Examination of Cantor's proofs for uncountability and axiom for counting infinite sets

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Abstract: An analysis of Cantor's theory of uncountable sets: The logic of his proofs has some weaknesses. Cantor assumes for both his proofs that all real numbers (set R) are in a list (list L). Considering L as a set this assumption assumes R belongs to L. This makes the claim "a real number is constructed but is not in the list L" questionable. We propose a solution to this problem, an axiom for counting infinite sets and a solution to continuum hypothesis.

1. Introduction

According to Georg Cantor there are sets that cannot be put into one-to-one correspondence with the set of natural numbers ℕ [1]. These sets are said uncountable. However, his theory of infinite sets has been criticized in several areas by mathematicians and philosophers [2].

To examine his proofs, we explain first his famous diagonal argument. A quotation from the page Wikipedia «Cantor's diagonal argument» [3] reads: "The proof starts by assuming that T is countable. Then all its elements can be written in an enumeration s_1, s_2, ... , s_n, ... Applying the previous lemma to this enumeration produces a sequence s that is a member of T, but is not in the enumeration. However, if T is enumerated, then every member of T, including this s, is in the enumeration. This contradiction implies that the original assumption is false. Therefore, T is uncountable."

2. Construction of the contradicting number

The sequence s above can be constructed using a method that we call diagonal construction. Let us explain it with the list of the 8 binary numbers s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 given in Table 1. For constructing a special number from this list we take the first bit of the first number, then the second bit of the second number and so on till s_8. The 8 chosen bits make the number s_{diag}. Then, we flip every bit of s_{diag} to construct the number s_{out}. That is, if a bit of s_{diag} equals 0, the same bit of s_{out} equals 1 and vice versa. Because of the flip of the 8 bits, s_{out} does not equal any of the 8 original numbers.

We can construct the number s_{out} from lists of numbers with n bits. Because every bit of s_{out} is flipped, s_{out} does not equal any of the numbers of the original list. The number of bits can increase indefinitely to infinity. In this case, the list becomes infinitely long, but the diagonal construction can still construct s_{out} and ensure that it does not equal any of the numbers of the original list.

List s_i i=1…8	Bits b_j j=1…8
s_1	01100110
s_2	00000101
s_3	10000100
s_4	10010001
s_5	01101010
s_6	01110110
s_7	01110100
s_8	00111101
s_{diag}	00011101
s_{out}	11100010

Table 1

Let the list used to construct s_{out} be referred as L_c. When the number of entries of L_c is n we refer it as $L_c(n)$. Other than his diagonal argument Cantor has provided a method to construct s_{out} using nested intervals. For this method we refer the used list also as L_c.

3. Cantor's two proofs

Cantor's diagonal argument and nested intervals method are both proof by contradiction which we summarize as the 4 steps deduction below:

1. Assumption A: All real numbers are in a list.
2. s_{out} is constructed using diagonal or nested intervals and is a real number.
3. s_{out} is not in the list, contradicting thus the Assumption A.
4. Conclusion: the set of all real numbers cannot be put into a list.

- Meaning of "All real numbers are in a list"

The Assumption A is "All real numbers are in a list" where "All real numbers" means the set ℝ and the "list" is an infinite list of all real numbers which we name L. From the Assumption A we deduce that if r is a real number, r belongs to the list L, see (1).

$$\forall r \in \mathbb{R} \Rightarrow r \in L \quad (1)$$

In other words, the set ℝ is a subset of the list L, see (2).

$$(\forall r \in \mathbb{R} \Rightarrow r \in L) \Rightarrow \mathbb{R} \subseteq L \quad (2)$$

4. Real numbers as points on the number line

a. Cantor's diagonal argument

Let us see the 4 steps deduction above. The step 3 states: "s_{out} is not in the list". The "list" here is in fact the list used to construct s_{out} and we have referred it as L_c in section 2. So, the step 3 claims in fact "s_{out} is not in L_c". Does L_c equal the list L mentioned by the Assumption A? According to the Assumption A ℝ is a subset of L, see (2). s_{out} is a real number and thus, belongs to ℝ and then to L. But s_{out} is not in the list L_c. Having not the same members, the lists L and L_c are not the same list.

Because L and L_c are different lists, the claim "s_{out} is not in L_c" does not contradict that s_{out} is in L, which means that the contradiction claimed by the step 3 does not exist. Because there is no contradiction, Cantor's diagonal argument does not prove that ℝ is uncountable.

Lemma 1: Cantor's diagonal argument does not prove that ℝ is uncountable.

b. Cantor's nested intervals proof

Cantor's diagonal argument expresses real numbers in binary or digital form. But irrational numbers cannot be expressed, for example, $\sqrt{2}$ or π. On the other hand, Cantor has proposed the idea of using a sequence of nested intervals to construct a real number that is out of the list while representing real numbers only as points on the number line, see his first set theory article [4] and « Cantor's first set theory article»[5].

In this proof Cantor assumed also that "All real numbers are in a list". So, we have ℝ ⊆ L too, see (2). Because the constructed real number belongs to ℝ, it forcefully belongs to L. So, the constructed real number s_{out} is in the list L and the Assumption A is not contradicted. Then the nested intervals proof does not prove that ℝ is uncountable.

Lemma 2: The nested intervals proof does not prove that ℝ is uncountable.

c. The logic of Cantor's two proofs

Cantor's nested intervals proof and diagonal argument both follow the scheme of the 4 steps deduction exposed in "Introduction". Using nested intervals or diagonal construction the second step constructs the real number s_{out} which is out of the list. The third step declares the contradiction.

Cantor's both proofs rely on the Assumption A "All real numbers are in a list" where "All real numbers" means the set ℝ and the "list" is an infinite list of all real numbers which we name L. So, ℝ is a subset of the list L, see (2). From the Assumption A we deduce that if r is a real number, r belongs to ℝ and thus, to the list L. So, being a real number s_{out} belongs to the list L.

In the opposite way, Cantor's both proofs claim that s_{out} is not in the list. We derive from Assumption A that if r is not in L, r is not in ℝ and is not a real number. So, Cantor's claim becomes: the s_{out} he constructed is not a real number. Do his proofs go out of ℝ?

In fact, Cantor constructs s_{out} from lists named L_c which he has not proven to be the list L. The section 4.a shows that L_c is different from L. So, Cantor's claim "s_{out} is not in the list" is in reality "s_{out} is not in L_c". Since L_c is not the list L, even s_{out} is not in L_c, s_{out} still belongs to the list L. So, s_{out} does not contradict the Assumption A and Cantor fails to prove R is uncountable.

The hole in Cantor's logic for both proofs is that, s_{out} being a real number and ℝ ⊆ L, s_{out} is in the list L, see (3).

$$s_{out} \in \mathbb{R} \text{ and } \mathbb{R} \subseteq L \Rightarrow s_{out} \in L \quad (3)$$

5. Real numbers as binary numbers

a. n-bits numbers

Why did Cantor claim that the flipped number s_{out} does not belong to the list? Let us explain with binary numbers. Let n be the number of bits of the binary numbers that make a list. The diagonal construction imposes that the list contains n binary numbers.

On the other hand, we know that there are 2^n possible n-bits numbers in all. We write them randomly in a list named $L_{2^\wedge n}$. The number s_{out} is constructed with the first n numbers, see Table 2. So, $L_{2^\wedge n}$ is split into 2 parts:

1. L_n: The list of the first n numbers which are used by the diagonal construction.
2. $L_{2^\wedge n-n}$: The list of the remaining 2^n-n numbers.

s_{out} does not equal any of the numbers in L_n, which respects Cantor's claim. But, because the list $L_{2^\wedge n}$ is longer than L_n, s_{out} can sit after the n^{th} number in $L_{2^\wedge n}$, say at the position n+p, that is, $s_{out} = s_{n+p}$. So, s_{out} belongs to the list $L_{2^\wedge n}$ while being not in L_n. I have explained this phenomenon in the paper «Hidden assumption of the diagonal argument»[6].

d. Constructing numbers smaller than one

Let $b_1b_2b_3 \dots b_n$ be a natural number in binary form that range from 0 to 2^n-1. We divide it by 2^n in equation (4) and obtain the binary number $0.b_1b_2b_3 \dots b_n$ which is smaller than one.

$$0 \leq b_1 b_2 b_3 \cdots b_n \leq 2^n - 1, \qquad \frac{b_1 b_2 b_3 \cdots b_n}{2^n} = 0.b_1 b_2 b_3 \cdots b_n \quad (4)$$

By dividing with 2^n all the numbers in Table 2, we obtain a list of all the n-bits numbers that are smaller than one. These numbers are denoted as f_i with i being their subscript, see Table 3. The flipped number for f_i is f_{out} and equals s_{out} divided by 2^n, see (5). f_{out} does not equal any of the first n numbers in Table 3.

$$f_{out} = \frac{s_{out}}{2^n} \quad (5)$$

Let us make the change of variable in (6).

$$m = 2^n \quad (6)$$

Then the number of bits equals $\log_2(m)$, see (7).

$$n = \log_2(m) \quad (7)$$

With (6) the subscript of the last number in Table 3 becomes m.

$i=1\dots2^n$	Bits b_j, $j=1\dots n$
s_1	011001…1
s_2	000001…0
s_3	100001…1
⋮	⋮
s_n	001111…1
s_{n+1}	000110…1
⋮	⋮
$s_{out} = s_{n+p}$	111000…0
⋮	⋮
$s_{2^\wedge n}$	111111…1

Table 2

$i=1\dots m$	Bits b_j, $j=1\dots n$
f_1	0.011001…1
f_2	0.000001…0
f_3	0.100001…1
⋮	⋮
f_n	0.001111…1
f_{n+1}	0.000110…1
⋮	⋮
$f_{out} = f_{n+p}$	0.111000…0
⋮	⋮
f_m	0.111111…1

Table 3

$i=1\dots\infty$	Bits b_j, $j=1\dots\log_2(\infty)$
r_1	0.011001…
r_2	0.000001…
r_3	0.100001…
⋮	⋮
$r_{\log2(\infty)}$	0.001111…
$r_{\log2(\infty)+1}$	0.000110…
⋮	⋮
$r_{out} = r_{\log2(\infty)+p}$	0.111000…
⋮	⋮
r_∞	0.111111…

Table 4

b. List of real numbers

When m increases indefinitely to reach infinity, we write m=∞ and the number of bits equals $\log_2(\infty)$, see (7). In this case, the numbers f_i in Table 3 will have infinitely many bits because $\log_2(\infty)$ is the numbers of bits. Then f_i become the real numbers r_i, which equal $0.b_1b_2b_3 \dots$, see (8). Equation (8) is in fact (4) with n=∞.

$$\frac{b_1 b_2 b_3 \cdots}{2^{\infty}} = 0.b_1 b_2 b_3 \cdots \quad (8)$$

The list in Table 3 becomes the infinitely long list of r_i in Table 4 which we label as L_∞. Like for the list $L_{2^\wedge n}$, L_∞ is split into 2 parts:

1. $L_{\log2(\infty)}$: The list of the first $\log_2(\infty)$ real numbers which are used by the diagonal construction.
2. $L_{\infty-\log2(\infty)}$: The list of the remaining ∞-$\log_2(\infty)$ real numbers in L_∞.

The flipped number for r_i is named r_{out} and equals f_{out} with m=∞. r_{out} is a real number but does not equal any number of the first list $L_{\log2(\infty)}$, which respects Cantor's diagonal argument. But the list L_∞ being longer than $L_{\log2(\infty)}$, we would find r_{out} at the position $\log_2(\infty)$+p, $r_{out} = r_{\log2(\infty)+p}$. So, r_{out} belongs to the list L_∞, see Table 4.

The list of f_i contains all possible n-bits numbers smaller than one and this is true for all n while n increasing indefinitely. The list L_∞ being the list $L_{2^\wedge n}$ for n=∞, it contains all possible real numbers smaller than one, that is, L_∞ contains $\mathbb{R}_1$ which is the set of all real numbers in the interval [0, 1]. As $\mathbb{R}$ can be put into one-to-one correspondence with $\mathbb{R}_1$ and L_∞ is in one-to-one correspondence with ℕ, ℝ can be put into one-to-one correspondence with ℕ and is countable.

Lemma 3: The set ℝ is countable.

In the paper «Cardinality of the set of decimal numbers »[7] I have created a list of all real numbers of the interval [0, 1] in decimal form using (8) except the divisor was 10^∞.

c. About Cantor's conclusion

Why r_{out} was out of the list for Cantor? In fact, because the real numbers have $\log_2(\infty)$ bits the diagonal covers only the first $\log_2(\infty)$ real numbers, which excludes all the real numbers of the list $L_{\infty-\log2(\infty)}$. In consequence, r_{out} is in the list L_∞ while being out of the list $L_{\log2(\infty)}$.

Because the list $L_{\infty-\log2(\infty)}$ is ignored by Cantor's diagonal argument, the latter reaches naturally wrong conclusion. I have already explained this error in the paper «Building set and counting set»[8].

Lemma 4: The error of Cantor's diagonal argument is: the real numbers in the list $L_{\infty-\log2(\infty)}$ are ignored.

6. Axiom for counting infinite set

For studying the countability of infinite sets whose members do not have mathematical expression, for example, the set of all functions, we should better have a practical tool. Let us imagine infinitely many points in a magic box, the volume of each point being zero. In the wall of the box is a small hole through which the points go out. The points are scattered evenly and randomly inside the box so that every point has an equal chance to exit. The hole is guarded by a demon that lets the points go out only one by one. When a point passes through the hole the demon marks it with its number of exit.

If each point represents a real number and the number of exit its subscripts, the points that have come out will form a list of subscripted real numbers: $\{s_1, s_2, s_3, ...\}$. Let us put the interval [0, 1] into the magic box and shake the real numbers loose. This way all real numbers of the set $\mathbb{R}_1$ are scattered in the magic box. Through the hole the real numbers will come out forever and the demon will make an infinite list of subscripted real numbers. Because each point has an equal chance of exit, all real numbers will be out at the end of time. As the demon will never run out of natural numbers, his infinite list will be a one-to-one correspondence between $\mathbb{R}_1$ and ℕ.

But time never ends and the one-to-one correspondence between $\mathbb{R}_1$ and ℕ cannot be seen. However, as the correspondence between individual real number and natural number is forever true, we can devise the axiom for counting infinite sets below.

Axiom A: Let S be any infinite set whose members are distinct individual entities. S can be put into one-to-one correspondence with the set of natural numbers ℕ and is countable.

An axiom cannot be proven but must be verified by all known cases. The Axiom A is verified by ℕ and all multidimensional $\mathbb{N}^n$ which include the set of rational numbers ℚ. The set ℝ verifies this axiom too, see Lemma 3.

For 2D real space $\mathbb{R}^2$ Cantor has shown that $\mathbb{R}^2$ can be put into one-to-one correspondence with $\mathbb{R}$ using the following technique [9]:

> Let (x, y) = (0.abcde …, 0.αβγδ…) be a point in $\mathbb{R}^2$, a corresponding real number z is constructed with (x, y) and equals 0.aαbβcγdδe… .

The set of z is $\mathbb{R}$ which is thus in one-to-one correspondence with $\mathbb{R}^2$. So, $\mathbb{R}^2$ verifies the Axiom A.

The power set of $\mathbb{N}$ has been shown countable in the paper «On the uncountability of the power set of $\mathbb{N}$»[10] and thus, verifies the Axiom A.

According to the Axiom A, sets of infinitely many anything are countable provided that the things can be represented by distinct individual points. Mathematical functions are such things, then the set of all functions is countable too.

Using the Axiom A we deduce that infinity is unique and there is no cardinality higher than $\aleph_0$. That is, the cardinalities of all infinite sets are $\aleph_0$ and there are no higher cardinalities such as $\aleph_1$, $\aleph_2$, $\aleph_3$, ... So, the Continuum hypothesis is solved because $\aleph_0$ is the highest cardinality.

Lemma 5: The cardinality of all infinite sets is $\aleph_0$**.**

References

[1] Georg Cantor (1891). "Ueber eine elementare Frage der Mannigfaltigkeitslehre". Jahresbericht der Deutschen Mathematiker-Vereinigung. 1: 75–78. English translation: Ewald, William B., ed. (1996). From Immanuel Kant to David Hilbert: A Source Book in the Foundations of Mathematics, Volume 2. Oxford University Press. pp. 920–922. ISBN 0-19-850536-1.

[2] Wikipedia «Controversy over Cantor's theory», https://en.wikipedia.org/w/index.php?title=Controversy_over_Cantor%27s_theory&oldid=1097352064

[3] Wikipedia «Cantor's diagonal argument», https://en.wikipedia.org/w/index.php?title=Cantor%27s_diagonal_argument&oldid=1087488305

[4] In letter to Dedekind dated December 25, 1873, Cantor states that he has written and submitted "a short paper" titled On a Property of the Set of All Real Algebraic Numbers. (Noether & Cavaillès 1937, p. 17; English translation: Ewald 1996, p. 847.)

[5] Wikipedia «Cantor's first set theory article», https://en.wikipedia.org/w/index.php?title=Cantor%27s_first_set_theory_article&oldid=1102795380

[6] Kuan Peng, 2016, « Hidden assumption of the diagonal argument», https://www.academia.edu/20805963/Hidden_assumption_of_the_diagonal_argument

[7] Kuan Peng, 2016, «Cardinality of the set of decimal numbers », https://www.academia.edu/23155464/Cardinality_of_the_set_of_decimal_numbers

[8] Kuan Peng, 2018 , «Building set and counting set», https://www.academia.edu/37590687/Building_set_and_counting_set

[9] Fernando Q. Gouvea, «Was Cantor Surprised?» THE MATHEMATICAL ASSOCIATION OF AMERICA [Monthly 118 March 2011], https://www.maa.org/sites/default/files/pdf/pubs/AMM-March11_Cantor.pdf

[10] Kuan Peng, 2016, «On the uncountability of the power set of $\mathbb{N}$», https://www.academia.edu/21601620/On_the_uncountability_of_the_power_set_of_N