From electron to magnetism

5. Plasma under Coulomb magnetic force

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Abstract: Plasma is confined in fusion reactors using magnetic force. Coulomb magnetic force law for plasma is derived.

1. Magnetic force in plasma

Nuclear fusion reactors such as Tokamak or Stellarator use strong magnetic field to confine plasma in reaction chambers. The magnetic field is so designed that plasma should follow field lines which do not encounter the chambers' wall. But it seems that a mysterious force pushes plasma off its track. This force could come from the error of Lorentz force law that «Coulomb magnetic force» has shown in deriving Coulomb magnetic force law. However, Coulomb magnetic force law is for current in wires and cannot be used for plasma which is a cloud of individual ions. So, we derive here a magnetic force law for plasma using the result of «Length-contraction magnetic-force between arbitrary currents» and «Changing distance effect». In the following, these 2 articles will be respectively referred to by their abridged names «LC effect» and «CD effect» for saving place in the explanation.

Plasma is a bunch of charged particles each of which moves freely in space. We will compute the magnetic force exerted by a current element $d\boldsymbol{I}_b$ on one moving charged particle q_a. The velocity of q_a is $\boldsymbol{v}_a$ and that of a free electron in $d\boldsymbol{I}_b$ is $\boldsymbol{v}_b$. The radial vector pointing from $d\boldsymbol{I}_b$ to q_a is $\boldsymbol{r}$, whose magnitude r is the distance between $d\boldsymbol{I}_b$ and q_a, see **Figure 1**. We will first derive the force due to relativistic length contraction effect.

2. Length contraction effect

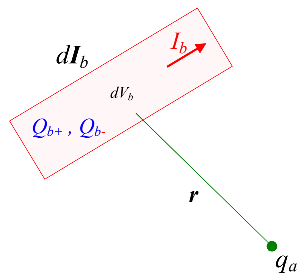

Figure 1

The current element $d\boldsymbol{I}_b$ contains free electrons whose charge is Q_{b-} and positive charges from atoms' nucleus whose charge is Q_{b+}. The forces on the charged particle q_a are the Coulomb forces $d\boldsymbol{F}_3$ exerted by the positive charge Q_{b+} and $d\boldsymbol{F}_4$ exerted by the negative charge Q_{b-}, see **Figure 1**. $d\boldsymbol{F}_3$ and $d\boldsymbol{F}_4$ are given in (1). The total force on q_a is the sum of $d\boldsymbol{F}_3$ and $d\boldsymbol{F}_4$ given in (2). We use the notation $d\boldsymbol{F}_3$, $d\boldsymbol{F}_4$ and $d\boldsymbol{I}_b$ from «LC effect» to facilitate the use of its formulas.

The principle of length contraction effect is that a volume moving relative to a frame appears smaller due to length contraction. In the frame of q_a which moves at the velocity $\boldsymbol{v}_a$, $d\boldsymbol{I}_b$ and its volume dV_b move at the velocity $-\boldsymbol{v}_a$. The reduced volume dV'_b is computed using «LC effect»'s equation (4), see (3).

$$d\boldsymbol{F}_3 = Q_{b+}q_a\frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$
$$d\boldsymbol{F}_4 = Q_{b-}q_a\frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \quad (1)$$

$$d\boldsymbol{F}_{lc} = d\boldsymbol{F}_3 + d\boldsymbol{F}_4 = Q_{b+}q_a\frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3} + Q_{b-}q_a\frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \quad (2)$$

$$dV'_b = dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2} \quad (3)$$

An element of a wire dl_b with no current is neutral and the charge density of the free electrons ρ_{b-} equals minus the charge density of the positive charges, $-\rho_{b+}$, see (4). The free electrons of $d\boldsymbol{I}_b$ move at the velocity $\boldsymbol{v}_b - \boldsymbol{v}_a$ in the frame of q_a and, due to the reduction of volume, the charge density of the free electrons increases from $-\rho_{b+}$ to ρ'_{b-}, see (5), which is computed using «LC effect»'s equation (7).

The value of a charge equals its charge density multiplied by its volume. So, the actual charge of the free electrons Q_{b-} equals the increased charge density ρ'_{b-} multiplied by reduced volume dV'_b , see (6). The Coulomb force $d\boldsymbol{F}_4$ that Q_{b-} exerts on q_a is computed in (7) using (1).

$$\rho_{b-} = -\rho_{b+} \quad (4)$$

$$\rho'_{b-} = \frac{-\rho_{b+}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}} \quad (5)$$

$$Q_{b-} = \rho'_{b-}dV'_b = \frac{-\rho_{b+}dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}} \quad (6)$$

$$d\boldsymbol{F}_4 = \frac{-\rho_{b+}dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}}\frac{q_a\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \quad (7)$$

Although Q_{b+} moves in the frame of q_a, the number of atoms in the volume dV_b is constant whatever the velocity. So, the charge Q_{b+} equals ρ_{b+} multiplies by dV_b and is computed in (8). The force $\boldsymbol{dF_3}$ that Q_{b+} exerts on q_a is computed in (9) using (1).

$$Q_{b+} = \rho_{b+} dV_b \tag{8}$$

$$\boldsymbol{dF_3} = \rho_{b+} dV_b q_a \frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \tag{9}$$

The total force due to length contraction effect $\boldsymbol{dF_{lc}}$ equals the sum of $\boldsymbol{dF_3}$ and $\boldsymbol{dF_4}$ which is computed in (10) for which we define the term G in (11). The 2 square roots in G are expanded using linear approximation formulas (12) and (13) and G is computed in (14) where the underlined term written in red is neglected. Then, the result of (14) is introduced into (10) to express the total force $\boldsymbol{dF_{lc}}$ in (15) where ρ_{b+}, dV_b and $\boldsymbol{v_b}$ are collected together to make $\boldsymbol{dI_b}$, see (16).

$$\begin{aligned}
\boldsymbol{dF_{lc}} &= \boldsymbol{dF_3} + \boldsymbol{dF_4} \\
&= \rho_{b+} dV_b q_a \frac{\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \\
&\quad + \frac{-\rho_{b+} dV_b \sqrt{1 - \boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1 - (\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})/c^2}} \frac{q_a \boldsymbol{r}}{4\pi\varepsilon_0 r^3} \\
&= \frac{\rho_{b+} dV_b q_a \boldsymbol{r}}{4\pi\varepsilon_0 r^3}\left(1 - \frac{\sqrt{1 - \boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1 - (\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})/c^2}}\right) \\
&= \frac{\rho_{b+} dV_b q_a \boldsymbol{r}}{4\pi\varepsilon_0 r^3} G
\end{aligned} \tag{10}$$

$$G = 1 - \frac{\sqrt{1 - \boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1 - (\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})/c^2}} \tag{11}$$

$$\frac{1}{\sqrt{1 - \boldsymbol{v}\cdot\boldsymbol{v}/c^2}} \approx 1 + \frac{1}{2}\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2} \tag{12}$$

$$\sqrt{1 - \boldsymbol{v}\cdot\boldsymbol{v}/c^2} \approx 1 - \frac{1}{2}\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2} \tag{13}$$

$$\begin{aligned}
G &= 1 - \left(1 + \frac{1}{2}\frac{(\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})}{c^2}\right)\left(1 - \frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}}{c^2}\right) \\
&= 1 - \left(1 - \frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}}{c^2}\right) - \left(\frac{1}{2}\frac{(\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})}{c^2}\right) \\
&\quad + \underline{\frac{1}{2}\frac{(\boldsymbol{v_b} - \boldsymbol{v_a})\cdot(\boldsymbol{v_b} - \boldsymbol{v_a})}{c^2}\frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}}{c^2}}) \\
&= \frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}}{c^2} - \frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a} - 2\boldsymbol{v_a}\cdot\boldsymbol{v_b} + \boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2} \\
&= \frac{1}{2}\frac{2\boldsymbol{v_a}\cdot\boldsymbol{v_b} - \boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2} \\
&= \frac{\boldsymbol{v_a}\cdot\boldsymbol{v_b}}{c^2} - \frac{1}{2}\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\boldsymbol{dF_{lc}} &= \frac{\rho_{b+} dV_b q_a \boldsymbol{r}}{4\pi\varepsilon_0 r^3}\left(\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_b}}{c^2} - \frac{1}{2}\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2}\right) \\
&= \frac{\rho_{b+} dV_b q_a \boldsymbol{r}}{4\pi\varepsilon_0 c^2 r^3}\left(\boldsymbol{v_a}\cdot\boldsymbol{v_b} - \frac{1}{2}\boldsymbol{v_b}\cdot\boldsymbol{v_b}\right) \\
&= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left(\boldsymbol{v_a}\cdot\rho_{b+} dV_b \boldsymbol{v_b} - \frac{1}{2}\boldsymbol{v_b}\cdot\rho_{b+} dV_b \boldsymbol{v_b}\right)\boldsymbol{r}
\end{aligned} \tag{15}$$

dV_b equals the cross section of the wire S_b multiplied by the length of the current element dl_b. The velocity of free electrons $\boldsymbol{v_b}$ is contrary to the sense of the current $\boldsymbol{I_b}$, so $\rho_{b+} dV_b\, \boldsymbol{v_b}$ equals minus $\boldsymbol{dI_b}$, see (16). For later use, $\boldsymbol{v_b}$ is expressed as a function of I_b and the unit vector $\boldsymbol{e_b}$ in (17). By introducing (16) into (15), the length contraction effect force, $\boldsymbol{dF_{lc}}$, is expressed in (18).

$$\begin{aligned}
dV_b &= S_b dl_b \\
\rho_{b+} dV_b \boldsymbol{v_b} &= \rho_{b+} S_b \boldsymbol{v_b} dl_b = -\boldsymbol{I_b} dl_b = -\boldsymbol{dI_b}
\end{aligned} \tag{16}$$

$$\boldsymbol{v_b} = -\frac{\boldsymbol{I_b} dl_b}{\rho_{b+} S_b dl_b} = -\frac{\boldsymbol{I_b}}{\rho_{b+} S_b} = -\frac{I_b}{\rho_{b+} S_b}\boldsymbol{e_b} \tag{17}$$

$$\boxed{\boldsymbol{dF_{lc}} = \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left(-(\boldsymbol{v_a}\cdot\boldsymbol{dI_b})\boldsymbol{r} + \frac{1}{2}(\boldsymbol{v_b}\cdot\boldsymbol{dI_b})\boldsymbol{r}\right)} \tag{18}$$

3. Changing distance effect

Changing distance effect force is the action between pairs of charges, but there is no pair of charges in plasma. So, we take one pair of charges from the current element $\boldsymbol{dI_b}$ formed by one positive charge ⊕b from an atom's nucleus and the nearest free electron ⊖b and compute its action on an individual charged particle q_a.

$$\boldsymbol{dF_3} = +\frac{q_a e}{4\pi\varepsilon_0}\frac{\boldsymbol{r_3}}{r_3^3} \tag{19}$$

$$\boldsymbol{dF_4} = -\frac{q_a e}{4\pi\varepsilon_0}\frac{\boldsymbol{r_4}}{r_4^3} \tag{20}$$

$$\begin{aligned}
\boldsymbol{dF_{pair}} &= \boldsymbol{dF_3} + \boldsymbol{dF_4} \\
&= \frac{q_a e}{4\pi\varepsilon_0}\frac{\boldsymbol{r_3}}{r_3^3} - \frac{q_a e}{4\pi\varepsilon_0}\frac{\boldsymbol{r_4}}{r_4^3} \\
&= \frac{q_a e}{4\pi\varepsilon_0}\left(\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3}\right)
\end{aligned} \tag{21}$$

The positive charge ⊕b acts the Coulomb forces $\boldsymbol{dF_3}$ on q_a, see (19), where $\boldsymbol{r_3}$ is the radial vector pointing from ⊕b to q_a. The electron ⊖b acts the Coulomb force $\boldsymbol{dF_4}$ on q_a, see (20), where $\boldsymbol{r_4}$ is the radial vector pointing from ⊖b to q_a. The total force $\boldsymbol{dF_{pair}}$ that the pair of charge acts on q_a equals $\boldsymbol{dF_3} + \boldsymbol{dF_4}$ which is computed in (21). The notations $\boldsymbol{dF_3}$ and $\boldsymbol{dF_4}$ are taken from «CD effect».

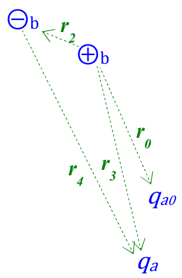

Figure 2

Radial vector ratio $\boldsymbol{r}/r^3$ in (21) will be expanded using linear approximation formula (22) which is «CD effect»'s equation (7). For doing so, we write $\boldsymbol{r_3}$ and $\boldsymbol{r_4}$ as the sum of a reference vector $\boldsymbol{r_0}$ and a variation vector $\Delta\boldsymbol{r}$, as shown in (23).

$$\frac{\boldsymbol{r}}{r^3} = \frac{1}{r_0{}^3}\left(1 - \frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0{}^2}\right)(\boldsymbol{r_0} + \Delta\boldsymbol{r}) \tag{22}$$

$$\boldsymbol{r_3} = \boldsymbol{r_0} + \Delta\boldsymbol{r_3}, \qquad \boldsymbol{r_4} = \boldsymbol{r_0} + \Delta\boldsymbol{r_4} \tag{23}$$

As **Figure 2** shows, ⊕b is immobile, the position of q_a is defined by the vector $\boldsymbol{r_3}$ pointing from ⊕b to q_a and the position of ⊖b by the vector $\boldsymbol{r_2}$ pointing from ⊕b to ⊖b. We define a reference state named state *0*. In this state, *t=0*, the moving ⊖b coincides with ⊕b in space and the moving q_a is at the position pointed by the reference vector $\boldsymbol{r_0}$.

The sum of radial vector ratios of (21), $\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3}$, is expanded in (24) using the formula (22) and becomes a function of $\Delta\boldsymbol{r_3}$ and $\Delta\boldsymbol{r_4}$. By expressing $\Delta\boldsymbol{r_4}$ with the vectors $\boldsymbol{r_2}$ and $\Delta\boldsymbol{r_3}$, we will transform (24) into a function of $\boldsymbol{r_2}$ and $\Delta\boldsymbol{r_3}$. $\boldsymbol{r_4}$ is the radial vector pointing from ⊖b to q_a which is given in (25), see **Figure 2**. Combining (25) with (23), we express $\Delta\boldsymbol{r_4}$ in (26) and $\Delta\boldsymbol{r_3}-\Delta\boldsymbol{r_4}$ in (27). Introducing (27) into (24) simplifies (24) into (28). Using (26) we develop the underlined term of (28) into (29), which is introduced into (28) to simplifies (28) into (30).

$$\begin{aligned}\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3} &= \frac{1}{r_0{}^3}\left(1-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}}{r_0{}^2}\right)(\boldsymbol{r_0}+\Delta\boldsymbol{r_3}) - \frac{1}{r_0{}^3}\left(1-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4}}{r_0{}^2}\right)(\boldsymbol{r_0}+\Delta\boldsymbol{r_4})\\ &= \frac{1}{r_0{}^3}\left(\boldsymbol{r_0}+\Delta\boldsymbol{r_3}-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}}{r_0{}^2}(\boldsymbol{r_0}+\Delta\boldsymbol{r_3}) - \left(\boldsymbol{r_0}+\Delta\boldsymbol{r_4}-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4}}{r_0{}^2}(\boldsymbol{r_0}+\Delta\boldsymbol{r_4})\right)\right)\\ &= \frac{1}{r_0{}^3}\left(\Delta\boldsymbol{r_3}-\Delta\boldsymbol{r_4}-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}}{r_0{}^2}\boldsymbol{r_0}-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}}{r_0{}^2}\Delta\boldsymbol{r_3}+\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4}}{r_0{}^2}\boldsymbol{r_0}+\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4}}{r_0{}^2}\Delta\boldsymbol{r_4}\right)\\ &= \frac{1}{r_0{}^3}\left(\Delta\boldsymbol{r_3}-\Delta\boldsymbol{r_4}+\frac{3}{r_0{}^2}[\boldsymbol{r_0}\cdot(\Delta\boldsymbol{r_4}-\Delta\boldsymbol{r_3})]\boldsymbol{r_0}-\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}}{r_0{}^2}\Delta\boldsymbol{r_3}+\frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4}}{r_0{}^2}\Delta\boldsymbol{r_4}\right)\end{aligned} \tag{24}$$

$$\boldsymbol{r_4} = \boldsymbol{r_3} - \boldsymbol{r_2} \tag{25}$$

$$\begin{aligned}\Delta\boldsymbol{r_4} &= \boldsymbol{r_3}-\boldsymbol{r_2}-\boldsymbol{r_0}\\ &= \boldsymbol{r_3}-\boldsymbol{r_0}-\boldsymbol{r_2}\\ &= \Delta\boldsymbol{r_3}-\boldsymbol{r_2}\end{aligned} \tag{26}$$

$$\Delta\boldsymbol{r_3}-\Delta\boldsymbol{r_4} = \boldsymbol{r_2} \tag{27}$$

$$\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3} = \frac{1}{r_0{}^3}\left(\boldsymbol{r_2}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_0}+\frac{3}{r_0{}^2}\left[-(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}+\underline{(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4})\Delta\boldsymbol{r_4}}\right]\right) \tag{28}$$

$$\begin{aligned}(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_4})\Delta\boldsymbol{r_4} &= \big(\boldsymbol{r_0}\cdot(\Delta\boldsymbol{r_3}-\boldsymbol{r_2})\big)(\Delta\boldsymbol{r_3}-\boldsymbol{r_2})\\ &= (\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3}-\boldsymbol{r_0}\cdot\boldsymbol{r_2})(\Delta\boldsymbol{r_3}-\boldsymbol{r_2})\\ &= (\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}-(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\Delta\boldsymbol{r_3}-(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\boldsymbol{r_2}+(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_2}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3} &= \frac{1}{r_0{}^3}\Big(\boldsymbol{r_2}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_0}\\ &\qquad+\frac{3}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}+(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}-(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\Delta\boldsymbol{r_3}-(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\boldsymbol{r_2}+(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_2}]\Big)\\ &= \frac{1}{r_0{}^3}\left(\boldsymbol{r_2}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_0}+\frac{3}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\Delta\boldsymbol{r_3}-(\boldsymbol{r_0}\cdot\Delta\boldsymbol{r_3})\boldsymbol{r_2}+(\boldsymbol{r_0}\cdot\boldsymbol{r_2})\boldsymbol{r_2}]\right)\end{aligned} \tag{30}$$

Like in «CD effect», we compute in (31) the time average value of $d\boldsymbol{F}_{pair}$ which is the average force $d\boldsymbol{F}_{avg}$ that the pair of charges acts on q_a. For doing so, we compute the velocities $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ by noticing that at state *0*, *t=0*, the position of q_a is $\boldsymbol{r_0}$ and that of ⊖b is *0*. At time *t*, q_a is at $\boldsymbol{r_3}$ and ⊖b at $\boldsymbol{r_2}$. Then, $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ are expressed in (32) and $\boldsymbol{r_2}$ and $\Delta\boldsymbol{r_3}$ in (33).We introduce (33) into (30) transforming it into a function of time in (34).

$$\begin{aligned}d\boldsymbol{F_{avg}} &= \frac{1}{t_2-t_1}\int_{t1}^{t2} d\boldsymbol{F_{pair}}dt\\ &= \frac{1}{t_2-t_1}\int_{t1}^{t2}\frac{q_a e}{4\pi\varepsilon_0}\left(\frac{\boldsymbol{r_3}}{r_3^3}-\frac{\boldsymbol{r_4}}{r_4^3}\right)dt\\ &= \frac{q_a e}{4\pi\varepsilon_0}\frac{1}{t_2-t_1}\int_{t1}^{t2}\left(\frac{\boldsymbol{r_3}}{r_3^3}-\frac{\boldsymbol{r_4}}{r_4^3}\right)dt\end{aligned} \tag{31}$$

The sum of radial vector ratios (34) is integrated over the time interval $[t_1, t_2]$ in (35). By making the time interval symmetric about zero, that is, $[t_1, t_2] = [-\Delta t, +\Delta t]$, where Δt is the half length of the interval, the value of (35) is computed in (36) which is a function of the velocities $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ and Δt.

$$\begin{aligned}\boldsymbol{v_a} &= \frac{\boldsymbol{r_3}-\boldsymbol{r_0}}{t-0} = \frac{\Delta\boldsymbol{r_3}}{t}\\ \boldsymbol{v_b} &= \frac{\boldsymbol{r_2}-0}{t-0} = \frac{\boldsymbol{r_2}}{t}\end{aligned} \tag{32}$$

$$\begin{aligned}\boldsymbol{r_2} &= \boldsymbol{v_b}t\\ \Delta\boldsymbol{r_3} &= \boldsymbol{v_a}t\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3} &= \frac{1}{r_0{}^3}\left(\boldsymbol{v_b}t-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{v_b}t)\boldsymbol{r_0}+\frac{3}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b}t)\boldsymbol{v_a}t-(\boldsymbol{r_0}\cdot\boldsymbol{v_a}t)\boldsymbol{v_b}t+(\boldsymbol{r_0}\cdot\boldsymbol{v_b}t)\boldsymbol{v_b}t]\right)\\ &= \frac{1}{r_0{}^3}\left(\left[\boldsymbol{v_b}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{r_0}\right]t+\frac{3}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a}-(\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}]t^2\right)\end{aligned} \tag{34}$$

$$\begin{aligned}\int_{t1}^{t2}\left(\frac{\boldsymbol{r_3}}{r_3^3}-\frac{\boldsymbol{r_4}}{r_4^3}\right)dt &= \int_{t1}^{t2}\frac{1}{r_0{}^3}\left(\left[\boldsymbol{v_b}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{r_0}\right]t+\frac{3}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a}-(\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}]t^2\right)dt\\ &= \frac{1}{r_0{}^3}\left(\frac{1}{2}\left[\boldsymbol{v_b}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{r_0}\right][t^2]_{t1}^{t2}+\frac{1}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a}-(\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}][t^3]_{t1}^{t2}\right)\end{aligned} \tag{35}$$

$$\begin{aligned}\int_{t1}^{t2}\left(\frac{\boldsymbol{r_3}}{r_3^3}-\frac{\boldsymbol{r_4}}{r_4^3}\right)dt &= \frac{1}{r_0{}^3}\left(\frac{1}{2}\left[\boldsymbol{v_b}-\frac{3}{r_0{}^2}(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{r_0}\right](\Delta t^2-\Delta t^2)+\frac{1}{r_0{}^2}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a}-(\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}](\Delta t^3+\Delta t^3)\right)\\ &= \frac{2\Delta t^3}{r_0{}^5}[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a}-(\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}]\end{aligned} \tag{36}$$

The average value of $\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3}$ is computed in (37). As explained in «CD effect» the value of Δt is r_0/c, which is introduced into (37) to give (38), which is then introduced into (31) to express the average force $d\boldsymbol{F}_{avg}$ in (39).

$$\frac{1}{t_2 - t_1}\int_{t1}^{t2}\left(\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3}\right)dt = \frac{2\Delta t^3}{2\Delta t {r_0}^5}\left[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}\right] = \frac{\Delta t^2}{{r_0}^5}\left[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}\right] \quad (37)$$

$$\frac{1}{t_2 - t_1}\int_{t1}^{t2}\left(\frac{\boldsymbol{r_3}}{r_3^3} - \frac{\boldsymbol{r_4}}{r_4^3}\right)dt = \frac{1}{c^2{r_0}^3}\left[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}\right] \quad (38)$$

$$d\boldsymbol{F_{avg}} = \frac{q_a e}{4\pi\varepsilon_0 c^2{r_0}^3}\left[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}\right] \quad (39)$$

The total average force $d\boldsymbol{F}_{cd}$ that $d\boldsymbol{I}_b$ exerts on q_a equals the sum of all average forces from all the pairs in $d\boldsymbol{I}_b$. Let n_b be the number of free electrons in $d\boldsymbol{I}_b$, the sum of n_b forces $d\boldsymbol{F}_{avg}$ is computed in (40).

$$\begin{aligned} d\boldsymbol{F}_{cd} &= \sum_{n_b} d\boldsymbol{F_{avg}} \\ &= \sum_{n_b}\frac{q_a e}{4\pi\varepsilon_0}\frac{1}{c^2{r_0}^3}\left[-(\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r_0}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_0}\cdot\boldsymbol{v_b})\boldsymbol{v_b}\right] \\ &= \frac{q_a e}{4\pi\varepsilon_0 c^2}\sum_{n_b}\left[-\left(\frac{\boldsymbol{r_0}}{{r_0}^3}\cdot\boldsymbol{v_b}\right)\boldsymbol{v_a} - \left(\frac{\boldsymbol{r_0}}{{r_0}^3}\cdot\boldsymbol{v_a}\right)\boldsymbol{v_b} + \left(\frac{\boldsymbol{r_0}}{{r_0}^3}\cdot\boldsymbol{v_b}\right)\boldsymbol{v_b}\right] \\ &= \frac{q_a e}{4\pi\varepsilon_0 c^2}\left[-\left(\boldsymbol{v_b}\cdot\sum_{n_b}\frac{\boldsymbol{r_0}}{{r_0}^3}\right)\boldsymbol{v_a} - \left(\boldsymbol{v_a}\cdot\sum_{n_b}\frac{\boldsymbol{r_0}}{{r_0}^3}\right)\boldsymbol{v_b} + \left(\boldsymbol{v_b}\cdot\sum_{n_b}\frac{\boldsymbol{r_0}}{{r_0}^3}\right)\boldsymbol{v_b}\right] \end{aligned} \quad (40)$$

The vectors $\boldsymbol{r_0}$ in (40) is slightly different from one pair to another. So, we define $\boldsymbol{r}_m$ the average vector of $\boldsymbol{r_0}$ in (41) which is in fact the radial vector between $d\boldsymbol{I}_b$ and q_a used for length contraction effect force in section 2, that is, $\boldsymbol{r}_m = \boldsymbol{r}$, see (42). Introducing (42) into (40) gives (43) where n_b, e and $\boldsymbol{v}_b$ are collected together to make $d\boldsymbol{I}_b$, see (44). Replacing $d\boldsymbol{I}_b$ for $n_b e\boldsymbol{v}_b$ in (43), we express the Changing distance effect force $d\boldsymbol{F}_{cd}$ on an individual charged particle q_a in (45).

$$\sum_{n_b}\frac{\boldsymbol{r_0}}{{r_0}^3} = n_b\frac{\boldsymbol{r_m}}{{r_m}^3} \quad (41)$$

$$\frac{\boldsymbol{r_m}}{{r_m}^3} = \frac{\boldsymbol{r}}{r^3} \rightarrow \sum_{n_b}\frac{\boldsymbol{r_0}}{{r_0}^3} = n_b\frac{\boldsymbol{r}}{r^3} \quad (42)$$

$$\begin{aligned} d\boldsymbol{F}_{cd} &= \frac{q_a e}{4\pi\varepsilon_0 c^2}\left[-\left(n_b\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{v_b}\right)\boldsymbol{v_a} - \left(n_b\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{v_a}\right)\boldsymbol{v_b} + \left(n_b\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{v_b}\right)\boldsymbol{v_b}\right] \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[-(\boldsymbol{r}\cdot n_b e\boldsymbol{v_b})\boldsymbol{v_a} - (\boldsymbol{r}\cdot\boldsymbol{v_a})n_b e\boldsymbol{v_b} + (\boldsymbol{r}\cdot n_b e\boldsymbol{v_b})\boldsymbol{v_b}\right] \end{aligned} \quad (43)$$

$$n_b e\boldsymbol{v_b} = -d\boldsymbol{I}_b \quad (44)$$

$$\boxed{d\boldsymbol{F}_{cd} = \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[(\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} + (\boldsymbol{r}\cdot\boldsymbol{v_a})d\boldsymbol{I}_b - (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_b}\right]} \quad (45)$$

4. Complete magnetic force

As explained in «Coulomb magnetic force», the complete Coulomb magnetic force is the sum of length contraction effect force and changing distance effect force. This stays true for Coulomb magnetic force on individual charged particle $d\boldsymbol{F}_{cm}$ which equals the sum of $d\boldsymbol{F}_{lc}$ (18) and $d\boldsymbol{F}_{cd}$ (45) and is computed in (46). The underlined terms in (46) are transformed into the double cross product in (48) using the vector identity (47). Then, the terms with $\boldsymbol{v}_b$ are expressed in function of current $\boldsymbol{I}_b$ using equation (17). Finally, the Coulomb magnetic force that the current element $d\boldsymbol{I}_b$ exerts on a moving individual charged particle q_a is expressed in (49).

$$\begin{aligned} d\boldsymbol{F}_{cm} &= d\boldsymbol{F}_{lc} + d\boldsymbol{F}_{cd} \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[-(\boldsymbol{v_a}\cdot d\boldsymbol{I}_b)\boldsymbol{r} + \frac{1}{2}(\boldsymbol{v_b}\cdot d\boldsymbol{I}_b)\boldsymbol{r}\right] + \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[(\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} + (\boldsymbol{r}\cdot\boldsymbol{v_a})d\boldsymbol{I}_b - (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_b}\right] \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[-(\boldsymbol{v_a}\cdot d\boldsymbol{I}_b)\boldsymbol{r} + \frac{1}{2}(\boldsymbol{v_b}\cdot d\boldsymbol{I}_b)\boldsymbol{r} + (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} + (\boldsymbol{r}\cdot\boldsymbol{v_a})d\boldsymbol{I}_b - (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_b}\right] \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[\underline{(\boldsymbol{v_a}\cdot\boldsymbol{r})d\boldsymbol{I}_b - (\boldsymbol{v_a}\cdot d\boldsymbol{I}_b)\boldsymbol{r}} + (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} - (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_b} + \frac{1}{2}(\boldsymbol{v_b}\cdot d\boldsymbol{I}_b)\boldsymbol{r}\right] \end{aligned} \quad (46)$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \quad (47)$$

$$\begin{aligned} d\boldsymbol{F}_{cm} &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[\boldsymbol{v_a}\times(d\boldsymbol{I}_b\times\boldsymbol{r}) + (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a}\underline{-(\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_b} + \frac{1}{2}(\boldsymbol{v_b}\cdot d\boldsymbol{I}_b)\boldsymbol{r}}\right] \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[\boldsymbol{v_a}\times(d\boldsymbol{I}_b\times\boldsymbol{r}) + (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} + (\boldsymbol{r}\cdot\boldsymbol{I}_b dl_b)\frac{\boldsymbol{I}_b}{\rho_{b+}S_b} - \frac{1}{2}\left(\frac{\boldsymbol{I}_b}{\rho_{b+}S_b}\cdot\boldsymbol{I}_b dl_b\right)\boldsymbol{r}\right] \\ &= \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[\boldsymbol{v_a}\times(d\boldsymbol{I}_b\times\boldsymbol{r}) + (\boldsymbol{r}\cdot d\boldsymbol{I}_b)\boldsymbol{v_a} + \frac{dl_b}{\rho_{b+}S_b}\left((\boldsymbol{r}\cdot\boldsymbol{I}_b)\boldsymbol{I}_b - \frac{1}{2}(\boldsymbol{I}_b\cdot\boldsymbol{I}_b)\boldsymbol{r}\right)\right] \end{aligned} \quad (48)$$

$$\boxed{d\boldsymbol{F}_{cm} = \frac{q_a}{4\pi\varepsilon_0 c^2 r^3}\left[\boldsymbol{v_a}\times(d\boldsymbol{I}_b\times\boldsymbol{r}) + \boldsymbol{v_a}(\boldsymbol{r}\cdot d\boldsymbol{I}_b) + \frac{dl_b}{\rho_{b+}S_b}\left((\boldsymbol{r}\cdot\boldsymbol{I}_b)\boldsymbol{I}_b - \frac{1}{2}{\boldsymbol{I}_b}^2\boldsymbol{r}\right)\right]} \quad (49)$$

5. Coulomb magnetic force for plasma

The Coulomb magnetic force in (49) is acted by one current element. For computing the Coulomb magnetic force for plasma $\boldsymbol{F}_{plm}$, we integrate (49) over a closed circuit, which gives (50). In (50) $\boldsymbol{e}_r$ and $\boldsymbol{e}_b$ are unit vectors of $\boldsymbol{r}$ and $\boldsymbol{I}_b$, that is, $\boldsymbol{r} = r\,\boldsymbol{e}_r$ and $\boldsymbol{I}_b = I_b\,\boldsymbol{e}_b$. The underlined term in (50) is the closed line integral of a conservative field which is zero. Then, $\boldsymbol{F}_{plm}$ become (51) where the underlined term is the classic magnetic field $\boldsymbol{B}$. So, the Coulomb magnetic force for plasma equals the classic Lorentz force $q_a\boldsymbol{v}\times\boldsymbol{B}$ plus a second term $\boldsymbol{F}_{sp}$, which is defined in (52).

$$\begin{aligned}\boldsymbol{F}_{plm} &= \oint d\boldsymbol{F}_{cm}\\ &= \oint \frac{q_a}{4\pi\varepsilon_0 c^2}\left[\frac{\boldsymbol{v}_a\times(d\boldsymbol{I}_b\times\boldsymbol{r})}{r^3}+\frac{\boldsymbol{v}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b)}{r^3}+\frac{1}{r^3}\frac{dl_b}{\rho_{b+}S_b}\left((\boldsymbol{r}\cdot\boldsymbol{I}_b)\boldsymbol{I}_b-\frac{1}{2}{\boldsymbol{I}_b}^2\boldsymbol{r}\right)\right]\\ &= \frac{q_a}{4\pi\varepsilon_0 c^2}\left[\boldsymbol{v}_a\times\oint\frac{d\boldsymbol{I}_b\times\boldsymbol{r}}{r^3}+\boldsymbol{v}_a\underline{\oint\frac{\boldsymbol{r}\cdot d\boldsymbol{I}_b}{r^3}}+\frac{{I_b}^2}{\rho_{b+}S_b}\oint\frac{r\,dl_b}{r^3}\left((\boldsymbol{e}_r\cdot\boldsymbol{e}_b)\boldsymbol{e}_b-\frac{1}{2}\boldsymbol{e}_r\right)\right]\end{aligned} \tag{50}$$

$$\oint\frac{\boldsymbol{r}\cdot d\boldsymbol{I}_b}{r^3}=0\rightarrow\boldsymbol{F}_{plm}=q_a\boldsymbol{v}_a\times\underline{\oint\frac{d\boldsymbol{I}_b\times\boldsymbol{r}}{4\pi\varepsilon_0 c^2 r^3}}+\frac{q_a{I_b}^2}{\rho_{b+}S_b}\frac{1}{4\pi\varepsilon_0 c^2}\oint\left((\boldsymbol{e}_r\cdot\boldsymbol{e}_b)\boldsymbol{e}_b-\frac{1}{2}\boldsymbol{e}_r\right)\frac{dl_b}{r^2} \tag{51}$$

In (52) R is the size of the coil, for example the radius of a round coil. $d\bar{l}_b$ and $\bar{r}$ are dimensionless length and radius and equal dl_b/R and r/R, which make the integral in (52) dimensionless. $\boldsymbol{F}_{sp}$ in (52) is a force which is constant at every point in space and independent of $\boldsymbol{v}_a$ the velocity of the particle. We call $\boldsymbol{F}_{sp}$ the spatial force.

$$\begin{aligned}\boldsymbol{F}_{plm} &= q_a\boldsymbol{v}_a\times\boldsymbol{B}+\boldsymbol{F}_{sp}\\ &= \boldsymbol{F}_{Lorentz}+\boldsymbol{F}_{sp}\\ \boldsymbol{F}_{sp} &= \frac{q_a{I_b}^2}{\rho_{b+}S_bR}\frac{1}{4\pi\varepsilon_0 c^2}\oint\left((\boldsymbol{e}_r\cdot\boldsymbol{e}_b)\boldsymbol{e}_b-\frac{\boldsymbol{e}_r}{2}\right)\frac{d\bar{l}_b}{\bar{r}^2}\\ &= \frac{q_a{I_b}^2}{\rho_{b+}S_bR}\frac{1}{4\pi\varepsilon_0 c^2}\left(\oint\frac{(\boldsymbol{e}_r\cdot\boldsymbol{e}_b)\boldsymbol{e}_b d\bar{l}_b}{\bar{r}^2}-\oint\frac{\boldsymbol{e}_r d\bar{l}_b}{2\bar{r}^2}\right)\end{aligned} \tag{52}$$

- $\boldsymbol{F}_{sp}$ at the center of a round coil

If q_a is at the center of a round coil, the radial vector from the coil to q_a is $\boldsymbol{r}=R\boldsymbol{e}_r$ with R being the radius of the coil. In this case, the current is always perpendicular to $\boldsymbol{r}$ and the scalar product $(\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b)$ is zero. Also, the integral of $\boldsymbol{e}_r$ around the coil is zero. Then at the center of a round coil, the spatial force $\boldsymbol{F}_{sp}$ is zero and the Coulomb magnetic force for plasma equals Lorentz force, see (53).

$$\left\{\begin{aligned}&\boldsymbol{e}_r\cdot\boldsymbol{e}_b=0\rightarrow\oint\frac{(\boldsymbol{e}_r\cdot\boldsymbol{e}_b)\boldsymbol{e}_b d\bar{l}_b}{\bar{r}^2}=0\\ &\oint\boldsymbol{e}_r d\bar{l}_b=0\rightarrow\oint\frac{\boldsymbol{e}_r d\bar{l}_b}{2\bar{r}^2}=0\end{aligned}\right.\rightarrow\boldsymbol{F}_{sp}=0 \qquad \rightarrow\boldsymbol{F}_{plm}=d\boldsymbol{F}_{Lorentz} \tag{53}$$

- When the particle is off center

Let us analyze the integral of (52) in the case where a positive charged particle $+q_a$ is off center, for example above the center as shown in **Figure 3**. The scalar product $\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b$ equals the cosine of the angle θ between the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_b$, that is, $\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b=\cos\theta$. In the left case of **Figure 3**, the angle θ is bigger than $\pi/2$ and $\cos\theta<0$. Then, the vector $(\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b)\boldsymbol{e}_b=-|\cos\theta|\boldsymbol{e}_b$ and is up tilted. The minus radial vector $-\boldsymbol{e}_r$ being up tilted too, the vector $(\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b)\boldsymbol{e}_b-\boldsymbol{e}_r/2$ is up tilted. In the right case which is the symmetry of the left case, the angle θ' is smaller than $\pi/2$ and $\cos\theta'>0$ and the vector $(\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b)\boldsymbol{e}_b-\boldsymbol{e}_r/2$ is also up tilted, see **Figure 3**. Then, the integral of (52) for the upper part of the coil results an up pointing vector.

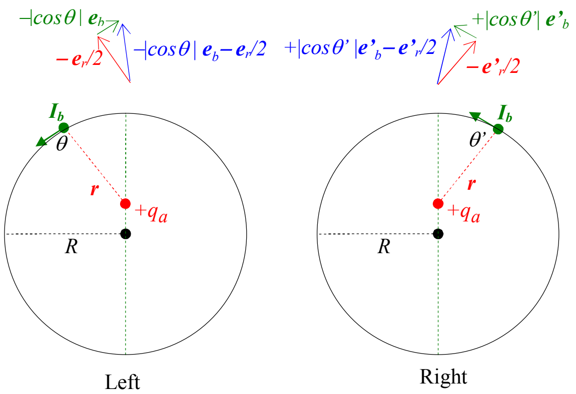

Figure 3

The same analysis for the lower part of the coil results a down pointing vector. But as the radial vector $\boldsymbol{r} = r\,\boldsymbol{e}_r$ is longer in the lower part, the vector $((\boldsymbol{e}_r{\cdot}\boldsymbol{e}_b)\boldsymbol{e}_b-\boldsymbol{e}_r/2)/r^2$ is smaller and then, the total closed integral of (52) results the vector $\mathbf{V}_{up}$ which is up pointing. As the charge $+q_a$ is positive, the spatial force $\boldsymbol{F}_{sp}\propto+q_a\mathbf{V}_{up}$ and pushes $+q_a$ toward the upper wire of the coil. So, in a fusion reaction chamber $\boldsymbol{F}_{sp}$ will always push $+q_a$ toward the wall despite the Lorentz force which tends to confine the plasma around the center.

6. Reducing plasma's deviation

Nuclear fusion reactors are so designed that plasmas should not touch the chamber's wall. But, as shown in the videos of experiments of Wendelstein 7-X **Figure 4**, and MAST Tokamak **Figure 5**, the plasmas steadily hit the wall at the bright burned spots. The atoms scraped off the wall pollute the plasma and the installations are damaged. The force that deviates the plasma out of the intended path could well be the spatial force $\boldsymbol{F}_{sp}$ which may also create instability. The expression for $\boldsymbol{F}_{sp}$ is in (52) from which we can derive some methods to reduce the deviation of plasma.

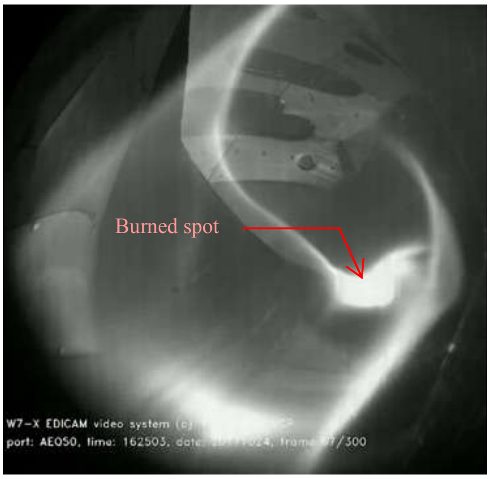

Figure 4

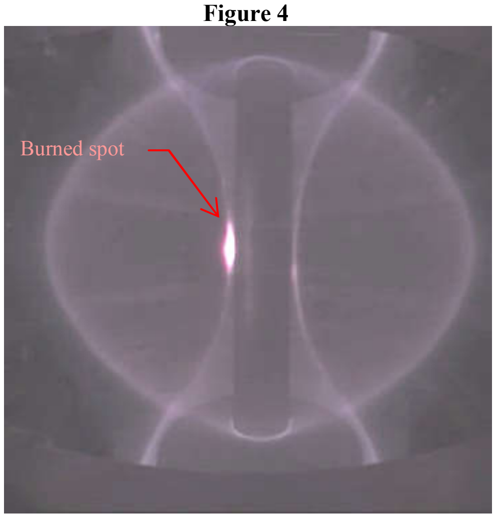

Figure 5

1. Reducing the velocity of electrons in conductor

The underlined and blue terms in (48) are the origin of $\boldsymbol{F}_{sp}$ which are proportional to the velocity $\boldsymbol{v}_b$ of the free electrons in the conductor that creates the magnetic field. If we reduce $\boldsymbol{v}_b$, $\boldsymbol{F}_{sp}$ will decrease. One can reduce $\boldsymbol{v}_b$ by increasing the cross section of the wires, by using conductor material with higher rate of free electron per unit volume. If copper has more free electrons than superconductor, then copper is better.

2. Bigger reaction chamber

(52) shows that $\boldsymbol{F}_{sp}$ is inversely proportional to the size of the coil R. So, $\boldsymbol{F}_{sp}$ is inversely proportional to the size of the section of the reaction chamber and the bigger the section is, the weaker $\boldsymbol{F}_{sp}$ is.

3. Torus-shaped chamber

The analysis in the precedent section has shown that $\boldsymbol{F}_{sp}$ is 0 at the center of a circular coil and stronger near the wire. In a reaction chamber positive particles will be pushed by stronger $\boldsymbol{F}_{sp}$ near the wall. So, plasma should stay as far as possible from the wall. As circle is the shape that is the farthest from the points inside, it is better to use chamber with round section, that is, a torus.

4. Augment electrons in plasma

$\boldsymbol{F}_{sp}$ is proportional to the charge of q_a and positive particles are pushed toward the wall. In the contrary, negative charged electrons are pulled back toward the center. If we inject electrons into the plasma or withdraw a quantity of positive ions, the plasma becomes negatively charged and the overall spatial force points to the center. Then the negatively charged plasma is drawn to the center and is prevented from hitting the wall.

This method is an active method which introduces a centre pointing force to plasma. Moreover, it is the easiest to realize. As Tokamak installations are rarely insulated from the ground, the injected electrons will surely leak out of the chamber in the form of electric current into the ground and the plasma will rapidly lose its negative charge and get deviated again.

To overcome this problem, I propose 2 solutions. The first is to insulate the whole installation from the ground. This method is suitable for small Tokamak and can be tested quickly. The second is for big reaction chamber for which it is simpler to insulate the chamber's wall with ceramic tiles or coating rather than insulate the whole installation.

However, there would always be leakage current out of the chamber. So, I think it will be necessary to maintain a constant current of injection to keep the plasma negatively charged at a constant level. The injection could be done using cathode ray electron gun that works constantly.

5. Positively charged coils for confinement

Since $\boldsymbol{F}_{sp}$ is created by the currents that produce the magnetic field for confining plasma, $\boldsymbol{F}_{sp}$ is a force that attracts the positive particles q_a toward the currents. So, we can create in the wires for confinement a positive electric field $\boldsymbol{E}$ that pushes the positive particles q_a off the wires, see **Figure 6**. Because $\boldsymbol{E}$ is contrary to $\boldsymbol{F}_{sp}$, the positive ions of the plasma are kept off the wall.

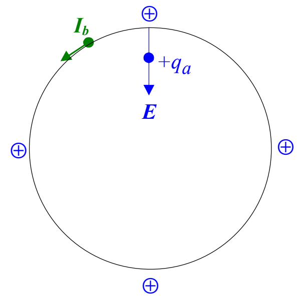

Figure 6

As the circuit for confinement magnetic field is already well insulated, the positive electric field can be created just by putting a high voltage on the whole circuit for confinement after disconnecting the circuit from the ground.

However, the reaction chamber must be insulated from the ground because the positive charge in the wire will create a capacitance with the metallic chamber which will be charged negatively by a current from the ground and annihilate the electric field $\boldsymbol{E}$ inside the chamber.

The voltage should be adjusted such that the force on the electrons from the electric field $\boldsymbol{E}$ balances the force $\boldsymbol{F}_{sp}$ to avoid the contact of the electron with the wall of the chamber.