Step by step rotation in normal and high dimensional space and meaning of quaternion

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Abstract: The orientation of body in space is defined 3 by angles. The step by step rotation process and chain of three-dots multiplication give an easy way to compute pile of rotations in 3D and high dimensional space and give a general orientation system. A visualization of quaternion is proposed.

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1. Introduction

The orientation of a rigid body in space is defined by 3 angles, for example the angles of pitch, roll and yaw of an airplane. The most commonly used orientation systems are Euler angles and Tait–Bryan angles. Quaternion is often used for computing rotation of body.

In this article we will present a new system for computing rotation in space that consists of rotating the proper frame of reference of the rigid body with respect to that of the space. This system is more general than Euler angles and Tait–Bryan angles and easier to use than rotation matrix and quaternion.

Below we will refer 2-dimensional space as 2D and 3-dimensional space as 3D. First, we will explain our approach using 2D body, then apply it to 3D body and body of even higher dimensions. We will use our system to give a visualization of the geometrical meaning of quaternion in 3D space.

Let us begin with defining the frame of reference of a rotated body and that of the space. A frame is defined by its basis that is a set of orthogonal unit vectors. The frame of the space is fixed. The frame of the rotated body is solidary to the body and rotates with it. The orientation in space of the body is defined by the angles its frame makes with that of the space.

We refer the basis of the frame of the space as space basis and that of the body the proper basis. The set of unit vectors of the space basis is ($\mathbf{e}_x$, $\mathbf{e}_y$) for 2D space and ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) for 3D space. The set of unit vectors of the proper basis is ($\mathbf{e}_x^i$, $\mathbf{e}_y^i$) and ($\mathbf{e}_x^i$, $\mathbf{e}_y^i$, $\mathbf{e}_z^i$) when it has gone *i* rotations. *i=0* at the start.

2. 2D body orientation

a. Basis rotation

Let us take a flat body, for example the smiling face in Figure 1, and rotate it in its plane. Before being rotated the proper basis of the body is ($\mathbf{e}_x^0$, $\mathbf{e}_y^0$) and coincides with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$), see equation (1).

The angle of the first rotation is θ_1 and the proper basis becomes ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$) , see Figure 2. The vectors $\mathbf{e}_x^1$ and $\mathbf{e}_y^1$ are expressed with $\mathbf{e}_x$, $\mathbf{e}_y$ and θ_1 in equation (2). By writing $\mathbf{e}_x$, $\mathbf{e}_y$ with complex number in (3), the vectors $\mathbf{e}_x^1$, $\mathbf{e}_y^1$ are expressed in (4). Using the Euler's exponential formula for complex number given in (5), $\mathbf{e}_x^1$ and $\mathbf{e}_y^1$ are expressed in the compact form (6).

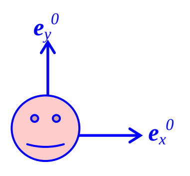

Figure 1

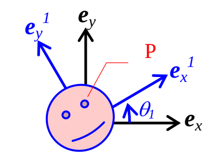

Figure 2

After the first rotation we rotate the body an angle θ_2. The proper basis becomes ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$) which is at the angle θ_2 with respect to ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$), see Figure 3. We refer ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$) as first proper basis and ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$) the second one. Since the angle θ_2 is with respect to ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$) but not to the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$), the second proper basis ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$) is expressed with the first proper basis ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$) in (7). Using (6), ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$) is expressed with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$) in (8).

$$(\boldsymbol{e}_x^0, \boldsymbol{e}_y^0) = (\boldsymbol{e}_x, \boldsymbol{e}_y) \tag{1}$$

$$\begin{aligned} \boldsymbol{e}_x^1 &= \cos\theta_1\, \boldsymbol{e}_x + \sin\theta_1\, \boldsymbol{e}_y \\ \boldsymbol{e}_y^1 &= -\sin\theta_1\, \boldsymbol{e}_x + \cos\theta_1\, \boldsymbol{e}_y \end{aligned} \tag{2}$$

$$i\boldsymbol{e}_x = \boldsymbol{e}_y, \qquad i\boldsymbol{e}_y = -\boldsymbol{e}_x \tag{3}$$

$$\begin{aligned} \boldsymbol{e}_x^1 &= (\cos\theta_1 + i\sin\theta_1)\boldsymbol{e}_x \\ \boldsymbol{e}_y^1 &= (\cos\theta_1 + i\sin\theta_1)\boldsymbol{e}_y \end{aligned} \tag{4}$$

$$e^{i\theta_1} = \cos\theta_1 + i\sin\theta_1 \tag{5}$$

$$\begin{aligned} (\boldsymbol{e}_x^1, \boldsymbol{e}_y^1) &= (e^{i\theta_1}\boldsymbol{e}_x, e^{i\theta_1}\boldsymbol{e}_y) \\ &= e^{i\theta_1}(\boldsymbol{e}_x, \boldsymbol{e}_y) \end{aligned} \tag{6}$$

$$(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2) = e^{i\theta_2}(\boldsymbol{e}_x^1, \boldsymbol{e}_y^1) \tag{7}$$

$$(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2) = e^{i\theta_2}e^{i\theta_1}(\boldsymbol{e}_x, \boldsymbol{e}_y) \tag{8}$$

We see that a step of rotation gives an Euler's exponential for the angle of rotation in (8). Then we extend the number of steps to n. Let θ_n be the n^th^ angle and ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$) the n^th^ proper basis. According to (8), ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$) is expressed with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$) in (9) with all the angles from θ_1 to θ_n.

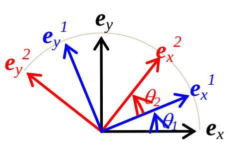

Figure 3

We refer this process of rotating upon the last proper basis as step by step rotation. The successive product of all the Euler's exponential of angles makes the transformation factor T_n in (10), of which the recurrent relation between n^th^ step and n+1^th^ step is (11).

$$(\boldsymbol{e}_x^n, \boldsymbol{e}_y^n) = e^{i\theta_n} \cdots e^{i\theta_1}(\boldsymbol{e}_x, \boldsymbol{e}_y) \tag{9}$$

$$\begin{aligned} &(\boldsymbol{e}_x^n, \boldsymbol{e}_y^n) = T_n(\boldsymbol{e}_x, \boldsymbol{e}_y) \\ &with\ T_n = e^{i\theta_n} \cdots e^{i\theta_1} \end{aligned} \tag{10}$$

$$\begin{aligned} T_{n+1} &= e^{i\theta_{n+1}}e^{i\theta_n} \cdots e^{i\theta_1} \\ &= e^{i\theta_{n+1}}T_{n+1} \end{aligned} \tag{11}$$

On the other hand, the transformation factor for θ_1 and θ_2 equals the Euler's exponential of the angle $\theta_1 + \theta_2$, see equation (14) and (12). This proves that the transformation factor T_n equals the Euler's exponential of the sum of all the angle from θ_1 to θ_n, see equation (10) and (13).

$$\begin{aligned} e^{i\theta_2}e^{i\theta_1} &= (\cos\theta_2 + i\sin\theta_2)(\cos\theta_1 + i\sin\theta_1) \\ &= \cos(\theta_2 + \theta_1) + i\sin(\theta_2 + \theta_1) \\ &= e^{i(\theta_2+\theta_1)} \end{aligned} \tag{14}$$

$$(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2) = e^{i(\theta_2+\theta_1)}(\boldsymbol{e}_x, \boldsymbol{e}_y) \tag{12}$$

$$T_n = e^{i(\theta_n + \cdots + \theta_1)} \tag{13}$$

Step by step rotation seems ordinary in 2D space, but in 3D space it makes the computation of angles of orientation easy and straightforward.

b. Coordinates transformation

With the transformation factor T_n it is easy to compute the coordinates of a point of the rotated body in the space. Let *P* be a point of the rotated body, see Figure 2. Before being rotated *P*'s coordinates are x_0, y_0 with respect to the proper basis ($\mathbf{e}_x^0$, $\mathbf{e}_y^0$) which equals the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$), *P* is expressed in (15) with ($\mathbf{e}_x$, $\mathbf{e}_y$).

$$\begin{aligned} P &= x_0\boldsymbol{e}_x^0 + y_0\boldsymbol{e}_y^0 \\ &= x_0\boldsymbol{e}_x + y_0\boldsymbol{e}_y \end{aligned} \tag{15}$$

See (1)

After n rotations the n^th^ proper basis is ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$) and the point *P* arrives at the point P_n in the space but P_n has the same coordinates x_0 and y_0 with respect to ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$), see (16). Using (10) P_n is expressed with ($\mathbf{e}_x$, $\mathbf{e}_y$) in (17).

$$\begin{aligned} P_n &= x_0\boldsymbol{e}_x^n + y_0\boldsymbol{e}_y^n \\ &= (x_0 + iy_0)\boldsymbol{e}_x^n \end{aligned} \tag{16}$$

$$P_n = (x_0 + iy_0)T_n\boldsymbol{e}_x \tag{17}$$

See (10) and (16)

$$P_n = (x_n + iy_n)\boldsymbol{e}_x \tag{18}$$

$$x_n + iy_n = (x_0 + iy_0)T_n \tag{19}$$

See (17) and (18)

Let the coordinates of P_n with respect to ($\mathbf{e}_x$, $\mathbf{e}_y$) be x_n and y_n and P_n is expressed in (18). Because P_n =P_n , we have the equality (19) which computes x_n and y_n. So, the coordinates in the space of any point of the rotated body can be computed this way.

3. 3D body orientation

a. Basis rotation

For 3D space the space basis vectors are ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$). Before being rotated the proper basis vectors of a rigid body are ($\mathbf{e}_x^0$, $\mathbf{e}_y^0$, $\mathbf{e}_z^0$) and equal ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$), see (20). Let us rotate this body around the axes of z, x and y successively. These axes are not of the frame of the space, but that of the proper frame of the body.

The first rotation is the rotation of the x-y plane, that is, the rotation of the 2 basis vectors $\mathbf{e}_x^0$ and $\mathbf{e}_y^0$ around the z axis. The rotation angle is θ_1 and the 2 basis vectors $\mathbf{e}_x^0$, $\mathbf{e}_y^0$ become $\mathbf{e}_x^1$, $\mathbf{e}_y^1$, see Figure 4. This rotation is exactly the same as for the 2D basis vectors above. So, $\mathbf{e}_x^0$, $\mathbf{e}_y^0$ are expressed in (21) exactly as (6).

$(\boldsymbol{e}_x^0, \boldsymbol{e}_y^0, \boldsymbol{e}_y^0) = (\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$	(20)
$(\boldsymbol{e}_x^1, \boldsymbol{e}_y^1) = e^{i\theta_1}(\boldsymbol{e}_x, \boldsymbol{e}_y)$ See (6)	(21)
$\boldsymbol{e}_z^1 = \boldsymbol{e}_z$	(22)

But the unit vector $\mathbf{e}_z^0$ does not change because the rotation is around the z axis, that is, around $\mathbf{e}_z^0$ itself. So, $\mathbf{e}_z^1$ equals $\mathbf{e}_z$, see (22). In sum, the 2 basis vectors orthogonal to the axis of rotation are rotated while the basis vector of the axis of the rotation stays unchanged. Using this property we define a special multiplication to simplify notation for rotation in multi-dimensional space, which is denoted by the operator "$\because$", see equation (23). We will refer this special product as three-dots product.

Definition

$$e^{i\theta_1} \because (\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z) = (e^{i\theta_1}\boldsymbol{e}_x, e^{i\theta_1}\boldsymbol{e}_y, e^{i\theta_1}\boldsymbol{e}_z) = (e^{i\theta_1}\boldsymbol{e}_x, e^{i\theta_1}\boldsymbol{e}_y, \boldsymbol{e}_z) \quad (23)$$

with $e^{i\theta_1}\boldsymbol{e}_z = \boldsymbol{e}_z$

$$(\boldsymbol{e}_x^1, \boldsymbol{e}_y^1, \boldsymbol{e}_z^1) = \begin{pmatrix} \cos\theta_1\, \boldsymbol{e}_x + \sin\theta_1\, \boldsymbol{e}_y, \\ -\sin\theta_1\, \boldsymbol{e}_x + \cos\theta_1 \boldsymbol{e}_y, \\ \boldsymbol{e}_z \end{pmatrix} \quad (24)$$

See (4)

So, the first proper basis ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$, $\mathbf{e}_z^1$) is expressed as the three-dots product of the Euler's exponential for the angle θ_1 and the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) (23). Equation (24) expressed it with $\sin(\theta_1)$ and $\cos(\theta_1)$ for further use.

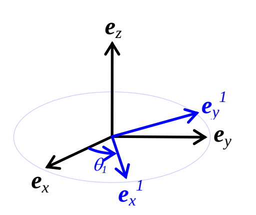

Figure 4

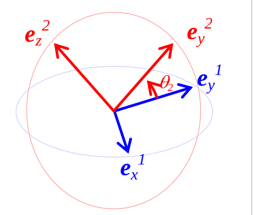

Figure 5

The second step rotation is a rotation of the y-z plane around the x axis, the angle of rotation is θ_2, see Figure 5. This is a rotation of the proper basis with respect to the first proper basis ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$, $\mathbf{e}_z^1$), of which the 2 unit vectors $\mathbf{e}_y^1$ and $\mathbf{e}_z^1$ are rotated and $\mathbf{e}_x^1$ unchanged.

$$(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2, \boldsymbol{e}_z^2) = e^{i\theta_2} \because (\boldsymbol{e}_x^1, \boldsymbol{e}_y^1, \boldsymbol{e}_z^1) = (\boldsymbol{e}_x^1, e^{i\theta_2}\boldsymbol{e}_y^1, e^{i\theta_2}\boldsymbol{e}_z^1) \quad (25)$$

$$\boldsymbol{e}_x^2 = \boldsymbol{e}_x^1$$

$$e^{i\theta_2} = \cos\theta_2 + i\sin\theta_2 \quad (26)$$

The second proper basis is ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$) which is expressed by the three-dots product of the Euler's exponential for the angle θ_2 with ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$, $\mathbf{e}_z^1$) , see (25). The Euler's exponential for the angle θ_2 is given in (26) using which we express ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$) with $\sin(\theta_2)$ and $\cos(\theta_2)$ in (27). We use the expression of ($\mathbf{e}_x^1$, $\mathbf{e}_y^1$, $\mathbf{e}_z^1$) given in (24) for expressing ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$) with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$), see (28).

$$(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2, \boldsymbol{e}_z^2) = \begin{pmatrix} \boldsymbol{e}_x^1, \\ \boldsymbol{e}_y^1 \cos\theta_2 + \boldsymbol{e}_z^1 \sin\theta_2, \\ \boldsymbol{e}_z^1 \cos\theta_2 - \boldsymbol{e}_y^1 \sin\theta_2 \end{pmatrix} \quad (27)$$

$$\begin{aligned}(\boldsymbol{e}_x^2, \boldsymbol{e}_y^2, \boldsymbol{e}_z^2) &= \begin{pmatrix} \boldsymbol{e}_x^1, \ (-\boldsymbol{e}_x \sin\theta_1 + \cos\theta_1 \boldsymbol{e}_y)\cos\theta_2 + \boldsymbol{e}_z \sin\theta_2, \\ \cos\theta_2\, \boldsymbol{e}_z - (-\boldsymbol{e}_x \sin\theta_1 + \cos\theta_1 \boldsymbol{e}_y)\sin\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_1\, \boldsymbol{e}_x + \sin\theta_1\, \boldsymbol{e}_y, \\ -\sin\theta_1 \cos\theta_2\, \boldsymbol{e}_x + \cos\theta_1 \cos\theta_2\, \boldsymbol{e}_y + \sin\theta_2\, \boldsymbol{e}_z, \\ \sin\theta_1 \sin\theta_2\, \boldsymbol{e}_x - \cos\theta_1 \sin\theta_2\, \boldsymbol{e}_y + \cos\theta_2\, \boldsymbol{e}_z \end{pmatrix}\end{aligned} \quad (28)$$

See (27) and (24)

$$(\boldsymbol{e}_x^3, \boldsymbol{e}_y^3, \boldsymbol{e}_z^3) = e^{i\theta_3} \because (\boldsymbol{e}_x^2, \boldsymbol{e}_y^2, \boldsymbol{e}_z^2) = (e^{i\theta_3}\boldsymbol{e}_x^2, \boldsymbol{e}_y^2, e^{i\theta_3}\boldsymbol{e}_z^2) \quad (29)$$

$$\boldsymbol{e}_y^3 = \boldsymbol{e}_y^2$$

The third step rotation is a rotation of the x-z plane around the y axis, the angle of rotation is θ_3, see Figure 6. This is a rotation of the proper basis with respect to the second proper basis ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$), of which the 2 unit vectors $\mathbf{e}_x^2$ and $\mathbf{e}_z^2$ are rotated and $\mathbf{e}_y^2$ unchanged. The third proper basis is ($\mathbf{e}_x^3$, $\mathbf{e}_y^3$, $\mathbf{e}_z^3$) which is expressed by the three-dots product of the Euler's exponential for the angle θ_3 with ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$) , see (29).

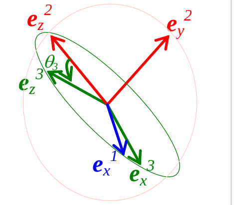

Figure 6

Notice the pattern of the step by step rotation, a rotation upon the last proper basis:

1. The third step rotation upon the second proper basis.
2. The second step rotation upon the first proper basis.
3. The first step rotation upon the zeroth proper basis.

Then,

1. The third proper basis equals the second proper basis three-dots-multiplied by the angle’s Euler's exponential
2. The second proper basis equals the first proper basis three-dots-multiplied by the angle’s Euler's exponential
3. The first proper basis equals the zeroth proper basis three-dots-multiplied by the angle’s Euler's exponential

By chaining up the three-dots multiplication of the angles θ_1, θ_2 and θ_3, we obtain the third proper basis express with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$), see (30). In equation (31) ($\mathbf{e}_x^3$, $\mathbf{e}_y^3$, $\mathbf{e}_z^3$) is expressed with sin(θ_3) and cos (θ_3) and ($\mathbf{e}_x^2$, $\mathbf{e}_y^2$, $\mathbf{e}_z^2$). By using (28) in (31), we get (32) which is rearranged in (33) to express ($\mathbf{e}_x^3$, $\mathbf{e}_y^3$, $\mathbf{e}_z^3$) with the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) and all the 3 angles θ_1, θ_2 and θ_3.

$$\left(\boldsymbol{e}_x^3, \boldsymbol{e}_y^3, \boldsymbol{e}_z^3\right) = e^{i\theta_3} \because e^{i\theta_2} \because e^{i\theta_1} \because \left(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\right) \tag{30}$$

$$\left(\boldsymbol{e}_x^3, \boldsymbol{e}_y^3, \boldsymbol{e}_z^3\right) = \left(\cos\theta_3\, \boldsymbol{e}_x^2 + \sin\theta_3\, \boldsymbol{e}_z^2,\ \ \boldsymbol{e}_y^2,\ \ \cos\theta_3\, \boldsymbol{e}_z^2 - \sin\theta_3 \boldsymbol{e}_x^2\right) \tag{31}$$

$$\left(\boldsymbol{e}_x^3, \boldsymbol{e}_y^3, \boldsymbol{e}_z^3\right) = \begin{pmatrix} \cos\theta_3\left(\cos\theta_1\, \boldsymbol{e}_x + \boldsymbol{e}_y \sin\theta_1\right) + \left(\cos\theta_2\, \boldsymbol{e}_z + \boldsymbol{e}_x \sin\theta_1 \sin\theta_2 - \cos\theta_1 \sin\theta_2\, \boldsymbol{e}_y\right)\sin\theta_3\,, \\ -\boldsymbol{e}_x \sin\theta_1 \cos\theta_2 + \cos\theta_1 \cos\theta_2\, \boldsymbol{e}_y + \boldsymbol{e}_z \sin\theta_2\,, \\ \cos\theta_3\left(\cos\theta_2\, \boldsymbol{e}_z + \boldsymbol{e}_x \sin\theta_1 \sin\theta_2 - \cos\theta_1 \sin\theta_2\, \boldsymbol{e}_y\right) - \left(\cos\theta_1\, \boldsymbol{e}_x + \boldsymbol{e}_y \sin\theta_1\right)\sin\theta_3 \end{pmatrix} \tag{32}$$

See (31) and (28)

$$\left(\boldsymbol{e}_x^3, \boldsymbol{e}_y^3, \boldsymbol{e}_z^3\right) = \begin{pmatrix} (\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3)\boldsymbol{e}_x + (\sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3)\boldsymbol{e}_y + \cos\theta_2 \sin\theta_3\, \boldsymbol{e}_z, \\ -\sin\theta_1 \cos\theta_2\, \boldsymbol{e}_x + \cos\theta_1 \cos\theta_2\, \boldsymbol{e}_y + \sin\theta_2\, \boldsymbol{e}_z, \\ (\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3)\boldsymbol{e}_x - (\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3)\boldsymbol{e}_y + \cos\theta_2 \cos\theta_3\, \boldsymbol{e}_z \end{pmatrix} \tag{33}$$

$$[T_3] = \begin{bmatrix} \cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3 & \sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3 & \cos\theta_2 \sin\theta_3 \\ -\sin\theta_1 \cos\theta_2 & \cos\theta_1 \cos\theta_2 & \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3 & -\cos\theta_1 \sin\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_3 & \cos\theta_2 \cos\theta_3 \end{bmatrix} \tag{34}$$

In (34) we define the 3×3 matrix [T_3] whose 9 coefficients are from equation (33) and are expressed in terms of the 3 angle of rotation θ_1, θ_2, θ_3. The matrix [T_3] transforms ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) into ($\mathbf{e}_x^3$, $\mathbf{e}_y^3$, $\mathbf{e}_z^3$) in (35) and is called transformation matrix of the third step.

$$\begin{bmatrix} \boldsymbol{e}_x^3 \\ \boldsymbol{e}_y^3 \\ \boldsymbol{e}_z^3 \end{bmatrix} = [T_3] \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} \tag{35}$$

See (33) and (34)

b. n rotations and orientation angles

Since each rotation is upon the last step’s proper basis, the step by step rotation process allows to go beyond 3 rotations and each rotation adds simply a three-dots multiplication of the angles of rotation on the left of the chain of three-dots products in (37) which expresses the n[th] step’s proper basis in terms of the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$).

$$\begin{bmatrix} \boldsymbol{e}_x^n \\ \boldsymbol{e}_y^n \\ \boldsymbol{e}_z^n \end{bmatrix} = [T_n] \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} \tag{36}$$

$$\left(\boldsymbol{e}_x^n, \boldsymbol{e}_y^n, \boldsymbol{e}_z^n\right) = e^{i\theta_n} \because \cdots \because e^{i\theta_1} \because \left(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\right) \tag{37}$$

So, the step by step rotation is an easy way to piles up multiple rotations around all the axes of proper basis and then to compute the last step’s proper basis in terms of the space basis ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$). Like for 3 rotations in (35), the n[th] step’s proper basis ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$, $\mathbf{e}_z^n$) is expressed with the n[th] transformation matrix [T_n] which is a 3×3 matrix, see (36).

$$[T_n] = \begin{bmatrix} t_{11}^n & t_{12}^n & t_{13}^n \\ t_{21}^n & t_{22}^n & t_{23}^n \\ t_{31}^n & t_{32}^n & t_{33}^n \end{bmatrix} \tag{38}$$

How can we compute the 3 orientation angles θ_1, θ_2 and θ_3 of the n[th] step proper basis? The matrix [T_n], see (38), transforms the space basis into the n[th] step’s proper basis. The matrix [T_3] transforms the space basis into the proper basis of any rotated body, see (35), for example the n[th] step proper basis ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$, $\mathbf{e}_z^n$). So, by equating [T_n] with [T_3], see (39), we obtain (40) which allows to compute the 3 angles θ_1, θ_2 and θ_3 in (41), (42) and (43).

$$[T_n] = [T_3] \tag{39}$$

$$\begin{bmatrix} t_{11}^n & t_{12}^n & t_{13}^n \\ t_{21}^n & t_{22}^n & t_{23}^n \\ t_{31}^n & t_{32}^n & t_{33}^n \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3 & \sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3 & \cos\theta_2 \sin\theta_3 \\ -\sin\theta_1 \cos\theta_2 & \cos\theta_1 \cos\theta_2 & \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3 & -\cos\theta_1 \sin\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_3 & \cos\theta_2 \cos\theta_3 \end{bmatrix} \tag{40}$$

See (38) and (34)

So, no matter how complex and numerous the rotations of a body are, the step by step rotation process allows to computed the set of the 3 angles (θ_1, θ_2, θ_3) from the n[th] transformation matrix [T_n]. As this set of 3 angles uniquely defines the orientation of a proper basis in the space, it can serve as an

$$t_{23}^n = \sin\theta_2 \Rightarrow \theta_2 = \sin^{-1}(t_{23}^n) \tag{41}$$

$$t_{22}^n = \cos\theta_1 \cos\theta_2 \Rightarrow \cos\theta_1 = \frac{t_{22}^n}{\cos\theta_2} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{t_{22}^n}{\cos\theta_2}\right) \tag{42}$$

$$t_{33}^n = \cos\theta_2 \cos\theta_3 \Rightarrow \cos\theta_3 = \frac{t_{33}^n}{\cos\theta_2} \Rightarrow \theta_3 = \cos^{-1}\left(\frac{t_{33}^n}{\cos\theta_2}\right) \tag{43}$$

orientation system to define the orientation of a rigid body just like Euler angles. So, we propose that the above defined set of 3 angles (θ_1, θ_2, θ_3) as a system of orientation.

c. Euler angles and Tait–Bryan angles

Euler angles and Tait–Bryan angles are the most commonly used orientation system in 3D space. The Euler angles of a body are obtained by rotating the body and its proper basis in the 3 steps below, see Figure 7:

1. Rotate the angle α around the z axis.
2. Rotate the angle β around the x axis.
3. Rotate the angle γ around the z axis.

The 3 Euler angles (α,β,γ) are the angles (θ_1, θ_2, θ_3) in our orientation system. The order of rotation is around z axis, x axis and z axis. We see that Euler angles rotate the proper basis around the the z axis twice, this is why Euler angles suffers from gimbal lock.

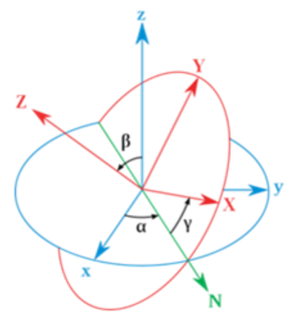

Figure 7
Euler angles

The Tait–Bryan angles of a body are obtained by rotating a body and its proper basis in the 3 steps below, see Figure 8:

1. Rotate the angle ψ around the z axis.
2. Rotate the angle θ around the y axis.
3. Rotate the angle φ around the x axis.

The 3 Tait–Bryan angles (ψ,θ,φ) are the angles (θ_1, θ_2, θ_3) in our orientation system. The order of rotation is around z axis, y axis and x axis. Since Tait–Bryan angles rotate around each axis once, it does not suffer from gimbal lock.

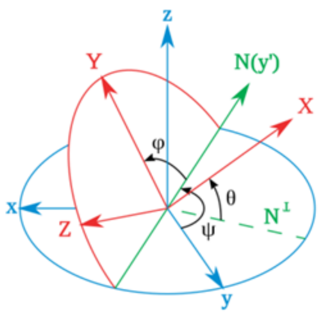

Figure 8
Tait–Bryan angles

For x, y and z axes, there are 6 possible orders of axes without repeating: (x, y, z), (x, z, y), (y, x, z), (y, z ,x), (z, x ,y), (z, y ,x)

The one we used is (z, x ,y), Tait–Bryan angles is (z, y ,x). As Euler angles repeats z axis, it is not included. So, step by step rotation defines all possible orders of axes and thus, it is a general orientation system. Also it allows indefinite chain of rotations and easy computation of the orientation angles.

$P = [x_0, y_0, z_0]\begin{bmatrix} \boldsymbol{e}_x^n \\ \boldsymbol{e}_y^n \\ \boldsymbol{e}_z^n \end{bmatrix}$	(44)
$P = [x_0, y_0, z_0][T_n]\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}$	(45)
$P = [x_n, y_n, y_n]\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}$	(46)
$[x_n, y_n, y_n] = [x_0, y_0, z_0][T_n]$	(47)
$[x_0, y_0, z_0] = [x_n, y_n, y_n][T_n]^{-1}$	(48)

d. Coordinates conversion

How can we compute the coordinates of an arbitrary point P of an n time rotated body with respect to the fixed space? Let its coordinates with respect to the proper basis be (x_0, y_0, z_0). Because the proper basis rotates with the body, the coordinates of P are still (x_0, y_0, z_0) with respect to the nth step proper basis ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$, $\mathbf{e}_z^n$), see (44). Using (36), ($\mathbf{e}_x^n$, $\mathbf{e}_y^n$, $\mathbf{e}_z^n$) is substituted with [T_n]($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) in (44) and we obtain (45).

On the other hand, let the P's coordinates with respect to the fixed space's basis be (x_n, y_n, z_n) which expresses P in (46) directly with ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$). Equating (45) with (46) we obtain (47) which computes (x_n, y_n, z_n). Equation (48) inverses [T_n] to compute the coordinates of P in its proper basis from its coordinates in the fixed space.

4. Geometrical meaning of quaternion

Quaternion is largely used to rotate vectors but its geometrical meaning is difficult to visualize since it is in 4D space. The step by step rotation process rotates vectors by rotating the whole proper basis of a body, which gives us a tool to visualize the geometrical meaning of quaternion in 3D space.

$q = a + bi + cj + dk$	(49)
$q_u = \cos\theta + \sin\theta\, \boldsymbol{v}$ with $\lVert\boldsymbol{v}\rVert = 1$	(50)
$\lVert q \rVert = l$	(51)
$q = l \cdot q_u$	(52)

A general quaternion is given in (49). A unit quaternion, which we denote by q_u, is a quaternion whose norm is one. Let $\boldsymbol{v}$ be a unit vector, a unit quaternion q_u can be written in the form given in (50). We denote the norm of q by l, see (51). Then q can be expressed as the product of l and q_u, see (52).

Let us see equation (31) in which the unit vector $\mathbf{e}_x^3$ is expressed in the form given in (53). As $\mathbf{e}_x^2$, $\mathbf{e}_y^2$ and $\mathbf{e}_z^2$ are orthogonal and $\mathbf{e}_z^2$ equals the vector product of $\mathbf{e}_x^2$ and $\mathbf{e}_y^2$, see (54), $\mathbf{e}_x^3$ is expressed in (55).

In (56) we factor out $\mathbf{e}_x^2$ while keeping in mind that the product of $\mathbf{e}_x^2$ and $\mathbf{e}_y^2$ is a cross product. Since $\mathbf{e}_y^2$ is a unit vector we can write (57) and we find that the parenthesis in (56) is in fact a unit quaternion and we write (58), which gives $\mathbf{e}_x^3$ as a quaternion product, see (59).

$\mathbf{e}_x^3$ equals $\mathbf{e}_x^2$ rotated by the angle θ around $\mathbf{e}_y^2$ and $\mathbf{e}_y^2$ is the vector $\boldsymbol{v}$ of the unit quaternion q_u. So, the quaternion product in (59) means that q_u rotates the vector $\mathbf{e}_x^2$ around the vector $\boldsymbol{v}$ by the angle θ. We see in (58) that $\boldsymbol{v}$ and θ are explicitly inscribed in q_u.

Let us make the quaternion q whose norm is l and whose unit vector is $\boldsymbol{v}$, see (60). The quaternion product $\mathbf{e}_x^2 q$ equals $l\mathbf{e}_x^3$, see (61), which means that q rotates $\mathbf{e}_x^2$ around the vector $\boldsymbol{v}$ by the angle θ and scales it by the factor l which is its norm.

$\boldsymbol{e}_x^3 = \boldsymbol{e}_x^2 \cos\theta + \sin\theta\, \boldsymbol{e}_z^2$	(53)
$\boldsymbol{e}_z^2 = \boldsymbol{e}_x^2 \times \boldsymbol{e}_y^2$	(54)
$\boldsymbol{e}_x^3 = \boldsymbol{e}_x^2 \cos\theta + \sin\theta \left(\boldsymbol{e}_x^2 \times \boldsymbol{e}_y^2\right)$	(55)
$\boldsymbol{e}_x^3 = \boldsymbol{e}_x^2\left(\cos\theta + \sin\theta\, \boldsymbol{e}_y^2\right)$	(56)
$\boldsymbol{v} = \boldsymbol{e}_y^2$	(57)
$\cos\theta + \sin\theta\, \boldsymbol{e}_y^2 = \cos\theta + \sin\theta\, \boldsymbol{v} = q_u$	(58)
$\boldsymbol{e}_x^3 = \boldsymbol{e}_x^2(\cos\theta + \sin\theta\, \boldsymbol{v}) = \boldsymbol{e}_x^2 q_u$	(59)
$q = l q_u = l(\cos\theta + \sin\theta\, \boldsymbol{v})$	(60)
$\boldsymbol{e}_x^2 q = l\boldsymbol{e}_x^2 q_u = l\boldsymbol{e}_x^3$	(61)

So, to the vector $\mathbf{e}_x^2$ the quaternion q gives the axis of rotation $\boldsymbol{v}$ and the angle of rotation θ and the scaling factor l. Thus, q is an operator for 3D vectors perpendicular to it. The resulting vector can reach any point in the plane perpendicular to $\boldsymbol{v}$ because l ranges from 0 to infinity.

θ and l are 2 free scalars, $\boldsymbol{v}$ has 3 coefficients, should not we need 5 coefficients for the operator quaternion which has only 4? In fact, although $\boldsymbol{v}$ has 3 coefficients, it is a unit vector and stays on the unit sphere which is a 2 dimensional surface. So, $\boldsymbol{v}$ is a 2D object. With θ and l we have well 4 free scalars in a quaternion, see (60).

So, quaternion can be visualized as an operator in 3D space. This is the geometrical meaning we find for quaternion. This visualization is much more natural and realistic for us 3D creatures then a 4D object such as the one presented in this Video.

In the same way, 2D complex number can also be seen as 2D space operator since by multiplication it rotates any vector an angle and scale it with its norm, see (4).

5. High dimensional body rotation

Since in 2D and 3D spaces an object can rotate and be oriented, can object do so in space with higher than 3 dimensions? In fact, object with higher dimensions can rotate in plane just like in 2D and 3D space. But in high-dimension space, a plane does not have a normal vector. For example, in 4D space a plane made by 2 basis vectors is perpendicular to the 2 other basis vectors. So, a plane does not rotate around an axis, but with respect to its 2 basis vectors.

Let us take a space with m dimensions and specify a rotation of an object in a plane. Let the plane be made by the 2 unit vectors $\mathbf{e}_\alpha^n$ and $\mathbf{e}_\beta^n$ which are basis vectors of the proper basis of the object. α and β are integers between 1 and m without being equal.

Let θ_{n+1} be the angle of the n+1[th] step rotation which is in the specified plane. The 2 basis vectors $\mathbf{e}_\alpha^n$ and $\mathbf{e}_\beta^n$ once rotated become $\mathbf{e}_\alpha^{n+1}$ and $\mathbf{e}_\beta^{n+1}$ and equal the product of $\mathbf{e}_\alpha^n$ and $\mathbf{e}_\beta^n$ with the Euler's exponential of θ_{n+1}, see (62). Because rotation concerns only $\mathbf{e}_\alpha^n$ and $\mathbf{e}_\beta^n$, all the other basis vectors stay unchanged, see (63).

By using the three-dots product in m-dimensional space which is defined in (64), the proper basis at the n[th] step is $(\mathbf{e}_1^n, \ldots, \mathbf{e}_m^n)$ which is expressed with the chain of three-dots products of the Euler's exponential and the space basis vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_m)$, see (65).

$\boldsymbol{e}_\alpha^{n+1} = e^{i\theta_{n+1}}\boldsymbol{e}_\alpha^n$ $\boldsymbol{e}_\beta^{n+1} = e^{i\theta_{n+1}}\boldsymbol{e}_\beta^n$ with $e^{i\theta_{n+1}} = \cos\theta_{n+1} + i\sin\theta_{n+1}$	(62)
$\boldsymbol{e}_\gamma^{n+1} = \boldsymbol{e}_\gamma^n$ $\gamma = 1 \cdots m$, except α and β	(63)

The matrix of transformation of the n[th] step proper basis is the m×m matrix $[T_n]$, see equation (66). Then $[T_n]$ serves to compute the coordinates in the space of a point P of the body at the n[th] step, which

$\left(\boldsymbol{e}_1^{n+1}, \cdots \boldsymbol{e}_\alpha^{n+1}, \cdots, \boldsymbol{e}_\beta^{n+1} \cdots \boldsymbol{e}_m^{n+1}\right) = e^{i\theta_{n+1}} \because \left(\boldsymbol{e}_1^n, \cdots \boldsymbol{e}_\alpha^n, \cdots, \boldsymbol{e}_\beta^n \cdots, \boldsymbol{e}_m^n\right) = \left(\boldsymbol{e}_1^n, \cdots e^{i\theta_{n+1}}\boldsymbol{e}_\alpha^n, \cdots, e^{i\theta_{n+1}}\boldsymbol{e}_\beta^n \cdots, \boldsymbol{e}_m^n\right)$	(64)
$\left(\boldsymbol{e}_1^n, \cdots, \boldsymbol{e}_m^n\right) = e^{i\theta_n} \because \cdots \because e^{i\theta_1} \because (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m)$	(65)

is (x_1^n, …, x_m^n). The initial coordinates of P at the 0th step being (x_1^0, …, x_m^0), the point P is defined with respect to the nth step proper basis and the space basis in (67). Introducing the m×m matrix [T_n] into (67) gives (68), then the coordinates (x_1^n, …, x_m^n) is computed in (69).

$$\begin{bmatrix} \boldsymbol{e}_1^n \\ \cdots \\ \boldsymbol{e}_m^n \end{bmatrix} = [T_n] \begin{bmatrix} \boldsymbol{e}_1 \\ \cdots \\ \boldsymbol{e}_m \end{bmatrix} \quad (66)$$

In m-dimensional space the number of planes is $\binom{m}{2}$ which is the number of 2-combination of m elements. The three-dots product computes only 2 vectors for each rotation against m vectors if computed using matrix, which makes big economy of computation if the number of rotation is big.

$$P = [x_1^0, \cdots, x_m^0] \begin{bmatrix} \boldsymbol{e}_1^n \\ \cdots \\ \boldsymbol{e}_m^n \end{bmatrix} = [x_1^n, \cdots, x_m^n] \begin{bmatrix} \boldsymbol{e}_1 \\ \cdots \\ \boldsymbol{e}_m \end{bmatrix} \quad (67)$$

$$P = [x_1^0, \cdots, x_m^0][T_n] \begin{bmatrix} \boldsymbol{e}_1 \\ \cdots \\ \boldsymbol{e}_m \end{bmatrix} = [x_1^n, \cdots, x_m^n] \begin{bmatrix} \boldsymbol{e}_1 \\ \cdots \\ \boldsymbol{e}_m \end{bmatrix} \quad (68)$$

$$[x_1^n, \cdots, x_m^n] = [x_1^0, \cdots, x_m^0][T_n] \quad (69)$$

Notice that there is no longer right-handedness in high dimension space. If we label the dimensions of a 3D space by 1, 2 and 3, the right-hand ordering of the basis planes is (1,2), (2,3) and (3,1), which enumerates all the basis planes of the space. But in a 4D space whose dimensions are labeled by 1, 2, 3 and 4, all the basis planes are (1,2), (1,3), (1,4), (2,3), (2,4) and (3,4). If we make the circular right-hand chain of the planes which is (1,2), (2,3), (3,4) and (4,1), the planes (1,3) and (2,4) are left over.

6. Discussion

In this article we have proposed a process of rotation in space called step by step rotation in which the proper basis of a body is rotated with respect to that of the previous step. This process allows piling up rotations of the body indefinitely. For computing each rotation we have defined a special multiplication called three-dots multiplication that transforms the proper basis at each step. A chain of three-dots multiplication computes a pile of rotations in the space. Three-dots multiplication economizes computation even for 3D space.

Using step by step rotation we have created a more general orientation system which includes Euler angles, Tait–Bryan angles and all possible sets of orientation angles in 3D space. The orientation angles of a body after any complex set of rotations can be easily computed using the step by step rotation and orientation system. Once the proper basis of a rotated body computed the coordinates of all the points of the body can be easily computed.

With the acquired better understanding of rotation we have found that quaternion is an operator of vector in 3D space and explained how the 4 free scalars of a quaternion work in 3D space, which visualizes quaternion in 3D space.

The step by step rotation process allows rotation of body in high dimensional space, which opens a new field of mathematical research.