

01/04/2023

## UNIT-2

### FOURIER TRANSFORMS:

- Fourier Integral Theorem (without proof)
- Fourier Sine integral
- Fourier Cosine integral
- Complex form of a Fourier integral
- Fourier transform
- Inverse Fourier Transform
- Fourier Sine Transform
- Inverse of Fourier Sine Transform
- Fourier Cosine Transform
- Inverse of Fourier Cosine Transform
- Properties of Fourier Transform
- Problems related to above topics

#### \* INTRODUCTION:

A transformation is a mathematical device which transforms one fn to another fn.

- A periodic function  $f(x)$  defined in a finite interval  $(-l, l)$  can be expressed in terms of Fourier series.

→ In many engineering applications, it is required to expand an arbitrary function like impressed force, voltage etc in a series of sinusoidal functions.

→ By extending this concept, non-periodic function defined in the interval  $-\infty < x < \infty$  & all  $x$  can be expressed as a Fourier Integral; since all fns are not always periodic.

\* So, FOURIER TRANSFORM is to reduce the no. of independent variables by one.

→ Fourier transforms is an integral transform to reduce the no. of independent variables by one.

→ Fourier transforms are widely used to solve various boundary value problems of engineering such as, vibrations of a string, conduction of heat, transverse oscillations of an electric beam, free and forced vibrations of a membrane etc.

→ Integral Transform:  
The integral transform of a function  $f(x)$  denoted by

$I[f(x)]$  (or)  $f(s)$  is defined as

$$I[f(x)] = f(s) = \int_{x_1}^{x_2} f(x) K(s, x) dx$$

where  $K(s, x)$  is the fn of ~~s and x~~ and  $x$  is the "kernel of the transform".

→ Depending upon the kernel we obtain the following various transforms:

(i) If  $K(s, x) = e^{-sx}$ ; we obtain the Laplace transform of  $f(x)$

$$\text{i.e. } L[f(x)] = F(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

(ii) If  $K(s, x) = e^{isx}$ ; we obtain the Fourier transform of  $f(x)$

$$\text{i.e. } F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

(iii) If  $K(s, x) = x^{s-1}$ ; we obtain the Mellin transform of  $f(x)$

$$\text{i.e. } M[f(x)] = M(s) = \int_0^{\infty} f(x) x^{s-1} dx$$

(ii) If  $k(s, x) = x J_n(sx)$ , which is the Bessel's fn, then we obtain the Hankel transform of  $f(x)$ .

$$\text{i.e. } H[f(x)] = H(s) = \int_0^\infty f(x) x J_n(sx) dx$$

In the same process by taking various kernels we obtain various transforms  
 → Before going to define the Fourier transform; first we have to know about the Fourier integral by the Fourier Integral Theorem. Before proving this theorem we must see the well known Dirichlet's conditions which play an important role in study of Fourier transforms & covers engineering problems.

#### \* DIRICHLET'S CONDITIONS:

A function  $f(x)$  is said to satisfy Dirichlet's conditions in the interval  $(a, b)$  if

(i)  $f(x)$  is defined and is single-valued function except possibly at a finite no. of points in the interval  $(a, b)$

(ii)  $f(x)$  &  $f'(x)$  are piecewise continuous in  $(a, b)$ .

#### \* FOURIER INTEGRAL THEOREM:

1) A function  $f(x)$  which is piecewise continuous in every finite interval and is absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |f(x)| dx \text{ be convergent; then the}$$

Fourier integral representation of  $f(x)$  is,

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt dx$$

#### Proof:

→  $f(x)$  be a fn satisfying the Dirichlet conditions in every  $(-l, l)$  and defined as  $f(x) = \frac{1}{2} [f(x_0+0) + f(x_0-0)]$  at every point of discontinuity

→ Fourier Series of  $f(x)$  in  $(-l, l)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{n\pi t}{l} dt$$

$$b_n = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{n\pi t}{l} dt$$

Substitute  $a_0, a_n, b_n$  in ①

$$f(t) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l \cos\left(\frac{n\pi t}{l}\right) \cos\left(\frac{n\pi t}{l}\right) f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l \sin\left(\frac{n\pi t}{l}\right) \sin\left(\frac{n\pi t}{l}\right) f(t) dt$$

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l \cos\left(\frac{n\pi(t-x)}{l}\right) f(t) dt$$

- ②

Let  $\int_{-\infty}^{\infty} |f(x)| dx$  converges i.e.  
has a finite value.

→ assume limits as  $l \rightarrow \infty$  in ②

first term

$$\lim_{l \rightarrow \infty} \left[ \frac{1}{2l} \int_{-l}^l f(t) dt \right] = 0$$

$$\therefore \left| \frac{1}{2l} \int_{-l}^l f(t) dt \right| \leq \frac{1}{2l} \int_{-l}^l |f(t)| dt$$

put  $\frac{\pi}{l} = \delta x$  in second term of ②

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-l}^l \cos(n\delta x(t-x)) f(t) dt \delta x - ③$$

$\lim_{l \rightarrow \infty}$  (or)  $\delta x \rightarrow 0$  eq. ③ takes the form

$$\frac{1}{\pi} \int_0^{\infty} \left[ \int_{-\infty}^{\infty} \cos(\lambda(t-x)) f(t) dt \right] \delta x$$

$$\therefore dt \sum_{n=1}^{\infty} \phi(n\delta x) \delta x = \int_0^{\infty} \phi(\lambda) d\lambda$$

Thus as  $l \rightarrow \infty$  eq. ② takes the form

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \cos \lambda(t-x) f(t) d\lambda} - ④$$

01/04/23

## \* FOURIER SINE & COSINE INTEGRALS

By Fourier Integral Theorem

$$\text{We have } f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{t=-\infty}^{\infty} f(t) \cos \lambda(t-x) dt \right] dx$$

Expand  $\cos \lambda(t-x)$  from eqn ① we get

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{t=-\infty}^{\infty} f(t) \left[ \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x \right] dt \right] dx$$

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos \lambda x \left[ \int_{-\infty}^{\infty} f(t) \cos \lambda t dt \right] dx$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \sin \lambda x \left[ \int_{-\infty}^{\infty} f(t) \sin \lambda t dt \right] dx \quad \text{②}$$

(Case i) If  $f(t)$  is an odd fn, we get

Fourier Sine Integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \int_0^{\infty} f(t) \sin \lambda t dt \right] dx$$

$\because f(t)$  is odd,  
 $\cos \lambda t$  is even  
 $f(t) \cos \lambda t$  is odd.  
do 1<sup>st</sup> term vanishes from ②

case(ii)

If  $f(t)$  is an even fn, we get  
Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[ \int_{-\infty}^{\infty} f(t) \cos \lambda t dt \right] dx$$

$\because f(t)$  is even,  
 $\sin \lambda t$  is odd;  
 $f(t) \sin \lambda t$  is odd

do 2<sup>nd</sup> term vanishes from ②

\* Using Fourier Integrals, show that

$$\int_0^{\infty} \frac{1 - \cos \pi x}{x} \sin \lambda x dx = \begin{cases} \pi/2 & 0 < x < \pi \\ 0 & x > \pi \end{cases}$$

Given integrand: Fourier Sine Integral  
contains  $\sin \lambda x$  formula:

\* Fourier Line Integral formula:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \int_0^{\infty} f(t) \sin \lambda t dt \right] dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \int_0^{\infty} f(t) \sin \lambda t dt \right] dx$$

consider  $f(t) = \begin{cases} \pi/2 & \text{if } t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \left[ \int_0^{\pi} \frac{\pi}{2} \sin \lambda t dt \right] + \left[ \int_{\pi}^{\infty} 0 \sin \lambda t dt \right] \right] dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \frac{\pi}{2} \cdot \left[ -\frac{\cos \lambda t}{\lambda} \right]_0^{\pi} dx$$

$$= \int_0^{\infty} \sin \lambda x \left( \frac{\cos \lambda \pi - 1}{\lambda} \right) dx$$

$$= \int_0^{\infty} \sin \lambda x \left( \frac{1 - \cos \pi \lambda}{\lambda} \right) dx$$

$$f(x) = \int_0^{\infty} \left( \frac{1 - \cos \lambda x}{\lambda} \right) \sin \lambda x \cdot dx$$

\* Using Fourier Integral

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \lambda x \cdot dx}{x^2 + a^2}$$

$$f(x) = e^{-ax}$$

$$f(t) = e^{-at}$$

\* ∵ integrand is having  $\cos \lambda x$

→ Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[ \int_0^{\infty} f(t) \cos \lambda t dt \right] dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[ \int_0^{\infty} f(t) \cos \lambda t dt \right] dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[ \int_0^{\infty} e^{-at} \cos \lambda t dt \right] dx$$

$$\boxed{\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)}$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \left[ \frac{e^{-at}}{a^2 + x^2} [-a \cos xt + x \sin xt] \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ 0 + \frac{a}{a^2 + x^2} \right] dx$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \lambda x}{a^2 + x^2} dx$$

$e^{i\lambda x}$

\* Using F.I.

$$\text{P.T. } \int_0^\infty \frac{\sin \lambda \pi \sin \lambda x}{1 - \lambda^2}$$

$$= \begin{cases} \pi/2 \sin x & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases}$$

01/04/23

## \* FOURIER SINE & COSINE INTEGRALS

By Fourier Integral Theorem

$$\text{we have } f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{t=-\infty}^{\infty} f(t) \cos \lambda(t-x) dt \right] dx$$

Expand  $\cos \lambda(t-x)$  from eqn ① we get

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{t=-\infty}^{\infty} f(t) \left[ \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x \right] dt \right] dx$$

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos \lambda x \left[ \int_{t=-\infty}^{\infty} f(t) \cos \lambda t dt \right] d\lambda$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \sin \lambda x \left[ \int_{t=-\infty}^{\infty} f(t) \sin \lambda t dt \right] d\lambda \quad \text{②}$$

Case ii) If  $f(t)$  is an odd fn; we get

Fourier Sine Integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[ \int_0^\infty f(t) \sin xt dt \right] dx$$

$\because f(t)$  is odd,  
 $\cos xt$  is even

$f(t) \cos xt$  is odd.

So 1<sup>st</sup> term vanishes from (2)

case(ii)

If  $f(t)$  is an even fn; we get

Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ \int_{-\infty}^\infty f(t) \cos xt dt \right] dx$$

$\because f(t)$  is even;  
 $\sin xt$  is odd;

$f(t) \sin xt$  is odd

So 2<sup>nd</sup> term vanishes from (2)

\*→ Using Fourier Integrals; show that

$$\int_0^\infty \frac{1 - \cos \pi x}{x} \underbrace{\sin \lambda x}_{\downarrow} dx = \begin{cases} \frac{\pi}{2} & 0 < \lambda < \pi \\ 0 & \lambda > \pi \end{cases}$$

Given integrand: Fourier Sine Integral  
 contains  $\sin \lambda x$  formula

\*→ Fourier Sine Integral formula:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \int_0^{\infty} f(t) \sin \lambda t dt \right] dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \int_0^{\infty} f(t) \sin \lambda t dt \right] dx$$

Consider  $f(t) = \begin{cases} \frac{\pi}{2} & \alpha t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \left[ \left[ \int_0^{\pi} \frac{\pi}{2} \sin \lambda t dt \right] + \left[ \int_{\pi}^{\infty} 0 \sin \lambda t dt \right] \right] dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \cdot \frac{\pi}{2} \cdot \left[ \frac{\cos \lambda t}{\lambda} \right]_0^{\pi} dx$$

$$= \int_0^{\infty} \sin \lambda x \left( \frac{\cos \lambda \pi - 1}{\lambda} \right) dx$$

$$= \int_0^{\infty} \sin \lambda x \left( \frac{1 - \cos \pi \lambda}{\lambda} \right) dx$$

$$f(x) = \int_0^\infty \left( \frac{1 - \cos \pi x}{\pi} \right) \sin \lambda x \cdot dx$$

\* Using Fourier Integral

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \lambda x \cdot dx}{x^2 + a^2}$$

$$f(x) = e^{-ax}$$

$$f(t) = e^{-at}$$

\* ∵ Integrand is having  $\cos \lambda x$

→ Fourier Cosine Integral.

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ \int_0^\infty f(t) \cos \lambda t \cdot dt \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ \int_0^\infty f(t) \cos \lambda t \cdot dt \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ \int_0^\infty e^{-at} \cos \lambda t \cdot dt \right] dx$$

$$\boxed{\int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)}$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ \frac{e^{-at}}{a^2 + x^2} [-a \cos xt + x \sin xt] \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \left[ 0 + \frac{a}{a^2 + x^2} \right] dx$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \lambda x}{a^2 + x^2} dx$$

$e^{-\infty}$

\* Using F.I.

$$\text{P.T. } \int_0^\infty \frac{\sin \lambda \pi \sin x}{1 - \lambda^2}$$

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x & 0 \leq x \leq \pi \\ 0 & x > \pi \end{cases}$$

$$f(t) = \begin{cases} \frac{\pi}{2} \sin t & 0 \leq t \leq \pi \\ 0 & t > \pi \end{cases}$$

∴ eq. contains  $\sin \lambda x$

∴ it is Fourier Sine Integral:

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[ \int_0^\infty f(t) \sin \lambda t dt \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[ \int_0^\pi \frac{\pi}{2} \sin t \sin \lambda t dt + \int_\pi^\infty 0 \sin \lambda t dt \right] dx$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[ \frac{\pi}{4} \left( \int_0^{\pi/2} 2 \sin t \sin \lambda t dt \right) \right] dx$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4} \int_0^\infty \sin \lambda x \left[ \cos(t - \lambda t) - \cos(t + \lambda t) \right] dt dx$$

$$= \frac{1}{2} \int_0^\infty \sin \lambda x \left[ \frac{\sin(t - \lambda t)}{1 - \lambda} - \frac{\sin(t + \lambda t)}{1 + \lambda} \right] dx$$

$$= \frac{1}{2} \int_0^\infty \sin \lambda x \left[ \frac{\sin((1-\lambda)\pi)}{1 - \lambda} - \frac{\sin((1+\lambda)\pi)}{1 + \lambda} \right] dx$$

$$= \frac{1}{2} \int_0^\infty \sin \lambda x \left[ \frac{\sin \lambda \pi}{1 - \lambda} + \frac{\sin \lambda \pi}{1 + \lambda} \right] dx$$

$$= \frac{1}{2} \int_0^\infty \frac{\sin \lambda x \sin \lambda \pi}{1 - \lambda^2} (1 + x + 1 - x) dx$$

$$= \int_0^\infty \sin \lambda x \sin \lambda \pi dx$$

03/04/2023

Series:

\* Complex form of Fourier ~~Integral~~

→ We know that Fourier Integral of  $f(x)$  is

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) \cos(\lambda(t-x)) dt \right] dx$$

∴  $\cos \lambda(t-x)$  is an even fn of  $\lambda$ ,

the above eqn can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt \right] d\lambda$$

Also ∵  $\sin \lambda(t-x)$  is an odd fn of  $\lambda$  ①  
we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) \cdot \sin \lambda(t-x) dt \right] d\lambda \quad \text{--- ②}$$

Now multiplying ② by  $i$  & add eq ①.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) \left[ \cos \lambda(t-x) + i \sin \lambda(t-x) \right] dt \right] d\lambda$$

$$f(x) = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) e^{ix(t-x)} dt \right] dx \right]$$

\* Fourier Transform of  $f(x)$ :

→ Writing the exponential function  $e^{ix(t-x)}$   
as a product of 2 exponential fn  
 $e^{ix(t-x)} = e^{ixt}, e^{-ixx}$  the

Fourier integral of  $f(x)$  in  
complex form may be in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\lambda} \left[ \int_{-\infty}^{\infty} f(t) e^{ixt} dt \right] d\lambda \quad \text{--- (1)}$$

→ Hence the Fourier transform from eq: (1)

is of the form:

$$F[f(x)] = \int_{-\infty}^{\infty} f(t) e^{ixt} dt$$

→ Inverse Fourier Transform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F[f(x)] e^{-ix\lambda} d\lambda$$

## Fourier Sine Transform:

We know that Fourier sine integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[ \int_0^{\infty} \sin x t \sin \lambda t f(t) dt \right] dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin x \lambda \left[ \int_0^{\infty} \sin x t f(t) dt \right] d\lambda \quad -①$$

From eq: ①  $\rightarrow$  Fourier Sine Transform

$$F_s[f(x)] = \int_0^{\infty} f(t) \sin \lambda t dt \quad -②$$

Sub ② in ① Inverse Sine Transform

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} F_s[f(x)] \sin x \lambda d\lambda}$$

## Fourier Cosine Transform:

We know that Fourier Cosine integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos x \lambda \left[ \int_0^{\infty} \cos x t f(t) dt \right] dx \quad -①$$

from eq: ①

## Fourier Cosine Transform

$$F_C[f(x)] = \int_0^\infty f(t) \cos \lambda t dt \quad \text{--- (2)}$$

Sub (2) in (1) Inverse Cosine Transform.

$$f(x) = \frac{2}{\pi} \int_0^\infty F_C[f(x)] \cos \lambda x d\lambda$$

\* Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

then evaluate  $\int \frac{\sin x}{x} dx$

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \text{ i.e. } -1 < x < 1 \\ 0 & \text{for } |x| > 1 \text{ i.e. } 1 < x < 8 \end{cases}$$

Fourier Transform of  $f(x)$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \int_{-1}^1 1 \cdot e^{isx} dx$$

$$\underline{e^{is\theta} - e^{-is\theta} = 2 \sin s\theta}$$

$$= \left[ \frac{e^{isx}}{is} \right]_{-1}^1 = \frac{1}{is} (e^{is} - e^{-is}) = \frac{2 \sin s}{s}$$

## Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{2}{2\pi} \int_0^{\infty} F(s) e^{-isx} ds$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx} ds$$

$\boxed{n=0}$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin s}{s} ds$$

$$\downarrow \quad \because \text{even fn.} \quad -\frac{\sin s}{s} = \frac{\sin s}{s}$$

$$\frac{\pi}{2} f(0) \Rightarrow f(0)$$

$$\boxed{f(0) =}$$

$$\boxed{\frac{\pi}{2} = \int_0^{\infty} \frac{\sin s}{s} ds}$$

\* Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

hence evaluate  $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

06/04/2023

\* Fourier Transform of  $f(x)$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F(s) = \int_{-1}^1 e^{isx} (1-x^2) dx$$

$$= \int_{-1}^1 e^{isx} dx - \int_{-1}^1 x^2 e^{isx} dx$$

$$= \left[ \frac{e^{isx}}{is} \right]_{-1}^1 - \left[ x^2 \frac{e^{isx}}{is} - \int x^2 \frac{e^{isx}}{is} dx \right]_{-1}^1$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[ \frac{x^2 e^{isx}}{is} - 2 \left( 2 \frac{e^{isx}}{(is)^2} - \frac{e^{isx}}{(is)^3} \right) \right]_{-1}^1$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[ \frac{x^2 e^{isx}}{is} - \frac{2x e^{isx}}{(is)^2} \right]_{-1}^1 + \frac{2e^{isx}}{(is)^3} \Big|_1$$

$$= \frac{e^{is} - e^{-is}}{is} - \left[ \left( \frac{e^{is}}{is} - \frac{2e^{is}}{(is)^2} + \frac{2e^{is}}{(is)^3} \right) - \left( \frac{e^{-is}}{is} + \frac{2e^{-is}}{(is)^2} + \frac{2e^{-is}}{(is)^3} \right) \right]$$

$$= \left( \frac{e^{is}}{is} - \frac{2e^{is}}{(is)^2} + \frac{2e^{is}}{(is)^3} \right) - \left( \frac{e^{-is}}{is} + \frac{2e^{-is}}{(is)^2} + \frac{2e^{-is}}{(is)^3} \right)$$

$$= \frac{2is\sin s}{is} - \left[ \frac{2(s\sin s)i}{is} \cdot \frac{2}{\left(\frac{e^{is} + e^{-is}}{(is)^2}\right)} \right. \\ \left. + \frac{2(s\sin s)i}{(is)^3} \right]$$

$$= \cancel{\frac{2s\sin s}{s}} - \frac{2\sin s}{s} - \frac{2(2\cos s)}{(is)^2} + \frac{4\sin s}{i^2 s^3}$$

$$= -\frac{2\cos s}{i^2 s^2} + \frac{4\sin s}{i^2 s^3}$$

$$= \frac{2\cos s}{s^2} - \frac{4\sin s}{s^3}$$

$$= 4 \left[ \frac{s\cos s - 4\sin s}{s^3} \right]$$

Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} F(s) ds$$

$$= \frac{4}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[ \frac{s\cos s - 4\sin s}{s^3} \right] ds$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-isx} \left[ \frac{s \cos s - \sin s}{s^3} \right] ds$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[ \frac{s \cos s - \sin s}{s^3} \right] ds$$

$$1 - \frac{1}{4} = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{is/2} \left[ \frac{s \cos s - \sin s}{s^3} \right] ds$$

$$\frac{1}{2} \times \frac{3}{4} \times \frac{\pi}{2} = \int_0^{\infty} \left( \cos \frac{s}{2} - i \sin \frac{s}{2} \right) \left( \frac{s \cos s - \sin s}{s^3} \right) ds$$

$$-\frac{3\pi}{16} = \int_0^{\infty} \cos \frac{s}{2} \left( \frac{s \cos s - \sin s}{s^3} \right) ds$$

\* Find the Fourier Transform of

$e^{-|x|}$  and hence evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$e^{-|x|} = \begin{cases} e^x & x < 0 \\ e^{-x} & x > 0 \end{cases}$$

$$F.T = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$= \int_{-\infty}^0 e^{isx} f(x) dx + \int_0^{\infty} e^{isx} f(x) dx$$

$$= \int_{-\infty}^0 e^{isx} e^x dx + \int_0^\infty e^{isx} \cdot e^{-x} dx$$

$$= \int_{-\infty}^0 e^{(is+1)x} dx + \int_0^\infty e^{x(is-1)} dx$$

$$= \left[ \frac{e^{(is+1)x}}{(1+is)} \right]_{-\infty}^0 + \left[ \frac{e^{x(is-1)}}{(is-1)} \right]_0^\infty$$

$$= \left[ \frac{1}{1+is} - 0 \right] + \left[ 0 + \frac{1}{1-is} \right]$$

$$= \frac{1-is + 1+is}{1-i^2s^2} = \frac{2}{1+s^2}$$

Inverse Fourier Transform.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixa} F(s) ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixa}}{1+s^2} ds$$

$$\frac{1}{2} \left[ e^x + e^{-x} \right]_{-\infty}^{\infty} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixa}}{1+s^2} ds$$

$$\frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{1}{1+s^2} ds$$

\* Show that the Fourier Transform of  $e^{-x^2/2}$  is  $\sqrt{2\pi} e^{-s^2/2}$  by finding the F.T. of  $e^{-a^2x^2}$

08/04/2023

Consider  $f(x) = e^{-a^2x^2}$

Fourier Transform

$$= \int_{-\infty}^{\infty} e^{isx} f(x) dx = \int_{-\infty}^{\infty} e^{isx} e^{-a^2x^2} dx$$

$$= \int_{-\infty}^{\infty} e^{(is-a^2x)x} dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} dx$$

$$a^2x^2 - isx$$

$$= (ax)^2$$

$$- 2ax \left(\frac{is}{2a}\right)$$

$$+ \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$$

$$= \int_{-\infty}^{\infty} e^{\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}} dx$$

$$= \left(ax - \frac{is}{2a}\right)^2 - \frac{i^2 s^2}{4a^2}$$

$$= \left( \int_{-\infty}^{\infty} e^{-(ax - \frac{is}{2a})^2} dx \right) \cdot e^{-\frac{s^2}{4a^2}}$$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a} e^{-s^2/4a^2} \sqrt{\pi}$$

$$ax - \frac{is}{2a} = t$$

$$adx = dt$$

$$dx = \frac{dt}{a}$$

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$F(s) = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi}$$

$$F[e^{-a^2x^2}] = e^{-s^2/4a^2} \frac{\sqrt{\pi}}{a}$$

$$a = 1/\sqrt{2}$$

$$F[e^{-x^2/2}] = e^{-s^2/2} \sqrt{2\pi}$$

\* Find the F.T  $f(x) = \begin{cases} e^{-x^2/2} & -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases}$

(OR) S.T. the F.T. of  $e^{-x^2/2}$  is

reciprocal.

$$F[e^{-x^2/2}] = \sqrt{2\pi} e^{-s^2/2}$$

$$\text{F.T. of } e^{-x^2/2} = \int_{-\infty}^{\infty} e^{isx} e^{-x^2/2} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2x is)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2x is + (is)^2 - (is)^2)} dx$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[x-is]^2} \cdot e^{+s^2/2} dx \\
 &= e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2} dx \quad \frac{x-is}{\sqrt{2}} = t \\
 &\quad \frac{dx}{\sqrt{2}} = dt \\
 &= \sqrt{2} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt = \underline{\underline{\sqrt{2}\pi e^{-s^2/2}}}
 \end{aligned}$$

\* S.T. F.T. of  $f(x) = \begin{cases} a - |x| & |x| < a \\ 0 & |x| > a \end{cases}$

is  $\frac{1 - \cos as}{s^2}$ . Hence evaluate  
 $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ .

$\therefore$  Fourier Transform of  $f(x)$

$$f(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$f(x) = \begin{cases} a+x & -a < x < 0 \\ a-x & 0 < x < a \end{cases}$$

$$F(s) = \int_{-a}^0 e^{isx} (a+x) dx + \int_0^a e^{isx} (a-x) dx.$$

$$= \left[ (a+\alpha) \frac{e^{isx}}{is} - (1) \frac{e^{isx}}{(is)^2} \right]_0^\alpha$$

$$+ \left[ (a-\alpha) \frac{e^{isx}}{is} + (1) \frac{e^{isx}}{(is)^2} \right]_0^\alpha$$

$$= \left[ \left( \frac{\alpha(1)}{is} + \frac{1}{s} \right) - \left( 0 + e^{\frac{is(-\alpha)}{s^2}} \right) \right]$$

$$+ \left[ \left( 0 - e^{\frac{isa}{s^2}} \right) - \left( \frac{\alpha(1)}{is} - \frac{1}{s^2} \right) \right]$$

$$F(s) = \frac{2}{s^2} - \frac{1}{s^2} \left[ e^{isa} + e^{-isa} \right]$$

$$= \frac{2}{s^2} - \frac{1}{s^2} \left[ 2 \cos(sa) \right]$$

$$= \frac{2 - 2 \cos(sa)}{s^2}$$

Inverse Fourier Transform of  $f(x)$

\* Find the F.T of  $e^{-ax}$  ( $a > 0$ )

hence S.T  $\int_0^\infty \frac{\cos ax}{a^2 + s^2} = \frac{\pi}{2a} e^{-as}$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} F(s) ds$$

$$\begin{aligned} a &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \frac{2(1-\cos(sa))}{s^2} ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-isx} \frac{2\sin^2(\frac{sa}{2})}{s^2} ds. \\ x=0; a=2 \end{aligned}$$

$$2 = \frac{2}{\pi} \int_0^{\infty} \frac{2\sin^2(s)}{s^2} ds.$$

$$\boxed{\int_0^{\infty} \left( \frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}}$$

### \*Sine and Cosine Transforms:

\* Find the Fourier Sine and Cosine transform

\* Find the integral of  $e^{-ax}$  ( $a > 0$ ) & deduce the integral

$$(i) \int_0^{\infty} \frac{s \sin s \alpha}{s^2 + a^2} ds \quad (ii) \int_0^{\infty} \frac{\cos s \alpha}{s^2 + a^2} ds.$$

\*Fourier Sine Transform.

$$f_s(s) = \int_0^\infty f(x) \sin sx dx$$

$$= \int_0^\infty e^{-ax} \sin sx dx$$

$$\left[ \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

$$= \left[ \frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \frac{e^{-\infty}}{a^2+s^2} \underset{x \rightarrow \infty}{dt} (-a \sin sx - s \cos sx)$$

$$- \frac{e^0}{s^2+a^2} [-a \sin(0) - s \cos(0)]$$

$$= \frac{-1}{s^2+a^2} (s)$$

$$f_s(s) = \frac{s}{s^2+a^2}$$

→ Inverse Fourier Sine Transform.

$$f(x) = \int_0^\infty F_s(s) \sin s x \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2 + a^2} \sin s x \, ds$$

$$\frac{\pi e^{-ax}}{2} = \int_0^\infty \frac{s \sin s x}{s^2 + a^2} \, ds$$

\* Fourier Cosine Transform.

$$f_c(s) = \int_0^\infty f(x) \cos s x \, dx$$

$$= \int_0^\infty e^{-ax} \cos s x \, dx$$

$$= \left[ \frac{e^{-ax}}{a^2 + s^2} (-a \cos s x + s \sin s x) \right]_0^\infty$$

$$= 0 - \frac{1}{a^2 + s^2} (-a)$$

$$\therefore F_c(s) = \frac{a}{s^2 + a^2}$$

Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(s) \cos s x \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos s x}{a^2 + s^2} \, ds$$

$$\frac{\pi e^{-ax}}{2a} = \int_0^\infty \frac{\cos s x}{a^2 + s^2} \, ds$$

\* Find the Fourier Sine & Cosine Transform of  $\frac{e^{-ax}}{x}$  and deduce that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$$

$$= \tan^{-1}\left(\frac{s}{a}\right) + \tan^{-1}\left(\frac{s}{b}\right)$$

\* Fourier Sine Transform:

$$F_s(s) = \int_0^\infty f(x) \sin sx dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx dx \quad \text{---(1)}$$

$$F_s(s) = \int_0^\infty \frac{e^{-ax}}{x} \sin sx dx$$

diff. w.r.t 's' on both sides.

$$\frac{d}{ds} F_s(s) = \int_0^\infty \frac{e^{-ax}}{x} \cos sx \cdot x dx$$

$$\frac{d}{ds} F_s(s) = \int_0^\infty e^{-ax} \cos sx dx$$

$$= \left[ \frac{e^{-ax}}{x^2+s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2} (-a) = \frac{a}{s^2+a^2}$$

$$F_s(s) = \int \frac{a}{s^2+x^2} dx$$

$$\boxed{F_s(s) = \tan^{-1}\left(\frac{s}{a}\right)}$$

→ Inverse Fourier Transform:

$$\int_0^\infty \frac{e^{-bx}}{x} \sin sx dx = \tan^{-1}\left(\frac{s}{b}\right)$$

$$\int_0^\infty \left( \frac{e^{-ax} - e^{-bx}}{x} \right) \sin sx dx = \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$$

\* Fourier Cosine Transform:

$$F_c(s) = \int_0^\infty f(x) \cos sx dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \cos sx dx \quad \text{---(1)}$$

diff. w.r.t 's' on both sides

$$\frac{d}{ds} F_c(s) = \int_0^\infty \frac{e^{-ax}}{x} -\sin sx \cdot x dx$$

$$\frac{d}{ds} F_c(s) = - \int_0^\infty e^{-ax} \sin sx dx$$

$$= - \left[ \frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \left[ \frac{e^{-ax}}{s^2+a^2} (a \sin sx + s \cos sx) \right]_0^\infty$$

$$\frac{d}{ds} F_c(s) = 0 - \frac{s}{s^2+a^2}$$

$$\frac{d}{ds} F_c(x) = \frac{-s}{s^2 + a^2}$$

$$F_c(x) = -\frac{1}{2} \log(a^2 + s^2)$$

$$\int_0^\infty \cos sx \frac{e^{-bx}}{x} = -\frac{1}{2} \log(b^2 + s^2)$$

$$\Rightarrow \int_0^\infty \left( \frac{e^{-ax} - e^{-bx}}{x} \right) dx \cos sx = \frac{1}{2} \log \left( \frac{b^2 + a^2}{a^2 + s^2} \right)$$

$$\int_0^\infty \left( \frac{e^{-ax} - e^{-bx}}{x} \right) dx \cdot \cos sx = \frac{1}{2} \left( \log \left( \frac{b^2 + s^2}{a^2 + s^2} \right) \right)$$

\* Find the Fourier sine transform of

$$\frac{x}{x^2 + a^2} f_s(s) = \int_0^\infty \frac{x}{x^2 + a^2} \sin sx \, dx$$

$$f_s(e^{-ax}) = \frac{s}{s^2 + a^2}$$

then inverse fourier sine

transform of  $e^{-ax}$  is

$$f(x) = \frac{2}{\pi} \int_0^\infty f_s(s) \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2 + a^2} \sin sx \, ds$$

$$\frac{\pi e^{-ax}}{2} = \int_0^\infty \frac{s}{s^2 + a^2} \sin sx \, dx$$

$\downarrow$

$x = s$        $s = x$

$$\boxed{\frac{\pi e^{-as}}{2} = \int_0^\infty \frac{x}{x^2 + a^2} \sin sx \, dx}$$

\* Find the Fourier Cosine Transform

$$\text{of } \frac{1}{x^2 + a^2} \quad f_c(e^{-ax}) = \frac{a}{s^2 + a^2}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty f_c(s) \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds$$

$$\frac{\pi e^{-ax}}{2a} = \int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds$$

$$\boxed{\frac{\pi e^{-as}}{2a} = \int_0^\infty \frac{\cos sx}{s^2 + a^2} \, ds}$$

15/04/23

\* Find Fourier Sine Transform of  $e^{-|x|}$ , hence evaluate  $\int_0^\infty \frac{x \sin sx}{1+x^2} dx$

$$F_s[f(x)] = F_s(s) = \int_0^\infty f(x) \sin sx dx$$

$$\begin{aligned} F_s[e^{-|x|}] &= \int_0^\infty e^{-|x|} \sin sx dx \\ &= \int_0^\infty e^{-x} \sin sx dx \end{aligned}$$

$$a = -1, b = s$$

$$F_s[e^{-|x|}] = \int_0^\infty e^{-|x|} \left[ \frac{e^{-x}}{1+s^2} \left[ -\sin x \right] - s \cos x \right] dx$$

$$F_s[e^{-|x|}] = \left[ 0 + \frac{s}{s^2+1} \right]$$

$$F_s[e^{-|x|}] = \frac{s}{s^2+1}$$

Inverse Fourier Sine Transform.

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx F_s(s) ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx \cdot \frac{s}{s^2+1} ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+1} ds$$

$$\frac{\pi e^{-x}}{2} = \int_0^\infty \frac{s \sin sx}{s^2+1} ds$$

$$\begin{array}{l} \downarrow \\ x=s \end{array} \quad \begin{array}{l} \downarrow \\ s=x \end{array}$$

$$\boxed{\int_0^\infty \frac{x \sin sx}{x^2+1} dx = \frac{\pi e^{-s}}{2}}$$

$$* F_s(e^{-ax}) = \frac{s}{s^2+a^2} \quad f(\frac{x}{x+a^2}) = \frac{\pi}{2} e^{-as}$$

$$F_c(e^{-ax}) = \frac{a}{s^2+a^2} \quad f(\frac{1}{x^2+a^2}) = \frac{\pi}{2a} e^{-as}$$

\* Properties of Fourier Transform:

1) Linearity property:

If  $F(s)$  and  $G(s)$  are Fourier transform of  $f(x)$  &  $g(x)$  respectively then

$$F[a f(x) \pm b g(x)] = a F[f(x)] \pm b F[g(x)]$$

$$= a F(s) \pm b G(s)$$

\* Given that  $F(s)$  &  $G(s)$  are Fourier Transforms of  $f(x)$  &  $g(x)$  resp.

$$F(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{isx} dx \quad \text{--- (1)}$$

$$G(s) = F(g(x)) = \int_{-\infty}^{\infty} g(x) e^{isx} dx \quad \text{--- (2)}$$

LHS:

$$F[a f(x) \pm b g(x)]$$

( $\because$  by def<sup>n</sup> of F.T.)

$$= \int_{-\infty}^{\infty} [a f(x) \pm b g(x)] e^{isx} dx$$

$$= \int_{-\infty}^{\infty} a f(x) e^{isx} dx \pm \int_{-\infty}^{\infty} b g(x) e^{isx} dx$$

$$= a \int_{-\infty}^{\infty} f(x) e^{isx} dx \pm b \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

by (1) & (2)

$$= a F(s) \pm b G(s)$$

~~2) Change of Scale Property:~~

\* Note:

$$(i) F_s [a f(x) \pm b g(x)] = a F_s(s) \pm b G_s(s)$$

$$(ii) F_c [a f(x) \pm b g(x)] = a F_c(s) \pm b G_c(s)$$

~~2) Change of Scale Property:~~

$\Rightarrow$  If  $F(s)$  is the Fourier transform of

$f(x)$  then

$$(i) F(f(ax)) = \frac{1}{a} F(s/a)$$

$$(ii) F(f(x/a)) = a F(as)$$

\* Given  $F(s)$  is the Fourier transform of

$$f(x) \text{ i.e. } F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$F(f(ax)) = \int_{-\infty}^{\infty} e^{isx} f(ax) dx \quad ax=t$$

$$x=t/a$$

$$= \int_{-\infty}^{\infty} e^{is(t/a)} f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt$$

$$= \frac{1}{a} F(s/a)$$

∴ we can prove  $F(f(\gamma/a)) = a F(\gamma_s)$

Note: a)  $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

b)  $F_s(f(\gamma/a)) = a F_s(as)$

$\rightarrow F_c(f(ax)) = \frac{1}{a} F_c(s/a)$

$F_c(f(\gamma/a)) = a F_c(as)$

$$= \frac{1}{a} F(s/a)$$

iii) we can prove  $F(f(ax)) = a F(as)$

$$\text{then i)} F[f(x-a)] = e^{ias} F(s)$$

$$\text{ii)} F[f(x+a)] = e^{-ias} F(s)$$

Note: a)  $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

b)  $\bar{F}_s(f(ax)) = a F_s(as)$



$$F_c(f(ax)) = \frac{1}{a} F_c(s/a)$$

$$F_c(f(ax)) = a F_c(as)$$

$$F[f(ax)] = \int_{-\infty}^{\infty} f(ax) e^{isax} dx$$

$$= \int_{-\infty}^{\infty} f(t) e^{isat} dt$$

$\begin{matrix} t \\ a \\ x = at \\ dx = adt \end{matrix}$

$$= \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$\begin{matrix} x \\ t+a \\ dx = dt \end{matrix}$

$$= e^{isa} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{ias} F(s)$$

$$F[f(x+a)] = \int_{-\infty}^{\infty} f(x+a) e^{isx} dx$$

$\begin{matrix} x+a \\ t \\ dx = dt \end{matrix}$

$$= \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{ist} \cdot e^{-ias} dt$$

$$= e^{-ias} F(s)$$

17/04/2023

3) Shifting Property:

If  $F(s)$  is the Fourier Transform of  $f(x)$

#### 4) Modulation Property:

20/04/2023

= RHS; hence the modulation property.

If  $F(s)$  is the F.T. of  $f(x)$

$$\text{then } F[f(x)\cos ax]$$

$$= \frac{1}{2} [f(s-a) + f(s+a)]$$

→ By defn of F.T. of  $f(x)$

$$\text{i.e. } F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = F[s] - \textcircled{1}$$

$$\text{LHS: } F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x) \cos ax e^{isx} dx$$

$$= \int_{-\infty}^{\infty} f(x) \cos ax e^{isx} dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[ \frac{e^{iax} + e^{-iax}}{2} \right] e^{isx} dx$$

$$\left( \because \cos x = \frac{e^{ix} + e^{-ix}}{2} \right)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx + \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} dx$$

$$= \frac{1}{2} [F(s+a) + F(s-a)] \quad (\because \text{by } \textcircled{1})$$

Note:  
(i)  $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(ii)  $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(iii)  $F_c[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

iv)  $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

v) Show that  $\boxed{F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)}$

→ We know that

Fourier Transform of  $f(x)$

$$\text{R.H.S: } F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$$

$$\text{Consider } F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx - \textcircled{1}$$

diff  $\textcircled{1}$  w.r.t 's' on both sides

$$\frac{d}{ds} F(s) = \int_{-\infty}^{\infty} f(x) \frac{di}{ds}(e^{isx}) dx$$

$$\frac{d}{ds} F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} ix dx$$

$$= i \int_{-\infty}^{\infty} f(x) e^{isx} x dx$$

$$\frac{d}{ds} F[s] = \int_{-\infty}^{\infty} [x f(x)] e^{isx} dx \quad (2)$$

consider  $F_s(s) = \int_0^\infty f(x) \sin sx dx$  - (1)

d iff (1) wrt 's' on both sides

$$\frac{d}{ds} F_s(s) = \int_0^\infty f(x) \cos sx dx$$

$$= \int_0^\infty (x f(x)) \cos sx dx$$

again diff (2) wrt 's' on both sides  
 $\frac{d^2}{ds^2} F[s] = (-i)^2 \int_{-\infty}^{\infty} [x^2 f(x)] e^{isx} dx$   
 $\rightarrow$  continuing diff; we get the general form:

$$\frac{d^n}{ds^n} F(s) = \int_{-\infty}^{\infty} [x^n f(x)] e^{isx} dx$$

$$\boxed{\frac{d}{ds} F_s(s) = F_C(x f(x))}$$

7) Show that  $F[\alpha f(x)] = -\frac{d}{ds} F_C[f(x)]$

$$F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F[s]$$

c) Show that  $\boxed{F_C[x f(x)] = \frac{d}{ds} F_s[s]}$

RHS: F.S.T. of  $f(x)$  is given by

$$F_s[f(x)] = \int_0^\infty f(x) \sin sx dx = F_s[s]$$

\* Find the Fourier sine Transform of

$$\frac{1}{x}$$

→ Fourier sine transform of  $\frac{1}{x}$

$$F_S(s) = \int_0^\infty f(x) \sin sx dx$$

$$= \int_0^\infty \frac{1}{x} \sin sx dx - ①$$

$$\text{Consider } \int_0^\infty e^{-ax} \sin sx dx = \frac{s}{s^2 + a^2} - ②$$

Integrate ② wrt 'a' taking  $a_1$  to  $a_2$   
on limit on both sides.

$$\int_0^\infty \left[ \int_{a_1}^{a_2} e^{-ax} dx \right] \sin sx dx = \int_{a_1}^{a_2} \frac{s}{s^2 + a^2} da.$$

$$\int_0^\infty \left[ \frac{e^{-ax}}{-a} \right]_{a_1}^{a_2} s \sin s x dx = \left[ \tan^{-1} \left( \frac{a}{s} \right) \right]_{a_1}^{a_2}$$

$$\int_0^\infty \left[ e^{-a_1 x} - e^{-a_2 x} \right] \frac{\sin s x}{s} dx = \tan^{-1} \frac{a_2}{s} - \tan^{-1} \frac{a_1}{s}$$

put  $a_1 \rightarrow 0$ ;  $a_2 \rightarrow \infty$  on both sides

$$\boxed{\int_0^\infty \frac{\sin s x}{x} dx = \frac{\pi}{2}}$$

\* Find the F.C.T. of  $\frac{1}{1+x^2}$  & hence find

$$F.S.T. \text{ of } \frac{x}{1+x^2}$$

→ Fourier cosine transform  
 $F_C(s) = \int_0^\infty f(x) \cos sx dx$  - ③

$$F_C \left[ \frac{1}{1+x^2} \right] = \int_0^\infty \frac{1}{1+x^2} \cos sx dx - ③$$

$$I = \int_0^\infty \frac{1}{1+x^2} \cos sx dx - ①$$

diff. wrt 's' on both sides

$$\frac{dI}{ds} = \int_0^\infty \frac{1}{1+x^2} - \sin sx (x) dx$$

$$= - \int_0^\infty \frac{x}{1+x^2} \sin sx dx$$

$$\frac{dI}{ds} = - \int_0^\infty \frac{x}{1+x^2} \sin sx dx \text{ (M&N, Dr. by 2)}$$

$$\frac{dI}{ds} = - \int_0^\infty \frac{x^2}{x(1+x^2)} \sin s x \, dx.$$

$$= - \int_0^\infty \left[ \frac{(x^2+1)^{-1}}{x(1+x^2)} \right] \sin s x \, dx$$

$$= - \left[ \int_0^\infty \frac{\sin s x}{x} \, dx - \int_0^\infty \frac{\sin s x}{x(1+x^2)} \, dx \right]$$

$$= - \frac{\pi}{2} + \int_0^\infty \frac{\sin s x}{x(1+x^2)} \, dx \quad \text{--- (2)}$$

diff. (2) wrt 's'

$$\frac{dI}{ds^2} = 0 + \int_0^\infty \frac{\cos s x \cdot x}{x(1+x^2)} \, dx$$

$$= \int_0^\infty \frac{\cos s x}{1+x^2} \, dx$$

$$\frac{d^2I}{ds^2} = I \quad (\text{from (1)})$$

$$\frac{d^2I}{ds^2} - I = 0 \quad 0 \cdot I \Rightarrow (D^2 - 1)I = 0$$

$$A \cdot E \quad m^2 - 1 = 0$$

$$C_1 e^s + C_2 e^{-s} \quad \text{--- (3)}$$

roots are real & distinct

$$\frac{dI}{ds} = C_1 e^s - C_2 e^{-s} \quad \text{--- (4)}$$

$$\text{put } s = 0 \text{ in (1) & (3)}$$

$$\text{from (1)} \quad I = C_1 + C_2 = \pi/2 \quad \text{--- (5)}$$

$$= (\tan^{-1} x)_0 = \pi/2$$

$$(I = \pi/2)$$

$$\text{from (3)}$$

$$I = C_1 + C_2 = \pi/2 \quad \text{--- (6)}$$

$$\text{put } s = 0 \text{ in (2) & (4)}$$

$$\text{from (2)} \quad \frac{dI}{ds} = -\pi/2$$

$$\text{from (4)} \quad \frac{dI}{ds} = C_1 - C_2 = -\pi/2 \quad \text{--- (7)}$$

$$\text{Solve (6) & (7).}$$

$$C_1 = 0; C_2 = \pi/2$$

Sub  $C_1, C_2$  in (3)

$$I = \pi/2 e^{-s}$$

$$\int_0^\infty \frac{1}{1+x^2} \cos s x = \frac{\pi}{2} e^{-s}$$

$$\frac{dI}{ds} = \int_0^\infty \frac{x \sin sx}{1+x^2} dx = \frac{\pi}{2} e^{-s}$$

\* Find  $f(x)$  if its Fourier direct Transform

$$\frac{df}{dx} = -\frac{2}{\pi} \int_0^\infty \frac{\cos sx}{1+s^2} ds \quad (2)$$

$$\frac{s}{1+s^2} \cdot \cancel{4}$$

$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+1} \sin s x ds$  (We have find  $f(x)$  by Inverse F.S.T.)

Multiply N.R.E.Dr by s.

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2}{s(s^2+1)} \sin s x ds$$

~~$\frac{2}{\pi} \int_0^\infty (1+s^2)^{-1} \sin s x ds$~~

$$= \frac{2}{\pi} \int_0^\infty ((1+s^2)-1) \sin s x ds$$

$$f = A e^x + B e^{-x} \quad (3)$$

$$\frac{df}{dx} = A e^x - B e^{-x} \quad (4)$$

from (1) put  $x=0$ ;  $f=1$

from (2) put  $x=0$ ;  $f=A+B=1$   $\star$

from (3) put  $x=0$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} - \frac{2}{\pi} \int_0^\infty \frac{\sin s x}{s(1+s^2)} ds$$

$$= 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin s x}{s(1+s^2)} ds$$

$$f(x) = 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin s x}{s(1+s^2)} ds \quad (1)$$

$$\frac{df}{dx} = -\frac{2}{\pi} \int_0^\infty \frac{\cos s x}{1+s^2} ds$$

$$\frac{df}{dx} = -\frac{2}{\pi} \cdot \frac{\pi}{2} = -1$$

from (4) put  $x=0$

$$A-B=-1 \quad \star$$

$$\boxed{A=0; B=1}$$

$$\boxed{f(x) = e^{-x}}$$