Velocity, mass, momentum and energy of an accelerated object in relativity

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Abstract: Analytical derivation of relativistic velocity, mass, momentum and kinetic energy of an accelerated object.

For Special relativity the momentum of an object of rest mass m_0 and velocity u is expressed by equation (1) which is infinite when u equals c. Is it physically meaningful that the momentum of an object becomes infinite while its velocity stays finite? On the other hand, the principle of mass–energy equivalence proposed by Albert Einstein has not been rigorously demonstrated, hence it is called a principle not a law. In the contrary, in my theory of Time relativity momentum and kinetic energy are derived by direct integration and stays limited when u=c.

$$P = \frac{m_0 u}{\sqrt{1-\frac{u^2}{c^2}}} \quad (1)$$

m_0 is the rest mass of an object q, u its velocity, P its momentum. P is infinite when u equals c.

$$\begin{cases} x_2 = x_1 - ut_1 \\ t_2 = t_1\sqrt{1-\frac{u^2}{c^2}} \end{cases} \quad (2)$$

Transformation of coordinates

$$v_2 = \frac{v_1 - u}{\sqrt{1-\frac{u^2}{c^2}}} \quad (3)$$

Transformation of velocity

1. Velocity

Let us derive the expression of the velocity of an object q by integration. Let q be an object getting a momentum impulse which begins at time t_a and ends at time t_b. At time t_a, q moves at the velocity u in the stationary frame of reference frame 1 and at v=0 in the inertial frame 2, see Figure 1. At time t_b, q moves at the velocity v_1 in frame 1 and v_2 in frame 2, see Figure 2. So, v_1 and v_2 are related through equation (3) which is the transformation of velocity, the equation (19) in «Time relativity transformation of velocity».

The indices 1 and 2 indicate the frame 1 and 2. u is the velocity of frame 2 in frame 1. Frame 2' moves at velocity v_1.

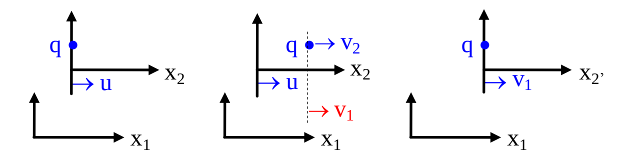

Figure 1

At time t_a, q moves at velocity u in frame 1 and static in frame 2.

Figure 2

At time t_b, q moves at velocity v_1 in frame 1 and at v_2 in frame 2.

Figure 3

Frame 2' moves at velocity v_1 in frame 1 such that q is static in frame 2'

Due to the impulse, the change of velocity of q is $\Delta u = v_1 - u$ in frame 1 (see equation (4)) and $\Delta v_2 = v_2$ in frame 2 because the velocity of q is zero at the beginning of the impulse (see equation (5)). We introduce equations (4) and (5) in equation (3) to obtain equation (6).

$$\Delta u = v_1 - u \quad (4)$$

$$\Delta v_2 = v_2 \quad (5)$$

$$\Delta v_2 = \frac{\Delta u}{\sqrt{1-\frac{u^2}{c^2}}} \quad (6)$$

We create the inertial frame 2' which moves momentarily with q at time t_b, that is, the velocity of frame 2' is v_1 in frame 1, see Figure 3. At this moment the object q is given a second impulse and we are back to the situation of the beginning of the first impulse, with frame 2' at the place of frame 2 and v_1 at the place of u. Then, we can compute a second Δu for q, then a third and so on. The value of Δu is infinitesimal, so we can integrate Δu step by step.

The duration of the impulse in frame 2 is Δt_2 with which we divide both sides of equation (6) to get equation (7), in which we have introduced Δt_1, the duration of the same impulse but in frame 1. The ratio $\frac{\Delta t_1}{\Delta t_2}$ is given in equation (8) which is computed using equation (2), the transformation of coordinates derived in « Time relativity transformation of coordinates». We substitute equation (8) for $\frac{\Delta t_1}{\Delta t_2}$ in equation (7), after rewriting Δt_1, Δt_2, Δu and Δv_2 in differential form as dt_1, dt_2, du and dv_2, we obtain equation (9), where the ratio $\frac{dv_2}{dt_2}$ is in fact the acceleration of q in frame 2, α_2.

$$\frac{\Delta v_2}{\Delta t_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\Delta u}{\Delta t_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\Delta u}{\Delta t_1}\frac{\Delta t_1}{\Delta t_2} \quad (7)$$

$$\frac{\Delta t_1}{\Delta t_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \quad (8)$$

$$\frac{1}{1-\frac{u^2}{c^2}}\frac{du}{dt_1} = \frac{dv_2}{dt_2} = \alpha_2 \quad (9)$$

Δt_1 and Δt_2 are the durations of the impulse in frame 1 and 2. α2 is the acceleration of q in frame 2.

Equation (9) is then integrated in equation (10). Using the integration formula (11), equation (10) becomes equation (12), which we inverse to get equation (13). Equation (13) is the expression of the velocity of the object q in frame 1 expressed in terms of its acceleration in frame 2, α_2. If the integration is over infinite time for positive α_2, the velocity of q equals the speed of light c, see equation (14).

$$\int_0^u \frac{d\frac{u}{c}}{1-\frac{u^2}{c^2}} = \int_0^{t_1} \frac{\alpha_2}{c} dt_1 \quad (10)$$

$$\int_0^x \frac{dx}{1-x^2} = \tanh^{-1}(x) \quad (11)$$

$$\tanh^{-1}\left(\frac{u}{c}\right) = \int_0^{t_1} \frac{\alpha_2}{c} dt_1 \quad (12)$$

$$u = c \cdot \tanh\left(\int_0^{t_1} \frac{\alpha_2}{c} dt_1\right) \quad (13)$$

$$\begin{cases} t_1 = \infty \\ \alpha_2 > 0 \end{cases} \Rightarrow u = c \quad (14)$$

u is the velocity of q in frame 1 obtained by integration.

2. Mass

When a force F acts on the object q which is free to move, its velocity and momentum increase. F equals the time derivative of its momentum P_1, see equation (15). The acceleration of q in frame 1 is denoted by α and equals the time derivative of its velocity u, see equation (16). The quantities F, P_1 and α are in the x direction.

The resistance to acceleration of q is its inertial mass m which is defined as the instantaneous ratio of F to α for whatever velocity of q, including relativistic velocity. m is expressed in equation (17), in which we have substituted F by $\frac{dP_1}{dt}$ and α by $\frac{du}{dt}$. Then equation (17) becomes equation (18), which is the inertial mass of the object q expressed in terms of the derivative of momentum with respect to velocity.

$$F = \frac{dP_1}{dt} \quad (15)$$

$$\alpha = \frac{du}{dt} \quad (16)$$

$$m = \frac{F}{\alpha} = \frac{dP_1}{dt}\frac{dt}{du} \quad (17)$$

$$m = \frac{dP_1}{du} \quad (18)$$

P_1 is the momentum of q in frame 1, F the force on q, m the inertial mass of q, α the acceleration of q in frame 1.

At the beginning of an impulse, the momentum of q is P_1 in frame 1 but equals 0 in frame 2. After the impulse, the momentum becomes P'_1 in frame 1 and P'_2 in frame 2. So, the change of momentum is $\Delta P_1 = P'_1 - P_1$ in frame 1 and $\Delta P_2 = P'_2 - 0$ in frame 2. Because v_2 is infinitesimal in frame 2, ΔP_2 obeys classical mechanics and equals $m_0 v_2$, with m_0 being the rest mass of q, see equation (19)

Since ΔP_1 and ΔP_2 are the same impulse of momentum, ΔP_1 equals ΔP_2, then $\Delta P_1 = m_0 v_2$, see equation (20). We substitute equation (3) for v_2 in equation (20), which gives equation (21), in which we substitute $v_1 - u$ by Δu (see equation (4)).

After rewriting ΔP_1 and Δu in differential form as dP_1 and du, equation (21) becomes equation (22), which is transformed into equation (23) where the ratio $\frac{dP_1}{du}$ is the definition of inertial mass (see equation (18)). In consequence, the relativistic inertial mass of an object having velocity u is expressed by equation (24) and is denoted by m(u).

$$\Delta P_2 = m_0 v_2 \quad (19)$$

$$\Delta P_1 = \Delta P_2 = m_0 v_2 \quad (20)$$

$$\Delta P_1 = m_0 \frac{v_1 - u}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_0 \Delta u}{\sqrt{1-\frac{u^2}{c^2}}} \quad (21)$$

$$dP_1 = \frac{m_0 du}{\sqrt{1-\frac{u^2}{c^2}}} \quad (22)$$

$$\frac{dP_1}{du} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \quad (23)$$

$$m(u) = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \quad (24)$$

P_2 is the momentum of q in frame 2, m_0 its rest mass, m(u) its relativistic inertial mass in frame 1.

3. Momentum

We integrate equation (22) in equation (25), which is transformed into equation (27) by using the integration formula (26). Equation (27) is the expression of the momentum of an object moving at velocity u. When u equals c, P_1 equals $\frac{\pi}{2} m_0 c$, a constant proportional to the rest mass m_0, see equation (28).

Notice that for Time relativity the value of momentum (equation (27)) has a maximum value, which makes more physical sense than equation (1) which becomes infinity when u=c.

$$P_1 = \int_0^P dP_1 = m_0 \int_0^u \frac{du}{\sqrt{1-\frac{u^2}{c^2}}} \quad (25)$$

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \quad (26)$$

$$P_1 = m_0 c \cdot \sin^{-1}\left(\frac{u}{c}\right) \quad (27)$$

$$u = c \Rightarrow P_1 = \frac{\pi}{2} m_0 c \quad (28)$$

P_1 is the momentum of q in frame 1 obtained by integration.

4. Conservation of momentum

Let us take an isolated system made by 2 objects: a and b. The momentum in the x direction of the object a is P_a and that of b is P_b. The rest mass of the object a is $m_{0,a}$ and that of the object b is $m_{0,b}$. Let v_a and v_b be the velocities of a and b before a collision. After the collision the velocities of a and b become v'_a and v'_b.

According to the law of conservation of momentum the total momentum of an isolated system is constant, that is, the change of momentum is zero. So, the sum of P_a and P_b is constant during a collision and we have equation (29), in which ΔP_a and ΔP_b are infinitesimal. ΔP_a and ΔP_b are expressed in equation (30) by using equation (21): for object a, u is replaced by v_a and v_1 by v'_a, for object b, u is replaced by v_b and v_1 by v'_b. By introducing equation (30) in equation (29) we obtain equation (31) that we transform into equation (32).

$$\Delta(P_a + P_b) = \Delta P_a + \Delta P_b = 0 \tag{29}$$

$$\Delta P_a = m_{0,a}\frac{v'_a - v_a}{\sqrt{1-\frac{v_a{}^2}{c^2}}} \qquad \Delta P_b = m_{0,b}\frac{v'_b - v_b}{\sqrt{1-\frac{v_b{}^2}{c^2}}} \tag{30}$$

P_a and P_b are the momentums of a and b, $m_{0,a}$ $m_{0,b}$ their rest masses, v_a and v_b their velocities before collision, v'_a and v'_b their velocities after collision.

$$m_{0,a}\frac{v'_a - v_a}{\sqrt{1-\frac{v_a{}^2}{c^2}}} + m_{0,b}\frac{v'_b - v_b}{\sqrt{1-\frac{v_b{}^2}{c^2}}} = 0 \tag{31}$$

We put the momentum conservation law of Special relativity in equation (33) for comparison. Equation (32) is almost identical to (33), except that in the right hand side, the denominators are in terms of v_a and v_b instead of the primed v'_a and v'_b.

$$m_{0,a}\frac{v_a}{\sqrt{1-\frac{v_a{}^2}{c^2}}} + m_{0,b}\frac{v_b}{\sqrt{1-\frac{v_b{}^2}{c^2}}} = m_{0,a}\frac{v'_a}{\sqrt{1-\frac{v_a{}^2}{c^2}}} + m_{0,b}\frac{v'_b}{\sqrt{1-\frac{v_b{}^2}{c^2}}} \tag{32}$$

Conservation of momentum for infinitesimal collision for time relativity

$$m_{0,a}\frac{v_a}{\sqrt{1-\frac{v_a{}^2}{c^2}}} + m_{0,b}\frac{v_b}{\sqrt{1-\frac{v_b{}^2}{c^2}}} = m_{0,a}\frac{v'_a}{\sqrt{1-\frac{v'_a{}^2}{c^2}}} + m_{0,b}\frac{v'_b}{\sqrt{1-\frac{v'_b{}^2}{c^2}}} \tag{33}$$

Conservation of momentum for collision for Special relativity.

The difference between the two formulas is the primed $\frac{v'_a{}^2}{c^2}$ and $\frac{v'_b{}^2}{c^2}$ for Special relativity in the right hand side.

For prolonged interaction between the 2 objects, the change of momentum is large and must be integrated. In this case, we rewrite ΔP_a and ΔP_b in the differential form dP_a and dP_b and integrate equation (29) in equation (34). By using equation (27) we express $\int_1^2 dP_a$ and $\int_1^2 dP_b$ in equations (35), which we introduce in equation (34) to get equation (36).

The law of conservation of momentum for Time relativity is then equation (36). Notice that this time, the second part of the equation is expressed in terms of the primed v'_a and v'_b only, but this equation is completely different from the equation (33). For a system with n objects, the law of conservation of momentum is equation (37).

$$\int_1^2 dP_a + \int_1^2 dP_b = 0 \tag{34}$$

$$\int_1^2 dP_a = m_{0,a}c\left[\sin^{-1}\left(\frac{v'_a}{c}\right) - \sin^{-1}\left(\frac{v_a}{c}\right)\right] \qquad \int_1^2 dP_b = m_{0,b}c\left[\sin^{-1}\left(\frac{v'_b}{c}\right) - \sin^{-1}\left(\frac{v_b}{c}\right)\right] \tag{35}$$

$$m_{0,a}\sin^{-1}\left(\frac{v_a}{c}\right) + m_{0,b}\sin^{-1}\left(\frac{v_b}{c}\right) = m_{0,a}\sin^{-1}\left(\frac{v'_a}{c}\right) + m_{0,b}\sin^{-1}\left(\frac{v'_b}{c}\right) \tag{36}$$

$$\sum_{i=1}^{n} m_{0,i}\sin^{-1}\left(\frac{v_i}{c}\right) = \sum_{i=1}^{n} m_{0,i}\sin^{-1}\left(\frac{v'_i}{c}\right) \tag{37}$$

Integration of the momentums of a and b before and after collision. For more than 2 objects, $m_{0,i}$ is the rest mass of the i^{th} object, v_i and v'_i their velocities before and after collision.

5. Kinetic Energy

The kinetic energy of an object equals all the work done on it by external force while its velocity increases from v=0 to u. When a force F in the x direction pushes an object q to move a distance *dx*, it does a work dW which equals F·*dx*, see equation (38). For an object that moves at the velocity u, *dx* equals u·dt, with dt being the time of action, see equation (39). The force F equals the time derivative of the momentum of the object q which is P_1 (see equation (15)). We substitute in equation (38) equation (15) for F and equation (39) for *dx* and obtain equation (40). In equation (40) we substitute equation (22) for dP_1 and obtain equation (41).

$$dW = F\cdot dx \tag{38}$$

$$dx = u\cdot dt \tag{39}$$

$$dW = \frac{dP_1}{dt}\cdot u\cdot dt = u\cdot dP_1 \tag{40}$$

dW is the infinitesimal work of the force F, dP_1 the change of momentum, dt time of action.

The total kinetic energy of the object is denoted by E_1 and equals the integral of dW over the velocity range v=0 to v=u, see equation (42). By using the integration formula (43), equation (42) becomes equation (44) which is the expression of the total kinetic energy of an object moving at the velocity u for Time relativity.

When u equals the speed of light the total kinetic energy equals m_0c^2, see equation (45), which demonstrates mathematically the mass–energy equivalence principle. As it is now mathematically demonstrated, we can call this principle the mass–energy equivalence law.

$$dW = u\frac{m_0 du}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_0c^2}{2}\frac{d\frac{u^2}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{41}$$

$$E_1 = \int_0^W dW = \frac{m_0c^2}{2}\int_0^u \frac{d\frac{u^2}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{42}$$

$$\int_0^x \frac{dx}{\sqrt{1-x}} = -2(\sqrt{1-x}-1) \tag{43}$$

$$E_1 = m_0c^2\left(1-\sqrt{1-\frac{u^2}{c^2}}\right) \tag{44}$$

$$u = c \Rightarrow E_1 = m_0c^2 \tag{45}$$

E_1 is the kinetic energy of q moving at u in frame 1.

6. Momentum-kinetic energy relation

We transform equation (44) into equation (46). Equation (27) is transformed into equation (47), which we substitute for $\frac{u}{c}$ in equation (46), which then becomes equation (48), then (49). Equation (49) is the momentum-kinetic-energy relation for Time relativity.

For small velocity, P_1 equals rest mass multiplied by velocity, see equation (50). Equation (51) simplifies $\cos\frac{P_1}{m_0c}$ for small $\frac{P_1}{m_0c}$, which we introduced in equation (49) to obtain equation (52), which is identical to the expression of kinetic energy in classical mechanics.

For comparison, the momentum-energy relation of Special relativity is put in equation (53), which becomes equation (54) for small velocity. Notice that equation (54) does not equal the expression of kinetic energy in classical mechanics and when u=c equation (53) becomes infinite because P is infinite, see equation (1).

$$\left(1-\frac{E_1}{m_0c^2}\right)^2 = 1-\left(\frac{u}{c}\right)^2 \tag{46}$$

$$\sin\left(\frac{P_1}{m_0c}\right) = \frac{u}{c} \tag{47}$$

$$\left(1-\frac{E_1}{m_0c^2}\right)^2 = 1-\sin^2\left(\frac{P_1}{m_0c}\right) = \cos^2\left(\frac{P_1}{m_0c}\right) \tag{48}$$

$$E_1 = m_0c^2\left(1-\cos\frac{P_1}{m_0c}\right) \tag{49}$$

Momentum-kinetic-energy relation for Time relativity.

$$P_1 = m_0u \tag{50}$$

$$\cos\frac{P_1}{m_0c} \approx 1-\frac{1}{2}\left(\frac{u}{c}\right)^2 \tag{51}$$

$$E_1 \approx \frac{m_0u^2}{2} \tag{52}$$

Newtonian limit for Time relativity.

$$E^2 = (Pc)^2 + (m_0c^2)^2 \tag{53}$$

$$E = m_0c^2\sqrt{\left(\frac{P}{m_0c}\right)^2+1} \approx m_0c^2\left(1+\frac{1}{2}\left(\frac{u}{c}\right)^2\right) \tag{54}$$

Newtonian limit for Special relativity.

7. Comments

The expression of velocity equation (13) is obtained by integration and thus is mathematically exact. In the contrary, Einstein's velocity-addition formula cannot be analytically integrated and an approximation was used to compute the velocity of an object. Thus, the so obtained velocity is not exact (see section 5.3 of « Introduction to Special Relativity » by James H. Smith).

In Special relativity the expression of relativistic mass is derived with the help of a collision between 2 objects (see section 9.4 of « Introduction to Special Relativity » by James H. Smith). For Time relativity relativistic mass is the derivative of momentum with respect to velocity, which is exactly the definition of mass.

In Special relativity the expression of momentum was derived with the help of a collision between 2 objects (see section 9 of « Introduction to Special Relativity » by James H. Smith) and is infinite when the velocity equals c (see equation (1)). For Time relativity momentum is the integral of infinitesimal change of momentum (see equation (25)). When the velocity of the object equals c its momentum equals the constant $\frac{\pi}{2}m_0c$. So, to the question at the beginning: "Is it physically meaningful that the momentum of an object becomes infinite while its velocity stays finite?", our answer is negative.

For Time relativity the total kinetic energy of an object is the integral of the work done on it and thus, its expression is mathematically exact. Moreover, when the velocity of the object equals c, its expression equals m_0c^2 (see equation (45)), proving mathematically the principle of mass–energy equivalence, while this principle has not been mathematical proven in Special relativity.

Note: The above work is based on « Time relativity transformation of velocity » and « Time relativity transformation of coordinates » which are the basic transformations of the theory of Time relativity. I have developed this theory because Special relativity contains a flaw, which I have demonstrated in section 3 of « Length, distance and Michelson–Morley experiment » and in « Analysis of Einstein's derivation of the Lorentz Transformation ».

However, I am unable to get my work published because no reviewer accepts questioning Special relativity which is absolutely true for them. So, I call for help from experts willing to recommend my work for publishing and to whom I will be very grateful. Because of its groundbreaking nature there is no doubt that the advocates of my work will gain fame too.

References

1) « Introduction to Special Relativity » by James H. Smith