Analysis of Einstein's derivation of the Lorentz Transformation

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Abstract: Einstein's derivation of the Lorentz Transformation is purely theoretical. This study shows how it is related to the physical phenomenon of time dilation and length contraction.

1. Einstein's derivation

The Lorentz Transformation was first derived using the conditions of time dilation and length contraction. Later, Albert Einstein has given a different derivation of the Lorentz Transformation by using constancy of the speed of light only, making time dilation and length contraction subsequent to the Lorentz Transformation which then acquired the status of fundamental law.

Einstein exposed his derivation in the Appendix 1 of his 1920 book « Relativity: The Special and General Theory ». But the physical significance of this derivation is blurry. I will analyze Einstein's theoretical derivation to find out how it is related to the physical conditions of time dilation and length contraction. First, let us see his derivation step by step.

As shown in Figure 1, K is a static co-ordinate system and K' a co-ordinate system moving at velocity v with respect to K. Suppose that at time t=0 the origins of K and K' coincide and a light flash is sent out from the origins. At time t > 0, the light is at "Light signal +" and "Light signal -".

In K, the position of "Light signal +" is x=ct, which gives equation (1). Because light travels in K' at the speed c too, the position of "Light signal +" in K' is x'=ct', which gives equation (2). Einstein claimed $(x' - ct') = \lambda(x - ct)$, which is equation (3). The position of "Light signal -" is x=−ct in K and x'=−ct' in K', which gives equations (4) and (5). Einstein claimed $(x' + ct') = \mu(x + ct)$, which is equation (6).

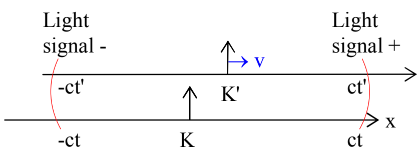

Figure 1

In order to separate x' and ct', we add side by side equation (3) to (6) in equation (7) and explicitly express x' from x and ct in equation (8). By subtracting side by side equation (6) from (3) in equation (9) we explicitly express ct' from x and ct in equation (10).

$$x - ct = 0 \tag{1}$$

$$x' - ct' = 0 \tag{2}$$

$$(x' - ct') = \lambda(x - ct) \tag{3}$$

$$x + ct = 0 \tag{4}$$

$$x' + ct' = 0 \tag{5}$$

$$(x' + ct') = \mu(x + ct) \tag{6}$$

For convenience we introduce the constants a and b to replace the constants λ and μ, see equation (11), and rewrite equations (8) and (10) into the simpler equations (12) and (13).

For finding out the constant $\frac{bc}{a}$, we apply equation (12) to the origin of K' which is permanently x' = 0. In K, the origin of K' moves the distance $vt_{k'}$ in time $t_{k'}$. We substitute 0 for x', $vt_{k'}$ for x and $t_{k'}$ for t in equation (12) and obtain $\frac{bc}{a} = v$ in equation (14).

$$(x' - ct') + (x' + ct') = \lambda(x - ct) + \mu(x + ct) \tag{7}$$

$$2x' = (\lambda + \mu)x - (\lambda - \mu)ct \Rightarrow x' = \frac{\lambda + \mu}{2}x - \frac{\lambda - \mu}{2}ct \tag{8}$$

$$(x' - ct') - (x' + ct') = \lambda(x - ct) - \mu(x + ct) \tag{9}$$

$$-2ct' = (\lambda - \mu)x - (\lambda + \mu)ct \Rightarrow ct' = \frac{\lambda + \mu}{2}ct - \frac{\lambda - \mu}{2}x \tag{10}$$

For finding out the constant a, we compute the length of a stick static in K' which moves at velocity v in K. The static length of the stick is denoted by Δx' while its length measured in K is denoted by Δx. The duration of the measurement is Δt=0. We substitute Δx' for x', Δx for x and 0 for t in equation (12) and obtain equation (15). By setting the static length Δx' to 1, the length measured in K $(\Delta x)_{\Delta x' moving}$ is computed in equation (16).

$$a = \frac{\lambda + \mu}{2}, b = \frac{\lambda - \mu}{2} \tag{11}$$

$$x' = ax - bct = a\left(x - \frac{bc}{a}t\right) \tag{12}$$

$$ct' = act - bx = a\left(ct - \frac{b}{a}x\right) \tag{13}$$

$$0 = vt_{k'} - \frac{bc}{a}t_{k'} \Rightarrow \frac{bc}{a} = v \tag{14}$$

In an inverse case where the stick is static in K, we compute the length of the stick measured in K' where it moves. We measure the length at the time t'=0 and use equation (13) to find the relation between x and t for t'=0, which is $t = \frac{bx}{ac}$, see equation (17). By substituting $\frac{bx}{ac}$ for t in equation (12), we obtain the relation between x and x', which is equation (18). By setting the static length Δx to 1 and substituting 1 for x in equation (18), the length in K' $(\Delta x')_{\Delta x\, moving}$ is computed in equation (19).

$$\Delta x' = a\left(\Delta x - \frac{bc}{a}0\right) \Rightarrow \Delta x' = a\Delta x \quad (15)$$

$$\Delta x' = 1 \Rightarrow (\Delta x)_{\Delta x' moving} = \frac{1}{a} \quad (16)$$

$$\begin{cases} ct' = act - bx \\ t' = 0 \end{cases} \Rightarrow t = \frac{bx}{ac} \quad (17)$$

$$x' = ax - b\frac{bx}{a} = ax\left(1 - \frac{b^2}{a^2}\right) \quad (18)$$

$$\Delta x = 1 \Rightarrow (\Delta x')_{\Delta x\, moving} = a\left(1 - \frac{b^2}{a^2}\right) \quad (19)$$

In summary, $(\Delta x)_{\Delta x' moving}$ is the length of a unity stick moving in K and $(\Delta x')_{\Delta x\, moving}$ the length of a unity stick moving in K'. By the principle of relativity, these 2 lengths must have the same value, so we have equation (20).

$$(\Delta x)_{\Delta x' moving} = (\Delta x')_{\Delta x\, moving} \Rightarrow \frac{1}{a} = a\left(1 - \frac{b^2}{a^2}\right) \quad (20)$$

In equation (21) we combine equation (14) with (20) and find the value of the constant a, which is expressed by equation (22). By introducing the constants a and $\frac{bc}{a}$ into equations (12) and (13), we obtain equations (23) and (24) which are the space and time equations of the Lorentz Transformation.

$$\begin{cases} \frac{1}{a} = a\left(1 - \frac{b^2}{a^2}\right) \\ v = \frac{bc}{a} \end{cases} \Rightarrow a^2 = \frac{1}{1 - \frac{v^2}{c^2}} \quad (21)$$

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (22)$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (23)$$

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (24)$$

2. Proportionality assumption

In his derivation Einstein only used the fact that the speed of light is c in K and K'. However, another claim was used without being properly declared, which is: (x,t) the co-ordinates of an event in K and (x',t') the co-ordinates of the same event in K' satisfy equations (3) and (6) in general. As these equations are in direct proportional form, we give this claim the name "Proportionality assumption".

This claim is an assumption because it is not proven. Although at "Light signal +" $(x' - ct') = \lambda(x - ct)$ reduces to 0=0 and $(x' + ct') = \mu(x + ct)$ reduces to 0=0 at "Light signal -", the quantities $d = (x - ct), e = (x' - ct'), f = (x + ct)$ and $g = (x' + ct')$ are not zero elsewhere and there is no physical principle that makes equations (3) and (6) naturally true in general or $e = \lambda \cdot d$ and $f = \mu \cdot g$. For example, the co-ordinates of the origin of K' are (x'=0,t') in K' and ($x=vt_{K'}$, $t_{K'}$) in K which give $(x' - ct') = -ct' \neq 0$ and $(x - ct) = vt_{k'} - ct_{k'} \neq 0$ respectively. But we do not naturally have $-ct' = \lambda(vt_{k'} - ct_{k'})$.

How did Einstein get the "Proportionality assumption"? His words reveal his reasoning: "Obviously this will be the case when the relation is fulfilled in general" (see paragraph 2 of Appendix 1 of «Relativity: The Special and General Theory »). The said "relation" refers to equation (3) and "this will be the case" refers to that "$(x' - ct') = 0$ and $(x - ct) = 0$ at once". So, his idea was: if equation (3) is naturally "fulfilled in general", then $(x' - ct') = 0$ and $(x - ct) = 0$ at "Light signal +". But this reasoning does not prove equations (3) and (6) true in general.

If we follow this reasoning, we can say for example: when the relation $(x'^3 - (ct')^3) = \lambda(x^3 - (ct)^3)$ is fulfilled in general, we have $x - ct = 0$ and $x' - ct' = 0$ at "Light signal +" and we can make non-linear relativistic co-ordinate transformation using $(x'^3 - (ct')^3) = \lambda(x^3 - (ct)^3)$, which is arbitrary and surely wrong.

"Proportionality assumption" contains a sub-assumption: the direct proportional form of the relation $(x' - ct') = \lambda(x - ct)$, which excludes all non proportional relations that satisfies constancy of the speed of light. For example, we can make the general linear relation (25) to give x=ct and x'=ct' by substituting ct and ct' for x and x' in equation (25), which gives equation (26). In equation (27) we divide equation (25) by equation (26) and we have x=ct and x'=ct' at once.

Why did Einstein choose from the outset the direct proportional form and exclude the general form of linear relation? The only reason I see is that $(x' - ct') = \lambda(x - ct)$ and $(x' + ct') = \mu(x + ct)$ give exactly the Lorentz Transformation after rearrangement (see equations (12) and (13)), while the general form does not. This suggests that Einstein could have known the Lorentz Transformation while deriving it by his own and thus, makes his derivation based on the Lorentz Transformation.

$$x' = \alpha x + \beta ct \tag{25}$$

$$ct' = \alpha ct + \beta ct \Rightarrow t' = (\alpha + \beta)t \tag{26}$$

$$c = \frac{x'}{t'} = \frac{\alpha x + \beta ct}{(\alpha + \beta)t} \Rightarrow x = ct \tag{27}$$

In consequence, "Proportionality assumption" being not naturally true in general, its validity must come from the Lorentz Transformation. So, Einstein's derivation does not make the Lorentz Transformation a purely theoretical law, which then is subject to defect.

3. Distance discrepancy

And indeed, here is an example of discrepancy of the Lorentz Transformation. Take the origin O of K in Figure 2, where x = 0. By substituting 0 for x and t_o for t in equation (23), the position of O in K' x'_1 is computed in equation (28).

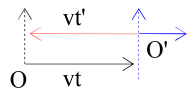

Figure 2

On the other hand, we compute the distance that O has traveled in K'. At time t_o, O' is at the position $x_1 = vt_o$ in K. By substituting vt_o for x and t_o for t in equation (24), we obtain in equation (29) the time at O' in K' which is denoted by t'_o. So, time has passed t'_o after O coinciding with O'.

$$x'_1 = \frac{0 - vt_o}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{vt_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{28}$$

$$ct'_o = \frac{ct_o - \frac{v}{c}vt_o}{\sqrt{1 - \frac{v^2}{c^2}}} = ct_o\sqrt{1 - \frac{v^2}{c^2}} \tag{29}$$

O' moves at the velocity v in K. According to relativity principle, O moves at the velocity −v in K'. Because time has passed t'_o at O', O has traveled $-vt'_o$ in K' to the position $x'_2 = -vt'_o$, see equation (30).

$$x'_2 = -vt'_o = -vt_o\sqrt{1 - \frac{v^2}{c^2}} \neq x'_1 \tag{30}$$

Equations (23) and (24) are the Lorentz Transformation. When the position of O in K' is computed using equation (23), we get x'_1, when it is computed using equation (24), we get x'_2. Because x'_1 does not equal x'_2, the Lorentz Transformation gives self-conflicting results.

How does this self-conflict come about? Let us denote the distance between O and O' by L in K and L' in K'. Due to time dilation, time flows slower in K' than in K, which by the way has caused the twin paradox, see « Twin paradox when Earth is the moving frame ». But this shorter time makes a shorter travel to O in K'. So, L' is shorter than L because of time dilation.

On the other hand, due to length contraction, a moving stick looks shorter than its static length. For this case, the static length is L' and the moving length is L. Thus, L is shorter than L' because of length contraction. So, it is time dilation and length contraction that give different values to the position of O in K'.

4. Cause of the discrepancy

Let us search what is the cause of this discrepancy by searching the common point of the several discrepancies that I have shown in previous articles.

a) Relativistic jets

Relativistic jets are streams of matter in space traveling near the speed of light. So they should be length contracted. But we do not see any evidence of length contraction in photographs of relativistic jets. See « Astrophysical jet and length contraction ». This indicates experimentally that length contraction may not exist.

b) Synchronization of clocks of GPS satellites

GPS satellites move very fast and should be affected by relativity of simultaneity. So they could not be synchronized with one another because all satellites are synchronized with one master clock on Earth. But they have to be so synchronized in order to correctly function. The good functioning of the GPS system shows that the clocks aboard the satellites are in reality synchronized with one another, which indicates that relativity of simultaneity may not exist.

Since length contraction is equivalent to relativity of simultaneity (see « Testing relativity of simultaneity using GPS satellites » and « Synchronizing moving GPS clocks »), non existence of relativity of simultaneity is an experimental evidence of non existence of length contraction.

c) Accelerating length

In chapter 6 "Acceleration of a ruler" of « From Michelson–Morley experiment to length contraction », I have shown that the far end of an accelerating ruler would go backward. Because someone has argued that this could be caused by mechanical compression of the ruler, I will explain this discrepancy by using 2 distant rabbits who, by a magical coincidence, jump at the same time to the right with the same acceleration, avoiding any interaction between them.

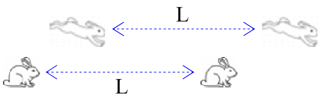

Figure 3

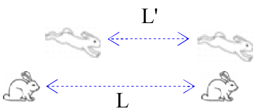

Figure 4

Take 2 sitting rabbits, one on the left, one on the right, see Figure 3. The distance between them is L. During their jump they are still separated by the distance L in their proper referential frame.

In the static frame, the distance L should contract and become L', see Figure 4. So, the right rabbit should go slightly less far than the left one. Since length contraction is proportional to L, if L is sufficiently long, L' would contract to such a short length that the right rabbit should go back to match L', see Figure 4.

This discrepancy is that due to length contraction the right rabbit would go left while jumping right. This is a theoretical evidence that length contraction should not exist.

d) Particles in circular accelerator

Chapter 4 "Circular accelerator" of « Testing relativity of simultaneity using GPS satellites » explains that due to length contraction the accelerated particles in a circular accelerator would squeeze into a train shorter than the circumference of the accelerator leaving the rest of the accelerator empty. If no experiment shows such phenomenon, this would be an experimental evidence that length contraction may not exist.

e) Faster than light contraction

In chapter 3 "Head and Tail's velocities" of « Astrophysical jet and length contraction » I have shown that due to length contraction the accelerating basketball could shrink so fast that its edge could exceed the speed of light. This is a theoretical evidence that length contraction could be wrong.

5. Conclusion

All the cited discrepancies point to length contraction. In Einstein's derivation length contraction is created by the "Proportionality assumption" because time dilation can be derived from constancy of the speed of light only. In « From Michelson–Morley experiment to length contraction » I have shown a way to derive the Lorentz Transformation without using length contraction, which then becomes non necessary. Next, I will correct all these discrepancies by getting rid of length contraction in relativistic transformation of co-ordinates.

Reference

Albert Einstein, «Relativity: The Special and General Theory»