Relativistic dynamics: force, mass, kinetic energy, gravitation and dark matter

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Abstract: Special relativity does not deal with acceleration, general relativity does not deal with non gravitational acceleration, which leave the theory of relativity imperfect. We will demonstrate some relativistic dynamical laws that specify relativistic acceleration, force and kinetic energy. Also, based on equivalence principle does gravitational mass vary with inertial mass?

1. Introduction

Newtonian kinematics defines motions of objects with velocity and acceleration, Newtonian dynamics defines force with acceleration and mass, which makes Newtonian mechanics the most complete theory in physics. Special and general relativity are extremely successful, but they lack the capability of dealing with acceleration and force. For relativity mass increases to infinity when $u=c$, which makes energy and momentum to become incorrectly infinity. Also, general relativity is based on equivalence principle according to which inertial mass is equivalent to gravitational mass. Then does gravitational mass increases when inertial mass increases? So, relativity needs new laws to deal with acceleration and force.

In previous studies of relativity [1][2][3][4][5], we have already correctly treated acceleration, inertial mass, kinetic energy. Below we will demonstrate the laws that describe them. For setting the demonstrations on a strong base, we begin with rigorously proving the equality of differential momentum in 2 relatively moving frames of reference.

Notation convention:

Bold letters: vectors
(i): equation index
M, m: masses
M_0, m_0: rest masses of M, m
$\boldsymbol{\alpha}$: acceleration
c: speed of light
$\boldsymbol{F}$: Force
$\boldsymbol{P}$: momentum
t: time
u: velocity of a frame
$\boldsymbol{v}$: velocity of an object

Subscript:
1: Stationary frame
2: Mobile frame
g: Gravitational
u: Moving at velocity u
a, b: Object a and b

Note: In this article the word "relativistic" means "in the theory of relativity", not moving at extremely high speed. For example, "relativistic mass" means that a mass increases with velocity but not that it moves near the speed of light.

2. Differential momentum equality

a. Differential momentum

The momentum of an object equals the product of its mass m and its velocity $\boldsymbol{u}$ as defined in equation (1). If the object gets an infinitesimal impulse, it gets an infinitesimal change of momentum, which we call differential momentum. If we see the object in 2 relatively moving frames of reference, the value of its momentum is different in each frame. But, for Newtonian mechanics a differential momentum has the same value in all frames.

In relativity, a change of velocity has different value in relatively moving frames. However, differential momentum has the same value in such frames, which we call equality of differential momentum and have explained in « Velocity, mass, momentum and energy of an accelerated object in relativity » [2].

For rigorously proving this equality, let us take 2 identical objects labeled a and b. The object b moves at the velocity $\boldsymbol{u}$ with respect to a, the frame of reference of the object b is labeled *frm. b*, see Figure 1. If the object b gets an infinitesimal impulse, it gets a differential momentum and a differential change of velocity labeled $d\boldsymbol{u}_b$.

b. Proper inertial frame

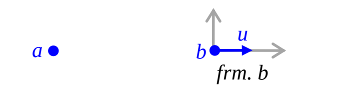

Figure 1

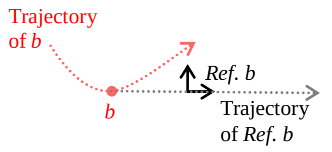

Figure 2

Notice that in the frame *frm. b* the velocity of b is constantly zero. Then, how can its change of velocity $d\boldsymbol{u}_b$ be nonzero? In fact, $d\boldsymbol{u}_b$ is with respect to an inertial frame, not to *frm. b*. For defining $d\boldsymbol{u}_b$ we have to create a new type of inertial frame that coincides with b. We name this type of frame "Proper inertial frame".

For example, the object b moves at the instant velocity $\boldsymbol{u}_t$ at a given time t. At this time we create the proper inertial frame of b labeled *Ref. b* which moves at constant velocity that equals $\boldsymbol{u}_t$. The trajectory of *Ref. b* is a straight line while that of b is a curve, see Figure 2. After the infinitesimal impulse b moves at the instant velocity $\boldsymbol{u}'_t$ and the change of velocity equals $d\boldsymbol{u}_b = \boldsymbol{u}'_t - \boldsymbol{u}_t$ with $\boldsymbol{u}_t$ being the velocity of the inertial frame *Ref. b*. In the same way the proper inertial frame of the object a is labeled *Ref. a*.

Proper inertial frame is a new notion, so we give its definition below:

Definition 1: Proper inertial frame of an accelerated object

At a given time t an accelerated object moves at the instant velocity $\boldsymbol{u}_t$. The proper inertial frame of this object at this time is a frame that moves at the constant velocity $\boldsymbol{u}_t$.

c. Equality of differential momentum

$\boldsymbol{P} = m\boldsymbol{u}$	(1)
$d\boldsymbol{P}_b = m_0 d\boldsymbol{u}_b$	(2)
$d\boldsymbol{P}_a = m_0 d\boldsymbol{u}_a$	(3)
Interaction $d\boldsymbol{u}_a = -d\boldsymbol{u}_b$	(4)
$m_0 d\boldsymbol{u}_a = -m_0 d\boldsymbol{u}_b$	(5)
Using (2) and (3) $d\boldsymbol{P}_a + d\boldsymbol{P}_b = 0$	(6)

Back to our 2 identical objects a and b and let them interact dynamically with each other. During an infinitesimal time the 2 objects acquire the differential momentums $d\boldsymbol{P}_a$ and $d\boldsymbol{P}_b$ in their respective proper inertial frame. Suppose that a and b are alone in space, then $d\boldsymbol{P}_a$ comes from b and $d\boldsymbol{P}_b$ comes from a.

In *Ref. b* Newtonian mechanics applies because the velocity of b is zero. The infinitesimal change of velocity of b is $d\boldsymbol{u}_b$, its differential momentum is $d\boldsymbol{P}_b$ which equals $m_0 d\boldsymbol{u}_b$ with m_0 being its rest mass, see equation (2). In the same way, the differential momentum of a in *Ref. a* is $d\boldsymbol{P}_a$ and is expressed in (3).

The frames *Ref. a* and *Ref. b* are both inertial. According to the principle of relativity, *Ref. a* is not privileged over *Ref. b* and vice versa. Also, because the objects a and b are identical, their dynamical behavior are exactly symmetrical. That is, if someone travels with one object and looks at the trajectories of the other, he has no way to tell if he is with a or with b. So, the change of velocity of one object must be exactly opposing that of the other. Then, we have $d\boldsymbol{u}_a = -d\boldsymbol{u}_b$, see (4). We multiply (4) with m_0 and obtain (5), which gives (6) by using (2) and (3).

On the other hand, the differential momentum of the object *b* can be specified in the proper inertial frame of the object *a* which is *Ref. a*. We label this special differential momentum with $d\boldsymbol{P'}_b$. So, $d\boldsymbol{P'}_b$ is in the same frame as $d\boldsymbol{P}_a$ and because of momentum conservation $d\boldsymbol{P}_a$ and $d\boldsymbol{P'}_b$ cancel out in the frame *Ref. a*, see (7). By combining (6) and (7) we obtain (8) which means that the value of the differential momentum of *b* is the same in *Ref. b* and *Ref. a*. This is the equality of differential momentum for the object *b* and is proven by using 2 identical objects. Is this equality valid in general?

Momentum conservation $d\boldsymbol{P}_a + d\boldsymbol{P'}_b = 0$	(7)
Using (6) and (7) $d\boldsymbol{P}_b = d\boldsymbol{P'}_b$	(8)
$d\boldsymbol{P}_1 = d\boldsymbol{P'}_b$ $d\boldsymbol{P}_2 = d\boldsymbol{P}_b$	(9)
Using (8) and (9) $d\boldsymbol{P}_1 = d\boldsymbol{P}_2$	(10)

For the object *b* there is no constrain about the position and velocity or about the kind of force of interaction in the frame *Ref. a*. That is, *b* can be at any position and moves at any velocity in *Ref. a*. So, *b* is in fact an arbitrary object in *Ref. a* and reversly, *Ref. a* an arbitrary inertial frame with respect to *b*. Since $d\boldsymbol{P}_b$ is specified in the proper inertial frame of *b* and $d\boldsymbol{P'}_b$ in *Ref. a*, the equality $d\boldsymbol{P}_b = d\boldsymbol{P'}_b$ means that the differential momentum of *b* has the same value whether specified in its proper inertial frame or in another arbitrary inertial frame. That is, this equality is valid in general.

For general use, we will refer the frame *Ref. a* as *Ref. 1* and the frame *Ref. b* as *Ref. 2*, $d\boldsymbol{P'}_b$ as $d\boldsymbol{P}_1$ and $d\boldsymbol{P}_b$ as $d\boldsymbol{P}_2$, see equation (9). Using (9) in (8) we get the equality $d\boldsymbol{P}_1 = d\boldsymbol{P}_2$ in (10), with $d\boldsymbol{P}_2$ being the differential momentum of an accelerated object specified in its proper inertial frame and $d\boldsymbol{P}_1$ that specified in any other inertial frame.

We call equation (10) equality of differential momentum and state the following law:

Law 1: Equality of differential momentum

An object moves at the velocity $\boldsymbol{u}$ in the inertial frame *Ref. 1*. The proper inertial frame of the object is the frame *Ref. 2*. When the object gets an infinitesimal impulse, the resulting differential momentum of the object specified in *Ref. 1* and *Ref. 2* has the same value.

The equality of differential momentum can be demonstrated directly from differential momentum as below without using rest mass and velocity:

> During an infinitesimal interaction between the objects *a* and *b* which are identical, the behavior of the objects *a* and *b* are exactly symmetrical. According to relativistic principle, the proper inertial frame of *a* is not privileged over that of *b* and vice versa. So, the differential momentum that the object *a* gets and that the object *b* gets are exactly opposing, which is expressed in equation (6). After that, the equality of differential momentum is derived from (7) to (10).

As differential momentum is more abstract than velocity, this demonstration is independent from the other one and makes the equality of differential momentum stronger.

$\boldsymbol{F} = \dfrac{d\boldsymbol{P}}{dt}$	(11)
$\boldsymbol{\alpha} = \dfrac{d\boldsymbol{u}}{dt}$	(12)
m constant $\dfrac{d\boldsymbol{P}}{dt} = m\dfrac{d\boldsymbol{u}}{dt}$ $= m\boldsymbol{\alpha}$	(13)
$\boldsymbol{F} = m\boldsymbol{\alpha}$	(14)

3. Inertial force

a. Newtonian inertial force

Inertial force is defined as the time derivative of momentum, see equation (11). For Newtonian mechanics, the mass *m* of an object is constant and the time derivative of its momentum is in (13) with $\boldsymbol{u}$ being its velocity and $\boldsymbol{\alpha}$ its acceleration, see (12). Then, we get the Newton's formula for inertial force in (14).

But in relativity *m* is not constant, then how is inertial force defined?

b. Relativistic inertial force

Using (1) and (12) $\dfrac{d\boldsymbol{P}}{dt} = \dfrac{d(m\boldsymbol{u})}{dt}$ $= m\dfrac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\dfrac{dm}{dt}$ $= m\boldsymbol{\alpha} + \boldsymbol{u}\dfrac{dm}{dt}$	(15)
$\dfrac{dm}{dt} \neq 0, \boldsymbol{u} \neq 0$ $\Rightarrow \boldsymbol{u}\dfrac{dm}{dt} \neq 0$ $\Rightarrow \dfrac{d\boldsymbol{P}}{dt} \neq m\boldsymbol{\alpha}$	(16)
Proper inertial frame $\boldsymbol{u} = 0$, using (15) $\boldsymbol{F} = \dfrac{d\boldsymbol{P}}{dt} = m_0\boldsymbol{\alpha}$	(17)

For relativity, momentum is still the product of *m* and $\boldsymbol{u}$ whereas *m* is not constant. Inertial force is still defined as the time derivative of momentum, see (1) and (11). Now we develop completely the time derivative of the product $m\boldsymbol{u}$ in (15). Because the mass *m* is not constant, if $\boldsymbol{u} \neq 0$ the term $\boldsymbol{u}\frac{dm}{dt}$ is nonzero and the time derivative of momentum does not equal $m\boldsymbol{\alpha}$, see (16). So, the Newton's formula for inertial force (14) is not valid for relativity in general.

However, in the proper inertial frame of an object its velocity is always zero and the term $\boldsymbol{u}\frac{dm}{dt}$ in (15) is zero and the inertial force of the object equals $m_0\boldsymbol{\alpha}$ with m_0 being its rest mass, see (17). So, in relativity the Newton's formula for inertial force (14) is valid in the proper inertial frame of the object. As this inertial force changes its value across frames, we call it "relativistic inertial force".

Law 2: Relativistic inertial force

Relativistic inertial force of an accelerated object is defined as $\frac{d\boldsymbol{P}}{dt}$ and equals $\boldsymbol{F} = m_0\boldsymbol{\alpha}$ in the proper inertial frame of the object with m_0 being its rest mass and $\boldsymbol{\alpha}$ its acceleration in this frame.

c. Transformation of relativistic inertial force

Since relativistic inertial force changes its value, we need a transformation for relativistic inertial force to know its value in inertial frames other than the proper inertial frame. Let *Ref. 2* be the proper inertial frame of an object which moves at the instant velocity $\boldsymbol{u}$ in another inertial frame *Ref. 1*. Let $\boldsymbol{P}_1$ be the momentum of the object in *Ref. 1* and t_1 the time in *Ref. 1*. The inertial force of the object in *Ref. 1* is $\boldsymbol{F}_1$ and equals the derivative of $\boldsymbol{P}_1$ with respect to t_1, see equation (18). In the same way, the force in *Ref. 2* is $\boldsymbol{F}_2$ and expressed in (19).

We transform the derivative $\frac{d\boldsymbol{P}_1}{dt_1}$ in (20) where $\frac{dt_2}{dt_1}$ is the ratio of time between *Ref. 1* and *Ref. 2*, see (21). The expression of $\frac{dt_2}{dt_1}$ was derived in « Velocity, mass, momentum and energy of an accelerated object in relativity » [2] , see the equation (8) of this article.

Using the equality of differential momentum (10) we replace $d\boldsymbol{P}_1$ with $d\boldsymbol{P}_2$ in (20), then we apply (21) for $\frac{dt_2}{dt_1}$ in (20) and obtain (22). Using (18) and (19) in (22) we obtain (23) which expresses $\boldsymbol{F}_1$ in terms of $\boldsymbol{F}_2$. Equation (23) is the transformation for relativistic inertial force between inertial frames.

Using (11) in *Ref. 1* $\boldsymbol{F}_1 = \frac{d\boldsymbol{P}_1}{dt_1}$	(18)
Using (11) in *Ref. 2* $\boldsymbol{F}_2 = \frac{d\boldsymbol{P}_2}{dt_2}$	(19)
$\frac{d\boldsymbol{P}_1}{dt_1} = \frac{d\boldsymbol{P}_1}{dt_2}\frac{dt_2}{dt_1}$	(20)
$\frac{dt_2}{dt_1} = \sqrt{1 - \frac{u^2}{c^2}}$	(21)
Using (10) and (21) in (20) $\frac{d\boldsymbol{P}_1}{dt_1} = \frac{d\boldsymbol{P}_2}{dt_2}\sqrt{1 - \frac{u^2}{c^2}}$	(22)
Using (18) and (19) in (22) $\boldsymbol{F}_1 = \boldsymbol{F}_2\sqrt{1 - \frac{u^2}{c^2}}$	(23)

Law 3: Transformation of relativistic inertial force
An accelerated object gets an infinitesimal impulse. The resulting relativistic inertial force of the object in the inertial frame *Ref. 1* and the proper inertial frame of the object *Ref. 2* are F_1 and F_2 respectively. The transformation of the relativistic inertial forces between *Ref. 1* and *Ref. 2* is $\boldsymbol{F}_1 = \boldsymbol{F}_2\sqrt{1 - \frac{u^2}{c^2}}$ with u being the relative velocity between the 2 frames.

4. Gravitational force

a. Transformation of gravitational force

Gravitational force between 2 bodies is defined by Newton's universal law of gravitation. Let M be an attracting body and m an attracted body. The gravitational force on m is $\boldsymbol{F}_g$ and is expressed in (24) with M_0 and m_0 being the rest masses of M and m respectively, R the distance between M and m, $\boldsymbol{e}_r$ the unit vector pointing from M to m.

Gravitational force is not inertial force but creates an inertial force on the object it accelerates. Let *Ref. 1* and *Ref. 2* be the proper inertial frames of M and m respectively. In the case where m moves at the velocity $\boldsymbol{u}$ with respect to M, $\boldsymbol{F}_{g2}$ the gravitational force on m in *Ref. 2* creates the relativistic inertial force $\boldsymbol{F}_2$ which equals exactly $\boldsymbol{F}_{g2}$, see (25). The transformation of relativistic inertial force (23) applied to $\boldsymbol{F}_{g2}$ gives $\boldsymbol{F}_1$ which is the relativistic inertial force specified in *Ref. 1*, see (26), which is in fact created by the gravitational force in *Ref. 1* labeled $\boldsymbol{F}_{g1}$, see (27). Using (27) in (26) we get (28), which transforms the gravitational force $\boldsymbol{F}_{g2}$ into $\boldsymbol{F}_{g1}$. So, gravitational force also changes its value across frames.

$\boldsymbol{F}_g = -\frac{GM_0m_0}{R^2}\boldsymbol{e}_r$	(24)
$\boldsymbol{F}_{g2} = \boldsymbol{F}_2$	(25)
Using (25) in (23) $\boldsymbol{F}_1 = \boldsymbol{F}_{g2}\sqrt{1 - \frac{u^2}{c^2}}$	(26)
$\boldsymbol{F}_{g1} = \boldsymbol{F}_1$	(27)
Using (27) in (26) $\boldsymbol{F}_{g1} = \boldsymbol{F}_{g2}\sqrt{1 - \frac{u^2}{c^2}}$	(28)

But Newton's universal law of gravitation gives only one value. Which of $\boldsymbol{F}_{g1}$ and $\boldsymbol{F}_{g2}$ is the correct one?

b. Gravitational force on moving and fixed bodies

In fact, Newton's universal law of gravitation treats all bodies as stationary even they move in space, for example, the moon and the planets. But for relativity the velocities of bodies matter and the relative velocity between attracting and attracted bodies has to be taken into account for gravitation.

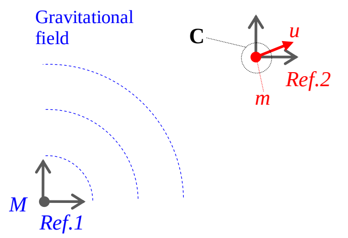

Figure 3

Let m move at the velocity $\boldsymbol{u}$ with respect to M and *Ref. 2* be the proper inertial frame of m. The proper inertial frame of M is *Ref. 1*. M and its gravitational field are stationary in *Ref. 1*, see Figure 3. As gravitational force equals the time derivative of momentum, gravitational field is a constant flow of momentum. The attracted body collects a quantity of momentum per unit time which equals the gravitational force on it.

The circle C in Figure 3 is a small region fixed in the frame *Ref. 1*. Suppose that m moves across the region in the time dt_1 within which m is inside this region and collects the quantity of momentum $d\boldsymbol{P}_{g1}$. Suppose now that m is stationary inside the region. In this case m collects the quantity of momentum $d\boldsymbol{P}'_g$ within the same time dt_1. When the size of the circle C is reduced indefinitely, the positions of the moving and stationary m become the same, $d\boldsymbol{P}_{g1}$, $d\boldsymbol{P}'_g$ and dt_1 become infinitesimal. Because M delivers the same quantity of momentum during dt_1 whether m moves or not, we should have $d\boldsymbol{P}_{g1}=d\boldsymbol{P}'_g$, see (29). Dividing (29) with dt_1 gives (30). The 2 sides of this equation are the gravitational forces $\boldsymbol{F}_{g1}$ on the moving m and $\boldsymbol{F}_g$ on the stationary m, see (31). $\boldsymbol{F}_{g1}$ and $\boldsymbol{F}_g$ are in *Ref. 1* because $d\boldsymbol{P}_{g1}$ and $d\boldsymbol{P}'_g$ are in *Ref. 1*. So, we obtain $\boldsymbol{F}_{g1} = \boldsymbol{F}_g$ in (32), with $\boldsymbol{F}_g$ defined in (24).

$d\boldsymbol{P}_{g1} = d\boldsymbol{P}'_g$	(29)
$\dfrac{d\boldsymbol{P}_{g1}}{dt_1} = \dfrac{d\boldsymbol{P}'_g}{dt_1}$	(30)
$\dfrac{d\boldsymbol{P}_{g1}}{dt_1} = \boldsymbol{F}_{g1}$ $\dfrac{d\boldsymbol{P}'_g}{dt_1} = \boldsymbol{F}_g$	(31)
Using (31) in (30), then (24) $\boldsymbol{F}_{g1} = \boldsymbol{F}_g$ $= -\dfrac{GM_0m_0}{R^2}\boldsymbol{e}_r$	(32)

The above demonstration is rather tedious. Here is a simpler demonstration. Let A and B be 2 fixed points in the gravitational field of M, see Figure 4. The body m does a round trip from A to B then back to A. The velocity on AB trip is u_o and that on BA trip is u_i. Let the gravitational force on the AB trip be $\boldsymbol{F}(u_o)$ and that on the BA trip $\boldsymbol{F}(u_i)$.

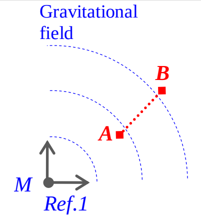

Figure 4

The differential work done by a force $\boldsymbol{F}$ on a differential distance $\boldsymbol{dx}$ is dw, see equation (33). On the AB trip and the BA trip the differential works done by the gravitational force are defined by (34). The total work done over the round trip is defined by (35). As gravitational field is conservative the total work done over the round trip is zero, see (36), which is true for the velocities u_o and u_i of whatever value and $\boldsymbol{F}(u_o)$ cancels $\boldsymbol{F}(u_i)$ out, see (36). Thus, $\boldsymbol{F}(u_o) = \boldsymbol{F}(u_i)$ for whatever velocity, see (37). In other words, whether m moves or not, the gravitational force on m has the same value in *Ref. 1*.

As (37) is true for the straight line AB in any direction, $\boldsymbol{F}(u_o) = \boldsymbol{F}(u_i)$ is true for the body m moving in any direction. So, equation (37) and the statement "gravitational force on a body is independent of its velocity" is true in general.

Law 4: Gravitational force is independent of velocity

Let M be an attracting body, m an attracted body. The gravitational force on m is independent of the velocity of m in the proper inertial frame of M.

Applying (32) in (28) gives (38) which expresses $\boldsymbol{F}_{g2}$ the gravitational force on m in *Ref. 2*.

Law 5: Gravitational force in the proper inertial frame of the attracted body

Let M be an attracting body, m an attracted body. m moves at the velocity $\boldsymbol{u}$ with respect to M. Let M_0 and m_0 be the rest masses of M and m respectively. The gravitational force on m specified in the proper inertial frame of m is

$\boldsymbol{F}_{g2} = -\dfrac{GM_0m_0}{R^2\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{e}_r$.

$dw = \boldsymbol{F}\cdot\boldsymbol{dx}$	(33)
$dw(u_o) = \boldsymbol{F}(u_o)\cdot\boldsymbol{dx}$ $dw(u_i) = \boldsymbol{F}(u_i)\cdot\boldsymbol{dx}$	(34)
$W = \int_A^B \boldsymbol{F}(u_o)\cdot\boldsymbol{dx} + \int_B^A \boldsymbol{F}(u_i)\cdot\boldsymbol{dx}$ $= \int_A^B \big(\boldsymbol{F}(u_o) - \boldsymbol{F}(u_i)\big)\cdot\boldsymbol{dx}$	(35)
$W = 0$ $\Rightarrow \int_A^B \big(\boldsymbol{F}(u_o) - \boldsymbol{F}(u_i)\big)\cdot\boldsymbol{dx} = 0$ True for whatever u_o and u_i $\boldsymbol{F}(u_o) - \boldsymbol{F}(u_i) = 0$	(36)
$\boldsymbol{F}(u_o) = \boldsymbol{F}(u_i)$	(37)
Applying (24) in (28) $\boldsymbol{F}_{g2} = -\dfrac{GM_0m_0}{R^2\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{e}_r$	(38)

It is important to underline the significances of (32) and (28):

- In (32) $\boldsymbol{F}_{g1}$ is the force on the moving m, $\boldsymbol{F}_g$ is the force on the stationary m. So, (32) expresses the equality of the forces on 2 different bodies.
- In (28), $\boldsymbol{F}_{g1}$ is specified in *Ref. 1* and $\boldsymbol{F}_{g2}$ specified in *Ref. 2*, they are the same force on the same body but specified in 2 different frames. $\boldsymbol{F}_{g1}$ and $\boldsymbol{F}_{g2}$ are the transformation of each other.

c. Gravitational mass of a moving body

In the proper inertial frame of the moving m, *Ref. 2*, the force on m is $\boldsymbol{F}_{g2}$ which equals its rest mass m_0 multiplied by its acceleration $\boldsymbol{\alpha}$ because Newtonian mechanics applies, see equation (39). Combining (38) and (39) gives (40), which expresses the acceleration $\boldsymbol{\alpha}$ in terms of M_0 without m_0. By reducing the term $\frac{M_0}{\sqrt{1-\frac{u^2}{c^2}}}$ into M_u, see (41), $\boldsymbol{\alpha}$ is expressed in (42) where M_u acts as a gravitational mass.

In *Ref. 2* $\boldsymbol{F}_{g2} = m_0\boldsymbol{\alpha}$	(39)
Using (38) and (39) $\boldsymbol{\alpha} = -\dfrac{GM_0}{R^2\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{e}_r$	(40)

Because M_u increases with the relative velocity, we call M_u relativistic gravitational mass of M. Reversely, in the proper inertial frame of M the relativistic gravitational mass of m is $m_u = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$, see (43).

M_u is bigger than M, which means that in the proper inertial frame of m the gravitational mass of M increases with velocity. So, for 2 gravitational bodies, the relativistic gravitational mass of the attracting body increases with velocity. This increase is symmetrical between the 2 bodies. Indeed, the term $\sqrt{1-\frac{u^2}{c^2}}$ in (38) can be associated either with M_0 or with m_0.

Law 6: Relativistic gravitational mass

In the gravitational field of an attracting body M, an attracted body m moves at the velocity $\boldsymbol{u}$. In the proper inertial frame of m the relativistic gravitational mass of the attracting body is $M_u = \frac{M_0}{\sqrt{1-\frac{u^2}{c^2}}}$ with M_0 being the rest mass of M.

$$M_u = \frac{M_0}{\sqrt{1-\frac{u^2}{c^2}}}$$	(41)
Using (41) in (40) $$\boldsymbol{\alpha} = -\frac{GM_u}{R^2}\boldsymbol{e}_r$$	(42)
Reversely for M $$m_u = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$	(43)
$$m_u \approx m_0\left(1+\frac{1}{2}\frac{u^2}{c^2}\right) = m_0 + \frac{m_0}{2}\frac{u^2}{c^2}$$	(44)

d. Gravitational magnetic force and rotation curves of disc galaxies

Because $\boldsymbol{F}_{g2}$ is bigger than the Newtonian gravitational force $\boldsymbol{F}_g$, see equations (38) and (24), the gravitational force between 2 relatively moving bodies is bigger than when the 2 bodies are stationary. The increased part of the gravitational force is proportional to the relative velocity squared, see equation (44), which is sort of like magnetic force. This is the gravitational magnetic force that we proposed in «Magnetism and dark matter» [5]. In a disc galaxy the stars rotate at big velocity and the gravitational force the stars feel should be bigger than if the galaxy were still. So, the gravitational magnetic force makes the disc of galaxies rotating faster than expected.

If we suppose that most mass of a disc galaxy is concentrated in its center and use Newtonian mechanics to compute the rotation curve of a disc galaxy, we will find a big difference between the computed curve and the observed one. But, it was shown in «How galaxies make their rotation curves flat and what about dark matter?» [3] that the Newtonian gravitational force of a mass distributed in the shape of a disc makes the rotation curves already quite flat, which is the case of a disc galaxy. Thus, the difference between the computed Newtonian flat curves and the observed flat curves is in fact quite small and could be explained with the gravitational magnetic force.

5. Inertial mass and kinetic energy

a. Relativistic inertial mass

In Newtonian mechanics the inertial mass of an object is the coefficient m in equation (14). We can also write equation (14) for relativity, but in this case the coefficient m varies with the velocity of the object. We call this varying m the relativistic inertial mass of the object and derive its variation law below.

Suppose that an object b moves at the velocity $\boldsymbol{u}$ in an inertial frame labeled *Ref. 1* and gets an infinitesimal impulse. In its proper inertial frame *Ref. 2* the resulting differential momentum and velocity are $d\boldsymbol{P}_2$ and $d\boldsymbol{v}_2$ respectively. Because Newtonian mechanics applies in *Ref. 2* we write equation (45). The differential velocity of b specified in *Ref. 1* is $d\boldsymbol{v}_1$ which is transformed from $d\boldsymbol{v}_2$ using (46) the transformation for differential velocity derived in « Relativistic kinematics » [1], see the equation (15) of this article.

Combining the equations (45) and (46) gives the expression of $d\boldsymbol{P}_2$ in (47). According to (10), $d\boldsymbol{P}_2 = d\boldsymbol{P}_1$, with $d\boldsymbol{P}_1$ being the differential momentum of b specified in *Ref. 1*. Then we replace $d\boldsymbol{P}_2$ with $d\boldsymbol{P}_1$ in (47) to obtain the expression of $d\boldsymbol{P}_1$ in (48). By dividing (48) with the differential time dt_1 we obtain (49) which is written into (50) with $\boldsymbol{F}_1$ being the relativistic inertial force of b in *Ref. 1* and $\boldsymbol{\alpha}_1$ its acceleration in *Ref. 1*.

From (50) we extract the term $\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ and label it as m_u, see (51). m_u is the

In proper inertial frame $$d\boldsymbol{P}_2 = m_0 d\boldsymbol{v}_2$$	(45)
$$d\boldsymbol{v}_2 = \frac{d\boldsymbol{v}_1}{\sqrt{1-\frac{u^2}{c^2}}}$$	(46)
$$d\boldsymbol{P}_2 = m_0\frac{d\boldsymbol{v}_1}{\sqrt{1-\frac{u^2}{c^2}}}$$	(47)
Using (10) in (47) $$d\boldsymbol{P}_1 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}d\boldsymbol{v}_1$$	(48)
$$\frac{d\boldsymbol{P}_1}{dt_1} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1}$$	(49)
$$\boldsymbol{F}_1 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{\alpha}_1 = m_u\boldsymbol{\alpha}_1$$	(50)
$$m_u = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$	(51)

relativistic inertial mass of b moving at the instant velocity $\boldsymbol{u}$. Equation (51) is in the same form as that given in « Introduction to Special Relativity » by James H. Smith [5], except that the latter was derived using a complex process of shock.

Equation (51) is the variation law of relativistic inertial mass. Notice that equation (51) transforms m_0 into m_u which are the inertial masses of b in 2 relatively moving frames. So, equation (51) is in fact the transformation for relativistic inertial mass the same way as the transformation for time in relativity, see (21). For relativistic gravitational mass also, equation (43) is the transformation for relativistic gravitational mass because it has the same form as equation (51).

Since the relativistic inertial and gravitational masses of an object are transformed the same way, they will always have the same value for whatever velocity of the object. This result illustrates well the equivalence principle which stipulates that inertial and gravitational mass are equivalent. Imagine a space rock flies by the Earth at any velocity, if we measure its inertial mass with one method and its gravitational mass with another method, we will always find the same value for the 2 masses.

b. Relativistic kinetic energy

Since we have the expression of relativistic inertial force in equation (50), we can compute the relativistic kinetic energy of a moving object by integrating (50). An accelerated object acquires kinetic energy as work is continuously done by the inertial force. Equation (52) expresses the relativistic differential work dw that the relativistic inertial force $\boldsymbol{F}_1$ does over the differential distance $d\boldsymbol{x}$, with $\boldsymbol{F}_1$ being given by (50).

We transform the term $\boldsymbol{\alpha}_1 \cdot d\boldsymbol{x}$ from (52) in (53). The proper inertial frame of the accelerated object is *Ref. 2*, $\boldsymbol{u}$ is its velocity and $\boldsymbol{v}_1$ that of the object, so $\boldsymbol{u}$ equals $\boldsymbol{v}_1$, see (54). Then, the term $\frac{d(\boldsymbol{v}_1{}^2)}{2}$ in (53) is transformed in (55). Combining (55), (53) and (52) we obtain the expression of dw in (56).

When the object is accelerated from velocity zero to velocity $\boldsymbol{u}$, the total work equals the integral of (56) over the velocity zero to $\boldsymbol{u}$ and this work is completely stored as kinetic energy in the object. The integration of (56) gives (57) which is the expression of the relativistic kinetic energy of an object with rest mass m_0 moving at the velocity u. This expression was already derived in « Velocity, mass, momentum and energy of an accelerated object in relativity » [2], but the equality of differential momentum (10) was not proven at the time.

The constant of integration for (57) was determined to satisfy the condition of the lower limit: $u=0$, $E_0 = 0$. But one constant should not satisfy 2 conditions at once. However, the upper limit condition: $u=c$, $E_c = m_0c^2$ which is the energy-mass equivalence is satisfied nevertheless, see (58). This is an amazing but consistent result which makes (57) reliable.

Law 7: Relativistic kinetic energy
The relativistic kinetic energy of an object with rest mass m_0 and moving at the velocity $\boldsymbol{u}$ equals $m_0c^2\left(1-\sqrt{1-\frac{u^2}{c^2}}\right)$

Using (50) $dw = \boldsymbol{F}_1 \cdot d\boldsymbol{x} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\boldsymbol{\alpha}_1 \cdot d\boldsymbol{x}$	(52)
$\boldsymbol{\alpha}_1 \cdot d\boldsymbol{x} = \frac{d\boldsymbol{v}_1}{dt_1}\cdot d\boldsymbol{x} = \frac{d\boldsymbol{x}}{dt_1}\cdot d\boldsymbol{v}_1 = \boldsymbol{v}_1 \cdot d\boldsymbol{v}_1 = \frac{d(\boldsymbol{v}_1{}^2)}{2}$	(53)
$\boldsymbol{v}_1 = \boldsymbol{u}$	(54)
$\frac{d(\boldsymbol{v}_1{}^2)}{2} = \frac{d(u^2)}{2} = -\frac{c^2}{2}d\left(1-\frac{u^2}{c^2}\right)$	(55)
Using (52) and (55) $dw = -\frac{m_0c^2}{2}\frac{d\left(1-\frac{u^2}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}} = -m_0c^2 d\sqrt{1-\frac{u^2}{c^2}}$	(56)
$E_u = \int_{u=0}^{u} dw = m_0c^2\left(1-\sqrt{1-\frac{u^2}{c^2}}\right)$	(57)
$u = c \Rightarrow E_c = m_0c^2$	(58)

c. Relativity and virial theorem

The Caltech professor Fritz Zwicky was first to notice that, "The galaxies (in the neighboring Coma Cluster of galaxies) were moving too fast within the cluster for the amount of illuminated stuff", wrote Marlene Gotz in her Presentation «Dark matter»[7]. Fritz Zwicky used the virial theorem [9] to draw this conclusion. Marlene Gotz continued: "Analyzing the motions of all kinds of clusters shows that they cannot be stable unless there is a large amount of mass than visible", which marks the birth of the notion "Dark matter".

The virial theorem that Fritz Zwicky used was derived using Newtonian kinetic energy. However the correct expression of kinetic energy is the relativistic one given in (57). Could a relativistic virial theorem explain the excess of velocity in clusters of galaxies?

Let us compare the velocities given by Newtonian kinetic energy and relativistic kinetic energy in the case where the velocity of an object decreases from u_o to u_N. Decrease of velocity and kinetic energy happens when objects leave a gravitational field. For example, Oumuamua[10] and the Pioneer spacecrafts[11] leaving the solar system. The Newtonian kinetic energy is expressed by (59) using which we compute the variation of velocity squared $\Delta {u_N}^2$ and the resulting difference of Newtonian kinetic energy ΔE in (60).

$E_{\text{Newton}} = m_0 \dfrac{u^2}{2}$	(59)
Newtonian ΔE, using (59) $\Delta E = \dfrac{m_0 c^2}{2}\left(\dfrac{{u_N}^2}{c^2} - \dfrac{{u_0}^2}{c^2}\right) = \dfrac{m_0 c^2}{2}\dfrac{\Delta {u_N}^2}{c^2}$	(60)

In (61) we compute the variation of velocity squared $\Delta {u_r}^2$ using relativistic kinetic energy (57). $\Delta {u_r}^2$ is so computed to give the same difference of energy ΔE. Dividing (61) by (60) we obtain (63). Then, we transform (63) into (64) which we multiply with $\Delta {u_N}^2$ to obtain (65).

Because kinetic energy decreases, $\Delta E<0$, $\Delta {u_N}^2$ is negative, see (66). Starting from the same velocity u_0 for the 2 computations, we obtain (67), which shows that ${u_r}^2$ the final velocity squared given by relativistic kinetic energy is bigger than ${u_N}^2$ the final velocity squared given by Newtonian kinetic energy. So, if a galaxy is far from the center of its cluster, the velocity given by relativistic kinetic energy would be bigger than that predicted by the Newtonian virial theorem. So, the result of a relativistic virial theorem would be closer to the observation.

Relativistic ΔE, using (57) $\Delta E = m_0 c^2\left(1 - \sqrt{1-\dfrac{{u_r}^2}{c^2}} - 1 + \sqrt{1-\dfrac{{u_0}^2}{c^2}}\right) = \dfrac{m_0 c^2\left(\dfrac{{u_r}^2}{c^2} - \dfrac{{u_0}^2}{c^2}\right)}{\sqrt{1-\dfrac{{u_0}^2}{c^2}} + \sqrt{1-\dfrac{{u_r}^2}{c^2}}} = \dfrac{m_0 c^2 \dfrac{\Delta {u_r}^2}{c^2}}{\sqrt{1-\dfrac{{u_0}^2}{c^2}} + \sqrt{1-\dfrac{{u_r}^2}{c^2}}}$	(61)
$\Delta {u_r}^2 = {u_r}^2 - {u_0}^2, \quad \Delta {u_N}^2 = {u_N}^2 - {u_0}^2$	(62)
Dividing (61) by (60) $\dfrac{\dfrac{m_0 c^2 \dfrac{\Delta {u_r}^2}{c^2}}{\sqrt{1-\dfrac{{u_0}^2}{c^2}} + \sqrt{1-\dfrac{{u_r}^2}{c^2}}}}{\dfrac{m_0 c^2}{2}\dfrac{\Delta {u_N}^2}{c^2}} = \dfrac{2}{\sqrt{1-\dfrac{{u_0}^2}{c^2}} + \sqrt{1-\dfrac{{u_r}^2}{c^2}}}\dfrac{\Delta {u_r}^2}{\Delta {u_N}^2} = \dfrac{\Delta E}{\Delta E} = 1$	(63)

In fact, equation (67) was already derived in « Analytical equation for Space-Time geodesics and relativistic orbit equation » [4] which explained the unexpected "boost in speed" or the bigger than expected velocity of Oumuamua[10] and the Pioneer anomaly[11].

If relativistic kinetic energy could explain the excess of velocity in clusters of galaxies, could the clusters be held stable by gravitational force? In the previous chapter we have shown that relativistic gravitational force is bigger than Newtonian gravitational force, see (38). In a cluster of galaxies, the gravitational attraction on one galaxy is acted by all the other galaxies and increased by the big relative velocities with respect to each one. So, the relativistic gravitational attraction could hold a cluster of galaxies stable.

$\dfrac{\Delta {u_r}^2}{\Delta {u_N}^2} = \dfrac{\sqrt{1-\dfrac{{u_0}^2}{c^2}} + \sqrt{1-\dfrac{{u_r}^2}{c^2}}}{2} \approx \dfrac{2 - \left(\dfrac{{u_0}^2}{2c^2} + \dfrac{{u_r}^2}{2c^2}\right)}{2} < 1$	(64)
Using (64) $\Delta {u_r}^2 < \Delta {u_N}^2$	(65)
Using (65) and (62), $\Delta E<0 \Rightarrow$ $\Delta {u_N}^2 = {u_N}^2 - {u_0}^2 < 0$ $\Rightarrow \Delta {u_r}^2 > \Delta {u_N}^2$	(66)
Using (62) in (66) ${u_r}^2 - {u_0}^2 > {u_N}^2 - {u_0}^2$ $\Rightarrow {u_r}^2 > {u_N}^2$	(67)

In consequence, relativistic kinetic energy as well as the increase of gravitational force could explain at least part of the problem of fast moving galaxies in clusters. In addition, it's worth deriving the relativistic virial theorem for space sciences.

6. Discussion

We have introduced the new notion "Proper inertial frame of an accelerated object" that allows defining acceleration for relativity while special relativity cannot. Using this new frame of reference, the equality of differential momentum is rigorously proven, new relativistic dynamical laws such as transformations for inertial and gravitational forces and transformations for inertial and gravitational masses have been demonstrated. Also, we have derived the relativistic expression of kinetic energy which satisfies the conditions of both the lower and upper limits, which strengthens the consistency of this new expression.

We have derived relativistic inertial mass directly from the new relativistic dynamics, which shows the true nature of the increase of relativistic inertial mass and gives a better understanding of special relativity. Also, we have demonstrated that gravitational mass increases with velocity just like inertial mass does, which satisfies the equivalence principle. It turns out that the variation of mass with velocity is the transformation of mass between 2 relatively moving frames like the transformation of time.

We find that our relativistic dynamical laws could in principle explain part of the fast rotation of disc galaxies and fast moving galaxies in cluster of galaxies. In fact relativistic dynamical effects increase the velocity of space objects without emitting any electromagnetic wave, which is exactly the property of Dark matter. So, the effect of the invisible "Dark matter" could be in reality relativistic effect. Of cause, this hypothesis needs to be verified through further theoretical and experimental researches.

References

[1] « Relativistic kinematics » https://www.academia.edu/44582027/Relativistic_kinematics, https://pengkuanonphysics.blogspot.com/2020/11/relativistic-kinematics.html

[2] « Velocity, mass, momentum and energy of an accelerated object in relativity » https://www.academia.edu/42616126/Velocity_mass_momentum_and_energy_of_an_accelerated_object_in_relativity, https://pengkuanonphysics.blogspot.com/2020/04/velocity-mass-momentum-and-energy-of.html

[3] «How galaxies make their rotation curves flat and what about dark matter?» https://www.academia.edu/46903516/How_galaxies_make_their_rotation_curves_flat_and_what_about_dark_matter https://pengkuanonphysics.blogspot.com/2021/04/how-galaxies-make-their-rotation-curves.html

[4] « Analytical equation for Space-Time geodesics and relativistic orbit equation » https://www.academia.edu/44540764/Analytical_orbit_equation_for_relativistic_gravity_without_using_Space_Time_geodesics https://pengkuanonphysics.blogspot.com/2020/11/analytical-orbit-equation-for.html

[5] «Magnetism and dark matter» https://www.academia.edu/10183169/Magnetism_and_dark_matter, https://pengkuanem.blogspot.com/2015/01/magnetism-and-dark-matter.html

[6] « Introduction to Special Relativity » by James H. Smith

[7] «Dark matter» Marlene Gotz (Jena) https://www.haus-der-astronomie.de/3440769/09darkmatter.pdf

[8] https://en.wikipedia.org/wiki/Equivalence_principle

[9] https://en.wikipedia.org/wiki/Virial_theorem

[10] https://en.wikipedia.org/wiki/%CA%BBOumuamua

[11] https://en.wikipedia.org/wiki/Pioneer_anomaly