General equation for Space-Time geodesics and orbit equation in relativistic gravity (equation layout)

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Abstract: Orbital precession of the planet Mercury was computed using Space-Time curvature which is extremely difficult to understand. This article exposes an analytical orbit equation for relativistic gravity and explains how it is derived without Space-Time geodesics. The value of orbital precession that this orbit equation gives is identical to that computed using General Relativity. This orbit equation is an analytical form of Space-Time geodesics for space objects and makes everyone able to compute the orbit of any object in gravitational field which obeys General Relativity using personal computer and its direct derivation from gravitational acceleration gives a new insight to General Relativity without the need of knowing Einstein tensor.

1. Orbit equation and orbital precession

General Relativity explains gravity as Space-Time curvature and orbits of planets as geodesics of curved Space-Time. However, this concept is extremely hard to understand and geodesics hard to compute. If we can find an analytical orbit equation for planets like Newtonian orbit equation, relativistic gravity will become intuitive and straightforward so that most people can understand.

From gravitational force and acceleration, I have derived the analytical orbit equation for relativistic gravity which is equation (1). Below I will explain the derivation of this equation. Albert Einstein had correctly predicted the orbital precession of planet Mercury which had definitively validated General Relativity. Equation (2) is the angle of orbital precession that this orbit equation gives, which is identical to the one Albert Einstein had given [1][2].

If this orbit equation gave the same result than Space-Time geodesics, then everyone can compute the orbit of any object in gravitational field which obeys General Relativity using personal computer rather than big or super computer. The direct derivation from gravitational acceleration allows everyone to see the mechanism linking gravity with curved Space-Time, which brings a new insight to General Relativity.

Notation convention

Bold letter: vector variables.
Number in parentheses (*i*): equation number

Subscript:
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0: A known point
m: Proper frame of m
l: Local frame
∞: Frame at infinitely far
N: Newtonian quantity
a: Near perihelion
b: Away from perihelion
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Relativistic orbit equation

$$r = \frac{a(1-\varepsilon^2)}{1-\varepsilon\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)} \qquad (1)$$

a: semi-major axis, ε: eccentricity

Angle of precession per turn

$$|\Delta\Phi| \approx \frac{6\pi GM}{a(1-\varepsilon^2)c^2} \qquad (2)$$

2. Relativistic dynamics

a) Velocity in local frame

Take an attracting body of mass M around which orbits a small body of mass m, see Figure 1. We work with a polar coordinate system of which the body M sits at the origin. The position of the body m with respect to M is specified by the radial position vector $\boldsymbol{r}$, of which the magnitude is r and the polar angle is θ.

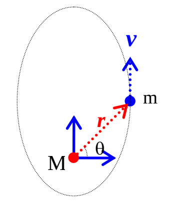

Figure 1
m orbits at velocity *v* around M. Polar coordinates: *r* and *θ*.

Let the frame of reference "frame_m" be an inertial fame that instantaneously moves with m. Frame_m is the proper frame of m where the velocity of m is 0. So, Newton's laws apply in this frame. Let $\boldsymbol{\alpha}_m$ be the acceleration vector of m in frame_m and the inertial force of m is m·$\boldsymbol{\alpha}_m$, see equation (3). The gravitational force on m is given by equation (4). Equating (4) with (3), we get equation (5), the proper acceleration of m caused by gravitational force in frame_m.

Let "frame_*l*" be the local frame of reference in which M is stationary. In frame_*l* m is under the effect of gravity of M, the velocity vector of m is $\boldsymbol{v}_l$ and the acceleration of m is $\boldsymbol{\alpha}_l$. As frame_m moves with m, it moves at the velocity $\boldsymbol{v}_l$ in frame_l.

In frame_m $\boldsymbol{F}=m\boldsymbol{\alpha}_m$	(3)	$\boldsymbol{F}=-\frac{GMm}{r^2}\boldsymbol{e}_r$	(4)	(3)=(4) : $\boldsymbol{\alpha}_m=-\frac{GM}{r^2}\boldsymbol{e}_r$	(5)

Acceleration transform $\boldsymbol{\alpha}_m=\frac{\boldsymbol{\alpha}_l}{1-\frac{\boldsymbol{v}_l^2}{c^2}}$	(6)	(5) = (6): $-\frac{GM}{r^2}\boldsymbol{e}_r=\frac{\boldsymbol{\alpha}_l}{1-\frac{\boldsymbol{v}_l^2}{c^2}}$	(7)

The acceleration of m in frame_m and frame_*l* are respectively $\boldsymbol{\alpha}_m$ and $\boldsymbol{\alpha}_l$. To transform $\boldsymbol{\alpha}_l$ into $\boldsymbol{\alpha}_m$ we use the transformation of acceleration between relatively moving frames which is the equation (18) in « Relativistic kinematics » [3][4], in which we replace $\boldsymbol{\alpha}_1$ with $\boldsymbol{\alpha}_l$, $\boldsymbol{\alpha}_2$ with $\boldsymbol{\alpha}_m$ and $\boldsymbol{u}$ with $\boldsymbol{v}_l$. Then the transformation between $\boldsymbol{\alpha}_m$ and $\boldsymbol{\alpha}_l$ is equation (6).

(7) dotted with $v_l\cdot dt_l$ $-\frac{GM}{r^2}\boldsymbol{e}_r\cdot\boldsymbol{v}_l dt_l=\frac{\boldsymbol{\alpha}_l\cdot\boldsymbol{v}_l dt_l}{1-\frac{\boldsymbol{v}_l^2}{c^2}}$ t_l: time in frame_*l*	(8)

Equating (5) with (6) we get equation (7), both sides of which are then dotted by the vector $\boldsymbol{v}_l\cdot dt_l$, see equation (8). On the left hand side of (8), we find *dr* the variation of the radial distance *r*, see (9). On the right hand side, we find the variation vector of velocity $d\boldsymbol{v}_l$, see (10), which, dotted by the velocity vector $\boldsymbol{v}_l$, gives $d\boldsymbol{v}_l^2/2$ in (11).

Plugging (9) and (11) into (8), we get (12), both sides of which are differential expressions, see (13) and (15). Then, plugging (13) and (15) into (12) gives (16) which is a differential equation. (16) is integrated to give (17), with *K* being the integration constant. Then, we rearrange (17) to express $\boldsymbol{v}_l^2/c^2$ in (18), which relates the local orbital velocity $\boldsymbol{v}_l$ to the gravitational field of M.

$\boldsymbol{e}_r\cdot\boldsymbol{v}_l dt_l=dr$	(9)
$d\boldsymbol{v}_l=\boldsymbol{\alpha}_l dt_l$	(10)
(10) makes $\boldsymbol{\alpha}_l\cdot\boldsymbol{v}_l dt_l=\frac{d\boldsymbol{v}_l^2}{2}$	(11)
Using (9) and (11) in (8) $-\frac{GM}{r^2}dr=\frac{1}{1-\frac{\boldsymbol{v}_l^2}{c^2}}\frac{d\boldsymbol{v}_l^2}{2}$	(12)
Left hand side of (12) $-\frac{GM}{r^2}dr=d\left(\frac{GM}{r}\right)$	(13)

The value of e^K is determined at a known point 0 at which the velocity is $\boldsymbol{v}_0$ and the radial distance is r_0, see (19).

Formula $\frac{d\boldsymbol{v}_l^2}{2}=-\frac{c^2}{2}d\left(1-\frac{\boldsymbol{v}_l^2}{c^2}\right)$	(14)
Right hand side of (12), using (14) $\frac{1}{1-\frac{\boldsymbol{v}_l^2}{c^2}}\frac{d\boldsymbol{v}_l^2}{2}=-\frac{c^2}{2}d\ ln\left(1-\frac{\boldsymbol{v}_l^2}{c^2}\right)$	(15)

Using (13) and (15) in (12) $d\left(\frac{GM}{r}\right)=-\frac{c^2}{2}d\ ln\left(1-\frac{\boldsymbol{v}_l^2}{c^2}\right)$ Integration $ln\left(1-\frac{\boldsymbol{v}_l^2}{c^2}\right)+\frac{2GM}{c^2r}=K$ *K*: integration constant	(16)

$1-\frac{\boldsymbol{v}_l^2}{c^2}=e^{K-\frac{2GM}{c^2r}}$	(17)
$\frac{\boldsymbol{v}_l^2}{c^2}=1-e^{K-\frac{2GM}{c^2r}}$	(18)
Value of e^K $e^K=\left(1-\frac{\boldsymbol{v}_0^2}{c^2}\right)e^{\frac{2GM}{c^2r_0}}$	(19)

b) Velocity in the frame at infinitely far

The orbit of a planet is the curve drawn by its position with respect to a frame infinitely far from M which is labeled "frame_∞" and where the gravitational effect of M is 0. In this frame the time is t_∞, the radial position vector is $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}_\infty$ equals the derivative of the radial position vector $\boldsymbol{r}$ with respect to the time t_∞, see (20).

In frame_∞ $\boldsymbol{v}_\infty=\frac{d\boldsymbol{r}}{dt_\infty}$	(20)

The vector $\boldsymbol{v}_l$ is the velocity in frame_*l* and equals the derivative of the same $\boldsymbol{r}$ with respect to t_l, the time in frame_*l*, see (21), where $\boldsymbol{v}_l$ is transformed into the product of $\boldsymbol{v}_\infty$ and $\frac{dt_\infty}{dt_l}$ which is the time dilation factor of the Schwarzschild metric around M, see (22). By plugging (22) into (21) we obtain (23), which relates $\boldsymbol{v}_l$ to $\boldsymbol{v}_\infty$. We substitute (23) for $\boldsymbol{v}_l$ in (18) to get (24), which relates $\boldsymbol{v}_\infty$ to the gravitational field of M.

In frame_*l* $\boldsymbol{v}_l=\frac{d\boldsymbol{r}}{dt_l}=\frac{d\boldsymbol{r}}{dt_\infty}\frac{dt_\infty}{dt_l}=\boldsymbol{v}_\infty\frac{dt_\infty}{dt_l}$	(21)

Time dilation in Schwarzschild metric $t_l = t_\infty\sqrt{1-\frac{2GM}{c^2r}} \Rightarrow \frac{dt_\infty}{dt_l} = \frac{1}{\sqrt{1-\frac{2GM}{c^2r}}}$	(22)	Using (22) in (21) $\boldsymbol{v}_l = \frac{\boldsymbol{v}_\infty}{\sqrt{1-\frac{2GM}{c^2r}}}$	(23)	Substituting (23) for $\boldsymbol{v}_l$ in (18) $\frac{\boldsymbol{v}_\infty{}^2}{c^2\left(1-\frac{2GM}{c^2r}\right)} = 1 - e^{K-\frac{2GM}{c^2r}}$	(24)

Equation (24) gives (25) the right hand side of which is the product of two parentheses, the first comes from (18) which relates velocity to gravitational force, the second comes from (22) which expresses the time dilation of Schwarzschild metric.

(24) gives $\frac{\boldsymbol{v}_\infty{}^2}{c^2} = \left(1-e^{K-\frac{2GM}{c^2r}}\right)\left(1-\frac{2GM}{c^2r}\right)$	(25)
For very small $\frac{GM}{c^2r}$ $e^{K-\frac{2GM}{c^2r}} \approx e^K\left(1-\frac{2GM}{c^2r}+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2\right)$	(26)

In the case of planets, $\frac{GM}{c^2r}$ is very small and the term $e^{K-\frac{2GM}{c^2r}}$ is simplified till the second order of $\frac{GM}{c^2r}$ in (26). Substituting (26) for $e^{K-\frac{2GM}{c^2r}}$ in (25) gives (27). We drop off the third order term $4\left(\frac{GM}{c^2r}\right)^3$ from (27) and obtain the expression of $\frac{\boldsymbol{u}_\infty{}^2}{c^2}$ in (28). $\boldsymbol{u}_\infty$ is the velocity vector of m in frame_∞ correct to the 2nd order of $\frac{GM}{c^2r}$.

Substituting (26) for $e^{K-\frac{2GM}{c^2r}}$ in (25) $\frac{\boldsymbol{v}_\infty{}^2}{c^2} \approx \left(1-e^K\left(1-\frac{2GM}{c^2r}+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2\right)\right)\left(1-\frac{2GM}{c^2r}\right)$ $= 1-\frac{2GM}{c^2r}-e^K\left(\left(1-\frac{2GM}{c^2r}\right)^2+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2\left(1-\frac{2GM}{c^2r}\right)\right)$ $= 1-\frac{2GM}{c^2r}-e^K\left(1-\frac{4GM}{c^2r}+\left(\frac{2GM}{c^2r}\right)^2+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2-\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^3\right)$	(27)
Dropping off $\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^3$ from (27) $\frac{\boldsymbol{u}_\infty{}^2}{c^2} = 1-\frac{2GM}{c^2r}-e^K\left(1-\frac{4GM}{c^2r}+6\left(\frac{GM}{c^2r}\right)^2\right)$ $\boldsymbol{u}_\infty$ is the orbit velocity vector in frame_∞ correct to the 2nd order of $\frac{GM}{c^2r}$	(28)

c) Velocity squared

The general expression of velocity vector in polar coordinates is given in equation (110) and Kepler's second law in (115). The derivation of (110) and (115) are in equations (107) through (115). Readers can find them in textbooks or other references [5][6].

Velocity squared is, see (110) $\boldsymbol{u}_\infty{}^2 = \left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2 = \left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2$	(29)	Formula $\frac{dr}{d\theta} = -r^2\frac{d}{d\theta}\left(\frac{1}{r}\right)$	(30)

Using (30) and (115) for $\frac{dr}{d\theta}\frac{d\theta}{dt}$ $\frac{dr}{d\theta}\frac{d\theta}{dt} - r^2\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d\theta}{dt} = -\left(r^2\frac{d\theta}{dt}\right)\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{d}{d\theta}\left(\frac{h}{r}\right)$	(31)

Let us express the velocity squared $\boldsymbol{u}_\infty{}^2$ with the constant h of Kepler's second law. Using (110) we express $\boldsymbol{u}_\infty{}^2$ in (29), of which the term $\frac{dr}{d\theta}\frac{d\theta}{dt}$ is expressed with h in (31) by using (30) and (115). Then using (115) and (31) , (29) is transformed into (32), which expresses $\boldsymbol{u}_\infty{}^2$ in terms of $\frac{h}{r}$ instead of r.

Using (115) and (31) in (29) $\boldsymbol{u}_\infty{}^2 = \left(\frac{d}{d\theta}\left(\frac{h}{r}\right)\right)^2+\left(\frac{h}{r}\right)^2$	(32)

d) Differential equation for orbit

We divide (32) with c^2 to express $\frac{\boldsymbol{u}_\infty{}^2}{c^2}$ in terms of $\frac{h}{cr}$, see (33). We plug (33) and (34) into (28) to get (36), which is the differential equation for relativistic orbit in frame_∞. Solving (36) will give the analytical relativistic orbit equation.

(32) divide by c^2 $\frac{\boldsymbol{u}_\infty{}^2}{c^2} = \frac{1}{c^2}\left(\frac{d}{d\theta}\left(\frac{h}{r}\right)\right)^2 + \frac{1}{c^2}\left(\frac{h}{r}\right)^2 = \left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2$	(33)
Formula $\frac{GM}{c^2 r} = \frac{GM}{hc}\frac{h}{cr}$	(34)
Using (33) and (34) in (28) $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2 = 1 - 2\frac{GM}{hc}\frac{h}{cr} - e^K\left(1 - 4\frac{GM}{hc}\frac{h}{cr} + 6\left(\frac{GM}{hc}\frac{h}{cr}\right)^2\right)$ $= 1 - e^K - 2\frac{GM}{hc}\frac{h}{cr} + 4e^K\frac{GM}{hc}\frac{h}{cr} - 6e^K\left(\frac{GM}{hc}\frac{h}{cr}\right)^2$ Then $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = 1 - e^K + 2(2e^K - 1)\frac{GM}{hc}\frac{h}{cr} - 6e^K\left(\frac{GM}{hc}\right)^2\left(\frac{h}{cr}\right)^2 - \left(\frac{h}{cr}\right)^2$	(35)
From (35) $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = 1 - e^K + 2(2e^K - 1)\frac{GM}{hc}\frac{h}{cr} - \left(1 + 6e^K\left(\frac{GM}{hc}\right)^2\right)\left(\frac{h}{cr}\right)^2$ This is the Relativistic orbit differential equation	(36)
Transformation of (36) $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = -\left(1 + 6e^K\left(\frac{GM}{hc}\right)^2\right)\left(-\frac{1 - e^K}{1 + 6e^K\left(\frac{GM}{hc}\right)^2} - 2\left(\frac{2e^K - 1}{1 + 6e^K\left(\frac{GM}{hc}\right)^2}\right)\frac{GM}{hc}\frac{h}{cr} + \left(\frac{h}{cr}\right)^2\right)$	(37)

3. Relativistic orbit

a) Solving the differential equation for orbit

For solving the differential equation (36) we transform it into (37). As the right hand side of (37) is a messy polynomial of degree 2, we replace the coefficients with constants *A*, *B* and *C* of which the expressions are in (38). Then (37) is transformed into the much simpler (40).

Here, we do a change of variable which is defined in (41) for *s* and (42) for $\frac{ds}{d\theta}$, with which we transform (40) into the simple differential equation (44). By using the formula of integration (45), we integrate (44) and find (46), with θ_0 being the integration constant.

Replacing coefficients by constants *A*, *B* and *C* $C = 1 + 6e^K\left(\frac{GM}{hc}\right)^2, A = \frac{2e^K - 1}{C}\frac{GM}{hc}, B = \frac{1 - e^K}{C}$	(38)
(37) is expressed using *A*, *B* and *C* $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = -C\left(-B - 2A\frac{h}{cr} + \left(\frac{h}{cr}\right)^2\right)$ $= C(A^2 + B)\left(1 - \frac{\left(\frac{h}{cr} - A\right)^2}{A^2 + B}\right)$	(39)
(39) divided by $A^2 + B$ $\left(\frac{d}{d\theta}\left(\frac{\frac{h}{cr}}{\sqrt{A^2 + B}}\right)\right)^2 = C\left(1 - \left(\frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}}\right)^2\right)$	(40)

Change of variables $s = \frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}} = \frac{\frac{h}{Acr} - 1}{\sqrt{1 + \frac{B}{A^2}}}$	(41)	$\frac{ds}{d\theta} = \frac{d}{d\theta}\left(\frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}}\right) = \frac{d}{d\theta}\left(\frac{\frac{h}{cr}}{\sqrt{A^2 + B}}\right)$	(42)	Using (41) and (42) in (40) $\left(\frac{ds}{d\theta}\right)^2 = C(1 - s^2)$ $\Rightarrow \frac{ds}{d\theta} = \pm\sqrt{C}\sqrt{1 - s^2}$	(43)

Then $\frac{ds}{\sqrt{1 - s^2}} = \pm\sqrt{C}d\theta$	(44)	Formula of integration $\int \frac{ds}{\sqrt{1 - s^2}} = \sin^{-1} s$	(45)	Using (45) to integrate (44) $\sin^{-1} s = \pm\sqrt{C}(\theta + \theta_0)$ θ_0: integration constant	(46)

The solution of the differential equation (44) is not unique. We choose the one with θ_0=0 and negative sign in (46). Taking the sine of this solution, we obtain (47). Then, for returning to the original variable we substitute (41) for s in (47) and obtain (49) which is the relativistic orbit equation expressed with A, B and C. For getting rid of the constants A, B and C, we use their expressions given in (38) to express the terms $\frac{h}{Ac}$ and $\frac{B}{A^2}$ in (51) and (53), which are plugged into (49) to obtain the final equation for r in (54).

θ_0=0 in (46) $s = -\sin\theta\sqrt{C}$	(47)
Using (41) in (47) $$\frac{\frac{h}{Acr}-1}{\sqrt{1+\frac{B}{A^2}}} = -\sin\theta\sqrt{C}$$	(48)

(48) transformed $$r = \frac{\frac{h}{Ac}}{1-\sqrt{1+\frac{B}{A^2}}\sin(\theta\sqrt{C})}$$ Solution of (40)	(49)	Using A from (38) $$\frac{h}{Ac} = \frac{h}{c}\left(\frac{2e^K-1}{C}\frac{GM}{hc}\right)^{-1} = \frac{C}{2e^K-1}\frac{h^2}{GM}$$	(50)	Using C from (38) in (50) $$\frac{h}{Ac} = \frac{1+6e^K\left(\frac{GM}{hc}\right)^2}{2e^K-1}\frac{h^2}{GM}$$	(51)

Using A and B from (38) for $\frac{B}{A^2}$ $$\frac{B}{A^2} = \frac{1-e^K}{C}\left(\frac{2e^K-1}{C}\frac{GM}{hc}\right)^{-2} = C\frac{1-e^K}{(2e^K-1)^2}\left(\frac{hc}{GM}\right)^2$$	(52)	Using C from (38), (52) becomes $$\frac{B}{A^2} = \frac{1-e^K}{(2e^K-1)^2}\left(\frac{hc}{GM}\right)^2\left(1+6e^K\left(\frac{GM}{hc}\right)^2\right) = \frac{1-e^K}{(2e^K-1)^2}\left(\left(\frac{hc}{GM}\right)^2+6e^K\right)$$	(53)

Using (51), (53) and (38) for C, (49) becomes $$r = \frac{\frac{1+6e^K\left(\frac{GM}{hc}\right)^2}{2e^K-1}\frac{h^2}{GM}}{1-\sqrt{1+\frac{1-e^K}{(2e^K-1)^2}\left(\left(\frac{hc}{GM}\right)^2+6e^K\right)}\sin\left(\theta\sqrt{1+6e^K\left(\frac{GM}{hc}\right)^2}\right)}$$ This is the solution of the relativistic orbit differential equation (36) and the relativistic orbit equation in the gravitational field of the body M.	(54)

(54) is the solution of the differential equation (36) and is the analytical relativistic orbit equation in the gravitational field of the attracting body M. Equation (54) expresses r in terms of θ, G, M, c, K the integration constant in (17) and h the constant of Kepler's second law, see (17) and (115).

b) Orbit as conic section curve

Orbits of planets are usually expressed as conic section curve. Let us express the relativistic orbit equation with the parameters of conic section which are l: the semi-latus rectum, a: semi-major axis, ε: eccentricity [6].

A general conic section curve is defined in polar coordinates by the function (55). By comparing (49) with (55), we find the l, ε and ϕ of the orbit, which are expressed in (56). Then, the relativistic orbit equation is expressed as a conic section curve in (57).

Now, we have to express the constant C with conic section parameters. For doing so, we compare (57) with (54) and find the expression of l in (58), with which we express the term $\left(\frac{hc}{GM}\right)^2$ in (59). Plugging (59) into (60) gives the expression of C in (61).

Conic section $$r = \frac{l}{1-\varepsilon\sin\phi}$$	(55)
Using (55) and (49) $$l = \frac{h}{Ac}$$ $$\varepsilon = \sqrt{1+\frac{B}{A^2}}$$ $$\phi = \theta\sqrt{C}$$	(56)
(49) becomes $$r = \frac{l}{1-\varepsilon\sin(\theta\sqrt{C})}$$	(57)

Using (57) and (54) $$l = \frac{1+6e^K\left(\frac{GM}{hc}\right)^2}{2e^K-1}\frac{h^2}{GM} = \frac{GM}{c^2}\frac{\left(\frac{hc}{GM}\right)^2+6e^K}{2e^K-1}$$	(58)	(58) gives $\left(\frac{hc}{GM}\right)^2$ $$\left(\frac{hc}{GM}\right)^2 = \frac{lc^2}{GM}(2e^K-1)-6e^K$$	(59)	Using (59) in (38) for C $$C = 1+6e^K\left(\frac{GM}{hc}\right)^2 = \frac{\frac{lc^2}{6GM}\frac{2e^K-1}{e^K}-1+1}{\frac{lc^2}{6GM}\frac{2e^K-1}{e^K}-1}$$	(60)

The term $\frac{e^K}{2e^K-1}$ in (61) has to be expressed with conic section parameters. For doing so we take the square of ε (see (56)) in (62) to express the term $\frac{B}{A^2}$. We find another expression of $\frac{B}{A^2}$ in (64) using (63). Equating (62) with (64) gives the expression of $\frac{1-e^K}{2e^K-1}$ with conic section parameters in (67), which allow us to express $\frac{e^K}{2e^K-1}$ with conic section parameters in (69).

(60) becomes $C=\left(1-\frac{e^K}{2e^K-1}\frac{6GM}{lc^2}\right)^{-1}$	(61)
$\frac{B}{A^2}=\varepsilon^2-1$	(62)

(53) divided by (50) $\frac{\frac{B}{A^2}}{\frac{h}{Ac}}=\frac{C\frac{1-e^K}{(2e^K-1)^2}\left(\frac{hc}{GM}\right)^2}{\frac{C}{2e^K-1}\frac{h^2}{GM}}$	(63)	Using $l=\frac{h}{Ac}$ from (56), (63) gives $\frac{B}{A^2}=\frac{h}{Ac}\frac{c^2}{GM}\frac{1-e^K}{2e^K-1}=\frac{lc^2}{GM}\frac{1-e^K}{2e^K-1}$	(64)	Equating (62) with (64) $\frac{lc^2}{GM}\frac{1-e^K}{2e^K-1}=\varepsilon^2-1$ $\Rightarrow\frac{1-e^K}{2e^K-1}=\frac{(\varepsilon^2-1)GM}{lc^2}$	(65)

Conic section formula $l=a(1-\varepsilon^2)$ a: semi-major axis	(66)	Using (66) in (65) $\frac{1-e^K}{2e^K-1}=\frac{(\varepsilon^2-1)GM}{a(1-\varepsilon^2)c^2}=-\frac{GM}{ac^2}$	(67)	Searching $\frac{e^K}{2e^K-1}$ in $\frac{1-e^K}{2e^K-1}$ $\frac{1-e^K}{2e^K-1}=\frac{1-2e^K+e^K}{2e^K-1}=\frac{e^K}{2e^K-1}-1$	(68)

Introducing (69) into (61) gives the expression of C with conic section parameters in (70), which is then introduced into (57) to obtain (71).

Using (67) in (68) $\frac{e^K}{2e^K-1}=\frac{1-e^K}{2e^K-1}+1=1-\frac{GM}{ac^2}$	(69)
Using (69) and (66) in (61) $C=\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-1}$	(70)

Equation (71) is the final analytical relativistic orbit equation expressed with conic section parameters. For making sure that (71) is the correct orbit equation, we check its validity using 3 cases.

Using (66) and (70), (57) becomes $r=\frac{a(1-\varepsilon^2)}{1-\varepsilon\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)}$ This is the Relativistic orbit as conic section curve and solution of the differential equation (36)	(71)

4. Validation of the orbit equation

a) Orbital precession

The first check is to see if the relativistic orbit equation (71) gives the correct angle of precession for planets. Due to General Relativistic effect elliptic orbit rotates in space, which is materialized by the angle that the major axis of elliptic orbit rotates from one revolution to the next. We denote this angle with $\Delta\Phi$ and call it the angle of precession per turn.

Orbit starts from θ=0, (71) gives $\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$	(72)
Orbit ends at θ_1. For the same radial distance θ_1 must make $\sin\left(\theta_1\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$	(73)

For computing $\Delta\Phi$, suppose a first turn of orbit starts at the polar angle θ=0 and the radial distance is $r_0=a(1-\varepsilon^2)$, see (71). θ=0 makes $\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$, see (72). At the exact end of this turn the polar angle is θ_1. The radial distance is again r_0 because the second orbit starts here. As the radial distance is the same as the first turn, θ_1 must make $\sin\left(\theta_1\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$ too, see (71) and (73). So, $\theta_1\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}=2\pi$, see (74).

$\theta_1\neq 0\Rightarrow$ $\frac{\theta_1}{\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{\frac{1}{2}}}=2\pi$	(74)	(74) gives $\theta_1=2\pi\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{\frac{1}{2}}$	(75)

The polar angle of the starting point of the first turn is 0 or 2π and that of the second turn is θ_1. The difference of these 2 polar angles equals $\Delta\Phi = \theta_1 - 2\pi$, see (76), which means that the starting point of the second turn is at the angle $\Delta\Phi$ with respect to starting point of the first turn. Or, seeing each turn as an ellipse, the major axis of the second ellipse has rotated the angle $\Delta\Phi$ with respect to the first ellipse.

The end of the orbit is at θ_1 and has rotated the angle $\Delta\Phi$ with respect to the starting point at 2π $\Delta\Phi = \theta_1 - 2\pi$ This is the angle of precession per turn	(76)
Simplification of (75) for $\left\|\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right\| \ll 1$ $\theta_1 \approx 2\pi - \frac{2\pi}{2}\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}$	(77)
Using (77) in (76) $\Delta\Phi \approx 2\pi - \frac{2\pi}{2}\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2} - 2\pi$	(78)
$\Delta\Phi \approx -\left(1-\frac{GM}{ac^2}\right)\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ Angle of precession per turn	(79)

Because $\left|\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right| \ll 1$, we simplify the expression of θ_1 (75) in (77) and get the value of $\Delta\Phi$ in (79). Because $1-\frac{GM}{ac^2} \approx 1$ for planets, then $\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$, see (80). The absolute value of $\Delta\Phi$ is $\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$, which is identical to the one Albert Einstein had found [1], see (81).

For planets, $1-\frac{GM}{ac^2} \approx 1$ $\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$	(80)
$\|\Delta\Phi\| \approx \frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ A. Einstein's value	(81)

So, the relativistic orbit equation (71) is the general equation for geodesics in curved Space-Time and has successively passed the check of angle of precession.

b) Variation of orbital velocity

The second check is to see if my theory of relativistic gravity predicts correctly the rate of variation of the velocity of a celestial body moving along its orbit. The method of this check is to compare the rate of variation given by my theory of relativistic gravity with that given by Newtonian gravity. For doing this comparison, we use the expression of the orbital velocity (82) but to the first order. So, the second order term $6\left(\frac{GM}{c^2r}\right)^2$ is dropped out in (83).

Orbital velocity in frame_∞, (28) multiplied by c^2 $\boldsymbol{u}_\infty{}^2 = c^2 - \frac{2GM}{r} - c^2e^K\left(1-\frac{4GM}{c^2r}+6\left(\frac{GM}{c^2r}\right)^2\right)$	(82)
Neglecting the second order $6\left(\frac{GM}{c^2r}\right)^2$ in (82) $\boldsymbol{u}_\infty{}^2 \approx c^2 - \frac{2GM}{r} - c^2e^K + c^2e^K\frac{4GM}{c^2r}$ $= c^2 - c^2e^K - \frac{2GM}{r}(1-2e^K)$	(83)
Relativistic orbit velocities at points a and b are $\boldsymbol{u}_a$ and $\boldsymbol{u}_b$. Difference of velocity squared computed using (83) $(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Relativity}$ $= c^2 - c^2e^K - \frac{2GM(1-2e^K)}{r_b}$ $-\left(c^2 - c^2e^K - \frac{2GM(1-2e^K)}{r_a}\right)$	(84)

For finding a variation of orbital velocity for my relativistic gravity, we take 2 points on the orbit, a and b, compute the velocities at these 2 points which are $\boldsymbol{u}_a$ and $\boldsymbol{u}_b$, then compute the difference of velocity squared in (85). When we make $1-2e^K = -1$ in (85), it gives that for Newtonian gravity, see (86). For comparing these 2 values, we divide (85) with (86) in (87) and the ratio of them is expressed with e^K in (88).

Then (84) becomes $(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Relativity} = -2GM(1-2e^K)\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$	(85)	$1-2e^K = -1$, (85) gives that for Newtonian orbit $(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Newtonian} = 2GM\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$	(86)

Ratio between (85) and (86) $\frac{(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Relativity}}{(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Newtonian}} = \frac{-2GM(1-2e^K)\left(\frac{1}{r_b}-\frac{1}{r_a}\right)}{2GM\left(\frac{1}{r_b}-\frac{1}{r_a}\right)}$	(87)	Then $\frac{(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Relativity}}{(\boldsymbol{u}_b{}^2 - \boldsymbol{u}_a{}^2)_{Newtonian}} = 2e^K - 1$	(88)

We do the comparison for hyperbolic orbit, for which the value of e^K at infinitely far is given in (90). Introducing (90) into (88) gives (91) which shows that the ratio $\frac{(u_b{}^2-u_a{}^2)_{Relativity}}{(u_b{}^2-u_a{}^2)_{Newtonian}}$ is smaller than 1.

Infinitely far, $r_0=\infty$ $e^K=\left(1-\frac{v_0{}^2}{c^2}\right)e^{\frac{2GM}{c^2r_0}}=\left(1-\frac{v_0{}^2}{c^2}\right)e^0$ See (19)	(89)	Then $e^K=1-\frac{v_0{}^2}{c^2}$	(90)

For real observation, we measure u_a the velocity of a celestial object at point a, with respect to which we compare the differences of velocity squared. So, for this comparison my relativistic gravity and Newtonian gravity take the measured u_a as the velocity at point a, see (92). Then, we get the ratio $\frac{(u_b{}^2)_{Relativity}-u_a{}^2}{(u_b{}^2)_{Newtonian}-u_a{}^2}$ in (93), which is smaller than 1 because of (91), see (94).

For an object that leaves perihelion, its velocity decreases and the difference of velocity squared is negative, see (95). By multiplying (94) with the negative $(u_b{}^2)_{Newtonian}-u_a{}^2$, we obtain (97), which shows that the u_b given by my relativistic gravity is bigger than that given by Newtonian gravity.

Introducing (90) into (88) $\frac{(u_b{}^2-u_a{}^2)_{Relativity}}{(u_b{}^2-u_a{}^2)_{Newtonian}}=2\left(1-\frac{v_0{}^2}{c^2}\right)-1$ $=1-2\frac{v_0{}^2}{c^2}<1$	(91)
u_a is the observed velocity at point a $u_a{}^2=(u_a{}^2)_{Relativity}=(u_a{}^2)_{Newtonian}$	(92)
Using (92) we have $\frac{(u_b{}^2-u_a{}^2)_{Relativity}}{(u_b{}^2-u_a{}^2)_{Newtonian}}=\frac{(u_b{}^2)_{Relativity}-u_a{}^2}{(u_b{}^2)_{Newtonian}-u_a{}^2}$	(93)
Then, (91) makes (93) to be smaller than 1 $\frac{(u_b{}^2)_{Relativity}-u_a{}^2}{(u_b{}^2)_{Newtonian}-u_a{}^2}<1$	(94)
Leaving perihelion, velocity decreases $(u_b{}^2)_{Newtonian}-u_a{}^2<0$	(95)
(94) multiplied by the negative (95) $(u_b{}^2)_{Relativity}-u_a{}^2>(u_b{}^2)_{Newtonian}-u_a{}^2$	(96)
Then, velocities away from perihelion $(u_b{}^2)_{Relativity}>(u_b{}^2)_{Newtonian}$	(97)

Was bigger velocity on hyperbolic orbit really been observed? Yes. Fortunately, we have 2 real observations of bigger than expected velocity. The first is Oumuamua[7] which has got an unexpected "boost in speed" or a bigger than expected velocity when leaving perihelion. Equation (97) explains well this "boost in speed". The second is Pioneer anomaly[8], which is the unexplained resistance to slow down of the probes. Resistance to slow down means that the velocities of these probes are bigger than expected at a second point of measurement, which is well explained by (97) too. A third candidate with hyperbolic orbit is 2I/Borissov[9]. But there is no information about the rate of variation of its velocity.

So, we can say that my relativistic gravity has successively passed the check of the rate of variation of orbital velocity.

As for General Relativity, because the observed bigger than expected velocities are unexpected, it is possible that no one has computed the variation of orbital velocity using General Relativity or that the result given by General Relativity does not match observation.

c) Direction of precession

The third check is to see if my relativistic gravity predicts correctly the direction of precession. Today, the rotation of the major axis of elliptic orbit due to precession is thought to be in the same direction as the velocity of the planet, as shown in Figure 2.

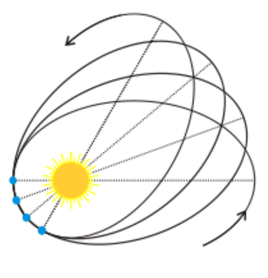

Figure 2
Today's view of precession: ellipse of orbit rotates in the same direction as the planet (Image Wikipedia)

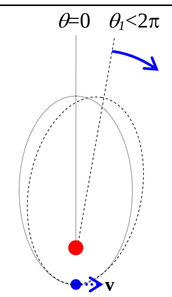

Figure 3

Negative precession

However, equation (80) shows that ΔΦ is negative, that is, the major axis of an elliptic orbit rotates a negative angle in the polar coordinate system. I call this rotation negative precession, which is shown with the 2 ellipses in Figure 3. The first ellipse is vertical, the second is tilted to the right. An object with counterclockwise velocity begins on the first ellipse from the polar angle θ=0. At the end of the orbit, it arrives at the polar angle θ_l<2π, which gives the major axis a clockwise rotation. So, the major axis of the ellipse rotates against the velocity of the object. This result is contrary to the current image of orbital precession.

Is the negative ΔΦ correct? For answering this question, let us compute the orbital velocity of a circular orbit, with r being the radius of the orbit. Because the orbit is circular, the orbital acceleration α is expressed with the orbital velocity v in (98). For Newtonian gravity, gravitational acceleration equals the orbital acceleration α_N, see (99). By equating (98) with (99) while replacing v with v_N we get the Newtonian orbital velocity v_N in (100).

Acceleration of circular orbit $\alpha = -\frac{v^2}{r}$ **v**: orbital velocity	(98)
For Newtonian gravity, gravitational acceleration equals orbital acceleration α_N $-\frac{GM}{r^2} = \alpha_N$	(99)
Using (98) in (99), Newtonian orbital velocity is $v_N{}^2 = \frac{GM}{r}$	(100)
For relativistic gravity, gravitational acceleration equals, see (7) $-\frac{GM}{r^2} = \frac{\alpha_l}{1-\frac{v_l{}^2}{c^2}}$ v_l : local orbital velocity	(101)

For relativistic gravity, the orbital acceleration is α_l which is related to the gravitational acceleration through equation (101) with v_l being the local orbital velocity, see (7). By substituting (98) for α_l in (101) and replacing v with v_l, we get (102), which is then transformed in (103) to express v_l^2 in (104).

For comparing the relativistic and Newtonian orbital velocities, we divide (100) with (104) in (105), which shows that v_l is smaller than v_N, see (106). So, on a circular orbit my relativistic gravity gives a velocity smaller than that given by Newtonian gravity.

Substituting (98) for α_l in (101) with v being replaced by v_l $-\frac{GM}{r^2} = \frac{-\frac{v_l{}^2}{r}}{1-\frac{v_l{}^2}{c^2}}$	(102)	(102) is transformed into $\frac{GM}{c^2 r} - \frac{v_l{}^2}{c^2}\frac{GM}{c^2 r} = \frac{v_l{}^2}{c^2}$ $\Rightarrow \frac{v_l{}^2}{c^2}\left(1+\frac{GM}{c^2 r}\right) = \frac{GM}{c^2 r}$	(103)	(103) gives relativistic orbital velocity $v_l{}^2 = \frac{GM}{r}\frac{1}{1+\frac{GM}{c^2 r}}$	(104)

Dividing (104) with (100) $\frac{v_l{}^2}{v_N{}^2} = \frac{\frac{GM}{r}}{\left(1+\frac{GM}{c^2 r}\right)\frac{GM}{r}} = \frac{1}{1+\frac{GM}{c^2 r}} < 1$	(105)	(105) gives $v_l < v_N$	(106)

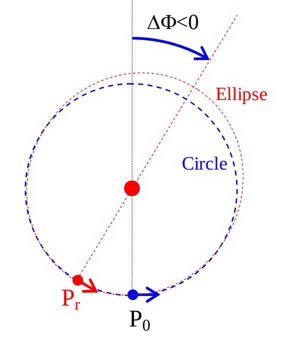

Figure 4

In one circular orbit, planet goes from P_0 to P_0 for Newtonian gravity, but from P_0 to P_r for relativistic gravity. On the elliptic orbit planet arrives near P_r making the ellipse to rotate clockwise.

What the velocity of a circular orbit can tell about precession of elliptic orbit? Let us see Figure 4 which shows a circular orbit with a similar but slightly elliptical orbit. Suppose that a body m and a body n start doing the circular orbit from the point P_0 in the direction of the arrows. m moves at the velocity v_N while n moves at v_l. Because v_l is smaller than v_N, n is slower than m. When m is back at the point P_0, n arrives only at the point P_r.

Now, let m and n do the slightly elliptical orbit starting from P_0. When m is back at the point P_0, n arrives only at a point near P_r. At the start the major axis of the ellipse passed across P_0 and was vertical. But at the end, the major axis passes across P_r and tilts to the right, which makes the angle ΔΦ negative. That is, the major axis rotates against the orbital velocity. So, it is the slower velocity v_l that makes elliptical orbit to begin earlier on each turn, resulting in a negative rotation of the major axis of the ellipse. So, the slower relativistic velocity is the explanation of negative precession.

Why v_l is slower than v_N? If an object gets a quantity of kinetic energy ΔE, it will move at a higher velocity and the increased part of velocity Δv carries the kinetic energy ΔE. For Newtonian laws, the velocity is increased by Δv_N while its mass stays the same. For the same quantity of energy increase ΔE, according to Special Relativity the mass of the object is bigger than before, which makes the increase of velocity Δv_l smaller than Δv_N. So, the slower velocity for relativistic orbit is due to Special Relativity effect and this way Special Relativity corroborates the negative angle shown in Figure 4 and negative precession. So, negative precession is correct and my relativistic gravity has successively passed the theoretical check of direction of precession.

For General Relativity orbits are geodesics of a curved Space-Time which are not related to velocity or energy. If geodesics respect Special Relativity, then the velocity of objects on geodesics would be slower than that the Newtonian laws would give and thus, General Relativity would probably give negative precession too. If General Relativity really gives positive precession, then it would violate Special Relativity because the velocity on geodesics would be faster than the Newtonian one. But as no information exists about negative precession in General Relativity. I suspect that the direction of precession has never been computed using General Relativity.

However, the explanation with slower orbital velocity is not a rigorous proof. For really checking the validity of my relativistic gravity, we have to search for negative precession in the observations of celestial bodies, such as in the old data of the planet Mercury, in the recent data of the star S_2 that orbits the supermassive black hole Sagittarius A* at the center of Milky Way. We can also use the navigation data of Parker Solar Probe[10] which will reach 200 km/s near the Sun.

5. Comments

What benefit the new theory of relativistic gravity provides with respect to General Relativity? Geodesics of curved Space-Time are so obscure that one cannot see how velocity and energy are related to orbit. In the contrary, the analytical relativistic orbit equation is derived from velocity and gravitational force, which makes relativistic gravity intuitive and easy to understand, but also more powerful for studying gravity.

For example, I have already found with this equation that in relativistic gravity the asymptote of hyperbolic orbit will deviate from the Newtonian asymptote like orbital precession, that the difference between the maximum and minimum velocities of elliptic orbit deviates from the Newtonian one in the same order as orbital precession.

The derivation of the relativistic orbit equation shows clearly the physical meaning of orbital precession and has pinpointed the two contributors: 1) the slower velocity due to effect of Special Relativity, 2) the time dilation of Schwarzschild metric, see the 2 parentheses in (25). The smaller rate of variation of velocity on hyperbolic orbit (see (94)) explains well the “boost in speed” of Oumuamua[7] and Pioneer anomaly[8]. Such insights cannot be found easily with General Relativity.

6. Appendix: Velocity vector in polar coordinates and Kepler's second law

Radial position vector $\boldsymbol{r} = r\boldsymbol{e}_r$	(107)
Velocity vector is the time derivative of $\boldsymbol{r}$ $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\boldsymbol{e}_r}{dt}$	(108)
Derivatives of unit vectors $\frac{d\boldsymbol{e}_r}{dt} = \frac{d\theta}{dt}\boldsymbol{e}_\theta, \quad \frac{d\boldsymbol{e}_\theta}{dt} = -\frac{d\theta}{dt}\boldsymbol{e}_r$	(109)
Using (109) in (108) velocity vector is $\boldsymbol{v} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\theta}{dt}\boldsymbol{e}_\theta$	(110)
Acceleration vector is the second time derivative of $\boldsymbol{r}$ $\boldsymbol{\alpha} = \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right) = \left(\frac{d^2r}{dt^2}\boldsymbol{e}_r + \frac{dr}{dt}\frac{d\boldsymbol{e}_r}{dt}\right) + \left(\frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{e}_\theta + r\frac{d^2\theta}{dt^2}\boldsymbol{e}_\theta + r\frac{d\theta}{dt}\frac{d\boldsymbol{e}_\theta}{dt}\right)$	(111)
Using (109), (111) becomes $\boldsymbol{\alpha} = \frac{d^2r}{dt^2}\boldsymbol{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{e}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{e}_\theta + r\frac{d^2\theta}{dt^2}\boldsymbol{e}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\boldsymbol{e}_r = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\boldsymbol{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\boldsymbol{e}_\theta$	(112)

Tangent acceleration is 0 $2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0$ Division by $r\frac{d\theta}{dt}$ $\frac{2}{r}\frac{dr}{dt} + \left(\frac{d\theta}{dt}\right)^{-1}\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = 0$ $\Rightarrow 2\frac{d\,ln(r)}{dt} + \frac{d}{dt}\left(ln\left(\frac{d\theta}{dt}\right)\right) = 0$	(113)
Integration of (113) $ln(r^2) + ln\left(\frac{d\theta}{dt}\right) = ln(h)$ $ln(h)$: integration constant	(114)
From (114), Kepler's second law $r^2\frac{d\theta}{dt} = h = Constant$	(115)

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