10.1 设$f(x)$, $g(x)$在$[a,b]$上连续，证明至少存在一点$\xi\in(a,b)$, s.t. $f(\xi)\int_\xi^b g(x)dx=g(\xi)\int_a^\xi f(x)dx$

证明: $G(x)=f(x)\int_x^b g(t)dt-g(x)\int_a^x f(t)dt$

$F(x)=\int_a^x f(t)dt\cdot\int_x^b g(t)dt$

$\because F(a)=F(b)\Rightarrow$ 罗尔定理 $\Rightarrow F'(\xi)=0$

10.2 设$\varphi(x)$是可微函数$f(x)$的反函数，且$f(1)=0$,

证明: $\int_0^1\left[\int_0^{f(x)}\varphi(t)dt\right]dx=2\int_0^1 xf(x)dx$

证明: 分部积分法 $\cdot f'(x)dx$

$\int_0^1\left[\int_0^{f(x)}\varphi(t)dt\right]dx=x\int_0^{f(x)}\varphi(t)dt\Big|_0^1-\int_0^1 x\underbrace{\varphi[f(x)]}_{x}$

$=\underline{\int_0^{f(1)}\varphi(t)dt}^{\,=0}-\int_0^1 x^2f'(x)dx$

$=-\int_0^1 x^2df(x)=-x^2f(x)\Big|_0^1+2\int_0^1 xf(x)dx$

$$=2\int_0^1 xf(x)dx$$

10.3 设$f(x)$在$[a,b]$上连续且严格单调增加，证明

$$(a+b)\int_a^b f(x)dx < 2\int_a^b xf(x)dx$$

证明：设$F(x)=(a+x)\int_a^x f(t)dt-2\int_a^x tf(t)dt$

$$F'(x)=\int_a^x f(t)dt+(a+x)f(x)-2xf(x)$$

$$=\int_a^x f(t)dt+(a-x)f(x)$$

$$F''(x)=f(x)+(-f(x)+(a-x)f'(x))$$

$$=(a-x)f'(x)<0$$

$\therefore F'(x)$在$[a,b]$上$\downarrow$

$\because F'(a)=0 \quad \therefore F'(x)\leqslant 0$

$\therefore F(x)$在$[a,b]$上$\downarrow$

$\therefore F(a)=0$

$\therefore F(x)<0$ 在 $[a,b]$ 成立

$\therefore F(b)<0 \longrightarrow$ 证毕

10.4 设 $f(x)$ 在 $[0,1]$ 上连续且单调减少，证明：

当 $0<\lambda<1$ 时，$\int_0^\lambda f(x)dx \geq \lambda\int_0^1 f(x)dx$

证明：$\dfrac{\int_0^\lambda f(x)dx}{\lambda} \geq \dfrac{\int_0^1 f(x)dx}{1}$

设 $F(x)=\dfrac{\int_0^x f(t)dt}{x}$

$F'(x)=\dfrac{f(x)\cdot x-\int_0^x f(t)dt}{x^2}$

设 $G(x)=xf(x)-\int_0^x f(t)dt$

$G'(x)=f(x)+xf'(x)-f(x)=xf'(x)<0$

$\therefore G(x)$ 在 $[0,1]$ 上 $\downarrow$

$\because G(0)=0$

$\therefore G(x)<0$ 在$(0,1)$上恒成立

$\therefore F'(x)<0$

$\because F(0)=0$

$\therefore F(x) \geqslant F(1)$

10.5 证明：$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+x^2} dx \geqslant \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+x^2} dx$

证明：$\sin(x+\frac{\pi}{2}) = \cos x$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+x^2} dx \geqslant 0$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{1+x^2} dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+x^2} dx$$

$$= \frac{1}{1+\xi^2} \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \frac{1}{1+\eta^2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) dx$$

$$= (\sqrt{2}-1)\left(\frac{1}{1+\xi^2} - \frac{1}{1+\eta^2}\right) > 0$$