Proof of the remaining resultant Lorentz force internal to a triangular coil

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The Lorentz force respects the third Newton's law between closed loops. Is the Lorentz force internal to one coil consistent with the third Newton's law ? Let us analyze the triangular coil ABC in the Figure 1; the current is I . Each side feels a Lorentz force from the magnetic field of the coil itself. The resultant force of all the Lorentz forces on the 3 sides is the double integrated $\boldsymbol{F}$:

$$\boldsymbol{F} = \int_{coil}\int_{coil} \frac{\mu_0}{4\pi} d\boldsymbol{I}_q \times \left(d\boldsymbol{I}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right)$$

with p the point that generates magnetic field and q the point that feels the Lorentz force, $\boldsymbol{e}_{pq}$ the unit vector pointing from p to q and r the distance between p and q.

I have done a numerical computation for a triangular coil with base length of 1 and height of 10. The calculated force is dimensionless and the value of the overall resultant force is (see the Figure 2):

$$\boldsymbol{S} = 35.21\ \boldsymbol{e}_y$$

This force is not 0, violating the third Newton's law. In general, this value suffices to prove that the Lorentz force law is flawed, because only one counter example is sufficient to topple a general law. However, to exclude any doubt about the accuracy of this numerical calculation, I have done a rigorous analytical proof, which gives the expression of the dimensionless resultant force for a isosceles triangular coil, $\boldsymbol{F}_{res}$:

$$\boldsymbol{F}_{res} = \frac{\mu_0}{2\pi} I^2 \sin(\theta) \left(\int_{\substack{AB\\(q)}} \int_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2} \cos(\beta) \right) dl_p + \int_{\substack{CA\\(q')}} \int_{\substack{BC\\(p')}} \left(\frac{dl_{q'}}{r_{p'q'}^2} \cos(\gamma) \right) dl_{p'} \right) \boldsymbol{e}_y$$

Thus, the analytical method proves without a doubt that the Lorentz force law is flawed. The mathematical derivation of the proof is given in the mathematical proof (see pdf link).

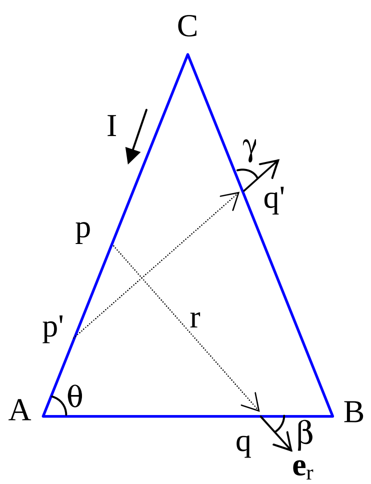

Figure 1

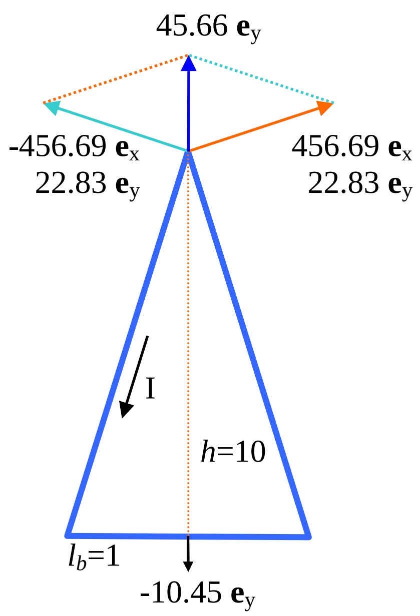

Figure 2

Mathematical proof

The integration over the triangle can be seen as over 3 domains: AB, BC and CA. Let us see how a double integral over multiple domains is expanded. For a function *s* integrated over 2 domains, 1 and 2, the integral is expanded into 2 integrals:

$$f = \int_{1+2} s(x, y)dx = \int_1 s(x, y)dx + \int_2 s(x, y)dx$$

The function *f* itself is integrated over 2 domains, 3 and 4, the double integral is:

$$g = \int_{3+4} fdy = \int_{3+4}\left(\int_1 s(x, y)dx + \int_2 s(x, y)dx\right)dy$$

And the expanded form of the double integral is:

$$g = \int_3\int_1 s(x, y)dxdy + \int_3\int_2 s(x, y)dxdy + \int_4\int_1 s(x, y)dxdy + \int_4\int_2 s(x, y)dxdy$$

For the integral of the Lorentz force over the triangle, the Domain 1 is AB, Domain 2 is BCA, Domain 3 is AB, Domain 4 is BCA. The double integral is expanded as below:

$$\boldsymbol{F} = \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{F}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{F}_{pq}$$

$$\text{with } d^2\boldsymbol{F}_{pq} = \frac{\mu_0}{4\pi} d\boldsymbol{I}_q \times \left(d\boldsymbol{I}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)$$

We define the dimensionless force $\boldsymbol{S}$:

$$d^2\boldsymbol{S}_{pq} = d\boldsymbol{l}_q \times \left(d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)$$

with ***dl*** the differential length vector in the direction of the local current.

The integrated dimensionless force $\boldsymbol{S}$ is then :

$$\boldsymbol{S} = \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq}$$

$$\text{with } \boldsymbol{F} = \frac{\mu_0}{4\pi} I^2 \boldsymbol{S}$$

For points p and q on the side AB, they are on the same straight, the vectors $\boldsymbol{dl_p}$ and $\boldsymbol{e}_{pq}$ are parallel, and their cross product is 0. So, the double integral over AB is 0:

$$d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = 0 \Rightarrow \int_{\substack{AB\\(q)}}\int_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = 0$$

We expand the double integral over BCA in the same way and obtain:

$$\int_{\substack{BCA\\(q)}}\int_{\substack{BCA\\(p)}} d^2\boldsymbol{S}_{pq} = \int_{\substack{BC\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq}$$

$$= \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq}$$

The double integrals over the same straight are dropped and the force $\boldsymbol{S}$ becomes:

$$\boxed{\boldsymbol{S} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq}} \quad (1)$$

- ## Elimination of the direct force

By using the relation:

a×(b×c)= (a·c)b- (a·b)c

We expand the elementary **S** force $d^2\boldsymbol{S}$ as follow:

$$d^2\boldsymbol{S}_{pq} = d\boldsymbol{l}_q \times \left(d\boldsymbol{l}_p \times \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right) = \left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right) d\boldsymbol{l}_p - \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}$$

Let us reduce the pair AB-BC, which is the following sum of integrals:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = \left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right) d\boldsymbol{l}_p + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right) d\boldsymbol{l}_p \right)$$
$$- \left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} \right)$$

The differential forces in the last parentheses satisfies the condition of action and reaction, that is, they lie on the line joining the 2 points, have the same magnitude and are opposite. This is a direct force that can be eliminated because their sum is 0. To prove this for the double integrals, we use discrete calculus. The double integrals are the limit of the following discrete sums:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{n} \left(\Delta\boldsymbol{l}_j \cdot \Delta\boldsymbol{l}_i \right) \frac{\boldsymbol{e}_{ij}}{r_{ij}^2}$$

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{r}}{r_{pq}^2} = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\Delta\boldsymbol{l}_i \cdot \Delta\boldsymbol{l}_j \right) \frac{\boldsymbol{e}_{ji}}{r_{ji}^2}$$

with i corresponding to BC and j to AB.

As the distance vector r_{pq} is the reverse of r_{qp}, we have:

$$\boldsymbol{r}_{ij} = -\boldsymbol{r}_{ji} \Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{n} \left(\Delta\boldsymbol{l}_j \cdot \Delta\boldsymbol{l}_i \right) \frac{\boldsymbol{e}_{ij}}{r_{ij}^2} = -\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\Delta\boldsymbol{l}_i \cdot \Delta\boldsymbol{l}_j \right) \frac{\boldsymbol{e}_{ji}}{r_{ji}^2}$$

We have proven that the double integrals in reverse order are opposite:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = -\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}$$

And:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} \left(d\boldsymbol{l}_q \cdot d\boldsymbol{l}_p \right) \frac{\boldsymbol{e}_{pq}}{r_{pq}^2} = 0$$

So, the pair AB-BC becomes:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = \left(\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p\right)$$

Let us define the differential ***G*** force as follow:

$$d^2\boldsymbol{G}_{pq} = \left(d\boldsymbol{l}_q \cdot \frac{\boldsymbol{e}_{pq}}{r_{pq}^2}\right)d\boldsymbol{l}_p \qquad (2)$$

And the pair AB-BC is written as:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} \qquad (3)$$

In the same way we obtain:

For the pair BC-CA
$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{S}_{pq} = \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} \qquad (4)$$

For the pair CA-AB
$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{S}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{S}_{pq} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} \qquad (5)$$

From the equations (1), (3), (4) and (5) we obtain the expression of ***S***:

$$\boxed{\boldsymbol{S} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}} \qquad (6)$$

- **Reduction of the sum** $\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq}$

We can simplify this expression by using symmetry. The differential $d^2\boldsymbol{G}_{pq}$ for the double integral over AB then BC is (see the equation (2) and Figure 3):

$$d^2\boldsymbol{G}_{pq} = \left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)d\boldsymbol{l}_p \quad \text{for} \quad \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over AB then CA is:

$$d^2\boldsymbol{G}_{p'q'} = \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)d\boldsymbol{l}_{p'} \quad \text{for} \quad \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq}$$

Then:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)d\boldsymbol{l}_p + \int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{AB\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)d\boldsymbol{l}_{p'}$$

The integration of this sum by keeping the symmetry (see the Figure 3) gives:

$$\cos(\alpha) = -\cos(\alpha')$$

For the scalar length dl_q , the vector length $\boldsymbol{dl}_p$ and the distances we have the equalities:

$$dl_q = dl_{q'}\,,\, \boldsymbol{dl}_p = \boldsymbol{dl}_{p'}\,,\, r_{pq} = r_{p'q'}$$

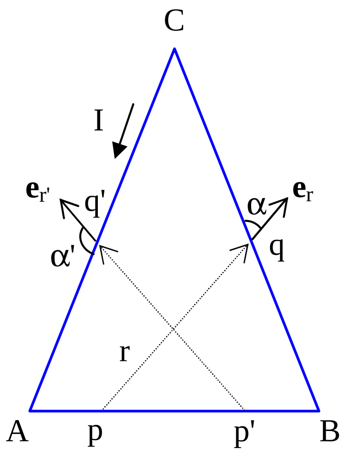

Figure 3

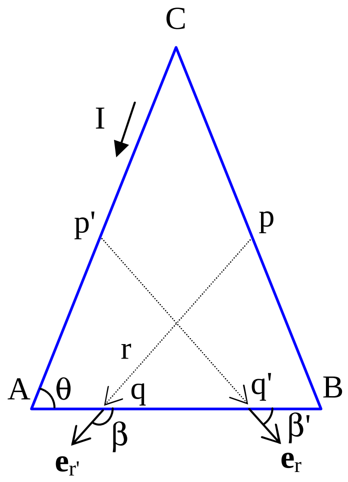

Figure 4

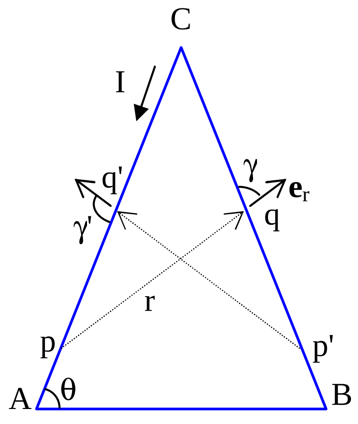

Figure 5

Thus, the double integral over AB then CA has the same absolute value than that over AB then BC. So, we have:

$$\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\alpha)\right)\boldsymbol{dl}_p = -\int\limits_{\substack{CA\\(q')}}\int\limits_{\substack{AB\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\alpha')\right)\boldsymbol{dl}_{p'}$$

And:

$$\boxed{\int\limits_{\substack{BC\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{CA\\(q)}}\int\limits_{\substack{AB\\(p)}} d^2\boldsymbol{G}_{pq} = 0} \qquad (7)$$

- **Reduction of the sum** $\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$

$d^2\boldsymbol{G}_{pq}$ for the double integral over BC then AB is (see the equation (2) and Figure 4):

$$d^2\boldsymbol{G}_{pq} = \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)\boldsymbol{dl}_p \text{ for } \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over CA then AB is:

$$d^2\boldsymbol{G}_{p'q'} = \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right)\boldsymbol{dl}_{p'} \text{ for } \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$$

We have:

$$\int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = \int\limits_{\substack{AB\\(q)}}\int\limits_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)\boldsymbol{dl}_p + \int\limits_{\substack{AB\\(q')}}\int\limits_{\substack{CA\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right)\boldsymbol{dl}_{p'}$$

The vector lengths $\boldsymbol{dl}_p$ and $\boldsymbol{dl}_{p'}$ are :

$$\boldsymbol{dl}_p = dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right), \boldsymbol{dl}_{p'} = dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

Then the sum becomes:

$$\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = \int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right)$$
$$+ \int_{\substack{AB\\(q')}}\int_{\substack{CA\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right)dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The integration by keeping the symmetry gives (see the Figure 4):

$$\cos(\beta) = -\cos(\beta')$$

For the scalar length dl_q and the distances we have the equalities:

$$dl_q = dl_{q'}\,,\; r_{pq} = r_{p'q'}$$

Thus, the double integral over BC then AB has the same absolute value than that over CA then AB. So, we have:

$$\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)dl_p = -\int_{\substack{AB\\(q')}}\int_{\substack{CA\\(p')}}\left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\beta')\right)dl_{p'}$$

And:

$$\boxed{\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} + \int_{\substack{AB\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} = 2\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}}\left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right)dl_p\sin(\theta)\boldsymbol{e}_y} \qquad (8)$$

- **Reduction of the sum** $\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$

$d^2\boldsymbol{G}_{pq}$ for the double integral over CA then BC is (see the equation (2) and Figure 5):

$$d^2\boldsymbol{G}_{pq} = \left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right)\boldsymbol{dl}_p \text{ for } \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq}$$

$d^2\boldsymbol{G}_{pq}$ over BC then CA is:

$$d^2\boldsymbol{G}_{p'q'} = \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right)\boldsymbol{dl}_{p'} \text{ for } \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq}$$

The vector lengths $\boldsymbol{dl}_p$ and $\boldsymbol{dl}_{p'}$ are :

$$\boldsymbol{dl}_p = dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right), \boldsymbol{dl}_{p'} = dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The sum becomes:

$$\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} = \int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right) dl_p\left(-\cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y\right)$$
$$+ \int_{\substack{CA\\(q')}}\int_{\substack{BC\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right) dl_{p'}\left(-\cos(\theta)\boldsymbol{e}_x - \sin(\theta)\boldsymbol{e}_y\right)$$

The integration by keeping the symmetry gives (see the Figure 5):

$$\cos(\gamma) = -\cos(\gamma')$$

For the scalar length dl_q and the distances we have the equalities:

$$dl_q = dl_{q'}\,,\ r_{pq} = r_{p'q'}$$

Thus, the double integral over CA then BC has the same absolute value than that over BC then CA. So, we have:

$$\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right) dl_p = -\int_{\substack{CA\\(q')}}\int_{\substack{BC\\(p')}} \left(\frac{dI_{q'}}{r_{p'q'}^2}\cos(\gamma')\right) dl_{p'}$$

And:

$$\boxed{\int_{\substack{BC\\(q)}}\int_{\substack{CA\\(p)}} d^2\boldsymbol{G}_{pq} + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} d^2\boldsymbol{G}_{pq} = 2\int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} \left(\frac{dI_q}{r_{pq}^2}\cos(\gamma)\right) dI_p \sin(\theta)\boldsymbol{e}_y} \qquad (9)$$

- **Final expression of *S***

Finally we obtain the dimensionless resultant force $\boldsymbol{S}$ from equations (6), (7), (8) and (9):

$$\boxed{\boldsymbol{S} = 2\sin(\theta)\left(\int_{\substack{AB\\(q)}}\int_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\beta)\right) dl_p + \int_{\substack{CA\\(q)}}\int_{\substack{BC\\(p)}} \left(\frac{dl_q}{r_{pq}^2}\cos(\gamma)\right) dl_p\right)\boldsymbol{e}_y}$$

$$\text{with } \boldsymbol{F} = \frac{\mu_0}{4\pi} I^2 \boldsymbol{S}$$

When the height of the triangle increases, the absolute value of the double integrals over BC then CA increases because the distance r decreases in average. In the contrary, that over BC then AB decreases because r increases in average. The sum of 2 functions varying in opposite direction cannot be 0. Thus, the overall resultant Lorentz force internal to the triangular coil is not 0. This analytical result confirms the numerical result which gives the following value of the dimensionless resultant force:

$$\boldsymbol{S} = 35.21\ \boldsymbol{e}_y$$

And the flaw of the Lorentz force law is proven.