17/07/2022

LINEAR DIFFERENTIAL EQUATIONS OF SECOND & HIGHER ORDER:

$$\to \frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + K_n y = 0 \quad -(1)$$

$$[D^n + K_1 D^{n-1} + \cdots\cdots + K_n]\, y = 0 - (2)$$

→ Operator / Standard form.

Let $y = e^{mx}$

$Dy = me^{mx}$

$D^2y = m^2 e^{mx}$

$\vdots$

$D^n y = m^n e^{mx}$

Auxiliary equation:

$$e^{mx}[m^n + K_1 m^{n-1} + \cdots\cdots + K_n] = 0$$

$\because e^{mx} \neq 0$

$\therefore m^n + K_1 m^{n-1} + \cdots\cdots + K_n = 0$

$f(m) = 0$

$$[(D-m_1)(D-m_2)\cdots\cdots(D-m_n)]y = 0$$

$$\because [(D-m_1)(D-m_2)]y = 0$$

* If roots are real & distinct

(i) $(D-m_1)y = 0$

$\frac{dy}{dx} - m_1 y = 0$

$\frac{dy}{y} = m_1 dx$

$\log_e y = m_1 x + \log_e c_1$

$\boxed{y = c_1 e^{m_1 x}}$

(ii) $(D-m_2)y = 0$

$\frac{dy}{dx} - m_2 y = 0$

$\frac{dy}{y} = m_2 dx$

$\log_e y = m_2 x + \log_e c_2$

$\boxed{y = c_2 e^{m_2 x}}$

III^{ly} $y = c_3 e^{m_3 x}$ - - - - -

∴ General solution of Linear Homogenous Differential Equation is a complementary function only.

i.e. $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots\cdots + c_n e^{m_n x}}$

* If roots are equal & real.

$$[(D-m_1)^2(D-m_1)]y = 0$$

Let $(D-m_1)y = z$

$(D-m_1)z = 0$

$\frac{dz}{dx} - m_1 z = 0$

$\int \frac{dz}{z} = \int m_1 \, dx$

$\log_e z = m_1 x + \log c_1$

$z = c_1 e^{m_1 x}$

$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x} \to$ LDE

$I.F = e^{-m_1 x}$

$ye^{-mx} = \int c_1 \, dx + c_2$

$\boxed{y = (c_1 x + c_2) e^{mx}}$

* If roots are imaginary

Let roots be $a \pm ib$

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$

$$y = c_1 e^{ax} e^{ibx} + c_2 e^{ax} e^{-ibx}$$
$$= e^{ax}\left[c_1 e^{ibx} + c_2 e^{-ibx}\right]$$
$$= e^{ax}\left[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)\right]$$
$$= e^{ax}\left[\cos bx(c_1+c_2) + \sin bx(i(c_1-c_2))\right]$$
$$\boxed{y = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right]}$$

* A DE is said to be LDE if its dependent variable & its derivatives occurs in the first degree and are not multiplied together.

* Non-Homogeneous Linear DE of Higher order:

NHLDE are always having complete solution i.e. complementary function + Particular Integral.

$$\left[D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n\right] y = Q(x)$$

(i) To find CF
$Q(x) = 0$
Similar to HLDE

(ii) To find PI:
$$\boxed{f(D)\, y = Q(x)}$$

a) $Q(x) = e^{ax}$

$$y = \frac{e^{ax}}{f(D)}$$ put $D = a$; $f(a) \neq 0$

b) $Q(x) = \sin ax / \cos ax$

$$y = \frac{\sin ax/\cos ax}{f(D)}$$ put $D^2 = -a^2$; $f(-a^2) \neq 0$

c) $Q(x) = x^K$ $K>0$

$$y = [f(D)]^{-1} x^K \rightarrow \text{Binomial Expansion.}$$
$$y = [1 \pm \phi(D)]^{-1} x^K$$

formulae:

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$

d) $Q(x) = e^{ax}.V(x)$ → $\sin bx/\cos bx$; → x^k

$$y = \frac{1}{f(D)} e^{ax} \sin bx/\cos bx$$
put $D = D + a$
$$y = e^{ax}\left[\frac{\sin bx/\cos bx}{\phi(D)}\right]$$ put $D^2 = -b^2$

Similar to (b)

$$y = \frac{e^{ax} x^k}{f(D)}$$
put $D = D + a$
$$y = e^{ax}\left[\frac{x^k}{\phi(D)}\right]$$

Similar to (c)

e) $Q(x) = x\,V(x)$ → $\sin ax$, $\cos ax$

$$y_P = \left[x - \frac{f'(D)}{f(D)}\right]\frac{V(x)}{f(D)} \rightarrow D^2 = -a^2$$

$\underbrace{}$ first simplify

*** Direct Method to find PI:**

$Q \rightarrow$ function of x $\quad \alpha \rightarrow$ const

*→ P.I. of

$$\frac{1}{D-\alpha}Q = e^{\alpha x}\int Q e^{-\alpha x}\,dx$$

$$\frac{1}{D+\alpha}Q = e^{-\alpha x}\int Q e^{\alpha x}\,dx$$

→ $\frac{1}{D-\beta}$; $\frac{1}{D-\alpha}$ → 2 inverse operators.

$\alpha, \beta \rightarrow$ constants

$Q \rightarrow$ fn of 'x'

$$\frac{1}{(D-\beta)(D-\alpha)}Q = \frac{1}{D-\beta}\left[\frac{Q}{D-\alpha}\right]$$

$$\frac{1}{(D-\beta)}\left[\frac{Q}{D-\alpha}\right] = \frac{1}{D-\beta}\left[e^{\alpha x}\int Q e^{-\alpha x}\,dx\right]$$

$$= e^{\beta x}\int e^{-\beta x}\left\{e^{\alpha x}\int Q e^{-\alpha x}\,dx\right\}dx$$

*** Method of Variation of Parameters:**

Find C.F →
- Real & equal: $(C_1x+C_2)e^{mx}$
- Real & distinct: $C_1e^{m_1x}+C_2e^{m_2x}$
- Imaginary: $e^{ax}[C_1\cos bx + C_2\sin bx]$

→ Coeff of $C_1 = U(x)$; coeff of $C_2 = V(x)$.

$U(x) =$, $V(x) =$ } known ⇒ { $U'(x)$, $V'(x)$

*** Wronskain function:**

$$W(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$$

$$W(x) = u(x)\cdot v'(x) - v(x)\cdot u'(x) \neq 0$$

then $u(x)$ & $v(x)$ are linearly independent fns. (known functions)

$$y_P = A(x)\cdot u(x) + B(x)\cdot v(x)$$

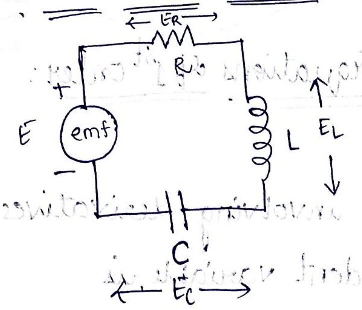
Unknown functions

$$A(x) = -\int \frac{v(x)\,R(x)}{W(x)}\,dx$$

$$B(x) = \int \frac{u(x)\,R(x)}{W(x)}\,dx$$

$$R(x) = \frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y$$

***LCR SERIES CIRCUIT:**



$E_R = IR$

$E_L = L\frac{di}{dt}$

$E_C = \frac{q}{C}$

$$E = E_R + E_L + E_C$$

$$= IR + L\frac{di}{dt} + \frac{q}{C}$$

$$E = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = E$$

$$\boxed{D = \frac{d}{dt}}$$

$$LD^2q + RDq + \frac{q}{C} = E$$

$$\left(LD^2 + RD + \frac{1}{C}\right)q = E \quad \text{--- (3)}$$

2^{nd} order LDE with const. coeff.

O. form of DE

$\therefore$ L, C, R are constant for any given circuit

* Ordinary Differential Equations of 1^{st} order:

→ **Ordinary DE**:

A differential equation involving derivatives wrto a single independent variable is called ordinary DE.

→ **Partial DE**:

A differential equation involving derivatives wrto more than one independent variable is called partial DE.

→ **Order of DE**: The order of the DE is the order of the highest order derivative occuring in the DE.

→ **Degree of DE**: The degree of the DE is the degree of the highest order derivative which occurs in it provided the equation has been made free of radical signs & fractional powers.

→ **General (or) Complete Solution**:

The differential equation is that in which the no. of independent arbitrary constants is equal to the order of the DE.

→ **Particular Solution**:

This is obtained from general solution of DE giving particular values to arbitrary constants.

→ **Exact Differential Equation**:

A differential equation obtained from its primitive directly by differentiation without any operation is said to be an exact differential equation.

$$U(x, y) = C$$

$$M\,dx + N\,dy = 0$$

Necessary condition for DE to be exact.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

General Solution:

$$\boxed{\int M dx + \int N dy = C}$$

* Integrating factor:

Non-exact DE can be made into exact DE by multiplying a suitable non-zero factor is said to be an integrating factor.

→ A DE can have more than 1 I.F and all the I.F gives same general solution

5 standard types to find an I.F:

1) Direct Method:

* 1) $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$; $\frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$

3) $\frac{x dy - y dx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

4) $\frac{x dy - y dx}{xy} = d\left(\log\left(\frac{y}{x}\right)\right)$

5) $\frac{y dx + x dy}{xy} = d(\log(xy))$

6) $\frac{x dx + y dy}{x^2 + y^2} = d\left(\frac{1}{2}\log(x^2 + y^2)\right)$

7) $\frac{x dy - y dx}{x^2 - y^2} = d\left(\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right)$

2) If the given eq is non-exact & it is HLDE (degree of x & y → same); then

$$\boxed{IF = \frac{1}{Mx + Ny}}$$

3) If the given eq is non-exact & it is of the form $y f(x,y) dx + x g(x,y) dy = 0$

then $\boxed{I.F = \frac{1}{Mx - Ny}}$

4) If DE is non exact and:

(i) $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is divisible by N i.e.

$$\boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)}$$

then $\boxed{I.F = e^{\int f(x) dx}}$

(ii) $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ is divisible by M i.e.

$$\boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)}$$

then $\boxed{I.F = e^{\int g(y) dy}}$

* LDE of first order:

A DE is said to be linear if the dependent variable & its derivatives occur only in 1st degree and not multiplied together

* Standard forms of Linear first order equation:

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \quad -(1)$$

$$\frac{dx}{dy} + P(y)\cdot x = Q(y) \quad -(2)$$

$$I\cdot F = e^{\int (\text{function of } I\cdot V) \text{ wrt } I\cdot V}$$

$$I\cdot F = e^{\int P(x)dx}, \quad I\cdot F = e^{\int P(y)\,dy}$$

General Solution:

$$\boxed{(D\cdot V) I\cdot F = \int Q(I\cdot V)(I\cdot F) \text{ wrt } I\cdot V + C}$$

$$ye^{\int P(x)dx} = \int Q(x)\cdot e^{\int P(x)dx}\cdot dx + C$$

$$xe^{\int P(y)dy} = \int Q(y)\cdot e^{\int P(y)dy}\cdot dy + C$$

* NLDE of 1st order: Bernoulli's Equation

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

* Clairaut's Equation:

$$p = \frac{dy}{dx} \qquad \boxed{y = x\frac{dy}{dx} + f(p)}$$

$$\boxed{y = px + f(p)}$$

Diff. wrto 'x' on both sides.

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0$$

General Solⁿ — Particular Solⁿ

* Orthogonal Trajectory:

$$f(x,y,c) = 0 \quad -(1)$$

diff (1) wrto 'x' and eliminate c

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad -(2)$$

DE of family of curves

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$f\left(x, y, -\frac{dx}{dy}\right) = 0 \quad -(3)$$

$g(x, y, t) = 0$ is orthogonal trajectory of given family of curves.

→ Trajectory: A curve which cuts every member of a given family of curves according to some definite law is called trajectory of the family.

→ Orthogonal Trajectory: A curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family

* Electric Circuits:

V → Voltage (V); L → inductance (H)
R → resistance (Ω) C → capacitance (F)
I → current q → charge

Voltage drop across

→ resistance (R) = IR
→ inductance (L) = $L\frac{di}{dt}$
→ capacitance (C) = $\frac{q}{C}$

LR circuit:

$$L\frac{di}{dt} + iR = V(E)$$

$$\frac{di}{dt} + \frac{iR}{L} = \frac{V}{L}$$

$$I.F. = e^{\int \frac{R}{L} dt} = e^{\left(\frac{Rt}{L}\right)}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)}$$

RC circuit:

$$\frac{q}{c} + iR = V(E)$$

$$\frac{1}{c}q + \frac{dq}{dt} \cdot R = V$$

$$\frac{dq}{dt} + \frac{q}{CR} = \frac{V}{R}$$

$$I.F = e^{\int \frac{1}{CR} dt} = e^{\left(\frac{t}{CR}\right)}$$

$$\boxed{q = EC\left(1 - e^{-t/RC}\right)}$$