From electron to magnetism

3. Changing distance effect

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Abstract: The motion of electrons in current modifies Coulomb force between charges by changing the distance between them. This effect creates magnetic force.

1. Force between charges in currents

Magnetic force is an interaction between moving charges which, if immobile, exert Coulomb force. Can we derive magnetic force directly from Coulomb's law and velocities of charges? In «Length-contraction-magnetic-force between arbitrary currents» I have derived a magnetic force from Coulomb's law and relativistic length contraction effect. But this force is zero for perpendicular currents.

The magnetic force between perpendicular currents is created by another effect: moving individual charge modifies the individual Coulomb force by changing the distance between charges. For example, the Coulomb force an electron exerts on an immobile charge depends on the distance between them. When the electron moves, the distance changes, which modifies the Coulomb force.

In **Figure 1**, the individual Coulomb force between one charge of the current element $d\boldsymbol{I}_a$ and one charge of the current element $d\boldsymbol{I}_b$ varies differently from charge to charge. Although the difference is tiny, because of the huge amount of free electrons, all the individual Coulomb forces in the current element $d\boldsymbol{I}_a$ still sum up to a net force, $\boldsymbol{F}_{cd}$, which is in fact a magnetic force. I call the creation of $\boldsymbol{F}_{cd}$ the "Changing distance effect".

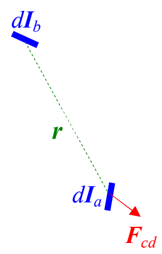

Figure 1

2. Force from a moving charge

Coulomb's law, equation (1), defines the force $\boldsymbol{F}$ that a charge q_b exerts on another charge q_a. In (1) the boldface $\boldsymbol{r}$ is the radial vector pointing from q_b to q_a. The magnitude of $\boldsymbol{r}$ equals the distance between q_b and q_a and is denoted by the regular r. In the following, q_b will be the moving charge and q_a the immobile charge. To locate q_b with respect to q_a we will use $\boldsymbol{r}$ which we will call the radial vector of q_b.

$$\boldsymbol{F} = \frac{q_a q_b}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3} \tag{1}$$

As the charge q_b moves, $\boldsymbol{r}$ varies and the value of equation (1) $\boldsymbol{F}$ varies too. We will expand equation (1) into a first order Taylor's series to compute $\boldsymbol{F}$.

3. First order expansion

For expanding the term $\boldsymbol{r}/r^3$ of (1) we write the vector $\boldsymbol{r}$ in the form of a constant vector $\boldsymbol{r_0}$ plus a small variation vector $\Delta\boldsymbol{r}$, that is, $\boldsymbol{r} = \boldsymbol{r_0} + \Delta\boldsymbol{r}$, see (2). $\boldsymbol{r_0}$ is called the reference radial vector. The magnitude r is computed using $\boldsymbol{r_0} + \Delta\boldsymbol{r}$ and transformed in (3) which gives (4). The parenthesis in (4) equals r_0^3/r^3 and is in the form $f(1+x)$ which can be expanded using the formulas in (5). The expansion of r_0^3/r^3 is given in (6) where we drop the red and underlined terms because they are of second order of $\Delta\boldsymbol{r}$ and thus, negligible. In (6) we use the sign "≈" to mark this approximation. Then, combining (6) with (2), we express the expansion of $\boldsymbol{r}/r^3$ about $\boldsymbol{r_0}$ in (7).

$$\boldsymbol{r} = \boldsymbol{r_0} + \Delta\boldsymbol{r} \tag{2}$$

$$\begin{aligned} r &= \sqrt{(\boldsymbol{r_0}+\Delta\boldsymbol{r})\cdot(\boldsymbol{r_0}+\Delta\boldsymbol{r})} \\ &= (\boldsymbol{r_0}\cdot\boldsymbol{r_0} + 2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r} + \Delta\boldsymbol{r}\cdot\Delta\boldsymbol{r})^{1/2} \\ &= r_0\left(1 + \frac{2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2} + \frac{\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{r}}{r_0^2}\right)^{1/2} \end{aligned} \tag{3}$$

$$\frac{1}{r^3} = \frac{1}{r_0^3}\left(1 + \frac{2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2} + \frac{\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{r}}{r_0^2}\right)^{-3/2} \tag{4}$$

$$\begin{aligned} &f(1+x) = f(1) + f'(1)x + o^2(x) \\ &x = \frac{2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2} + \frac{\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{r}}{r_0^2}, \qquad f'(1) = -\frac{3}{2} \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{r_0^3}{r^3} &= 1 - \frac{3}{2}\left(\frac{2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2} + \underline{\frac{\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{r}}{r_0^2}}\right) + \underline{o^2(x)} \\ &\approx 1 - \frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2} \end{aligned} \tag{6}$$

$$\frac{\boldsymbol{r}}{r^3} = \frac{1}{r_0^3}\left(1 - \frac{3\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{r_0^2}\right)(\boldsymbol{r_0} + \Delta\boldsymbol{r}) \tag{7}$$

Notice that in (7) we use the sign "=" instead of "≈", which means that all expressions in the following are in first order of $\Delta\mathbf{r}$.

4. Pairs of charges

a. Four forces in one pair

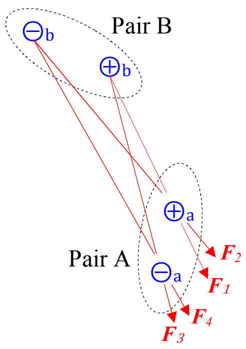

Figure 2

Coulomb force is created by net charges while current element is neutral. As magnetic force is created by current, we must ensure the neutrality of current in the derivation. So, we take from the current element $d\mathbf{I}_a$ a pair of charges which is formed by ⊖a a free electron and ⊕a a unit positive charge of an atom's nucleus; from $d\mathbf{I}_b$ we take the pair ⊖b and ⊕b. The pair in $d\mathbf{I}_a$ is called pair A and that in $d\mathbf{I}_b$ pair B, see **Figure 2**. The net charge of each pair is zero, which guarantees the neutrality.

Pair B exerts 4 Coulomb forces on pair A:

- $\mathbf{F}_1$ from the positive charge ⊕b to the positive charge ⊕a, the charge-product $q_a\, q_b$ equals $+e^2$, the radial vectors is $\mathbf{r}_1$.
- $\mathbf{F}_2$ from the electron ⊖b to the positive charge ⊕a, the charge-product is $-e^2$, the radial vectors is $\mathbf{r}_2$.
- $\mathbf{F}_3$ from the positive charge ⊕b to the electron ⊖a, the charge-product is $-e^2$, the radial vectors is $\mathbf{r}_3$.
- $\mathbf{F}_4$ from the electron ⊖b to the electron ⊖a, the charge-product is $+e^2$, the radial vectors is $\mathbf{r}_4$.

Radial vectors	Charges products	Coulomb forces
$\mathbf{r}_1$	$+e^2$	$\mathbf{F}_1 = +\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_1}{r_1^3}$
$\mathbf{r}_2$	$-e^2$	$\mathbf{F}_2 = -\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_2}{r_2^3}$
$\mathbf{r}_3$	$-e^2$	$\mathbf{F}_3 = -\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_3}{r_3^3}$
$\mathbf{r}_4$	$+e^2$	$\mathbf{F}_4 = +\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_4}{r_4^3}$

Tab. 1

These 4 forces are shown in **Figure 2** and are computed in Tab. 1 column 3. The total force that pair B exerts on pair A, $\mathbf{F}_a$, equals the sum of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ and is computed in (8).

$$\begin{aligned}\mathbf{F}_a &= \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4\\ &= \frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_1}{r_1^3}-\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_2}{r_2^3}-\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_3}{r_3^3}+\frac{e^2}{4\pi\varepsilon_0}\frac{\mathbf{r}_4}{r_4^3}\\ &= \frac{e^2}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}_1}{r_1^3}-\frac{\mathbf{r}_2}{r_2^3}-\frac{\mathbf{r}_3}{r_3^3}+\frac{\mathbf{r}_4}{r_4^3}\right)\end{aligned} \qquad (8)$$

b. Expansion of radial vector ratios

For expanding the expression in (8), the reference radial vector $\mathbf{r}_0$ must be the same for the 4 radial vector ratios $\frac{\mathbf{r}_1}{r_1^3}, \frac{\mathbf{r}_2}{r_2^3}, \frac{\mathbf{r}_3}{r_3^3}$ and $\frac{\mathbf{r}_4}{r_4^3}$. Among $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ only $\mathbf{r}_1$ is constant because ⊕b and ⊕a are both immobile. So, we choose $\mathbf{r}_1$ as the reference radial vector and replace $\mathbf{r}_1$ for $\mathbf{r}_0$ in equation (7) to expand the 4 radial vector ratios in (9).

$$\begin{aligned}\frac{\mathbf{r}_1}{r_1^3} &= \frac{\mathbf{r}_1}{r_1^3}\\ \frac{\mathbf{r}_2}{r_2^3} &= \frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_2}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_2)\\ \frac{\mathbf{r}_3}{r_3^3} &= \frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_3}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_3)\\ \frac{\mathbf{r}_4}{r_4^3} &= \frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_4}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_4)\end{aligned} \qquad (9)$$

We sum the 4 expressions of (9) to compute the sum of radial vector ratios and transform the result in (10), which is a function of $\Delta\mathbf{r}_2$, $\Delta\mathbf{r}_3$ and $\Delta\mathbf{r}_4$. The positions of the 2 electrons of pair A and B determine $\Delta\mathbf{r}_2$, $\Delta\mathbf{r}_3$ and $\Delta\mathbf{r}_4$, which constitute a system of 2 conditions and 3 variables from which we can extract a relation that links $\Delta\mathbf{r}_2$, $\Delta\mathbf{r}_3$ and $\Delta\mathbf{r}_4$.

$$\begin{aligned}&\frac{\mathbf{r}_1}{r_1^3}-\frac{\mathbf{r}_2}{r_2^3}-\frac{\mathbf{r}_3}{r_3^3}+\frac{\mathbf{r}_4}{r_4^3}\\ &= \frac{\mathbf{r}_1}{r_1^3}-\frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_2}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_2)-\frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_3}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_3)+\frac{1}{r_1^3}\left(1-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_4}{r_1^2}\right)(\mathbf{r}_1+\Delta\mathbf{r}_4)\\ &= \frac{1}{r_1^3}\left(\mathbf{r}_1-(\mathbf{r}_1+\Delta\mathbf{r}_2)-(\mathbf{r}_1+\Delta\mathbf{r}_3)+(\mathbf{r}_1+\Delta\mathbf{r}_4)+\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_2}{r_1^2}(\mathbf{r}_1+\Delta\mathbf{r}_2)+\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_3}{r_1^2}(\mathbf{r}_1+\Delta\mathbf{r}_3)-\frac{3\mathbf{r}_1\cdot\Delta\mathbf{r}_4}{r_1^2}(\mathbf{r}_1+\Delta\mathbf{r}_4)\right)\\ &= \frac{1}{r_1^3}\left(-\Delta\mathbf{r}_2-\Delta\mathbf{r}_3+\Delta\mathbf{r}_4+\frac{3}{r_1^2}(\mathbf{r}_1\cdot\Delta\mathbf{r}_2+\mathbf{r}_1\cdot\Delta\mathbf{r}_3-\mathbf{r}_1\cdot\Delta\mathbf{r}_4)\mathbf{r}_1+\frac{3}{r_1^2}\big((\mathbf{r}_1\cdot\Delta\mathbf{r}_2)\Delta\mathbf{r}_2+(\mathbf{r}_1\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}_3-(\mathbf{r}_1\cdot\Delta\mathbf{r}_4)\Delta\mathbf{r}_4\big)\right)\\ &= \frac{1}{r_1^3}\left(-\Delta\mathbf{r}_2-\Delta\mathbf{r}_3+\Delta\mathbf{r}_4+\frac{3}{r_1^2}\mathbf{r}_1\cdot(\Delta\mathbf{r}_2+\Delta\mathbf{r}_3-\Delta\mathbf{r}_4)\mathbf{r}_1+\frac{3}{r_1^2}[(\mathbf{r}_1\cdot\Delta\mathbf{r}_2)\Delta\mathbf{r}_2+(\mathbf{r}_1\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}_3-(\mathbf{r}_1\cdot\Delta\mathbf{r}_4)\Delta\mathbf{r}_4]\right)\end{aligned} \qquad (10)$$

c. Eliminating Δr_4

Figure 3 shows that the 4 charges of pair A and B are located by the 4 position vectors $\boldsymbol{P}_{a+}$,$\boldsymbol{P}_{a-}$, $\boldsymbol{P}_{b+}$, $\boldsymbol{P}_{b-}$. Then the radial vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$ are expressed with $\boldsymbol{P}_{a+}$,$\boldsymbol{P}_{a-}$, $\boldsymbol{P}_{b+}$, $\boldsymbol{P}_{b-}$, see (11), and the variation vectors $\Delta\boldsymbol{r}_2$, $\Delta\boldsymbol{r}_3$ and $\Delta\boldsymbol{r}_4$ are expressed in (12). The relation (13) is derived from (12).

$$\begin{aligned}\boldsymbol{r_1} &= \boldsymbol{P}_{a+} - \boldsymbol{P}_{b+}\\ \boldsymbol{r_2} &= \boldsymbol{P}_{a+} - \boldsymbol{P}_{b-}\\ \boldsymbol{r_3} &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{b+}\\ \boldsymbol{r_4} &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{b-}\end{aligned} \tag{11}$$

$$\begin{aligned}\Delta\boldsymbol{r_2} &= \boldsymbol{r_2} - \boldsymbol{r_1}\\ &= \boldsymbol{P}_{a+} - \boldsymbol{P}_{b-} - (\boldsymbol{P}_{a+} - \boldsymbol{P}_{b+})\\ &= \boldsymbol{P}_{b+} - \boldsymbol{P}_{b-}\\ \Delta\boldsymbol{r_3} &= \boldsymbol{r_3} - \boldsymbol{r_1}\\ &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{b+} - (\boldsymbol{P}_{a+} - \boldsymbol{P}_{b+})\\ &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{a+}\\ \Delta\boldsymbol{r_4} &= \boldsymbol{r_4} - \boldsymbol{r_1}\\ &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{b-} - (\boldsymbol{P}_{a+} - \boldsymbol{P}_{b+})\\ &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{a+} + \boldsymbol{P}_{b+} - \boldsymbol{P}_{b-}\end{aligned} \tag{12}$$

$$\Delta\boldsymbol{r_4} = \Delta\boldsymbol{r_2} + \Delta\boldsymbol{r_3} \tag{13}$$

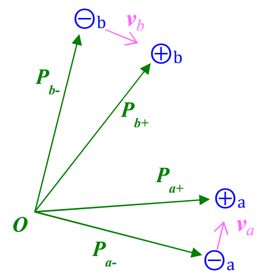

Figure 3

(13) gives $\Delta\boldsymbol{r_2} + \Delta\boldsymbol{r_3} - \Delta\boldsymbol{r_4} = 0$, which eliminates the first 2 terms in (10), which becomes (14). Then, we apply (13) to the term $(\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_4})\Delta\boldsymbol{r_4}$ in (15), which simplifies (14) into (16).

$$\frac{\boldsymbol{r_1}}{r_1^3} - \frac{\boldsymbol{r_2}}{r_2^3} - \frac{\boldsymbol{r_3}}{r_3^3} + \frac{\boldsymbol{r_4}}{r_4^3} = \frac{1}{r_1{}^3}\left(\frac{3}{r_1{}^2}\big((\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3} - (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_4})\Delta\boldsymbol{r_4}\big)\right) \tag{14}$$

$$\begin{aligned}(\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_4})\Delta\boldsymbol{r_4} &= \big(\boldsymbol{r_1}\cdot(\Delta\boldsymbol{r_2} + \Delta\boldsymbol{r_3})\big)(\Delta\boldsymbol{r_2} + \Delta\boldsymbol{r_3})\\ &= (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_3} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\boldsymbol{r_1}}{r_1^3} - \frac{\boldsymbol{r_2}}{r_2^3} - \frac{\boldsymbol{r_3}}{r_3^3} + \frac{\boldsymbol{r_4}}{r_4^3} &= \frac{3}{r_1{}^5}\Big((\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}\\ &\quad - \big((\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_3} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_3}\big)\Big)\\ &= -\frac{3}{r_1{}^5}\big((\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_3}\big)\end{aligned} \tag{16}$$

d. Force function of Δr_2, Δr_3

(16) is introduced into (8) which becomes (17). (17) expresses the total force on pair A, $\boldsymbol{F}_a$, as a function of $\Delta\boldsymbol{r}_2$ and $\Delta\boldsymbol{r}_3$ only. As the electrons of pair A and B move, $\Delta\boldsymbol{r}_2$ and $\Delta\boldsymbol{r}_3$ vary with time and $\boldsymbol{F}_a$ varies with time too.

$$\boldsymbol{F_a} = -\frac{3e^2}{4\pi\varepsilon_0 r_1{}^5}\big((\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_3})\Delta\boldsymbol{r_2} + (\boldsymbol{r_1}\cdot\Delta\boldsymbol{r_2})\Delta\boldsymbol{r_3}\big) \tag{17}$$

5. $\boldsymbol{F}_a$ function of time

For expressing $\boldsymbol{F}_a$ as a function of time, $\Delta\boldsymbol{r}_2$ and $\Delta\boldsymbol{r}_3$ are expressed with the velocities of the electrons ⊖a and ⊖b, respectively $\boldsymbol{v_a}$ and $\boldsymbol{v_b}$. We assume that the current elements are sufficiently small that electrons move inside in straight line at constant velocity. We define that ⊖a passes by ⊕a at time t_a and ⊖b passes by ⊕b at time t_b. Then, $\boldsymbol{v_a}$ and $\boldsymbol{v_b}$ are expressed in (18). Combining (18) with (12), we express $\Delta\boldsymbol{r}_2$ and $\Delta\boldsymbol{r}_3$ in (19), which is then introduced into (17) to express $\boldsymbol{F}_a$ as a function of time in (20).

$$\boldsymbol{v_a} = \frac{\boldsymbol{P}_{a+} - \boldsymbol{P}_{a-}}{t_a - t}, \qquad \boldsymbol{v_b} = \frac{\boldsymbol{P}_{b+} - \boldsymbol{P}_{b-}}{t_b - t} \tag{18}$$

$$\begin{aligned}\Delta\boldsymbol{r_2} &= \boldsymbol{P}_{b+} - \boldsymbol{P}_{b-} = -\boldsymbol{v_b}(t - t_b)\\ \Delta\boldsymbol{r_3} &= \boldsymbol{P}_{a-} - \boldsymbol{P}_{a+} = \boldsymbol{v_a}(t - t_a)\end{aligned} \tag{19}$$

$$\begin{aligned}\boldsymbol{F_a} &= \frac{3e^2}{4\pi\varepsilon_0 r_1{}^5}\Big(\big(\boldsymbol{r_1}\cdot\boldsymbol{v_a}(t - t_a)\big)\boldsymbol{v_b}(t - t_b) + \big(\boldsymbol{r_1}\cdot\boldsymbol{v_b}(t - t_b)\big)\boldsymbol{v_a}(t - t_a)\Big)\\ &= \frac{e^2}{4\pi\varepsilon_0 r_1{}^5}\big((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\big) * 3(t - t_a)(t - t_b)\end{aligned} \tag{20}$$

6. Average force

For deriving magnetic force which is constant from $\boldsymbol{F}_a$ which is variable, we computed the average value of $\boldsymbol{F}_a$ over a time interval $[t_1, t_2]$, which equals the integral of $\boldsymbol{F}_a$ divided by the length of the interval $t_2 - t_1$. $\boldsymbol{F}_a$ is integrated in (21). As said before, the velocities $\boldsymbol{v_a}$ and $\boldsymbol{v_b}$ are constant in current elements, then the terms in $\boldsymbol{v_a}$ and $\boldsymbol{v_b}$ are extracted from the integral. $\boldsymbol{F}_{\text{avg}}$ the average value of $\boldsymbol{F}_a$ is expressed in (22) with T, the integral part of (21). T is computed in (23).

$$\int_{t1}^{t2} \boldsymbol{F_a} dt = \int_{t1}^{t2} \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right) * 3(t-t_a)(t-t_b)dt = \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right)\int_{t1}^{t2} 3(t-t_a)(t-t_b)dt \quad (21)$$

$$\boldsymbol{F_{avg}} = \frac{1}{t_2-t_1}\int_{t1}^{t2}\boldsymbol{F_a}dt = \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right)\frac{T}{t_2-t_1} \quad (22)$$

$$\begin{aligned} T &= \int_{t1}^{t2} 3(t-t_a)(t-t_b)dt \\ &= \int_{t1}^{t2} 3(t-t_a)(t-t_a+t_a-t_b)dt \\ &= \int_{t1}^{t2} \left(3(t-t_a)^2 + 3(t_a-t_b)(t-t_a)\right)d(t-t_a) \\ &= \left[(t-t_a)^3 + \frac{3}{2}(t_a-t_b)(t-t_a)^2\right]_{t1}^{t2} \\ &= (t_2-t_a)^3 - (t_1-t_a)^3 + \frac{3}{2}(t_a-t_b)[(t_2-t_a)^2 - (t_1-t_a)^2] \end{aligned} \quad (23)$$

The interval $[t_1 , t_2]$ can be symmetric about the time t_a , with $t_1 = t_a - \Delta t$ and $t_2 = t_a + \Delta t$. Its length is $t_2 - t_1 = 2\Delta t$. The value of $T / (t_2 - t_1)$ is computed in (24) which is introduce into (22) to express $\boldsymbol{F}_{\text{avg}}$ in (25).

$$\begin{aligned} &t_2 - t_a = -(t_1-t_a) = \Delta t, \qquad t_2 - t_1 = 2\Delta t \\ &T = \Delta t^3 + \Delta t^3 + \frac{3}{2}(t_a-t_b)[\Delta t^2 - \Delta t^2] = 2\Delta t^3 \\ &\frac{T}{t_2-t_1} = \frac{2\Delta t^3}{2\Delta t} = \Delta t^2 \end{aligned} \quad (24)$$

$$\boldsymbol{F_{avg}} = \frac{\Delta t^2 e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right) \quad (25)$$

The interval $[t_1 , t_2]$ can also simply start from t_a and have length Δt. In this case, the interval is $[t_a, t_a + \Delta t]$ and T is computed in (26). The average value of $\boldsymbol{F}_a$ is $\boldsymbol{F}_{\text{avg2}}$ of (27). As $t_a - t_b$ is not necessarily zero, $\boldsymbol{F}_{\text{avg2}}$ does not equal $\boldsymbol{F}_{\text{avg}}$ of (25).

$$\begin{aligned} &t_1 = t_a, \qquad t_2 - t_a = \Delta t \\ &T = \Delta t^3 + \frac{3}{2}(t_a-t_b)\Delta t^2 \end{aligned} \quad (26)$$

$$\boldsymbol{F_{avg2}} = \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right)\frac{\Delta t^3 + \frac{3}{2}(t_a-t_b)\Delta t^2}{\Delta t} \quad (27)$$

However, $\boldsymbol{F}_{\text{avg2}}$ is the force on one pair of charges and can be averaged in a current element which contains billions of pairs. $\boldsymbol{F'}_{\text{avg2}}$ the average value of $\boldsymbol{F}_{\text{avg2}}$ in a current element is computed in (28). The number of pair of charges n is very large, we can reasonably assume that the sum of $t_a - t_b$ equals zero, $\Sigma(t_a - t_b)=0$ in (28). Then $\boldsymbol{F'}_{\text{avg2}} = \boldsymbol{F}_{\text{avg}}$, see (29) and (25). So, the 2 intervals give the same average value for $\boldsymbol{F}_a$ in current elements.

$$\begin{aligned} \boldsymbol{F'_{avg2}} &= \frac{1}{n}\sum \boldsymbol{F_{p\ air2}} \\ &= \frac{1}{n}\sum \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right)\frac{\Delta t^3 + \frac{3}{2}(t_a-t_b)\Delta t^2}{\Delta t} \\ &= \frac{e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right)\left(\frac{n\Delta t^3}{n\Delta t} + \frac{3\Delta t^2}{2n\Delta t}\sum(t_a-t_b)\right) \end{aligned} \quad (28)$$

$$\begin{aligned} &n \to \infty, \qquad \sum(t_a-t_b) \to 0 \\ &\boldsymbol{F'_{avg2}} = \frac{\Delta t^2 e^2}{4\pi\varepsilon_0 {r_1}^5}\left((\boldsymbol{r_1}\cdot\boldsymbol{v_a})\boldsymbol{v_b} + (\boldsymbol{r_1}\cdot\boldsymbol{v_b})\boldsymbol{v_a}\right) = \boldsymbol{F_{avg}} \end{aligned} \quad (29)$$

$\boldsymbol{F}_{\text{avg}}$ is proportional to Δt^2, but Δt is still undetermined. What is the value of Δt?

7. Length of the time interval

If Δt is big, $\boldsymbol{F}_{\text{avg}}$ is big and loses its physical meaning. Indeed, at the end of the time interval the distance between the electron and the positive charge of a pair is $\boldsymbol{v}\Delta t$ and a long distance makes the pair of charges inconsistent with current element. So, the smaller Δt is, the more accurate $\boldsymbol{F}_{\text{avg}}$ is. But Δt should not be zero because if so, $\boldsymbol{F}_{\text{avg}}$ would be zero and we would not find any magnetic force. So, there must exist a lower limit for Δt.

In fact, Δt depends on the relativistic principle “nothing travels faster than light” which Coulomb force must obey. Let us call the signal that carries Coulomb force Coulomb signal and assume that Coulomb signal travels at the speed of light c. How does the speed of Coulomb signal affect Coulomb force?

Let us explain with the example shown in **Figure 4** where the charge q_b moves at the velocity $\boldsymbol{v}$ and interacts with the immobile charge q_a. The total Coulomb force $\boldsymbol{F}_{\mathrm{t}}$ that q_b exerts on q_a within the time interval $[t_1, t_2]$ equals the integral of Coulomb force. The length of the interval is $\Delta t = t_2 - t_1$ during which q_b moves from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$. So $\boldsymbol{r_2} - \boldsymbol{r_1} = \boldsymbol{v}\Delta t$, see (30). As Δt is very small, we can consider expressing $\boldsymbol{F}_{\mathrm{t}}$ using the first radial vector $\boldsymbol{r_1}$. But this is allowed only if $\boldsymbol{F}_{\mathrm{t}}$ is physically linked to $\boldsymbol{r_1}$. Is this true when Δt approaches zero?

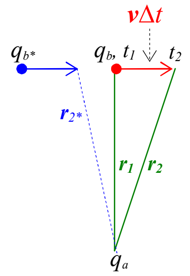

Figure 4

The answer is no and let me explain why. Coulomb signal is not instantaneous and for covering a distance r, it takes the time r/c. In our case, S_2 is the Coulomb signal at q_a at time t_2. S_2 has covered the radial vector $\boldsymbol{r_2}$ from q_b to at q_a and is delayed by the time r_2/c. So, it was radiated r_2/c time earlier when q_b was at $\boldsymbol{r_{2*}} = \boldsymbol{r_2} - \boldsymbol{v}\, r_2/c$, see (31). So, the entire Coulomb force integrated into $\boldsymbol{F}_{\mathrm{t}}$ is radiated before the radial vector $\boldsymbol{r_{2*}}$ in the trajectory of q_{b*}, see **Figure 4**.

$$\begin{aligned} \boldsymbol{r}_2 - \boldsymbol{r}_1 &= \boldsymbol{v}\Delta t \\ \boldsymbol{r}_1 &= \boldsymbol{r}_2 - \boldsymbol{v}\Delta t \end{aligned} \tag{30}$$

$$\boldsymbol{r}_{2*} = \boldsymbol{r}_2 - \boldsymbol{v}\frac{r_2}{c} \tag{31}$$

$$\begin{aligned} \Delta\boldsymbol{r}_* &= \boldsymbol{r}_{2*} - \boldsymbol{r}_1 \\ &= \boldsymbol{r}_2 - \boldsymbol{v}\frac{r_2}{c} - (\boldsymbol{r}_2 - \boldsymbol{v}\Delta t) \\ &= \left(\Delta t - \frac{r_2}{c}\right)\boldsymbol{v} \end{aligned} \tag{32}$$

Let us compare $\boldsymbol{r_1}$ which is going to be used to express $\boldsymbol{F}_{\mathrm{t}}$ to the radial vector $\boldsymbol{r_{2*}}$ where the last Coulomb signal is radiated. (32) computes the difference vector $\Delta\boldsymbol{r_*} = \boldsymbol{r_{2*}} - \boldsymbol{r_1}$.

When $\Delta t < r_2/c$, (33) shows that $\Delta t - \frac{r_2}{c}$ is negative and $\Delta\boldsymbol{r_*}$ is contrary to $\boldsymbol{v}$, which means that $\boldsymbol{r_{2*}}$ is before $\boldsymbol{r_1}$ in the trajectory of q_b and the entire Coulomb signal integrated into $\boldsymbol{F}_{\mathrm{t}}$ is radiated strictly before $\boldsymbol{r_1}$. In other words, the total Coulomb force $\boldsymbol{F}_{\mathrm{t}}$ has no physical relation with $\boldsymbol{r_1}$. If $\Delta t < r_2/c$, $\boldsymbol{r_1}$ is inconsistent for expressing $\boldsymbol{F}_{\mathrm{t}}$, see **Figure 4**.

$$\Delta t < \frac{r_2}{c} \Rightarrow \begin{cases} \Delta t - \frac{r_2}{c} < 0 \\ \Delta\boldsymbol{r}_* = -\left|\Delta t - \frac{r_2}{c}\right|\boldsymbol{v} \end{cases} \tag{33}$$

$$\Delta t = \frac{r_2}{c} \Rightarrow \begin{cases} \Delta t - \frac{r_2}{c} = 0 \\ \boldsymbol{r}_{2*} = \boldsymbol{r}_1 \end{cases} \tag{34}$$

When $\Delta t = r_2/c$, the coefficient $\Delta t - \frac{r_2}{c}$ equals zero, see (34). Then $\boldsymbol{r_{2*}} = \boldsymbol{r_1}$ and the last Coulomb signal in $\boldsymbol{F}_{\mathrm{t}}$ is radiated at $\boldsymbol{r_1}$, see **Figure 5**. This is the physical link that makes $\boldsymbol{r_1}$ consistent for expressing $\boldsymbol{F}_{\mathrm{t}}$. So, r_2/c is the lower limit for Δt.

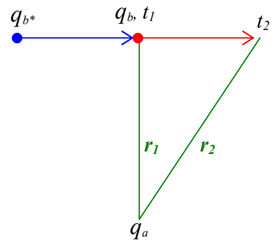

Figure 5

In consequence, $\Delta t = r/c$ is the general formula for the total Coulomb force $\boldsymbol{F}_{\mathrm{t}}$ to be consistently expressed with the distance r. For the case of $\boldsymbol{F}_{\mathrm{avg}}$, r equals the distance between the immobile charges of pair A and pair B which is r_1 and $\Delta t = r_1/c$, see (35).

8. $\boldsymbol{F_{avg}}$ function of c^2

So, we replace r_1/c for Δt in (25) which becomes (36) then (37), which is a function of c^2.

$$\Delta t = \frac{r_1}{c} \tag{35}$$

$$\boldsymbol{F}_{avg} = \frac{r_1^{\,2}}{c^2}\frac{e^2}{4\pi\varepsilon_0 r_1^{\,5}}[(\boldsymbol{r}_1\cdot\boldsymbol{v}_a)\boldsymbol{v}_b + (\boldsymbol{r}_1\cdot\boldsymbol{v}_b)\boldsymbol{v}_a] \tag{36}$$

$$\boldsymbol{F}_{avg} = \frac{e^2}{4\pi\varepsilon_0 c^2 r_1^{\,3}}[(\boldsymbol{r}_1\cdot\boldsymbol{v}_a)\boldsymbol{v}_b + (\boldsymbol{r}_1\cdot\boldsymbol{v}_b)\boldsymbol{v}_a] \tag{37}$$

$\boldsymbol{F}_{avg}$ is the force on one pair. The sum of all $\boldsymbol{F}_{avg}$ in the current element $d\boldsymbol{I}_a$ is the force that the current element $d\boldsymbol{I}_b$ exerts on the current element $d\boldsymbol{I}_a$.

9. Force on current element

One pair B exerts the individual force $\boldsymbol{F}_{avg,i}$ on one pair A. Let n_b be the number of pair B in $d\boldsymbol{I}_b$, then there are n_b individual forces $\boldsymbol{F}_{avg,i}$ on <u>one</u> pair A. Let n_a be the number of pair A in $d\boldsymbol{I}_a$, then $\boldsymbol{F}_{cd}$ the total force on the current element $d\boldsymbol{I}_a$ is the sum of $n_a n_b$ individual forces $\boldsymbol{F}_{avg,i}$, which is computed in (38). $\boldsymbol{F}_{cd}$ is the net force shown in **Figure 1**.

$\boldsymbol{r}_\mathrm{i}$ is the distance between 2 immobile positive charges which is slightly different from one pair to another. We define the mean distance $\boldsymbol{r}_\mathrm{m}$ in (39) as the distance between $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ which is introduced into (38) to express $\boldsymbol{F}_{cd}$ in terms of $n_a n_b$ and $\boldsymbol{r}_\mathrm{m}$ in (40). In (41) we gather n_a with $\boldsymbol{v_a}$ and n_b with $\boldsymbol{v_b}$ for expressing $\boldsymbol{F}_{cd}$ as a function of $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$.

One pair contains one free electron, so the number of pair equals the number of free electrons. Then, $-n_a e$ equals the total charge of all the free electrons in $d\boldsymbol{I}_a$ and $n_a e\,\boldsymbol{v_a}$ equals $-I_a d\boldsymbol{l}_a$ where $d\boldsymbol{l}_a$ is the length vector of $d\boldsymbol{I}_a$ and I_a the intensity of current, see (42). $I_a d\boldsymbol{l}_a$ is the current element vector $d\boldsymbol{I}_a$. In the same way, $n_b e\,\boldsymbol{v_b}$ equals $-d\boldsymbol{I}_b$, see (43). Combining (43) with (41) we obtain in (44) the expression for $\boldsymbol{F}_{cd}$ the total Coulomb force that all the charges of $d\boldsymbol{I}_b$ exert on all the charges of $d\boldsymbol{I}_a$ which is a function of $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$.

$$\begin{aligned}\boldsymbol{F}_{cd} &= \sum_{n_a}\sum_{n_b}\boldsymbol{F}_{\boldsymbol{avg,i}}\\ &= \sum_{n_a}\sum_{n_b}\frac{e^2}{4\pi\varepsilon_0 c^2 r_i^3}[(\boldsymbol{r_i}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_i}\cdot\boldsymbol{v_b})\boldsymbol{v_a}]\\ &= \frac{e^2}{4\pi\varepsilon_0 c^2}\left[\left(\sum_{n_a}\sum_{n_b}\frac{\boldsymbol{r_i}}{r_i^3}\cdot\boldsymbol{v_a}\right)\boldsymbol{v_b}+\left(\sum_{n_a}\sum_{n_b}\frac{\boldsymbol{r_i}}{r_i^3}\cdot\boldsymbol{v_b}\right)\boldsymbol{v_a}\right]\end{aligned} \tag{38}$$

$$\frac{\boldsymbol{r_m}}{r_m^3} = \frac{1}{n_a n_b}\sum_{n_a}\sum_{n_b}\frac{\boldsymbol{r_i}}{r_i^3} \tag{39}$$

$$\boldsymbol{F}_{cd} = \frac{n_a n_b e^2}{4\pi\varepsilon_0 c^2 r_m^3}[(\boldsymbol{r_m}\cdot\boldsymbol{v_a})\boldsymbol{v_b}+(\boldsymbol{r_m}\cdot\boldsymbol{v_b})\boldsymbol{v_a}] \tag{40}$$

$$\boldsymbol{F}_{cd} = \frac{1}{4\pi\varepsilon_0 c^2 r_m^3}[(\boldsymbol{r_m}\cdot n_a e\,\boldsymbol{v_a})n_b e\,\boldsymbol{v_b}+(\boldsymbol{r_m}\cdot n_b e\,\boldsymbol{v_b})n_a e\,\boldsymbol{v_a}] \tag{41}$$

$$\begin{aligned}&n_a e = dq_a, \qquad \boldsymbol{v_a} = -\frac{d\boldsymbol{l}_a}{dt}\\ &\Rightarrow\ n_a e\,\boldsymbol{v_a} = -dq_a\frac{d\boldsymbol{l}_a}{dt} = -\frac{dq_a}{dt}d\boldsymbol{l}_a = -I_a d\boldsymbol{l}_a = -d\boldsymbol{I}_a\end{aligned} \tag{42}$$

$$n_a e\,\boldsymbol{v_a} = -I_a d\boldsymbol{l}_a = -d\boldsymbol{I}_a, \qquad n_b e\,\boldsymbol{v_b} = -I_b d\boldsymbol{l}_b = -d\boldsymbol{I}_b \tag{43}$$

$$\boxed{\boldsymbol{F}_{cd} = \frac{1}{4\pi\varepsilon_0 c^2 r_m^3}[d\boldsymbol{I}_b(\boldsymbol{r_m}\cdot d\boldsymbol{I}_a)+d\boldsymbol{I}_a(\boldsymbol{r_m}\cdot d\boldsymbol{I}_b)]} \tag{44}$$

$$\begin{aligned}\boldsymbol{F}_{45} &= \frac{1}{4\pi\varepsilon_0 c^2 r_m^2}\left[\cos\frac{\pi}{4}dI_a d\boldsymbol{I}_b+\cos\frac{\pi}{4}dI_b d\boldsymbol{I}_a\right]\\ &= \frac{dI_a dI_b}{4\pi\varepsilon_0 c^2 r_m^2}\cos\frac{\pi}{4}[\boldsymbol{e_a}+\boldsymbol{e_b}]\end{aligned} \tag{45}$$

10. Comments

$\boldsymbol{F}_{cd}$ is magnetic because the current elements $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ are maintained neutral in the derivation. The expression (44) for $\boldsymbol{F}_{cd}$ does not look like Lorentz force law because $\boldsymbol{F}_{cd}$ is a partial magnetic force which will be combined with the force derived in «Length-contraction-magnetic-force between arbitrary currents» to form complete magnetic force.

$\boldsymbol{F}_{cd}$ is not zero when $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ are perpendicular to each other. For example, (45) expresses the force $\boldsymbol{F}_{45}$ when $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ make both 45° angle with the radial vector $\boldsymbol{r}_\mathrm{m}$, which is not zero because the unit vectors of $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ do not sum to zero, $\boldsymbol{e}_\mathrm{a} + \boldsymbol{e}_\mathrm{b} \neq 0$.