Radius of a black hole for relativity and Newtonian mechanics

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1. For Newtonian mechanics

The Schwarzschild radius is the radius of the event horizon of a black hole. Amazingly, we can compute it with Newtonian mechanics, which is explained below. Consider a big mass M which creates the gravitational acceleration α for a small mass m at the distance r from M, see figure 1 and equation (1).

$$\alpha = \frac{GM}{r^2} \quad (1)$$

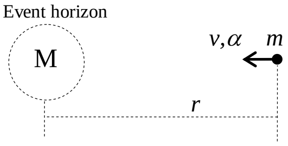

Figure 1

For computing v the radial velocity of m in the gravitational field of M we integrate equation (1). For doing so, we write the equations (2).

$$v = \frac{dr}{dt}, \quad \alpha = \frac{dv}{dt}, \quad v\alpha = v\frac{dv}{dt} = \frac{1}{2}\frac{dv^2}{dt} \quad (2)$$

Applying (2) into (1) we obtain (3).

$$\text{(1) and (2)} \Rightarrow \quad v\alpha = \frac{GM}{r^2}\frac{dr}{dt} = \frac{1}{2}\frac{dv^2}{dt} \quad (3)$$

Equation (4) integrates (3).

$$\int \frac{1}{2} dv^2 = \int \frac{GM}{r^2} dr \quad (4)$$

The right and left hand sides of (4) give (5), then (6).

$$\text{(4)} \Rightarrow \quad \int_1^2 \frac{1}{2} dv^2 = \frac{{v_2}^2 - {v_1}^2}{2}, \quad \int_1^2 \frac{GM}{r^2} dr = \frac{GM}{r_2} - \frac{GM}{r_1} \quad (5)$$

$$\text{(4) and (5)} \Rightarrow \quad \frac{{v_2}^2 - {v_1}^2}{2} = \frac{GM}{r_2} - \frac{GM}{r_1} \quad (6)$$

We compute for the case where m freefalls from infinitely far starting with zero velocity, see (7).

$$r_1 = \infty, \quad v_1 = 0 \quad (7)$$

With these conditions the radial velocity of m at the distance r_2 from M equals the v_2 computed in (8) from (6).

$$\text{(6) and (7)} \Rightarrow \quad \frac{{v_2}^2}{2} = \frac{GM}{r_2} \quad (8)$$

Reversing (8) r_2 is express with v_2 in (9).

$$\text{(8)} \Rightarrow \quad r_2 = \frac{2GM}{{v_2}^2} \quad (9)$$

The Schwarzschild radius of the event horizon of M is r_s such that the Schwarzschild factor equals infinity, see (10).

$$\text{Schwarzschild radius} \quad \frac{1}{\sqrt{1 - \frac{2GM}{c^2 r_s}}} = \infty \rightarrow r_s = \frac{2GM}{c^2} \quad (10)$$

When v_2 equals the speed of light c, we apply $v_2 = c$ into (9) and we obtain (11) where $r_2 = r_s$. So, r_2 equals r_s and the Schwarzschild radius is computed with Newtonian mechanics.

$$\text{(9) and } v_2 = c \;\Rightarrow\; r_2 = \frac{2GM}{c^2} = r_s \quad (11)$$

2. For relativity

Although the Schwarzschild radius r_s is a relativistic quantity, in the above it is derived completely with Newtonian mechanics, which is somewhat weird. What will be its value if we apply relativistic principle?

In the following derivation we will use the formula for relativistic transformation of acceleration which is derived in the paper « Relativistic kinematics » linked here: https://www.academia.edu/44582027/Relativistic_kinematics
The formula is the equation (18) of the paper.

Here, this formula is given by (12) in which the gravitational acceleration of m is α and the acceleration in space is α_r.

$$\alpha = \frac{\alpha_r}{1-\frac{v^2}{c^2}}, \quad \alpha_r = \frac{dv}{dt} \quad \Rightarrow \alpha = \frac{1}{1-\frac{v^2}{c^2}}\frac{dv}{dt} \quad (12)$$

We apply (12) into (1) and obtain (13).

$$\text{(12) and (1)} \Rightarrow \quad \frac{1}{1-\frac{v^2}{c^2}}\frac{dv}{dt} = \frac{GM}{r^2} \quad (13)$$

We multiply (13) with v to obtain (14).

$$(13) \Rightarrow \quad \frac{vdv}{1-\frac{v^2}{c^2}} = \frac{GM}{r^2}vdt \Rightarrow \frac{1}{2}\frac{dv^2}{1-\frac{v^2}{c^2}} = \frac{GM}{r^2}dr \quad (14)$$

The integration of (14) gives (15) where k is the constant of integration.

$$d\,ln\left(1-\frac{v^2}{c^2}\right) = -d\left(\frac{2GM}{c^2 r}\right) \Rightarrow 1-\frac{v^2}{c^2} = e^{-\frac{2GM}{c^2 r}+k} \quad (15)$$

With the same conditions as (7), the constant of integration k equals 0, see (16).

$$r = \infty,\, v = 0 \;\Rightarrow\; k = 0 \quad (16)$$

Then, using k = 0 in (15), v is expressed with r in (17).

$$\text{(15) and } k = 0 \Rightarrow \quad 1-\frac{v^2}{c^2} = e^{-\frac{2GM}{c^2 r}} \quad (17)$$

In the case where the small mass m approaches M, the distance r approaches 0, the radial velocity of m approaches the speed of light c, see (18).

$$\text{(17) and } r \to 0 \Rightarrow \quad 1-\frac{v^2}{c^2} = e^{-\frac{2GM}{c^2 r}} \to 0 \Rightarrow v^2 \to c^2 \quad (18)$$

So, v the radial velocity of m does not become bigger than the speed of light c for r > 0. The Schwarzschild radius r_s is the radius such that v = c. So, $r_s = r = 0$, see (19).

$$\text{Radius of a black hole: (18)} \Rightarrow \quad v^2 = c^2 \Rightarrow r = 0 \Rightarrow r_s = 0 \quad (19)$$

3. Conclusion

When the relativistic principles are applied correctly, the Schwarzschild radius r_s equals zero. The gravitational force on m approaches infinity near M. But the speed of m never reaches the speed of light c, which is true for any force however strong it is and for time of acceleration however long it is. The Schwarzschild radius r_s must obey relativistic principle and is shown to be zero.

That the Schwarzschild radius r_s equals zero means that the geometrical size of a black hole should be zero and thus, a black hole should not have an interior. I have reached this conclusion and explained it in the paper « Gravitational time dilation and black hole » in which I have also shown that point masses could not coalesce to form a black hole whose volume is zero and that observations support this conclusion. This paper is linked here:
https://www.academia.edu/45434676/Gravitational_time_dilation_and_black_hole

4. Comment

What is surprising is that Newtonian mechanics gives the exact value of the Schwarzschild radius. There surely is a reason for that. I find that in the equivalence principle, Einstein used Newtonian gravitation constant and acceleration when he imagined construction worker falling with his tools and both stayed in the same inertial frame.

So, the reason that Newtonian mechanics gives the exact value of the Schwarzschild radius is that his conception of equivalence principle used Newtonian mechanics, then Schwarzschild radius is the result of this principle and cannot be other value but Newtonian Schwarzschild radius.

This is an example of mixing relativistic principle with Newtonian mechanics which gives forcefully dubious result.

The point of this paper is that in relativity, any velocity must be less than the speed of light and thus, gravitational acceleration should decrease as the speed of the falling mass increases such that the final velocity does not reach the speed of light. So, my result that $v = c$ at the position $r = 0$ violates no physical principle and make the physical theory valid inside black holes, contrary to GR which crushed physics inside black holes because for $r <$ Newtonian Schwarzschild radius $v > c$.

For more detailed information, I invite you to read the two cited papers:
« Gravitational time dilation and black hole » https://www.academia.edu/45434676/Gravitational_time_dilation_and_black_hole
« Relativistic kinematics » https://www.academia.edu/44582027/Relativistic_kinematics