# Computing orientation with complex multiplication but without trigonometric function

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30 May 2022

**Abstract**: Today's methods for computing orientation are quaternion and rotation matrix. However, their efficiencies are tarnished by the complexity of the rotation matrix and the counterintuitivity of quaternion. A better method is presented here. It uses complex multiplication for rotating vectors in 3D space and can compute orientation without angle and trigonometric functions, which is simple, intuitive and fast.

## 1. Basic orientation

A rigid body can rotate around 3 orthogonal axes in space, see Figure 1. The state of these 3 rotations is the orientation of this body. The commonly used orientation systems are Euler angles and Tait–Bryan angles[1]. Two methods are usually used to compute orientation: quaternion [2] and rotation matrix [3]. But they have their drawbacks. For quaternion, the computation for rotating a vector **p** needs to multiply **p** on the left by the rotation vector **q** and on the right by its conjugate $\mathbf{q}^{-1}$, which implies two 4D multiplications with **p** in between, see equation (1). The weird thing is that the rotation vector **q** is not computed with the rotation angle θ but its half θ/2, see (2), [4]. The half angle and the "sandwich multiplication" make the quaternion method counterintuitive.

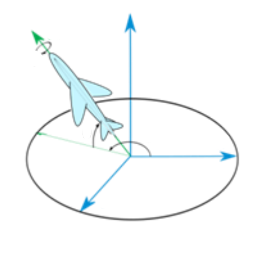

Figure 1

By Juansempere [9][1]

| | |
|---|---|
| $\boldsymbol{p}' = \boldsymbol{q}\boldsymbol{p}\boldsymbol{q}^{-1}$ | (1) |
| $\boldsymbol{q} = e^{\frac{\theta}{2}(u_x\boldsymbol{i}+u_y\boldsymbol{j}+u_z\boldsymbol{k})}$ | (2) |

$$\begin{bmatrix} \cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3 & \sin\theta_1\cos\theta_3-\cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & \sin\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3 & -\cos\theta_1\sin\theta_2\cos\theta_3-\sin\theta_1\sin\theta_3 & \cos\theta_2\cos\theta_3 \end{bmatrix} \quad (3)$$

For rotation matrix, the 3 angles of orientation are mingled in the 9 elements of the matrix where one gets easily lost. For example, in «Step by step rotation in normal and high dimensional space and meaning of quaternion»[5], the transformation matrix is equation (3) which is confusing with the messed trigonometric functions.

Can we find a better method? In fact, I have constructed a 3D complex number system in «Extending complex number to spaces with 3, 4 or any number of dimensions» [6] which computes easily the rotation of a vector in 3D space. The combination of this system with the step by step rotation described in «Step by step rotation in normal and high dimensional space and meaning of quaternion»[5] gives birth to a new method. This method can use directly the coordinates of points to rotate an object without using trigonometric function. We will explain first this method that uses complex multiplication.

## 2. Rotation using complex multiplication

### a. Complex multiplication

Complex multiplication is very practical to compute rotation in a plane. In Figure 2 a smiling face is presented on the complex plane. The axes of this plane are h and i, with h being the real number 1 and i the imaginary unit.

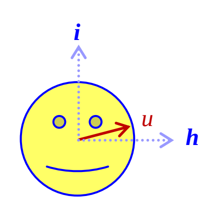


Figure 2

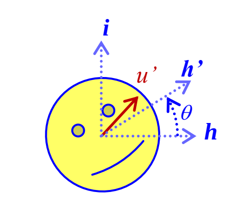


Figure 3

The complex number u expressed in (4) will be rotated by the angle θ. For doing so, we write the unit complex number $e^{i\theta}$ in (5), multiply u with $e^{i\theta}$ and obtain the complex product $ue^{i\theta}$ in (6). Equation (6) shows that $ue^{i\theta}$ equals u rotated by the angle θ. $ue^{i\theta}$ is label as u' in (6) and in Figure 3. Graphically u and the axis h are both rotated by the angle θ and become u' and h' in Figure 3.

| | |
|---|---|
| $u = r_u(\cos\theta_u + \sin\theta_u\, i)$ | (4) |
| $e^{i\theta} = \cos\theta + \sin\theta\, i$ | (5) |

For rotating the smiling face as a whole, we just multiply all the points of the smiling face with $e^{i\theta}$. Then the resulting smiling face is rotated, see Figure 3.

$$\begin{aligned} ue^{i\theta} &= r_u(\cos\theta_u + \sin\theta_u\, i)(\cos\theta + \sin\theta\, i) \\ &= r_u(\cos(\theta_u+\theta) + \sin(\theta_u+\theta)\, i) \\ &= u' \end{aligned} \quad (6)$$

As it is so easy to compute the rotation of vectors in a complex plane, we will use complex multiplication to compute rotation for orientation. But we will multiply 3D vectors with 2D complex numbers because the objects we rotate are in 3D space.

## b. Mixed multiplication

We notice that when a 3D vector rotates around an axis that is perpendicular to it, the vector rotates in a plane and we call it the rotation plane. For using 2D complex multiplication in 3D space, we consider the rotation plane as a complex plane and use the 2D complex multiplication to rotate a vector in this plane.

Let (**e**$_1$, **e**$_2$) be a plane in 3D space, **e**$_1$ and **e**$_2$ the base vectors of the plane, so they 3D vectors. We make this plane equivalent to the complex plane using equation (7), that is, **e**$_1$ corresponds to the real axis and **e**$_2$ to the imaginary axis. **u** is a vector in the plane and is expressed in (8), with a and b being its components**.** Because the plane (**e**$_1$, **e**$_2$) is equivalent to the complex plane, **u** has an equivalent complex number u, which is expressed in (9). Let v be an other complex number which is expressed in (10).

| | |
|---|---|
| $\boldsymbol{e}_1 \to 1 \quad \boldsymbol{e}_2 \to i$ | (7) |
| $\boldsymbol{u} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ | (8) |
| $u = a + bi$ | (9) |
| $v = c + di$ | (10) |

We multiply u with v and the complex product uv is given in (11). The real and imaginary parts of uv are written in (12). Because the complex plane is equivalent to the plane (**e**$_1$, **e**$_2$), we replace the 1 and i that are in (11) with **e**$_1$ and **e**$_2$ and obtain **u**·v in (13) which is a vector in the plane (**e**$_1$, **e**$_2$).

| | |
|---|---|
| $uv = (a+bi)(c+di) = (ac-bd)\cdot 1 + (ad+cb)\cdot i$ | (11) |
| $real(uv) = ac - bd \quad imag(uv) = ad + cb$ | (12) |
| $\boldsymbol{u}\cdot v = (ac-bd)\boldsymbol{e}_1 + (ad+cb)\boldsymbol{e}_2 = real(uv)\boldsymbol{e}_1 + imag(uv)\boldsymbol{e}_2$ | (13) |

So, we have created a new type of multiplication: the vector **u** multiplied by the complex number v. The vector **u** is a 3D vector because **e**$_1$ and **e**$_2$ are 3D vectors. The product of this multiplication is **u**·v which is a 3D vector too. We call this multiplication "mixed multiplication" and the product "mixed product". We state the Definition 1.

Definition 1: Mixed multiplication and mixed product

The plane (**e**$_1$, **e**$_2$) is equivalent to the complex plane. **u** is a vector in this plane and equals a**e**$_1$+b**e**$_2$. u is a complex number and equals a+bi. v is an other complex number. **u** is multiplied by v and the result of this multiplication is denoted as **u**·v and equals real(uv)**e**$_1$+ imag (uv)**e**$_2$, with real(uv) and imag (uv) being the real and imaginary parts of the complex product uv. **u**·v is a vector in the plane (**e**$_1$, **e**$_2$). This multiplication is called the mixed multiplication and **u**·v the mixed product.

Let us use the newly defined mixed multiplication to rotate the vector **u** given in (14). The complex number u given in (4) is the complex equivalent of **u** and the complex number v is the $e^{i\theta}$ given in (5). The complex product uv is computed in (6). Then the mixed product **u**·v is given in (15) whose components equal the real and imaginary parts in (6). So, **u**·v is well the vector **u** rotated by the angle θ. We see that mixed multiplication is a very easy way to compute the rotation of a 3D vector in the rotation plane.

| | |
|---|---|
| $\boldsymbol{u} = r_u(\cos\theta_u\, \boldsymbol{e}_1 + \sin\theta_u\, \boldsymbol{e}_2)$ | (14) |
| $\boldsymbol{u}e^{i\theta} = r_u(\cos(\theta_u+\theta)\, \boldsymbol{e}_1 + \sin(\theta_u+\theta)\, \boldsymbol{e}_2)$ | (15) |

# 3. Reference frames

## a. Ground frame and proper frame

An object is oriented in space with respect to a fixed frame of reference which we call the ground frame. For example, the object in Figure 4 is a brick on the floor and the base vectors of the ground frame are **g**$_x$, **g**$_y$ and **g**$_z$, which are represented by the 3 orthogonal black arrows.

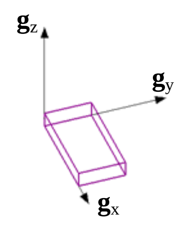


Figure 4
The base vectors of the ground frame **g**$_x$, **g**$_y$ and **g**$_z$

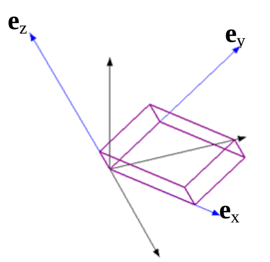


Figure 5
The base vectors of the proper frame **e**$_x$, **e**$_y$ and **e**$_z$

The position of a point in an object is determined with respect to a frame attached to the object, which we call the proper frame. In Figure 5 the blue arrows denoted by **e**$_x$, **e**$_y$ and **e**$_z$ are the base vectors of the proper frame which are parallel to the edges of the brick. So, the orientation of the object is that of the proper frame. Because vector can rotate but not a point, we will refer to a point of the object with the position vector that points to it when convenient.

An object has a main direction like the direction of an airplane. The proper frame is so positioned with respect to the object that **e**$_x$ indicates its main direction. **e**$_x$ is called the direction vector of the object.

## b. Direction frame

The orientation of an object can be defined in different ways. As we use the 3D complex number system of «Extending complex number to spaces with 3, 4 or any number of dimensions»[6], we will use its angle system for orienting an object.

First we make the proper frame of the object coincide with the ground frame, that is, their base vectors are equal: ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) = ($\mathbf{g}_x$, $\mathbf{g}_y$, $\mathbf{g}_z$). Then the proper frame will be rotated two times. The first rotation is around the vector $\mathbf{e}_z$, so $\mathbf{e}_z$ stays the same while $\mathbf{e}_x$ and $\mathbf{e}_y$ are rotated by the angle θ. After the rotation $\mathbf{e}_x$ becomes the vector $\mathbf{p}$ and $\mathbf{e}_y$ becomes the vector $\mathbf{d}_y$, both are in the horizontal plane ($\mathbf{g}_x$, $\mathbf{g}_y$), see Figure 6.

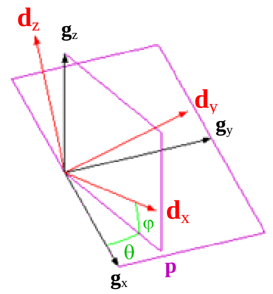


Figure 6
Rotations in the horizontal and vertical planes.

Then the proper frame rotates around the vector $\mathbf{d}_y$ which is $\mathbf{e}_y$ of the moment. So, $\mathbf{e}_y$ stays the same while the vectors $\mathbf{p}$ and $\mathbf{e}_z$ are rotated by the angle φ in the vertical plane ($\mathbf{p}$, $\mathbf{g}_z$). After the rotation, $\mathbf{p}$ becomes the vector $\mathbf{d}_x$ and $\mathbf{e}_z$ becomes the vector $\mathbf{d}_z$, see Figure 6.

Now we have a new reference frame whose base vectors are the 3 red arrows labeled as $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$, see Figure 6. This frame is also the proper frame after the 2 rotations. So, ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) equal ($\mathbf{d}_x$, $\mathbf{d}_y$, $\mathbf{d}_z$) in the present position. The vectors $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ have the following properties:

1) $\mathbf{d}_x = \mathbf{e}_x$, so $\mathbf{d}_x$ is the direction vector of the object.
2) $\mathbf{d}_y = \mathbf{e}_y$ and is in the horizontal plane ($\mathbf{g}_x$, $\mathbf{g}_y$).
3) $\mathbf{d}_z = \mathbf{e}_z$ and is in the vertical plane ($\mathbf{p}$, $\mathbf{g}_z$).

Because $\mathbf{d}_x$ is the direction vector we call this frame the direction frame. The base vectors of the three principal frames are listed in Table 1.

| Proper frame | Direction frame | Ground frame |
|---|---|---|
| $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ | $\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_z$ | $\mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z$ |

**Table 1**

# 4. Base vectors of the direction frame

## a. Rotation around the z axis

Using the mixed multiplication we can easily compute the base vectors of the direction frame which are rotated twice in the ground frame, see Section 3.b. During the first rotation, $\mathbf{e}_x$ and $\mathbf{e}_y$ are rotated by the angle θ around the z axis which is perpendicular to them. So, $\mathbf{e}_x$ and $\mathbf{e}_y$ stay in the plane ($\mathbf{g}_x$, $\mathbf{g}_y$), see Figure 7.

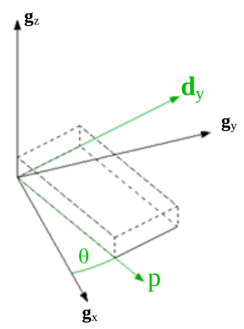


Figure 7
Rotation of the base vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ in the plane ($\mathbf{g}_x$, $\mathbf{g}_y$)

We make the plane ($\mathbf{g}_x$, $\mathbf{g}_y$) equivalent to the complex plane by equation (16) and we write the rotation complex number $e^{i\theta}$ in (17). $\mathbf{e}_x$ equals $\mathbf{g}_x$ before the rotation, so after the rotation the vector $\mathbf{e}_x$ equals the mixed product $\mathbf{g}_x \cdot e^{i\theta}$ which we name $\mathbf{p}$, see (18).

For the vector $\mathbf{e}_y$ we know that $\mathbf{e}_y$ is perpendicular to $\mathbf{e}_x$, which is now the vector $\mathbf{p}$. So, after the rotation $\mathbf{e}_y$ equals the mixed product $\mathbf{p}\cdot$i. Because after the rotation $\mathbf{e}_y$ equals $\mathbf{d}_y$, see Figure 7, $\mathbf{d}_y$ is expressed instead of $\mathbf{e}_y$ in (19).

| | |
|---|---|
| $\mathbf{g}_x \to 1,\ \mathbf{g}_y \to \mathrm{i}$ | (16) |
| $e^{i\theta} = \cos\theta + \sin\theta\, i$ | (17) |
| $\boldsymbol{p} = \boldsymbol{g}_x \cdot e^{i\theta} = \cos\theta\, \boldsymbol{g}_x + \sin\theta\, \boldsymbol{g}_y$ | (18) |
| $\boldsymbol{d}_y = \boldsymbol{p} \cdot i = -\sin\theta\, \boldsymbol{g}_x + \cos\theta\, \boldsymbol{g}_y$ | (19) |

## b. Rotation around the y axis

During the second rotation $\mathbf{e}_x$ and $\mathbf{e}_z$ rotate around $\mathbf{d}_y$ which is perpendicular to them, see Figure 8. So, the rotation plane is ($\mathbf{p}$, $\mathbf{g}_z$). Before this rotation the vector $\mathbf{e}_x$ equals $\mathbf{p}$ and $\mathbf{e}_z$ equals $\mathbf{g}_z$.

| | |
|---|---|
| $\mathbf{p} \to 1,\ \mathbf{g}_z \to \mathrm{i}$ | (20) |
| $e^{i\varphi} = \cos\varphi + \sin\varphi\, i$ | (21) |
| $\boldsymbol{d}_x = \boldsymbol{p} \cdot e^{i\varphi} = \cos\varphi\, \boldsymbol{p} + \sin\varphi\, \boldsymbol{g}_z$ | (22) |
| $\boldsymbol{d}_z = \boldsymbol{d}_x \cdot i = -\sin\varphi\, \boldsymbol{p} + \cos\varphi\, \boldsymbol{g}_z$ | (23) |

Using the mixed multiplication the plane ($\mathbf{p}$, $\mathbf{g}_z$) is made equivalent to the complex plane by equation (20). The angle of rotation is φ and we write the rotation complex number $e^{i\varphi}$ in (21).

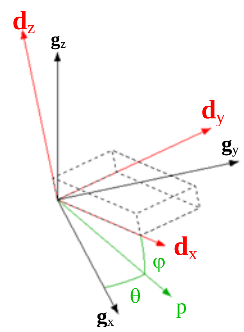


Figure 8
Rotation of the base vectors $\mathbf{e}_x$ and $\mathbf{e}_z$ in the plane ($\mathbf{p}$, $\mathbf{g}_z$)

After the rotation $\mathbf{e}_x$ equals the mixed product $\mathbf{p}\cdot e^{i\varphi}$ and equals $\mathbf{d}_x$, so $\mathbf{d}_x$ is expressed instead of $\mathbf{e}_x$ in (22), see Figure 8. For $\mathbf{e}_z$ we know that $\mathbf{e}_z$ is perpendicular to $\mathbf{e}_x$ which is now $\mathbf{d}_x$. So, after the rotation $\mathbf{e}_z$ equals the mixed product $\mathbf{d}_x\cdot$i. Because $\mathbf{e}_z$ equals $\mathbf{d}_z$ after this rotation $\mathbf{d}_z$ is expressed instead of $\mathbf{e}_z$ in (23), see Figure 8.

Because the direction frame is the rotated proper frame, we have $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = (\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_z)$, see Figure 8.

### c. The 3 axes of the direction frame

The expressions of $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ in (19), (22) and (23) are recollected in (24). But the vector $\mathbf{p}$ which is in (24) is not one of the vectors $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$. For eliminating $\mathbf{p}$ we replace it with its expression given in (18) and obtain (25). Then $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ are expressed with $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$ in (25).

$$\begin{cases} \boldsymbol{d_x} = & \cos\varphi\,\boldsymbol{p} + \sin\varphi\,\boldsymbol{g_z} \\ \boldsymbol{d_y} = & -\sin\theta\,\boldsymbol{g_x} + \cos\theta\,\boldsymbol{g_y} \\ \boldsymbol{d_z} = & -\sin\varphi\,\boldsymbol{p} + \cos\varphi\,\boldsymbol{g_z} \end{cases} \qquad (24)$$

### d. Direction frame and 3D complex number

$$\begin{cases} \boldsymbol{d_x} = & (\cos\theta\,\boldsymbol{g_x} + \sin\theta\,\boldsymbol{g_y})\cos\varphi + \sin\varphi\,\boldsymbol{g_z} \\ \boldsymbol{d_y} = & -\sin\theta\,\boldsymbol{g_x} + \cos\theta\,\boldsymbol{g_y} \\ \boldsymbol{d_z} = & -(\cos\theta\,\boldsymbol{g_x} + \sin\theta\,\boldsymbol{g_y})\sin\varphi + \cos\varphi\,\boldsymbol{g_z} \end{cases} \qquad (25)$$

$$\boldsymbol{v} = (\cos\theta\,\boldsymbol{h} + \sin\theta\,\boldsymbol{i})\cos\varphi + \sin\varphi\,\boldsymbol{j} \qquad (26)$$

The vector $\mathbf{d}_x$ can point to any direction in space because the angles θ and φ are free variables. So, $\mathbf{d}_x$ expresses the unit sphere in the ground frame and is the vector equivalent of a unit 3D complex number. Indeed the expression of $\mathbf{d}_x$ in (25) is exactly that of the unit 3D complex number I have defined in «Extending complex number to spaces with 3, 4 or any number of dimensions» [6], which is written in (26), with **h**, **i** and **j** corresponding to $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$. For 3D complex number, **h** is the real number 1, **i** and **j** the first and second imaginary units. So, we have given here a demonstration of the use of the 3D complex number system.

## 5. Roll, Pitch and Yaw

### a. Roll

Once the direction of the object is determined, it can still rotate around the vector $\mathbf{d}_x$ and this rotation is called roll, [7]. Figure 9 shows the roll of a brick. The brick in dashed line is parallel to the direction frame while the pink brick is tilted because rolled. The angle of the roll is measured between $\mathbf{d}_y$ and $\mathbf{e}_y$ and labeled as ψ.

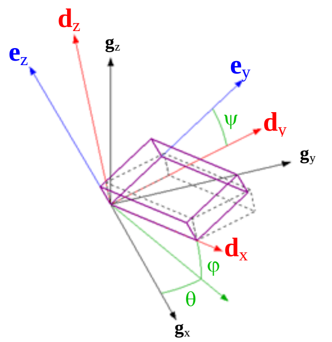


Figure 9
Rotation of the base vectors $\mathbf{e}_y$ and $\mathbf{e}_z$ in the plane ($\mathbf{d}_y$, $\mathbf{d}_z$)

We use the mixed multiplication to compute the roll of the vectors $\mathbf{e}_y$ and $\mathbf{e}_z$ around $\mathbf{d}_x$ which is perpendicular to them. So, the rotation plane ($\mathbf{d}_y$, $\mathbf{d}_z$) is made equivalent to the complex plane by equation (27). We write the rotation complex number $e^{i\psi}$ in (28). After the rotation $\mathbf{e}_y$ equals the mixed product $\mathbf{d}_y\cdot e^{i\psi}$, see (29). For $\mathbf{e}_z$ we know that $\mathbf{e}_z$ is perpendicular to $\mathbf{e}_y$, so $\mathbf{e}_z$ equals the mixed product $\mathbf{e}_y\cdot$i and is expressed in (30), see Figure 9.

| | |
|---|---|
| $\mathbf{d}_y \to 1,\ \mathbf{d}_z \to$ i | (27) |
| $e^{i\psi} = \cos\psi + \sin\psi\, i$ | (28) |
| $\boldsymbol{e_y} = \boldsymbol{d_y}\cdot e^{i\psi} = \cos\psi\,\boldsymbol{d_y} + \sin\psi\,\boldsymbol{d_z}$ | (29) |
| $\boldsymbol{e_z} = \boldsymbol{e_y}\cdot i = -\sin\psi\,\boldsymbol{d_y} + \cos\psi\,\boldsymbol{d_z}$ | (30) |

### b. Pitch and Yaw

In flight dynamics orientation has 3 angles: pitch, roll and yaw, [7]. The roll angle is the ψ explained above. But what are the angles of pitch and yaw in our system? Let us see the stylized airplane in Figure 10 whose 3 angles of orientation are: θ, φ and ψ.

θ and φ are the angles of rotations of the proper frame around the z and y axes. The blue arrows in Figure 10 represent the base vectors of the proper frame $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. By comparing these vectors with the axes of pitch, roll and yaw for Flight dynamics [7], we find that the unit vectors of pitch and yaw axes are in fact the vectors $-\mathbf{e}_y$ and $-\mathbf{e}_z$. So, the pitch and yaw angles are the angles φ and -θ for the direction frame. In summary, the angles of pitch, roll and yaw are the angles φ, ψ and -θ in our system.

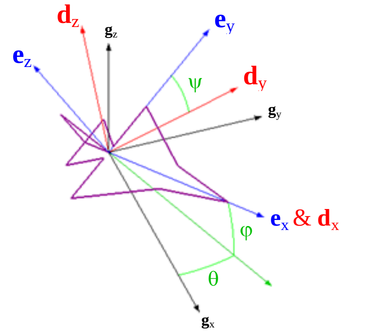


Figure 10

## 6. Determination of the angles

### a. Direction angles

Apart from airplane and boat, orientation is used in many other areas, for example, robotic arms that put tool at precise position, video games that make characters move on screen. Sometimes the 3 orientation angles θ, φ and ψ are unknown. In this case we have to derive them from the coordinates of some particular points which are known.

For doing so, let us determine the direction vector of an object from a particular point that indicates the direction of the object. If the object is an airplane, this point will be its head and the arrow going from the origin of its proper frame to its head will be the vector of the direction, which we label by **D**. We suppose that the origin of the proper frame coincides with that of the ground frame.

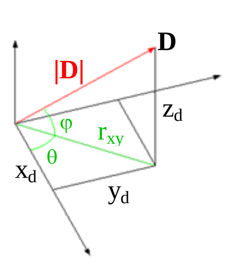

Figure 11

Let ($x_d$, $y_d$, $z_d$) be the coordinates of this point in the ground frame and the vector **D** is expressed in (31). We divide **D** with its modulus |**D**| and obtain the unit vector **D**/|**D**|. Because **D** indicates the direction of the object, **D**/|**D**| equals the direction vector $\mathbf{d}_x$ of the direction frame, see (32). We rewrite in (33) the expression of $\mathbf{d}_x$ which was given in (25).

$$\boldsymbol{D} = x_d\boldsymbol{g}_x + y_d\boldsymbol{g}_y + z_d\boldsymbol{g}_z \tag{31}$$

$$\frac{\boldsymbol{D}}{|\boldsymbol{D}|} = \boldsymbol{d}_x \tag{32}$$

$$\boldsymbol{d}_x = (\cos\theta\,\boldsymbol{g}_x + \sin\theta\,\boldsymbol{g}_y)\cos\varphi + \sin\varphi\,\boldsymbol{g}_z \tag{33}$$

For computing the angles θ and φ, let us see Figure 11. The components $x_d$, $y_d$ and $z_d$ of the vector **D** will give the value of cosθ, sinθ, cosφ and sinφ. We first compute the hypotenuse $r_{xy}$ and |**D**| in (34). The value of cosθ, sinθ, cosφ and sinφ are computed with $x_d$, $y_d$ and $z_d$, $r_{xy}$ and |**D**| in (35).

The angle φ is directly solved in (36). For θ we have to decide whether it is positive or negative. So, we impose that cosφ is always positive, then the sign of θ equals that of $y_d$ and θ is solved in (37).

$$r_{xy} = \sqrt{x_d^2 + y_d^2} \qquad |\boldsymbol{D}| = \sqrt{x_d^2 + y_d^2 + z_d^2} \tag{34}$$

$$\cos\theta = \frac{x_d}{r_{xy}} \quad \sin\theta = \frac{y_d}{r_{xy}} \qquad \cos\varphi = \frac{r_{xy}}{|\boldsymbol{D}|} \quad \sin\varphi = \frac{z_d}{|\boldsymbol{D}|} \tag{35}$$

$$\varphi = \sin^{-1}\left(\frac{z_d}{|\boldsymbol{D}|}\right) \tag{36}$$

$$y_d > 0 \quad \theta = \cos^{-1}\left(\frac{x_d}{r_{xy}}\right) \qquad y_d < 0 \quad \theta = -\cos^{-1}\left(\frac{x_d}{r_{xy}}\right) \tag{37}$$

## b. Roll angle

For determining the roll angle ψ, we use the coordinates of an other particular point which indicates the roll. In the case of an airplane this point could be the left wingtip. We label this point by **S** and express it in (38), with $x_s$, $y_s$ and $z_s$ being its coordinates in the ground frame.

Because roll is a rotation around $\mathbf{d}_x$, the component of **S** on $\mathbf{d}_x$ stays constant during the roll. So, the roll is determined by $\mathbf{S}_{yz}$ which is the projection of **S** on the plane ($\mathbf{d}_y$, $\mathbf{d}_z$), see Figure 12. Since the vector $\mathbf{S}_{yz}$ determines the roll, it must be parallel to the vector $\mathbf{e}_y$ of the proper frame.

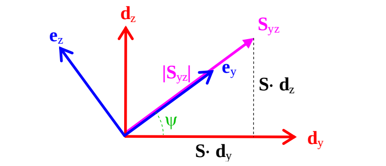

Figure 12

$$\boldsymbol{S} = x_s\boldsymbol{g}_x + y_s\boldsymbol{g}_y + z_s\boldsymbol{g}_z \tag{38}$$

$$\boldsymbol{S}_{yz} = (\boldsymbol{S}\cdot\boldsymbol{d}_y)\boldsymbol{d}_y + (\boldsymbol{S}\cdot\boldsymbol{d}_z)\boldsymbol{d}_z \tag{39}$$

The components of $\mathbf{S}_{yz}$ on $\mathbf{d}_y$ equals the dot product $\mathbf{S}\cdot\mathbf{d}_y$ and that on $\mathbf{d}_z$ equals $\mathbf{S}\cdot\mathbf{d}_z$, see (39). We divide $\mathbf{S}_{yz}$ with its modulus |$\mathbf{S}_{yz}$| and obtain the unit vector $\mathbf{S}_{yz}$/|$\mathbf{S}_{yz}$|.

$$\frac{\boldsymbol{S}_{yz}}{|\boldsymbol{S}_{yz}|} = \boldsymbol{e}_y = \cos\psi\,\boldsymbol{d}_y + \sin\psi\,\boldsymbol{d}_z \tag{40}$$

$$\cos\psi = \frac{\boldsymbol{S}\cdot\boldsymbol{d}_y}{|\boldsymbol{S}_{yz}|} \quad \sin\psi = \frac{\boldsymbol{S}\cdot\boldsymbol{d}_z}{|\boldsymbol{S}_{yz}|} \tag{41}$$

$$\boldsymbol{S}\cdot\boldsymbol{d}_z > 0 \quad \psi = \cos^{-1}\left(\frac{\boldsymbol{S}\cdot\boldsymbol{d}_y}{|\boldsymbol{S}_{yz}|}\right) \qquad \boldsymbol{S}\cdot\boldsymbol{d}_z < 0 \quad \psi = -\cos^{-1}\left(\frac{\boldsymbol{S}\cdot\boldsymbol{d}_y}{|\boldsymbol{S}_{yz}|}\right) \tag{42}$$

Because $\mathbf{S}_{yz}$ is parallel to $\mathbf{e}_y$, the unit vector $\mathbf{S}_{yz}$/|$\mathbf{S}_{yz}$| equals $\mathbf{e}_y$. So, $\mathbf{S}_{yz}$/|$\mathbf{S}_{yz}$| equals the expression of $\mathbf{e}_y$ which was given in (29), see (40). Then the expressions of cosψ and sinψ are extracted from (40) and written in (41). The cosψ is reversed in (42) to solve for the roll angle ψ.

## c. Direction angles without trigonometric functions

Trigonometric functions take more computer time and annoy some people. We can get rid of them from the formulae and instead work only with the coordinates of some particular points of the object. For doing so, let us denote cosθ and sinθ with the letters a and b, cosφ and sinφ with the letters d and f, see (43).

Combining (35) with (43) we express the numbers a, b, d and f in (44)

$$\cos\theta = a \quad \sin\theta = b \qquad \cos\varphi = d \quad \sin\varphi = f \tag{43}$$

$$a = \frac{x}{r_{xy}} \quad b = \frac{y}{r_{xy}} \qquad d = \frac{r_{xy}}{|\boldsymbol{D}|} \quad f = \frac{z}{|\boldsymbol{D}|} \tag{44}$$

$$\begin{cases} \boldsymbol{d}_x = & (a\boldsymbol{g}_x + b\boldsymbol{g}_y)d + f\boldsymbol{g}_z \\ \boldsymbol{d}_y = & -b\boldsymbol{g}_x + a\boldsymbol{g}_y \\ \boldsymbol{d}_z = & -(a\boldsymbol{g}_x + b\boldsymbol{g}_y)f + d\boldsymbol{g}_z \end{cases} \tag{45}$$

with $x_d$, $y_d$ and $z_d$ the components of **D**. By replacing the cosθ, sinθ, cosφ and sinφ that are in (25) with a, b, d and f, we express $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ with a, b, d and f in (45) which do not contain trigonometric function.

$$g = \cos\psi \quad h = \sin\psi \tag{46}$$

$$\boldsymbol{e}_y = \boldsymbol{d}_y \cdot e^{i\psi} = g\boldsymbol{d}_y + h\boldsymbol{d}_z \tag{47}$$

$$\boldsymbol{e}_z = \boldsymbol{e}_y \cdot i = -h\boldsymbol{d}_y + g\boldsymbol{d}_z \tag{48}$$

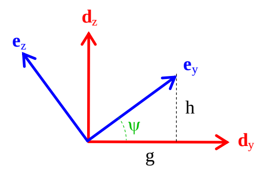

Figure 13

### d. Roll angle without trigonometric functions

The vectors $\mathbf{e}_y$ and $\mathbf{e}_z$ are expressed with cosψ and sinψ in (29) and (30). For getting rid of the trigonometric functions, we denote cosψ and sinψ by the letters g and h and replace them with g and h in (29) and (30). Then $\mathbf{e}_y$ and $\mathbf{e}_z$ are expressed with g and h in (47) and (48) in which there is no trigonometric function.

This change of symbol is not just for form, but has fundamental meaning. In Figure 13 we see that g and h are in fact the coordinates of the vector $\mathbf{e}_y$ in the plane ($\mathbf{d}_y$, $\mathbf{d}_z$). So, using g and h instead of cosψ and sinψ means that this rotation is not computed with the angle ψ but with the components of the vector $\mathbf{e}_y$. This is also true for replacing cosθ, sinθ, cosφ and sinφ with the number a, b, d and f. The change of symbol means that orientation can be computed directly with the coordinates of the points **D** and **S** which are ($x_d$, $y_d$, $z_d$) and ($x_s$, $y_s$, $z_s$), see (31) and (38).

## 7. Computation for oriented points

### a. Position of one point

$$\boldsymbol{v} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z \tag{49}$$

$$\boldsymbol{e}_x = \boldsymbol{d}_x \tag{50}$$

$$\begin{cases} \boldsymbol{e}_x = \boldsymbol{d}_x \\ \boldsymbol{e}_y = \cos\psi\,\boldsymbol{d}_y + \sin\psi\,\boldsymbol{d}_z \\ \boldsymbol{e}_z = -\sin\psi\,\boldsymbol{d}_y + \cos\psi\,\boldsymbol{d}_z \end{cases} \tag{51}$$

For a virtual object on screen, all the points of the object should be computed. Let us take the point indicated by the position vector **v** which is expressed in (49), with x, y and z being its components in the proper frame.

We will compute its coordinates in the ground frame which are labeled as ($x_g$, $y_g$, $z_g$). First we have to express the base vectors of the proper frame $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ in the direction frame. We know that $\mathbf{e}_x$ equals $\mathbf{d}_x$, see (50). The expressions of $\mathbf{e}_y$ and $\mathbf{e}_z$ are given in (29) and (30). Putting (50), (29) and (30) together gives the expressions of $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ in (51). We replace $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ that are in (49) with the expressions in (51) and obtain the vector $\mathbf{v}_d$ in (52).

$$\boldsymbol{v}_d = x\boldsymbol{d}_x + y(\cos\psi\,\boldsymbol{d}_y + \sin\psi\,\boldsymbol{d}_z) + z(-\sin\psi\,\boldsymbol{d}_y + \cos\psi\,\boldsymbol{d}_z) \tag{52}$$

$$\boldsymbol{v}_g = x_g\boldsymbol{g}_x + y_g\boldsymbol{g}_y + z_g\boldsymbol{g}_z \tag{53}$$

The vector $\mathbf{v}_d$ is the vector **v** given in (49) but expressed in the direction frame. By replacing $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ that are in (52) with their expressions given in (25) we obtain the vector $\mathbf{v}_g$ in (53), with $\mathbf{v}_g$ being **v** expressed in the ground frame. But the components $x_g$, $y_g$ and $z_g$ are not developed because they are uselessly complicated large formulae with plenty of trigonometric functions like those in the rotation matrix (3). In fact, since all rotations are computed using mixed multiplication, $x_g$, $y_g$ and $z_g$ can be easily computed using the built-in complex number functions in computer.

### b. Computation without trigonometric functions

For getting rid of trigonometric function, we have replaced cosθ, sinθ, cosφ, sinφ, cosψ and sinψ with the numbers a, b, d, f, g and h. The expressions of a, b, d and f are given in (35), that of g and h in (46). By replacing cosψ and sinψ with g and h in (52), we obtain $\mathbf{v}_d$ in (54) which is expressed with g and h and x, y and z. Then using (45) in (54), we obtain the expression of $\mathbf{v}_g$ in (55), with $x'_g$, $y'_g$ and $z'_g$ being its components in the ground frame. Although $x'_g$, $y'_g$ and $z'_g$ are not developed, we know for sure that they do not contain any trigonometric function because (45) does not contain any.

$$\boldsymbol{v}_d = x\boldsymbol{d}_x + y(g\boldsymbol{d}_y + h\boldsymbol{d}_z) + z(-h\boldsymbol{d}_y + g\boldsymbol{d}_z) \tag{54}$$

$$\boldsymbol{v}_g = x'_g\boldsymbol{g}_x + y'_g\boldsymbol{g}_y + z'_g\boldsymbol{g}_z \tag{55}$$

The advantages of using a, b, d, f, g and h instead of cosθ, sinθ, cosφ, sinφ, cosψ and sinψ are numerous. One of them is that we can get around the angles when they are not available. For example, for objects on screen the only available data are the coordinates of the points, no angle are known. When computing with trigonometric functions, we have to derive the angles from the coordinates first. In this case the process of computation for reorientation is:

1) Deriving the old angles from the old coordinates.
2) Reorienting the object with the new angles.
3) Computing the new cosθ, sinθ, cosφ, sinφ, cosψ and sinψ.
4) Computing the new coordinates for all points with the new cosθ, sinθ, cosφ, sinφ, cosψ and sinψ.

When using the numbers a, b, d, f, g and h, the process will be:

1) Computing the new numbers a, b, d, f, g and h from the new coordinates of two particular points.
2) Computing the coordinates for all the points with the new numbers a, b, d, f, g and h.

We see that the process using the numbers a, b, d, f, g and h is simpler and straightforward because the step with angles have disappeared. In comparison, rotation using quaternion needs to know the angle ψ but uses the half of its value and put the rotated vector **p** in sandwich between **q** and $\mathbf{q}^{-1}$, see (1), which is much less practical.

### c. Procedure of computation without trigonometric function

Case 1: The angles θ, φ and ψ are known. Compute a, b, d and f using (56) and go to step 2.

$$a = \cos\theta \quad b = \sin\theta \qquad d = \cos\varphi \quad f = \sin\varphi \tag{56}$$

Case 2: The angles θ, φ and ψ are unknown.

1. Determine a, b, d and f, using (57), see (34) and (44):

$$r_{xy} = \sqrt{x^2+y^2} \quad a = \frac{x}{r_{xy}} \quad b = \frac{y}{r_{xy}} \qquad |\boldsymbol{D}| = \sqrt{x^2+y^2+z^2} \quad d = \frac{r_{xy}}{|\boldsymbol{D}|} \quad f = \frac{z}{|\boldsymbol{D}|} \tag{57}$$

Expressing $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ with $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$:

2. Auxiliary vector **p** equals the mixed product $\mathbf{g}_x$·(a+bi):
   $\boldsymbol{p} = \boldsymbol{g}_x \cdot (a+bi) = a\boldsymbol{g}_x + b\boldsymbol{g}_y$ , see (18).
3. Vector $\mathbf{d}_y$ equals the mixed product **p**·i:
   $\boldsymbol{d}_y = \boldsymbol{p} \cdot i = -b\boldsymbol{g}_x + a\boldsymbol{g}_y$, see (19).
4. Vector $\mathbf{d}_x$ equals the mixed product **p**·(d+fi):
   $\boldsymbol{d}_x = \boldsymbol{p} \cdot (d+fi) = (a\boldsymbol{g}_x + b\boldsymbol{g}_y)d + f\boldsymbol{g}_z$ , see (22)
5. Vector $\mathbf{d}_z$ equals the mixed product $\mathbf{d}_x$·i, see (23).
   $\boldsymbol{d}_z = \boldsymbol{d}_x \cdot i = -(a\boldsymbol{g}_x + b\boldsymbol{g}_y)f + d\boldsymbol{g}_z$
6. $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ are expressed with $\mathbf{g}_x$, $\mathbf{g}_y$ and $\mathbf{g}_z$ in (45).

Expressing $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ with $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$:

7. Vector $\mathbf{e}_x$ equals $\mathbf{d}_x$, see (50).
8. Determine g and h using (58), see (28) and (41):

$$g = \cos\psi = \frac{\boldsymbol{S}\cdot\boldsymbol{d}_y}{|\boldsymbol{S}_{yz}|} \quad h = \sin\psi = \frac{\boldsymbol{S}\cdot\boldsymbol{d}_z}{|\boldsymbol{S}_{yz}|} \tag{58}$$

9. Vector $\mathbf{e}_y$ equals the mixed product $\mathbf{d}_y$·(g+hi), see (47):
   $\boldsymbol{e}_y = \boldsymbol{d}_y \cdot (g+hi) = g\boldsymbol{d}_y + h\boldsymbol{d}_z$
10. Vector $\mathbf{e}_z$ equals the mixed product $\mathbf{e}_y$·i, see (48):
    $\boldsymbol{e}_z = \boldsymbol{e}_y \cdot i = -h\boldsymbol{d}_y + g\boldsymbol{d}_z$
11. $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are expressed with $\mathbf{d}_x$, $\mathbf{d}_y$ and $\mathbf{d}_z$ in (59):

$$\begin{cases} \boldsymbol{e}_x = & \boldsymbol{d}_x \\ \boldsymbol{e}_y = & g\boldsymbol{d}_y + h\boldsymbol{d}_z \\ \boldsymbol{e}_z = & -h\boldsymbol{d}_y + g\boldsymbol{d}_z \end{cases} \tag{59}$$

Computation for all the points of the object:

12. In the proper frame the coordinates of a point are x, y and z. Compute the coordinates of this point in the ground frame which are $x'_g$, $y'_g$ and $z'_g$, see (55).
13. Repeat the step 12 for all the points of the object.

## 8. Discussion

In this article we have defined a new type of multiplication that multiplies a 3D vector with a 2D complex number. We call it mixed multiplication and it has been demonstrated very practical for computing the rotation of vectors in 3D space. The drawings of 3D rotated frames in this article are computed with mixed multiplication, which shows the efficiency of the mixed multiplication.

We have constructed a new method for computing the orientation of objects and shown that the new method was very intuitive because the base vectors are rotated like familiar complex number. It seems that this method does not suffer from gimbal lock because of mixed multiplication. The only critical location is the z axis where the hypotenuse $r_{xy}$ is zero for the number a and b while cosθ=0 and sinθ=1. In comparison with the quaternion method which works with the “sandwich multiplication”, half the angle and 4 dimensions in 3D space, the new method is much more natural and simple.

Mixed multiplication is essentially a complex multiplication, which will reduce the number of lines of code when programming for orientation because complex functions are often internal in programming languages.

This method can work without trigonometric function. It computes the coordinates of points without knowing the rotation angles and without the complicated rotation matrix. This possibility is very interesting for video game because if one wants to reach a point on the screen, for example, moving an axe to cut a tree, one does not have to derive the angles of orientation of the point of cut in the ground frame. One just uses the coordinates of the point of cut and move the axe to it, without knowing the angles.

An other advantage of this method is that the computation of orientation will be faster when the time consuming trigonometric functions are not used and because the process of computation has fewer steps. This is very beneficial for video games, computer-aided design (CAD) or mathematical graphics plotting. In fact, this acceleration of computation speed has been noticed by Edgar Malinovsky who used the 3D complex number system I constructed [6] to generate 3D fractals objects. He noticed a real acceleration of computation speed when computing without trigonometric function. The objects he generated can be seen in the document "Rendering of 3D Mandelbrot, Lambda and other sets using 3D complex number system" [10]. The formulae he used are in the document "Procedure to convert 2D formula into 3D complex formula" [11].

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