CENTROIDS AND CENTERS OF GRAVITY

6-1 INTRODUCTION

A body of weight W is supported by a string attached at A, as shown in Fig. 6-1.1. The only external forces acting on the body are its weight and the reaction exerted by the string. Equilibrium of the body can exist only if these two forces are equal, opposite, and collinear. The line of action of the weight W can be determined, therefore, by the line of action of the support reaction or the line of the string.

Let the body be supported in a new position by the string now attached to B. The body will shift its position so that the line of action of the weight is again collinear with the string. Thus two positions of the line of action of the weight are determined experimentally. The intersection of these positions of the line of action determines a point which is defined as the center of gravity of the body; this is the point through which the action line of the weight always passes.

From the above discussion it is apparent that the problem of locating the center of gravity of a body reduces to determining the point through which the resultant force of gravity of the body acts.

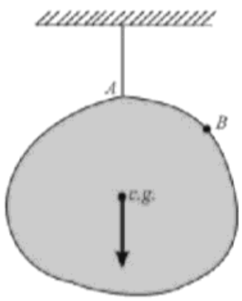


Figure 6.1.1

6-2 CENTRE OF GRAVITY OF FLAT PLATE

The analytical location of the center of gravity is simply an application of the principle of moments; that is, the moment of the resultant is equal to the moment sum of its parts. As an example, consider the flat plate of irregular section shown in Fig. 6-2.1. *A* pictorial as well as front and side views is shown. The network shown divides the plate into small elements having weights w_1, w_2, etc., which act at the center of each element. These gravity forces form a parallel force system, the resultant of which is the total weight *W* of the plate.

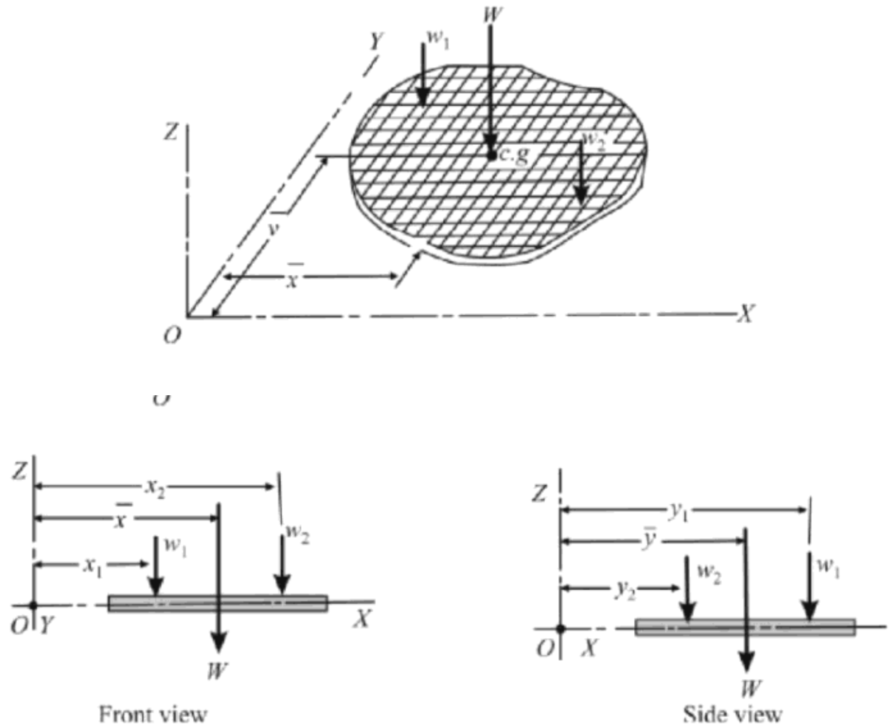


Figure 6-2.1 Coordinates of the center of gravity.

Let the coordinates of each elemental weight be (x_1, y_1), (x_2, y_2), etc., and the coordinates of the resultant weight be $(\bar{x}, \bar{y})$, as shown in Fig. 6-2.1. Note the use of the bar sign. In this book the coordinates of a resultant—be it force, weight, or area—are always distinguished by a bar sign. The coordinates are read as "bar *x*," etc. Taking moments of the weights about the Y axis, we get

$$W\,\bar{x} = w_1 x_1 + w_2 x_2 + \ldots = \sum w_i x_i \qquad (a)$$

With respect to the *X* axis, we have

$$W\,\bar{y} = w_1 y_1 + w_2 y_2 + \ldots = \sum w_i y_i \qquad (b)$$

These equations merely state that the moment of a weight *W* about an axis is equal to the moment sum of its elemental weights.

6-3 CENTROIDS OF AREAS AND LINES

If the material of the plate in Fig. 6-2.1 is homogeneous, the weight W may be expressed as the product of its density γ (i.e., weight per unit volume) multiplied by tA, where t is the thickness of the plate and A is its area. Similarly the weight w of an element is given by γta, where a is the cross-sectional area of the element. Substituting these values in Eq. (a) in Section 6-2 results in

$$\gamma tA\overline{x} = \gamma ta_1 x_1 + \gamma ta_2 x_2 + \ldots = \gamma t \sum ax$$

whence, canceling the constant terms γ and t, we get

$$\left.\begin{aligned} \boldsymbol{A\overline{x}} &= \sum \boldsymbol{a_i x_i} \\ \boldsymbol{A\overline{y}} &= \sum \boldsymbol{a_i y_i} \end{aligned}\right\} \qquad (6\text{-}3.1)$$

By analogy with Eqs. (a) and (b) in Section 6-2, the expression $A\,\overline{x}$, as well as $A\,\overline{y}$, is called the moment of area. It is equivalent to the sum of the moments of the elemental areas composing the total area. Note that the moment of area is therefore defined as the product of the area multiplied by the perpendicular distance from the center of area to the axis of moments.

If Eq. (6-3.1) is rewritten in the following form:

$$\left.\begin{aligned} \overline{x} &= \frac{\sum a_i x_i}{A} \\ \overline{y} &= \frac{\sum a_i y_i}{A} \end{aligned}\right\} \qquad (6\text{-}3.1a)$$

this gives a method of locating a point called the centroid of area. The *centroid* of area is defined as the point corresponding to the center of gravity of a plate of infinitesimal thickness. The term “centroid” rather than “center of gravity” is used when referring to areas (as well as to lines and volumes) because such figures do not have weight. The term “center of gravity” is widely used, although it is a misnomer. Strictly speaking, it should refer to the center of weight of actual bodies.

When referring to lines, we may determine the centroid by similar means. A line may be assumed to be the axis of a homogeneous slender wire. Thus Fig 6-3.1 represents the center line of a homogeneous wire of length L and constant cross-sectional area a lying in the XY plane. The weight W is given by the equation $W = \gamma aL$ and the weight w of an elemental length l by $w = \gamma al$.

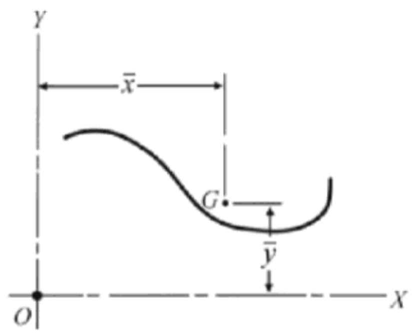


Figure 6.3.1 Homogeneous, slender and uniform wire.

Substituting these values in Eqs. (*a*) and (*b*) in Section 6-2, we have

$$\gamma aL\bar{x} = \gamma al_1 x_1 + \gamma al_2 x_2 + \ldots = \gamma a \sum lx$$

$$\gamma aL\bar{y} = \gamma al_1 y_1 + \gamma al_2 y_2 + \ldots = \gamma a \sum ly$$

whence, canceling the constant terms γ and a, we get

$$\left.\begin{aligned} \boldsymbol{L\bar{x}} &= \boldsymbol{\sum l_i x_i} \\ \boldsymbol{L\bar{y}} &= \boldsymbol{\sum l_i y_i} \end{aligned}\right\} \tag{6-3.2}$$

6-4 Importance of Centroids and Moments of Area

In subsequent work on strength of materials, the student will find the location of the centroid of an area is of great importance. For example, he will learn that in order to produce uniform stress distribution, the loads must be placed so that the line of action of their resultant coincides with the centroid of the cross section of the member. The position of the centroid of an area is also important for determining the location of the neutral axis in the bending of beams, for in strength of materials it is shown that the neutral axis (line of zero stress) passes through the centroid of the cross section of the beam.

An axis passing through the centroid of an area is known as a centroidal axis. The next chapter, which deals with moments of inertia, will make clear the great importance of the position of centroidal axes of areas. Many other instances of their importance will come to the student's attention in his engineering studies.

Of equal importance to the position of a centroid is the moment of an area. We recall that the moment of an area with respect to an axis was defined as the product of the area multiplied by the perpendicular distance from its centroid to the axis. In dynamics, the moment of area is used to determine the displacement of a body subjected to variable forces (see Prob. 11-3.1). In strength of materials, it is used to determine shearing stresses in beams. In addition, the moment of an area is extensively used for determining the deflection of beams by the area-moment method. These instances, as well as many others that the student will encounter, should indicate the importance of a permanent, not a temporary, knowledge of centroids and moments of area.

6-5 CENTROIDS DETERMINED BY INTEGRATION

We recall that integration is the process of summing up infinitesimal quantities. Except for a change in symbols and procedure, integration is equivalent to a finite summation. In the preceding sections, for example, if the area of an element had been expressed as the differential dA (i.e., a small part of the total area A), the equations for determining the centroid of an area would have become

$$\left.\begin{aligned} A\bar{x} &= \int x\, dA \\ A\bar{y} &= \int y\, dA \end{aligned}\right\} \qquad (6\text{-}5.1)$$

and for determining the centroid of a line, we could have used

$$\left.\begin{aligned} L\bar{x} &= \int x\, dL \\ L\bar{y} &= \int y\, dL \end{aligned}\right\} \qquad (6\text{-}5.2)$$

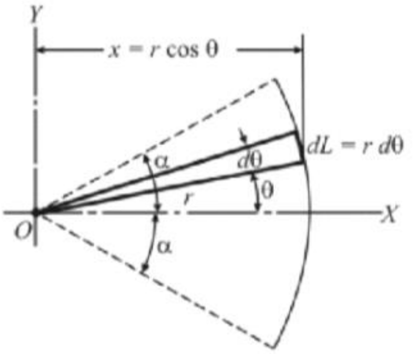
When we determine the centroid by integration, the figure is divided into differential elements so that:

1. All points of the element are located the same distance from the axis of moments, or
2. The position of the centroid of the element is known so that the moment of the element about the axis of moments is the product of the element and the distance of its centroid from the axis.

If a plane figure has a line of symmetry, its centroid is located on that line. This statement may be demonstrated by balancing a plate on its line of symmetry, whence the moments of the weights (also areas if the plate has constant thickness) on either side of the line of symmetry must be numerically equal and of opposite sign. If a plane figure has two lines of symmetry, the centroid is located at the point of intersection of the lines.

6-5.1 *Centroid of the Arc of a Circle.*

Determine the centroid of the line which is an arc of a circle, as shown in Fig. 6-5.1.



Solution

Let the axis of symmetry be chosen as the X axis. Then $\bar{y} = 0$. If the radius of the arc is denoted by r and the subtended angle by 2α, the element of arc dL and its distance from the Y axis are $dL = r\,d\theta$ and $x = r\cos\theta$. Applying Eq. (6-5.2), we have

$$[L\bar{x} = \int x\,dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{+\alpha} (r\cos\theta)\,r\,d\theta = r^2 \int_{-\alpha}^{+\alpha} \cos\theta\,d\theta$$

$$= 2r^2 \sin\alpha$$

Finally,

$$\bar{x} = \frac{2r^2 \sin\alpha}{2\alpha r} = \frac{r\sin\alpha}{\alpha} \qquad \textbf{\textit{Ans.}}$$

If the arc is a semicircle, as in Fig. 6-5.2, $\alpha = 90^{O} = \pi/2$ radians and $\sin\alpha = 1$. Substituting these values in the above result gives

$$\bar{x} = \frac{2r}{\pi}$$

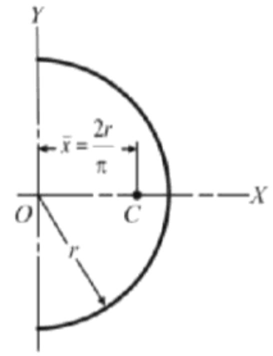


Figure 6.5.2

6-5.2 *Centroid of the Area of a Triangle.*

The triangle shown in Fig. 6-5.3 has a base b and an altitude h. Locate the centroid of the triangular area with respect to the base.

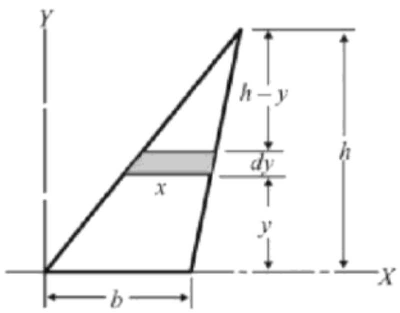


Figure 6.5.3

Solution

In accordance with Rule 1 given earlier, select strips parallel to the base as the differential elements of area. The area of any differential element is then $dA = x\,dy$. Applying Eq. (6-5.1), we obtain

$$[A\overline{y} = \int y\,dA] \qquad \left(\frac{1}{2}bh\right)\overline{y} = \int_0^h xy\,dy \qquad (a)$$

From similar triangles, $x = \frac{b}{h}(h - y)$, so that Eq. (a) becomes

$$\left(\frac{1}{2}bh\right)\overline{y} = \frac{b}{h}\int_0^h (h-y)y\,dy = \frac{1}{6}bh^2$$

$$\overline{y} = \frac{1}{3}h \qquad \textbf{\textit{Ans.}}$$

Observe that for any triangle the distance from the centroid to any side is equal to one-third of the altitude measured from that side. Furthermore, the centroid of a triangle is located on a median because the median to any side contains the centroids of all strips drawn parallel to that side. Therefore the centroid is at the intersection of the median

6-5.3 *Centroid of the Area of a Circular Sector.*

Determine the location of the centroid of the area of the sector of the circle shown in Fig. 6-5.4. Let the radius of the circle be r and the subtended angle be 2α.

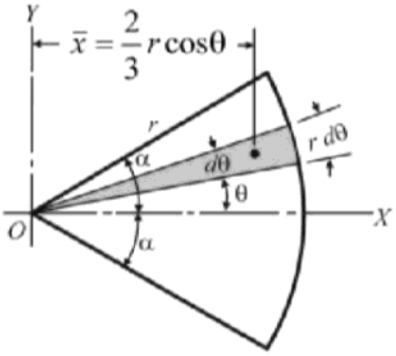


Figure 6.5.4

Solution

Let the axis of symmetry be taken as the X axis; then $\bar{y} = 0$. Select as the element of area the shaded triangle the position of whose centroid is known from the answer to Prob. 6-5.2 to be $x = \frac{2}{3} r \cos\theta$. The area of the element is $dA = \left(\frac{1}{2}\mathrm{r}\right) r\, d\theta = \frac{1}{2} r^2\, d\theta$. Applying Eq. (6-5.1), we obtain

$$[A\bar{x} = \int x\, dA] \qquad \bar{x} \times \int_{-\alpha}^{+\alpha} \frac{1}{2} r^2\, d\theta = \int_{-\alpha}^{+\alpha} \frac{2}{3} r \cos\theta \left(\frac{1}{2} r^2\, d\theta\right)$$

$$\bar{x}(r^2\alpha) = \frac{1}{3} r^3 \int_{-\alpha}^{+\alpha} \cos\theta\, d\theta = \frac{2}{3} r^3 = \sin\alpha$$

$$\bar{x} = \frac{2}{3}\, \frac{r \sin\alpha}{\alpha}$$ ***Ans.***

If the sector is a *semicircular area* as in Fig. 6-5.5, $\alpha = 90^\circ = \pi/2$ radians, whence by substituting in the last equation above we find the distance of the centroid from the diameter to be

$$\bar{x} = \frac{4r}{3\pi} = 0.424r$$

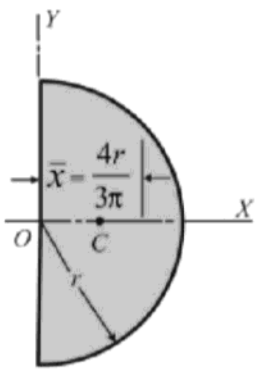


Figure 6.5.5

6-5.4 *Centroid of the Area of a Parabolic Segment.*

In Fig 6-5.6 is shown a parabolic segment bounded by the X axis, the line $x = a$, and the parabola $y = kx^2$. Determine the coordinates of the centroid.

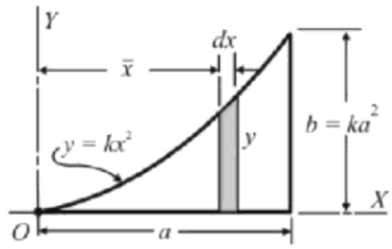


Figure 6.5.6

Solution

Select the element of area as the shaded strip parallel to the Y axis. All points in this element are the same distance from the Y axis. The area of the element is $dA = y\,dx$. The area A of the entire parabolic segment is found from

$$[A = \int dA] \qquad A = \int_0^a y\,dx = \int_0^a kx^2\,dx = \frac{1}{3}ka^3$$

To determine $\bar{x}$ we apply Eq. (6-5.1) as follows:

$$[A\bar{x} = \int x\,dA] \qquad \left(\frac{1}{3}ka^3\right)\bar{x} = \int_0^a xy\,dx = \int_0^a kx^3\,dx$$

$$\left(\frac{1}{3}ka^3\right)\bar{x} = \frac{1}{4}ka^4$$

$$\bar{x} = \frac{3}{4}a$$

To determine $\overline{y}$, we use the same elementary strip; but since each point of the element is not the same distance from the X axis, we must use Rule 2 above; i.e., the moment of the differential element about the X axis is the product of its centroidal coordinate, $\frac{1}{2}y$ multiplied by the area $y\ dx$. Applying Eq. (6-5.1) again, we obtain

$$[A\overline{y} = \int y\ dA] \qquad \left(\frac{1}{3}ka^3\right)\overline{y} = \int_0^a \frac{1}{2}y\ (y\ dx) = \frac{1}{2}k^2\int_0^a x^4\ dx$$

$$\left(\frac{1}{3}ka^3\right)\tilde{y} = \frac{k^2a^5}{10}$$

$$\overline{y} = \frac{3}{10}ka^2 = \frac{3}{10}b \qquad \textbf{\textit{Ans.}}$$

6-5.5 Determine the centroid of the shaded area shown in Fig. P-6-5.5, which is bounded by the X axis, the line $x = a$ and the parabola $y^2 = kx$.

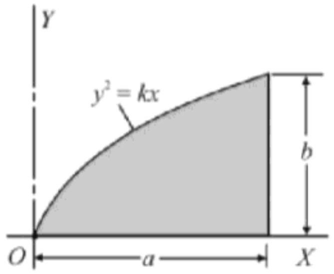


Figure P-6.5.5

$$\overline{x} = \frac{3}{5}a;\ \overline{y} = \frac{3}{8}b \qquad \textbf{\textit{Ans.}}$$

6-5.6 Determine the centroid of the quarter circle shown in Fig. P-6-5.6 whose radius is r.

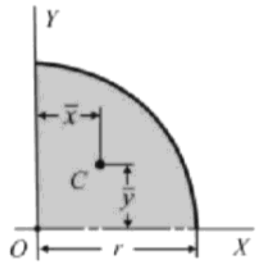


Figure P-6.5.6

$$\bar{x} = \bar{y} = \frac{4r}{3\pi} \qquad \textbf{\textit{Ans.}}$$

6-5.8 Compute the area of the spandrel in Fig. P-6-5.8 bounded by the X axis, the line x = b, and the curve $y = kx^n$ where $n \geqq 0$. What is the location of its centroid from the line $x = b$? Prepare a table of areas and location of centroid for values of n = 0, 1, 2, and 3.

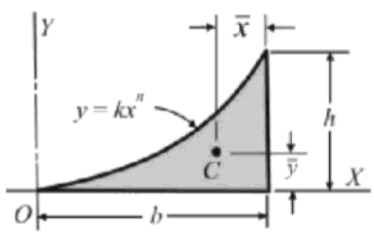


Figure P-6.5.8

$$A = \frac{1}{n+1}(bh); \quad \bar{x} = \frac{b}{n+2}; \text{ refer to Table 9-4.1.} \qquad \textbf{\textit{Ans.}}$$

6-6 CENTROIDS OF COMPOSITE FIGURES

Many of the figures used in engineering are composed of combinations of the geometrical shapes discussed in the last section. Still other figures are composed of structural shapes. The location of the centroids for structural elements is given in handbooks.

When the given figure can be divided into the finite elements discussed above, these elements can be treated in the same manner as were the infinitesimal elements. When this is done, the process is called finite summation, as contrasted to integration, or the summation of infinitesimal elements.

If given area can be divided into parts, each centroid of which is known, the moment of the total area will be the sum of the moments of area of its parts. This statement is really an extension of the principle of moments (Section 2-9).

The centroid of the composite figure is determined by applying the following equations, which were developed in Section 6-3. In these equations the elemental areas become the areas of the geometrical shapes into which the entire area has been divided.

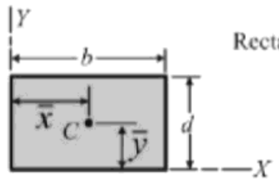
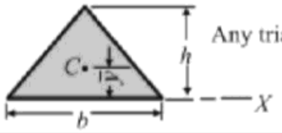
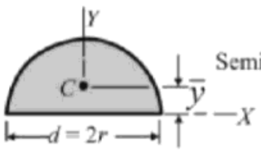
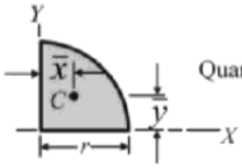
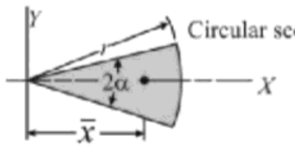
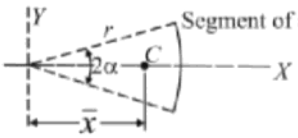
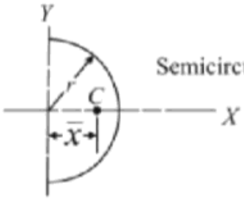
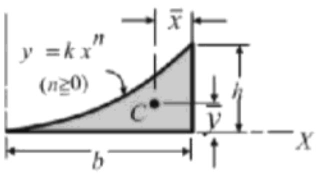
$$\left.\begin{aligned} A\bar{x} &= \sum a_i x_i \\ A\bar{y} &= \sum a_i y_i \end{aligned}\right\} \qquad (6\text{-}3.1)$$

A similar process may be applied to lines. The given line may be divided into finite segments whose centroids are known, and the following equations may be used:

$$\left.\begin{aligned} L\bar{x} &= \sum l_i x_i \\ L\bar{y} &= \sum l_i y_i \end{aligned}\right\} \qquad (6\text{-}3.2)$$

Before these equations are applied to illustrative problems, it will be convenient to summarize the location of centroids for common geometrical shapes (determined in preceding problems) given in Table 6-6.1.

Table 6-6.1 Centroids for Common Geometric Shapes.

Shape	Area or length	$\bar{x}$	$\bar{y}$
 Rectangle	bd	$\frac{1}{2}b$	$\frac{1}{2}d$
 Any triangle	$\frac{1}{2}bh$		$\frac{1}{3}h$
 Semicircle	$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$ or $0.424r$
 Quarter circle	$\frac{1}{4}\pi r^2$	$\frac{4r}{3\pi}$ or $0.424r$	$\frac{4r}{3\pi}$ or $0.424r$
 Circular sector	$r^2\alpha$	$\frac{2}{3}\frac{r\sin\alpha}{\alpha}$	0
 Segment of arc	$2r\alpha$	$\frac{r\sin\alpha}{\alpha}$	0
 Semicircular arc	πr	$\frac{2r}{\pi}$	0
Area under spandrel 	$\frac{1}{n+1}bh$	$\frac{1}{n+2}b$	$\frac{n+1}{n+2}h$

SIGNS

The following discussion indicates the method of determining which sign to give to an area or coordinate. Consider a plate balanced on a knife edge denoted

by Y-Y in Fig. 6-6.1. It is evident that the moment of W_1 about Y-Y is opposite in effect to the moment of W_2. Since the directions of W_1 and W_2 are both down, the difference in moment effect is caused by W_1 and W_2 being located on opposite sides of the axis. Or if x_1 is considered to be positive in terms of coordinates with respect to the X-Y axes, x_2 must be considered to be negative. Proceeding to replace weight in terms of area, we may eliminate the items γ and t from the product $W = \gamma\, tA$, whence we observe that the sign of the moment of area similarly depends on the position of the centroid of the area relative to the coordinate axes.

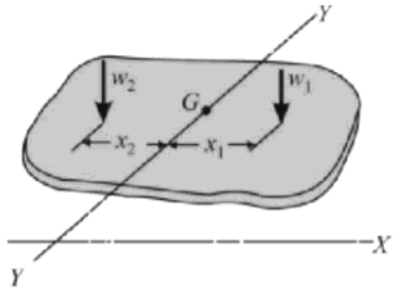


Figure 6-6.1

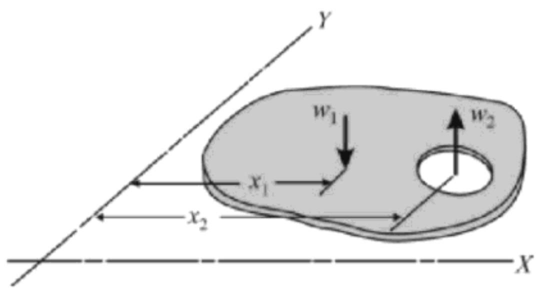


Figure 6-6.2

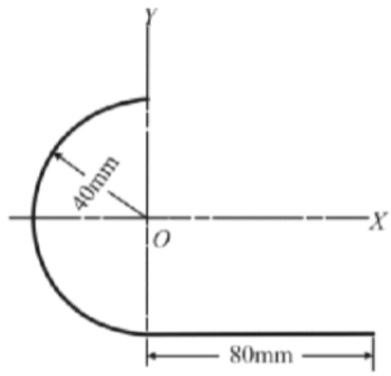
In the case of a plate with a hole cut in it as in Fig. 6-6.2, the resultant weight W may be considered equivalent to the weight W_1 of the solid plate minus the weight W_2 of the cut-out portion. In this instance, the directions of W_1 and W_2 are opposite, so their moments with respect to the Y axis are of opposite sign. If, as before, these weights are expressed in terms of equivalent areas, the moment effect of the area of the cut-out is opposite to the moment effect of the area of the original plate, even though the centroid of each area is now on the same side of the axis.

Hence the sign of the moment of an area, expressed as ax for example, depends on the signs of a and of x, the positive sign for area being associated with area that adds to the net area of the figure, and the negative sign to area that reduces the net area. Moreover, the sign of the coordinate of the centroid of an area may be plus or minus, depending on the location of the centroid with respect to the axis of moments.

ILLUSTRATIVE PROBLEMS

6-6.1 A slender homogeneous wire of uniform cross section is bent into the form shown in Fig. 6-6.3. Determine the position of the centroid of the wire with respect to the given axes.

CHAPTER 6 CENTROIDS AND CENTERS OF GRAVITY



6.6.2 Determine the position of the centroid of the shaded area shown in Fig. 6-6.4.

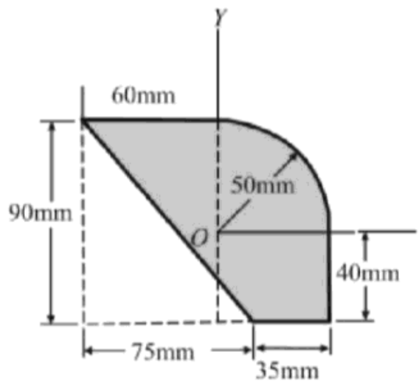


Figure 6-6.4

Computation of Results

	Area (mm^2)	X (mm)	Ax (mm^3)	y (mm)	ay (mm^3)
Quarter circle	+ 1963	+ 21.2	+ 41615	+ 21.2	+ 41615
Rectangle 50 × 40	+ 2000	+ 25	+ 50000	– 20	– 40000
Rectangle 60 × 90	+ 5400	– 30	– 162000	+ 5	+ 27000
Triangle	– 3375	– 35	+ 118125	– 10	+ 33750
Totals	+ 5988		+ 47740		62365

Taking the sums from the tabulation, Eq. (6-3. la) gives the following results:

$$\left.\begin{array}{ll} \left[\bar{x} = \frac{\sum ax}{a}\right] & \bar{x} = \frac{47740}{5988} = 7.97 \text{ mm} \\ \left[\bar{y} = \frac{\sum ay}{a}\right] & \bar{y} = \frac{62365}{5988} = 10.41 \text{ mm} \end{array}\right\} \quad \textit{Ans.}$$

6-6.4 A slender homogeneous wire of uniform cross section is bent into the shape shown in Fig. P.6-6.4. Determine the coordinates of its center of gravity.

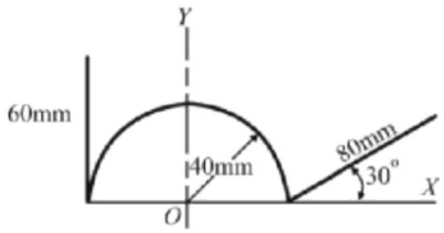


Figure P-6-6.4

$\bar{x}$ = 13.4 mm.; $\bar{y}$ = 24.8 mm

6-6.5 Locate the center of gravity of the bent wire shown in Fig. P.6-6.5. The wire is homogeneous and of uniform cross section.

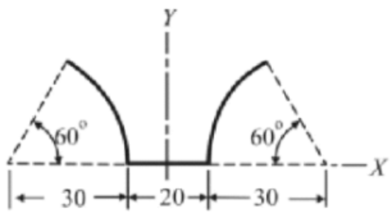
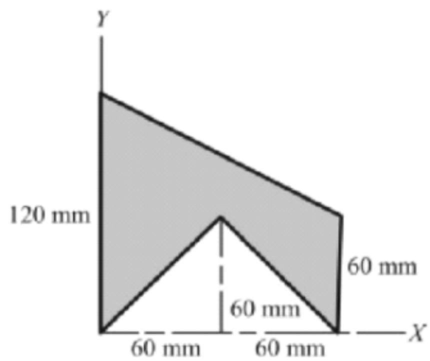


Figure P-6-6.5

6.6.6 Determine the centroid of the lines that form the boundary of the shaded area in Fig. P-6.6.6.



$\bar{x} = 52.6$ mm; $\bar{y} = 54.1$ mm ***Ans.***

6-6.7 Locate the centroid of the shaded area shown in Fig. P.6-6.6.

6-6.8 Locate the centroid of the shaded area in Fig. P-6-6.8 created by cutting a semicircle of diameter r from a quarter circle of radius r.

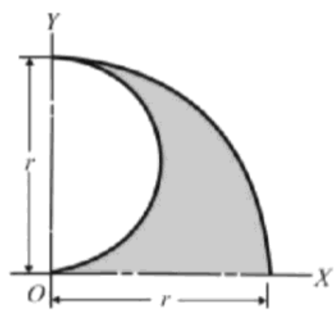


Figure P-6-6.8

$\bar{x} = 0.636\, r$; $\bar{y} = 0.348\, r$ ***Ans.***

6-6.9 In order to fully utilize the different values of the compressive stress S_c and the tensile stress S_t in cast iron beams, it is desirable to locate the centroidal axis so that the ratio of its distance to the top of a section to its distance from the bottom equals S_c/S_t. Using the section shown in Fig. P.6-6.9, find the dimension *b* to satisfy this criterion if $S_c/S_t = 3$.

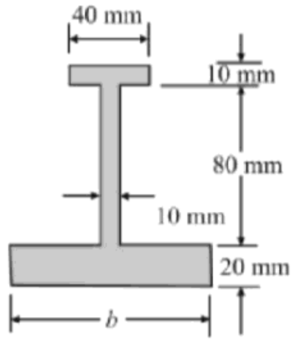


Figure P-6-6.9

6-6.10 Determine the dimension *b* that will locate the centroidal axis at 40 mm above the base of the section shown in Fig. P-6.6.10.

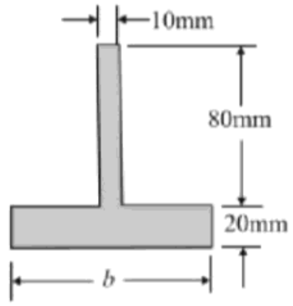


Figure P-6-6.10

$b = 70$ mm **Ans.**

6-6.14 The surface of a plate of uniform thickness is given by the shaded area in Fig. P.6-6.14. If this plate is suspended from a hinge at A, what angle will line AB make with the vertical?

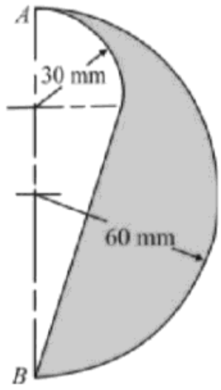


Figure P-6-6.14

$\theta = 23.7°$ ***Ans.***

6.6.15 With respect to the given axes, locate the centroid of the shaded area shown in Fig. P-6-6.15.

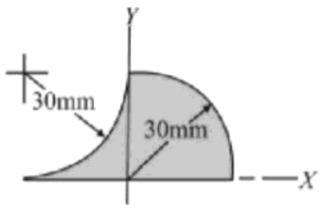


Figure P-6-6.15

$\overline{x} = 8.58\ mm.; \overline{y} = 11.42\ mm$ ***Ans.***

6-6.16 Locate the centroid of the shaded area shown in Fig. P.6-8.16 resulting from removing the circular segment of 60 mm radius from the circular plate of 80 mm radius.

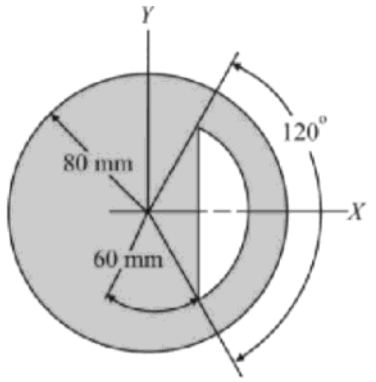


Figure P-6-6.16

$\bar{x} = -\ 3.35\ mm$

6-6.17 Locate the centroid of the shaded area shown in Fig. P-6-6.17.

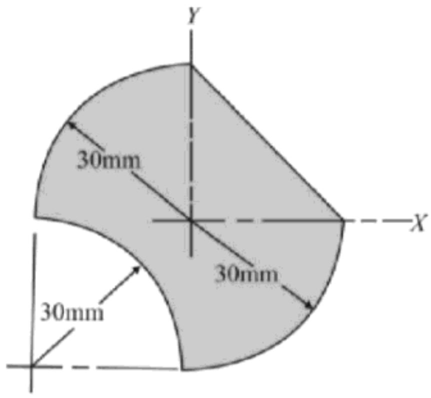


Figure P-6-6.17

6-7 THEOREM OF PAPPUS[1]

Pappus developed two simple theorems for determining the surface area or the volume generated by revolving respectively a plane curve or a plane area about a *nonintersecting* axis lying in its plane. The first theorem states that the surface area is the product of the length of the generating curve multiplied by the distance traveled by its centroid. Thus, let the curve *AB* of length *L* in Fig. 6-7.1 be revolved about *OX* through an angle of 2π radians. The differential length dL sweeps through the distance $2\pi y$ thereby generating a hoop whose area is $2\pi y\ dL$. The total area generated by *AB* is the area of all such hoops or

$$A = \int 2\pi\ y\ dL = 2\pi \int y\ dL = 2\pi \bar{y} L \qquad (6\text{-}7.1)$$

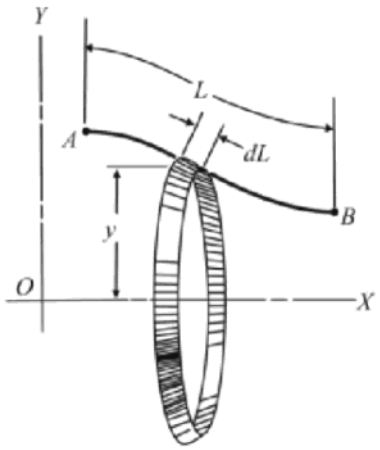


Figure 6-7.1 First theorem of Pappus surface generated by revolving a line about a nonintersecting axis

If the generating line L is composed of several segments, the centroid $\overline{y}$ of that line need not be found since the product $L\,\overline{y}$ is equivalent to and may be replaced by the sum of the moments of length (i.e., $\sum ly$) of those segments.

When the centroid of an area is known, the volume generated by revolving the area about a *nonintersecting* axis can be found by applying the second theorem. This theorem states that the volume is the product of the area of the figure multiplied by the length of the path described by the centroid of the area. To demonstrate this, let the area A in Fig. 6-7.2 be rotated about the axis OX through an angle of 2π radians. The differential area dA sweeps through the distance $2\pi y$ and generates a ring whose volume is $2\pi y\ dA$. The total volume generated is the sum of the volumes of all such rings or

$$V = \int 2\pi\ y\ dA = 2\pi \int y\ dA = 2\pi A\overline{y} \qquad (6\text{-}7.2)$$

If the generating area A is composed of several parts, the centroid $\overline{y}$ of that area need not be found since the product $A\,\overline{y}$ is equivalent to the sum of the moments of area (i.e., Σay) of the several parts.

If the generating line or area is revolved through an angle θ less than 2π radians, the generated surface or volume may be found by substituting θ for 2π in Eqs. (6-7.1) and (6-7.2).

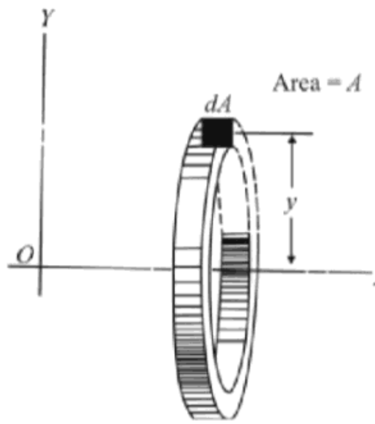


Figure 6-7.2 Second theorem of pappus. Volume generated by revolving an area about a nonintersecting axis.

6.7.1 Compute the surface area of the cone generated by revolving the line in Fig. 6-7.3 about the *Y* axis.

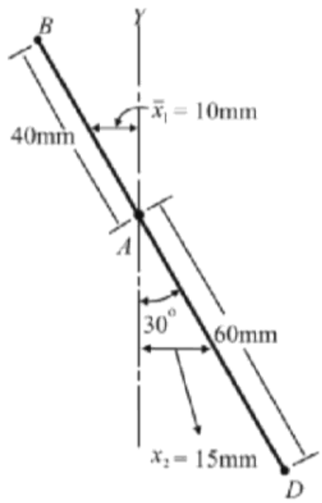


Figure 6-7.3

Solution

Two cones are generated by the line; one by the 40 mm, length above *A* and the other by the 60 mm length below *A*. The .*x* coordinate of the centroid for each segment of the line is given by $\bar{x} = \frac{1}{2} L \sin\ 30°$; hence $\bar{x}_1 = 10mm$ and $\bar{x}_2 = 15mm$. Applying Eq. (6-7.1), we obtain

$[A = 2\pi \bar{x} L]$ 40 mm segment: $A_1 = 2\pi \times 10 \times 40 = 2513.2$ mm^2

60 mm segment: $A_2 = 2\pi \times 15 \times 60 = 5654.8$ mm^2

Total surface area = $A_1 + A_2 = 2513.2 + 5654.8$

= 8168.0 mm^2 ***Ans.***

6-7.2 The shaded area in Fig. 6-7.4 is composed of a second-degree parabola and a semicircle. Determine the volume generated by rotating the area through one revolution about the Y_1-Y_1 axis.

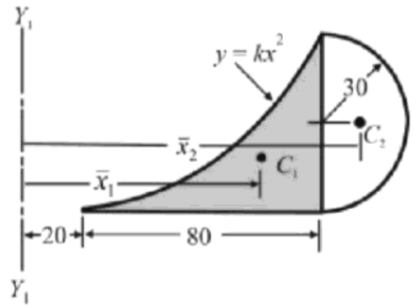


Figure 6-7.4

Solution

From Table 6-6.1, the x coordinate of the centroid for each part of the total area is

$$\bar{x}_1 = 20 + \frac{3}{4} \times 80 = 80 \text{ mm}$$

$$\bar{x}_2 = 100 + 0.424 \times 30 = 112.7 \text{ mm}$$

Applying Eq. (6-7.2) we now find the volume to be

$[V = 2\pi A\bar{x} = 2\pi \sum a x]$ $$V = 2\pi \left[\left(\frac{80 \times 60}{3} \right)(80) + \left(\frac{\pi \times 30^2}{2} \right)(112.7) \right]$$

$V = 2\pi\, [128 \times 10^3 + 159 \times 10^3] = 1805 \times 10^3$ mm ***Ans.***

6-7.8 Determine the volume and surface area of the solid shown in Fig. P- 6-7.8.

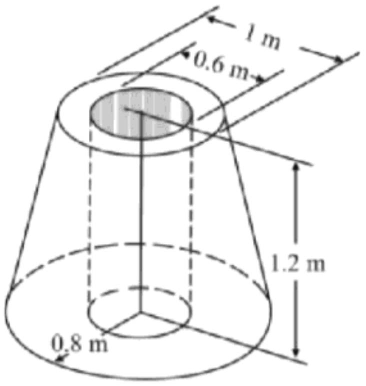


Figure P-6-7.8

6-7.10 Determine the difference in volume (if any) generated by rotating the shaded area shown in Fig. P-6-7.10 about the *X-X* axis and then about the *Y-Y* axis.

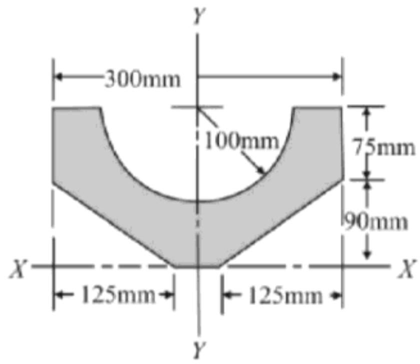


Figure P-6-7.10