Technical Appendix for "Expected Hypervolume Improvement Is a Particular Hypervolume Improvement"

A. Proofs

A.1 The Proof of **Theorem 2**

Theorem 2. *Given $A = \{\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(n)}\} \subset \mathbb{R}^m$, $\boldsymbol{\mu} \in \mathbb{R}^{qm}$, $\boldsymbol{\Sigma} \in \mathcal{S}_{++}^{qm\times qm}$ in the qEHVI problem. For each index set $I \subset \{1, \ldots, q\}$, define $\hat{A}(I) = \{\hat{\mathbf{f}}^{(1)}(I), \ldots, \hat{\mathbf{f}}^{(n)}(I)\} \subset \mathbb{R}^m, \hat{\mathbf{r}}(I) \in \mathbb{R}^m$ by*

$$\hat{\mathbf{r}}_j(I) = q\mathrm{MinEI}(0, \boldsymbol{\mu}_I^{(j)}, \boldsymbol{\Sigma}_{I\times I}^{(j)}), \forall 1 \leq j \leq m$$

and for $1 \leq i \leq n, 1 \leq j \leq m$,

$$\hat{\mathbf{f}}_j^{(i)}(I) = \hat{\mathbf{r}}_j(I) - q\mathrm{MinEI}(\mathbf{f}_j^{(i)}, \boldsymbol{\mu}_I^{(j)}, \boldsymbol{\Sigma}_{I\times I}^{(j)}),$$

where $\boldsymbol{\mu}_I^{(j)}, \boldsymbol{\Sigma}_{I\times I}^{(j)}$ are the mean and covariance matrix of $(\mathbf{y}_j^{(i)})_{i\in I}$ (see the definition of qEHVI), respectively. Then,

$$q\mathrm{EHVI}(A; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{I\subset\{1,\ldots,q\}} (-1)^{|I|+1}\mathrm{HVI}(\{\hat{\mathbf{r}}(I)\}, \hat{A}(I)).$$

Proof. For $\{\mathbf{y}^{(i)}\}_{i=1}^q \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in the qEHVI problem, define

$$\mathbf{Y} := (\mathbf{y}_1^{(1)}, \ldots, \mathbf{y}_1^{(q)}, \mathbf{y}_2^{(1)}, \ldots, \mathbf{y}_2^{(q)}, \ldots, \mathbf{y}_m^{(1)}, \ldots, \mathbf{y}_m^{(q)}).$$

Then, $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ after rearranging some rows and columns of $\boldsymbol{\mu}, \boldsymbol{\Sigma}$. We still use $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ without causing confusion.

Further, we define

$$\mathbf{Y}^{(i)} := (\mathbf{y}_i^{(1)}, \ldots, \mathbf{y}_i^{(q)}), \forall 1 \leq i \leq m.$$

Then, $\mathbf{Y} = (\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)})$. Under the assumption that different objective functions are mutually independent, $\{\mathbf{Y}^{(i)}\}_{i=1}^m$ are mutually independent. Therefore,

$$\xi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(z_1^{(1)}, \ldots, z_m^{(q)}) = \prod_{i=1}^m \xi_{\boldsymbol{\mu}_{(i-1)m+1:im}, \boldsymbol{\Sigma}_{(i-1)m+1:im\times(i-1)m+1:im}}(z_i^{(1)}, \ldots, z_i^{(q)}).$$

Let $\boldsymbol{\mu}^{(i)} := \boldsymbol{\mu}_{(i-1)m+1:im}$ and $\boldsymbol{\Sigma}^{(i)} := \boldsymbol{\Sigma}_{(i-1)m+1:im\times(i-1)m+1:im}, \forall 1 \leq i \leq m$. Then,

$$\xi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(z_1^{(1)}, \ldots, z_m^{(q)}) = \prod_{i=1}^m \xi_{\boldsymbol{\mu}^{(i)},\boldsymbol{\Sigma}^{(i)}}(z_i^{(1)}, \ldots, z_i^{(q)}).$$

In the following, we formulate qEHVI as a repeated integral:

$$\begin{aligned} q\mathrm{EHVI}(A; \boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \int_{\mathbb{R}^{qm}} \mathrm{HVI}(\{\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(q)}\}, A)\xi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(z_1^{(1)}, \ldots, z_m^{(q)})\mathrm{d}z_1^{(1)} \cdots \mathrm{d}z_m^{(q)} \\ &= \int_{\mathbb{R}^{qm}} \left[\int_{\mathbb{R}^m} \mathbf{1}_{\cup_{i=1}^q[\mathbf{0},\mathbf{z}^{(i)}]\setminus\cup_{i=1}^n[\mathbf{0},\mathbf{f}^{(i)}]}(\mathbf{w})\mathrm{d}\mathbf{w}\right] \xi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(z_1^{(1)}, \ldots, z_m^{(q)})\mathrm{d}z_1^{(1)} \cdots \mathrm{d}z_m^{(q)}. \end{aligned} \tag{S1}$$

Using the inclusion-exclusion principle, we have

$$\mathbf{1}_{\cup_{i=1}^q[\mathbf{0},\mathbf{z}^{(i)}]}(\mathbf{w}) = \sum_{I=\{j_1,\ldots,j_s\}\subset S:=\{1,\ldots,q\}} (-1)^{|I|+1}\mathbf{1}_{[\mathbf{0},\mathbf{z}^*(I)]}(\mathbf{w})$$

where $\mathbf{z}^*(I)$ means the element-wise minimum of points in $\{\mathbf{z}^i\}_{i\in I}$, that is,

$$\mathbf{z}^*(I) = (\min_{i\in I}\{\mathbf{z}_1^{(i)}\}, \min_{i\in I}\{\mathbf{z}_2^{(i)}\}, \ldots, \min_{i\in I}\{\mathbf{z}_m^{(i)}\}).$$

Then,

$$\begin{aligned} \mathbf{1}_{\cup_{i=1}^q[\mathbf{0},\mathbf{z}^{(i)}]\setminus\cup_{i=1}^n[\mathbf{0},\mathbf{f}^{(i)}]}(\mathbf{w}) &= \sum_{I=\{j_1,\ldots,j_s\}\subset S} (-1)^{|I|+1}\mathbf{1}_{[\mathbf{0},\mathbf{z}^*(I)]\setminus\cup_{i=1}^n[\mathbf{0},\mathbf{f}^{(i)}]}(\mathbf{w}) \\ &= \sum_{I=\{j_1,\ldots,j_s\}\subset S} (-1)^{|I|+1} \begin{cases} 1, & \mathbf{z}^*(I) \in [\mathbf{w}, +\infty) \text{ and } \mathbf{w} \in [\mathbf{0}, +\infty) \setminus \cup_{i=1}^n[\mathbf{0}, \mathbf{f}^{(i)}] \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \tag{S2}$$

Applying similar proofs as in **Theorem 1**, qEHVI can be computed by:

$$\begin{aligned}
&\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\int_{\Psi}\left[\int_{\mathbb{R}^{qm}}\begin{cases}\xi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(z_1^{(1)},\ldots,z_m^{(q)})\mathrm{d}z_1^{(1)}\cdots\mathrm{d}z_m^{(q)}, & \min_{1\le k\le s}\left\{z_i^{(j_k)}\right\}\ge w_i, \forall 1\le i\le m\\ 0, & \text{otherwise}\end{cases}\right]\mathrm{d}w_1\cdots\mathrm{d}w_m\\
&=\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\int_{\Psi}\left[\int_{\mathbb{R}^{qm}}\prod_{i=1}^{m}\begin{cases}\xi_{\boldsymbol{\mu}^{(i)},\boldsymbol{\Sigma}^{(i)}}(z_i^{(1)},\ldots,z_i^{(q)})\mathrm{d}z_i^{(1)}\cdots\mathrm{d}z_i^{(q)}, & \min_{1\le k\le s}\left\{z_i^{(j_k)}\right\}\ge w_i\\ 0, & \text{otherwise}\end{cases}\right]\mathrm{d}w_1\cdots\mathrm{d}w_m \qquad \text{(S3)}\\
&=\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\int_{\Psi}\prod_{i=1}^{m}\left[\int_{\mathbb{R}^{q}}\begin{cases}\xi_{\boldsymbol{\mu}^{(i)},\boldsymbol{\Sigma}^{(i)}}(z_i^{(1)},\ldots,z_i^{(q)})\mathrm{d}z_i^{(1)}\cdots\mathrm{d}z_i^{(q)}, & \min_{1\le k\le s}\left\{z_i^{(j_k)}\right\}\ge w_i\\ 0, & \text{otherwise}\end{cases}\right]\mathrm{d}w_i
\end{aligned}$$

where $\Psi := [\mathbf{0},+\boldsymbol{\infty})\setminus\cup_{i=1}^{n}[\mathbf{0},\mathbf{f}^{(i)}]$.

Define $\hat{\mathbf{w}}(I)\in\mathbb{R}^m$ depended on I such that for $1\le j\le m$,

$$\begin{aligned}
\hat{w}_j(I)&=\int_{w_j}^{+\infty}\left[\int_{\mathbb{R}^q}\begin{cases}\xi_{\boldsymbol{\mu}^{(j)},\boldsymbol{\Sigma}^{(j)}}(z_j^{(1)},\ldots,z_j^{(q)})\mathrm{d}z_j^{(1)}\cdots\mathrm{d}z_j^{(q)}, & \min_{1\le k\le s}\left\{z_j^{(j_k)}\right\}\ge t\\ 0, & \text{otherwise}\end{cases}\right]\mathrm{d}t\\
&=\int_{\mathbb{R}^q}\left[\int_{w_j}^{+\infty}\begin{cases}1\mathrm{d}t, & t\le\min_{1\le k\le s}\left\{z_j^{(j_k)}\right\}\\ 0\mathrm{d}t, & \text{otherwise}\end{cases}\right]\xi_{\boldsymbol{\mu}^{(j)},\boldsymbol{\Sigma}^{(j)}}(z_j^{(1)},\ldots,z_j^{(q)})\mathrm{d}z_j^{(1)}\cdots\mathrm{d}z_j^{(q)}\\
&=\int_{\mathbb{R}^q}\left(\min_{1\le k\le s}\left\{z_j^{(j_k)}\right\}-w_j\right)_+\xi_{\boldsymbol{\mu}^{(i)},\boldsymbol{\Sigma}^{(i)}}(z_j^{(1)},\ldots,z_j^{(q)})\mathrm{d}z_j^{(1)}\cdots\mathrm{d}z_j^{(q)} \qquad \text{(S4)}\\
&=\mathbb{E}_{\boldsymbol{\mu}^{(j)},\boldsymbol{\Sigma}^{(j)}}\left[\left(\min_{1\le k\le s}\left\{z_j^{(j_k)}\right\}-w_j\right)_+\right]\\
&=q\mathrm{MinEI}\left(w_j,\boldsymbol{\mu}_I^{(j)},\boldsymbol{\Sigma}_{I\times I}^{(j)}\right).
\end{aligned}$$

Then, $\hat{w}_j(I)$ is a strictly decreasing function of w_j and

$$\mathrm{d}\hat{w}_j(I)=-\left[\int_{\mathbb{R}^q}\begin{cases}\xi_{\boldsymbol{\mu}^{(j)},\boldsymbol{\Sigma}^{(j)}}(z_j^{(1)},\ldots,z_j^{(q)})\mathrm{d}z_j^{(1)}\cdots\mathrm{d}z_j^{(q)}, & \min_{1\le k\le s}\{z_j^{(j_k)}\}\ge w_j\\ 0, & \text{otherwise}\end{cases}\right]\mathrm{d}w_j. \qquad \text{(S5)}$$

Transforming the coordinate system in Equation (S3) from $\mathbf{w}$ to $\hat{\mathbf{w}}(I)$ for each $I\subset S$, we can obtain:

$$\begin{aligned}
q\mathrm{EHVI}(A;\boldsymbol{\mu},\boldsymbol{\Sigma})&=\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\int_{\hat{\Psi}(I)}\prod_{j=1}^{m}\mathrm{d}\hat{w}_j(I)\\
&=\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\lambda_m(\hat{\Psi}(I)) \qquad \text{(S6)}\\
&=\sum_{I=\{j_1,\ldots,j_s\}\subset S}(-1)^{|I|+1}\mathrm{HVI}(\{\hat{\mathbf{r}}(I)\},\hat{A}(I))
\end{aligned}$$

where $\hat{\Psi}(I):=[\mathbf{0},\hat{\mathbf{r}}(I)]\setminus\cup_{i=1}^{n}[\mathbf{0},\hat{\mathbf{f}}^{(i)}(I)]$ and $\hat{A}(I)=\{\hat{\mathbf{f}}^{(1)}(I),\ldots,\hat{\mathbf{f}}^{(n)}(I)\}$ with

$$\hat{\mathbf{r}}_j(I)=q\mathrm{MinEI}(0,\boldsymbol{\mu}_I^{(j)},\boldsymbol{\Sigma}_{I\times I}^{(j)}),\forall 1\le j\le m$$

and

$$\hat{\mathbf{f}}_j^{(i)}(I)=\hat{\mathbf{r}}_j(I)-q\mathrm{MinEI}(\mathbf{f}_j^{(i)},\boldsymbol{\mu}_I^{(j)},\boldsymbol{\Sigma}_{I\times I}^{(j)}),\forall 1\le i\le n,1\le j\le m.$$

$\square$

*A.2 Some Details for **Corollary 2** and **Corollary 4***

For **Corollary 2**, we provide an explicit construction of reducing an arbitrary HVI of one single point to an EHVI problem. Given an arbitrary HVI of one single point:

$$\begin{aligned}
\mathrm{HVI}(\{\mathbf{z}\},A)&=\lambda_m([\mathbf{0},\mathbf{z}]\setminus\cup_{i=1}^{n}[\mathbf{0},\mathbf{f}^{(n)}])\\
&=\lambda_m([\mathbf{0},\mathbf{z}]\setminus\cup_{i=1}^{n}\{[\mathbf{0},\mathbf{z}]\cap[\mathbf{0},\mathbf{f}^{(n)}]\}).
\end{aligned}$$

Define $\mathbf{f}_{|\mathbf{z}}^{(i)}:=(\min\{\mathbf{z}_1,\mathbf{f}_1^{(i)}\},\ldots,\min\{\mathbf{z}_m,\mathbf{f}_m^{(i)}\})$ as the element-wise minimum of $\mathbf{f}^{(i)}$ and $\mathbf{z}$ for $1\le i\le n$. Then, $(\mathbf{f}_{|\mathbf{z}}^{(i)})_j\le\mathbf{z}_j,\forall 1\le i\le n,1\le j\le m$, and

$$\mathrm{HVI}(\{\mathbf{z}\},A)=\mathrm{HVI}\left(\{\mathbf{z}\},\left\{\mathbf{f}_{|\mathbf{z}}^{(1)},\ldots,\mathbf{f}_{|\mathbf{z}}^{(n)}\right\}\right).$$

As a result, we can always construct an EHVI problem whose result is equal to the right hand side of the above equation, according to **Theorem 1** and the inverse of EI function. This suggests that an algorithm for EHVI leads to an algorithm for HVI of one single point with the same time complexity.

For **Corollary 4**, if treating q as a variable, based on **Lemma 1**, the cost of computing c.d.f's of k-variate in our formulation is of $O(mnT_k)$, where for all $1 \leq k \leq q$, $T_k := k \binom{q}{k} + k^2 \binom{q}{k+1}$ is the number of c.d.f.'s to be computed for k-variate normal distributions. The detailed values of T_k are listed as follows.

TABLE S1
THE ASYMPTOTIC VALUE OF T_k AS $q \to \infty$

k	1	2	$\cdots$	$\frac{q}{2}$	$\cdots$	$q-1$	q
T_k	q^2	q^3	$\cdots$	$q^{\frac{3}{2}}2^q$	$\cdots$	q^2	q

B. Experimental Results

We provide all the experimental results in Tables S2-S7. The Wilcoxon rank sum test is conducted on the running time with a significance level of 0.01, where the symbols '$+$', '$\approx$' and '$-$' denote that the result of other algorithms is significantly faster, statistically similar and significantly slower than our qEHVI-HVI, respectively. We have the following observations:

- Our qEHVI-HVI is much more efficient for computing EHVI and gradient of EHVI than DBB and qDBB-MC. The larger m and n are, the more significant its advantages become.
- For qEHVI ($q = 2$), our qEHVI-HVI is generally faster than qDBB-MC using 128 samples for $m \geq 3$.
- For qEHVI ($q = 4$), qEHVI-HVI is slower than qDBB-MC using 128 samples when $m \leq 5$, but faster for $m \geq 6$.
- For qEHVI ($q = 6$), qEHVI-HVI is significantly slower than qDBB-MC in many experiments, except the case that m is quite large ($m \geq 8$).
- The accuracy of qDBB-MC using 128 samples may not provide satisfactory results, especially for small values of m. It is recommended to use more samples when m is small.

TABLE S2

RESULTS OF COMPUTING EHVI AND GRADIENT OF EHVI IN THE CASE OF $n = 10$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '$+$', '$\approx$', '$-$' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE ERROR OF qDBB-MC IS THE AVERAGE RELATIVE ERROR BETWEEN qDBB-MC AND THE EXACT METHOD DBB, IN WHICH THE BRACKETS UNDER qDBB-MC IS THE NUMBER OF SAMPLES USED, WHILE THAT OF qEHVI-HVI IS THE MAXIMUM RELATIVE ERROR BETWEEN qEHVI-HVI AND DBB

d			2	3	4	5	6	7	8
EHVI	Time	qDBB-MC (128)	0.0040 (−) (0.0006)	0.0237 (−) (0.0024)	0.063 (−) (0.044)	0.169 (−) (0.010)	1.059 (−) (0.029)	6.794 (−) (0.061)	33.818 (−) (0.184)
		qDBB-MC (1280)	0.0043 (−) (0.0004)	0.0229 (−) (0.0028)	0.065 (−) (0.044)	0.200 (−) (0.029)	1.078 (−) (0.021)	6.861 (−) (0.183)	34.137 (−) (2.119)
		qDBB-MC (12800)	0.0061 (−) (0.0007)	0.0262 (−) (0.0027)	0.074 (−) (0.052)	0.216 (−) (0.012)	1.223 (−) (0.030)	7.231 (−) (0.089)	35.028 (−) (2.026)
		DBB	0.0027 (−) (0.0002)	0.020 (−) (0.002)	0.058 (−) (0.016)	0.203 (−) (0.018)	1.095 (−) (0.041)	6.982 (−) (0.066)	34.138 (−) (0.188)
		qEHVI-HVI	0.0023 (0.0006)	0.0054 (0.0004)	0.010 (0.001)	0.015 (0.003)	0.015 (0.002)	0.038 (0.004)	0.063 (0.007)
	Error	qDBB-MC (128)	0.2120 (0.1730)	0.0660 (0.0502)	0.060 (0.059)	0.034 (0.028)	0.030 (0.021)	0.031 (0.029)	0.017 (0.016)
		qDBB-MC (1280)	0.0269 (0.0239)	0.0092 (0.0078)	0.007 (0.006)	0.006 (0.005)	0.005 (0.003)	0.003 (0.002)	0.002 (0.002)
		qDBB-MC (12800)	0.0033 (0.0024)	0.0009 (0.0006)	0.001 (0.001)	0.001 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
		qEHVI-HVI	<2e-14	<1e-14	<1e-14	<1e-14	<1e-14	<2e-14	<3e-14
Gradient of EHVI	Time	qDBB-MC (128)	0.0054 (−) (0.0007)	0.0244 (−) (0.0017)	0.061 (−) (0.007)	0.186 (−) (0.009)	1.074 (−) (0.026)	6.859 (−) (0.053)	33.873 (−) (0.146)
		qDBB-MC (1280)	0.0076 (−) (0.0012)	0.0279 (−) (0.0021)	0.079 (−) (0.014)	0.198 (−) (0.011)	1.155 (−) (0.026)	7.130 (−) (0.221)	34.356 (−) (0.189)
		qDBB-MC (12800)	0.0127 (−) (0.0010)	0.0393 (−) (0.0019)	0.127 (−) (0.021)	0.364 (−) (0.021)	2.022 (−) (0.035)	9.708 (−) (0.320)	57.007 (−) (1.583)
		DBB	0.0040 (−) (0.0003)	0.022 (−) (0.002)	0.083 (−) (0.015)	0.180 (−) (0.032)	1.139 (−) (0.025)	7.134 (−) (0.112)	34.638 (−) (0.141)
		qEHVI-HVI	0.0036 (0.0003)	0.0091 (0.0006)	0.034 (0.003)	0.067 (0.025)	0.108 (0.018)	0.185 (0.070)	0.441 (0.192)
	Error	qDBB-MC (128)	0.2274 (0.1310)	0.0886 (0.0527)	0.049 (0.028)	0.085 (0.041)	0.042 (0.021)	0.033 (0.020)	0.025 (0.011)
		qDBB-MC (1280)	0.0287 (0.0176)	0.0123 (0.0068)	0.008 (0.005)	0.014 (0.008)	0.006 (0.003)	0.005 (0.003)	0.003 (0.002)
		qDBB-MC (12800)	0.0038 (0.0022)	0.0014 (0.0008)	0.001 (0.000)	0.002 (0.001)	0.001 (0.000)	0.001 (0.000)	0.000 (0.000)
		qEHVI-HVI	<7e-15	<6e-15	<6e-15	<2e-14	<7e-15	<1e-14	<7e-15

TABLE S3

RESULTS OF COMPUTING EHVI IN THE CASE OF $n > 10$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '+', '≈', '−' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE ERROR OF qDBB-MC IS THE AVERAGE RELATIVE ERROR BETWEEN qDBB-MC AND THE EXACT METHOD DBB, WHILE THAT OF qEHVI-HVI IS THE MAXIMUM RELATIVE ERROR BETWEEN qEHVI-HVI AND DBB. TESTED ALGORITHMS ARE TERMINATED IF THEY SPEND MORE THAN 100 SECONDS. IN THIS CASE, THE RELATIVE ERROR IS NOT APPLICABLE

d			2	3	4	5	6	7	8
$n = 20$	Time	DBB	0.0035 (−) (0.0005)	0.033 (−) (0.003)	0.120 (−) (0.009)	0.536 (−) (0.010)	3.992 (−) (0.237)	42.866 (−) (1.370)	>100
		qDBB-MC	0.0026 (−) (0.0004)	0.0321 (−) (0.0025)	0.117 (−) (0.004)	0.532 (−) (0.012)	3.945 (−) (0.162)	42.806 (−) (1.309)	>100
		qEHVI-HVI	0.0023 (0.0002)	0.0075 (0.0008)	0.015 (0.001)	0.032 (0.002)	0.053 (0.013)	0.164 (0.046)	0.455 (0.080)
	Error	qDBB-MC	0.3702 (0.0189)	0.0715 (0.0040)	0.028 (0.003)	0.067 (0.007)	0.074 (0.005)	0.012 (0.003)	N/A
		qEHVI-HVI	<2e-14	<6e-15	<2e-14	<1e-14	<3e-14	<3e-14	N/A
$n = 40$	Time	DBB	0.0036 (−) (0.0005)	0.063 (−) (0.003)	0.236 (−) (0.008)	1.671 (−) (0.049)	20.970 (−) (0.678)	>100	>100
		qDBB-MC	0.0026 (−) (0.0002)	0.0610 (−) (0.0027)	0.236 (−) (0.012)	1.665 (−) (0.043)	21.073 (−) (0.560)	>100	>100
		qEHVI-HVI	0.0024 (0.0003)	0.0121 (0.0008)	0.029 (0.002)	0.087 (0.012)	0.227 (0.026)	0.713 (0.091)	2.132 (0.245)
	Error	qDBB-MC	0.2326 (0.0251)	0.0708 (0.0037)	0.029 (0.004)	0.067 (0.007)	0.074 (0.005)	N/A	N/A
		qEHVI-HVI	<1e-14	<9e-15	<2e-14	<1e-14	<2e-14	N/A	N/A
$n = 80$	Time	DBB	0.0041 (−) (0.0008)	0.126 (−) (0.006)	0.579 (−) (0.016)	4.445 (−) (0.081)	>100	>100	>100
		qDBB-MC	0.0030 (−) (0.0004)	0.1245 (−) (0.0061)	0.578 (−) (0.021)	4.432 (−) (0.083)	>100	>100	>100
		qEHVI-HVI	0.0026 (0.0004)	0.0204 (0.0015)	0.057 (0.003)	0.202 (0.008)	0.828 (0.046)	3.277 (0.224)	6.749 (0.262)
	Error	qDBB-MC	0.2599 (0.0186)	0.0711 (0.0040)	0.029 (0.005)	0.068 (0.006)	N/A	N/A	N/A
		qEHVI-HVI	<1e-14	<7e-15	<1e-14	<1e-14	N/A	N/A	N/A

TABLE S4

RESULTS OF COMPUTING GRADIENT OF EHVI IN THE CASE OF $n > 10$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '+', '≈', '−' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE ERROR OF qDBB-MC IS THE AVERAGE RELATIVE ERROR OF THE DISTANCE BETWEEN GRADIENTS PROVIDED BY qDBB-MC AND THE EXACT METHOD DBB, WHILE THAT OF qEHVI-HVI IS THE MAXIMUM RELATIVE ERROR BETWEEN qEHVI-HVI AND DBB. TESTED ALGORITHMS ARE TERMINATED IF THEY SPEND MORE THAN 100 SECONDS. IN THIS CASE, THE RELATIVE ERROR IS NOT APPLICABLE

d			2	3	4	5	6	7	8
$n = 20$	Time	DBB	0.0050 (−) (0.0005)	0.0369 (−) (0.0019)	0.124 (−) (0.004)	0.565 (−) (0.044)	4.046 (−) (0.160)	43.344 (−) (0.620)	> 100
		qDBB-MC	0.0037 (−) (0.0003)	0.0350 (−) (0.0022)	0.119 (−) (0.003)	0.559 (−) (0.053)	4.017 (−) (0.177)	43.765 (−) (1.699)	> 100
		qEHVI-HVI	0.0035 (0.0004)	0.0144 (0.0011)	0.066 (0.004)	0.158 (0.021)	0.397 (0.053)	0.878 (0.073)	2.140 (1.449)
	Error	qDBB-MC	0.3137 (0.1618)	0.0923 (0.0361)	0.043 (0.017)	0.034 (0.016)	0.057 (0.026)	0.027 (0.014)	N/A
		qEHVI-HVI	<8e-15	<6e-15	<1e-14	<1e-14	<1e-14	<1e-14	N/A
$n = 40$	Time	DBB	0.0054 (−) (0.0006)	0.0661 (−) (0.0020)	0.246 (−) (0.008)	1.828 (−) (0.233)	21.462 (−) (0.684)	> 100	> 100
		qDBB-MC	0.0039 (−) (0.0005)	0.0620 (−) (0.0010)	0.240 (−) (0.008)	1.747 (−) (0.158)	21.476 (−) (0.703)	> 100	> 100
		qEHVI-HVI	0.0037 (0.0004)	0.0276 (0.0007)	0.146 (0.019)	0.479 (0.045)	1.634 (0.117)	3.392 (0.125)	9.082 (0.432)
	Error	qDBB-MC	0.3414 (0.1463)	0.1002 (0.0390)	0.043 (0.019)	0.037 (0.020)	0.054 (0.028)	N/A	N/A
		qEHVI-HVI	<1e-14	<1e-14	<1e-14	<1e-14	<2e-14	N/A	N/A
$n = 80$	Time	DBB	0.0062 (−) (0.0006)	0.1299 (−) (0.0055)	0.670 (−) (0.141)	4.978 (−) (0.500)	> 100	> 100	> 100
		qDBB-MC	0.0041 (−) (0.0004)	0.1237 (−) (0.0044)	0.651 (−) (0.131)	5.207 (−) (0.504)	> 100	> 100	> 100
		qEHVI-HVI	0.0038 (0.0003)	0.0744 (0.0156)	0.392 (0.057)	1.333 (0.212)	8.844 (5.320)	25.066 (8.029)	73.635 (6.348)
	Error	qDBB-MC	0.3439 (0.1600)	0.0942 (0.0390)	0.042 (0.018)	0.038 (0.013)	N/A	N/A	N/A
		qEHVI-HVI	<6e-15	<2e-14	<3e-14	<2e-14	N/A	N/A	N/A

TABLE S5

RESULTS OF COMPUTING qEHVI IN THE CASE OF $q = 2$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '$+$', '$\approx$', '$-$' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE BRACKETS UNDER qDBB-MC IS THE NUMBER OF SAMPLES USED. THE ERROR OF qDBB-MC IS THE RELATIVE ERROR BETWEEN RESULTS OF qDBB-MC AND qEHVI-HVI. TESTED ALGORITHMS ARE TERMINATED IF THEY SPEND MORE THAN 100 SECONDS OR THEY REQUIRE TOO MUCH MEMORY. IN THIS CASE, THE RELATIVE ERROR IS NOT APPLICABLE

d			2	3	4	5	6	7	8
$n = 10$	Time	qDBB-MC (128)	0.0050 (+) (0.0046)	0.0213 (+) (0.0015)	0.055 (−) (0.004)	0.172 (−) (0.038)	1.196 (−) (0.025)	7.421 (−) (0.253)	34.092 (−) (0.409)
		qDBB-MC (1280)	0.0129 (−) (0.0047)	0.0242 (+) (0.0021)	0.061 (−) (0.007)	0.191 (−) (0.020)	1.153 (−) (0.022)	7.359 (−) (0.180)	35.229 (−) (0.426)
		qDBB-MC (12800)	0.0127 (−) (0.0013)	0.0400 (−) (0.0055)	0.089 (−) (0.007)	0.245 (−) (0.006)	1.619 (−) (0.032)	8.584 (−) (0.114)	38.421 (−) (1.128)
		qEHVI-HVI	0.0118 (0.0018)	0.0247 (0.0014)	0.037 (0.002)	0.066 (0.004)	0.090 (0.009)	0.109 (0.017)	0.220 (0.042)
	Error	qDBB-MC (128)	0.1373 (0.1037)	0.0514 (0.0144)	0.119 (0.019)	0.037 (0.016)	0.034 (0.024)	0.021 (0.015)	0.022 (0.006)
		qDBB-MC (1280)	0.0250 (0.0121)	0.0084 (0.0064)	0.005 (0.002)	0.003 (0.001)	0.004 (0.003)	0.003 (0.002)	0.004 (0.003)
		qDBB-MC (12800)	0.0033 (0.0010)	0.0011 (0.0006)	0.001 (0.000)	0.001 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
$n = 20$	Time	qDBB-MC (128)	0.0045 (+) (0.0004)	0.0346 (≈) (0.0029)	0.122 (−) (0.005)	0.560 (−) (0.050)	3.843 (−) (0.069)	41.714 (−) (0.541)	>100
		qDBB-MC (1280)	0.0071 (+) (0.0006)	0.0397 (−) (0.0043)	0.135 (−) (0.007)	0.602 (−) (0.024)	3.927 (−) (0.055)	41.282 (−) (0.193)	>100
		qDBB-MC (12800)	0.0168 (−) (0.0019)	0.0686 (−) (0.0101)	0.233 (−) (0.010)	0.880 (−) (0.030)	5.242 (−) (0.047)	Out of memory	>100
		qEHVI-HVI	0.0123 (0.0023)	0.0347 (0.0035)	0.066 (0.004)	0.113 (0.005)	0.217 (0.038)	0.478 (0.048)	1.210 (0.157)
	Error	qDBB-MC (128)	0.1285 (0.1053)	0.0510 (0.0158)	0.119 (0.016)	0.037 (0.017)	0.030 (0.019)	0.036 (0.007)	N/A
		qDBB-MC (1280)	0.0251 (0.0124)	0.0098 (0.0079)	0.004 (0.002)	0.003 (0.001)	0.004 (0.003)	0.005 (0.003)	N/A
		qDBB-MC (12800)	0.0033 (0.0013)	0.0012 (0.0007)	0.001 (0.000)	0.001 (0.000)	0.000 (0.000)	N/A	N/A
$n = 40$	Time	qDBB-MC (128)	0.0049 (+) (0.0004)	0.0690 (−) (0.0069)	0.263 (−) (0.043)	1.725 (−) (0.115)	20.597 (−) (0.688)	>100	>100
		qDBB-MC (1280)	0.0087 (+) (0.0009)	0.0724 (−) (0.0034)	0.273 (−) (0.014)	1.822 (−) (0.060)	21.961 (−) (2.388)	>100	>100
		qDBB-MC (12800)	0.0238 (−) (0.0038)	0.1210 (−) (0.0070)	0.448 (−) (0.015)	3.343 (−) (0.132)	Out of memory	>100	>100
		qEHVI-HVI	0.0123 (0.0017)	0.0482 (0.0021)	0.116 (0.009)	0.345 (0.021)	0.864 (0.221)	2.442 (0.301)	4.892 (0.619)
	Error	qDBB-MC (128)	0.1436 (0.1151)	0.0498 (0.0149)	0.118 (0.017)	0.038 (0.014)	0.021 (0.006)	N/A	N/A
		qDBB-MC (1280)	0.0233 (0.0110)	0.0080 (0.0062)	0.004 (0.002)	0.003 (0.001)	0.006 (0.005)	N/A	N/A
		qDBB-MC (12800)	0.0035 (0.0011)	0.0013 (0.0006)	0.001 (0.000)	0.001 (0.000)	N/A	N/A	N/A
$n = 80$	Time	qDBB-MC (128)	0.0060 (+) (0.0005)	0.1272 (−) (0.0062)	0.607 (−) (0.056)	4.747 (−) (0.349)	>100	>100	>100
		qDBB-MC (1280)	0.0107 (+) (0.0007)	0.1400 (−) (0.0068)	0.661 (−) (0.036)	4.848 (−) (0.111)	>100	>100	>100
		qDBB-MC (12800)	0.0395 (−) (0.0046)	0.2396 (−) (0.0120)	1.211 (−) (0.043)	13.985 (−) (1.718)	>100	>100	>100
		qEHVI-HVI	0.0155 (0.0055)	0.0842 (0.0057)	0.213 (0.021)	1.034 (0.248)	2.735 (0.323)	10.564 (4.881)	22.249 (1.760)
	Error	qDBB-MC (128)	0.1503 (0.1098)	0.0522 (0.0195)	0.114 (0.018)	0.033 (0.019)	N/A	N/A	N/A
		qDBB-MC (1280)	0.0233 (0.0116)	0.0101 (0.0062)	0.005 (0.002)	0.003 (0.001)	N/A	N/A	N/A
		qDBB-MC (12800)	0.0037 (0.0012)	0.0011 (0.0007)	0.001 (0.000)	0.001 (0.000)	N/A	N/A	N/A

TABLE S6

RESULTS OF COMPUTING qEHVI IN THE CASE OF $q = 4$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '$+$', '$\approx$', '$-$' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE BRACKETS UNDER qDBB-MC IS THE NUMBER OF SAMPLES USED. THE ERROR OF qDBB-MC IS THE RELATIVE ERROR BETWEEN RESULTS OF qDBB-MC AND qEHVI-HVI. TESTED ALGORITHMS ARE TERMINATED IF THEY SPEND MORE THAN 100 SECONDS OR THEY REQUIRE TOO MUCH MEMORY. IN THIS CASE, THE RELATIVE ERROR IS NOT APPLICABLE

d			2	3	4	5	6	7	8
$n = 10$	Time	qDBB-MC (128)	0.0060 (+) (0.0007)	0.0239 (+) (0.0046)	0.056 (+) (0.002)	0.165 (+) (0.007)	1.138 (+) (0.015)	6.903 (−) (0.094)	34.296 (−) (0.493)
		qDBB-MC (1280)	0.0188 (+) (0.0018)	0.0501 (+) (0.0046)	0.121 (+) (0.007)	0.316 (+) (0.023)	1.257 (−) (0.024)	7.772 (−) (0.199)	35.332 (−) (0.246)
		qDBB-MC (12800)	0.0480 (+) (0.0042)	0.0952 (+) (0.0045)	0.221 (+) (0.010)	0.707 (+) (0.022)	3.178 (−) (0.043)	61.591 (−) (4.819)	Out of memory
		qEHVI-HVI	0.4135 (0.0239)	0.6607 (0.0369)	0.844 (0.048)	1.038 (0.057)	1.212 (0.074)	1.967 (0.115)	2.468 (0.139)
	Error	qDBB-MC (128)	0.0449 (0.0377)	0.0486 (0.0308)	0.057 (0.031)	0.014 (0.011)	0.029 (0.023)	0.012 (0.009)	0.013 (0.010)
		qDBB-MC (1280)	0.0250 (0.0196)	0.0104 (0.0073)	0.011 (0.008)	0.005 (0.004)	0.004 (0.003)	0.005 (0.003)	0.005 (0.004)
		qDBB-MC (12800)	0.0047 (0.0031)	0.0019 (0.0012)	0.001 (0.001)	0.001 (0.001)	0.001 (0.001)	0.001 (0.001)	N/A
$n = 20$	Time	qDBB-MC (128)	0.0071 (+) (0.0006)	0.0377 (+) (0.0022)	0.129 (+) (0.010)	0.571 (+) (0.035)	4.027 (−) (0.147)	41.695 (−) (0.489)	>100
		qDBB-MC (1280)	0.0272 (+) (0.0024)	0.0822 (+) (0.0035)	0.258 (+) (0.015)	0.845 (+) (0.027)	5.003 (−) (0.356)	59.128 (−) (41.148)	>100
		qDBB-MC (12800)	0.0663 (+) (0.0044)	0.1542 (+) (0.0055)	0.624 (+) (0.024)	2.660 (−) (0.045)	86.728 (−) (6.273)	Out of memory	>100
		qEHVI-HVI	0.5108 (0.0185)	0.8839 (0.0794)	1.253 (0.081)	1.797 (0.146)	3.231 (0.200)	4.234 (2.181)	8.994 (0.343)
	Error	qDBB-MC (128)	0.0502 (0.0445)	0.0530 (0.0364)	0.059 (0.034)	0.013 (0.009)	0.013 (0.011)	0.013 (0.009)	N/A
		qDBB-MC (1280)	0.0237 (0.0199)	0.0140 (0.0123)	0.009 (0.007)	0.006 (0.005)	0.005 (0.004)	0.005 (0.004)	N/A
		qDBB-MC (12800)	0.0036 (0.0025)	0.0018 (0.0017)	0.002 (0.001)	0.001 (0.001)	0.001 (0.001)	N/A	N/A
$n = 40$	Time	qDBB-MC (128)	0.0081 (+) (0.0006)	0.0707 (+) (0.0102)	0.254 (+) (0.014)	1.761 (+) (0.058)	20.490 (−) (0.516)	>100	>100
		qDBB-MC (1280)	0.0401 (+) (0.0033)	0.1434 (+) (0.0047)	0.438 (+) (0.027)	3.021 (+) (1.607)	25.012 (−) (0.943)	>100	>100
		qDBB-MC (12800)	0.0981 (+) (0.0033)	0.2743 (+) (0.0071)	1.360 (+) (0.039)	26.916 (−) (5.220)	Out of memory	>100	>100
		qEHVI-HVI	0.6943 (0.0376)	1.3151 (0.1547)	1.988 (0.194)	3.916 (1.927)	6.756 (0.568)	13.444 (0.551)	33.460 (1.943)
	Error	qDBB-MC (128)	0.0507 (0.0424)	0.0616 (0.0316)	0.054 (0.030)	0.012 (0.010)	0.017 (0.011)	N/A	N/A
		qDBB-MC (1280)	0.0227 (0.0190)	0.0112 (0.0086)	0.007 (0.006)	0.008 (0.007)	0.003 (0.003)	N/A	N/A
		qDBB-MC (12800)	0.0034 (0.0032)	0.0018 (0.0013)	0.002 (0.001)	0.001 (0.001)	N/A	N/A	N/A
$n = 80$	Time	qDBB-MC (128)	0.0104 (+) (0.0008)	0.1349 (+) (0.0068)	0.641 (+) (0.074)	4.754 (+) (0.272)	>100	>100	>100
		qDBB-MC (1280)	0.0665 (+) (0.0040)	0.2467 (+) (0.0107)	0.995 (+) (0.055)	6.746 (+) (3.363)	>100	>100	>100
		qDBB-MC (12800)	0.1716 (+) (0.0045)	0.6833 (+) (0.0265)	3.486 (≈) (0.061)	Out of memory	>100	>100	>100
		qEHVI-HVI	1.0711 (0.0912)	2.2128 (0.3439)	3.663 (0.400)	8.026 (3.073)	17.757 (0.971)	52.321 (1.360)	>100
	Error	qDBB-MC (128)	0.0545 (0.0457)	0.0645 (0.0364)	0.063 (0.030)	0.014 (0.011)	N/A	N/A	N/A
		qDBB-MC (1280)	0.0253 (0.0203)	0.0121 (0.0092)	0.010 (0.008)	0.005 (0.005)	N/A	N/A	N/A
		qDBB-MC (12800)	0.0038 (0.0031)	0.0017 (0.0015)	0.001 (0.001)	N/A	N/A	N/A	N/A

TABLE S7

RESULTS OF COMPUTING qEHVI IN THE CASE OF $q = 6$. THE AVERAGE RUNNING TIME (IN SECONDS) AND STANDARD DEVIATION (IN BRACKETS BELOW) ARE SUMMERIZED. THE SYMBOLS '+', '≈', '−' (IN BRACKETS ON THE RIGHT) MEAN THAT ONE OTHER ALGORITHM IS SIGNIFICANTLY FASTER, STATISTICALLY SIMILAR AND SIGNIFICANTLY SLOWER THAN qEHVI-HVI, RESPECTIVELY, UNDER THE WILCOXON RANK SUM TEST WITH A SIGNIFICANCE LEVEL OF 0.01. THE BRACKETS UNDER qDBB-MC IS THE NUMBER OF SAMPLES USED. THE ERROR OF qDBB-MC IS THE RELATIVE ERROR BETWEEN RESULTS OF qDBB-MC AND qEHVI-HVI. TESTED ALGORITHMS ARE TERMINATED IF THEY SPEND MORE THAN 100 SECONDS OR THEY REQUIRE TOO MUCH MEMORY. IN THIS CASE, THE RELATIVE ERROR IS NOT APPLICABLE

d			2	3	4	5	6	7	8
$n = 10$	Time	qDBB-MC (128)	0.0110 (+) (0.0023)	0.0319 (+) (0.0024)	0.117 (+) (0.012)	0.223 (+) (0.019)	1.151 (+) (0.067)	6.918 (+) (0.048)	36.437 (−) (18.344)
		qDBB-MC (1280)	0.0279 (+) (0.0032)	0.0614 (+) (0.0048)	0.123 (+) (0.004)	0.327 (+) (0.014)	1.871 (+) (0.038)	Out of memory	Out of memory
		qDBB-MC (12800)	0.1051 (+) (0.0029)	0.2558 (+) (0.0208)	0.629 (+) (0.019)	2.033 (+) (0.069)	88.911 (−) (5.454)	Out of memory	Out of memory
		qEHVI-HVI	8.9377 (1.1694)	14.4453 (0.9963)	17.047 (1.240)	21.349 (1.030)	27.029 (2.252)	28.260 (1.966)	33.133 (9.638)
	Error	qDBB-MC (128)	0.1737 (0.1531)	0.0608 (0.0396)	0.042 (0.033)	0.035 (0.023)	0.023 (0.019)	0.027 (0.022)	0.016 (0.012)
		qDBB-MC (1280)	0.0297 (0.0232)	0.0147 (0.0085)	0.008 (0.007)	0.006 (0.005)	0.004 (0.003)	N/A	N/A
		qDBB-MC (12800)	0.0051 (0.0039)	0.0021 (0.0014)	0.002 (0.001)	0.001 (0.001)	0.001 (0.001)	N/A	N/A
$n = 20$	Time	qDBB-MC (128)	0.0141 (+) (0.0011)	0.0637 (+) (0.0042)	0.190 (+) (0.021)	0.669 (+) (0.016)	4.100 (+) (0.039)	43.711 (+) (5.800)	>100
		qDBB-MC (1280)	0.0405 (+) (0.0037)	0.1009 (+) (0.0090)	0.282 (+) (0.012)	1.153 (+) (0.020)	6.644 (+) (0.120)	Out of memory	>100
		qDBB-MC (12800)	0.1597 (+) (0.0037)	0.4484 (+) (0.0344)	2.412 (+) (0.112)	Out of memory	Out of memory	Out of memory	>100
		qEHVI-HVI	11.9476 (1.3227)	19.4322 (1.3155)	25.466 (2.103)	31.210 (2.612)	46.463 (4.861)	54.649 (10.433)	71.731 (4.583)
	Error	qDBB-MC (128)	0.1881 (0.1244)	0.0779 (0.0535)	0.036 (0.029)	0.031 (0.027)	0.024 (0.017)	0.015 (0.009)	N/A
		qDBB-MC (1280)	0.0340 (0.0237)	0.0132 (0.0086)	0.007 (0.006)	0.007 (0.005)	0.004 (0.003)	N/A	N/A
		qDBB-MC (12800)	0.0058 (0.0049)	0.0023 (0.0020)	0.002 (0.001)	N/A	N/A	N/A	N/A
$n = 40$	Time	qDBB-MC (128)	0.0188 (+) (0.0050)	0.1332 (+) (0.0693)	0.327 (+) (0.015)	1.943 (+) (0.064)	20.821 (+) (0.158)	>100	>100
		qDBB-MC (1280)	0.0622 (+) (0.0138)	0.2009 (+) (0.0932)	0.544 (+) (0.013)	3.830 (+) (0.209)	Out of memory	>100	>100
		qDBB-MC (12800)	0.3006 (+) (0.1072)	1.1282 (+) (0.5041)	4.165 (+) (0.561)	Out of memory	Out of memory	>100	>100
		qEHVI-HVI	18.6394 (4.4472)	31.6869 (4.3929)	40.343 (3.328)	50.573 (4.313)	74.137 (9.218)	>100	>100
	Error	qDBB-MC (128)	0.1496 (0.1294)	0.0649 (0.0502)	0.050 (0.037)	0.033 (0.022)	0.028 (0.020)	N/A	N/A
		qDBB-MC (1280)	0.0307 (0.0236)	0.0110 (0.0093)	0.008 (0.005)	0.005 (0.004)	N/A	N/A	N/A
		qDBB-MC (12800)	0.0056 (0.0036)	0.0023 (0.0013)	0.002 (0.002)	N/A	N/A	N/A	N/A
$n = 80$	Time	qDBB-MC (128)	0.0218 (+) (0.0017)	0.1598 (+) (0.0109)	0.799 (+) (0.316)	5.196 (+) (0.129)	>100	>100	>100
		qDBB-MC (1280)	0.1457 (+) (0.0131)	0.3933 (+) (0.0222)	1.670 (+) (0.706)	17.004 (+) (2.571)	>100	>100	>100
		qDBB-MC (12800)	0.6428 (+) (0.0193)	2.3412 (+) (0.0962)	58.120 (+) (3.613)	Out of memory	>100	>100	>100
		qEHVI-HVI	31.0208 (3.1915)	47.9797 (4.9003)	73.826 (7.427)	>100	>100	>100	>100
	Error	qDBB-MC (128)	0.0726 (0.0630)	0.0564 (0.0470)	0.044 (0.040)	N/A	N/A	N/A	N/A
		qDBB-MC (1280)	0.0256 (0.0163)	0.0176 (0.0141)	0.008 (0.007)	N/A	N/A	N/A	N/A
		qDBB-MC (12800)	0.0045 (0.0040)	0.0024 (0.0016)	0.002 (0.001)	N/A	N/A	N/A	N/A