Procedure to convert 2D formula into 3D complex formula

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Let z_2=x+yi be the 2D complex variable, we can replace it with the 3D complex z = x+yi+z_*j in any equation.

1. Basics of the 3D complex number algebra

a. Forms of a 3D complex number

Trigonometric form	$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$	(1)
abdf form	$z = r\big((a + bi)d + fj\big)$ $with \cos\theta = a, \sin\theta = b, \cos\varphi = d, \sin\varphi = f$	(2)
Cartesian form	$z = x + yi + z_* j = r(ad + bdi + fj)$	(3)

b. Conversion from Cartesian form into abdf form

The transformed z in abdf form is in (6).

$z = x + yi + z_* j$		$r_{xy} = \sqrt{x^2 + y^2}$			$r = \sqrt{x^2 + y^2 + {z_*}^2}$	(4)
$a = \frac{x}{r_{xy}}$	$b = \frac{y}{r_{xy}}$	$d = \frac{r_{xy}}{r}$	$f = \frac{z_*}{r}$	(5)	$z = r\big((a + bi)d + fj\big)$	(6)

c. Conversion from Cartesian form into trigonometric form

1. Compute the moduli r_{xy} and r using (4)
2. Compute the angles in the order φ first, θ second using (7)
3. z in trigonometric form is in (8)

$z = x + yi + z_* j$		$r_{xy} = \sqrt{x^2 + y^2}$		$r = \sqrt{x^2 + y^2 + {z_*}^2}$	(4)
$\varphi = \sin^{-1}\left(\frac{z_*}{r}\right)$	If y>0	$\theta = \cos^{-1}\left(\frac{x}{r_{xy}}\right)$	If y<0	$\theta = -\cos^{-1}\left(\frac{x}{r_{xy}}\right)$	(7)
$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$					(8)

d. Addition and subtraction

In Cartesian form

$z = x_z + y_z i + z_{*z} j$	$c = x_c + y_c i + z_c j$	(9)
$z \pm c = (x_z + y_z i + z_{*z} j) + (x_c + y_c i + z_{*c} j) = (x_z \pm x_c) + (y_z \pm y_c)i + (z_{*z} \pm z_{*c})j$		(10)

In trigonometric form

$z = r_z\big((\cos\theta_z + \sin\theta_z\, i)\cos\varphi_z + \sin\varphi_z\, j\big)$	$c = r_c\big((\cos\theta_c + \sin\theta_c\, i)\cos\varphi_c + \sin\varphi_c\, j\big)$	(11)
$z \pm c = (r_z\cos\theta_z\cos\varphi_z \pm r_c\cos\theta_c\cos\varphi_c) + (r_z\sin\theta_z\cos\varphi_z \pm r_c\sin\theta_c\cos\varphi_c)i + (r_z\sin\varphi_z \pm r_c\sin\varphi_c)j$		(12)

e. Multiplication

- In trigonometric form

1. Take z and c given in (11)
2. Add the angles, the angles of the product equal the sum of that of the factors
3. Multiply the moduli, the product is in (13)

$z \cdot c = r_z r_c\big((\cos(\theta_z + \theta_c) + \sin(\theta_z + \theta_c)\, i)\cos(\varphi_z + \varphi_c) + \sin(\varphi_z + \varphi_c)\, j\big)$	(13)

- In Cartesian form

1. Take z and c given in (9)
2. Convert z and c into abdf form using (4), (5)
3. Use formula (14)

$z = x_z + y_z i + z_{*z} j$	$c = x_c + y_c i + z_c j$	(9)
$z \cdot c = r_z r_c\Big(\big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)j\Big)$		(14)

- Proprieties of multiplication

Multiplication is commutative: $r_c r_z = r_z r_c$ and $\theta_c + \theta_z = \theta_z + \theta_c$, $\varphi_c + \varphi_z = \varphi_z + \varphi_c$
Multiplication is associative: $(r_c\, r_z)r_+ = r_c\,(r_z\, r_+)$ and $(\theta_c+\theta_z)+\theta_+ = \theta_c+(\theta_z+\theta_+)$, $(\varphi_c+\varphi_z)+\varphi_+ = \varphi_c+(\varphi_z+\varphi_+)$
Multiplication is not distributive over addition, (15). If you have (A+B)C in a 2D equation, you have to compute AC+BC, but not D=A+B then D·C, because multiplication has to be done before addition, see (16).

$(A + B)C \neq A \cdot C + B \cdot C$	(15)	$(A + B)C \Rightarrow A \cdot C + B \cdot C$	(16)

f. Division z/c

- In trigonometric form

1. Take z and c given in (11)
2. Subtract the angles, angles of the product equal the difference of that of the factors
3. Divide the moduli, the quotient is in (17)

$z = r_z\big((\cos\theta_z + \sin\theta_z\, i)\cos\varphi_z + \sin\varphi_z\, j\big)$	$c = r_c\big((\cos\theta_c + \sin\theta_c\, i)\cos\varphi_c + \sin\varphi_c\, j\big)$	(11)
$\frac{z}{c} = \frac{r_z}{r_c}\big((\cos(\theta_z - \theta_c) + \sin(\theta_z - \theta_c)\, i)\cos(\varphi_z - \varphi_c) + \sin(\varphi_z - \varphi_c)\, j\big)$		(17)

- In Cartesian form

1. Compute the numbers a, b, d, f of z using (18), (19) and those of c using (18), (20)
2. Use formula (21)

$a = x + yi + z_* j$	$r_{xy} = \sqrt{x^2 + y^2}$		$r = \sqrt{x^2 + y^2 + {z_*}^2}$		(18)
$z = x_z + y_z i + z_{*z} j$	$a_z = \left(\frac{x}{r_{xy}}\right)_z$	$b_z = \left(\frac{y}{r_{xy}}\right)_z$	$d_z = \left(\frac{r_{xy}}{r}\right)_z$	$f_z = \left(\frac{z_*}{r}\right)_z$	(19)
$c = x_c + y_c i + z_c j$	$a_c = \left(\frac{x}{r_{xy}}\right)_c$	$b_c = -\left(\frac{y}{r_{xy}}\right)_c$	$d_c = \left(\frac{r_{xy}}{r}\right)_c$	$f_c = -\left(\frac{z_*}{r}\right)_c$	(20)
$\frac{z}{c} = \frac{r_z}{r_c}\Big(\big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c) i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c) j\Big)$					(21)

g. 3D complex number to the power 2 and 3: z^2 and z^3 in abdf form

1. Compute the numbers a, b, d, f of the number z using (18), (19)
2. Use formulae (22), (23), (24), (25)
3. Resulting z^2 and z^3 are in (23), (25)

Power 2	$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(22)
	$z^2 = (r^2 a_{p2} d_{p2}) + (r^2 b_{p2} d_{p2}) i + (r^2 f_{p2}) j = x_2 + y_2 i + z_{*2} j$				(23)
Power 3	$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	(24)
	$z^3 = (r^3 a_{\therefore} d_{\therefore}) + (r^3 b_{\therefore} d_{\therefore}) i + (r^3 f_{\therefore}) j = x_3 + y_3 i + z_{*3} j$				(25)

h. 3D complex number to the power n

- z^n in trigonometric form

$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$	$z^n = r^n\big((\cos n\theta + \sin n\theta\, i)\cos n\varphi + \sin n\varphi\, j\big)$	(26)

- z^n in Cartesian form $z = x + yi + z_* j$

1. Convert z into trigonometric form using (4), (7), (8)
2. Use formula (26)

i. n^{th} Root

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + \sin\frac{\theta}{n} i\right)\cos\frac{\varphi}{n} + \sin\frac{\varphi}{n} j$	(27)

j. Conjugate

$z = x + yi + z_* j$		$\bar{z} = x - yi - z_* j$		(28)
$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$		$\bar{z} = r\big((\cos\theta + \sin(-\theta)\, i)\cos\varphi + \sin(-\varphi)\, j\big)$		(29)
$z = r\big((a + bi)d + fj\big)$	$\bar{z} = r\big((a - bi)d - fj\big)$	(30)	$\bar{z} = \frac{\lvert z\rvert^2}{z}$	(31)

k. Sign of d for iteration

For φ to rotate in the same direction at n^{th} and $n+1^{th}$ step:

$z_n = x_n + y_n i + z_{*n} j = r_n\big((a_n + b_n i)d_n + f_n j\big)$	(32)	$z_{n+1} = f(z_n)$	(33)
$z_{n+1} = x_{n+1} + y_{n+1} i + z_{*n+1} j = r_{n+1}\big((a_{n+1} + b_{n+1} i)d_{n+1} + f_{n+1} j\big)$	(34)	$sign(d_{n+1}) = sign(d_n)$	(35)

l. Exponential and logarithm

Exponential	$z = x + yi + z_* j$	$e^z = e^x(\cos y + \sin y\, i)\cos z_* + \sin z_*\, j$	(36)	See (80)(81)
Natural logarithm	$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$	$\log z = \log r + i\theta + j\varphi$	(37)	See (82)(83)

m. Non trigonometric large power computation

Multiplication and Complex number z^n with modulus r

$z \cdot c = \big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)j$ see (66)	(38)	$z^n = r^n\big((a_n + b_n i)d_n + f_n j\big)$	(39)

Large Power for unit complex number z

$z^4 = (z^2)^2 = \big((2{a_2}^2 - 1) + 2a_2 b_2 i\big)({d_2}^2 - 1) + 2d_2 f_2 j$	$z^5 = z^2 z^3$	(40)	See (38), (22), (23), (24), (25)
$z^6 = (z^3)^2 = \big((2{a_3}^2 - 1) + 2a_3 b_3 i\big)({d_3}^2 - 1) + 2d_3 f_3 j$	$z^7 = z^4 z^3$	(41)	
$z^8 = (z^4)^2 = \big((2{a_4}^2 - 1) + 2a_4 b_4 i\big)({d_4}^2 - 1) + 2d_4 f_4 j$		(42)	
$z^9 = (z^3)^3 = \big(a_3(4{a_3}^2 - 3) + b_3(3 - 4{b_3}^2)i\big)d_3(4{d_3}^2 - 3) + f_3(3 - 4{f_3}^2)j$		(43)	

Recurrent formula for unit complex number z using multiplication formula (38)

$z^1 = (a_1 + b_1 i)d_1 + f_1 j$	(44)	see (38)
$z^2 = (a_2 + b_2 i)d_2 + f_2 j = \big((2{a_1}^2 - 1) + 2a_1 b_1 i\big)({d_1}^2 - 1) + 2d_1 f_1 j$	(45)	
$z^3 = (a_3 + b_3 i)d_3 + f_3 j = \big((a_2 a_1 - b_2 b_1) + (b_2 a_1 + a_2 b_1)i\big)(d_2 d_1 - f_2 f_1) + (f_2 d_1 + d_2 f_1)j$	(46)	
$z^{n+1} = (a_{n+1} + b_{n+1} i)d_{n+1} + f_{n+1} j = \big((a_n a_1 - b_n b_1) + (b_n a_1 + a_n b_1)i\big)(d_n d_1 - f_n f_1) + (f_n d_1 + d_n f_1)j$	(47)	

n. Forms of a 4D complex number

Trigonometric form	$z = r\Big(\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)\cos\phi + \sin\phi\, k\Big)$	(48)
abdf form	$z = r\big((a + bi)d + fj\big)g + lj$ $with \cos\theta = a, \sin\theta = b, \cos\varphi = d, \sin\varphi = f, \cos\phi = g, \sin\phi = l$	(49)
Cartesian form	$z = x_1 + x_2 i + x_3 j + x_4 k = r(adg + bdgi + fgj + lj)$	(50)

2. Derivation of the formulae, 9 April 2022

1. Common form of a 3D complex number is (51):

$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big) = r\big((a + bi)d + fj\big) = r(ad + bdi + fj)$	(51)

The form z= x+yi+z*j is converted into expression with r, a, b, d, f

$z = x + yi + z_* j = rad + rbdi + rfj$	$x = rad$	$y = rbd$	$z_* = rf$	(52)

2. Relation between r, a, b, d, f and θ, φ, obtained by comparing the terms in (51)

$r = \sqrt{x^2 + y^2 + z_*{}^2}$	$\cos\theta = a$	$\sin\theta = b$	$\cos\varphi = d$	$\sin\varphi = f$	(53)

Computing the number d from x+yi+z*j, see (52), (53)

$r_{xy} = \sqrt{x^2 + y^2} = \sqrt{(rad)^2 + (rbd)^2} = rd\sqrt{a^2 + b^2} = r\cos\varphi\sqrt{\cos^2\theta + \sin^2\theta} = r\cos\varphi$	$\cos\varphi = \frac{r_{xy}}{r} = d$	(54)

Computing a,b,f from x+yi+z*j, see (52), $\sqrt{a^2 + b^2} = 1$

$\sin\varphi = f = \frac{z_*}{r}$	$\cos\theta = \frac{x}{r_{xy}} = \frac{rad}{\sqrt{(rad)^2 + (rbd)^2}} = \frac{a}{\sqrt{a^2 + b^2}} = a$	$\sin\theta = \frac{y}{r_{xy}} = \frac{b}{\sqrt{a^2 + b^2}} = b$	(55)

Conversion to a,b,d,f

$z = x + yi + z_* j$	$r_{xy} = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2 + z_*{}^2}$			(56)
$z = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big) = r\big((a + bi)d + fj\big)$	$x = rad$	$y = rbd$	$z_* = rf$		(57)
$a = \frac{x}{r_{xy}}$	$b = \frac{y}{r_{xy}}$	$d = \frac{r_{xy}}{r}$	$f = \frac{z_*}{r}$	see (54), (55)	(58)

3. Multiplication of two 3D complex numbers

$z = r_z\big((\cos\theta_z + \sin\theta_z\, i)\cos\varphi_z + \sin\varphi_z\, j\big)$	$c = r_c\big((\cos\theta_c + \sin\theta_c\, i)\cos\varphi_c + \sin\varphi_c\, j\big)$	(59)

Multiplication consists of multiplying the moduli and adding the angles:

$z \cdot c = r_z r_c\big((\cos(\theta_z + \theta_c) + \sin(\theta_z + \theta_c)\, i)\cos(\varphi_z + \varphi_c) + \sin(\varphi_z + \varphi_c)\, j\big)$	(60)

Correspondence sine cosine ↔a,b,d,f

$\cos\theta_z = a_z$	$\sin\theta_z = b_z$	$\cos\varphi_z = d_z$	$\sin\varphi_z = f_z$	$\cos\theta_c = a_c$	$\sin\theta_c = b_c$	$\cos\varphi_c = d_c$	$\sin\varphi_c = f_c$	(61)

Trigonometric sum in (60) expressed in terms of a,b,d,f

$\cos(\theta_z + \theta_c) = \cos\theta_z\cos\theta_c - \sin\theta_z\sin\theta_c$ $\sin(\theta_z + \theta_c) = \sin\theta_z\cos\theta_c + \cos\theta_z\sin\theta_c$	(62)	$\cos(\theta_z + \theta_c) = a_z a_c - b_z b_c$ $\sin(\theta_z + \theta_c) = b_z a_c + a_z b_c$	(63)
$\cos(\varphi_z + \varphi_c) = \cos\varphi_z\cos\varphi_c - \sin\varphi_z\sin\varphi_c$ $\sin(\varphi_z + \varphi_c) = \sin\varphi_z\cos\varphi_c + \cos\varphi_z\sin\varphi_c$	(64)	$\cos(\varphi_z + \varphi_c) = d_z d_c - f_z f_c$ $\sin(\varphi_z + \varphi_c) = f_z d_c + d_z f_c$	(65)

Expression of multiplication of z·c (60) in terms of a,b,d,f using (63), (65)

$z \cdot c = r_z r_c\Big(\big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)j\Big)$	(66)

4. Division

$\cos(-\theta_c) = \frac{x}{r_{xy}} = a_c$	$\sin(-\theta_c) = \frac{-y}{r_{xy}} = b_c$	$\cos(-\varphi_c)\, d = \frac{r_{xy}}{r} = d_c$	$\sin(-\varphi_c)\, f = \frac{-z_*}{r} = f_c$	(67)

Division of z/c (60)

$\frac{z}{c} = \frac{r_z}{r_c}\Big(\big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)j\Big)$	(68)

5. Expression of multiplication of z·c (60) in terms of a,b,d,f using (63), (65)

$z \cdot c = r_z r_c\Big(\big((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)i\big)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)j\Big)$	(69)

6. Power of a 3D complex number

$z \cdot z \cdots z = r_z{}^n\big((\cos(\theta_z + \cdots + \theta_z) + \sin\cos(\theta_z + \cdots + \theta_z)\, i)\cos(\varphi_z + \cdots + \varphi_z) + \sin(\varphi_z + \cdots + \varphi_z)\, j\big)$	
$z^n = r_z{}^n\big((\cos n\theta_z + \sin n\theta_z\, i)\cos n\varphi_z + \sin n\varphi_z\, j\big)$	(70)

Coefficients a, b, d, f for Power 2 and 3: z_{n+1}^{2}, z_{n+1}^{3}

Power 2	$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(71)
$z^2 = (r_z{}^2 a_{p2} d_{p2}) + (r_z{}^2 b_{p2} d_{p2}) i + (r_z{}^2 f_{p2}) j = x_{n+1} + y_{n+1} i + z_{*n+1} j$					(72)
Power 3	$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	(73)
$z_n{}^3 = (r_n{}^3 a_{\therefore} d_{\therefore}) + (r_n{}^3 b_{\therefore} d_{\therefore}) i + (r_n{}^3 f_{\therefore}) j = x_{n+1} + y_{n+1} i + z_{*n+1} j$					(74)

7. nth Root

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{j\frac{\varphi}{n}} = r^{\frac{1}{n}} \left(\cos\frac{\theta}{n} + \sin\frac{\theta}{n} i\right) \cos\frac{\varphi}{n} + \sin\frac{\varphi}{n} j$	(75)

8. Addition of 3D complex numbers

$z + c = (x_z + y_z i + z_{*z} j) + (x_c + y_c i + z_{*c} j) = (x_z + x_c) + (y_z + y_c) i + (z_{*z} + z_{*c}) j$	(76)

9. Order of algebraic operations

$(A + B)C = D \cdot C$	$D = A + B$	(77)	$A \cdot B + C = D + C$	$D = A \cdot B$	(78)	$(A + B)C \neq A \cdot C + B \cdot C$	(79)

The addition A+B should be done before multiplication D·C in (77).
The multiplication A·B should be done before addition D+C in (78).
Multiplication is not distributive over addition, see (79).

a. Exponential function, 18 April 2022

$z = x + yi + z_* j$	$e^z = e^{x + yi + z_* j} = e^x e^{yi} e^{z_* j} = e^x (\cos y + \sin y\, i)(\cos z_* + \sin z_* j)$	(80)
	$e^z = e^x (\cos y + \sin y\, i) \cos z_* + \sin z_* j$	(81)

b. Natural logarithm

$z = x + yi + z_* j = r(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j) = r e^{i\theta} e^{j\varphi}$	(82)
$\log z = \log(r e^{i\theta} e^{j\varphi}) = \log r + i\theta + j\varphi$	(83)

3. Summary of the procedure

- 3D complex number z and r, r_{xy}, a, b, d, f and angles

$z = x + yi + z_* j = r_z((\cos\theta_z + \sin\theta_z\, i) \cos\varphi_z + \sin\varphi_z\, j)$		$a = \frac{x}{r_{xy}} = \cos\theta_z$	$b = \frac{y}{r_{xy}} = \sin\theta_z$	(84)
$r_{xy} = \sqrt{x^2 + y^2}$	$r = \sqrt{x^2 + y^2 + z_*{}^2}$	$d = \frac{r_{xy}}{r} = \cos\varphi_z$	$f = \frac{z_*}{r} = \sin\varphi_z$	(85)

- Computation of the angles must respect this order: φ first, θ second.
  1. Compute φ_z with z_* using (86), because $\cos\varphi_z$ is defined as positive.
  2. Compute θ_z with x and y using (87) and (88) because $-\pi < \theta_z < \pi$

$\cos\varphi_z > 0$	$-\frac{\pi}{2} < \varphi_z < \frac{\pi}{2}$	$\varphi_z = \sin^{-1}\left(\frac{z_*}{r}\right)$	(86)
If y>0,	$0 < \theta_z < \pi$	$\theta_z = \cos^{-1}\left(\frac{x}{r_{xy}}\right)$	(87)
If y<0,	$-\pi < \theta_z < 0$	$\theta_z = -\cos^{-1}\left(\frac{x}{r_{xy}}\right)$	(88)

- Multiplication of z·c

Multiplication is done in terms of a, b, d, f in (89), that in terms of moduli and angles is done in (90). The moduli and angles are computed using (86), (87) and (88). One chooses the convenient formula according to the case.

With a, b, d, f	$z \cdot c = r_z r_c \left(((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c) i)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c) j\right)$	(89)
With angles	$z \cdot c = r_z r_c ((\cos(\theta_z + \theta_c) + \sin(\theta_z + \theta_c)\, i) \cos(\varphi_z + \varphi_c) + \sin(\varphi_z + \varphi_c)\, j)$	(90)

Multiplication is commutative: $r_c\, r_z = r_z\, r_c$ and $\theta_c + \theta_z = \theta_z + \theta_c$, $\varphi_c + \varphi_z = \varphi_z + \varphi_c$

Multiplication is associative: $(r_c\, r_z)\, r_+ = r_c\, (r_z\, r_+)$ and $(\theta_c+\theta_z)+\theta_+ = \theta_c+(\theta_z+\theta_+)$, $(\varphi_c+\varphi_z)+\varphi_+ = \varphi_c+(\varphi_z+\varphi_+)$

- Power of a 3D complex number

$z \cdot z \cdots z = r_z{}^n\big((\cos(\theta_z + \cdots + \theta_z) + \sin\cos(\theta_z + \cdots + \theta_z)\,i)\cos(\varphi_z + \cdots + \varphi_z) + \sin(\varphi_z + \cdots + \varphi_z)\,j\big)$ $= r_z{}^n\big((\cos n\theta_z + \sin n\theta_z\,i)\cos n\varphi_z + \sin n\varphi_z\,j\big)$	
$z^n = (r_z{}^n \cos n\theta_z \cos n\varphi_z) + (r_z{}^n \sin n\theta_z \cos n\varphi_z)i + (r_z{}^n \sin n\varphi_z)j = x_n + y_n i + z_{*n} j$	(91)

Coefficients a, b, d, f for Power 2 and 3: for computing z_{n+1}

Power 2	$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(92)
$z^2 = \big(r^2 a_{p2} d_{p2}\big) + \big(r^2 b_{p2} d_{p2}\big)i + \big(r^2 f_{p2}\big)j = x_2 + y_2 i + z_{*2} j$					(93)
Power 3	$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	(94)
$z^3 = (r^3 a_{\therefore} d_{\therefore}) + (r^3 b_{\therefore} d_{\therefore})i + (r^3 f_{\therefore})j = x_3 + y_3 i + z_{*3} j$					(95)

Example of Mandelbrot sets to the power 2 and 3

Power 2	$z_{n+1} = z^2 + c = x_2 + y_2 i + z_{*2} j + x_c + y_c i + z_{*c} j = x_{n+1} + y_{n+1} i + z_{*n+1} j$	See (93)	(96)
Power 3	$z_{n+1} = z^3 + c = x_3 + y_3 i + z_{*3} j + x_c + y_c i + z_{*c} j = x_{n+1} + y_{n+1} i + z_{*n+1} j$	See (95)	(97)

Go back to (84)

- Order of algebraic operations

Addition must be done before being multiplied, that is, multiplication is not distributive.

$(A + B)C = D \cdot C$	$D = A + B$	(98)	$A \cdot B + C = D + C$	$D = A \cdot B$	(99)	$(A + B)C \neq A \cdot C + B \cdot C$	(100)