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Why EM wave equation does not conform to relativity?

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This question is not about real electromagnetic wave. Physical wave does not violate relativity principle and always travel at constant speed c. It is the electromagnetic wave equation that does not conform to relativity. By computing the speed of the solution of the wave equation, we find that its velocity is faster than that of light and varies with distance and frequency. This result proves that the wave equation does not faithfully describe the physical phenomenon of electromagnetic wave.

A rigorous mathematical proof is given below. The annex is provided just in case where someone wants to check the validity of the used equations and is not necessary to the proof.

The general wave equation is:

$$\nabla \times \nabla \times \boldsymbol{H} + \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0 \qquad (1)$$

We study the spherical wave sent by an element of antenna of length *dl* which carries a current *I* (see the Figure 1). As the magnetic field of this element is only in the ϕ direction, in spherical coordinates, the above wave equation simplifies to the polar wave equation for the ϕ component (see annex for the derivation):

$$\mu_0 \varepsilon_0 \frac{\partial^2 H_\phi}{\partial t^2} - \frac{1}{r}\left(\frac{\partial^2 r H_\phi}{\partial r^2} + \frac{\partial}{\partial \theta}\left(\frac{1}{r^2 \sin\theta}\left(\frac{\partial r H_\phi \sin\theta}{\partial \theta} \right)\right)\right) = 0 \qquad (2)$$

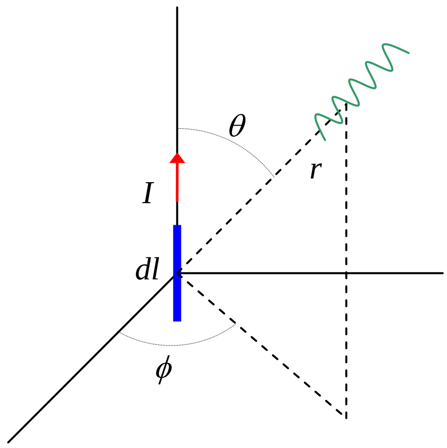

Figure 1

This equation admits the following analytical solution (ref. Clayton R. Paul, Keith W. Whites and Syed A. Nasar, Introduction to Electromagnetic Fields, Mcgraw-Hill College; 3 Sub edition (December 9, 1997), p.590):

$$H_\phi = \frac{Idl}{4\pi} \beta_0^2 \sin\theta \left(\frac{j r^{-1}}{\beta_0} + \frac{r^{-2}}{{\beta_0}^2} \right) e^{j(\omega t - \beta_0 r)} \qquad (3)$$

In the following we call this solution “antenna wave”. This expression can be simplified by defining an additional phase α as follow:

$$\cos\alpha = \frac{1}{\sqrt{1+\beta_0{}^2 r^2}}, \tan\alpha = \beta_0 r \tag{4}$$

$$H_\phi = \frac{Idl}{4\pi} r^{-2}\sqrt{1+\beta_0{}^2 r^2}\sin\theta(\cos\alpha + j\sin\alpha)e^{j(\omega t-\beta_0 r)}$$

Then, the equation (3) is transformed into the complex exponential form below:

$$H_\phi = \frac{Idl}{4\pi} r^{-2}\sqrt{1+\beta_0{}^2 r^2}\sin\theta e^{j(\omega t-\beta_0 r+\arctan(\beta_0 r))} \tag{5}$$

The definition of β_0 is $\beta_0 = \omega\sqrt{\mu_0\varepsilon_0}$ and we can write $\beta_0 r$ as a function of wave length λ:

$$\beta_0 = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}, \quad \beta_0 r = 2\pi\frac{r}{\lambda} \tag{6}$$

The equation (5) is then a function of r/λ:

$$H_\phi = \frac{Idl}{4\pi} r^{-2}\sqrt{1+\left(2\pi\frac{r}{\lambda}\right)^2}\sin\theta e^{j\left(\omega t-2\pi\frac{r}{\lambda}+\arctan\left(2\pi\frac{r}{\lambda}\right)\right)} \tag{7}$$

The argument of the complex exponential permits us to compute the time a wave takes to cover the distance between the source, $r=0$, and an arbitrary point r. For doing so, we mark a point on the wave by its phase, Φ. Take a point at the source, its phase is:

$$t_0 = 0,\, r_0 = 0 \Rightarrow \Phi_0 = \omega t_0 - 2\pi\frac{r_0}{\lambda} + \arctan\left(2\pi\frac{r_0}{\lambda}\right) = 0$$

After a period of time t_1, this point arrives at r_1, the phase is:

$$\Phi_1 = \omega t_1 - 2\pi\frac{r_1}{\lambda} + \arctan\left(2\pi\frac{r_1}{\lambda}\right)$$

Since the point (t_1, r_1) is the point (t_0, r_0) that has traveled the distance r_1, their phases Φ_0 and Φ_1 have the same value. Then we have:

$$\Phi_0 = \Phi_1 \Rightarrow \omega t_1 - 2\pi\frac{r_1}{\lambda} + \arctan\left(2\pi\frac{r_1}{\lambda}\right) = 0$$

So, the time at which the point arrives at the distance r_1 is:

$$t_1 = \frac{2\pi}{\omega}\frac{r_1}{\lambda} - \frac{1}{\omega}\arctan\left(2\pi\frac{r_1}{\lambda}\right)$$

With $T = \frac{2\pi}{\omega}$ the period of the antenna wave, $t_c = \frac{2\pi}{\omega}\frac{r_1}{\lambda} = \frac{r_1}{c}$ the time that light takes to cover the same distance, the time the wave takes is:

$$t_1 = t_c - \frac{T}{2\pi}\arctan\left(2\pi\frac{r_1}{\lambda}\right) \tag{8}$$

When r_1 tends to infinity, the time becomes t_∞:

$$r_1 \to \infty, \quad \arctan\left(2\pi\frac{r_1}{\lambda}\right) \to \frac{\pi}{2} \quad \Rightarrow \quad t_\infty = t_c - \frac{T}{4} \tag{9}$$

Here we obtain a key result: the antenna wave travels a fourth of a period ahead of light. Can light be overtaken by electromagnetic wave? No. Why this solution is faster than light? Let us compute the velocity of the antenna wave. Consider 2 points on the wave, of phases Φ_1 and Φ_2. The expressions of the phases are (see equation (5)):

$$\Phi_1 = \omega t_1 - \beta_0 r_1 + \alpha_1, \quad \Phi_2 = \omega t_2 - \beta_0 r_2 + \alpha_2$$

The point 2 is in fact the point 1 moved to r_2. Thus, $\Phi_1 = \Phi_2$ and we obtain the equation:

$$(\omega t_1 - \beta_0 r_1 + \alpha_1) - (\omega t_2 - \beta_0 r_2 + \alpha_2) = 0$$

So we have:

$$\omega \Delta t - \beta_0 \Delta r + \Delta\alpha = 0$$

$$\omega = \frac{\beta_0 \Delta r - \Delta\alpha}{\Delta t} = \frac{\beta_0 \Delta r}{\Delta t} - \frac{\Delta\alpha}{\Delta r}\frac{\Delta r}{\Delta t}$$

For point 1 very close to point 2, we have:

$$\omega = \left(\beta_0 - \frac{\partial\alpha}{\partial r}\right)\frac{\partial r}{\partial t}$$

The phase velocity of the antenna wave is:

$$v = \frac{\partial r}{\partial t} = \frac{\omega}{\beta_0 - \dfrac{\partial\alpha}{\partial r}} \tag{10}$$

Using the equation (4) we have:

$$\frac{\partial \tan\alpha}{\partial r} = \frac{1}{\cos^2\alpha}\frac{\partial\alpha}{\partial r} \Rightarrow \frac{\partial\alpha}{\partial r} = \frac{\beta_0}{1+\beta_0^{\,2} r^2}$$

Then the phase velocity of the wave is:

$$v = \frac{\omega}{\beta_0 - \beta_0\left(1+\beta_0^{\,2} r^2\right)^{-1}} = \frac{\omega}{\beta_0}\frac{1}{1-\left(1+\beta_0^{\,2} r^2\right)^{-1}}$$

$$v = \frac{\omega}{\beta_0}\left(\frac{1}{\beta_0^{\,2} r^2} + 1\right)$$

By using the equation (6), the phase velocity becomes:

$$v = c\left(1 + \frac{1}{4\pi^2}\left(\frac{\lambda}{r}\right)^2\right) \tag{11}$$

We see that v is greater than the speed of light c; v tends even to infinity if r tends to 0. This is why antenna wave is in advance of light. What about a non sinusoidal signal? As phase velocity of sinusoidal antenna wave of any frequency is higher than that of light (equation (11)), the signal which can be considered as a Fourier series of sinusoidal waves travels faster than light too.

In order to give a visual feeling of this effect, I have computed the magnitude of antenna wave and drawn it against a light signal carrying a signal of varying intensity at the same frequency. At the source, the antenna wave and the light signal are at the same phase. After 3 periods, antenna wave has already an advance of ¼ of period ahead of the light signal (see Figure 2).

So, the solution of electromagnetic wave equation travels faster than light. Its velocity is even infinitely large near the source, that is, for small r (see equation (11)). Should we verify by experiment that the relativity principle is inviolable?

I have written the mathematical derivation of the polar wave equation and the mathematical verification of the analytical solution (the equations (2) and (3)) in the annex for anyone who wants to check the validity of these equations.

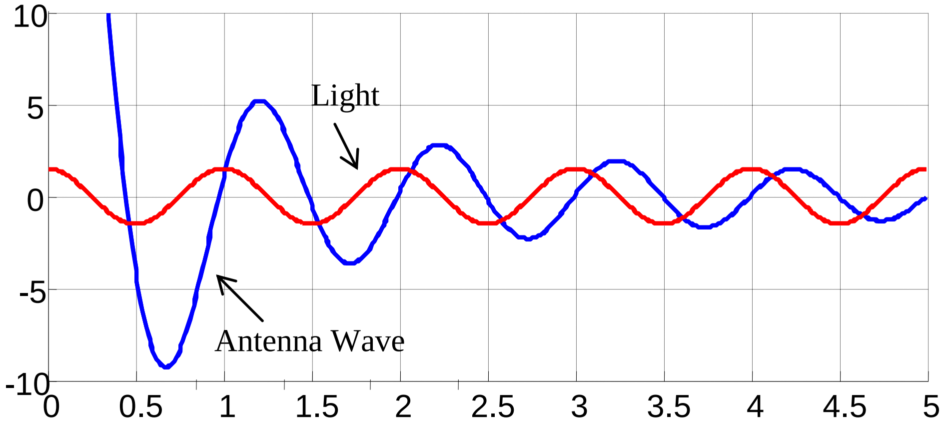

Figure 2

Annex 1. Wave equations

General wave equation is the equation (1). The expression for curl in spherical coordinates is:

$$\nabla\times\boldsymbol{H}=\frac{1}{r\sin\theta}\left(\frac{\partial H_\phi\sin\theta}{\partial\theta}-\frac{\partial H_\theta}{\partial\phi}\right)\boldsymbol{e}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi}-\frac{\partial rH_\phi}{\partial r}\right)\boldsymbol{e}_\theta+\frac{1}{r}\left(\frac{\partial rH_\theta}{\partial r}-\frac{\partial H_r}{\partial\theta}\right)\boldsymbol{e}_\phi$$

The magnetic field is only in the ϕ direction. So the expression for curl of magnetic field is:

$$H_r=0,\ H_\theta=0\Rightarrow\nabla\times\boldsymbol{H}=\frac{1}{r\sin\theta}\left(\frac{\partial H_\phi\sin\theta}{\partial\theta}\right)\boldsymbol{e}_r-\frac{1}{r}\frac{\partial rH_\phi}{\partial r}\boldsymbol{e}_\theta$$

Spherical symmetry makes:

$$\frac{\partial}{\partial\phi}=0\Rightarrow(\nabla\times\nabla\times\boldsymbol{H})_\phi=-\frac{1}{r}\left(\frac{\partial^2 rH_\phi}{\partial r^2}+\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\left(\frac{\partial rH_\phi\sin\theta}{\partial\theta}\right)\right)\right)$$

The final expression of the wave equation is:

$$(\nabla\times\nabla\times\boldsymbol{H})_\phi+\mu_0\varepsilon_0\frac{\partial^2 H_\phi}{\partial t^2}=0\Leftrightarrow\mu_0\varepsilon_0\frac{\partial^2 H_\phi}{\partial t^2}-\frac{1}{r}\left(\frac{\partial^2 rH_\phi}{\partial r^2}+\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\left(\frac{\partial rH_\phi\sin\theta}{\partial\theta}\right)\right)\right)=0$$

Annex 2. Verification of the solution

In order to simplify the derivation, a variable h is used:

$$H_\phi=\frac{Idl}{4\pi}\beta_0^2\sin\theta\cdot h\Rightarrow h=\left(\frac{jr^{-1}}{\beta_0}+\frac{r^{-2}}{\beta_0{}^2}\right)e^{j(\omega t-\beta_0 r)}$$

Introducing the analytical solution (equation (5)):

$$\frac{\partial rh}{\partial r}=\frac{\partial}{\partial r}\left[\left(\frac{j}{\beta_0}+\frac{r^{-1}}{\beta_0{}^2}\right)e^{j(\omega t-\beta_0 r)}\right]=-\frac{r^{-2}}{\beta_0{}^2}e^{j(\omega t-\beta_0 r)}-j\beta_0\left(\frac{j}{\beta_0}+\frac{r^{-1}}{\beta_0{}^2}\right)e^{j(\omega t-\beta_0 r)}$$

$$=\left[-\frac{r^{-2}}{\beta_0{}^2}+\frac{-jj\beta_0}{\beta_0}+\frac{-j\beta_0 r^{-1}}{\beta_0{}^2}\right]e^{j(\omega t-\beta_0 r)}=\left[-\frac{r^{-2}}{\beta_0{}^2}+1-\frac{jr^{-1}}{\beta_0}\right]e^{j(\omega t-\beta_0 r)}$$

$$\frac{\partial^2 rh}{\partial r^2} = \frac{\partial}{\partial r}\left(\left(-\frac{r^{-2}}{\beta_0{}^2} + 1 - \frac{jr^{-1}}{\beta_0}\right)e^{j(\omega t - \beta_0 r)}\right)$$

$$= \left[2\frac{r^{-3}}{\beta_0{}^2} + \frac{jr^{-2}}{\beta_0}\right]e^{j(\omega t-\beta_0 r)} - j\beta_0\left[-\frac{r^{-2}}{\beta_0{}^2} + 1 - \frac{jr^{-1}}{\beta_0}\right]e^{j(\omega t-\beta_0 r)}$$

$$= \left[2\frac{r^{-3}}{\beta_0{}^2} + \frac{jr^{-2}}{\beta_0} + j\beta_0\frac{r^{-2}}{\beta_0{}^2} - j\beta_0 + j\beta_0\frac{jr^{-1}}{\beta_0}\right]e^{j(\omega t-\beta_0 r)}$$

The second term of the equation (2) becomes then:

$$\frac{\partial^2 rH_\phi}{\partial r^2} = \frac{Idl}{4\pi}\beta_0^2 \sin\theta\left[2\frac{r^{-3}}{\beta_0{}^2} + 2\frac{jr^{-2}}{\beta_0} - j\beta_0 - r^{-1}\right]e^{j(\omega t-\beta_0 r)}$$

Derivation of the third term in the equation (2):

$$\frac{\partial rH_\phi \sin\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{Idl}{4\pi}\beta_0^2 \sin^2\theta\left(\frac{j}{\beta_0} + \frac{r^{-1}}{\beta_0{}^2}\right)e^{j(\omega t-\beta_0 r)}\right) = \left(\frac{j}{\beta_0} + \frac{r^{-1}}{\beta_0{}^2}\right)\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}\cdot 2\sin\theta\cos\theta$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\left(\frac{\partial rH_\phi \sin\theta}{\partial\theta}\right)\right) = \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\left(\frac{j}{\beta_0} + \frac{r^{-1}}{\beta_0{}^2}\right)\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}\cdot 2\sin\theta\cos\theta\right)$$

$$= \frac{\partial}{\partial\theta}\left(\left(\frac{jr^{-2}}{\beta_0} + \frac{r^{-3}}{\beta_0{}^2}\right)\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}\cdot 2\cos\theta\right) = -\left(\frac{jr^{-2}}{\beta_0} + \frac{r^{-3}}{\beta_0{}^2}\right)\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}\cdot 2\sin\theta$$

The expression of $(\nabla\times\nabla\times\boldsymbol{H})_\phi$ (see equation (1) and (2)) is then:

$$(\nabla\times\nabla\times\boldsymbol{H})_\phi = -r^{-1}\left(\begin{array}{l}\left(2\dfrac{r^{-3}}{\beta_0{}^2} + 2\dfrac{jr^{-2}}{\beta_0} - j\beta_0 - r^{-1}\right)\sin\theta\dfrac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)} \\ -\left(\dfrac{jr^{-2}}{\beta_0} + \dfrac{r^{-3}}{\beta_0{}^2}\right)\dfrac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}\cdot 2\sin\theta\end{array}\right)$$

$$= -r^{-1}\left(\left(2\left(\frac{r^{-3}}{\beta_0{}^2} + \frac{jr^{-2}}{\beta_0}\right) - j\beta_0 - r^{-1}\right) - 2\left(\frac{jr^{-2}}{\beta_0} + \frac{r^{-3}}{\beta_0{}^2}\right)\right)\sin\theta\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}$$

$$(\nabla\times\nabla\times\boldsymbol{H})_\phi = -r^{-1}\left(-j\beta_0 - r^{-1}\right)\sin\theta\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)} = \left(j\beta_0 r^{-1} + r^{-2}\right)\sin\theta\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)}$$

The term of time derivative is:

$$\frac{\partial^2 H_\phi}{\partial t^2} = \frac{Idl}{4\pi}\beta_0^2 \sin\theta\left(\frac{jr^{-1}}{\beta_0} + \frac{r^{-2}}{\beta_0{}^2}\right)\left(-\omega^2\right)e^{j(\omega t-\beta_0 r)}$$

Introducing the above 2 expressions into the equation (2):

$$(\nabla\times\nabla\times\boldsymbol{H})_\phi + \mu_0\varepsilon_0\frac{\partial^2 H_\phi}{\partial t^2} = \left(j\beta_0 r^{-1} + r^{-2}\right)\sin\theta\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)} - \omega^2\mu_0\varepsilon_0\frac{Idl}{4\pi}\beta_0^2\sin\theta\left(\frac{jr^{-1}}{\beta_0} + \frac{r^{-2}}{\beta_0{}^2}\right)e^{j(\omega t-\beta_0 r)}$$

By using $\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = \dfrac{\omega}{c}$, the analytical solution makes the wave equation (2) equal to 0:

$$(\nabla\times\nabla\times\boldsymbol{H})_\phi + \mu_0\varepsilon_0\frac{\partial^2 H_\phi}{\partial t^2} = \left[\left(j\beta_0 r^{-1} + r^{-2}\right) - \beta_0^2\left(\frac{jr^{-1}}{\beta_0} + \frac{r^{-2}}{\beta_0{}^2}\right)\right]\sin\theta\frac{Idl}{4\pi}\beta_0^2 e^{j(\omega t-\beta_0 r)} = 0$$