Analysis of the proof of Cantor's theorem

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Abstract: Cantor's theorem states that the power set of ℕ is uncountable. This article carefully analyzes this proof to clarify its logical reasoning.

Cantor's theorem states that the power set of the set of natural numbers ℕ is uncountable and there is no largest infinity. Georg Cantor proved this theorem using a contradiction, which is somehow intricate. I have carefully analyzed this proof and clarified its logical reasoning. First, let us recall this proof.

1. Recall of the proof

The power set of ℕ is the set of all subsets of ℕ, which we denote by ℘. For creating the contradiction, we assume that all members of ℘ can be put in a list, which we denote by ***L***. Each subset has an index which is a natural number. It can happen that the index is a member of the subset. In this case the subset and index are described as selfish. On the other hand, if a subset does not contain its index, the subset and index are non-selfish. So, each subset of ℕ is either selfish or non-selfish.

Then we create the contradiction subset called subset ***K***. ***K*** contains all the non-selfish indexes, which makes it a subset of ℕ and a member of ***L***. Let us determine whether ***K*** is selfish or non-selfish.

- Let *k* be the index of ***K***. If ***K*** is selfish, ***K*** should contain *k*. By definition, ***K*** contains only non-selfish index, but not selfish index such as *k*. So, ***K*** is not selfish.
- If ***K*** is non-selfish, ***K*** does not contain *k*. By definition ***K*** contains all the non-selfish indexes such as *k* that ***K*** does not contain. So, ***K*** should contain *k* and is not non-selfish.

In consequence, ***K*** is neither selfish nor non-selfish and cannot be a subset of ℕ or a member of ***L***. This contradicts the assumption that ***L*** contains all members of ℘ and thus, ℘ is uncountable.

2. Analysis of the contradiction

Why is the situation "***K*** neither selfish nor non-selfish" contradictory? In fact, it is accepted that every subset of ℕ is in one of these 2 states: selfish or non-selfish, while ***K*** is in none of them. But what if the number of states is not 2? For example, in coin tossing the outcomes will be heads or tails. But if the coin is allowed to stand on its edge, the coin has a third possible state. If subsets of ℕ had a third possible state, then "neither selfish nor non-selfish" will not be contradictory. In order to explain this point, let us start with a finite set ***B***.

1) Case of finite set

B = (1, 2, 3, 4, 5, 6, 7, 8) (1)

{2, 5, 6}⇔(0,1,0,0,1,1,0 ,0) (2)

	1	2	3	4	5	6	7	8	***K'***
1	1	0	0	0	1	0	0	0	0
2	0	0	0	0	0	1	0	0	1
3	1	0	1	0	1	0	0	0	0
4	0	0	1	1	0	0	0	0	0
5	0	0	0	0	0	0	0	0	1
6	0	1	0	0	0	0	0	0	1
7	0	0	1	0	1	0	1	0	0
8	0	1	0	1	0	0	0	1	0
9	0	1	0	0	1	1	0	0	
…	…								

Table 1 Finite list ***X***

Let us take the 8-members-set ***B*** shown in (1) and build the power set of ***B***. For deduction, each subset of ***B*** is represented by a binary string as shown in **(2)**. As subsets of ***B*** can have 8 members, all strings will have 8 bits. The i^{th} bit of a string equals 1 if i is a member of the subset, otherwise, the i^{th} bit equals 0 (See section 2 of «On the uncountability of the power set of ℕ»).

We create a list of binary strings ***X*** which is in the middle of Table 1. The first line of Table 1 is the set ***B*** and the first column is the indexes. Let us find the selfish and non-selfish subsets of ***X***. The subsets n° 1, 3, 4, 7, 8 contain their indexes because the diagonal bits of their strings are 1. So, these subsets are selfish. On the other hand, the diagonal bits of the strings n° 2, 5, 6 are 0. So, these subsets are non-selfish.

Now let us create the subset ***K*** which contains all the non-selfish indexes and write the string ***K'*** that represents ***K***. ***K*** contains the non-selfish indexes 2, 5 and 6. So, the 2nd, 5th and 6th bits of ***K'*** are 1. ***K*** does not contain the selfish indexes 1, 3, 4, 7 and 8. So, the 1st, 3rd, 4th, 7th and 8th bits of ***K'*** are 0. The bits of ***K'*** are written in the last column of Table 1. Comparing with the diagonal bits of ***X***, we notice that the bits of ***K'*** are complementary to the diagonal bits. So, ***K'*** does not equal any string in the lines 1 to 8 and thus, ***K'*** is in none of these lines.

For finding whether ***K*** is selfish or non-selfish, we have to determine the index of ***K***, that is, in which line is ***K'***? Since ***K'*** is not in the lines 1 to 8, the index of ***K*** is higher than 8. Let us put ***K'*** in the line 9 and give it the index 9 (see line 9 of Table 1). But, because 9 is not a member of ***B***, ***K*** is not allowed to contain its index 9. Is ***K*** non-selfish then? Not quite.

In fact, because ***K*** contains 8 members, ***K'*** has only 8 bits. On the other hand, ***K'*** is in the 9th line where the diagonal bit is the 9th bit. So ***K'*** does not have the diagonal bit. As the value of the diagonal bit indicates if a subset is selfish or non-selfish, without the diagonal bit ***K*** is not qualified to be selfish or non-selfish. Indeed, can you determine the value of a diagonal bit which does not exist? In consequence, for finite set the assumption that ***K*** is either selfish or non-selfish is self-contradictory.

However, if ***K*** is not required to be selfish or non-selfish, ***K*** is a perfect subset of ***B***. Its index 9 is correct because 9 is smaller than the number of subsets of ***B***, 2^8.

2) Case of infinite set

We will use the result above to analyze the set ℕ which contains ∞ members. A subset of ℕ can contain ∞ members, so each subset is represented by a binary string that has ∞ bits. The list ***L*** contains all subsets of ℕ and is represented by ***L'*** the list of binary strings in the middle of Table 2. Like for finite set, the diagonal bit is 0 if the string is non-selfish and 1 if the string is selfish.

ℕ→	1	2	3	4	5	6	7	8	…	*n*	…	∞	***K'***
1	1	0	0	0	1	0	0	0	…	*	…		0
2	0	0	0	0	0	1	0	0	…	*	…		1
3	1	0	1	0	1	0	0	0	…	*	…		0
4	0	0	1	1	0	0	0	0	…	*	…		0
5	0	0	0	0	0	0	0	0	…	*	…		1
6	0	1	0	0	0	0	0	0	…	*	…		1
7	0	0	1	0	1	0	1	0	…	*	…		0
8	0	1	0	1	0	0	0	1	…	*	…		0
…					…								*
n	*	*	*	*	*	*	*	*	*	*	*		*
…					…								*
∞					…								*
∞+1	0	1	0	0	1	1	0	0	…	*	…		

Table 2 Infinite list ***L'***

Let us build the subset ***K*** and write its binary string ***K'***. ***K'*** has ∞ bits and is in the last column of Table 2. Like for finite set, the bits of ***K'*** are complementary to that of the diagonal, so ***K'*** does not equal any of the ∞ strings of ***L'***. We put ***K'*** in the line ∞+1 and give it the index ∞+1 (See the ∞+1st line of Table 2).

Is ***K*** selfish or non-selfish? ***K'*** has ∞ bits and is in the ∞+1st line whose diagonal bit is the ∞+1st bit. So, ***K'*** does not have the diagonal bit and makes ***K*** not qualified to be selfish or non-selfish. In consequence, the assumption that ***K*** is either selfish or non-selfish is self-contradictory and "the contradiction" of Cantor's proof cannot prove that the power set of ℕ is uncountable.

However, if ***K*** is not required to be selfish or non-selfish, ***K*** is a perfect subset of ℕ. Its index ∞+1 is correct because ∞+1 is smaller than 2^∞, the number of subsets of ℕ. But ∞+1 is bigger than the length of ℕ which is ∞. ℕ is in the first line of Table 2 whose length equals the length of the diagonal, which in turn equals the length of the first column. The length of the first column is the length of ***L'*** and ***L***. So, the length of ***L*** equals the length of ℕ, that is, ∞.

In this case, ***K*** is not in the list ***L*** but out of ***L***. So, in addition to the 2 states of subset: selfish or non-selfish, we define the third valid state: out of ***L***. This way, "***K*** neither selfish nor non-selfish" is no longer contradictory because ***K*** is in the third valid state.

3) Diagonal argument for irrational numbers

By the way, the analysis above shows that Cantor's proof for the power set of ℕ is exactly equivalent to the diagonal argument proposed to prove that the set of irrational numbers is uncountable. The diagonal argument is presented by this article of Wikipedia.

When we compare the list ***L*** of Table 2 and ***K'*** with the sequences of binary digits (s_1, s_2, s_3, …) and the complementary diagonal *s* in diagonal argument, we see that ***K'*** is not in the list ***L***, just as *s* is not in the sequences (s_1, s_2, s_3, …). So, Table 2 shows the same contradiction as that in the diagonal argument.

For the same reason that ***K'*** is legitimately not in the list ***L***, *s* is legitimately not in (s_1, s_2, s_3, …). So, the contradiction in diagonal argument "*s* does not equal any of the members in (s_1, s_2, s_3, …)" is not contradictory and the set of irrational numbers is not proven uncountable.

3. Implicit conditions

How did the contradiction of Cantor's proof get accepted by mathematical community if it is not true? In fact, this contradiction is the consequence of several conditions which are not true. As they are implicitly used, these errors are not noticed. These conditions are explained below.

1) Length of *L*

The first implicit condition is about the length of the list ***L***. The assumption that $\wp$ can be completely put in ***L*** is accepted as consistent simply because ***L*** is infinitely long. However, the subset ***K*** is shown to be out of ***L*** in the section "Case of infinite set", which means that a member of $\wp$ is not in ***L***, in spite that ***L*** is infinitely long. Why an infinite list cannot contain another infinite set?

For illustrating the answer, let us count the members of $\wp_n$, the power set of a finite set S_n. S_n has n members and $\wp_n$ has 2^n members, so $\wp_n$ is bigger than S_n. When n increases indefinitely $n = \infty$, $S_n = \mathbb{N}$ and $\wp_n = \wp$. $\mathbb{N}$ has ∞ members and $\wp$ has 2^∞ members, so $\wp$ is bigger than $\mathbb{N}$. Because ***L*** and $\mathbb{N}$ have the same length (see section "Case of infinite set"), $\wp$ is bigger than ***L***. Therefore, the assumption that $\wp$ can be completely put in ***L*** is wrong.

2) Diagonal bit

The second implicit condition is the claim that "each subset of $\mathbb{N}$ is either selfish or non-selfish", which is equivalent to say "each subset of $\mathbb{N}$ has the diagonal bit". The subset ***K*** does not have the diagonal bit while being a correct subset of $\mathbb{N}$ (see section "Case of infinite set"). Therefore, the above claim is wrong.

3) Only 2 states

The third implicit condition is the assumption that it exists only 2 possible states for subset of $\mathbb{N}$: selfish or non-selfish. The subset ***K*** is in none of them, it is in the third state: out of ***L*** (see section "Case of infinite set"). Therefore, this assumption is wrong.

4) Subset *K*

The fourth implicit condition is the assumption that ***K*** is a member of the list ***L***. But ***K*** is not a member of ***L*** (see section "Case of infinite set"). Therefore, this assumption is wrong.

4. Building set and counting set

The assumption that $\wp$ can be completely put in the list ***L*** has led to error. Then, how should we count $\wp$? Let us see the counting of the set of integers $\mathbb{Z}$ in Table 3. First, we put $\mathbb{Z}$ into the list (1, -1, 2, -2, 3, -3, ..., ∞, -∞) in column 1. Then we put this list in one-to-one correspondence with the set of natural numbers {1, 2, 3, ..., 2∞} in column 3. This set is called the counting set and is denoted by $\mathbb{N}_c$. On the other hand, $\mathbb{Z} = \{-\infty,\ldots,-3,-2,-1,1, 2, 3, ..., \infty\}$ and is built from the set {1, 2, 3, ..., ∞}, which we call the building set and denote by $\mathbb{N}_b$. So, $\mathbb{Z}$ involves 2 different sets of natural numbers, $\mathbb{N}_b$ and $\mathbb{N}_c$.

$\mathbb{Z}$		$\mathbb{N}$
1	$\rightarrow$	1
-1	$\rightarrow$	2
2	$\rightarrow$	3
-2	$\rightarrow$	4
3	$\rightarrow$	5
-3	$\rightarrow$	6
...	$\rightarrow$	...
i	$\rightarrow$	$2i-1$
$-i$	$\rightarrow$	$2i$
...	$\rightarrow$	...
∞	$\rightarrow$	$2\infty-1$
$-\infty$	$\rightarrow$	2∞

Table 3

Why do we count $\mathbb{Z}$ with $\mathbb{N}_c$ but not $\mathbb{N}_b$? Let us explain with the help of Table 4 where $\mathbb{Z}$ is so arranged that $\mathbb{N}_b$ is on the upper part and $-\mathbb{N}_b$ on the lower part. If we count $\mathbb{Z}$ using $\mathbb{N}_b$, the one-to-one correspondence between $\mathbb{Z}$ and $\mathbb{N}_b$ ends at ∞ as shown in the columns 2 and 3. In this case the members of $\mathbb{Z}$ from -1 to $-\infty$ are not counted, so $\mathbb{N}_b$ is not suitable. Rather, we use $\mathbb{N}_c$ which is independent of $\mathbb{Z}$ and free to count beyond 2∞. As the columns 3 and 4 show, $\mathbb{N}_c$ is correctly in one-to-one correspondence with $\mathbb{Z}$.

In Cantor's proof the error is to count $\wp$ with its building set. For explaining this error, let us count the finite power set $\wp_n$ whose building set is S_n. S_n has n members and $\wp_n$ has 2^n members. If we put $\wp_n$ and S_n side by side, we will never obtain one-to-one correspondence. If we want to catch up $\wp_n$ by increasing S_n to 2^n members, $\wp_n$ will have 2^{2^n} members and there is still no match. In the contrary, if we count with the set N_c which is independent of $\wp_n$, N_c will increase from 1 to 2^n members while $\wp_n$ stays constant. The one-to-one correspondence between $\wp_n$ and N_c will be achieved when $N_c = \{1,2,3,\ldots, 2^n\}$.

$-\mathbb{N}_b$	$\mathbb{N}_b$	$\mathbb{Z}$	$\mathbb{N}_c$
	1	1	1
	2	2	2
	3	3	3
	…	…	…
	∞	∞	∞
-1		-1	$\infty+1$
-2		-2	$\infty+2$
-3		-3	$\infty+3$
…		…	…
$-\infty$		$-\infty$	2∞

Table 4

For counting $\wp$ too we must not use the building set of $\wp$. $\mathbb{N}$ is the building set of $\wp$, but $\mathbb{N}$ is also the generic name for set of natural numbers. For avoiding confusion, we declare that $\mathbb{N}_b$ is the building set of $\wp$ and $\mathbb{N}_b = \{1, 2, 3, ..., \infty\}$. As $\mathbb{N}_b$ has ∞ members and $\wp$ has 2^∞ members, counting $\wp$ with $\mathbb{N}_b$ will surely fail. Anyway, no rule imposes to count $\wp$ with $\mathbb{N}_b$.

We count $\wp$ with the set $\mathbb{N}_c = \{1,2,3\ldots 2^\infty\}$, which is independent of $\wp$ and free to count till 2^∞. $\mathbb{N}_c$ is a set of natural numbers because 2^∞ is also infinity. $\wp$ can be put in one-to-one correspondence with $\mathbb{N}_c$ and is countable. So, $\wp$ can be put into a list.

5. Lists of $\wp$ and $\mathbb{R}$

$$\infty_0 = 2^\infty \tag{3}$$

1) List of the power set

In Table 5, the first line is the building set $\mathbb{N}_b$ which has ∞ members, the list of binary strings in the middle represents $\wp$. Because $\wp$ contains all subsets of $\mathbb{N}_b$, the list contains all possible binary strings with ∞ bits. For the list to be more familiar, we make the change of notation shown in **(3)**. Every binary string is indexed in the first column by a natural number that runs from 1 to ∞_0. So, the first column is the counting set $\mathbb{N}_c = \{1,2,3\ldots\infty_0\}$ and is in one-to-one correspondence with the list of $\wp$.

	1	2	3	4	…
1	1	0	0	0	…
2	0	1	0	0	…
3	1	1	0	0	…
4	0	0	1	0	…
5	1	0	1	0	…
6	0	1	1	0	…
7	1	1	1	0	…
…	…	…	…	…	…
∞_0	1	1	1	1	…

Table 5 List $\wp$

2) List of the set of real numbers

Table 5 can be used to write the set of real numbers $\mathbb{R}$ in a list. For doing so, we transform the binary strings of Table 5 into real numbers by adjoining "0." to each string as shown in Table 6. This way, the list of Table 5 becomes the list of real numbers in Table 6.

		2^{-1}	2^{-2}	2^{-3}	2^{-4}	…
1	0.	1	0	0	0	…
2	0.	0	1	0	0	…
3	0.	1	1	0	0	…
4	0.	0	0	1	0	…
5	0.	1	0	1	0	…
6	0.	0	1	1	0	…
7	0.	1	1	1	0	…
…	0.	…	…	…	…	…
∞_0	0.	1	1	1	1	…

Table 6 List $\mathbb{R}$

As Table 5 contains all possible binary strings, Table 6 contains all possible real numbers smaller than 1, which is in one-to-one correspondence with $\mathbb{N}_c$ in the first column. The real numbers have ∞ bits and thus, contain all irrational numbers smaller than 1 and the set of irrational numbers is countable.

This list does not order the real numbers according to their values. I have given a well ordered list in «Lists of binary sequences and uncountability» section 4.a "Creation of the numbers".

6. Conclusion

In « Lists of binary sequences and uncountability » I have given a counter-proof to Cantor's theorem. Here, the thorough analysis of the reasoning of Cantor's proof reveals several neglected errors. We have shown that the contradiction claimed in Cantor's proof is invalid because the assumptions about the subset ***K*** and the list ***L*** are inconsistent. Also, we have put the power set of ℕ and the set of real numbers in one-to-one correspondence with $\{1,2,3\ldots\infty_0\}$, showing that they are countable.

Since the power set of ℕ is countable, it has the same cardinality than ℕ. In consequence, the power set of a countable set does not create higher cardinality. Then, the claim that there is no largest infinity is not proven.

Note

I think that my idea in this article is important and correct. If this idea was accepted, there will be a large field for new studies and papers in set theory. But I lack the training for theoretical mathematics and I am not able to write in suitable form for publishing in mathematical journal. Nevertheless, this idea is worth spreading. So, I authorize you, mathematician, to rewrite this article in suitable form to publish under your name, at the condition that you cite my article as the original source and give in your publication the internet links to my article which are:
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