From electron to magnetism

2. Length-contraction-magnetic-force between arbitrary currents

Peng Kuan 彭宽 *titang78@gmail.com*

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Abstract: Formula for magnetic force between 2 arbitrary current elements derived from Coulomb's law and relativistic length contraction formula.

In ≪Relativistic length contraction and magnetic force≫ I have explained the mechanism of creation of magnetic force from Coulomb force and relativistic length contraction. For facilitating the understanding of this mechanism I used parallel current elements because the lengths are contracted in the direction of the currents. But real currents are rarely parallel, for example, $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ of the two circuits in **Figure 1**. For correctly applying length contraction on currents in any direction, we will consider conductor wires in their volume and apply length contraction on volume elements of the wires.

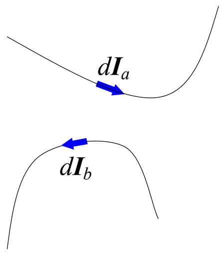

Figure 1

Note: boldface letters denote vectors. $d\boldsymbol{I}_a$, $d\boldsymbol{I}_b$ and all boldfaced variables below are vectors.

1. Contraction of volume

The volume of an object suffers length contraction and will appear shorter in the direction of its velocity. For example, when moving the sphere A in **Figure 2** will become the ellipsoid A', with its small axis in the direction of its velocity $\boldsymbol{v}$. So, the volume of the object shrinks due to length contraction.

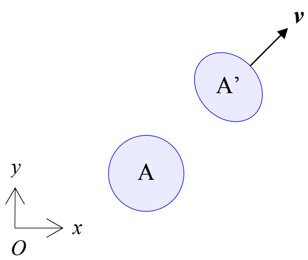

Figure 2

The ratio of contraction of the volume equals that of length contraction. Take the moving rectangular volume element dV in **Figure 3** whose velocity $\boldsymbol{v}$ is not parallel to the x axis of the frame of reference O. We rotate the frame O such that its x axis becomes parallel to $\boldsymbol{v}$. The so created frame is O_1 in which the sides of dV are dx_1, dy_1 and dz_1 and the volume equals $dV=dx_1\cdot dy_1\cdot dz_1$ of equation **(1)**.

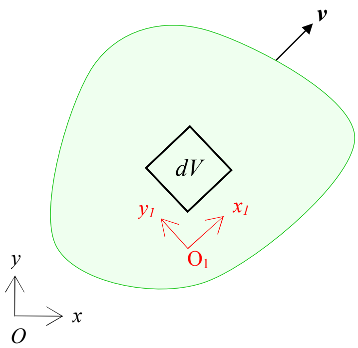

Figure 3

By length contraction, the side dx_1 is contracted and becomes dx'_1 which is expressed by equation **(2)**, where v is the magnitude of the vector $\boldsymbol{v}$. Then, the volume element dV becomes dV' whose volume is expressed by equation **(3)**.

The factor of contraction $\sqrt{1-v^2/c^2}$ is a function of the scalar v. In order to express this factor as a function of the vector velocity $\boldsymbol{v}$, we express v^2 in the form of the scalar product of $\boldsymbol{v}$**:** $\boldsymbol{v}\cdot\boldsymbol{v}$. Then, the formula of volume contraction becomes equation **(4)**, function of the vector $\boldsymbol{v}$, velocity in any direction.

Equation **(4)** is valid for volume element of any shape. For example, the sphere A in **Figure 2** is contracted into the ellipsoid A', the volume of A and A' obey equation **(4)**.

$$dV = dx_1\cdot dy_1\cdot dz_1 \tag{1}$$

$$dx'_1 = dx_1\sqrt{1-v^2/c^2} \tag{2}$$

$$\begin{aligned} dV' &= dx'_1\cdot dy_1\cdot dz_1 \\ &= dx_1\sqrt{1-v^2/c^2}\cdot dy_1\cdot dz_1 \\ dV' &= dV\sqrt{1-v^2/c^2} \end{aligned} \tag{3}$$

$$dV' = dV\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2} \tag{4}$$

2. Increase of volume density of charge

If a moving object is uniformly charged, its volume density of charge will increase due to the contraction of volume. Let dQ be the quantity of charge in the volume element A of **Figure 2** and dV_a

its volume. The volume charge density of A, ρ_a , is expressed by equation **(5)**. The apparent volume of the moving A' is dV'_a and its volume charge density is ρ'_a which is expressed by equation **(6)**.

$$\rho_a = \frac{dQ}{dV_a} \tag{5}$$

$$\rho'_a = \frac{dQ}{dV'_a} = \frac{dQ}{dV_a\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2}} = \frac{\rho_a}{\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2}} \tag{6}$$

$$\rho'_- = \frac{-\rho_+}{\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2}} \tag{7}$$

A conductor is uniformly filled with free electrons, whose volume charge density is denoted by ρ_- . Also, positive charge is uniformly dispersed within the conductor, whose volume charge density is denoted by ρ_+ . Because the conductor is neutral, the two charge densities are equal in magnitude, that is, $\rho_- = -\rho_+$. When a current circulates, the free electrons move at velocity $\boldsymbol{v}$ and their charge density ρ_- increases according to equation **(6)** and becomes ρ'_- of equation **(7)**.

Note that the increase of charge density is a relativistic effect, the wires are neutral and not actually charged, that is, there is **not** excess of positive or negative charges. This effect is like the length contraction of a moving ruler which seems shorter without actually being cut away a portion. In ≪Relativistic length contraction and magnetic force≫ I have explained in detail how magnetic force is created by the increase of charge density in current carrying wire. Below, we will derive the magnetic force between two current carrying volume elements.

3. Two current carrying volume elements

Let A and B be two conductor wires in which currents I_a and I_b circulate. We make one spherical volume element in each wire, dV_a in A and dV_b in B, as **Figure 4** shows. The spherical shape is chosen to facilitate imagining the volume elements contract into ellipsoids, but any shape is suitable for the derivation of the force between dV_a and dV_b.

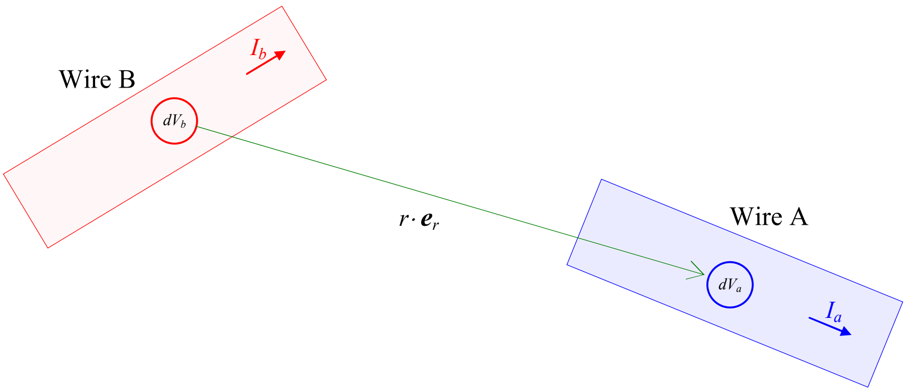

Figure 4

Let us compute the sum of electrostatic forces between the electric charges of dV_a and dV_b . The positive and negative charges enclosed in dV_a are Q_{a+}, Q_{a-} and those in B are Q_{b+}, Q_{b-} . Let r be the distance between dV_a and dV_b , $\boldsymbol{e}_r$ the unit radial vector pointing from dV_b to dV_a. From dV_b , Q_{b+} and Q_{b-} exert on Q_{a+} the two Coulomb forces $d\boldsymbol{F}_1$ and $d\boldsymbol{F}_2$ of equations **(8)**. In the same way, the two forces that Q_{b+} and Q_{b-} exert on Q_{a-} are the Coulomb forces $d\boldsymbol{F}_3$ and $d\boldsymbol{F}_4$ of equations **(8)**. The sum of these 4 forces is expressed by equation **(9)** and is the resultant force that the volume element dV_b exerts on the volume element dV_a**:** $d\boldsymbol{F}_a$.

$$d\boldsymbol{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b+}Q_{a+}}{r^2}\boldsymbol{e}_r, \quad d\boldsymbol{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b-}Q_{a+}}{r^2}\boldsymbol{e}_r$$
$$d\boldsymbol{F}_3 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b+}Q_{a-}}{r^2}\boldsymbol{e}_r, \quad d\boldsymbol{F}_4 = \frac{1}{4\pi\varepsilon_0}\frac{Q_{b-}Q_{a-}}{r^2}\boldsymbol{e}_r \tag{8}$$

$$d\boldsymbol{F}_a = d\boldsymbol{F}_1 + d\boldsymbol{F}_2 + d\boldsymbol{F}_3 + d\boldsymbol{F}_4 = \frac{1}{4\pi\varepsilon_0}(Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-})\frac{\boldsymbol{e}_r}{r^2} \tag{9}$$

If no current circulates, the Coulomb's forces $d\boldsymbol{F}_1$, $d\boldsymbol{F}_2$, $d\boldsymbol{F}_3$ and $d\boldsymbol{F}_4$ will cancel out and $d\boldsymbol{F}_a$ will be zero. When there are currents in the volume elements, the values of Q_{a+}, Q_{a-}, Q_{b+} and Q_{b-} increase differently, then the quantities $Q_{b+}Q_{a+}$, $Q_{b-}Q_{a+}$, $Q_{b+}Q_{a-}$ and $Q_{b-}Q_{a-}$, that we name charge-product, will go out of balance and make a non zero sum. The charge-products are carefully computed below.

1. $Q_{b+}Q_{a+}$

Q_{b+} and Q_{a+} are both stationary and their densities are the constants ρ_{a+} and ρ_{b+}. Their volumes are that of the volume elements dV_a and dV_b . So, the values of Q_{b+} and Q_{a+} are the products given in equation **(10)**. The charge-product $Q_{b+}Q_{a+}$ is given in equation **(11)**.

$$Q_{a+} = \rho_{a+}dV_a, \qquad Q_{b+} = \rho_{b+}dV_b \qquad (10)$$

$$Q_{b+}Q_{a+} = \rho_{a+}\rho_{b+}dV_a dV_b \qquad (11)$$

2. $Q_{b-}Q_{a+}$

Due to the current I_b, Q_{b-} moves at the velocity $\boldsymbol{v}_b$ in the stationary frame. According to equation **(7)**, the density of Q_{b-} increases and becomes ρ'_{b-} of equation **(12)**. The volume of Q_{b-} equals that of dV_b and the value of Q_{b-} is the product $\rho'_{b-}\cdot dV_b$ given by equations **(13)**. Q_{a+} is stationary and keeps the same density and value. Then, the charge-product $Q_{b-}Q_{a+}$ is given in equation **(14)**.

$$\rho'_{b-} = \frac{-\rho_{b+}}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} \qquad (12)$$

$$Q_{a+} = \rho_{a+}dV_a,$$

$$Q_{b-} = \rho'_{b-}dV_b = \frac{-\rho_{b+}dV_b}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} \qquad (13)$$

$$Q_{b-}Q_{a+} = -\frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} \qquad (14)$$

3. $Q_{b+}Q_{a-}$

Due to the current I_a, Q_{a-} moves at the velocity $\boldsymbol{v}_a$ in the stationary frame. According to equation **(7)**, the density of Q_{a-} increases and becomes ρ'_{a-} of equation **(15)**. The volume of Q_{a-} equals that of dV_a and the value of Q_{a-} is the product $\rho'_{a-}\cdot dV_a$ given by equations **(16)**. Q_{b+} is stationary and keeps the same density and value. Then, the charge-product $Q_{b+}Q_{a-}$ is given in equation **(17)**.

$$\rho'_{a-} = \frac{-\rho_{a+}}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}}} \qquad (15)$$

$$Q_{a-} = \rho'_{a-}dV_a = \frac{-\rho_{a+}dV_a}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}, \qquad (16)$$

$$Q_{b+} = \rho_{b+}dV_b$$

$$Q_{b+}Q_{a-} = -\frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}} \qquad (17)$$

4. $Q_{b-}Q_{a-}$

This charge-product is a little tricky to compute because both Q_{a-} and Q_{b-} are moving. In which frame will Q_{b-} be evaluated? According to the principle of relativity, the length contraction that Q_{b-} suffers depends only on the velocity relative to the observer. In this case, the observer is an electron of Q_{a-} on which the force from Q_{b-} acts on. The correct frame is then a frame moving with Q_{a-} because only in such frame the velocity of Q_{b-} , which equals $\boldsymbol{v}_b$ -$\boldsymbol{v}_a$, is relative to Q_{a-} . So, the length contraction factor of Q_{b-} is $1/\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}$ and the charge density of Q_{b-} becomes ρ'_{b-} given by equation **(18)**.

$$\rho'_{b-} = \frac{-\rho_{b+}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}} \qquad (18)$$

$$dV'_b = dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2} \qquad (19)$$

$$Q_{b-} = \rho'_{b-}\cdot dV'_b = \frac{-\rho_{b+}dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}} \qquad (20)$$

Furthermore, because the frame is moving at the velocity $\boldsymbol{v}_a$, the volume element dV_b moves at the opposite velocity $-\boldsymbol{v}_a$ and contracts to dV'_b whose value is given by **(19)**. In consequence, the charge Q_{b-} equals the increased charge density ρ'_{b-} multiplied by the contracted volume dV'_b and equals the value given by equation **(20)**.

The charge Q_{a-} is evaluated in the stationary frame because it is the charge of all electrons in the volume element dV_a which is stationary. In this case, the values of ρ_{a-} and Q_{a-} are already computed by equations **(15)** and **(16)**. Finally, the charge-product $Q_{b-}Q_{a-}$ is given by equation **(21)**.

$$\begin{aligned} Q_{b-}Q_{a-} &= \frac{-\rho_{b+}dV_b\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}}\frac{-\rho_{a+}dV_a}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}} \\ &= \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}} \end{aligned} \qquad (21)$$

5. $Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-}$

Having obtained all the 4 charge-products necessary for computing the resultant force expressed by equation **(9)**, we compute the sum of the 4 charge-products which is expressed by equation **(22)**:

$$Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-}$$
$$= \rho_{a+}\rho_{b+}dV_a dV_b - \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} - \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}} + \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}}$$
$$= \rho_{a+}\rho_{b+}dV_a dV_b\left(1 - \frac{1}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} - \frac{1}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}} + \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})/c^2}}\right) \quad (22)$$

$$(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a}) = \boldsymbol{v_a}\cdot\boldsymbol{v_a} - \boldsymbol{v_a}\cdot\boldsymbol{v_b} - \boldsymbol{v_b}\cdot\boldsymbol{v_a} + \boldsymbol{v_b}\cdot\boldsymbol{v_b} = \boldsymbol{v_a}\cdot\boldsymbol{v_a} - \boldsymbol{2v_a}\cdot\boldsymbol{v_b} + \boldsymbol{v_b}\cdot\boldsymbol{v_b} \quad (23)$$

The scalar product $(\boldsymbol{v_b}-\boldsymbol{v_a})\cdot(\boldsymbol{v_b}-\boldsymbol{v_a})$ is expanded by equation **(23)**. Equation **(22)** can be expanded into first-order Taylor series using equation **(24)** since the scalar product $\boldsymbol{v}\cdot\boldsymbol{v}$ is a scalar. Below, we will use the sign "=" instead of the sign "≈"because $\boldsymbol{v}\cdot\boldsymbol{v}/c^2$ is extremely small with respect to 1. So, using "=" does not harm precision while simplifying the expansion.

Then the parenthesis in equation **(22)** reduces to equation **(25)** and the sum of the 4 charge-products to equation **(26)**.

$$\frac{1}{\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2}} \approx 1 + \frac{1}{2}\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2} \quad (24)$$

$$\left(1 - \frac{1}{\sqrt{1-\boldsymbol{v_b}\cdot\boldsymbol{v_b}/c^2}} - \frac{1}{\sqrt{1-\boldsymbol{v_a}\cdot\boldsymbol{v_a}/c^2}} + \frac{\rho_{a+}\rho_{b+}dV_a dV_b}{\sqrt{1-(\boldsymbol{v_a}\cdot\boldsymbol{v_a}-\boldsymbol{2v_a}\cdot\boldsymbol{v_b}+\boldsymbol{v_b}\cdot\boldsymbol{v_b})/c^2}}\right)$$
$$= 1 - \left(1+\frac{1}{2}\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2}\right) - \left(1+\frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}}{c^2}\right) + \left(1+\frac{1}{2}\frac{\boldsymbol{v_a}\cdot\boldsymbol{v_a}-\boldsymbol{2v_a}\cdot\boldsymbol{v_b}+\boldsymbol{v_b}\cdot\boldsymbol{v_b}}{c^2}\right) \quad (25)$$
$$= -\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_a}}{c^2}$$

$$Q_{b+}Q_{a+} + Q_{b-}Q_{a+} + Q_{b+}Q_{a-} + Q_{b-}Q_{a-} = -\rho_{a+}\rho_{b+}dV_a dV_b\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_a}}{c^2} \quad (26)$$

6. Resultant force

Introducing equation **(26)** into equations **(9)** gives the expression for the resultant force on the volume element dV_a, which is equation **(27)**. For transforming this expression into a function of current element $\boldsymbol{I}dl$=$d\boldsymbol{I}$, we consider a segment of each wire as the volume elements dV_a and dV_b, with S_a and S_b the wires' sections, dl_a and dl_b the lengths of the segments. The values of dV_a and dV_b , which equal section multiplied by length, are expressed by equations **(28)**.

$$d\boldsymbol{F}_a = -\frac{1}{4\pi\varepsilon_0}\rho_{a+}\rho_{b+}dV_a dV_b\frac{\boldsymbol{v_b}\cdot\boldsymbol{v_a}}{c^2}\frac{\boldsymbol{e}_r}{r^2} \quad (27)$$
$$dV_a = S_a dl_a, \quad dV_b = S_b dl_b \quad (28)$$
$$\rho_{a+}dV_a\boldsymbol{v}_a = \rho_{a+}S_a dl_a\boldsymbol{v}_a = -\boldsymbol{I}_a dl_a = -d\boldsymbol{I}_a$$
$$\rho_{b+}dV_b\boldsymbol{v}_b = \rho_{b+}S_b dl_b\boldsymbol{v}_b = -\boldsymbol{I}_b dl_b = -d\boldsymbol{I}_b \quad (29)$$
$$d\boldsymbol{F}_a = -\frac{1}{4\pi\varepsilon_0 c^2}\frac{(d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b)\boldsymbol{e}_r}{r^2} \quad (30)$$
$$d\boldsymbol{F}_a = -\frac{1}{4\pi\varepsilon_0 c^2 r^3}(d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b)\boldsymbol{r} \quad (31)$$

The products $\rho_{a+}dV_a\boldsymbol{v}_a$ and $\rho_{b+}dV_b\boldsymbol{v}_b$ in equation **(27)** equal the vector current elements $-d\boldsymbol{I}_a$ and $-d\boldsymbol{I}_b$ given by equation **(29)**. Then, combining equations **(27)** and **(29)**, the expression for the resultant force $d\boldsymbol{F}_a$ becomes equation **(30)** or **(31)** in another notation. We see that $d\boldsymbol{F}_a$ is proportional to the scalar product of the vector current elements $d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b$ and collinear to the radial vector $\boldsymbol{e}_r$. This formula is the differential law for magnetic force that I proposed in ≪Correct differential magnetic force law≫.

4. Comparison with Lorentz force

Equation **(30)** resembles to Lorentz force between the two current elements $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$, which is denoted by $d\boldsymbol{F}_l$ and given by equation **(32)**. For comparing $d\boldsymbol{F}_a$ with $d\boldsymbol{F}_l$, we integrate the Lorentz force $d\boldsymbol{F}_l$ over a closed circuit which is the wire A.

For doing so, we expand the double cross product $d\boldsymbol{I}_a\times(d\boldsymbol{I}_b\times\boldsymbol{e}_r)$ using the vector identity given by equation **(33)**.The integral of $d\boldsymbol{F}_l$ becomes the two terms integral of equation **(34)**. The current element $d\boldsymbol{I}_b$ being constant, the integral of the first term with respect to $d\boldsymbol{I}_a$ equals zero because it is a closed line integral of the gradient field $\boldsymbol{e}_r/r^2$, see equation **(35)**. Then, the total Lorentz force $\boldsymbol{F}_{la}$ that the vector current element $\boldsymbol{I}_b dl_b$ exerts on

$$d\boldsymbol{F}_l = \frac{\mu_0}{4\pi}\frac{d\boldsymbol{I}_a\times(d\boldsymbol{I}_b\times\boldsymbol{e}_r)}{r^2} \quad (32)$$
$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \quad (33)$$
$$\boldsymbol{F}_{la} = \oint d\boldsymbol{F}_l$$
$$= \frac{\mu_0}{4\pi}\left(\oint\frac{d\boldsymbol{I}_b(d\boldsymbol{I}_a\cdot\boldsymbol{e}_r) - \boldsymbol{e}_r(d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b)}{r^2}\right) \quad (34)$$
$$\oint\frac{\boldsymbol{e}_r}{r^2}\cdot d\boldsymbol{I}_a = 0 \Rightarrow \oint\frac{d\boldsymbol{I}_b(d\boldsymbol{I}_a\cdot\boldsymbol{e}_r)}{r^2} = d\boldsymbol{I}_b\oint\frac{d\boldsymbol{I}_a\cdot\boldsymbol{e}_r}{r^2} = 0 \quad (35)$$
$$\boldsymbol{F}_{la} = \oint\left(-\frac{\mu_0}{4\pi}\frac{(d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b)\boldsymbol{e}_r}{r^2}\right) \quad (36)$$

the closed circuit A equals the integral of equation **(36)**. See page 199 of ≪ Introduction to Electromagnetic Fields ≫ by Clayton R. Paul, Keith W. Whites, Syed A. Nasar.

The integrant of equation **(36)** is identical to the resultant force given by equation **(30)** with $\mu_0 = 1/\varepsilon_0 c^2$. So, for closed circuits, the force derived from Coulomb's law and length contraction formula give the same result as Lorentz force law. This shows the correctness of this mechanism of creation of magnetic force.

Although the resultant force expressed by equation **(30)** is correct for closed circuit, it lacks the force that two perpendicular current elements exert on each other because in this case $d\boldsymbol{I}_a \cdot d\boldsymbol{I}_b = 0$. This force will be studied in the next chapter.

5. About displacement current

The expression for the resultant force, equation **(30)**, respects Newton's third law. Indeed, when we reverse the role of dV_a and dV_b, we will get the opposite force that dV_a exerts on dV_b . The reaction force on dV_b arises from Coulomb's law in which each charge is the source of the force on the other, that is, Newton's third law is respected because dV_b is the source of the force on dV_a and vice versa.

In the classic theory of electromagnetism, magnetic force is not directly linked to a source object, but to magnetic field which can be created by a varying electric field which is a displacement current, $\partial \boldsymbol{E}/\partial t$. Displacement current is not a physical current of flowing charges which would act as the source object of a physical force. By lacking the source object, a displacement current cannot bear the reaction force of the magnetic force $\boldsymbol{F}$ that it exerts on a current $\boldsymbol{I}$, see **Figure 5**. If no reaction force exists, Newton's third law is violated and the description of displacement current seems questionable.

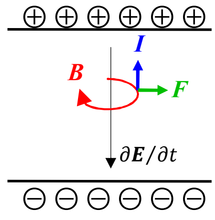

Figure 5

Based on the fact that the resultant force on dV_a possesses the source object dV_b while displacement current does not, I have designed the two paradoxes ≪Phantom Lorentz force Paradox≫ and ≪Displacement Current Paradox≫ to explain this inconsistency.