Time relativity transformation of velocity

Kuan Peng 彭宽 titang78@gmail.com

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Abstract: Einstein's velocity-addition formula creates a discrepancy. A new transformation of velocity is derived using the Time relativity transformation of coordinates.

1. Discrepancy of double position

Einstein's velocity-addition formula is equation (2), where v_1 is the velocity of an object q in frame 1, v_2 its velocity in frame 2 and u the velocity of frame 2 in frame 1. Suppose that in frame 2 q starts to move from the origin, after a while, q is at the position x_q where the time is t_q. x_q is computed in equation (3) using the Lorentz transformation equation (1). t_q is expressed in terms of t_1 the time in frame 1.

For someone at the origin of frame 2, he sees that time is t_{2o} when q arrives at x_q. So, he computes the position of q using t_{2o} in equation (4). The so computed position is x_{2o} in equation (5). But x_{2o} does not equal x_q. So, Einstein's velocity-addition formula gives 2 different positions to the object q for the same time in frame 1, t_1, hence discrepancy. We call it the discrepancy of double position and its cause is the difference between the time at the origin and at the abscissa x_q.

Lorentz transformation and Einstein's velocity-addition formula.

$$\begin{cases} x_2 = \dfrac{x_1 - ut_1}{\sqrt{1-\dfrac{u^2}{c^2}}} \\ ct_2 = \dfrac{ct_1 - \dfrac{u}{c}x_1}{\sqrt{1-\dfrac{u^2}{c^2}}} \end{cases} \quad (1)$$

$$v_2 = \frac{v_1 - u}{1 - \dfrac{v_1 u}{c^2}} \quad (2)$$

The indices 1 and 2 indicate the frame 1 and 2. u is the velocity of frame 2 in frame 1. v_1 is the velocity of object q in frame 1, v_2 its velocity in frame 2,

Below, we will derive a transformation of velocity using system (6), the Time relativity transformation of coordinates derived in «Time relativity transformation of coordinates».

Time relativity transformation of coordinates

$$\begin{cases} x_2 = x_1 - ut_1 \\ t_2 = t_1\sqrt{1-\dfrac{u^2}{c^2}} \end{cases} \quad (6)$$

x_1 and t_1 are the abscissa and time in frame 1, x_2, and t_2 those in frame 2.

2 ways of computing the position of q in frame 2 with Special relativity.

$$\begin{aligned} x_q = v_2 t_q &= \frac{v_1 - u}{1-\dfrac{v_1 u}{c^2}} t_q \\ &= \frac{v_1 - u}{1-\dfrac{v_1 u}{c^2}}\left(\frac{t_1 - \dfrac{u}{c^2}x_1}{\sqrt{1-\dfrac{u^2}{c^2}}}\right) \\ &= t_1 \frac{v_1 - u}{\sqrt{1-\dfrac{v^2}{c^2}}} \end{aligned} \quad (3)$$

t_q is the time in frame 2 at q, t_1 time in frame 1. x_q is computed using t_q.

$$t_{2o} = t_1\sqrt{1-\frac{u^2}{c^2}} \quad (4)$$

$$\begin{aligned} x_{2o} &= v_2 t_{2o} \\ &= \frac{v_1 - u}{1-\dfrac{v_1 u}{c^2}} t_1\sqrt{1-\frac{u^2}{c^2}} \\ &\neq x_q \end{aligned} \quad (5)$$

t_{2o} is the time at origin of frame 2. x_{2o} the position of q in frame 2 computed using t_{2o}.

$x_{2o} \neq x_q$, hence discrepancy.

2. Derivative of x_2

Take 2 inertial frames of reference, frame 1 and 2. Frame 2 moves at the velocity u in frame 1. An object q moves at the velocity v_1 in frame 1 and v_2 in frame 2. When observed in frame 1, q's abscissa is x_1, time is t_1 and v_1 is the derivative of x_1 with respect to t_1, see equation (7). When observed in frame 2, its abscissa is x_2, time is t_2 and v_2 is the derivative of x_2 with respect to t_2, see equation (8). We have multiplied and divided t_1 to create the terms $\frac{dx_2}{dt_1}$ and $\frac{dt_1}{dt_2}$.

We compute $\frac{dx_2}{dt_1}$ using system (6) and express it in terms of v_1 and u in equation (9). $\frac{dt_1}{dt_2}$ is computed using system (6), see equation (10). Equations (9) and (10) are introduced into equation (8) to get equation (11), which is the Time relativity transformation of velocity that relates v_2 to v_1 and u.

Transformation of velocity is the time derivative of x_2.

$$v_1 = \frac{dx_1}{dt_1} \quad (7)$$

$$v_2 = \frac{dx_2}{dt_2} = \frac{dx_2}{dt_1}\frac{dt_1}{dt_2} \quad (8)$$

$$\begin{aligned} \frac{dx_2}{dt_1} &= \frac{d(x_1 - ut_1)}{dt_1} \\ &= \frac{dx_1}{dt_1} - \frac{d(ut_1)}{dt_1} \\ &= v_1 - u \end{aligned} \quad (9)$$

$$\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} \quad (10)$$

$$v_2 = \frac{v_1 - u}{\sqrt{1-\dfrac{u^2}{c^2}}} \quad (11)$$

3. Meaning of equation (11)

Let us see a displacement of the object q in frame 1. During a time interval, q moves from the point *a* to point *b*, see Figure 1 and Figure 2. The abscissa of *a* is $x_1(a)$ and that of b is $x_1(b)$. The duration of the time interval is Δt_1. So, the velocity of q is the v_1 that is expressed in equation (12).

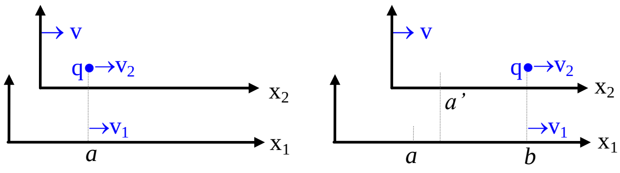

Figure 1 Figure 2

a and b are the start and end points of the displacement of q. a' is the start point in frame 2 which moves in frame 1.

In frame 2, the start point of q is the point *a'*. q moves from *a'* to point *b*, see Figure 2. The abscissa of *a'* is $x_2(a')$ and that of b is $x_2(b)$. The duration of the time interval is Δt_2. So, the velocity of q is the v_2 that is expressed in equation (13).

Derivation of transformation of velocity using the points a, a' and b in Figure 1 and Figure 2

$$v_1 = \frac{x_1(b) - x_1(a)}{\Delta t_1} \tag{12}$$

$$v_2 = \frac{x_2(b) - x_2(a')}{\Delta t_2} \tag{13}$$

$$x_1(b) - x_1(a') = x_2(b) - x_2(a') \tag{14}$$

$$x_1(a') - x_1(a) = u\Delta t_1 \tag{15}$$

$$\begin{aligned} v_1 &= \frac{x_1(b) - x_1(a') + x_1(a') - x_1(a)}{\Delta t_1} \\ &= \frac{x_2(b) - x_2(a') + u\Delta t_1}{\Delta t_1} \\ &= \frac{x_2(b) - x_2(a')}{\Delta t_2}\frac{\Delta t_2}{\Delta t_1} + u \end{aligned} \tag{16}$$

Δt_1 and Δt_2 are time interval in frame 1 and 2. $x_1(a)$ $x_1(a')$ and $x_1(b)$ are abscissa of a, a' and b in frame 1, $x_2(a')$ and $x_2(b)$ are those in frame 2

Let $x_1(a')$ be the abscissa of *a'* in frame 1. The distance from *a'* to *b* is $x_1(b)-x_1(a')$. For converting this distance in frame 1 to a distance in frame 2, we recall the law of length conservation in «Time relativity transformation of coordinates» section 1, which states that, if the length of a ruler is Δx_1 when static and Δx_2 when moving, Δx_1 and Δx_2 are equal. So, we have equation (14).

Because *a'* moves with frame 2, the distance between *a* and *a'* is the distance moved by frame 2, which equals $u{\cdot}\Delta t_1$ in frame 1, see equation (15). Then, we transform equation (12) into (16) by adding and subtracting $x_1(a')$ and using equations (14) and (15). The ratio $\frac{\Delta t_2}{\Delta t_1}$ is computed using system (6), see equation (17), the ratio $\frac{x_2(b)-x_2(a')}{\Delta t_2}$ is v_2, see equation (13). Then, equation (16) is transformed into equation (18), which is identical to equation (11).

$$\frac{\Delta t_2}{\Delta t_1} = \sqrt{1 - \frac{u^2}{c^2}} \tag{17}$$

$$v_2 = \frac{v_1 - u}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{18}$$

4. Reverse transformations

Equation (18) transforms velocity from frame 1 to frame 2. What is the transformation from frame 2 to frame 1? For doing so, we must distinguish 2 cases:

$$v_1 = v_2\sqrt{1 - \frac{u^2}{c^2}} + u \tag{19}$$

Inverse rearrangement of equation (11).

$$v_1 = \frac{v_2 + u}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{20}$$

Replacing u by –u and swapping v_1 and v_2 in equation (11).

1. Frame 2 moves in frame 1

In this case, we use equation (19) which is the inverse rearrangement of equation (18).

2. Frame 1 moves in frame 2

In this case, we use equation (20) which is equation (18) in which we have replaced u by –u and swapped v1 and v2.

- What is the difference?

We see that equation (20) is different from equation (19). Why are they different while both transform from frame 2 to frame 1? For explanation, let us see the example of a train moving past a platform. The platform is the frame 1 and the train the frame 2. On the platform there is a flying arrow named Pla, in the train is a flying arrow named Tra.

In case 1, the velocity of Pla is v_1 on the platform. The the velocity of Pla with respect to the train is v_2 which is computed using equation (18). This is the transformation of velocity from frame 1 to frame 2.

If a traveler in the train has measured v_2 and want to know v_1, he will compute v_1 using equation (19). Notice that v_2 is not the velocity of Tra, but that of Pla.

In case 2, frame 1 is still the platform and frame 2 the train. But this time the traveler measures the velocity of Tra in the train, not the velocity of Pla. When he computes the velocity of Tra with respect to the platform by knowing that the platform moves at the velocity –u with respect to the train, he uses equation (18) after having replaced u with –u and swapped v1 and v2, which gives equation (20). He must use equation (20) because the velocity he computes is that of Tra, not that of Pla.

For distinguishing the 2 transformations, we name case 1 inverse transformation and case 2 back transformation. Inverse transformation converts the velocity of Pla on the platform, back transformation converts the velocity of Tra in the train. Because they are not the same object, their transformations are different.

5. Comments

$$v_1 = \frac{v_2 + u}{1 + \frac{v_2 u}{c^2}} \qquad (21)$$

Inverse rearrangement of Einstein's velocity-addition formula equation (2).

Contrary to equations (19) and (20), Einstein's velocity-addition formula equation (2) has only one reverse transformation which is equation (21). When we inverse mathematically equation (2), we obtain equation (21). When we replace u with –u and swap v1 and v2, we obtain also equation (21). If we convert the velocity of Pla using equation (21), we will obtain the velocity of Tra with respect to the platform, which is wrong and makes Special relativity inconsistent.

We have shown the discrepancy of double position of Special relativity which is caused by relativity of simultaneity that makes time in frame 2 different from point to point. Time relativity does not suffer from such discrepancy because frame 2 has only one time.

Time relativity transformation of velocity is equation (18) which is the derivative of x_2. So, this transformation is valid for accelerating frame and object. In the contrary, Special relativity is not valid under acceleration.

Next, I will write about the conservation of momentum, mass and energy based on Time relativity transformation.