Lists of binary sequences and uncountability

Peng Kuan 彭宽

24 November 2016

Abstract: Creation of binary lists, discussion about the power set of ℕ, the diagonal argument, Cantor's first proof and uncountability.

Binary system is kind of magic because it can express natural numbers, real numbers and subsets of natural numbers. Below, we will create lists of binary sequences to study the uncountability of the power set of ℕ and real numbers.

1. Infinite list of binary sequences

A binary sequence is a string of bits each of which takes the value 0 or 1, for example 011. In the following, a natural number written in binary system will be called a binary number, for example, the number 45 is the binary number 101101; binary sequences with n bits will be called n-bits sequences; lists of n-bits sequences will be called n-bits lists.

Binary numbers have different numbers of bits, for example 101 or 101101. But the sequences in n-bits lists should have all the same number of bits. To make an n-bits sequence with a binary number having fewer bits, we complete with enough "0" on the left of the number. For example, on the left of the 6 bits 101101 we add 4 "0" to make the 10-bits sequence 0000101101.

The biggest binary number with n bits is the sequence n times 1: …1111, which equals the decimal number 2^n-1. For creating an n-bits list, the list $(0,1,2,3…2^n-1\ )$ is written in binary system to make a list of binary numbers, then we complete with "0" all the binary numbers to make n-bits sequences. Table 1 shows examples of n-bits lists.

1 bit		2 bits		3 bits		…	∞ bits	
i_2	L_2	i_4	L_4	i_8	L_8	…	i_∞	L_∞
						…	∞	…11111111
						…	…	…
						…	8	…00001000
				7	111	…	7	…00000111
				6	110	…	6	…00000110
				5	101	…	5	…00000101
				4	100	…	4	…00000100
		3	11	3	011	…	3	…00000011
		2	10	2	010	…	2	…00000010
1	1	1	01	1	001	…	1	…00000001
0	0	0	00	0	000	…	0	…00000000

Table 1

The lengths of a sequence or a list are respectively the numbers of bits or elements. The number of elements of an n-bit list is m:

$$m=2^n \qquad (1)$$

The name of n-bits lists are indexed by length, for example, L_2, L_4 in Table 1; the lists i_2 and i_4 are their values. Since the list $i_m=(0,1,2,3\ldots2^n-1)$ is the list of values of L_m, the list L_m is comprehensive, meaning that it contains all possible n-bit sequences. When n increases indefinitely, $n\to\infty$, the lists L_m and i_m become L_∞ and i_∞. So, the list L_∞ is comprehensive and contains all ∞-bits binary sequences. L_∞ is in one-to-one correspondence with $i_\infty=(0,1,2,3\ldots2^\infty-1)$ which equals $\mathbb{N}$, making L_∞ countable.

2. About the Power set of $\mathbb{N}$

The power set of $\mathbb{N}$ contains all and only subsets of $\mathbb{N}$. $\mathbb{N}$ is said to be the base set. A subset of $\mathbb{N}$ can be represented by a sequence of bits, in which 1 represents a natural number, 0 signals the absence of number. For example, the subset $\{10,7,5,4,2,1\}$ is represented by the binary sequence $\ldots001001011011$, as shown in Table 2.

ℕ	...	10	9	8	7	6	5	4	3	2	1
Subset	...	10	-	-	7	-	5	4	-	2	1
Binary	...	1	0	0	1	0	1	1	0	1	1

Table 2

The size of a set is the number of its elements, the size of $\mathbb{N}$ is ∞. So, a subset of $\mathbb{N}$ will be represented by a ∞-bits sequence. The power set of $\mathbb{N}$ can then be represented by a list of ∞-bits sequence, which is in fact L_∞. The power set of $\mathbb{N}$ is in one-to-one correspondence with L_∞ which is in one-to-one correspondence i_∞ which is $\mathbb{N}$. The table of one-to-one correspondences of i_∞, L_∞ and the power set of $\mathbb{N}$ is shown in Table 3. Then, could the power set of $\mathbb{N}$ be countable?

i_∞	Binary L_∞				Subsets of $\mathbb{N}$				
∞	...11111	1	1	1	...9876	5	4321		
...	...				...				
15	... 1	1	1	1	... 4	3	2	1
14	... 1	1	1	0	... 4	3	2	-
13	... 1	1	0	1	... 4	3	-	1
12	... 1	1	0	0	... 4	3	-	-
11	... 1	0	1	1	... 4	-	2	1
10	... 1	0	1	0	... 4	-	2	-
9	... 1	0	0	1	... 4	-	-	1
8	... 1	0	0	0	... 4	-	-	-
7	... 0	1	1	1	... -	3	2	1
6	... 0	1	1	0	... -	3	2	-
5	... 0	1	0	1	... -	3	-	1
4	... 0	1	0	0	... -	3	-	-
3	... 0	0	1	1	... -	-	2	1
2	... 0	0	1	0	... -	-	2	-
1	... 0	0	0	1	... -	-	-	1
0	... 0	0	0	0	... -	-	-	-
Base	... 2^3	2^2	2^1	2^0 $\mathbb{N}$	..	4	3	2	1

Table 3

Cantor's theorem states that the power set of $\mathbb{N}$ is uncountable. How did Georg Cantor get this conclusion? Let us examine step by step this theorem, which is a proof by contradiction:

1. Object of the theorem: List containing all subsets of $\mathbb{N}$ does not exist.
2. Assumption of the contrary: Let X be a list containing all the subsets of $\mathbb{N}$.
   - In X each subset has an index which is a natural number. So, subsets can contain indices. All the indices of X belong to 2 categories:
     – In-index: An index that is in the subset it indexes.
     – Out-index: An index that is not in the subset it indexes.
3. A subset of $\mathbb{N}$ "key of the proof" is created. This subset and its index are named K-subset and K-index. It has the 2 properties below:
   3.1. K-subset contains **only** out-indices. Then, K-index is not in K-subset for K-subset not to contain an in-index, K-index. So, K-index is not an in-index.
   3.2. K-subset contains **all** the out-indices. Since K-index is not an in-index, it should be an out-index. As K-subset contains **all** the out-indices, it should contain K-index. But step 3.1 does not allow K-subset to contain K-index. So, K-index is not an out-index.
4. Contradiction: K-index is neither an in-index nor an out-index. So, K-index is not an index in the list X, making X fail to contain a subset of $\mathbb{N}$, K-subset.
5. Conclusion: Then, the assumption is false, meaning that list containing all the subsets of $\mathbb{N}$ does not exist. So, the power set of $\mathbb{N}$ is uncountable.

Where is the bug? Cantor's theorem **claims** that K-subset is a subset of $\mathbb{N}$. But the above 2 properties make this impossible. To illustrate our explanation, let us see the power set of the set $A=\{1,2,3\}$. A subset of A has at most 3 elements which can be 1,2 or 3. So, an in-index can be 1,2 or 3 only. How many out-indices are there? The power set has $2^3=8$ elements and 8 indices. If 1,2 and 3 are all in-

indices, we will have 8–3=5 out-indices which are the numbers 4,5,6,7 or 8. So, a subset of A **cannot** contain **all** the out-indices due to smaller size, but also because the out-indices are even not elements of the set A. So, K-subset is not a subset of A.

This demonstration is valid for the power set of the set $B=\{1,2,3…n\}$. The size of the power set is 2^n. The largest out-indices are the numbers $n+1$, $n+2…2^n$ which are elements of the set $P=\{1,2,3…2^n\}$, making K-subset a subset of P. We interpret such K-subset in 2 ways:

1. The size of B is n, which is smaller than $2^n–n$ the minimal
   size of K-subset. So, K-subset is **not** a subset of B but of P.
2. The number $n+1$, $n+2…2^n$ are the largest out-indices, they are
   not elements of B. So, K-subset is **not** a subset of B but of P.

Let X_n be the power set of B which is a finite set. So, X_n is a finite list and contains all and only the subsets of B. In the case of Cantor's theorem, n increases indefinitely. For no n K-subset is a subset of B or is in the list X_n. Then, when $n\to\infty$, $B\to\mathbb{N}$, $X_n\to X$, K-subset is not in the list $X=X_n(n\to\infty)$ because K-subset is **not** a subset of $\mathbb{N}=B(n\to\infty)$, which is **not in contradiction** with the assumption, invalidating the proof for the uncountability of the power set of $\mathbb{N}$.

For metaphor, we liken subsets of $\mathbb{N}$ as peaches and the power set of $\mathbb{N}$ a basket of peaches. K-subset is not a subset of $\mathbb{N}$ and is likened as an apple. The apple is not in the basket not because the basket is too small, but because an apple is not a peach. In the same way, K-subset is not in the power set of $\mathbb{N}$ not because the power set is uncountable, but because it is not a subset of $\mathbb{N}$. So, the **claim** of Cantor's theorem is wrong. In addition, the power set of $\mathbb{N}$ is put in one-to-one correspondence with i_∞ which lists all natural numbers. Then, the power set of $\mathbb{N}$ must be countable. I have given another account on this topic in «On the uncountability of the power set of $\mathbb{N}$».

Why is K-subset not a subset of $\mathbb{N}$ while being made of natural numbers? In fact, for all n, $B\neq P$. When $n\to\infty$, B and P tend to different sets of natural numbers, that is, $B\to\mathbb{N}_b$, $P\to\mathbb{N}_p$ with $\mathbb{N}_b\neq\mathbb{N}_p$. In the above, $\mathbb{N}$ is in reality $\mathbb{N}_b$ and "K-subset is not a subset of $\mathbb{N}$" means "K-subset is not a subset of $\mathbb{N}_b$", while K-subset is a subset of $\mathbb{N}_p$.

3. Frame of Natural Infinity

However, we have a problem: the size of the power set of $\mathbb{N}$ is 2^∞, the size of $\mathbb{N}$ is ∞. How can a set of size 2^∞ be in one-to-one correspondence with a set of size ∞? In order to solve this problem, we will appeal the philosophical absolute infinity which is bigger than all things. In the following, the absolute infinity will be named "Natural Infinity" and denoted by ∞_0. Because ∞_0 is absolute, it should be respected absolutely, that is, no mathematical object should be bigger than it, for example, the generic infinity ∞ is not bigger than ∞_0. To respect ∞_0, the size of the power set of $\mathbb{N}$ must be ∞_0, but not 2^{∞_0}.

Then, how will we deal with infinity? Let us create the infinitely long list L_∞ from the finite list L_m which is a list of n-bits sequences. The length of L_m is $m=2^n$. When we increase the number of bits of L_m, we do not increase directly n but increase m instead. This way, when $m\to\infty_0$, L_m becomes L_∞ whose length is surely ∞_0, avoiding 2^{∞_0} that exceeds the absolute infinity ∞_0. This is why the names of lists are indexed with their length such as for L_2, L_4 or L_m.

The number of bits of the list L_m is n. For finding n with m, we reverse equation **(1)** to get n:

$$n = \log_2 m \qquad \textbf{(2)}$$

For $m \to \infty_0$, the length of L_∞ is m_∞ and the number of bits is $n_\infty = \log_2 m_\infty$ which we write as $\log_2 \infty_0$. The quantity $\log_2 \infty_0$ is also infinitely big but smaller than ∞_0. It will be named Logarithmic Infinity and denoted by ∞_l. We see that both m_∞ and n_∞ respect absolutely ∞_0. Note that the length of the list L_∞ is ∞_0, the number of bits of L_∞ is ∞_l, and $\infty_0 = 2^{\infty_l}$.

The logarithmic infinity ∞_l is one of the many smaller-than-∞_0 infinities, for example, $\frac{\infty_0}{2}$ or $\sqrt[2]{\infty_0}$. Smaller-than-∞_0 infinities make a class called Sub-Infinity. To respect the absolute infinity, ∞_0 will serve as a frame out of which no number should exist. Within this frame Sub-Infinity allows infinitely big numbers to exist without contradiction. This frame is referred to as the **Frame of Natural Infinity**.

Let us see two uses of this frame. Take 2 natural numbers p and q such that $q=2p$. What will be the value of q when p becomes infinity? If we write $p=\infty_0$, then $q=2\infty_0$, breaking the frame. To maintain q below ∞_0, the upper bound of p must be $\frac{\infty_0}{2}$. This way, p and q can be infinitely big while staying within the frame. The infinitely big number $\frac{\infty_0}{2}$ is a sub-infinity.

Sub-Infinity is also used in set theory. Take the set of couples of natural numbers $\{i,j\}$ which is denoted by 2D$\mathbb{N}$. This set is countable and the number of its elements is ∞_0. For counting the couples in 2D$\mathbb{N}$, we make the gray square in Table 3 and find i^2 couples in it. When the square becomes the entire 2D$\mathbb{N}$, i is infinitely big. If we write $i=\infty_0$, the number of couples will be ∞_0^2, breaking the frame. So, the maximal value of i must be $\sqrt{\infty_0}$, which is a sub-infinity.

∞	...	...	...	...	...	...
i	1,i	2,i	3,i	...	i,i	...
...	...	...	...	...	...	...
3	1,3	2,3	3,3	...	i,3	...
2	1,2	2,2	3,2	...	i,2	...
1	1,1	2,1	3,1	...	i,1	...
	1	2	3	...	i	∞

Table 3

Remark: I do not use $\aleph_0$ instead of ∞_0 because I think the Frame of Natural Infinity makes more sense than $\aleph_0, \aleph_1, \aleph_2 \ldots$ Indeed, $\aleph_1, \aleph_2 \ldots$ are bigger than ∞_0, in utter contradiction with the philosophical absolute infinity. The Frame of Natural Infinity is free of such contradiction.

Return to the problem about the sizes of sets. The power set of $\mathbb{N}$ is represented by the list L_∞ which is in one-to-one correspondence with $i_\infty = \{0,1,2,3 \ldots 2^{\infty_l} - 1\}$ whose size is 2^{∞_l} and which is the set $\mathbb{N}_p$. So, the power set of $\mathbb{N}$ is in one-to-one correspondence with $\mathbb{N}_p$, not with the base set $\mathbb{N}_b$ whose size equals ∞_l the number of bits of L_∞. Natural Infinity and Logarithm of Infinity have clarified the sizes of sets and dissipated the problem.

This problem shows why Cantor's theorem fails. The list X is made of subsets of $\mathbb{N}_b$ whose size is ∞_l while the size of K-subset is $2^{\infty_l} - \infty_l$. Cantor's theorem needs X to contain K-subset. This requires the members of X and K-subset to have the same size, that is, $k = 2^k - k$ for $k \to \infty_l$, which is impossible. So, the real flaw of Cantor's theorem is to compare $\mathbb{N}_b$ with the power set of $\mathbb{N}_b$ while they are related. In general, two sets must be independent from each other when searching for one-to-one correspondence. $\mathbb{N}_p$ is independent of the power set of $\mathbb{N}_b$, so the one-to-one correspondence between them is succesful.

4. List of numbers smaller than 1

a. Creation of the numbers

A binary sequence can be transformed into a smaller-than-1 number by adjoining "0." to its left. For example, adjoining "0." to 1000 transforms it into 0.1000, which is the decimal 0.5. So, the list of binary sequences L_∞ can be transformed into a list of

i_∞	L_∞ (∞_l bits)
∞_0-1	...11111111
...	...
2	...00000010
1	...00000001
0	...00000000

Table 4

smaller-than-1 numbers by adjoining "0." to the left of all its elements. The last elements of L_∞ is $...111111$ (∞_l times 1), whose binary values is $\infty_0-1=2^{\infty_l}-1$. We recall that the binary sequences in L_∞ equal the numbers $(0,1,2,3...\infty_0-1)$ written in binary system, see Table 4.

After adjoining "0." to every binary sequence, L_∞ is transformed into a list of smaller-than-1 numbers, the list R_∞ which is shown in Table 5. The list R_∞ inherits the length and number of bits of L_∞ which are ∞_0 and ∞_l. So, the elements of R_∞ are numbers with infinitely many bits after the radix point. Numbers in R_∞ can be irrational if they do not have repeating pattern of bits. For example, the binary number $0.01010000011...$ in Table 5 is the irrational $\frac{\pi}{10}$ Since the list L_∞ is comprehensive, the list R_∞ is also comprehensive, that is, R_∞ contains all possible binary numbers that are smaller than 1. So, it is possible that R_∞ contains all the real numbers in the interval $[0,1[$.

i_∞	R_∞ (∞_l bits)
∞_0-1	0.11111111111...
...	...
$\infty_0 \times 0.314...$	0.01010000011...
...	...
2	0.000000...00010
1	0.000000...00001
0	0.000000...00000

Table 5

b. Denseness of R_∞

For R_∞ to contain all the real numbers in $[0,1[$, R_∞ must have the same properties than $\mathbb{R}$. The set $\mathbb{R}$ is dense. So, R_∞ must be dense too, that is, between any 2 different numbers of R_∞ there exist infinitely many numbers in R_∞.

Since R_∞ is a list, this seems impossible. Indeed, even in an infinite list such as $(0,1,2,3...\infty_0)$, between 2 arbitrary numbers k_1 and k_2 there are k_2-k_1 numbers but not infinitely many. However, if k_2 is ∞_0, there are infinitely many numbers between them. The bound ∞_0 is rather an extreme example, but infinite intervals such as $[\infty_0/3,\infty_0/2]$ or $[\log_2 \infty_0,\sqrt{\infty_0}]$ exist and contain infinitely many numbers.

The list of indices of R_∞ is $i_\infty=(0,1,2,3...\infty_0-1)$, in which sub-infinite numbers will be bounds of intervals. For example, $0.5\infty_0$ and $0.75\infty_0$ are 2 sub-infinite indices of R_∞ which point to the decimal numbers 0.5 and 0.75 which are the binary numbers 0.1 and 0.11.

In order to get a sense of an infinite index, we take a forever increasing big number as handle for ∞_0, for example, the elapsed time from the Big Bang counted in nanosecond, which will be denoted by T. The infinite index $0.5\infty_0$ will be handled with $0.5T$ at the instant of use. As $0.5T$ is a natural number, it indexes a true element of the list R_∞, but only at this instant. When we use $0.5\infty_0$ slightly later, $0.5T$ has increased and is bigger. T is a very big increasing number with which no normal number can catch up and as such, is a good handle for working with the infinity ∞_0.

Indices	R_∞
$0.75\infty_0$	0.1100...0000
$0.75\infty_0-1$	0.1011...1111
...	...
$0.5\infty_0+1$	0.1000...0001
$0.5\infty_0$	0.1000...0000

Table 6

The interval $[0.5\infty_0,0.75\infty_0]$ of i_∞ contains infinitely many indices. How many are they? Subtract $0.5\infty_0$ from $0.75\infty_0$ and we obtain $0.25\infty_0$ indices, which are effectively infinitely many. These indices and the indexed numbers in R_∞ are illustrated in Table 6. In general, take two different numbers in R_∞: s and t, their indices are respectively $s\times\infty_0$ and $t\times\infty_0$. With $s<t$, for whatever small $(t-s)$, there are $(t-s)\infty_0$ indices between $s\times\infty_0$ and $t\times\infty_0$ and each of which indexes a number in R_∞. So, between any 2 different numbers of R_∞ there are infinitely many numbers in R_∞ and the list R_∞ is dense, as the set $\mathbb{R}$ is.

c. Completeness of R_∞

The set $\mathbb{R}$ is complete, meaning that real numbers have least-upper-bound property. So, R_∞ must be complete, that is, any nonempty subset of R_∞ has a least-upper-bound within the list R_∞.

Let D be a nonempty subset of $i_\infty=(0,1,2,3\ldots\infty_0-1)$ the list of indices of R_∞. Let C be the set of the numbers indexed by the indices in D, then C is a nonempty subset of R_∞. D has a maximal number d which is the least-upper-bound of D because i_∞ is a list of natural numbers. The sequences of bits of the numbers of R_∞ equal elements of i_∞ written binary system, then the number indexed by d is the least-upper-bound of C. So, any nonempty subset of R_∞ has a least-upper-bound in R_∞. Since R_∞ is a dense set, R_∞ has the least-upper-bound property and is complete, similar to the set $\mathbb{R}$.

d. Real numbers in [0,1[

We have shown that the list R_∞ contains all possible binary numbers that are smaller than 1, including irrational numbers, that R_∞ is dense and complete. So, R_∞ contains all the real numbers in the real interval $[0,1[$, which constitute $\mathbb{R}_1$ the set of real numbers in this interval, then $R_\infty=\mathbb{R}_1$. R_∞ is in one-to-one correspondence with i_∞ which is $\mathbb{N}$, making R_∞ and $\mathbb{R}_1$ countable. The set $\mathbb{R}_1$ is in one-to-one correspondence with the set of real numbers $\mathbb{R}$, so the set $\mathbb{R}$ is countable.

5. About Cantor's first proof

In the contrary, Georg Cantor proved that $\mathbb{R}$ is uncountable with his first proof of the uncountability of $\mathbb{R}$. This proof shows that a real number z lies out of any sequence of real numbers, proving that real numbers are uncountable. Here is the proof in 3 steps:

1. Let X_0 be any sequence of real numbers in the interval $[0,1]$.
2. The first two elements of X_0 create the first interval $[a_1, b_1]$. The next sequence X_1 contains the elements of X_0 that fall into $[a_1, b_1]$. In the same way, create the second interval and sequence $[a_2, b_2]$ and X_2 from X_1. Repetition of this process creates the nested intervals $[a_1, b_1]\ldots[a_i, b_i]\ldots$. When i reaches infinity, all elements of X_0 are excluded from the interval $[a_\infty, b_\infty]$ which contains at least one number, z, the limit of a_i and b_i. So z does not belong to the sequence X_0.
3. Conclusion: Because X_0 is **any** sequence of real numbers and does not contain z, **no** sequence can contain its proper limit z. In other words, sequence that contains all real numbers does not exist. So, $\mathbb{R}$ is uncountable.

Let us examine this proof. In the conclusion, X_0 is any sequence of real numbers. Then, X_0 can contain all the real numbers and equals $\mathbb{R}$. The limit z is inevitably a real number because $\mathbb{R}$ is complete. The nested intervals cannot exclude the real number z from X_0 **because** $X_0=\mathbb{R}$. So, z is an element of the sequence X_0, invalidating Cantor's first proof.

This flaw slipped into the proof due to the mix-up of strict subsets of $\mathbb{R}$ with the set $\mathbb{R}$ itself. Take $\mathbb{Q}$ as an example, if a sequence of rational number is an X_0, the limit z can be irrational and lies out of the sequence. So, the limit of any sequence in strict subsets of $\mathbb{R}$ can lie out of the sequence. Any sequence in strict subsets of $\mathbb{R}$ is mixed up with any sequence in the whole $\mathbb{R}$, giving the impression that any sequence of real numbers does not contain z. But when the set $\mathbb{R}$ itself is exclusively used as X_0, one finds that any sequence of **all** real numbers contains z. So, Cantor's first proof is in fact a confusion. I have given another account on this topic in «On Cantor's first proof of uncountability».

Remark: I think that the method of nested intervals was invented to show the existence of transcendental numbers. Since the set of algebraic numbers is a strict subset of $\mathbb{R}$, it confused Cantor's first proof when this method was extended to the whole $\mathbb{R}$.

6. About the diagonal argument

The diagonal argument is a proof by contradiction. First, one assumes that X is a sequence that lists all real numbers. Then, using this assumption, one creates a real number F that is not in X, contradicting the assumption and proves that $\mathbb{R}$ is uncountable. This proof is explained below:

1. Radom real numbers are expressed in binary system and put in a column, which is the sequence X. This sequence is supposed to contain all the real numbers of the interval $[0,1]$.
2. One creates the flipped diagonal number F in the following way: take the first bit of the first real number, flip it to its opposite. This value will be the first bit of F. The ith bit of F equals the flipped value of the ith bit of the ith real number.
3. Because the ith bits of the ith real number and of F are different, F does not equal the ith real number. This is true for any i, so F does not equal any of the real numbers in the sequence X. Thus, X does not contain a real number, F, proving that $\mathbb{R}$ is uncountable.

4 bits	i
0.1111	15
0.1**11**0	14
0.110**1**	13
0.110**0**	12
0.1011	11
0.1010	10
0.1001	9
0.1000	8
0.0111	7
0.0110	6
0.0101	5
0.0100	4
0.0011	3
0.0010	2
0.0001	1
0.0000	0

Table 7

It is true that F is not in the sequence X. But, could X supposedly contain all the real numbers of the interval $[0,1]$? Table 7 shows all the 16 possible 4-bits binary numbers. When we apply the diagonal argument to this list, only the first 4 numbers enter in the diagonal. So, the first 4 numbers are the sequence X_4, the equivalent sequence of X. The flipped diagonal is $F_4=0.0011$. Effectively, F_4 does not equal any of the first 4 numbers. But, F_4 is the number with index $i=3$ in the lower part of the list.

If the binary numbers have n bits, the sequence X_n will contain n numbers to make a square of $n\times n$ bits to possess a diagonal of bits. There exist 2^n n-bits numbers, of which 2^n-n are not in the square of X_n. So, it is not contradictory that the flipped diagonal number F_n is not in X_n.

Figure 1 shows the diagonal of bits and the $n\times n$ bits square made by the sequence X_n. The black square and the blue rectangle together contain all the 2^n n-bits numbers. The flipped diagonal number F_n is one of them. Since F_n is not in the black square, it is necessarily in the blue rectangle. This is not contradictory.
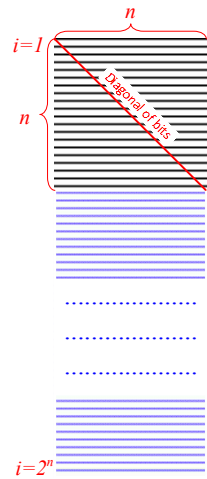

Figure 1

When n increases indefinitely, $n=\infty_l$ for the length of R_∞ to respect the Frame of Natural Infinity. X_n becomes X which makes a $\infty_l\times\infty_l$ bits square like the black square in Figure 1. The sequence X does not contain all the real numbers because for whatever n, the sequence X_n never contains the numbers in the blue rectangle. There are at least 2^{∞_l} real numbers, most of them are in the blue rectangle where the flipped diagonal number F necessarily is. So, it is not contradictory that F is not in the sequence X, invalidating the diagonal argument. I have given another account on this topic in «Hidden assumption of the diagonal argument».

There is a similarity between the diagonal argument and Cantor's theorem. Let the sequence X and real numbers represent the power set of $\mathbb{N}$ and subsets of $\mathbb{N}$ respectively. Subsets of $\mathbb{N}$ are represented by real numbers such as $0.0101101\ldots$, the bits of which are sub-sequences of the infinitely long sequence $0.1111111\ldots$ which represents $\mathbb{N}$. The diagonal of bits serves to put the bits of $0.1111111\ldots$ in one-to-one correspondence with $\mathbb{R}$, which fails because the diagonal compares the bits of $0.1111111\ldots$ with its sub-sequences. This is equivalent to the incorrect comparison of $\mathbb{N}$ and the power set of $\mathbb{N}$ explained in section 3. In consequence, the flaws of the diagonal argument and Cantor's theorem are the same.

7. Conclusion

In Cantor's theorem we find that it is **not** contradictory that K-subset is not in X, invalidating the proof that the power set of $\mathbb{N}$ is uncountable. The power set of $\mathbb{N}$ is put in one-to-one correspondence with the list of natural numbers i_∞, proving that the power set of $\mathbb{N}$ is countable. This proof and the negation of the proof for the uncountability make it doubly credible that the power set of $\mathbb{N}$ is countable.

In the diagonal argument we find that the flipped diagonal number is **in** the list of real numbers X, invalidating the proof that $\mathbb{R}$ is uncountable. In Cantor's first proof we find the confusion due to the mix-up of strict subsets of $\mathbb{R}$ with $\mathbb{R}$ itself. When $\mathbb{R}$ itself is exclusively used, the limit z is **in** the sequence X_0, invalidating Cantor's first proof.

The list R_∞ is created, it contains all the real numbers of the interval $[0,1[$. Previously in <<Cardinality of the set of binary-expressed real numbers>>, I have proposed another list of all the real numbers of the interval $]0,1[$, shown in Figure 2. The number of bits is on the y axis, the real numbers are represented by the blue diamonds. Once y reaches logarithmic infinity ∞_l, the number of the diamonds reaches the natural infinity ∞_0 and all real numbers of $]0,1[$ will be in the list, but in another order than R_∞.
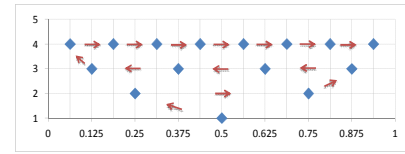

Figure 2

The above 2 lists, the denseness and completeness of R_∞ are 4 proofs for the countability of $\mathbb{R}$. Together with the negation of the diagonal argument and Cantor's first proof, they support 6 times that $\mathbb{R}$ is countable. Also, uncountability should not exist anymore since $\mathbb{R}$ and the power set of $\mathbb{N}$ are countable.

The proposed Natural Infinity ∞_0 is the philosophical absolute infinity above which nothing should exist. Sub–Infinity deals with infinitely big numbers within the Frame of Natural Infinity, free of contradiction. The system of Natural Infinity enters in competition with Georg Cantor's system of $\aleph_0, \aleph_1, \aleph_2....$ Contrary to Georg Cantor's absolute infinity which is never used, the absolute infinity ∞_0 is frequently used in the present article, showing the consistency of the system of Natural Infinity. In my opinion, $\aleph_1, \aleph_2...$ violate the philosophical absoluteness of infinity by being bigger than the absolute infinity. It is fundamentally for this reason that Georg Cantor's 3 proofs fail.