1. 用定积分表达和计算平面图形的面积

① 直角系下 直接算　　③ 极坐标系下

② 参数下 直接算 / 换元法 ☆

eg 9.2 (换元法) ☆

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \sin t) \end{cases}$$

[分析] 参数方程下的问题是重点:

$$S = \int_0^{2\pi a} f(x)\,dx \quad (\text{直角系下})$$

$$= \int_0^{2\pi} y(t)\,dx(t)$$

$$= \int_0^{2\pi} a(1-\cos t)\,a(1-\cos t)\,dt$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$$= a^2\left(2\pi - 0 + 4\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 3\pi a^2$$

eg 9.3 (极坐标)

2. 用定积分表达和计算旋转体的体积

(P148 − 149)

eg 9.8

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

$$V_x = \int_0^{2\pi a} \pi y^2(x)\, dx$$

$$= \int_0^{2\pi} \pi y^2(t)\, x'(t)\, dt$$

3.用定积分表达和计算函数的平均值 (P149)

积分等式与积分不等式

一、积分等式

eg 10.1 (推广的积分中值定理)

1.积中: $\int_a^b f(x)dx = f(\xi)(b-a)$

推广: $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$

证明: 令 $\int_a^x f(t)g(t)dx = F(x)$

$$\int_a^x g(t)dt = G(x)$$

不妨设 $g(x)>0$, 对 $F(x)$、$G(x)$ 在 $[a,b]$ 上用柯西中值定理 $\Rightarrow$

$$\frac{F(b)-F(a)}{G(b)-G(a)}=\frac{F'(\xi)}{G'(\xi)}$$,即

$$\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}=\frac{f(\xi)g(\xi)}{g(\xi)} \Rightarrow$$ 证毕

eg、设 $f(x)$ 在 $(1,2)$ 上连续,则计算 $\lim\limits_{n\to\infty}\int_1^2 f(x)e^{-x^n}dx$

[分析] $\int_1^2 f(x)e^{-x^n}dx = f(\xi_1)\int_1^2 e^{-x^n}dx = f(\xi_1)\cdot e^{\xi_2^n}$

其中 $f(\xi_1)$ 有界, $\xi_1\in(1,2)$

$e^{-\xi_2^n}$, $\xi_2\in(1,2)$

故原式 $=\lim\limits_{n\to\infty} f(\xi_1)\cdot e^{-\xi_2^n} = 0$ $\quad \xi_2^n \to +\infty$

2、夹逼准则

eg 10.3

$$\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+x}\,dx$$

$$0 \le \frac{x^n}{1+x} \le x^n$$

$$\underset{0}{\overset{\shortparallel}{0}} \qquad \underset{0}{\shortparallel} \qquad \int_0^1 x^n\,dx = \frac{1}{n+1} \longrightarrow 0$$

eg 10.4 $u_n = \int_0^1 |\ln t|[\ln(1+t)]^n\,dt \ (n=1,2,\cdots)$, $\lim_{n\to\infty} u_n$

(1) $\ln(1+t) \le t$

$$\Rightarrow \int_0^1 |\ln t|[\ln(1+t)]^n\,dt \le \int_0^1 |\ln t|\,t^n\,dt$$

(2) $$\int_0^1 |\ln t|\,t^n\,dt = -\int_0^1 \ln t\,d\frac{t^{n+1}}{n+1}$$

$$= -\frac{t^{n+1}}{n+1}\ln t\Big|_0^1 + \int_0^1 \frac{t^{n+1}}{n+1}\cdot\frac{1}{t}\,dt$$

$$= \frac{1}{n+1}\int_0^1 t^n\,dt = \frac{1}{(n+1)^2}$$

$$0 < u_n < \frac{1}{(n+1)^2} \longrightarrow 0$$

$$\shortparallel$$

$$0$$

3. 用积分法

eg 10.5 (区间再现公式) $f(x)=f(-x)$

[分析] (1) $\int_0^{nT} xf(x)dx \xlongequal{x=0+nT-t} \int_0^{nT}(nT-x)f(nT-x)dx$

$$= nT\int_0^{nT} f(x)dx - \int_0^{nT} xf(x)dx \Rightarrow$$

$$\int_0^{nT} xf(x)dx = \frac{nT}{2}\int_0^{nT} f(x)dx = \frac{n^2T}{2}\int_0^{T} f(x)dx$$

(2) $I=\int_0^{n\pi} x|\sin x|dx = \frac{n^2\pi}{2}\int_0^{\pi}\sin x\,dx = n^2\pi$

二、积分不等式 (2022 考题 12')

1. 用函数的单调性

$$\int_a^b \longrightarrow \int_a^x$$

eg 10.6 (P159)

eg 10.7 (P159-160)

(1) $0 \le \int_a^x g(t)\,dt \le \int_a^x dt = x-a$

(2) [分析] $\int_a^{a+\int_a^b g(t)dt} f(x)\,dx \le \int_a^b f(x)g(x)\,dx$

令 $F(x) = \int_a^{a+\int_a^b g(t)dt} f(u)\,du - \int_a^x f(u)\,g(u)\,du$

$F'(x) = f[a+\int_a^x g(t)dt] \cdot g(x) - f(x)g(x)$

$= \{f[\underbrace{a+\int_a^x g(t)dt}_{\le 0}] - f(x)\}\underbrace{g(x)}_{\ge 0} \le 0$

$F(x) \downarrow$

2. 拉格朗日中值定理

eg 10.8

[分析] 见到 $f, f' \longrightarrow$ 拉氏中值定理

$[0,x] \Rightarrow f(x)-f(0) = f'(\xi_1)x \quad 0<\xi_1<x$

$[x,1] \Rightarrow f(1)-f(x) = f'(\xi_2)(1-x), \quad x<\xi_2<1$

即 $\begin{cases} |f(x)| = |f'(\xi_1)|x, \quad 0<\xi_1<x & \le Mx \\ |f(x)| = |f'(\xi_2)|(1-x), \quad x<\xi_2<1 & \le M(1-x) \end{cases}$

$|f'(x)| \le M$

$$\Rightarrow \left|\int_0^1 f(x)\,dx\right| = \left|\int_0^x f(t)dt + \int_x^1 f(t)dt\right|$$

$$\le \left|\int_0^x f(t)dt\right| + \left|\int_x^1 f(t)dt\right|$$

$$\le \int_0^x |f(t)|\,dt + \int_x^1 |f(t)|\,dt \le \int_0^x Mt\,dt + \int_x^1 M(1-t)\,dt$$

$$= M\cdot\frac{x^2}{2} + M\cdot\frac{(1-x)^2}{2}$$

$$= M\cdot\left[\frac{x^2}{2} + \frac{(1-x)^2}{2}\right] \le \frac{1}{4}\max\{|f'(x)|\}$$

$$\sqrt{\frac{x^2+(1-x)^2}{2}} \ge \frac{x+1-x}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{x^2+(1-x)^2}{2} \ge \frac{1}{4}$$

3. 泰勒公式 (2022)

eg 10.9

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(\xi)}{2}(x-1)^2 \qquad \xi \in (x,1)$$

$$\int_0^2 f(x)\,dx = f'(1)\underbrace{\int_0^2 (x-1)\,dx}_{=0} + \frac{1}{2}\int_0^2 f''(\xi)(x-1)^2\,dx$$

$$\le \frac{1}{2}\int_0^2 \overbrace{|f''(\xi)|}^{\le M}(x-1)^2\,dx$$

$$\le M \cdot \frac{1}{2}\int_0^2 (x-1)^2\,dx = \frac{1}{3}M$$

4. 积分法

eg 10.10 (P160)