Which infinity for irrational numbers?

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Abstract: This article clarifies the kind of infinity used to quantify the number of digits of irrational numbers and try to check the cardinality of decimal numbers.

The value of a decimal number depends on the number of its digits. For irrational numbers that have infinity of digits, their values seem to be definitive. However, the meaning of infinity is ambiguous because there exist several kinds of infinities. If the infinity used to define the number of digits is not clear, the values of irrational numbers will not be well defined. This is why we have to answer the question of the title.

The first kind of infinity is "potential infinity" which sees infinity as non-attainable and to which a number can approach indefinitely. For potential infinity, the existence of infinity is not real but potential. The second kind is "actual infinity" which claims that infinity can actually be reached and is a real mathematical object, for example as the upper bounds of natural numbers or real line. There are more information about infinities in <<Potential vs. Completed Infinity>> of Eric Schechter or << The Meaning of Infinity>> of W. Mückenheim.

1. Irrational numbers under potential infinity

Under the logic of potential infinity, number of digits can increase indefinitely but never reaches infinity and thus, decimal numbers with infinity of digits are excluded. Because irrational numbers are limits of converging sequences of decimal numbers and limits are not in the sequences, the set of real numbers does not equal the set of decimal numbers.

For example, the irrational number π is usually written as 3.14159265359… Under the logic of potential infinity, this expression is not a decimal number with infinitely many digits, but the limit of the sequence π_1=3.1, π_2=3.14,…, π_n=3.141…n^{th}-digits when $n\to\infty$. The value of 3.14159265359… is π_n, an approximate value within a rounding error 10^{-n}. For example, when n=2, π_n=3.14, giving π between 3.14 and 3.15 with the rounding error 10^{-2}.

Because n is not allowed to reach infinity, the rounding error is never zero. So, an irrational number is not located as a precise point on the real line but as one of the infinitely many points that an interval of length 10^{-n} contains.

2. Irrational numbers under actual infinity

Under the logic of actual infinity, decimal numbers can actually have infinitely many digits making the rounding error $10^{-\infty}=0$. So, the decimal expansion of an irrational number has infinity of digits and is a distinct point on the real line. This property makes each irrational number equal a unique decimal number and the set of real numbers equal the set of decimal numbers.

Because decimal numbers are precise expressions, each one can be pinned and paired with natural numbers. Although the set of real numbers is known to be uncountable, it is still interesting to check its cardinality using the pairing of decimal numbers with natural numbers.

3. Pairing up decimal numbers with natural numbers

a. Terminating decimal numbers

These numbers have finite number of digits. In the unit interval]0, 1[these numbers are in the form of "0.abcd…" where "abcd…" is a finite sequence of digits which is a natural number. By removing "0." from "0.abcd…", we creates the natural number "abcd…" that is the counterpart of "0.abcd…" in the pairing. For example, 1234 is the counterpart of "0.1234". Table 1 shows examples of this pairing:

Terminating decimal numbers	0.1	0.2	0.3	…	0.15789	…	0.n
Natural numbers	1	2	3	…	15789	…	n

Table 1

"0.n" is the generic terminating decimal number and n the corresponding natural number.

When the first digit is not zero, each decimal number corresponds to a unique natural number. But when the first digits are zero, the uniqueness of correspondence is broken. For example 0.00123 corresponds to 00123 which is in fact the number 123. So, the natural number 123 corresponds to 0.123, 0.0123, 0.00123 and so on. In order to circumvent this problem, the zeros will be shifted.

b. Zero-shifting

In order to create unique natural numbers corresponding to decimal numbers such as 0.0123, 0.00123, the first zeros in the sequence of digits will be shifted from the left to the right. For example the zeros in 00123 will be shifted to the right creating the natural number 12300. This way, 0.123, 0.0123 and 0.00123 will correspond to 123, 1230 and 12300 respectively and the overlap is suppressed. Table 2 shows examples of zero-shifted natural numbers:

Decimal numbers	0.123	0.0123	0.00123	…	$0.n/10^i$
Zero-shifted numbers	123	123×10^1	123×10^2	…	$n\times10^i$

Table 2

In the last column, i is the number of zeros before the first non-zero digits, n is the significant natural number in the sequence of digits. Actually, zero-shifting is equivalent to multiplying or dividing the significant natural number by 10^i, as shown in the last column.

c. Nonterminating decimal numbers

Terminating decimal numbers are easily paired up with natural numbers because their significant digits are natural numbers. Nonterminating decimal numbers have infinity of digits that are not natural numbers. Nevertheless, they can be paired with hypernatural numbers that I borrow from Nonstandard Analysis.

A hypernatural number is a whole number that has infinitely many digits. For example the number 3333333… whose digits go for ever is a hypernatural number. The generic form of nonterminating decimal number is "$0.d_1d_2d_3…$" and the corresponding hypernatural number is "$d_1d_2d_3…$" Table 3 shows examples of such decimal numbers paired with hypernatural numbers:

Irrational numbers	$\pi/10$	$\sqrt{2}/10$	$e/10$	$e/1000$	r
Decimal expansions	0.314159…	0.141421…	0.271828…	0. 00271828…	$0.d_1d_2d_3…$
Hypernatural numbers	314159…	141421…	271828…	$271828…\times10^2$	$d_1d_2d_3…$

Table 3

Notice that the decimal expansion of the irrational numbers e/1000 is 0. 00271828… and its corresponding hypernatural numbers is 271828…×10^2 which is zero-shifted despite of the infinity of number of digits.

d. Pairing up the interval]0, 1[with the set of hypernatural numbers

The set of hypernatural numbers contains plain natural numbers and hypernatural numbers. Using the above pairings of terminating and nonterminating decimal numbers, we can make a list of correspondence between the decimal numbers of the unite interval]0, 1[and hypernatural numbers. Table 4 shows this correspondence arranged in the order of increasing hypernatural numbers:

Ordered hypernatural numbers	Corresponding decimal numbers
1	0.1
2	0.2
…	…
123	0. 123
…	…
1230	0. 0123
…	…
314159265359…	0.314159265359…
…	…
999999999999…	0. 999999999999…

Table 4

Ordered decimal numbers	Corresponding hypernatural numbers
…	…
0. 0123	1230
…	…
0.1	1
…	…
0. 123	123
…	…
0.314159265359…	314159265359…
…	…
0. 999999999999…	999999999999…

Table 5

Is this pairing a bijection? The set of hypernatural numbers is completely listed in Table 4. So this table is an injection from the set of hypernatural numbers to the interval]0, 1[. In Table 5 the first column completely lists the decimal numbers of the interval]0, 1[in increasing order. So Table 5 is an injection from the interval]0, 1[to the set of hypernatural numbers. As Table 4 and Table 5 are the same function, this function is a bijection and a one-to-one correspondence between the interval]0, 1[and the set of hypernatural numbers.

A set having a bijection to the set of natural numbers is countable. As the interval]0, 1[has a bijection to the set of hypernatural numbers, the set of decimal numbers in this interval can be qualified as hyper-countable.

4. Hypernatural numbers and natural numbers

The set of hypernatural numbers seems bigger than the set of natural numbers. But, this comparison will not make sense unless the 2 sets obey the same rules. The set of natural numbers that does not contain infinity will be denoted by the barred letter ₦:

₦ = $\{i, \mid i=1, 2, 3,\ldots, n<\infty, n\rightarrow\infty\}$ (1)

The set of natural numbers that contains infinity will be denoted by $\mathbb{N}_\infty$:

$$\mathbb{N}_\infty = \{i, \mid i=1, 2, 3,\ldots, \infty\} \quad (2)$$

For correctly doing the comparison, the 2 sets must obey the same rules. ₦ obeys the rules of potential infinity and thus, cannot be compared with the set of hypernatural numbers which obeys the rules of actual infinity. So, only $\mathbb{N}_\infty$ is qualified for the comparison.

What is the infinite member of $\mathbb{N}_\infty$? Under the logic of actual infinity the number of digits of natural numbers is allowed to reach infinity. Take for example the natural number 999…9 with n digits. By increasing n to infinity, 999…9 becomes the infinite number 999… without crossing the upper bound of $\mathbb{N}_\infty$, that is, 999… is not greater than infinity. So, 999… is a legitimate member of $\mathbb{N}_\infty$. If 999… is a member, the numbers 111…, 222… and 456 456… are also members. Then, numbers having infinitely many non-repeating digits too, are member of $\mathbb{N}_\infty$.

In equation 2, infinity is represented by the symbol ∞, giving the impression that the infinite member of $\mathbb{N}_\infty$ is a single number. Here, we discover that $\mathbb{N}_\infty$ contains infinitely many infinite members. These members are whole numbers with infinity of digits and thus, hypernatural numbers. So, I propose that the set of natural numbers under the logic of actual infinity is identical to the set of hypernatural numbers.

5. Conclusion

I have shown that under the logic of actual infinity, a bijection exists between the decimal numbers of the interval]0, 1[and the set of hypernatural numbers. Also the set of decimal numbers of this interval equals the set of real numbers of the same interval. So, the set of real numbers of the interval]0, 1[has the same cardinality than the set of hypernatural numbers and is hyper-countable.

I have proposed that the set of natural numbers under the logic of actual infinity is identical to the set of hypernatural numbers. If this proposition were correct, the set of real numbers would be countable under the logic of actual infinity.

My second conclusion is that under the logic of potential infinity irrational numbers could not be pinned as points on the real line because decimal numbers have finite number of digits.