Complement for
“Extending complex number to spaces with 3, 4 or any number of dimensions”

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1. 13 April 2022: Direction of rotation of θ and φ

When we solve a and d of a vector of the step n+1, we use the equality (1).

$x_{n+1} = r_{n+1}a_{n+1}d_{n+1} = r_{n+1}\cos\theta_{n+1}\cos\varphi_{n+1}$	(1)	$\cos 2\varphi = 2\cos^2\varphi - 1$	$\sin 2\varphi = 2\sin\varphi\cos\varphi$	(2)

At the previous step n, the angles are θ_n and φ_n. The square z^2 gives the angles $2\theta_n$ and $2\varphi_n$. The rotation of θ in the multiplication is always counterclockwise. But the plane (θ, z*) rotates with θ, so, when we do z^2+c, the x_{n+1} can be positive or negative. Then, for making the rotation of φ at the n+1th step counterclockwise, we have to know the direction of the last rotation of φ, which is indicated by the sign of the last cos2φ.

$z^2 + c = x_2 + y_2 i + z_{*2} j + x_c + y_c i + z_c j = (x_2 + x_c) + (y_2 + y_c)i + (z_{*2} + z_c)j$	$z_{*2} = r_2 \sin 2\varphi$	(3)

When added with c, the value of x_c and y_c rotate the plane (θ, z*), z_c changes the value of $z_{*2}+z_c$, but the value of z_{*2} is not changed neither that of cos2φ. So, if we keep the sign of cos2φ, then we know the angle φ is in which quadrant, positive in the 1st and 4th, negative in the 2nd or 3rd, the the formula (2) makes z^2 to rotate in the counterclockwise direction of the plane (θ, z*). When the sign of d_{n+1} is known the sign of a and b are known, which gives the value of θ_{n+1} and the direction of the plane (θ, z*).

$a_{n+1} = \dfrac{x_{n+1}}{r_{n+1}d_{n+1}} = \dfrac{x_{n+1}}{r_{n+1}\cos\varphi_{n+1}} = \cos\theta_{n+1}$	$b_{n+1} = \dfrac{y_{n+1}}{r_{n+1}d_{n+1}} = \dfrac{y_{n+1}}{r_{n+1}\cos\varphi_{n+1}} = \sin\theta_{n+1}$	(4)

If we do not keep the sign of cos2φ, we have to arbitrarily make $d_{n+1}>0$ or $a_{n+1}>0$. When the sign of d_{n+1} does not equal that of cos2 φ, then d_{n+1} , a_{n+1} , b_{n+1} change sign and φ rotates in the opposite direction, which results in the rotation of θ is always counterclockwise, but φ rotates back and forth. This is not correct.

When I made $a_{n+1}>0$ for making the cut of the plane y=0 to be a correct Mandelbrot power 2, I have made an arbitrary choice and this created the discontinuity at x=0. So, the correct Mandelbrot power 3 is that made with the correct sign of d_{n+1}, not with $a_{n+1}>0$ nor with $d_{n+1}>0$.

2. 11 April 2022: Keeping the rotation of φ in the same direction

I have tried two methods to compute the values of a, b, d, f, see (5):

1. d>0, which results in having the 3D Mandelbrot set z^2+c to be a heart Mandelbrot set in the plane y=0
2. a>0, which results a cut in the plane x=0, which gives the 3D Mandelbrot set a size for x<0 different from that for x>0.

$z = r\big((a+bi)d + fj\big) = r(ad + bdi + fj) = x + yi + z_* j$	$\cos\theta = a$	$\sin\theta = b$	$\cos\varphi = d$	$\sin\varphi = f$	(5)

I think the problem is caused by the change of the direction of rotation of the angle φ. The rotation of θ is always counterclockwise. But if we define that cos φ be always positive, then φ will rotate toward $\mathbf{e}_z$ and the direction of rotation will reverse if φ>π/2, see Figure 1 and Figure 2.

For avoiding the change of the direction of rotation, we have to keep the sign of cos φ steady. So, when computing the numbers a, b, d, f of z_{n+1}, see (10), we have to make the sign of d_{n+1} equal that of cos 2φ of z^2 which is d_{p2}, see (6), (7).

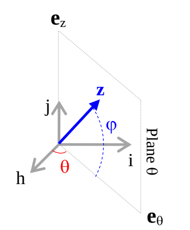

Figure 1

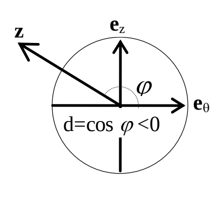

Figure 2

Power 2	$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(6)
$z^2 = (r^2 a_{p2} d_{p2}) + (r^2 b_{p2} d_{p2}) i + (r^2 f_{p2}) j = x_2 + y_2 i + z_{*2} j$					(7)
$z_{n+1} = z^2 + c = x_2 + y_2 i + z_{*2} j + c = x_{n+1} + y_{n+1} i + z_{*n+1} j$					(8)

z_{n+1} is computed in (8). We compute r_{xy} and r of z_{n+1} and the signed modulus r_{xys} in (9). We compute a_{n+1}, b_{n+1}, d_{n+1}, f_{n+1} in (10) and check that the product r*a_{n+1}*d_{n+1} equals well x_{n+1}, see (5) and (8).

$r_{xy} = \sqrt{{x_{n+1}}^2 + {y_{n+1}}^2}$		$r = \sqrt{{x_{n+1}}^2 + {y_{n+1}}^2 + {z_{*n+1}}^2}$		$r_{xys} = sign(d_{p2})\sqrt{{x_{n+1}}^2 + {x_{n+1}}^2}$	(9)
$a_{n+1} = \frac{x_{n+1}}{r_{xys}}$	$b_{n+1} = \frac{y_{n+1}}{r_{xys}}$	$d_{n+1} = \frac{r_{xys}}{r}$	$f_{n+1} = \frac{z_{*n+1}}{r}$	$r a_{n+1} d_{n+1} = \frac{x_{n+1}}{r_{xys}} \frac{r_{xys}}{r} r = x_{n+1}$	(10)

3. 9 April 2022, Talis formula

$z = x + yi + z_* j = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big) = r\big((a + bi)d + fj\big) = r\boldsymbol{u}$						(11)
${r_{xy}}^2 = x^2 + y^2$ $r^2 = {r_{xy}}^2 + z^2$	$\varphi = \sin^{-1}\left(\frac{z_*}{r}\right)$	If y>0	$\theta = \cos^{-1}\left(\frac{x}{r_{xy}}\right)$	If y<0	$\theta = -\cos^{-1}\left(\frac{x}{r_{xy}}\right)$	(12)
$z^m = r^m \boldsymbol{u}^m = r^m\big((\cos m\theta + \sin m\theta\, i)\cos m\varphi + \sin m\varphi\, j\big)$						(13)
$z^{m-1} = r^{m-1}\boldsymbol{u}^{m-1} = r^{m-1}\big((\cos(m-1)\theta + \sin(m-1)\theta\, i)\cos(m-1)\varphi + \sin(m-1)\varphi\, j\big)$						(14)

Talis formula

$z_{n+1} = s\frac{z^m}{z^{m-1} + \alpha} + c$	$z_{n+1} = s\frac{r^m \boldsymbol{u}^m}{r^{m-1}\boldsymbol{u}^{m-1} + \alpha} + c = sr\left(\frac{\boldsymbol{u}^m}{\boldsymbol{u}^{m-1} + \frac{\alpha}{r^{m-1}}}\right) + c$	(15)

Denominator

$\boldsymbol{u}^{m-1} = (a_- + b_- i)d_- + f_- j$		$\cos(m-1)\theta = a_-$	$\sin(m-1)\theta = b_-$	$\cos(m-1)\varphi = d_-$	$\sin(m-1)\varphi = f_-$	(16)
$\boldsymbol{u}^{m-1} + \frac{\alpha}{r^{m-1}} = (a_- + b_- i)d_- + f_- j + \frac{\alpha}{r^{m-1}} = d_-\left(\frac{\alpha}{r^{m-1}d_-} + a_- + b_- i + \frac{f_-}{d_-} j\right)$						(17)
The term in the parenthesis						
${r_{xy}}^2 = \left(\frac{\alpha}{r^{m-1}d_-} + a_-\right)^2 + {b_-}^2$	${r_-}^2 = {r_{xy}}^2 + \left(\frac{f_-}{d_-}\right)^2$	$\cos\varphi = \frac{r_{xy}}{r_-}$	$\sin\varphi = \frac{f_-}{r_- \cdot d_-}$	$\cos\theta = \frac{\frac{\alpha}{r^{m-1}d_-} + a_-}{r_{xy}}$	$\sin\theta = \frac{b_-}{r_{xy}}$	(18)
$\varphi_- = \sin^{-1}\left(\frac{f_-}{r_- d_-}\right)$	If b_->0	$\theta_- = \cos^{-1}\left(\frac{\frac{\alpha}{r^{m-1}d_-} + a_-}{r_{xy}}\right)$	If b_-<0	$\theta_- = -\cos^{-1}\left(\frac{\frac{\alpha}{r^{m-1}d_-} + a_-}{r_{xy}}\right)$		(19)
$\frac{\alpha}{r^{m-1}d_-} + a_- + b_- i + \frac{f_-}{d_-} j = r_-\big((\cos\theta_- + \sin\theta_-\, i)\cos\varphi_- + \sin\varphi_-\, j\big)$						(20)

Computing z_{n+1}

$\frac{\boldsymbol{u}^m}{d_-\left(\frac{\alpha}{r^{m-1}d_-} + a_- + b_- i + \frac{f_-}{d_-} j\right)} = \frac{(\cos m\theta + \sin m\theta\, i)\cos m\varphi + \sin m\varphi\, j}{d_- r_-\big((\cos\theta_- + \sin\theta_-\, i)\cos\varphi + \sin\varphi_-\, j\big)}$				(21)
$z_{n+1} = sr\left(\frac{\boldsymbol{u}^m}{\boldsymbol{u}^{m-1} + \frac{\alpha}{r^{m-1}}}\right) + c = sr\left(\frac{(\cos(m\theta - \theta_-) + \sin(m\theta - \theta_-)\, i)\cos(m\varphi - \varphi_-) + \sin(m\varphi - \varphi_-)\, j}{d_- r_-}\right) + c$				(22)
$z_{n+1} = \frac{sr\cos(m\varphi - \varphi_-)}{d_- r_-}\left(\cos(m\theta - \theta_-) + \sin(m\theta - \theta_-)\, i + \frac{\sin(m\varphi - \varphi_-)}{\cos(m\varphi - \varphi_-)} j\right) + c$				(23)
$z_{n+1} = \frac{sr\cos(m\varphi - \varphi_-)}{d_- r_-}(x + yi + z_* j) + c$	$x = \cos(m\theta - \theta_-)$	$y = \sin(m\theta - \theta_-)$	$z_* = \frac{\sin(m\varphi - \varphi_-)}{\cos(m\varphi - \varphi_-)}$	(24)

Back to (11). The red lines are to be computed in code.

Summary for Talis

$$z_{n+1} = s\frac{z^m}{z^{m-1}+\alpha} + c$$

$z = x + yi + z_* j$		$r_{xy}{}^2 = x^2 + y^2$		$r^2 = r_{xy}{}^2 + z^2$	(25)
$\varphi = \sin^{-1}\left(\frac{z_*}{r}\right)$	If y>0	$\theta = \cos^{-1}\left(\frac{x}{r_{xy}}\right)$	If y<0	$\theta = -\cos^{-1}\left(\frac{x}{r_{xy}}\right)$	(26)
$\cos(m-1)\theta = a\_$	$\sin(m-1)\theta = b\_$	$\cos(m-1)\varphi = d\_$	$\sin(m-1)\varphi = f\_$		(27)
$r_{xy}{}^2 = \left(\frac{\alpha}{r^{m-1}d\_} + a\_\right)^2 + b\_{}^2$		$r\_{}^2 = r_{xy}{}^2 + \left(\frac{f\_}{d\_}\right)^2$			(28)
$\varphi\_ = \sin^{-1}\left(\frac{f\_}{r\_d\_}\right)$	If $b\_$>0	$\theta\_ = \cos^{-1}\left(\frac{\frac{\alpha}{r^{m-1}d\_} + a\_}{r_{xy}}\right)$	If $b\_$<0	$\theta\_ = -\cos^{-1}\left(\frac{\frac{\alpha}{r^{m-1}d\_} + a\_}{r_{xy}}\right)$	(29)
$x = \cos(m\theta - \theta\_)$		$y = \sin(m\theta - \theta\_)$		$z_* = \frac{\sin(m\varphi - \varphi\_)}{\cos(m\varphi - \varphi\_)}$	(30)

$$z_{n+1} = \frac{sr\cos(m\varphi - \varphi\_)}{d\_r\_}(x + yi + z_* j) + c$$

Back to (25)

4. 8 April 2022: Symmetry of power 2 Mandelbrot set

Mandelbrot set formula, z is 2D or 3D complex: $z_{n+1} = z^2 + c$

z in 3D complex

$z = r\big((a + bi)d + fj\big)$		$z^2 = r\Big(\big((2a^2-1) + 2abi\big)(2d^2-1) + 2dfj\Big)$		(31)
$z_{n+1} = z^2 + c = r\Big(\big((2a^2-1) + 2abi\big)(2d^2-1) + 2dfj\Big) + x_c + y_c i + z_{*c} j$				(32)
$\cos\theta = a$	$\sin\theta = b$	$\cos\varphi = d$	$\sin\varphi = f$	(33)

Cut plane for z=0 or y=0:

z_*=0	$\sin\varphi = f = 0$	$\cos\varphi = d = 1$	$z^2 = r^2\Big(\big((2a^2-1) + 2abi\big)1 + 0j\Big)$	(34)
y=0	$\sin\theta = b = 0$	$\cos\theta = a = 1$	$z^2 = r^2\big((1 + 0i)(2d^2-1) + 2dfj\big)$	(35)
z_{*c}=0	$z_{n+1} = z^2 + c = r^2\big((2a^2-1) + 2abi\big) + x_c + y_c i$		See (32)(34)	(36)
y_c=0	$z_{n+1} = z^2 + c = r^2\big((2d^2-1) + 2dfj\big) + x_c + z_{*c} j$		See (32)(35)	(37)

In plane z=0, cutout is a 2D Mandelbrot set in plane (h, i), see (36).
In plane y=0, cutout is a 2D Mandelbrot set in plane (h, j), see (37).
Equations (36) and (37) have the same form, thus, the cutouts for z=0 and y=0 will have same form, that is, a 2D Mandelbrot set.