Hidden assumption of the diagonal argument

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Abstract: This article uncovers a hidden assumption that the diagonal argument needs, then, explains its implications in matter of infinity.

1. The hidden assumption

For proving that the set of real numbers is bigger than that of natural numbers, Georg Cantor proposed the diagonal argument. For doing so, he creates a list of infinitely long binary sequences which are the digits of real numbers. For example, the sequence 01100110011101… corresponds to the real number 0.01100110011101…. If the real numbers are as many as the natural numbers, it should be possible to put all the sequences in a list. If a sequence is found out of the list, then the real numbers are more numerous than the natural numbers.

Table 1 is an example of such list that is supposed to contain all the infinitely long binary sequences. A diagonal sequence s_{diag} is constructed using the digits of the diagonal. Then, its digits are flipped to their opposites, 0 to 1 and 1 to 0, to create the flipped sequence s_{flip}. Because the i^{th} digit of s_{flip} differs from the i^{th} digit of the i^{th} sequence, s_{flip} equals none of the sequences in the list. So, s_{flip} is out of the list and the set of real numbers is proven to contain more members than the set of natural numbers.

$i=1\ldots\infty_s$	Sequences, $j=1\ldots\infty_d$
s_1	01100110011101…
s_2	00000101100101…
s_3	10000100101011…
s_4	10010001111011…
s_5	01101010000011…
s_6	01110110011001…
s_7	01110100000111…
s_8	00111101010010…
s_9	01100110011010…
s_{10}	00011010101111…
s_{11}	01101110010100…
s_{12}	00000100111110…
s_{13}	00101011010001…
s_{14}	00000111001101…
…	…
s_{diag}	00011101000101…
s_{flip}	11100010111010…

Table 1

This proof makes no restriction to the list and is commonly accepted. However, the use of the diagonal digits imposes a condition unnoticed until now: the list must possess a diagonal. This is the hidden assumption. It was tacitly supposed true because the numbers of digits and sequences are both infinite and infinite number should equal infinite number. Indeed, can natural numbers have more than one upper bound? But if these infinite numbers are ever different, the conclusion of the diagonal argument would be overturned. We will try to find the consequences of this assumption.

2. Searching for the flipped sequence

Classically, a sequence of numbers is written from left to right. For example, the first digit of the decimal sequence 123456 is the leftmost 1; the last digit is the rightmost 6. The sequences in Table 1 are written in this order and the three dots "…" are to the right.

But the digits of natural number are in reverse order. For example, the first digit of the number 123456 is the 6 to the right and the last digit is the 1 to the left. For making the explanation simpler, we will write our sequences in the same order than a natural number. This way a sequence will also be a natural number. For example, the decimal sequence 654321 is also the natural number 654321. If the sequence is infinitely long, three dots are added to its left. So, the infinite sequence …654321 corresponds to the infinite hypernatural number …654321.

a. Sequences with limited digits

The effect of the hidden assumption will be easy to show for finite binary sequences which are strings of 0 and 1. Table 2 shows a list of m sequences each of which has n digits. The i^{th} sequence is written as: $b_{in}\ldots b_{ij}\ldots b_{i3}b_{i2}b_{i1}$ where j is the rank of digits. Remark that i the ranks of the sequences are in the right column to indicate that j are in reverse order.

Sequences	i
$b_{1n}\ldots b_{13}b_{12}b_{11}$	1
$b_{2n}\ldots b_{23}b_{22}b_{21}$	2
…	…
$b_{mn}\ldots b_{m3}b_{m2}b_{m1}$	m

Table 2

With n digits, the total number of possible sequences is 2^n. They can be completely listed. For example, with 4 binary digits only 16 sequences exist, which are completely listed in the first column of Table 3. This is a complete list because all possible sequences are in it. As the digits are in reverse order, the sequences are also 16 binary numbers that equal the ranks i in the second column.

s_i	i
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

Table 3

But it is impossible to apply the diagonal argument to this list because it has no diagonal! This list is rectangular. Nevertheless, we can apply the diagonal argument to an extracted square list, for example the first 4 sequences. In creating the diagonal sequence, digits must be reversed too and the diagonal digits of the first 4 sequences are 0000. These digits are flipped to their opposite to create the flipped sequence: s_{flip}=1111, which equals 15 in decimal. So, s_{flip} is the 15^{th} sequence in the list. Fulfilling the request of the diagonal argument, s_{flip} equals none of the first 4 sequences but is still in the list. The flipped sequence for the last 4 sequences is 0011 which ranks 3 in the list. Applying the diagonal argument to any 4 sequences of Table 3 will give a sequence in the list because this list is complete.

And for the square list of Table 1, where is located the flipped sequence in the complete list of 14 digits sequences? The flipped sequence is 11100010111010 (given in last line of Table 1). We reverse the digits to respect our convention and obtain 01011101000111. This binary number is 5959 in decimal. There exist $16384=2^{14}$ sequences with 14 digits and 5959<16384. The flipped sequence is the 5959^{th} in the complete list.

In general, for sequences with n digits, the length of complete lists is 2^n. These lists are rectangles of dimension $n\times 2^n$. Rectangles have no diagonal making the hidden assumption wrong. If the diagonal argument is applied to a randomly arranged square list of $n\times n$, the flipped sequence is always in the complete lists. So, the diagonal argument fails for partial square list.

b. Infinitely many digits under the logic of potential infinity

When the number of digits is infinity, the length of complete lists is also infinity. What becomes the hidden assumption? In <<Which infinity for irrational numbers?>> I have explained that there were 2 logics of infinities: potential infinity and actual infinity. Depending on which logic is used, the result will be different. Under the logic of potential infinity, the number of digits n goes to infinity without reaching it. For all n while $n\to\infty$, the length of the complete lists equals 2^n. The form of the complete lists is shown in Table 2 with $m=2^n$, which is rectangular and have no diagonal. So, under the logic of potential infinity the hidden assumption is wrong and the diagonal argument fails.

c. Infinitely many digits under the logic of actual infinity

Under the logic of actual infinity, the number of digits is actually infinity or $\aleph_0$. In the first column of Table 4 is a list of sequences with infinitely many binary digits. The first sequence, denoted as s_0, is an infinite string of zero. As the sequences are whole binary numbers, $s_0=0$. The second sequence is: $s_1= s_0+1=1$. The entire list is generated by the recurrent relation $s_{i+1}= s_i+1$, and covers all natural numbers from 0 to infinity. While the last sequence does not exist, its form is known: an infinite string of 1, shown in the last line of Table 4.

Sequences with $\aleph_0$ digits	Rank i
…0000000000000000000000000000	0
…0000000000000000000000000001	1
…0000000000000000000000000010	2
…0000000000000000000000000011	3
………	…
…0001110111110101111001110110	31415926
………	…
…1111111111111111111111111111	$2^{\aleph_0}$

Table 4

By construction, this list is a complete list because it contains all possible sequences with $\aleph_0$ digits. Its width is the number of digits, $\aleph_0$, its length is the number of sequences, $2^{\aleph_0}$. This list is rectangular and has no diagonal even though its dimension is infinite. So, under the logic of actual infinity the hidden assumption is wrong and the diagonal argument does not apply. If the diagonal argument is applied to a square list of dimension $\aleph_0 \times \aleph_0$, as Georg Cantor did, the flipped sequence is in the complete list. So, the diagonal argument fails.

3. Ordering infinite numbers

The hidden assumption was not noticed due to the unconscious idea that infinite natural number is a single value. The uniqueness of infinity not only makes the diagonal argument invalid, it leads to other confusions too. For example, a set S_1 contains n members and another set S_2 contains 2n. Once n becomes infinite, S_1 contains ∞ and S_2 contains 2∞. As 2∞ is also infinity, we can write $2\infty=\infty$ and think that S_1 and S_2 have the same size. But clearly S_2 is bigger than S_1 and the ordering of these 2 sets is confused. In order to avoid such confusions, I propose a method to solve this problem.

a. An infinite number as reference

In <<Which infinity for irrational numbers?>> I have shown that infinity is not a single value but a domain of infinitely many numbers. For example, the decimal numbers 111… and 555… have infinitely many digits and are both infinity. But which is bigger? In order to compare them we can multiply 111… by 5 and obtain:

$$111\ldots \times 5 = 555\ldots \Rightarrow 111\ldots < 555\ldots \qquad (1)$$

But if we do another operation, we obtain the contrary:

$$555\ldots \times 2 + 1 = 111\ldots \Rightarrow 111\ldots > 555\ldots \qquad (2)$$

This confusion is due to the indetermination of digits numbers. The solution would be to choose an infinite number as reference. For example, we choose the number ‘First 111…’ of equation 1 as the reference and accept ‘First 111…’ < 555... . Then equation 2 makes ‘Second 111…’>555... and the ordering is:

$$\text{First } 111\ldots < 555\ldots < \text{Second } 111\ldots \qquad (3)$$

This ordering makes the 'First 111…' to have ∞ digits and the 'Second 111…' to have $\infty+1$ digits. Although ∞ and $\infty+1$ are both infinity, they are nevertheless different. This difference distinguishes the two 111… and orders the 3 infinite numbers.

The size of S_1 is $|S_1|=\infty$, using it as the reference, we can order the sets S_1 and S_2. The size of S_2 is $|S_2|=2\infty$ and $\frac{|S_2|}{|S_1|} = \frac{2\infty}{\infty} = 2$, then, these 2 sets are ordered, $S_2>S_1$.

The heart of this method is to compare infinite numbers with an infinite reference. Their exact values do not matter. In fact, this is the way physicists do in physical world. For example, they work with a chunk of gold without caring about how many atoms it contains exactly. A huge number that cannot be counted is physically infinity. For such infinity they have invented the infinite reference called Kilogram. If they cut the chunk into 2 pieces, they actually split a set of infinitely many atoms into 2 sets of infinitely many atoms. Without counting the atoms in each piece, they order the 2 pieces by weighing them.

Although for mathematicians, the number of atoms in the chunk is in fact finite, but it is sufficiently huge to behave like infinity, that is, so huge that its exact value does not matter. The way of thinking about infinity in physics could inspire mathematicians in matter of infinity.

b. Infinity of digits and infinity of sequences

In Table 1, I have denoted the number of digits by ∞_d and the number of sequences by ∞_s. The infinity of reference is ∞_d, then, ∞_s is found to be bigger by the comparison $\infty_s=2^{\infty_d} > \infty_d$, confirming that the complete list of sequences is rectangular. I have used the signs ∞_s and ∞_d to underline their difference. If Georg Cantor had not used a single sign for both infinite numbers, maybe he would notice that he had no diagonal for his argument.

4. Converting binary sequences into real numbers

The objective of the diagonal argument is to compare the sizes of 2 infinite sets, the real numbers and the natural numbers. The diagonal argument failed, but its objective can be achieved using the above result by converting the infinitely long binary sequences into real numbers. The sequences are written in reverse order with the three dots to the left. A binary real number is written in the right order with the three dots to the right. For the conversion, the digits and the three dots of a sequence are first reversed, then, the zero and the radix point are added. For example, the sequence …123456 is reversed into 654321…, "0." is added and the converted real number is 0.654321…

The first column of Table 5 is the complete list of infinitely long binary sequences already shown in Table 4. The third column is the converted binary real numbers which are all in the interval]0, 1[.The second column is the ranks of the sequences that equal the binary integers of the first column.

Because the first column is a complete list, the converted third column is also a complete list, which lists the real numbers of the interval]0, 1[without leaving one out. The real numbers in the third columns are in one-to-one correspondence with the rank numbers in the second column. So, the set of real numbers of the interval]0, 1[has the same number of members than the set of natural numbers.

Sequences in reverse order	Rank i	Real numbers in]0, 1[
…0000000000000000000000000000	0	0.0000000000000000000000000000…
…0000000000000000000000000001	1	0.1000000000000000000000000000…
…0000000000000000000000000010	2	0.0100000000000000000000000000…
…0000000000000000000000000011	3	0.1100000000000000000000000000…
………	…	………
…0001110111110101111001110110	31415926	0.0110111001111010111110111000…
………	…	………
…1111111111111111111111111111	$2^{\aleph_0}$	0.1111111111111111111111111111…

Table 5

5. Conclusion

For the diagonal argument the lists of sequences must have a diagonal. This is the hidden assumption. But complete lists are rectangular and have no diagonal. Thus, the diagonal argument is invalid.

This fail is due to the confusion that the infinite width and length of the lists cannot be different. In reality they are infinite numbers with different values. In order to correctly treat infinity and avoid such confusion in the future, I propose a method to distinguish and compare infinite numbers.

Although the diagonal argument fails, its objective of comparing the sets of real numbers and natural numbers should be achieved. The complete list of infinitely long binary sequences in this article is a complete list of natural numbers. Converted into a list of real number, we obtain a one-to-one correspondence between the natural numbers and the real numbers. So, the set of real numbers seems to be countable under the logic of actual infinity, confirming the conclusion of <<Which infinity for irrational numbers?>>.