Is Hilbert's Grand Hotel a paradox?

Kuan Peng 彭宽 *titang78@gmail.com*
21 May 2023, last modified: 5 July 2023

Note: the blue texts are the modified part.

Abstract: Hilbert's Grand Hotel shows that a fully occupied hotel with infinitely many rooms can accommodate additional guests. But our analyze finds that this is not true.

1. Hilbert's Grand Hotel

Hilbert's Grand Hotel [1] is an illustration of a counterintuitive property of infinite sets. It shows that a hotel with infinitely many rooms, all of which are occupied, may still accommodate additional guests, see Hilbert's paradox of the Grand Hotel [1]. This is thought to be a veridical paradox, that is, in mathematical sense, Hilbert's Grand Hotel can really accommodate additional guests.

Let us illustrate Hilbert's Grand Hotel in Figure 1 where each square is a room and is occupied. Suppose that the rooms of the hotel are numbered 1, 2, 3 … . We call the guest of the room 1 guest 1, the guest of the room 2 guest 2, the guest of the room n guest n and so on. The new guest is called guest G.

1	2	3	4	5	6	7	8	9	…	n	…

Figure 1 Rooms and guests in Hilbert's Grand Hotel

2. Analysis of the paradox

a) Each guest moves to the next room

Let us analyze this paradox with a first case where the occupant of each room accepts to shift to the next room. When the guest G arrives and asks for a room, according to David Hilbert, the hotelkeeper will move the guest 1 to room 2 and accommodate the guest G in room 1, then move the guest 2 to room 3 to accommodate the guest 1, and so on. The general way is to move the guest n-1 to the room n to accommodate the guest n-2. This way the guest G is accommodated while all the old guests still has a room.

Let us show this procedure of room shifting with Figure 2. The room shifting is done step by step. The guest G takes the room 1, the guest 1 takes the room 2 and so on. At the step n-1, the guest n-1 is before the door of the room n. At each step from 1 to n-1, the guests 1 to n-1 are successively out of room. This is true for all n, however big n is. So, at any step one guest is out, which is shown in Figure 2.

Note: Julio Di Egidio had written his view of this paradox and put it on line 2014. [2]

	1										
G	2	3	4	5	6	7	8	9	…	n	…
		2									
G	1	3	4	5	6	7	8	9	…	n	…
			3								
G	1	2	4	5	6	7	8	9	…	n	…
										n-1	
G	1	2	3	4	5	6	7	8	…	n	…

Figure 2 Each guest moves to the next room

b) All the guests are identical

Let us consider the case where all the guests are identical and indistinguishable from the guest G such that when the guest G gets into a room and the previous occupant gets out, we still see guest G before the room. Just imagine that rooms are boxes and the guests are Ping-Pong balls. Then this guest G knocks the next room and will do this

[1] Wikipedia (2023, April 17). Hilbert's paradox of the Grand Hotel. Retrieved 00:53, May 18, 2023, from https://en.wikipedia.org/w/index.php?title=Hilbert%27s_paradox_of_the_Grand_Hotel&oldid=1150312655

[2] Julio Di Egidio, 26 February 2014 “Hilbert's impossible hotel”, https://seprogrammo.blogspot.com/2014/02/hilberts-impossible-hotel.html

forever. This is illustrated by the letter G before the rows of rooms, see Figure 2. In consequence, the additional guest G is always out of room the same way as Hilbert's Grand Hotel does not accommodate him.

G											
1	2	3	4	5	6	7	8	9	…	n	…
	G										
1	2	3	4	5	6	7	8	9	…	n	…
		G									
1	2	3	4	5	6	7	8	9	…	n	…
										G	
1	2	3	4	5	6	7	8	9	…	n	…

Figure 3 Guest G check all the rooms

c) One guest always out

In the first case, it was the guests 1, 2, 3… that are out of room successively. In the second case it is the indistinguishable guest G who is out of room at each step. Suppose that the out-of-room-guest is before the n^{th} room asking for entry. We will always have a guest asking for the room n in the hotel while n increases with no end. So, an additional guest is not accommodated in a room even he goes to infinitely far. In other words, Hilbert's Grand Hotel cannot accommodate additional guest in its infinitely many rooms.

There is an argument that the hotelkeeper can magically inform the infinitely many guests for room shifting all at once. But the hotelkeeper cannot reach the last occupant of the hotel because there is none. The word infinite is a Latin word in which “in-” means “contrary to”, “-finite” means end. In mathematics infinite means “no end” and reaching infinity means reaching “no end”, or never reaching the end. So, the hotelkeeper cannot reach the last guest to make him and everybody to shift room and the guest G will not have a room.

On the other hand, for a guest to move into a room, the latter must be empty. As the hotel is full, there is no empty room from the first to infinitely far and no guest can shift room without expelling the next guest from his room, which makes him an out-of-room-guest. For example, the $n\text{-}1^{th}$ guest expels n^{th} guest who becomes the out-of-room-guest, then the $n+1^{th}$ guest becomes the out-of-room-guest and so on. There is always a guest out of room during this process whatsoever.

The magical room shifting all at once could be done only if there is a last guest with the room next to him empty. But there is not a last guest if the guests are infinitely many. A room after the last of the infinitely many rooms makes no sense. Also, if the hotel is full, all the rooms are occupied and there is no empty room. Thus, the assertion that a fully occupied infinite hotel can accommodate additional guest is wrong and Hilbert's Grand Hotel is broken.

d) A coach with real numbers as guests

It is said that Hilbert's Grand Hotel can accommodate infinitely many additional guests. For example, let a coach with infinitely many guests arrive at an empty Hilbert's Grand Hotel. All the rooms being available the hotelkeeper distribute the rooms to the guests one by one. Suddenly, he notices that the coach is the interval [0, 1] and each guest is a real number. He wonders if he has enough rooms to accommodate all the real numbers. I think he has because according to uniform distribution in probability, every real number has an equal chance to be chosen. As the probability to have a room for those still in the coach is identical to the probability of those who already have a room, those still in the coach will have a room.

There is another way to think about this case which is that the distribution of room will never end because the rooms are infinitely many. This never-ending process can be understood in two ways:

1. There always be empty rooms left for those still in the coach.
2. There always be guests in the coach to fill the left empty rooms.

Let A be the cardinality of the set of the rooms and B the cardinality of the set of the guests. The first way means:

$$A \geq B$$

The second way means:

$$A \leq B$$

Since we have $A \geq B$ and $A \leq B$ at once, we have $A = B$. Because the set of the rooms is equivalent to the set of natural numbers and the set of the guests in the coach equivalent to the set of real numbers, A is the cardinality of the set of natural numbers and B the cardinality of the set of real numbers. So, $A = B$ means the cardinality of the set of natural numbers equals the cardinality of the set of real numbers.

Conclusion: the set of real numbers is countable.

3. Discussion

In consequence, the paradox of the Hilbert's Grand Hotel is wrong because we have shown that no additional guest can be accommodated. In the contrary, this property shows that the cardinalities of the sets of natural numbers and real numbers are equal and the set of real numbers is countable.

Please read the following article where I have explained the idea that a real number is a point rather than a sequence of digits:
Kuan Peng, 2022, «Real numbers and points on the number line with regard to Cantor's diagonal argument»
https://www.academia.edu/88279926/Real_numbers_and_points_on_the_number_line_with_regard_to_Cantors_diagonal_argument

P.S. : For the convenience of the readers I put my articles about uncountability below

1. Kuan Peng, 2022, «Real numbers and points on the number line with regard to Cantor's diagonal argument»
   https://www.academia.edu/88279926/Real_numbers_and_points_on_the_number_line_with_regard_to_Cantors_diagonal_argument
2. Kuan Peng, 2022, « Construction of the diagonal flipped number »
   https://www.academia.edu/86917528/Construction_of_the_diagonal_flipped_numbe r
   We write the natural numbers 1,2,3, … in column 1,write them in binary form in column 2, invert the bits of all the numbers of column 2, the leftmost bit becomes the rightmost bit etc, then add 0. on the left of each inverted number in the column 3 to make them smaller than 1. The column 3 is the list L and does not end. As the bits of all these numbers do not ends, each number in the list L is a real number. So, L contains all the real numbers in the interval [0, 1].
3. Kuan Peng, 2022, «Examination of Cantor's proofs for uncountability and axiom for counting infinite sets»
   https://www.academia.edu/86410224/Examination_of_Cantors_proofs_for_uncountability_and_axiom_for_counting_infinite_sets!
   An analysis of Cantor's theory of uncountable sets: The logic of his proofs has some weaknesses. Cantor assumes for both his proofs that all real numbers (set R) are in a list (list L). Considering L as a set this assumption assumes R belongs to L. This makes the claim "a real number is constructed but is not in the list L" questionable. We propose a solution to this problem, an axiom for counting infinite sets and a solution to continuum hypothesis.
4. Kuan Peng, 2018, « Graphic of set counting and infinite number»
   https://www.academia.edu/37766761/Graphic_of_set_counting_and_infinite_number
   When counting a set, we can plot a graphic that represents the members of the set on the plane (x, y) to observe visually the counting. Also, graphic of counting of infinite set helps us to understand infinite natural number.
5. Kuan Peng, 2018, «Building set and counting set»
   https://www.academia.edu/37590687/Building_set_and_counting_set
   A counting set is the set of natural numbers with which a countable set is put in bijection. Is the counting set for the power set of ℕ the set that builds it?
6. Kuan Peng, 2018, « Analysis of the proof of Cantor's theorem »
   https://www.academia.edu/37356452/Analysis_of_the_proof_of_Cantors_theorem
   Cantor's theorem states that the power set of ℕ is uncountable. This article carefully analyzes this proof to clarify its logical reasoning
7. Kuan Peng, 2016, « Lists of binary sequences and uncountability »
   https://www.academia.edu/30072323/Lists_of_binary_sequences_and_uncountability
   Creation of binary lists, discussion about the power set of ℕ, the diagonal argument, Cantor's first proof and uncountability.
8. Kuan Peng, 2016, «Continuity and uncountability »
   https://www.academia.edu/28750869/Continuity_and_uncountability
   Discussion about continuity of line, how continuity is related to uncountability and the continuum hypothesis
9. Kuan Peng, 2016, «Cardinality of the set of decimal numbers »
   https://www.academia.edu/23155464/Cardinality_of_the_set_of_decimal_numbers
   Cardinalities of the set of decimal numbers and ℝ are discussed using denominator lines and rational plane.
10. Kuan Peng, 2016, «Prime numbers and irrational numbers »
    https://www.academia.edu/22457358/Prime_numbers_and_irrational_numbers
    The relation between prime numbers and irrational numbers are discussed using prime line and pre-irrationality.
11. Kuan Peng, 2016, « On Cantor's first proof of uncountability»
    https://www.academia.edu/22104462/On_Cantors_first_proof_of_uncountability
    Discussion about Cantor's first proof using the next-interval-function, potential and actual infinity.
12. Kuan Peng, 2016, «On the uncountability of the power set of ℕ»
    https://www.academia.edu/21601620/On_the_uncountability_of_the_power_set_of_N
    This article discusses the uncountability of the power set of ℕ proven by using the out-indexes subset contradiction.
13. Kuan Peng, 2016, « Hidden assumption of the diagonal argument»
    https://www.academia.edu/20805963/Hidden_assumption_of_the_diagonal_argument
    This article uncovers a hidden assumption that the diagonal argument needs, then, explains its implications in matter of infinity.
14. Kuan Peng, 2016, «Which infinity for irrational numbers? »
    https://www.academia.edu/20147272/Which_infinity_for_irrational_numbers
    This article clarifies the kind of infinity used to quantify the number of digits of irrational numbers and try to check the cardinality of decimal numbers.
15. Kuan Peng, 2015, «Continuous set and continuum hypothesis»
    https://www.academia.edu/19589645/Continuous_set_and_continuum_hypothesis
    This article explains why the cardinality of a set must be either $\aleph_0$ or $|\mathbb{R}|$.
16. Kuan Peng, 2015, «Cardinality of the set of binary-expressed real numbers»
    https://www.academia.edu/19403597/Cardinality_of_the_set_of_binary-expressed_real_numbers
    This article gives the cardinal number of the set of all binary numbers by counting its elements, analyses the consequences of the found value and discusses Cantor's diagonal argument, power set and the continuum hypothesis.