

# Building set and counting set

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**Abstract:** A counting set is the set of natural numbers with which a countable set is put in bijection. Is the counting set for the power set of  $\mathbb{N}$  the set  $\mathbb{N}$  that builds it?

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## 1. Two expressions of $\mathbb{N}$

In this article, we will study the power set of the set of natural numbers, which is often written as  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Usually, natural numbers are expressed in decimal system, but they can also be expressed in binary system. In Table 1, column  $\mathbb{D}$  is the decimal expression of  $\mathbb{N}$ , column  $\mathbb{B}$  is the binary expression of  $\mathbb{N}$ . The set of binary expressions of all natural numbers will be referred to as set  $\mathbb{B}$ . The binary expression of a natural number will be called binary number for short. We will use the set  $\mathbb{B}$  to build the power set of the set of natural numbers.

| $\mathbb{N}$ |              |
|--------------|--------------|
| $\mathbb{D}$ | $\mathbb{B}$ |
| 1            | 1            |
| 2            | 10           |
| 3            | 11           |
| 4            | 100          |
| ...          | ...          |

Table 1

## 2. Set $\mathbb{B}$ and power set

A binary number is a sequence of 0 and 1 that can be used to construct a subset of  $\mathbb{N}$ . For example in Table 2, the two 1 in the binary number 1010 select the numbers 4 and 2 from the sequence (...4, 3, 2, 1) just above to construct the set  $\{4, 2\}$  (See section 2 of «[On the uncountability](#) of [the power set of  \$\mathbb{N}\$](#) »).

|     |            |
|-----|------------|
| ... | 4, 3, 2, 1 |
| 1   | 0 1 0      |
|     | {4, 2}     |

Table 2

For building the power set of natural numbers, we put the sequence (...4, 3, 2, 1) in the first line of Table 3, below which we put the set  $\mathbb{B}$  in column  $\mathbb{B}$ . The members of the sequence (...4, 3, 2, 1) are infinitely many and constitute the set of natural numbers  $\mathbb{M}$ . The sequence (...4, 3, 2, 1) is written in reverse order to match the position of the binary numbers below. Each binary number selects corresponding numbers from the sequence (...4, 3, 2, 1) to construct a subset of natural numbers that we put in column  $\wp(\mathbb{M})$ .

| ... 4 3 2 1 ← $\mathbb{M}$ |              |              |  |
|----------------------------|--------------|--------------|--|
| $\wp(\mathbb{M})$          | $\mathbb{B}$ | $\mathbb{N}$ |  |
| $\{1\}$ ←                  | 1            | 1            |  |
| $\{2\}$ ←                  | 1 0          | 2            |  |
| $\{1, 2\}$ ←               | 1 1          | 3            |  |
| $\{3\}$ ←                  | 1 0 0        | 4            |  |
| $\{3, 1\}$ ←               | 1 0 1        | 5            |  |
| $\{3, 2\}$ ←               | 1 1 0        | 6            |  |
| $\{3, 2, 1\}$ ←            | 1 1 1        | 7            |  |
| $\{4, 3, 2, 1\}$ ←         | 1 0 0 0      | 8            |  |
| ...                        | ...          | ...          |  |

Table 3

Column  $\mathbb{B}$  contains all possible binary numbers, every binary number constructs a subset of  $\mathbb{M}$ . So, column  $\wp(\mathbb{M})$  contains all possible subsets of  $\mathbb{M}$  and is the power set of  $\mathbb{M}$ . On the other hand, the decimal expression of column  $\mathbb{B}$  is put in column  $\mathbb{N}$ , which is the set of natural numbers  $\mathbb{N}$ .

### 3. Building $\wp(\mathbb{N})$

We build  $\wp(\mathbb{N})$  in Table 4 and put in the first line the sequence (...8,7,6,5,4,3,2,1) which is the set  $\mathbb{N} = \{1,2,3,4,5,6,7,8,\dots\}$  in reverse order. Column  $\mathbb{B}_2$  contains all possible binary numbers and constructs all possible subsets of  $\mathbb{N}$  which are then put in column  $\wp(\mathbb{N})$ . So, column  $\wp(\mathbb{N})$  is the power set of  $\mathbb{N}$ . On the other hand, the decimal expression of column  $\mathbb{B}_2$  is column  $\mathbb{N}_2$  which is the set of natural numbers  $\mathbb{N}_2$ .

|                   |                |   |   |   |   |   |   |   |                |
|-------------------|----------------|---|---|---|---|---|---|---|----------------|
| ...               | 8              | 7 | 6 | 5 | 4 | 3 | 2 | 1 | ← $\mathbb{N}$ |
| $\wp(\mathbb{N})$ | $\mathbb{B}_2$ |   |   |   |   |   |   |   | $\mathbb{N}_2$ |
| $\{1\}$           | ←              | 1 |   |   |   |   |   |   | 1              |
| $\{2\}$           | ←              | 1 | 0 |   |   |   |   |   | 2              |
| $\{1,2\}$         | ←              | 1 | 1 |   |   |   |   |   | 3              |
| $\{3\}$           | ←              | 1 | 0 | 0 |   |   |   |   | 4              |
| $\{3,1\}$         | ←              | 1 | 0 | 1 |   |   |   |   | 5              |
| $\{3,2\}$         | ←              | 1 | 1 | 0 |   |   |   |   | 6              |
| $\{3,2,1\}$       | ←              | 1 | 1 | 1 |   |   |   |   | 7              |
| $\{4,3,2,1\}$     | ←              | 1 | 0 | 0 | 0 |   |   |   | 8              |
| ...               |                |   |   |   |   |   |   |   | ...            |

Table 4

Column  $\wp(\mathbb{N})$  is in one-to-one correspondence with column  $\mathbb{N}_2$  in Table 4, so the power set  $\wp(\mathbb{N})$  is countable. We will explain this result with respect to Cantor's theorem in the following. The method of putting  $\wp(\mathbb{N})$  and  $\mathbb{N}_2$  in one-to-one correspondence will be referred to as Linear-Counting.

### 4. About Cantor's theorem

#### 1) Counting $\wp(\mathbb{N})$

The result of Linear-Counting is quite surprising and seems to contradict Cantor's theorem because  $\wp(\mathbb{N})$  is known to be uncountable. To understand this result, let us explain why  $\wp(\mathbb{N})$  is uncountable for Cantor's theorem.

According to Cantor's theorem,  $\wp(\mathbb{N})$  is strictly bigger than  $\mathbb{N}$ . This is illustrated by Table 5 which put the singleton members of  $\wp(\mathbb{N})$  in one-to-one correspondence with  $\mathbb{N}$ . As  $\wp(\mathbb{N})$  possesses also subsets with more than 1 member like  $\{2,1\}$  or  $\{3,2,1\}$  which have no corresponding members in  $\mathbb{N}$ ,  $\wp(\mathbb{N})$  is uncountable. This method uses Cantor's theorem and tries to put  $\wp(\mathbb{N})$  in one-to-one correspondence with the set  $\mathbb{N}$ , which is equivalent to count the members of  $\wp(\mathbb{N})$  using the set  $\mathbb{N}$ .

|           |              |
|-----------|--------------|
| Singleton | $\mathbb{N}$ |
| $\{1\}$   | 1            |
| $\{2\}$   | 2            |
| $\{3\}$   | 3            |
| $\{4\}$   | 4            |
| ...       | ...          |

Table 5

But Linear-Counting method counts  $\wp(\mathbb{N})$  using the set  $\mathbb{N}_2$ . Because the used sets are different, the 2 methods do not get the same conclusion. Which of these 2 methods is correct? In article [Georg Cantor](#) of Wikipedia one reads: "In 1878, Cantor defined countable sets (or denumerable sets) as sets which can be put into a 1-to-1 correspondence with the natural numbers". But Georg Cantor has not specified which set of natural numbers should be used to count infinite set. Fortunately, he has given a model of counting: the counting of the set of rational numbers  $\mathbb{Q}$ .

#### 2) Counting $\mathbb{Q}$

$\mathbb{Q}$  is the set of ratios  $i/j$  where  $i$  and  $j$  are natural numbers. All values of the number  $i$  constitute the set of natural numbers  $\mathbb{N}$ . Table 6 shows the ratios  $i/j$  with the values of  $i$  in the column  $\mathbb{N}$  and the values of  $j$  in the first line.

To count the members of  $\mathbb{Q}$  we define the number  $n_{\mathbb{Q}}$ , which follows the back-and-forth diagonal line that we call counting line.  $n_{\mathbb{Q}}$  equals 1 at the start of the counting line, then it increases by 1 for each ratio  $i/j$ . As  $n_{\mathbb{Q}}$  increases indefinitely, the set of all values of  $n_{\mathbb{Q}}$  is a set of natural numbers, which we name  $\mathbb{N}_{\mathbb{Q}}$ . While the counting line passes across all ratios, the set  $\mathbb{N}_{\mathbb{Q}}$  counts all members of  $\mathbb{Q}$ .

Each value of  $i$  corresponds to a straight line of the counting line. The value of  $n_{\mathbb{Q}}$  corresponding to  $i$  equals  $i(i+1)/2$  which is the number of ratios already counted at the end of the  $i^{\text{th}}$  straight line. These values of  $n_{\mathbb{Q}}$  are given in column  $\mathbb{N}_{\mathbb{Q}}$ .

Table 7 compares  $\mathbb{N}_{\mathbb{Q}}$  with  $\mathbb{Q}$ . In the first line are the members of  $\mathbb{Q}$  along the counting line, in the second line are the corresponding values of  $n_{\mathbb{Q}}$ . We see that  $\mathbb{Q}$  is in one-to-one correspondence with  $\mathbb{N}_{\mathbb{Q}}$ . So,  $\mathbb{Q}$  is proven countable by being counted with  $\mathbb{N}_{\mathbb{Q}}$ .

|     |     |     |     |     |     |     |     |     |     |                |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----------------|
| ... | 8   | 7   | 6   | 5   | 4   | 3   | 2   | 1   | N   | N <sub>Q</sub> |
| ... | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 1/1 | 1   | 1              |
| ... | 2/8 | 2/7 | 2/6 | 2/5 | 2/4 | 2/3 | 2/2 | 2/1 | 2   | 3              |
| ... | 3/8 | 3/7 | 3/6 | 3/5 | 3/4 | 3/3 | 3/2 | 3/1 | 3   | 6              |
| ... | 4/8 | 4/7 | 4/6 | 4/5 | 4/4 | 4/3 | 4/2 | 4/1 | 4   | 10             |
| ... | 5/8 | 5/7 | 5/6 | 5/5 | 5/4 | 5/3 | 5/2 | 5/1 | 5   | 15             |
| ... | 6/8 | 6/7 | 6/6 | 6/5 | 6/4 | 6/3 | 6/2 | 6/1 | 6   | 21             |
| ... | 7/8 | 7/7 | 7/6 | 7/5 | 7/4 | 7/3 | 7/2 | 7/1 | 7   | 28             |
| ... | 8/8 | 8/7 | 8/6 | 8/5 | 8/4 | 8/3 | 8/2 | 8/1 | 8   | 36             |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ...            |

Table 6

|                |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |                |     |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----------------|-----|
| Q              | 1/1 | 2/1 | 1/2 | 1/3 | 2/2 | 3/1 | 4/1 | 3/2 | 2/3 | 1/4 | ... | 5/1 | ... | 1/6 | ... | 7/1 | ... | 1/8 | ... | $i/i$ or $i/1$ | ... |
| N <sub>Q</sub> | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | ... | 15  | ... | 21  | ... | 28  | ... | 36  | ... | $i(i+1)/2$     | ... |
| N              | 1   |     | 2   |     |     | 3   |     |     |     | 4   |     | 5   |     | 6   |     | 7   |     | 8   | ... | $i$            | ... |

Table 7

What if  $\mathbb{Q}$  is counted with  $\mathbb{N}$  the set of all values of  $i$ ? We rewrite the column  $\mathbb{N}$  of Table 6 in the third line of Table 7 where the values of  $i$  are written under their corresponding  $n_{\mathbb{Q}}$  leaving many cells void. So, the members of  $\mathbb{Q}$  above these void cells do not have corresponding members in  $\mathbb{N}$ . In fact,  $\mathbb{N}$  can be put in one-to-one correspondence with the set of  $i/i$  which is a strict subset of  $\mathbb{Q}$  (see Table 8). So,  $\mathbb{N}$  is less numerous than  $\mathbb{Q}$  and if we count  $\mathbb{Q}$  using  $\mathbb{N}$ , we will find that  $\mathbb{Q}$  cannot be put in one-to-one correspondence with  $\mathbb{N}$  and obtain the wrong conclusion that  $\mathbb{Q}$  is uncountable.

|       |     |
|-------|-----|
| $i/i$ | N   |
| 1/1   | 1   |
| 2/1   | 2   |
| 3/1   | 3   |
| 4/1   | 4   |
| ...   | ... |

Table 8

### 3) Building set and counting set

What have we learned from the counting of  $\mathbb{Q}$ ? We have learned that  $\mathbb{Q}$  is countable only because it is counted with  $\mathbb{N}_{\mathbb{Q}}$ . We call  $\mathbb{N}_{\mathbb{Q}}$  the counting set for  $\mathbb{Q}$ . The set  $\mathbb{Q}$  is built with the set  $\mathbb{N}$  through  $i$ , so  $\mathbb{N}$  is called the building set of  $\mathbb{Q}$ . From this observation we deduce a general rule for counting infinite set: an infinite set should not be counted with its proper building set. Otherwise, if a countable set is bigger than its building set, it will be found uncountable.

For  $\wp(\mathbb{N})$  the power set of  $\mathbb{N}$ , the correct counting set is under debate. The counting set for Linear-Counting method is  $\mathbb{N}_2$  while for the method using Cantor's theorem the counting set is  $\mathbb{N}$  the building set of  $\wp(\mathbb{N})$ . For deciding which method is correct, let us see how Cantor's theorem is used for proving the uncountability of  $\wp(\mathbb{N})$ .

In the article [Cantor's theorem](#) of Wikipedia, **Theorem (Cantor)** is stated as below:

Let  $f$  be a map from set  $A$  to its power set  $\wp(A)$ . Then  $f: A \rightarrow \wp(A)$  is not surjective.

Cantor's theorem stated above does not directly conclude that  $\wp(\mathbb{N})$  is uncountable, but only that the map from  $\mathbb{N}$  to  $\wp(\mathbb{N})$  is not surjective. A second element is needed for the proof.

The second element is the criterion of uncountability. In the article [Countable set](#) of Wikipedia one reads “A set is countable if it has the same cardinality as some subset of the set of natural numbers. Otherwise, it is uncountable.” That is: all sets more numerous than  $\mathbb{N}$  are uncountable.

So, the uncountability of  $\wp(\mathbb{N})$  is deduced in 3 steps:

- 1) All sets more numerous than  $\mathbb{N}$  are uncountable.
- 2)  $\wp(\mathbb{N})$  is more numerous than  $\mathbb{N}$  according to Cantor's theorem.
- 3) Then,  $\wp(\mathbb{N})$  is uncountable.

We see that the uncountability of  $\wp(\mathbb{N})$  does not only depend on Cantor's theorem, but also on the first step. If the first step were wrong, this deduction collapses.

Can a set more numerous than  $\mathbb{N}$  be countable? The section “Counting  $\mathbb{Q}$ ” shows that the set  $\mathbb{Q}$  is more numerous than its building set  $\mathbb{N}$  while being countable. This example proves that being more numerous than  $\mathbb{N}$  does not imply necessarily uncountability. So, the first step is wrong and  $\wp(\mathbb{N})$  is not proven uncountable by the above 3-step-deduction.

However, countable  $\wp(\mathbb{N})$  does not contradict Cantor's theorem because  $\wp(\mathbb{N})$  is bigger than  $\mathbb{N}$ . Also,  $\mathbb{N}_2$  is bigger than  $\mathbb{N}$  too. But can the set of natural numbers exist in more than one size?

## 5. Related Sets

For understanding why  $\mathbb{N}_2$  or  $\mathbb{N}_Q$  is bigger than  $\mathbb{N}$ , let us see equations (1) which express the relation between  $\mathbb{N}_Q$  and  $\mathbb{N}$ .  $n$  is a member of  $\mathbb{N}$  that corresponds to the member  $n_Q$  of  $\mathbb{N}_Q$ . The number  $n_Q$  is a function of the number  $n$ :  $n_Q = n(n+1)/2$  (see Table 7). As  $n_Q$  is always bigger than  $n$  for all  $n$ , the finite set  $\{1, 2, 3, \dots, n\}$  is always a strict subset of the set  $\{1, 2, 3, \dots, n_Q = n(n+1)/2\}$ . So,  $\mathbb{N}$  is a strict subset of  $\mathbb{N}_Q$  and thus, smaller than  $\mathbb{N}_Q$ . This is confirmed by the many void cells in the line  $\mathbb{N}$  comparing to the line  $\mathbb{N}_Q$  in Table 7.

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, \dots, n, \dots\} \\ \text{For all } n, n_Q &= n(n+1)/2 \\ \mathbb{N}_Q &= \{1, 2, 3, \dots, n_Q-2, n_Q-1, n_Q, \dots\} \end{aligned} \quad (1)$$

For  $\wp(\mathbb{N})$ , the relation between the building set  $\mathbb{N}$  and the counting set  $\mathbb{N}_2$  is expressed by equations (2) which show that because  $n < n_2$ , the set  $\mathbb{N}$  is a strict subset of  $\mathbb{N}_2$  and thus, smaller than  $\mathbb{N}_2$ .

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, \dots, n, \dots\} \\ \text{For all } n, n_2 &= 2^n \\ \mathbb{N}_2 &= \{1, 2, 3, \dots, n_2-2, n_2-1, n_2, \dots\} \end{aligned} \quad (2)$$

In general, a set of natural numbers  $\mathbb{N}_r$  can be related to another set of natural numbers  $\mathbb{N}$  through a function  $f$ . If the sets  $\mathbb{N}_r$  and  $\mathbb{N}$  are in the form expressed by equations (3), they are **Related Sets**.

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, \dots, n, \dots\} \\ \text{For all } n, n_r &= f(n) \\ \mathbb{N}_r &= \{1, 2, 3, \dots, n_r-2, n_r-1, n_r, \dots\} \end{aligned} \quad (3)$$

$\mathbb{N}$  was the one and only set of natural numbers for Georg Cantor. It was for this reason that, seeing  $\wp(\mathbb{N})$  bigger than  $\mathbb{N}$ , he concluded that  $\wp(\mathbb{N})$  was uncountable. He did not realize that this result contradicted his own proof for the countability of  $\mathbb{Q}$ , which he would not get if the counting set were the building set of  $\mathbb{Q}$ . Unknowingly, he used 2 sets of natural numbers of different sizes, that is, 2 related sets, to construct and count  $\mathbb{Q}$ .

## 6. Consequence

For keeping set theory coherent, we must apply the same rule of counting to all infinite sets. That is, we should count all infinite sets either with counting sets which may have different size than their building sets, or with their building sets. We are not allowed to count  $\mathbb{Q}$  with a counting set whilst we count  $\wp(\mathbb{N})$  with its building set. So, we have to choose one method of counting from the 2 options below:

- 1) We count all infinite sets with their building set  $\mathbb{N}$ . In this case,  $\wp(\mathbb{N})$  is uncountable. But  $\mathbb{Q}$  will also be uncountable because its building set is smaller than itself.
- 2) We count all infinite sets with counting sets. In this case,  $\mathbb{Q}$  is countable. But  $\wp(\mathbb{N})$  will also be countable because counted with a counting set bigger than its building set.

## Note

I think that my idea in this article is important and correct. But I lack the training for theoretical mathematics to write in suitable form for publishing in mathematical journals. Nevertheless, this idea is worth spreading because if accepted, new field of set theory will emerge, which will call for new studies and papers. So, I authorize you, mathematician, to rewrite this article in suitable form for publishing under your name, at the condition that you cite my article as the original source and give in your publication the internet links to my article which are:

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