6.1 $f(x)$在(a,b)内可导，x_1和x_2是(a,b)内任两点，且$x_1<x_2$，至少存在一点ξ，s.t.

C

$$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)\quad(x_1<\xi<x_2)$$

6.2 $f(x)$在$[a,b]$上连续，(a,b)内可导，$f(a)=f(b)=0$，证明：

(1) $\exists\,\xi\in(a,b)$，使$f(\xi)+\xi f'(\xi)=0$

(2) $\exists\,\eta\in(a,b)$，使$\eta f(\eta)+f'(\eta)=0$

(1) 证明：令$g(x)=xf(x)$

$$g(a)=af(a)=0\qquad g(b)=bf(b)=0$$

由罗尔定理，$g'(x)=f(x)+xf'(x)$

$\therefore\ \exists\,\xi\in(a,b)$，s.t. $g'(\xi)=f(\xi)+\xi f'(\xi)=0$

(2) 证明：设 $h(x) = f(x)e^{\frac{x^2}{2}}$

$$h'(x) = e^{\frac{x^2}{2}}\left(xf(x) + f'(x)\right)$$

$h(a) = f(a)e^{\frac{a^2}{2}} = 0$ $h(b) = f(b)e^{\frac{b^2}{2}} = 0$

由罗尔定理，$\exists\ \xi \in (a,b)$，s.t. $e^{\frac{\xi^2}{2}}(\xi f(\xi) + f'(\xi)) = 0$

即 $\xi f(\xi) + f'(\xi) = 0$

6.3 设不恒为常数的函数 $f(x)$ 在 $[a,b]$ 上连续，在 (a,b) 内可导，且 $f(a) = f(b)$，证明：在 (a,b) 内至少存在一点 ξ，s.t. $f'(\xi) > 0$

证明：由罗尔定理，$\exists\ \eta \in (a,b)$，s.t. $f'(\eta) = 0$

$\therefore$ η 为极值点，不妨设 η 为极大值点

$\because$ $f(x)$ 不恒为常数

$\therefore$ 在 $(\eta-\delta,\eta)$ 邻域内必有 ξ, s.t. $f'(\xi)>0$

$\because$ $f(a)=f(b)$ 且 $f(x)$ 不恒为常数，至少存在一点 $c\in(a,b)$,

s.t. $f(c)>f(a)$

由拉格朗日 $\Rightarrow f'(\xi)=\frac{1}{c-a}[f(c)-f(a)]>0$

6.4 设函数 $f(x)$ 在 $[0,1]$ 上连续，在 $(0,1)$ 内可导，

且 $f(0)=0$, $f(1)=1$，证明：

(1) $\exists\,\xi\in(0,1)$. s.t. $f(\xi)=\frac{1}{2}$

(2) 存在不同两点 $x_1,x_2\in(0,1)$，$\frac{1}{f'(x_1)}+\frac{1}{f'(x_2)}=2$

(1) 证明：由介值定理可得，$\exists\,\xi\in(0,1)$, s.t. $f(\xi)=\frac{1}{2}$

(2) 证明：由(1)知，$\exists\,\xi\in(0,1)$, s.t. $f(\xi)=\frac{1}{2}$

由拉氏定理：

$\frac{1}{2} = f'(x_1)\xi$ $\qquad$ $\frac{1}{2} = f'(x_2)(1-\xi)$

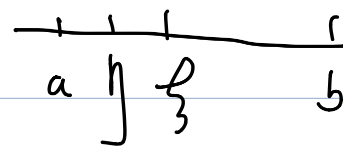
$\frac{1}{f'(x_1)} = 2\xi$ $\qquad$ $\frac{1}{f'(x_2)} = 2(1-\xi)$

∴存在两个不同的点 x_1, x_2, s.t. $\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2$

6.5 设函数 $f(x)$ 在 $[a,b]$ 上连续 $(a, b>0)$，在 (a,b) 内可导且 $f(a)\neq f(b)$，证明：存在 $\xi, \eta \in (a,b)$，s.t.

$$\frac{f'(\xi)}{2\xi} = \frac{f'(\eta)}{b+a}$$

证明：$f'(\xi)(b-a) = f(b) - f(a)$



$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{f'(\xi)}{2\xi} = \frac{f(b)-f(a)}{2\xi(b-a)}$$

$$f'(\eta)(\xi-a) = f(\xi) - f(a)$$

$$f'(\eta) = \frac{f(\xi)-f(a)}{\xi-a}$$

$$\frac{f'(\eta)}{b+a} = \frac{f(\xi)-f(a)}{(\xi-a)(b+a)}$$

对$f(x)$在$[a,b]$上用拉格朗日中值定理,

$$f(b)-f(a)=f'(\eta)(b-a), \quad \eta\in(a,b)$$

对$f(x)$, x^2在$[a,b]$上用柯西中值定理知

$$\frac{f(b)-f(a)}{b^2-a^2} = \frac{f'(\xi)}{2\xi}$$

$$\therefore f(b)-f(a) = \frac{f'(\xi)}{2\xi}(b^2-a^2)$$,则

$$f'(\eta)(b-a) = \frac{f'(\xi)}{2\xi}(b^2-a^2)$$

即 $$\frac{f'(\xi)}{2\xi} = \frac{f'(\eta)}{b+a}$$

6.6 设$f(x)$在$[0,1]$上二阶可导，且$f(0)=f(1)=0$，$\min\limits_{x\in(0,1)}\{f(x)\}=-1$，证明：存在$\xi\in(0,1)$, s.t. $f''(\xi)\geq 8$

由泰勒公式, $f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2}x^2 \geq -1$

$$\frac{f''(\xi)}{2}x^2 \geq -1 - f'(0)x$$

$$f''(\xi) \geq \frac{-2 - 2f'(0)x}{x^2}$$

$$f''(\xi) \geq -\frac{2}{x^2} - \frac{2f'(0)}{x}$$

令 $g(x) = -\frac{2}{x^2} - \frac{2f'(0)}{x}$

$$g'(x) = \frac{4}{x^3} + \frac{2f'(0)}{x^2} \quad \because f'(0)$$

令 $g'(x) = 0$ $\quad \frac{2f'(0)}{x^2} = -\frac{4}{x^3}$

$$2f'(0)x = -4 \qquad -\frac{2}{f'(0)}$$

$$x = -\frac{2}{f'(0)}$$

① $-\frac{2}{f'(0)} > 1 \Rightarrow -2 < f'(0)$

$$f''(\xi) \geq -2 - 2f'(0) \ngtr 2$$

由于$f(x)$在$[0,1]$上连续，因此存在$x_0\in[0,1]$

s.t. $f(x_0)=\min\limits_{x\in[0,1]}\{f(x)\}=-1$ $\because f(0)=f(1)=0>f(x_0)$

$\therefore x_0\in(0,1)$, $f'(x_0)=0$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\eta)}{2!}(x-x_0)^2$$

$$=-1+\frac{1}{2}f''(x-x_0)^2$$

$x=0$, $f(0)=-\frac{1}{2}+\frac{1}{2}f''(\xi_1)x_0^2$

$$f''(\xi_1)=\frac{2}{x_0^2}$$

$x=1$, $f(1)=-1+\frac{1}{2}f''(\xi_2)(1-x_0)^2$

$$f''(\xi_2)=\frac{2}{(1-x_0)^2}$$

记$f''(\xi)=\max\{f''(\xi_1),f''(\xi_2)\}$

$\exists\ \xi\in(0,1)$, s.t. $f''(\xi)=2\max\{\frac{1}{x_0^2},\frac{1}{(1-x_0)^2}\}\geq 2\frac{1}{(\frac{1}{2})^2}=8$

在x_0点展开没想到