Relativistic kinematics

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Abstract: Like in Newtonian kinematics, the relativistic change of reference frame must be a vector system of transformation laws for position, velocity and acceleration.

The Lorentz transformation transforms coordinates from one frame to another, Einstein's velocity-addition formula transforms the velocity of a moving point from one frame to another. But the velocity-addition formula is not a derivative of the Lorentz transformation like the transformation of coordinates and velocity in Galilean transformation. In addition, in special relativity there is not acceleration transformation at all. However, my theory of Time relativity provides a coherent vector system of transformation of position, velocity and acceleration.

$$\begin{aligned} x_2 &= x_1 - ut_1 \\ y_2 &= y_1 \\ z_2 &= z_1 \\ t_2 &= t_1\sqrt{1-\frac{u^2}{c^2}} \end{aligned} \qquad (1)$$

The indices 1 and 2 indicate the reference frame 1 and 2. u is the velocity of frame 2 in frame 1.

In Time relativity the transformation of coordinates is the system (1) which is the system (12) given in «Time relativity transformation of coordinates» with the identity of y and z added. The transformation of the x-component of velocity is the equation (4) which is the equation (19) given in «Time relativity transformation of velocity ». Below we will create the transformation of the vector velocity by deriving the transformation of the y and z components of velocity.

$$\boldsymbol{v}_1 = v_{1,x}\boldsymbol{i} + v_{1,y}\boldsymbol{j} + v_{1,z}\boldsymbol{k} \qquad (2)$$

$$\boldsymbol{v}_2 = v_{2,x}\boldsymbol{i} + v_{2,y}\boldsymbol{j} + v_{2,z}\boldsymbol{k} \qquad (3)$$

$\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the vector velocities of an object q in frame 1 and 2.

$$v_{2,x} = \frac{v_{1,x} - u}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (4)$$

$v_{1,x}$ and $v_{2,x}$ are the x components of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

a) Y and Z components of velocity

Let frame 1 be a stationary frame of reference in which the inertial frame 2 moves at the velocity u in the x direction, see Figure 1. Let q be an object moving in the frame 2 at the velocity $v_{2,y}$ in the y direction. During an infinitesimal time interval dt_2, q has moved the infinitesimal distance dy_2. So, the y-component of velocity in frame 2 equals dy_2/dt_2, see equation (5).

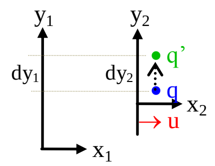

Figure 1

An object moves from q to q' in frame 2. The moved distance is dy_1 in frame 1 and dy_2 in frame 2.

For the same motion, q has moved the infinitesimal distance dy_1 during the time interval dt_1 in frame 1. Because there is not length contraction in the y direction, we have $dy_1=dy_2$ (see Figure 1) and write equation (6) where the term dy_1/dt_1 is the y-component of velocity in frame 1 $v_{1,y}$, see equation (7).

The ratio dt_1/dt_2 is given in equation (8) which is obtained using the time equation of the system (1). We introduce equation (8) in to (6), which becomes equation (9), which is the relation between the y-components of velocity in frames 1 and 2.

The z-components of velocity in frame 1 and 2 are $v_{1,z}$ and $v_{2,z}$ and are related by equation (10) in the same way as for y-components.

$$v_{2,y} = \frac{dy_2}{dt_2} \qquad (5)$$

$$\frac{dy_2}{dt_2} = \frac{dy_1}{dt_2} = \frac{dy_1}{dt_1}\frac{dt_1}{dt_2} \qquad (6)$$

$$\frac{dy_1}{dt_1} = v_{1,y} \qquad (7)$$

$$\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (8)$$

$$v_{2,y} = \frac{v_{1,y}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (9)$$

$$v_{2,z} = \frac{v_{1,z}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (10)$$

dt_1 and dt_2 are time intervals in which q moves dy_1 in frame 1 and dy_2 in frame 2.

b) Velocity vector

Let ***i, j*** and ***k*** be the unit vectors in the x, y and z directions. By convention, vectors are written in boldface letters. The velocity vector of q in frame 2 is written in equations (3) in which we substitute equations (4), (9) and (10) for the x, y and z components and obtain equation (11). The vector velocity of q in frame 1 is $\mathbf{v}_1$ written in equations (2), which we substitute for the corresponding terms in equation (11), we also substitute **u** for u***i***, with **u** being the vector velocity of the moving frame 2 in frame 1, see equation (12). Then, equation (11) becomes equation (13).

The velocity vectors of q in frame 1 and 2 are $\mathbf{v_1}$ and $\mathbf{v_2}$ and are related through equation (13).

$$\boldsymbol{v}_2 = \frac{v_{1,x} - u}{\sqrt{1 - \frac{u^2}{c^2}}}\boldsymbol{i} + \frac{v_{1,y}}{\sqrt{1 - \frac{u^2}{c^2}}}\boldsymbol{j} + \frac{v_{1,z}}{\sqrt{1 - \frac{u^2}{c^2}}}\boldsymbol{k} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\left(v_{1,x}\boldsymbol{i} + v_{1,y}\boldsymbol{j} + v_{1,z}\boldsymbol{k} - u\boldsymbol{i}\right) \quad (11)$$

$$\boldsymbol{u} = u \cdot \boldsymbol{i} \quad (12)$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (13)$$

$\boldsymbol{u}$ is the vector velocity of the frame 2 in frame 1.

c) Acceleration vector

Suppose that the velocity of q is 0 at time t_2 in frame 2 and is $\mathbf{u}$ in frame 1. After an infinitesimal time interval dt_2, q's velocity becomes $\mathbf{v_2}$ in frame 2. At the same moment, q's velocity becomes $\mathbf{v_1}$ in frame 1.

Because dt_2 is infinitesimal the vectors $\mathbf{v_2}$ and $\mathbf{v_1}-\mathbf{u}$ are infinitesimal too and can be written in differential form as $d\mathbf{v_2}$ and $d\mathbf{v_1}$ in equations (14), which we introduce into equation (13) to obtain equation (15), of which both sides are divided by dt_2 to make equation (16), in which the ratio dt_1/dt_2 is substituted by equation (8) to make equation (17), in which we substitute $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ for $\frac{d\boldsymbol{v}_1}{dt_2}$ and $\frac{d\boldsymbol{v}_2}{dt_2}$ to get equation (18), the relation between the vector accelerations, $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$, of the object q in frame 1 and 2.

$$d\boldsymbol{v}_2 = \boldsymbol{v}_2 - 0, \quad d\boldsymbol{v}_1 = \boldsymbol{v}_1 - \boldsymbol{u} \quad (14)$$

$$d\boldsymbol{v}_2 = \frac{d\boldsymbol{v}_1}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (15)$$

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1}\frac{dt_1}{dt_2} \quad (16)$$

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{1}{1 - \frac{u^2}{c^2}}\frac{d\boldsymbol{v}_1}{dt_1} \quad (17)$$

$$\boldsymbol{\alpha}_2 = \frac{\boldsymbol{\alpha}_1}{1 - \frac{u^2}{c^2}} \quad (18)$$

$\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are vector accelerations of q in frame 1 and 2.

d) Derivative of vectors

The velocity vector of q in frame 2 is the time derivative of its vector position $\mathbf{X}_2$, see equation (20). The derivative of $\mathbf{X}_2$ is derived in equation (21), where the coordinates (x_2, y_2, z_2) are written in terms of (x_1, y_1, z_1) the coordinates of q in frame 1 using system (1).

The vector position of q in frame 1 is $\mathbf{X}_1$, which is given in equation (19). We substitute $\mathbf{X}_1$ for the corresponding terms in equation (21), which then becomes equation (22). Since the derivatives of $\mathbf{X}_1$ and $\mathbf{X}_2$ are the vector velocity in frame 1 and 2, we write equation (22) into equation (25) which is the equation (13).

$$\boldsymbol{X}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k} \quad (19)$$

$$\boldsymbol{X}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k} \quad (20)$$

$$\frac{d\boldsymbol{X}_2}{dt_2} = \frac{d(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k})}{dt_2} = \frac{d\big((x_1 - ut_1)\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}\big)}{dt_1}\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{d(x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k} - ut_1\boldsymbol{i})}{dt_1} \quad (21)$$

$$\frac{d\boldsymbol{X}_2}{dt_2} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\left(\frac{d\boldsymbol{X}_1}{dt_1} - \boldsymbol{u}\right) \quad (22)$$

$\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are the vector positions of q in frame 1 and 2.

We can derive the acceleration vector also by taking the time derivative of the vector velocity of q, see equation (23). Because frame 2 is inertial, $\mathbf{u}$ is constant. The derivative of $\mathbf{v_1}$ and $\mathbf{v_2}$ are the vector accelerations $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$, then, equation (23) becomes (26), which is the equation (18).

In summary, the transformation of vector position is equation (24), the time derivative of which is the transformation of vector velocity equation (25), the time derivative of which is the transformation of vector acceleration equation (26). We see that the transformations in Time relativity form a consistent vector system. In the contrary, the transformations in Special relativity do not. For example, the velocity-addition formula is not the time derivative of the Lorentz transformation.

$$\frac{d\boldsymbol{v}_2}{dt_2} = \frac{d}{dt_2}\left(\frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1 - \frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{d\boldsymbol{v}_1}{dt_1}\frac{dt_1}{dt_2} = \frac{1}{1 - \frac{u^2}{c^2}}\frac{d\boldsymbol{v}_1}{dt_1} \quad (23)$$

$$\boldsymbol{X}_2 = \boldsymbol{X}_1 - \boldsymbol{u}t_1 \quad (24)$$

$$\left.\begin{aligned}\frac{d\boldsymbol{X}_1}{dt_1} &= \boldsymbol{v}_1 \\ \frac{d\boldsymbol{X}_2}{dt_2} &= \boldsymbol{v}_2\end{aligned}\right\} \Rightarrow \boldsymbol{v}_2 = \frac{\boldsymbol{v}_1 - \boldsymbol{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (25)$$

$$\left.\begin{aligned}\frac{d\boldsymbol{v}_1}{dt_1} &= \boldsymbol{\alpha}_1 \\ \frac{d\boldsymbol{v}_2}{dt_2} &= \boldsymbol{\alpha}_2\end{aligned}\right\} \Rightarrow \boldsymbol{\alpha}_2 = \frac{\boldsymbol{\alpha}_1}{1 - \frac{u^2}{c^2}} \quad (26)$$