2022 Yulia's Dream entrance problem set

Due April 12, 2022

Program description. Yulia's Dream is a math enrichment and research program for exceptional high school students (grades 9-11) from Ukraine. Under this program, students would meet online once a week in small groups to study advanced math topics beyond high school curriculum or work on math research projects under the guidance of academic mentors from among MIT Math Department graduate students and MIT math majors. The instruction will be available in Ukrainian, English and Russian.

Yulia's Dream is an initiative under PRIMES – Program for Research in Mathematics, Engineering and Science for High School Students at the Massachusetts Institute of Technology. The working groups at Yulia's Dream will operate similarly to those at PRIMES-USA, the remote section of PRIMES.

Yulia's Dream is dedicated to the memory of Yulia Zdanovska, a 21-year-old graduate of the National University of Kyiv, a silver medalist at the 2017 European Girls' Mathematical Olympiad, and a teacher for the "Teach for Ukraine" program who was killed by a Russian-fired missile in her home city of Kharkiv. We hope to help other Ukrainian boys and girls fulfill her dream.

General advice. Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps over several days. Try to solve as many problems as you can, but we encourage you to apply if you have solved at least three problems if you are in 10th/11th grade, and at least two if you are in 9th grade, in whole or in part.

Велику частину програми займає вивчення наукових статей, які написані англійською мовою. Більшість з них не перекладені ніякою іншою мовою. Через це ми вважаємо необхідним для участі у програмі базове розуміння математичної англійської школярами, і тому вирішили не перекладати умови вступного завдання українською мовою. Якщо у Вас є будь-які запитання щодо перекладу, ми будемо раді відповісти за нашою контактною e-mail адресою yuliasdream@mit.edu

Важной частью программы является изучение англоязычных математических текстов. Большинство из них не переведены ни на какой другой язык. Поэтому мы считаем необходимым для участия в программе базовое понимание математического английского школьниками, и решили не переводить условия задач на русский язык. Если у Вас есть какие-либо вопросы насчет условий, пишите на наш контактный e-mail адрес yuliasdream@mit.edu и мы будем рады Вам ответить.

Solutions format. You may write your solutions in Ukrainian, Russian, or English - whichever language you choose.

You can type the solutions or write them up by hand and then scan them or take a picture. Please save your solutions as a file (preferably, PDF), upload it to the Web, and submit the link along with your application. Make sure the file is accessible by link and does not require special permission to access. The name of the attached file must start with your last name, for example, "lastname-solutions.pdf". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted.

PDFs produced by LaTeX are preferred, but PDFs produced by Word and scans of handwritten solutions are also accepted.

Academic integrity. You are allowed to use any resources to solve these problems, except other people's help. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Contact. If you have any questions, please contact Yulia's Dream program coordinator Dmytro Matvieievskyi at yuliasdream@mit.edu

Problems

1. Consider a plane passing through the midpoints of two opposite edges of a regular tetrahedron. The projection of the tetrahedron to this plane is a quadrilateral of area A with one of the angles $60°$. Find the surface area of the tetrahedron.

2. For an m-digit number A and $(n-m)$-digit number B let $A \circ B$ be the n-digit number obtained by concatenation of A and B (where we allow the leftmost digit to be zero). For example, if $m = 2, n = 5$, $A = 23$, $B = 045$, then $A \circ B = 23045$ and $B \circ A = 04523$.

From now on assume that $m = 2$. Let k be a 2-digit number, and consider the equation

$$\frac{B \circ A}{A \circ B} = k$$

with $A > 0$ and any $n \geq 3$. It is clear that if $X := A \circ B$ is a solution of this equation then so is $X \circ X$, $X \circ X \circ X$, etc. We say that a solution X

is *primitive* if it is not obtained in this way, by concatenating a smaller solution with itself several times.

(i) Find all primitive solutions for $k = 09$ and $k = 15$.

(ii) Describe all primitive solutions for a general 2-digit number k. Are there finitely many?

3. Let m be a fixed positive integer, and consider the following game. At each move, you pick uniformly at random an integer $0 \le k \le m$. Then you score k points, but only if k does not exceed the smallest previously picked number (otherwise you don't score any points on that move). For example, if $m = 3$ and your random numbers are $2, 3, 1, 2, 1, 0, 3, \ldots$ then you score only on the 1st, 3rd and 5th move and don't score anything after the 5th move, so you total score is $2 + 1 + 1 = 4$.

(i) How much will you score on average if you play indefinitely?

(ii) Let $a(n, m)$ be the average amount you score in n steps. Find an explicit formula for $a(n, 1)$ and $a(n, 2)$.

(iii) Find an explicit formula for $a(n, m)$.

4. A street is lit by n street lights arranged in a row ($n \ge 2$). Eventually the lights start to burn out, one at a time, in random order. As long as for each broken light, its neighbors are still working[1], the Department of Public Works (DPW) does not do anything. However, once two consecutive lights are out, the DPW immediately replaces the light bulbs in all broken lights. For example:

$$\circ \quad \circ \quad \circ \quad \circ \quad \circ$$

$$\circ \quad \circ \quad \bullet \quad \circ \quad \circ$$

do nothing

$$\circ \quad \circ \quad \bullet \quad \circ \quad \bullet$$

still do nothing

$$\circ \quad \bullet \quad \bullet \quad \circ \quad \bullet$$

replace all broken lights

(i) What is the chance that the DPW will have to replace k lights, if lights break independently and with equal probability?

(ii) What is the average number of lights that they have to replace in each repair?

Compute the answers for general n, k, then compute them for $n = 9, k = 4$ with two digits precision after the decimal point.

[1] Both neighbors, or only one for the first or the last light.

5. (i) Describe an algorithm to find the closed ball (disk) of smallest radius containing a given finite set of points (x_i, y_i), $i = 1, ..., n$, in $\mathbb{R}^2$.

(ii) Do the same for points (x_i, y_i, z_i), $i = 1, ..., n$, in $\mathbb{R}^3$.

(iii) Show that the ball in (i),(ii) is unique.