Real numbers and points on the number line with regard to Cantor's diagonal argument

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Abstract: Cantor's diagonal argument shows that ℝ is uncountable. But our analysis shows that ℝ is in fact the set of points on the number line which can be put into a list. We will explain what the property that supports uncountability really is and propose a new scheme to count points.

1. Introduction

According to Georg Cantor there are uncountable sets that cannot be put into one-to-one correspondence with the set of natural numbers ℕ, [1]. The set of real numbers ℝ is a such set. However, his theory of infinite sets has been criticized in several areas by mathematicians and philosophers, [2]. Here we will analyze the uncountability of ℝ and explain a new method of counting ℝ.

The most famous demonstration of the uncountability of ℝ is Cantor's diagonal argument [1], [3], [4]. It is a proof by contradiction which we summarize by the Contradiction deduction below:

1. Assumption A: All real numbers are put in a list which we name L.
2. s_{out} is a real number constructed using diagonal construction.
3. s_{out} is not in the list L, contradicting the Assumption A.
4. Conclusion: the set of all real numbers cannot be put into a list.

- Diagonal construction

For explaining the diagonal construction, let us see the list of 8 binary numbers in Table 1. We take the first bit of the first number s_1, the second bit of the second number s_2 and so on till s_8. The 8 selected bits make the number s_{diag}. Then, the number s_{out} equals the sequence of flipped bits of s_{diag}, that is, if a bit of s_{diag} equals 0, the same bit of s_{out} equals 1 and vice versa. Because the 8 bits of s_{out} are flipped, s_{out} does not equal any of the 8 original numbers.

We can construct the number s_{out} from lists that contain infinitely many numbers. In this case, s_{out} has infinitely many bits. Because every bit of s_{out} is flipped, s_{out} does not equal any of the numbers of the original infinitely long list. We call s_{out} the out-of-the-list-number.

List s_i i=1…8	Bits b_j j=1…8
s_1	0.01100110
s_2	0.00000101
s_3	0.10000100
s_4	0.10010001
s_5	0.01101010
s_6	0.01110110
s_7	0.01110100
s_8	0.00111101
s_{disg}	0.00011101
s_{out}	0.11100010

Table 1

According to Cantor's diagonal argument, if the set ℝ is put into a list, which we call L, an out-of-the-list-number s_{out} can be constructed by applying the diagonal construction to L. Because s_{out} is not in L, ℝ is not completely in the list L and thus, uncountable.

2. Points on the number line and real numbers

a. Composite list

For going beyond Cantor's diagonal argument, we split ℝ into two sets S_a and S_b, each contains a half of ℝ which then equals the union of S_a and S_b, see (1). We create a list of real numbers by picking one member from S_a and one member from S_b alternately and forever. The resulting list is called composite list and shown in Table 2 with the a_i's being the members of S_a and b_i's the members of S_b. The composite list would form a list that contains ℝ, but Cantor's diagonal argument has shown that such list cannot exist.

$$\mathbb{R} = S_a \cup S_b \qquad (1)$$

Composite list	a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6, a_7, b_7, …

Table 2

On the other hand, real numbers are in reality the positions of points on the number line and equal the distances between the origin 0 and the points. If each point had a name we can put all the names into a list which is then equivalent to a list of real numbers. Because Cantor's diagonal argument does not deal with names, the list of names can exist without contravening Cantor's diagonal argument.

Let the a_i's and b_i's in the composite list be the expressions of real numbers except that the a_i's are in binary system and b_i's in decimal system. Because the a_i's and b_i's are sequences of digits the composite list is a table of digits, but its diagonal is a sequence of binary and decimal digits alternately. This diagonal is not a valid expression of real number, then out-of-the-list-number cannot be constructed from it.

So, for Cantor's diagonal argument to work it needs the condition below:

Condition 1: All the real numbers in the list must be expressed in the same numeral system.

The Condition 1 has restricted the list of real numbers, which is materialized by the out-of-the-list-number. When this constraint is dropped as the composite list is expressed in 2 numeral systems, the out-of-the-list-number cannot be constructed.

b. Multi-composite list

In fact, composite list can be created in splitting ℝ into many subsets in numeral systems of different bases. Let x_n be a real number expressed in base n, equation (2) is its expression, with p(i) being its i^{th} bit.

$$x_n = \sum_{i=1}^{\infty} \frac{p(i)}{n^i} \qquad (2)$$

Suppose ℝ is split into the subsets $S_2, S_3, \ldots, S_j$ which are expressed in numeral systems of bases 2,3,…,j, see (3). The set ℝ equals the union of these subsets, see (4).

$$S_n = \{ x \mid x = \textstyle\sum_{i=1}^{\infty} \frac{p(i)}{n^i} \},\ n=2,3,\ldots,j \qquad (3)$$

$$\mathbb{R} = S_2 \cup S_3 \cup \ldots \cup S_j \qquad (4)$$

We construct the Table 3 with the lines $S_2, S_3, S_4, \ldots, S_j$ whose cells are members picked from each subset.

S_j	d_1		d_2	→	d_3		d_4	→	d_5	…
⋮			⋮		⋮		⋮		⋮	
	↑	↘		↖		↘		↖		↘
S_4	c_1		c_2		c_3		c_4		c_5	…
		↖		↘		↖		↘		↖
S_3	b_1		b_2		b_3		b_4		b_5	…
	↑	↘		↖		↘		↖		↘
S_2	a_1		a_2	→	a_3		a_4	→	a_5	…

Table 3

For creating a composite list from the Table 3, we follow the arrows to construct the "List of points" in Table 4.

List of points	$a_1, b_1, a_2, a_3, b_2, c_1, \ldots, c_2, b_3, a_4, a_5, b_4, c_3, \ldots$

Table 4

Because the lines of Table 3 are expressed in numeral systems of different bases, the diagonal of the "List of points" contains digits of many bases and it is impossible to create out-of-the-list-number from it. If any flipped number was created it cannot be compared digit by digit with real numbers because the flipped number is made by digits of many bases while real numbers are expressions in one numeral system. So, no real number can be proven excluded from the "List of points". Since this list is constructed from the whole ℝ and no real number is found outside, we can conclude that the "List of points" contains ℝ.

If there is one list that contains ℝ we can already conclude that ℝ is countable. But the Table 3 contains j-1 subsets of ℝ the permutation of which can create a huge number of different "Lists of points" that contain ℝ. So, we conclude with confidence that ℝ is countable.

By restricting the numeral system of the list to only one, Cantor's diagonal argument shows in fact that one numeral system can produce out-of-the-list-number, but not that ℝ is uncountable. Here, we find another feature from the composite list which is: out-of-the-list-number is not a property of $\mathbb{R}$ but of numeral system. This is the flaw that breaks Cantor's diagonal argument.

Apart from $\mathbb{R}$ other infinite sets exist and we need a practical tool to count them too.

3. Axiom for counting infinite set

Let us imagine a magic box which can contain infinitely many points. In the wall of the box is a small hole through which the points go out. The points are scattered randomly inside the box so that every point has an equal chance to exit. The hole is guarded by a demon who lets the points go out only one by one and marks each point with its number of exit.

Let us put the interval [0, 1] into the magic box and shake the real numbers loose. Then each point in the magic box is a real number. Through the hole the points come out and are marked with their number of exit. As the points come out forever and the demon will never run out of natural numbers, we obtain an infinite list of real numbers marked with their number of exit. Because each point has an equal chance to exit, each real number can get out. All the real numbers of the interval [0, 1] together make the set $\mathbb{R}_1$. So, the infinite list will be a one-to-one correspondence between the sets $\mathbb{R}_1$ and $\mathbb{N}$.

Let us see the infinite list from another perspective. Suppose that the demon is tired of marking the real numbers and want to play a new game. He puts all the real numbers in one magic box and all the natural numbers in another. The game consists of determining which box contains natural numbers without looking at the numbers. So, he scrubs all the numbers off the points so that the points coming out from both boxes are bare points.

Since real numbers could be more numerous than natural numbers, the demon expects that a box will stop to give out points before the other and this box will be the box of natural numbers. But as natural numbers and real numbers are both infinitely many, points come out from both boxes endlessly and he has no way to determine which box contains natural numbers. He swaps the two boxes in his hands to look for difference but there is none. The two boxes are identical and deliver endlessly the same flux of points. Finally, he settles that the two boxes contain the same number of points, that is, there are as many real numbers as natural numbers.

From these two thought experiments we extract the axiom for counting infinite sets below [19].

Axiom A: If S is an infinite set whose members can be picked one by one, S can be put into one-to-one correspondence with the set of natural numbers $\mathbb{N}$ and is countable.

The **Axiom A** is an abstraction of what we do when picking things one by one: one thing, two thing, etc. This is the principle of counting things and the origin of natural numbers. If a set contains infinitely many things and the picking is random, each thing can be selected and things will come out forever, which put the set in one-to-one correspondence with natural numbers.

Because real numbers can be picked one by one, points in n-dimensional real space $\mathbb{R}^n$ can be picked one by one, mathematical functions can be picked one by one, the **Axiom A** implies that the sets $\mathbb{R}$, $\mathbb{R}^n$ and the set of all functions are countable. The **Axiom A** is then the practical tool that we searched for in this section.

From the **Axiom A** we deduce that infinite sets of anything are countable. So, the cardinality of all infinite sets is $\aleph_0$ and then, Continuum hypothesis is solved because the cardinalities of $\mathbb{R}$ and $\mathbb{N}$ are equal. From this we deduce that $\aleph_0$ is the highest cardinality. The **Axiom A** may solve the Banach–Tarski paradox [20] [21] and give birth to new useful theorems.

4. My reasoning

I want to explain the reasoning that led me to the present study. At first, I believe that when a sort of thing can be picked one by one, the set of infinitely many these things should be countable because picking is counting. As points on the number line can be picked one by one and are equivalent to real numbers, real numbers should be countable. It remains to prove this idea. So, I have written many articles analyzing uncountability from multiple standpoints and proposing diverse schemes for counting real numbers, see [5] , [6] , [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19].

But analyses and new schemes of counting are not strong enough against the well established Cantor's diagonal argument. I have to break it first by finding out an undeniable flaw in it. For doing so, I have split the Contradiction deduction given in section 1 into thinner steps as below:

1. Assumption A: All real numbers can be put into a list.
2. Construct a list of all real numbers which we name L.
3. Construct the out-of-the-list-number s_{out} from L using diagonal construction.
4. Confirm that s_{out} is a real number.
5. Confirm that s_{out} is not in the list L.
6. Observe that a real number is not in the list L.
7. Contradiction: the Assumption A is contradicted by step 6.
8. Conclusion: the set of all real numbers cannot be put into a list.

And I searched for flaw in each step. The step 1 is the Assumption A which is correct. Could the step 2 contain a flaw? Constructing an infinitely long list could be a flaw because this cannot be finished. But infinitely long lists exist, for example, the list of all rational numbers. So, the flaw is not in the step 2.

Could the step 3 contain a flaw? The out-of-the-list-number s_{out} is an infinitely long sequence of digits and cannot be finished to construct. But, an existing infinitely long list possesses an infinitely long diagonal without constructing it. So, the flaw is not in the step 3.

For the steps 4 and 5, s_{out} is no doubt a real number and is truly not in the list. So, the flaw is not in the steps 4 and 5.

The step 6 claims that a real number is not in the list. But what is this list really? The list used by Cantor's diagonal argument is expressed for example in binary system. If a binary list is shown not to contain $\mathbb{R}$, this can be caused either by "list" or by "binary". Because Cantor has focused only on "list" overlooking "binary", the flaw could be here.

In keeping the conviction that points are countable, I have done much work and finally discovered that Cantor's diagonal argument expresses real numbers only in one numeral system, which restricts the used list. This is the flaw that breaks Cantor's diagonal argument which then does not prove $\mathbb{R}$ uncountable, see section 2.

5. Summarizing the results

Section 2.a:	Splitting $\mathbb{R}$ into 2 subsets and creating composite list. Finding the Condition 1: lists must be in one numeral system for Cantor's diagonal argument.
Section 2.b:	Creating multi-composite list on many numeral systems. Multi-composite list proves that $\mathbb{R}$ is countable. Out-of-the-list-number is not a property of $\mathbb{R}$ but of numeral system.
Section 3:	Creation of the Axiom for counting infinite sets.
Section 4:	Searching for flaw in Cantor's diagonal argument.

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