Extending complex number to spaces with 3, 4 or any number of dimensions

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Abstract: Multidimensional complex systems with 3, 4 or more dimensions are constructed. They possess algebraic operations which have geometrical meanings. Multidimensional complex numbers can be written in Cartesian, trigonometric and exponential form and can be converted from one form to another. Each complex numbers has a conjugate. Multidimensional complex systems are extensions of the classical complex number system.

In about 500 years after the birth of complex number, there were several attempts to extend complex number to more than 2 dimensions, for example we have theories such as quaternions, tessarines, coquaternions, biquaternions, and octonions. But none has reached the success of the classical complex number in 2 dimensions. Among these theories the most famous is quaternion which has found use in computational geometry. But quaternion is a 4 dimensional complex number but is used in 3 dimensional vector space, which is somewhat awkward.

In this article we will show that multidimensional complex number with 3, 4 or more dimensions exist and will explain how to construct them. Like classical complex number system, a multidimensional complex number system possesses algebraic operations in its complex space that have geometrical meaning in the corresponding vector space.

In the following exposition, spaces with 3, 4 or n dimensions will be referred to as 3D, 4D and nD spaces and the corresponding complex numbers as 3D, 4D and nD complex numbers. Since a complex number corresponds to a vector, a complex number will be referred to as a vector when convenient. We will begin with constructing 3D complex number system. Then we will generalize to spaces with 4 and more dimensions.

The 3D complex number system is constructed from a 2D complex number system which is the classical complex number system. So, let us see how classical complex number works.

1. Concept of multidimensional complex number

The proposed concept of multidimensional complex number is simple but the demonstration is rather long. For the readers not to get lost, we present beforehand the guiding idea which, kept in mind, will make the reading easier.

Let us start with a familiar unit 2D complex number $e^{i\theta}$, which is written with its argument θ in equation (1). 2D complex space has two dimensions, the first one is the real line labeled as h, the second one is the imaginary line labeled as i. 3D complex space is an extension of the 2D complex space by appending with the dimension labeled j. Figure 1 shows the 3D complex space in which u is a 2D complex number and v a 3D complex number.

$A_i = e^{i\theta}$	(1)

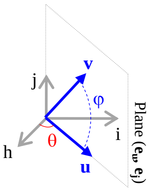

Figure 1

A 3D complex number has imaginary parts in i and j. Let us represent the imaginary part in i with the 2D complex number $e^{i\theta}$ and that in j with the 2D complex number $e^{j\varphi}$. A unit 3D complex number is defined as the product of $e^{i\theta}$ and $e^{j\varphi}$, see (3). So, the unit 3D complex number is $e^{i\theta}\cdot e^{j\varphi}$ see (3).

In the same way, a unit 4D complex number is defined as the product of the 3D complex number (3) with the 2D complex number $e^{k\phi}$ representing the imaginary part in k, see (4). The expression of a unit 4D complex number is $e^{i\theta}\cdot e^{j\varphi}\cdot e^{k\phi}$, see (5).

3D complex	
$A_j = e^{j\varphi}$	(2)
$v = e^{i\theta} e^{j\varphi}$	(3)

4D complex	
$A_k = e^{k\phi}$	(4)
$w = e^{i\theta} e^{j\varphi} e^{k\phi}$	(5)

3D and 4D complex numbers can be multiplied. In equations (6) are written two 3D complex numbers v_1 and v_2 whose arguments are $(i\theta_1, j\varphi_1)$ and $(i\theta_2, j\varphi_2)$.

3D complex multiplication	
$v_1 = e^{i\theta_1} e^{j\varphi_1}$ $v_2 = e^{i\theta_2} e^{j\varphi_2}$	(6)
$v_1 v_2 = e^{i\theta_1} e^{j\varphi_1} e^{i\theta_2} e^{j\varphi_2}$ $= e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)}$	(7)

4D complex multiplication	
$w_1 = e^{i\theta_1} e^{j\varphi_1} e^{k\phi_1}$ $w_2 = e^{i\theta_2} e^{j\varphi_2} e^{k\phi_2}$	(8)
$w_1 w_2 = e^{i\theta_1} e^{j\varphi_1} e^{k\phi_1} e^{i\theta_2} e^{j\varphi_2} e^{k\phi_2}$ $= e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)} e^{k(\phi_1+\phi_2)}$	(9)

The product of v_1 and v_2 is expressed in (7) and whose arguments are the sum $i(\theta_1+\theta_2)$ and $j(\varphi_1+\varphi_2)$. In the same way, in (8) are written two 4D complex numbers w_1 and w_2 and their product is the exponential of the sum of the arguments of the two factors, $i(\theta_1+\theta_2)$, $j(\varphi_1+\varphi_2)$ and $k(\phi_1+\phi_2)$, see (9).

$$u_1 = e^{i\theta_1} \quad u_2 = e^{i\theta_2} \tag{10}$$

$$u_1 u_2 = e^{i\theta_1} e^{i\theta_2} \tag{11}$$

The so defined 3D and 4D complex numbers multiply just like the multiplication of 2D complex numbers. Let us see the two 2D complex numbers u_1 and u_2 in (10). Their product is u_1u_2 and is shown in (11). The argument of the product equals the sum of that of the factors u_1 and u_2, that is, $i(\theta_1+\theta_2)$, see (12). The product of the 3D v_1 and v_2 is shown in (13) and that of the 4D w_1 and w_2 is shown in (14). We find the same pattern in (12), (13) and (14).

$$u_1 u_2 = e^{i(\theta_1+\theta_2)} \tag{12}$$

$$v_1 v_2 = e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)} \tag{13}$$

$$w_1 w_2 = e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)} e^{k(\phi_1+\phi_2)} \tag{14}$$

Unit 3D and 4D complex numbers are converted into trigonometric form by using Euler's formula (15) which transforms (3) and (5) into (16) and (17). Unit 2D complex number in trigonometric form is the Euler's formula (15). We find the same pattern in (15), (16) and (17).

$$e^{i\theta} = \cos\theta + \sin\theta\, i \tag{15}$$

$$v = (\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j) \tag{16}$$

$$w = (\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)(\cos\phi + \sin\phi\, k) \tag{17}$$

$$v = \cos\theta\cos\varphi + \sin\theta\cos\varphi\, i + \sin\varphi\, j \tag{18}$$

3D complex number is a vector in 3D space, for example, the 3D complex number v in Figure 1 is defined by its Cartesian components which is converted from (16) into (18) in the following section. So, 3D complex number is well a vector in 3D Cartesian space.

2. Classical complex number

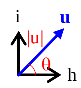

Figure 2

Classical complex space is a plane with two orthogonal axes, see Figure 2:

1. The axis of real numbers which is labeled as h.
2. The axis of imaginary numbers which is labeled as i.

This plane is labeled as (h, i). On this plane a complex number is both a point and a vector, for example the vector u in Figure 2. u makes the angle θ with the axis h and and its length is |u|. As complex number, u's argument is θ and its modulus is |u|. In polar coordinate system the complex number u is expressed in equation (19), where i is the imaginary unit, see (21). Equation (19) is referred to as the trigonometric form of u.

$$u = |u|(\cos\theta + \sin\theta\, i) \tag{19}$$

$$u = |u|\cos\theta + |u|\sin\theta\, i \tag{20}$$

$$ii = -1 \tag{21}$$

$$a = |u|\cos\theta \quad b = |u|\sin\theta \tag{22}$$

$$u = a + bi \tag{23}$$

We develop (19) into (20) in which we introduce (22) and obtain (23) where the numbers 'a' and b are the Cartesian coordinates of u. So, equation (23) is referred to as the Cartesian form of u.

Equation (24) is the Euler's formula for θ and i, and is introduced into (19) which becomes (25). Equation (25) expresses u in the form of an exponential function and is referred to as the exponential form of u. So, a classical complex number can be expressed in Cartesian, trigonometric or exponential form and has a geometrical meaning which is the vector u in Figure 2.

$$e^{i\theta} = \cos\theta + \sin\theta\, i \tag{24}$$

$$u = |u| e^{i\theta} \tag{25}$$

3. 3D complex number

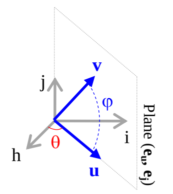

Figure 3

a. 3D space and vector

A 3D complex number is also a vector, which we will construct from the 2D plane (h, i). For doing so, we add the axis j perpendicularly to the plane (h, i) and obtain the 3D space whose axes are labeled as h, i and j, see Figure 3. We refer to this space as (h, i, j).

We attach the unit vectors $\mathbf{e}_h$, $\mathbf{e}_i$ and $\mathbf{e}_j$ to the axes h, i and j respectively. The 3D space based on these vectors is referred to as ($\mathbf{e}_h$, $\mathbf{e}_i$, $\mathbf{e}_j$). We have then two 3D spaces: the complex space (h, i, j) and the vector space ($\mathbf{e}_h$, $\mathbf{e}_i$, $\mathbf{e}_j$). We will create a vector labeled as v in ($\mathbf{e}_h$, $\mathbf{e}_i$, $\mathbf{e}_j$) which corresponds to a 3D complex number in (h, i, j) labeled also as v.

$$u = |u|(\cos\theta\, \boldsymbol{e}_h + \sin\theta\, \boldsymbol{e}_i) \tag{26}$$

$$\boldsymbol{e}_u = \frac{u}{|u|} = \cos\theta\, \boldsymbol{e}_h + \sin\theta\, \boldsymbol{e}_i \tag{27}$$

With the help of Figure 3 we create the vector v in the desired form by starting with a vector u which is expressed in (26) with |u| being its modulus and θ the angle it makes with the axis h. So, u is in the horizontal plane ($\mathbf{e}_h$, $\mathbf{e}_i$). Dividing u by |u| gives the unit vector $\mathbf{e}_u$, see (27). The unit vectors $\mathbf{e}_u$ and $\mathbf{e}_j$ are the basis vectors of the vertical plane ($\mathbf{e}_u$, $\mathbf{e}_j$), see Figure 3. The vector v is created by rotating the vector u in this plane toward the axis j. The angle of rotation is φ, so v is expressed with the angle φ on the basis vectors $\mathbf{e}_u$ and $\mathbf{e}_j$ in (28).

As the length of u stays the same during the rotation, the modulus of u and v are equal, see (29). Introducing the expression of $\mathbf{e}_u$ (27) into (28) gives (30) which is developed into (31) using (29). The vector v is expressed with its modulus |v| and the angles θ and φ on the basis vectors $\mathbf{e}_h$, $\mathbf{e}_i$ and $\mathbf{e}_j$, see (31).

$v = \|u\|(\cos\varphi\, \boldsymbol{e}_u + \sin\varphi\, \boldsymbol{e}_j)$	(28)	$\|u\| = \|v\|$	(29)
$v = \|u\|[(\cos\theta\, \boldsymbol{e}_h + \sin\theta\, \boldsymbol{e}_i)\cos\varphi + \sin\varphi\, \boldsymbol{e}_j]$			(30)
$v = \|v\|\cos\theta\cos\varphi\, \boldsymbol{e}_h + \|v\|\sin\theta\cos\varphi\, \boldsymbol{e}_i + \|v\|\sin\varphi\, \boldsymbol{e}_j$			(31)

Notice that the angle φ is between the vector v and the horizontal plane ($\mathbf{e}_h$, $\mathbf{e}_i$), see Figure 3, which is different from the usual spherical coordinate system where the angle φ is between the vector v and the axis j. So, when u is horizontal the angle φ equals zero rather then π/2.

$\boldsymbol{e}_h \Rightarrow 1$ $\boldsymbol{e}_i \Rightarrow i$ $\boldsymbol{e}_j \Rightarrow j$	(32)	$ii = -1$ $jj = -1$	(33)

b. 3D complex space and number

The vector v corresponds to the 3D complex number v whose expression is obtained by replacing the basis vectors $\mathbf{e}_h$, $\mathbf{e}_i$ and $\mathbf{e}_j$ with 1, i and j in (31), see (32), with i and j being the imaginary units of the axes i and j in the space (h, i, j), see (33). The expression of the complex number v is given in (34).

$$\begin{aligned} v &= |v|\cos\theta\cos\varphi + |v|\sin\theta\cos\varphi\, i + |v|\sin\varphi\, j \\ &= |v|\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big) \end{aligned} \quad (34)$$

Because (31) and (34) are the same equation, the 3D complex space (h, i, j) is in one to one correspondence with the 3D vector space ($\mathbf{e}_h$, $\mathbf{e}_i$, $\mathbf{e}_j$) and any vector has a corresponding 3D complex number.

- Cartesian form

Equation (34) gives the Cartesian coordinates of the 3D complex number v with |v|, θ and φ, see (35). So, equation (34) can be written as (36) which is the Cartesian form of v.

$$\begin{aligned} a &= |v|\cos\theta\cos\varphi \\ b &= |v|\sin\theta\cos\varphi \\ c &= |v|\sin\varphi \end{aligned} \quad (35)$$

$$v = a + bi + cj \quad (36)$$

- Trigonometric form

We will create the expression of the trigonometric form of a 3D complex number such that it makes the multiplication of 3D complex numbers to operate in the same way as 2D complex number. When multiplying together the 2 complex numbers u_1 and u_2 given in (37) their complex product u_1u_2 is (38). u_1, u_2 and u_1u_2 are in trigonometric form. This multiplication corresponds to a rotation of the multiplied vector. As the above vector v is obtained by rotating the vector u, the complex number v would be the result of a multiplication of the complex number u.

$$\begin{aligned} u_1 &= |u_1|(\cos\theta_1 + \sin\theta_1\, i) \\ u_2 &= |u_2|(\cos\theta_2 + \sin\theta_2\, i) \end{aligned} \quad (37)$$

$$u_1u_2 = |u_1||u_2|(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\, i) \quad (38)$$

$$\begin{aligned} v &= \cos\varphi\, |u|\boldsymbol{e}_u + \sin\varphi\, |u|\boldsymbol{e}_j \\ &= u\cos\varphi + \sin\varphi\, |u|\boldsymbol{e}_j \end{aligned} \quad (39)$$

$$v = u\cos\varphi + \sin\varphi\, |u| j \quad (40)$$

This rotation is done in the vertical plane ($\mathbf{e}_u$, $\mathbf{e}_j$) and by developing (28) and using (27), v is expressed with u in (39). We make the vertical plane ($\mathbf{e}_u$, $\mathbf{e}_j$) to correspond to a 2D complex plane which we label as (h_u, j), with h_u and j being its real and imaginary axes and corresponding to the unit vectors $\mathbf{e}_u$ and $\mathbf{e}_j$ respectively. So, the vector v can be expressed as a complex number whose expression is (40) which is obtained by replacing $\mathbf{e}_u$ and $\mathbf{e}_j$ with 1 and j in (39) respectively.

Then the vector v can be operated as a 2D complex number in this plane and we search for the expression of the complex number v to be the product of the complex number u with another complex number. For doing so, we have to transform the term |u|j of the equation (40) into a factor of u.

Let us apply the multiplication formula for 2D complex number (38) to the two particular complex numbers given in (41) and (42), with u_1 being pure real and u_2 unit complex. The product u_1u_2 can be written either as (43) or as (44). Because (43) equals (44), the equality in (45) is true for the given u_1 and u_2.

$$\theta_1 = 0 \Rightarrow u_1 = |u_1| \quad (41)$$

$$|u_2| = 1 \Rightarrow u_2 = \cos\theta_2 + \sin\theta_2\, i \quad (42)$$

$$u_1u_2 = u_1\cos\theta_2 + \sin\theta_2\, u_1 i \quad (43)$$

$$u_1u_2 = u_1\cos\theta_2 + \sin\theta_2\, |u_1| i \quad (44)$$

$$u_1 i = |u_1| i \quad (45)$$

The equality (45) applies to any complex plane and thus, to the complex plane (h_u, j) in which the complex number u corresponds to u_1 and the imaginary unit j corresponds to the imaginary unit i of (45). So, we can write equation (46) for (h_u, j).

But the complex plane (h_u, j) is particular because it corresponds to the vertical plane ($\mathbf{e}_u$, $\mathbf{e}_j$) which makes the angle θ with the unit vector $\mathbf{e}_h$, that is, the real axis, see Figure 3. The angle θ can have any value, which means that the equality (46) is true for any vector in the horizontal plane ($\mathbf{e}_h$, $\mathbf{e}_i$).

$$uj = |u|j \tag{46}$$

$$v = u\cos\varphi + \sin\varphi\, uj \tag{47}$$

$$v = u(\cos\varphi + \sin\varphi\, j) \tag{48}$$

$$\cos\varphi + \sin\varphi\, j = e^{j\varphi} \tag{49}$$

Because equation (46) defines the multiplication in 3D complex space, we state Definition 1.
Definition 1: The complex number u is a vector perpendicular to the imaginary axis j. The complex product uj equals the modulus |u| multiplied by the imaginary unit j, |u|j, see (46).

We use (46) for equation (40) which give (47). We factor out the complex number u and obtain (48). Finally, the complex number v is expressed as the product of u and the Euler's formula for the argument φ and the imaginary unit j, see (49), which we state as the Rule 1.
Rule 1: u is a complex number perpendicular to the imaginary axis j, when u is rotated toward the axis j by the angle φ, the resulting complex number equals u multiplied by the Euler's formula for the argument φ and the imaginary unit j, see (48).

The 3D complex number v is better expressed in (50) which is obtained by introducing the expression of u (19) and the equality (29) into (48). Equation (50) is the trigonometric form of a 3D complex number, with |v| being its modulus, θ and φ its arguments about the imaginary units i and j respectively.

- Exponential form

We can easily transform (50) into exponential form by replacing the Euler's formulas (24) and (49) in (50) and have (51), which is the exponential form of v.

$$v = |v|(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j) \tag{50}$$

$$v = |v|e^{i\theta}e^{j\varphi} = |v|e^{i\theta + j\varphi} \tag{51}$$

c. Geometrical meaning of 3D complex number

- Commutation of the arguments θ and φ

The vector meaning of the arguments θ and φ helps us to see the working of 3D complex numbers which is also vectors in the 3D space (h, i, j), see Figure 4. We define a 3D complex number v and the 3 following planes:

1. The vertical plane (h, j) made by the axes h and j
2. The vertical plane (v, j) made by the the vector v and axis j
3. The horizontal plane (h, i) made by the axes h and i

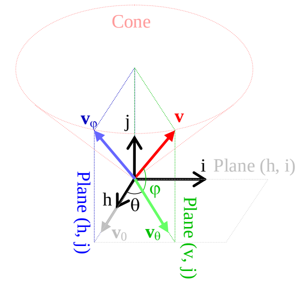

Figure 4

The first argument θ is the angle between the plane (h, j) and the plane (v, j). When θ varies, the plane (v, j) rotates. The second argument φ is the angle between the vector v and the horizontal plane (h, i). When φ varies, the vector v rotates within the plane (v, j). When the angle θ increases from 0 to 2π while keeping the angle φ constant, the vector v draws a cone, see Figure 4. The vector v is on the intersection line of the plane (v, j) and the cone.

The vector v can be obtained in two ways:

1. Take the vector v_0 on the real axis h, rotate it horizontally by the angle θ which gives the vector v_θ. Then, rotate the vector v_θ toward the axis j by the angle φ and v_θ becomes v.
2. Take the vector v_0 and rotate it toward the axis j by the angle φ which gives the vector v_φ. Then, rotate the vector v_φ on the cone by the angle θ, and v_φ becomes v.

The two ways give the same vector v because both final vectors are on the intersection line of the plane (v, j) and the cone. So, we conclude that the order of the rotations does not matter. This property makes the 2 arguments θ and φ commutative mathematically. Indeed, let us write the trigonometric and exponential forms (52) and (53) which respect the order of the rotations, with v_1 and v_2 corresponding to the vectors obtained in the first and second way. The two expressions in (52) are in reverse order and also those in (53).

Because the two ways give the same vector, the 3D complex number v_1 equals v_2. So, the 2 expressions in (52) are equal and those in (53) too. In consequence, the arguments for i and j commute when v_1 and v_2 multiply together.

$v_1 = \lvert v\rvert(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)$ $v_2 = \lvert v\rvert(\cos\varphi + \sin\varphi\, j)(\cos\theta + \sin\theta\, i)$	(52)	$v_1 = \lvert v\rvert e^{i\theta + j\varphi}$ $v_2 = \lvert v\rvert e^{j\varphi + i\theta}$	(53)

So, we state the Definition 2.

Definition 2: Imaginary arguments of different dimensions commute within trigonometric and exponential forms.

- Geometrical meaning of the Definition 1

By the way, the vectors v_1 and v_φ show well the geometrical meaning of the Definition 1. Suppose that the angle φ equals π/2. In this case, the vector v obtained in the first way is parallel to the axis j and equals |u|j. On the other hand, for the second way the vector v_φ obtained after the first rotation is already |u|j which will not change with the rotation by the angle θ.

So, for whatever angle θ the rotation of a horizontal vector u toward the axis j by the angle φ=π/2 makes the resulting vector to coincides with the axis j which corresponds to the complex number |u|j. Because a rotation of vector corresponds to a multiplication of complex number, the resulting vector corresponds to the complex number uj. That is, uj equals |u|j geometrically, which illustrates well the Definition 1.

d. Conversion from trigonometric to Cartesian form

Now a 3D complex number can be expressed in Cartesian form (36), trigonometric form (50) and exponential form (51). Since these 3 forms represent the same number, we must be able to convert them. Equation (34) gives the expressions of the 3 components of the Cartesian form, see also (35). So, (34) is the formula that converts the trigonometric form of a 3D complex number into its Cartesian form.

e. Conversion from Cartesian to trigonometric and exponential forms

$\frac{v}{\lvert v\rvert} = (\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j$	(54)
$\sqrt{(\cos\theta\cos\varphi)^2 + (\sin\theta\cos\varphi)^2 + \sin^2\varphi} = 1$	(55)

- The number's modulus

When writing a 3D complex number in trigonometric form we need its modulus which is computed with the 3 Cartesian components of the 3D complex number, see (36) and (56).

- The arguments

For deriving the arguments of a 3D complex number we eliminate the modulus from its trigonometric form, which is done by dividing (34) with its modulus |v|, see (54).

By the way, the modulus of the expression in (54) is 1, see (55). So, the modulus of (34) equals |v|, which shows that the trigonometric form (34) is correctly constructed.

On the other hand, the Cartesian form (56) divided by |v| gives (57). By equating (54) with (57), we obtain (58), (59) and (60). We derive the angle φ in (61). The sign of sin φ is that of c and the value of φ has no ambiguity. In (62) we derive cos φ which can be positive or negative. We decide that cos φ is positive to remove any ambiguity about the value of the argument θ which is derived in (63).

Then the 3D complex number v can be written in trigonometric and exponential forms with the above derived |v|, θ and φ. So, a 3D complex number in Cartesian form can be converted into trigonometric as well as exponential forms.

$\lvert v\rvert = \sqrt{a^2 + b^2 + c^2}$	(56)
$\frac{v}{\lvert v\rvert} = \frac{a + bi + cj}{\lvert v\rvert}$ $= \frac{a}{\lvert v\rvert} + \frac{b}{\lvert v\rvert}i + \frac{c}{\lvert v\rvert}j$	(57)
$\frac{a}{\lvert v\rvert} = \cos\theta\cos\varphi$	(58)
$\frac{b}{\lvert v\rvert} = \sin\theta\cos\varphi$	(59)
$\frac{c}{\lvert v\rvert} = \sin\varphi$	(60)
$\varphi = \sin^{-1}\left(\frac{c}{\lvert v\rvert}\right)$	(61)
$\cos\varphi = \sqrt{1 - \sin^2\varphi}$	(62)
$\sin\theta = \frac{b}{\lvert v\rvert\cos\varphi}$ $\cos\theta = \frac{a}{\lvert v\rvert\cos\varphi}$	(63)
$v_1 = a_1 + b_1 i + c_1 j$ $v_2 = a_2 + b_2 i + c_2 j$	(64)

Since the sign of cos φ is our choice we state the Rule 2.

Rule 2: The cosines of all the imaginary arguments except the first one are positive.

4. Algebraic operations

a. Addition and subtraction

$v_1 + v_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_1)j$	(65)
$v_1 - v_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_1)j$	(66)

Two 3D complex numbers can be added together. Let us take the two 3D complex numbers v_1 and v_2 in (64). The addition of v_1 and v_2 is done by adding their components dimension by dimension, see (65).

Since addition is defined, subtraction comes naturally by inverting the sign of v_2 in (65) , which gives (66).

If v_1 and v_2 are in trigonometric or exponential form, we have to convert them first into Cartesian form, add their components dimension by dimension, then convert the resulting number back into trigonometric or exponential form if necessary. So, trigonometric and exponential forms are not suited for addition.

- Geometrical meaning of addition

In classical 2D complex number system the addition of two complex number corresponds to the vector addition of the two corresponding vectors in the plane (h, i), see Figure 5. In the same way, the addition of two 3D complex numbers corresponds to the vector addition of the two corresponding vector in the 3D space (h, i, j).

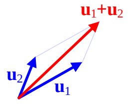

Figure 5

b. Multiplication

- Cartesian form and multiplication

Two 3D complex numbers can be multiplied together, but not in Cartesian form. Let us try to multiply the two 3D complex numbers in Cartesian form v_1 and v_2 given in (64). If we develop their product v_1v_2 the usual way, we get (67). Since ij and ji are not defined in the 3D complex space (h, i, j), equation (67) is wrong.

$$\begin{aligned} v_1v_2 &= (a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) \\ &= (a_1a_2 + b_1b_2 ii + c_1c_2 jj) + (a_1b_2 + b_1a_2)i + (a_1c_2 + c_1a_2)j + (c_1b_2 ij + b_1c_2 ji) \end{aligned} \tag{67}$$

So, we state the Rule 3 and Rule 4.

Rule 3: The products of imaginary units ij and ji are not defined in the 3D complex space (h, i, j).

Rule 4: Multiplication of complex numbers except 2D complex numbers should not be done in Cartesian form.

- Distributivity

Since 3D complex numbers should not be multiplied in Cartesian form, multiplication of 3D complex numbers is not distributive over addition. So, we state the Rule 5.

Rule 5: Multiplication of complex numbers except 2D complex numbers is not distributive over addition.

- Multiplication in trigonometric form

Let us write v_1 and v_2 in trigonometric form, see (68), and multiply them together in (69). The product v_1v_2 is simplified into (70) which is a valid 3D complex number. So, multiplication done in trigonometric form is correct.

$$\begin{aligned} v_1 &= |v_1|(\cos\theta_1 + \sin\theta_1\, i)(\cos\varphi_1 + \sin\varphi_1\, j) \\ v_2 &= |v_2|(\cos\theta_2 + \sin\theta_2\, i)(\cos\varphi_2 + \sin\varphi_2\, j) \end{aligned} \tag{68}$$

$$\begin{aligned} v_1v_2 &= |v_1|(\cos\theta_1 + \sin\theta_1\, i)(\cos\varphi_1 + \sin\varphi_1\, j)|v_2|(\cos\theta_2 + \sin\theta_2\, i)(\cos\varphi_2 + \sin\varphi_2\, j) \\ &= |v_1||v_2|(\cos\theta_1 + \sin\theta_1\, i)(\cos\theta_2 + \sin\theta_2\, i)(\cos\varphi_1 + \sin\varphi_1\, j)(\cos\varphi_2 + \sin\varphi_2\, j) \end{aligned} \tag{69}$$

$$v_1v_2 = |v_1||v_2|[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\, i][\cos(\varphi_1 + \varphi_2) + \sin(\varphi_1 + \varphi_2)\, j] \tag{70}$$

v_1 and v_2 are written in exponential form (71), their product v_1v_2 is computed in (72) which is simplified into (73). The imaginary exponents in (73) are added dimension by dimension, which gives $i(\theta_1 + \theta_2) + j(\varphi_1 + \varphi_2)$ corroborating well with the trigonometric form in (70). So, multiplication done in exponential form is also correct.

$$\begin{aligned} v_1 &= |v_1| e^{i\theta_1} e^{j\varphi_1} \\ v_2 &= |v_2| e^{i\theta_2} e^{j\varphi_2} \end{aligned} \tag{71}$$

One legitimate question is that, are we allowed to shuffle the arguments? In fact, the Definition 2 states that imaginary arguments commute within trigonometric and exponential forms. The shuffle of the arguments in (70) and (73) is a commutation of the arguments. So, the product v_1v_2 are correctly expressed in (70) and (73).

$$v_1v_2 = |v_1| e^{i\theta_1} e^{j\varphi_1} |v_2| e^{i\theta_2} e^{j\varphi_2} \tag{72}$$

$$v_1v_2 = |v_1||v_2| e^{i(\theta_1+\theta_2)+j(\varphi_1+\varphi_2)} \tag{73}$$

From the above we extract the Rule 6.

Rule 6: Multiplication of complex numbers except 2D complex numbers should be done in trigonometric or exponential forms.

In summary, for 3D complex numbers addition is done in Cartesian form and multiplication in trigonometric or exponential forms. Both operations are within the 3D complex space (h, i, j), which supports that the formulation of 3D complex number is correctly constructed.

c. Geometrical meaning of multiplication

Equations (70) and (73) show that in multiplication the arguments of the two 3D complex numbers add together. Let us see how the corresponding vectors behave. Take the two unit 3D complex numbers in (74) and multiply them together. First, v_1 is multiplied by $e^{i\theta_2}$ in (75) which gives the 3D complex number w for which the arguments θ_1 and θ_2 are added together, see (75).

$v_1 = e^{j\varphi_1}e^{i\theta_1} \qquad v_2 = e^{i\theta_2}e^{j\varphi_2}$	(74)
$w = v_1 e^{i\theta_2} = e^{j\varphi_1}e^{i\theta_1}e^{i\theta_2} = e^{j\varphi_1}e^{i(\theta_1+\theta_2)}$	(75)
$we^{j\varphi_2} = e^{i(\theta_1+\theta_2)}e^{j\varphi_1}e^{j\varphi_2}$	(76)
$v_1v_2 = we^{j\varphi_2} = e^{i(\theta_1+\theta_2)}e^{j(\varphi_1+\varphi_2)}$	(77)

This operation is shown geometrically in Figure 6 where v_1 was on the intersection line of the plane θ_1 and the cone φ_1. The multiplication by $e^{i\theta_2}$ rotates the plane θ_1 by the angle θ_2 which becomes the plane $\theta_1+\theta_2$. So, w is on the intersection line of the plane $\theta_1+\theta_2$ and the cone φ_1.

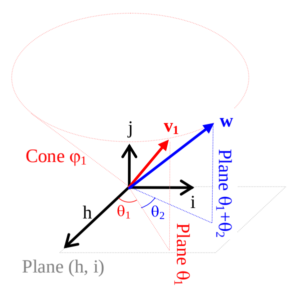

Figure 6

Then w is multiplied by $e^{j\varphi_2}$ in (76), which gives the product v_1v_2 in (77). The second imaginary argument of w equals φ_1 which is added with φ_2. This operation is shown geometrically in Figure 7 where w passes from the cone φ_1 to the cone $\varphi_1+\varphi_2$, and w equals the product v_1v_2 geometrically.

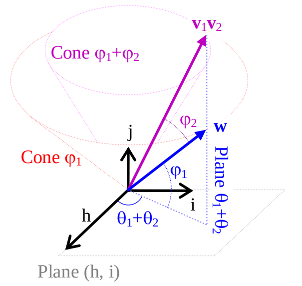

Figure 7

We see that the multiplication by $e^{i\theta_2}$ adds the arguments θ_1 and θ_2 together giving $\theta_1+\theta_2$ and the multiplication by $e^{j\varphi_2}$ adds the arguments φ_1 and φ_2 together giving $\varphi_1+\varphi_2$. So, the geometrical meaning of the multiplication of two 3D complex numbers shows: not only their arguments add together in the same way as for 2D complex numbers, but also the arguments of the two imaginary dimensions add independently.

d. Division

$$\frac{v_1}{v_2} = \frac{|v_1|e^{i\theta_1}e^{j\varphi_1}}{|v_2|e^{i\theta_2}e^{j\varphi_2}} = \frac{|v_1|}{|v_2|}e^{i(\theta_1-\theta_2)}e^{j(\varphi_1-\varphi_2)} \quad (78)$$

The reverse operation of multiplication is division. Since multiplication is well defined, division is also well defined. Let us divide the 3D complex number v_1 by v_2 given in (71). The ratio v_1/v_2 is computed in (78), which is converted using Euler's formula into trigonometric form in (79).

$$\frac{v_1}{v_2} = \frac{|v_1|}{|v_2|}[\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)\,i][\cos(\varphi_1-\varphi_2)+\sin(\varphi_1-\varphi_2)\,j] \quad (79)$$

e. Conjugate

For finding out the conjugate of a 2D complex number we just flip the sign of the imaginary part. But, what is the conjugate of a 3D complex number that has two imaginary arguments? Let us see what the function of conjugate is for 2D complex number. The product of a 2D complex number u with its conjugate $\bar{u}$ equals the modulus of u squared $|u|^2$, see (80). We extend this concept to complex numbers with more than two dimensions, which makes the conjugate of a 3D complex number to equal its modulus squared divide by itself, see (81).

So, we state the Definition 3.
Definition 3: The conjugate of a complex number equals its modulus squared divided by itself.

$u\bar{u} = \|u\|^2$	(80)
$\bar{v} = \frac{\|v\|^2}{v}$	(81)

Let us compute the conjugate of the 3D complex number v whose exponential and trigonometric forms are in (82). Dividing the modulus squared $|v|^2$ by v, we obtain the exponential form of the conjugate of v in (83) which is then converted into trigonometric form in (84).

The trigonometric form of v and $\bar{v}$ are converted into Cartesian form in (85) and (86) respectively.

$v = \|v\|e^{i\theta+j\varphi} = \|v\|(\cos\theta+\sin\theta\, i)(\cos\varphi+\sin\varphi\, j)$	(82)
$\bar{v} = \frac{\|v\|^2}{\|v\|e^{i\theta+j\varphi}} = \|v\|e^{-(i\theta+j\varphi)}$	(83)
$\bar{v} = \|v\|(\cos\theta-\sin\theta\, i)(\cos\varphi-\sin\varphi\, j)$	(84)
$v = \|v\|\cos\theta\cos\varphi + \|v\|\sin\theta\cos\varphi\, i + \|v\|\sin\varphi\, j$	(85)
$\bar{v} = \|v\|\cos\theta\cos\varphi - \|v\|\sin\theta\cos\varphi\, i - \|v\|\sin\varphi\, j$	(86)

By comparing (86) with (85), we find that (86) and (85) are in the same form except that all the imaginary terms are flipped to negative, which corroborates with the form of conjugate for 2D complex number. Then the general Cartesian form of a 3D complex number v and its conjugate are in (87).

We define that the sum of all the imaginary components of a complex number will be referred to as imaginary part and will be labeled as ι (the Greek letter iota). Then we state the Definition 4.
Definition 4: In Cartesian form the sum of all the imaginary components of a complex number is called imaginary part and is labeled as ι (the Greek letter iota).

Then we can write a complex number and its conjugate in (88) and we state the Rule 7.
Rule 7: In Cartesian form the conjugate of a complex number equals its real part minus its imaginary part.

$v = a + bi + cj$ $\bar{v} = a - (bi + cj)$	(87)	$v = a + \iota$ $\bar{v} = a - \iota$	(88)

f. Exponentiation

When we multiply v by itself again and again we will obtain the powers of v. Let us multiply the 3D complex number v in exponential form (51) by itself m times, we obtain v to the m^{th} power, see (89). v^m is converted into trigonometric form in (90), then using (34) into Cartesian form in (91). Notice that the values of (89), (90) and (91) are all in the complex space (h, i, j).

$v^m = vv \cdots v = \|v\|^m e^{m(i\theta + j\varphi)} = \|v\|^m e^{im\theta + jm\varphi}$	(89)
$v^m = \|v\|^m (\cos m\theta + \sin m\theta\, i)(\cos m\varphi + \sin m\varphi\, j)$	(90)
$v^m = \|v\|^m (\cos m\theta \cos m\varphi + \sin m\theta \cos m\varphi\, i + \sin m\varphi\, j)$	(91)

$\sqrt[m]{v} = \sqrt[m]{\|v\|} e^{\frac{i\theta + j\varphi}{m}}$	(92)

g. m^{th} root

Since exponentiation of a 3D complex number is well defined, its reverse operation m^{th} root is well defined too. The m^{th} root of the 3D complex number v in exponential form (51) is given in (92) which is obtained by reversing (89). The trigonometric and Cartesian forms of the complex number in (92) are converted from (92) and are in the same form as (90) and (91) except that m will be replaced by m^{-1}.

h. Polynomial

Using v to the m^{th} power, we construct the m^{th} degree polynomial of v, see (93). The value of this polynomial is within the 3D complex space (h, i, j) because all the terms are within this space.

$P(v) = a_m v^m + \cdots + a_1 v + a_0 = a_m \|v\|^m e^{m(i\theta + j\varphi)} + \cdots + a_1 \|v\| e^{i\theta + j\varphi} + a_0$	(93)

5. 3D complex number system

We have constructed the 3D complex space (h, i, j), constructed 3D complex number in Cartesian, trigonometric and exponential forms, given the methods that convert a 3D complex number from one form to another, defined the conjugate of a 3D complex number and the algebraic operations which operate within the constructed 3D complex space and have geometrical meanings. So, the 3D complex number system is well built.

- Multiplication formula for Cartesian form

Let us search for a formula that directly computes the Cartesian components of the product of two 3D complex numbers v_1 and v_2. That is, to express the the product $v_1 v_2$ with the Cartesian components of v_1 and v_2.

The Cartesian form of a 3D complex numbers is $v = a + bi + cj$, see (36). But here we write v in the form of (94) with 4 real numbers A, B, C and D rather than the 3 components in (36) because this form makes the following derivation much simpler. Because A, B, C and D are not the Cartesian components of v, let us give them the name sub-component.

$v = (A + Bi)C + Dj$ $= AC + BCi + Dj$		(94)
$\|v\| = \sqrt{A^2C^2 + B^2C^2 + D^2}$		(95)
$\frac{v}{\|v\|} = (a + bi)c + dj$		(96)
$A = \|v\|a$ $B = \|v\|b$	$C = c$ $D = \|v\|d$	(97)

The modulus of v is |v| and is computed in (95). We divide (94) with |v|, which gives the unit 3D complex number in (96) expressed with the 4 sub-components a, b, c and d. By equating (94) with (96), we define the relation between A, B, C and D and a, b, c and d in (97).

The unit 3D complex number in (96) is also written with its arguments in (98), which was derived in (54). By comparing (96) with (98), we find that the 4 sub-components a, b, c and d are the sin and cos of the 2 arguments, see (99).

$\frac{v}{\|v\|} = (\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j$		(98)
$\cos\theta = a$ $\sin\theta = b$	$\cos\varphi = c$ $\sin\varphi = d$	(99)

Now, let us write the two 3D complex numbers v_1 and v_2 in (100) which we will multiply. Using (98) v_1 and v_2 are also expressed with their arguments in (101). As it is convenient to multiply in exponential form, we write $\frac{v_1}{|v_1|}$ and $\frac{v_2}{|v_2|}$ in exponential form in (102), see (51), and multiply them in (103). Using equation (54), the expression (103) is converted into (104).

$$v_1 = (A_1 + B_1 i)C_1 + D_1 j \qquad v_2 = (A_2 + B_2 i)C_2 + D_2 j \tag{100}$$

$$\frac{v_1}{|v_1|} = (\cos\theta_1 + \sin\theta_1\, i)\cos\varphi_1 + \sin\varphi_1\, j \qquad \frac{v_2}{|v_2|} = (\cos\theta_2 + \sin\theta_2\, i)\cos\varphi_2 + \sin\varphi_2\, j \tag{101}$$

$$\frac{v_1}{|v_1|} = e^{i\theta_1}e^{j\varphi_1} \qquad \frac{v_2}{|v_2|} = e^{i\theta_2}e^{j\varphi_2} \tag{102}$$

We develop the sin and cos of the equation (104) in (105) and (106), which are expressed with the 8 sub-components a_1, b_1, c_1 and d_1, a_2, b_2, c_2 and d_2 by using (99). Then, (105) and (106) are introduced into (104) to give (107). The expression of the product v_1v_2 is obtained in (108) by multiplying (107) with the modulus $|v_1||v_2|$.

$$\frac{v_1v_2}{|v_1||v_2|} = e^{i\theta_1}e^{j\varphi_1}\, e^{i\theta_2}e^{j\varphi_2} = e^{i(\theta_1+\theta_2)}e^{j(\varphi_1+\varphi_2)} \tag{103}$$

$$\frac{v_1v_2}{|v_1||v_2|} = (\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\, i)\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\, j \tag{104}$$

Using the formula (97) in equations (109) and (110), which are introduced into (108), the final expression of the product v_2v_2 with the original sub-components of v_1 and v_2 is (111) which can be expressed with the Cartesian components of v_1 and v_2.

$$\begin{aligned}\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = a_1a_2 - b_1b_2\\ \sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = b_1a_2 + a_1b_2\end{aligned} \tag{105}$$

$$\begin{aligned}\cos(\varphi_1+\varphi_2) &= \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 = c_1c_2 - d_1d_2\\ \sin(\varphi_1+\varphi_2) &= \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 = d_1c_2 + c_1d_2\end{aligned} \tag{106}$$

$$\frac{v_1v_2}{|v_1||v_2|} = \big((a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i\big)(c_1c_2 - d_1d_2) + (d_1c_2 + c_1d_2)j \tag{107}$$

$$v_1v_2 = |v_1||v_2|\Big(\big((a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i\big)(c_1c_2 - d_1d_2) + (d_1c_2 + c_1d_2)j\Big) \tag{108}$$

$$\begin{aligned}|v_1||v_2|(a_1a_2 - b_1b_2) &= A_1A_2 - B_1B_2\\ |v_1||v_2|(b_1a_2 + a_1b_2) &= B_1A_2 + A_1B_2\end{aligned} \tag{109}$$

$$\begin{aligned}c_1c_2 - d_1d_2 &= C_1C_2 - \frac{D_1D_2}{|v_1||v_2|}\\ |v_1||v_2|(d_1c_2 + c_1d_2) &= (|v_1||v_2|d_1c_2 + c_1|v_1||v_2|d_2)\\ &= D_1|v_2|C_2 + |v_1|C_1D_2\end{aligned} \tag{110}$$

$$\begin{aligned}v_1v_2 &= (A_1C_1 + B_1C_1 i + D_1 j)(A_2C_2 + B_2C_2 i + D_2 j)\\ &= \big((A_1A_2 - B_1B_2) + (B_1A_2 + A_1B_2)i\big)\left(C_1C_2 - \frac{D_1D_2}{|v_1||v_2|}\right) + (D_1|v_2|C_2 + |v_1|C_1D_2)j\\ |v_1| &= \sqrt{{A_1}^2{C_1}^2 + {B_1}^2{C_1}^2 + {D_1}^2} \qquad |v_2| = \sqrt{{A_2}^2{C_2}^2 + {B_2}^2{C_2}^2 + {D_2}^2}\end{aligned} \tag{111}$$

Let us explain (111) with the two simple 3D complex numbers v_1 and v_2 given in (112). For using equation (111), we write v_1 and v_2 with sub-components in (113) which gives the values of C_1, D_1, B_2 and C_2 in (114). We apply (114) in (111) and get (115). The modulus of v_1 is computed in (116), which is introduced into (115) to give (117). The final expression of the product v_1v_2 is (117) which is expressed with the Cartesian components of v_1 and v_2.

$$v_1 = A_1 + B_1 i \qquad v_2 = A_2 + D_2 j \tag{112}$$

$$v_1 = (A_1 + B_1 i)1 + 0j \qquad v_2 = (A_2 + 0i)1 + D_2 j \tag{113}$$

$$C_1 = 1 \qquad D_1 = 0 \qquad B_2 = 0 \qquad C_2 = 1 \tag{114}$$

$$\begin{aligned}v_1v_2 &= \big((A_1A_2 - 0) + (B_1A_2 + 0)i\big)\left(1 - \frac{0}{|v_1||v_2|}\right) + (0 + |v_1|D_2)j\\ &= A_1A_2 + B_1A_2 i + |v_1|D_2 j\end{aligned} \tag{115}$$

$$|v_1| = \sqrt{{A_1}^2 + {B_1}^2} \tag{116}$$

$$(A_1 + B_1 i)(A_2 + D_2 j) = A_1A_2 + B_1A_2 i + D_2\sqrt{{A_1}^2 + {B_1}^2}\, j \tag{117}$$

6. Complex number systems with higher dimension

$$\begin{aligned}ii &= -1\\ jj &= -1\\ kk &= -1\end{aligned} \tag{118}$$

a. Extension to 4 dimensions

- 4D complex space

For explaining the new concept of multidimensional complex number in more detail, let us construct the 4D complex space and number using the new definitions and rules stated above. The 4 axes of the 4D complex space are h, i, j and k, with h being the axis of real numbers, i, j and k being the axes of the first, second and third imaginary numbers. This 4D complex space is labeled as (h, i, j, k). The 3 imaginary units are also labeled as i, j and k respectively and their square equal -1, see (118). In Cartesian coordinate system the 4 components of a 4D complex number are ‘a’, b, c and d, see (119), with ‘a’ being the real component, b, c and d the first, second and third imaginary components.

We construct the 4D complex space (h, i, j, k) by adding the third imaginary dimension k perpendicularly to the 3D complex space the (h, i, j). In Figure 8 the horizontal vector v is in the 3D complex space (h, i, j) and the vertical vector k is the third imaginary unit. As 4D complex space cannot be visualized geometrically, the horizontal line represents the whole 3D space (h, i, j). The axis k is perpendicular to the horizontal line and thus, perpendicular to the axes h, i and j.

$$w = a + bi + cj + dk \qquad (119)$$

- Trigonometric form

A 4D complex number is constructed in the same way as for 3D complex number, see equations from (39) to (50). For explaining the process of construction in a general manner, we start the construction from the 1D space which is the real number line h. We take a 1D vector on the axis h and rotate it into 2D, 3D and 4D spaces. The 4D complex space and number are constructed by the following Procedure 1.

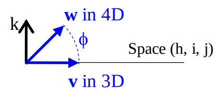

Figure 8

Procedure 1

1. The starting space is labeled as S_1 which is the real number line h.
2. Adding the imaginary axis i perpendicularly to S_1 gives the 2D space which is labeled as S_2, see Figure 2.
3. The starting vector is r and is positively pointing on the axis h. Its length is |r|.
4. Rotate the vector r toward the axis i by the angle θ. The resulting vector is the 2D complex number u whose modulus is |u| and argument θ.
5. According to Rule 1, the trigonometric form of u equals r multiplied by the Euler’s formula for θ and i, see (120) and (19).
6. Add the imaginary axis j perpendicularly to the space S_2, see Figure 3. The resulting space has 3 dimensions and is labeled as S_3.
7. Rotate the vector u toward the axis j by the angle φ. The resulting vector is the 3D complex number v whose modulus is |v| and whose argument about axis i is θ, that about the axis j is φ.
8. According to Rule 1, the trigonometric form of v equals u multiplied by the Euler’s formula for φ and j, see (121) and (48).
9. Add the imaginary axis k perpendicularly to the space S_3, see Figure 8. The resulting space has 4 dimensions and is labeled as S_4.
10. Rotate the vector v toward the axis k by the angle ϕ. The resulting vector is the 4D complex number w whose modulus is |w| and whose argument about the axis i is θ, that about the axis j is φ, that about the axis k is ϕ.
11. According to Rule 1, the trigonometric form of w equals v multiplied by the Euler’s formula for ϕ and k, see (122).

As the modulus of the rotated vector stays constant during the rotations, the moduli of the 4 vectors r, u, v and w are equal, see the equality (123). By combining the equations (120), (121), (122) and (123), we obtain (124) which is the trigonometric form of w, a 4D complex number in the 4D complex space (h, i, j, k).

$$u = r(\cos\theta + \sin\theta\, i) \qquad (120)$$

$$v = u(\cos\varphi + \sin\varphi\, j) \qquad (121)$$

$$w = v(\cos\phi + \sin\phi\, k) \qquad (122)$$

$$|w| = |v| = |u| = |r| = r \qquad (123)$$

By comparing (124) with (50) we notice that the trigonometric form of a 4D as well as 3D complex number is a chain product of Euler's formulas for the imaginary dimensions.

$$w = |w|(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)(\cos\phi + \sin\phi\, k) \qquad (124)$$

- Exponential form

Replacing each Euler's formula in (124) with the corresponding exponential function, we obtain the exponential form of w in (125).

- Cartesian form

The Cartesian form of w is converted from its trigonometric form (124). Let us write each Euler's formula in (124) as unit vectors which are u_θ ,u_φ and u_ϕ in (126), (127) and (128). Then w is expressed with u_θ ,u_φ and u_ϕ in (129).

First, the product $u_\theta u_\varphi$ is developed in (130) where we have the term $u_\theta j$.

$$w = |w|e^{i\theta}e^{j\varphi}e^{k\phi} \qquad (125)$$

$$u_\theta = \cos\theta + \sin\theta\, i \qquad (126)$$

$$u_\varphi = \cos\varphi + \sin\varphi\, j \qquad (127)$$

$$u_\phi = \cos\phi + \sin\phi\, k \qquad (128)$$

$$w = |w|u_\theta u_\varphi u_\phi \qquad (129)$$

According to the Definition 1, see (46), the term $u_\theta j$ equals $|u_\theta|j$, see (132). Because u_θ is a unit vector, its modulus equals 1, see (133), then $u_\theta j=j$, see (134), which is introduced into (130) to give (131). By introducing (126) into (131) we get (138).

$u_\theta u_\varphi = u_\theta(\cos\varphi + \sin\varphi\, j) = u_\theta\cos\varphi + \sin\varphi\, u_\theta j$	(130)
$u_\theta u_\varphi = u_\theta\cos\varphi + \sin\varphi\, j$	(131)

$u_\theta j = \lvert u_\theta\rvert j$	(132)
$\lvert u_\theta\rvert=1$	(133)
$u_\theta j = j$	(134)

$u_\theta u_\varphi k = \lvert u_\theta u_\varphi\rvert k$	(135)
$\lvert u_\theta u_\varphi\rvert = 1$	(136)
$u_\theta u_\varphi k = k$	(137)

In the same way, we write $u_\theta u_\varphi u_\phi$ in (139), which contains the term $u_\theta u_\varphi k$. According to the Definition 1, the term $u_\theta u_\varphi k$ equals $|u_\theta u_\varphi|k$, see (135). The modulus of $u_\theta u_\varphi$ was computed in (55) and $|u_\theta u_\varphi|$ equals 1, see (136), then the terms $|u_\theta u_\varphi|k$ and $u_\theta u_\varphi k$ equal k, see (137), which is introduced into (139) to give (140). By introducing (138) into (140) we get (141). Applying (141) to (129) gives (142) which is the Cartesian form of w. Then the Cartesian components a, b, c and d of w are expressed in (143).

$u_\theta u_\varphi = (\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j$	(138)
$u_\theta u_\varphi u_\phi = u_\theta u_\varphi(\cos\phi + \sin\phi\, k) = u_\theta u_\varphi\cos\phi + \sin\phi\, u_\theta u_\varphi k$	(139)
$u_\theta u_\varphi u_\phi = u_\theta u_\varphi\cos\phi + \sin\phi\, k$	(140)
$u_\theta u_\varphi u_\phi = \big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)\cos\phi + \sin\phi\, k$	(141)
$w = \lvert w\rvert(\cos\theta\cos\varphi\cos\phi + \sin\theta\cos\varphi\cos\phi\, i + \sin\varphi\cos\phi\, j + \sin\phi\, k)$	(142)

- Conversion from Cartesian form to trigonometric form

We can compute the modulus of a 4D complex number w from its components a, b, c and d, see (144). From (143) we get the equation (145), (146) and (147). According to the Rule 2 cos φ and cos ϕ are positive. The solution of (145) will give θ, that of (146) will give φ and that of (147) will give ϕ. Then the trigonometric form of w can be written with |w|, θ, φ and ϕ.

$a = \lvert w\rvert\cos\theta\cos\varphi\cos\phi$ $b = \lvert w\rvert\sin\theta\cos\varphi\cos\phi$ $c = \lvert w\rvert\sin\varphi\cos\phi$ $d = \lvert w\rvert\sin\phi$	(143)
$\lvert w\rvert = \sqrt{a^2+b^2+c^2+d^2}$	(144)
$\sin\phi = \dfrac{d}{\lvert w\rvert}$	(145)
$\sin\varphi = \dfrac{c}{\lvert w\rvert\cos\phi}$	(146)
$\sin\theta = \dfrac{b}{\lvert w\rvert\cos\varphi\cos\phi}$ $\cos\theta = \dfrac{a}{\lvert w\rvert\cos\varphi\cos\phi}$	(147)

b. Complex number with any number of dimensions

- Extension to n dimensions

In the Procedure 1 above, we have constructed 4D complex space from 3D complex space; 3D complex space from 2D complex space; 2D complex space from 1D complex space. Using recurrent method, if we could show that nD complex space can be constructed from n-1D complex space, we would have proven that complex space with any number of dimensions can be constructed.

Let us construct the complex space with n dimensions from the complex space S_{n-1} which has n-1 dimensions. Its first dimension is the real numbers line labeled as h. Its following dimensions are labeled as $i_2, i_3, \ldots, i_{n-1}$ with the subscripts being the rank of the dimensions. The imaginary units corresponding to these dimensions are also labeled as $i_2, i_3, \ldots, i_{n-1}$, see (148). A vector called v_{n-1} has been constructed in S_{n-1} up from the vector r on the real axis h, see the Procedure 1. v_{n-1} corresponds to complex number v_{n-1} whose arguments are labeled as $\theta_2, \theta_3, \ldots, \theta_{n-1}$ meaning that v_{n-1} has been rotated by these angles.

$i_2 i_2 = -1$ $i_3 i_3 = -1$ $\ldots$ $i_{n-1}i_{n-1} = -1$	(148)

This construction is done with the Procedure 2 which is the steps 9, 10 and 11 of the Procedure 1.

Procedure 2

1. Add the imaginary axis i_n perpendicularly to the complex space S_{n-1}. The resulting space has n dimensions and is labeled as S_n, see Figure 9.
2. Rotate the vector v_{n-1} toward the axis i_n by the angle θ_n. The resulting vector is the nD complex number v_n whose modulus is $|v_n|$ and whose arguments about the axes $i_2, i_3, \ldots, i_{n-1}, i_n$ are $\theta_2, \theta_3, \ldots, \theta_{n-1}, \theta_n$ respectively.
3. According to Rule 1, the trigonometric form of v_n equals v_{n-1} multiplied by the Euler's formula for θ_n and i_n, see (149).

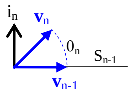

Figure 9

$v_n = v_{n-1}(\cos\theta_n + \sin\theta_n\, i_n)$	(149)

So, the nD complex space S_n and complex number v_n are constructed from the n-1D complex space S_{n-1} and the complex number v_{n-1}. Then, complex space and number with any number of dimensions can be constructed.

- Trigonometric and exponential form

Using (149), we can express all complex number in the complex spaces from S_1 to S_n. Equation (150) expresses the chain of vectors created in each space as the product of the vector of a lower space with the Euler's formulas corresponding to the rotation into the higher space. The chain of vectors exposes the rotations from the 1D vector r to the nD vector v_n which results in the trigonometric form of v_n which is a chain product of Euler's formulas, see (151).

$$\begin{aligned} v_1 &= r \\ v_2 &= v_1(\cos\theta_2 + \sin\theta_2\, i_2) \\ &\dots \\ v_{n-1} &= v_{n-2}(\cos\theta_{n-1} + \sin\theta_{n-1}\, i_{n-1}) \\ v_n &= v_{n-1}(\cos\theta_n + \sin\theta_n\, i_n) \end{aligned} \quad (150)$$

By replacing all the Euler's formulas in (151) with their corresponding exponential functions we obtain (152) which is the exponential form of the nD complex number v_n.

$$v_n = r(\cos\theta_2 + \sin\theta_2\, i_2)\cdots(\cos\theta_n + \sin\theta_n\, i_n) \quad (151)$$

$$v_n = |v_n| e^{i_2\theta_2} \dots e^{i_n\theta_n} = |v_n| e^{i_2\theta_2+\dots+i_n\theta_n} \quad (152)$$

- Cartesian form

For expressing a nD complex number in Cartesian form, we have to convert from its trigonometric form. Using Rule 1, see (48), we get the Cartesian form of the complex numbers v_2 in (153) and the incomplete Cartesian form of v_3 in (154) from (150). Then by introducing (153) into (154), we get the complete Cartesian form of v_3 in (155), which is simplified into (157) with (156).

$$v_2 = v_1(\cos\theta_2 + \sin\theta_2\, i_2) = v_1\cos\theta_2 + \sin\theta_2\,|v_1| i_2 \quad (153)$$

$$v_3 = v_2(\cos\theta_3 + \sin\theta_3\, i_3) = v_2\cos\theta_3 + \sin\theta_3\,|v_2| i_3 \quad (154)$$

$$v_3 = (v_1\cos\theta_2 + \sin\theta_2\,|v_1| i_2)\cos\theta_3 + \sin\theta_3\,|v_2| i_3 \quad (155)$$

$$|r| = |v_1| = |v_2| \quad (156)$$

$$v_3 = |r|\big((\cos\theta_2 + \sin\theta_2\, i_2)\cos\theta_3 + \sin\theta_3\, i_3\big) \quad (157)$$

$$v_n = v_{n-1}\cos\theta_n + \sin\theta_n\,|v_{n-1}| i_n \quad (158)$$

$$|r| = |v_1| = |v_2| = |v_3| = \dots = |v_n| \quad (159)$$

For getting the incomplete Cartesian form of v_n we do the same operation as for v_3 and obtain it in (158) by using Rule 1 and (150). We get the incomplete Cartesian form of all the vectors v_{n-1}, v_{n-2},…, v_4 in the same way. By introducing the incomplete Cartesian form of v_{n-1} into that of v_n, the one of v_{n-2} into that of v_{n-1} and so on till v_1, we get the Cartesian form of v_n in (160) whose modulus equals |r| because v_2, …, v_n are constructed by rotations which do not change their moduli, see (159).

$$v_n = |r|\Big(\big(\cdots\big((\cos\theta_2 + \sin\theta_2\, i_2)\cos\theta_3 + \sin\theta_3\, i_3\big)\cdots\big)\cos\theta_n + \sin\theta_n\, i_n\Big) \quad (160)$$

We insist that in the trigonometric form the dimensions must be in increasing order. For example, if the two factors in (154) were reversed as (161) shows and if Rule 1 were applied bluntly, we would get (162) which is wrong because different from (155). So, we state the Rule 8.

$$\begin{aligned} v_{3i} &= (\cos\theta_3 + \sin\theta_3\, i_3) v_2 \\ &= (\cos\theta_3 + \sin\theta_3\, i_3) v_1 (\cos\theta_2 + \sin\theta_2\, i_2) \end{aligned} \quad (161)$$

$$v_{3i} = (v_2\cos\theta_3 + \sin\theta_3\,|v_2| i_3)\cos\theta_2 + \sin\theta_2\,|v_1| i_2 \quad (162)$$

Rule 8: The dimensions in the trigonometric form must be in increasing order before conversion into Cartesian form.

- Conversion into trigonometric form

For converting the nD complex number v_n (163) from Cartesian form into trigonometric form we should first compute its modulus as shown in (164). Then we divide v_n with its modulus and obtain the components free of modulus in (165). On the other hand when we divide (160) with its modulus we obtain (166). By equating (165) with (166) we obtain (167) which expresses the components free of modulus with all the arguments θ_2, θ_3, …, θ_n.

$$v_n = A + B_2 i_2 + B_3 i_3 + \dots + B_n i_n \quad (163)$$

$$|v_n| = \sqrt{A^2 + {B_2}^2 + {B_3}^2 + \dots + {B_n}^2} \quad (164)$$

$$\frac{v_n}{|v_n|} = a + b_2 i_2 \cdots + b_n i_n \quad (165)$$

According to the Rule 2, the cosines of θ_3, …, θ_n are all positive. So, we can derive θ_3, …, θ_n with equation (168) and θ_2 with (169). Then the trigonometric and exponential forms of the nD complex number v_n can be expressed with the above derived modulus $|v_n|$ and arguments θ_2, θ_3, …, θ_n.

$$\frac{v_n}{|v_n|} = (\cos\theta_2 \cos\theta_3 \cdots \cos\theta_n) + \sin\theta_2 (\cos\theta_3 \cdots \cos\theta_n) i_2 + \cdots + \sin\theta_{n-1} \cos\theta_n \, i_{n-1} + \sin\theta_n \, i_n \quad (166)$$

$$\begin{aligned} a &= \cos\theta_2 \cos\theta_3 \cdots \cos\theta_n \\ b_2 &= \sin\theta_2 (\cos\theta_3 \cdots \cos\theta_n) \\ b_3 &= \sin\theta_3 (\cos\theta_4 \cdots \cos\theta_n) \\ &\dots \\ b_{n-1} &= \sin\theta_{n-1} \cos\theta_n \\ b_n &= \sin\theta_n \end{aligned} \quad (167)$$

$$\begin{aligned} \sin\theta_n &= b_n \\ \sin\theta_{n-1} &= \frac{b_{n-1}}{\cos\theta_n} \\ &\dots \\ \sin\theta_3 &= \frac{b_3}{\cos\theta_4 \cdots \cos\theta_n} \end{aligned} \quad (168)$$

$$\begin{aligned} \sin\theta_2 &= \frac{b_2}{\cos\theta_3 \cdots \cos\theta_n} \\ \cos\theta_2 &= \frac{a}{\cos\theta_3 \cdots \cos\theta_n} \end{aligned} \quad (169)$$

So, nD complex numbers with components of arbitrary values can be converted into trigonometric and exponential forms, which shows that the nD complex space S_n is completely represented by the above constructed nD complex numbers.

- Addition and subtraction

Addition and subtraction of nD complex numbers is done in Cartesian form. Take the two nD complex numbers in (170) and add them together or subtract v_a by v_b, the result is expressed in (171).

$$\begin{aligned} v_a &= a_1 + a_2 i_1 \cdots + a_n i_n \\ v_b &= b_1 + b_2 i_1 \cdots + b_n i_n \end{aligned} \quad (170)$$

$$v_a \pm v_b = (a_1 \pm b_1) + (a_2 \pm b_2) i_1 \cdots + (a_n \pm b_n) i_n \quad (171)$$

- Multiplication and division

Multiplication of nD complex numbers is done in trigonometric or exponential forms. For example, the two nD complex numbers v_a and v_b in (172) are multiplied together and the product $v_a v_b$ is computed in (173) where the modulus $|v_a|$ and $|v_b|$ are multiplied together and the arguments are added dimension by dimension in the exponent. When v_a is divide by v_b their ratio is shown in (174).

For expressing the product or the ratio in trigonometric form, we just have to replace each exponential function with the corresponding Euler's formula and the trigonometric form of the product and the ratio is expressed in (175). Equation (175) can be converted into Cartesian form using (160), which shows that multiplication operates well within the nD complex space.

$$\begin{aligned} v_a &= |v_a| e^{i_2\alpha_2 + \cdots + i_n\alpha_n} \\ v_b &= |v_b| e^{i_2\beta_2 + \cdots + i_n\beta_n} \end{aligned} \quad (172)$$

$$v_a v_b = |v_a||v_b| e^{i_2(\alpha_2+\beta_2)} \dots e^{i_n(\alpha_n+\beta_n)} \quad (173)$$

$$\frac{v_a}{v_b} = \frac{|v_a|}{|v_b|} e^{i_2(\alpha_2-\beta_2)} \dots e^{i_n(\alpha_n-\beta_n)} \quad (174)$$

$$v_a (v_b)^{\pm 1} = |v_a||v_b|^{\pm 1} (\cos(\alpha_2 \pm \beta_2) + \sin(\alpha_2 \pm \beta_2) i_2) \cdots (\cos(\alpha_n \pm \beta_n) + \sin(\alpha_n \pm \beta_n) i_n) \quad (175)$$

- Conjugate

The Definition 3 state that the conjugate of a complex number equals its modulus squared divided by itself which gives (176). The nD complex number v_n is expressed in exponential form in (177), of which the conjugate is expressed in (178) using (176).

$$\overline{v_n} = \frac{|v_n|^2}{v_n} \quad (176) \qquad v_n = |v_n| e^{i_2\theta_2 + \cdots + i_n\theta_n} \quad (177)$$

$$\overline{v_n} = \frac{|v_n|^2}{|v_n| e^{i_2\theta_2 + \cdots + i_n\theta_n}} = |v_n| e^{-(i_2\theta_2 + \cdots + i_n\theta_n)} \quad (178)$$

The Rule 7 states that in Cartesian form the conjugate of a complex number equals its real part minus its imaginary part. The imaginary part of v_n is ι_n which is shown in (179), then v_n and its conjugate are expressed in (180).The exponential form of v_n (178) can be converted into the Cartesian form using (160).

$$\iota_n = b_1 i_1 \cdots + b_n i_n \quad (179)$$

$$\begin{aligned} v_n &= a + \iota_n \\ \overline{v_n} &= a - \iota_n \end{aligned} \quad (180)$$

- Exponentiation, m^{th} root and polynomial

The nD complex number v_n to the m^{th} power equals v_n multiplied m time by itself. The expression of v_n in (152) multiplied by itself m times gives v_n to the m^{th} power in (181). Reversing the expression in (181) gives the m^{th} root of v_n in (182). The m^{th} degree polynomial of v_n is in the same form as (93).

$${v_n}^m = |v_n|^m e^{m(i_2\theta_2 + \cdots + i_n\theta_n)} \quad (181)$$

$$\sqrt[m]{v_n} = \sqrt[m]{|v_n|} e^{\frac{i_2\theta_2 + \cdots + i_n\theta_n}{m}} \quad (182)$$

c. nD complex number system

We have constructed the nD complex space, constructed nD complex number in Cartesian, trigonometric and exponential forms, given the methods that convert a nD complex number from one form to another, defined the conjugate of a nD complex number and the algebraic operations which operate within the constructed nD complex space $(h, i_2, i_3, \dots, i_n)$. So, the nD complex number system is well built.

7. Discussion

Above we have explained the multidimensional complex number systems. The pattern of construction of multidimensional complex spaces and numbers is the following:

- Add an axis of imaginary number perpendicularly to an existing complex space with n dimensions. The result is a complex space with n+1 dimensions.
- The vectors of the higher space are created by rotating the vectors of the lower space toward the newly added axis.
- The trigonometric expression of a created complex number in the higher space equals the complex number in the lower space multiplied by the Euler's formula for the rotated angle and the imaginary unit newly added.

Let us see how the classical complex space and number are constructed:

- Add the axis of imaginary number perpendicularly to the real line which is a complex space with 1 dimension. The result is the complex space with 1+1=2 dimensions.
- The vectors of the higher space, the 2D complex space, are created by rotating the vectors of the lower space, the real line, toward the newly added axis, the axis of i.
- The trigonometric expression of a created complex number in the 2D complex space equals the complex number in the real line multiplied by the Euler's formula for the rotated angle and i.

We see that the patterns of construction of multidimensional complex spaces and numbers and that of the classical 2D complex space and number are the same. Let us compare the trigonometric and exponential forms of 2D, 3D, 4D and nD complex numbers:

- 2D complex number (183), see (19) and (25)
- 3D complex number (184), see (50) and (51)
- 4D complex number (185), see (124) and (125)
- nD complex number (186), see (151) and (152)

$u = \|u\|(\cos\theta + \sin\theta\, i)$	$= \|u\|e^{i\theta}$	(183)
$v = \|v\|(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)$	$= \|v\|e^{i\theta + j\varphi}$	(184)
$w = \|w\|(\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)(\cos\phi + \sin\phi\, k)$	$= \|w\|e^{i\theta}e^{j\varphi}e^{k\phi}$	(185)
$v_n = r(\cos\theta_2 + \sin\theta_2\, i_2)\cdots(\cos\theta_n + \sin\theta_n\, i_n)$	$= \|v_n\|e^{i_2\theta_2 + \cdots + i_n\theta_n}$	(186)

We see that the trigonometric and exponential forms of 2D, 3D, 4D and nD complex number follow the same patterns. Let us compare the multiplication formulas:

- 2D complex number (187), see (38)
- 3D complex number (188), see (70)
- nD complex number (189), see (175)

u_1u_2	$= \|u_1\|\|u_2\|(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\, i)$	(187)
v_1v_2	$= \|v_1\|\|v_2\|[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\, i][\cos(\varphi_1 + \varphi_2) + \sin(\varphi_1 + \varphi_2)\, j]$	(188)
$v_a(v_b)^{\pm 1}$	$= \|v_a\|\|v_b\|^{\pm 1}(\cos(\alpha_2 \pm \beta_2) + \sin(\alpha_2 \pm \beta_2)\, i_2)\cdots(\cos(\alpha_n \pm \beta_n) + \sin(\alpha_n \pm \beta_n)\, i_n)$	(189)

We see that all the 3 multiplication formulas consist of multiplying the modulus and adding the argument dimension by dimension which are the same pattern.

So, for 2D, 3D, 4D and nD complex numbers, the patterns of construction are the same, the patterns of the trigonometric and exponential forms are the same and the patterns of multiplication formulas are the same. Also, the patterns of the algebraic operations, the conjugate element and the correspondence between the nD complex and vector spaces are the same. In consequence 3D, 4D and nD complex number systems are exact extensions of the classical 2D complex number system.

I have to declare that I'm not a professional mathematician and I apology for the poor exposition of this theory and the possible mathematical errors in this article. However, since I'm confident that my theory contains valuable discovery for mathematics, I have decided to publish it and hope that professional mathematicians will benefit from it.

8. Appendix: another concrete example of multiplication

It is clearer to explain the working of the 3D complex number system with concrete number. Let us take an actual product: $(1i+4h)(1i+5j)$, what is it value? For converting $(1i+4h)$ into trigonometric form, we first compute its modulus and arguments in equations (190), (191) and (192). Then, we compute the modulus and arguments of $(1i+5j)$ in (193), (194) and (195).

$1i+4h=4+1i$ $=\lvert u\rvert(\cos\theta+\sin\theta\, i)$ $=\lvert u\rvert\big((\cos\theta+\sin\theta\, i)*1+0j\big)$ $=\lvert u\rvert\big((\cos\theta+\sin\theta\, i)\cos 0+\sin 0\, j\big)$	(190)
$\lvert u\rvert=\sqrt{4^2+1}$	(191)
$\cos\theta=\frac{4}{\sqrt{4^2+1}},\sin\theta=\frac{1}{\sqrt{4^2+1}}$	(192)

Equation (190) and (193) are in Cartesian form but also in the form of (34). Equation (34) is converted into trigonometric form by (50). So, equation (190) and (193) are converted the same way in (196) and (197).

$1i+5j$ $=\lvert v\rvert(\cos\varphi\, i+\sin\varphi\, j)$ $=\lvert v\rvert\big((0+i)\cos\varphi+\sin\varphi\, j\big)$ $=\lvert v\rvert\left(\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}i\right)\cos\varphi+\sin\varphi\, j\right)$	(193)
$\lvert v\rvert=\sqrt{5^2+1}$	(194)
$\cos\varphi=\frac{1}{\sqrt{5^2+1}},\sin\varphi=\frac{5}{\sqrt{5^2+1}}$	(195)

$1i+4h=\lvert u\rvert(\cos\theta+\sin\theta\, i)(\cos 0+\sin 0\, j)$	(196)
$1i+5j=\lvert v\rvert\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}i\right)(\cos\varphi+\sin\varphi\, j)$	(197)

Then, we multiply (196) and (197) together in (198). The formula of the product of two 3D complex numbers in trigonometric form is given by (69) and (70), which consists of adding the arguments of the numbers dimension by dimension and multiplying the modulus in the resulting number. So, we do the same for $(1i+4h)(1i+5j)$ in (199).

$(1i+4h)(1i+5j)=\lvert u\rvert(\cos\theta+\sin\theta\, i)(\cos 0+\sin 0\, j)*\lvert v\rvert\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}i\right)(\cos\varphi+\sin\varphi\, j)$	(198)
$(1i+4h)(1i+5j)=\lvert v\rvert\lvert u\rvert\left(\cos\left(\theta+\frac{\pi}{2}\right)+\sin\left(\theta+\frac{\pi}{2}\right)i\right)(\cos(0+\varphi)+\sin(0+\varphi)\, j)$	(199)

By the way, it is easier to use exponential form for multiplication. Equation (200) transforms (196) and (197) using Euler's formula, equation (201) multiplies them together. In (201) and (199) we have the same number.

$1i+4h=\lvert u\rvert e^{i\theta+j0}$ $1i+5j=\lvert v\rvert e^{i\frac{\pi}{2}+j\varphi}$	(200)	$(1i+4h)(1i+5j)=\lvert u\rvert e^{i\theta+j0}\lvert v\rvert e^{i\frac{\pi}{2}+j\varphi}=\lvert u\rvert\lvert v\rvert e^{i\left(\theta+\frac{\pi}{2}\right)+j\varphi}$	(201)

Then, we develop equation (199) into Cartesian form in (202) by using (34).

$(1i+5j)(1i+4h)=\lvert u\rvert\lvert v\rvert\big((-\sin\theta+\cos\theta\, i)\cos\varphi+\sin\varphi\, j\big)$	(202)

Introducing the expressions of the moduli and arguments given in (191) and (192), (194) and (195) into (202), the product $(1i+4h)(1i+5j)$ becomes (203) which is then simplified into (204), which is the resulting product in 3D complex number.

$(1i+5j)(1i+4h)=\sqrt{4^2+1}\sqrt{5^2+1}\left(\left(\frac{-1}{\sqrt{4^2+1}}+\frac{4}{\sqrt{4^2+1}}i\right)\frac{1}{\sqrt{5^2+1}}+\frac{5}{\sqrt{5^2+1}}j\right)$ $=\left(\frac{-1}{\sqrt{4^2+1}}+\frac{4}{\sqrt{4^2+1}}i\right)\frac{\sqrt{4^2+1}\sqrt{5^2+1}}{\sqrt{5^2+1}}+\frac{5\sqrt{4^2+1}\sqrt{5^2+1}}{\sqrt{5^2+1}}j$	(203)
$(1i+5j)(1i+4h)=-1+4i+5\sqrt{4^2+1}j$	(204)