COSMOS Cluster 3
Homework 4
UC Davis

Contents

1 Mass Spring System

1.1 Mass-Spring System

First recall from physics Newton's second law of motion

$$m\frac{d^2y}{dt^2} = F(y,t)$$

where F denotes the sum of all forces acting on the objective mass. In a mass-spring system we want to consider three forces: restoring, damping and external. We define these respectively as:

$$F_{restoring} = -ky,\ F_{damping} = -c\frac{dy}{\Delta t},\ F_{external} = f(t)$$

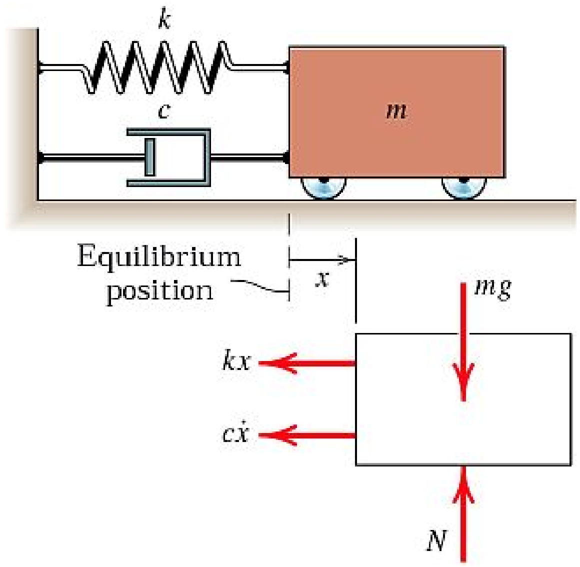
Taking these forces together in $F(t)$ we get a governing equation for the mass-spring system

$$m\frac{d^2y}{dt^2} = -c\frac{dy}{dt} - ky + f(t)$$

which is typically rearranged in the classical way

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f(t).$$

It is a second order, constant coefficient, non-homogeneous linear differential equation. Physically, it really only makes sense to consider cases: $m > 0, k > 0, b \geq 0$. Mathematically one can consider other parameter values. Additionally, m, c, k can all be considered as functions of space or time and this leads to interesting examples. If $f(t) = 0$, the dynamics are unforced, we then say it is homogeneous.



In modeling vibration of building foundations, vehicle suspension, seismometers and accelerometers, a damped forced vibration arises. In the event some vibration motion is forcing the system, the model becomes a forced-mass-spring system of the form

$$m\frac{d^2y}{dt^2} + ky + c\frac{dy}{dt} = \text{Vibration Force.}$$

The vibration force can take many forms depending on the its source, but the most essential to understand first is called harmonic forcing, like that introduced by a regular and periodic motion. In this case the equation of motion becomes

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + kx = F\sin(\omega t)$$

One solves this equation by finding:

$$y(t) = y_{homogeneous} + y_{particular} = y_h + y_p$$

First we set:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

and find $y_h(t)$ depending on whether the characteristic equation has real unequal (overdamped), real repeated (critically damped), or imaginery (underdamped) roots. Then depending on the forcing function $f(t)$ we use the method of undetermined coefficients to find y_p. It is typically several weeks worth of a first course in differential equations to learn to do this.

1.2 Stability Analysis*

Start by guessing a solution of the form $y = e^{\lambda t}$. Then, when $f(y,t) = 0$ it follows

$$m\lambda^2 + c\lambda + k = 0$$

and by the quadratic equation, the solutions are

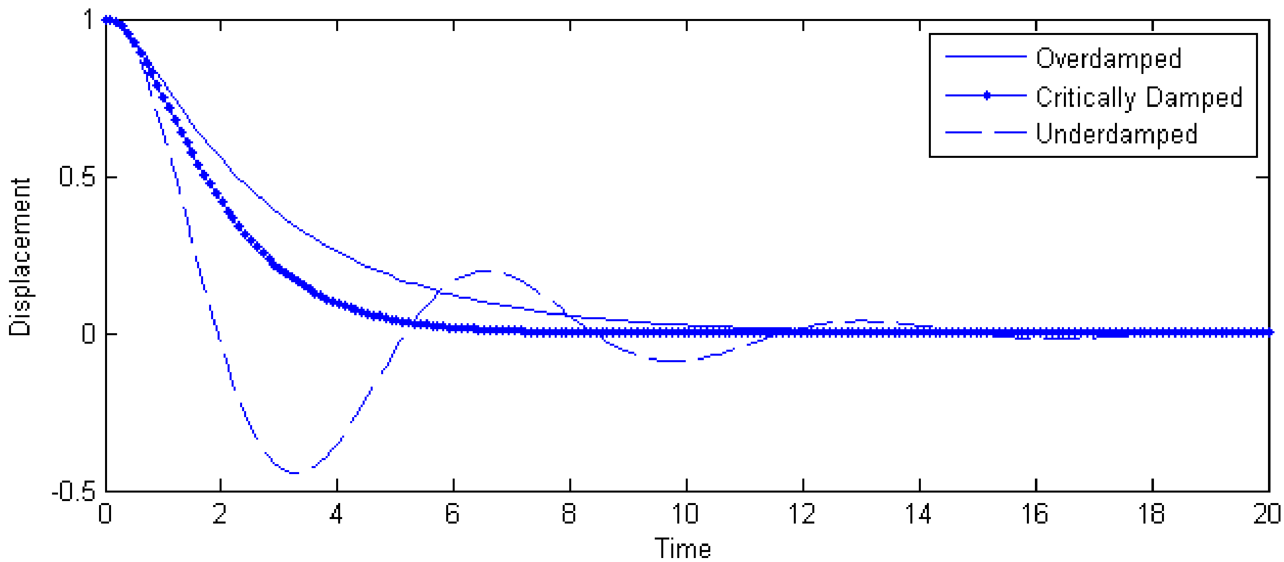
$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

There are three cases

$c^2 > 4mk$ Overdamped: Distinct real roots ($\lambda_{1,2}$)

$c^2 = 4mk$ Critically damped: Real repeated roots ($\lambda_1 = \lambda_2$)

$c^2 < 4mk$ Underdamped: Complex conjugate roots ($\lambda_{1,2} = \alpha \pm i\beta$)



The stability analysis can also be considered by writing the second order differential equation as a system of first order equations.

1.3 Scalar Non-Homogeneous Damped Mass Spring System

First let's first let $f(t) = 10\cos(3t)$, m = 1, c = 4, and k = 5 to get a differential equation of the form:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 10\cos(3t)$$

subject to the initial conditions:

$$y(0) = 0, \frac{dy(0)}{\Delta t} = 0.$$

Use $h = \Delta t = 0.1$ and solve out to time $t = 2$. To check your answers the true solution is given by:

$$y(t) = e^{-2t}\left(\frac{1}{4}\cos(t) - \frac{7}{4}\sin(t)\right) - \frac{1}{4}\cos(3t) + \frac{3}{4}\sin(3t)$$

We will implement finite difference methods to numerically solve this initial value problem. To solve the general equation we can descretize the system as follows:

(a)

$$m\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} + c\frac{y_{n+1} - y_{n-1}}{2\Delta t} + ky_n = f(t_n)$$

(b)

$$m\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} + c\frac{y_{n+1} - y_{n-1}}{2\Delta t} + k\frac{y_{n+1} - y_{n-1}}{2\Delta t} = f(t_n)$$

(c)

$$m\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} + c\frac{y_{n+1} - y_{n-1}}{2\Delta t} + k\frac{y_{n+1} + 2y_n + y_{n-1}}{4} = f(t_n)$$

Find the general formula for: $y_{n+1} = fnc(y_n, y_{n-1})$. Notice y_0 is given. To find y_1 use from the initial condition that:

$$\frac{y_1 - y_{-1}}{2\Delta t} = 0$$

in the general formula for y_{n+1} to solve for $y_1 = fnc(y_0, y_{-1})$ and use this relation to write $y_1 = fnc(y_0)$ only. You should get something like:

$$y_1 = \frac{10\cos(3t_0)}{2/(\Delta t)^2}$$

We can also reduce the order by rewriting the differential equation as a system of first order equations:

$$y' = v; \quad mv' + cv + ky = f(t)$$

Discretizing this system we get:

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{v_{n+1} + v_n}{2}; \quad m\frac{v_{n+1} - v_n}{\Delta t} + c\frac{v_{n+1} + v_n}{2} + k\frac{y_{n+1} - y_n}{2} = \sin(\omega_n t_n)$$

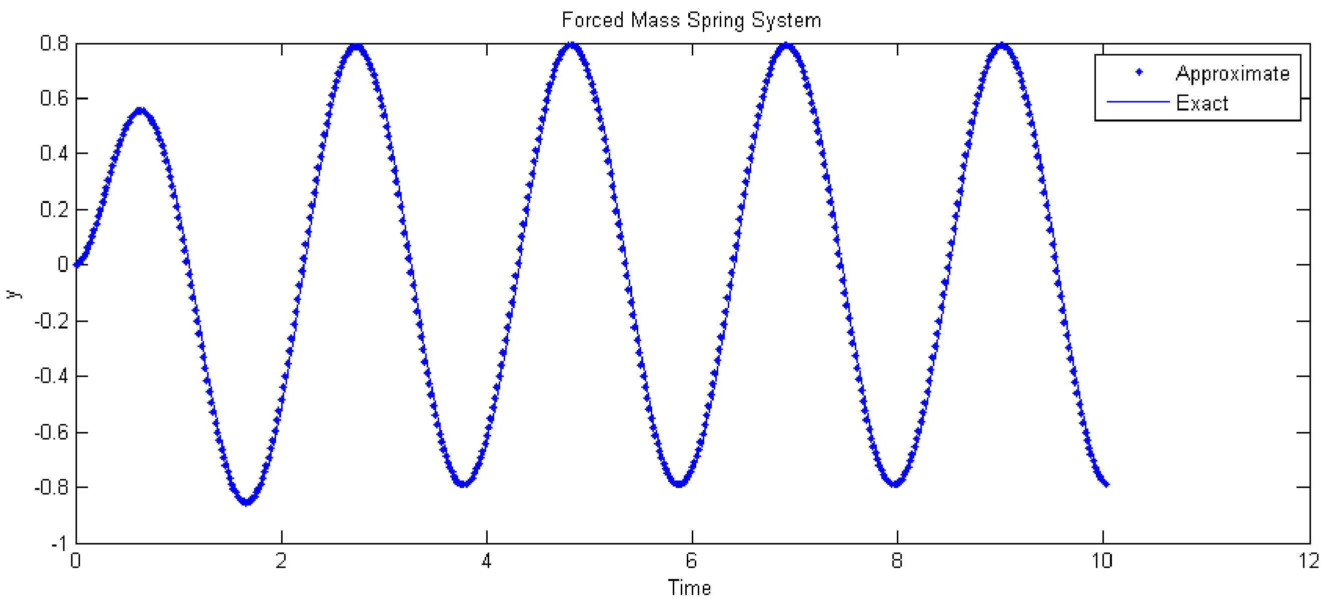
We will implement this subject to initial conditions:

$$y(0) = 0, \frac{dy(0)}{\Delta t} = v_0 = 0$$

How do we implement this? Again notice:

$$\frac{y_1 - y_{-1}}{2\Delta t} = v_0 = 0 \Rightarrow y_1 = y_{-1}$$

We will then use this to solve for y_1. Take careful note of how we used the boundary condition to handle finding y_1. This procedure is very important to understand.



Above is a classical damped mass-spring system. Several other interesting examples are available. The case of beats and pure resonance are of particular interest.

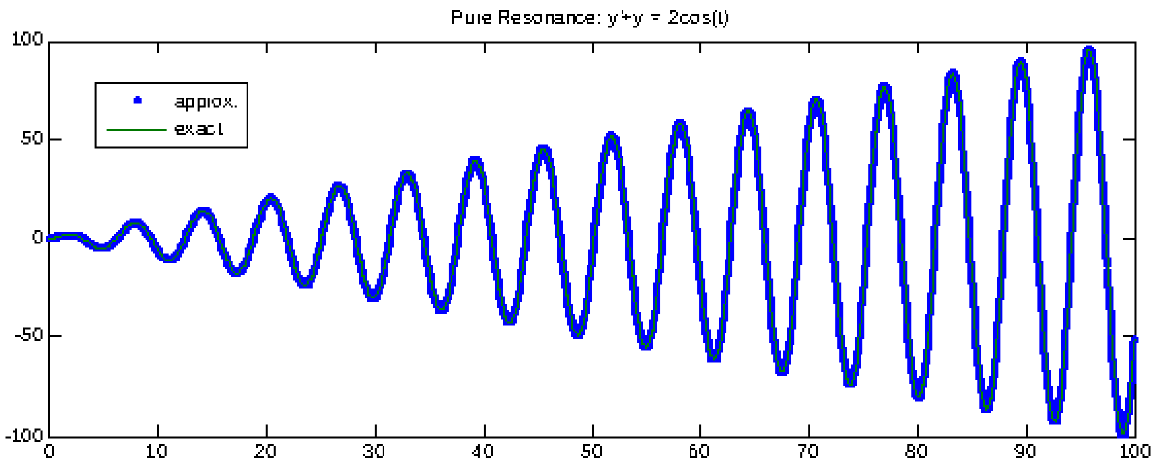
1.4 Pure resonance

In the case of pure resonance we have the natural frequency of the system matching the forcing frequency of the forcing function. In this case oscillations in the solutions are self-reinforcing and the solution always grows in amplitude with time (until, as occurs in the natural world, they become unsustainable). An example of this phenomenon is:

$$\frac{d^2y}{\Delta t^2} + y = 2\cos(t), \quad y(0) = 0, \frac{dy(0)}{\Delta t} = 0$$

Here, discretize this equation as above and numerically solve it. Use $h = 0.5$ and numerically solve to $t = 10$. To check your answer use:

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + t\sin(t) = t\sin(t) \quad c_1 = c_2 = 0 \text{ from the initial conditions}$$



1.5 Beat Effect

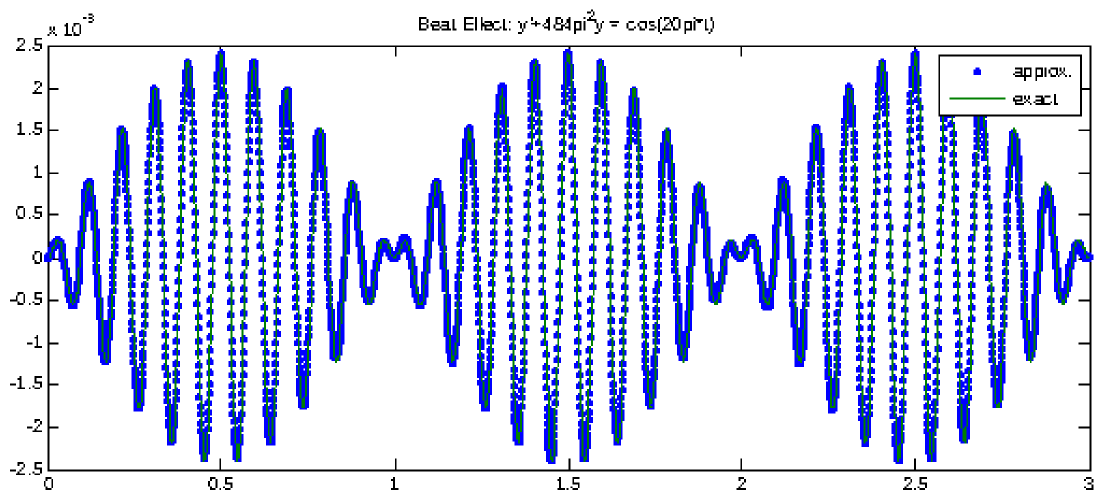
The case of beats is related to the phenomenon of the solution reinforcing itself and diminishing itself periodically. The regularity, or periodicity, of this phenomenon is what makes it special. Here we will consider:

$$\frac{d^2y}{dt^2} + 484\pi^2 y = \cos(20\pi t), \quad y(0) = 0, \frac{dy(0)}{\Delta t} = 0$$

Next discretize this equation as above and numerically solve it. Use $h = 0.005$ and numerically solve to $t = 1$ (this will take alot of rows!). Experiment with how much accuracy you lose with a large step size. To check your answer use:

$$y(t) = \frac{2}{(22\pi)^2 - (20\pi)^2} \sin(0.5(22\pi - 20\pi)t) \sin(0.5(22\pi + 20\pi)t)$$

Note that because of the highly oscillatory nature of this problem that a very fine discretization is required to get accurate results. This is a common and important problem that often needs to be dealt with computer aided mathematics.



1.6 Electric Circuits, Josephson Junction and Analog Computers

To analyze electric circuits a charge on a capacitor $Q(t)$ is modeled. The rate of change of charge Q is called the current I. The circuit is supplied an electromotive force E, from a battery for example. A typical electromotive force model is sinusoidal, for example $E(t) = E_0 \sin(\omega t)$. The circuit also contains an

Ohm's Resistor, R, Inductor L and Capacitor C. A second order equation can be derived from Kirkoff's Law to model these components in series around a circuit by

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

subject to the initial conditions

$$Q(0) = Q_0, \quad Q'(0) = I(0) = I_0$$

Noting that $I = dQ/dt$, this can be used to obtain by differentiation of the above equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

In the special case of an RC-Circuit (where $L = 0$), this reduces to a first order model for current due to the electromotive force as a function of time

$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}, \qquad I(0) = c$$

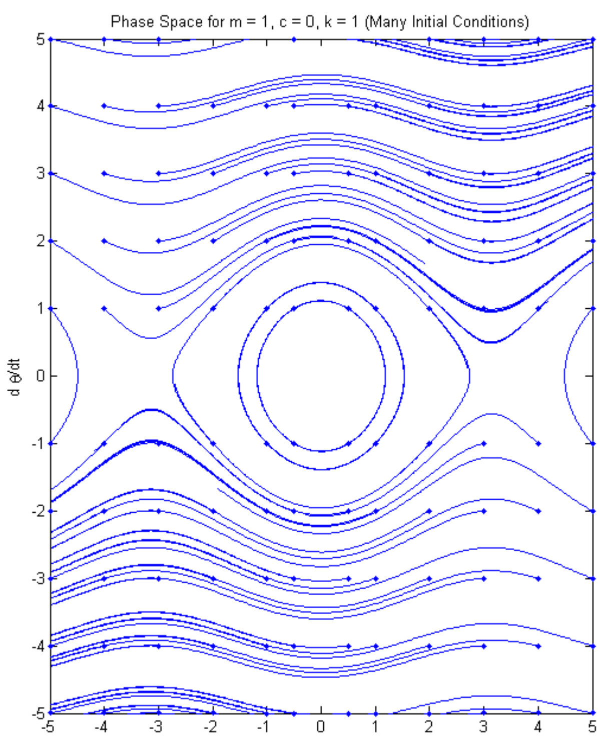
where c is the initial charge. There is an analogy between the circuit and the Mass-Spring System: displacement is charge, velocity is current, mass is inductance, dampening is resistance, spring constant is elastance (1/C), and external forcing is electromotive forcing.

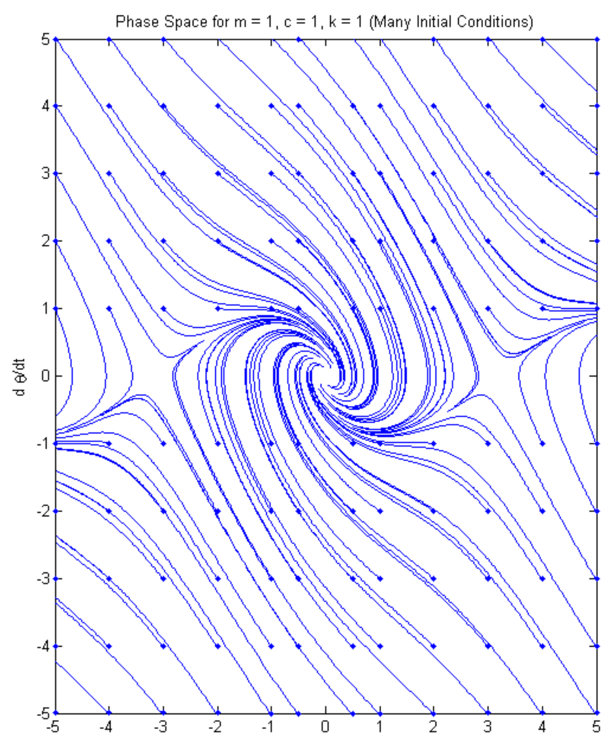
1.7 Nonlinear Spring: Pendulum

The damped pendulum is a classical and very intuitive problem:

$$\frac{d^2\theta}{dt^2} - c\frac{d\theta}{dt} + k\sin\theta = 0, \quad k \geq 0, c \geq 0$$

First the problem is studied by plotting the phase space for several values of θ and $d\theta/dt$. The dots in the plot indicate where the starting position was in phase space, which gives both of these initial conditions on the axes respectively.





Next the problem is studied in a damped case for a single trajectory. Each trajectory in the phase space corresponds with a displacement solution in time. This is a plot of the solution and the phase plane together