

# General equation for Space-Time geodesics and orbit equation in relativistic gravity

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**Abstract:** Orbital precession of the planet Mercury was computed using Space-Time curvature which is extremely difficult to understand. This article exposes an analytical orbit equation for relativistic gravity and explains how it is derived without Space-Time geodesics. The value of orbital precession that this orbit equation gives is identical to that computed using General Relativity. This orbit equation is an analytical form of Space-Time geodesics for space objects and makes everyone able to compute the orbit of any object in gravitational field which obeys General Relativity using personal computer and its direct derivation from gravitational acceleration gives a new insight to General Relativity without the need of knowing Einstein tensor.

## 1. Orbit equation and orbital precession

General Relativity explains gravity as Space-Time curvature and orbits of planets as geodesics of curved Space-Time. However, this concept is extremely hard to understand and geodesics hard to compute. If we can find an analytical orbit equation for planets like Newtonian orbit equation, relativistic gravity will become intuitive and straightforward so that most people can understand.

From gravitational force and acceleration, I have derived the analytical orbit equation for relativistic gravity which is equation (1). Below I will explain the derivation of this equation. Albert Einstein had correctly predicted the orbital precession of planet Mercury which had definitely validated General Relativity. Equation (2) is the angle of orbital precession that this orbit equation gives, which is identical to the one Albert Einstein had given [1][2].

If this orbit equation gave the same result than Space-Time geodesics, then everyone can compute the orbit of any object in gravitational field which obeys General Relativity using personal computer rather than big or super computer. The direct derivation from gravitational acceleration allows everyone to see the mechanism linking gravity with curved Space-Time, which brings a new insight to General Relativity.

The derivation of the orbit equation is rather tedious and lengthy. So, for clarity of the reasoning and explanation, I have collected all the mathematical equations in the last section “Derivation of equations”, in which full details are provided to help readers for checking the validity of my mathematics.

### Notation convention

Bold letter: vector variables.  
Number in parentheses (*i*):  
equation number

### Subscript:

0: A known point  
m: Proper frame of m  
l: Local frame  
 $\infty$ : Frame at infinitely far  
N: Newtonian quantity  
a: Near perihelion  
b: Away from perihelion

### Relativistic orbit equation

$$r = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon \sin\left(\theta \left(1 - \left(1 - \frac{GM}{ac^2}\right) \frac{6GM}{a(1 - \varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)} \quad (1)$$

a: semi-major axis,  $\varepsilon$ : eccentricity

### Angle of precession per turn

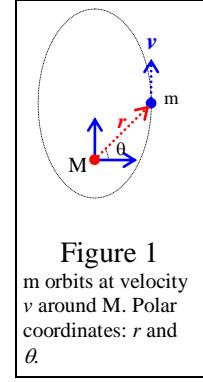
$$|\Delta\Phi| \approx \frac{6\pi GM}{a(1 - \varepsilon^2)c^2} \quad (2)$$

## 2. Relativistic dynamics

### a) Velocity in local frame

Take an attracting body of mass  $M$  around which orbits a small body of mass  $m$ , see Figure 1. We work with a polar coordinate system of which the body  $M$  sits at the origin. The position of the body  $m$  with respect to  $M$  is specified by the radial position vector  $\mathbf{r}$ , of which the magnitude is  $r$  and the polar angle is  $\theta$ .

Let the frame of reference “frame\_m” be an inertial frame that instantaneously moves with  $m$ . Frame\_m is the proper frame of  $m$  where the velocity of  $m$  is 0. So, Newton’s laws apply in this frame. Let  $\boldsymbol{\alpha}_m$  be the acceleration vector of  $m$  in frame\_m and the inertial force of  $m$  is  $m \cdot \boldsymbol{\alpha}_m$ , see equation (3). The gravitational force on  $m$  is given by equation (4). Equating (4) with (3), we get equation (5), the proper acceleration of  $m$  caused by gravitational force in frame\_m.



Let “frame\_l” be the local frame of reference in which  $M$  is stationary. In frame\_l  $m$  is under the effect of gravity of  $M$ , the velocity vector of  $m$  is  $\mathbf{v}_l$  and the acceleration of  $m$  is  $\boldsymbol{\alpha}_l$ . As frame\_m moves with  $m$ , it moves at the velocity  $\mathbf{v}_l$  in frame\_l.

The acceleration of  $m$  in frame\_m and frame\_l are respectively  $\boldsymbol{\alpha}_m$  and  $\boldsymbol{\alpha}_l$ . To transform  $\boldsymbol{\alpha}_l$  into  $\boldsymbol{\alpha}_m$  we use the transformation of acceleration between relatively moving frames which is the equation (18) in « [Relativistic kinematics](#) » [3][4], in which we replace  $\boldsymbol{\alpha}_l$  with  $\boldsymbol{\alpha}_l$ ,  $\boldsymbol{\alpha}_2$  with  $\boldsymbol{\alpha}_m$  and  $\mathbf{u}$  with  $\mathbf{v}_l$ . Then the transformation between  $\boldsymbol{\alpha}_m$  and  $\boldsymbol{\alpha}_l$  is equation (6).

Equating (5) with (6) we get equation (7), both sides of which are then dotted by the vector  $\mathbf{v}_l \cdot dt_l$ , see equation (8). On the left hand side of (8), we find  $dr$  the variation of the radial distance  $r$ , see (9). On the right hand side, we find the variation vector of velocity  $d\mathbf{v}_l$ , see (10), which, dotted by the velocity vector  $\mathbf{v}_l$ , gives  $d\mathbf{v}_l^2/2$  in (11).

Plugging (9) and (11) into (8), we get (12), both sides of which are differential expressions, see (13) and (15). Then, plugging (13) and (15) into (12) gives (16) which is a differential equation. (16) is integrated to give (17), with  $K$  being the integration constant. Then, we rearrange (17) to express  $\mathbf{v}_l^2/c^2$  in (18), which relates the local orbital velocity  $\mathbf{v}_l$  to the gravitational field of  $M$ .

The value of  $e^K$  is determined at a known point 0 at which the velocity is  $\mathbf{v}_0$  and the radial distance is  $r_0$ , see (19).

### b) Velocity in the frame at infinitely far

The orbit of a planet is the curve drawn by its position with respect to a frame infinitely far from  $M$  which is labeled “frame\_∞” and where the gravitational effect of  $M$  is 0. In this frame the time is  $t_\infty$ , the radial position vector is  $\mathbf{r}$  and the velocity vector  $\mathbf{v}_\infty$  equals the derivative of the radial position vector  $\mathbf{r}$  with respect to the time  $t_\infty$ , see (20).

The vector  $\mathbf{v}_l$  is the velocity in frame\_l and equals the derivative of the same  $\mathbf{r}$  with respect to the time in frame\_l,  $t_l$ . In (21)  $\mathbf{v}_l$  is transformed into the product of  $\mathbf{v}_\infty$  and  $\frac{dt_\infty}{dt_l}$  which is the time dilation factor of the Schwarzschild metric around  $M$ , see (22). By plugging (22) into (21) we obtain (23), which relates  $\mathbf{v}_l$  to  $\mathbf{v}_\infty$ . We substitute (23) for  $\mathbf{v}_l$  in (18) to get (25), which relates  $\mathbf{v}_\infty$  to the gravitational field of  $M$ .

Notice that the right hand side of (25) is the product of two parentheses, the first comes from (18) which relates velocity to gravitational force, the second comes from (22) which expresses the time dilation of Schwarzschild metric.

In the case of planets,  $\frac{GM}{c^2r}$  is very small and the term  $e^{K-\frac{2GM}{c^2r}}$  is simplified till the second order of  $\frac{GM}{c^2r}$  in (26). Substituting (26) for  $e^{K-\frac{2GM}{c^2r}}$  in (25) gives (27). We drop out the third order term  $4\left(\frac{GM}{c^2r}\right)^3$  from (27) and obtain the expression of  $\frac{u_\infty^2}{c^2}$  in (28).  $\mathbf{u}_\infty$  is the velocity vector of m in frame<sub>∞</sub> correct to the 2nd order of  $\frac{GM}{c^2r}$ .

### c) Velocity squared

The general expression of velocity vector in polar coordinates is given in equation (32) and Kepler's second law in (37). The derivation of (32) and (37) are in equations (29) through (37). Readers can find them in books or other references [5][6].

Let us express the velocity squared  $\mathbf{u}_\infty^2$  with the constant  $h$  of Kepler's second law. Using (32) we express  $\mathbf{u}_\infty^2$  in (38), of which the term  $\frac{dr d\theta}{d\theta dt}$  is expressed with  $h$  in (40) by using (39) and (37). Then using (37) and (40), (38) is transformed into (41), which expresses  $\mathbf{u}_\infty^2$  in terms of  $\frac{h}{r}$  instead of  $r$ .

### d) Differential equation for orbit

We divide (41) with  $c^2$  to express  $\frac{u_\infty^2}{c^2}$  in terms of  $\frac{h}{cr}$ , see (42). We plug (42) and (43) into (28) to get (45), which is the differential equation for relativistic orbit in frame<sub>∞</sub>. Solving (45) will give the analytical relativistic orbit equation.

## 3. Relativistic orbit

### a) Solving the differential equation for orbit

For solving the differential equation (45) we transform it into (46). As the right hand side of (46) is a messy polynomial of degree 2, we replace the coefficients with constants  $A$ ,  $B$  and  $C$  of which the expressions are in (47). Then (46) is transformed into the much simpler (49).

Here, we do a change of variable which is defined in (50) for  $s$  and (51) for  $\frac{ds}{d\theta}$ , with which we transform (49) into the simple differential equation (53). By using the formula of integration (54), we integrate (53) and find (55), with  $\theta_0$  being the integration constant.

The solution of the differential equation (53) is not unique. We choose the one with  $\theta_0=0$  and negative sign in (55). Taking the sine of this solution, we obtain (56). Then, for returning to the original variable we substitute (50) for  $s$  in (56) and obtain (58) which is the relativistic orbit equation expressed with  $A$ ,  $B$  and  $C$ . For getting rid of the constants  $A$ ,  $B$  and  $C$ , we use their expressions given in (47) to express the terms  $\frac{h}{Ac}$  and  $\frac{B}{A^2}$  in (60) and (62), which are plugged into (58) to obtain the final equation for  $r$  in (63).

(63) is the solution of the differential equation (45) and is the analytical relativistic orbit equation in the gravitational field of the attracting body M. Equation (63) expresses  $r$  in terms of  $\theta$ ,  $G$ ,  $M$ ,  $c$ ,  $K$  the integration constant in (17) and  $h$  the constant of Kepler's second law, see (17) and (37).

### b) Orbit as conic section curve

Orbits of planets are usually expressed as conic section curve. Let us express the relativistic orbit equation with the parameters of conic section which are  $l$ : the semi-latus rectum,  $a$ : semi-major axis,  $\varepsilon$ : eccentricity [6].

A general conic section curve is defined in polar coordinates by the function (64). By comparing (58) with (64), we find the  $l$ ,  $\varepsilon$  and  $\phi$  of the orbit, which are expressed in (65). Then, the relativistic orbit equation is expressed as a conic section curve in (66).

Now, we have to express the constant  $C$  with conic section parameters. For doing so, we compare (66) with (63) and find the expression of  $l$  in (67), with which we express the term  $\left(\frac{hc}{GM}\right)^2$  in (68). Plugging (68) into (69) gives the expression of  $C$  in (70).

The term  $\frac{e^K}{2e^K - 1}$  in (70) has to be expressed with conic section parameters. For doing so we take the square of  $\varepsilon$  (see (65)) in (71) to express the term  $\frac{B}{A^2}$ . We find another expression of  $\frac{B}{A^2}$  in (73) using (72). Equating (71) with (73) gives the expression of  $\frac{1-e^K}{2e^K - 1}$  with conic section parameters in (76), which allow us to express  $\frac{e^K}{2e^K - 1}$  with conic section parameters in (78). Introducing (78) into (70) gives the expression of  $C$  with conic section parameters in (79), which is then introduced into (66) to obtain (80).

Equation (80) is the final analytical relativistic orbit equation expressed with conic section parameters. For making sure that (80) is the correct orbit equation, we check its validity using 3 cases.

#### 4. Check of the orbit equation

##### a) Orbital precession

The first check is to see if the relativistic orbit equation (80) gives the correct angle of precession for planets. Due to General Relativistic effect elliptic orbit rotates in space, which is materialized by the angle that the major axis of elliptic orbit rotates from one revolution to the next. We denote this angle with  $\Delta\Phi$  and call it the angle of precession per turn.

For computing  $\Delta\Phi$ , suppose a first turn of orbit starts at the polar angle  $\theta=0$  and the radial distance is  $r_0 = a(1 - \varepsilon^2)$ , see (80).  $\theta=0$  makes  $\sin\left(\theta\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right) = 0$ , see (81). At the exact end of this turn the polar angle is  $\theta_1$ . The radial distance is again  $r_0$  because the second orbit starts here. As the radial distance is the same as the first turn,  $\theta_1$  must make  $\sin\left(\theta_1\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right) = 0$  too, see (80) and (82). So,  $\theta_1\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}} = 2\pi$ , see (83).

The polar angle of the starting point of the first turn is 0 or  $2\pi$  and that of the second turn is  $\theta_1$ . The difference of these 2 polar angles equals  $\Delta\Phi = \theta_1 - 2\pi$ , see (85), which means that the starting point of the second turn is at the angle  $\Delta\Phi$  with respect to starting point of the first turn. Or, seeing each turn as an ellipse, the major axis of the second ellipse has rotated the angle  $\Delta\Phi$  with respect to the first ellipse.

Because  $\left|1 - \frac{GM}{ac^2}\right| \frac{6GM}{a(1-\varepsilon^2)c^2} \ll 1$ , we simplify the expression of  $\theta_1$  (84) in (86) and get the value of  $\Delta\Phi$  in (88). Because  $1 - \frac{GM}{ac^2} \approx 1$  for planets, then  $\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ , see (89). The absolute value of  $\Delta\Phi$  is  $\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$ , which is identical to the one Albert Einstein had found [1], see (90).

So, the relativistic orbit equation (80) is the general equation for geodesics in curved Space-Time and has successively passed the check of angle of precession.

##### b) Variation of orbital velocity

The second check is to see if my theory of relativistic gravity predicts correctly the rate of variation of the velocity of a celestial body moving along its orbit. The method of this check is to compare the rate of variation given by my theory of relativistic gravity with that given by Newtonian gravity. For doing this comparison, we use the expression of the orbital velocity (91) but to the first order. So, the second order term  $6\left(\frac{GM}{c^2 r}\right)^2$  is dropped out in (92).

For finding a variation of orbital velocity for my relativistic gravity, we take 2 points on the orbit,  $a$  and  $b$ , compute the velocities at these 2 points which are  $\mathbf{u}_a$  and  $\mathbf{u}_b$ , then compute the difference of

velocity squared in (94). When we make  $1 - 2e^K = -1$  in (94), it gives that for Newtonian gravity, see (95). For comparing these 2 values, we divide (94) with (95) in (96) and the ratio of them is expressed with  $e^K$  in (97).

We do the comparison for hyperbolic orbit, for which the value of  $e^K$  at infinitely far is given in (99). Introducing (99) into (97) gives (100) which shows that the ratio  $\frac{(u_b^2 - u_a^2)_{\text{Relativity}}}{(u_b^2 - u_a^2)_{\text{Newtonian}}}$  is smaller than 1.

For real observation, we measure  $u_a$  the velocity of a celestial object at point  $a$ , with respect to which we compare the differences of velocity squared. So, for this comparison my relativistic gravity and Newtonian gravity take the measured  $u_a$  as the velocity at point  $a$ , see (101). Then, we get the ratio  $\frac{(u_b^2)_{\text{Relativity}} - u_a^2}{(u_b^2)_{\text{Newtonian}} - u_a^2}$  in (102), which is smaller than 1 because of (100), see (103).

For an object that leaves perihelion, its velocity decreases and the difference of velocity squared is negative, see (104). By multiplying (103) with the negative  $(u_b^2)_{\text{Newtonian}} - u_a^2$ , we obtain (106), which shows that the  $u_b$  given by my relativistic gravity is bigger than that given by Newtonian gravity.

Was bigger velocity on hyperbolic orbit really been observed? Yes. Fortunately, we have 2 real observations of bigger than expected velocity. The first is [Oumuamua](#)<sup>[7]</sup> which has got an unexpected “boost in speed” or a bigger than expected velocity when leaving perihelion. Equation (106) explains well this “boost in speed”. The second is [Pioneer anomaly](#)<sup>[8]</sup>, which is the unexplained resistance to slow down of the probes. Resistance to slow down means that the velocities of these probes are bigger than expected at a second point of measurement, which is well explained by (106) too. A third candidate with hyperbolic orbit is [2I/Borisov](#)<sup>[9]</sup>. But there is no information about the rate of variation of its velocity.

So, we can say that my relativistic gravity has successively passed the check of the rate of variation of orbital velocity.

As for General Relativity, because the observed bigger than expected velocities are unexpected, it is possible that no one has computed the variation of orbital velocity using General Relativity or that the result given by General Relativity does not match observation.

### c) Direction of precession

The third check is to see if my relativistic gravity predicts correctly the direction of precession. Today, the rotation of the major axis of elliptic orbit due to precession is thought to be in the same direction as the velocity of the planet, as shown in Figure 2.

However, equation (89) shows that  $\Delta\Phi$  is negative, that is, the major axis of an elliptic orbit rotates a negative angle in the polar coordinate system. I call this rotation negative precession, which is shown with the 2 ellipses in Figure 3. The first ellipse is vertical, the second is tilted to the right. An object with counterclockwise velocity begins on the first ellipse from the polar angle  $\theta=0$ . At the end of the orbit, it arrives at the polar angle  $\theta_f < 2\pi$ , which gives the major axis a clockwise rotation. So, the major axis of the ellipse rotates against the velocity of the object. This result is contrary to the current image of orbital precession.

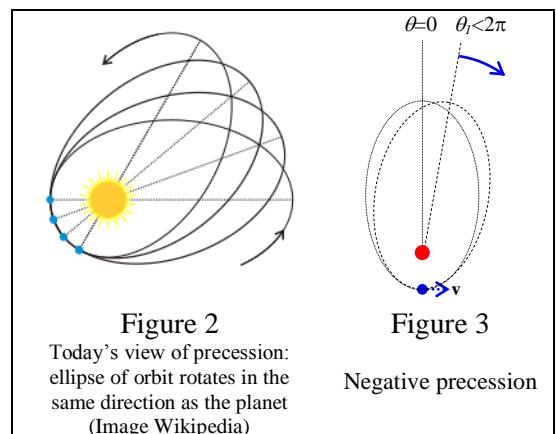


Figure 2  
Today's view of precession:  
ellipse of orbit rotates in the  
same direction as the planet  
(Image Wikipedia)

Figure 3  
Negative precession

Is the negative  $\Delta\Phi$  correct? For answering this question, let us compute the orbital velocity of a circular orbit, with  $r$  being the radius of the orbit. Because the orbit is circular, the orbital acceleration  $\alpha$  is expressed with the orbital velocity  $v$  in (107). For Newtonian gravity, gravitational acceleration

equals the orbital acceleration  $\alpha_N$ , see (108). By equating (107) with (108) while replacing  $v$  with  $v_N$  we get the Newtonian orbital velocity  $v_N$  in (109).

For relativistic gravity, the orbital acceleration is  $\alpha_l$  which is related to the gravitational acceleration through equation (110) with  $v_l$  being the local orbital velocity, see (7). By substituting (107) for  $\alpha_l$  in (110) and replacing  $v$  with  $v_l$ , we get (111), which is then transformed in (112) to express  $v_l^2$  in (113).

For comparing the relativistic and Newtonian orbital velocities, we divide (109) with (113) in (114), which shows that  $v_l$  is smaller than  $v_N$ , see (115). So, on a circular orbit my relativistic gravity gives a velocity smaller than that given by Newtonian gravity.

What the velocity of a circular orbit can tell about precession of elliptic orbit? Let us see Figure 4 which shows a circular orbit with a similar but slightly elliptical orbit. Suppose that a body m and a body n start doing the circular orbit from the point  $P_0$  in the direction of the arrows. m moves at the velocity  $v_N$  while n moves at  $v_l$ . Because  $v_l$  is smaller than  $v_N$ , n is slower than m. When m is back at the point  $P_0$ , n arrives only at the point  $P_r$ .

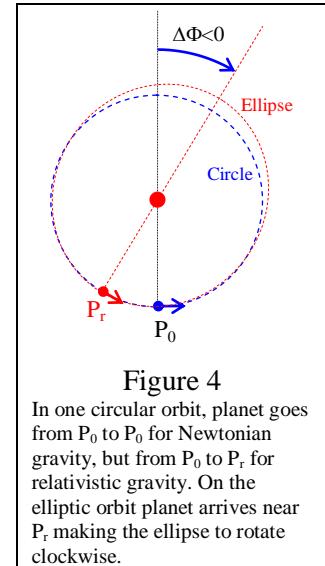


Figure 4

In one circular orbit, planet goes from  $P_0$  to  $P_0$  for Newtonian gravity, but from  $P_0$  to  $P_r$  for relativistic gravity. On the elliptic orbit planet arrives near  $P_r$  making the ellipse to rotate clockwise.

Now, let m and n do the slightly elliptical orbit starting from  $P_0$ . When m is back at the point  $P_0$ , n arrives only at a point near  $P_r$ . At the start the major axis of the ellipse passed across  $P_0$  and was vertical. But at the end, the major axis passes across  $P_r$  and tilts to the right, which makes the angle  $\Delta\Phi$  negative. That is, the major axis rotates against the orbital velocity. So, it is the slower velocity  $v_l$  that makes elliptical orbit to begin earlier on each turn, resulting in a negative rotation of the major axis of the ellipse. So, the slower relativistic velocity is the explanation of negative precession.

Why  $v_l$  is slower than  $v_N$ ? If an object gets a quantity of kinetic energy  $\Delta E$ , it will move at a higher velocity and the increased part of velocity  $\Delta v$  carries the kinetic energy  $\Delta E$ . For Newtonian laws, the velocity is increased by  $\Delta v_N$  while its mass stays the same. For the same quantity of energy increase  $\Delta E$ , according to Special Relativity the mass of the object is bigger than before, which makes the increase of velocity  $\Delta v_l$  smaller than  $\Delta v_N$ . So, the slower velocity for relativistic orbit is due to Special Relativity effect and this way Special Relativity corroborates the negative angle shown in Figure 4 and negative precession. So, negative precession is correct and my relativistic gravity has successively passed the theoretical check of direction of precession.

For General Relativity orbits are geodesics of a curved Space-Time which are not related to velocity or energy. If geodesics respect Special Relativity, then the velocity of objects on geodesics would be slower than that the Newtonian laws would give and thus, General Relativity would probably give negative precession too. If General Relativity really gives positive precession, then it would violate Special Relativity because the velocity on geodesics would be faster than the Newtonian one. But as no information exists about negative precession in General Relativity. I suspect that the direction of precession has never been computed using General Relativity.

However, the explanation with slower orbital velocity is not a rigorous proof. For really checking the validity of my relativistic gravity, we have to search for negative precession in the observations of celestial bodies, such as in the old data of the planet Mercury, in the recent data of the star S<sub>2</sub> that orbits the supermassive black hole Sagittarius A\* at the center of Milky Way. We can also use the navigation data of [Parker Solar Probe](#)<sup>[10]</sup> which will reach 200 km/s near the Sun.

## 5. Comments

What benefit the new theory of relativistic gravity provides with respect to General Relativity? Geodesics of curved Space-Time are so obscure that one cannot see how velocity and energy are

related to orbit. In the contrary, the analytical relativistic orbit equation is derived from velocity and gravitational force, which makes relativistic gravity intuitive and easy to understand, but also more powerful for studying gravity.

For example, I have already found with this equation that in relativistic gravity the asymptote of hyperbolic orbit will deviate from the Newtonian asymptote like orbital precession, that the difference between the maximum and minimum velocities of elliptic orbit deviates from the Newtonian one in the same order as orbital precession.

The derivation of the relativistic orbit equation shows clearly the physical meaning of orbital precession and has pinpointed the two contributors: 1) the slower velocity due to effect of Special Relativity, 2) the time dilation of Schwarzschild metric, see the 2 parentheses in (25). The smaller rate of variation of velocity on hyperbolic orbit (see (103)) explains well the “boost in speed” of [Oumuamua](#)<sup>[7]</sup> and [Pioneer anomaly](#)<sup>[8]</sup>. Such insights cannot be found easily with General Relativity.

## 6. Derivation of equations

### a) Acceleration and velocity in local frame

Inertial force in frame_m $\mathbf{F} = m\boldsymbol{\alpha}_m$ $\boldsymbol{\alpha}_m$ : acceleration vector in frame_m	(3)	Left hand side of (12) $-\frac{GM}{r^2}dr = d\left(\frac{GM}{r}\right)$	(13)
Gravitational Force $\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r$	(4)	Formula $d\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right) = -\frac{d\mathbf{v}_l^2}{c^2}$ $\Rightarrow \frac{d\mathbf{v}_l^2}{2} = -\frac{c^2}{2}d\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right)$	(14)
Acceleration equals force/m in frame_m $\boldsymbol{\alpha}_m = -\frac{GM}{r^2}\mathbf{e}_r$	(5)	Right hand side of (12), using (14) $\frac{1}{1 - \frac{\mathbf{v}_l^2}{c^2}}\frac{d\mathbf{v}_l^2}{2} = -\frac{c^2}{2}\frac{1}{1 - \frac{\mathbf{v}_l^2}{c^2}}d\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right)$ $= -\frac{c^2}{2}d\ln\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right)$	(15)
Relativistic transformation of acceleration between frame_l and frame_m [3][4] $\boldsymbol{\alpha}_m = \frac{\boldsymbol{\alpha}_l}{1 - \frac{\mathbf{v}_l^2}{c^2}}$ Subscript l: in local frame, $\mathbf{v}_l$ : velocity vector, $\boldsymbol{\alpha}_l$ : acceleration vector	(6)	Using (13) and (15), (12) becomes $d\left(\frac{GM}{r}\right) = -\frac{c^2}{2}d\ln\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right)$	(16)
Equating (5) with (6) $-\frac{GM}{r^2}\mathbf{e}_r = \frac{\boldsymbol{\alpha}_l}{1 - \frac{\mathbf{v}_l^2}{c^2}}$	(7)	(16) integrated $\ln\left(1 - \frac{\mathbf{v}_l^2}{c^2}\right) + \frac{2GM}{c^2r} = K$ K: integration constant	
(7) dotted with $\mathbf{v}_l \cdot dt_l$ $-\frac{GM}{r^2}\mathbf{e}_r \cdot \mathbf{v}_l dt_l = \frac{\boldsymbol{\alpha}_l \cdot \mathbf{v}_l dt_l}{1 - \frac{\mathbf{v}_l^2}{c^2}}$	(8)	Then $1 - \frac{\mathbf{v}_l^2}{c^2} = e^{K - \frac{2GM}{c^2r}}$	(17)
$t_l$ : time in frame_l		Rearrangement of (17) $\frac{\mathbf{v}_l^2}{c^2} = 1 - e^{K - \frac{2GM}{c^2r}}$	(18)
On left hand side we have dr $\mathbf{e}_r \cdot \mathbf{v}_l dt_l = \mathbf{e}_r \cdot d\mathbf{r} = dr$	(9)	Value of $e^K$ at point 0, see (17) $e^K = \left(1 - \frac{\mathbf{v}_0^2}{c^2}\right)e^{\frac{2GM}{c^2r_0}}$	(19)
On right hand side we have $d\mathbf{v}_l$ $d\mathbf{v}_l = \boldsymbol{\alpha}_l dt_l$	(10)		
(10) makes			
$\boldsymbol{\alpha}_l \cdot \mathbf{v}_l dt_l = \mathbf{v}_l \cdot d\mathbf{v}_l = \frac{d\mathbf{v}_l^2}{2}$	(11)		
Using (9) and (11), equation (8) becomes			
$-\frac{GM}{r^2}dr = \frac{1}{1 - \frac{\mathbf{v}_l^2}{c^2}}\frac{d\mathbf{v}_l^2}{2}$	(12)		

### b) Velocity in the frame at infinitely far

Velocity vector in frame <sub>∞</sub>	
$\mathbf{v}_\infty = \frac{d\mathbf{r}}{dt_\infty}$	(20)
$t_\infty$ : time in frame <sub>∞</sub>	
Velocity vector in frame <sub>l</sub>	
$\mathbf{v}_l = \frac{d\mathbf{r}}{dt_l} = \frac{d\mathbf{r}}{dt_\infty} \frac{dt_\infty}{dt_l}$	(21)
$= \mathbf{v}_\infty \frac{dt_\infty}{dt_l}$	
$t_l$ : time in frame <sub>l</sub>	
Time dilation in Schwarzschild metric	
$t_l = t_\infty \sqrt{1 - \frac{2GM}{c^2 r}}$	(22)
$\Rightarrow \frac{dt_\infty}{dt_l} = \frac{1}{\sqrt{1 - \frac{2GM}{c^2 r}}}$	

Plugging (22) into (21)	
$\mathbf{v}_l = \frac{\mathbf{v}_\infty}{\sqrt{1 - \frac{2GM}{c^2 r}}}$	(23)
Substituting (23) for $\mathbf{v}_l$ in (18)	
$\frac{\mathbf{v}_\infty^2}{c^2 \left(1 - \frac{2GM}{c^2 r}\right)} = 1 - e^{K - \frac{2GM}{c^2 r}}$	(24)
(24) gives	
$\frac{\mathbf{v}_\infty^2}{c^2} = \left(1 - e^{K - \frac{2GM}{c^2 r}}\right) \left(1 - \frac{2GM}{c^2 r}\right)$	(25)
For very small $\frac{GM}{c^2 r}$	
$e^{K - \frac{2GM}{c^2 r}} \approx e^K \left(1 - \frac{2GM}{c^2 r} + \frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^2\right)$	(26)

Substituting (26) for  $e^{K - \frac{2GM}{c^2 r}}$  in (25)

$$\begin{aligned} \frac{\mathbf{v}_\infty^2}{c^2} &\approx \left(1 - e^K \left(1 - \frac{2GM}{c^2 r} + \frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^2\right)\right) \left(1 - \frac{2GM}{c^2 r}\right) \\ &= 1 - \frac{2GM}{c^2 r} - e^K \left(\left(1 - \frac{2GM}{c^2 r}\right)^2 + \frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^2 \left(1 - \frac{2GM}{c^2 r}\right)\right) \\ &= 1 - \frac{2GM}{c^2 r} - e^K \left(1 - \frac{4GM}{c^2 r} + \left(\frac{2GM}{c^2 r}\right)^2 + \frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^2 - \frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^3\right) \end{aligned} \quad (27)$$

By dropping out  $\frac{1}{2} \left(\frac{2GM}{c^2 r}\right)^3$  from (27)

$$\frac{\mathbf{u}_\infty^2}{c^2} = 1 - \frac{2GM}{c^2 r} - e^K \left(1 - \frac{4GM}{c^2 r} + 6 \left(\frac{GM}{c^2 r}\right)^2\right) \quad (28)$$

$\mathbf{u}_\infty$  is the orbit velocity vector in frame<sub>∞</sub> correct to the 2<sup>nd</sup> order of  $\frac{GM}{c^2 r}$

### c) Velocity vector in polar coordinates and Kepler's second law

Radial position vector	
$\mathbf{r} = r \mathbf{e}_r$	(29)
Velocity vector is the time derivative of $\mathbf{r}$	
$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\mathbf{e}_r}{dt}$	(30)
Derivatives of unit vectors	
$\frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt} \mathbf{e}_\theta, \quad \frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt} \mathbf{e}_r$	(31)
Using (31) in (30) velocity vector is	
$\mathbf{v} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta$	(32)
Acceleration vector is the second time derivative of $\mathbf{r}$	
$\mathbf{a} = \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right) = \left( \frac{d^2 r}{dt^2} \mathbf{e}_r + \frac{dr}{dt} \frac{d\mathbf{e}_r}{dt} \right) + \left( \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta + r \frac{d^2 \theta}{dt^2} \mathbf{e}_\theta + r \frac{d\theta}{dt} \frac{d\mathbf{e}_\theta}{dt} \right)$	(33)

Using (31), (33) becomes

$$\begin{aligned} \mathbf{a} &= \frac{d^2 r}{dt^2} \mathbf{e}_r + \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta \\ &\quad + \frac{dr}{dt} \frac{d\theta}{dt} \mathbf{e}_\theta + r \frac{d^2 \theta}{dt^2} \mathbf{e}_\theta - r \frac{d\theta}{dt} \frac{d\theta}{dt} \mathbf{e}_r \\ &= \left( \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + \left( 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \right) \mathbf{e}_\theta \end{aligned} \quad (34)$$

Tangent acceleration is 0

$$2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} = 0$$

Division by  $r \frac{d\theta}{dt}$

$$\begin{aligned} \frac{2 dr}{r dt} + \left( \frac{d\theta}{dt} \right)^{-1} \frac{d}{dt} \left( \frac{d\theta}{dt} \right) &= 0 \\ \Rightarrow 2 \frac{d \ln(r)}{dt} + \frac{d}{dt} \left( \ln \left( \frac{d\theta}{dt} \right) \right) &= 0 \end{aligned} \quad (35)$$

Integration of (35)

$$\ln(r^2) + \ln \left( \frac{d\theta}{dt} \right) = \ln(h) \quad (36)$$

$\ln(h)$ : integration constant

From (36), Kepler's second law

$$r^2 \frac{d\theta}{dt} = h = \text{Constant} \quad (37)$$

#### d) Velocity squared using Kepler's second law

Velocity squared is, see (32)

$$\begin{aligned}\mathbf{u}_\infty^2 &= \left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2 \\ &= \left(\frac{dr}{d\theta} \frac{d\theta}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2\end{aligned}\quad (38)$$

Formula

$$\frac{dr}{d\theta} = -r^2 \frac{d}{d\theta} \left(\frac{1}{r}\right)\quad (39)$$

Using (39) and (37) for  $\frac{dr}{d\theta} \frac{d\theta}{dt}$

$$\begin{aligned}\frac{dr}{d\theta} \frac{d\theta}{dt} &= -r^2 \frac{d}{d\theta} \left(\frac{1}{r}\right) \frac{d\theta}{dt} \\ &= -\left(r^2 \frac{d\theta}{dt}\right) \frac{d}{d\theta} \left(\frac{1}{r}\right) \\ &= -\frac{d}{d\theta} \left(\frac{h}{r}\right)\end{aligned}\quad (40)$$

Using (37) and (40), (38) becomes

$$\mathbf{u}_\infty^2 = \left(\frac{d}{d\theta} \left(\frac{h}{r}\right)\right)^2 + \left(\frac{h}{r}\right)^2\quad (41)$$

#### e) Differential equation for orbit

(41) divide by  $c^2$

$$\frac{\mathbf{u}_\infty^2}{c^2} = \frac{1}{c^2} \left(\frac{d}{d\theta} \left(\frac{h}{r}\right)\right)^2 + \frac{1}{c^2} \left(\frac{h}{r}\right)^2 = \left(\frac{d}{d\theta} \left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2\quad (42)$$

Formula

$$\frac{GM}{c^2 r} = \frac{GM}{hc} \frac{h}{cr}\quad (43)$$

Using (42) and (43) in (28)

$$\begin{aligned}\left(\frac{d}{d\theta} \left(\frac{h}{cr}\right)\right)^2 + \left(\frac{h}{cr}\right)^2 &= 1 - 2 \frac{GM}{hc} \frac{h}{cr} - e^K \left(1 - 4 \frac{GM}{hc} \frac{h}{cr} + 6 \left(\frac{GM}{hc} \frac{h}{cr}\right)^2\right) \\ &= 1 - e^K - 2 \frac{GM}{hc} \frac{h}{cr} + 4e^K \frac{GM}{hc} \frac{h}{cr} - 6e^K \left(\frac{GM}{hc} \frac{h}{cr}\right)^2\end{aligned}$$

Then

$$\left(\frac{d}{d\theta} \left(\frac{h}{cr}\right)\right)^2 = 1 - e^K + 2(2e^K - 1) \frac{GM}{hc} \frac{h}{cr} - 6e^K \left(\frac{GM}{hc}\right)^2 \left(\frac{h}{cr}\right)^2 - \left(\frac{h}{cr}\right)^2\quad (44)$$

From (44)

$$\left(\frac{d}{d\theta} \left(\frac{h}{cr}\right)\right)^2 = 1 - e^K + 2(2e^K - 1) \frac{GM}{hc} \frac{h}{cr} - \left(1 + 6e^K \left(\frac{GM}{hc}\right)^2\right) \left(\frac{h}{cr}\right)^2\quad (45)$$

This is the Relativistic orbit differential equation

#### f) Solving the differential equation for orbit

Transformation of (45)

$$\left(\frac{d}{d\theta} \left(\frac{h}{cr}\right)\right)^2 = -\left(1 + 6e^K \left(\frac{GM}{hc}\right)^2\right) \left(-\frac{1 - e^K}{1 + 6e^K \left(\frac{GM}{hc}\right)^2} - 2 \left(\frac{2e^K - 1}{1 + 6e^K \left(\frac{GM}{hc}\right)^2}\right) \frac{GM}{hc} \frac{h}{cr} + \left(\frac{h}{cr}\right)^2\right)\quad (46)$$

Replacing coefficients by constants  $A, B$  and  $C$

$$C = 1 + 6e^K \left(\frac{GM}{hc}\right)^2, A = \frac{2e^K - 1}{C} \frac{GM}{hc}, B = \frac{1 - e^K}{C}\quad (47)$$

(46) is expressed using  $A$ ,  $B$  and  $C$

$$\begin{aligned} \left( \frac{d}{d\theta} \left( \frac{h}{cr} \right) \right)^2 &= -C \left( -B - 2A \frac{h}{cr} + \left( \frac{h}{cr} \right)^2 \right) \\ &= -C \left( -A^2 - B + A^2 - 2A \frac{h}{cr} + \left( \frac{h}{cr} \right)^2 \right) \\ &= -C \left( -(A^2 + B) + \left( \frac{h}{cr} - A \right)^2 \right) \\ &= C(A^2 + B) \left( 1 - \frac{\left( \frac{h}{cr} - A \right)^2}{A^2 + B} \right) \end{aligned} \quad (48)$$

(48) divided by  $A^2 + B$

$$\left( \frac{d}{d\theta} \left( \frac{h}{\sqrt{A^2 + B}} \right) \right)^2 = C \left( 1 - \left( \frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}} \right)^2 \right) \quad (49)$$

Change of variable

$$s = \frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}} = \frac{\frac{h}{cr} - 1}{\sqrt{1 + \frac{B}{A^2}}} \quad (50)$$

$$\frac{ds}{d\theta} = \frac{d}{d\theta} \left( \frac{\frac{h}{cr} - A}{\sqrt{A^2 + B}} \right) = \frac{d}{d\theta} \left( \frac{\frac{h}{cr}}{\sqrt{A^2 + B}} \right) \quad (51)$$

Using (50) and (51) to transform (49)

$$\left( \frac{ds}{d\theta} \right)^2 = C(1 - s^2) \Rightarrow \frac{ds}{d\theta} = \pm \sqrt{C} \sqrt{1 - s^2} \quad (52)$$

Then

$$\frac{ds}{\sqrt{1 - s^2}} = \pm \sqrt{C} d\theta \quad (53)$$

Formula of integration

$$\int \frac{ds}{\sqrt{1 - s^2}} = \sin^{-1} s \quad (54)$$

Using (54), the integration of (53) gives

$$\sin^{-1} s = \pm \sqrt{C}(\theta + \theta_0) \quad (55)$$

$\theta_0$ : integration constant

By choosing  $\theta_0=0$ , keeping the negative sign and taking sine of (55)

$$s = -\sin \theta \sqrt{C} \quad (56)$$

Returning to the original variable by using (50), (56) becomes

$$\frac{\frac{h}{cr} - 1}{\sqrt{1 + \frac{B}{A^2}}} = -\sin \theta \sqrt{C} \quad (57)$$

(57) is transformed into

$$r = \frac{\frac{h}{Ac}}{1 - \sqrt{1 + \frac{B}{A^2}} \sin(\theta \sqrt{C})} \quad (58)$$

Solution of the differential equation (49)

Using  $A$  from (47)

$$\begin{aligned} \frac{h}{Ac} &= \frac{h}{c} \left( \frac{2e^K - 1}{C} \frac{GM}{hc} \right)^{-1} \\ &= \frac{C}{2e^K - 1} \frac{h^2}{GM} \end{aligned} \quad (59)$$

Using  $C$  from (47), (59) becomes

$$\frac{h}{Ac} = \frac{1 + 6e^K \left( \frac{GM}{hc} \right)^2}{2e^K - 1} \frac{h^2}{GM} \quad (60)$$

Using  $A$  and  $B$  from (47) for  $\frac{B}{A^2}$

$$\begin{aligned} \frac{B}{A^2} &= \frac{1 - e^K}{C} \left( \frac{2e^K - 1}{C} \frac{GM}{hc} \right)^{-2} \\ &= C \frac{1 - e^K}{(2e^K - 1)^2} \left( \frac{hc}{GM} \right)^2 \end{aligned} \quad (61)$$

Using  $C$  from (47), (61) becomes

$$\begin{aligned} \frac{B}{A^2} &= \frac{1 - e^K}{(2e^K - 1)^2} \left( \frac{hc}{GM} \right)^2 \left( 1 + 6e^K \left( \frac{GM}{hc} \right)^2 \right) \\ &= \frac{1 - e^K}{(2e^K - 1)^2} \left( \left( \frac{hc}{GM} \right)^2 + 6e^K \right) \end{aligned} \quad (62)$$

Using (60), (62) and (47) for  $C$ , (58) becomes

$$r = \frac{\frac{1 + 6e^K \left( \frac{GM}{hc} \right)^2}{2e^K - 1} \frac{h^2}{GM}}{1 - \sqrt{1 + \frac{1 - e^K}{(2e^K - 1)^2} \left( \left( \frac{hc}{GM} \right)^2 + 6e^K \right)}} \sin \left( \theta \sqrt{1 + 6e^K \left( \frac{GM}{hc} \right)^2} \right) \quad (63)$$

This is the solution of the relativistic orbit differential equation (45) and the relativistic orbit equation in the gravitational field of the body M.

### g) Orbit as conic section curve

Conic section in polar coordinates <sup>[5][6]</sup> $r = \frac{l}{1 - \varepsilon \sin \phi} \quad (64)$ <p><math>l</math>: the semi-latus rectum, <math>\varepsilon</math>: eccentricity</p>	
Comparing (58) with (64) we have $l = \frac{h}{Ac}, \quad \varepsilon = \sqrt{1 + \frac{B}{A^2}}, \quad \phi = \theta\sqrt{C} \quad (65)$	
(58) becomes $r = \frac{l}{1 - \varepsilon \sin(\theta\sqrt{C})} \quad (66)$	
Comparing (66) with (63), we obtain $\begin{aligned} l &= \frac{1 + 6e^K \left(\frac{GM}{hc}\right)^2 h^2}{2e^K - 1} GM \\ &= \frac{\frac{h^2}{GM} + 6e^K \frac{GM}{c^2}}{2e^K - 1} \\ &= \frac{GM \left(\frac{hc}{GM}\right)^2 + 6e^K}{c^2 \frac{2e^K - 1}{2e^K - 1}} \end{aligned} \quad (67)$	
(67) gives $\left(\frac{hc}{GM}\right)^2$ $\left(\frac{hc}{GM}\right)^2 = \frac{lc^2}{GM} (2e^K - 1) - 6e^K \quad (68)$	
Using (68) in (47) for $C$ $\begin{aligned} C &= 1 + 6e^K \left(\frac{GM}{hc}\right)^2 \\ &= 1 + 6e^K \left(\frac{lc^2}{GM} (2e^K - 1) - 6e^K\right)^{-1} \\ &= 1 + \left(\frac{lc^2}{GM} \left(\frac{2e^K - 1}{6e^K}\right) - 1\right)^{-1} \\ &= \frac{lc^2}{6GM} \frac{2e^K - 1}{e^K} - 1 + 1 \\ &= \frac{lc^2}{6GM} \frac{2e^K - 1}{e^K} - 1 \end{aligned} \quad (69)$	
(69) becomes $C = \left(1 - \frac{e^K}{2e^K - 1} \frac{6GM}{lc^2}\right)^{-1} \quad (70)$	

Search for $\frac{e^K}{2e^K - 1}$ , using $\varepsilon = \sqrt{1 + \frac{B}{A^2}}$ from (65) $\frac{B}{A^2} = \varepsilon^2 - 1 \quad (71)$	
(62) divided by (59) $\frac{B}{A^2} = \frac{C}{(2e^K - 1)^2} \left(\frac{hc}{GM}\right)^2 \quad (72)$	
Using $l = \frac{h}{Ac}$ from (65), (72) gives $\frac{B}{A^2} = \frac{h}{Ac} \frac{c^2}{GM} \frac{1 - e^K}{2e^K - 1} = \frac{lc^2}{GM} \frac{1 - e^K}{2e^K - 1} \quad (73)$	
Equating (71) with (73) $\frac{lc^2}{GM} \frac{1 - e^K}{2e^K - 1} = \varepsilon^2 - 1$	
Then $\frac{1 - e^K}{2e^K - 1} = \frac{(\varepsilon^2 - 1)GM}{lc^2} \quad (74)$	
Conic section formula $l = a(1 - \varepsilon^2) \quad (75)$ $a$ : semi-major axis	
Using (75) in (74) $\frac{1 - e^K}{2e^K - 1} = \frac{(\varepsilon^2 - 1)GM}{a(1 - \varepsilon^2)c^2} = -\frac{GM}{ac^2} \quad (76)$	
Searching $\frac{e^K}{2e^K - 1}$ in $\frac{1 - e^K}{2e^K - 1}$ $\frac{1 - e^K}{2e^K - 1} = \frac{1 - 2e^K + e^K}{2e^K - 1} = \frac{e^K}{2e^K - 1} - 1 \quad (77)$	
Using (76) in (77) $\frac{e^K}{2e^K - 1} = \frac{1 - e^K}{2e^K - 1} + 1 = 1 - \frac{GM}{ac^2} \quad (78)$	
Using (78) and (75) in (70) $C = \left(1 - \left(1 - \frac{GM}{ac^2}\right) \frac{6GM}{a(1 - \varepsilon^2)c^2}\right)^{-1} \quad (79)$	

Using (75) and (79), (66) becomes $a(1 - \varepsilon^2)$ $r = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon \sin\left(\theta \left(1 - \left(1 - \frac{GM}{ac^2}\right) \frac{6GM}{a(1 - \varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right)} \quad (80)$	
This is the <b>Relativistic orbit as conic section curve</b> and solution of the differential equation (45)	

## h) Orbital precession

Orbit starts from $\theta=0$ , (80) gives	
$\sin\left(\theta\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right) = 0$	(81)
Orbit ends at $\theta_1$ . For the same radial distance $\theta_1$ must make	
$\sin\left(\theta_1\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}}\right) = 0$	(82)
As $\theta_1 \neq 0$	
$\theta_1\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{-\frac{1}{2}} = 2\pi$	(83)
(83) gives	
$\theta_1 = 2\pi\left(1 - \left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}\right)^{\frac{1}{2}}$	(84)
The end of the orbit is at $\theta_1$ and has rotated the angle $\Delta\Phi$ with respect to the starting point at $2\pi$	
$\Delta\Phi = \theta_1 - 2\pi$	(85)
This is the angle of precession per turn	

Simplification of (84) for $\left 1 - \frac{GM}{ac^2}\right  \ll 1$	
$\theta_1 \approx 2\pi - \frac{2\pi}{2}\left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}$	(86)
Using (86), (85) becomes	
$\Delta\Phi \approx 2\pi - \frac{2\pi}{2}\left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2} - 2\pi$	(87)
Then	
$\Delta\Phi \approx -\left(1 - \frac{GM}{ac^2}\right)\frac{6GM}{a(1-\varepsilon^2)c^2}$	(88)
Angle of precession per turn	
For planets, $1 - \frac{GM}{ac^2} \approx 1$	
$\Delta\Phi \approx -\frac{6\pi GM}{a(1-\varepsilon^2)c^2}$	(89)
Absolute value	
$ \Delta\Phi  \approx \frac{6\pi GM}{a(1-\varepsilon^2)c^2}$	(90)
The value Albert Einstein had given	

## i) Variation of orbital velocity

Orbital velocity in frame $\infty$ , (28) multiplied by $c^2$	
$\mathbf{u}_\infty^2 = c^2 - \frac{2GM}{r} - c^2 e^K \left(1 - \frac{4GM}{c^2 r} + 6 \left(\frac{GM}{c^2 r}\right)^2\right)$	(91)
Neglecting the second order $6 \left(\frac{GM}{c^2 r}\right)^2$ in (91)	
$\mathbf{u}_\infty^2 \approx c^2 - \frac{2GM}{r} - c^2 e^K + c^2 e^K \frac{4GM}{c^2 r}$	(92)
$= c^2 - c^2 e^K - \frac{2GM}{r} (1 - 2e^K)$	
Relativistic orbit velocities at points $a$ and $b$ are $\mathbf{u}_a$ and $\mathbf{u}_b$ . Difference of velocity squared computed using (92)	
$(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}$	(93)
$= c^2 - c^2 e^K - \frac{2GM(1 - 2e^K)}{r_b}$	
$- \left(c^2 - c^2 e^K - \frac{2GM(1 - 2e^K)}{r_a}\right)$	
Then (93) becomes	
$(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}$	(94)
$= -2GM(1 - 2e^K) \left(\frac{1}{r_b} - \frac{1}{r_a}\right)$	
When $1 - 2e^K = -1$ , (94) gives that for Newtonian orbit	
$(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Newtonian}} = 2GM \left(\frac{1}{r_b} - \frac{1}{r_a}\right)$	(95)
Ratio between (94) and (95)	
$\frac{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}}{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Newtonian}}} = \frac{-2GM(1 - 2e^K) \left(\frac{1}{r_b} - \frac{1}{r_a}\right)}{2GM \left(\frac{1}{r_b} - \frac{1}{r_a}\right)}$	(96)
Then	
$\frac{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}}{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Newtonian}}} = 2e^K - 1$	(97)

Hyperbolic orbit at infinitely far, $r_0 = \infty$	
$e^K = \left(1 - \frac{\mathbf{v}_0^2}{c^2}\right) e^{\frac{2GM}{c^2 r_0}} = \left(1 - \frac{\mathbf{v}_0^2}{c^2}\right) e^0$	(98)
See (19)	
Then	
$e^K = 1 - \frac{\mathbf{v}_0^2}{c^2}$	(99)
Introducing (99) into (97)	
$\frac{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}}{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Newtonian}}} = 2 \left(1 - \frac{\mathbf{v}_0^2}{c^2}\right) - 1$	(100)
$= 1 - 2 \frac{\mathbf{v}_0^2}{c^2} < 1$	
$\mathbf{u}_a$ is the observed velocity at point $a$	
$\mathbf{u}_a^2 = (\mathbf{u}_a^2)_{\text{Relativity}} = (\mathbf{u}_a^2)_{\text{Newtonian}}$	(101)
Using (101) we have	
$\frac{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Relativity}}}{(\mathbf{u}_b^2 - \mathbf{u}_a^2)_{\text{Newtonian}}} = \frac{(\mathbf{u}_b^2)_{\text{Relativity}} - \mathbf{u}_a^2}{(\mathbf{u}_b^2)_{\text{Newtonian}} - \mathbf{u}_a^2}$	(102)
Then, (100) makes (102) to be smaller than 1	
$\frac{(\mathbf{u}_b^2)_{\text{Relativity}} - \mathbf{u}_a^2}{(\mathbf{u}_b^2)_{\text{Newtonian}} - \mathbf{u}_a^2} < 1$	(103)
Leaving perihelion, velocity decreases	
$(\mathbf{u}_b^2)_{\text{Newtonian}} - \mathbf{u}_a^2 < 0$	(104)
(103) multiplied by the negative (104)	
$(\mathbf{u}_b^2)_{\text{Relativity}} - \mathbf{u}_a^2 > (\mathbf{u}_b^2)_{\text{Newtonian}} - \mathbf{u}_a^2$	(105)
Then, velocities away from perihelion	
$(\mathbf{u}_b^2)_{\text{Relativity}} > (\mathbf{u}_b^2)_{\text{Newtonian}}$	(106)

### j) Circular orbit

Acceleration of circular orbit $\alpha = -\frac{v^2}{r}$ v: orbital velocity	(107)
For Newtonian gravity, gravitational acceleration equals orbital acceleration $\alpha_N$ $-\frac{GM}{r^2} = \alpha_N$	(108)
Using (107) in (108), Newtonian orbital velocity is $v_N^2 = \frac{GM}{r}$	(109)
For relativistic gravity, gravitational acceleration equals, see (7) $-\frac{GM}{r^2} = \frac{\alpha_l}{1 - \frac{v_l^2}{c^2}}$ $v_l$ : local orbital velocity	(110)
Substituting (107) for $\alpha_l$ in (110) with v being replaced by $v_l$ $-\frac{GM}{r^2} = \frac{-\frac{v_l^2}{r}}{1 - \frac{v_l^2}{c^2}}$	(111)

(111) is transformed into

$$\frac{GM}{c^2 r} - \frac{v_l^2 GM}{c^2 c^2 r} = \frac{v_l^2}{c^2} \\ \Rightarrow \frac{v_l^2}{c^2} \left(1 + \frac{GM}{c^2 r}\right) = \frac{GM}{c^2 r} \quad (112)$$

(112) gives relativistic orbital velocity

$$v_l^2 = \frac{GM}{r} \frac{1}{1 + \frac{GM}{c^2 r}} \quad (113)$$

Dividing (113) with (109)

$$\frac{v_l^2}{v_N^2} = \frac{\frac{GM}{r}}{\left(1 + \frac{GM}{c^2 r}\right) \frac{GM}{r}} \\ = \frac{1}{1 + \frac{GM}{c^2 r}} < 1 \quad (114)$$

$$(114) \text{ gives} \\ v_l < v_N \quad (115)$$

### References

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