Relativistic kinematics and gravitation

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Abstract: Like in Newtonian kinematics, the relativistic change of reference frame must be a vector system of transformation laws for position, velocity and acceleration.

The Lorentz transformation transforms coordinates from one frame to another, Einstein's velocity-addition formula transforms the velocity of a moving point from one frame to another. But the velocity-addition formula is not a derivative of the Lorentz transformation like the transformation of coordinates and velocity in Galilean transformation. In addition, in special relativity there is not acceleration transformation at all. However, my theory of Time relativity provides a coherent vector system of transformation of position, velocity and acceleration.

$$\begin{aligned} x_2 &= x_1 - ut_1 \\ y_2 &= y_1 \\ z_2 &= z_1 \\ t_2 &= t_1\sqrt{1-\frac{u^2}{c^2}} \end{aligned} \qquad (1)$$

The indices 1 and 2 indicate the reference frame 1 and 2. u is the velocity of frame 2 in frame 1.

In Time relativity the transformation of coordinates is the system (1) which is the system (12) given in «Time relativity transformation of coordinates» with the identity of y and z added. The transformation of the x-component of velocity is the equation (4) which is the equation (19) given in «Time relativity transformation of velocity ». Below we will create the transformation of the vector velocity by deriving the transformation of the y and z components of velocity.

$$\boldsymbol{v}_1 = v_{1,x}\boldsymbol{i} + v_{1,y}\boldsymbol{j} + v_{1,z}\boldsymbol{k} \qquad (2)$$

$$\boldsymbol{v}_2 = v_{2,x}\boldsymbol{i} + v_{2,y}\boldsymbol{j} + v_{2,z}\boldsymbol{k} \qquad (3)$$

$\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the vector velocities of an object q in frame 1 and 2.

$$v_{2,x} = \frac{v_{1,x} - u}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (4)$$

$v_{1,x}$ and $v_{2,x}$ are the x components of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

1. Relativistic kinematics

a) Y and Z components of velocity

Let frame 1 be a stationary frame of reference in which the inertial frame 2 moves at the velocity u in the x direction, see Figure 1. Let q be an object moving in the frame 2 at the velocity $v_{2,y}$ in the y direction. During an infinitesimal time interval dt_2, q has moved the infinitesimal distance dy_2. So, the y-component of velocity in frame 2 equals dy_2/dt_2, see equation (5).

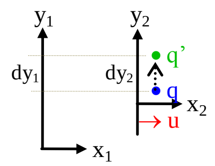

Figure 1

An object moves from q to q' in frame 2. The moved distance is dy_1 in frame 1 and dy_2 in frame 2.

For the same motion, q has moved the infinitesimal distance dy_1 during the time interval dt_1 in frame 1. Because there is not length contraction in the y direction, we have $dy_1=dy_2$ (see Figure 1) and write equation (6) where the term dy_1/dt_1 is the y-component of velocity in frame 1 $v_{1,y}$, see equation (7).

The ratio dt_1/dt_2 is given in equation (8) which is obtained using the time equation of the system (1). We introduce equation (8) in to (6), which becomes equation (9), which is the relation between the y-components of velocity in frames 1 and 2.

The z-components of velocity in frame 1 and 2 are $v_{1,z}$ and $v_{2,z}$ and are related by equation (10) in the same way as for y-components.

$$v_{2,y} = \frac{dy_2}{dt_2} \qquad (5)$$

$$\frac{dy_2}{dt_2} = \frac{dy_1}{dt_2} = \frac{dy_1}{dt_1}\frac{dt_1}{dt_2} \qquad (6)$$

$$\frac{dy_1}{dt_1} = v_{1,y} \qquad (7)$$

$$\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (8)$$

$$v_{2,y} = \frac{v_{1,y}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (9)$$

$$v_{2,z} = \frac{v_{1,z}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (10)$$

dt_1 and dt_2 are time intervals in which q moves dy_1 in frame 1 and dy_2 in frame 2.

b) Velocity vector

Let ***i, j*** and ***k*** be the unit vectors in the x, y and z directions. By convention, vectors are written in boldface letters. The velocity vector of q in frame 2 is written in equations (3) in which we substitute equations (4), (9) and (10) for the x, y and z components and obtain equation (11). The vector velocity of q in frame 1 is $\mathbf{v}_1$ written in equations (2), which we substitute for the corresponding terms in equation

(11), we also substitute **u** for u***i***, with **u** being the vector velocity of the moving frame 2 in frame 1, see equation (12). Then, equation (11) becomes equation (13).

$$\boldsymbol{v}_2 = \frac{v_{1,x}-u}{\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{i} + \frac{v_{1,y}}{\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{j} + \frac{v_{1,z}}{\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{k} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(v_{1,x}\boldsymbol{i} + v_{1,y}\boldsymbol{j} + v_{1,z}\boldsymbol{k} - u\boldsymbol{i}\right) \quad (11)$$

$$\boldsymbol{u} = u \cdot \boldsymbol{i} \quad (12)$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}} \quad (13)$$

u *is the vector velocity of the frame 2 in frame 1.*

The velocity vectors of q in frame 1 and 2 are $\mathbf{v_1}$ and $\mathbf{v_2}$ and are related through equation (13).

c) Acceleration vector

Suppose that the velocity of q is 0 at time t_2 in frame 2 and is **u** in frame 1. After an infinitesimal time interval dt_2, q's velocity becomes $\mathbf{v_2}$ in frame 2. At the same moment, q's velocity becomes $\mathbf{v_1}$ in frame 1.

Because dt_2 is infinitesimal the vectors $\mathbf{v_2}$ and $\mathbf{v_1}-\mathbf{u}$ are infinitesimal too and can be written in differential form as $d\mathbf{v_2}$ and $d\mathbf{v_1}$ in equations (14), which we introduce into equation (13) to obtain equation (15), of which both sides are divided by dt_2 to make equation (16), in which the ratio dt_1/dt_2 is substituted by equation (8) to make equation (17), in which we substitute $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ for $\frac{d\boldsymbol{v}_1}{dt_2}$ and $\frac{d\boldsymbol{v}_2}{dt_2}$ to get equation (18), the relation between the vector accelerations, $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$, of the object q in frame 1 and 2.

$$d\boldsymbol{v}_2 = \boldsymbol{v}_2 - 0, \quad d\boldsymbol{v}_1 = \boldsymbol{v}_1 - \boldsymbol{u} \quad (14)$$

$$d\boldsymbol{v}_2 = \frac{d\boldsymbol{v}_1}{\sqrt{1-\frac{u^2}{c^2}}} \quad (15)$$

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1}\frac{dt_1}{dt_2} \quad (16)$$

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{1}{1-\frac{u^2}{c^2}}\frac{d\boldsymbol{v}_1}{dt_1} \quad (17)$$

$$\boldsymbol{\alpha}_2 = \frac{\boldsymbol{\alpha}_1}{1-\frac{u^2}{c^2}} \quad (18)$$

$\boldsymbol{\alpha}_1$ *and* $\boldsymbol{\alpha}_2$ *are vector accelerations of q in frame 1 and 2.*

d) Derivative of vectors

The velocity vector of q in frame 2 is the time derivative of its vector position $\mathbf{X}_2$, see equation (20). The derivative of $\mathbf{X}_2$ is derived in equation (21), where the coordinates (x_2, y_2, z_2) are written in terms of (x_1, y_1, z_1) the coordinates of q in frame 1 using system (1).

The vector position of q in frame 1 is $\mathbf{X}_1$, which is given in equation (19). We substitute $\mathbf{X}_1$ for the corresponding terms in equation (21), which then becomes equation (22). Since the derivatives of $\mathbf{X}_1$ and $\mathbf{X}_2$ are the vector velocity in frame 1 and 2, we write equation (22) into equation (25) which is the equation (13).

$$\boldsymbol{X}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k} \quad (19)$$

$$\boldsymbol{X}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k} \quad (20)$$

$$\frac{d\boldsymbol{X}_2}{dt_2} = \frac{d(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k})}{dt_2} = \frac{d\big((x_1-ut_1)\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}\big)}{dt_1}\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d(x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k} - ut_1\boldsymbol{i})}{dt_1} \quad (21)$$

$$\frac{d\boldsymbol{X}_2}{dt_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\frac{d\boldsymbol{X}_1}{dt_1} - \boldsymbol{u}\right) \quad (22)$$

$\boldsymbol{X}_1$ *and* $\boldsymbol{X}_2$ *are the vector positions of q in frame 1 and 2.*

We can derive the acceleration vector also by taking the time derivative of the vector velocity of q, see equation (23). Because frame 2 is inertial, **u** is constant. The derivative of $\mathbf{v_1}$ and $\mathbf{v_2}$ are the vector accelerations $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$, then, equation (23) becomes (26), which is the equation (18).

In summary, the transformation of vector position is equation (24), the time derivative of which is the transformation of vector velocity equation (25), the time derivative of which is the transformation of vector acceleration equation (26). We see that the transformations in Time relativity form a consistent vector system. In the contrary, the transformations in Special relativity do not. For example, the velocity-addition formula is not the time derivative of the Lorentz transformation.

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{d}{dt_2}\left(\frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1}\frac{dt_1}{dt_2} = \frac{1}{1-\frac{u^2}{c^2}}\frac{d\boldsymbol{v}_1}{dt_1} \quad (23)$$

$$\boldsymbol{X}_2 = \boldsymbol{X}_1 - \boldsymbol{u}t_1 \quad (24)$$

$$\left.\begin{aligned}\frac{d\boldsymbol{X}_1}{dt_1} &= \boldsymbol{v}_1\\ \frac{d\boldsymbol{X}_2}{dt_2} &= \boldsymbol{v}_2\end{aligned}\right\} \Rightarrow \boldsymbol{v}_2 = \frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}} \quad (25)$$

$$\left.\begin{aligned}\frac{d\boldsymbol{v}_1}{dt_1} &= \boldsymbol{\alpha}_1\\ \frac{d\boldsymbol{v}_2}{dt_2} &= \boldsymbol{\alpha}_2\end{aligned}\right\} \Rightarrow \boldsymbol{\alpha}_2 = \frac{\boldsymbol{\alpha}_1}{1-\frac{u^2}{c^2}} \quad (26)$$

2. Gravitation

a) Relativistic orbit

Warning: I have done a mistake in equation (27), which makes the following chapters wrong. I will give the correct theory later. But I keep this erroneous part below for showing a faithful historical image of my publications.

During small time intervals the gravitation of a mass M transfers momentum impulse to other mass. According to Newton's gravitational law, the impulse that M transfers to the mass m_0 during an infinitesimal time interval dt_1 in frame 1 is the $d\mathbf{P}_1$ which is defined by equation (27), with M and m_0 being the respective rest masses, r_M the distance from M to m_0, $\mathbf{e}_r$ the unit vector pointing from M to m_0 and G the gravitational constant.

$$d\boldsymbol{P}_1 = \left(-\frac{GM\cdot m_0}{{r_M}^2}\boldsymbol{e}_r\right)dt_1 \quad (27)$$

$$d\boldsymbol{P}_2 = m_0 d\boldsymbol{v}_2 = \frac{m_0 d\boldsymbol{v}_1}{\sqrt{1-\frac{u^2}{c^2}}} \quad (28)$$

$\mathbf{P}_1$ and $\mathbf{P}_2$ are the vector momentum of q in frame 1 and 2. G is the gravitational constant, M the mass of the attracting body, m_0 the mass of the orbiting body, $\boldsymbol{e}_r$ the unit radial vector, r_M the distance between M and m_0.

Let frame 2 be an inertial frame which moves momentarily with m_0. The impulse that m_0 feels in frame 2 is $d\mathbf{P}_2$ and equals $m_0 d\mathbf{v}_2$ with $d\mathbf{v}_2$ being the change of velocity of m_0 in frame 2, see equation (28), where we have substituted equation (15) for $d\mathbf{v}_2$.

The impulse on m_0 in frames 1 and 2 refer to the same impulse. So, $d\mathbf{P}_1 = d\mathbf{P}_2$, see equation (29). Then, we equate equation (27) with (28) and obtain equation (30) which we transform into equation (31).

$\frac{dv_1}{dt_1}$ is the acceleration of m_0 in frames 1, of which the magnitude is $-\alpha_R$, see equation (32). Combining Equations (31) and (32) we obtain the expression of α_R in equation (33).

$$d\boldsymbol{P}_1 = d\boldsymbol{P}_2 \quad (29)$$

$$-\frac{GM\cdot m_0\cdot dt_1}{{r_M}^2}\boldsymbol{e}_r = \frac{m_0 d\boldsymbol{v}_1}{\sqrt{1-\frac{u^2}{c^2}}} \quad (30)$$

$$-\frac{GM}{{r_M}^2}\boldsymbol{e}_r = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1} \quad (31)$$

$$\frac{d\boldsymbol{v}_1}{dt_1} = \boldsymbol{\alpha}_r = -\alpha_R\boldsymbol{e}_r \quad (32)$$

$$\alpha_R = \frac{GM}{{r_M}^2}\sqrt{1-\frac{u^2}{c^2}} \quad (33)$$

$$\alpha_N = \frac{GM}{{r_M}^2} \quad (34)$$

$$\alpha_R = \alpha_N\sqrt{1-\frac{u^2}{c^2}} \quad (35)$$

$\boldsymbol{\alpha}_r$ is the vector acceleration of m_0 on the relativistic orbit, $-\alpha_R$ the magnitude of $\boldsymbol{\alpha}_r$, $-\alpha_N$ the magnitude of the acceleration on the Newtonian orbit.

The theoretical orbit of m_0 in the gravitational field of M that obeys Newtonian law is called Newtonian orbit. The one that obeys Time relativity is called relativistic orbit. The orbital acceleration of the relativistic orbit is α_R, that of the Newtonian orbit is α_N which is given in equation (34), which we introduce in to equation (33) to express α_R in terms of α_N in equation (35).

Notice that the coefficient $\sqrt{1-\frac{u^2}{c^2}}$ in equation (35) comes from the conversion of time from frame 1 to 2 and has nothing to do with relativistic mass.

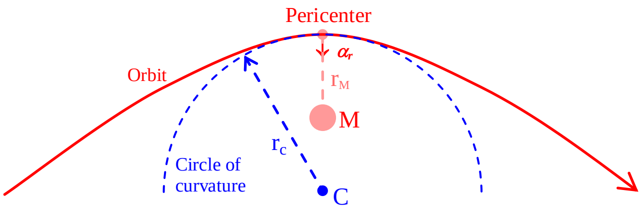

Figure 2

b) Pericenter of orbit

Pericenter is the point of an orbit that is closest to the attracting body. The center of curvature of the orbit at the pericenter is C and the radius of curvature is r_c which equal that of the circle of curvature at the pericenter, see Figure 2.

m_0 follows momentarily the circle of curvature at the pericenter. On a relativistic orbit, m_0's velocity and acceleration are u_R and α_R which are related by equation (36). On a Newtonian orbit, m_0's velocity and acceleration are u_N and α_N which are related by equation (37). Introducing equations (36) and (37) in to (35), we obtain equation (38), which is then transformed into equation (39) then (40). The solution of equation (40) is given in equation (41), with the negative solution been dropped off because $\frac{u_R^2}{c^2}$ is always positive. Equation (41) expresses the velocity of m_0 at the pericenter of its relativistic orbit in the gravitational field of M.

$$\frac{{u_R}^2}{r_c} = \alpha_R \quad (36)$$

$$\frac{{u_N}^2}{r_c} = \frac{GM}{{r_M}^2} \Rightarrow \frac{1}{{u_N}^2} = \frac{{r_M}^2}{GMr_c} \quad (37)$$

$$\frac{{u_R}^2}{r_c} = \frac{{u_N}^2}{r_c}\sqrt{1-\frac{{u_R}^2}{c^2}} \quad (38)$$

$$\left(\frac{{u_R}^2}{c^2}\right)^2 = \left(\frac{{u_N}^2}{c^2}\right)^2\left(1-\frac{{u_R}^2}{c^2}\right) \quad (39)$$

u_R is the magnitude of the orbital velocity of object on the relativistic orbit, u_N that on the Newtonian orbit, r_c the radius of curvature of the orbit at the pericenter.

$$\left(\frac{c^2}{u_N{}^2}\right)^2\left(\frac{u_R{}^2}{c^2}\right)^2+\frac{u_R{}^2}{c^2}-1=0 \quad (40)$$

$$\frac{u_R{}^2}{c^2}=\frac{\sqrt{1+4\left(\frac{c^2}{u_N{}^2}\right)^2}-1}{2\left(\frac{c^2}{u_N{}^2}\right)^2} \quad (41)$$

- u_R for large r_M

When r_M is very large such that $\left(\frac{c^2}{u_N{}^2}\right)^2$ is much larger than 1 (see equation (37)), the square root in equation (41) can be simplified as in equation (42), which makes equation (41) to become (43). We see that, when $r_M \to \infty$, $u_R{}^2$ reduces to the Newtonian solution while being always smaller than it, see equation (44).

$$\left(\frac{2c^2}{u_N{}^2}\right)^2 \gg 1 \Rightarrow \sqrt{\left(\frac{2c^2}{u_N{}^2}\right)^2+1} \approx \frac{2c^2}{u_N{}^2} \quad (42)$$

$$\frac{u_R{}^2}{c^2} \approx \frac{\frac{2c^2}{u_N{}^2}-1}{2\left(\frac{c^2}{u_N{}^2}\right)^2}=\frac{u_N{}^2}{c^2}-\frac{1}{2}\left(\frac{u_N{}^2}{c^2}\right)^2 \quad (43)$$

$$u_R \xrightarrow[r_M\to\infty]{} u_N \quad (44)$$

$$\left(\frac{2c^2}{u_N{}^2}\right)^2 \ll 1 \Rightarrow$$
$$\sqrt{1+\left(\frac{2c^2}{u_N{}^2}\right)^2} \approx 1+\frac{1}{2}\left(\frac{2c^2}{u_N{}^2}\right)^2-\frac{1}{8}\left(\frac{2c^2}{u_N{}^2}\right)^4 \quad (45)$$

$$\frac{u_R{}^2}{c^2} \approx \frac{\frac{1}{2}\left(\frac{2c^2}{u_N{}^2}\right)^2-\frac{1}{8}\left(\frac{2c^2}{u_N{}^2}\right)^4}{2\left(\frac{c^2}{u_N{}^2}\right)^2}=1-\left(\frac{c^2}{u_N{}^2}\right)^2<1 \quad (46)$$

$$u_R<c, u_R \xrightarrow[r_M\to 0]{} c \quad (47)$$

- u_R for small r_M

When r_M is very small such that $\left(\frac{c^2}{u_N{}^2}\right)^2$ is much smaller than 1, the square root can be simplified as in equation (45), which makes equation (41) to become equation (46), which shows that, when $r_M \to 0$, $\frac{u_R{}^2}{c^2}$ reduces to nearly 1 while being always smaller than 1. So, when r_M the distance between M and m_0 decreases towards zero, the orbital velocity of m_0 tends to the speed of light c without ever reaching it, see equation (47).

- Circular orbit

M coincides with C in Figure 2 and r_c=r_M in the above discussion.

3. Solar mass

The mass of the Sun is not measured by weighting, but derived from the parameters of Earth's orbit which is nearly circular. Let r_M be the radius of the Earth's orbit and u_E its orbital velocity. By equating the orbital acceleration of the Earth (see equation (36)) with its Newtonian gravitational acceleration (see equation (34)), we obtain equation (48) which gives the today's used mass of the Sun, M_0, in equation (49).

$$\frac{u_E{}^2}{r_M}=\frac{GM_0}{r_M{}^2} \quad (48)$$

$$M_0=\frac{r_M u_E{}^2}{G} \quad (49)$$

$$\frac{u_E{}^2}{r_M}=\frac{GM_R}{r_M{}^2}\sqrt{1-\frac{u_E{}^2}{c^2}} \quad (50)$$

r_M is the radius of the Earth's orbit , u_E its orbital velocity, M_0 the mass of the Sun according to Newton's law, M_R the mass of the Sun according to relativistic law.

According to Time relativity, the orbital acceleration of the Earth is given by equation (33). So, by equating equation (36) with (33), we obtain equation (50) which gives the mass of the Sun M_R in equation (51), which is derived using Time relativity.

$$M_R=\frac{r_M u_E{}^2}{G\sqrt{1-\frac{u_E{}^2}{c^2}}} \quad (51)$$

$$\frac{M_R}{M_0}=\frac{1}{\sqrt{1-\frac{u_E{}^2}{c^2}}} \approx 1+\frac{u_E{}^2}{2c^2}>1 \quad (52)$$

$$u_E=29.8\ \mathrm{km/s} \Rightarrow$$
$$\frac{M_R}{M_0}-1 \approx \frac{u_E{}^2}{2c^2}=4.94\times 10^{-9} \quad (53)$$

The ratio M_R/M_0 is given in equation (52), which shows that M_R is bigger than M_0, that is, the mass of the Sun derived using Time relativity is bigger than its today's value. The relative gap between M_R and M_0 is defined as $\frac{M_R}{M_0}-1$, of which the numerical value is computed in equation (53) using the orbital velocity of the Earth which is 29.8 km/s.

Since bigger mass makes bigger orbital velocity, the bigger mass M_R may explain the unexpected Speed Boost of 'Oumuamua.