Complement for
"Extending complex number to spaces with 3, 4 or any number of dimensions"

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1. 7 April 2022: Correction to Mandelbrot set to the power 2 and 3

I have done a mistake in the computation of power 2 of 3D complex number z using a, b, d, f. So, I correct the equations from (37) to (41). In fact, d is positive in all z, but once z^2 is done, the resulting angle can be bigger than $\pi/2$, and in consequence, $d_{\therefore}$ can be negative. In this case, for z_{n+1} to have positive d, we have to reverse the sing of $d_{\therefore}$ and consequently, that of a, b, f too. Let us derive the correct equations.

$z^2 = r_z^2\left((\cos 2\theta_z + \sin 2\theta_z\, i)\cos 2\varphi_z + \sin 2\varphi_z\, j\right)$	See (36)	(1)	$\cos 2\theta_z = 2\cos^2\theta_z - 1$	$\sin 2\theta_z = 2\sin\theta_z\cos\theta_z$	(2)
If $\cos 2\varphi_z < 0$, then,	$\cos\varphi_{z+1} = -\cos 2\varphi_z > 0$		$\cos\theta_{z+1} + \sin\theta_{z+1}\, i = -\cos 2\theta_z - \sin 2\theta_z\, i$		(3)
$-\cos 2\theta_z - \sin 2\theta_z\, i = \cos(2\theta_z + \pi) + \sin(2\theta_z + \pi)\, i$			$\cos\theta_{z+1} + \sin\theta_{z+1}\, i = \cos(2\theta_z + \pi) + \sin(2\theta_z + \pi)\, i$		

Coefficients a, b, d, f

Power 2	z^2 in terms of $a_{p2}, b_{p2}, d_{p2}, f_{p2}$		$z^2 = r_z^2\left(a_{p2}d_{p2} + b_{p2}d_{p2}\, i + f_{p2}\, j\right)$		(4)
$d_{p2*} = 2d^2 - 1 > 0$	$d_{p2} = 2d^2 - 1$	$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$f_{p2} = 2df$	(5)
$d_{p2*} < 0$	$d_{p2} = -(2d^2 - 1)$	$a_{p2} = -(2a^2 - 1)$	$b_{p2} = -2ab$	$f_{p2} = 2df$	(6)
Power 3	z^3 in terms of $a_{\therefore}, b_{\therefore}, d_{\therefore}, f_{\therefore}$		$z_n^3 = r_n^3(a_{\therefore}d_{\therefore} + b_{\therefore}d_{\therefore}\, i + f_{\therefore}\, j)$		(7)
$d_{\therefore *} = 4d^3 - 3d > 0$	$d_{\therefore} = 4d^3 - 3d$	$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$f_{\therefore} = 3f - 4f^3$	(8)
$d_{\therefore *} < 0$	$d_{\therefore} = -(4d^3 - 3d)$	$a_{\therefore} = -(4a^3 - 3a)$	$b_{\therefore} = -(3b - 4b^3)$	$f_{\therefore} = 3f - 4f^3$	(9)

In code

Power 2	$d_{p2*} = 2d^2 - 1$	sd=sign(d_{p2*})	$a_{p2} = \mathrm{sd}(2a^2 - 1)$	$b_{p2} = \mathrm{sd}(2ab)$	$f_{p2} = 2df$	$d_{p2} = \mathrm{sd}\cdot d_{p2*}$	(10)
Power 3	$d_{\therefore *} = 4d^3 - 3d$	sd=sign($d_{\therefore *}$)	$a_{\therefore} = \mathrm{sd}(4a^3 - 3a)$	$b_{\therefore} = \mathrm{sd}(3b - 4b^3)$	$f_{\therefore} = 3f - 4f^3$	$d_{\therefore} = \mathrm{sd}\cdot d_{\therefore *}$	(11)

2. 3 April 2022: True Lambda

$z_{n+1} = cz - cz^2$	(12)

c and z in terms of a,b,d,f

$c = x_c + y_c\, i + z_c\, j$	$a_c = \left(\frac{x}{r_{xy}}\right)_c$	$b_c = \left(\frac{y}{r_{xy}}\right)_c$	$d_c = \left(\frac{r_{xy}}{r}\right)_c$	$f_c = \left(\frac{z_*}{r}\right)_c$	(13)
$z = x_z + y_z\, i + z_z\, j$	$a_z = \left(\frac{x}{r_{xy}}\right)_z$	$b_z = \left(\frac{y}{r_{xy}}\right)_z$	$d_z = \left(\frac{r_{xy}}{r}\right)_z$	$f_z = \left(\frac{z_*}{r}\right)_z$	(14)

Multiplication in terms of a,b,d,f

$z\cdot c = r_z r_c\left(\left((a_z a_c - b_z b_c) + (b_z a_c + a_z b_c)\, i\right)(d_z d_c - f_z f_c) + (f_z d_c + d_z f_c)\, j\right)$ $= r_{z\cdot c}\left((a_{z\cdot c} + b_{z\cdot c}\, i)d_{z\cdot c} + f_{z\cdot c}\, j\right)$					(15)
$r_{z\cdot c} = r_z r_c$	$a_{z\cdot c} = a_z a_c - b_z b_c$	$b_{z\cdot c} = b_z a_c + a_z b_c$	$d_{z\cdot c} = d_z d_c - f_z f_c$	$f_{z\cdot c} = f_z d_c + d_z f_c$	(16)

z^2 using (38), (39)

a,b,d,f of power 2, a,b,d,f with subscript z					z^2 in terms of $a_{p2}, b_{p2}, d_{p2}, f_{p2}$	
$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(17)	$z^2 = r_z^2\left((a_{p2} + b_{p2}\, i)d_{p2} + f_{p2}\, j\right)$	(18)

cz^2 using (13) , (18), for multiplication (15), (16)

$c \cdot z^2 = r_c r_z{}^2\Big(\big((a_{p2}a_c - b_{p2}b_c) + (b_{p2}a_c + a_{p2}b_c)i\big)(d_{p2}d_c - f_{p2}f_c) + (f_{p2}d_c + d_{p2}f_c)j\Big) = \Big(r\big((a+bi)d + fj\big)\Big)_{z2\cdot c}$	(19)

$r_{z2\cdot c} = r_c r_z{}^2$	$a_{z2\cdot c} = a_{p2}a_c - b_{p2}b_c$	$b_{z2\cdot c} = b_{p2}a_c + a_{p2}b_c$	$d_{z2\cdot c} = d_{p2}d_c - f_{p2}f_c$	$f_{z2\cdot c} = f_{p2}d_c + d_{p2}f_c$	(20)

Subtraction, (15), (16), (19), (20)

$z_{n+1} = cz - cz^2 = r_{z\cdot c}\big((a_{z\cdot c} + b_{z\cdot c}i)d_{z\cdot c} + f_{z\cdot c}j\big) - \Big(r\big((a+bi)d + fj\big)\Big)_{z2\cdot c}$	See (15), (16) for cz and (19), (20) for cz^2	(21)

Back to (14)

3. 31 March 2022: Half-angle formulae

The sign of the sine and cosine of half-angle depends on the quadrant where the angle is, see Figure 1.

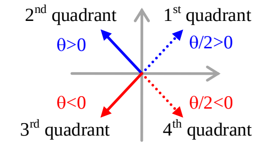

Figure 1

1st and 2nd quadrant		3rd and 4th quadrant		1st and 4th quadrant		2nd and 3rd quadrant	
$\sin\theta > 0$	(22)	$\sin\theta < 0$	(23)	$\cos\theta > 0$	(24)	$\cos\theta < 0$	(25)

When the angle is allowed to be negative, then -π<θ<π:

$0 < \theta < \pi \rightarrow \frac{\theta}{2} < \frac{\pi}{2} \rightarrow \frac{\theta}{2} \in$ 1st quadrant			$-\pi < \theta < 0 \rightarrow \frac{\theta}{2} > -\frac{\pi}{2} \rightarrow \frac{\theta}{2} \in$ 4th quadrant		
$\sin\theta > 0$ $\sin\frac{\theta}{2} > 0$	$\sin\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1-\cos\theta}{2}}$	(26)	$\sin\theta < 0$ $\sin\frac{\theta}{2} < 0$	$\sin\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1-\cos\theta}{2}}$	(27)
$\cos\theta > 0$	$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$	(28)	$\cos\theta > 0$	$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$	(29)

Case where the angle is only positive: $0 < \theta < 2\pi$

$0 < \theta < \pi \rightarrow \frac{\theta}{2} < \frac{\pi}{2} \rightarrow \frac{\theta}{2} \in$ 1st quadrant			$\pi < \theta < 2\pi \rightarrow \frac{\pi}{2} < \frac{\theta}{2} < \pi \rightarrow \frac{\theta}{2} \in$ 2nd quadrant		
$\sin\frac{\theta}{2} > 0$	$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$	(30)	$\sin\frac{\theta}{2} > 0$	$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$	(31)
$\sin\theta > 0$ $\cos\theta > 0$	$\cos\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1+\cos\theta}{2}}$	(32)	$\sin\theta < 0$ $\cos\theta < 0$	$\cos\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1+\cos\theta}{2}}$	(33)

Half-angle formulae

Angle positive or negative $-\pi < \theta < \pi$			Angle only positive: $0 < \theta < 2\pi$		
$\sin\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1-\cos\theta}{2}}$	$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$	(34)	$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$	$\cos\frac{\theta}{2} = sgn(\sin\theta)\sqrt{\frac{1+\cos\theta}{2}}$	(35)

As the sine and cosine of half-angle have not the same value for the two cases, the corresponding Mandelbrot set could be different.

These old equations are not completely correct. See (4) to (11) for complete and correct ones.

Power 2 of 3D complex number z

$z^2 = r_z{}^2\big((\cos 2\theta_z + \sin 2\theta_z\, i)\cos 2\varphi_z + \sin 2\varphi_z\, j\big)$	(36)	$\cos 2\theta_z = 2\cos^2\theta_z - 1$	$\sin 2\theta_z = 2\sin\theta_z\cos\theta_z$	(37)

a,b,d,f of power 2					z^2 in terms of a_{p2},b_{p2},d_{p2},f_{p2}	
$a_{p2} = 2a^2 - 1$	$b_{p2} = 2ab$	$d_{p2} = 2d^2 - 1$	$f_{p2} = 2df$	(38)	$z^2 = r_z{}^2\big((a_{p2} + b_{p2}i)d_{p2} + f_{p2}j\big)$	(39)
a,b,d,f of power 3					z^3 in terms of $a_{\therefore}$,$b_{\therefore}$,$d_{\therefore}$,$f_{\therefore}$	
$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	(40)	$z_n{}^3 = r_n{}^3 a_{\therefore}d_{\therefore} + r_n{}^3 b_{\therefore}d_{\therefore}i + r_n{}^3 f_{\therefore}j$	(41)

4. 23 March 2022: Other power Mandelbrot

Power two

$z_n{}^2 = r_n{}^2\cos 2\theta_n\cos 2\varphi_n + r_n{}^2\sin 2\theta_n\cos 2\varphi_n\, i + r_n{}^2\sin 2\varphi_n\, j$ $= r_n{}^2 a_{\therefore}d_{\therefore} + r_n{}^2 b_{\therefore}d_{\therefore}i + r_n{}^2 f_{\therefore}j$				See (74)		(42)
$\cos 2\theta = 2\cos^2\theta - 1$	$\sin 2\theta = 2\sin\theta\cos\theta$	$a = \cos\theta$	$b = \sin\theta$	$d = \cos\varphi$	$f = \sin\varphi$	(43)
$a_{\therefore} = 2a^2 - 1$	$b_{\therefore} = 2ab$	$d_{\therefore} = 2d^2 - 1$		$f_{\therefore} = 2df$		(44)

Power m

$a_m = \cos m\theta$	$b_m = \sin m\theta$	$d_m = \cos m\varphi$	$f_m = \sin m\varphi$	(45)
$\cos(m+1)\theta = \cos(m\theta+\theta) = \cos m\theta \cos\theta - \sin m\theta \sin\theta$		$\sin(m+1)\theta = \sin m\theta \cos\theta + \cos m\theta \sin\theta$		(46)
$a_{m+1} = \cos(m\theta+\theta) = a_m a_1 - b_m b_1$		$d_{m+1} = \cos(m\varphi+\varphi) = d_m d_1 - f_m f_1$		(47)
$b_{m+1} = \sin(m\theta+\theta) = b_m a_1 + a_m b_1$		$f_{m+1} = \sin(m\varphi+\varphi) = f_m d_1 + d_m f_1$		

Figure 1, recurrent formulae power two:

$a = \frac{x}{r_{xy}}$	$b = \frac{y}{r_{xy}}$	$d = \frac{r_{xy}}{r}$	$f = \frac{z}{r}$	See (65)	(48)
$a_{\therefore} = 2a^2 - 1$	$b_{\therefore} = 2ab$	$d_{\therefore} = 2d^2 - 1$	$f_{\therefore} = 2df$	See (84)	(49)
$z_n{}^2 = r_n{}^2 a_{\therefore} d_{\therefore} + r_n{}^2 b_{\therefore} d_{\therefore} i + r_n{}^2 f_{\therefore} j$	$z_{n+1} = z_n{}^2 + c = r_n{}^2 a_{\therefore} d_{\therefore} + r_n{}^2 b_{\therefore} d_{\therefore} i + r_n{}^2 f_{\therefore} j + x_c + y_c i + z_c j$			See (67)	(50)

Figure 2, recurrent formulae power m:

$a = \frac{x}{r_{xy}}$	$b = \frac{y}{r_{xy}}$	$d = \frac{r_{xy}}{r}$	$f = \frac{z}{r}$	See (65)	(51)
Do m-1 time (52) from m = 2 to m, see (45), (46), (47)					(52)
$a_m = a_{m-1} a_1 - b_{m-1} b_1$	$b_m = b_{m-1} a_1 + a_{m-1} b_1$	$d_m = d_{m-1} d_1 - f_{m-1} f_1$	$f_m = f_{m-1} d_1 + d_{m-1} f_1$		
$z_n{}^m = r_n{}^m a_m d_m + r_n{}^m b_m d_m i + r_n{}^m f_m j$	$z_{n+1} = z_n{}^m + c = r_n{}^m a_m d_m + r_n{}^m b_m d_m i + r_n{}^m f_m j + x_c + y_c i + z_c j$			See (67)	(53)

For high power, that is, big m, it is time-consuming to do m-1 time (52) from m = 2 to m. It is less long to compute first the angles and do m*angles.

$r_{xy} = \sqrt{x^2+y^2}$	$\cos\theta = \frac{x}{r_{xy}}$	$\sin\theta = \frac{y}{r_{xy}}$	$\cos\varphi = \frac{r_{xy}}{r}$	$\sin\varphi = \frac{z_*}{r}$	$r = \sqrt{x^2+y^2+z_*{}^2}$	(54)
$x > 0: \theta = \sin^{-1}\left(\frac{y}{r_{xy}}\right)$		$\varphi = \sin^{-1}\left(\frac{z_*}{r}\right)$	$a_m = \cos m\theta$		$d_m = \cos m\varphi$	(55)
$x < 0: \theta = \pi - \sin^{-1}\left(\frac{y}{r_{xy}}\right)$	See (58)		$b_m = \sin m\theta$		$f_m = \sin m\varphi$	
$z_n{}^m = r_n{}^m a_m d_m + r_n{}^m b_m d_m i + r_n{}^m f_m j$		$z_{n+1} = z_n{}^m + c = r_n{}^m a_m d_m + r_n{}^m b_m d_m i + r_n{}^m f_m j + x_c + y_c i + z_c j$				(56)

Value of arcsin(y/x) and arctan(y/x) for x<0

$x < 0: \theta > \frac{\pi}{2}, \theta = \pi - \theta', with\ \theta' < \frac{\pi}{2}$	$\sin\theta = \frac{y}{r_{xy}} = \sin(\pi - \theta') = \sin\pi\cos(-\theta') + \cos\pi\sin(-\theta') = 0 + \sin\theta'$		(57)
$\sin^{-1}(\sin\theta) = \sin^{-1}\frac{y}{r_{xy}} = \sin^{-1}(\sin\theta')$	$\sin^{-1}\frac{y}{r_{xy}} = \theta' = \pi - \theta$	$\theta = \pi - \theta' = \pi - \sin^{-1}\frac{y}{r_{xy}}$	(58)
$x < 0: \theta > \frac{\pi}{2}, \theta = \pi - \theta'$	$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \frac{\sin(\pi-\theta')}{\cos(\pi-\theta')} = \frac{\sin\pi\cos\theta' - \cos\pi\sin\theta'}{\cos\pi\cos\theta' - \sin\pi\sin\theta'} = \frac{0+\sin\theta'}{-\cos\theta' - 0} = -\tan\theta'$		(59)
$\tan^{-1}\tan\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}(-\tan\theta')$	$\tan^{-1}\frac{y}{x} = -\theta' = \theta - \pi$	$\theta = \tan^{-1}\frac{y}{x} + \pi$	(60)

5. 21 March 2022: Computation of 3D Mandelbrot set with 3D complex in vector form (x,y,z)

Derivation of the formulae

$z = r((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j) = r\cos\theta\cos\varphi + r\sin\theta\cos\varphi\, i + r\sin\varphi\, j = x + yi + z_* j$					See (73)	(61)
$x^2 + y^2 = (r\cos\theta\cos\varphi)^2 + (r\sin\theta\cos\varphi)^2 = ((r\cos\theta)^2 + (r\sin\theta)^2)(\cos\varphi)^2 = (r)^2(\cos\varphi)^2 = (r\cos\varphi)^2$						(62)
$\cos\varphi = \frac{\lvert r\cos\varphi\rvert}{r} = \lvert\cos\varphi\rvert$						(63)
$r_{xy} = \sqrt{x^2+y^2}$	$\cos\theta = \frac{r\cos\theta\cos\varphi}{r_{xy}} = \frac{x}{r_{xy}}$	$\sin\theta = \frac{r\sin\theta\cos\varphi}{r_{xy}} = \frac{y}{r_{xy}}$	$\cos\varphi = \frac{r_{xy}}{r}$ see (63)	$\sin\varphi = \frac{r\sin\varphi}{r} = \frac{z_*}{r}$		(64)
$\cos\theta = a = \frac{x}{r_{xy}}$	$\sin\theta = b = \frac{y}{r_{xy}}$	$\cos\varphi = d = \frac{r_{xy}}{r}$	$\sin\varphi = f = \frac{z_*}{r}$			(65)
$z = rad + rbdi + rfj$ $= x + yi + zj$	$x = rad$	$y = rbd$	$z = rf$	See (61)		(66)
$z_n{}^3 = r_n{}^3\cos 3\theta_n\cos 3\varphi_n + r_n{}^3\sin 3\theta_n\cos 3\varphi_n\, i + r_n{}^3\sin 3\varphi_n\, j$ $= r_n{}^3 a_{\therefore} d_{\therefore} + r_n{}^3 b_{\therefore} d_{\therefore} i + r_n{}^3 f_{\therefore} j$					See (74)	(67)

Recurrent formulae:

$c = x_c + y_c i + z_c j$	$z = x + yi + z_* j$	$r = \sqrt{x^2 + y^2 + z_*^2}$	$r_{xy} = \sqrt{x^2 + y^2}$		(68)
$a = \frac{x}{r_{xy}}$	$b = \frac{y}{r_{xy}}$	$d = \frac{r_{xy}}{r}$	$f = \frac{z}{r}$	See (65)	(69)
$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	See (84)	(70)
$z_n^3 = x_{*n+1} + y_{*n+1} i + z_{*n+1} j$	$x_{*n+1} = r_n^3 a_{\therefore} d_{\therefore}$	$y_{*n+1} = r_n^3 b_{\therefore} d_{\therefore}$	$z_{*n+1} = r_n^3 f_{\therefore}$	See (67)	(71)
$z_{n+1} = z_n^3 + c = x_{*n+1} + y_{*n+1} i + z_{*n+1} j + x_c + y_c i + z_c j$					(72)

6. 19 March 2022: 3D Mandelbrot set

Computing the points of a 3D Mandelbrot set, 3D complex with angles.

$z = re^{i\theta} e^{j\varphi}$ $= r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big)$	$c = r_c\big((\cos\theta_c + \sin\theta_c\, i)\cos\varphi_c + \sin\varphi_c\, j\big)$	See (30) of Original Paper	(73)
$z^3 + c = r^3 e^{i3\theta} e^{j3\varphi} + r_c e^{i\theta_c} e^{j\varphi_c}$ $= r^3\big((\cos 3\theta + \sin 3\theta\, i)\cos 3\varphi + \sin 3\varphi\, j\big) + r_c\big((\cos\theta_c + \sin\theta_c\, i)\cos\varphi_c + \sin\varphi_c\, j\big)$ $= r^3(\cos 3\theta \cos 3\varphi + \sin 3\theta \cos 3\varphi\, i) + (r^3 \sin 3\varphi + r_c \sin\varphi_c) j + r_c(\cos\theta_c \cos\varphi_c + \sin\theta_c \cos\varphi_c\, i)$ $= (r^3 \cos 3\theta \cos 3\varphi + r_c \cos\theta_c \cos\varphi_c) + (r^3 \sin 3\theta \cos 3\varphi + r_c \sin\theta_c \cos\varphi_c) i + (r^3 \sin 3\varphi + r_c \sin\varphi_c) j$			(74)
$z_{n+1} = (r_n^3 \cos 3\theta_n \cos 3\varphi_n + r_c \cos\theta_c \cos\varphi_c) + (r_n^3 \sin 3\theta_n \cos 3\varphi_n + r_c \sin\theta_c \cos\varphi_c) i + (r_n^3 \sin 3\varphi_n + r_c \sin\varphi_c) j$			(75)
$\frac{z_{n+1}}{r_{n+1}} = \frac{r_n^3 \cos 3\theta_n \cos 3\varphi_n + r_c \cos\theta_c \cos\varphi_c}{r_{n+1}} + \frac{r_n^3 \sin 3\theta_n \cos 3\varphi_n + r_c \sin\theta_c \cos\varphi_c}{r_{n+1}} i + \frac{r_n^3 \sin 3\varphi_n + r_c \sin\varphi_c}{r_{n+1}} j$			(76)
$\sin 3\theta_n = 3\sin\theta_n - 4\sin^3\theta_n$	$\cos 3\theta_n = 4\cos^3\theta_n - 3\cos\theta_n$	Formulae for triple angles	(77)

Recurrent formula for z_n.

$z_1 = 0$	$z_2 = c$	$z_3 = c^3 + c$	$z_{n+1} = z_n^3 + c$	(78)
$r_{n+1} = \lvert z_{n+1} \rvert$ $= \sqrt{(r_n^3 \cos 3\theta_n \cos 3\varphi_n + r_c \cos\theta_c \cos\varphi_c)^2 + (r_n^3 \sin 3\theta_n \cos 3\varphi_n + r_c \sin\theta_c \cos\varphi_c)^2 + (r_n^3 \sin 3\varphi_n + r_c \sin\varphi_c)^2}$				(79)
$\sin\varphi_{n+1} = \frac{r_n^3 \sin 3\varphi_n + r_c \sin\varphi_c}{r_{n+1}}$		$\cos\varphi_{n+1} = \sqrt{1 - \sin^2\varphi_{n+1}}$		(80)
$\sin\theta_{n+1} = \frac{r_n^3 \sin 3\theta_n \cos 3\varphi_n + r_c \sin\theta_c \cos\varphi_c}{r_{n+1}\cos\varphi_{n+1}}$		$\cos\theta_{n+1} = \frac{r_n^3 \cos 3\theta_n \cos 3\varphi_n + r_c \cos\theta_c \cos\varphi_c}{r_{n+1}\cos\varphi_{n+1}}$		(81)

We apply (77) to $\cos 3\theta_n, \sin 3\theta_n, \cos 3\varphi_n, \sin 3\varphi_n$ in (79) for r_{n+1} and to (80) and (81) for $\cos\theta_{n+1}, \sin\theta_{n+1}, \cos\varphi_{n+1}, \sin\varphi_{n+1}$, which gives z_{n+1}. Then, we go back to (79) for next step.

3D complex with sub-components a, b, d, f

$v = r\big((\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j\big) = r\big((a + bi)d + fj\big) = r(ad + bdi + fj)$		See (73)	(82)
$z^3 + c = r^3\big((a_{\therefore} + b_{\therefore} i)d_{\therefore} + f_{\therefore} j\big) + r_c\big((a_c + b_c i)d_c + f_c j\big)$ $= r^3(a_{\therefore} d_{\therefore} + b_{\therefore} d_{\therefore} i + f_{\therefore} j) + r_c(a_c d_c + b_c d_c i + f_c j)$ $= (r^3 a_{\therefore} d_{\therefore} + r^3 b_{\therefore} d_{\therefore} i + r^3 f_{\therefore} j) + (r_c a_c d_c + r_c b_c d_c i + r_c f_c j)$ $= (r^3 a_{\therefore} d_{\therefore} + r_c a_c d_c) + (r^3 b_{\therefore} d_{\therefore} + r_c b_c d_c) i + (r^3 f_{\therefore} + r_c f_c) j$		See (73), (74), (82)	(83)
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = a_{\therefore} = 4a^3 - 3a$ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = b_{\therefore} = 3b - 4b^3$	$\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi = d_{\therefore} = 4d^3 - 3d$ $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi = f_{\therefore} = 3f - 4f^3$	Formulae for triple angles, See (77).	(84)

Recurrent formulae

$z_1 = 0$	$z_2 = c$	(85)	$z_{n+1} = z_n^3 + c$		(86)
$z_n = r_n(a_n d_n + b_n d_n i + f_n j)$		$c = r_c(a_c d_c + b_c d_c i + f_c j)$		See (82)	(87)
$a_{\therefore} = 4a^3 - 3a$	$b_{\therefore} = 3b - 4b^3$	$d_{\therefore} = 4d^3 - 3d$	$f_{\therefore} = 3f - 4f^3$	See (84)	(88)
$r_{n+1} = \sqrt{(r_n^3 a_{\therefore n} d_{\therefore n} + r_c a_c d_c)^2 + (r_n^3 b_{\therefore n} d_{\therefore n} + r_c b_c d_c)^2 + (r_n^3 f_{\therefore n} + r_c f_c)^2}$				See (83)	(89)
$f_{n+1} = \frac{r_n^3 f_{\therefore n} + r_c f_c}{r_{n+1}}$	$d_{n+1} = \sqrt{1 - f_{n+1}^2}$	$b_{n+1} = \frac{r_n^3 b_{\therefore n} d_{\therefore n} + r_c b_c d_c}{r_{n+1} d_{n+1}}$	$a_{n+1} = \frac{r_n^3 a_{\therefore n} d_{\therefore n} + r_c a_c d_c}{r_{n+1} d_{n+1}}$ See (76), (80), (81)		(90)
$z_{n+1} = r_{n+1}(a_{n+1} d_{n+1} + b_{n+1} d_{n+1} i + f_{n+1} j)$				See (87)	(91)

We apply (88) to (89) for r_{n+1} and to (90) for z_{n+1} in (91). Then, we go back to (89) for next step.

7. 18 March 2022: Multiplication of 3D complex

A unit 3D complex is like the product of 2 unit vectors of 2D complex, see (92) and (93). In 3D space, we have a horizontal plane which is defined by the 2D complex vector A_i and a vertical plane defined by the 2D complex vector A_j, see (95). Notice that the vertical plane rotates around the vertical axis and reaches the entire 3D space. The 3D complex vector that defines all the points of the entire 3D space is v and equals and the product A_iA_j, see (96).

$u_1 = e^{i\theta_1}$ $u_2 = e^{i\theta_2}$	(92)	$u_1u_2 = e^{i\theta_1}e^{i\theta_2}$	(93)	$v_1 = e^{i\theta_1}e^{j\varphi_1}$ $v_2 = e^{i\theta_2}e^{j\varphi_2}$	(94)
$A_i = e^{i\theta}$ $A_j = e^{j\varphi}$	(95)	$v = A_iA_j = e^{i\theta}e^{j\varphi}$	(96)	$v_1v_2 = e^{i\theta_1}e^{j\varphi_1}e^{i\theta_2}e^{j\varphi_2}$ $= e^{i(\theta_1+\theta_2)}e^{j(\varphi_1+\varphi_2)}$	(97)

For multiplying v_1 and v_2 written in (94), two unit 3D complexes, we just apply the formula of multiplication of exponential, which give the product of v_1 and v_2 in (97).

When expressing 3D complex v in the form of 3D vector, that is, in the form a+bi+cj, we use the conversion law (98), which is the equation (39) of the original paper.

$v = A_i \cos\varphi + \sin\varphi\,\lvert A_i\rvert j$	(98)
$v_1 = e^{i\theta_1}\cos\varphi_1 + \sin\varphi_1\,\lvert e^{i\theta_1}\rvert j$ $v_2 = e^{i\theta_2}\cos\varphi_2 + \sin\varphi_2\,\lvert e^{i\theta_2}\rvert j$	(99)

After conversion, the 2 factors v_1 and v_2 in (94) are in 3D vector form (99), the product v_1v_2 in (97) is in 3D vector form (100).

$v_1v_2 = e^{i(\theta_1+\theta_2)}\left(\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\,\lvert e^{i(\theta_1+\theta_2)}\rvert j\right)$	(100)

The moduli equal 1, see (101), (102) and (103), and $e^{i\theta}$ is the Euler's formula (104).

$\lvert e^{i\theta_1}\rvert = 1$	(101)	$\lvert e^{i\theta_2}\rvert = 1$	(102)
$\lvert e^{i(\theta_1+\theta_2)}\rvert = 1$	(103)	$e^{i\theta} = \cos\theta + \sin\theta\, i$	(104)

Applying (101), (102), (103) and (104) to (99) and (100), the multiplication formula in 3D vector form is (105). Using the change of variable (106) in (105), (105) becomes (107), see (135) below, which does not contain cross product ij.

$v_1v_2 = \left((\cos\theta_1 + \sin\theta_1\, i)\cos\varphi_1 + \sin\varphi_1\, j\right)\left((\cos\theta_2 + \sin\theta_2\, i)\cos\varphi_2 + \sin\varphi_2\, j\right)$ $= \left((\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\, i)\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\, j\right)$	(105)
$\cos\theta = a \qquad \sin\theta = b \qquad \cos\varphi = c \qquad \sin\varphi = d$	(106)
$v_1v_2 = (a_1c_1 + b_1c_1 i + d_1 j)(a_2c_2 + b_2c_2 i + d_2 j)$ $= \left((a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i\right)(c_1c_2 - d_1d_2) + (d_1c_2 + c_1d_2)j$ $= A_1 + B_1 i + C_1 j$	(107)

Notice that (107) is valid for unit 3D vectors only. For non-unit 3D vectors, we have to apply (141).

8. 15 March 2022: Concept of multidimensional complex number

The proposed concept of multidimensional complex number is new and the demonstration is rather long. For the readers not to get lost, we present beforehand the guiding idea which, kept in mind, will make the reading easier.

Let us start with a familiar unit 2D complex number $e^{i\theta}$, which is written with its argument θ in equation (108). 2D complex space has two dimensions, the first one is the real line labeled as h, the second one is the imaginary line labeled as i. 3D complex space is an extension of the 2D complex space by appending with the dimension labeled j. Figure 2 shows the 3D complex space in which u is a 2D complex number and v a 3D complex number.

$A_i = e^{i\theta}$	(108)

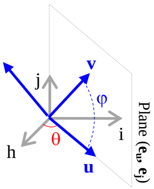

Figure 2

3D complex number is defined such that it can be multiplied. A 3D complex number has imaginary parts in i and j. Let us represent the imaginary part in i with the 2D complex number $e^{i\theta}$ and that in j with the 2D complex number $e^{j\varphi}$. A unit 3D complex number is defined as the product of $e^{i\theta}$ and $e^{j\varphi}$, see (112). So, the unit 3D complex number is $e^{i\theta}\cdot e^{j\varphi}$ see (112).

$u_1 = e^{i\theta_1}$ $u_2 = e^{i\theta_2}$	(109)
$u_1 u_2 = e^{i\theta_1} e^{i\theta_2}$	(110)

In the same way, a unit 4D complex number is defined as the product of the 3D complex number (112) with the 2D complex number $e^{k\phi}$ representing the imaginary part in k, see (113). The expression of a unit 4D complex number is $e^{i\theta}\cdot e^{j\varphi}\cdot e^{k\phi}$, see (114).

3D complex	
$A_j = e^{j\varphi}$	(111)
$v = e^{i\theta} e^{j\varphi}$	(112)

4D complex	
$A_k = e^{k\phi}$	(113)
$w = e^{i\theta} e^{j\varphi} e^{k\phi}$	(114)

3D and 4D complex numbers can be multiplied. In equations (115) are written two 3D complex numbers v_1 and v_2 whose arguments are $(i\theta_1, j\varphi_1)$ and $(i\theta_2, j\varphi_2)$.

3D complex multiplication	
$v_1 = e^{i\theta_1} e^{j\varphi_1}$ $v_2 = e^{i\theta_2} e^{j\varphi_2}$	(115)
$v_1 v_2 = e^{i\theta_1} e^{j\varphi_1} e^{i\theta_2} e^{j\varphi_2}$ $= e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)}$	(116)

4D complex multiplication	
$w_1 = e^{i\theta_1} e^{j\varphi_1} e^{k\phi_1}$ $w_2 = e^{i\theta_2} e^{j\varphi_2} e^{k\phi_2}$	(117)
$w_1 w_2 = e^{i\theta_1} e^{j\varphi_1} e^{k\phi_1} e^{i\theta_2} e^{j\varphi_2} e^{k\phi_2}$ $= e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)} e^{k(\phi_1+\phi_2)}$	(118)

The product of v_1 and v_2 is expressed in (116) and whose arguments are the sum $i(\theta_1+\theta_2)$ and $j(\varphi_1+\varphi_2)$. In the same way, in (117) are written two 4D complex numbers w_1 and w_2 and their product is the exponential of the sum of the arguments of the two factors, $i(\theta_1+\theta_2)$, $j(\varphi_1+\varphi_2)$ and $k(\phi_1+\phi_2)$, see (118).

The so defined 3D and 4D complex numbers multiply just like the multiplication of 2D complex numbers. Let us see the two 2D complex numbers u_1 and u_2 in (109). Their product is u_1u_2 and is shown in (110). The argument of the product equals the sum of that of the factors u_1 and u_2, that is, $i(\theta_1+\theta_2)$, see (119). The product of the 3D v_1 and v_2 is shown in (120) and that of the 4D w_1 and w_2 is shown in (121). We find the same pattern in (119), (120) and (121).

$u_1 u_2 = e^{i(\theta_1+\theta_2)}$	(119)
$v_1 v_2 = e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)}$	(120)
$w_1 w_2 = e^{i(\theta_1+\theta_2)} e^{j(\varphi_1+\varphi_2)} e^{k(\phi_1+\phi_2)}$	(121)

Unit 3D and 4D complex numbers are converted into trigonometric form by using Euler's formula (122) which transforms (112) and (114) into (123) and (124). Unit 2D complex number in trigonometric form is the Euler's formula (122). We find the same pattern in (122), (123) and (124).

$e^{i\theta} = \cos\theta + \sin\theta\, i$	(122)
$v = (\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)$	(123)
$w = (\cos\theta + \sin\theta\, i)(\cos\varphi + \sin\varphi\, j)(\cos\phi + \sin\phi\, k)$	(124)

3D complex number is a vector in 3D space, for example, the 3D complex number v in Figure 2 is defined by its Cartesian components which will be converted from (123) in the following section. So, 3D complex number is well a vector in 3D Cartesian space.

9. 24 February 2022

The two 3D complex numbers are v_1 and v_2 in (125) and we search for a formula of v_1v_2 with the Cartesian components A, B, C and D.

$v_1 = (A_1 + B_1 i)C_1 + D_1 j = \lvert v_1\rvert(\cos\theta_1 + \sin\theta_1\, i)(\cos\varphi_1 + \sin\varphi_1\, j)$ $v_2 = (A_2 + B_2 i)C_2 + D_2 j = \lvert v_2\rvert(\cos\theta_2 + \sin\theta_2\, i)(\cos\varphi_2 + \sin\varphi_2\, j)$	(125)

Using the equation 52 of the original paper, we write the product divided by the modulus in (126). Equation (126) is in trigonometric form which we convert into Cartesian form in (127) using the equations 32 and 16 of the original paper, equation 32 being the trigonometric form and 16 being the Cartesian form of the same complex number.

$\frac{v_1 v_2}{\lvert v_1\rvert\lvert v_2\rvert} = (\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\,i)(\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\,j)$	(126)
$\frac{v_1 v_2}{\lvert v_1\rvert\lvert v_2\rvert} = (\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\,i)\cos(\varphi_1+\varphi_2) + \sin(\varphi_1+\varphi_2)\,j$	(127)

For converting the Cartesian components A, B, C and D of the 3D complex number in (128) into sin and cos of the arguments, we have derived the formulas below. In (129) v is written with a, b, c and d its Cartesian components without modulus and in (130) the correspondence between A, B, C and D and a, b, c and d.

$v = (A+Bi)C + Dj$ $\lvert v\rvert = \sqrt{A^2C^2 + B^2C^2 + D^2}$	(128)	$\frac{v}{\lvert v\rvert} = (a+bi)c + dj$	(129)	$A = \lvert v\rvert a$ $B = \lvert v\rvert b$	$C = c$ $D = \lvert v\rvert d$	(130)

In (131) v without modulus is written with the sin and cos of its arguments and in (132) the correspondence between a, b, c and d and its arguments.

$\frac{v}{\lvert v\rvert} = (\cos\theta + \sin\theta\, i)\cos\varphi + \sin\varphi\, j$	(131)	$\cos\theta = a$ $\sin\theta = b$	$\cos\varphi = c$ $\sin\varphi = d$	(132)

The sin and cos of the equation (127) are expressed with the Cartesian components without modulus a, b, c and d of the two 3D complex numbers v_1 and v_2 in (133) and (134) using (132).

$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = a_1a_2 - b_1b_2$ $\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = b_1a_2 + a_1b_2$	(133)

$\cos(\varphi_1+\varphi_2) = \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 = c_1c_2 - d_1d_2$ $\sin(\varphi_1+\varphi_2) = \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 = d_1c_2 + c_1d_2$	(134)

Then, (127) is expressed in (135) with the Cartesian components without modulus using (133) and (134).

$\frac{v_1 v_2}{\lvert v_1\rvert\lvert v_2\rvert} = \big((a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i\big)(c_1c_2 - d_1d_2) + (d_1c_2 + c_1d_2)j$	(135)

Multiplying (135) with the modulus gives (136).

$v_1v_2 = \lvert v_1\rvert\lvert v_2\rvert\big((a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i\big)(c_1c_2 - d_1d_2) + \lvert v_1\rvert\lvert v_2\rvert(d_1c_2 + c_1d_2)j$	(136)

Using (130) we write the terms in the parentheses of (136) with the Cartesian components A, B, C and D of the two 3D complex numbers v_1 and v_2 in (137) and (138). Introducing (137) and (138) into (136) gives the expression of the product v_2v_2 in (141).

$\lvert v_1\rvert\lvert v_2\rvert(a_1a_2 - b_1b_2) = A_1A_2 - B_1B_2$ $\lvert v_1\rvert\lvert v_2\rvert(b_1a_2 + a_1b_2) = B_1A_2 + A_1B_2$	(137)	$c_1c_2 - d_1d_2 = C_1C_2 - \frac{D_1D_2}{\lvert v_1\rvert\lvert v_2\rvert}$ $\lvert v_1\rvert\lvert v_2\rvert(d_1c_2 + c_1d_2) = (\lvert v_1\rvert\lvert v_2\rvert d_1c_2 + c_1\lvert v_1\rvert\lvert v_2\rvert d_2)$ $= D_1\lvert v_2\rvert C_2 + \lvert v_1\rvert C_1D_2$	(138)

So, the final formula for expressing the product v_2v_2 of the two 3D complex numbers v_1 and v_2 in (139) is (141).

$v_1 = (A_1 + B_1i)C_1 + D_1j$	$v_2 = (A_2 + B_2i)C_2 + D_2j$	(139)
$\lvert v_1\rvert = \sqrt{{A_1}^2{C_1}^2 + {B_1}^2{C_1}^2 + {D_1}^2}$	$\lvert v_2\rvert = \sqrt{{A_2}^2{C_2}^2 + {B_2}^2{C_2}^2 + {D_2}^2}$	(140)

$v_1v_2 = \big((A_1A_2 - B_1B_2) + (B_1A_2 + A_1B_2)i\big)\left(C_1C_2 - \frac{D_1D_2}{\lvert v_1\rvert\lvert v_2\rvert}\right) + (D_1\lvert v_2\rvert C_2 + \lvert v_1\rvert C_1D_2)j$	(141)

For example, let's compute the product of the two 3D complex numbers v_1 and v_2 in (142). For using the equation (141), we find the values of the Cartesian components in (143) which gives (144).

$v_1 = A_1 + B_1 i$	$v_2 = A_2 + D_2 j$	(142)
$v_1 = (A_1 + B_1 i)1 + 0j$	$v_2 = (A_2 + 0i)1 + D_2 j$	(143)
$C_1 = 1 \qquad D_1 = 0$	$B_2 = 0 \qquad C_2 = 1$	(144)

We apply (144) in (141), which gives (145). The modulus of v_1 is computed in (146), which is introduced into (145) to give (147), which is the final formula of the product v_1v_2.

$v_1v_2 = \big((A_1A_2 - 0) + (B_1A_2 + 0)i\big)\left(1 - \frac{0}{\lvert v_1\rvert\lvert v_2\rvert}\right) + (0 + \lvert v_1\rvert D_2)j$ $= A_1A_2 + B_1A_2 i + \lvert v_1\rvert D_2 j$			(145)
$\lvert v_1\rvert = \sqrt{A_1{}^2 + B_1{}^2}$	(146)	$(A_1 + B_1 i)(A_2 + D_2 j)$ $= A_1A_2 + B_1A_2 i + D_2\sqrt{A_1{}^2 + B_1{}^2}\, j$	(147)

10. 21 February 2022

I have constructed the multidimensional complex systems with 3, 4 or more dimensions in the paper «Extending complex number to spaces with 3, 4 or any number of dimensions». In this system multiplication is not distributive over addition. For example, 3D complex number multiplication does develop like in (148), g being another complex number, while for 2D complex number multiplication is distributive, see (149).

$g(a + bi + cj) \neq ga + gbi + gcj$	(148)
$g(a + bi) = ga + gbi$	(149)

Complex numbers in (148) and (149) are in Cartesian form and then, multiplication of 3D complex numbers should not be done in Cartesian form. For multiplying two 3D complex numbers we have to convert them into trigonometric form. For example, the complex numbers $a + bi + cj$ and g in (148) must be converted into trigonometric form.

It is clearer to explain with concrete number. Let us take an actual product: $(1i + 4h)(1i + 5j)$, h being the unit of the real line, what is it value? If we develop this product like for 2D complex number we get equation (150), which has the term ij that does not belong to the 3D space (h, i, j).

$(1i + 4h)(1i + 5j) = (1i + 4h)1i + (1i + 4h)5j$ $= 1ii + 4hi + 5ij + 20hj$ $= -1 + 4i + 5ij + 20j$	(150)

For converting $(1i + 4h)$ into trigonometric form, we first compute its modulus and arguments in equations (151), (152) and (153). Then, we compute the modulus and arguments of $(1i + 5j)$ in (154), (155) and (156).

$1i + 4h = 4 + 1i$ $= \lvert u\rvert(\cos\theta + \sin\theta\, i)$ $= \lvert u\rvert\big((\cos\theta + \sin\theta\, i) * 1 + 0j\big)$ $= \lvert u\rvert\big((\cos\theta + \sin\theta\, i)\cos 0 + \sin 0\, j\big)$	(151)
$\lvert u\rvert = \sqrt{4^2 + 1}$	(152)
$\cos\theta = \frac{4}{\sqrt{4^2 + 1}}, \sin\theta = \frac{1}{\sqrt{4^2 + 1}}$	(153)

$1i + 5j$ $= \lvert v\rvert(\cos\varphi\, i + \sin\varphi\, j)$ $= \lvert v\rvert\big((0 + i)\cos\varphi + \sin\varphi\, j\big)$ $= \lvert v\rvert\left(\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2} i\right)\cos\varphi + \sin\varphi\, j\right)$	(154)
$\lvert v\rvert = \sqrt{5^2 + 1}$	(155)
$\cos\varphi = \frac{1}{\sqrt{5^2 + 1}}, \sin\varphi = \frac{5}{\sqrt{5^2 + 1}}$	(156)

Equation (151) and (154) are in Cartesian form but also in the form of the equation 16 of the original paper «Extending complex number to spaces with 3, 4 or any number of dimensions».
https://www.academia.edu/71708344/Extending_complex_number_to_spaces_with_3_4_or_any_number_of_dimensions
https://pengkuanonmaths.blogspot.com/2022/02/extending-complex-number-to-spaces-with.html

The equation 16 of the original paper is converted into trigonometric form by the equation 32 of the original paper. So, equation (151) and (154) are converted the same way in (157) and (158).

$1i + 4h = \lvert u\rvert(\cos\theta + \sin\theta\, i)(\cos 0 + \sin 0\, j)$	(157)
$1i + 5j = \lvert v\rvert\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2} i\right)(\cos\varphi + \sin\varphi\, j)$	(158)

Then, we multiply (157) and (158) together in (159).

$$(1i+4h)(1i+5j)=|u|(\cos\theta+\sin\theta\, i)(\cos 0+\sin 0\, j)*|v|\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}i\right)(\cos\varphi+\sin\varphi\, j) \quad (159)$$

The formula of the product of two 3D complex numbers in trigonometric form is given by the equations 51 and 52 of the original paper, which consists of adding the arguments of the numbers dimension by dimension and multiplying the modulus in the resulting number. So, we do the same for $(1i+4h)(1i+5j)$ in (160).

$$(1i+4h)(1i+5j)=|v||u|\left(\cos\left(\theta+\frac{\pi}{2}\right)+\sin\left(\theta+\frac{\pi}{2}\right)i\right)(\cos(0+\varphi)+\sin(0+\varphi)\, j) \quad (160)$$

By the way, it is easier to use exponential form. Equation (161) transforms (157) and (158) using Euler's formula, equation (162) multiplies them together. By comparing (162) with (160), we find the same number.

$$\begin{aligned}1i+4h&=|u|e^{i\theta+j0}\\ 1i+5j&=|v|e^{i\frac{\pi}{2}+j\varphi}\end{aligned} \quad (161) \qquad (1i+4h)(1i+5j)=|u|e^{i\theta+j0}|v|e^{i\frac{\pi}{2}+j\varphi}=|u||v|e^{i\left(\theta+\frac{\pi}{2}\right)+j\varphi} \quad (162)$$

Then, we develop equation (160) into Cartesian form in (163) by using the equation 16 of the original paper.

$$(1i+5j)(1i+4h)=|u||v|\big((-\sin\theta+\cos\theta\, i)\cos\varphi+\sin\varphi\, j\big) \quad (163)$$

Introducing the expression of the moduli and arguments given in (152), (153), (155) and (156), the product becomes (164) which is then simplified in (165), which is the resulting product $(1i+5j)(1i+4h)$ in 3D complex number.

$$\begin{aligned}(1i+5j)(1i+4h)&=\sqrt{4^2+1}\sqrt{5^2+1}\left(\left(\frac{-1}{\sqrt{4^2+1}}+\frac{4}{\sqrt{4^2+1}}i\right)\frac{1}{\sqrt{5^2+1}}+\frac{5}{\sqrt{5^2+1}}j\right)\\ &=\left(\frac{-1}{\sqrt{4^2+1}}+\frac{4}{\sqrt{4^2+1}}i\right)\frac{\sqrt{4^2+1}\sqrt{5^2+1}}{\sqrt{5^2+1}}+\frac{5\sqrt{4^2+1}\sqrt{5^2+1}}{\sqrt{5^2+1}}j\end{aligned} \quad (164)$$

$$(1i+5j)(1i+4h)=-1+4i+5\sqrt{4^2+1}j \quad (165)$$