零点问题

1. 零点问题

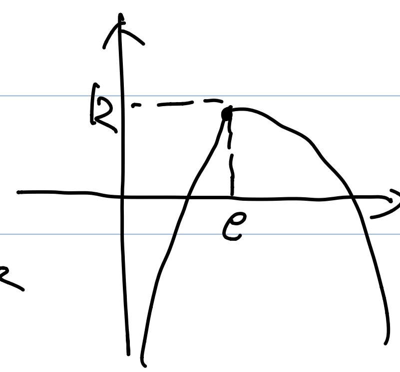
2. 单调性

eg7.4 $k>0$, $f(x)=\ln x-\frac{x}{e}+k$ $(0,+\infty)$

[分析] $f(x)=\ln x-\frac{x}{e}+k$

$f'(x)=\frac{1}{x}-\frac{1}{e}$ (导数不含参)

$\frac{1}{x}-\frac{1}{e}=0 \Rightarrow x=e$



$0<x<e$, $f'(x)>0$, $f(x)\uparrow$

$f(e)=k$

$x>e$, $f'(x)<0$, $f(x)\downarrow$

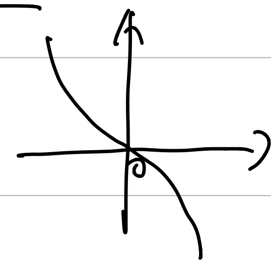
$\lim\limits_{x\to 0^+}(\ln x-\frac{x}{e}+k)=-\infty$

$\lim\limits_{x\to +\infty}(\ln x-\frac{x}{e}+k)=-\infty$

eg7.7 $k\arctan x - x = 0$

$f(x) = k\arctan x - x$ 奇

$f'(x) = k\cdot\frac{1}{1+x^2} - 1 = \frac{k-1-x^2}{1+x^2}$

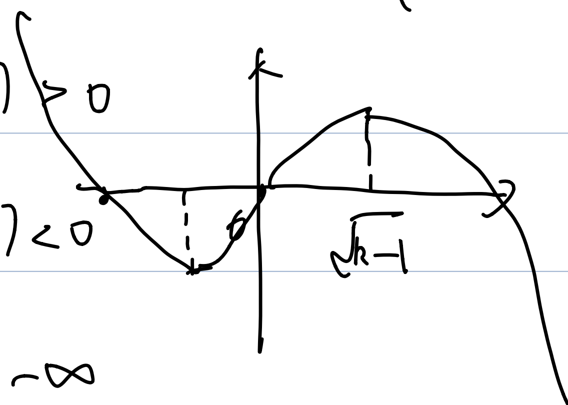


① $k-1\le 0$，即 $k\le 1$ 时，$f'(x)<0$

恰有一个实根

② $k-1>0$，即 $k>1$，令 $f'(x)=0 \Rightarrow x=\sqrt{k-1}$

$$\begin{cases} 0<x<\sqrt{k-1}, & f'(x)>0 \\ x>\sqrt{k-1}, & f'(x)<0 \end{cases}$$



$\lim_{x\to+\infty}(k\arctan x - x) = -\infty$

3. 罗尔原话（罗尔定理推论）

若 $f^{(n)}(x)=0$ 至多有 k 个根，则 $f(x)=0$ 至多有 $k+n$

个根

4. 实系数奇次方程至少有一个实根

eg 7.3 $3a^2-5b<0$, $x^5+2ax^3+3bx+4c=0$

$f(x)=x^5+2ax^3+3bx+4c$

$f'(x)=5x^4+6ax^2+3b$

$x^2=y \Rightarrow 5y^2+6ay+3b \Rightarrow$ 无实根

二、微分不等式

讲义 P101 - 102

eg 7.8 $x>0$, $\ln(1+\frac{1}{x}) < \frac{1}{\sqrt{x(x+1)}}$

【分析】 $\ln(1+\frac{1}{x}) < \frac{1}{\sqrt{x(x+1)}} \xLeftrightarrow{\frac{1}{x}=t} \ln(1+t) < \frac{t}{\sqrt{1+t}}$

$\Leftrightarrow \sqrt{1+t}\ln(1+t) - t < 0$

令 $f(t) = \sqrt{1+t}\ln(1+t) - t$

则 $f'(t) = \frac{1}{2\sqrt{1+t}}\ln(1+t) + \frac{\sqrt{1+t}}{1+t} - 1$

$= \frac{\ln(1+t) + 2 - 2\sqrt{1+t}}{2\sqrt{1+t}}$

令 $g(t) = \ln(1+t) + 2 - 2\sqrt{1+t}$

$g'(t) = \frac{1}{1+t} - \frac{1}{\sqrt{1+t}} < 0$

$\Rightarrow g(t)\downarrow \Rightarrow g(t) < g(0) = 0$

$\Rightarrow f'(t) < 0 \Rightarrow f(t)\downarrow$

$f(t) < f(0) = 0$

eg7.9 $0 < x < \frac{\pi}{2}$, $\sin x > \frac{2x}{\pi}$

[分析] $f(x) = \sin x - \frac{2x}{\pi} \quad x \in (0, \frac{\pi}{2})$

$f'(x) = \cos x - \frac{2}{\pi}$

$f''(x) = -\sin x < 0$

$f(0) = 0$, $f(\frac{\pi}{2}) = 0$

eg7.10 $1 + x\ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2}$

令 $f(x) = 1 + x\ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$

$f'(x) = \ln(x + \sqrt{1+x^2}) + x \cdot \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} \cdot x$

$= \ln(x + \sqrt{1+x^2}) \overset{令}{=} 0 \Rightarrow x = 0$

且 $f'(x)=\frac{1}{\sqrt{1+x^2}}>0$

$\Rightarrow f(x)\geq f(0)=0$

常数变量化

eg7.11 $0<a<b$, $\ln\frac{b}{a}>2\frac{b-a}{a+b}$

[分析] $\ln\frac{b}{a}>2\cdot\frac{\frac{b}{a}-1}{1+\frac{b}{a}}$ $\quad x=\frac{b}{a}>1$

中值定理证明不等式

eg7.12 $0<a<b<1$, $\arctan b-\arctan a<\frac{b-a}{2ab}$

令 $f(x)=\arctan x$

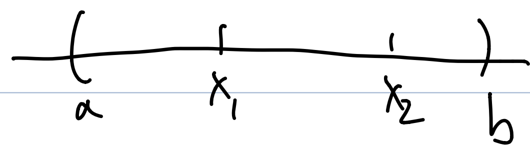
$\arctan b-\arctan a=\frac{1}{1+\xi^2}(b-a)$, $a<\xi<b$

其中 $\frac{1}{1+b^2}<\frac{1}{1+\xi^2}<\frac{1}{1+a^2}$

$$\Rightarrow \arctan b - \arctan a < \frac{b-a}{1+a^2} < \frac{b-a}{b^2+a^2} < \frac{b-a}{2ab}$$

eg7.14 题目 讲义 P105

[分析] f 与 $f^{(n)}$ $(n \geq 2)$ 的关系 $\Rightarrow$ 泰勒



$$\lambda x_1 + (1-\lambda) x_2 = x$$

$$f(x_1) = f(x) + f'(x)(x_1 - x) + \frac{f''(\xi_1)}{2}(x_1 - x)^2$$

$$f(x_2) = f(x) + f'(x)(x_2 - x) + \frac{f''(\xi_2)}{2}(x_2 - x)^2$$

$$\Rightarrow \lambda f(x_1) = \lambda f(x) + f'(x)\lambda(x_1 - x) + \frac{f''(\xi_1)}{2}(x_1 - x)^2 \cdot \lambda$$

$$\Rightarrow (1-\lambda) f(x_2) = (1-\lambda) f(x) + f'(x)(1-\lambda)(x_2 - x)$$

$$+ \frac{f''(\xi_2)}{2}(x_2 - x)^2 (1-\lambda)$$