Prime numbers and irrational numbers

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Abstract: The relation between prime numbers and irrational numbers are discussed using prime line and relative irrationality. A alternative definition of irrational numbers is proposed.

1. Prime line

A rational number is the quotient of 2 whole numbers *i* and *j*, coordinates of a points (*j, i*) in the plane of 2 dimensional natural numbers shown in Figure 1. Each points (*j, i*) represents a rational number whose value is $\frac{i}{j}$ that equals the slope of the straight line connecting the point (*j,i*) to the origin (0,0). For example, the green line in Figure 1 represents the number $\frac{1}{2}$ and the red line the number $\frac{1}{3}$. The green line crosses the points (2,1) , (4,2) and (6,3) showing that these 3 points share the same slope and represent the same rational number, $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}$.

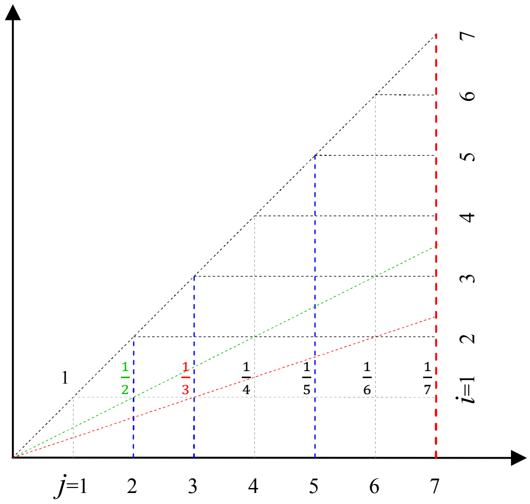

Figure 1

Points on a vertical line, (*P*, i_p), represent the rational numbers r_p that have the same denominator *P*. We will work on rational numbers smaller than 1 and then, the numerators will be smaller than *P*:

$$r_P=\frac{i_P}{P}<1\Rightarrow i_P<P \tag{1}$$

A vertical line whose abscissa number *P* is prime will be named **Prime line of *P*.** For example, the prime lines of 2, 3, 5 and 7 which are highlighted in blue and red in Figure 1. The rational numbers on the prime line of *P* will be named **Prime line numbers of P.**

2. Relative irrationality

When a prime line number equals a rational number with non-prime denominator *n*, we have:

$$\frac{i_n}{n}=\frac{i_P}{P}\Rightarrow n=P\frac{i_n}{i_P} \tag{2}$$

P is a prime number and cannot be divided by i_p. *n* is a natural number and so, the quotient $\frac{i_n}{i_P}$ must be a whole number and bigger than 1. If *n*<*P*, equation (**2**) cannot be satisfied and $\frac{i_P}{P}$ cannot equal any $\frac{i_n}{n}$:

$$n<P\Rightarrow\frac{i_n}{n}\neq\frac{i_P}{P} \tag{3}$$

This property is illustrated in Figure 1 by the green line of $\frac{1}{2}$ which crosses the prime lines of 3, 5 and 7 at points whose ordinates are not whole numbers. Also, the red line of $\frac{1}{3}$ crosses the prime lines of 5 and 7 at non whole ordinates.

Figure 2 illustrates geometrically this property for very big P. One can imagine millions of straight lines radiating from **A**, each one passes through a point (j,i) in the pink triangle, but magically none of them touches a prime line number of P.

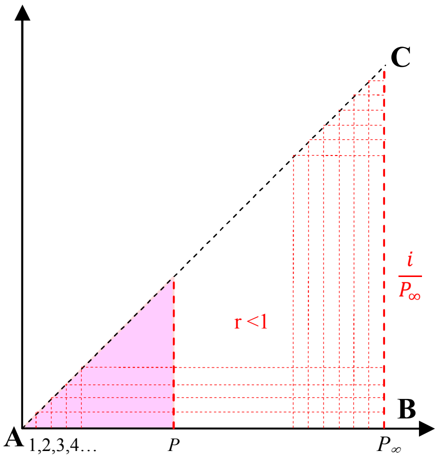

Figure 2

It is not hard to imagine a infinitely big prime P_∞ to be the denominator of $\frac{i_P}{P_\infty}$. P_∞ exists because there are infinitely many prime numbers. In this case, a prime line number of P_∞ does not equal any quotient of natural numbers for $n<P_\infty$, that is, $\frac{i_P}{P_\infty} \neq \frac{i_n}{n}$ for $n < \infty$. This is equivalent to say: the prime line numbers of P_∞ are irrational. However, irrational numbers have infinitely many non repeating digits. Can the fraction $\frac{i_P}{P_\infty}$ satisfy this criterion?

For finite P, the property that the prime line numbers of P do not equal quotients of natural numbers for $n<P$ will be named **Relative irrationality** because they are "irrational" relatively to the rational numbers whose denominators are smaller than P, that is, those in the pink triangle only.

3. Digits of prime line numbers

Using the equation $\frac{i_P}{P} = i_P \times \frac{1}{P}$, the digits of $\frac{i_P}{P}$ can be computed from $\frac{1}{P}$. The digital expansion of $\frac{1}{P}$ equals a whole number T divided by B^m where B is the numeral base and m the number of digits:

$$\frac{1}{P} = \frac{T}{B^m} \Rightarrow B^m = P \times T \qquad (4)$$

In case where $B = P \times T$, the number of digit is 1. For example, $P = 2, T = 5, B = 10 \Rightarrow \frac{1}{P} = \frac{1}{2} = 0.5$. When $T = P^l$, the number of digits is $l+1$ in the system of base $B=P$:

$$B^m = P \times P^l \Rightarrow m = l + 1 \qquad (5)$$

For big P and small B, no definite T will satisfy equation (**4**) and the digital expansion of $\frac{1}{P}$ will endlessly repeat a group of digits, the repetend, and the total number of digits is infinite, $m=\infty$.

What is the number of digits of the repetend? Let us see the example of a real number:

$$0.123\ 123\ 123\ 123\ 123 \ldots = 0.123(1 + 10^{-3} + 10^{-6} \ldots)$$
$$= 0.123\left(\frac{1}{1 - 10^{-3}}\right) = \frac{123}{10^3 - 1} \qquad (6)$$

By substituting R for the repetend 123, B for the base 10 and k for the repetend's number of digits 3, we get the digital expansion of $\frac{1}{P}$, which gives the number of digits k:

$$\frac{1}{P} = \frac{R}{B^k - 1} \Rightarrow B^k = P \times R + 1 \Rightarrow k = \frac{\log(P \times R + 1)}{\log \mathrm{B}}$$
$$\Rightarrow k = \log_B(P \times R + 1) > \log_B(P) \qquad (7)$$

In numeral system of any base, the repetend's number of digits increases with P. For an infinitely big prime P_∞, the repetend's number of digits is infinite:

$$k > log_B(P_\infty) = \infty \tag{8}$$

Because the repetend of $\frac{1}{P_\infty}$ has infinitely many digits, the second repetend and further will be ignored. So, the digits of the prime line numbers of P_∞, $\frac{i_P}{P_\infty}$, would have infinitely many non-repeating digits and are irrational. But, does $\frac{i_P}{P_\infty}$ express all irrational numbers? This is not obvious because the prime line numbers of P_∞ are:

$$\left\{\frac{1}{P_\infty}, \frac{2}{P_\infty}, \dots, \frac{P_\infty}{P_\infty}\right\} \tag{9}$$

These numbers are apparently not continuous.

4. Continuity

Irrational numbers are continuous because 2 irrational numbers can get infinitely close to one another. What are the neighbors of an irrational number? If the decimal expansions of the neighbors are written to the last decimal, they would terminate with one of the 9 numbers: 0,1,2,3,4,5,6,7,8,9. But irrational numbers do not have last decimal, do they?

The number of decimals of an irrational number is infinite. Infinite number of digits is usually represented by a number n that increases indefinitely, that is, infinity of number of digits is the increase without end of n. This is the logic of potential infinity. Under this logic, at each step of the increase n is a definite number. Any decimal number smaller than 1 has at most n decimals and can be expressed as a natural number smaller than 10^n divided by 10^n. The set of decimal numbers at step n is:

$$\left\{\frac{0}{10^n}, \frac{1}{10^n}, \frac{2}{10^n}, \dots, \frac{10^n}{10^n}\right\} = \{0,1,2,\dots,10^n\}\frac{1}{10^n} \tag{10}$$

To get coherent conclusion, prime line numbers must be checked under the same logic of potential infinity. For doing so, we take the prime number P_n immediately superior to 10^n and create the set of prime line numbers of P_n:

$$\left\{\frac{0}{P_n}, \frac{1}{P_n}, \frac{2}{P_n}, \dots, \frac{P_n}{P_n}\right\} = \{0,1,2,\dots,P_n\}\frac{1}{P_n} \tag{11}$$

The members of the sets in the right parts of equations (**10**) and (**11**) for 10^n and P_n are both natural numbers. So, the prime line numbers of P_∞ and decimals numbers have the same type of continuity under the logic of potential infinity.

Table 1	Table 2
$0\times\frac{1}{10^\infty}$	$0\times\frac{1}{P_\infty}$
$1\times\frac{1}{10^\infty}$	$1\times\frac{1}{P_\infty}$
…	…
$5\ldots(\infty \text{ of d})\ldots 8\times\frac{1}{10^\infty}$	$5\ldots(\infty \text{ of d})\ldots 8\times\frac{1}{P_\infty}$
…	…
$10^\infty\times\frac{1}{10^\infty}$	$P_\infty\times\frac{1}{P_\infty}$

Table 1 **Table 2**

The other way to consider infinity is the logic of actual infinity which takes infinity as a definite value. Under this logic the number of digits of an irrational number is an infinite but determined number and thus, last decimal exists. In this case, irrational numbers are in the form of a whole number smaller than 10^∞ having infinitely many decimal divided by 10^∞, as Table 1 shows.

We compare prime line numbers with decimal numbers under the same logic of actual infinity which allows an infinitely big prime to exist, P_∞. The prime line numbers of P_∞ are shown in Table 2, which has the same type of continuity than the irrational numbers in Table 1.

In consequence, prime line numbers are equivalent to decimal numbers with regard to continuity, despite that continuity is not well defined.

5. Prime line numbers compared with irrational numbers

Prime line numbers of P_∞ have the following properties:

1) Not a quotients of whole numbers
2) Having infinitely many non repeating digits
3) Being equivalently continuous as decimal numbers

In consequence, the set of prime line numbers of P_∞ is equivalent to the set of decimal expansions of irrational numbers. Also, as I have shown in section 2 "Relative irrationality", the prime line numbers of P_∞ are all irrational numbers, well separated from the rational numbers that are points (*j,i*) in the triangle **ABC** in Figure 3. Decimal numbers are real numbers which mix rational and irrational numbers and make confusion. So, prime line numbers of P_∞ are better representation of irrational numbers than infinite decimal numbers.

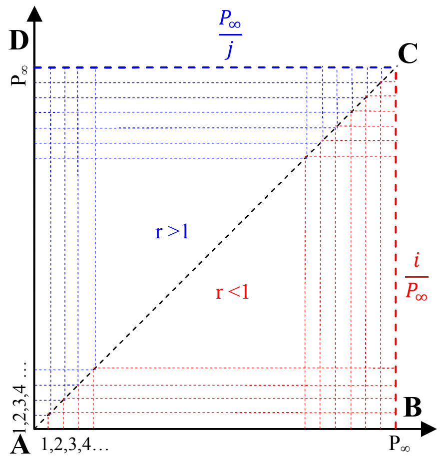

Figure 3

6. Numbers bigger than 1

The above study concerns numbers smaller than 1. For bigger than 1 numbers, the prime lines will be horizontal lines above the unit line **AC** in Figure 3. The prime line numbers bigger than 1 will be the quotients of a prime number *P* and abscissa *1,2,3…P*:

$$\left\{\frac{P}{1},\frac{P}{2},\frac{P}{3},\dots,\frac{P}{P}\right\} \qquad (12)$$

Irrational numbers bigger than 1 are on the line **DC** in Figure 3. Rational numbers bigger than 1 are $\frac{i}{j}$ for *i*>*j*, which are represented by points (*j,i*) in the upper triangle **ADC** in Figure 3.