On the uncountability of the power set of ℕ

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Abstract: This article discusses the uncountability of the power set of ℕ proven by using the out-indexes subset contradiction.

1. Cantor's theorem

Cantor's theorem proves that the power set of ℕ is uncountable. This is a proof by contradiction. Suppose that the power set of ℕ is countable. This allows us to put all subsets of ℕ in a list. An example of the list is given in the second column of Table 1. The contradiction will come from the indexes, which are given in the first column.

As the indexes are natural numbers like the members of the subsets, they can appear in the subsets they indicate. We call the indexes that appear in their subsets “in-indexes” and those that are not in their subsets “out-indexes”. For example, in Table 1 the index of the subset {1,5,6}, which is 1, appears in the subset. So, 1 is an in-index. In the contrary, the index of the subset {3,4}, which is 0, is not in the subset. So, 0 is an out-index.

	ℕ={0,1,2,... }
i	Subsets
0	{3, 4}
1	{1, 5, 6}
2	{0, 9,14...}
...	...

Table 1

Now, we create the special subset which contains all the out-indexes of the list. Naturally, the special subset is a subset of ℕ and a member of the power set of ℕ. Its index is called the special index. Is the special index an in-index? If so, it should appear in the special subset. But this breaks a rule of the special subset: all members are out-indexes. So, the special index is not an in-index. Then, is the special index an out-index? If so, it should not appear in the special subset so that an out-index is missing, the special index. But this breaks another rule of the special subset: ALL the out-indexes are in the special subset. So, the special index is not an out-index.

Finally, the special index, which is neither in-index nor out-index, cannot exist. Without an index, the special subset is not in the list of subsets. This contradicts our supposition: that the members of the power set of ℕ are all in the list. Thus, the power set of ℕ is uncountable.

Although the above reasoning is correct, its result is somehow counter-intuitive. Indeed, the power set of a finite set is finite. How will power set become uncountable when the original set becomes infinite? It will be interesting to find out the mathematics behind the transition from countable to uncountable.

2. Power sets of finite sets

How are power sets created? As an example, we take the 4 members set {0,1,2,3} as the original set and create its power set. A subset is formed with members chosen from the original set. The selection of a member can be noted by a binary digit, 1 if the member is chosen, 0 if not. So, a subset can be represented by a binary sequence, the sequence of the selection digits. For example, in Table 2 the subset {1,2,3} is represented by the sequence (0,1,1,1) and {0,2} by (1,0,1,0).

Original set {0,1,2,3}	0	1	2	3
Subsets	Binary sequences			
{1,2,3}	0	1	1	1
{0,2}	1	0	1	0

Table 2

Table 3 presents the power set of {0,1,2,3} in binary sequences. The original set has in total 4 members and 16 subsets. The 16 members of the power set can be listed. Two different lists of subsets are presented in columns 2 and 3. The indexes range from 0 to 15 as shown in column 1.

The first 4 lines of the 2 lists have the same diagonal, four digits of 1's, meaning that the indexes of these subsets appear in the subsets and these indexes are in-indexes. But the first 4 lines of the 2 lists have different subsets, indicating that, except the null set {}, any subset can be put in the top 4 lines so that it has in-index.

Having arranged the first 4 lines, we find that there can be at most 4 in-indexes because the original set has only 4 members. The indexes bigger than 3 are not members of the original set and thus, are all out-indexes. The set of the out-indexes is:

$$\{4,5,6,7,8,9,10,11,12,13,14,15\}$$

What about the contradiction? In fact, the out-indexes are not members of the original set and are much more numerous than the original-set's members. So, the set of the out-indexes is not a subset of the original set. Being not a member of the power set, the set of the out-indexes does not need index. We have no contradiction.

i	List 1	List 2
	0 1 2 3	**0 1 2 3**
0	1 0 0 0	1 0 1 0
1	0 1 0 0	1 1 0 1
2	0 0 1 0	0 1 1 0
3	0 0 0 1	0 0 1 1
4	0 0 0 0	0 0 0 0
5	0 1 0 1	0 0 0 1
6	0 1 1 0	0 0 1 0
7	0 1 1 1	0 1 0 0
8	0 0 1 1	0 1 0 1
9	1 0 0 1	0 1 1 1
10	1 0 1 0	1 0 0 0
11	1 0 1 1	1 0 0 1
12	1 1 0 0	1 0 1 1
13	1 1 0 1	1 1 0 0
14	1 1 1 0	1 1 1 0
15	1 1 1 1	1 1 1 1

Table 3

3. Under the logic of potential infinity

If the original set is potentially infinite, it has n members while n goes to infinity. As the power set has 2^n members, it stays a finite set for all $n \to \infty$. Thus the power set will not become uncountable under the logic of potential infinity.

4. ℕ under the logic of actual infinity

ℕ is the infinite set {0,1,2,… } and obeys the logic of actual infinity. The number of members is actually infinite or alternatively, equals $\aleph_0$. Like for finite sets, the subsets of ℕ can be represented by binary sequences, which are listed in Table 4. The subsets in the top lines of the list contain only one member, their proper indexes. This way, the first indexes are in-indexes and have values 0,1,2,… This is corroborated by the diagonal whose digits are only 1's.

The remaining subsets such as 2 members, 3 members ... continue down the list with indexes which are not 0,1,2,… Being not members of the original set ℕ, these indexes are out-indexes. But will the out-indexes be bigger than $\aleph_0$? Can numbers bigger than $\aleph_0$ ever exist? If not, have we found a new way to prove that the power set of ℕ is uncountable?

ℕ	0 1 2 3 4 5 6 7 8 …
i	Subsets binary sequences
0	1 0 0 0 0 0 0 0 0 …
1	0 1 0 0 0 0 0 0 0 …
2	0 0 1 0 0 0 0 0 0 …
3	0 0 0 1 0 0 0 0 0 …
4	0 0 0 0 1 0 0 0 0 …
5	0 0 0 0 0 1 0 0 0 …
6	0 0 0 0 0 0 1 0 0 …
7	0 0 0 0 0 0 0 1 0 …
8	0 0 0 0 0 0 0 0 1 …
…	…

Table 4

I do not think so. Just think of the set of rational numbers. If we put the rationales $\frac{1}{i}$ for i=1 $\to \infty$ on top of the list, the indexes of the rational numbers $\frac{2}{i}$ and beyond will have indexes bigger than $\aleph_0$. This is due to the arrangement of the list. If this is a proof of the uncountability of the listed set, then, the set of rational numbers would be uncountable. This is not true.

Anyway, the out-indexes are not members of ℕ and are $2^{\aleph_0}$-$\aleph_0$ in number, which is much bigger than $\aleph_0$. Even if we arrange the list in such a way that subsets have out-indexes in the first lines, there will still be more than $\aleph_0$ out-indexes down the list which are not members of ℕ. Thus, the set of the out-indexes is not a subset of ℕ. Being not a member of the power set of ℕ, the set of the out-indexes does not need index and we have no contradiction. So, Cantor's theorem has not proven that the power set of ℕ is uncountable.

Note: Numbers bigger than $\aleph_0$ are explained in <<Which infinity for irrational numbers?>>.

5. About transfinite numbers

The diagonal of 1's in Table 4 recalls the diagonal argument in which the flipped sequence should be out of the list to create contradiction. In <<Hidden assumption of the diagonal argument>> I have shown that the flipped sequence is in fact in the list. For proving the uncountability of the power set of ℕ, the set of the out-indexes should be a subset of ℕ to create contradiction, but in fact it is not. So, we have no contradiction to prove the uncountability of the power set of ℕ.

Georg Cantor used power set to create sets with cardinality bigger than $\aleph_0$ and hoped that sets with still bigger cardinality can be created. For example, the cardinality of ℕ is $\aleph_0$, the cardinality of the power set of ℕ is $\aleph_1$, the cardinality of the power set of the power set of ℕ is $\aleph_2$ and so on. This way, transfinite numbers will be in the order $\aleph_0<\aleph_1<\aleph_2<\ldots$ and have no upper limit. Now transfinite numbers seem to be in trouble. If ever the existence of transfinite numbers were denied, the set of real numbers will no longer be the continuum and the continuum hypothesis will lose its reason to exist.