Cardinality of the set of decimal numerals

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Abstract: Cardinalities of the set of decimal numerals and ℝ are discussed using denominator lines and rational plane.

1. Rational plane

A rational number is the quotient of 2 natural numbers: $\frac{i}{j}$ where i is the numerator and j the denominator. On the rational plane where j is the abscissa and i the ordinate, the point (j,i) represents the number $\frac{i}{j}$. Figure 1 shows the correspondence between (j,i) and $\frac{i}{j}$.

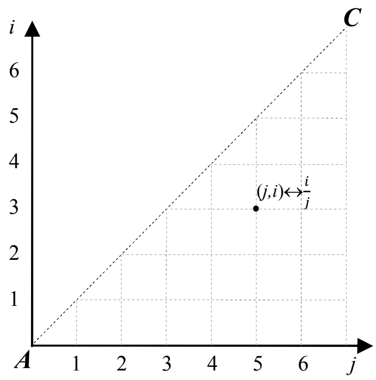

Figure 1

In Figure 1, AC is the unit line on which all quotients equal 1. The rational plane is divided into 2 parts by the unit line. The quotients in the upper part are bigger than 1 and those in the lower part are smaller than 1. We work with quotients smaller than or equal to 1, which are in the lower part under the unit line.

2. Denominator line

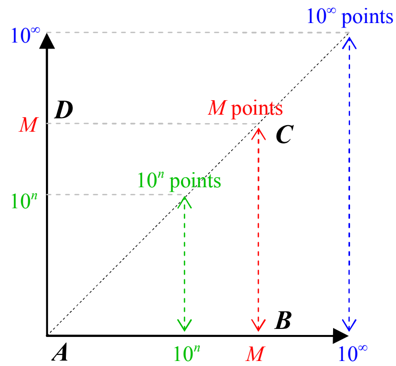

Figure 2

On the rational plane, the points of a vertical line at abscissa M represent the quotients $\frac{i}{M}$. As we work with $\frac{i}{M} \leq 1$ and $\frac{i}{M}$ have the same denominator M, the vertical line at abscissa M below the unit line is named denominator line of M, which is shown by the dashed gray vertical lines below the unit line in Figure 1.

So, the numerators of the points on denominator lines are between 1 and M. For example, on the denominator line of 2 the numerators are 1 and 2; those on the denominator line of 5 are 1,2,3,4 and 5 (see Figure 1). In general, the quotients on the denominator line of M are:

$$\left\{\frac{i}{M} | i = 1,2,3,\ldots,M\right\} \tag{1}$$

Each quotient corresponds to a natural number on the ordinate axis. So, the denominator line of M has the same number of points than the ordinate line AD in Figure 2. That is, BC and AD are equal and both have M points.

When the denominator M increases indefinitely, M becomes M_∞ and AD extends indefinitely to equal the ordinate axis which is ℕ. Because BC equals AD which equals ℕ, the denominator line of M_∞ contains the same number of points than ℕ:

$$\left|\left\{\frac{i}{M_\infty} | i = 1,2,3,\ldots\right\}\right| = |\mathbb{N}| = \aleph_0 \tag{2}$$

When M=10^n, the denominator line of M contains only decimal numerals.

3. Decimal numerals

The set of decimal numerals is the set of decimal expansions of real numbers in the interval [0,1]. A decimal numeral with n digits is written as $D=0.d_1d_2d_3...d_n$ that can also be expressed as the quotient of the natural number $d_1d_2d_3...d_n$ and 10^n:

$$D = \frac{d_1 d_2 d_3 \dots d_n}{10^n} \tag{3}$$

With 1 digit, $D = \frac{d_1}{10}$ and D has 10 possible values. If D has 2 digits, the number of possible values is $10\times10=10^2$. Each time a digit is added, the number of possible values is multiplied by 10. For D with n digits, the number of possible values is 10^n. So, the set of decimal numerals with n digits has 10^n members, which is written as below:

$$S_n = \left\{\frac{1}{10^n}, \frac{2}{10^n}, \frac{3}{10^n}, \dots, \frac{d_1 d_2 d_3 \dots d_n}{10^n}, \dots, \frac{10^n}{10^n}\right\} \tag{4}$$

In the set S_n, the denominator is 10^n, the numerators contain every natural number between 1 and 10^n. So, S_n constitutes the denominator line of 10^n (see Figure 2). When the number of digits n increases, S_n lengthens by adding 9 times more points with each added digit, as the numerators of S_n show below:

$$1,2,3,4,5,6,7,8,9,10 \rightarrow 1,2,3,\dots,100 \rightarrow 1,2,3,\dots,1000 \rightarrow \dots\dots\dots \rightarrow 1,2,3,\dots,10^n \tag{5}$$

When $n\rightarrow\infty$, S_n becomes the denominator line of 10^∞. In terms of quotients of whole numbers, we replace n by ∞ in equation (4) and obtain the set of quotients on the denominator line of 10^∞:

$$S_\infty = \left\{\frac{1}{10^\infty}, \frac{2}{10^\infty}, \frac{3}{10^\infty}, \dots, \frac{d_1 d_2 d_3 \dots d_\infty}{10^\infty}, \dots, \frac{10^\infty}{10^\infty}\right\} \tag{6}$$

The members of S_∞ are $\frac{d_1 d_2 d_3 \dots d_\infty}{10^\infty} = 0.d_1 d_2 d_3 \dots d_\infty$. The number of digits of $0.d_1d_2d_3...d_\infty$ is infinite because $n\rightarrow\infty$. If the numerators $d_1d_2d_3...d_\infty$ has infinitely many non repeating digits, $0.d_1d_2d_3...d_\infty$ is an irrational number. If only a few digits of $d_1d_2d_3...d_\infty$ are non-zero, $0.d_1d_2d_3...d_\infty$ is a terminating decimal numeral. But as the number of digits is ∞, the remaining digits of $d_1d_2d_3...d_\infty$ are all zero. So, S_∞ contains every decimal numerals D for $0< D \leq 1$ and thus, equals the set of decimal numerals.

Geometrically speaking, S_∞ is the denominator line of 10^∞ in Figure 2, which contains 10^∞ points, the number of digits of each point is ∞.

4. Countability of the set of decimal numerals

S_∞ constitutes the denominator line of 10^∞. Like the denominator line of M_∞ in section 2, S_∞ has the same size than the ordinate axis (see Figure 2). As the ordinate axis equals $\mathbb{N}$, the cardinality of S_∞ is $\aleph_0$. Because the set of decimal numerals equals S_∞, the set of decimal numerals is countable.

The countability of S_∞ is also shown by equation (6) in which the sequence of numerators equals the set of natural numbers $\mathbb{N}$ (see equation (2)). So, S_∞ and the set of decimal numerals are countable. Because the set of decimal numerals is the set of decimal expansions of real numbers in the interval [0,1], $\mathbb{R}$ should be countable too.

But, has not Georg Cantor proven that ℝ is uncountable? Yes, he has. For doing so, he used the diagonal argument and created a flipped diagonal decimal numeral from a list of decimal numerals. He showed that this number is not in the list. However, I have shown in <<Hidden assumption of the diagonal argument>> that his list was not complete. This flaw invalidates the diagonal argument.

Georg Cantor has also provided a proof of the uncountability of the power set of ℕ which, representing the subsets of ℕ using binary numerals, showed that the subset of all out-indexes cannot be indexed. But this proof too, is flawed. I have shown in <<On the uncountability of the power set of ℕ>> that the subset of all out-indexes was not a subset of the original ℕ used to create the power set. So this proof of uncountability is also invalid. In the contrary, using the denominator line of 2^n and making $n \to \infty$, we can prove that the set of binary numerals is countable. In consequence, the power set of ℕ is countable.

The diagonal argument and the power-set-of-ℕ-proof are based on positional writing systems of numbers such as decimal or binary systems. The above denominator line argument is also based on decimal system. Could countability be a property of the writing system rather than a property of ℝ? I do not think so. In fact, Cantor's first proof used a sequence of nested intervals to prove uncountability without writing real numbers down. But this proof fails too. I have explained in << On Cantor's first proof of uncountability>> that the sequence of nested intervals did not isolate a non listed real number.

So, countability cannot be attributed to the writing system of numbers. ℝ has gained a proof for countability with the denominator line argument, but lost its most important proofs for uncountability with the papers cited above. If ℝ is countable, it cannot be the continuum and the continuum hypothesis should not have to exist.