

Continuity and uncountability

Peng Kuan 彭寬

27 September 2016

Abstract: Discussion about continuity of line, how continuity is related to uncountability and the continuum hypothesis.

1. What is uncountability for?

The real line is made of real numbers which are points. Points are discrete objects, but lines are continuous objects. Can continuity arise out of discreteness when points make line? The idea of uncountability seems to solve this problem. Rational numbers are countable, the line they make contains holes. Real numbers are uncountable, the line they make is continuous. So, continuity may arise if the points of a continuous line are uncountable. One can imagine that uncountable points are so numerous on the real line that real numbers are squeezed together.

Georg Cantor called the set of real numbers continuum, so he probably thought of creating continuity with discreteness when inventing uncountability. But, what does continuity really mean?

2. Cauchy's continuity

I haven't found existing definition for continuity of line but definitions for continuous function instead. For example, using a Cauchy's sequence $s = (x_i \mid x_i \in \mathbb{R})_{i \in \mathbb{N}}$ which converges to a point a , the continuity of a function $f(x)$ at point a is defined as follow:

$$\lim_{i \rightarrow \infty} x_i = a \Rightarrow \lim_{i \rightarrow \infty} f(x_i) = f(a) \quad (1)$$

The real line is the function $f=0$, satisfies this definition at any real numbers and is continuous. I call this definition Cauchy's continuity.

However, if a and x_i are rational numbers, the converging sequence will be entirely in \mathbb{Q} and Cauchy's continuity will allow the set of rational numbers to be continuous, which is wrong. So, Cauchy's continuity is inadequate to define continuity of line.

3. Geometric continuity

Line is a geometric form that represents the form of real objects, for example conductive wire, water pipe, trajectory of planets etc. To illustrate the continuity of line, imagine the lines in Figure 1 as a conductive wire interrupted between points A and B . When electric current flows in the wire and the interruption, electrons move in the conductive medium of the wire and make an electric arc through the air in the interruption. To cross the interruption an electron must quit the conductive medium from point A , pass through the air and enter the point B . Following this image, continuous line is a mathematical medium in the form of line in which a point can move without quitting. An interruption is a location where a moving point must quit the medium.



Figure 1

So, I propose the following definition of continuity:

A line is continuous between 2 points C and D if the space between them is zero. Equivalently, the line is continuous between C and D if a moving point can go from C to D without crossing other point else than C and D . If all points of a line satisfy this condition, then the line is everywhere continuous.

C and D are said to be in contact and adjacent to each other. In the following, this kind of continuity will be referred to as geometric continuity.

4. Real line

Is the real line geometrically continuous? No interruption can be found on the real line, but the condition of geometric continuity is not satisfied. Take 2 different real numbers a and b and bring them close to each other, no matter how close they are, they are always separated by infinitely many other numbers. If an imaginary electron goes from a to b , it must cross many other points else than a and b . So, the real line is interrupted between a and b but not geometrically continuous.

Also, being not in contact with other point, a is an isolated point. As a can be any real numbers, all real numbers are isolated and the set \mathbb{R} is discrete. So, \mathbb{R} is not a continuum.

5. Constructing continuous line

Why are real numbers discrete? Let us see Figure 2. The points on the right are in contact to each other and they are continuous. The distance between the centers of adjacent points is denoted by d and the width of points by w . These points are continuous because $w=d$.

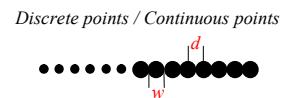


Figure 2

On the left, the distance between the points is still d but the width of points is smaller, $w < d$, this makes them discrete. However, we can shrink the distance d on the left to make the points continuous again.

If the points were real numbers, the width of points equals zero and, however small the value of d is, the points are always separated by a distance because $d > 0$. Therefore, that the width of points is zero is the reason that makes real numbers discrete. This also proves that uncountability is unrelated to continuity. Indeed, real numbers are uncountable and discrete at the same time.

On the other hand, if one puts a real number s in contact with another number r , they will occupy the same point because their widths are zero. If t is put in contact with s , the 3 numbers r , s and t will occupy the point of r . We can repeat this operation uncountably many times, we will obtain only one point, not a line. So, uncountably many points of zero width do not make continuous line.

So, to construct a geometrically continuous line the constructing points must have nonzero width, that is, $w > 0$. What would be the value of w ? Let us deconstruct the continuous line in the interval $[0,1]$ by splitting, as shown in Figure 3 and Figure 4. The first splitting point is $\frac{1}{2}$, then the resulted 2 segments are split at $\frac{1}{4}$ and $\frac{3}{4}$. And then, the 4 resulted segments are split at $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$. The splitting goes forever and we obtain an infinite sequence of splitting points $s_{\text{split}} = (a_i \in \mathbb{R})_{i \in \mathbb{N}}$ and an infinite sequence of segments.

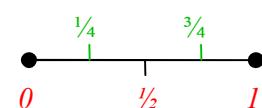


Figure 3

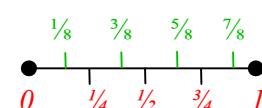


Figure 4

The segments are in contact with one another, securing continuity. Their length equals the infinitesimal number $\varepsilon = \frac{1}{2^\infty}$. These segments are the constructing blocks of the original line, each one starts at its splitting point $a_{i \in \mathbb{N}}$ and has the length ε .

Remark: The construction of geometrically continuous line proves that the controversial infinitesimal number ε really exist, otherwise, continuity cannot arise.

6. General model

For a general line in space such as the one shown in Figure 5, a constructing segment is determined by 6 quantities: 3 coordinates for starting position, 2 angles for direction and ε for length. This segment, S_ε in Figure 5, will be referred to as infinitesimal vector-segment and is the constructing blocks for general line.

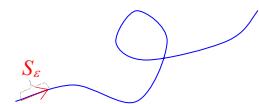


Figure 5

Real numbers are discrete points that are 0-dimensional objects. In the contrary, infinitesimal vector-segment has nonzero length and is a one-dimensional object. So, we have the following property:

One-dimensional geometrically continuous line is constructed only with one-dimensional objects.

Consequently, 0-dimensional points cannot construct one-dimensional line, even they are uncountably many. In general, continuous objects in higher dimension are not constructed with objects of lower dimension. For example, 2-dimensional surfaces are constructed with infinitesimal surface ε^2 and n-dimensional volumes with infinitesimal n-volume ε^n .

7. Uncountability

How did Georg Cantor link uncountability to continuity? In fact, he constructed the continuum \mathbb{R} in two steps: 1) \mathbb{R} is uncountable; 2) Uncountability of \mathbb{R} creates continuity for the real line.

He concentrated himself on proving that \mathbb{R} is uncountable. The first proof he gave was based on nested intervals $[a_0, b_0], \dots [a_n, b_n]$, as shown in Figure 6. Because $a_n \rightarrow b_n$ when $n \rightarrow \infty$, Georg Cantor claims that the limit of a_n and b_n is a number not included in the lists $a_0 \dots a_\infty$ and $b_0 \dots b_\infty$, thus real numbers are uncountable.



Figure 6

However, does the limit of a_n and b_n really exist? A limit is a real number which must be fully determined, that is, all the digits from 1st to ∞^{th} are fixed, for example π . When n increases, the first m digits of a_n and b_n get fixed and make a number that seems to converge. The first m digits of the limit may equal this number, but the limit's last digits, from $m+1^{\text{st}}$ to ∞^{th} , will never be determined. In fact, when n increases, a_n and b_n both vary and the points within the interval $[a_n, b_n]$ are all undetermined. So, the limit that Georg Cantor claims cannot exist and this proof is invalid.

In addition to this flaw which is explained in «[On Cantor's first proof of uncountability](#)», Georg Cantor's later proofs, the [power-set argument](#) and the [diagonal argument](#), contain also flaws, which are explained in «[On the uncountability of the power set of \$\mathbb{N}\$](#) » and «[Hidden assumption of the diagonal argument](#)». So, all 3 proofs that Georg Cantor provided fail and uncountability possibly does not exist.

About the second step Georg Cantor did nothing but simply claim that \mathbb{R} is a continuum; probably he assumed that uncountability really created continuity. But it is shown above that uncountability is not related to continuity. So, uncountability has lost its utility and becomes useless except for itself.

8. Continuum hypothesis

The [continuum hypothesis](#) states that there is no set whose cardinality is strictly between that of the integers which is a discrete set and the real numbers which is a continuum. The idea behind this hypothesis is that there cannot be set that is discrete and continuous at the same time. Georg Cantor tried hard to find such set; the [Cantor's ternary set](#) is probably one of his attempts.

However, it is shown that uncountability is not proven, then the cardinality of real numbers is questionable. Anyway, \mathbb{R} is not a continuum and the continuum hypothesis makes no longer sense.