On Fermat's last theorem

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1. Fermat's last theorem

This theorem states that for any n except 2, the equation $X^n + Y^n = Z^n$ is not true for all positive integer triplets X, Y and Z. Fermat's "*I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*" has fascinated mathematicians from 1637 but no one has found what his proof was. Let us try to understand this theorem better.

First, write Z^n as function of the integers X and Y:

$$Z^n = X^n + Y^n \tag{1}$$

Then write the n^{th} root of equation (**1**) and divide it by Y:

$$\frac{Z}{Y} = \sqrt[n]{\left(\frac{X}{Y}\right)^n + 1} \tag{2}$$

Choose X to be smaller than or equal to Y to represent the general case in an ordered way:

$$X \leq Y \;\Rightarrow\; (X/Y)^n \leq 1 \;\text{ and } 1 < \sqrt[n]{(X/Y)^n + 1} \leq \sqrt[n]{2} \tag{3}$$

$(X/Y)^n + 1$ is a rational number and the n^{th} root of a rational number between 1 and 2 is very likely irrational. If this is true, Z/Y would be always irrational and no integer Z will satisfy equation (**2**) and $X^n + Y^n = Z^n$.

2. Comments

Before discussing the irrationality of $\sqrt[n]{(X/Y)^n + 1}$, I would like to make some comments about Fermat and his theorem.

1. As Fermat qualified his proof as "*truly marvelous*", it must be elegant, simple and clear-cut. The above derivation has only 3 steps to reach the irrationality of Z and fulfills this criterion.
2. Why did not Fermat write his proof elsewhere when the margin was too narrow? If the above 3-steps derivation were his proof, he may deem it as too easy to be worth writing.
3. Why did Fermat write his note in the margin? Perhaps he had quickly derived his theorem in mind while reading the book. The 3-steps derivation can be easily imagined in mind without writing a word.
4. Fermat's proof must use solely the mathematical knowledge of his time. The 3-steps derivation uses only irrationality of numbers that was well known in the 17th century.
5. For him, the fact that $\sqrt[n]{(X/Y)^n + 1}$ is irrational could be self-evident. So, if this derivation were his proof, he could very well drop the demonstration here and think the proof to be definitely correct making his proof too short for a paper but too long for a margin.

It is impossible to know exactly Fermat's real proof, but considering the above reasons, that the 3-steps derivation is near his theorem makes sense.

3. Irrationality of $\sqrt[n]{(X/Y)^n + 1}$

a. Using Taylor's series

Let us develop the expression $\sqrt[n]{(X/Y)^n + 1}$ into Taylor's series:

$$\begin{aligned}\sqrt[n]{1+(X/Y)^n} &= (1+\delta)^{1/n}\\ &= 1+\frac{\delta}{n}+\frac{1/n\,(1/n-1)}{2!}\delta^2+\frac{1/n\,(1/n-1)(1/n-2)}{3!}\delta^3\cdots\\ &= 1+\frac{1}{n}\frac{X^n}{Y^n}+\sum_{i=2}^{\infty}\frac{1/n\,(1/n-1)(1/n-2)\cdots(1/n-i+1)}{i!}\left(\frac{X^n}{Y^n}\right)^i\\ &= 1+\frac{1}{n}\frac{X^n}{Y^n}+\sum_{j=1}^{\infty}\frac{1/n\,(1/n-1)(1/n-2)\cdots(1/n-2j+1)}{2j!}\left(1+\frac{1/n-2j}{2j+1}\frac{X^n}{Y^n}\right)\left(\frac{X^{2n}}{Y^{2n}}\right)^j\\ &= 1+\frac{1}{n}\frac{X^n}{Y^n}+\sum_{j=1}^{\infty}\frac{\sigma_j}{Y^{2nj}} = 1+\frac{1}{n}\frac{X^n}{Y^n}+\sum_{j=1}^{\infty}\tau_i\end{aligned} \tag{4}$$

We have added consecutive terms of indices $i=2j$ and $i=2j+1$ to form terms of indices j that have all the same sign. The j^{th} term is: $\tau_j = \frac{\sigma_j}{Y^{2nj}}$. This series is infinite and is not a finite digit rational.

In order to determine if the digits of $\sqrt[n]{(X/Y)^n + 1}$ has a repeating sequence, we write τ_j in the numeral system of base Y^{2n} and σ_j is a rational number relative to the j^{th} digit. If the digits of $\sqrt[n]{(X/Y)^n + 1}$ repeat σ_j must be constant after a critical digit j_c, but it is variable with j. So, the digits of $\sqrt[n]{(X/Y)^n + 1}$ do not repeat and $\sqrt[n]{(X/Y)^n + 1}$ must be irrational.

b. Using continued fraction

Any real number x can be expressed in the form of general continued fraction with coefficients a_i, b_i:

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{5}$$

Fortunately, mathematicians have already found an infinite general continued fraction expression for $\sqrt[n]{(X/Y)^n + 1}$. In *"General Method for Extracting Roots using (Folded) Continued Fractions" by Manny Sardina, section "2.7. General formula for roots of the form* $(y)^{\frac{1}{n}}$" we find:

$$\sqrt[n]{\alpha^n+\beta} = \alpha + \cfrac{\beta}{n\alpha^{n-1} + \cfrac{\beta(n-1)}{2\alpha + \cfrac{\beta(n+1)}{3n\alpha^{n-1} + \cfrac{\beta(2n-1)}{2\alpha + \cfrac{\beta(2n+1)}{5n\alpha^{n-1} + \cfrac{\beta(3n-1)}{2\alpha + \cfrac{\beta(3n+1)}{7n\alpha^{n-1} + \cfrac{\beta(4n-1)}{2\alpha + \cdots}}}}}}}} \tag{6}$$

As X is smaller than Y, we choose:

$$\alpha = 1, \beta = (X/Y)^n < 1 \tag{7}$$

Substituting these coefficients into equation (**6**) gives:

$$\sqrt[n]{1+(X/Y)^n} = 1 + \cfrac{(X/Y)^n}{n + \cfrac{(n-1)(X/Y)^n}{2 + \cfrac{(n+1)(X/Y)^n}{3n + \cfrac{(2n-1)(X/Y)^n}{2 + \cfrac{(2n+1)(X/Y)^n}{5n + \cfrac{(3n-1)(X/Y)^n}{2 + \cfrac{(3n+1)(X/Y)^n}{7n + \cfrac{(4n-1)(X/Y)^n}{2+\cdots}}}}}}}} \tag{8}$$

This general continued fraction is infinite. So it could be irrational. We transform (**8**) to make a_k and b_k integers:

$$\sqrt[n]{1+(X/Y)^n} = 1 + \cfrac{X^n}{nY^n + \cfrac{(n-1)X^n}{2 + \cfrac{(n+1)X^n}{3nY^n + \cfrac{(2n-1)X^n}{2 + \cfrac{(2n+1)X^n}{5nY^n + \cfrac{(3n-1)X^n}{2 + \cfrac{(3n+1)X^n}{7nY^n + \cfrac{(4n-1)X^n}{2+\cdots}}}}}}}} \tag{9}$$

The coefficients are:

$$\begin{aligned}
a_1 &= X^n & b_1 &= nY^n \\
a_2 &= (n-1)X^n & b_2 &= 2 \\
a_3 &= (n+1)X^n & b_3 &= 3nY^n \\
a_4 &= (2n-1)X^n & b_4 &= 2 \\
a_5 &= (2n+1)X^n & b_5 &= 5nY^n \\
a_6 &= (3n-1)X^n & b_6 &= 2 \\
a_7 &= (3n+1)X^n & b_7 &= 7nY^n \\
&\cdots & &\cdots
\end{aligned} \tag{10}$$

All a_i and b_i are integers and, as $X < Y$, $a_k < b_k$ for all odd k. We find in "*Irrationality and Transcendence - Lambert's Irrationality Proofs*", *Angell, David (2007), School of Mathematics, University of New South Wales* that, if $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are positive integers with $a_k \le b_k$ for all sufficiently large k, then $x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$ converges to an irrational limit (see Conditions_for_irrationality). But this theorem is not suitable for us. We need to prove that for a_i and b_i integers and $a_k < b_k$ for all odd k, $\sqrt[n]{1+(X/Y)^n}$ converges to an irrational limit for $n>2$. Alas, I haven't found such theorem. For $X = Y$, $\sqrt[n]{(X/Y)^n + 1} = \sqrt[n]{2}$ is irrational.

4. Conclusion

Using Taylor's series, we find that $\sqrt[n]{(X/Y)^n + 1}$ must be irrational. However, for n=2 and particular cases of Pythagorean triplets, this proof fails. Using continued fraction, we need a new theorem for the proof. So, that all Z satisfying $X^n + Y^n = Z^n$ is irrational for all n>2 is not definitely proven.

Being not mathematician myself, I'm unable to carry out the missing theorems for definite proof. My deductions only indicate direction of search for final proof and I call for good mathematicians to carry it out to complete the demonstration of this particular proof of Fermat's last theorem.