Determination of the relative roll, pitch and yaw between arbitrary objects using 3D complex number

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Abstract: The roll, pitch and yaw of an object relative to another is complex to compute. We use 3D complex number to compute them which makes the computation easier and more intuitive.

1. Frames of objects

Roll, pitch and yaw are angles of orientation of an object in space and the conversion of these angles among different reference frames is not easy, as the example "How to calculate relative pitch, roll and yaw given absolutes" [1] illustrates. This example is about the roll, pitch and yaw of a telephon relative to a car.

We will use 3D complex number to compute roll, pitch and yaw. We label the car as object a and the telephon as object b. Their reference frames are labeled as frame A and B and these frames are orientated with respect to the ground whose frame is labeled as frame G. The base vectors of the frame A are $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, that of the frame B are $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$ and that of the frame G are $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$. The vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ and $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$ are expressed with $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$ in equations (1) and (2) with $\mathbf{M}_a$ and $\mathbf{M}_b$ being the matrices of transformation.

$$\begin{bmatrix}\boldsymbol{a}_x\\ \boldsymbol{a}_y\\ \boldsymbol{a}_z\end{bmatrix} = \boldsymbol{M}_a \begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} \quad (1) \qquad \begin{bmatrix}\boldsymbol{b}_x\\ \boldsymbol{b}_y\\ \boldsymbol{b}_z\end{bmatrix} = \boldsymbol{M}_b \begin{bmatrix}\boldsymbol{g}_x\\ \boldsymbol{g}_y\\ \boldsymbol{g}_z\end{bmatrix} \quad (2)$$

2. Angles of orientation

The roll, pitch and yaw of the object b relative to the object a define the orientation of the frame B in the frame A. This orientation is defined by the angles of rotation θ, φ and ψ, see Figure 1. For computing these angles we express the vectors $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$ with the vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ in (3), (4) and (5).

$$\boldsymbol{b}_x = \cos\theta\cos\varphi\,\boldsymbol{a}_x + \cos\varphi\sin\theta\,\boldsymbol{a}_y + \sin\varphi\,\boldsymbol{a}_z \quad (3)$$

$$\boldsymbol{b}_y = (-\sin\theta\cos\psi - \sin\psi\sin\varphi\cos\theta)\boldsymbol{a}_x + (\cos\theta\cos\psi - \sin\psi\sin\varphi\sin\theta)\boldsymbol{a}_y + \sin\psi\cos\varphi\,\boldsymbol{a}_z \quad (4)$$

$$\boldsymbol{b}_z = (\sin\psi\sin\theta - \cos\psi\sin\varphi\cos\theta)\boldsymbol{a}_x + (-\sin\psi\cos\theta - \cos\psi\sin\varphi\sin\theta)\boldsymbol{a}_y + \cos\psi\cos\varphi\,\boldsymbol{a}_z \quad (5)$$

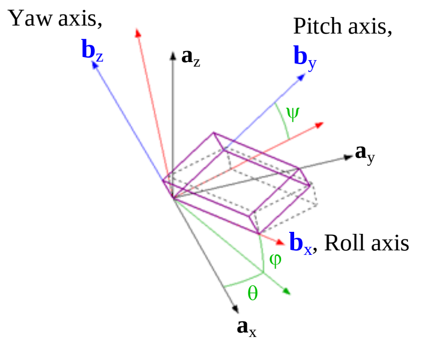

Figure 1
Rotation of the base vectors

The equations (3), (4) and (5) are identical to the equations (6), (7) and (8) which are derived from the equation (51) of « Computing orientation with complex multiplication but without trigonometric function »[3], with $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ being the base vectors of an arbitrary frame E relative to the ground frame G.

$$\boldsymbol{e}_x = \cos\theta\cos\varphi\,\boldsymbol{g}_x + \cos\varphi\sin\theta\,\boldsymbol{g}_y + \sin\varphi\,\boldsymbol{g}_z \quad (6)$$

$$\boldsymbol{e}_y = (-\sin\theta\cos\psi - \sin\psi\sin\varphi\cos\theta)\boldsymbol{g}_x + (\cos\theta\cos\psi - \sin\psi\sin\varphi\sin\theta)\boldsymbol{g}_y + \sin\psi\cos\varphi\,\boldsymbol{g}_z \quad (7)$$

$$\boldsymbol{e}_z = (\sin\psi\sin\theta - \cos\psi\sin\varphi\cos\theta)\boldsymbol{g}_x + (-\sin\psi\cos\theta - \cos\psi\sin\varphi\sin\theta)\boldsymbol{g}_y + \cos\psi\cos\varphi\,\boldsymbol{g}_z \quad (8)$$

I have created the system of 3D complex number and explained it in «Extending complex number to spaces with 3, 4 or any number of dimensions»[2]. In this system a unit vector **u** is expressed by equation (9), with θ and φ being the angles of rotation of **u** in the 3D complex space (1, i, j).

$\boldsymbol{u} = \cos\theta\cos\varphi + \sin\theta\cos\varphi\, i + \sin\varphi\, j$	(9)

We see that the expression of the vector $\mathbf{b}_x$ is identical to the 3D complex number **u,** see (3) and (9). So, finding the angles θ and φ for the object b is equivalent to find the angles θ and φ in the 3D complex space (1, i, j).

3. Computation of roll, pitch and yaw

For computing the angles of rotation θ, φ and ψ of the frame B relative to the frame A, we express $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$ with $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ in (10), (11) and (12) where the coefficients in the parentheses are dot products between $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ and $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$.

$\boldsymbol{b}_x = (\boldsymbol{b}_x \cdot \boldsymbol{a}_x)\boldsymbol{a}_x + (\boldsymbol{b}_x \cdot \boldsymbol{a}_y)\boldsymbol{a}_y + (\boldsymbol{b}_x \cdot \boldsymbol{a}_z)\boldsymbol{a}_z$	(10)
$\boldsymbol{b}_y = (\boldsymbol{b}_y \cdot \boldsymbol{a}_x)\boldsymbol{a}_x + (\boldsymbol{b}_y \cdot \boldsymbol{a}_y)\boldsymbol{a}_y + (\boldsymbol{b}_y \cdot \boldsymbol{a}_z)\boldsymbol{a}_z$	(11)
$\boldsymbol{b}_z = (\boldsymbol{b}_z \cdot \boldsymbol{a}_x)\boldsymbol{a}_x + (\boldsymbol{b}_z \cdot \boldsymbol{a}_y)\boldsymbol{a}_y + (\boldsymbol{b}_z \cdot \boldsymbol{a}_z)\boldsymbol{a}_z$	(12)

The equations (10) , (11) and (12) are written in matrix form in (13) from which we extract the matrix of transformation $\mathbf{M}_{ba}$ shown in (14), see (3), (4) and (5).

$$\begin{bmatrix}\boldsymbol{b}_x\\ \boldsymbol{b}_y\\ \boldsymbol{b}_z\end{bmatrix} = \begin{bmatrix}\boldsymbol{b}_x\cdot\boldsymbol{a}_x & \boldsymbol{b}_x\cdot\boldsymbol{a}_y & \boldsymbol{b}_x\cdot\boldsymbol{a}_z\\ \boldsymbol{b}_y\cdot\boldsymbol{a}_x & \boldsymbol{b}_y\cdot\boldsymbol{a}_y & \boldsymbol{b}_y\cdot\boldsymbol{a}_z\\ \boldsymbol{b}_z\cdot\boldsymbol{a}_x & \boldsymbol{b}_z\cdot\boldsymbol{a}_y & \boldsymbol{b}_z\cdot\boldsymbol{a}_z\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_x\\ \boldsymbol{a}_y\\ \boldsymbol{a}_z\end{bmatrix} \quad (13)$$

$$\boldsymbol{M}_{ba} = \begin{bmatrix}\underline{\boldsymbol{b}_x\cdot\boldsymbol{a}_x} & \underline{\boldsymbol{b}_x\cdot\boldsymbol{a}_y} & \underline{\boldsymbol{b}_x\cdot\boldsymbol{a}_z}\\ \boldsymbol{b}_y\cdot\boldsymbol{a}_x & \boldsymbol{b}_y\cdot\boldsymbol{a}_y & \underline{\boldsymbol{b}_y\cdot\boldsymbol{a}_z}\\ \boldsymbol{b}_z\cdot\boldsymbol{a}_x & \boldsymbol{b}_z\cdot\boldsymbol{a}_y & \underline{\boldsymbol{b}_z\cdot\boldsymbol{a}_z}\end{bmatrix} = \begin{bmatrix}\underline{\cos\theta\cos\varphi} & \underline{\cos\varphi\sin\theta} & \underline{\sin\varphi}\\ -\sin\theta\cos\psi - \sin\psi\sin\varphi\cos\theta & \cos\theta\cos\psi - \sin\psi\sin\varphi\sin\theta & \underline{\sin\psi\cos\varphi}\\ \sin\psi\sin\theta - \cos\psi\sin\varphi\cos\theta & -\sin\psi\cos\theta - \cos\psi\sin\varphi\sin\theta & \underline{\cos\psi\cos\varphi}\end{bmatrix} \quad (14)$$

From the 5 underscored coefficients in (14) we derive the angles θ, φ and ψ using (15), (16) and (17), knowing that $\cos\varphi$ is always positive because φ is between -π/2 and π/2, see Figure 1.

$\varphi = \sin^{-1}(\boldsymbol{b}_x\cdot\boldsymbol{a}_z)$	(15)	$\theta = sign(\boldsymbol{b}_x\cdot\boldsymbol{a}_y)\cos^{-1}\left(\frac{\boldsymbol{b}_x\cdot\boldsymbol{a}_x}{\cos\varphi}\right)$	(16)	$\psi = sign(\boldsymbol{b}_y\cdot\boldsymbol{a}_z)\cos^{-1}\left(\frac{\boldsymbol{b}_z\cdot\boldsymbol{a}_z}{\cos\varphi}\right)$	(17)

Roll, pitch and yaw of an object are the rotation angles of the object around the x, y and z axis respectively, see Figure 1 and the page "Flight dynamics"[4]. So, the conversion formula between roll, pitch and yaw and the angles θ, φ and ψ is equation (18).

roll= ψ, pitch = -φ, yaw = θ	(18)

4. Numerical validation

Equations (15), (16) and (17) give the values of the angles θ, φ and ψ. Are these values correctly computed by the method using 3D complex number? Let us check the validity of this method with numerical examples. For doing so, we take the 2 matrices with numerical coefficients $\mathbf{M}_a$ and $\mathbf{M}_b$ shown in (19) and (20), $\mathbf{M}_a$ and $\mathbf{M}_b$ transform the frame G into the frames A and B respectively.

Using (10) , (11) and (12) we obtain the matrix $\mathbf{M}_{ba}$ that transforms the base vectors $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ into $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{b}_z$. Using (15), (16) and (17) we obtain the values of the angles θ, φ and ψ which are shown in the first line of Table 1. Using (18) we obtain the angles of roll, pitch and yaw of the frame B relative to the frame A which are shown in the second line of Table 1.

$$\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} 0.75 & 0.4330127 & 0.5 \\ -0.6495191 & 0.625 & 0.4330127 \\ -0.125 & -0.6495191 & 0.75 \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = \boldsymbol{M}_a \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}$$	(19)
$$\begin{bmatrix} \boldsymbol{b}_x \\ \boldsymbol{b}_y \\ \boldsymbol{b}_z \end{bmatrix} = \begin{bmatrix} 0.25 & 0.4330127 & 0.8660254 \\ -0.8080127 & -0.3995191 & 0.4330127 \\ 0.5334936 & -0.8080127 & 0.25 \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix} = \boldsymbol{M}_b \begin{bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{bmatrix}$$	(20)
$$\begin{bmatrix} \boldsymbol{b}_x \\ \boldsymbol{b}_y \\ \boldsymbol{b}_z \end{bmatrix} = \begin{bmatrix} 0.8080127 & 0.4832532 & 0.3370191 \\ -0.5625 & 0.4626202 & 0.6852564 \\ 0.1752405 & -0.7432691 & 0.6456329 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix} = \boldsymbol{M}_{ba} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}$$	(21)

1	θ = 30.88264°	φ = 19.695362°	ψ = 46.705314°
2	roll= ψ = 46.705314°	pitch = -φ = -19.695362°	yaw = θ = 30.88264°

Table 1

For checking the validity of the computed angles θ, φ and ψ, we have introduced them into the matrix (14) to compute its coefficients. The so computed coefficients are identical to the original values shown in (21). We have also computed many other numerical examples which all gave correct values. So, the method using 3D complex number is well validated.

As a countercheck we have computed the matrix $\mathbf{M}_{ba}$ using equation (22) which is proposed by the replyer in the page "How to calculate relative pitch, roll and yaw given absolutes" [1]. We have obtained the same values for the coefficients of $\mathbf{M}_{ba}$. So, our method using 3D complex number is equivalent to the method proposed by the replyer.

$\boldsymbol{M}_{ba} = \boldsymbol{M}_b \boldsymbol{M}_a{}^{-1}$	(22)

We have also computed some examples for direct and backward roll, pitch and yaw, that is, for the object B in the frame A and the object A in the frame B and have proven that the numerical values of these angles in the two directions are not the same.

5. Discussion

The main advancement given by our method using 3D complex number is to have related the roll, pitch and yaw of an object to the 3D complex number that represent the x axis of the object. This mathematical discovery makes the conversion of roll, pitch and yaw between arbitrary frames easier and more intuitive.

This method completes the mathematical system made by 3D complex number and roll, pitch and yaw:

1. The article «Extending complex number to spaces with 3, 4 or any number of dimensions»[2] creates and explains the system of 3D complex number.
2. The article « Computing orientation with complex multiplication but without trigonometric function »[3] gives the equations (6), (7) and (8) that compute the angles of orientation of object relative to the ground.
3. The present article explains how to compute roll, pitch and yaw between any two objects in 3D space.

The angles of rotation θ, φ and ψ are computed with the 5 underscored coefficients of the matrix $\mathbf{M}_{ba}$ in (14) saving thus the computation time for the 4 other coefficients which are not needed. This will make the computation faster and the motion of moving images on screen smoother. So, our method using 3D complex number would be beneficial to applications such as video games or street view of digital map etc.

References

[1] "How to calculate relative pitch, roll and yaw given absolutes" https://math.stackexchange.com/questions/1884215/how-to-calculate-relative-pitch-roll-and-yaw-given-absolutes

[2] Kuan Peng, 2022, «Extending complex number to spaces with 3, 4 or any number of dimensions» https://www.academia.edu/71708344/Extending_complex_number_to_spaces_with_3_4_or_any_number_of_dimensions

[3] Kuan Peng, 2022, « Computing orientation with complex multiplication but without trigonometric function » https://www.academia.edu/80277267/Computing_orientation_with_complex_multiplication_but_without_trigonometric_function

[4] https://en.wikipedia.org/wiki/Flight_dynamics_(fixed-wing_aircraft)