Scalar Field Reheating - Project Description

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1 Introduction

The main purpose of this project, is to simulate on lattice the scalar field reheating due to an $O(4)$ scalar field thermalization. In order to do so, we will follow the approach by *R. Micha and I. I. Tkachev* [1], which consist in studying the decay of coherent field oscillations and the subsequent thermalization of the resulting stochastic classical wave-field. The main motivation of this study is to try to describe the reheating of the Universe after inflation, which may be described at the end of the inflationary phase by the expectation value of a scalar field. Our implementation, however, differ from the previously cited one for some technical aspects, in particular, our momenta will be initialized at zero while in the paper they will have a starting kick on the momenta, furthermore the parameters that we will use during our simulation will differ from the previous one.

2 Methodology

The Lagrangian of the system, is as follows:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \Phi(\eta, x, y, z) - \frac{\lambda \Phi^4(\eta, x, y, z)}{4!} - \frac{m^2 \Phi^2(\eta, x, y, z)}{2} \tag{1}$$

where we have $g^{\mu\nu} = Diag(1, -1, -1, -1)$, and the field is an $O(4)$ field described by:

$$\Phi(\eta, x, y, z) = \begin{pmatrix} \Phi^0(\eta, x, y, z) \\ \Phi^1(\eta, x, y, z) \\ \Phi^2(\eta, x, y, z) \\ \Phi^3(\eta, x, y, z) \end{pmatrix} \tag{2}$$

In the following, we will drop the dependance of the field on the 4 *Minkowskian* coordinates, hence we will from now on write $\Phi(\eta, x, y, z) \rightarrow \Phi$. Eqt. 1, describes a scalar field on a 4-dimensional space with a self coupling $O(4)$

invariant term. We may obtain the Hamiltonian of the system from the lagrangian by using the relation :

$$\mathcal{H} = \pi\dot{\Phi} - \mathcal{L} \tag{3}$$

where we defined $\pi = \partial_\eta \Phi$. It is easy to demonstrate that, after some algebra, we obtatin the following Hamiltonian to describe our system :

$$\mathcal{H} = \frac{1}{2}\pi^\mu \pi_\mu + \frac{1}{2}\partial_i \Phi^\mu \partial^i \Phi_\mu + \frac{\lambda(\Phi^\mu \Phi_\mu)(\Phi^\nu \Phi_\nu)}{4!} + \frac{m^2\Phi^2}{2} \tag{4}$$

with $i \in [1,3]$ and $\mu, \nu \in [0,3]$. The evolution of the field, is hence governed by the Hamilton equations :

$$\begin{array}{rcl} \dot{\pi}_\mu & = & -\frac{\partial \mathcal{H}}{\partial \Phi^\mu} = -\partial_i \partial^i \Phi_\mu - \frac{\lambda(\Phi^\nu \Phi_\nu)\Phi_\mu)}{3!} - \frac{m^2 \Phi_\mu}{2} \\ \dot{\Phi}_\mu & = & \frac{\partial \mathcal{H}}{\partial \pi^\mu} = \pi_\mu \end{array} \tag{5}$$

For what concern the initial conditions of the field samples, we are going to implement *Free theory* initial conditions. The latter, as described by *J. Berges* [2], are given in Fourier space by the following equation :

$$F(0,0;k) = \frac{1/2 + n_k(0)}{\sqrt{k^2 + M^2(0)}} \tag{6}$$

where n_k is the occupation number of particles having momenta equals to k and M is the mass of the field, we observe that at $\eta = 0$ the mass $M(0) = m$, where m is the same mass that appear in eqt. 1. Furthermore, if we impose void initial conditions we have $n_k(0) = 0$. The term $F(0,0;k)$ instead describes the two point function for the free theory between $< \Phi(k)\Phi(-k) >$, and the general form is given by the *Green* function :

$$F(\eta, \eta'; \vec{x} - \vec{y}) = < \Phi(\vec{x}, \eta)\Phi(\vec{y}, \eta') > = \int \frac{dk^3}{(2\pi)^3} \Phi_k(\eta)\Phi_{-k}(\eta') e^{-ik(\vec{x}-\vec{y})} \tag{7}$$

From the previous equation, we may observe that the evolution of the real part of the function $F(\eta, \eta'; k)$ is as follows :

$$Real(F(\eta, \eta'; k)) = Real(< \Phi_k(\eta)\Phi_{-k}(\eta') >) \approx \frac{1/2 + n_k(\eta)}{e_k(\eta)} cos((\eta - \eta')e_k(\eta)) \tag{8}$$

where the energy $e_k(\eta)$, may be obtained by using the recursive properties of the cosine function. We hence obtain the following relation :

$$e_k(\eta) = \sqrt{\frac{\partial_\eta \partial_{\eta'} F(\eta, \eta'; k)}{F(\eta, \eta'; k)}|_{\eta=\eta'}} \tag{9}$$

Finally, we may obtain the occupation number of particle having momenta k at time t by inverting eqt. 8 , we obtain :

$$n_k(\eta) = Real(F(\eta, \eta'; k))e_k(\eta) - 1/2 \tag{10}$$

3 Numerical Implementation

In order to simulate the evolution on the scalar field numerically on lattice, first of all we need to discretize the space volume by introducing the lattice space a and the lattice size N, governed by the following relation with the spatial lenght L:

$$L = Na \tag{11}$$

consequently, we will have the following coordinate transformation :

$$\begin{cases} x \rightarrow ia, & : i \in [0, N-1] \\ y \rightarrow ja, & : j \in [0, N-1] \\ z \rightarrow ka, & : k \in [0, N-1] \end{cases} \tag{12}$$

where we implemented periodic boundary condition such that $i, j, k = N \rightarrow i, j, k = 0$.

In the same way, the time coordinate η will evolve till the ending time of the simulation toward small steps $d\eta$, while the four component of the field will be indicized by the letter l. Our field and momenta will be then described by :

$$\Phi[i][j][k][l] \rightarrow \tilde{\Phi} \tag{13}$$

$$\pi[i][j][k][l] \rightarrow \tilde{\pi} \tag{14}$$

where we dropped the index in the RHS of the equations.

The discrete Hamiltonian equations for the system, will hence be given by :

$$\tilde{\Phi}_{new} = \tilde{\Phi}_{act} + d\eta\tilde{\pi} \tag{15}$$

$$\tilde{\pi}_{new} = \tilde{\pi}_{act} + d\eta(\Delta^2_{i,j,k}\tilde{\Phi} - \frac{m^2\tilde{\Phi}}{2} - \frac{\lambda Norm(\tilde{\Phi})\tilde{\Phi}}{4!}) \tag{16}$$

where the function $Norm(\tilde{\Phi})$ return the squared norm for a 4 dimensional vector, and $\Delta^2_{i,j,k}\tilde{\Phi}$ is the numerical second derivative of the field along the 3 spatial directions. We may obtain the numerical second derivative of the field by estimating the derivative of the first derivatives, in the case of the direction i we obtain :

$$\begin{aligned} \Delta^2_i\Phi[i][j][k][l] &= \frac{\Delta^+_i\Phi[i][j][k][l] - \Delta^-_i\Phi[i][j][k][l]}{a} \\ &= \frac{\Phi[i+1][j][k][l] - 2\Phi[i][j][k][l] + \Phi[i-1][j][k][l]}{a^2} \end{aligned}$$

with Δ^+ representing a forward derivative of the function and Δ^- representing a backward derivative.

For what concern the initial conditions of the field, we start by initializing in the coordinate phase some fluctuations of the field behaving like normal noise, by imposing :

$$\begin{aligned} \tilde{\Phi}^{\mu}|_{\eta=0} &= Gauss(0,1) \\ \tilde{\pi}^{\mu}|_{\eta=0} &= 0 \end{aligned} \tag{17}$$

As we have seen from eqt. 6, the free theory initial conditions are given in Fourier space, we will now need to fourier transform the field obtaining :

$$\Phi^{\mu}(0,x,y,z) \rightarrow \mathcal{FFT} \rightarrow \Phi^{\mu}(0,k_x,k_y,k_z) \tag{18}$$

and the free theory initial conditions, given by eqt. 6 in the void case, may easily be imposed on the field in fourier space by multiplying the latter by the factor :

$$\Phi^{\mu}_{FreeTheory}(0,k_x,k_y,k_z) = \frac{\Phi^{\mu}(0,k_x,k_y,k_z)}{2\omega} \tag{19}$$

where $\omega = \sqrt{k^2 + M^2(0)}$ and $k_{x,y,z}$ goes from 0 to π/a. The initial conditions on the coordinate space, will hence be given by the inverse fourier transform of the free theory initial conditions in Fourier Space :

$$\Phi^{\mu}_{FreeTheory}(0,k_x,k_y,k_z) \rightarrow \mathcal{IFFT} \rightarrow \Phi^{\mu}_{FreeTheory}(0,x,y,z) \tag{20}$$

We will also sum to the 0 component an expectation value equals to $1/\lambda$ to check in time the difference of evolution among the 0 component of the field and the other 3 ones.

The evolution, in particular, will be done using a symplectic solver algorithm to preserve better the energy of the system during the evolution, we choosed to adopt in particular a *Leap Frog* algorithm. We will estimate the values of the occupation number only at some certain instants during the time evolution, we choosed to save the values for :

$$\eta = [\eta_{max}/100, \eta_{max}/20, \eta_{max}/10, \eta_{max}/5, \eta_{max}/2, \eta_{max}] \tag{21}$$

The occupation number, in particular, will be given numerically in Fourier space by :

$$n^{\mu}_k(\eta) = (F^{\mu}(\eta,\eta';k))e^{\mu}_k(\eta) - 1/2 \tag{22}$$

where we will estimate the n_k for k along the diagonal in order to simplify the implementation. This simplification, will however cause a loss of precision on the relation among occupation number $n^{\mu}_k(\eta)$ and the related momenta k, as we will only estimate $k = \sqrt{(k_x^2 + k_y^2 + k_z^2)}$ for $k_x = k_y = k_z$, dropping all the intermediate cases. The two terms $F(\eta,\eta';k)$ and $e_k(\eta)$ will be given numerically by :

$$F^{\mu}(\eta,\eta;k) = \Phi^{\mu}[i_k][i_k][i_k](\eta) * \Phi^{\mu}[i_{-k}][i_{-k}][i_{-k}](\eta) \tag{23}$$

$$e^{\mu}_k(\eta) = \sqrt{\frac{(\Delta^{-}\Phi^{\mu}[i_k][i_k][i_k](\eta))(\Delta^{-}\Phi^{\mu}[i_{-k}][i_{-k}][i_{-k}](\eta))}{F^{\mu}(\eta,\eta;k)}} \tag{24}$$

where i_k is the index of the field corresponding to momenta k and i_{-k} is the index of the field corresponding to momenta $-k$.

Quantity	Value
m	1
λ	0.1
a	0.05
N	20
$d\eta$	0.01
η_{max}	1000

Table 1: The values of the input parameters for the simulation.

4 Results and Conclusions

In our simulation, we used the values reported in tab. 1 for the parameters. As we adopted a symplectic algorithm in order to conserve the energy during the evolution of the field, first of all we need to check that the energy is conserved throughout all the simulation, this is shown in figure 1.

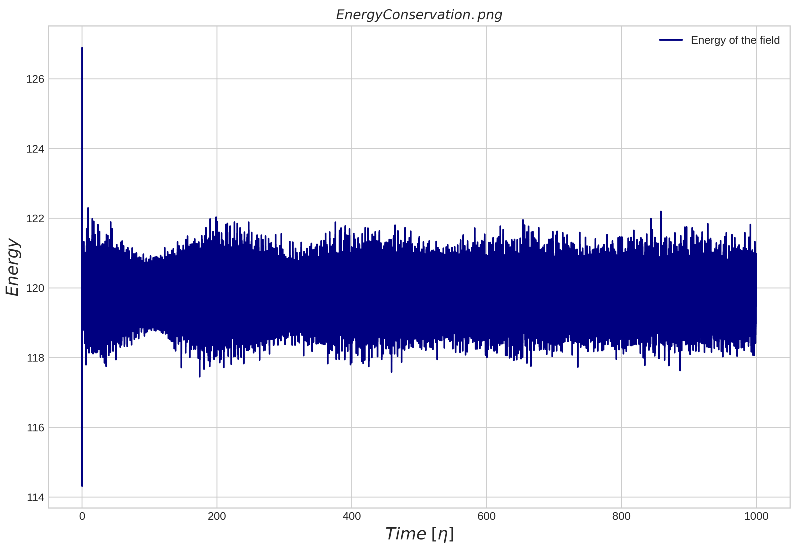

Figure 1: Energy of the field configuration throughout the whole evolution of the simulation.

We observe that after a starting spike needed to stabilize, even though the energy fluctuate around 2% of the average value during the simulation time, there is no increase or decrease of the latter in agreement with what expected.

Furthermore, we may observe that due to the small values for m, λ, the expectation values of the fields components $< \Phi^\mu >$ are supposed to behave similarly to an armonic oscillator throughout all the evolution of the system, this is shown in figs 2,3. We decided to use different values for η_{max} in the

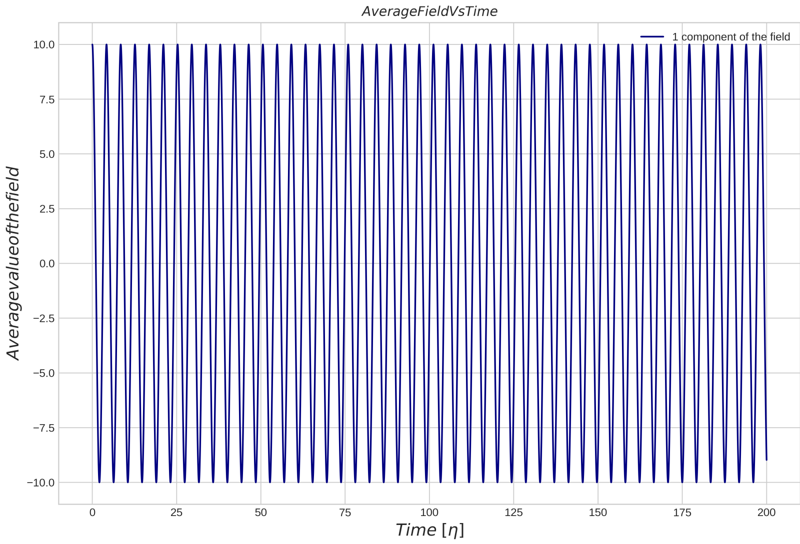

Figure 2: Behaviour in function of time of the value of $< \Phi^0 >$.

two plots, and compared to the whole evolution time, in order to emphasize the features of the latter. In particular, in the case of fig 2 we see that the expectation value of the field oscillates with a high frequency between $-1/\lambda$ and $1/\lambda$. This oscillations are extremely stable in time, and the values of oscillation are not surprising considering that the 0 component of the field was actually the one where we sum the $1/\lambda$ term after imposing the free theory initial conditions. Furthermore, the high frequency of the oscillations compared to figure 3 is expected considering that a bigger value on the field turns into bigger values for the derivatives obtained by the Hamiltonian equation 16, and of course a bigger change in the momenta is linked to faster fluctuations of the field over time.

In the case of the 3 other components of the field, shown in fig. 3, we observe that the situation is more colourful. In particular we observe that the fields, initialized as pure stochastic fluctuations and then rescaled in order to match the free theory, behaves differently one from the other. We see that the second component of the field increase a lot in amplitude before damping off with a low frequency, while the others two keep a more contained amplitude which however oscillates in amplitude more frequently.

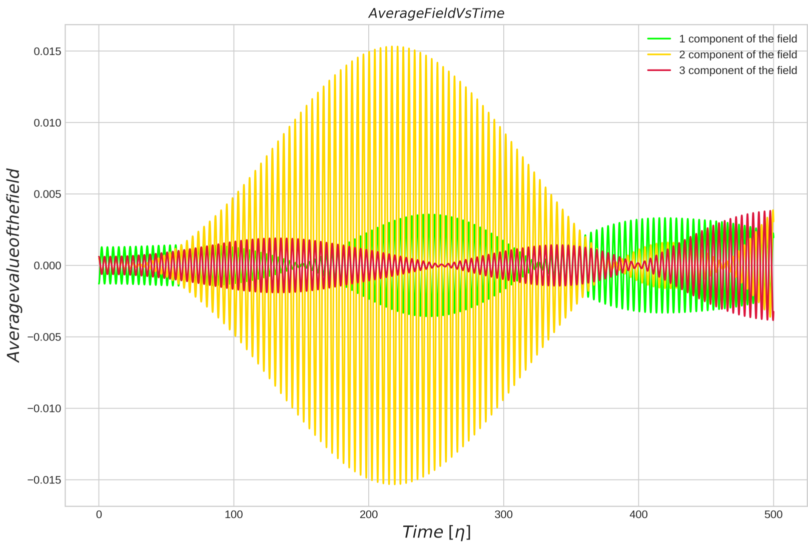

Figure 3: Behaviour in function of time of the value of $< \Phi^i >$.

For what concerns the occupation number in function of η we will show in figure 4,5 and 6 the obtained results at the values of $\eta = 10, 100, 1000$. We also report in figure 7 the average of the occupation number among the components $1, 2, 3$ of the field in function of the time η. It has to be emphasized that, as we estimated the two point function only along the diagonal entry of the matrix, the scale of the momenta k on the x axis is not reliable, de facto, jumping two momenta points among each of the shown ones. Even though the scale of the plot is kinda deceptive, it appear clears from fig. 7 that the occupation number, after the initial spike due to the fact that the average value of $< \Phi^\mu(k) >< \Phi^\mu(-k) >$ wasn't subtracted, shows commonly a spike around $k = 50$ for all the 3 components of the field. Even though as stated before the scale of the momenta is not reliable, this is in agreement with what expected, with the position of our peak differing from the case of *R. Micha and I. I. Tkachev* [1] as we are using different parameters and some deviations in the procedure from the latter.

References

[1] Raphael Micha and Igor I. Tkachev. Turbulent thermalization. *Phys. Rev. D*, 70:043538, 2004.

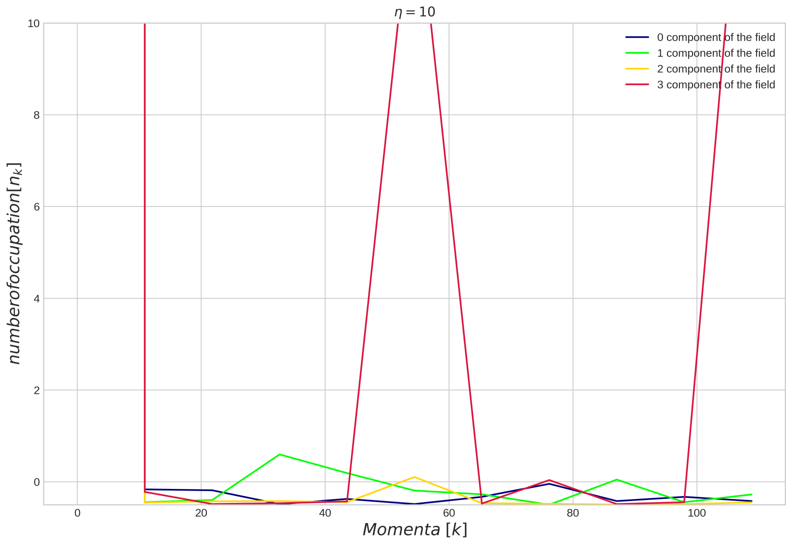

Figure 4: Occupation number n_k of the 4 components field at the time $\eta = 10$.

[2] Juergen Berges. Introduction to nonequilibrium quantum field theory. *AIP Conf. Proc.*, 739(1):3–62, 2004.

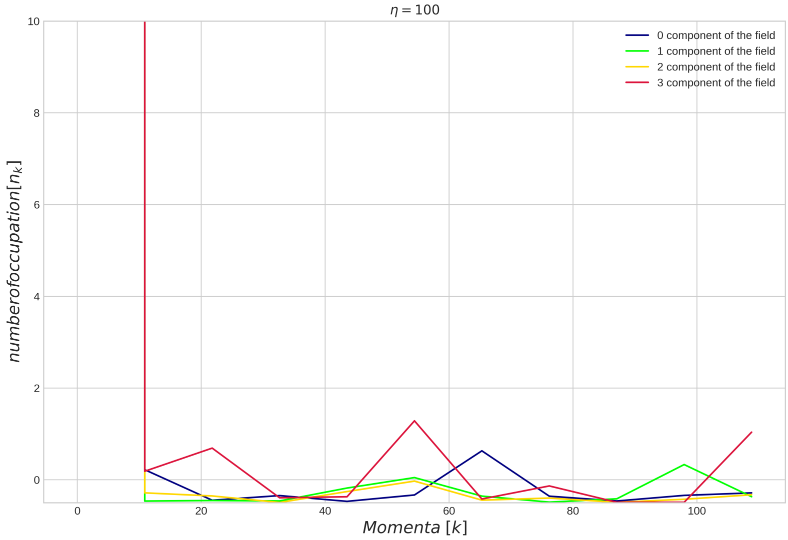

Figure 5: Occupation number n_k of the 4 components field at the time $\eta = 100$.

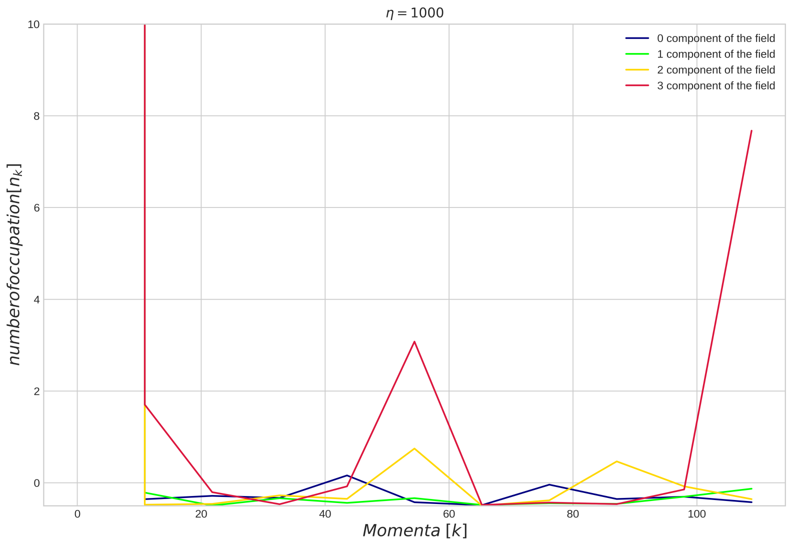

Figure 6: Occupation number n_k of the 4 components field at the time $\eta = 1000$.

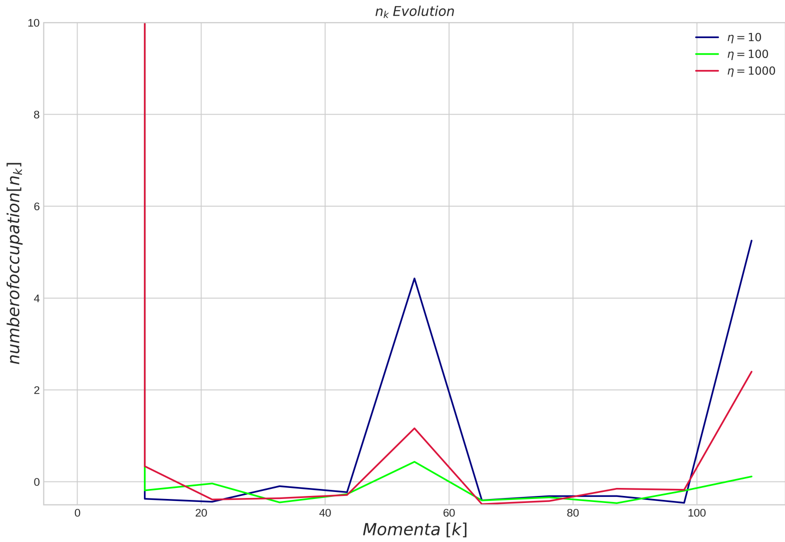

Figure 7: Occupation number n_k of the average among the components 1 to 3 at several η.