Coursework of "Convex Optimization"

1. Given the three points $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ of $\mathbb{R}^3$, provide the expressions of the *affine hull, convex hull, conic hull* of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

2. *Convexity of norm balls* Given set C defined as: let $p > 0$

$$C \overset{\text{def}}{=} \Big\{ \boldsymbol{x} \in \mathbb{R}^n \mid \big(\textstyle\sum_{i=1}^n |x_i|^p \big)^{1/p} \leq 1 \Big\}.$$

What is the condition required on p such that C is a convex set.

3. Show that if $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are convex sets in $\mathbb{R}^m \times \mathbb{R}^n$, then so is their partial sum

$$\boldsymbol{S} \overset{\text{def}}{=} \big\{ (\boldsymbol{x}, \boldsymbol{y}_1 + \boldsymbol{y}_2) \mid \boldsymbol{x} \in \mathbb{R}^m,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^n,\ (\boldsymbol{x}, \boldsymbol{y}_1) \in \boldsymbol{S}_1,\ (\boldsymbol{x}, \boldsymbol{y}_2) \in \boldsymbol{S}_2 \big\}.$$

4. Let C be a nonempty convex subset of $\mathbb{R}^n$ and let $\alpha, \beta \in \mathbb{R}_+$. Show that

$$\alpha C + \beta C = (\alpha + \beta) C$$

and that this property fails if C is not convex.

5. Given three sets $A, B, C \subset \mathbb{R}^n$, suppose they satisfy

$$A + C \subset B + C.$$

- Suppose A, B are convex, B is closed, and C is bounded, prove

$$A \subset B.$$

- Show that the result can fail if B is not convex.

6. *Relative interior*

- Find convex sets $A \subseteq B$, but $\mathrm{relint}(A) \nsubseteq \mathrm{relint}(B)$.
- Prove that for a point $\boldsymbol{x}$ in C, the following are equivalent
  - (i) $\boldsymbol{x} \in \mathrm{relint}(C)$.
  - (ii) For any point $\boldsymbol{y}$ in C there exists a real $\epsilon > 0$ with $\boldsymbol{x} + \epsilon(\boldsymbol{x} - \boldsymbol{y}) \in C$.
- Let $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and $C \subset \mathbb{R}^n$ is a convex set, prove that

$$\boldsymbol{A}\mathrm{relint}(C) \subset \mathrm{relint}(\boldsymbol{A}C).$$