Course Work of “Convex Optimization”

1 Problems

1. Let function $f : \mathbb{R}^n \to \mathbb{R}$, $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$. Show that

$$\bar{\boldsymbol{x}} + \mathrm{argmin}_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}) = \mathrm{argmin}_{\boldsymbol{y}\in\Omega'} f(\boldsymbol{y})$$

where $\Omega' = \{\boldsymbol{y} \,|\, \boldsymbol{y} - \bar{\boldsymbol{x}} \in \Omega\}$.

2. Consider the following function

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1.$$

In what direction does the function f decreases most rapidly at the point $\boldsymbol{x}^{(0)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

3. Considering the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x - c)^2/2$ with $c \in \mathbb{R}$. We are interested in computing the minimizer of f using the iterative algorithm

$$x^{(k+1)} = x^{(k)} - \alpha_k f'(x^{(k)}),$$

where f' is the derivative of f at $x^{(k)}$ and α_k is a step-size satisfying $0 < \alpha_k < 1$.

a). Derive a formula relating $f(x^{(k+1)})$ with $f(x^{(k)})$, involving α_k.

b). Show that the algorithm is globally convergence if and only if

$$\sum_{k=0}^{\infty} \alpha_k = +\infty.$$

Hint: use part a) and the fact that for any sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset [0, 1]$, we have

$$\prod_{k=0}^{\infty} (1 - \alpha_k) = 0 \iff \sum_{k=0}^{\infty} \alpha_k = +\infty.$$

4. Given $f : \mathbb{R}^n \to \mathbb{R}$, consider the general iterative scheme

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \gamma_k \boldsymbol{d}^{(k)}$$

where $\boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}, \ldots$ are given vectors in $\mathbb{R}^n$ and γ_k is chosen to minimize $f(\boldsymbol{x}^{(k)} + \gamma \boldsymbol{d}^{(k)})$; that is

$$\gamma_k = \mathrm{argmin}_{\gamma\geq 0} f(\boldsymbol{x}^{(k)} + \gamma \boldsymbol{d}^{(k)}).$$

Show that for each k, the vector $\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}$ is orthogonal to $\nabla f(\boldsymbol{x}^{(k+1)})$ (assuming that the gradient exists).

5. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-c)^4$ with $c \in \mathbb{R}$. Suppose we apply Newton's method to solve the problem of minimizing f.

a). Write down the update equation of the Newton's method.

b). Let $y^{(k)} = |x^{(k)} - c|$, where $x^{(k)}$ is the k'th iterate in Newton's method. Show that the sequence $\{y^{(k)}\}_{k\in\mathbb{N}}$ satisfies $y^{(k+1)} = \frac{2}{3}y^{(k)}$.

c). Show that $x^{(k)} \to c$ for any initial point $x^{(0)} \in \mathbb{R}$.

6. Consider the modified Newton's method

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \gamma_k \nabla^2 f^{-1}(\boldsymbol{x}^{(k)}) \nabla f(\boldsymbol{x}^{(k)}),$$

Where $\gamma_k = \operatorname{argmin}_{\gamma \geq 0} f\big(\boldsymbol{x}^{(k)} - \gamma \nabla^2 f^{-1}(\boldsymbol{x}^{(k)}) \nabla f(\boldsymbol{x}^{(k)})\big)$. Suppose that we apply the algorithm to a quadratic function $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{b}$ where $\boldsymbol{Q}$ is symmetric positive definite. Recall that the standard Newton's method reaches point $\boldsymbol{x}^\star$ such that $\nabla f(\boldsymbol{x}^\star) = \boldsymbol{0}$ in just one step starting from any initial point $\boldsymbol{x}^{(0)}$. Does the modified Newton's method above possess the same property?

2 Optional problems

1. Let function $f : \mathbb{R}^n \to \mathbb{R}$ be proper convex and real-valued (*i.e.* $\mathrm{dom}(f) = \mathbb{R}^n$). Suppose that the set of optimal points X_{opt} of f is non-empty, show that X_{opt} is convex.

2. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be non-empty subsets of $\mathbb{R}^n$ with intersection being non-empty. Provide at least 3 equivalent formulations of the feasibility problem for finding a common point of $\boldsymbol{C}$ and $\boldsymbol{D}$.

3. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \;\; x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

a). $f_0(x_1, x_2) = x_1 + x_2$.
b). $f_0(x_1, x_2) = -x_1 - x_2$.
c). $f_0(x_1, x_2) = x_1$.
d). $f_0(x_1, x_2) = \max\{x_1,\, x_2\}$.
e). $f_0(x_1, x_2) = x_1^2 + 4x_2^2$.

4. (*Linear programs*) Give an explicit solution of each of the following LPs

a). Minimizing a linear function over a halfspace

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}^T\boldsymbol{x} \geq b. \end{array}$$

where $\boldsymbol{a} \neq \boldsymbol{0}$.

b). Minimizing a linear function over a rectangle

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{l} \preceq \boldsymbol{x} \preceq \boldsymbol{u} \end{array}$$

where $\boldsymbol{l}$ and $\boldsymbol{u}$ satisfy $\boldsymbol{l} \preceq \boldsymbol{u}$.