Continuous set and continuum hypothesis

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Abstract: This article explains why the cardinality of a set must be either $\aleph_0$ or $|\mathbb{R}|$.

Georg Cantor has shown that cardinal numbers increases from that of the natural numbers, $\aleph_0$:

$$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 \ldots \qquad 1$$

The cardinality of the real numbers is $|\mathbb{R}|$. For Georg Cantor, $|\mathbb{R}|$ equals the cardinal number of the power set of the natural numbers:

$$|\mathbb{R}| = \aleph_1 = 2^{\aleph_0} \qquad 2$$

He felt that cardinal numbers are discrete and the first two are $\aleph_0$ and $\aleph_1=|\mathbb{R}|$, but he was unable to prove this. So he proposed his continuum hypothesis: "there is no set with cardinality strictly between $\aleph_0$ and $|\mathbb{R}|$". Determination of the truthfulness of the continuum hypothesis ranks number 1 in the list of Hilbert's 23 problems.

The continuum hypothesis was not proven from the point of view of set theory which emphasizes number of elements. Indeed, Kurt Gödel and Paul Cohen have proven that this hypothesis is independent of set theory. So, let us try from the point of view of topology. In Cardinality of the set of binary-expressed real numbers I have seen that binary numbers cannot fill the unit real interval in spite of infinity of digits. In general, the members of a discrete set can only occupy isolated points in a continuous space leaving empty intervals behind. So, the fundamental difference between a continuum and a discrete set is the continuity, not the number of elements.

1. Discreteness of sets

Discreteness is the property that qualifies sets that are formed by isolated points. Isolation means these points do not touch one another. Take the set {0, 1}. Because there is a void space between 0 and 1, they do not touch each other. This void space is named (0, 1). Then we put the number ½ in its middle to make a set with three elements {0, ½, 1}. The new point ½ cut the void space (0, 1) into 2 void spaces, (0, ½) and (½, 1). The set {0, ½, 1} is discrete because ½ is separated from the points 0 and 1by the 2 void spaces it created.

We keep adding new numbers to the above set and obtain: {0, 1/4, ½, 1},{0, 1/4, ½, 3/4, 1}….Each new number splits a void space into two void spaces which keep the number isolated. For a set with n elements, there are n-1 void spaces. Each number is surrounded by the 2 void spaces it creates. Like water surrounding isolated islands, the void spaces surrounding each element make the elements isolated. So, the resulting set is entirely discrete.

Suppose we have an discrete set with i ordered elements: $\{ x_0, x_1 , x_2 ,\ldots , x_i\}$. The next set having i+1 elements is created by putting an element x in the void space (x_k, x_{k+1}) such that $x_k < x < x_{k+1}$. By repeating this process indefinitely, the resulting set will have infinitely many elements. As the elements are added one by one without end, the cardinality of this set is $\aleph_0$.

This set is also discrete because there is a void space between any 2 elements. In fact, discrete sets do not need to be constructed this way. It is enough that any element is surrounded by void spaces on both sides and the set will be discrete, no matter how the elements are arranged. For example, any rational number is surrounded by void spaces. In spite of the ever shrinking interval between 2 rational numbers that makes them infinitely close, the set of rational numbers is discrete. Figure 1 shows three elements x_{i-1}, x_i, x_{i+1} of a set that are separated by the void spaces (x_{i-1}, x_i) and (x_i, x_{i+1}). This property defines the discreteness of a set.

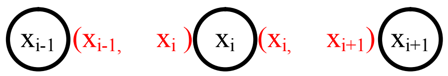

Figure 1

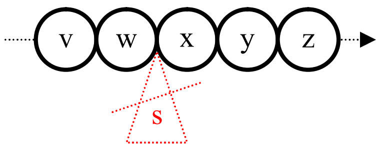

Figure 2

2. Continuity of sets

In the contrary, the set of real numbers is continuous. The elements of continuous set touch one another, that is, between any 2 points in the real line there is no void space in which an external point can be inserted. Figure 2 shows that, lacking void space, the point s cannot be inserted between the elements v, w, x, y, z. It is equivalent to say that except of boundary points, both side of a point x are immediately filled with other elements of the set, no void space exists there. As x can be any point in the real line, real numbers are in contact with one another and the set of real numbers is entirely continuous. This property defines the continuity of a set.

3. Collectively exhaustive and mutually exclusive events

When tossing a coin, all possible outcomes are heads or tails. The values heads or tails are said to be collectively exhaustive, that is, there is no other possibility. Also, when heads occurs, tails can't occur and vice versa. These two values are said to be mutually exclusive, that is, the outcome is either heads or tails, no mixed value is allowed, for example half heads and half tails.

If the elements of a set have void spaces on both sides, the set is discrete. If the interior elements of a set have no void spaces surrounding them, the set is continuous. The presence or not of the void spaces makes discreteness and continuity of a set the 2 outcomes of a collectively exhaustive and mutually exclusive game, no other possibility exists.

4. Continuum hypothesis

The cardinality of an infinite discrete set is $\aleph_0$, like the rational numbers. The cardinality of a continuous set is $|\mathbb{R}|$, like the real numbers. As shown above, a set must be exhaustively and exclusively discrete or continuous. So, its cardinality must be $\aleph_0$ or $|\mathbb{R}|$ but not strictly between $\aleph_0$ and $|\mathbb{R}|$. In consequence, the continuum hypothesis is true.

5. Cardinality of discontinuous subsets of real numbers

Can discontinuous subset of real numbers have cardinality smaller than $|\mathbb{R}|$?

The first possibility is to remove all rational numbers from the real line. Depleted of all rational numbers, the set of the real numbers becomes the set of irrational numbers. This set is not continuous because the rational numbers left holes in the real line. The cardinality of the rational numbers is $\aleph_0$. So, the cardinality of the irrational is $|\mathbb{R}|-\aleph_0$. As we have $|\mathbb{R}|-\aleph_0=|\mathbb{R}|$, the cardinality of the irrational numbers is not smaller than $|\mathbb{R}|$.

Second possibility is to remove real numbers proportionally, for example, removing 9 points in 10 from the unit interval [0, 1]. This way, it seems that the left points are only 1/10 of the original points. But removing 9 points in 10 is equivalent to removing 9 parts in 10 from the unit interval and the leftover numbers are equinumerous to 1/10 of the unit interval, that is, the interval [0, 0.1]. As the cardinality of [0, 0.1] is also $|\mathbb{R}|$, the cardinality of this subset is not smaller than $|\mathbb{R}|$.

Another possibility is Cantor ternary set which is constructed in splitting all intervals in 3 and removing the central one, then applying this process to the remaining intervals forever. This process seems to reduce the number of points indefinitely. But it turns out that each time a third of interval is removed, two third continuous intervals are left. So, the cardinality of this set stays $|\mathbb{R}|$.

Then we can remove all the irrational numbers from the unit interval [0, 1]. This way the left set is that of the rational numbers whose cardinality is $\aleph_0$. So, the cardinalities of the above subsets of the real numbers are either $\aleph_0$ or $|\mathbb{R}|$ but not strictly in between.

In consequence, discontinuous subsets constructed from the real line cannot have cardinality bigger than $\aleph_0$ and smaller than $|\mathbb{R}|$ at once and the continuum hypothesis holds.