# Course Work of "Convex Optimization"

**1.** (**Optional**) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex, and $a, b \in \mathrm{dom}(f)$ with $a < b$,

a). Show that

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

b). Show that

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

for all $x \in (a, b)$.

**2.** Provide examples of a function $f : \mathbb{R} \to \mathbb{R}$ and a non-empty set $C$ of $\mathbb{R}$ illustrating each of the following

a). $f$ is not convex, $C$ is convex, and $f$ is convex on $C$.

b). $f$ is not convex, $C$ is not convex, and $f$ is convex on $C$.

**3.** Let $f : \mathbb{R}^n \to ]-\infty, +\infty[$ be convex, denote $X = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = \inf_{\boldsymbol{u} \in \mathbb{R}^n} f(\boldsymbol{u})\}$ the set of global minimizers of $f$. Prove that $X$ is convex.

**4.** (*Arithmetic mean – geometric mean inequality*) Let $x_1, x_2, ..., x_m > 0$. Show that

$$\sqrt[m]{x_1 \cdots x_m} \le \frac{x_1 + x_2 + \cdots + x_m}{m},$$

and that equality occurs if and only if $x_1 = \cdots = x_m$.

**5.** The distance of a point $\boldsymbol{x}$ to a set $\boldsymbol{S} \subseteq \mathbb{R}^n$, in the norm $\|\cdot\|$, is defined as

$$\mathrm{dist}(\boldsymbol{x}, \boldsymbol{S}) \stackrel{\text{def}}{=} \inf_{\boldsymbol{y} \in \boldsymbol{S}} \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Show that the function is convex if the set $\boldsymbol{S}$ is convex. *This is Example 3.16 of the book (page 88), please prove the result NOT using the $\epsilon$-approach and the epigraph of the book*

**6.** (**Optional**) Let $f$ and $g$ be functions from $\mathbb{R}^n$ to $]-\infty, +\infty]$ and let $\theta \in (0, 1)$. Show that

$$\big(\theta f + (1-\theta) g\big)^* \le \theta f^* + (1-\theta) g^*.$$

**7.** Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be non-empty subsets of $\mathbb{R}^n$ such that $\boldsymbol{D}$ is closed and convex. Show that

$$\boldsymbol{C} \subset \boldsymbol{D} \quad \Longleftrightarrow \quad S_{\boldsymbol{C}}(\boldsymbol{x}) \le S_{\boldsymbol{D}}(\boldsymbol{x}).$$

**8.** Derive the conjugates of the following functions.

a). $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(\boldsymbol{x}) = \max_{i=1,\ldots,n} x_i$$

b). Let $p > 1$ and set $q = p/(p-1)$. Then

$$\left(\frac{|x|^p}{p}\right)^* = \frac{|x|^q}{q}.$$

c). (Optional) Let $f(x) = \begin{cases} 1/x & if\ x > 0 \\ +\infty & if\ x \le 0 \end{cases}$. Then

$$f^*(y) = \begin{cases} -2\sqrt{-y} & if\ y \le 0 \\ +\infty & if\ y > 0 \end{cases}$$

d). (Optional) *(negative Burg entropy)* Let $f(x) = \begin{cases} -\ln(x) & if\ x > 0 \\ +\infty & if\ x \le 0 \end{cases}$. Then

$$f^*(y) = \begin{cases} -\ln(-y) - 1 & if\ y < 0 \\ +\infty & if\ y \ge 0 \end{cases}$$

**9.** Derive the conjugates of the following functions

a). $f : \mathbb{R}^n \to \mathbb{R}$, $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \frac{\alpha}{2}\|\boldsymbol{x}\|^2$ where $\alpha > 0$.

b). (Optional) Let $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\lambda > 0$, $f(\boldsymbol{x}) = \lambda\|\boldsymbol{x}\|_p + \frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ where $p \ge 1$.