Analytical equation for Space-Time geodesics and relativistic orbit equation

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Abstract: Orbits for General Relativity are Space-Time geodesics which are extremely difficult to understand. This article exposes an analytical orbit equation for relativistic gravity and explains how it is derived without using Space-Time curvature. This orbit equation is an analytical expression of Space-Time geodesics because the orbital precession it gives is identical to the one given by Albert Einstein. Analytical geodesic is so simple that using personal computer everyone can compute the orbit of space object for General Relativity. Its direct derivation from gravitational acceleration gives a new insight into General Relativity without the need of knowing Einstein tensor.

1. Orbit equation and orbital precession

General Relativity explains gravity as Space-Time curvature and orbits of space objects as Space-Time geodesics. This concept is extremely hard to understand and geodesics hard to compute. However, if we can express Space-Time geodesics in analytical form, that is, as orbit equation like Newtonian orbit equation for planets, relativistic gravity will be much simpler that most people can understand.

From gravitational and inertial accelerations, I have derived the orbit equation for relativistic gravity, see equation (58), of which I will explain the derivation below. I will refer this equation as the relativistic orbit equation. Albert Einstein had correctly predicted the orbital precession of planet Mercury which had definitively validated General Relativity. The orbital precession that this orbit equation gives is identical to the one Albert Einstein had given [1][2], see equation (66).

Because the relativistic orbit equation gives the same result than Space-Time geodesics, it is an analytical expression of Space-Time geodesics which is so simple that one can use personal computer rather than big or super computer to compute geodesics. The direct derivation from gravitational and inertial accelerations is a new insight into General Relativity.

2. Relativistic dynamics

a) Velocity in local frame

Figure 1 shows a body of mass m orbiting around a body of mass M which sits at the origin of the polar coordinate system. The position of m with respect to M is specified by the radial position vector $\boldsymbol{r}$, of which the magnitude is r and the polar angle is θ.

Let frame_m be an inertial frame of reference that instantaneously moves with m. Newton's laws apply in frame_m because m's velocity is 0, so the inertial acceleration vector of m $\boldsymbol{\alpha}_m$ equals the gravitational acceleration $\boldsymbol{\alpha}_g$, see (1).

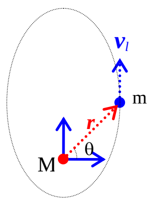

Figure 1
m orbits at velocity v_l around M. r, θ: polar coordinates

Notation convention

Bold letter: vectors.
Number in parentheses (i): equation number

Subscript:
- `0`: A known point
- `m`: Proper frame of m
- `l`: Local frame
- `∞`: Frame at infinitely far
- `N`: Newtonian quantity
- `a`: Near perihelion
- `b`: Away from perihelion

Let frame_l be the local frame of reference in which M is stationary, m move at the velocity $\boldsymbol{v}_l$ and the acceleration of m is $\boldsymbol{\alpha}_l$. As frame_m moves with m, its velocity is $\boldsymbol{v}_l$ in frame_l. We transform $\boldsymbol{\alpha}_l$ into $\boldsymbol{\alpha}_m$ using the transformation of accelerations between relatively moving frames, see equation (18) in « Relativistic kinematics »[3][4]. This transformation is in (2) where we have replaced $\boldsymbol{\alpha}_1$ with $\boldsymbol{\alpha}_l$, $\boldsymbol{\alpha}_2$ with $\boldsymbol{\alpha}_m$ and $\boldsymbol{u}$ with $\boldsymbol{v}_l$.

Equating (1) with (2) we get (3), both sides of which are dotted with the vector $\boldsymbol{v}_l \cdot dt_l$ in (4) to give $d\left(\frac{GM}{r}\right)$ on the left hand side, see (5), and $-\frac{c^2}{2}d\,ln\left(1-\frac{v_l{}^2}{c^2}\right)$ on the right hand side, see (6). Plugging (5) and (6) into (4) gives (7), which we integrate to give (8), with K being the integration constant. The value of e^K is given in (9). Rearranging (8) gives (10), which expresses the local orbital velocity $\boldsymbol{v}_l$ due to the gravitational field of M.

b) Velocity in the frame at infinitely far

The orbit of a planet is the curve drawn by its position with respect to a frame in which the gravitational effect of M is 0. This frame is infinitely far from M and is labeled frame_∞. In this frame the time is t_∞, the radial position vector is $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}_\infty$ equals the derivative of $\boldsymbol{r}$ with respect to t_∞, see (11).

The velocity vector $\boldsymbol{v}_l$ in frame_l equals the derivative of the same $\boldsymbol{r}$ with respect to the time in frame_l, t_l. We transform $\boldsymbol{v}_l$ into the product of $\boldsymbol{v}_\infty$ and $\frac{dt_\infty}{dt_l}$ in (12). $\frac{dt_\infty}{dt_l}$ is the time dilation factor of the Schwarzschild metric around M, see (13). By plugging (13) into (12) we obtain the relation between $\boldsymbol{v}_l$ and $\boldsymbol{v}_\infty$ in (14). Substituting (14) for $\boldsymbol{v}_l$ in (10) gives (15), which expresses $\boldsymbol{v}_\infty$ with the gravitational field of M. In the case of planets, $\frac{GM}{c^2r}$ is very small and the term $e^{-\frac{2GM}{c^2r}}$ is simplified into the Taylor's series in (16). Substituting (16) for $e^{-\frac{2GM}{c^2r}}$ in (15) gives (17), from which we drop out the third order term $4\left(\frac{GM}{c^2r}\right)^3$ to get $\frac{\boldsymbol{u}_\infty{}^2}{c^2}$ in (18), where $\boldsymbol{u}_\infty$ is the velocity vector of m in frame_∞ correct to the 2nd order of $\frac{GM}{c^2r}$.

Notice that the right hand side of (15) is the product of two parentheses, the first represents gravitational acceleration, see (10), the second represents the time dilation of Schwarzschild metric, see (13).

c) Differential equation for orbit

The general expression of velocity in polar coordinates is given in (19) and the velocity squared $\boldsymbol{u}_\infty{}^2$ in (20). For expressing $\boldsymbol{u}_\infty{}^2$ with h the constant of Kepler's second law given in (21) [5][6], we transform $\frac{dr}{dt}$ in (22), which with (21) are plugged into (20) to give (23), which expresses $\boldsymbol{u}_\infty{}^2$ in terms of $\frac{h}{r}$ instead of r. We divide (23) with c^2 to express $\frac{\boldsymbol{u}_\infty{}^2}{c^2}$ in terms of $\frac{h}{cr}$, see (24). For transforming (18), we plug (24) and $\frac{GM}{c^2r} = \frac{GM}{hc}\frac{h}{cr}$ into (18) to get (25) and (26), which is the differential equation for orbit in frame_∞. Solving (26) will give the relativistic orbit equation.

3. Relativistic orbit

a) Solving the differential equation for orbit

We transform (26) into (27), the right hand side of which is a messy polynomial of degree 2. So, we replace the coefficients with constants A, B and C which are expressed in (28). Then (27) is transformed into the much simpler (30). Here, we do the change of variable defined in (31) for s and (32) for $\frac{ds}{d\theta}$, with which we transform (30) into the simple differential equation (34), which is integrated into (36), with θ_0 being the integration constant.

Equation (36) contains several solutions of the differential equation (34). We choose the one with $\theta_0=0$ and negative sign. Taking the sine of this solution, we obtain (37). Then, for returning to the original variable we substitute (31) for s in (37) and obtain (39) which is the relativistic orbit equation expressed with A, B and C. For getting rid of A, B and C, we use their expressions given in (28) to express the terms $\frac{h}{Ac}$ and $\frac{B}{A^2}$ in (41) and (42), which are plugged into (39) to obtain the final equation for r in (43). Equation (43) is the relativistic orbit equation and expresses r in terms of θ, G, M, c, K and h.

b) Orbit as conic section curve

Orbits of planets are usually expressed as conic section curve. Let us express the relativistic orbit equation with the parameters of conic section which are l: the semi-latus rectum, a: semi-major axis, ε: eccentricity [6]. General conic section curve in polar coordinates is the function (44). By comparing (39) with (44), we find the l, ε and ϕ of the orbit, which are expressed in (45). Then, the relativistic orbit equation is expressed as a conic section curve in (46).

For expressing the constant C with conic section parameters, we compare (46) with (43) and find the expression of l in (47), which we transform in (48) to express $\left(\frac{hc}{GM}\right)^2$. Plugging (48) in the expression of C in (28) gives (49). For expressing the term $\frac{e^K}{2e^K-1}$, we take the square of ε in (50) to express $\frac{B}{A^2}$, see (45). We find another expression of $\frac{B}{A^2}$ in (52). Equating (50) with (52) gives the expression of $\frac{1-e^K}{2e^K-1}$ in (55), which allow us to express $\frac{e^K}{2e^K-1}$ in (56).

Plugging (56) into (49) gives the expression of C in (57), which is then introduced into (46) to obtain (58), which is the final relativistic orbit equation expressed with conic section parameters. For making sure that (58) is correct, we check its validity with 3 cases.

4. Check of the orbit equation

a) Orbital precession

The first check is to see if (58) gives the correct orbital precession for planets. Due to General Relativistic effect elliptic orbit rotates in space, which is materialized by the angle that the major axis of elliptic orbit rotates from one revolution to the next. We denote this angle with $\Delta\Phi$ and call it the angle of precession per turn.

For computing $\Delta\Phi$, let us suppose a first turn of orbit starts at the polar angle θ=0 or 2π with the radial distance $r_0 = a(1-\varepsilon^2)$, see (58) and (59). The second turn starts at the exact end of the first turn where the polar angle is θ_1 such that the radial distance is r_0 again. So, $\sin\left(\theta_1\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right) = 0$ too, see (60). Then, $\theta_1 = 2\pi\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{\frac{1}{2}}$, see (61). The polar angle between the starting point of the second turn and that of the first turn at θ=2π is $\Delta\Phi = \theta_1 - 2\pi$, see (62). So, seeing each turn as an ellipse, the major axis of the second ellipse has rotated the angle $\Delta\Phi$ with respect to the first ellipse.

For $\left|\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right| \ll 1$, we simplify (61) the expression of θ_1 into the Taylor's series in (63) and get the value of $\Delta\Phi$ in (64), which is the general expression of the angle of precession per turn. For planets $\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ because $1-\frac{GM}{ac^2} \approx 1$, see (65). The absolute value of $\Delta\Phi$ is $\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$, which is identical to the one Albert Einstein had found [1], see (66).

Because the relativistic orbit equation (58) gives the correct orbital precession, it would give the same curve in space than geodesics and analytically express the geodesics of Schwarzschild metric.

b) Variation of orbital velocity

The second check is to see if my theory of relativistic gravity predicts correctly the rate of variation of orbital velocity. The method of this check is to compare the rates given by my theory of relativistic gravity with that given by Newtonian gravity. So, we express orbital velocity to the first order in (67) by multiplying (18) with c^2 and dropping out the second order term $6\left(\frac{GM}{c^2 r}\right)^2$.

For computing the variation of orbital velocity we take 2 points on the orbit, a and b, compute the velocities at these 2 points, $\boldsymbol{u}_a$ and $\boldsymbol{u}_b$. The difference of these 2 velocities squared is computed with my relativistic gravity in (68). When we make $1-2e^K = -1$ in (68), we get that for Newtonian gravity, see (69). We compare these 2 values by dividing (68) with (69) in (70) where the resulting ratio is expressed with e^K. The comparison is done for hyperbolic orbit, for which the constant e^K is given for $r_0=\infty$ in (71). Plugging (71) into (70) gives (72) which shows that the resulting ratio is smaller than 1.

For real observation, we measure $\boldsymbol{u}_a$ the velocity of a celestial object at point a, with respect to which we compute the differences of velocity squared for relativistic gravity as well as for Newtonian gravity, see (73). Then, we get the ratio $\frac{(\boldsymbol{u}_b{}^2)_{Relativity}-\boldsymbol{u}_a{}^2}{(\boldsymbol{u}_b{}^2)_{Newtonian}-\boldsymbol{u}_a{}^2}$ in (74), which is shown to be smaller than 1. For an object that leaves perihelion, its velocity decreases and the difference of velocity squared is negative, see (75). By multiplying (74) with the negative value $(\boldsymbol{u}_b{}^2)_{Newtonian} - \boldsymbol{u}_a{}^2$, we obtain (76) and (77), which shows that the $\boldsymbol{u}_b$ given by my relativistic gravity is bigger than that given by Newtonian gravity.

Was any hyperbolic orbit velocity really measured? Yes. Fortunately, we have 2 real observations of bigger than expected velocity. The first is Oumuamua[7] which has got an unexpected "boost in speed" or a bigger than expected velocity when leaving the perihelion. Equation (77) gives an explanation to the "boost in speed". The second is Pioneer anomaly[8], which shows that the velocities of these probes are bigger than expected at a second point of measurement. This anomaly is also explained by (77). A third candidate with hyperbolic orbit is 2I/Borissov[9]. But there is no information about the rate of variation of its velocity.

As 2 real observations support the result given in equation (77), my relativistic gravity has successively passed the check of the rate of variation of orbital velocity. As the observed velocities are bigger than the expected ones, the latter were possibly computed with Newtonian gravity or the ones computed using General Relativity does not match observation.

c) Direction of precession

The third check is to see if my relativistic gravity predicts correctly the direction of precession. Today, the major axis of elliptic orbit is thought to rotate in the same direction as the velocity of the planet, as shown in Figure 2. However, equation (65) shows that $\Delta\Phi$ is negative, that is, the major axis rotates a negative angle in the polar coordinate system.

I call the negative $\Delta\Phi$ negative precession, which I explain with the 2 ellipses in Figure 3. The first ellipse is vertical, the second is tilted to the right. An object with counterclockwise velocity begins an orbit on the first ellipse from the polar angle θ=0. At the end of the orbit, it arrives at the polar angle $\theta_1<2\pi$ on the second ellipse. Because θ_1 is at the right of θ=0, the major axis has done a clockwise rotation against the velocity of the object, in contradiction with the current image of orbital precession.

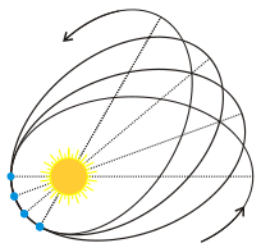

Figure 2
Today's view of precession: ellipse of orbit rotates in the same direction as the planet (Image Wikipedia)

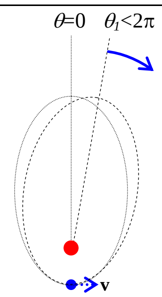

Figure 3

Negative precession

Is the negative $\Delta\Phi$ correct? For answering this question, let us compute the orbital velocity of a circular orbit of radius r on which the orbital acceleration α is expressed with the orbital velocity v in (78). For Newtonian gravity, gravitational acceleration equals the orbital acceleration α_N, see (79). Equating (78) with (79) while replacing v with v_N and α with α_N, we get the Newtonian orbital velocity squared v_N^2 in (80).

For relativistic gravity, the orbital acceleration is α_l in (81) with v_l being the local orbital velocity, see (3). By substituting (78) for α_l in (81) and replacing v with v_l, we get (82), which gives the local orbital velocity squared v_l^2 in (83). For comparing the relativistic and Newtonian orbital velocities, we divide (80) with (83) in (84), which shows that v_l is smaller than v_N, see (85). So, for a circular orbit relativistic gravity gives a velocity smaller than that given by Newtonian gravity.

What the velocity of a circular orbit can tell about precession of elliptic orbit? Let us see Figure 4 which shows a circular orbit with a similar but slightly elliptical orbit. Suppose that a body m and a body n do a counterclockwise turn on the circular orbit starting from the point P_0. m moves at the velocity v_N while n moves at v_l. Because v_l is smaller than v_N, n is slower than m. Then when m is back at the point P_0, n arrives only at the point P_r.

Now, let m and n do the slightly elliptical orbit starting from P_0. When m is back at the point P_0, n arrives only at a point near P_r. At the start the major axis of the ellipse passed across P_0 and was vertical. But at the end, the major axis passes across P_r and tilts to the right, which makes the angle $\Delta\Phi$ negative. That is, the major axis does a clockwise rotation against the velocity of n. In consequence, as n is slower than m the elliptical orbit begins earlier each turn, resulting in a negative precession.

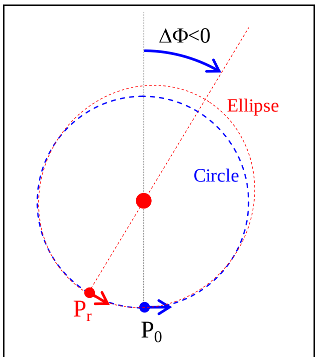

Figure 4
In one circular orbit, planet goes from P_0 to P_0 for Newtonian gravity, but from P_0 to P_r for relativistic gravity. On the elliptic orbit planet arrives near P_r making the ellipse to rotate clockwise.

Why v_l is slower than v_N? If an object gets a quantity of kinetic energy ΔE, it will move at a higher velocity and the increased part of velocity Δv carries the kinetic energy ΔE. For Newtonian laws, the velocity increase is Δv_N while its mass stays the same. For the same quantity of energy increase ΔE, according to Special Relativity the mass of the object is bigger than before, which makes the velocity increase Δv_l smaller than Δv_N. So, relativistic gravity gives a slower orbital velocity due to the effect of Special Relativity, which supports the negative $\Delta\Phi$ in Figure 4 and negative precession.

For General Relativity orbits are geodesics which are not related to velocity or energy. If geodesics respect Special Relativity, then the velocity on geodesics would be slower than the Newtonian one and thus, General Relativity would probably give negative precession too. If General Relativity really gave positive precession, for which I think no one has ever done the computation, the velocity on geodesics should be faster than the Newtonian one and this way General Relativity would violate Special Relativity. As my relativistic gravity gives negative precession in agreement with Special Relativity, it has successively passed the theoretical check of direction of precession.

However, the reasoning with orbital velocity is a theoretical explanation. For really checking negative precession, we have to search in the observations of celestial bodies, such as in the old data of the planet Mercury, in the recent data of the star S_2 that orbits the supermassive black hole Sagittarius A* at the center of Milky Way. We can also use the navigation data of Parker Solar Probe[10] which will reach 200 km/s near the Sun.

5. Comments

What are the benefits of the new theory of relativistic gravity with respect to General Relativity? Geodesics of curved Space-Time are so obscure that one cannot see how velocity and energy behave on orbit. In the contrary, the general equation for Space-Time geodesics (which is the relativistic orbit equation) is derived from gravitational and inertial acceleration, which gives an intuitive way to see Space-Time geodesics with velocity and energy and makes relativistic gravity easy to understand. Also, the relativistic orbit equation is more powerful for studying gravity because its analytical form allows us to easily get many more properties of gravity than with General Relativity by doing research numerically with huge computational resources for individual orbit and one by one.

For example, I have already computed with the relativistic orbit equation the deviation of the asymptote of hyperbolic orbit with respect to the Newtonian asymptote; I have also computed the difference between the maximum and minimum velocities of an elliptic orbit, which deviates from the Newtonian one in the same order as orbital precession.

The derivation of the relativistic orbit equation shows clearly the physical meaning of Space-Time geodesics by showing the action of the two contributors with the 2 parentheses in (15): 1) The gravitational acceleration in relativity; 2) The time dilation of Schwarzschild metric. The above discussed rate of variation of orbital velocity and positive/negative precession are new properties. These 3 discoveries cannot be found with General Relativity easily.

6. Derivation of equations

a) Acceleration and velocity in local frame

$\boldsymbol{\alpha}_g$: gravitational acceleration vector $\boldsymbol{\alpha}_m$: inertial acceleration vector in frame_m $\boldsymbol{\alpha}_g = -\frac{GM}{r^2}\boldsymbol{e}_r = \boldsymbol{\alpha}_m$	(1)
Relativistic transformation of inertial acceleration between frame_l and frame_m [3][4] $\boldsymbol{\alpha}_m = \frac{\boldsymbol{\alpha}_l}{1-\frac{\boldsymbol{v}_l{}^2}{c^2}}$ Subscript l: in local frame, $\boldsymbol{v}_l$: velocity vector, $\boldsymbol{\alpha}_l$: acceleration vector	(2)
(1) and (2) makes $-\frac{GM}{r^2}\boldsymbol{e}_r = \frac{\boldsymbol{\alpha}_l}{1-\frac{\boldsymbol{v}_l{}^2}{c^2}}$	(3)
(3) dotted with $v_l \cdot dt_l$ $-\frac{GM}{r^2}\boldsymbol{e}_r \cdot \boldsymbol{v}_l dt_l = \frac{\boldsymbol{\alpha}_l \cdot \boldsymbol{v}_l dt_l}{1-\frac{\boldsymbol{v}_l{}^2}{c^2}}$ t_l: time in frame_l	(4)
Left hand side of (4) $\boldsymbol{e}_r \cdot \boldsymbol{v}_l dt_l = \boldsymbol{e}_r \cdot d\boldsymbol{r} = dr$ Then $-\frac{GM}{r^2}\boldsymbol{e}_r \cdot \boldsymbol{v}_l dt_l = -\frac{GM}{r^2}dr = d\left(\frac{GM}{r}\right)$	(5)
On right hand side of (4) $d\boldsymbol{v}_l = \boldsymbol{\alpha}_l dt_l$ Then $\frac{\boldsymbol{\alpha}_l \cdot \boldsymbol{v}_l dt_l}{1-\frac{\boldsymbol{v}_l{}^2}{c^2}} = \frac{d\boldsymbol{v}_l{}^2}{2\left(1-\frac{\boldsymbol{v}_l{}^2}{c^2}\right)}$ $= -\frac{c^2}{2}\frac{1}{1-\frac{\boldsymbol{v}_l{}^2}{c^2}}d\left(1-\frac{\boldsymbol{v}_l{}^2}{c^2}\right)$ $= -\frac{c^2}{2}d\,ln\left(1-\frac{\boldsymbol{v}_l{}^2}{c^2}\right)$	(6)
Plugging (5) and (6) into (4) $d\left(\frac{GM}{r}\right) = -\frac{c^2}{2}d\,ln\left(1-\frac{\boldsymbol{v}_l{}^2}{c^2}\right)$	(7)
Integration of (7) $ln\left(1-\frac{\boldsymbol{v}_l{}^2}{c^2}\right) + \frac{2GM}{c^2r} = K$ K: integration constant Then $1-\frac{\boldsymbol{v}_l{}^2}{c^2} = e^K e^{-\frac{2GM}{c^2r}}$	(8)
(8) gives the value of e^K at point 0 $e^K = \left(1-\frac{\boldsymbol{v}_0{}^2}{c^2}\right)e^{\frac{2GM}{c^2r_0}}$	(9)
Rearrangement of (8) $\frac{\boldsymbol{v}_l{}^2}{c^2} = 1 - e^K e^{-\frac{2GM}{c^2r}}$	(10)

b) Velocity in the frame at infinitely far

Velocity vector in frame_∞ $\boldsymbol{v}_\infty = \frac{d\boldsymbol{r}}{dt_\infty}$ t_∞: time in frame_∞	(11)
Velocity vector in frame_l $\boldsymbol{v}_l = \frac{d\boldsymbol{r}}{dt_l} = \frac{d\boldsymbol{r}}{dt_\infty}\frac{dt_\infty}{dt_l}$ $= \boldsymbol{v}_\infty \frac{dt_\infty}{dt_l}$ t_l: time in frame_l	(12)
Time dilation in Schwarzschild metric $t_l = t_\infty\sqrt{1-\frac{2GM}{c^2r}}$ $\Rightarrow \frac{dt_\infty}{dt_l} = \frac{1}{\sqrt{1-\frac{2GM}{c^2r}}}$	(13)
Plugging (13) into (12) $\boldsymbol{v}_l = \frac{\boldsymbol{v}_\infty}{\sqrt{1-\frac{2GM}{c^2r}}}$	(14)
Substituting (14) for $\boldsymbol{v}_l$ in (10) gives $\frac{\boldsymbol{v}_\infty{}^2}{c^2} = \left(1-e^K e^{-\frac{2GM}{c^2r}}\right)\left(1-\frac{2GM}{c^2r}\right)$	(15)
Taylor's series for very small $\frac{GM}{c^2r}$ $e^{-\frac{2GM}{c^2r}} \approx 1 - \frac{2GM}{c^2r} + \frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2$	(16)
Substituting (16) for $e^{-\frac{2GM}{c^2r}}$ in (15) $\frac{\boldsymbol{v}_\infty{}^2}{c^2} \approx \left(1-e^K\left(1-\frac{2GM}{c^2r}+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2\right)\right)\left(1-\frac{2GM}{c^2r}\right)$ $= 1 - \frac{2GM}{c^2r}$ $-e^K\left(1-\frac{4GM}{c^2r}+\left(\frac{2GM}{c^2r}\right)^2+\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^2-\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^3\right)$	(17)
By dropping out $\frac{1}{2}\left(\frac{2GM}{c^2r}\right)^3$ from (17) $\frac{\boldsymbol{u}_\infty{}^2}{c^2} = 1 - \frac{2GM}{c^2r} - e^K\left(1-\frac{4GM}{c^2r}+6\left(\frac{GM}{c^2r}\right)^2\right)$ $\boldsymbol{u}_\infty$: orbit velocity in frame_∞ correct to the 2nd order of $\frac{GM}{c^2r}$	(18)

c) Velocity squared using Kepler's second law

Velocity vector in polar coordinates $\boldsymbol{v} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\theta}{dt}\boldsymbol{e}_\theta$	(19)
Velocity squared using (19) $\boldsymbol{u}_\infty{}^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$	(20)
Kepler's second law $r^2\frac{d\theta}{dt} = h$ h:constant of Kepler's second law	(21)

Formula for transforming $\frac{dr}{dt}$ $\frac{dr}{d\theta} = -r^2\frac{d}{d\theta}\left(\frac{1}{r}\right)$ Using (21) and above formula $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{d}{d\theta}\left(\frac{h}{r}\right)$	(22)
Plugging (21) and (22) into (20) $\boldsymbol{u}_\infty{}^2 = \left(\frac{d}{d\theta}\left(\frac{h}{r}\right)\right)^2 + \left(\frac{h}{r}\right)^2$	(23)

d) Differential equation for orbit

(23) divided by c^2 $\frac{\boldsymbol{u}_\infty{}^2}{c^2} = \frac{1}{c^2}\left(\frac{d}{d\theta}\left(\frac{h}{r}\right)\right)^2 + \frac{1}{c^2}\left(\frac{h}{r}\right)^2 = \left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2$	(24)
For transforming (18), we plug (24) and $\frac{GM}{c^2 r} = \frac{GM}{hc}\frac{h}{cr}$ into (18) $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2 = 1 - 2\frac{GM}{hc}\frac{h}{cr} - e^K\left(1 - 4\frac{GM}{hc}\frac{h}{cr} + 6\left(\frac{GM}{hc}\frac{h}{cr}\right)^2\right)$	(25)
Derived from (25), Differential equation for orbit in frame_∞ $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = 1 - e^K + 2(2e^K - 1)\frac{GM}{hc}\frac{h}{cr} - \left(1 + 6e^K\left(\frac{GM}{hc}\right)^2\right)\left(\frac{h}{cr}\right)^2$	(26)

e) Solving the differential equation for orbit

Transformation of (26) $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = -\left(1 + 6e^K\left(\frac{GM}{hc}\right)^2\right)\left(-\frac{1 - e^K}{1 + 6e^K\left(\frac{GM}{hc}\right)^2} - 2\left(\frac{2e^K - 1}{1 + 6e^K\left(\frac{GM}{hc}\right)^2}\right)\frac{GM}{hc}\frac{h}{cr} + \left(\frac{h}{cr}\right)^2\right)$	(27)

Replacing coefficients with constants A, B and C $C = 1 + 6e^K\left(\frac{GM}{hc}\right)^2, A = \frac{2e^K - 1}{C}\frac{GM}{hc}, B = \frac{1 - e^K}{C}$	(28)
Expressing (27) using A, B and C $\left(\frac{d}{d\theta}\left(\frac{h}{cr}\right)\right)^2 = -C\left(-B - 2A\frac{h}{cr} + \left(\frac{h}{cr}\right)^2\right)$ $= C(A^2 + B)\left(1 - \frac{\left(\frac{h}{cr} - A\right)^2}{A^2 + B}\right)$	(29)
(29) divided by $A^2 + B$ $\left(\frac{d}{d\theta}\left(\frac{\frac{h}{cr}}{\sqrt{A^2 + B}}\right)\right)^2 = C\left(1 - \left(\frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}}\right)^2\right)$	(30)
Change of variable $s = \frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}} = \frac{\frac{h}{Acr} - 1}{\sqrt{1 + \frac{B}{A^2}}}$	(31)

$\frac{ds}{d\theta} = \frac{d}{d\theta}\left(\frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}}\right) = \frac{d}{d\theta}\left(\frac{\frac{h}{cr}}{\sqrt{A^2 + B}}\right)$	(32)
Plugging (31) and (32) into (30) $\left(\frac{ds}{d\theta}\right)^2 = C(1 - s^2)$	(33)
(31) becomes $\frac{ds}{\sqrt{1 - s^2}} = \pm\sqrt{C}d\theta$	(34)
Formula of integration $\int\frac{ds}{\sqrt{1 - s^2}} = \sin^{-1}s$	(35)
Integration of (34) using (35) $\sin^{-1}s = \pm\sqrt{C}(\theta + \theta_0)$ θ_0: integration constant	(36)
For θ_0=0 and negative $\pm\sqrt{C}d\theta$, the sine of (36) $s = -\sin\theta\sqrt{C}$	(37)

Plugging (31) into (37) $$\frac{\frac{h}{Acr}-1}{\sqrt{1+\frac{B}{A^2}}}=-\sin\theta\sqrt{C}$$	(38)
(38) is transformed into $$r=\frac{\frac{h}{Ac}}{1-\sqrt{1+\frac{B}{A^2}}\sin(\theta\sqrt{C})}$$ Solution of the differential equation (30)	(39)
Using A from (28) $$\frac{h}{Ac}=\frac{h}{c}\left(\frac{2e^K-1}{C}\frac{GM}{hc}\right)^{-1}=\frac{C}{2e^K-1}\frac{h^2}{GM}$$	(40)

Using C from (28), (40) becomes $$\frac{h}{Ac}=\frac{1+6e^K\left(\frac{GM}{hc}\right)^2}{2e^K-1}\frac{h^2}{GM}$$	(41)
Using A, B and C from (28) for $\frac{B}{A^2}$ $$\frac{B}{A^2}=\frac{1-e^K}{C}\left(\frac{2e^K-1}{C}\frac{GM}{hc}\right)^{-2}=\frac{1-e^K}{(2e^K-1)^2}\left(\left(\frac{hc}{GM}\right)^2+6e^K\right)$$	(42)

Using (41), (42) and (28) for C in (39) $$r=\frac{\frac{1+6e^K\left(\frac{GM}{hc}\right)^2}{2e^K-1}\frac{h^2}{GM}}{1-\sqrt{1+\frac{1-e^K}{(2e^K-1)^2}\left(\left(\frac{hc}{GM}\right)^2+6e^K\right)}\sin\left(\theta\sqrt{1+6e^K\left(\frac{GM}{hc}\right)^2}\right)}$$ Relativistic orbit equation	(43)

f) Orbit as conic section curve

Conic section in polar coordinates[5][6] $$r=\frac{l}{1-\varepsilon\sin\phi}$$ l : the semi-latus rectum, ε: eccentricity	(44)
Comparing (39) with (44) we have $$l=\frac{h}{Ac},\qquad \varepsilon=\sqrt{1+\frac{B}{A^2}},\qquad \phi=\theta\sqrt{C}$$	(45)
(39) becomes $$r=\frac{l}{1-\varepsilon\sin(\theta\sqrt{C})}$$	(46)
Comparing (46) with (43), we obtain $$l=\frac{GM}{c^2}\frac{\left(\frac{hc}{GM}\right)^2+6e^K}{2e^K-1}$$	(47)
(47) gives $\left(\frac{hc}{GM}\right)^2$ $$\left(\frac{hc}{GM}\right)^2=\frac{lc^2}{GM}(2e^K-1)-6e^K$$	(48)
Plugging (48) in C, see (28) $$C=\left(1-\frac{e^K}{2e^K-1}\frac{6GM}{lc^2}\right)^{-1}$$	(49)
For expressing $\frac{e^K}{2e^K-1}$, (45) gives $$\frac{B}{A^2}=\varepsilon^2-1$$	(50)
(42) divided by (40) $$\frac{\frac{B}{A^2}}{\frac{h}{Ac}}=\frac{C\frac{1-e^K}{(2e^K-1)^2}\left(\frac{hc}{GM}\right)^2}{\frac{C}{2e^K-1}\frac{h^2}{GM}}$$	(51)

(51) with $l=\frac{h}{Ac}$ from (45) $$\frac{B}{A^2}=\frac{lc^2}{GM}\frac{1-e^K}{2e^K-1}$$	(52)
Equating (50) with (52) $$\frac{lc^2}{GM}\frac{1-e^K}{2e^K-1}=\varepsilon^2-1$$ Then $$\frac{1-e^K}{2e^K-1}=\frac{(\varepsilon^2-1)GM}{lc^2}$$	(53)
Conic section formula $l=a(1-\varepsilon^2)$ a: semi-major axis	(54)
Using (54) in (53) $$\frac{1-e^K}{2e^K-1}=\frac{(\varepsilon^2-1)GM}{a(1-\varepsilon^2)c^2}=-\frac{GM}{ac^2}$$	(55)
Expressing $\frac{e^K}{2e^K-1}$ using (55) $$\frac{e^K}{2e^K-1}=1+\frac{1-e^K}{2e^K-1}=1-\frac{GM}{ac^2}$$	(56)
Using (56) and (54) in (49) $$C=\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-1}$$	(57)
Relativistic orbit as conic section curve is derived be plugging (54) and (57) into (46) $$r=\frac{a(1-\varepsilon^2)}{1-\varepsilon\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)}$$	(58)

g) Orbital precession

Orbit starts from θ=0, (58) gives $\sin\left(\theta\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$	(59)
Orbit ends at θ_1 which makes $\sin\left(\theta_1\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)=0$	(60)
As $\theta_1 \neq 0$, (60) gives $\theta_1 = 2\pi\left(1-\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{\frac{1}{2}}$	(61)
The end of the first turn θ_1 makes with that of starting point θ=2π this angle $\Delta\Phi = \theta_1 - 2\pi$	(62)

Taylor's series of (61) for $\left\lvert\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right\rvert \ll 1$ $\theta_1 \approx 2\pi - \frac{2\pi}{2}\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}$	(63)
Plugging (63) into (62) $\Delta\Phi \approx -\left(1-\frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}$ Angle of precession per turn	(64)
For planets, $1-\frac{GM}{ac^2} \approx 1$ $\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$	(65)
Absolute value $\lvert\Delta\Phi\rvert \approx \frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ The value Albert Einstein had given	(66)

h) Variation of orbital velocity

Orbital velocity in frame_∞, (18) multiplied by c^2, then neglecting the second order $6\left(\frac{GM}{c^2r}\right)^2$ $\boldsymbol{u}_\infty{}^2 \approx c^2(1-e^K) - \frac{2GM}{r}(1-2e^K)$	(67)
Δ velocity-squared using (67) $(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Relativity} = -2GM(1-2e^K)\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$ $\boldsymbol{u}_a$ and $\boldsymbol{u}_b$: orbit velocities at points a and b for relativistic gravity.	(68)
When $1-2e^K = -1$, (68) gives that for Newtonian orbit $(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Newtonian} = 2GM\left(\frac{1}{r_b}-\frac{1}{r_a}\right)$	(69)
Ratio between (68) and (69) $\frac{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Relativity}}{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Newtonian}} = \frac{-2GM(1-2e^K)\left(\frac{1}{r_b}-\frac{1}{r_a}\right)}{2GM\left(\frac{1}{r_b}-\frac{1}{r_a}\right)} = 2e^K-1$	(70)

Hyperbolic orbit at infinitely far, $r_0 = \infty$, see (9) $e^K = \left(1-\frac{\boldsymbol{v}_0{}^2}{c^2}\right)e^{\frac{2GM}{c^2r_0}} = 1-\frac{\boldsymbol{v}_0{}^2}{c^2}$	(71)
Introducing (71) into (70) $\frac{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Relativity}}{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Newtonian}} = 1-2\frac{\boldsymbol{v}_0{}^2}{c^2} < 1$	(72)
Observed velocity is $\boldsymbol{u}_a$ for relativistic and Newtonian orbits $(\boldsymbol{u}_a{}^2)_{Relativity} = \boldsymbol{u}_a{}^2 = (\boldsymbol{u}_a{}^2)_{Newtonian}$	(73)
Using (73) and (72) $\frac{(\boldsymbol{u}_b{}^2)_{Relativity}-\boldsymbol{u}_a{}^2}{(\boldsymbol{u}_b{}^2)_{Newtonian}-\boldsymbol{u}_a{}^2} = \frac{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Relativity}}{(\boldsymbol{u}_b{}^2-\boldsymbol{u}_a{}^2)_{Newtonian}} < 1$	(74)
Leaving perihelion, velocity decreases $(\boldsymbol{u}_b{}^2)_{Newtonian}-\boldsymbol{u}_a{}^2 < 0$	(75)
(74) multiplied by the negative (75) $(\boldsymbol{u}_b{}^2)_{Relativity}-\boldsymbol{u}_a{}^2 > (\boldsymbol{u}_b{}^2)_{Newtonian}-\boldsymbol{u}_a{}^2$	(76)
Then, velocities away from perihelion $(\boldsymbol{u}_b{}^2)_{Relativity} > (\boldsymbol{u}_b{}^2)_{Newtonian}$	(77)

i) Circular orbit

Acceleration of circular orbit $\alpha = -\frac{v^2}{r}$ $\boldsymbol{v}$: orbital velocity	(78)
Orbital acceleration on Newtonian orbit $\alpha_N = -\frac{GM}{r^2}$	(79)
(78) with (79) give Newtonian orbital velocity $v_N{}^2 = \frac{GM}{r}$	(80)

Relativistic accelerations, see (3) $-\frac{GM}{r^2} = \frac{\alpha_l}{1-\frac{v_l{}^2}{c^2}}$ v_l : orbital velocity, α_l:orbital acceleration	(81)
Substituting (78) for α_l in (81) with v being replaced by v_l $-\frac{GM}{r^2} = \frac{-\frac{v_l{}^2}{r}}{1-\frac{v_l{}^2}{c^2}}$	(82)

Transformation of (82) gives the relativistic orbital velocity $${v_l}^2=\frac{GM}{r}\frac{1}{1+\frac{GM}{c^2r}}$$	(83)
Dividing (83) with (80) $$\frac{{v_l}^2}{{v_N}^2}=\frac{\frac{GM}{r}}{\left(1+\frac{GM}{c^2r}\right)\frac{GM}{r}}$$ $$=\frac{1}{1+\frac{GM}{c^2r}}<1$$	(84)

(84) gives $v_l<v_N$	(85)

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