Cardinality of the set of binary-expressed real numbers

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Abstract: This article gives the cardinal number of the set of all binary numbers by counting its elements, analyses the consequences of the found value and discusses Cantor's diagonal argument, power set and the continuum hypothesis.

The cardinal number of the set of real numbers is $|\mathbb{R}|$, that of the natural numbers is $\aleph_0$. Georg Cantor has shown that:

$$|\mathbb{R}|= 2^{\aleph_0} \tag{1}$$

The binary expansion of a real number is a sequence of 0 and 1 with a point. On the right of the point is the fractional part and on the left the integer part. As every real number has at least one binary expansion, the set of binary expansions of all real numbers must have the same cardinality than the real numbers. In fact, we can actually count the elements of this set which will be denoted by **B.**

1. Counting the fractional binary numbers

First let us count the elements of the set of all fractional binary numbers which will be denoted by $\mathbf{B_F}$. We can write two fractional numbers with one digit**:** 0.0 and 0.1, four with 2 digits: 0.00, 0.01, 0.10, and 0.11. With n digits, there can be 2^n fractional numbers. For counting all the fractional binary numbers, we let n go to the cardinal number of the set of natural numbers, $\aleph_0$, and obtain $2^{\aleph_0}$. So, the set $\mathbf{B_F}$ contains $2^{\aleph_0}$ elements and its cardinal number is:

$$|\mathbf{B_F}|= 2^{\aleph_0} \tag{2}$$

This result is in conformity with equation 1.

2. Fractional binary numbers on the real line

Figure 1 shows where the fractional binary numbers lie on the unit interval of real line, [0, 1]. The one-digit binary number 0.1 lies at the center of the unit interval and cut it into 2 half intervals. We call the number 0.1 and the unit interval the first rank cutting number and interval, the 2 half intervals second rank intervals. In turn the second rank intervals are cut by the second rank cutting numbers 0.01 and 0.11 giving 4 third rank intervals. The process of interval cutting is shown in Figure 1 where the blue diamonds represent the first 4 ranks of cutting numbers, which are given in Table 1.

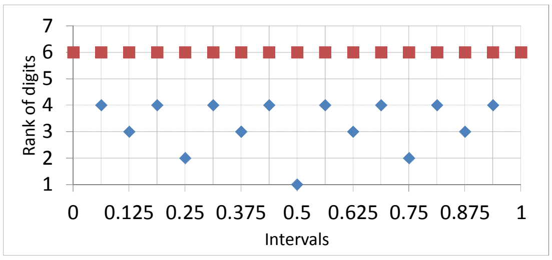

Figure 1

Rank	Cutting numbers							
1	0.1							
2	0.01	0.11						
3	0.001	0.011	0.101	0.111				
4	0.0001	0.0011	0.0101	0.0111	0.1001	0.1011	0.1101	0.1111

Table 1

The cutting numbers plus 0 and 1 are the 17 numbers with 4 or less digits represented by the red squares in Figure 1. So, the red squares show the 2^4+1 positions of the 4-or-less-digit numbers and between them the 5^{th} rank intervals. In the same way, the n-or-less-digit numbers lie discretely at 2^n+1 points and leave the intervals empty between them.

Because the intervals are always empty when n→∞, the set of all fractional binary numbers is not continuous but discrete. Thus, $\mathbf{B_F}$ should be countable. This is an unusual result.

3. Countability of $\mathbf{B_F}$

The countability of the set $\mathbf{B_F}$ can be shown in the same manner than for the rational numbers. In Figure 2 the blue dots {m, n} representing the rational numbers $\frac{m}{n}$ are connected successively by red arrows making a path that puts all $\frac{m}{n}$ in a one-to-one correspondence with the natural numbers. Similarly in Figure 3, the blue diamonds representing the 4-or-less-digit numbers are connected by red arrows making a path that, connecting successively the cutting numbers of rank 1, 2, 3, …, ∞, puts all the fractional numbers in a one-to-one correspondence with the natural numbers. Thus, the cardinal number of the set $\mathbf{B_F}$ equals that of the natural numbers:

$$|\mathbf{B_F}| = \aleph_0 \tag{3}$$

As equation 2 shows that $|\mathbf{B_F}| = 2^{\aleph_0}$, we have:

$$2^{\aleph_0} = \aleph_0 \tag{4}$$

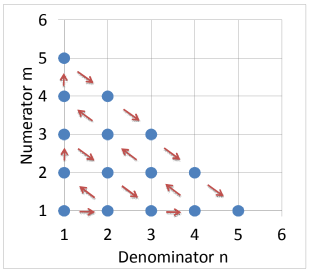

Figure 2

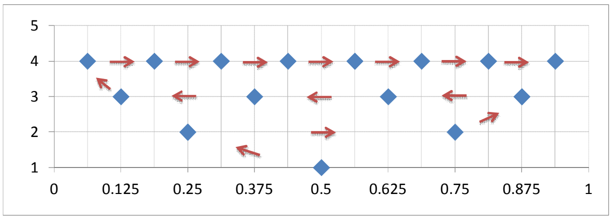

Figure 3

Another way to see the countability of $\mathbf{B_F}$ is to list all the 4-or-less-digit numbers in decimal fraction. Then we see that the numerators constitute the sequence of natural numbers from 0 to 2^4:

$$\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} \tag{5}$$

In the same way, the n-or-less-digit numbers constitute the sequence $\frac{0}{2^n}, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \cdots, \frac{2^n}{2^n}$, the numerators of this sequence are 0,1, 2, 3, … , n. When n=$\aleph_0$, this sequence equals the set $\mathbf{B_F}$ and the numerators are 0, 1, 2, 3, … , ∞. So, each binary number of $\mathbf{B_F}$ corresponds to a unique natural number and thus, $\mathbf{B_F}$ is countable. Also, as all members of $\mathbf{B_F}$ are rational numbers, $\mathbf{B_F}$ is a subset of the rational numbers.

4. Set of all binary numbers, B

A general n-digit binary number is a sequence of n 0 or 1 with a separation point. The sequence has 2^n possibility, the point has n+1 possibility of locations. So, n-digit binary numbers can take $2^n \times (n+1)$ values and the number of elements of the set of n-digit binary numbers equals $2^n \times (n+1)$. When n=$\aleph_0$ we get the cardinal number of **B** (see equation **4**):

$$|\mathbf{B}| = 2^{\aleph_0} \times (\aleph_0 + 1) = \aleph_0 \times \aleph_0 = \aleph_0 \tag{6}$$

5. On Cantor's diagonal argument

In some sense, Cantor's diagonal argument consists of creating a new binary number from a complete list of the fractional binary numbers and showing that the new number cannot be in the list. This seems to prove that $\mathbf{B_F}$ has more members than the natural numbers. However, the above work shows that $\mathbf{B_F}$ is countable. So, this argument must be wrong.

But where is the flaw? When a new guest arrives at Hilbert's Grand Hotel which is full, David Hilbert is able to find him a room because the number of room is infinite. The diagonal number created by Cantor's diagonal argument is the "new guest" in the "hotel" of the list which has infinite members. So, it can be fitted in the list. Concretely, create a diagonal number with the first n numbers of the list and let n go to infinity. The diagonal number is a binary number and thus, a member of $\mathbf{B_F}$. As $\mathbf{B_F}$ is completely listed, the diagonal number is in the list. Then, Cantor's diagonal argument does not reach contradiction.

From another angle of view, the last element of the list must be used to create the diagonal number. As the last element does not exist, the diagonal number cannot be created. Without diagonal number, Cantor's diagonal argument does not reach contradiction.

Cantor's diagonal argument is frequently used to prove the uncountability of the real numbers. Since it is wrong, the uncountability of the real numbers relies on Cantor's first uncountability proof.

6. On Cantor's theorem

Cantor's theorem of set theory states that, for any set A, the power set of A has a strictly greater cardinality than A itself. The cardinal number of the power set of natural numbers is $2^{\aleph_0}$. But according equation 4 we have $2^{\aleph_0} = \aleph_0$. So, the cardinality of the power set of natural numbers actually equals the cardinality of the natural numbers but is not greater. In consequence, Cantor's theorem does not hold for infinite sets. As the real numbers is uncountable, it cannot be created using the power set of natural numbers.

7. On infinite digital expansion of irrational number

Georg Cantor made these errors because he thought that a number with infinite many non-repeating digits is irrational. This is wrong because infinity is not a definite value but the process of the forever increase of value. A number with infinite many non-repeating digits is not a definite number but a converging rational number whose digits expand indefinitely without repeating. Take the n-digit decimal expansion of the irrational number π:

$$\pi_n = \frac{31415926535 \ldots \ldots \ldots \ldots \ldots (\text{n} + 1 \text{ digits})}{10^n} \qquad 7$$

π is irrational because it is the limit of the sequence π_n when $n \rightarrow \infty$, but π_n is never irrational. π and π_n illustrate the fundamental difference between an irrational number and its digital expansion. Because a converging function never equals its limit, the infinite digital expansion of an irrational number is not the irrational number but a rational number. In consequence, the set of all digital numbers is a subset of the rational numbers and its cardinality is $\aleph_0$. It is not the continuum of real numbers.

Remark:

- This is true not only for binary or decimal numbers, but for numbers in all numeral systems.
- The power set of natural numbers is equivalent to the set of all digital numbers. Thus, its cardinality is $\aleph_0$, confirming that Cantor's theorem does not hold for infinite sets.

8. On the continuum hypothesis

As $2^{\aleph_0}$ and $\aleph_0$ are proven to be identical, the continuum hypothesis can no longer be interpreted as "there is no cardinal number strictly between $\aleph_0$ and $2^{\aleph_0}$", but "there is no set with cardinality strictly between $\aleph_0$ and $|\mathbb{R}|$, the cardinality of the continuum".

Since $|\mathbb{R}| \neq 2^{\aleph_0}$, is $|\mathbb{R}|$ still related to $\aleph_0$? I do not think so. In fact, in the section 2 "Fractional binary numbers on the real line" I have shown that binary numbers cannot fill the unit real interval in spite of infinity of digits. In general, the members of a discrete set can only occupy isolated points in a continuous space leaving the intervals empty. So, the fundamental difference between a continuum and a discrete set is the continuity, not the number of elements.

A set whose members are in contact with one another is continuous. In the set of real numbers, an interval however small is always wider than the gap between 2 real numbers that are in contact. This property defines the continuity of a set. In the contrary, the set of rational numbers is discrete. Between 2 neighboring rational numbers, $\frac{i}{j}$ and $\frac{k}{l}$, the gap $\frac{kj-il}{jl}$ has a minimal value $\frac{1}{jl}$. However big jl is, there is always the number $\frac{i}{j}+\frac{1}{2jl}$ in the gap such that $\frac{i}{j}<\frac{i}{j}+\frac{1}{2jl}<\frac{k}{l}$. So, $\frac{i}{j}$ and $\frac{k}{l}$ are always disjoint while becoming infinitely close. This property defines the discreteness of a set.

The cardinality of an infinite discrete set is $\aleph_0$, that of a continuous set is $|\mathbb{R}|$. Because neighboring members of a set have no other possibility than being disjoint or in contact, the set has no other possibility than being discrete or continuous. So, the cardinality of a set must be $\aleph_0$ or $|\mathbb{R}|$, but not strictly between $\aleph_0$ and $|\mathbb{R}|$. In consequence, the continuum hypothesis is true.