26/03/2022

INFINITE SERIES:

→ Sequence :- definition - its convergence - divergence.

→ Series - definition - its convergence - divergence.

→ Geometric Series - statement - problems on
- Convergence
- Divergence
- Oscillatory
  - Oscilates finitely
  - oscillates infinitely

→ P- series (or) auxiliary series (or) Harmonic Series
- statement - problems on convergence & divergence.

→ D'Alembert's Ratio test - statement - problems on
- Convergence
- divergence
- Test fails

→ Cauchy's n^{th} root test - statement
- problems on
  - Conv.
  - Div.
  - Test fails.

→ Alternating Series:
(i) Leibnitz test - st. of convergence series
(ii) Absolutely convergent.
(iii) conditionally convergent.

Sequence: A set arranged in a well-defined manner is called sequence. $\boxed{<S_n> ; \{S_n\}}$

⇒ **Convergence of Sequence:**

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \text{finite (according to problem)}.$$

if the limit is not finite
⇒ Divergence of Sequence.

* **Series:** An expression in sequence is called series.

seq: $\{u_1, u_2, \text{-------}\}$

positive Series:

$S_n = u_1 + u_2 + \text{-------}$

$S_1 = u_1$; $S_2 = u_1 + u_2$; ~~$S_n = u_n +$~~

$S_{n-1} = u_1 + u_2 + \text{------} + u_{n-1}$

$S_n = u_1 + u_2 + \text{------} + u_{n-1} + u_n.$

This is called Partial Sum.

$S_n \Rightarrow n^{th}$ partial sum.

$$\boxed{U_n = S_n - S_{n-1}}$$

* Convergence in Series:

⇒ $\sum_{n=1}^{\infty} U_n$ where $\frac{1}{n} = U_n$ in $1+\frac{1}{2}+\frac{1}{3}+\dots$

$$\boxed{\sum_{n=1}^{\infty} U_n = \text{Series}}$$

Ex: $\frac{1\cdot 3}{2\cdot 4} + \frac{3\cdot 5}{4\cdot 6} + \frac{5\cdot 7}{6\cdot 8} + \dots$

1, 3, 5 ......

$$\boxed{\sum_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{2n(2n+2)}}$$

* If sequence is convergence ; then series is also convergent.

Note: Do not use the definitions for solving the problems.

Series is having osillatory property.

$1-1+1-1+ \dots$ ⇒ Osuillates finitely.

$S_1 = 1$ | $S_3 = 1$

$S_2 = 0$ | $S_4 = 0$

*NECESSARY CONDITION : (Theorem)

If ΣU_n convergence then $\lim_{n\to\infty} U_n = 0$.

Proof: Given that ΣU_n convergence

By the definition of convergence

Series $\boxed{\lim_{n\to\infty} S_n = l \text{ (finite)}}$

$$\boxed{\lim_{n\to\infty} S_{n-1} = l \text{ (finite)}}$$

By partial sum;

$\cancel{S_n - U_n - U_n} \; \cancel{U_n = S_n - S_{n-1}}$

$\cancel{= l - l}$

$\cancel{\boxed{U_n = 0}}$

* $U_n = S_n - S_{n-1}$

apply limit on both sides

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= l - l = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty} U_n = 0}$$

THEOREM: 2

Converse of neccessary condition need not be true.

If $\lim_{n\to\infty} U_n = 0$, then ΣU_n may (or) may not be true.

• Note: Theorem 2 is not applicable for problems.

THEOREM-3:

If $\lim_{n\to\infty} U_n \neq 0$; then ΣU_n diverges.

***Geometric Series:**

If the given series

$1 + r + r^2 + \cdots + $ (It is a G. series with common ratio r) $\ni$ it satisfies the following conditions.

(i) $|r| \le 1$ $(-1 < r < 1)$ then the series converges

(ii) $r \ge 1$ $\left(\text{ie } r>1,\ r=1\right)$ then the series diverges

To check for if the series is in G. series

$$\Sigma (\text{const})^n = \Sigma \frac{1}{(\text{const})^n}$$

(iii) $r = -1$; Oscillates finitely

(iv) $r < -1$; oscillates infinitely

***P-Series (Auxiliary Series)**

I: Comparison Test-1:

If ΣU_n and ΣV_n be a +ve term series and $U_n \le V_n$ $\ni$ ΣV_n converges then ΣU_n also converges.

II: Comparison Test-2:

If ΣU_n and ΣV_n be a +ve term series and $U_n \ge V_n$ $\ni$ ΣV_n diverges then ΣU_n also diverges.

III: Limit form of comparison test:

If ΣU_n and ΣV_n be two +ve series and $\lim_{n\to\infty} \frac{U_n}{V_n} = l$ (finite & non-zero).

Then ΣU_n and ΣV_n converges together. and ΣU_n and ΣV_n diverges together.

⇒ P series

If $\Sigma \frac{1}{n^p}$ is the standard form (or) +ve term series

(i) If $p > 1 \longrightarrow$ converges

(ii) If $p \le 1 \longrightarrow$ Diverges.

$\lim_{n\to\infty} \Sigma U_n = 0 \longrightarrow$ May (or) may not be convergent.

* Test for convergence $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$

$$\Sigma U_n = \frac{(2n\mp1)}{n(n+1)(n+2)}$$

$$\Sigma U_n = \frac{(2n-1)}{n(n+1)(n+2)}$$

$$\underset{n\to\infty}{Lt}\ \Sigma U_n = \underset{n\to\infty}{Lt}\ \frac{n\left(\frac{2}{\cancel{2}}-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

~~converges~~

$$\Sigma V_n = \Sigma \frac{1}{n^2}$$

$\therefore P = 2 > 1$

$\Rightarrow$ converges.

By P-series; ΣV_n converges $\Rightarrow \Sigma U_n$ converges.

By limit form of comparison test

~~$\Rightarrow U_n$ converges~~

$$\underset{n\to\infty}{Lt}\ \frac{U_n}{V_n} = \underset{n\to\infty}{Lt}\ \frac{\frac{\cancel{n}\left(2-\frac{1}{n}\right)}{\cancel{n^3}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}}{\frac{1}{\cancel{n^2}}}$$

$$= \underset{n\to\infty}{Lt}\left(\frac{2}{1}\right) = 2$$

* $\frac{1\cdot2\cdot3}{1} + \frac{2\cdot3\cdot4}{3} + \cdots$

$$\Sigma U_n = \Sigma \frac{n(n+1)(n+2)}{2n-1}$$

$$= \Sigma \frac{n^{\cancel{3}2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\cancel{n}\left(2-\frac{1}{n}\right)}$$

$$= \Sigma \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{2-\frac{1}{n}}$$

$$\underset{n\to\infty}{Lt}\ \Sigma U_n = \infty \neq 0 \Rightarrow \text{diverges.}$$

* $\Sigma \frac{1}{2^n+3^n}$

$$\Sigma U_n = \Sigma \frac{1}{2^n+3^n} = \Sigma \frac{1}{3^n\left(1+\left(\frac{2}{3}\right)^n\right)}$$

$$\Sigma V_n = \Sigma \frac{1}{3^n}$$

It is G-series with $r = \frac{1}{3} < 1$

ΣV_n converges.

By limit form of comparison test

$$\underset{n\to\infty}{Lt}\ \frac{U_n}{V_n} = \underset{n\to\infty}{Lt}\ \frac{\frac{1}{\cancel{3^n}\left(1+\left(\frac{2}{3}\right)^n\right)}}{\frac{1}{\cancel{3^n}}} = 1$$

* $\Sigma \frac{\log n}{2n^3-1}$

$$\Sigma U_n = \Sigma \frac{\log n}{2n^3-1} = \Sigma \frac{\log n}{n^3\left(2-\frac{1}{n^3}\right)}$$

$$\Sigma V_n = \Sigma \frac{1}{n^3}$$

$P = 3 > 1$

By p series $\Rightarrow$ ~~converges~~ ΣV_n converges.

By limit form of comparison.

$$\underset{n\to\infty}{Lt}\ \frac{U_n}{V_n} = \underset{n\to\infty}{Lt}\ \frac{\log n}{2-\frac{1}{n^3}}$$

$U_n = \frac{\log n}{2n^3 - 1}$

$\frac{1}{2n^3-1} \leq \frac{1}{n^3} \quad \forall n \geq 2$

$\Sigma U_n \leq \Sigma V_n$

$\Sigma V_n = \frac{1}{n^3}$

$3 > 1 \Rightarrow$ By P-series

ΣV_n converges.

$\Rightarrow$ By comparison test - 1;

ΣU_n converges.

$* \ \Sigma \sin \frac{1}{n}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin \frac{1}{n} = \frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} - \cdots$

$\Sigma U_n = \frac{1}{n}\left[1 - \frac{1}{3!n^2} + \frac{1}{n^4 5!} - \cdots\right]$

$\Sigma V_n = \frac{1}{n}$

$P = 1 \Rightarrow$ By P-series $\Rightarrow$ diverges.

$\Rightarrow$ By Limit form of comparison test

$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{\cancel{\frac{1}{n}}\left[1 - \frac{1}{3!n^2} + \cdots\right]}{\cancel{\frac{1}{n}}}$

$= 1$ (finite & non-zero).

ΣU_n also diverges. By limit form of comparison test.

$*$ H.W:

$\frac{1\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 3}{4\cdot 5\cdot 6} + \cdots$

$* \ \Sigma \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$

$\Sigma \frac{1}{n^3}\cdot\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^n$

$\Sigma V_n = \Sigma \frac{1}{n^3}$

$\Rightarrow P = 3 > 1$

By P-series;

ΣV_n converges.

$\Rightarrow$ By limit form of comparison test.

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{e^{n\left(\frac{2}{n}\right)}}{e^{n\left(\frac{3}{n}\right)}}$

$= \frac{1}{e}$

$\Rightarrow$ finite & non-zero

$\Rightarrow \Sigma U_n$ converges.

$* \ \frac{2^q}{1^p} + \frac{3^q}{2^p} + \cdots$

$\Sigma U_n = \Sigma \frac{(n+1)^q}{n^p} = \Sigma \frac{n^q\left(1+\frac{1}{n}\right)^q}{n^p}$

$\Sigma U_n = \Sigma \frac{\left(1+\frac{1}{n}\right)^q}{n^{p-q}} \Rightarrow$ $p - q > 1 \Rightarrow$ converges; $p - q \leq 1 \Rightarrow$ diverges

$* \sum \tan\frac{1}{n}$

$* \sum \sqrt{n^2+1} - \sqrt{n^2-1}$

$$\sum \frac{n^2+1-n^2+1}{\sqrt{n^2+1}-\sqrt{n^2-1}} = \sum \frac{2}{\sqrt{n^2+1}-\sqrt{n^2-1}} = \sum \frac{2}{n\left[\sqrt{1+\frac{1}{n^2}}-\sqrt{1-\frac{1}{n^2}}\right]}$$

$\sum V_n = \sum \frac{1}{n}$

$\Rightarrow p = 1 \Rightarrow$ P-series $\Rightarrow \sum V_n$ diverges.

By limit form of comparison method.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^2}}-\sqrt{1-\frac{1}{n^2}}}$$

$= \infty$

$\Rightarrow \sum U_n$ diverges.

$* \quad 1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+ \cdots$

$$\sum U_n = \frac{n^n}{(n+1)^{n+1}}$$

$$= \sum \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \sum \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

$\sum V_n = \frac{1}{n} \Rightarrow p = 1$

By P-series; $\Rightarrow \sum V_n$ diverges.

By limit form of Comparison test.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e}$$

$\Rightarrow \sum U_n$ diverges.

$$* \sum \left(\sqrt[3]{n^3+1} - n\right)$$

$$\sum U_n = \sum\left(\sqrt[3]{n^3+1} - n\right)$$

$$= \sum n\left(\sqrt[3]{1+\frac{1}{n^3}} - 1\right)$$

$$= \sum n\left[\cancel{1} - \cancel{1} + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 \cdots\right]$$

$$= \sum \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$$

$\sum V_n = \frac{1}{n^2}$

$\Rightarrow p = 2 \Rightarrow$ By P-series; V_n converges.

By limit form of comparison method:

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$$

$= \frac{1}{3}$ (finite & non-zero).

$\Rightarrow \sum U_n$ converges.

$$* \sum \frac{1}{\sqrt{n}+\sqrt{n+1}} \Rightarrow \sum U_n = \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

$p = 1/2 < 1 \Rightarrow$ diverges

ΣV_n diverges.

* $\Sigma \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$

$$\Sigma \frac{n\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{n^3\left[4-\frac{7}{n}+\frac{2}{n^3}\right]} \Rightarrow \Sigma V_n = \frac{1}{n^2}$$

$p = 2 \Rightarrow$ By P-series, ΣV_n converges

By Limit form of C.T

$$= \lim_{n\to\infty} \frac{U_n}{V_n} = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} \text{ (finite \& non-zero)}$$

$\Rightarrow \Sigma U_n$ converges.

*** D'ALEMBERT's RATIO TEST:** (!, x, Repeated Terms complicated)

If ΣU_n be a +ve series and $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lambda$ (say)

then the series (i) converges when $\lambda > 1$,

(ii) Diverges when $\lambda < 1$.

(iii) Test fails when $\lambda = 1$

Repeated terms/series

$$\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots$$

$$U_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)}$$

* $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

By D'Alembert's ratio test:

$$U_n = \frac{1}{n!}, \quad U_{n+1} = \frac{1}{(n+1)!}$$

By D'Alembert's ratio test.

$$\lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} = \infty > 1$$

$\Rightarrow$ converges.

* $\Sigma \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}$

$$\Sigma U_n = \Sigma \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

$$\frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots 2n+1} \times \frac{3\cdot 5\cdot 7 \cdots 2(n+1)}{1\cdot 2\cdot 3 \cdots n+1}$$

$$U_{n+1} = \Sigma \frac{1\cdot 2\cdot 3 \cdots n+1}{3\cdot 5\cdot 7 \cdots 2(n+1)}.$$

By D'Alembert's ratio test

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{2(n+1)\cdot 1}{(n+1)} = 2 > 1$$

$\Rightarrow$ converges.

* $1 + \frac{x}{2} + \frac{x^2}{5} + \cdots$

$$\Sigma U_n = \frac{x^n}{n^2+1} \qquad \Sigma U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\cancel{x^n}}{n^2+1} \cdot \frac{n^2\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}{\cancel{x^n}\cdot x}$$

$$\lim_{n\to\infty} \frac{\cancel{n^2}\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}{x\cdot \cancel{n^2}\left(1+\frac{1}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{1}{x}\left(\frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)}\right)$$

$$= \frac{1}{x} = (\lambda \text{ say})$$

(i) $\lambda > 1 \Rightarrow \frac{1}{x} > 1 \Rightarrow x < 1 \Rightarrow$ converges.

(ii) $\lambda < 1 \Rightarrow \frac{1}{x} < 1 \Rightarrow x > 1 \Rightarrow$ diverges

(iii) $\lambda = 1 \Rightarrow x = 1 \Rightarrow$ test fails.

put $x = 1$ in U_n

$$U_n = \frac{1}{n^2+1} = \frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$$

$\Sigma V_n = \frac{1}{n^2}$

$\Rightarrow \Sigma V_n$ converges.

$\Rightarrow$ at $x = 1$ & $x < 1$; ΣU_n converges

$\lambda > 1 \Rightarrow \frac{1}{x} > 1 \Rightarrow x < 1 \Rightarrow$ converges.

(ii) $\lambda < 1 \Rightarrow \frac{1}{x} < 1 \Rightarrow x > 1 \Rightarrow$ diverges

(iii) $\lambda = 1 \Rightarrow x = 1 \Rightarrow$ test fails.

put $x = 1$ in U_n

$$U_n = \frac{1}{n^2+1} = \frac{1}{n^2\left(1+\frac{1}{n^2}\right)} \qquad \Sigma V_n = \frac{1}{n^2}$$

$\Rightarrow \Sigma V_n$ converges.

$\Rightarrow$ at $x = 1$ & $x < 1$; ΣU_n converges.

* $\frac{1\cdot 2}{3\cdot 4\cdot 5} + \frac{2\cdot 3}{4\cdot 5\cdot 6} + \cdots$

$$U_n = \frac{n(n+1)}{(n+2)(n+3)(n+4)}$$

$$\Sigma U_n = \Sigma \frac{n^2\left(1+\frac{1}{n}\right)}{n^3\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

$$= \Sigma \frac{1}{n} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

$$\Sigma V_n = \frac{1}{n}.$$

$P = 1 \Rightarrow$ By P- ~~test~~ series test

$\Rightarrow \Sigma V_n$ diverges.

By Limit form of Comparison test $\Rightarrow$

$$\lim_{n\to\infty} \frac{U_n}{V_n} \Rightarrow \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)} = 1$$

(finite & non-zero)

$\Rightarrow \Sigma U_n$ diverges.

* $\Sigma \tan\frac{1}{n}$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$\tan\frac{1}{n} = \frac{1}{n} + \frac{1}{3n^3} + \frac{2}{15}\cdot\frac{1}{n^5} + \frac{17}{315}\cdot\frac{1}{n^7} + \cdots$$

$$\Sigma U_n = \frac{1}{n}\left[1 + \frac{1}{3n^2} + \frac{2}{15n^4} + \cdots\right]$$

$$\Sigma V_n = \frac{1}{n}$$

$P = 1 \Rightarrow$ By P-series test

ΣV_n diverges.

$\Rightarrow$ By Limit form of comparison test

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} 1 + \frac{1}{3n^2} + \frac{2}{15n^4} + \cdots$$

$= 1$ (finite & non-zero)

$\Rightarrow \Sigma U_n$ diverges.

28/03/2022

*CAUCHY'S n^{th} root test:

series

(i) converges when $l<1$

(ii) diverges when $l>1$

(iii) Test fails when $l=1$

* $\Sigma \frac{1}{n^n}$ $\Rightarrow$ Verify Cauchy's n^{th} root Test.

$U_n = \frac{1}{n^n}$

By Cauchy's n^{th} root test.

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = 0 = l \text{ (say)}.$$

$\Rightarrow \Sigma U_n$ converges

* $\Sigma \frac{1}{(\log n)^n}$

$U_n = \frac{1}{(\log n)^n}$ $\qquad$ ΣU_n converges

By cauchy's n^{th} root test.

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{(\log n)^n}\right)^{1/\cancel{n}} = \frac{1}{\infty} = 0$$

~~* $\Sigma \frac{1}{n(\log n)^n}$~~ $\qquad$ * $\Sigma \frac{1}{(\log(\log n))^n}$

$U_n = \frac{1}{(\log(\log n))^n}$

By Cauchy's n^{th} root test $\qquad$ ΣU_n converges

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log(\log n)} = 0$$

* $\Sigma\left(\frac{n}{n+1}\right)^n = \Sigma \frac{\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n}$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

test fails.

* $\Sigma \frac{2^n}{n^3}$

$U_n = \frac{2^n}{n^3}$ $\Rightarrow \lim_{n\to\infty} \Sigma (U_n)^{1/n}$

$$\lim_{n\to\infty}\left(\frac{2^n}{n^3}\right)^{1/n} = \lim_{n\to\infty}\frac{2}{(n^{1/n})^3} = 2 > 1$$

ΣU_n diverges

* ~~$1+\frac{1}{-n^2}$~~ $\Sigma\left(1+\frac{1}{n}\right)^{-n^2} = \Sigma\left(1+\frac{1}{n}\right)^{-n^2}$

$$\lim_{n\to\infty} \Sigma(U_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\Rightarrow \Sigma U_n$ Converges.

* $\Sigma\left(\frac{nx}{n+1}\right)^n = \Sigma \frac{n^n x^n}{n^n \left(1+\frac{1}{n}\right)^n}$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty}\left(\frac{x^n}{\left(1+\frac{1}{n}\right)^n}\right)^{1/n} = \frac{x}{1}$$

$x < 1 \longrightarrow$ converges.

$x > 1 \longrightarrow$ diverges

$x = 1 \longrightarrow$ fails

$x = 1$.

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty}\left(\frac{n}{1+n}\right)^n$$

$$= \lim_{n\to\infty}\frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq$$

Diverges.

$x<1 \rightarrow$ converges ; $x \geq 1 \rightarrow$ diverges

* $\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}$

$$\lim_{n\to\infty} (U_n)^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{n^{-3/2} \times \frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{n^{-5/2}}$$

$$= \cancel{\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{5/2}}}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{n^{-5/2}}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n^{1/2}}\right)^{n^{-5/2}}$$

$$= \lim_{n\to\infty}$$

* $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$U_n^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n\sqrt{n}}\right]^{1/n}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{1/\sqrt{n}} = \frac{1}{e} < 1$$

$\therefore$ converges.

* $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \cdots$

$$U_n = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}$$

$$= \left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-n}$$

$$(U_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(\frac{n}{n+1}\right)\left[\frac{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}{\cancel{n^n}} - 1\right]^{-1}$$

$$= \frac{n}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n - 1}$$

$$\lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n - 1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n - 1}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$= (e-1)^{-1} = \frac{1}{e-1} < 1$$

$\Rightarrow$ converges

*** ALTERNATING SERIES :**

A series whose terms are alternatively positive and negative is called an alternating series.

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} U_n \quad (\text{or}) \quad \sum_{n=0}^{\infty} (-1)^n U_{n+1}}$$

Ex: 1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

2) $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \cdots = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n}{\log n}$

applicable to alternate series

*** LEIBNITZ TEST:** only convergence occurs in this

If ΣU_n is an alternating series such that

(i) $U_1 \geq U_2 \geq U_3 \geq \cdots$ (or) $\boxed{U_n \geq U_{n+1}}$

(ii) $\lim_{n\to\infty} U_n = 0$; then given series is said to be convergent by Leibnitz test

* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ $\quad \left(1 > \frac{1}{2} > \frac{1}{3} \cdots\right)$

(i) $\frac{1}{n} > \frac{1}{n+1}$

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow$ converges.

Leibnitz Test Convergence.

- Absolutely Converge
- Conditionally converge.

* 1) convert the alternating series U_n into +ve series;

2) Then by any other test;

if U_n converges by the Leibnitz and other test is called absolutely converge.

3) if U_n diverges by other test and Leibnitz test U_n converges $\Rightarrow$ conditionally converge.

* If $\lim_{n\to\infty} U_n \neq 0$ by Leibnitz test, U_n is said to be oscillatory.

* Absolutely & conditionally convergent.