MA random-effects Gibbs sampling

Distributional assumptions

$$
\begin{aligned}
Y_i &\sim N(\theta_i, \sigma^2_{Y_i}) \quad i = 1, \dots, n \\
\theta_i &\sim N(\mu, \sigma^2_\theta) \\
\mu &\sim N(\theta_o, \sigma^2_0) \\
\sigma^2_\theta &\sim IG(\alpha, \beta)
\end{aligned}
$$

Assuming $Y_i \perp \mu$, $Y_i \perp \sigma^2_\theta$, $\mu \perp \sigma^2_\theta$, we have

$$
P(\theta_i, \mu, \sigma^2_\theta, Y_i) = P(Y_i \mid \theta_i, \mu, \sigma^2_\theta) P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu) P(\sigma^2_\theta) = P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu) P(\sigma^2_\theta) \tag{1}
$$

Conditional posterior of θ_i

From (1), we have

$$
\begin{aligned}
P(\theta_i \mid \mu, \sigma^2_\theta, Y_i) = \frac{P(\theta_i, \mu, \sigma^2_\theta, Y_i)}{P(\mu, \sigma^2_\theta, Y_i)} = \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu) P(\sigma^2_\theta)}{P(Y_i \mid \mu, \sigma^2_\theta) P(\mu) P(\sigma^2_\theta)} &= \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta)}{P(Y_i \mid \mu, \sigma^2_\theta)} \\
&= \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta)}{\int P(Y_i \mid \theta_i, \mu, \sigma^2_\theta) p(\theta_i \mid \mu, \sigma^2_\theta) d\theta_i} \\
&= \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta)}{\int P(Y_i \mid \theta_i) p(\theta_i \mid \mu, \sigma^2_\theta) d\theta_i},
\end{aligned}
$$

and therefore

$$
\theta_i \mid \mu, \sigma^2_\theta, Y_i \sim N(\hat{\mu}_\theta, \hat{\sigma}^2_\theta),
$$

where $\hat{\mu}_\theta = \dfrac{\mu \sigma^2_{Y_i} + Y_i \sigma^2_\theta}{\sigma^2_{Y_i} + \sigma^2_\theta}$, $\hat{\sigma}^2_\theta = \dfrac{\sigma^2_{Y_i} \sigma^2_\theta}{\sigma^2_{Y_i} + \sigma^2_\theta}$.

Conditional posterior of μ

From (1), we have

$$
\begin{aligned}
P(\mu \mid \theta_i, \sigma^2_\theta, Y_i) = \frac{P(\mu, \theta_i, \sigma^2_\theta, Y_i)}{P(\theta_i, \sigma^2_\theta, Y_i)} = \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu) P(\sigma^2_\theta)}{P(Y_i \mid \theta_i, \sigma^2_\theta) P(\theta_i \mid \sigma^2_\theta) P(\sigma^2_\theta)} &= \frac{P(Y_i \mid \theta_i) P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu)}{P(Y_i \mid \theta_i) P(\theta_i \mid \sigma^2_\theta)} \\
&= \frac{P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu)}{\int P(\theta_i \mid \mu, \sigma^2_\theta) P(\mu) d\mu},
\end{aligned}
$$

and therefore

$$
\mu \mid \theta_i, \sigma^2_\theta, Y_i \sim N(\hat{\mu}_\mu, \hat{\sigma}^2_\mu),
$$

where $\hat{\mu}_\mu = \dfrac{\sigma^2_0 \sum_{i=1}^n \theta_i + \theta_0 \sigma^2_\theta}{n\sigma^2_0 + \sigma^2_\theta}$, $\hat{\sigma}^2_\mu = \dfrac{\sigma^2_0 \sigma^2_\theta}{n\sigma^2_0 + \sigma^2_\theta}$.

Conditional posterior of σ^2_θ

From (1), we have

$$P(\sigma_\theta^2 \mid \theta_i, \mu, Y_i) = \frac{P(\sigma_\theta^2, \theta_i, \mu, Y_i)}{P(\theta_i, \mu, Y_i)} = \frac{P(Y_i \mid \theta_i)P(\theta_i \mid \mu, \sigma_\theta^2)P(\mu)P(\sigma_\theta^2)}{P(Y_i \mid \theta_i, \mu)P(\theta_i \mid \mu)P(\mu)} = \frac{P(Y_i \mid \theta_i)P(\theta_i \mid \mu, \sigma_\theta^2)P(\mu)P(\sigma_\theta^2)}{P(Y_i \mid \theta_i)P(\theta_i \mid \mu)P(\mu)}$$
$$= \frac{P(\theta_i \mid \mu, \sigma_\theta^2)P(\sigma_\theta^2)}{P(\theta_i \mid \mu)}$$
$$= \frac{P(\theta_i \mid \mu, \sigma_\theta^2)P(\sigma_\theta^2)}{\int P(\theta_i \mid \mu, \sigma_\theta^2)d\sigma_\theta^2},$$

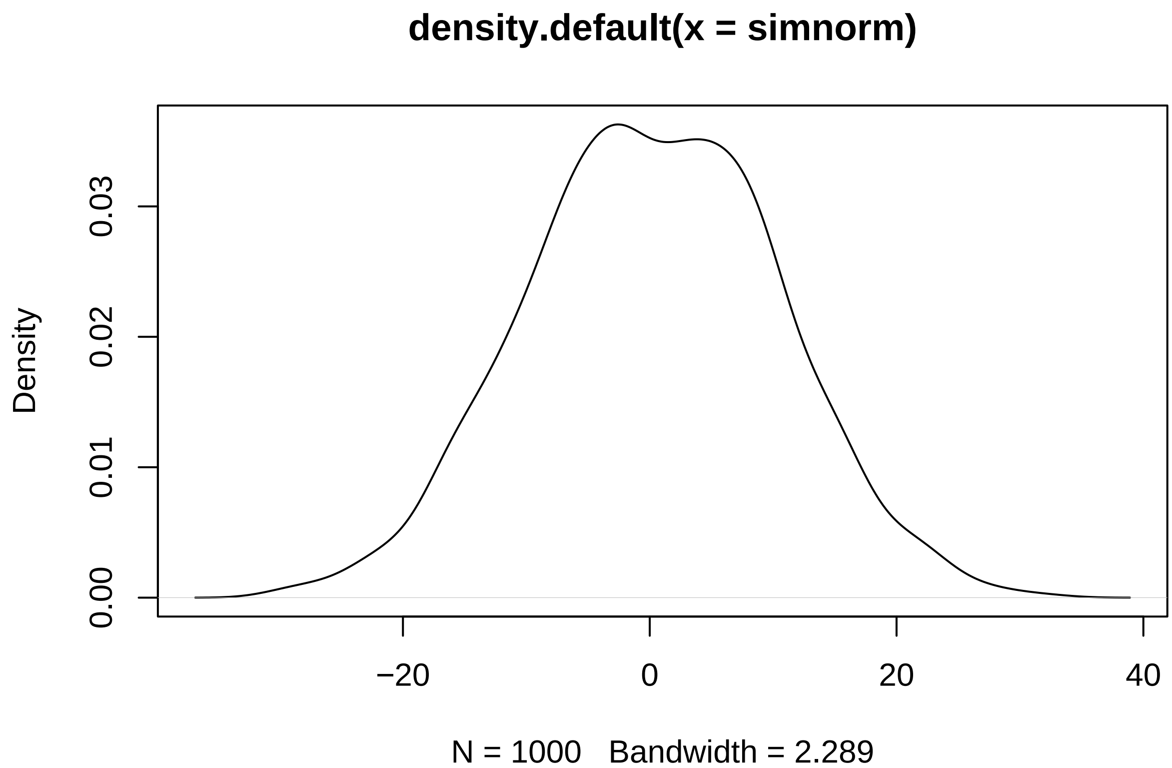
and therefore

$$\sigma_\theta^2 \mid \theta_i, \mu, Y_i \sim IG\left(\alpha + \frac{n}{2}, \beta + \frac{1}{2}\sum_{i=1}^{n}(\theta_i - \mu)^2\right).$$

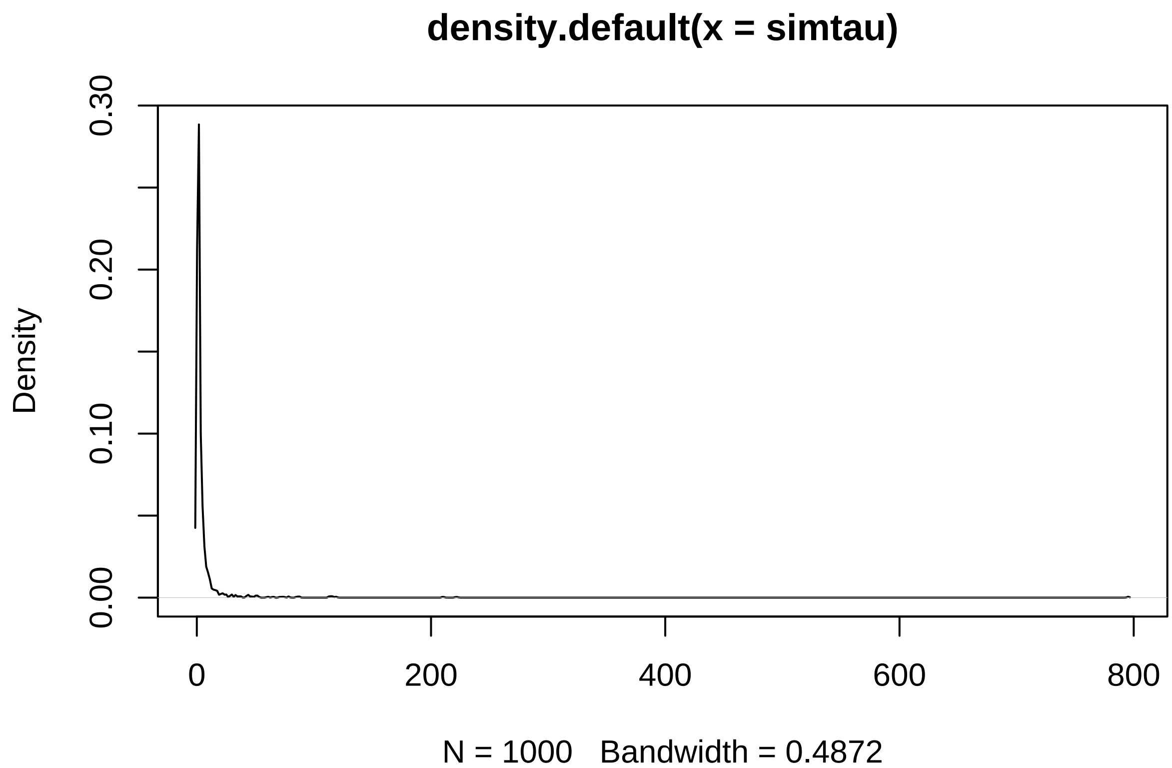
```
library(metafor)
dat <- dat.raudenbush1985
dat
```

```
##    study               author year weeks setting tester n1i n2i      yi     vi
## 1      1     Rosenthal et al. 1974     2   group  aware  77 339  0.0300 0.0156
## 2      2          Conn et al. 1968    21   group  aware  60 198  0.1200 0.0216
## 3      3          Jose & Cody 1971    19   group  aware  72  72 -0.1400 0.0279
## 4      4   Pellegrini & Hicks 1972     0   group  aware  11  22  1.1800 0.1391
## 5      5   Pellegrini & Hicks 1972     0   group  blind  11  22  0.2600 0.1362
## 6      6    Evans & Rosenthal 1969     3   group  aware 129 348 -0.0600 0.0106
## 7      7      Fielder et al. 1971    17   group  blind 110 636 -0.0200 0.0106
## 8      8             Claiborn 1969    24   group  aware  26  99 -0.3200 0.0484
## 9      9               Kester 1969     0   group  aware  75  74  0.2700 0.0269
## 10    10              Maxwell 1970     1   indiv  blind  32  32  0.8000 0.0630
## 11    11               Carter 1970     0   group  blind  22  22  0.5400 0.0912
## 12    12              Flowers 1966     0   group  blind  43  38  0.1800 0.0497
## 13    13              Keshock 1970     1   indiv  blind  24  24 -0.0200 0.0835
## 14    14            Henrikson 1970     2   indiv  blind  19  32  0.2300 0.0841
## 15    15                 Fine 1972    17   group  aware  80  79 -0.1800 0.0253
## 16    16              Grieger 1970     5   group  blind  72  72 -0.0600 0.0279
## 17    17 Rosenthal & Jacobson 1968     1   group  aware  65 255  0.3000 0.0193
## 18    18   Fleming & Anttonen 1971     2   group  blind 233 224  0.0700 0.0088
## 19    19             Ginsburg 1970     7   group  aware  65  67 -0.0700 0.0303
```

```
J <- nrow(dat)

y <- dat$yi

sigma <- sqrt(dat$vi)

sigma0 <- 10
mu0 <- 0
alpha0 <- 1
beta0 <- 1

# Priors

simnorm <- rnorm(1000,mu0,sigma0)
plot(density(simnorm))
```

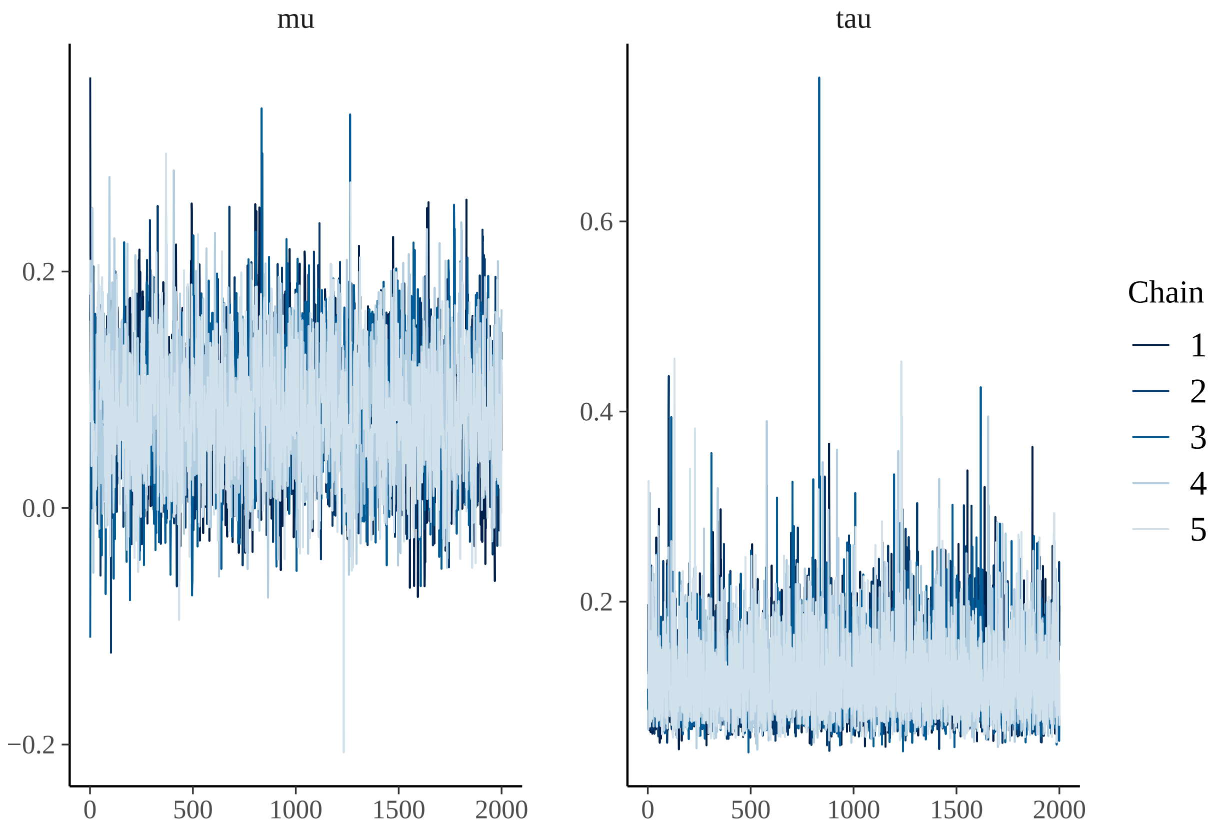



```
simtau <-1/rgamma(1000,alpha0,beta0)
plot(density(simtau))
```

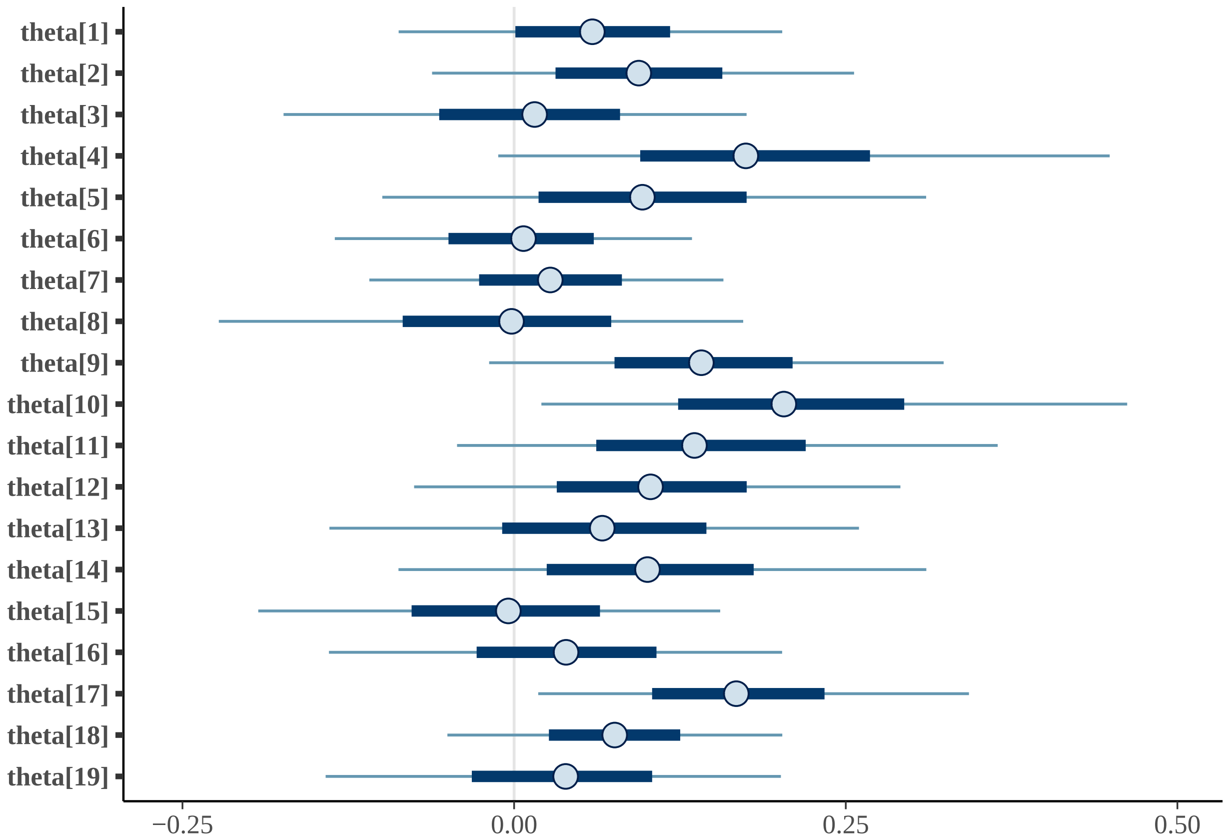



```r
theta_update <- function(){
  #theta_hat <- (mu/tau^2 + y/sigma^2)/(1/tau^2 + 1/sigma^2)
  theta_hat <- (mu*sigma^2+y*tau^2)/(sigma^2+tau^2)

  #V_theta <- 1/(1/tau^2 + 1/sigma^2)
  V_theta <- (sigma^2*tau^2)/(sigma^2+tau^2)

  rnorm(J, theta_hat, sqrt(V_theta))
}

mu_update <- function(){
  mu_hat <- (sigma0^2*sum(theta)+mu0*tau^2)/(J*sigma0^2+tau^2)
  V_mu <- (sigma0^2*tau^2)/(J*sigma0^2+tau^2)

  #rnorm(1, mean(theta), tau/sqrt(J))
  rnorm(1, mu_hat, sqrt(V_mu))
}

tau_update <- function(){
  alpha <- alpha0 + J/2
  beta <- beta0 + (sum((theta-mu)^2)/2)
  #sqrt(sum((theta-mu)^2)/rchisq(1,J-1))
  1/rgamma(1,alpha,beta)
}
```

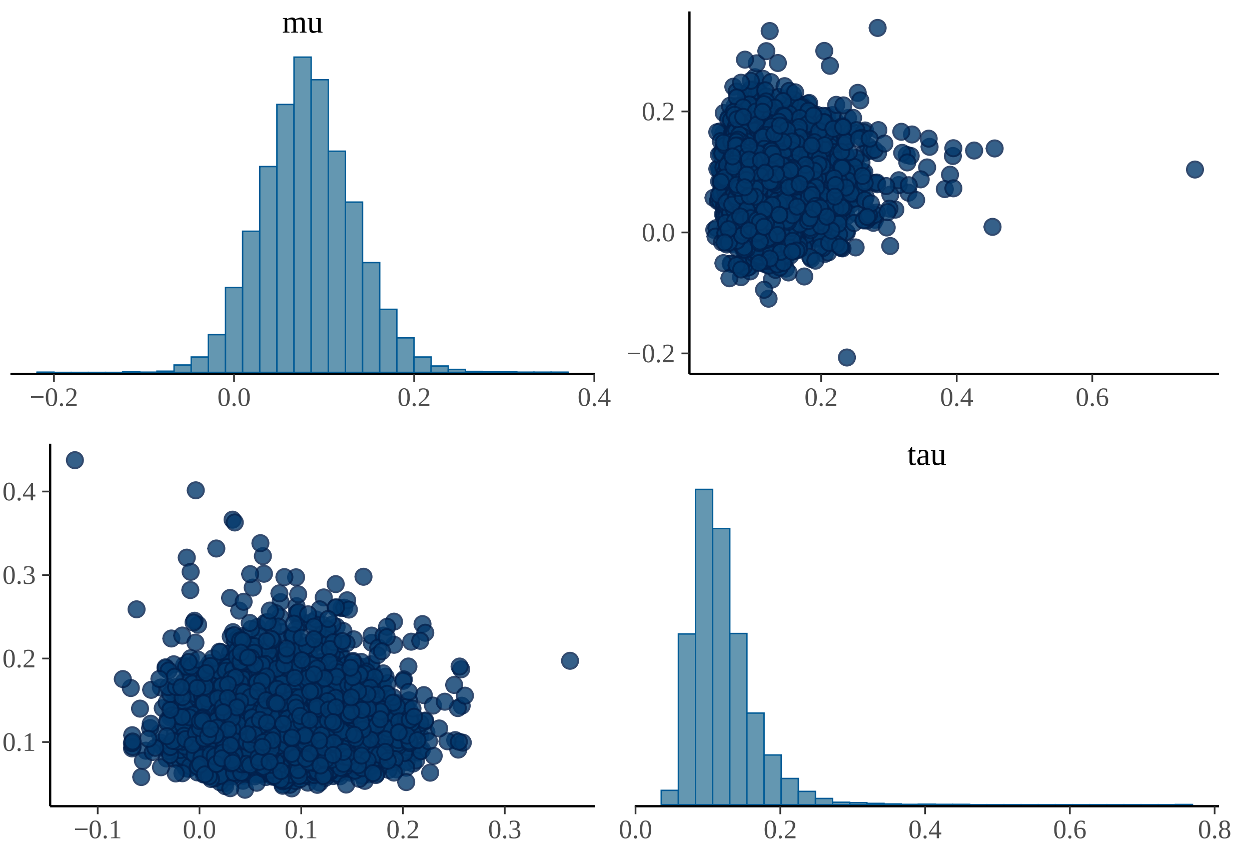
```r
chains <- 5

iter <- 2000

sims <- array(NA, c(iter, chains, J+2))

dimnames(sims) <- list(NULL, NULL,
                       c(paste("theta[", 1:nrow(dat), "]", sep=""), "mu", "tau"))
for (m in 1:chains){
  mu <- rnorm(1, mean(y), sd(y))

  tau <- runif(1, 0, sd(y))

  for (t in 1:iter){
    theta <- theta_update()

    mu <- mu_update()

    tau <- tau_update()

    sims[t,m,] <- c(theta, mu, tau)
  }
}

### Diagnostics ###
library(bayesplot)



#Traceplots
mcmc_trace(sims,pars = c("mu","tau"))
```




```
# Forest plots theta
mcmc_intervals(sims,regex_pars = "theta")
```




```
# mu, tau scatterplot
mcmc_pairs(sims,pars = c("mu","tau"))
```


mu
tau
−0.2
0.0
0.2
0.4
0.6
0.8
−0.1
0.1
0.3