2.1 设 $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 1$, 则

B

2.2 设 $x_n = \frac{1}{3} + \frac{1}{15} + \cdots + \frac{1}{4n^2-1}$, $n = 1, 2, \cdots$

则 $\lim_{n\to\infty} x_n$

$$x_n = \sum_{i=1}^{n} \frac{1}{4n^2-1} = \sum_{i=1}^{n} \frac{1}{(2n-1)(2n+1)} = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

$$\lim_{n\to\infty} x_n = \frac{1}{2} \lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$$

2.3 极限 $\lim_{n\to\infty}\left(\frac{n+1}{n-2}\right)^n$

解: $\lim_{n\to\infty}\left(\frac{n+1}{n-2}\right)^n = \lim_{n\to\infty} e^{\ln\frac{n+1}{n-2}} = e^{\lim_{n\to\infty}\ln\frac{n+1}{n-2}} = 1$

$\therefore \lim_{n\to\infty}\left(\frac{n+1}{n-2}\right)^n = 1$

$$\lim_{n\to\infty}\left(\frac{n+1}{n-2}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{2}{n}\right)^n} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n} = \frac{e}{e^{-2}}$$

e^3

$= e$

2.4 求极限 $\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}}\right)$

解：$\sqrt{n+\sqrt{n}}-\sqrt{n-\sqrt{n}} = \dfrac{2\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}}$

$$\lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n-\sqrt{n}}} = 1$$

2.5 设数列 $\{x_n\}$ 满足 $x_n>0$，且 $\lim_{n\to\infty}\dfrac{x_{n+1}}{x_n}=\dfrac{1}{2}$，则

$\lim_{n\to\infty} x_n = 0$ A

2.6 设 $\lim_{n\to\infty}\dfrac{n^{99}}{n^k-(n-1)^k}$ 存在且不为零，则常数 k

$k=100$

2.7 求$\lim\limits_{n\to\infty}(1+2^n+3^n)^{\frac{1}{n}}$

解：$\lim\limits_{n\to\infty}(1+2^n+3^n)^{\frac{1}{n}} = e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln 1+2^n+3^n}$

$= e^{\ln 3} = 3$

2.8 求极限$\lim\limits_{n\to\infty}\sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}$

解 $\lim\limits_{n\to\infty}\sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} = \lim\limits_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n})^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\sum\limits_{i=1}^{n}\frac{1}{i}}$

$\sum\limits_{i=1}^{n}\frac{1}{n} < \sum\limits_{i=1}^{n}\frac{1}{i} < \sum\limits_{i=1}^{n}1$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\sum\limits_{i=1}^{1}\frac{1}{n}} = 1 \qquad e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln\sum\limits_{i=1}^{1}1} = 1$

$\therefore \lim\limits_{n\to\infty}\sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} = 1$

2.9 设$x_{n+1}=\sqrt{2+x_n}$ $(n=1,2,\cdots)$，$x_1=\sqrt{2}$，

证明$\lim\limits_{n\to\infty}x_n$存在，并求$\lim\limits_{n\to\infty}x_n$

$$x_{n+1}-x_n=\sqrt{2+x_n}-\sqrt{2+x_{n-1}}$$

$$=\frac{x_n-x_{n-1}}{\sqrt{2+x_n}+\sqrt{2+x_{n-1}}}$$

$\because x_1=\sqrt{2}$ $\quad x_2=\sqrt{2+\sqrt{2}}>x_1$

$\therefore\{x_n\}$单调递增

由于$x_1=\sqrt{2}<2$，$x_2=\sqrt{2+x_1}<\sqrt{2+2}=2$

设$x_k<2$，则$x_{k+1}=\sqrt{2+x_k}<\sqrt{2+2}=2$

$\therefore\{x_n\}$有上界

$\therefore\{x_n\}$单调且有上界，$\lim\limits_{n\to\infty}x_n$存在

设$\lim\limits_{n\to\infty}x_n=A$

$A=\sqrt{2+A}$ $\quad A^2=A+2$

$A^2-A-2=0$ $\quad A=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm3}{2}$

由保号性，$A=2$

$\therefore \lim_{n\to\infty} x_n=2$

2.10 设$x_1=25$，$x_{n+1}=\arctan x_n (n=1,2,\cdots)$

(1) 证明数列$\{x_n\}$有极限，并求此极限；

(2) 求$\lim_{n\to\infty}\frac{x_n-x_{n+1}}{x_n^3}$.

(1) 证明：$\because x_1=25$

$x_2=\arctan 25<x_1$

$x_{n+1}-x_n=\arctan x_n-x_n<0$

$\therefore \{x_n\}$单减

$\because x_n = \arctan x_{n-1} > 0$

$\therefore \{x_n\}$ 单调有界

设 $\lim\limits_{n\to\infty} x_n = A$ $\quad \therefore \lim\limits_{n\to\infty} x_n = 0$

$A = \arctan A$

$A = 0$

(2) 解: $\lim\limits_{n\to\infty} \frac{x_n - x_{n+1}}{x_n^3} = \lim\limits_{n\to\infty} \frac{x_n - \arctan x_n}{x_n^3}$

$\underline{\underline{x_n = t}} \lim\limits_{t\to 0} \frac{t - \arctan t}{t^3} = \lim\limits_{t\to 0} \frac{1 - \frac{1}{1+t^2}}{3t^2}$

$= \lim\limits_{t\to 0^+} \frac{t^2}{3t^2(1+t^2)} = \frac{1}{3}$