EBERHARD KARLS UNIVERSITÄT TÜBINGEN

PHILOSOPHISCHE FAKULTÄT

Seminar für Sprachwissenschaft

Kuan's Logic Tutorial 03

2021.11.26

Nr	Date	Content	Main reading	Additional Reading
1	08. Nov	Overview & Sets	Script Sets, ... Section 1	
2	15. Nov	Proofs	Script Sets, ... Section 2	
3	22. Nov	Relations & Functions	Script Sets, ... Sections 3 & 4	
4	29. Nov	Propositional Logic: Syntax & Truth Tables		Gamut 2.1 - 2.3
5	06. Dec	Propositional Logic: Semantics		Gamut 2.5
6	13. Dec	Logical Validity		Gamut 4.2 (up to page 118)
7	20. Dec	Predicate Logic: Syntax & Semantics		Gamut 3.1 - 3.3, 3.6
		XMAS		
8	10. Jan	Predicate Logic: Semantics, Identity & Validity		Gamut 3.6, 3.7, 4.21
9	17. Jan	Natural Deduction: Propositional Logic		Gamut 4.3.1 - 4.3.5
10	24. Jan	Natural Deduction: Predicate Logic; Meta Logic		Gamut 4.3.6 - 4.3.7 & 4.4
11	31. Jan	trade-in or something fun (informal argumentation, probability logic ..)		
12	07. Feb	spill-over, reflection Q&A		
	14. Feb	Exam		

Review of Tutorial 02: Proofs

3. Relations and Functions

Key points:

- Relations
- Functions

3.1. Relations

In toturial 1, we use notation with curly braces "$\{\dots\}$" which means the order of representation of elements is *irrelevant* to represent such a collection.

- $\{a, b\} = \{b, a\}$

We use **angle bracket "$\langle\dots\rangle$"**, which means the order of the elements is *relevant* , in this case $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$

- $\langle a, b\rangle \neq \langle b, a\rangle$

3.1.1 Ordered triples

- $\{\{a\}, \{a. b\}\} = \{\{b\}, \{a. b\}\}$ (that is $\langle a, b\rangle = \langle b, a\rangle$), if and only if we have $a = b$.
- Ordered triples are defined as: $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle$

3.1.2. Cartesian products

The *Cartesian product* of sets $X_1, X_2, \dots, X_n$ is the set of all *n*-tuples $\langle x_1, x_2, \dots, x_n\rangle$ such that x_i is an element from set X_i:

- $X_1 \times X_2 \times \dots X_n = \{\langle x_1, x_2, \dots, x_n\rangle | x_1 \in X_1, x_2 \in X_2, \dots, x_n \in X_n\}$

If we have two sets A and B, we can form ordered pairs from them by taking an element of A as the first member of the pair and an element of B:

- $A \times B = \{\langle x, y\rangle | x \in A \; and \; y \in B\}$

Exercise 01: Let $K = \{a, b, c\}$ and $L = \{1, 2\}$, then

$K \times L =$
$L \times K =$
$L \times L =$

3.1.3. Relations

The relation *R* holds between X and Y, we write *Rxy* or *xRy*, i.e. $\{\langle x, y\rangle | xRy\}$ or $R \subseteq X \times Y$.

The ***domain*** of *R* is

$$dom(R) = \{x \in X | there\ is\ some\ y \in Y\ with\ Rxy\}$$

The ***range*** of *R* is

$$range(R) = \{y \in Y | there\ is\ some\ x \in X\ with\ Rxy\}$$

The ***negation*** of *n*-place relation: (or the complement of *n*-place relation R')

$$\overline{R} \subseteq X_1 \times \cdots \times X_n\ is\ R = (X_1 \times \cdots \times X_n) \setminus R$$

The ***converse*** of *n*-place relation:

$$R^{-1} \subseteq X_1 \times \cdots \times X_n\ is\ R^{-1} = \{\langle y, x\rangle | Rxy\}$$

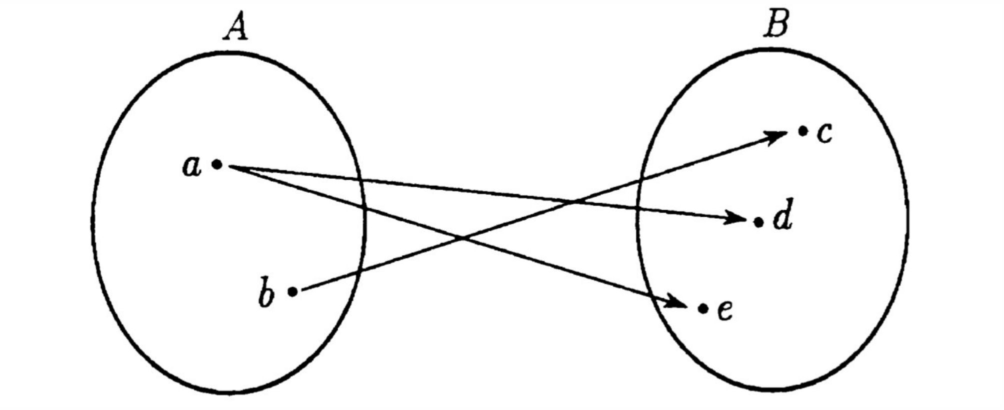
Notice: $\overline{\overline{R}} = R$, $(R^{-1})^{-1} = R$, and if $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$, but $\overline{R} \subseteq A \times B$

3.1.3.1 Set-theoretic relation

Let $A = \{a, b\}$, $B = \{c, d, e\}$, then the relation $R : A \to B$

the arrows represent a set-theoretic relation $R = \{\langle a, d\rangle, \langle a, e\rangle, \langle b, c\rangle\}$

the **domain** of R the **range** of R



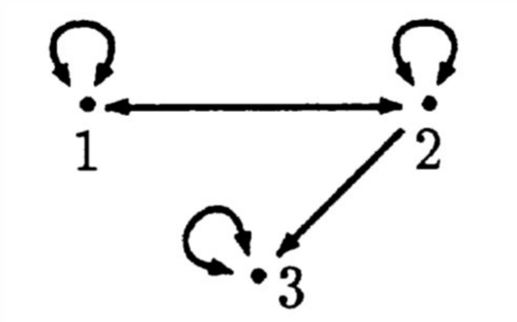
Exercise 02: Let $A = \{1, 2, 3\}$ and let $R \subseteq A \times A$ be $\{\langle 3, 2\rangle, \langle 3, 1\rangle, \langle 2, 1\rangle\}$. What is the negation of R, $\overline{R}$, and the inverse of R, R^{-1}?

$\overline{R}$

R^{-1}

3.1.3.2. Diagrams of relations

The members of the relevant set are represented by labeled points (the particular spatial arrangement of them is irrelevant). If x is related to y, i.e. $\langle 2, 3\rangle \in R$, an arrow connects the corresponding points.



- $R = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 1, 1\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\}$

Exercise 03: The following is false: If $R \subseteq X \times X$, then dom(R) = range(R). (Try to use a set-theoretic proof?)

Proof.

3.1.4. Properties of binary relations

Certain properties of binary relations are so frequently encountered that it is useful to have names for them. The properties we shall consider are:

- reflexivity
- symmetry
- transitivity
- connectedness.

3.1.4.1. Reflexivity

Binary relation $R \subseteq X \times X$ is

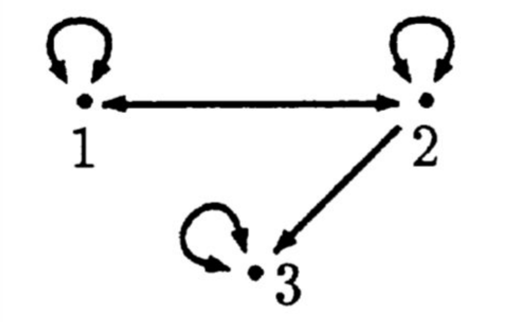
- *reflexive* iff Rxx for all $x \in X$
- *irreflexive* iff Rxx for no $x \in X$

Exercise 04: Take the set $A = \{1, 2, 3\}$ and the relation $R_1 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$ in A. The relation R_1 in A is reflexive

Is $R_2 = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$ in A reflexive ?

Is $R_3 = \{\langle 1, 2\rangle, \langle 3, 2\rangle\}$ in A reflexive?

Exercise 05: Is the relation in the diagram reflexive?



3.1.4.2. Symmetry

Binary relation $R \subseteq X \times X$ is

- **symmetric** iff for all $x, y \in X$ if Rxy then also Ryx. Examples of **symmetric** relations in $\{1, 2, 3\}$:

$\{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle\}$

$\{\langle 1, 3\rangle, \langle 3, 1\rangle\}$

$\{\langle 2,2\rangle\}$

- **asymmetric** iff for no $x, y \in X$ both Rxy and Ryx. Examples of **asymmetric** relatiolls in $\{1,2,3\}$:

$\{\langle 2,3\rangle, \langle 1,2\rangle\}$

$\{\langle 1,3\rangle, \langle 2,3\rangle, \langle 1,2\rangle\}$

$\{\langle 3,2\rangle\}$

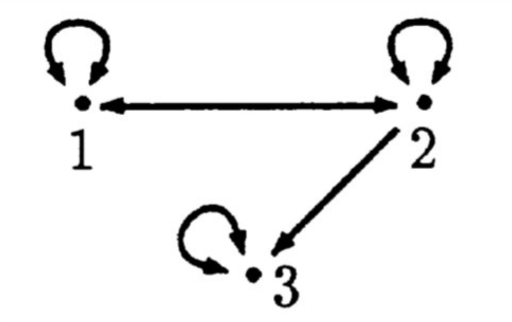
- **anti-symmetric** iff for all $x, y \in X$ if Rxy and Ryx, then $x = y$. Examples of **anti-symmetric** relations in $\{1,2,3\}$:

$\{\langle 2,3\rangle, \langle 1,1\rangle\}$

$\{\langle 1,1\rangle, \langle 2,2\rangle\}$

$\{\langle 1,2\rangle, \langle 2,3\rangle\}$

Exercise 06: Is the relation in the diagram symmetric, asymmetric, or anti-symmetric?



3.1.4.3. Transitivity

Binary relation $R \subseteq X \times X$ is

- *transitive* iff for all $x, y \in X$ if Rxy and Ryz, then also Rxz. Examples of **transitive** relations in $\{1,2,3\}$:

$\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$

$\{\langle 1,2\rangle, \langle 2,1\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$

$\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$

Exercise 07: Are the relations in $\{1,2,3\}$ below transitive?

$\{\langle 2,2\rangle\}$

$\{\langle 2,3\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$

$\{\langle 1,2\rangle, \langle 2,3\rangle\}$

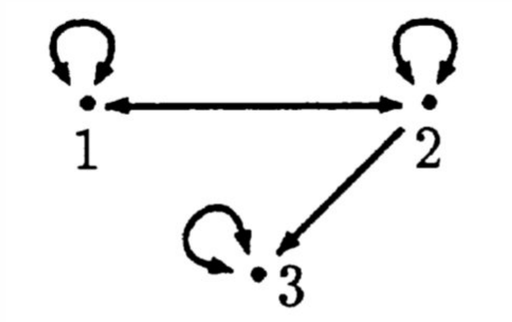
$\{\langle 2,3\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$

- *intransitive* iff for all $x, y \in X$ if Rxy and Ryz, then not Rxz. Examples of **intransitive** relations in $\{1,2,3\}$

$\{\langle 3,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle\}$

$\{\langle 3,2\rangle, \langle 1,3\rangle\}$

Exercise 08: Is the relation in the diagram transitive, nontransitive or intransitive?



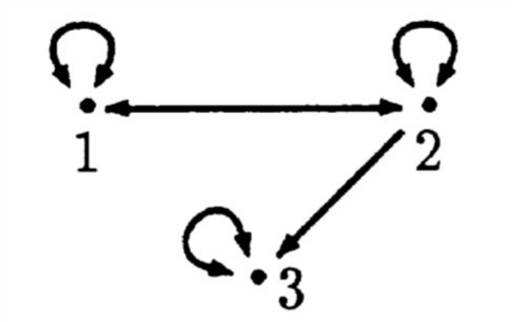
3.1.4.4. Connectedness

Binary relation $R \subseteq X \times X$ is

- connected iff for all $x, y \in X$ either Rxy or Ryx or $x = y$. Examples of **intransitive** relations in $\{1,2,3\}$:

$\{\langle 1,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$

Exercise 09: Is the relation in the diagram connected?



3.1.4.5 Summary

R (not $\emptyset$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric ($R^{-1} = R$)	symmetric
asymmetric	asymmetric	nonsymmetric
anti-symmetric	anti-symmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Example: R_f is the relation '***is father of***' in the set H of all human beings.

R_f is

- irreflexive (no one is his own father);
- asymmetric (if x is $y's$ father, then it is never true that y is x's father);
- intransitive (if x is y's father and y is z's father. then x is z's grandfather but not z's father);
- nonconnected (there are distinet iudividuals x and y in H such that neither 'x is the father of y' nor $'y$ is the father of x' is true).

Example: R is the relation '***greater than***' defined in the set $Z = \{1, 2, 3, 4, \ldots\}$ of all the positive integers. Z contains an infinite number of members and so does R, but we are able to determine the relevant properties of R from our knowledge of the properties of numbers in general.

R is

- irreflexive {no number is greater than itself);
- asymmetric (if $x > y$, then $y \not> x$;
- transitive (if x > y and y > z, then x > z),
- connected (for every distinct pair of integers x and y, either x > y or y > x.

Example: R_a is the relation defined by *'**x is the same age as y,**'* in the set H of all living human beings.

R_a is

- reflexive (everyone is the same age as himself or herself);
- symmetric (if x is the same age as y, then y is the same age as x);
- transitive (if x and yare the same age and so are y and z, then x is the same age as z);

- nonconnected (there are distinct individuals in H who are not of the same age).

3.2. Functions

A function is generally represented in set-theoretic terms as a special kind of relation. A relation R from A to B is a function if and only if it meets both of the following conditions:

1. Each element in the domain is paired with just one element in the range (One-to-one mapping)
2. The domain of R is equal to A.

Example of a function: $F = \{\langle x, y\rangle | y = 2x + 1\}$ (where x and y are integers)

- A function $F : A \to B$ is used for 「F is a function from A to B」
- A function $F : A \to A$ such that $F = \{\langle x, x\rangle | x \in A\}$ is called the *identity function*, written id_A
- the successor function: $f(x) = x + 1$ alt. $x \mapsto x + 1$
- the identity function: $f(x) = x$ alt. $x \mapsto x$

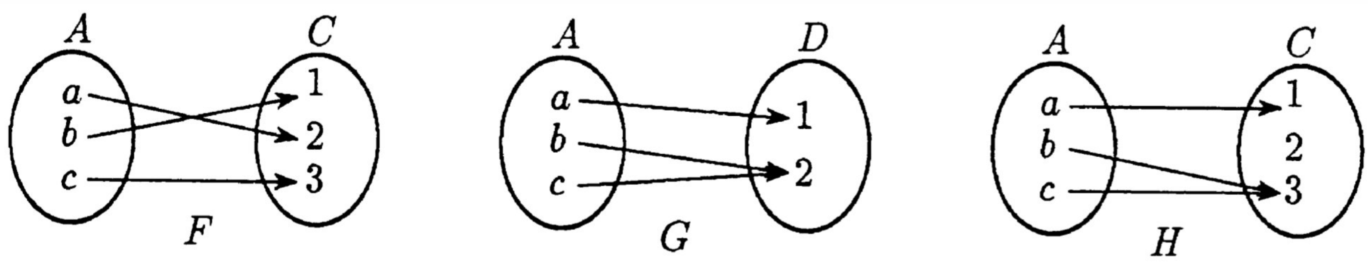
Exercise 10: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. The following relations from A to B are functions or not:

1. $P = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle\}$
2. $Q = \{\langle a, 3\rangle, \langle b, 4\rangle, \langle c, 1\rangle\}$
3. $R = \{\langle a, 3\rangle, \langle b, 2\rangle, \langle c, 2\rangle\}$
4. $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$
5. $T = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 3\rangle, \langle c, 1\rangle\}$
6. $V = \{\langle a, 2\rangle, \langle a, 3\rangle, \langle b, 4\rangle\}$

3.2.1. Properties of Functions

A function $f : X \to Y$ is

- *injective* iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$
- *surjective* iff for each $y \in Y$ there is an $x \in X$ with $f(x) = y$
- *bijective* iff f is injective and a surjective



- $|X| \leq |Y|$ iff there exists an injection $f : X \to Y$
- $|X| = |Y|$ iff there exists a bijection $f : X \to Y$
- $|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$ iff there exists injection $f : X \to Y$ but no surjection $g : X \to Y$

We can now prove that there are "different infinities". In particular, $|N| \not< |Q|$, but that $|N| < |R|$.

Exercise 11: Which of the following natural language expressions is such that its meaning could be captured by a function $f : X \to Y$ (rather than just a relation which is *not* a function)?

- a. x admires y
- b. x is the father of y
- c. x is the same person as y
- d. x self-identifies as gender y
