

# Twin paradox when Earth is the moving frame

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**Abstract:** We analyze the mathematical mechanism that slows the time of the traveler in the twin paradox and explain what distinguishes the traveler's frame from the Earth's frame.

## 1. What is twin paradox?

Let Alice and Betty be two twins and Betty goes on a high-speed journey that brings her to a distant star then back on Earth, while Alice stays on Earth. When Alice sees Betty back, Betty will be younger than Alice.

Why time runs slow for Betty the traveler? Figure 1 shows the journey of Betty in the frame of the Earth, where S is the star at the distance  $l$  from the Earth. Betty travels toward S at the speed  $v$  and returns to the Earth at the speed  $-v$ . The outgoing journey makes Betty to travel the distance  $l$  and takes the time  $t_b = l/v$ , the return journey travels also the distance  $l$  and takes the time  $t_{b-c} = l/v$ , see equations (1) and (2). So, on Earth Alice sees the round journey of Betty takes the time  $t_{Alice} = 2l/v$  in total, see equation (3).

For computing the duration of the round journey in the frames of Betty, we use the time formula of the Lorentz transformation which is given by equations (4) and (5). An event in the frame of the Earth is represented by its coordinates  $(x, t)$ , with  $x$  and  $t$  being the position and time of the event. In the moving frame of Betty this same event is represented by the primed coordinates  $(x', t')$ . In Figure 1, the event  $a$  is the departure of Betty from the Earth, the event  $b$  is the arrival of Betty at the star S, the event  $c$  is the reunion of Betty with Alice on Earth.

In the frame of the Earth the event  $b$  is  $(x_b, t_b)$ , see equation (6) and (1). The time of this event in Betty's frame is computed by substituting  $x_b$  and  $t_b$  for  $x$  and  $t$  in equation (4) which gives  $t'_b$ , see equation (7).

On the return journey, Betty travels at the speed  $-v$  toward the Earth and the times of the events  $b$  and  $c$  must be computed using  $-v$  in equation (4). In the frame of the Earth the event  $b$  is  $(x_b, t_b)$  and the event  $c$  is  $(0, t_{Alice})$ , see equations (6), (1) and (3). Then, the times of events  $b$  and  $c$  in Betty's frame are  $t'_{b2}$  and  $t'_c$  computed in equations (8) and (9). The duration of the return journey in Betty's frame is the difference between the times of the events  $c$  and  $b$  which is  $\Delta t'$  computed in equation (10).

So, in Betty's frame the duration of the round journey is the sum of  $t'_b$  and  $\Delta t'$  which is  $t'_{Betty}$  computed in equation (11).

For comparing the duration of the round journey in Betty's frame with that in Earth's frame, we compute the ratio between  $t'_{Betty}$  and  $t_{Alice}$  which equals  $\sqrt{1 - \frac{v^2}{c^2}}$ , see equation (12). As this ratio is smaller than 1, time in Betty's frame runs slower than in Earth's frame and Betty would be younger than Alice at the reunion.

The above computation is done in Earth's frame and time in Betty's frame runs slow. If we compute in Betty's frame, could the duration in Earth's frame be shorter than in Betty's frame?

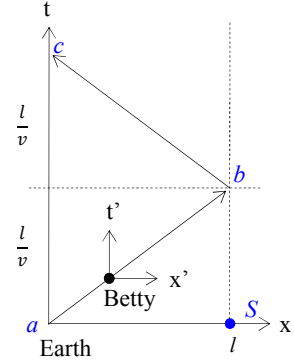


Figure 1

$$t_b = \frac{l}{v} \quad (1)$$

$$t_{b-c} = \frac{l}{v} \quad (2)$$

$$t_{Alice} = t_b + t_{b-c} = 2\frac{l}{v} \quad (3)$$

$$t' = \gamma \left( t - x \frac{v}{c^2} \right) \quad (4)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (5)$$

$$x_b = l \quad (6)$$

$$t'_b = \gamma \left( t_b - x_b \frac{v}{c^2} \right) = \frac{1}{\gamma} \frac{l}{v} \quad (7)$$

$$t'_{b2} = \gamma \left( \frac{l}{v} + l \frac{v}{c^2} \right) \quad (8)$$

$$t'_c = \gamma \left( 2\frac{l}{v} + 0 \right) \quad (9)$$

$$\Delta t' = t'_c - t'_{b2} = \frac{1}{\gamma} \frac{l}{v} \quad (10)$$

$$t'_{Betty} = \Delta t' + t'_b = 2\frac{l}{v} \sqrt{1 - \frac{v^2}{c^2}} \quad (11)$$

$$\frac{t'_{Betty}}{t_{Alice}} = \sqrt{1 - \frac{v^2}{c^2}} \quad (12)$$

## 2. In the frame of the traveler

For answering the above question, let us compute the duration of the round journey in Betty's frame where Betty is immobile and the Earth is moving, see Figure 2. On the outgoing journey which is represented by the straight  $ab$ , the Earth and the star S move to the left at the speed  $-v$ . On the return journey which is represented by the straight  $bc$ , the Earth and the star S move to the right at the speed  $v$ .

In Earth's frame the distance between the Earth and S is  $l$ . In Betty's frame the straight line Earth-S is moving and  $l$  is length-contracted to  $l'$ , see equation (13).

On the outgoing journey, the star S will reach Betty after traveling the distance  $l'$  in time  $t_s$ , see equation (14). Then, on the return journey the Earth will reach Betty after a journey of distance  $l'$  and time  $t_e$ , see equation (15). The total time taken by the Earth for the round journey is the sum of  $t_s$  and  $t_e$ , which is  $t_{\text{Betty}}$  computed in equation (16).

In Earth's frame the duration of this journey is still  $t_{\text{Alice}}$  given by equation (3). So, the ratio of times in Betty's frame and Earth's frame is  $\sqrt{1 - \frac{v^2}{c^2}}$  again, see equation (17). It is still in Betty's frames that time runs slow. Why?

The explanation given by the video [Twin paradox: the real explanation](#) from Fermilab's channel is that Betty passes through 2 frames in the round journey while the Earth stays in only one frame. But in Betty's frame, the Earth is moving and passes through 2 frames, while Betty is in only one frame. So, it is not the changing of frame that distinguishes Betty's frame from the Earth's frame.

## 3. One unique traveling frame

In order to find what distinguishes Betty from the Earth, let us see the case shown in Figure 3 where the traveler makes a straight line journey without changing direction, which makes the traveler stay in only one inertial frame.

The traveler of this straight journey is Betty 2 who starts from the location  $S_2$  which is at the distance  $2l$  from the Earth. Betty 2 travels toward the Earth at the speed  $-v$ . The distance between  $S_2$  and S is  $l$ , so, the journey of Betty 2 from  $S_2$  to S takes the time  $l/v$ , which is identical to that of the outgoing journey of Betty represented by the straight line  $ab$ .

The straight journey from  $S_2$  to the Earth is represented by the straight line  $S_2c$ , the round journey of Betty is represented by the straight lines  $ab+bc$ . These two journeys are equivalent in that the durations and traveled distances are identical, except that Betty 2 stays in one unique inertial frame while Betty changes frame at S.

In the frame of the Earth the journey of Betty 2 takes the time  $t_{S_2}$ , see equation (18). In the frame of Betty 2, the distance between the Earth and  $S_2$  is length-contracted to  $2l'$  (equation (19)) and the Earth travels this distance in time  $t''_c$ , see equation (20). So, the ratio of times in the frames of Betty 2 and

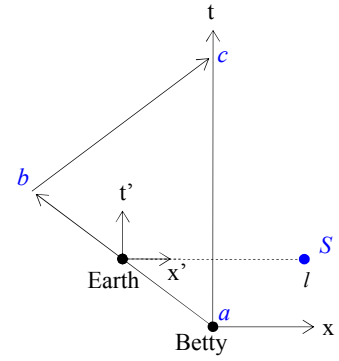


Figure 2

$$l' = \frac{l}{\gamma} \quad (13)$$

$$t_s = \frac{l'}{v} = \frac{l}{\gamma v} \quad (14)$$

$$t_e = \frac{l'}{v} = \frac{l}{\gamma v} \quad (15)$$

$$t_{\text{Betty}} = t_s + t_e = 2 \frac{l}{\gamma v} \quad (16)$$

$$\frac{t_{\text{Betty}}}{t_{\text{Alice}}} = \sqrt{1 - \frac{v^2}{c^2}} \quad (17)$$

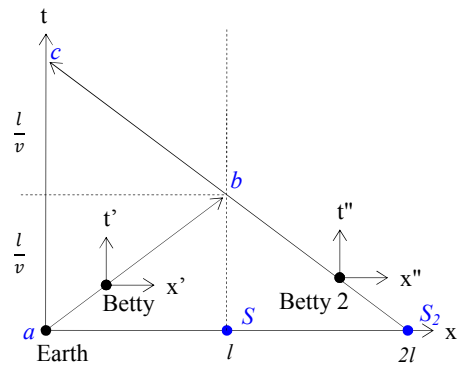


Figure 3

the Earth is  $t''_c / t_{s2}$ , which equals  $\sqrt{1 - \frac{v^2}{c^2}}$ , see equation (21). This ratio equals that of the twin paradox shown in equation (12).

Remark: Since Betty 2 starts from  $S_2$  but not from the Earth, she cannot see the time of departure on Earth. So, we will put a clock at  $S_2$  which is synchronized with a clock on Earth. When Betty 2 starts from  $S_2$ , she notes the time shown on the clock at  $S_2$ . Once arrived on Earth, she notes the time on the clock on Earth. By subtracting the 2 times, she obtains the duration of the straight journey seen from the Earth.

$$t_{s2} = \frac{2l}{v} \quad (18)$$

$$2l' = \frac{2l}{\gamma} \quad (19)$$

$$t''_c = \frac{2l'}{v} = \frac{2l}{\gamma v} \quad (20)$$

$$\frac{t''_c}{t_{s2}} = \sqrt{1 - \frac{v^2}{c^2}} \quad (21)$$

Now, Betty 2 is in one unique inertial frame, the Earth is also in one inertial frame. But time still runs slow for Betty 2. Why? Because in Betty 2's frame the distance between the Earth and  $S_2$  is length-contracted and the Earth travels a shorter distance. So, the same journey seen from Betty 2's frame is shorter in time.

For twin paradox, the distance between the Earth and S is length-contracted in Betty's frames for the outgoing journey and the return journey, which makes the round journey shorter seen from Betty's frames. So, it is the length contraction that causes time to run slow for the traveler.

The fact that length contraction shortens time is well proven by experiment. For example, Steve Adams<sup>[1]</sup> cited that the journey of muons from the top of the atmosphere to the surface of the Earth take 200  $\mu$ s on Earth, which is 133 times the lifetime of muons. If this time were true for the muons, no muon could survive till Earth's surface. But since muons travel near the speed of light, the height of the atmosphere is length-contracted such that the journey of the muons last only 4.5  $\mu$ s in their frame and large quantity of muons reach the surface of the Earth.

However, because special relativity is symmetrical, muons should see time running slow on Earth too. Since this is not the case, the symmetry of special relativity seems to be broken.

#### 4. Making time run slow on Earth

What makes that Betty's time must run slow rather than Earth's time? To answer this question, let us imagine a situation where time runs slow on Earth. In Figure 4, we stay in the frame of Betty where the Earth moves to the left at the speed  $-v$ . In this situation, Betty has a sister spaceship named R on the left. The motion of R is synchronized with the motion of Betty. So, in the frame of Betty the distance between Betty and R is  $l$  and is constant.

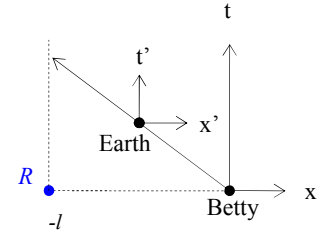


Figure 4

In the frame of Betty the Earth moves to the left and will reach R after the time  $t_R$ , see equation (22). Since Betty and R are moving in the frame of the Earth, the distance  $l$  is length-contracted to  $l'$ , see equation (23). So, the Earth will travel the contracted distance  $l'$  in the time  $t'_{Earth}$ , see equation (24). The ratio of times of the journey in Earth's frame and Betty's frame is  $t'_{Earth} / t_R$ , see equation (25), which is  $\sqrt{1 - \frac{v^2}{c^2}}$  and smaller than 1. Then, time runs slow in Earth's frame.

$$t_R = \frac{l}{v} \quad (22)$$

$$l' = \frac{l}{\gamma} \quad (23)$$

$$t'_{Earth} = \frac{l}{\gamma v} \quad (24)$$

$$\frac{t'_{Earth}}{t_R} = \sqrt{1 - \frac{v^2}{c^2}} \quad (25)$$

What is the difference between this situation and the twin paradox? Here, the Earth encounters the spaceship R, while in the twin paradox Betty encounters the star S. We notice that time runs slow in the frame of the object that encounters a third object.

In fact, the encounter with a third object can be any event. If A and B are 2 objects moving relative to one another, B is moving in the frame of A and A is moving in the frame of B and we cannot tell in which frame time runs slow. But if an event C occurs on B, for example B explodes or a moving muon

decays, then, time runs slow for B or for the muon. If the event C occurs on A, then time runs slow for A. So, it is what happens to A or B that determines which object's time runs slow.

Why does a third event other than A or B make which object's time runs slow? For knowing about any event, its location in space should be determined with respect to an object. If an event occurs on B, its location in the frame of B is always  $x=0$ . The only meaningful coordinates of this event are the distance of B from A and the time, both measured in the frame of A. As this distance is length-contracted in the frame of B, the duration of the journey in the frame of B is shorter than in the frame of A. Thus, time runs slow in the frame of B. It is the frame in which the third event is measured that determines in which frame time runs slow.

In the twin paradox Betty encounters the star S on the outgoing journey and Earth on the return journey, so Betty's time runs slow for the round journey. In the case of muons, the muons encounter the surface of the Earth, so time runs slow for the muons but not for the Earth.

## 5. Comments

- 1) We have shown that it is the length contraction that makes time run slow for the traveling twin, not her acceleration.
- 2) We have shown that the shortening of time occurs in the frame of an object because something happens to it, not because it passes through 2 or more different frames.
- 3) Since a third event makes time run slow, if no event occurs to any of the 2 relatively moving objects, can we tell which object's time runs slow?

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Reference:

- 1) «Relativity: An Introduction to Spacetime Physics» by Steve Adams