Mathematical cause of the existence of the remaining resultant internal Lorentz force

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I have given a rigorous proof of the existence of a remaining resultant Lorentz force internal to a triangular coil and a numerical computation that confirms this proof:
Proof of the remaining resultant Lorentz force internal to a triangular coil
http://pengkuanem.blogspot.com/2012/04/analyze-of-lorentz-forces-internal-to.html

Why the Lorentz force law cannot respect the third Newton's law? What is the mathematical cause that leads to this inconsistency? Let us examine the effect of the characteristic perpendicularity of the Lorentz force with the current. Take a triangle with height h and base a+b (see the Figure 1). The unit normal vectors to each side are $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\mathbf{n_0} = -\mathbf{e_y}, \mathbf{n}_1 = \frac{h\mathbf{e}_x + b\mathbf{e_y}}{\sqrt{h^2+b^2}}, \mathbf{n}_2 = \frac{-h\mathbf{e}_x + a\mathbf{e_y}}{\sqrt{h^2+a^2}}$$

The force on the 3 sides are:

$$\mathbf{F_0} = f_0\mathbf{n_0}, \mathbf{F}_1 = f_1\mathbf{n}_1, \mathbf{F}_2 = f_2\mathbf{n}_2$$

with f the magnitude of the forces.

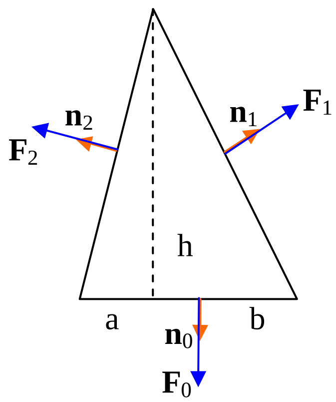

Figure 1

The sum of these 3 forces should be 0 because they are the internal forces. That is:

$$\boldsymbol{F_0} + \boldsymbol{F}_1 + \boldsymbol{F}_2 = f_0\boldsymbol{n_0} + f_1\boldsymbol{n}_1 + f_2\boldsymbol{n}_2$$

$$0 = -f_0\boldsymbol{e}_y + f_1\frac{h\boldsymbol{e}_x + b\boldsymbol{e}_y}{\sqrt{h^2+b^2}} + f_2\frac{-h\boldsymbol{e}_x + a\boldsymbol{e}_y}{\sqrt{h^2+a^2}}$$

The x and y components give 2 equations:

$$f_1\frac{h}{\sqrt{h^2+b^2}} + f_2\frac{-h}{\sqrt{h^2+a^2}} = 0$$

$$f_1\frac{b}{\sqrt{h^2+b^2}} + f_2\frac{a}{\sqrt{h^2+a^2}} = f_0$$

The magnitude of the forces are the solution of this system, that is:

$$f_1 = f_0 \frac{\sqrt{h^2+b^2}}{a+b},\ f_2 = f_0 \frac{\sqrt{h^2+a^2}}{a+b} \qquad (1)$$

This is the solution that secures the remaining resultant force to be 0 and each force to be perpendicular to the current. Let us call this result the perpendicularity solution. We remark that this solution do not involve the Lorentz force law. It should be solution of this law. But is it?

For comparing with the Lorentz force law, we calculate the differential Lorentz force on a elementary current $\boldsymbol{dl_1}$ using the following equation (see the Figure 2):

$$d\boldsymbol{F} = d\boldsymbol{I}_1 \times \left(d\boldsymbol{I}_2 \times \frac{\boldsymbol{e}_r}{r^2} \right)$$

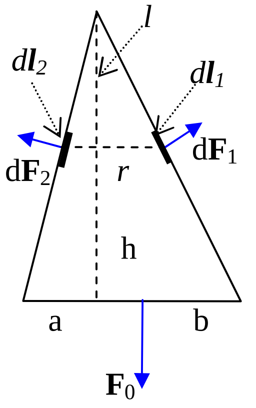

Figure 2

The $\boldsymbol{dl_1}$ and $\boldsymbol{dl_2}$ are at a vertical distance l from the summit. The distance between $\boldsymbol{dl_1}$ and $\boldsymbol{dl_2}$ is r, which is:

$$r = \frac{l}{h}(a+b)$$

So,

$$d\boldsymbol{F} = d\boldsymbol{l}_1 \times (d\boldsymbol{l}_2 \times \boldsymbol{e}_r) \frac{I^2 h^2}{l^2 (a+b)^2}$$

Let us calculate an incomplete sum of this differential force, whose magnitude is:

$$i = 1 \to n, \quad l_i = i \cdot \Delta l, \quad f_{part} = \frac{I^2 h^2}{(a+b)^2} \sum_{i=1}^{n} \frac{1}{l_i^2} = \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2}$$

with n and Δl constant

This force is only part of the total Lorentz force. So, the magnitude of the Lorentz force on one side is greater:

$$f_{Lorentz} > \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2} \qquad (2)$$

Now we compare this magnitude with the perpendicular solution, the equation (1). When the height h is very large, the 2 upper sides become more and more vertical and the force f_0 become nearly constant. In this case, the magnitude f_1 becomes asymptotic to the following linear function of h:

$$f_1 \xrightarrow[h>>a+b]{} f_0 \frac{h}{a+b}$$

On the other hand, the solution of the Lorentz law is greater than a parabolic function of h, the equation (2). Because a linear function cannot be equal to a parabolic function, the perpendicularity solution cannot be solution of the Lorentz force law, that is:

$$f_1 \xrightarrow[h>>a+b]{} f_0 \frac{h}{a+b} \text{ and } f_{Lorentz} > \frac{I^2 h^2}{\Delta l^2 (a+b)^2} \sum_{i=1}^{n} \frac{1}{i^2}$$
$$\Rightarrow f_{Lorentz} \neq f_1$$

The Lorentz force must satisfy the Lorentz law, this involves x and y components equations; the Lorentz force must also satisfy the perpendicularity system, this involves 2 different x and y components equations. We have 4 equations, but only 2 unknown, f_1 and f_2 . There cannot be any solution.

Since we always calculate Lorentz force with the Lorentz force law, the perpendicularity system is not satisfied. In consequence, the sum of $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ is not 0. This is the mathematical cause of the existence of the remaining resultant force.