Rotation with complex multiplication but not trigonometric function

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Theory about computing orientation, Download the theory paper

Discussion in GameDev forum

1. Man's angle goes from θ_a to θ_b

Rotation matrix

$\begin{bmatrix} \cos\theta\cos\psi+\sin\theta\sin\varphi\sin\psi & \sin\theta\cos\psi-\cos\theta\sin\varphi\sin\psi & \cos\varphi\sin\psi \\ -\sin\theta\cos\varphi & \cos\theta\cos\varphi & \sin\varphi \\ \sin\theta\sin\varphi\cos\psi-\cos\theta\sin\psi & -\cos\theta\sin\varphi\cos\psi-\sin\theta\sin\psi & \cos\varphi\cos\psi \end{bmatrix}$	(1)

a. Computing trigonometric way

$\theta_a \to \theta_b \quad \theta_b-\theta_a=N\cdot\Delta\theta$ $n=0\to N \quad \theta_n=\theta_0+n\cdot\Delta\theta$	(2)
Compute N times cosine and sine $C_n=\cos(\theta_0+n\cdot\Delta\theta) \quad S_n=\sin(\theta_0+n\cdot\Delta\theta)$	(3)

b. Computing complex number way

Z_n=complex number of the angular position of the step n

Step 0	$Z_0=A_0+B_0 i=\cos\theta_0+\sin\theta_0\, i \quad A_0=\cos\theta_0 \quad B_0=\sin\theta_0$	(4)
Multiplicative increment	$Z_\mathrm{d}=A_\mathrm{d}+B_\mathrm{d} i=\cos\Delta\theta+\sin\Delta\theta\, i \quad A_\mathrm{d}=\cos\Delta\theta \quad B_\mathrm{d}=\sin\Delta\theta$	(5)
Addition of angles	$\cos(\theta_0+\Delta\theta)+\sin(\theta_0+\Delta\theta)\, i=(\cos\theta_0+\sin\theta_0\, i)(\cos\Delta\theta+\sin\Delta\theta\, i)$	(6)
Step 1=(Step 0)*(Increment)	$Z_1=\cos(\theta_0+\Delta\theta)+\sin(\theta_0+\Delta\theta)\, i=(\cos\theta_0+\sin\theta_0\, i)(\cos\Delta\theta+\sin\Delta\theta\, i)$ $=(A_0+B_0 i)(A_\mathrm{d}+B_\mathrm{d} i)=Z_0\cdot Z_\mathrm{d}=A_1+B_1 i$	(7)
Recurrent formula Step 2=(Step 1)*(Increment) … Step n+1=(Step n)*(Increment)	$Z_2=Z_1\cdot Z_\mathrm{d}=(A_1+B_1 i)(A_\mathrm{d}+B_\mathrm{d} i)=\cos(\theta_0+2\Delta\theta)+\sin(\theta_0+2\Delta\theta)\, i$ … $Z_{n+1}=Z_n\cdot Z_\mathrm{d}=(A_n+B_n i)(A_\mathrm{d}+B_\mathrm{d} i)$ $=\cos(\theta_0+(n+1)\cdot\Delta\theta)+\sin(\theta_0+(n+1)\cdot\Delta\theta)\, i$	(8)
cosine and sine are in the complex number Z_n	$Z_n=\cos(\theta_0+n\cdot\Delta\theta)+\sin(\theta_0+n\cdot\Delta\theta)\, i=A_n+B_n i$ $A_n=\cos(\theta_0+n\cdot\Delta\theta) \quad B_n=\sin(\theta_0+n\cdot\Delta\theta)$	(9)

c. Rotation of point X_g around vector u

Axis of rotation vector **u**= $\mathbf{d}_x$ $\boldsymbol{u}=u_x\boldsymbol{g}_x+u_y\boldsymbol{g}_y+u_z\boldsymbol{g}_z=\boldsymbol{d}_x=ad\boldsymbol{g}_x+bd\boldsymbol{g}_y+f\boldsymbol{g}_z$		(10)
Direction vector $\mathbf{d}_x$=**u** $\boldsymbol{d}_x=(a\boldsymbol{g}_x+b\boldsymbol{g}_y)d+f\boldsymbol{g}_z=(\cos\theta\,\boldsymbol{g}_x+\sin\theta\,\boldsymbol{g}_y)\cos\varphi+\sin\varphi\,\boldsymbol{g}_z$	$a=\cos\theta \quad b=\sin\theta$ $d=\cos\varphi \quad f=\sin\varphi$	(11)
Rotated point, X_{1g}= X_g (Original point) * transpose [A] [rotation angle] * [A] $[x_{1g}\ \ y_{1g}\ \ z_{1g}]=[x_g\ \ y_g\ \ z_g]\begin{bmatrix} da & -b & -fa \\ db & a & -fb \\ f & 0 & d\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix} da & db & f \\ -b & a & 0 \\ -fa & -fb & d\end{bmatrix}$		(12)
Matrix of rotation [B] $[B(\boldsymbol{u},\psi)]=\begin{bmatrix} da & -b & -fa \\ db & a & -fb \\ f & 0 & d\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix} da & db & f \\ -b & a & 0 \\ -fa & -fb & d\end{bmatrix}$		(13)
Rotation using direction frame: $[x_{1g}\ \ y_{1g}\ \ z_{1g}]=[x_g\ \ y_g\ \ z_g][B(\boldsymbol{u},\psi)]$		(14)
Quaternion rotation: $[x_{1g}\ \ y_{1g}\ \ z_{1g}]=q(\boldsymbol{u},\psi)[x_g\ \ y_g\ \ z_g]q(\boldsymbol{u},\psi)^{-1}$		(15)

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Point to be rotated (16)

$$[x_g \quad y_g \quad z_g]\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

Direction frame (17)

$$\begin{cases} \boldsymbol{d}_x = & (\cos\theta\,\boldsymbol{g}_x + \sin\theta\,\boldsymbol{g}_y)\cos\varphi + \sin\varphi\,\boldsymbol{g}_z \\ \boldsymbol{d}_y = & -\sin\theta\,\boldsymbol{g}_x + \cos\theta\,\boldsymbol{g}_y \\ \boldsymbol{d}_z = & -(\cos\theta\,\boldsymbol{g}_x + \sin\theta\,\boldsymbol{g}_y)\sin\varphi + \cos\varphi\,\boldsymbol{g}_z \end{cases} = \begin{cases} \boldsymbol{d}_x = & (a\boldsymbol{g}_x + b\boldsymbol{g}_y)d + f\boldsymbol{g}_z \\ \boldsymbol{d}_y = & -b\boldsymbol{g}_x + a\boldsymbol{g}_y \\ \boldsymbol{d}_z = & -(a\boldsymbol{g}_x + b\boldsymbol{g}_y)f + d\boldsymbol{g}_z \end{cases}$$

Inverse transformation –angle. Not so simple (18)

$$\begin{cases} \boldsymbol{g}_x = & (\cos-\theta\,\boldsymbol{d}_x + \sin-\theta\,\boldsymbol{d}_y)\cos-\varphi + \sin-\varphi\,\boldsymbol{d}_z \\ \boldsymbol{g}_y = & -\sin-\theta\,\boldsymbol{d}_x + \cos-\theta\,\boldsymbol{d}_y \\ \boldsymbol{g}_z = & -(\cos-\theta\,\boldsymbol{d}_x + \sin-\theta\,\boldsymbol{d}_y)\sin-\varphi + \cos-\varphi\,\boldsymbol{d}_z \end{cases} = \begin{cases} \boldsymbol{g}_x = & (\cos\theta\,\boldsymbol{d}_x - \sin\theta\,\boldsymbol{d}_y)\cos\varphi - \sin\varphi\,\boldsymbol{d}_z \\ \boldsymbol{g}_y = & \sin\theta\,\boldsymbol{d}_x + \cos\theta\,\boldsymbol{d}_y \\ \boldsymbol{g}_z = & (\cos\theta\,\boldsymbol{d}_x - \sin\theta\,\boldsymbol{d}_y)\sin\varphi + \cos\varphi\,\boldsymbol{d}_z \end{cases}$$

NO. Try to reverse angle to find reverse matrix failed

Rotation 1 (19)

$$\begin{cases} \boldsymbol{d}_{1x} = & \cos\theta\,\boldsymbol{g}_x + \sin\theta\,\boldsymbol{g}_y \\ \boldsymbol{d}_{1y} = & -\sin\theta\,\boldsymbol{g}_x + \cos\theta\,\boldsymbol{g}_y \\ \boldsymbol{d}_{1z} = & \boldsymbol{g}_z \end{cases} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}$$

Rotation 2 (20)

$$\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}\begin{bmatrix} \boldsymbol{d}_{1x} \\ \boldsymbol{d}_{1y} \\ \boldsymbol{d}_{1z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = [A]\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}$$

Inverse (21)

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}^{-1}\begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos-\theta & \sin-\theta & 0 \\ -\sin-\theta & \cos-\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos-\varphi & 0 & \sin-\varphi \\ 0 & 1 & 0 \\ -\sin-\varphi & 0 & \cos-\varphi \end{bmatrix}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

The order of the matrices is reversed. So, we do not change the angles in negative only, but reverse the order of the matrices.

Let rotate one vector in ground (20) (22)

$$\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix} = [A]\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = [A]^{-1}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

(23)

$$[x_d \quad y_d \quad z_d]\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix} = [x_d \quad y_d \quad z_d][A]\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}, \quad [x_g \quad y_g \quad z_g]\begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = [x_g \quad y_g \quad z_g][A]^{-1}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

$$[x_g \quad y_g \quad z_g] = [x_d \quad y_d \quad z_d][A] \qquad [x_d \quad y_d \quad z_d] = [x_g \quad y_g \quad z_g][A]^{-1}$$

Rotation around d_x, (24)

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix}$$

Rotation in the direction frame (24) (25)

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix} \quad \begin{bmatrix} \boldsymbol{d}_x \\ \boldsymbol{d}_y \\ \boldsymbol{d}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}$$

$$[x_d\quad y_d\quad z_d]\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} = [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix} = [x_e\quad y_e\quad z_e]\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix}$$

$$[x_d\quad y_d\quad z_d] = [x_e\quad y_e\quad z_e]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix},\quad [x_e\quad y_e\quad z_e] = [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}^{-1}$$

Rotation in ground frame (22) (26)

$$[x_g\quad y_g\quad z_g]\begin{bmatrix}g_x\\ g_y\\ g_z\end{bmatrix} = [x_g\quad y_g\quad z_g][A]^{-1}\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} = [x_d\quad y_d\quad z_d]\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix}$$

How a vector is rotated in a frame? (27)

$$\begin{aligned}y_{1d}+z_{1d}i = (y_d+z_d i)(\cos\psi+\sin\psi\, i) &= y_d(\cos\psi+\sin\psi\, i)+z_d i(\cos\psi+\sin\psi\, i)\\ &= \cos\psi\, y_d+\sin\psi y_d\, i+\cos\psi\, z_d i+\sin\psi\, i z_d i\\ &= y_d\cos\psi-\sin\psi\, z_d+(\sin\psi\, y_d+\cos\psi\, z_d)i\end{aligned}$$

$$[y_{1d}\quad z_{1d}] = [y_d\quad z_d]\begin{bmatrix}\cos\psi&\sin\psi\\-\sin\psi&\cos\psi\end{bmatrix}$$

Rotated 3D (27) (28)

$$[x_{1d}\quad y_{1d}\quad z_{1d}] = [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}$$

Rotation in ground direction frame, point 1 in direction frame (22) (29)

$$[x_{1d}\quad y_{1d}\quad z_{1d}]\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} = [x_{1d}\quad y_{1d}\quad z_{1d}][A]\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} = [x_{1g}\quad y_{1g}\quad z_{1g}]\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix}$$

d1 =after rotation, d= before rotation (30)

$$\begin{aligned}[x_{1d}\quad y_{1d}\quad z_{1d}]\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} &= [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix}\\ &= [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}[A]\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix}\end{aligned}$$

(25)

Transform d1 in to g1 in ground frame (31)

$$[x_{1g}\quad y_{1g}\quad z_{1g}] = [x_{1d}\quad y_{1d}\quad z_{1d}][A] = [x_d\quad y_d\quad z_d]\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}[A]$$

(23)(28)

Ground to direction, rotate, direction to ground (32)

$$[x_{1g}\quad y_{1g}\quad z_{1g}] = [x_g\quad y_g\quad z_g][A]^{-1}\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}[A]$$

(23)

Formula for rotating ψ in the ground frame around d_x. (33)

$$\begin{aligned}&[x_{1g}\quad y_{1g}\quad z_{1g}]\\ &= [x_g\quad y_g\quad z_g]\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}\cos\varphi&0&\sin\varphi\\0&1&0\\-\sin\varphi&0&\cos\varphi\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&\cos\psi&\sin\psi\\0&-\sin\psi&\cos\psi\end{bmatrix}\begin{bmatrix}\cos\varphi&0&\sin\varphi\\0&1&0\\-\sin\varphi&0&\cos\varphi\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\end{aligned}$$

Ground to direction (34)

$$[x_g\quad y_g\quad z_g]\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} = [x_g\quad y_g\quad z_g]\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}\cos\varphi&0&\sin\varphi\\0&1&0\\-\sin\varphi&0&\cos\varphi\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix}$$

Direction rotated

$$= [x_g\quad y_g\quad z_g]\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}\cos\varphi&0&\sin\varphi\\0&1&0\\-\sin\varphi&0&\cos\varphi\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&\cos-\psi&\sin-\psi\\0&-\sin-\psi&\cos-\psi\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix} = [x_{1d}\quad y_{1d}\quad z_{1d}]\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix}$$

(21)

Rotated in the base of direction frame (35)

$$\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix} = \begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix}$$

Rotated in the direction frame to ground frame (36)

$$[x_{1g}\quad y_{1g}\quad z_{1g}]\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} = [x_{1d}\quad y_{1d}\quad z_{1d}]\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} = [x_{1d}\quad y_{1d}\quad z_{1d}]\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix}$$

I cannot beat quaternion

What is ground frame in direction frame?

(21)

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}^{-1} = \begin{bmatrix}\cos-\theta & \sin-\theta & 0\\ -\sin-\theta & \cos-\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos-\varphi & 0 & \sin-\varphi\\ 0 & 1 & 0\\ -\sin-\varphi & 0 & \cos-\varphi\end{bmatrix}$$

$$= \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\varphi & 0 & -\sin\varphi\\ 0 & 1 & 0\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\varphi & 0 & -\sin\varphi\\ 0 & 1 & 0\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi & -\sin\theta & -\sin\varphi\cos\theta\\ \sin\theta\cos\varphi & \cos\theta & -\sin\varphi\sin\theta\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix}$$

$$\begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi & -\sin\theta & -\sin\varphi\cos\theta\\ \sin\theta\cos\varphi & \cos\theta & -\sin\varphi\sin\theta\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix}\begin{bmatrix}\boldsymbol{d}_x\\ \boldsymbol{d}_y\\ \boldsymbol{d}_z\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\varphi\,\boldsymbol{d}_x & -\sin\theta\boldsymbol{d}_y & -\sin\varphi\cos\theta\boldsymbol{d}_z\\ \sin\theta\cos\varphi\,\boldsymbol{d}_x & \cos\theta\,\boldsymbol{d}_y & -\sin\varphi\sin\theta\,\boldsymbol{d}_z\\ \sin\varphi\,\boldsymbol{d}_x & 0 & \cos\varphi\,\boldsymbol{d}_z\end{bmatrix}$$

$$= \begin{bmatrix}\cos\theta\cos\varphi\,\boldsymbol{d}_x - \sin\varphi\cos\theta\boldsymbol{d}_z - \sin\theta\boldsymbol{d}_y\\ \sin\theta\cos\varphi\,\boldsymbol{d}_x + \cos\theta\,\boldsymbol{d}_y - \sin\varphi\sin\theta\,\boldsymbol{d}_z\\ \sin\varphi\,\boldsymbol{d}_x + \cos\varphi\,\boldsymbol{d}_z\end{bmatrix} = \begin{bmatrix}(\cos\varphi\,\boldsymbol{d}_x - \sin\varphi\,\boldsymbol{d}_z)\cos\theta - \sin\theta\boldsymbol{d}_y\\ (\cos\varphi\,\boldsymbol{d}_x - \sin\varphi\,\boldsymbol{d}_z)\sin\theta + \cos\theta\,\boldsymbol{d}_y\\ \sin\varphi\,\boldsymbol{d}_x + \cos\varphi\,\boldsymbol{d}_z\end{bmatrix}$$

Compare with (17) g_x and g_z are x and y, rotation -φ, g_x and g_y are x and z, rotation -θ.

We rotate First are g_x and g_z -φ first, second are g_x and g_y, -θ second. Correct.

What is direction matrix (37)

$$\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}d & 0 & f\\ 0 & 1 & 0\\ -f & 0 & d\end{bmatrix}\begin{bmatrix}a & b & 0\\ -b & a & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}da & db & f\\ -b & a & 0\\ -fa & -fb & d\end{bmatrix}$$

Formula for rotating ψ in the ground frame around d_x.(33) (38)

$$[x_{1g}\quad y_{1g}\quad z_{1g}] = [x_g\quad y_g\quad z_g]\begin{bmatrix}da & db & f\\ -b & a & 0\\ -fa & -fb & d\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi & \sin\psi\\ 0 & -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}da & db & f\\ -b & a & 0\\ -fa & -fb & d\end{bmatrix}$$

Direction vector (39)

$$\boldsymbol{d}_x = u_x\boldsymbol{g}_x + u_y\boldsymbol{g}_y + u_z\boldsymbol{g}_z$$
$$= (a\boldsymbol{g}_x + b\boldsymbol{g}_y)d + f\boldsymbol{g}_z = ad\boldsymbol{g}_x + bd\boldsymbol{g}_y + f\boldsymbol{g}_z$$

(40)

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}^{-1} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\varphi & 0 & -\sin\varphi\\ 0 & 1 & 0\\ \sin\varphi & 0 & \cos\varphi\end{bmatrix}$$

$$= \begin{bmatrix}a & -b & 0\\ b & a & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}d & 0 & -f\\ 0 & 1 & 0\\ f & 0 & d\end{bmatrix} = \begin{bmatrix}da & -b & -fa\\ db & a & -fb\\ f & 0 & d\end{bmatrix} = tranpose\begin{bmatrix}da & db & f\\ -b & a & 0\\ -fa & -fb & d\end{bmatrix}$$

Rotated X_g=Original Xg * transpose [A] rotation angle * [A] (41)

$$[x_{1g}\quad y_{1g}\quad z_{1g}] = [x_g\quad y_g\quad z_g]\begin{bmatrix}da & -b & -fa\\ db & a & -fb\\ f & 0 & d\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi & \sin\psi\\ 0 & -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}da & db & f\\ -b & a & 0\\ -fa & -fb & d\end{bmatrix}$$