3.1 求极限 $\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{x^2}}}{x^{100}}$

解：$\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{x^2}}}{x^{100}} \xlongequal{令t=\frac{1}{x^2}} \lim\limits_{t\to +\infty}\dfrac{e^{-t}}{t^{50}}$

$=\lim\limits_{t\to+\infty}\dfrac{t^{50}}{e^{t}}=\lim\limits_{t\to+\infty}\dfrac{50t^{49}}{e^{t}}=\cdots=0$

3.2 已知 $I=\lim\limits_{x\to 0}\left(\dfrac{\ln(1+e^{\frac{2}{x}})}{\ln(1+e^{\frac{1}{x}})}+a[x]\right)$ 存在，$[\cdot]$ 为取整函数，求 I, a

解：$\lim\limits_{x\to 0^+}\dfrac{\ln(1+e^{\frac{2}{x}})}{\ln(1+e^{\frac{1}{x}})} \xlongequal{令t=\frac{1}{x}} \lim\limits_{t\to+\infty}\dfrac{\ln(1+e^{2t})}{\ln(1+e^{t})}$

$=\lim\limits_{t\to+\infty}\dfrac{\frac{2e^{2t}}{e^{2t}+1}}{\frac{e^{t}}{e^{t}+1}}=\lim\limits_{t\to+\infty}\dfrac{2e^{2t}}{e^{2t}+1}\times\dfrac{e^{t}+1}{e^{t}}$

$$=\lim_{t\to+\infty}\frac{2e^t(e^t+1)}{e^{2t}+1}=\lim_{t\to+\infty}\frac{2e^{2t}+2e^t}{e^{2t}+1}=2$$

$$\lim_{t\to-\infty}\frac{2e^{2t}+2e^t}{e^{2t}+1}=0$$

$$\because \lim_{x\to0^+}a[x]=0 \qquad 2=0-a$$

$$\lim_{x\to0^-}a[x]=-a \qquad a=-2$$

$$I=2$$

3.3 已知 $a>0$, $b>0$, 则 $\lim\limits_{x\to+\infty}x(a^{\frac{1}{x}}-b^{\frac{1}{x}})$

解: $$\lim_{x\to+\infty}\frac{a^{\frac{1}{x}}-b^{\frac{1}{x}}}{\frac{1}{x}}\overset{令t=\frac{1}{x}}{=\!=\!=}\lim_{t\to0^+}\frac{a^t-b^t}{t} \qquad \frac{0}{0}$$

$$\lim_{t\to0^+}(a^t\ln a-b^t\ln b)' \overset{?}{=} \ln\frac{a}{b}$$

~~3.4~~ 求极限 $\lim\limits_{x\to0}\left(\dfrac{e^x+xe^x}{e^x-1}-\dfrac{1}{x}\right)$ $\quad \infty-\infty$

$$e^x\sim 1+x+\frac{x^2}{2}+o(x^2)$$

解: $\lim\limits_{x\to0}\dfrac{(e^x+xe^x)x-(e^x-1)}{x(e^x-1)}$ $\quad \dfrac{0}{0}$

$$=\lim_{x\to0}\frac{xe^x(x+1)-(e^x-1)}{x(e^x-1)}=\lim_{x\to0}\frac{xe^x(x+1)-e^x+1}{x^2}$$

$$=\lim_{x\to0}\frac{(x^2e^x+xe^x)'-e^x}{2x}=\lim_{x\to0}\frac{(2xe^x+x^2e^x+e^x+xe^x)-e^x}{2x}$$

$$=\lim_{x\to0}\frac{x^2e^x+3xe^x}{2x}=\lim_{x\to0}\frac{3e^x+3xe^x+2xe^x+x^2e^x}{2}$$

$$=\frac{3}{2}$$

3.5 设 $a\neq\frac{1}{2}$，计算 $\lim\limits_{n\to\infty}\ln\left[\frac{n-2na+1}{n(1-2a)}\right]^n$

解：$\lim\limits_{n\to\infty}\ln\left[\frac{n-2an+1}{n-2an}\right]^n$ $\quad 1^\infty$

$\xlongequal{令 x=\frac{1}{n}} \lim\limits_{x\to 0}\ln\left[\frac{\frac{1-2a}{x}+1}{\frac{1-2a}{x}}\right]^{\frac{1}{x}} = \lim\limits_{x\to 0}\ln\left[\frac{1-2a+x}{x}\cdot\frac{x}{1-2a}\right]^{\frac{1}{x}}$

$= \lim\limits_{x\to 0}\ln\left[\frac{1-2a+x}{1-2a}\right]^{\frac{1}{x}} = \ln e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\frac{1-2a+x}{1-2a}}$ $\quad \ln(1+x)\sim x$

$= \ln e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot\frac{x}{1-2a}} = \ln e^{\frac{1}{1-2a}} = \frac{1}{1-2a}$

3.6 求极限 $\lim\limits_{x\to 0^+}\frac{x^x-(\sin x)^x}{x^2\ln(1+x)}$ $\quad \frac{0}{0}$ $\quad \ln(1+x)\sim x$

$= \lim\limits_{x\to 0^+}\frac{x^x\left(1-\left(\frac{\sin x}{x}\right)^x\right)}{x^3} = \lim\limits_{x\to 0^+}\frac{x^x\left(\left(\frac{\sin x}{x}\right)^x-1\right)}{x^3}$

$\lim\limits_{x\to 0^+}\frac{e^{x\ln\frac{\sin x}{x}}-1}{x^3}$ $\quad \ln\frac{\sin x}{x}\sim\left(\frac{\sin x-x}{x}\right)$

$$\lim_{x\to 0^+} \frac{e^{\sin x - x} - 1}{x^3} = \lim_{x\to 0^+} \frac{e^{-\frac{x^3}{6}} - 1}{x^3} = -\frac{1}{6}$$

$$-\lim_{x\to 0^+} \frac{1}{6} x^x = \frac{1}{6}$$

3.7 求极限 $\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots a_n^x}{n} \right)^{\frac{n}{x}}$，其中 $a_i > 0$，

$i = 1, 2, \cdots, n$ $\quad 1^\infty$

$$\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{\frac{n}{x}}$$

$$= e^{\lim_{x\to 0} \frac{n}{x} \cdot \ln \frac{a_1^x + a_2^x + \cdots + a_n^x}{n}}$$

$$= e^{\lim_{x\to 0} \frac{n}{x} \cdot \frac{(a_1^x - 1) + (a_2^x - 1) + \cdots + (a_n^x - 1)}{n}}$$

$$= e^{\lim_{x\to 0} \frac{(a_1^x - 1) + (a_2^x - 1) + \cdots + (a_n^x - 1)}{x}}$$

$$= e^{\lim\limits_{x \to 0} \frac{\sum_{i=1}^{n} a_i^x - 1}{x}}$$

$$\lim_{x \to 0} \frac{\sum_{i=1}^{n} a_i^x - 1}{x}$$

$$\lim_{x \to 0} \frac{a_1^x + a_2^x + \cdots + a_n^x - n}{x} = \ln(a_1 a_2 \cdots a_n) \ ?$$

$\therefore$ 原式 $= a_1 a_2 \cdots a_n$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{e^{x\ln a} - 1}{x} \qquad e^x \sim 1 + x$$

$$= \lim_{x \to 0} \frac{x \ln a}{x} = \ln a$$

3.8 设 $\lim\limits_{x \to 0} \frac{\ln\left[1 + \frac{f(x)}{\sin x}\right]}{2^x - 1} = 2$，求 $\lim\limits_{x \to 0} \frac{f(x)}{x^2}$

解：$\lim\limits_{x \to 0} \frac{\ln\left[1 + \frac{f(x)}{\sin x}\right]}{2^x - 1} = 2 \qquad \frac{0}{0}$ 同阶无穷小

$\ln(1+x) \sim x$

$$\lim_{x \to 0} \frac{\frac{f(x)}{\sin x}}{2^x - 1} = 2$$

$$\lim_{x \to 0} 2^x - 1 = e^{x\ln 2} - 1 \sim x\ln 2$$

$$= \lim_{x \to 0} \frac{f(x)}{\ln 2\, x^2} = 2$$

$$\lim_{x \to 0} f(x) = 2\ln 2\, x^2$$

$$\lim_{x \to 0} \frac{f(x)}{x^2} = 2\ln 2$$

3.9 设 $f(x)$ 在 $(-\infty, +\infty)$ 内有定义，且 $\lim_{x \to \infty} f(x) = a$，$g(x) = \begin{cases} f(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(A) (B) $x = 0$

$\lim_{x\to 0^-} g(x) = f(-\infty)$ $\lim_{x\to 0^+} g(x) = f(+\infty)$

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3.10 设函数 $f(x) = \lim_{n\to\infty} \frac{1+x}{1+x^{2n}}$ 讨论函数间断点

$f(x) = \lim_{n\to\infty} \frac{1+x}{1+x^{2n}}$

$|x|<1$, $f(x) = 1+x$

$|x|>1$, $\lim_{n\to\infty} \frac{1+x}{1+x^{2n}} = 0$

$f(1) = 1$, $f(-1) = 0$

$$f(x) = \lim_{n\to\infty} \frac{1+x}{1+x^{2n}} = \begin{cases} 0, & x \le -1 \\ 1+x, & -1<x<1 \\ 1, & x=1 \\ 0, & x>1 \end{cases}$$

$x=1$ 间断点 B