From electron to magnetism

1. Relativistic length contraction and magnetic force

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Abstract: Derivation of the expression for the magnetic force between parallel current elements from relativistic length contraction formula.

Magnetism is intimately related to special relativity. Maxwell's equations are invariant under a Lorentz transformation; the electromagnetic wave equation gives the speed of light *c*. Many have explained magnetic force as a consequence of relativistic length contraction, for example Richard Feynman in page 13-8 of his ≪The Feynman Lectures on Physics, Volume II≫ and Steve Adams in page 266 in his ≪Relativity: An Introduction to Spacetime Physic≫. If magnetic force is really created by relativistic length contraction, we should be able to derive the expression for magnetic force from the length contraction formula: $l = l_0\sqrt{1-v^2/c^2}$. And indeed we can, as I will show below.

1. Length contraction and charge density

Length contraction makes a moving rod to appear shorter in a stationary reference frame. Take a rod of proper length l_a and make it travel at velocity *v*. In the stationary frame its apparent length is l_b which is given by equation (**1**). If this rod is charged, the same charge will be squeezed within a shorter length and its charge density becomes higher.

$$l_b = l_a\sqrt{1-v^2/c^2} \qquad (1)$$

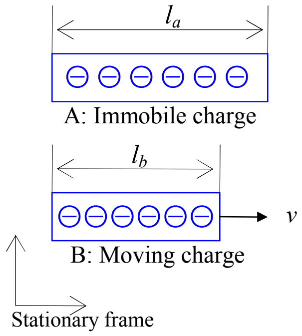

Figure 1

This is illustrated in **Figure 1** where A is an immobile rod of length l_a and uniformly charged with charge Q , which is schematically represented by the 6 electrons. B represents the rod A moving at velocity *v* and thus, is also charged with Q. Since the length of B is shorter the 6 electrons are more compact in B, making the charge density higher.

$$\rho_a = \frac{Q}{l_a} \qquad (2)$$

$$\rho_b = \frac{Q}{l_b} = \frac{Q}{l_a\sqrt{1-v^2/c^2}} = \frac{\rho_a}{\sqrt{1-v^2/c^2}} \qquad (3)$$

The linear charge densities of A and B, ρ_a and ρ_b , are given respectively by equations (**2**) and (**3**). ρ_b is higher than ρ_a by the factor $1/\sqrt{1-v^2/c^2}$.

2. Current carrying segment

Since the charge density of a moving charged rod increases with velocity, the free electrons in a conductor wire will also have a higher density when a current circulates in the wire. Indeed, electric current is a moving line of electrons or charge. Thus, the free electrons will have higher density than the positive charge of the wire. So, the wire will appear charged and exert a Coulomb force on a charge located nearby. Let us compute this force.

Take a segment of neutral wire in which no current circulates. The free electrons are **immobile** and their charge density equals that of positive charge ρ_+ with a minus sign, that is, $-\rho_+$. When a current circulates, the free electrons move at velocity *v*. In the frame of reference moving with the free electrons, their charge density still equals $-\rho_+$. In the stationary frame, the apparent charge density of the **moving** free electrons increases by the factor $1/\sqrt{1-v^2/c^2}$. Its value ρ_- is given by equation (**4**).

$$\rho_- = \frac{-\rho_+}{\sqrt{1-v^2/c^2}} \qquad (4)$$

Figure 2 illustrates this phenomenon. A is a neutral segment without current whose positive and negative charges are represented schematically by the 6 electrons and 6 positive charges. The segment B represents the segment A with a current circulating within. As the moving electrons suffer length contraction, more electrons are packed in B. Schematically, 7 electrons and 6 positive charges are in the segment and B appears charged with one more electron.

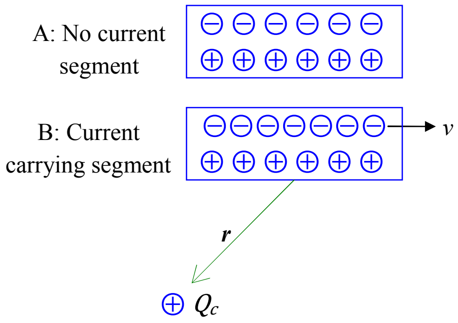

Figure 2

Q_c is a static charge near the segment B which exerts a Coulomb force on Q_c . The net charge in B equals $Q_b = (\rho_+ + \rho_-)\Delta l$ where Δl is B's length. The Coulomb force $\boldsymbol{F}_c$ is given by equation (**5**) where r is the distance between B and Q_c, $\boldsymbol{e}_r$ the unit radial vector pointing from B to Q_c. The boldface $\boldsymbol{F}$ and $\boldsymbol{e}$ are **vectors**. Richard Feynman has explained this phenomenon in page 13-8 of his ≪The Feynman Lectures on Physics, Volume II≫.

Note: For explanation, the electrons and positive charges are in 2 separate lines in **Figure 2**, but in real conductor electrons and positive charges are not separated.

$$\begin{aligned}\boldsymbol{F}_c &= \frac{1}{4\pi\varepsilon_0}\frac{Q_c Q_b}{r^2}\boldsymbol{e}_r \\ &= \frac{Q_c}{4\pi\varepsilon_0}(\rho_+ + \rho_-)\Delta l\frac{\boldsymbol{e}_r}{r^2} \\ &= \frac{Q_c}{4\pi\varepsilon_0}\rho_+\left(1 - \frac{1}{\sqrt{1 - v^2/c^2}}\right)\Delta l\frac{\boldsymbol{e}_r}{r^2}\end{aligned} \tag{5}$$

3. Two current carrying segments

Since the current carrying segment B exerts Coulomb force on the static charge, it can also exert Coulomb force on another current carrying segment because the latter too, will appear charged due to current. This force is not zero while the 2 segments are statically neutral. Thus, this force seems to be magnetic. But we compute it with Coulomb's law.

Let A and B be 2 parallel current carrying segments, which are shown in **Figure 3**. The quantities of positive and negative charges in A are Q_{a+}, Q_{a-} and those in B are Q_{b+}, Q_{b-} . We compute the two forces on the positive charge of the segment A exerted by the positive and negative charges of B, that is, the two Coulomb forces on Q_{a+} exerted by Q_{b+} and Q_{b-} which are $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in equations (**6**) where r is the distance between A and B, $\boldsymbol{e}_r$ the unit radial vector pointing from B to A. In the same way, the forces on the free electrons of A, Q_{a-} , are also exerted by Q_{b+} and Q_{b-} and are the Coulomb forces $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ in equations (**6**). The sum of these 4 forces, $\boldsymbol{F}_a = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4$ given by equation (**7**), is the resultant force that the segment B exerts on the segment A.

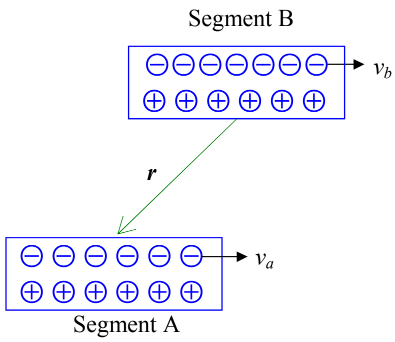

Figure 3

Due to length contraction, the values of Q_{a+}, Q_{a-}, Q_{b+} and Q_{b-} vary with their velocities. But depending on the reference frame in which they are evaluated, their values are not always the same. For example, let v_a and v_b be the velocities of free electrons in A and B respectively. In the stationary frame the velocity of Q_{b-} is v_b, in the frame of Q_{a-}, the velocities of Q_{b-} is $v_b - v_a$. The value of Q_{b-} will not be the same in these two frames because their velocities are different. This is why the products of charges in equation (7), $Q_{b+}Q_{a+}$, $Q_{b-}Q_{a+}$, $Q_{b+}Q_{a-}$ and $Q_{b-}Q_{a-}$, depend also on the frames in which they are evaluated. They are given the name charge-product and are carefully computed below.

$$\begin{aligned}\boldsymbol{F}_1 &= \frac{1}{4\pi\varepsilon_0}\frac{Q_{b+}Q_{a+}}{r^2}\boldsymbol{e}_r, \qquad \boldsymbol{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b-}Q_{a+}}{r^2}\boldsymbol{e}_r \\ \boldsymbol{F}_3 &= \frac{1}{4\pi\varepsilon_0}\frac{Q_{b+}Q_{a-}}{r^2}\boldsymbol{e}_r, \qquad \boldsymbol{F}_4 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b-}Q_{a-}}{r^2}\boldsymbol{e}_r\end{aligned} \tag{6}$$

$$\boldsymbol{F}_a = \frac{1}{4\pi\varepsilon_0}\left(Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-}\right)\frac{\boldsymbol{e}_r}{r^2} \tag{7}$$

1. $Q_{b+}Q_{a+}$

Q_{b+} and Q_{a+} are both stationary and their densities are the constants ρ_{a+} and ρ_{b+}. The values of Q_{b+} and Q_{a+} are given by equation (**8**) where dl_a and dl_b are the lengths of the segments A and B. The charge-product $Q_{b+}Q_{a+}$ is given by equation (**9**).

$$Q_{a+} = \rho_{a+}dl_a, \qquad Q_{b+} = \rho_{b+}dl_b \tag{8}$$

$$Q_{b+}Q_{a+} = \rho_{a+}\rho_{b+}\cdot dl_a dl_b \tag{9}$$

2. $Q_{b-}Q_{a+}$

Q_{b-} moves at the velocity v_b in the stationary frame and its density ρ_{b-} is increased. The value of ρ_{b-} is given by equation (**10**). Q_{a+} is stationary and keeps the same density. The values of Q_{a+} and Q_{b-} are given by equations (**11**) and the charge-product $Q_{b-}Q_{a+}$ by equation (**12**).

$$\rho_{b-} = \frac{-\rho_{b+}}{\sqrt{1-v_b^2/c^2}} \tag{10}$$

$$Q_{a+} = \rho_{a+}dl_a, \qquad Q_{b-} = \frac{-\rho_{b+}dl_b}{\sqrt{1-v_b^2/c^2}} \tag{11}$$

$$Q_{b-}Q_{a+} = -\frac{\rho_{a+}\rho_{b+}\cdot dl_a dl_b}{\sqrt{1-v_b^2/c^2}} \tag{12}$$

3. $Q_{b+}Q_{a-}$

Q_{a-} moves at the velocity v_a in the stationary frame and its density ρ_{a-} is increased. The value of ρ_{a-} is given by equation (**13**). Q_{b+} is stationary and keeps the same density. Q_{a-} and Q_{b+} are given by equations (**14**) and the charge-product $Q_{b+}Q_{a-}$ by equation (**15**).

$$\rho_{a-} = \frac{-\rho_{a+}}{\sqrt{1-v_a^2/c^2}} \tag{13}$$

$$Q_{a-} = \frac{-\rho_{a+}dl_a}{\sqrt{1-v_a^2/c^2}}, \qquad Q_{b+} = \rho_{b+}dl_b \tag{14}$$

$$Q_{b+}Q_{a-} = -\frac{\rho_{a+}\rho_{b+}\cdot dl_a dl_b}{\sqrt{1-v_a^2/c^2}} \tag{15}$$

4. $Q_{b-}Q_{a-}$

This charge-product is a little tricky to compute because both Q_{a-} and Q_{b-} are moving. We cannot use the stationary frame to evaluate them because we would obtain the same values for Q_{b-} and Q_{a-} as above and get a wrong value for $Q_{b-}Q_{a-}$. The correct frame is the one moving with Q_{a-} because the computed force is on Q_{a-}. In this frame Q_{b-} moves at the velocity v_b-v_a and its density ρ_{b-} is increased by the factor $1/\sqrt{1-(v_b-v_a)^2/c^2}$. The value of ρ_{b-} is given by equation (**16**).

In the frame of Q_{a-}, not only the charge density of Q_{b-} is increased, the segment B itself moves at the velocity -v_a and its length is contracted by the factor $\sqrt{1-v_a^2/c^2}$. So, for computing the charge Q_{b-} we do not multiply ρ_{b-} with the B's proper length dl_b, but with its contracted length dl'_b given by equation (**17**). Then, Q_{b-} is the more complex expression in equation (**18**).

$$\rho_{b-} = \frac{-\rho_{b+}}{\sqrt{1-(v_b-v_a)^2/c^2}} \tag{16}$$

$$dl'_b = dl_b\sqrt{1-v_a^2/c^2} \tag{17}$$

$$\begin{aligned} Q_{b-} &= \rho_{b-}\cdot dl'_b \\ &= \frac{-\rho_{b+}dl_b\sqrt{1-v_a^2/c^2}}{\sqrt{1-(v_b-v_a)^2/c^2}} \end{aligned} \tag{18}$$

For evaluating the charge Q_{a-} we stay in the stationary frame in which the value of Q_{a-} is already given by equation (**14**). Finally, the charge-product $Q_{b-}Q_{a-}$ is given by equation (**19**).

$$\begin{aligned} Q_{b-}Q_{a-} &= \frac{-\rho_{b+}dl_b\sqrt{1-v_a^2/c^2}}{\sqrt{1-(v_b-v_a)^2/c^2}}\,\frac{-\rho_{a+}dl_a}{\sqrt{1-v_a^2/c^2}} \\ &= \frac{\rho_{a+}\rho_{b+}dl_a dl_b}{\sqrt{1-(v_b-v_a)^2/c^2}} \end{aligned} \tag{19}$$

One can question why Q_{a-} is evaluated in the stationary frame while Q_{b-} is evaluated in the moving frame of Q_{a-}. According to the principle of relativity, the Coulomb force on an electron depends only on the relative velocity between this electron and the charge interacting with it. In the moving frame of Q_{a-} the interacting charge Q_{b-} moves at the velocity relative to Q_{a-}. So, we evaluate Q_{b-} in this frame.

Once the force on one electron is known, we have to compute the quantity of charge of all the free electrons within the segment A. Because the force on the segment A is measured in the stationary frame, the quantity of free electrons must be evaluated in the stationary frame too.

5. $Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-}$

Now we have obtained all the 4 charge-products necessary for computing the resultant force on the segment A as expressed by equation (**7**). This force is proportional to the sum of the 4 charge-products which is given by equation (**20**).

$$\begin{aligned} &Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-} \\ &= \rho_{a+}\rho_{b+}\cdot dl_a dl_b - \frac{\rho_{a+}\rho_{b+}\cdot dl_a dl_b}{\sqrt{1-v_b^2/c^2}} - \frac{\rho_{a+}\rho_{b+}\cdot dl_a dl_b}{\sqrt{1-v_a^2/c^2}} + \frac{\rho_{a+}\rho_{b+}\cdot dl_a dl_b}{\sqrt{1-(v_b-v_a)^2/c^2}} \\ &= \rho_{a+}\rho_{b+}\cdot dl_a dl_b\left(1 - \frac{1}{\sqrt{1-v_b^2/c^2}} - \frac{1}{\sqrt{1-v_a^2/c^2}} + \frac{1}{\sqrt{1-(v_b-v_a)^2/c^2}}\right) \end{aligned} \tag{20}$$

We expand the 3 relativistic factors into first-order Taylor series using equation (**21**). Then the parenthesis reduces to equation (**22**).

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1+\frac{1}{2}\frac{v^2}{c^2} \quad (21)$$

$$\begin{aligned}&\left(1-\frac{1}{\sqrt{1-v_b^2/c^2}}-\frac{1}{\sqrt{1-v_a^2/c^2}}+\frac{1}{\sqrt{1-(v_b-v_a)^2/c^2}}\right)\\&=1-\left(1+\frac{1}{2}\frac{v_b^2}{c^2}\right)-\left(1+\frac{1}{2}\frac{v_a^2}{c^2}\right)+\left(1+\frac{1}{2}\frac{(v_b-v_a)^2}{c^2}\right)\\&=-\frac{v_bv_a}{c^2}\end{aligned} \quad (22)$$

Below, we will use the sign "=" instead of the sign "≈" as in equation (**21**) because v^2/c^2 is extremely small with respect to 1. So, using "=" does not impair precision while simplifying explanation.

6. Resultant force

Combining equations (**7**), (**20**) and (**22**) we obtain the resultant force on the segment A which is expressed by equation (**23**). We recognize in this expression the intensities of currents in the segments A and B, I_a and I_b , which are expressed by equations (**24**). Then, $\boldsymbol{F}_a$ becomes the expression given by equation (**25**) which resembles strangely to Lorentz force.

$$\boldsymbol{F}_a=-\frac{1}{4\pi\varepsilon_0}\rho_{a+}\rho_{b+}\cdot dl_a dl_b\frac{v_bv_a}{c^2}\frac{\boldsymbol{e}_r}{r^2} \quad (23)$$

$$\rho_{a+}v_a=I_a, \qquad \rho_{b+}v_b=I_b \quad (24)$$

$$\boldsymbol{F}_a=-\frac{1}{4\pi\varepsilon_0c^2}\frac{I_bI_a\cdot dl_adl_b}{r^2}\boldsymbol{e}_r \quad (25)$$

4. Comparison with Lorentz force

The Lorentz force on the segment A exerted by the segment B is expressed by equation (**26**), where dl_a and dl_b are lengths of A and B, $\boldsymbol{I}_a$ and $\boldsymbol{I}_a$ the vector currents in A and B. If $\boldsymbol{I}_a$ and $\boldsymbol{I}_a$ are parallel to each other and perpendicular to the radial vector $\boldsymbol{e}_r$, the Lorentz force reduces to equation (**27**), which is exactly equation (**25**) because of the relation $\mu_0=1/\varepsilon_0c^2$.

$$\boldsymbol{F}_l=\frac{\mu_0}{4\pi}\frac{\boldsymbol{I}_adl_a\times(\boldsymbol{I}_bdl_b\times\boldsymbol{e}_r)}{r^2} \quad (26)$$

$$\boldsymbol{F}_l=-\frac{\mu_0}{4\pi}\frac{I_bI_a\cdot dl_adl_b}{r^2}\boldsymbol{e}_r \quad (27)$$

Note that the factor $1/\varepsilon_0c^2$ in equation (**25**) arises naturally out of Coulomb's law for ε_0 and length contraction for c^2, showing that the relation $\mu_0\varepsilon_0c^2=1$ does not necessarily depend on the electromagnetic wave equation. This discovery is important.

Although equation (**25**) is still not the correct expression for magnetic force, it shows clearly that magnetic force is created by Coulomb force and relativistic length contraction. In order to make the creation of magnetic force easy to explain, the segments A and B are parallel to each other in the above derivation. The formula for magnetic force of 2 segments pointing in any direction is the subject of the next chapter.