$\approx \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\text{Rank} = 2}$

* 3) $-1 \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$

$\approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_3 \to R_3 - R_2}{\approx} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{rank} = 2}$

* MATRIX:

An ordered set of 'mn' numbers (real/complex) arranged in a rectangular array of 'm' rows and n columns is called $m \times n$ matrix.

These mn numbers

$|A| = 0 \to$ singular matrix

$|A| \neq 0 \to$ Non-singular matrix

$A^T = A \to$ symmetric matrix

$A^T = -A \to$ non-symmetric matrix.

$A^2 = A \to$ Idempotent

$A^2 = I \to$ Involuntary

$A^m = 0 \to$ Nilpotent

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* Find the eigen values and Eigen vectors of matrix $\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$

$$A = \begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$$

$$|A-\lambda I| = 0$$

$$\begin{vmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{vmatrix} = 0$$

Eigen Values of matrix 'A' $= -2, 3, 6$
$\Rightarrow$ L.I.

$$\lambda^3 - (7)\lambda^2 + (0)\lambda + 36 = 0$$

$$\boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

① If $X_1 = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ be non-zero vector corresponding to λ_1.

$$[A - \lambda_1 I] X_1 = 0.$$

$$[A + 2I] X_1 = 0$$

$$\begin{bmatrix}3&1&3\\1&7&1\\3&1&3\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = 0$$

$R_3 \to R_3 - R_1$

$R_2 \to 3R_2 - R_1$

$$\approx \begin{bmatrix}3&1&3\\0&20&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = 0$$

$$\cancel{\approx 3\begin{bmatrix}1&1/3&1\\0&20&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}}$$

$$\approx 20\begin{bmatrix}3&1&3\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\cancel{3x_1 + y_1 + 3}$

$$\boxed{\text{rank} = 2}$$

$$20\begin{bmatrix}3x_1+y_1+3z_1\\y_1\\0\quad 0\quad 0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$y_1 = 0$

Eigen vector need not be a zero vector.

$3x_1 + 3z_1 = 0$

$x_1 + z_1 = 0$

$x_1 = -k$

$z_1 = +k$

$$X_1 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$AX = \lambda X$$

$$= \begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix} = -2\begin{bmatrix}-1\\0\\+1\end{bmatrix}.$$

② If $X_2 = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$ corresponding to λ_2

$$[A - \lambda_2 I] X_2 = 0$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_3 \to R_3 - 3R_2$

~~$R_2 \to R_2 + 2R_1$~~

$$\sim \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 0 & -5 & -5 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\sim -5\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim -25\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim -25\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank = 2

$$\begin{bmatrix} -2x_2 + y_2 + 3z_2 \\ y_2 + z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_2 + y_2 + 3z_2 = 0$

$y_2 = -z_2$

$y_2 = k$

$z_2 = -k$

$-2x_2 + k - 3k = 0$

$2x_2 = -2k$

$x_2 = -k$

$$X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$AX = \lambda X$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

(3) If $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ corresponding to λ_3

$[A - \lambda_3 I] = 0$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 4 & -8 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 + 5R_2$

$$4\begin{bmatrix} 0 & -4 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4\times4\begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\sim 16\begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank = 2

$$\sim \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -y_3 + 2z_3 \\ x_3 - y_3 + z_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y_3 + 2z_3 = 0$

$2z_3 = y_3 \Rightarrow y_3 = k$

$z_3 = \frac{k}{2}$ $\qquad z_3 = k$

$\cancel{x_3 = k - \frac{k}{2}}$ $\qquad y_3 = 2k$

$\cancel{x_3 = \frac{k}{2}}$ $\qquad x_3 = k$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$AX = \lambda X$

$$AX = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigenvalues are $-2, 3, 6$. Eigenvectors are

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

3 eigen ~~values~~ vectors corresponding to 3 eigenvalues.

*Find Eigen value & Eigen vectors of matrix

$$\begin{bmatrix} -2 & +2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$\det A = 24 + 12 + 9$
$= 45$

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

-12
-3
-6
$\overline{-21}$

$\cancel{-2(-12) - 2(-6) - 3(-3)}$

$$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - (45) = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -21 & -45 \\ & 0 & 1 & 4 & -34 \\ \hline & 1 & 2 & -17 & \neq 0 \end{array}$$

$$\begin{array}{r|cccc} 3 & 1 & 1 & -21 & -45 \\ & 0 & 3 & 12 & -27 \\ \hline & 1 & 4 & -9 & \neq 0 \end{array}$$

$$\begin{array}{r|cccc} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\lambda+3) \lambda^3+\lambda^2-21\lambda-45 (\lambda^2-2\lambda-15$

$\lambda^3+3\lambda^2$

$-2\lambda^2-21\lambda$

$-2\lambda^2-6\lambda$

$-15\lambda-45$

$-15\lambda-45$

0

$(\lambda-5)(\lambda+3)=0$

$\lambda=-3,-3,5.$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

case(i) $\boxed{\lambda=-3}$ $\quad X_1=\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$[A-\lambda_1 I]X_1=0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3+R_1$

$R_2 \to R_2-2R_1$

$$\sim\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank = 1

$\cancel{x_1+2y_1-3z_1=0}$ $\quad z_1=k_1$

$\cancel{k_1+2k_2-3z_1=0}$ $\quad y_1=k_2$

$\cancel{z_1=\frac{k_1+2k_2}{3}}$

$x_1+2y_1-3z_1=0$

$x_1+2k_2-3k_1=0$

$x=-2k_2+3k_1$

$$k_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}+k_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$X_1 \qquad X_2$

$\Rightarrow$ ~~2 eigen vector~~

~~$\lambda=5$~~ $[A-\lambda_2 I]X_2=0$

$$\cancel{\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & \end{bmatrix}}$$

Case (iii) $\lambda = 5$ $\quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$[A - \lambda_2 I] X_2 = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ 0 & -8 & -16 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8 \times 2 \begin{bmatrix} -7 & 2 & -3 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$R_2 \to 7R_2 + R_1$~~ $\quad R_2 \to 7R_2 + R_1$

$$\cancel{-16 \begin{bmatrix} -7 & 2 & -3 \\ 0 & -12 & \end{bmatrix}} \quad -16 \begin{bmatrix} -7 & 2 & -3 \\ 0 & -12 & -18 \\ 0 & 1 & 2 \end{bmatrix}$$

$$-96 \begin{bmatrix} -7 & 2 & -3 \\ 0 & +2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$96 \begin{bmatrix} -7 & 2 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_2 - R_1$

$-7x_2 + 2y_2 - 3z_2 = 0$

$2y_2 + 3z_2 = 0$

$z_2 = 0 \quad y_2 = 0$

$x_2 = 0$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$X_2 = \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$AX = -2\,2\,3$

* Find the eigen values and eigen vectors of matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$\det A = 6(8) + 2(-4) + 2(-4) = 32$

characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$8 + 14 + 14$

$\lambda^3 - (12)\lambda^2 + (36)\lambda - 32 = 0$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

1	1	-12	36	-32
	0	1	-11	25
	1	-11	25	$\neq 0$

$$3\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & 3 & -27 & 27\\ \hline 1 & -9 & 9 & \neq 0\end{array}\right.$$

$$-2\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & -2 & 28 & \\ \hline 1 & -14 & 58 & \end{array}\right.$$

$$-3\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & -3 & 45 & \\ \hline 1 & -15 & & \neq 0\end{array}\right.$$

$$-4\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & -4 & 64 & \\ \hline 1 & -16 & & \end{array}\right.$$

$$-1\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & -1 & 13 & -49\\ \hline 1 & -13 & 49 & \end{array}\right.$$

$$2\left|\begin{array}{cccc}1 & -12 & 36 & -32\\ 0 & 2 & -20 & 32\\ \hline 1 & -10 & 16 & 0\end{array}\right.$$

$\lambda-2)\lambda^3-12\lambda^2+36\lambda-32(\lambda^2-10\lambda+16$

$\lambda^3-2\lambda^2$

$-10\lambda^2+36\lambda-32$

$-10\lambda^2+20\lambda$

$16\lambda-32$

$16\lambda-32$

0

$(\lambda-2)(\lambda-8)=0$

$\lambda=2,2,8$

<u>Case(i)</u> $\lambda_1=2$

$X_1=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ be a non-zero matrix corresponding to λ_1

$[A-\lambda_1 I]X_1=0$

$$\begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3\to R_3+R_2$

$$2\begin{bmatrix}2 & -1 & 1\\ -2 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad R_2\to R_2+R_1$$

$$2\begin{bmatrix}2 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2x_1-y_1+z_1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$z_1=k_1$

$x_1=k_2$

$2x_1-y_1+z_1=0$

$y_1=2k_2+k_1$

$$k_1\begin{bmatrix}0\\1\\1\end{bmatrix}+k_2\begin{bmatrix}1\\2\\0\end{bmatrix}$$

<u>Case(ii)</u> $\lambda_2=8$ $\quad X_2=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$ be a non-zero matrix corresponding to λ_2

$[A-\lambda_2 I]X_2=0$

$$\begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$R_1\to R_1+R_2$

$$-2\begin{bmatrix}1 & 1 & -1\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$-2\begin{bmatrix} 1 & 1 & -1 \\ -2 & 5 & -1 \\ 0 & -6 & -6 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$-2\times-6\begin{bmatrix} 1 & 1 & -1 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 + y_2 - z_2 \\ 7y_2 + z_2 \\ y_2 + z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

* r

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$-6\times-3\times-2\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$x_2 + y_2 - z_2 = 0$ $\quad z_2 = k$

$y_2 + z_2 = 0$ $\quad y_2 = -k$

$x_2 = k + \cancel{2}k = 2k$ $\quad 2k + k$

* Find ~~product of~~ eigen values of adjoint A.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Eigen values $= 2, 4, 3$

$\because$ upper triangular matrix $\Rightarrow$ $\det A = 24$

Eigen values of adj A $= \frac{24}{2}, \frac{24}{4}, \frac{24}{3}$

$= 12, 6, 8$

* Find eigen values of $A^2 - 2A + I = 0$.

$= (4, 16, 9) - (4, 8, 6) + (1, 1, 1)$

$= (1, 9, 4)$ $\quad \det A = 1$

$$[I - \lambda I] = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$\cancel{\lambda^3(3) - \lambda^2(0) -}$

$\lambda^3 - 3\lambda^2 - 1 = 0$

* If $-4, 10, \sqrt{2}$ are the eigen values of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix}$$ find eigen values of A^{-1}

$1 + 1 + 2 + 2 = -4 + 10 + \sqrt{2} + x$

$\boxed{x = -\sqrt{2}}$

Eigen values of $A^{-1} = \frac{-1}{4}, \frac{1}{10}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

* Find Eigen values and eigen vectors

of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\det A = 0$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 + \lambda(0) - 0 = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

Case (i) $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $\lambda = 0, 0, 3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$x_1 + y_1 + z_1 = 0$ $z_1 = k_1$

$x_1 = -k_1 - k_2$ $y_1 = k_2$

$$k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii) $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$\cancel{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z_2 = k$

$-2x_2 + y_2 + z_2 = 0$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$-y_2 + z_2 = 0$

$y_2 = k$

$x_2 = k$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

* Find Eigen values and eigen vectors

$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Eigen values $= 2, 2, 1$

(i) $\lambda = 2$ $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$[A - \lambda_1 I] X_1 = 0$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3y_1 + 4z_1 = 0$

$z_1 = 0$

$y_1 = 0$

$x_1 = k$

$k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ii) $\lambda_2 = 1$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$z_2 = k$

$y_2 - z_2 = 0$

$y_2 = k$

$x_2 + 3y_2 + 4z_2 = 0$

$x_2 = -7k$

$$k \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Eigen values are 2, 2, 1

Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

* Similarity of matrices

$A = x_1 \; x_2 \; x_3$

$B = P^{-1}AP$

Modal Matrix

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

P is non-singular

$$(P^{-1}A)P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$D = P^{-1}AP$

$PDP^{-1} = P(P^{-1}AP)P^{-1}$

$= (PP^{-1})A(PP^{-1})$

$\boxed{PDP^{-1} = A}$

$(PDP^{-1})(PDP^{-1}) = A^2$

$PD(P^{-1}P)DP^{-1} = A^2$

$\boxed{PD^2P^{-1} = A^2}$

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* Determine the modal matrix of A

Verify that $P^{-1}AP$ is a diagonal matrix.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$\det A = -2(-12) - 2(-6) - 3(-3)$

-12
-3
-6
-36
-24

The charadteristic equation of A is given by $|A - \lambda I| = 0$.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\lambda^3 - (-1)\lambda^2 + \lambda(-21) - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$(\lambda+3)(\lambda^2-2\lambda-15)=0$

$\boxed{\lambda=-3,-3,5}$

(i) $\lambda=-3$ $\quad X_1=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$

$[A+3I]$

$$\begin{bmatrix}1&2&-3\\2&4&-6\\-1&-2&3\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=0$$

$$2\begin{bmatrix}1&2&-3\\1&2&-3\\-1&-2&+3\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=0$$

$R_2\to R_2-R_1$

$R_3\to R_3+R_1$

$$2\begin{bmatrix}1&2&-3\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}=0$$

$x_1+2y_1-3z_1=0$ $\quad x_1=k_1$

$y_1=k_2$

$k_1+2k_2=3z_1$

$$\boxed{z_1=\frac{k_1+2k_2}{3}}$$

$$k_1\begin{bmatrix}1\\0\\\frac{1}{3}\end{bmatrix}+k_2\begin{bmatrix}0\\1\\2/3\end{bmatrix}\Rightarrow\frac{1}{3}\begin{bmatrix}3\\0\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}0\\3\\2\end{bmatrix}$$

(ii) $\lambda=5$ $\quad X_2=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$

$[A-5I]X_2=0$

$$\begin{bmatrix}-7&2&-3\\2&-4&-6\\-1&-2&-5\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$2\begin{bmatrix}-7&2&-3\\1&-2&-3\\-1&-2&-5\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$R_2\leftrightarrow R_1$

$$2\begin{bmatrix}1&-2&-3\\-7&2&-3\\-1&-2&-5\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$R_3\to R_3+R_1$

$R_2\to R_2+7R_1$

$$2\begin{bmatrix}1&-2&-3\\0&-12&-24\\0&-4&-8\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$$2\times-12\times-4\begin{bmatrix}+1&-2&-3\\0&1&2\\0&1&2\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$R_3\to R_3-R_2$

$$96\begin{bmatrix}1&-2&-3\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=0$$

$x_2-2y_2-3z_2=0$

$y_2+2z_2=0$ $\quad z_2=k$

$y_2=-2z_2$ $\quad y_2=-2k$

$x_2=2y_2+3z_2=-4k+3k=-k$

$$X_3 = \begin{bmatrix}-1\\-2\\1\end{bmatrix}$$

Mutual Matrix $P = [X_1\ \ X_2\ \ X_3]$

$$= \begin{bmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3\end{bmatrix}$$

$$P = \begin{bmatrix}3&0&-1\\0&3&-2\\1&2&1\end{bmatrix}$$

P is non-singular so P^{-1} exists

$\det P = 3(7)-0-1(-3) = 24$

$$P^{-1} = \frac{\operatorname{adj} P}{\det P} = \cancel{\frac{-1}{8}\begin{bmatrix}-1&-2&3\\-2&4&6\\-1&-2&-5\end{bmatrix}}$$

$(P^{-1}A)P = D$

$$\cancel{D = \frac{-1}{8}\begin{bmatrix}-1&-2&3\\-2&4&6\\-1&-2&-5\end{bmatrix}\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}\begin{bmatrix}3&0&1\\0&3&2\\1&2&-1\end{bmatrix}}$$

$$\cancel{= \frac{-1}{8}\begin{bmatrix}-5&-10&15\\6&-12&-18\\3&6&15\end{bmatrix}\begin{bmatrix}3&0&1\\0&3&2\\1&2&-1\end{bmatrix}}$$

$$\cancel{= \frac{-1}{8}\times -5\times 6\times 3\begin{bmatrix}1&2&-3\\1&-2&-3\\1&2&5\end{bmatrix}\begin{bmatrix}3&0&1\\0&3&2\\1&2&-1\end{bmatrix}}$$

$$\cancel{= \frac{90}{8}\Big[\ \Big]}$$

$$\operatorname{adj} P = \begin{bmatrix}+7&-2&+-3\\-2&+4&-6\\+3&+6&+9\end{bmatrix}^T$$

$$= \begin{bmatrix}7&-2&3\\-2&4&6\\-3&-6&9\end{bmatrix}$$

$$P^{-1} = \frac{1}{24}\begin{bmatrix}7&-2&3\\-2&4&6\\-3&-6&9\end{bmatrix}$$

$$D = (P^{-1}A)P$$

$$= \frac{1}{24}\begin{bmatrix}7&-3&3\\-2&4&6\\-3&-6&9\end{bmatrix}\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}\begin{bmatrix}3&0&-1\\0&3&-2\\1&2&1\end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix}-21&6&-9\\6&-12&-18\\-15&-30&45\end{bmatrix}\begin{bmatrix}3&0&-1\\0&3&-2\\1&2&1\end{bmatrix}$$

$$\cancel{= \frac{1}{24}\times 3\begin{bmatrix}-7&2&-3\\2&-4&-6\\-5&-10&15\end{bmatrix}\begin{bmatrix}3&0&-1\\0&3&-2\\1&2&1\end{bmatrix}}$$

$$\cancel{= \frac{90}{24}\begin{bmatrix}-24&0&0\\0&-24&0\\0&0&40\end{bmatrix}} = \cancel{\frac{18}{24}\times +8\times 4\times 8\begin{bmatrix}-3&0&\\0&-30&\\0&0&\end{bmatrix}}$$

$$= \frac{9}{24}\times 80\times 8\times 8$$

$$\frac{1}{24}\begin{bmatrix}-72&0&0\\0&-72&0\\0&0&120\end{bmatrix} = \begin{bmatrix}-3&0&0\\0&-3&0\\0&0&5\end{bmatrix}\quad 72\times 8\begin{bmatrix}-3&0&0\\0&-3&0\\0&0&\end{bmatrix}$$

* Diagonalize $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ hence Evaluate A^4.

$\lambda = 1, 2, 3$

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$(P^{-1}A)P = \frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$P^{-1}AP = D$

By definition power of matrix

$$A^4 = PD^4P^{-1}$$

$$A^4 = \frac{1}{2}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ -65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

* $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Calculate A^5.

Characteristic equation $|A - \lambda I| = 0$

Note: If asked power of matrix, initially prove diagonalization of matrix

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

14
8
14
36

$\det A = 3(14) + 1(-2) + 1(-4) = 36$

$$\lambda^3 - (11)\lambda^2 + (36)\lambda - (36) = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

+3	1	-11	36	-36
	0	+3	~~12~~ -24	36
	1	~~-9~~ -8	12	0

$$(\lambda - 3)(\lambda^2 - 8\lambda + 12) = 0$$

$$(\lambda - 3)(\lambda - 6)(\lambda - 2) = 0$$

$$\lambda = 3, 6, 2 = 3, 2, 6$$

(i) $\lambda = 3$ $\quad X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$[A - 3I]X_1 = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

* Verify the given matrix is diagonalizable or not.

$$A = \begin{bmatrix} 6 & -3 & 3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

Method -2:

To test for diagonalizable matrix; if the given matrix is symmetric $A = I_3 A I_3$ (rows, columns)

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

diagonal form:

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & -2 \\ 0 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 2C_3 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -36 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\boxed{\cancel{D = P^{-1} A P}}$

$\boxed{D = P^T A P}$ $\Rightarrow$ symmetric

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y_1 + z_1 = 0$

$-x_1 + 2y_1 - z_1 = 0$

$x_1 = k$

$z_1 = k$

$y_1 = k$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ (ii) $\lambda = 6$: $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$R_3 \to R_3 + 3R_1$~~ $R_3 \to 3R_3 + R_1$

$$= -1 \cancel{\begin{bmatrix} -3 & -1 & 1 \\ +1 & +1 & +1 \end{bmatrix}}$$

$$= -1 \begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 4 \begin{bmatrix} -3 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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*INNER PRODUCT SPACES :

→ Dot Product on R^n:

Consider 2 vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ of R^n; the dot product of u&v is given by.

$$\boxed{u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}$$

→ length of vector in R^n : (or) NORM OF A VECTOR

The norm (or) length of vector V in R^n is denoted by $\boxed{\|V\|}$ where $V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and is defined as.

$$\boxed{\|V\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}}$$

Ex: $V = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\boxed{\|V\| = \sqrt{6}}$

$$\boxed{\begin{aligned} \|V\| &= \sqrt{V \cdot V} \\ \|V\|^2 &= V \cdot V \end{aligned}}$$

→ Distance between vectors in R^n:

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in R^n; the distance between the vectors is defined by $\boxed{\|u - v\| = \sqrt{(u-v) \cdot (u-v)}}$

$$\Rightarrow \boxed{\|u - v\| = \|v - u\|}$$

*** NOTE:** If V is a vector in R^n and c is a real number (const/scalar) then

$$\boxed{\|cV\| = |c|\,\|V\|}$$

* Let V be a non-zero vector in R^n then

$$\boxed{U_v = \frac{V}{\|V\|}} \Rightarrow \text{UNIT VECTOR of } U \text{ IN THE DIRECTION OF } V.$$

ex: $V = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ Find unit vector U_v

$$U_v = \frac{V}{\|V\|} = \frac{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\sqrt{1+2+1}} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{U_v = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$$

*** PREPOSITION / THEOREM:**

Let u, v, w be vectors in R_n and c is a scalar;

(i) $u \cdot u \geq 0$

(ii) $u \cdot u = 0$ iff $u = 0$

(iii) $u \cdot v = v \cdot u$.

(iv) $u(v+w) = u \cdot v + u \cdot w$ (and)

$(u+v) \cdot w = u \cdot w + v \cdot w$.

(v) $(cu) \cdot v = c(u \cdot v)$ → External composition with scalar multiplication

***THEOREM:**

Let u, v be vectors in R^n; ~~$(u+v)$~~ then

$$\boxed{(u+v)(u+v) = \|u\|^2 + 2u \cdot v + \|v\|^2.}$$

Proof:

$(u+v)(u+v) = (u+v) \cdot u + (u+v) \cdot v$.

$= u \cdot u + v \cdot u + u \cdot v + v \cdot v$.

$$\boxed{(u+v)(u+v) = \|u\|^2 + 2u \cdot v + \|v\|^2}$$

*** CAUCHY'S — SCHWARTZ INEQUALITY:**

If u & v be two vectors in R^n; then

$$\boxed{|u \cdot v| \leq \|u\|\,\|v\|}$$

→ Proof:

If $u = 0$; then $u \cdot v = 0$; we also know ; in this case; that $\|u\| \cdot \|v\| = 0\,\|v\| = 0$ so that equality holds; now suppose that $u \neq 0$ and k is a real number. ∵ vector need not be zero vector

Consider the dot product of the vectors $ku + v$ with itself.

By theorem, we have

$(ku+v)\cdot(ku+v) \geq 0$

Now, by theorem: distributive property:

LHS can be expanded to obtain:

$$\boxed{k^2(u\cdot u)+2k(u\cdot v)+k^2(u\cdot v)\geq 0}$$

Observe that the expression on LHS is quadratic in the variable k with real coeff.

$D<0 \rightarrow$ Ellipse

$D=0 \rightarrow$ Parabola.

$D>0 \rightarrow$ Hyperbola

Let : $a=u\cdot u$; $b=u\cdot v$ and $c=v\cdot v$.

$$\boxed{ak^2+2bk+c\geq 0}$$

∵ The vectors must be in closed contour ⇒ Ellipse

This inequality imposes conditions on the coefficients a, b, c; specifically the equation $\boxed{ak^2+2bk+c=0}$ must have atmost one real zero.

Thus by the quadratic formula;

the discriminant $\boxed{(2b)^2-4ac\leq 0}$ (or)

equivalently $(u\cdot v)^2 \leq (u\cdot u)(v\cdot v)$.

After taking the square root on both sides:

$$\boxed{|u\cdot v| \leq \|u\|\|v\|}$$

Q: Let $V=R^2$ with inner product defined by $\langle u\cdot v\rangle = u_1v_1+3u_2v_2$ and $u=\begin{bmatrix}2\\-2\end{bmatrix}$ $v=\begin{bmatrix}1\\4\end{bmatrix}$; Verify that Cauchy Schwartz inequality is upheld.

We know that

$$|u\cdot v| \leq \|u\|\cdot\|v\|$$

LHS: $|u\cdot v| = |2+3(-8)| = |2-24| = |-22| = \underline{\underline{22}}$

$\cancel{\|u\| = \sqrt{4+4} = 2\sqrt{2}}$

$\cancel{\|v\| = \sqrt{1+16}}$

$\|u\| = \sqrt{\langle u\cdot u\rangle} = \sqrt{(2)(2)+3(-2)(-2)} = \sqrt{16} = 4.$

$\|v\| = \sqrt{\langle v\cdot v\rangle} = \sqrt{1+3(16)} = 7$

$\|u\|\cdot\|v\| = 7\times 4 = 28.$

$$\Rightarrow \boxed{|u\cdot v| \leq \|u\|\cdot\|v\|.}$$

Cauchy Schwartz inequality holds.

*. Properties of NORM in R^n:

Let v be a vector ~~be~~ in R^n and c is a scalar:

i) $\|v\| \geq 0$

(ii) $\|v\| = 0$ iff $v = 0$

(iii) $\|cv\| = |c|\|v\|$

(iv) $\boxed{\|u+v\| \leq \|u\| + \|v\|} \longrightarrow$ Triangle Inequality

*TRIANGLE INEQUALITY:

$$\boxed{\|u+v\| \leq \|u\| + \|v\|}$$

Proof:

$$(\|u+v\|)^2 = (u+v)(u+v)$$

$$= \cancel{(u+v)\cdot u + 2(u\cdot v) +}$$

$$= u\cdot u + 2(u\cdot v) + v\cdot v$$

$$= \|u\|^2 + 2(u\cdot v) + \|v\|^2$$

By Cauchy Schwartz Inequality

$$|u\cdot v| \leq \|u\|\,\|v\|,$$

So that $|u\cdot v|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$$\leq (\|u + v\|)^2$$

after taking square root on both sides

$$\boxed{\|u+v\| \leq \|u\| + \|v\|}$$

$*\ u = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \langle u\cdot v\rangle = u_1v_1 + 3u_2v_2$

$$\|u\| = 4$$

$$\|v\| = 7$$

$u+v = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \langle (u+v)\cdot(u+v)\rangle$

$$= \sqrt{(u+v)\cdot(u+v)}$$

$$= \sqrt{9 + 3\times 4}$$

$$= \sqrt{21}$$

* Find Algebraic Multiplicity and Geometric multiplicity of matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ 22/03/

$\lambda = 2, 2, 2$

AM of 2 = 3

This matrix is not diagonalizable.

$\lambda = 2$

$[A - 2I]x = 0$

Case(i)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = 2; n = 3$

$3 - 2 = 1$

GM of 2 is 1 (No. of L.I vectors)

* Find symmetric matrix A; the eigen values of A are: 0, 3, 15 and corresponding eigen vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$; $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Symmetric Matrix:

$a + b + c = 18$

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$D = P^T A P$.

$$P = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}.$$

* INNER PRODUCT SPACES:

→ Angle between vectors:

If u & v are vectors in R^n then the cosine of angle 'θ' between vectors is defined as

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}}$$

* Find angle between 2 vectors $u = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$u \cdot v = -2 - 4 + 6 = -12$

$\|u\| = \sqrt{4+4+9} = \sqrt{17}$

$\|v\| = \sqrt{1+4+4} = 3.$

$\theta = \cos^{-1}\left(\frac{0}{3 \times \sqrt{17}}\right) = \frac{\pi}{2}.$

* $u = \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$; find $\|u\|$; distance b/w u & v ($\|u-v\|$) ; find the unit vector in the direction of u (w_u). Find $\cos\theta$ between 2 vectors ; are the vectors orthogonal? Explain. Find the vector in opp. direction of v of length 3.

Vector in opp. direction to v. $= \frac{3v}{\|v\|}$

$= \frac{3}{\sqrt{1+1+9}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\boxed{= \frac{3}{\sqrt{11}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}$$

Find a vector w i.e. orthogonal to both u & v.

* Let $V = P_2$ with inner product defined by $<p \cdot q> = \int_0^1 p(x) \cdot q(x)\, dx$.

a) Let $P(x) = 1 - x^2$
$q(x) = 1 - x + 2x^2$; Find $<p \cdot q>$

b) Let $P(x) = 1 - x^2$. verify $<p \cdot p> > 0$

b) $\int_0^1 1 + x^4 - 2x^2\, dx = 1 + \frac{1}{5} - \frac{2}{3} = \frac{15 + 3 - 10}{15} = \frac{8}{15}$

a) $<p \cdot q> = \int_0^1 (1 - x^2)(1 - x + 2x^2)\, dx$

$= \int_0^1 (1 - x + 2x^2 - x^2 + x^3 - 2x^4)\, dx$

$= \int_0^1 (1 - x + x^2 + x^3 - 2x^4)\, dx$

$= 1 - \frac{1}{2}(1 - 0) + \frac{1}{3}(1 - 0) - \frac{2}{5}(1 - 0) + \frac{1}{4}$

$= \frac{1}{2} + \frac{1}{3} - \frac{2}{5} + \frac{1}{4}$

$= \frac{5}{6} - \frac{2}{5} + \frac{1}{4}$

65
24

$= \frac{50 - 24 + 15}{60} = \frac{41}{60}$

* **Properties of Norm in inner product:**

Let u & v be vectors in inner product space & and c be a scalar.

① $||v|| \geq 0$

② $||v|| = 0$ iff $v = 0$

③ $||cv|| = |c|\,||v||$

④ $|u \cdot v| \leq ||u||\,||v||$ (Cauchy - Schwartz).

⑤ $||u + v|| \leq ||u|| + ||v||$ (Triangle Inequality)

→

~~If u & v 2 vectors in an~~

⇒ Cosine of the angle between

* The vectors U and V in an inner product space given by $\cos\theta = \frac{<u \cdot v>}{||u||\,||v||} = 0$ (or)

$\theta = \pi/2$ and also $<u \cdot v> = 0$; is called orthogonal vectors.

* Let $V = P_2$ with inner product defined by $\langle p \cdot q \rangle = \int_{-1}^{1} p(x) \cdot q(x)\, dx$.

a) S.T. the vectors in $S = \{1, x, \frac{1}{2}(3x^2-1)\}$ are mutually orthogonal.

b) Find the length of each vector in S.

$\|1\| = \sqrt{\langle 1 \cdot 1 \rangle} = \sqrt{\int_{-1}^{1} dx} = \sqrt{2}$

$\|x\| = \sqrt{\langle x \cdot x \rangle} = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{1}{3}(2)} = \sqrt{\frac{2}{3}}$

$\|\frac{1}{2}(3x^2-1)\| = \sqrt{\langle \frac{1}{2}(3x^2-1) \cdot \frac{1}{2}(3x^2-1) \rangle}$

$= \sqrt{\frac{1}{4} \int_{-1}^{1} (9x^4 + 1 - 6x^2)\, dx}$

$= \frac{1}{2}\sqrt{\frac{9}{5}(2) + 1(2) - \frac{6}{3}(1+1)}$

$= \frac{1}{2}\sqrt{\frac{18}{5} + 2 - 4}$

$= \frac{1}{2}\sqrt{\frac{18-10}{5}}$

$= \frac{1}{2} \cdot \frac{2\sqrt{2}}{\sqrt{5}} = \underline{\underline{\sqrt{\frac{2}{5}}}}$

* Define inner product of P_3 by $\langle p \cdot q \rangle$.

i.e. $\langle p \cdot q \rangle = \int_0^1 p(x) \cdot q(x) \cdot dx$ where $p(x) = x$ and $q(x) = x^2$.

23/03/2022

① Find $\text{proj}_q p$

② Find $p - \text{proj}_q p$ & verify that $p - \text{proj}_q p$ is orthogonal to $\text{proj}_q p$.

$\langle p \cdot q \rangle = \int_0^1 x^3 dx = \frac{1}{4}$

$\langle q \cdot q \rangle = \int_0^1 x^4 dx = \frac{1}{5}$

* ① $\text{proj}_q p = \frac{\langle p \cdot q \rangle}{\langle q \cdot q \rangle} \cdot q$

$\boxed{\text{proj}_q p = \frac{5}{4} x^2}$

② (i) $p - \text{proj}_q p = x - \frac{5x^2}{4}$

P.T. $\left(x - \frac{5x^2}{4}\right)$ is orthogonal to $\frac{5x^2}{4}$.

$= \int_0^1 \left(x - \frac{5x^2}{4}\right)\left(\frac{5x^2}{4}\right) dx = \int_0^1 \left(\frac{5x^3}{4} - \frac{25}{16}x^4\right) dx$

$= \frac{5}{4} \cdot \frac{1}{4} - \frac{25}{16} \cdot \frac{1}{5}$

$= \frac{5}{16}\left[1 - \frac{5}{5}\right] = \underline{\underline{0}}$

$= 0$

By definition of orthogonality dot product b/w the 2 vectors is 0.

⇒ Here the dot product is an inner product.

* $U = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ $V = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

1) Find $proj_V^u$

2) Find vector $u - proj_V^u$ and verify orthogonality

$$proj_V^u = \frac{\langle U \cdot V \rangle}{\langle V \cdot V \rangle} \cdot V$$

$\langle U \cdot V \rangle = -1$

$\langle V \cdot V \rangle = 1$

$$proj_V^u = \frac{-1}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$u - proj_V^u$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$proj_V^u$ is orthogonal to $u - proj_V^u$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

* GRAM SMIDTH PROCESS:

Given that $B = \{V_1, V_2, V_3\}$ is an ordered (or) standard basis.

* WORKING PROCEDURE:

* Step 1:

To construct orthogonal basis; consider $B' = \{W_1, W_2, W_3\}$

$W_1 = V_1$

$W_2 = V_2 - proj_{W_1}^{V_2} = V_2 - \frac{\langle V_2 \cdot W_1 \rangle}{\langle W_1 \cdot W_1 \rangle} \cdot W_1$

$W_3 = V_3 - proj_{W_1}^{V_3} - proj_{W_2}^{V_3}$

$= V_3 - \frac{\langle V_3 \cdot W_1 \rangle}{\langle W_1 \cdot W_1 \rangle} \cdot W_1 - \frac{\langle V_3 \cdot W_2 \rangle}{\langle W_2 \cdot W_2 \rangle} \cdot W_2$

* Step 2:

To construct orthonormal basis; consider $B'' = \left\{ \frac{W_1}{\|W_1\|}, \frac{W_2}{\|W_2\|}, \frac{W_3}{\|W_3\|} \right\}$

$W_1 = V_1$

$W_2 = V_2 - proj_{W_1}^{V_2}$

$W_3 = V_3 - proj_{W_1}^{V_3} - proj_{W_2}^{V_3}$.

* Use the standard inner product on R^n and use the basis B', and Gram-Smidth process to find an orthonormal basis.

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$$

$$W_1 = V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$W_2 = V_2 - \text{proj}_{W_1} V_2$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{\langle V_2 \cdot W_1 \rangle}{\langle W_1 \cdot W_1 \rangle} \cdot W_1$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0 - 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

$\frac{1}{4}+1+\frac{1}{4}$

$\frac{1}{2}+1$

$$W_3 = V_3 - \text{proj}_{W_1} V_3 - \text{proj}_{W_2} V_3$$

$$= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{\langle V_3 \cdot W_1 \rangle}{\langle W_1 \cdot W_1 \rangle} W_1 - \frac{\langle V_3 \cdot W_2 \rangle}{\langle W_2 \cdot W_2 \rangle} \cdot W_2$$

$$= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}} \cdot \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1/6 \\ -1/3 \\ \frac{1}{6} \end{bmatrix}$$

$-\frac{1}{2} - \frac{1}{6}$

$$= \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ -2/3 \end{bmatrix}$$

$$W_3 \cdot W_1 = \frac{2}{3} - \frac{2}{3} = 0.$$

$$W_3 \cdot W_2 = \frac{-1}{3} + \frac{2}{3} - \frac{1}{3} = 0$$

$$W_1 \cdot W_2 = \frac{-1}{2} + \frac{1}{2} = 0.$$

$$B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -2/3 \\ -2/3 \end{bmatrix} \right\} \Rightarrow \text{orthogonal basis}$$

$\frac{1}{\frac{2}{3}(\sqrt{1+1+1}}$

$= \frac{2}{3}\sqrt{3}$

$\frac{4}{9}+\frac{4}{9}+\frac{4}{9}$

$$B'' = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \end{bmatrix}, \frac{\sqrt{3}}{2} \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} \right\}$$

$\Rightarrow$ orthonormal basis.

$$* \; B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$