Paradoxes and solutions about Lorentz force law

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I. Internal Lorentz force paradox

The Lorentz force respects the third Newton's law. Is the Lorentz force internal to a coil consistent with the third Newton's law ? Let us analyze the triangular coil ABC in the Figure 1; the current is I . Each side feels a Lorentz force from the magnetic field of the coil itself. The resultant force of all the Lorentz forces on the 3 sides is the following double integral:

$$\boldsymbol{F} = \int_{coil}\int_{coil} \frac{\mu_0}{4\pi} d\boldsymbol{I}_q \times \left(d\boldsymbol{I}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right)$$

with p the point that generates magnetic field and q the point that feels the Lorentz force, $\boldsymbol{e}_{pq}$ the unit vector pointing from p to q and r the distance between p and q.

I have done a numerical computation for a triangular coil with base length of 1 and height of 10. The calculated force is dimensionless and the value of the overall resultant force is (see the Figure 2):

$$\boldsymbol{S} = 35.21\ \boldsymbol{e}_y$$

This force is not 0, violating the third Newton's law. This is not permitted by the fundamental laws of dynamic. $\boldsymbol{S}$ is the net force on the coil if it were isolated. If the real resultant internal force had non null value, one could make the following process:

> Make a coil shown above, let the coil move in the direction of the resultant force $\boldsymbol{S}$. Since the magnetic flux passing through the coil is constant, the current will not do any work. But $\boldsymbol{S}$ would do a work in the movement, creating a quantity of energy.

This is impossible because energy cannot be created.

In general, the numerical value suffices to prove that the Lorentz force law is flawed, because only one counter example is sufficient to topple a general law. However, to exclude any doubt about the accuracy of this numerical calculation, I have done a rigorous analytical proof, which gives the expression of the dimensionless resultant force for a isosceles triangular coil as follow (see the Figure 1):

$$\boldsymbol{F}_{res} = \frac{\mu_0}{2\pi} I^2 \sin(\theta) \left(\int_{\substack{AB \\ (q)}} \int_{\substack{BC \\ (p)}} \left(\frac{dl_q}{r_{pq}^2} \cos(\beta) \right) dl_p + \int_{\substack{CA \\ (q')}} \int_{\substack{BC \\ (p')}} \left(\frac{dl_{q'}}{r_{p'q'}^2} \cos(\gamma) \right) dl_{p'} \right) \boldsymbol{e}_y$$

Thus, the analytical method proves without a doubt that the Lorentz force law is flawed. The mathematical derivation of the proof and a different proof are given in the annex.

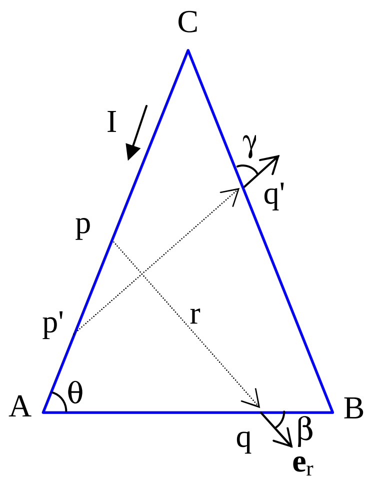

Figure 1

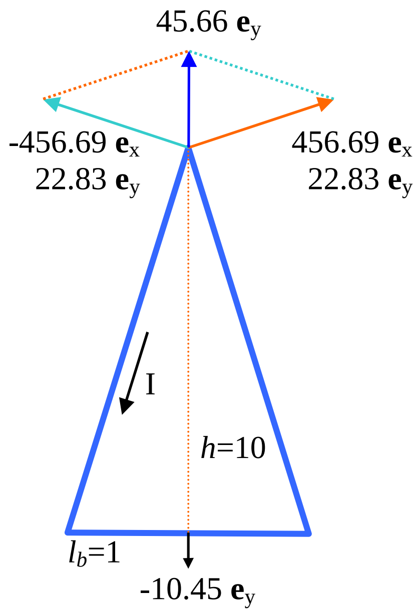

Figure 2

The analyze and the numerical example have shown that the Lorentz force law does not predict correct internal forces. I call this inconsistency the "Internal Lorentz force paradox".

II. B-cutting paradox

An electromotive force can be generated either by a variation of magnetic flux or by a moving conductor cutting the magnetic force lines (B-cutting). For example, in the set-up shown in the Figure 3 the bar conductor with length l is constrained to move at the velocity v cutting the force lines of the magnetic field B. The generated electromotive force is:

$$U = Blv$$

In the circuit, a current source provides a current I and maintains it constant. The current does an electrical work against U. When the bar moves from the position a to the position b, the quantity of work is:

$$W_{emf} = \int_a^b IU\,dt = Il\int_a^b Bdx$$

On the other hand, the current-carrying bar will experience the following Lorentz force:

$$F=IBl$$

Let us calculate the mechanical work this Lorentz force does against the constraint force when the bar during the same movement:

$$W_{mec} = \int_a^b Fdx = Il\int_a^b Bdx$$

We see that the mechanical work is equal to the electrical work:

$$W_{emf} = W_{mec}$$

So, no extra energy is left in the coil. However, the energy conservation law is not respected. Indeed, the magnetic energy stored in the coil depends on the positions of the bar. For the bar being at the positions *a* and *b*, the stored magnetic energies are respectively[1]:

$$E_a = \frac{1}{2}L_a I^2 \text{ and } E_b = \frac{1}{2}L_b I^2$$

As the self-inductance L_b is greater than L_a, the magnetic energy stored in the coil is greater for position *b* than *a*:

$$L_b > L_a \Rightarrow E_b > E_a$$

This implies that the coil has gained the following extra energy:

$$\Delta E = E_b - E_a > 0$$

But where does this energy come from? If this were true, we could make the following machine. Imagine the above coil functions in the following way:

1) The bar is at the position *a*, I=0, E_a=0.
2) The current increases to I, the current source does the electrical work $\int_0^t L_a \frac{dI}{dt} I dt$.
3) The magnetic energy stored in the coil is $E_a = \frac{1}{2}L_a I^2$.
4) The bar slides in the magnetic field created by the other 3 sides. The bar goes to the position *b*, the current source does the electric work W_{emf}, the Lorentz force does the mechanical work W_{mec}. W_{emf} and W_{mec} cancel out and the coil has not gained any energy.
5) But, the stored magnetic energy is now $E_b = \frac{1}{2}L_b I^2$.
6) The current decreases to 0, the coil gives back the stored magnetic energy to the current source. In this cycle, the current source has gained the energy $\Delta E = E_b - E_a$.
7) The bar goes back to the position *a* with no current, thus no work.
8) Restart from the step 2).

After *n* cycles, the quantity of energy that the current source would get is $n\Delta E$. This is impossible because energy cannot be created. Thus, the principle of this process is incorrect. I call this inconsistency the "B-cutting paradox".

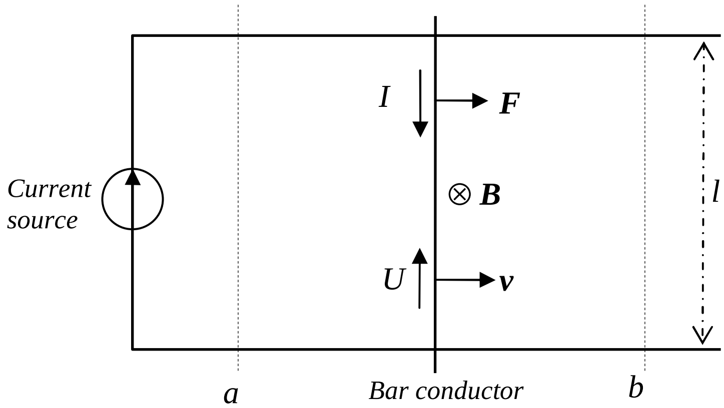

Figure 3

III. Lorentz' EMF paradox

The "B-cutting paradox" concerns the energy balance. Let us see the creation of electromotive force by the moving bar. The conductor bar's free electrons feel a vertical Lorentz force in moving horizontally, which is denoted $\boldsymbol{F}_e$. But, is there really an electric tension?

The force $\boldsymbol{F}_e$ is constant for steady velocity and constitutes an electrostatic field, $\boldsymbol{E}_e$. Obeying electrostatic laws the free electrons flow and distribute them-self to induce an electrostatic field so that the resultant electrostatic force on all free electrons inside the conductor bar is 0 and the surface becomes an equipotential. So the 2 points of contact with the 2 horizontal wires have the same potential and the tension is 0.

In order to understand this phenomenon, we imagine an identical conductor bar jointed on the one which is connected to the 2 horizontal wires. But this bar is insulated from the conducting bar and the horizontal wires. The 2 bars are called connected bar and non connected bar (See the Figure 4).

Evidently, the electric field in the non connected Bar is 0 and the surface is an equipotential. The connected bar is exactly in the same electromagnetic situation but caries the current of the coil. According to the superposition principle, the electric field is not influenced by the current. Thus, the surface of the connected bar is also an equipotential and the tension between the 2 contact points is 0.

Another point of view is that the potential is proportional to the energy that a free electron acquires in movement. Along any path inside the connected bar the line integral of the resultant electrostatic force is 0. Thus, the free electrons of the current get no energy at all. Again, the tension is 0.

This conclusion puts the Lorentz force law in opposition with the electrostatic law. I call this opposition the "Lorentz' EMF paradox".

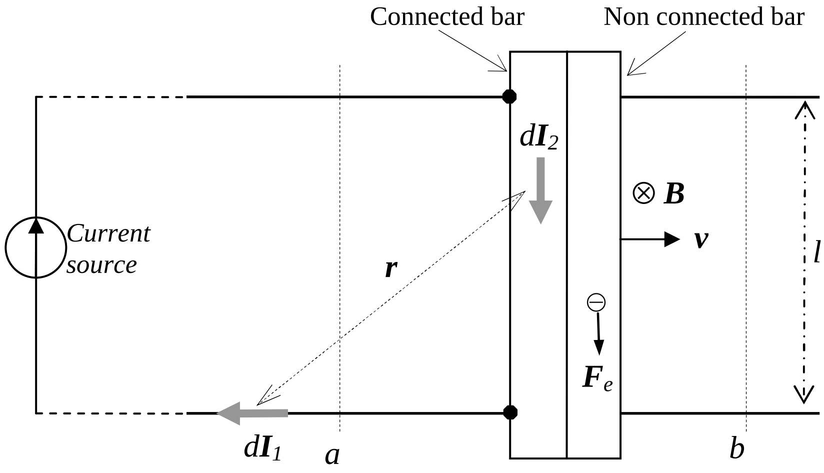

Figure 4

IV. Differential magnetic force

The Internal Lorentz force paradox is in contradiction with the Newton's third law, which is outside the electromagnetic theory. However, the B-cutting paradox and the Lorentz' EMF paradox are contrary to principles of electric energy conservation and electrostatic law, hence, the Lorentz force law is self-inconsistent and this is more troublesome.

To resolve these mysteries, let us calculate the Lorentz force between 2 closed coils without using the magnetic field, but instead, by integrating the differential force on infinitesimal current elements [2]. A current element vector is defined as the product of the current I, the length dl and the unit vector $\vec{n}$:

$$d\boldsymbol{I} = I\, dl\, \boldsymbol{n}$$

The current elements vectors taken from the coils c_1 and c_2 are indicated by the subscript 1 and 2 (see Figure 5):

$$d\boldsymbol{I}_1 = I\, dl_1\, \boldsymbol{n}\ ,\ d\boldsymbol{I}_2 = I\, dl_2\, \boldsymbol{n} \qquad (1)$$

The differential Lorentz force that $d\boldsymbol{I}_1$ exerts on $d\boldsymbol{I}_2$ is [3]:

$$d^2\boldsymbol{F}_{12} = \frac{\mu_0}{4\pi} d\boldsymbol{I}_2 \times \left(d\boldsymbol{I}_1 \times \frac{\boldsymbol{r}}{r^3} \right) \qquad (2)$$

We expand the double cross product by using the well known relation:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\, \boldsymbol{c}$$

And the equation (2) becomes:

$$d^2\boldsymbol{F}_{12} = \frac{\mu_0}{4\pi} \left[\left(d\boldsymbol{I}_2 \bullet \frac{\boldsymbol{r}}{r^3} \right) d\boldsymbol{I}_1 - \left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1 \right) \frac{\boldsymbol{r}}{r^3} \right] \qquad (3)$$

First, we integrate $d^2\boldsymbol{F}_{12}$ over c_2:

$$d\boldsymbol{F}_{1-c2} = \int_{c2} d^2\boldsymbol{F}_{12} = \frac{\mu_0}{4\pi} \left[\int_{c2} \left(d\boldsymbol{I} \bullet \frac{\boldsymbol{r}}{r^3} \right) d\boldsymbol{I}_1 - \int_{c2} \left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1 \right) \frac{\boldsymbol{r}}{r^3} \right] \qquad (4)$$

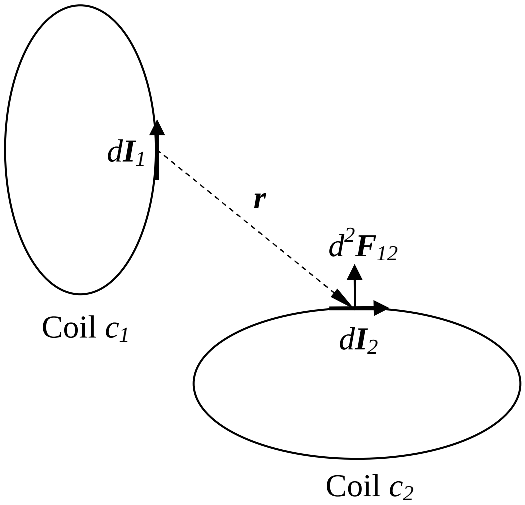

Figure 5

By using the equation (1), $d\boldsymbol{I}_1$ and I_2 being constant, the first integral becomes:

$$\int_{c2}\left(d\boldsymbol{I}_2 \bullet \frac{\boldsymbol{r}}{r^3}\right)d\boldsymbol{I}_1 = \int_{c2}\left(I_2 dl_2 \boldsymbol{n}_2 \bullet \frac{\boldsymbol{r}}{r^3}\right)d\boldsymbol{I}_1 = I_2 d\boldsymbol{I}_1 \int_{c2}\frac{\boldsymbol{r}}{r^3} \bullet \boldsymbol{n}_2 dl_2 = 0$$

Because, like an electrostatic field whose circulation over a closed path is 0, the integral of the vector field $\frac{\boldsymbol{r}}{r^3}$ is its circulation over the coil c_2 and is 0.

So, the equation (4) becomes:

$$d\boldsymbol{F}_{1-c2} = -\frac{\mu_0}{4\pi}\int_{c2}\frac{\boldsymbol{r}}{r^3}\left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1\right)$$

Now, we integrate $d\boldsymbol{F}_{1\text{-}c2}$ over the coil c_1 and obtain the total Lorentz force the coil 1 exerts on the coil 2[2]:

$$\boldsymbol{F}_{c1-c2} = \int_{c1} d\boldsymbol{F}_{1-c2} = \int_{c1}\int_{c2}\left(-\frac{\mu_0}{4\pi}\frac{\boldsymbol{r}}{r^3}\left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1\right)\right) \tag{5}$$

The integrand of this double integral is a differential magnetic force:

$$d^2\boldsymbol{F}_{amp} = -\frac{\mu_0}{4\pi}\frac{\boldsymbol{r}}{r^3}\left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1\right) \tag{6}$$

This force is proportional to the dot product of 2 current elements vectors that resembles the Ampere's force between 2 parallel currents. So, I call it the "differential Ampere's force" and the equation (6) the "differential Ampere's force law".

The equation (5) is interesting in that it proves that, although the differential Ampere's force has a different value than the differential Lorentz force, it gives still an integral force identical to the Lorentz force on closed coils (see the equation (2)):

$$d^2\boldsymbol{F}_{amp} \neq d^2\boldsymbol{F}_{12} \qquad\qquad \int_{c1}\int_{c2} d^2\boldsymbol{F}_{amp} = \int_{c1}\int_{c2} d^2\boldsymbol{F}_{12}$$

Thus the differential Ampere's force law satisfies all experiments that have served to define the Lorentz force law. Even more, the differential Ampere's force respects the third Newton's law for infinitesimal current elements. Indeed, the reaction force of $d^2\boldsymbol{F}_{amp}$ is obtained by reversing the radial vector $\vec{r}$ in the equation (6), and will have the same magnitude with the opposite sign.

In the following sections, we will try to solve the above paradoxes with differential Ampere's force law.

V. Solution of the Internal Lorentz force paradox

This paradox exists because the Lorentz force does not respect the third Newton's law on segments of current. Today, the accepted explanation for this inconsistency is that for segments of current the Lorentz force has not to obey the third Newton's law since it respects this law on complete coils [2]. But this argument fails to explain the Internal Lorentz force paradox.

In the contrary, the differential Ampere's force respect the third Newton's law on segments of current and will cancel them out on coil of any shape. For the triangular coil, the forces on the sides become opposite to each other, canceling them-self and giving a resultant force equal to 0.

VI. Solution of the B-cutting and the Lorentz' EMF paradoxes

The B-cutting paradox arises because the Lorentz force is in conflict with the stored magnetic energy. When we use the differential Ampere's force law, this conflict disappears.

Let us calculate the trouble making works and energy with the differential Ampere's force law. Take a very long horizontal coil with a sliding bar conductor (see the Figure 4). The calculation is simplified because the magnetic field from the left side is negligible and the sliding bar interacts with the 2 horizontal wires only. The differential Ampere's force is given by the equation (6):

$$d^2\boldsymbol{F}_{amp} = -\frac{\mu_0}{4\pi}\frac{\boldsymbol{r}}{r^3}\left(d\boldsymbol{I}_2 \bullet d\boldsymbol{I}_1\right)$$

As the current elements vectors $d\boldsymbol{I}_1$ and $d\boldsymbol{I}_2$ are perpendicular to one another, their dot product is 0; so the differential Ampere's force is 0 and the total magnetic force perpendicular to the bar is 0:

$$d\boldsymbol{I}_1 \bullet d\boldsymbol{I}_2 = 0 \Rightarrow d^2\boldsymbol{F}_{amp} = 0 \Rightarrow \boldsymbol{F}_{bar} = 0$$

In consequence, the differential Ampere's force will not do mechanical work. For the Lorentz force on the free electrons inside the conductor bar, we refer to the section "III. The Lorentz' EMF paradox". We note that we are using the differential Ampere's force law and can drop the idea that the Lorentz force generates electromotive force, that is, no tension is generated. So, the conductor bar does not participate in the building of the stored magnetic energy of the coil with neither mechanical and nor electrical work.

In reality, the electromotive force is generated by the increase of magnetic flux due to the expansion of the surface enclosed by the coil. The electromotive tension and the elementary electrical work are:

$$U_{emf} = -\frac{d\Phi}{dt}, \qquad dW_{emf} = -I\frac{d\Phi}{dt}dt$$

When the bar moves from the position *a* to *b*, *I* being constant, the electrical work is:

$$W_{emf} = \int_a^b Id\Phi = I\Phi_b - I\Phi_a \tag{7}$$

On the other hand, the stored magnetic energies of the coil for the bar being at the position *a* and *b* is the product of the coil's current *I* with the magnetic flux [4]:

$$E_a = I\Phi_a, \ E_b = I\Phi_b$$

Then the increase of the stored magnetic energy is equal to the electrical work:

$$E_b - E_a = W_{emf}$$

The energy conservation law is respected by the coil. But Lorentz force is not involved. The B-cutting and the Lorentz' EMF paradoxes are then solved.

Although this is only a particular case, it is nevertheless a counter-example that topples the general law: electromotive force is not generated by conductor cutting the magnetic force lines. This solution resolves the puzzle of Richard P. Feynman who has written in his "The Feynman Lectures on Physics"[5]:

> So the "flux rule" that the emf in a coil is equal to the rate of change of the magnetic flux through the coil - applies whether the flux changes because the field changes or because the coil moves (or both).... Yet in our explanation of the rule we have used two completely distinct laws for the two cases $-\vec{v}\times\vec{B}$ for "coil moves" and $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ for "field changes".
>
> We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena.

The truth is that there is not "two different phenomena" of electromotive force generation, but only one, that of "the flux changes". The electromotive force and stored magnetic energy are related to magnetic flux but not to Lorentz force. The remark of Richard P. Feynman was judicious.

VII. Lorentz torque experiment

The paradoxes have shown the deficiency of the Lorentz force law; the differential Ampere's force law has successfully solved all these paradoxes. It is however necessary to confirm by new experiments never carried out before. A success will show the flaw of the Lorentz force law and prove the new law experimentally. Below is the design of the experiment.

The suggested experiment makes 2 rectangular coils interact. The coil 1 is horizontal, the coil 2 is tilted at an angle with respect to the coil 1 (See the Figure 6). The magnetic force will create a torque on the 2 coils, which are calculated numerically.

The parameters for the calculation are as follow:

The dimensions of the horizontal coil:	lx=0.4 m, ly=0.8 m
The dimensions of the tilted coil:	lx=0.36 m, ly=0.144 m
The current in the 2 coils:	I=3000 A·turn

The torques in N·m predicted by the Lorentz force law and the differential Ampere's force law are the 2 curves drawn in the Figure 8. The torque varies with respect to the angle between the 2 coils. For angle between 0° and 180°, the torque predicted by the Lorentz force law draws a single-hump-shaped curve, whereas the prediction of the differential Ampere's force law draws a double-hump-shaped curve. The values of the predictions are very different. At 90°, the Lorentz force law predicts 1.2755 N·m, against 0.1877 N·m for the differential Ampere's force law.

The shapes of the curves are very distinguishable. If the measured data has a double-hump shape, the magnetic force follows the differential Ampere's force law; if the measured data has a single-hump shape, the magnetic force follows the Lorentz force law.

The quantity of wire is calculated here. The lengths of each turn of the 2 coils are:

$$l1=(0.4+0.8)*2=2.4 \text{ m}$$
$$l2=(0.36+0.144)*2=1.008 \text{ m}$$

For coils of 3 000 turns in each, the lengths of wire are:

$$L1 = 7\ 200 \text{ m}$$
$$L2 = 3\ 024 \text{ m}$$

The total length of wire needed is: 10 224 m

If we measure the force on the top of the coil 2, at angle 20°, the expected force is:

$$F_0 = 0.4450 \text{ N·m} / 0.072 \text{ m} = 6.18 \text{ N}$$

We can use currents of different values, coils with different number of turn. The expected forces at angle 20° with the corresponding currents, wire lengths are given in the Table 1.

The torque can be measured using diverse methods. In Figure 7 the torque on the coil 1 is measured by a force sensor, the signal of which is connected to the y connector of an oscilloscope. The angle between the 2 coils is measured by an angle sensor that is a force sensor on a spring which is pulled by the pulley fixed to the coil 2. The spring links the angle and the elastic force with a linear function. So, the spring's force signal which is linear to the angle is connected to the x connector of the oscilloscope.

The outcome of the experiment is the curve force-angle which will be drawn on the oscilloscope when the angle varies from 0° to 180° (see the Figure 7).

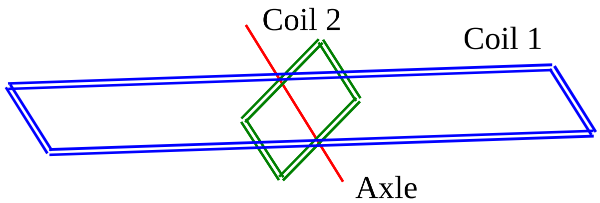

Figure 6

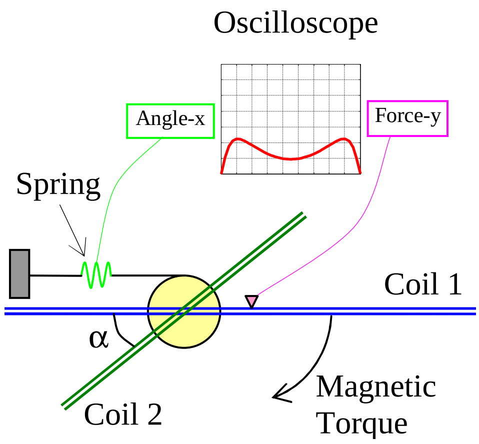

Figure 7

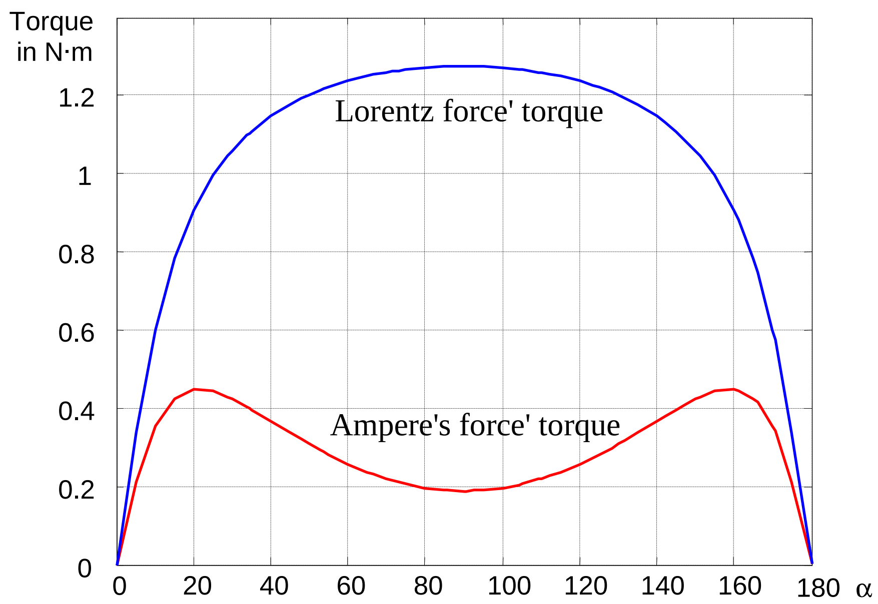

Figure 8

Turns	Current	Force for angle=20°			Wire length
	A	Newton	gram	pound	m
3000	1	6,18	630,18	1,39	10224
1000	1	0,69	70,02	0,15	3408
1000	2	2,75	280,08	0,62	3408
500	1	0,17	17,51	0,04	1704
500	2	0,69	70,02	0,15	1704
500	3	1,55	157,55	0,35	1704

Table 1

Annexes

A.I. Analytical proof of the Lorentz force law's flaw

The integration over the triangle can be seen as over 3 domains: AB, BC and CA. Let us see how a double integral over multiple domains is expanded. For a function *s* integrated over 2 domains, 1 and 2, the integral is expanded into 2 integrals:

$$f = \int_{1+2} s(x,y)dx = \int_1 s(x,y)dx + \int_2 s(x,y)dx$$

The function *f* itself is integrated over 2 domains, 3 and 4, the double integral is:

$$g = \int_{3+4} f dy = \int_{3+4}\left(\int_1 s(x,y)dx + \int_2 s(x,y)dx\right)dy$$

And the expanded form of the double integral is:

$$g = \int_3\int_1 s(x,y)dxdy + \int_3\int_2 s(x,y)dxdy + \int_4\int_1 s(x,y)dxdy + \int_4\int_2 s(x,y)dxdy$$

For the integral of the Lorentz force over the triangle, the Domain 1 is AB, Domain 2 is BCA, Domain 3 is AB, Domain 4 is BCA. The double integral is expanded as below:

$$\boldsymbol{F} = \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{F}_{pq}$$

$$\text{with } d^2\boldsymbol{F}_{pq} = \frac{\mu_0}{4\pi} d\boldsymbol{I}_q \times \left(d\boldsymbol{I}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)$$

We define the dimensionless force ***S***:

$$d^2\boldsymbol{S}_{pq} = d\boldsymbol{l}_q \times \left(d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)$$

with ***dl*** the differential length vector in the direction of the local current.

The integrated dimensionless force ***S*** is then :

$$\boldsymbol{S} = \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq}$$

$$\text{with } \boldsymbol{F} = \frac{\mu_0}{4\pi} I^2 \boldsymbol{S}$$

For points p and q on the side AB, they are on the same straight, the vectors $\boldsymbol{dl_p}$ and $\boldsymbol{e}_{pq}$ are parallel, and their cross product is 0. So, the double integral over AB is 0:

$$d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = 0 \Rightarrow \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = 0$$

We expand the double integral over BCA in the same way and obtain:

$$\int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq} = \int_{\substack{BC\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq}$$

$$= \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq}$$

The double integrals over the same straight are dropped and the force $\boldsymbol{S}$ becomes:

$$\boxed{\boldsymbol{S} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq}} \quad (1)$$

- ## Elimination of the direct force

By using the relation:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

We expand the elementary **S** force $d^2\boldsymbol{S}$ as follow:

$$d^2\boldsymbol{S}_{pq} = d\boldsymbol{l}_q \times \left(d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right) = \left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right) d\boldsymbol{l}_p - \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}$$

Let us reduce the pair AB-BC, which is the following sum of integrals:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = \left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right) d\boldsymbol{l}_p + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right) d\boldsymbol{l}_p\right)$$
$$- \left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)$$

The differential forces in the last parentheses satisfies the condition of action and reaction, that is, they lie on the line joining the 2 points, have the same magnitude and are opposite. This is a direct force that can be eliminated because their sum is 0. To prove this for the double integrals, we use discrete calculus. The double integrals are the limit of the following discrete sums:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\Delta\boldsymbol{l}_j \cdot \Delta\boldsymbol{l}_i\right)\frac{\boldsymbol{e}_{ij}}{r_{ij}^2}$$

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{r}}{r_{pq}^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Delta\boldsymbol{l}_i \cdot \Delta\boldsymbol{l}_j\right)\frac{\boldsymbol{e}_{ji}}{r_{ji}^2}$$

with i corresponding to BC and j to AB.

As the distance vector r_{pq} is the reverse of r_{qp}, we have:

$$\boldsymbol{r}_{ij} = -\boldsymbol{r}_{ji} \Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\Delta\boldsymbol{l}_j \cdot \Delta\boldsymbol{l}_i\right)\frac{\boldsymbol{e}_{ij}}{r_{ij}^2} = -\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\Delta\boldsymbol{l}_i \cdot \Delta\boldsymbol{l}_j\right)\frac{\boldsymbol{e}_{ji}}{r_{ji}^2}$$

We have proven that the double integrals in reverse order are opposite:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = -\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}$$

And:

$$\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q\cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q\cdot d\boldsymbol{l}_p\right)\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}=0$$

So, the pair AB-BC becomes:

$$\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{S}_{pq}+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{S}_{pq}=\left(\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q\cdot\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q\cdot\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p\right)$$

Let us define the differential ***G*** force as follow:

$$d^2\boldsymbol{G}_{pq}=\left(d\boldsymbol{l}_q\cdot\frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p \tag{2}$$

And the pair AB-BC is written as:

$$\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{S}_{pq}+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{S}_{pq}=\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq} \tag{3}$$

In the same way we obtain:

For the pair BC-CA
$$\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{S}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{S}_{pq}=\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{G}_{pq} \tag{4}$$

For the pair CA-AB
$$\int_{\substack{AB\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{S}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{S}_{pq}=\int_{\substack{AB\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq} \tag{5}$$

From the equations (1), (3), (4) and (5) we obtain the expression of ***S***:

$$\boxed{\boldsymbol{S}=\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{AB\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}}d^2\boldsymbol{G}_{pq}} \tag{6}$$

- **Reduction of the sum** $\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}+\int_{\substack{CA\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}$

We can simplify this expression by using symmetry. The differential $d^2\boldsymbol{G}_{pq}$ for the double integral over AB then BC is (see the equation (2) and Figure 9):

$$d^2\boldsymbol{G}_{pq}=\left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)d\boldsymbol{l}_p\quad\text{for}\quad\int_{\substack{BC\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over AB then CA is:

$$d^2\boldsymbol{G}_{p'q'}=\left(\frac{dl_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)d\boldsymbol{l}_{p'}\quad\text{for}\quad\int_{\substack{CA\\(q)}}\int_{\substack{AB\\(p)}}d^2\boldsymbol{G}_{pq}$$

Then:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)\boldsymbol{dl}_p + \int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{AB\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)\boldsymbol{dl}_{p'}$$

The integration of this sum by keeping the symmetry (see the Figure 9) gives:

$$\cos(\alpha) = -\cos(\alpha')$$

For the scalar length dl_q , the vector length $\boldsymbol{dl}_p$ and the distances we have the equalities:

$$dl_q = dl_{q'}\,,\ \boldsymbol{dl}_p = \boldsymbol{dl}_{p'}\,,\ r_{pq} = r_{p'q'}$$

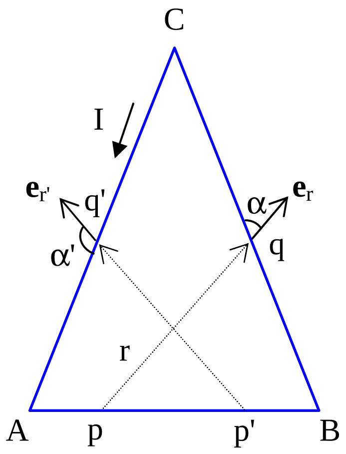

Figure 9

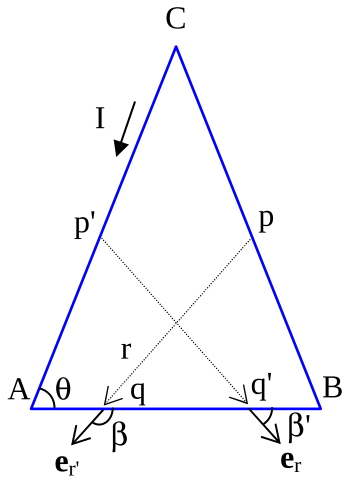

Figure 10

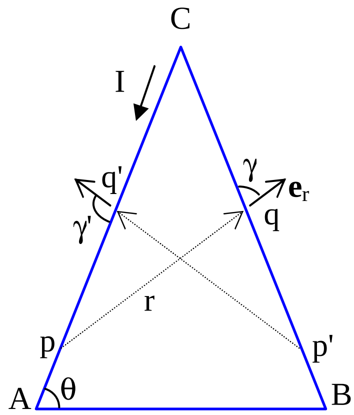

Figure 11

Thus, the double integral over AB then CA has the same absolute value than that over AB then BC. So, we have:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)\boldsymbol{dl}_p = -\int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{AB\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)\boldsymbol{dl}_{p'}$$

And:

$$\boxed{\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} = 0} \qquad (7)$$

- **Reduction of the sum** $\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$

$d^2\boldsymbol{G}_{pq}$ for the double integral over BC then AB is (see the equation (2) and Figure 10):

$$d^2\boldsymbol{G}_{pq} = \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)\boldsymbol{dl}_p \ \text{ for } \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over CA then AB is:

$$d^2\boldsymbol{G}_{p'q'} = \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right)\boldsymbol{dl}_{p'} \ \text{ for } \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$$

We have:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right) d\boldsymbol{l}_p + \int\limits_{\substack{AB\\(q')}}\int\limits_{\substack{CA\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right) d\boldsymbol{l}_{p'}$$

The vector lengths $d\boldsymbol{l}_p$ and $d\boldsymbol{l}_{p'}$ are :

$$d\boldsymbol{l}_p = dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right), d\boldsymbol{l}_{p'} = dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

Then the sum becomes:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right) dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right)$$
$$+ \int\limits_{\substack{AB\\(q')}}\int\limits_{\substack{CA\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right) dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The integration by keeping the symmetry gives (see the Figure 10):

$$\cos(\beta) = -\cos(\beta')$$

For the scalar length dl_q and the distances we have the equalities:

$$dl_q = dl_{q'},\; r_{pq} = r_{p'q'}$$

Thus, the double integral over BC then AB has the same absolute value than that over CA then AB. So, we have:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right) dl_p = -\int\limits_{\substack{AB\\(q')}}\int\limits_{\substack{CA\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right) dl_{p'}$$

And:

$$\boxed{\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = 2\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right) dl_p \sin(\theta)\boldsymbol{e}_y} \qquad (8)$$

- **Reduction of the sum** $\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$

$d^2\boldsymbol{G}_{pq}$ for the double integral over CA then BC is (see the equation (2) and Figure 11):

$$d^2\boldsymbol{G}_{pq} = \left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right) d\boldsymbol{l}_p \text{ for } \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over BC then CA is:

$$d^2\boldsymbol{G}_{p'q'} = \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right) d\boldsymbol{l}_{p'} \text{ for } \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$$

The vector lengths $\boldsymbol{dl}_p$ and $\boldsymbol{dl}_{p'}$ are :

$$\boldsymbol{dl}_p = dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right), \boldsymbol{dl}_{p'} = dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The sum becomes:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right)dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right)$$

$$+ \int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{BC\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right)dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The integration by keeping the symmetry gives (see the Figure 11):

$$\cos(\gamma) = -\cos(\gamma')$$

For the scalar length dl_q and the distances we have the equalities:

$$dl_q = dl_{q'}, \; r_{pq} = r_{p'q'}$$

Thus, the double integral over CA then BC has the same absolute value than that over BC then CA. So, we have:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right)dl_p = -\int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{BC\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right)dl_{p'}$$

And :

$$\boxed{\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} = 2\int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(\frac{dI_q}{r_{pq}^2}\cos(\gamma)\right)dI_p\sin(\theta)\boldsymbol{e}_y} \qquad (9)$$

- **Final expression of *S***

Finally we obtain the dimensionless resultant force $\boldsymbol{S}$ from equations (6), (7), (8) and (9):

$$\boxed{\boldsymbol{S} = 2\sin(\theta)\left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)dl_p + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right)dl_p\right)\boldsymbol{e}_y}$$

$$\text{with } \boldsymbol{F} = \frac{\mu_0}{4\pi}I^2\boldsymbol{S}$$

When the height of the triangle increases, the absolute value of the double integrals over BC then CA increases because the distance r decreases in average. In the contrary, that over BC then AB decreases because r increases in average. The sum of 2 functions varying in opposite direction cannot be 0. Thus, the overall resultant Lorentz force internal to the triangular coil is not 0. This analytical result confirms the numerical result which gives the following value of the dimensionless resultant force:

$$\boldsymbol{S} = 35.21\ \boldsymbol{e}_y$$

And the flaw of the Lorentz force law is proven.

A.II. Mathematical cause of the existence of the remaining resultant internal Lorentz force

I have given a rigorous proof of the existence of a remaining resultant Lorentz force internal to a triangular coil and a numerical computation that confirms this proof:
Proof of the remaining resultant Lorentz force internal to a triangular coil
https://docs.google.com/open?id=0B3YDEaOyRUwcT0ZfaXdpenFvSjA
http://pengkuanem.blogspot.com/2012/04/analyze-of-lorentz-forces-internal-to.html

Why the Lorentz force law cannot respect the third Newton's law? What is the mathematical cause that leads to this inconsistency? Let us examine the effect of the characteristic perpendicularity of the Lorentz force with the current. Take a triangle with height h and base a+b (see the Figure 12). The unit normal vectors to each side are $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\mathbf{n_0} = -\mathbf{e}_y , \mathbf{n}_1 = \frac{h\mathbf{e}_x + b\mathbf{e}_y}{\sqrt{h^2 + b^2}} , \mathbf{n}_2 = \frac{-h\mathbf{e}_x + a\mathbf{e}_y}{\sqrt{h^2 + a^2}}$$

The force on the 3 sides are:

$$\mathbf{F_0} = f_0\mathbf{n_0} , \mathbf{F}_1 = f_1\mathbf{n}_1 , \mathbf{F}_2 = f_2\mathbf{n}_2$$

with *f* the magnitude of the forces.

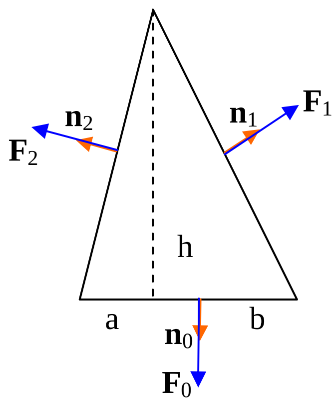

Figure 12

The sum of these 3 forces should be 0 because they are the internal forces. That is:

$$\boldsymbol{F_0} + \boldsymbol{F}_1 + \boldsymbol{F}_2 = f_0\boldsymbol{n_0} + f_1\boldsymbol{n}_1 + f_2\boldsymbol{n}_2$$

$$0 = -f_0\boldsymbol{e}_y + f_1\frac{h\boldsymbol{e}_x + b\boldsymbol{e}_y}{\sqrt{h^2 + b^2}} + f_2\frac{-h\boldsymbol{e}_x + a\boldsymbol{e}_y}{\sqrt{h^2 + a^2}}$$

The x and y components give 2 equations:

$$f_1\frac{h}{\sqrt{h^2 + b^2}} + f_2\frac{-h}{\sqrt{h^2 + a^2}} = 0$$

$$f_1\frac{b}{\sqrt{h^2 + b^2}} + f_2\frac{a}{\sqrt{h^2 + a^2}} = f_0$$

The magnitude of the forces are the solution of this system, that is:

$$f_1 = f_0 \frac{\sqrt{h^2+b^2}}{a+b},\ f_2 = f_0 \frac{\sqrt{h^2+a^2}}{a+b} \tag{10}$$

This is the solution that secures the remaining resultant force to be 0 and each force to be perpendicular to the current. Let us call this result the perpendicularity solution. We remark that this solution do not involve the Lorentz force law. It should be solution of this law. But is it?

For comparing with the Lorentz force law, we calculate the differential Lorentz force on a elementary current $\boldsymbol{dl_1}$ using the following equation (see the Figure 13):

$$d\boldsymbol{F} = d\boldsymbol{I}_1 \times \left(d\boldsymbol{I}_2 \times \frac{\boldsymbol{e}_r}{r^2} \right)$$

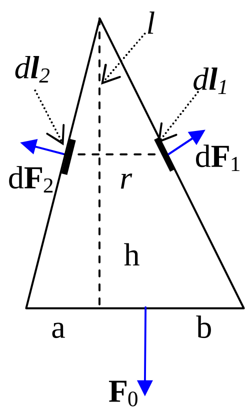

Figure 13

The $\boldsymbol{dl_1}$ and $\boldsymbol{dl_2}$ are at a vertical distance l from the summit. The distance between $\boldsymbol{dl_1}$ and $\boldsymbol{dl_2}$ is r, which is:

$$r = \frac{l}{h}(a+b)$$

So,

$$d\boldsymbol{F} = d\boldsymbol{l}_1 \times (d\boldsymbol{l}_2 \times \boldsymbol{e}_r) \frac{I^2 h^2}{l^2 (a+b)^2}$$

Let us calculate an incomplete sum of this differential force, whose magnitude is:

$$i = 1 \to n, \quad l_i = i \cdot \Delta l, \quad f_{part} = \frac{I^2 h^2}{(a+b)^2} \sum_{i=1}^{n} \frac{1}{l_i^2} = \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2}$$

with n and Δl constant

This force is only part of the total Lorentz force. So, the magnitude of the Lorentz force on one side is greater:

$$f_{Lorentz} > \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2} \tag{11}$$

Now we compare this magnitude with the perpendicular solution, the equation (10). When the height h is very large, the 2 upper sides become more and more vertical and the force f_0 become nearly constant. In this case, the magnitude f_1 becomes asymptotic to the following linear function of h:

$$f_1 \xrightarrow[h>>a+b]{} f_0 \frac{h}{a+b}$$

On the other hand, the solution of the Lorentz law is greater than a parabolic function of h, the equation (11). Because a linear function cannot be equal to a parabolic function, the perpendicularity solution cannot be solution of the Lorentz force law, that is:

$$f_1 \xrightarrow[h>>a+b]{} f_0 \frac{h}{a+b} \text{ and } f_{Lorentz} > \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2}$$
$$\Rightarrow f_{Lorentz} \neq f_1$$

The Lorentz force must satisfy the Lorentz law, this involves x and y components equations; the Lorentz force must also satisfy the perpendicularity system, this involves 2 different x and y components equations. We have 4 equations, but only 2 unknown, f_1 and f_2 . There cannot be any solution.

Since we always calculate Lorentz force with the Lorentz force law, the perpendicularity system is not satisfied. In consequence, the sum of $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ is not 0. This is the mathematical cause of the existence of the remaining resultant force.

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Abstract:
The Lorentz force law is a basic law of electromagnetism. However, in some particular circumstances, its predictions are incorrect, as shown by the paradoxes in these article. This deficiency was unknown and is mended by an improvement that this study proposes, which solves the paradoxes.