From electron to magnetism

4. Coulomb magnetic force

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28 March 2018

Abstract: The relativistic length contraction effect and changing distance effect produce 2 different magnetic forces. Together they form complete magnetic force.

1. Two magnetic forces

I have derived 2 magnetic forces with Coulomb's law and charges' velocity. The first force dF_{lc} is derived in «Length-contraction magnetic-force between arbitrary currents» and is expressed in equation (1) in terms of current elements dI_a and dI_b and the radial vector r pointing from dI_b to dI_a.

$$dF_{lc} = -\frac{1}{4\pi\varepsilon_0 c^2 r^3}(dI_a \cdot dI_b)r \qquad (1)$$

$$dF_{cd} = \frac{1}{4\pi\varepsilon_0 c^2 r^3}[dI_b(r \cdot dI_a) + dI_a(r \cdot dI_b)] \qquad (2)$$

$$\begin{aligned} dF_{cm} &= dF_{lc} + dF_{cd} \\ &= -\frac{1}{4\pi\varepsilon_0 c^2 r^3}(dI_a \cdot dI_b)r \\ &\quad + \frac{1}{4\pi\varepsilon_0 c^2 r^3}[dI_b(r \cdot dI_a) + dI_a(r \cdot dI_b)] \end{aligned} \qquad (3)$$

The second force dF_{cd} is derived in «Changing distance effect» and given in (2). dF_{lc} and dF_{cd} are added together in (3) to give the expression for complete magnetic force dF_{cm} in (4). In the following, we will call dF_{cm} Coulomb magnetic force and (4) Coulomb magnetic force law.

$$dF_{cm} = \frac{1}{4\pi\varepsilon_0 c^2 r^3}\left[-(dI_a \cdot dI_b)r + dI_b(r \cdot dI_a) + dI_a(r \cdot dI_b)\right] \qquad (4)$$

2. Comparison with Lorentz force law

Let us compare (4) with Lorentz force law (5), which is transformed in (6) using the identity $\mu_0 = 1/\varepsilon_0 c^2$. First, we transform the 2 underlined terms of (4) into a double cross product using the vector identity (7), which is done in (8). Then we combine (8) with (4) to express dF_{cm} in (9), then as the sum of 2 forces in (10).

$$dF_{Lorentz} = \frac{\mu_0}{4\pi r^3} dI_a \times (dI_b \times r) \qquad (5)$$

$$dF_{Lorentz} = \frac{1}{4\pi\varepsilon_0 c^2 r^3} dI_a \times (dI_b \times r) \qquad (6)$$
$$\text{With } \mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \qquad (7)$$

$$(dI_a \cdot r)dI_b - (dI_a \cdot dI_b)r = dI_a \times (dI_b \times r) \qquad (8)$$

$$dF_{cm} = \frac{1}{4\pi\varepsilon_0 c^2 r^3}[dI_a \times (dI_b \times r) + dI_a(r \cdot dI_b)] \qquad (9)$$

$$dF_{cm} = \frac{1}{4\pi\varepsilon_0 c^2 r^3} dI_a \times (dI_b \times r) + \frac{1}{4\pi\varepsilon_0 c^2 r^3} dI_a(r \cdot dI_b) \qquad (10)$$

We recognize the first term of (10) as the Lorentz force in (6). So, Coulomb magnetic force dF_{cm} equals Lorentz force plus a second force dF_a, which is named cross force A, see (11). dF_{cm} is different from Lorentz force for current elements.

$$dF_{cm} = dF_{Lorentz} + dF_a$$
$$\text{With } dF_a = \frac{1}{4\pi\varepsilon_0 c^2 r^3} dI_a(r \cdot dI_b) \qquad (11)$$

$$\begin{aligned} \oint dF_{cm} &= \oint (dF_{Lorentz} + dF_a) \\ &= \oint dF_{Lorentz} + \oint dF_a \end{aligned} \qquad (12)$$

Second, we compare the Coulomb magnetic force and the Lorentz force that a closed circuit C exerts on a current element dI_a. The Coulomb magnetic force that C exerts on dI_a equals the closed line integral of dF_{cm} over C, which is computed in (12) and equals the integral of Lorentz force plus the integral of dF_a. We compute the integral of dF_a in (13) where $dI_b = I_b dl_b$ is a current element of C, I_b is the current intensity and dl_b the length vector of dI_b.

$$\begin{aligned} \oint dF_a &= \oint \frac{1}{4\pi\varepsilon_0 c^2 r^3} dI_a(r \cdot dI_b) \\ &= \frac{dI_a}{4\pi\varepsilon_0 c^2} \oint \frac{r \cdot I_b dl_b}{r^3} \\ &= \frac{dI_a I_b}{4\pi\varepsilon_0 c^2} \oint \frac{r \cdot dl_b}{r^3} \end{aligned} \qquad (13)$$

In space, $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ are located by the position vectors $\boldsymbol{P}_a$ and $\boldsymbol{P}_b$ whose difference equals the radial vector $\boldsymbol{r}$, see (14). Then, (14) is differentiated in (15) to give $d\boldsymbol{r} = -d\boldsymbol{P}_b$. As $\boldsymbol{P}_b$ is the position of $d\boldsymbol{I}_b$, $d\boldsymbol{P}_b$ is an infinitesimal segment of C that equals the length vector $d\boldsymbol{l}_b$, that is, $d\boldsymbol{P}_b = d\boldsymbol{l}_b$. Then, $d\boldsymbol{r} = -d\boldsymbol{l}_b$, see (15).

$$\boldsymbol{r} = \boldsymbol{P}_a - \boldsymbol{P}_b \qquad (14)$$

$$d\boldsymbol{r} = -d\boldsymbol{P}_b = -d\boldsymbol{l}_b \qquad (15)$$

$$\oint d\boldsymbol{F}_a = \frac{d\boldsymbol{I}_a I_b}{4\pi\varepsilon_0 c^2} \oint \frac{\boldsymbol{r} \cdot d\boldsymbol{l}_b}{r^3} \qquad (16)$$

Because $d\boldsymbol{I}_a$ and I_b are constant, they are extracted from the integral of (13) to give (16). Using (15), the integral of (16) is transformed into (17) which equals zero. So, the integral of $d\boldsymbol{F}_a$ is zero and the integral of Coulomb magnetic force equal the integral of Lorentz force, see (18). So, Coulomb magnetic force law and Lorentz force law are equivalent for the magnetic force that a closed circuit exerts on a current element.

$$\oint \frac{\boldsymbol{r} \cdot d\boldsymbol{l}_b}{r^3} = -\oint \frac{\boldsymbol{r} \cdot d\boldsymbol{r}}{r^3} = 0 \qquad (17)$$

$$\oint d\boldsymbol{F}_a = \frac{d\boldsymbol{I}_a I_b}{4\pi\varepsilon_0 c^2} * 0 = 0$$

$$\Rightarrow \oint d\boldsymbol{F}_{cm} = \oint d\boldsymbol{F}_{Lorentz} \qquad (18)$$

Although Coulomb magnetic force law is equivalent with Lorentz force law, it is still not clear why the 2 magnetic forces $d\boldsymbol{F}_{lc}$ and $d\boldsymbol{F}_{cd}$ should be added together to make Coulomb magnetic force. Without a consistent explanation, there will always be room to questioning its validity.

3. One Coulomb's law for 2 effects

$d\boldsymbol{F}_{lc}$ and $d\boldsymbol{F}_{cd}$ are previously derived each from a separate expression for Coulomb's law. Now, we derive the complete expression for Coulomb magnetic force from a single expression for Coulomb's law (19). $\boldsymbol{F}$ in (19) is the Coulomb force that the charge q_b exerts on q_a, the boldface $\boldsymbol{r}$ is the radial vector pointing from q_b to q_a and the regular r its magnitude and the distance between q_a and q_b.

Magnetic force arises when the Coulomb force $\boldsymbol{F}$ varies due to the velocities of charges. $\Delta\boldsymbol{F}$, the variation of $\boldsymbol{F}$, is derived from (19) by using the formula for differentiating the product of 2 function fg: $\Delta(fg) = g\,\Delta f + f\,\Delta g$. $\Delta\boldsymbol{F}$ is computed in (20) and equals a term in $\Delta q_a q_b$ plus a term in $\Delta(\boldsymbol{r}/r^3)$. The term in $\Delta q_a q_b$ will give the magnetic force due to length contraction effect. The term in $\Delta(\boldsymbol{r}/r^3)$ will give the magnetic force due to changing distance effect. So, the adding of the 2 magnetic forces to make Coulomb magnetic force comes from the differentiation of the Coulomb force $\boldsymbol{F}$.

$$\boldsymbol{F} = \frac{q_a q_b}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{r^3} \qquad (19)$$

$$\Delta\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0} \Delta\left(q_a q_b \frac{\boldsymbol{r}}{r^3}\right)$$
$$= \frac{1}{4\pi\varepsilon_0} [\Delta(q_a q_b)\left(\frac{\boldsymbol{r}}{r^3}\right)_0 \qquad (20)$$
$$+ \Delta\left(\frac{\boldsymbol{r}}{r^3}\right)(q_a q_b)_0]$$

$$\Delta\boldsymbol{F} = \boldsymbol{F} - \boldsymbol{F}_0$$
$$\Delta(q_a q_b) = q_a q_b - (q_a q_b)_0 \qquad (21)$$
$$\Delta\left(\frac{\boldsymbol{r}}{r^3}\right) = \frac{\boldsymbol{r}}{r^3} - \left(\frac{\boldsymbol{r}}{r^3}\right)_0$$

$$\boldsymbol{F}_0 = \frac{1}{4\pi\varepsilon_0} (q_a q_b)_0 \left(\frac{\boldsymbol{r}}{r^3}\right)_0 \qquad (22)$$

$\Delta\boldsymbol{F}$, $\Delta q_a q_b$ and $\Delta(\boldsymbol{r}/r^3)$ are the variations of the functions $\boldsymbol{F}$, $q_a q_b$ and $\boldsymbol{r}/r^3$ about a reference state 0 where the values of these functions are $\boldsymbol{F}_0$, $(q_a q_b)_0$ and $(\boldsymbol{r}/r^3)_0$. We call these values reference values and the values of $\boldsymbol{F}$, $q_a q_b$ and $\boldsymbol{r}/r^3$ at instant t the actual values. The reference state 0 is the state where currents are zero in $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$.

The variation of a function equals the actual value minus the reference value. $\Delta\boldsymbol{F}$, $\Delta q_a q_b$ and $\Delta(\boldsymbol{r}/r^3)$ are expressed in this fashion in (21). $\boldsymbol{F}_0$ is expressed in terms of $(q_a q_b)_0$ and $(\boldsymbol{r}/r^3)_0$ in (22). We introduce (21) into (20) to express $\boldsymbol{F} - \boldsymbol{F}_0$ in (23), which is combined with (22) to give the expression for $\boldsymbol{F}$ (24). (24) is in terms of the actual values $q_a q_b$ and $\boldsymbol{r}/r^3$, which enables us to use the results from previous derivations of $d\boldsymbol{F}_{lc}$ and $d\boldsymbol{F}_{cd}$ for the following.

$$\boldsymbol{F} - \boldsymbol{F}_0 = \Delta\boldsymbol{F}$$
$$= \frac{1}{4\pi\varepsilon_0}\left((q_a q_b - (q_a q_b)_0)\left(\frac{\boldsymbol{r}}{r^3}\right)_0 + \left(\frac{\boldsymbol{r}}{r^3} - \left(\frac{\boldsymbol{r}}{r^3}\right)_0\right)(q_a q_b)_0\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\left(q_a q_b \left(\frac{\boldsymbol{r}}{r^3}\right)_0 - (q_a q_b)_0\left(\frac{\boldsymbol{r}}{r^3}\right)_0 + (q_a q_b)_0\frac{\boldsymbol{r}}{r^3} - (q_a q_b)_0\left(\frac{\boldsymbol{r}}{r^3}\right)_0\right) \qquad (23)$$
$$= \frac{1}{4\pi\varepsilon_0}\left(q_a q_b \left(\frac{\boldsymbol{r}}{r^3}\right)_0 + (q_a q_b)_0\frac{\boldsymbol{r}}{r^3}\right) - 2\boldsymbol{F}_0$$

$$\boldsymbol{F} = \boldsymbol{F}_0 + \frac{1}{4\pi\varepsilon_0}\left(q_a q_b \left(\frac{\boldsymbol{r}}{r^3}\right)_0 + (q_a q_b)_0\frac{\boldsymbol{r}}{r^3}\right) - 2\boldsymbol{F}_0$$
$$= \frac{1}{4\pi\varepsilon_0} q_a q_b \left(\frac{\boldsymbol{r}}{r^3}\right)_0 - \boldsymbol{F}_0 + \frac{1}{4\pi\varepsilon_0}(q_a q_b)_0\frac{\boldsymbol{r}}{r^3}$$
$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{\boldsymbol{r}}{r^3}\right)_0 (q_a q_b - (q_a q_b)_0) \qquad (24)$$
$$+ \frac{1}{4\pi\varepsilon_0}(q_a q_b)_0\frac{\boldsymbol{r}}{r^3}$$

4. Coulomb magnetic force law

Coulomb magnetic force is the sum of the Coulomb forces between positive and negative charges of the current elements dI_a and dI_b. When currents circulate in dI_a and dI_b, the total charge of all free electrons of dI_a is Q_{a-}, that of dI_b is Q_{b-}. The corresponding positive charge of dI_a is Q_{a+}, that of dI_b is Q_{b+}. In the reference state 0 where currents are zero, the total positive and negative charges of dI_a and dI_b are respectively $n_a e$, $-n_a e$, $n_b e$ and $-n_b e$, with n_b and n_a being the numbers of free electrons of dI_a and dI_b respectively. So, the 4 Coulomb forces between the charges of dI_a and dI_b are:

- F_1 from Q_{b+} to Q_{a+}, the actual charge-product $q_a q_b$ is $Q_{b+}Q_{a+}$, the reference charge-product $(q_a q_b)_0$ is $+n_a n_b e^2$, the radial vector is r_1.
- F_2 from Q_{b-} to Q_{a+}, the actual charge-product is $Q_{b-}Q_{a+}$, the reference charge-product is $-n_a n_b e^2$, the radial vector is r_2.
- F_3 from Q_{b+} to Q_{a-}, the actual charge-product is $Q_{b+}Q_{a-}$, the reference charge-product is $-n_a n_b e^2$, the radial vector is r_3.
- F_4 from Q_{b-} to Q_{a-}, the actual charge-product is $Q_{b-}Q_{a-}$, the reference charge-product is $+n_a n_b e^2$, the radial vector is r_4.

Tab. 1 column 4 summarizes the 4 Coulomb forces computed using equation (24) while columns 1, 2 and 3 summarize the interacting charges, actual and reference charge-product, $q_a q_b$ and $(q_a q_b)_0$.

Charges	$q_a q_b$	$(q_a q_b)_0$	Coulomb forces
$Q_{b+}{\to}Q_{a+}$	$Q_{b+}Q_{a+}$	$+n_b n_a e^2$	$F_1 = \dfrac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a+} + n_b n_a e^2)\dfrac{r_1}{r_1^3} + \dfrac{n_b n_a e^2}{4\pi\varepsilon_0}\dfrac{r_1}{r_1^3}$
$Q_{b-}{\to}Q_{a+}$	$Q_{b-}Q_{a+}$	$-n_b n_a e^2$	$F_2 = \dfrac{1}{4\pi\varepsilon_0}(Q_{b-}Q_{a+} - n_b n_a e^2)\dfrac{r_1}{r_1^3} - \dfrac{n_b n_a e^2}{4\pi\varepsilon_0}\dfrac{r_2}{r_2^3}$
$Q_{b+}{\to}Q_{a-}$	$Q_{b+}Q_{a-}$	$-n_b n_a e^2$	$F_3 = \dfrac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a-} - n_b n_a e^2)\dfrac{r_1}{r_1^3} - \dfrac{n_b n_a e^2}{4\pi\varepsilon_0}\dfrac{r_3}{r_3^3}$
$Q_{b-}{\to}Q_{a-}$	$Q_{b-}Q_{a-}$	$+n_b n_a e^2$	$F_4 = \dfrac{1}{4\pi\varepsilon_0}(Q_{b-}Q_{a-} + n_b n_a e^2)\dfrac{r_1}{r_1^3} + \dfrac{n_b n_a e^2}{4\pi\varepsilon_0}\dfrac{r_4}{r_4^3}$

Tab. 1

F_{s4}, the sum of the 4 Coulomb forces, is computed in (25) and transformed into a sum of 2 terms, $FQ+Fn$. FQ is the magnetic force due to relativistic length contraction effect derived in «Length-contraction magnetic-force between arbitrary currents» which is expressed in (26).

Fn is the magnetic force due to changing distance effect derived in «Changing distance effect». For using the derivation of this article, we must transform Fn back to the sum of individual Coulomb force between pairs of charges which is averaged over the time interval $[t_1 , t_2]$.

So, we dissociate the radial vectors r_1, r_2, r_3 and r_4 which are in fact the mean radial vectors defined using the formula in (27). The vectors $r_{1,i}$, $r_{2,i}$, $r_{3,i}$ and $r_{4,i}$ are the radial vectors between individual charges of pairs of charges. Then Fn is rearranged into the sum of individual forces on one pair of charges in (28).

For making the average value we integrate the whole expression for F_{s4} in (29), then divide the integral by the length of the time interval. FQ is unchanged in (29) because it is constant with respect to time. Now, the average value of Fn is the result force of «Changing distance effect» which is given in (30).

$$
\begin{aligned}
F_{s4} &= F_1 + F_2 + F_3 + F_4 \\
&= \frac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a+} + n_b n_a e^2)\frac{r_1}{r_1^3} + \frac{n_b n_a e^2}{4\pi\varepsilon_0}\frac{r_1}{r_1^3} \\
&+ \frac{1}{4\pi\varepsilon_0}(Q_{b-}Q_{a+} - n_b n_a e^2)\frac{r_1}{r_1^3} - \frac{n_b n_a e^2}{4\pi\varepsilon_0}\frac{r_2}{r_2^3} \\
&+ \frac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a-} - n_b n_a e^2)\frac{r_1}{r_1^3} - \frac{n_b n_a e^2}{4\pi\varepsilon_0}\frac{r_3}{r_3^3} \\
&+ \frac{1}{4\pi\varepsilon_0}(Q_{b-}Q_{a-} + n_b n_a e^2)\frac{r_1}{r_1^3} + \frac{n_b n_a e^2}{4\pi\varepsilon_0}\frac{r_4}{r_4^3} \\
&= \frac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-})\frac{r_1}{r_1^3} \\
&+ \frac{e^2}{4\pi\varepsilon_0}\left(n_b n_a \frac{r_1}{r_1^3} - n_b n_a \frac{r_2}{r_2^3} - n_b n_a \frac{r_3}{r_3^3} + n_b n_a \frac{r_4}{r_4^3}\right) \\
&= FQ + Fn
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
FQ &= \frac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-})\frac{r_1}{r_1^3} \\
&= -\frac{(dI_a \cdot dI_b)r_1}{4\pi\varepsilon_0 c^2 r_1^3}
\end{aligned}
\tag{26}
$$

$$
n_b n_a \frac{r_1}{r_1^3} = \sum_{n_a}\sum_{n_b}\frac{r_{1,i}}{r_{1,i}^3}, \qquad n_b n_a \frac{r_2}{r_2^3} = \sum_{n_a}\sum_{n_b}\frac{r_{2,i}}{r_{2,i}^3}
$$
$$
n_b n_a \frac{r_3}{r_3^3} = \sum_{n_a}\sum_{n_b}\frac{r_{3,i}}{r_{3,i}^3}, \qquad n_b n_a \frac{r_4}{r_4^3} = \sum_{n_a}\sum_{n_b}\frac{r_{4,i}}{r_{4,i}^3}
\tag{27}
$$

$$
\begin{aligned}
Fn &= \frac{e^2}{4\pi\varepsilon_0}\left(n_b n_a \frac{r_1}{r_1^3} - n_b n_a \frac{r_2}{r_2^3} - n_b n_a \frac{r_3}{r_3^3} + n_b n_a \frac{r_4}{r_4^3}\right) \\
&= \frac{e^2}{4\pi\varepsilon_0}\left(\sum_{n_a}\sum_{n_b}\frac{r_{1,i}}{r_{1,i}^3} - \sum_{n_a}\sum_{n_b}\frac{r_{2,i}}{r_{2,i}^3} - \sum_{n_a}\sum_{n_b}\frac{r_{3,i}}{r_{3,i}^3} + \sum_{n_a}\sum_{n_b}\frac{r_{4,i}}{r_{4,i}^3}\right) \\
&= \sum_{n_a}\sum_{n_b}\frac{e^2}{4\pi\varepsilon_0}\left(\frac{r_{1,i}}{r_{1,i}^3} - \frac{r_{2,i}}{r_{2,i}^3} - \frac{r_{3,i}}{r_{3,i}^3} + \frac{r_{4,i}}{r_{4,i}^3}\right)
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
F_{cm} &= \frac{1}{t_2 - t_1}\int_{t1}^{t2} F_{s4}\,dt \\
&= \frac{1}{t_2 - t_1}\int_{t1}^{t2} FQ\,dt + \frac{1}{t_2 - t_1}\int_{t1}^{t2} Fn\,dt \\
&= FQ + \frac{1}{t_2 - t_1}\int_{t1}^{t2} Fn\,dt
\end{aligned}
\tag{29}
$$

Combining (30) and (26) with (29), we obtain the expression for F_{cm} in (31) which is identical to equation (4), the Coulomb magnetic force law. This justifies the adding of the 2 partial magnetic forces at the beginning.

$$\frac{1}{t_2 - t_1}\int_{t1}^{t2} Fn\, dt$$
$$= \frac{e^2}{4\pi\varepsilon_0}\sum_{n_a}\sum_{n_b}\frac{1}{t_2 - t_1}\int_{t1}^{t2}\left(\frac{r_{1,i}}{r_{1,i}{}^3} - \frac{r_{2,i}}{r_{2,i}{}^3} - \frac{r_{3,i}}{r_{3,i}{}^3} + \frac{r_{4,i}}{r_{4,i}{}^3}\right)dt \qquad (30)$$
$$= \frac{1}{4\pi\varepsilon_0 c^2 r_1{}^3}[dI_b(r_1 \cdot dI_a) + dI_a(r_1 \cdot dI_b)]$$

$$F_{cm} = \frac{1}{4\pi\varepsilon_0 c^2 r_1{}^3}[-(dI_a \cdot dI_b)r_1 + dI_b(r \cdot dI_a) + dI_a(r \cdot dI_b)] \qquad (31)$$

5. Fundamental aspects

– About respecting Newton's third law

Lorentz force law is inconsistent because of the violation of Newton's third law. In the documents listed in Annex I have shown many examples of the inconsistency. In the contrary, Coulomb magnetic force law respects well Newton's third law. Let us compare $dF_{b\to a}$ of (32) to $dF_{a\to b}$ of (33),

$$dF_{b\to a} = \frac{1}{4\pi\varepsilon_0 c^2 r_{b\to a}{}^3}\left[\begin{array}{c}-(dI_a \cdot dI_b)r_{b\to a}\\+dI_b(r_{b\to a} \cdot dI_a) + dI_a(r_{b\to a} \cdot dI_b)\end{array}\right] \qquad (32)$$

$$dF_{a\to b} = \frac{1}{4\pi\varepsilon_0 c^2 r_{a\to b}{}^3}\left[\begin{array}{c}-(dI_b \cdot dI_a)r_{a\to b}\\+dI_a(r_{a\to b} \cdot dI_b) + dI_b(r_{a\to b} \cdot dI_a)\end{array}\right] \qquad (33)$$

$$r_{a\to b} = -r_{b\to a} \qquad (34)$$

$$dF_{a\to b} = \frac{1}{4\pi\varepsilon_0 c^2 r_{b\to a}{}^3}\left[\begin{array}{c}+(dI_b \cdot dI_a)r_{b\to a}\\-dI_a(r_{b\to a} \cdot dI_b) - dI_b(r_{b\to a} \cdot dI_a)\end{array}\right] \qquad (35)$$
$$= -dF_{b\to a}$$

which are the Coulomb magnetic force that current element dI_b exerts on dI_a and the reaction force that dI_a exerts on dI_b respectively. Both are expressed using Coulomb magnetic force law (4).

The radial vector from dI_b to dI_a is $r_{b\to a}$ and that from dI_a to dI_b is $r_{a\to b}$. These 2 vectors are opposite vectors, that is, $r_{a\to b} = - r_{b\to a}$, see (34). We replace $- r_{b\to a}$ for $r_{a\to b}$ in (33) to express $dF_{a\to b}$ in terms of $r_{b\to a}$, which results in (35). We see that $dF_{a\to b}$ equals $-dF_{b\to a}$. So, Coulomb magnetic force respects Newton's third law. The sum $dF_{a\to b} + dF_{b\to a}$ equals zero and all the inconsistencies in the documents cited in Annex disappear.

Cross force A makes the expression for Coulomb magnetic force symmetric about dI_a and dI_b, see (4). The reason for which Lorentz force law violates Newton's third law is that it lacks cross force A, which is the only difference between the 2 laws, see (11).

– Tangential force

Cross force A is not in the classical theory because it is parallel to the current element dI_a, see (11). In my experiment «Continuous rotation of a circular coil experiment» the round coil is rotated by a force parallel to the current. This force is exactly cross force A because it is the part of Coulomb magnetic force that is parallel to the current elements of the coil. We can measure cross force A by measuring the torque on the coil.

– About magnetic field

Magnetic field B is the source of Lorentz force because $dF_{Lorentz} = dI_a \times B$. But the source of Coulomb magnetic force is not magnetic field because it is not derived using B and μ_0. Anyway, cross force A cannot be expressed in terms of B, see (11). One may wonder if Coulomb magnetic force is magnetic or not. If no, the force on a current element should equal Lorentz force plus Coulomb magnetic force which is 2 times Lorentz force. Since this is not true, Coulomb magnetic force is the same force as Lorentz force, that is, a magnetic force.

But how can a magnetic force be not created by magnetic field B? In fact, magnetic phenomena are discovered in antiquity through magnetic force. Hans Christian Ørsted discovered in 1819 electro-magnetic force, which André-Marie Ampère later measured as a physical force. In 1831, Michael Faraday *invented* the notion of magnetic field which James Clerk Maxwell theorized in 1865 as the theoretical object B. So, magnetic field B is *invented* to explain physical magnetic force, but not the contrary, that magnetic force is created by a physical magnetic field B. The real physical sequence should be: electrostatic force creates magnetic force, which is theorized as magnetic field B.

– About $\mu_0 = 1/\varepsilon_0 c^2$

μ_0 is the coefficient of Lorentz force and $1/\varepsilon_0 c^2$ that of Coulomb magnetic force, see (4) and (5). So, in addition to prove that the closed line integral of Coulomb magnetic force equals that of Lorentz force, equation (18) also proves the equality $\mu_0 = 1/\varepsilon_0 c^2$, independently form electromagnetic wave equation. It may seem useless to prove a well known property, but the proof using wave equation makes this equality dependant on electromagnetic wave and thus, on magnetic field $\boldsymbol{B}$ which, as said above, is not the source of magnetic force. Then, the proof using wave equation is in trouble.

Equation (18) shows that this equality is a relativistic property because the speed of light c in Coulomb magnetic force comes directly from Relativity, as shown by the derivations in «Length-contraction magnetic-force between arbitrary currents» and «Changing distance effect». This property makes magnetic force a direct relativistic effect rather than a consequence of magnetic field $\boldsymbol{B}$.

6. Annex

«Lorentz force on open circuit, Blogspot word»
«Numerical computation of the Lorentz force internal to an asymmetric coil, Blogspot word »

Intuitive paradoxes
The circuits of these paradoxes give rise to non-zero self Lorentz force, violating Newton's third law. These self forces can be figured out intuitively using the Lorentz force law. These 5 contradictions are strong proofs of the wrongness of the Lorentz force law.

Analyze of the Lorentz forces internal to an equilateral triangle coil, html
2 straight wires making an angle exert a force on each other and their sum lies on the bisector. The 2 bisector-lying forces on the lower corners of a triangular coil sum to a downward self force, violating Newton's third law.
Paradoxical Lorentz force internal to a triangle coil, html
Place a wire inside a magnetic shield and it will not feel Lorentz force. Shield one side of a triangular coil, the non shielded sides will feel a Lorentz force which is a non-zero self force on the coil.
Lorentz forces internal to a polygon coil, analyze and computation, html
The pentagon coil has a sharp angular top and a high rectangular bottom. The top exerts on itself a force that depends on the angle, while the force on the rectangular bottom is constant. Varying independently, the forces on the top and the bottom do no cancel each other and make a non-zero self force.
Lorentz force on open circuit, Blogspot word
The 2 ball capacitors enable an alternate current to circulate in the angular wire that creates a Lorentz force on the wire. Since there is not straight wire connecting the 2 capacitors there is no counter force and the self force is not zero.
Self force of a 3D coil, pdf word
The 2 upward pointing corners of the 3D coil exert on themselves vertical forces that cannot be balanced by the forces on the 2 horizontal wires, violating Newton's third law.

Analytic computations of the self force of triangular coil
The 2 self forces computed analytically are non-zero and show that the Lorentz force law is wrong for any triangular coils.

Mathematical cause of the existence of the remaining resultant internal Lorentz force, pdf word
The relation between the forces on the 3 sides of a triangular coil must be linear if the self force is zero. But the Lorentz force law is non linear and the relation between the Lorentz forces on the 3 sides is non linear, giving rise to non-zero self force.
Proof of the remaining resultant Lorentz force internal to a triangular coil, pdf word
Analytical computation of the vertical self force a triangular coil exerts on itself. This force is always non-zero.
Synthesis of the inconsistency of the Lorentz force law, pdf word
2 examples that answer the objections to my theory: 1) The Lorentz force law gives zero self force for a system of 2 closed coils , but this self force becomes non-zero when the 2 coils merge into a single coil. 2) The angular top of a pentagon coil can rotate generating horizontal force that the bottom sides cannot cancel.

Numerical computation
The 3 self forces computed numerically are non-zero confirming the analytic computations.

Numerical computation of the Lorentz force internal to an asymmetric coil, Blogspot word
Numerical computation of the self force of coils formed with 2 half ellipses of different major axis. The computed values of the self force are all non-zero.
Computation of the self force of a coil, Blogspot word
The self force of an asymmetric coil (2 different half ellipses) is computed using different number of discretization, which converges to a definite value showing the consistency of the computation.
Unhappiness of Newton with Lorentz and triangular coil experiment, Blogspot word
The self force of a triangular coil is computed using the Lorentz force law and the corrected magnetic force law, which is non-zero for the former and zero for the latter.