Gravitational time dilation and black hole

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Abstract: In this article, we derive the gravitational time dilation factor in a new manner, which allows us to identify the mathematical cause of Schwarzschild radius, to give a theoretical way to avoid it and to compute properties of black hole. General relativity effects are computed as simple as in special relativity. Observability of black hole is discussed.

1. Introduction

In general relativity gravitation is attributed to the curvature of a 4D-manifold called spacetime in which the orbits of objects are the geodesics. Only one year after the publication of general relativity, Karl Schwarzschild has given an exact solution to the extremely complex Einstein field equations. This solution is named Schwarzschild metric, see equation (1). Albert Einstein was so amazed that he wrote a letter to Karl Schwarzschild saying: "*I would not have expected that the exact solution to the problem could be formulated so simply.*" [1],[2]. The wonderful work of Karl Schwarzschild describes the spacetime curved by a spherical gravitational field, which agree precisely with observation as shown by the orbital precession of the planet Mercury.

In addition, Schwarzschild metric possesses a completely new feature: a critical radius named Schwarzschild radius that marks the lower limit of describable space. This feature is weird because it splits space into 2 regions, the normal space is outside this radius and the region inside it cannot be mathematically defined. This region is called a black hole because no information, not even light, could escape from it.

Notation convention
Bold letters: vectors
Number in parentheses (*i*): equation index
Dilator: Time dilation factor
Subscript:
1: Stationary frame
2: Mobile frame
∞: Frame at infinitely far
r: Frame at the position ***r***
m: Proper inertial frame of m

The funding principle of physics is that the entire universe is governed by physical laws whatsoever. Such weird region out of the reach of physics hints that unknown physical laws are there. So, let us try to understand deeper Schwarzschild radius and find out where it comes from.

2. Schwarzschild time dilation

a. Schwarzschild radius

In spherical coordinate system, the Schwarzschild metric created by an attracting mass M is given by equation (1), [3], with r being the radial coordinate of a point, t_r the local time at this point and t_∞ the time at infinitely far. Because t_r is influenced by the gravity of M but t_∞ is not, time is dilated. By keeping r constant and θ=0 and ϕ=0 in equation (1), we obtain the time dilation factor $\sqrt{1-\frac{2GM}{c^2r}}$ in (2), which we call Schwarzschild dilator.

The term $\frac{2GM}{c^2}$ in (2) is called Schwarzschild radius and denoted by r_s, see (3). The Schwarzschild dilator is written with $\frac{r_s}{r}$ in (4), which equals the square root of a negative value $\sqrt{-\left|1-\frac{r_s}{r}\right|}$ when r is smaller than r_s, see (5). As relativity does not allow square root of negative value, Schwarzschild metric is not defined in the region below r_s. So, Schwarzschild radius is determined at $r=r_s$ to exclude this region.

Let us see the mathematics of this region.

Schwarzschild metric $-c^2dt_r{}^2=-\left(1-\frac{2GM}{c^2r}\right)c^2dt_\infty{}^2+\frac{dr^2}{1-\frac{2GM}{c^2r}}+(d\theta^2+\sin^2\theta\,d\phi^2)r^2$ r: radial coordinate, t_r: local time, t_∞: time at infinitely far θ and ϕ: angular coordinates in spherical system G: gravitational constant, M: mass	(1)
Schwarzschild dilator $\frac{dt_r}{dt_\infty}=\sqrt{1-\frac{2GM}{c^2r}}$ r kept constant and θ and ϕ being 0 in (1)	(2)
Schwarzschild radius $r_s=\frac{2GM}{c^2}$	(3)
Using (3) in (2) $\frac{dt_r}{dt_\infty}=\sqrt{1-\frac{r_s}{r}}$	(4)
When $r<r_s$ $1-\frac{r_s}{r}<0$ $\Rightarrow\frac{dt_r}{dt_\infty}=\sqrt{-\left\|1-\frac{r_s}{r}\right\|}$ Square root of negative value	(5)
Time dilation factor in special relativity $\frac{dt_2}{dt_1}=\sqrt{1-\frac{v^2}{c^2}}$ 1: stationary frame, 2: frame moving at v	(6)

b. Comparison with the time dilation in special relativity

The Schwarzschild dilator (2) is similar to the time dilation factor for special relativity (6), where t_1 and t_2 are the times in frame 1 and 2 respectively and v the velocity of frame 2 in frame 1. We see that the term $\frac{2GM}{r}$ in (2) corresponds to the velocity squared v^2 in (6). This correspondence means that $\frac{2GM}{r}$ equals a virtual velocity squared. What is this virtual velocity?

General relativity is based on the equivalence principle [3] which states that a person free falling in a gravitational field has no way to tell whether he is falling in a gravitational field or in free space. So, the virtual velocity could correspond to the velocity of this person, except that his velocity can have any value while the virtual velocity is determined. Let us give this person an initial condition such that his velocity is determined. We define this velocity as follow:

Definition 1: Far falling velocity

Initially, an object is at infinitely far from the attracting mass M; its velocity is set to zero. Then, it free falls in the gravitational field of M. We call the free fall with this initial condition a far fall. As the object falls, it gets bigger and bigger velocity. This velocity has a determined value at r and is called far falling velocity.

For computing the far falling velocity, we use the equality of the gravitational acceleration $-\frac{GM}{r^2}\boldsymbol{e}_r$ with the inertial acceleration of an object m $\frac{d\boldsymbol{v}}{dt}$, see equation (7). $\boldsymbol{v}$ and $-\frac{GM}{r^2}\boldsymbol{e}_r$ being vectors, (7) is dotted with $\boldsymbol{v}$ to give (8) which is transformed into (10) with the formulas (9). Equation (10) is integrated into (11) with C being the constant of integration. Initially the object was at infinitely far where r=∞ and v=0, which make C=0, see (12). Then, (11) gives the expression of v^2 in (13) with v being the far falling velocity at the distance r from M.

By comparing equation (13) with (2), we find that v^2 in (13) equals the term $\frac{2GM}{r}$ in (2). So, the far falling velocity is the virtual velocity corresponding to $\frac{2GM}{r}$. We use (13) in (6) and obtain in (14) the time dilation factor for special relativity between the local frame and an far falling object with v being the far falling velocity at r. We see that the time dilation factor for special relativity (14) equals Schwarzschild dilator (2).

This equality is very interesting because, imagine that we sit at the radial coordinate r and see an object far falling past us, amazingly the time dilation factor for special relativity between the object and us equals the Schwarzschild dilator, see (14).

Equality of accelerations $\frac{d\boldsymbol{v}}{dt} = -\frac{GM}{r^2}\boldsymbol{e}_r$ $\frac{d\boldsymbol{v}}{dt}$: inertial acceleration	(7)
(7) dotted with $\boldsymbol{v}$ $\boldsymbol{v}\cdot d\boldsymbol{v} = -\boldsymbol{v}dt\cdot\frac{GM}{r^2}\boldsymbol{e}_r$	(8)
Formulas for transforming (8) $\boldsymbol{v}\cdot d\boldsymbol{v} = d\left(\frac{v^2}{2}\right)$ $\boldsymbol{v}dt = d\boldsymbol{r}$	(9)
Using (9) in (8) $d\left(\frac{v^2}{2}\right) = d\frac{GM}{r}$	(10)
Integration of (10) $\frac{v^2}{2} = \frac{GM}{r} + C$	(11)
r=∞ and v=0 at infinitely far $C = 0$	(12)
Using (12) in (11) $v^2 = \frac{2GM}{r}$	(13)
Using (13) in (6) $\frac{dt_r}{dt_m} = \sqrt{1 - \frac{2GM}{c^2 r}}$ t_r: local time, t_m: time of far falling object	(14)
Using (3) in (13) $v^2 = c^2\frac{2GM}{c^2 r} = c^2\frac{r_s}{r}$	(15)
For $r < r_s$ $\frac{r_s}{r} > 1 \Rightarrow v > c$ See (15)	(16)

c. What makes Schwarzschild radius?

Below the Schwarzschild radius, the far falling velocity v is defined for mathematics, see equation (15) which is written with $\frac{r_s}{r}$. When r is smaller than r_s, $\frac{r_s}{r} > 1$ and v is bigger than the speed of light and violates relativity principle, see (16). Why is v bigger than c there?

The far falling velocity was derived by using the equality of the gravitational acceleration with the inertial acceleration of m, see (7). But, is this equality valid in the context of relativity? In relativity an acceleration has not the same value in different frames. Without specifying the reference frame the equality (7) is valid only in Newtonian context. So, v is a Newtonian velocity which allows $v > c$.

For eliminating Newtonian velocity in space, we must not use equality (7) but an equality of accelerations that is consistent with relativity, which will make the velocity v consistent with relativity.

3. Using relativistic transformation of acceleration

a. Gravitational frames

For finding out the equality of accelerations consistent with relativity, let us define 3 reference frames, see Figure 1.

Definition 2: Far frame

This frame is labeled frame_∞. It is at infinitely far from the attracting body M and is stationary with respect to M. The time of frame_∞ is labeled t_∞.

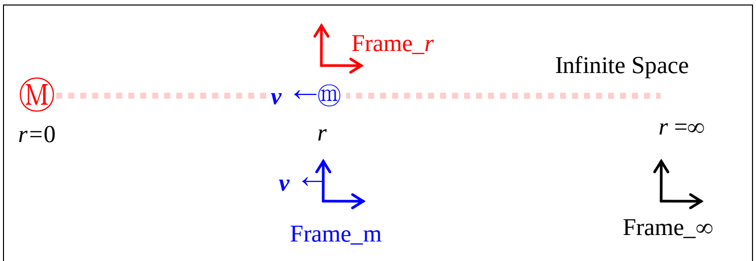

Figure 1

M is the attracting body, m far falls. Frame_m is attached to *m*, frame_*r* is stationary at *r*, frame_∞ is stationary at infinitely far.

Definition 3: Local frame

This frame is labeled frame_*r* and is stationary at the radial coordinate *r*. The time of frame_*r* is labeled t_r.

Definition 4: Proper inertial frame of a far falling object

Let m be a far falling object. Let frame_m be an inertial frame with constant velocity. Frame_m moves with m instantaneously. The time of frame_m is labeled t_m. Frame_m is called the proper inertial frame of m.

m is accelerated in frame_m because the velocity of frame_m is constant. So, m constantly changes its proper inertial frame, that is, a frame_m does not follow m. The velocity of a frame_m equals that of m only at the instant where the origin of the frame_m coincides with m. At this instant, the velocity of m is zero in the frame_m. So, the velocity of m is always zero in its proper inertial frame.

Frame_*r* and frame_∞ are both stationary and frame_m moves with respect to frame_*r*. How does the far falling m see frame_∞? Initially, m is at rest in frame_∞. So, frame_m equals frame_∞ at this instant. Then, m free falls. According to the equivalence principle, no experiment can tell m that it has moved or changed frame during the free fall. For m, its proper inertial frame has all the time been the same, that is, frame_∞. So, for the frame_m, frame_∞ is equivalent to frame_m.

Is this equivalence valid for frame_∞? According to the principle of relativity: "The laws of physics are the same in all inertial frames". Frame_∞ and frame_m being both inertial and equivalent for frame_m, then reciprocally, frame_m is equivalent to frame_∞ for frame_∞ too. So, we define the following law:

Law 1: Equivalence of frames

The proper inertial frame of a far falling object is equivalent to frame_∞, that is, frame_m is equivalent to frame_∞.

b. Equality of accelerations

For the equality of accelerations to be consistent with relativity, the gravitational acceleration and the inertial acceleration of m should be measured in the same frame. The gravitational acceleration $-\frac{GM}{r^2}\boldsymbol{e}_r$ is in frame_∞. According to Law 1, frame_m is equivalent to frame_∞. So, $-\frac{GM}{r^2}\boldsymbol{e}_r$ is in frame_m too.

The acceleration of m in frame_*r* is $\boldsymbol{\alpha}_r = \frac{dv}{dt_r}$, see (17) where $\boldsymbol{v}$ is the velocity of m and t_r the time in frame_*r*. So, we transform $\boldsymbol{\alpha}_r$ into frame_m using (18). The equation (18) is the transformation of acceleration for relativity which was derived in the article «Relativistic kinematics», [5], see the equation (18) of this article. The transformed $\boldsymbol{\alpha}_r$ is $\boldsymbol{\alpha}_m = \frac{\alpha_r}{1-\frac{v^2}{c^2}}$ which is the vector inertial acceleration of m in frame_m and *v* is the far falling velocity, see (18).

$\boldsymbol{\alpha}_m$ equals $-\frac{GM}{r^2}\boldsymbol{e}_r$ because both are in frame_m where the velocity of m is zero and the second law of Newton applies, see (19). The equality of accelerations consistent with relativity is (19) which is the equivalent of the Newtonian equality (7).

c. Far falling velocity

The far falling velocity in frame_*r* is obtained by integrating (19) where $\boldsymbol{v}$ and $-\frac{GM}{r^2}\boldsymbol{e}_r$ are vectors. So, we dot (19) with $\boldsymbol{v}$ to obtain (20) which is transformed into (21) using (9). The integration of (21) gives (22), with *k* being the constant of integration. At infinitely far $r=\infty$ and $v=0$, then *k* equals zero, see (23). Equation (22) gives in (24) the expression of $1-\frac{v^2}{c^2}$ with *v* being the far falling velocity in frame_*r* consistent with relativity.

d. Time dilation factors

The time dilation factor for special relativity between frame_*r* and frame_m is (25) which is obtained by using (24) in (6), with *v* being the far falling velocity. What is the time dilation factor between frame_*r* and frame_∞? Here, we use Law 1 whose consequence is that the times of frame_m and frame_∞ are equal: $t_m = t_\infty$ and $\frac{dt_r}{dt_\infty}=\frac{dt_r}{dt_m}$, see (26). Using (26) in (25) we obtain (27) which expresses the time dilation factor between frame_*r* and frame_∞. Because frame_*r* is influenced by the gravity of M while frame_∞ is not, (27) specifies the time dilation due to gravitational effect. We call the term in (27) $e^{-\frac{GM}{c^2r}}$ Gravitational time dilation factor.

e. Schwarzschild dilator

We see that (27) does not equal Schwarzschild dilator (2). But, using the Taylor's series (28), (27) is simplified into (29) which is identical to (2). So, the Schwarzschild dilator is a simplification of (27) for small $\frac{2GM}{c^2r}$. This shows that the gravitational time dilation factor (27) is the correct factor of time dilation due to gravitational effect which is consistent with relativity, while Schwarzschild dilator is not because the Einstein field equations are linked to the Newtonian equality (7), see section 2.c.

f. Consistence with relativity

In summary, Schwarzschild radius is caused by the use of the Newtonian equality (7), $\frac{d\boldsymbol{v}}{dt}=-\frac{GM}{r^2}\boldsymbol{e}_r$. In comparison, the gravitational time dilation factor (27) is derived using the equality (19), $\frac{1}{1-\frac{v^2}{c^2}}\frac{d\boldsymbol{v}}{dt_r}=-\frac{GM}{r^2}\boldsymbol{e}_r$, which is the equality of accelerations for relativity. Notice the difference between these 2 equations. So, the gravitational time dilation factor (27) is consistent with relativity, which is why its value is normal for $r < r_s$ except r=0, which eliminates Schwarzschild radius.

Acceleration in frame_*r* $\boldsymbol{\alpha}_r=\frac{d\boldsymbol{v}}{dt_r}$	(17)
Relativistic transformation of accelerations $\boldsymbol{\alpha}_m=\frac{\boldsymbol{\alpha}_r}{1-\frac{v^2}{c^2}}$ *v:* velocity of m in frame_*r*	(18)
Using (18) and (17) in (7) $\frac{1}{1-\frac{v^2}{c^2}}\frac{d\boldsymbol{v}}{dt_r}=-\frac{GM}{r^2}\boldsymbol{e}_r$	(19)
(19) dotted with $\boldsymbol{v}$ $\frac{\boldsymbol{v}\cdot d\boldsymbol{v}}{1-\frac{v^2}{c^2}}=-\frac{GM}{r^2}\boldsymbol{e}_r\cdot\boldsymbol{v}dt_r$	(20)
Using (9) in (20) $\frac{1}{2}\frac{dv^2}{1-\frac{v^2}{c^2}}=-\frac{GM}{r^2}dr$ Then $d\,ln\left(1-\frac{v^2}{c^2}\right)=-d\left(\frac{2GM}{c^2r}\right)$	(21)
Integration of (21) $1-\frac{v^2}{c^2}=e^{-\frac{2GM}{c^2r}+k}$	(22)
At infinitely far: $r=\infty$, $v=0$, (22) gives $k=0$	(23)
(22) and (23) give $1-\frac{v^2}{c^2}=e^{-\frac{2GM}{c^2r}}$	(24)
Using (24) in (6) $\frac{dt_r}{dt_m}=\sqrt{1-\frac{v^2}{c^2}}=\sqrt{e^{-\frac{2GM}{c^2r}}}$ Then $\frac{dt_r}{dt_m}=e^{-\frac{GM}{c^2r}}$	(25)
$t_\infty=t_m\Rightarrow$ $\frac{dt_r}{dt_\infty}=\frac{dt_r}{dt_m}$ t_∞: time of frame_∞	(26)
Using (26) in (25) $\frac{dt_r}{dt_\infty}=e^{-\frac{GM}{c^2r}}$	(27)
Taylor's series to the first order $e^{-\frac{2GM}{c^2r}}\approx 1-\frac{2GM}{c^2r}$	(28)
Using (28) in (27) $\frac{dt_r}{dt_\infty}=\sqrt{e^{-\frac{2GM}{c^2r}}}$ $\frac{dt_r}{dt_\infty}\approx\sqrt{1-\frac{2GM}{c^2r}}$ Same as Schwarzschild dilator (2)	(29)

Let us name the above theory Farfall theory of gravitational time dilation. Below, we explain how to use it for general relativity.

4. Gravitational relativistic dynamics

a. Transformations

In the Farfall theory, time, velocity and acceleration are transformed with the following transformations.

- Time transformation

The gravitational time dilation factor (27) transforms time between frame_*r* and frame_∞.

- Velocity transformation

In frame_*r* the vector velocity of m is $\boldsymbol{v}$, see equation (44). In frame_∞, the vector velocity of m is $\boldsymbol{v}_\infty$ which is written and transformed in (45). So, the velocity transformation is (45).

- Acceleration transformation

The acceleration transformation is (18) where $\boldsymbol{\alpha}_m$ is the inertial acceleration in frame_m, $\boldsymbol{\alpha}_r$ is the same acceleration but in frame_*r*, *v* is the far falling velocity.

b. Motion in gravitational field

In the Farfall theory the orbit of an object in a gravitational field is computed using (24) which gives *v* the scalar orbital velocity in frame_*r*. The orbit is computed in the article « Analytical equation for Space-Time geodesics and relativistic orbit equation [6]. This article gives the results below:

- The computed orbital precession of planet equals the value given by Albert Einstein, see (30).
- The computed hyperbolic orbit velocity shows that, when leaving the perihelion, the orbital velocity should decrease less than predicted by Newtonian mechanics, which explains the observed unexpected "boost in speed" of Oumuamua and the Pioneer anomaly, see (31).

Orbital precession of planet $$\lvert\Delta\Phi\rvert \approx \frac{6\pi GM}{a(1-\varepsilon^2)c^2}$$ a: semi-major axis, ε: eccentricity	(30)
Orbital velocities away from perihelion $$({\boldsymbol{u}_b}^2)_{Relativity} > ({\boldsymbol{u}_b}^2)_{Newtonian}$$	(31)

These results show that the Farfall theory is correct for general relativity.

In the gravitational field of n masses $$-\sum_{i=1}^{n}\frac{GM_i}{{r_i}^2}\boldsymbol{e}_{ri} = \frac{1}{1-\frac{v^2}{c^2}}\frac{d\boldsymbol{v}}{dt_l}$$	(32)
In the gravitational field of a continuous medium of mass $$-\int\frac{G}{r^2}\boldsymbol{e}_r dM = \frac{1}{1-\frac{v^2}{c^2}}\frac{d\boldsymbol{v}}{dt_l}$$	(33)

Note: In [6] the Schwarzschild time factor was used instead of the gravitational dilation time factor (27) because the latter was not available at the time. But, because the Schwarzschild time factor is a very good approximation for the computed orbits, the result in [6] was perfectly correct.

c. Gravitation of distributed mass

In the universe stars and galaxies are scattered in space and the gravitational field at one point is the sum of the gravitational fields of all the stars and galaxies. The combined gravitational acceleration at a point P in space is the left hand member of equation (32), with n being the number of stellar bodies in the universe, M_i the mass of the i^{th} body, r_i the distance from the point P to the i^{th} body and $\boldsymbol{e}_{ri}$ the unit vector along this distance.

The inertial acceleration of an object in its proper inertial frame is the right hand member of equation (32), with $\boldsymbol{v}$ being the velocity of the object in the universe, t_l the time in its proper inertial frame. The equality (32) being consistent with relativity the velocity of the object can be correctly computed using (32), under the assumption that the stellar bodies are stationary.

In the same way, if we consider a star or the whole universe as a continuous medium of mass, the combined gravitational field at point P equals the integral of the gravitational field of the continuous medium and the velocity of the object is computed using equation (33).

d. Gravitational red shift

In the Farfall theory, gravitational red shift is computed using the time transformation (27). Let Δt_r be the time of one period of a light wave in frame_r and Δt_∞ the time of the same period in frame_∞. By substituting Δt_r and Δt_∞ for dt_r and dt_∞ in (27) we obtain the ratio of time of the same period in frame_r and frame_∞ in (34) which expresses the time ratio of gravitational red shift.

In general relativity the time ratio of red shift is computed with Schwarzschild metric, see [3]. In the Farfall theory, the red shift between 2 different points a and b in the same gravitational field is $\frac{\Delta t_{ra}}{\Delta t_{rb}}$ and is expressed in (35). By using (28), (35) is simplified into (36) which is the well known time ratio of gravitational red shift in Schwarzschild metric, see [3]. This shows that the time ratio of gravitational red shift given by the Farfall theory is correct.

5. Below Schwarzschild radius

a. Free falling velocity

In the Farfall theory the region below the Schwarzschild radius does not collapse because the gravitational time dilation factor (27) is normal there. We call this region the central region.

The velocity of a free fall object is written with $\frac{r_s}{r}$ in (37) where the constant k can have any negative value. So, free falling velocity with any initial condition is smaller than c at any radial coordinate r except $r = 0$. At the Schwarzschild radius $r = r_s$, the far falling velocity (k=0) equals $0.795 \cdot c$, see (38). When $r \to 0$, v tends to c, and $v=c$ for $r = 0$, see (40) and (39), which is consistent with relativity. Indeed, the velocity of an accelerated object can be computed for example using the velocity addition formula, see (41). After infinitely long time, the velocity equals c as shown in «Introduction to Special Relativity» [4].

b. Circular orbital velocity

If an object is in a circular orbit in the central region, what is its velocity for the Farfall theory? The circular orbital velocity v_c is given by equation (42), with r_c being the radius of the orbit. This velocity is derived in « Analytical equation for Space-Time geodesics and relativistic orbit equation »,[6], see the equation (83) of this article. We see in equation (42) that v_c is always smaller than c. When $r_c \to 0$, v_c tends to c, and $v_c = c$ for $r_c = 0$, see (43). So, the circular orbital velocity is consistent with relativity.

c. Observed from infinitely far

Free falling velocity and circular orbital velocity are measured in local frame and are close to c in the central region. What would be their value measured from infinitely far? The velocity measured in frame_∞ is denoted by v_∞ and given by (46), with v being the velocity in frame_r. When $r \to 0$, v tends to c and the term $e^{-\frac{GM}{c^2 r}}$ tends to zero, see (47). So, when $r \to 0$, v_∞ tends to zero, see (48).

Gravitational red shift, see (27) $\frac{\Delta t_r}{\Delta t_\infty} = e^{-\frac{GM}{c^2 r}}$	(34)
Ratio of period at points a and b, see (34) $\frac{\Delta t_{ra}}{\Delta t_{rb}} = \frac{\Delta t_{ra}}{\Delta t_\infty}\frac{\Delta t_\infty}{\Delta t_{rb}} = \frac{e^{-\frac{GM}{c^2 r_a}}}{e^{-\frac{GM}{c^2 r_b}}}$ $= e^{\frac{GM}{c^2}\left(\frac{1}{r_b}-\frac{1}{r_a}\right)}$	(35)
Using (28) in (35) $\frac{\Delta t_{ra}}{\Delta t_{rb}} \approx \frac{\sqrt{1-\frac{2GM}{c^2 r_a}}}{\sqrt{1-\frac{2GM}{c^2 r_b}}}$	(36)
Using (22) in (6) $v = c\sqrt{1-e^{-\frac{2GM}{c^2 r}+k}}$ Using (3) $v = c\sqrt{1-e^{-\frac{r_s}{r}+k}} < c$	(37)
For $r = r_s$ and $k=0$, (37) gives $v = c\sqrt{1-e^{-\frac{r_s}{r_s}}} = c\sqrt{1-e^{-1}}$ $= 0.795 \cdot c$	(38)
For $r \to 0$ $e^{-\frac{2GM}{c^2 r}+k} \to e^{-\infty} = 0$	(39)
For $r \to 0$, using (39) in (37) $v \to c$	(40)
Velocity addition formula $v_1 = \frac{v_2+u}{1+\frac{v_2 u}{c^2}}$	(41)
Circular orbital velocity $v_c = \sqrt{\frac{1}{\frac{r_c}{GM}+\frac{1}{c^2}}}$ r_c: radius of the orbit	(42)
$r_c \to 0$, (42) gives $v_c \to \sqrt{\frac{1}{\frac{0}{GM}+\frac{1}{c^2}}} = c$	(43)
Vector velocity in frame_r $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt_r}$	(44)
Vector velocity in frame_∞, see (44) $\boldsymbol{v}_\infty = \frac{d\boldsymbol{r}}{dt_\infty} = \frac{d\boldsymbol{r}}{dt_r}\frac{dt_r}{dt_\infty} = \boldsymbol{v}\frac{dt_r}{dt_\infty}$ Velocity transformation $\boldsymbol{v}_\infty = \boldsymbol{v}\frac{dt_r}{dt_\infty}$	(45)

Using (27) in (45) $v_\infty = v\, e^{-\frac{GM}{c^2 r}}$	(46)
$r \to 0$ $v \to c,\ v_c \to c$ See (40) and (43) Also $e^{-\frac{GM}{c^2 r}} \to 0$	(47)
Using (47) in (46), $r \to 0$ $v_\infty \to c \cdot 0 = 0$	(48)

In consequence, seen from infinitely far, objects in the central region appear to stop moving. This result is very interesting for point mass.

d. Point masses

The radius of a point mass is zero. When 2 point masses join, the distance between them becomes zero and they form a unique point mass whose radius is also zero. If 2 point masses are infinitely close to each other, r is infinitely small and the gravitational time dilation is huge. Infinitely small r is valid for the Farfall theory because $r \neq 0$. For the outside, the velocities of the 2 point masses become infinitely small and the 2 point masses will not join together before infinitely long time. In other words they will never form a unique point mass. So, we conclude that point masses could not coalesce in space to form a unique point mass.

On the other hand, because point masses should not merge, but LIGO has observed several mergers of black holes, then black holes could not be point masses.

e. Light in the central region

Figure 2 shows how light travels in frame_r for the Farfall theory. An object m free falls at velocity v in frame_r in the gravitational field of M. At time t_1, the proper inertial frame of m is frame_m$_1$ and coincides with frame_r. At this instant, m sends a light flash. After an infinitely small time interval Δt_m, m arrives in frame_m$_2$. Because Δt_m is infinitely small, frame_m$_2$ has the same velocity as frame_m$_1$.

In frame_m$_2$ light has done the distance $c\Delta t_m$ and the light front is at $(c\text{-}v)\Delta t_m$ with respect to frame_r, see Figure 2. Because $v<c$ for any M and k, see (37), we have $(c\text{-}v)\Delta t_m > 0$, that is, the light front is farther than frame_r. In frame_r light has done the distance $c\Delta t_r$. Because $(c\text{-}v)\Delta t_m > 0$, we have
$c\Delta t_r > 0$ too. So, r_f the distance between the light front and M is bigger than that of the initial light flash r, which means that the light front is farther and farther from M as time flows and will finally get out into space.

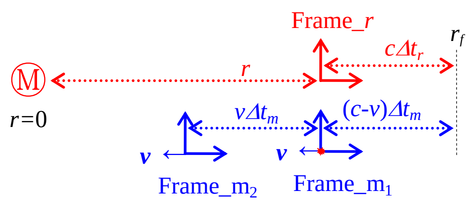

Figure 2

Velocity of frame_m is v. Light in frame_m travels $c\Delta t_m$. Light travels $(c\text{-}v)\Delta t_m$ more than frame_r. Light in frame_r travels $c\Delta t_r$. So, light front is at r_f, farther than r from M and goes out.

In Figure 2, M can have any mass. So, light emitted by m will escape from the gravitational field of M no matter how strong the field is. Below the Schwarzschild radius the gravitational field of a black hole is normal for the Farfall theory and the light emitted by any object free falling in it could escape. In conclusion, no stellar body could trap light and its radiation could be seen from the outside.

6. About black hole

a. A model

There is no physical force that can balance the gravitational attraction to prevent a black hole to shrink into a point. However, the density of point mass is infinite, which is fundamentally counter-physical. So, black hole is a paradox in physics and is called a singularity.

Above, we have shown that the radius of black holes could not be zero, which means that even without a force to balance the gravitational attraction, the particles in a black hole would stay apart and form a sort of super dense cloud of particles. What could be the form of this cloud? A possibility is that the cloud of particles is a rotating ring because it satisfies the following conditions:

1. **Stability**: The gravitational attraction that pulls the material of the ring toward the center is balanced by the relativistic centrifugal force of the material that rotates at nearly the speed of

light. Rotating ring could make black holes stable because circular orbit is stable for the smallest radius, see section 5.b.
2. **No singularity**: A ring is a distributed mass but not a point mass, which rules out singularity.

Could there be something at the center of the ring? Probably no because the center is an unstable place. Imagine an iron ball at the center of a ring magnet. At the instant the ball is set free, it will leave the center attracted by the magnet. Gravitational force is an attraction and any material at the center will leave the center attracted by the mass of the ring.

So, in contradiction with the idea that a black hole will collapse into a singularity, a rotating ring is a structure dynamically stable that can hold a black hole persistent in space without shrinking into a singularity.

b. Observational evidences

Let us call the idea of rotating ring of super dense material rotating ring hypothesis. For validating this hypothesis, we have to search for evidence by observation. But, we are in a logical dead end because by definition, black holes do not emit any radiation and are invisible. Either we see nothing and get no evidence or we see a radiation and deem it as not from the black hole itself. Would the hypothesis never be confirmed by observation?

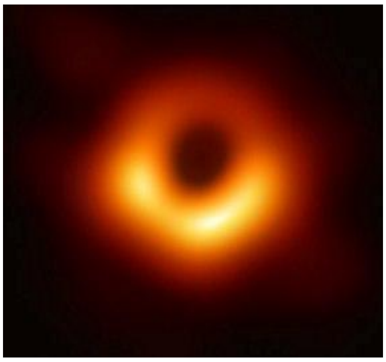

Figure 3

Direct image of a supermassive black hole, located at the galactic core of Messier 87.

In fact, according to the Farfall theory light could escape from the center of a strong gravitational field, see section 5.e. So, the core of black holes could be visible despite of the term “black” in the name. For avoiding confusion, let us emphasize that the radiation discussed here is emitted by the material inside a black hole, but not Hawking radiation which comes from the event horizon.

The first observational evidence is the image of the galactic core of Messier 87, Figure 3, which shows a ring structure. Galactic cores are Active galactic nucleus (AGN) [7] each of which hosts a supermassive black hole. Today, the ring is thought to be the accretion disk outside the Schwarzschild radius, but for the Farfall theory it could also be the supermassive black hole itself. The center of the ring is black, which suggests that the center emits nothing because void of material, which supports the rotating ring hypothesis. However, the black center could also support the idea that at the center there is a black hole that is really black.

Observation	Theory
Rotating ring hypothesis with the Farfall theory	
The ring is stable	*Rotating ring is dynamically stable*
The center of the ring is black	*The center is void of material*
The ring is visible	*Light can escape*
Black hole is really black with Schwarzschild metric	
The ring is stable	*Black holes shrink into a point*
The center of the ring is black	*Light is trapped in black hole. But there is no physical law that specifies the trapping of light.*
No clue about the masses of the ring and the black hole.	*Black hole would stay stably at the center only if the mass of the ring is tiny with respect to the mass of the black hole. Otherwise black hole is unstable there.*
The ring is visible	*Light cannot escape*

Table 1

Which of the rotating ring hypothesis and the idea that black hole is really black is more consistent with respect to the observation? Let us compare them in Table 1. The comparison shows that the theoretical prediction by “Rotating ring hypothesis with the Farfall theory” is more consistent than that by “Black hole is really black with Schwarzschild metric”. So, Figure 3 supports more the rotating ring hypothesis.

In the page Active galactic nucleus [7] we read: “Nuclear optical continuum emission … has a roughly power-law dependence on wavelength. … X-ray continuum emission … shows a power-law spectrum”.

According to the rotating ring hypothesis, the material of the ring rotates near the speed of light. Such velocity would heat the material to a temperature so high that the thermal radiation spectrum is beyond X-ray, see Figure 4. Thermal radiation obeys Planck's law whose low frequency part has the aspect of power-law spectrum. So, the observed optical continuum and X-ray continuum emissions from one AGN could be 2 parts of the same thermal radiation. Also, material rotating near the speed of light could be a stable source of high energy that emits stable thermal radiation beyond X-ray, which no other source of energy could do. So, the 2 continuum of emissions support the rotating ring hypothesis.

On the other hand, the thermal radiation spectrum would give us information about the temperature of the black hole. Its luminosity would inform us about the density of the material.

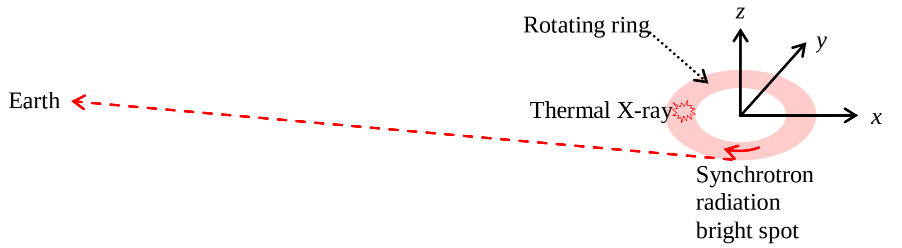

Figure 4

Model of black hole: Rotating ring with X-ray thermal radiation and Synchrotron radio radiation from a bright spot on the rim.

Also in the page Active galactic nucleus [7] we read: “Radio continuum emission… shows a spectrum characteristic of synchrotron radiation.” In the page Sagittarius A* [8] we read: “the Sagittarius A* radio emissions are not centered on the black hole, but arise from a bright spot in the region around the black hole”. Particles rotating near the speed of light emit synchrotron radiation from the rim of the ring, see Figure 4, which could explain the synchrotron radio emission and the location of the bright spot. Indeed, the material of a ring is out of the center. So, the synchrotron radio emission of Sagittarius A* supports the hypothesis of rotating ring.

On the other hand, the frequency of the synchrotron radiation would inform us about the velocity of the material and the radius of curvature of the ring, its polarization would inform us about the plane of rotation. The position of the bright spot with respect to the center would inform us about the size of the ring.

So, the above observational evidences support the rotating ring hypothesis and that black holes are in fact visible. Precise observational data would give us much more information about black holes than just their mass, for example temperature, density, speed of rotation, size, plane of rotation, etc.

7. Discussion

The Farfall theory gives correct result for general relativity, for example time dilation, red shift and orbit in gravitational field. It is valid even in the central region where the Schwarzschild metric collapses and shows that the velocities of objects are smaller than c and equal c for r=0, which is consistent with relativity. Two consequences of this result is that light could escape from black holes and that point masses could not unite with one another ruling out the collapse of stellar body into a singularity and getting rid of this paradox from physics, see section 5.d and 6.a.

We have shown that gravitational time dilation is equivalent to the time dilation for special relativity of a far falling frame, which makes the computation of gravitational effect as simple as in special relativity.

We have shown that, by using the equality of accelerations that is consistent with relativity, the far falling velocity v is not bigger than c anywhere, which means that the Schwarzschild radius is no longer a singularity in Schwarzschild spacetime and the region below the Schwarzschild radius is normal space.

We have proposed the hypothesis that the core of black holes could be a rotating ring which is stable for the smallest radius. The current observational evidences support this hypothesis. Possible new information about black holes could be obtained from observation using the Farfall theory.

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